Hyperbolic Differential Equations

by

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Introduction

Since Hadamard gave his Yale lectures in 1920 about the hyperbolic differential equations of second order, many important papers about the equations of any order have been published by Herglotz, Schauder, Petrowsky, Bureau, M. Riesz, and Gårding. They used various but interesting methods and their results are important but incomplete.

The first part of our lectures is related to the equations with constant coefficients; it goes beyond Petrowsky's and Gårding's results, and it improves their methods. Hadamard and Bureau used the Green's formula; by means of a duality it transforms a boundary value problem into the problem of finding a particular solution with a given singularity; this transformed problem is easy and this particular solution is handy and important when the given problem is very simple; but generally this method is a difficult one. Herglotz, Petrowsky and Gårding use another duality: that between the independent variables and the derivations, which gives rise to the Fourier and Laplace transformation. More precisely Herglotz and Petrowsky applied the Heaviside calculus, that is to say the Laplace transformation <u>to one</u> variable (as a matter of fact to the time) and the Fourier transformation <u>to the others</u> (as a matter of fact to the space); how shocking in a relativistic world! It is not astonishing that Gårding obtained more complete results by applying the Laplace transformation at once to all the variables. But he did not express all the results this transformation gives; for instance: he uses the director cones Γ of the convex domains Δ, without studying these important convex domains Δ; he defines some operators by means of the Laplace transformation and the others by means of the Riesz's analytical prolongation, whereas it is convenient to define and to study these operators all together by means of the Laplace transformation.

We do not transform any boundary problem; but by the Laplace transformation Chapter I defines and studies throughout the symbolic calculus of several variables; this calculus enables us to solve the Cauchy's boundary value problem for differential equations (and would also enable one to solve equations containing both derivatives and finite differences): see Chap. VII, §4, no. 107-108-109 and Chap. VIII, no. 113.

Chapter II gives in particular a new, general and concise expression of the inverse of $\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \dots - \frac{\partial^2}{\partial x^2}$ and of its powers.

Chapter III studies the inverse of any polynomial of $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x}$.

Chapter IV, assuming that this polynomial is homogeneous, achieves Herglotz-Petrowsky's calculation of its elementary solution.

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CHAPTER I

THE SYMBOLIC CALCULUS

§1. Fourier and Laplace transforms

(This whole §1 is classical: see Bochner's books [13] and [14].)

1. Fourier transforms. (See the bibliography: [13], [15], [21], [22], [23], [25].) Let X and Ξ be two vector spaces of the same finite dimension ℓ over the field of real numbers. Suppose they are dual: there is given a bilinear function

$$x \cdot \eta \quad \text{of } x \in X \text{ and } \eta \in \Xi,$$

the values of which are real and which satisfies: if $x \cdot \eta = 0$ for some x and all η, then $x = 0$; if $x \cdot \eta = 0$ for some η and all x, then $\eta = 0$. (See: Bourbaki, Algèbre linéaire.)

The coordinates $x_1, \dots x_\ell$ of x and $\eta_1 \dots \eta_\ell$ of η are real numbers and will be so chosen that

$$x \cdot \eta = x_1 \eta_1 + \dots + x_\ell \eta_\ell .$$

Let f(x), g(x) be functions with complex values, defined on X; let $\phi(\eta)$, $\psi(\eta)$ be functions with complex values, defined on Ξ; f(x)g(x) is the product of f(x) and g(x); $f(x)*g(x) = \int_X \dots \int f(x-y)g(y)dy_1 \dots dy_\ell$ is the convolution of f(x) and g(x). We use the norms

$$\|f(x)\|_q = \left[\int_X \dots \int |f(x)|^q dx_1 \dots dx_\ell\right]^{\frac{1}{q}} \text{ for } q = 1 \text{ or } q = 2$$

and

$$\|f(x)\|_\infty = \operatorname*{Sup.}_X |f(x)|.$$

We then have

$$(1.1)\qquad (\|f\|_2)^2 \leq \|f\|_1 \cdot \|f\|_\infty .$$

It is easy to prove the following:

If $\|f\|_1 < +\infty$ and $\|g\|_1 < +\infty$, then $\|f*g\|_1 \leq \|f\|_1 \|g\|_1 < +\infty$; and

If $\|f\|_1 < +\infty$ and $\|g\|_2 < +\infty$, then $\|f*g\|_2 \leq \|f\|_1 \|g\|_2 < +\infty$.

Thus, by use of the convolution, the f(x) such that $\|f\|_1 < +\infty$ constitute a ring and the f(x) such that $\|f\|_2 < +\infty$ constitute a vector space over this ring.

If $\|f(x)\|_1 < +\infty$, then its Fourier transform $\mathcal{F}[f(x)]$ is the continuous and bounded function $\phi(\eta)$, which is given by the formula [exp. λ = e^{λ}]:

$$(1.2)\qquad \mathcal{F}[f(x)] = \phi(\eta) = \int \cdots \int_X f(x)\, \exp.(-2\pi i\, x \cdot \eta)\, dx_1 \cdots dx_\ell .$$

It can be proved that, if $\|f\|_1$ and $\|g\|_1 < +\infty$, then $\mathcal{F}[g*f] = \mathcal{F}[g] \cdot \mathcal{F}[f]$, afterwards that $\|f(x)\|_2 = \|\phi(\eta)\|_2$ if $\phi = \mathcal{F}[f]$: this enables one to define $\mathcal{F}[f]$ whenever $\|f\|_2 < +\infty$. Planchérel proved the following regarding this extension:

$\mathcal{F}[f]$ is an isometric linear mapping of the Hilbert space of all functions f(x) such that $\|f\|_2 < +\infty$ onto the Hilbert space of all functions $\phi(\eta)$ such that $\|\phi(\eta)\|_2 < +\infty$ (thus it is a unitary mapping.) This mapping is given by (1.2) whenever both $\|f\|_1$ and $\|f\|_2 < +\infty$; its inverse is given by

$$(1.3)\qquad \mathcal{F}^{-1}[\phi(\eta)] = f(x) = \int \cdots \int_{\Xi} \phi(\eta)\, \exp.(2\pi i\, x \cdot \eta)\, d\eta_1 \cdots d\eta_\ell$$

whenever both $\|\phi\|_1$ and $\|\phi\|_2 < +\infty$.

We have also the following formulas:

(1.4) $\mathcal{F}[g * f] = \mathcal{F}[g]\mathcal{F}[f]$ if $\|g\|_1 < +\infty$ and $\|f\|_1$ or $\|f\|_2 < +\infty$;

likewise

(1.5) $\mathcal{F}[gf] = \mathcal{F}[g] * \mathcal{F}[f]$ if $\|g\|_2$ and $\|f\|_2 < +\infty$;

further, if $\mathcal{F}[f] = \phi$, then:

(1.6) $\mathcal{F}[f(Sx)] = \phi(\Sigma\eta)|\det.\Sigma|$ ($\det.\Sigma$ = determinant of Σ)

for any couple S, Σ of contragredient linear mappings of X and Ξ (that is to say: $x \cdot \eta = Sx \cdot \Sigma\eta$ for any $x \in X$ and $\eta \in \Xi$)

(1.7) $\mathcal{F}[f(x+y)] = \phi(\eta)\ \exp.\ (2\pi i y \cdot \eta)$ for any $y \in X$;

(1.8) $\mathcal{F}[f(x)\ \exp.\ (2\pi i x \cdot \xi)] = \phi(\eta - \xi)$ for any $\xi \in \Xi$;

(1.9) $$\mathcal{F}[x_1 f(x)] = \frac{i}{2\pi}\frac{\partial\phi}{\partial\eta_1};$$

(1.10) $$\mathcal{F}\left[\frac{\partial f}{\partial x_1}\right] = 2\pi i\eta_1\phi(\eta).$$

Remark 1.1. There is an easy extension of formula (1.5):

(1.11) If $\mathcal{F}[f(x, x')] = \phi(\eta, \eta')$, where x and $x' \in X$, η and $\eta' \in \Xi$,

then $\mathcal{F}[f(x, x)] = \int\!\dots\!\int_{\Xi} \phi(\eta - \eta', \eta')d\eta'_1 \dots d\eta'_\ell$.

(extension to "tensor prod").

Remark 1.2. The formula (1.10) shows that the Fourier transformation reduces the solution of a differential equation with constant coefficients to division by a polynomial ... if the solution of the differential equation has a Fourier transform, which happens rarely; therefore it is necessary to use the closely related Laplace transforms.

2. Laplace transformations. The Laplace transform of $f(x)$ is the function of $\zeta = \xi + i\eta$ ($\xi \in \Xi$, $\eta \in \Xi$, $i = \sqrt{-1}$; $\zeta_1 = \xi_1 + i\eta_1$, $\dots$, $\zeta_\ell = \xi_\ell + i\eta_\ell$)

(2.1) $$\mathcal{L}[f(x)] = \phi(\zeta) = \mathcal{F}[f(x)\ \exp.\ (-2\pi x\cdot\xi)].$$

If $\|f(x)\ \exp.\ (-2\pi x\cdot\xi)\|_1 < +\infty$ for some ξ, then

(2.2) $$\mathcal{L}[f] = \int_X\!\cdots\!\int f(x)\ \exp.\ (-2\pi x\cdot\zeta)\ dx_1 \ldots dx_\ell\ ;$$

$\mathcal{L}[f]$ is also defined if $\|f(x)\ \exp.\ (-2\pi x\cdot\xi)\|_2 < +\infty$ for some ξ. If $\|\phi(\xi + i\eta)\|_2 < +\infty$ for some fixed ξ, then $\mathcal{L}^{-1}[\phi]$ exists [but it could depend on ξ: see n°4]; if further $\|\phi(\xi + i\eta)\|_1 < +\infty$, then

(2.3) $$\mathcal{L}^{-1}[\phi] = \int_\Xi\!\cdots\!\int \phi(\xi + i\eta)\ \exp.\ [2\pi x\cdot(\xi + i\eta)]d\eta_1 \ldots d\eta_\ell.$$

The formulas of n°1 give the following ones upon application of (2.1):

(2.4) $\mathcal{L}[g * f] = \mathcal{L}[g]\mathcal{L}[f]$ if for some ξ, $\|g(x)\ \exp.\ (-2\pi x\cdot\xi)\|_1 < +\infty$ and $\|f(x)\ \exp.\ (-2\pi x\cdot\xi)\|_1$ or $\|f(x)\ \exp.\ (-2\pi x\cdot\xi)\|_2 < +\infty$;

(2.5) $\mathcal{L}[gf] = \mathcal{L}[g] * \mathcal{L}[f]$, if for some ξ and $\xi' \in \Xi$,

$\|g(x)\ \exp.\ [-2\pi x\cdot(\xi - \xi')]\|_2 < +\infty$, $\|f(x)\ \exp.\ (-2\pi x\cdot\xi')\|_2 < +\infty$;

$\phi(\zeta) * \psi(\zeta)$ means $\int_\Xi\!\cdots\!\int \phi(\xi + i\eta - \xi' - i\eta')\psi(\xi' + i\eta')d\eta'_1 \ldots d\eta'_\ell$;

(2.6) $$\mathcal{L}[f(Sx)] = \phi(\Sigma\zeta)|\det.\ \Sigma|$$

for any couple S, Σ of contragredient linear mappings of X and Ξ;

(2.7) $$\mathcal{L}[f(x + y)] = \phi(\zeta)\ \exp.\ (2\pi y\cdot\zeta)\ \text{for any } y \in X;$$

(2.8) $\mathcal{L}[f(x)\ \exp.\ (2\pi x\cdot\zeta')] = \phi(\zeta - \zeta')$ if $\zeta' \in \Xi + i\,\Xi$;

(2.9) $$\mathcal{L}[x_1 f(x)] = -\frac{1}{2\pi}\frac{\partial\phi(\zeta)}{\partial\zeta_1}$$

$$(2.10) \qquad \mathcal{L}\left[\frac{\partial f}{\partial x_1}\right] = 2\pi\, \zeta_1 \phi(\zeta).$$

Remark 2.1. The second part of these lectures will use the extension of formula (2.5) which follows from (1.11):

(2.11) If $\mathcal{L}[f(x,x')] = \phi(\zeta, \zeta')$, where x and $x' \in X$, ζ and $\zeta' \in \Xi + i\Xi$,

$$\text{then } \mathcal{L}[f(x,x)] = \int_{\Xi}\cdots\int \phi(\zeta - \zeta', \zeta')\, d\eta'_1 \ldots d\eta'_{\ell}.$$

Note. n°4 studies $\mathcal{L}^{-1}$, using the definitions given in n°3.

3. *The ring of distributions* $E(\Delta)$ *and the subring of functions* $F(\Delta)$.

Proposition 3.1. *$\log \| f(x) \exp.(-x \cdot \xi)\|_q$ is a convex function of ξ; therefore the set on which it is finite is convex.*

Proof. Let ξ and η be two points of Ξ; let μ and ν be two positive numbers such that $\mu + \nu = 1$; the classical Hölder's inequality (see [13], ch. III, L_p-spaces, §5)

$$(3.1) \qquad \| f \cdot g \|_q \leq \| f \|_{q/\mu} \, \| g \|_{q/\nu}$$

gives

$$\| f(x) \exp.(-x \cdot \mu\xi - x \cdot \nu\eta)\|_q =$$

$$\| \, |f(x) \exp.(-x \cdot \xi)|^{\mu} \, |f(x) \exp.(-x \cdot \eta)|^{\nu} \, \|_q \leq$$

$$\| \, |f(x) \exp.(-x \cdot \xi)|^{\mu} \|_{q/\mu} \; \| \, |f(x) \exp.(-x \cdot \eta)|^{\nu} \|_{q/\nu} =$$

$$[\| f(x) \exp.(-x \cdot \xi)\|_q]^{\mu} \cdot [\| f(x) \exp.(-x \cdot \eta)\|_q]^{\nu}$$

Definition of Δ. Besides X and Ξ *a convex domain* Δ *of* Ξ *is given.* [A domain is an open and connected set].

Definition of $F(\Delta)$. $F(\Delta)$ *is the set of functions* $f(x)$ *defined* on X and such that $\| f(x) \exp.(- \xi \cdot x) \|_2 < +\infty$ for any $\xi \in \Delta$.

Remark 3.1. The proposition 3.1 proves that, if Δ had not been supposed convex, then $F(\Delta)$ would not change by replacing Δ by its convex closure.

Remark 3.2. This proposition 3.1 proves also that $\| f(x) \exp.(- x \cdot \xi) \|_2$ is uniformly bounded in Δ (that means: on any compact subset of Δ).

Lemma 3.1. If $f(x) \in F(\Delta)$ and $\xi \in \Delta$, then $\| f(x) \exp.(- x \cdot \xi) \|_1 < +\infty$ (it is uniformly bounded in Δ).

Proof. It is sufficient to prove that $\| f(x) \|_1$ is finite for $0 \in \Delta$ and $f(x) = 0$ outside the domain $x_1 > 0, x_2 > 0, \ldots x_\ell > 0$. Let $h(x)$ be the function equal to 1 in this domain and 0 outside; (3.1) gives $(\| f(x) \|_1)^2 \leq \| f(x) \exp.[\varepsilon(x_1 + \ldots + x_\ell)] \|_2 \cdot \| h(x) \exp.[-\varepsilon(x_1 + \ldots + x_\ell)] \|_2$ where ε is a positive number, so small that $(-\varepsilon, \ldots -\varepsilon) \in \Gamma$.

This lemma and the formula (n°1) $\| f*g \|_2 \leq \| f \|_1 \| g \|_2$ prove that $f*g \in F(\Delta)$ if f and $g \in F(\Delta)$; thus

Proposition 3.2. $F(\Delta)$ is, by use of convolution, a ring.

Now let us use the Schwartz's theorie of distributions [21], which makes it possible to derive any function: the derivatives so obtained are functions or distributions.

Definition of $E(\Delta)$. The functions $f(x) \in F(\Delta)$ have derivatives of all orders; their convolutions are derivatives of functions of $F(\Delta)$, since $F(\Delta)$ is a ring. Therefore the finite sums of derivatives of elements of $F(\Delta)$ constitute a ring $E(\Delta)$; (the product to be used in this ring is the convolution). In other words: $E(\Delta)$ is the smallest ring which contains $F(\Delta)$ and is stable for the derivation.

4. The ring of analytic functions $\mathcal{L}[E]$ and its ideal A. Now we look

at the space $\Xi + i\,\Xi$: it is a vector space on the ring of complex numbers; its points are the points

$$\zeta = \xi + i\eta \quad (\xi = \text{real part of } \zeta;\ \eta = \text{imaginary part of } \zeta),$$

where $\xi \in \Xi$, $\eta \in \Xi$. The <u>tube with basis</u> Δ is the ~~set of~~ points $\zeta = \xi + i\eta$ such that $\xi \in \Delta$ (Bochner; see [14], ch. V, § 4, p. 90). Let the function $a(\zeta)$ be analytic in this tube; we denote

$$\|a(\xi + i\eta)\|_q = [\int\limits_{\Xi}\!\dots\!\int |a(\xi + i\eta)|^q \, d\eta_1 \dots d\eta_\ell]^{\frac{1}{q}};$$

$$\|a(\xi + i\eta)\|_\infty = \operatorname*{Sup.}_{\eta \in \Xi} |a(\xi + i\eta)|.$$

The Cauchy's formula

$$a(\zeta) = \frac{1}{(2\pi)^\ell} \int_0^{2\pi}\!\dots\!\int_0^{2\pi}$$

$$a[\zeta_1 + \rho_1 \exp.(i\phi_1), \dots \zeta_\ell + \rho_\ell \exp.(i\phi_\ell)]\, d\phi_1 \dots d\phi_\ell$$

enables us to express $a(\zeta)$ by an integral on a neighborhood of ζ, which proves this:

<u>Lemma</u> 4.1. If $\|a(\xi + i\eta)\|_q$ is bounded on any compact subset of Δ, then $\|a(\xi + i\eta)\|_\infty$ has the same property.

The Plancherel's theorem (n°1) has the following consequence [see the same proposition and a similar proof by [14], ch. VI, § 8, p. 128].

<u>Proposition</u> 4.1. <u>$\mathcal{L}$ maps $F(\Delta)$ one-one on the set $\Phi(\frac{1}{2\pi}\Delta)$ of the functions $\phi(\zeta)$ which are analytic in the tube with basis $\frac{1}{2\pi}\Delta$ and are such that $\|\phi(\xi + i\eta)\|_2$ is uniformly bounded in $\frac{1}{2\pi}\Delta$.</u>

<u>Moreover</u>

(4.1) $\| f(x) \exp.(-2\pi x \cdot \xi) \|_2 = \| \mathcal{L}[f] \|_2$.

<u>Note.</u> $\frac{1}{2\pi}\Delta$ is the set of the $\frac{1}{2\pi}\delta$, where $\delta \in \Delta$.

<u>Proof.</u> If $f(x) \in F(\Delta)$ and $\phi(\zeta) = \mathcal{L}[f]$, then we have, since $\mathcal{F}$ is an unitary mapping:

$$\| \phi(\xi + i\eta) \|_2 = \| f(x) \exp.(-2\pi x \cdot \xi) \|_2;$$

Therefore (see remark 3.2) $\phi \in \Phi(\frac{1}{2\pi}\Delta)$.

Conversely let $\phi \in \Phi(\frac{1}{2\pi}\Delta)$ and

$$f_\xi(x) = \mathcal{F}^{-1}[\phi(\xi + i\eta)] \exp.(2\pi x \cdot \xi);$$

let us prove that $f_\xi(x)$ does not depend on the choice of ξ in $\frac{1}{2\pi}\Delta$; because of the continuity of $\mathcal{F}^{-1}$, it is sufficient to prove this assertion after replacing $\phi(\zeta)$ by $\phi(\zeta)\exp.(\varepsilon\zeta^2)$ (where $\varepsilon > 0$), that is (see lemma 4.1) when f_ξ is given by (2.3):

$$f_\xi(x) = i^{-\ell} \int_{\xi + i\Xi} \cdots \int \phi(\zeta) \exp.(2\pi x \cdot \zeta)\, d\zeta_1 \ldots d\zeta_\ell;$$

a Cauchy's theorem asserts that this integral is independent of ξ.

<u>Proposition 4.2.</u> <u>$\mathcal{L}[E(\Delta)]$ is the ring whose elements are finite sums of products of polynomials of ζ by elements of $\Phi(\frac{1}{2\pi}\Delta) =$ $\mathcal{L}[F(\Delta)]$.</u>

<u>Proof.</u> $\mathcal{F}$ and therefore $\mathcal{L}$ are defined on $E(\Delta)$ (see Schwartz); we apply the definition of $E(\Delta)$ (see n°3) and the formula (2.10).

<u>Definitions.</u> $B(\Delta)$ is the ring of the functions $b(\zeta)$ which are analytic and regular in the tube with basis Δ and are such that

$\| b(\xi + i\eta) \|_\infty$ is uniformly bounded in Δ. And $A(\Delta)$ is the ring of the functions $a(\zeta)$ which are finite sums of products of polynomials of ζ by elements of $B(\Delta)$.

The Lemma 4.1 and the Proposition 4.1 prove that $\mathcal{L}[F(\Delta)]$ is an ideal of $B(\frac{1}{2\pi}\Delta)$; therefore, using Proposition 4.2:

Proposition 4.3. $\mathcal{L}[E(\Delta)]$ is an ideal of $A(\frac{1}{2\pi}\Delta)$.

This is the proposition which gives rise to the symbolic calculus.

Note. If Δ had not been supposed convex, then $\Phi(\frac{1}{2\pi}\Delta) = \mathcal{L}[F(\Delta)]$ would not change by replacing Δ by its convex closure (obvious by the remark 3.1).

Note. Let $\phi(\zeta) \in \Phi(\frac{1}{2\pi}\Delta)$;

$$\log \| \phi(\xi + i\eta) \|_2 = \log \| f(x) \exp. (-2\pi x \cdot \xi) \|_2$$

is a convex function of $\xi \in \frac{1}{2\pi}\Delta$ (see Proposition 3.1); let q be an integer > 0; $\phi^q(\zeta) \in \Phi(\frac{1}{2\pi}\Delta)$ (see Lemma 4.1); therefore

$$\log \| \phi(\xi + i\eta) \|_{2q} = \frac{1}{q} \log \| \phi^q(\xi + i\eta) \|_2$$

is also a convex function of $\xi \in \frac{1}{2\pi}\Delta$; but $\|\phi\|_q \longrightarrow \|\phi\|_\infty$ if $q \longrightarrow +\infty$; thus $\log \| \phi(\xi + i\eta) \|_\infty$ is also a convex function of $\xi \in \frac{1}{2\pi}\Delta$.

These two notes are sufficient for the following; but they are particular cases of two important convexity theorems, which deserve to be quoted (see: [14], ch. IV, p. 64, ch. V, § 4, p. 92; [18]).

A Bochner's theorem. If $a(\zeta)$ is analytic in the tube with basis Δ, it is analytic in the tube whose basis is the convex closure of Δ; it assumes the same values in both tubes.

A Hardy's theorem. $\log \| a(\xi + i\eta) \|_q$ is a convex function of ξ if $a(\zeta)$ is analytic.

More generally see M. Riesz's convexity theorem, Thorin's thesis and Lelong's theory of subharmonic functions.

§2. Definition and main properties of the symbolic calculus

(The extension of the symbolic calculus, which Heaviside had only defined for one variable [6].)

5. Definition of the symbolic calculus. Let f_x be a distribution and $a(\zeta)$ an analytic function such that

$$f_x \in E(\Delta), \; a(\zeta) \in A(\Delta);$$

the proposition 4.3 proves that

$$(5.1) \qquad a(p) \cdot f_x = \mathcal{L}^{-1}[a(2\pi\zeta)\mathcal{L}[f_x]]$$

is a distribution belonging to $E(\Delta)$; this distribution is said to be the symbolic product of f_x by $a(p)$.

Note. It often happens that $a(\zeta) \in A(\Delta_1)$ and $\in A(\Delta_2)$, without $\in A(\Delta_3)$ for some $\Delta_3 \supset \Delta_1 \cup \Delta_2$; then $a(p) \cdot f_x$ depends on the choice of Δ_α $(\alpha = 1,2)$ and it is necessary to be precise and say

$$(5.2) \qquad a(p) \cdot f_x \text{ for } p \in \Delta_1.$$

6. Properties of the symbolic calculus.

Theorem 6.1.

$$(6.1) \qquad a(p) \cdot f_x \text{ is bilinear}$$

$$(6.2) \qquad p_1 \cdot f_x = \frac{\partial}{\partial x_1} f_x$$

(6.3) $a(p) \cdot [f_x * g_x] = [a(p) \cdot f_x] * g_x = f_x * [a(p) \cdot g_x]$

(6.4) $[a(p)b(p)] \cdot f_x = a(p) \cdot [b(p) \cdot f_x] = b(p) \cdot [a(p) \cdot f_x]$

<u>Let</u> S <u>and</u> Σ <u>be contragredient isomorphisms of</u> X <u>and</u> Ξ; <u>let</u> y <u>be a fixed point of</u> X; <u>if</u> $a(p) \cdot f(x) = g(x)$ <u>or more generally if</u> $a(p) \cdot f_x = g_x$, <u>then</u>

(6.5) $a(\Sigma p) \cdot f(Sx) = g(Sx)$ <u>or</u> $a(\Sigma p) \cdot f_{Sx} = g_{Sx}$;

(6.6) $a(p) \cdot f(x + y) = g(x + y)$ <u>or</u> $a(p) \cdot f_{x+y} = g_{x+y}$.

<u>Let us give</u> $\zeta \in \Xi$; <u>then</u>

(6.7) $a(p + \zeta) \cdot f_x = \exp. (- x \cdot \zeta)\, a(p) \cdot [f_x \cdot \exp. (x \cdot \zeta)]$;

<u>If</u> $p_1, \dots p_\ell$ <u>and</u> $x_1, \dots x_\ell$ <u>are the coordinates of</u> p <u>and</u> x, <u>then it is allowed to treat</u>

(6.8) $a(p_2, \dots p_\ell) \cdot f(x_1, x_2, \dots x_\ell)$

<u>as if</u> x_1 <u>were a constant</u>.

<u>Proof of</u> (6.1) ... (6.7). See definition (5.1) and the properties of the Laplace transforms (n°2).

<u>Proof of</u> (6.8). The formula (2.2) shows that

$$\mathcal{L} = \mathcal{L}_1 \mathcal{L}_2,$$

where $\mathcal{L}_1$ is a Laplace transformation operating on the single variable x_1 and $\mathcal{L}_2$ a Laplace transformation operating on the other variables $x_2 \dots x_\ell$; therefore

$$a(p_2, \dots p_\ell) \cdot f(x_1, \dots x_\ell) = \mathcal{L}_2^{-1}\mathcal{L}_1^{-1}[a(2\pi\zeta_2, \dots 2\pi\zeta_\ell)$$

$$\mathcal{L}_1\mathcal{L}_2[f(x_1, \dots x_\ell)]] = \mathcal{L}_2^{-1}[a(2\pi\zeta_2, \dots 2\pi\zeta_\ell)\mathcal{L}_2[f(x_1, \dots x_\ell)]].$$

Theorem 6.2. We have

(6.9) $a(p) \cdot f_x = k_x * f_x$, where $k_x = \mathcal{L}^{-1}[a(2\pi\zeta)]$.

If $\|a(\xi + i\eta)\|_1$ is uniformly bounded in Δ, then the distribution k_x is the following function

(6.10) $k(x) = \dfrac{1}{(2\pi i)^{\ell}} \int_{\xi + i\Xi}\cdots\int a(\zeta)\,\exp.\,(x \cdot \zeta)\,d\zeta_1 \ldots d\zeta_{\ell}$, where $\xi \in \Delta$;

this function satisfies

$\|k(x)\,\exp.\,(-x \cdot \xi)\|_q < +\infty$ for $2 \leq q \leq +\infty$, $\xi \in \Delta$.

Proof. $\|a(\xi + i\eta)\|_\infty$ is uniformly bounded in Δ (see Lemma 4.1); therefore $\|a(\xi + i\eta)\|_2 < \sqrt{\|a\|_1 \|a\|_\infty}$ is also bounded; we use (2.3).

Theorem 6.3. If $a(\zeta) \in B(\Delta)$ and $f(x) \in F(\Delta)$, then $a(p) \cdot f(x) \in F(\Delta)$; more precisely

(6.11) $\|[a(p) \cdot f(x)]\,\exp.\,(-x \cdot \xi)\|_2 \leq \|a(\xi + i\eta)\|_\infty \cdot \|f(x)\,\exp.\,(-x \cdot \xi)\|_2$

for any $\xi \in \Delta$. In the Hilbert space, whose norm is $\|f(x)\,\exp.\,(-x \cdot \xi)\|_2$, we have (6.11bis). Bound of $a(p) \cdot = \|a(\xi + i\eta)\|_\infty$;

(6.12) Adjoint of $a(p) \cdot = \overline{a(2\xi - p)}$.

Proof of (6.11). Formula (4.1).

Proof of (6.12). $\mathcal{F}$ is unitary; therefore in the Hilbert space whose norm is $\|f(x)\,\exp.\,(-2\pi x \cdot \xi)\|_2$ the scalar product of $f(x)$ and $g(x) \in F(\Delta)$ is

$$< f, g > = \int_{\Xi}\cdots\int \phi(\xi + i\eta)\overline{\Psi(\xi + i\eta)}\,d\eta_1 \ldots d\eta_{\ell}$$

where $\phi(\zeta) = \mathcal{L}[f]$, $\Psi(\zeta) = \mathcal{L}[g]$, $2\pi\xi \in \Delta$, $\bar{z}$ = conjugate complex of z; therefore, if $a(\zeta) \in B(\Delta)$,

$$< a(p) \cdot f, g > = \int_{\Xi}\!\cdots\!\int a[2\pi(\xi + i\eta)]\phi(\xi + i\eta)\overline{\Psi(\xi + i\eta)}d\eta_1 \ldots d\eta_\ell$$

$$= \int_{\Xi}\!\cdots\!\int \phi(\xi + i\eta)\overline{b[2\pi(\xi + i\eta)]\Psi(\xi + i\eta)}d\eta_1 \ldots d\eta_\ell$$

$$= < f, b(p) \cdot g >,$$

if we take $\underline{b[2\pi(\xi + i\eta)] = \overline{a[2\pi(\xi + i\eta)]}} = \overline{a[4\pi\xi - 2\pi(\xi - i\eta)]}$, this is if $b(2\pi\zeta) = a[4\pi\xi - 2\pi\bar{\zeta}]$; the function $b(2\pi\zeta)$ is analytic and bounded in the tube with basis $\frac{1}{2\pi}(2\xi - \Delta)$; thus in this Hilbert space the adjoint operator of $a(p) \cdot$ is

$$b(p) \cdot = \overline{a(4\pi\xi - p)}.$$

7. <u>The dependence domain</u>. All this n°7 follows exclusively from (6.7).

<u>Definitions</u>. Let $a(\zeta) \in B(\Delta)$; let C be the <u>complement</u> of the subset of X each point x of which has such a neighborhood U that

(7.1) $$\inf_{\xi \in \Delta} \sup_{u \in U} [u \cdot \xi + \log\|a(\xi + i\eta)\|_\infty] = -\infty;$$

let D be the subset of X each point of which satisfies

(7.2) $$\inf_{\xi \in \Delta} [x \cdot \xi + \log\|a(\xi + i\eta)\|_\infty] > -\infty.$$

<u>Lemma</u> 7.1. Let K be a compact subset of X; the datum of the function f(x) on K - C determines the function $a(p) \cdot f(x)$ on K.

Note. These functions are defined almost everywhere.

Note. $K - C$ is the set of points $k - c$, where $k \in K$, $c \in C$; $K - C$ is closed, because C is obviously closed.

Proof. Let x' and $x'' \in X$; let U' and U'' be neighborhoods of x' and x''; suppose

$f(x) = 0$ outside U''; $\quad f(x) = 0 \;\; x''* U'' \qquad x' * U' \;\; g = a \circ f$

let $g(x)$ be the restriction of $a(p) \circ f(x)$ to U'; (6.7) and the mean value theorem give

$$\|g(x)\|_2 \exp.(-u' \circ \xi) \leq \|a(\xi + i\eta)\|_\infty \|f(x)\|_2 \exp.(-u'' \circ \xi)$$

for some $u' \in U'$ and $u'' \in U''$; that is

$$\log \|g(x)\|_2 \leq (u' - u'') \circ \xi + \log \|a(\xi + i\eta)\|_\infty + \log \|f(x)\|_2;$$

suppose $x'' \notin x' - C$: thus $x' - x''$ has a neighborhood U for which (7.1) holds; suppose U' and U'' so small that $U' - U'' \subset U$; by (7.1) the last inequality gives

$$\log \|g(x)\|_2 = -\infty;$$

it means: $a(p) \circ f(x)$ is zero in a neighborhood of x' if $f(x)$ is zero outside a neighborhood of $x'' \notin x' - C$. Thus: $a(p) \circ f(x)$ is not modified on some neighborhood of x' by setting $f(x) = 0$ in some neighborhood of $x'' \notin x' - C$. Therefore, more generally: $a(p) \circ f(x)$ is not modified on the compact subset K of X by setting $f(x) = 0$ in some neighborhood of $x'' \notin K - C$; therefore, again more generally, by setting $f(x) = 0$ outside the closed set $K - C$.

Lemma 7.2. C is closed and convex.

Proof. The complement of C is obviously open. Let

$$n(\xi) = \log \| a(\xi + i\eta) \|_\infty ;$$

$x \in C$ means that for any neighborhood U of x

$$(7.3) \qquad \underset{\xi \in \Delta}{\text{Inf.}} \ \underset{u \in U}{\text{Sup.}} [u \cdot \xi + n(\xi)] > -\infty .$$

Let x and $x' \in C$; let U and U' be neighborhoods of x and x': there is a number λ such that for any $\xi \in \Delta$ two points $u \in U$ and $u' \in U'$ can be found such that

$$u \cdot \xi + n(\xi) > \lambda, \ u' \cdot \xi + n(\xi) > \lambda ;$$

therefore

$$u^* \cdot \xi + n(\xi) > \lambda$$

for $u^* = \frac{u+u'}{2} \in \frac{U+U'}{2}$; thus (7.3) holds when U is replaced by $\frac{U+U'}{2}$, which is an arbitrarily small neighborhood of $x^* = \frac{x+x'}{2}$; therefore $\frac{x+x'}{2} \in C$.

<u>Lemma</u> 7.3. $D \subset C$; D is a convex F_σ . (F_σ means: the union of a denumberable system of closed subsets of X).

<u>Proof</u>. $D \subset C$ by definition. Let D_λ be the set of the points x such that

$$x \cdot \xi + n(\xi) \geq \lambda \text{ for any } \xi \in \Delta ;$$

$$D_\lambda \longrightarrow D \text{ when } \lambda \longrightarrow -\infty ;$$

D_λ , being an intersection of closed half spaces, is convex and closed.

<u>Lemma</u> 7.4. Each point $x \notin D$ is on the boundary of a closed half space containing C. [The boundary of a half space is a hyperplane.]

<u>Proof</u>. Because $x \notin D$, there is a sequence $\xi_1, \ldots \xi_q, \ldots$ of points $\in \Delta$ such that

$$(7.4) \qquad x \cdot \zeta_q + n(\zeta_q) \dashrightarrow -\infty;$$

by leaving out elements of this sequence, we can achieve that

$$\zeta_q \neq 0, \frac{\zeta_q}{\|\zeta_q\|} \dashrightarrow \theta, \text{ where } \|\theta\| = 1.$$

Let U be a bounded domain such that

$$(u - x) \cdot \theta \leq 0 \text{ for } u \in \overline{U}; \ (\overline{U} = \text{closure of } U);$$

thus for q sufficiently large,

$$(u - x) \cdot \zeta_q < 0 \text{ for any } u \in U;$$

therefore, by (7.4)

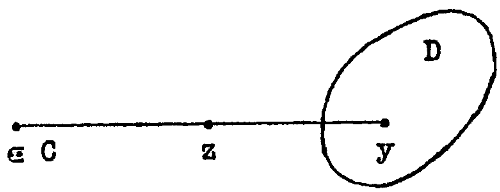
$$\operatorname{Sup.}_{u \in U}[u \cdot \zeta_q + n(\zeta_q)] < x \cdot \zeta_q + n(\zeta_q) \dashrightarrow -\infty;$$

thus U satisfies (7.1); therefore U is outside C: the points y of C satisfy

$$(y - x) \cdot \theta \geq 0.$$

Lemma 7.5. $\overline{D} = C$, when D has an interior ($\overline{D}$: closure of D).

Proof. Let $\overset{\circ}{D}$ be the interior of D; let



$y \in \overset{\circ}{D}$, $x \in C$, $x \notin \overline{D}$

On the segment joining x to y there is a point $z \notin D$ and $\neq x$; the lemmas 7.2 and 7.3 prove that z is on the boundary of a closed half space, which contains both x and D; but this is impossible.

Summing up these lemmas we obtain the following proposition, which is to be completed in chapter III under more particular hypotheses:

Proposition 7.1. Let $a(\zeta) \in B(\Delta)$; it means: $\|a(\xi + i\eta)\|_\infty$ is uniformly bounded in Δ. Let C and D be defined by (7.1) and (7.2).

Let f_x be a distribution $\in E(\Delta)$; let G be an open subset of X; the datum of f_x in the open subset G - C determines $a(p) \cdot f_x$ in G. C is closed and convex; D is a convex F_σ; $D \subset C$; $\bar{D} = C$ when D has an interior.

Remark 7.1. Thus in order to define $a(p) \cdot f_x$ in G it is sufficient to suppose that the restriction of f_x to G - C belongs to $E(\Delta)$.

Proposition 7.2. Let a(p) be independent of p_1; then C and D are both in the hyperplane X': $x_1 = 0$; and in the definitions (7.1) and (7.2) of C and D, X can be replaced by X'.

Proof. Let x be such that its first coordinate $x_1 > 0$; let U be a compact neighborhood of x on which $x_1 > \mathcal{E} > 0$; then, for $\xi_1 \to -\infty$, $\xi_2, \dots \xi_\ell$ fixed

$$\operatorname*{Sup.}_{u\in U} [u \cdot \xi + n(\xi_2, \dots \xi_\ell)] \leq \mathcal{E}\,\xi_1 + \operatorname*{Sup.}_{u\in U} [u_2\xi_2 + \dots u_\ell\,\xi_\ell + n(\xi_2, \dots \xi_\ell)] \longrightarrow -\infty;$$

thus (7.1) holds; therefore $C \subset X'$; therefore $D \subset X'$. It is now obvious that the definition (7.2) of D does not change by replacing X by X'. Moreover, if $x \in X'$ in the definition (7.1) of C and if U is a parallelepiped symetrical in respect to X', then

$$\operatorname*{Sup.}_{u\in U} [u \cdot \xi + n(\xi_2, \dots \xi_\ell)] = \mathcal{E} \cdot |\xi_1| + \operatorname*{Sup.}_{u\in U} [u_2\xi_2 + \dots u_\ell\,\xi_\ell + n(\xi_2, \dots \xi_\ell)];$$

$(\mathcal{E} > 0)$; therefore the definition of C does not change by replacing X by X'.

§3. Examples

8. The translation. Let $y \in X$;

(8.1) $$\exp.(-y \cdot p) \circ f(x) = f(x - y);\ C = D = y.$$

For instance:

$$\exp.(p_1) \circ f(x_1, x_2, \ldots x_{\ell}) = f(x_1 + 1, x_2, \ldots x_{\ell}).$$

Proof. By (5.1) and (2.7),

$$\exp.(-y \cdot p) \cdot f(x) = \mathcal{L}^{-1}[\exp.(-2\pi y \circ \zeta)\mathcal{L}[f(x)]] = \mathcal{L}^{-1}[\mathcal{L}[f(x - y)]]$$

$= f(x - y).$

$$\|\exp.[-y \cdot (\xi + i\eta)\|_{\infty} = \exp.(-y \cdot \xi);\ \text{thus } \Delta = \Xi,$$

$$\underset{\xi \in \Xi}{\text{Inf.}}\ \underset{u \in U}{\text{Sup.}}\ [(u - y) \cdot \xi] = 0 \text{ if } y \in \bar{U}$$

$$= -\infty \text{ if } y \notin \bar{U};$$

thus $C = y$. D is the set of the points x such that

$$\underset{\xi \in \Xi}{\text{Inf.}}\ [(x - y) \cdot \xi] > -\infty;$$

thus $D = y$.

9. The Gauss operator. For $\ell = 1 (p = p_1, x = x_1)$

(9.1) $$\exp.(\tfrac{a}{2} p^2) \circ f(x) = \frac{1}{\sqrt{2\pi a}} \exp.(-\frac{x^2}{2a}) * f(x)$$

where $\mathcal{R}(a) > 0$, $\mathcal{R}(\sqrt{a}) > 0$ ($\mathcal{R}$: real part).

10. Derivatives and primitives of complex orders. Let α be a complex number. We have ($\beta = -\alpha$ + positive integer)

(10.1) $$p_1^{\alpha} \cdot f(x) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^{x_1} (x_1 - t)^{\beta - 1} \frac{\partial^{\alpha+\beta} f(t, x_2, \ldots x_{\ell})}{\partial t^{\alpha+\beta}}\, dt$$

<u>for</u> $p_1 > 0$, $\mathcal{R}(\beta) > 0$; <u>the branches to be used for</u> p_1^{α} <u>and</u> $(x_1 - t)^{\beta}$ <u>have opposite arguments for</u> $p_1 > 0$, $x_1 - t > 0$. <u>Therefore</u>

$$(10.2) \quad (-p_1)^{\alpha} \cdot f(x) = \frac{1}{\Gamma(\beta)} \int_{x_1}^{+\infty} (t - x_1)^{\beta - 1} \frac{\partial^{\alpha+\beta} f(t, x_2, \dots x_{\ell})}{\partial t^{\alpha+\beta}} dt$$

<u>for</u> $p_1 < 0$, $\mathcal{R}(\beta) > 0$; <u>the branches to be used for</u> $(-p_1)^{\alpha}$ <u>and</u> $(t - x_1)^{\beta}$ <u>have opposite arguments for</u> $p_1 < 0$, $x_1 - t < 0$.

<u>For instance:</u>

$$\frac{1}{p_1 p_2} f(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(s, t)\, ds\, dt \text{ for } p_1 > 0,\ p_2 > 0,$$

$$= \int_{x_1}^{+\infty} \int_{x_2}^{+\infty} f(s, t)\, ds\, dt \text{ for } p_1 < 0,\ p_2 < 0,$$

$$= - \int_{-\infty}^{x_1} \int_{x_2}^{+\infty} f(s, t)\, ds\, dt \text{ for } p_1 > 0,\ p_2 < 0,$$

$$= - \int_{x_1}^{+\infty} \int_{-\infty}^{x_2} f(s, t)\, ds\, dt \text{ for } p_1 < 0,\ p_2 > 0.$$

<u>Proof of</u> (10.1). It is sufficient to prove (10.1) for $\mathcal{R}(\beta) > 1$, $\beta = -\alpha$ and, according to (6.8), $\ell = 1$. We have $p^{\alpha} \cdot f(x) = k(x) * f(x)$ where

$$k(x) = \frac{1}{2\pi i} \int_{\xi - i\infty}^{\xi + i\infty} \zeta^{\alpha} \exp.(x\zeta) d\zeta, \quad \xi > 0;$$

a classical residue calculation gives

$$k(x) = 0 \text{ for } x < 0; \ k(x) = \frac{x^{-\alpha - 1}}{\Gamma(-\alpha)} \text{ for } x > 0.$$

CHAPTER II

SYMBOLIC PRODUCT BY A FUNCTION OF $\pm p_1^2 \pm p_2^2 \pm \cdots \pm p_\ell^2$.

§1. Preliminary

(§1 proves that, if such a product is a bounded operator, then all $\pm$ are the same with at most one exception.)

11. The cases in which this product is a bounded operator. Let $a(\zeta)$ be an analytic function of $\zeta \in \Xi + i\,\Xi$.

Definition 11.1. $\Delta(a)$ is the interior of the set of the $\xi \in \Xi$ such that

$$\|a(\xi + i\eta)\|_\infty < +\infty;$$

$\Delta(a)$ is also the interior of the set of all ξ such that the symbolic product by $a(p)$ is a bounded operator for the norm

$$\|f(x)\ \exp.(-x\,.\,\xi)\|_2.$$

[See formula (6.11[bis]).]

Definition 11.2. $\delta(a)$ is the interior of the set of the real numbers ξ_1 such that

$$\|a(\xi_1 + i\eta_1, 0, \ldots, 0)\|_\infty < +\infty.$$

$\delta(a)$ is also the interior of the set of all ξ_1 such that the symbolic product by $a(p_1, 0, \ldots 0)$ is a bounded operator for the norm

$$\|f(x)\ \exp.(-x_1\,\xi_1)\|_2.$$

Proposition 11.1. Let $a(p)$ be a function of $\pm p_1^2 \pm p_2^2 \pm \cdots \pm p_\ell^2$.

If $\Delta(a)$ is non-void, then all the signs $\pm$ are the same with at most one exception.

Proof. Let

$$a(\zeta) = \phi(\zeta_1^2 + \varepsilon_2\zeta_2^2 + \dots \varepsilon_\ell \zeta_\ell^2), \text{ where } \varepsilon_2 = \pm 1, \dots \varepsilon_\ell = \pm 1.$$

The linear mappings of Ξ leaving $\xi_1^2 + \varepsilon_2\xi_2^2 + \dots + \varepsilon_\ell\xi_\ell^2$ invariant obviously leave $\Delta(a)$ invariant; therefore $\Delta(a)$ contains points of the coordinate axes, for instance the point

$$\xi_1 \neq 0, \quad \xi_2 = 0, \dots \xi_\ell = 0;$$

then

$$a(\xi_1 + i\eta_1, i\eta_2, \dots i\eta_\ell) = \phi(\xi_1^2 - \eta_1^2 - \varepsilon_2\eta_2^2 - \dots - \varepsilon_\ell\eta_\ell^2 + 2i\xi_1\eta_1)$$

remains defined and bounded when $\eta_1, \dots \eta_\ell$ vary arbitrarily; thus, if there are two different ε_λ, $\phi(z)$ remains defined, analytic and bounded for *all* complex numbers z; this can not happen.

§2. *Symbolic product by a function of $p_1^2 + \dots + p_\ell^2$.*

(An easy integration expresses such a product by means of the symbolic product by a function of *one* variable: see theorem 15.1).

12. *The relation between $\Delta(a)$ and $\delta(a)$.*

Proposition 12.1. Let $a(\zeta)$ be a function of $\zeta_1^2 + \dots + \zeta_\ell^2$; let $\|\xi\| = \sqrt{\xi_1^2 + \dots + \xi_\ell^2}$. *Then*

$$\|a(\xi + i\eta)\|_\infty = \sup_{|\xi_0| \leq \|\xi\|} \|a(\xi_0 + i\eta_0, 0, \dots 0)\|_\infty .$$

Therefore $\Delta(a)$ is the sphere

$$\|\xi\| < \text{const.},$$

whose intersection by the axis $\xi_2 = \dots = \xi_\ell = 0$ *is the connected component of* $\delta(a)$ *containing* 0.

Proof. The assumption is

$$a(\zeta_1, \zeta_2, \dots \zeta_\ell) = \phi(\zeta_1^2 + \dots + \zeta_\ell^2);$$

in particular

$$a(\xi_0 + i\eta_0, 0, \dots 0) = \phi(\xi_0^2 - \eta_0^2 + 2i\xi_0\eta_0)$$

$$a(\xi_1 + i\eta_1, i\eta_2, \dots i\eta_\ell) = \phi(\xi_1^2 - \eta_1^2 - \eta_2^2 \dots - \eta_\ell^2 + 2i\xi_1\eta_1).$$

Let $P(\xi_0)$ be the parabola described by

$$z = \xi_0^2 - \eta_0^2 + 2i\xi_0\eta_0 \quad (\xi_0: \text{ fixed; } \eta_0 \text{ variable});$$

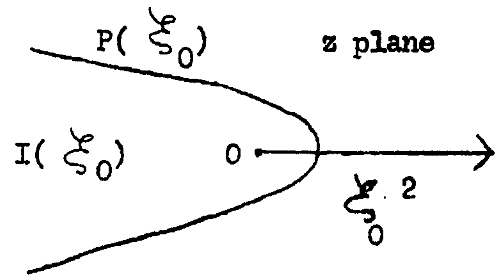
let $I(\xi_0)$ be the inside of this parabola; preceding formulae give

$$\|a(\xi_0 + i\eta_0, 0, \dots 0)\|_\infty = \operatorname*{Sup.}_{z \in P(\xi_0)} |\phi(z)|$$

$$\|a(\xi_1 + i\eta_1, i\eta_2, \dots i\eta_\ell)\|_\infty = \operatorname*{Sup.}_{z \in I(\xi_1)} |\phi(z)|;$$

but obviously

$$I(\xi_1) = \bigcup_{|\xi_0| \le |\xi_1|} P(\xi_0);$$



therefore

$$\|a(\xi_1 + i\eta_1, i\eta_2, \dots i\eta_\ell)\|_\infty = \operatorname*{Sup.}_{|\xi_0| \le |\xi_1|} \|a(\xi_0 + i\eta_0, 0, \dots 0)\|_\infty$$

this proves the proposition, since $\|a(\xi + i\eta)\|_\infty$ is obviously a function of $\|\xi\|$.

13. Fourier transforms of radial functions. The functions $f(x)$ and $\phi(\eta)$ are called radial and are denoted by $f(r)$ and $\phi(\rho)$ if they are functions of

$$r^2 = x_1^2 + \dots + x_\ell^2 \text{ and } \rho^2 = \eta_1^2 + \dots + \eta_\ell^2 .$$

It is obvious by (1.6) that the Fourier transform of a radial function $f(r)$ is a radial one $\phi(\rho)$. Suppose momentarily that $\int_0^{+\infty} |f(r)| r^{\ell-1} dr < +\infty$; the value of $\phi(\eta) = \mathcal{F}[f(x)]$ at the point $(\rho, 0, \dots 0)$ is

$$\phi(\rho) = \int \dots \int_x f(r) \exp.(-2\pi i x_1 \rho)\, dx_1 \dots dx_\ell ;$$

$$= \int_{-\infty}^{+\infty} \exp.(-2\pi x_1 \rho)\, dx_1 \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} f(r)\, dx_2 \dots dx_\ell ;$$

let

$$r_1^2 = x_2^2 + \dots + x_\ell^2 ;$$

since the measure of the sphere with dimension $\ell - 2$ and radius r_1 is $\frac{2\pi^{\frac{\ell-1}{2}}}{\Gamma(\frac{\ell-1}{2})} r_1^{\ell-2}$, we have

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} f(r)\, dx_2 \dots dx_\ell = \frac{2\pi^{\frac{\ell-1}{2}}}{\Gamma(\frac{\ell-1}{2})} \int_0^{+\infty} f(r) r_1^{\ell-2} dr_1, \text{ where } r^2 = x_1^2 + r_1^2 ;$$

thus

$$\phi(\rho) = \int_{-\infty}^{+\infty} \exp.(-2\pi i x_1 \rho)\, dx_1 \frac{\pi^{\frac{\ell-1}{2}}}{\Gamma(\frac{\ell-1}{2})} \int_{|x_1| < r < +\infty} f(r)(r^2 - x_1^2)^{\frac{\ell-3}{2}} d(r^2);$$

replacing x_1 and r by r and t we obtain

$$(13.1)\quad \phi(\rho) = \int_{-\infty}^{+\infty} \exp.(-2\pi i r\rho)dr\, \frac{\pi^{\frac{\ell-1}{2}}}{\Gamma(\frac{\ell-1}{2})} \int_{|r|<t<+\infty} (t^2-r^2)^{\frac{\ell-3}{2}} f(t)d(t^2).$$

<u>Let</u> $\mathfrak{F}^*$ <u>be the Fourier transformation of functions of</u> r <u>into functions of</u> ρ :

$$\mathfrak{F}^*[g(r)] = \int_{-\infty}^{+\infty} \exp.(-2\pi i r\rho)g(r)dr, \text{ if } \int_{-\infty}^{+\infty} g(r)\, dr < +\infty;$$

<u>let</u> Q <u>run over the space dual to the one over which</u> r^2 <u>runs</u>:

$$Q\,. = \frac{d}{d(r^2)}.$$

Let g(r) be the even function equal to

$$(-\frac{Q}{\pi})^{-\frac{\ell-1}{2}} .\, f(r) \text{ for } r > 0 \text{ (see n}^\circ\text{10);}$$

<u>the branch of</u> $(-\frac{Q}{\pi})^{\frac{\ell-1}{2}}$ <u>to be used is</u> > 0 <u>for</u> $Q < 0$.

Formula (13.1) becomes

$$\phi(\rho) = \mathfrak{F}^*[g(r)];$$

therefore, <u>defining</u> $\phi(\rho) = \phi(-\rho)$ <u>for</u> $\rho < 0$,

$$(13.2)\quad \boxed{f(r) = (-\frac{Q}{\pi})^{\frac{\ell-1}{2}} .\, \mathfrak{F}^{*-1}[\phi(\rho)] \text{ for } r > 0.}$$

Continuity shows that (13.2) holds whenever $\int_0^{+\infty} (1+\rho^{\ell-1})\phi^2(\rho)d\rho < +\infty$.

14. <u>Digression about a Bochner formula</u>. If ℓ is odd, (13.2) is a Bochner formula: [13], ch. II, §7. If ℓ is even, Bochner gives an apparently different formula using the Bessel function:

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp.(ix\cos s)\, ds = \frac{1}{\pi}\int_{-1}^{1} \frac{\exp.(ixt)}{\sqrt{1-t^2}}\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(x \cos s)\, ds = \frac{1}{\pi}\int_{-1}^{1} \frac{\cos(xt)}{\sqrt{1-t^2}}\, dt$$

$$= \sum_{\nu \geq 0} \frac{1}{(\nu !)^2}\left(-\frac{x^2}{4}\right)^{\nu} ;$$

indeed (13.1) can be written for $\ell = 2$

$$\phi(\rho) = 2\int_0^{+\infty} f(t)\, t\, dt \int_{-t}^{t} \frac{\exp.(-2\pi i r \rho)}{\sqrt{t^2 - r^2}}\, dr$$

or, replacing t and r by r and rt

$$\phi(\rho) = 2\int_0^{+\infty} f(r)\, r\, dr \int_{-1}^{1} \frac{\exp.(-2\pi i r\rho t)}{\sqrt{1-t^2}}\, dt = 2\pi\int_0^{+\infty} J_0(2\pi r \rho) r f(r) dr;$$

therefore, since the relation between $\phi(\rho)$ and $f(r)$ is symetrical,

$$f(r) = 2\pi\int_0^{+\infty} J_0(2\pi r\rho)\rho\,\phi(\rho)\, d\rho ;$$

comparing with (13.2), we obtain

$$\boxed{\left(-\frac{Q}{\pi}\right)^{\frac{1}{2}} \cdot \mathcal{F}^{*-1}[\phi(\rho)] = 2\pi\int_0^{+\infty} J_0(2\pi r \rho)\,\rho\,\phi(\rho)\, d\rho .} \tag{14.1}$$

15. Calculation of $a(p) \cdot f(x)$. Let $a(p)$ be a function of $p_1^2 + \dots + p_\ell^2$ such that $\Delta(a)$ is non-void. The Theorem 6.2 can not be directly applied; now let us use the convergence factor

$$\exp. [\varepsilon (\zeta_1^2 + \dots + \zeta_\ell^2)]$$

and define

$$a_\varepsilon(\zeta) = a(\zeta) \exp.[\varepsilon(\zeta_1^2 + \dots + \zeta_\ell^2)], \text{ where } \varepsilon > 0, \quad \varepsilon \to 0.$$

On the one hand the assumption $\| a(\xi + i\eta)\|_\infty < +\infty$ and the definition

(5.1) of the symbolic calculus show that

(15.1)
$$\lim_{\mathcal{E} \to 0} a_{\mathcal{E}}(p) \cdot f(x) = a(p) \cdot f(x)$$
$$\lim_{\mathcal{E} \to 0} a_{\mathcal{E}}(p_1, 0, \ldots)f(x) = a(p_1, 0, \ldots 0) \cdot f(x)$$

at each point x, if $p \in \Delta(a)$ and if f(x) is for instance an infinitely differentiable function with compact support.

On the other hand, Theorem 6.2 can be applied to $a_{\mathcal{E}}(p)$ and gives

$$a_{\mathcal{E}}(p) \cdot f(x) = \int_X \cdots \int k(x - y) f(y) dy_1 \ldots dy_{\ell}$$

where

$$k(x) = \mathcal{L}^{-1}[a_{\mathcal{E}}(2\pi\zeta)] = \mathcal{F}^{-1}[a_{\mathcal{E}}(2\pi i\eta)];$$

upon application of (13.2) k(x) is the following function of $r^2 = x_1^2 + \ldots x_{\ell}^2$

$$k(r) = \left(-\frac{Q}{\pi}\right)^{\frac{\ell-1}{2}} \cdot \mathcal{l}(r)$$

where

$$\mathcal{l}(r) = \int_{-\infty}^{+\infty} a_{\mathcal{E}}(2\pi i\rho, 0, \ldots 0) \exp. (2\pi i r\rho) d\rho$$

that is

(15.2)
$$\mathcal{l}(r) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} a_{\mathcal{E}}(\zeta_1, 0, \ldots 0) \exp. (r\zeta_1) d\zeta_1;$$

thus

(15.3)
$$a_{\mathcal{E}}(p) \cdot f(x) = \int_0^{+\infty} \left[\left(-\frac{Q}{\pi}\right)^{\frac{\ell-1}{2}} \cdot \mathcal{l}(r)\right] g(R) dR$$

if

$$R = r^2, \qquad g(R) = \int_{\Omega_R} \cdots \int f(y)\,\omega(y, dy)$$

$\omega(y, dy)$ is an exterior differential form of $y_1, \ldots, dy_\ell$ such that (see chapter IV, §1)

(15.4) $d[y_1 - x_1)^2 + \ldots + (y_\ell - x_\ell)^2]\,\omega(y, dy) = dy_1 \ldots dy_\ell$;

Ω_R is the sphere

$$(y_1 - x_1)^2 + \ldots + (y_\ell - x_\ell)^2 = R > 0; \tag{15.5}$$

with that orientation which makes $\omega(y, dy) > 0$. Let us define $g(R) = 0$ for $R < 0$; formulas (6.12) and (15.3) give, if $f(x)$ is for instance an infinitely differentiable function with compact support,

$$a_\varepsilon(p) \cdot f(x) = \int_0^{+\infty} \ell(r)\left[\left(\frac{Q}{\pi}\right)^{\frac{\ell-1}{2}} \cdot g(R)\right]dR;$$

that is

$$a_\varepsilon(p) \cdot f(x) = \int_0^{+\infty} \ell(r)\,\frac{1}{\pi}\,\frac{d}{dr}\left[\left(\frac{Q}{\pi}\right)^{\frac{\ell-3}{2}} \cdot g(R)\right]dr. \tag{15.6}$$

Now (15.2) shows that $\ell(r)$ is even since $a_\varepsilon(\zeta_1, 0, \ldots 0)$ is even; (15.2) and Theorem 6.2 prove that

$$\int_{-\infty}^{+\infty} \ell(r - s)h(s)ds = a_\varepsilon(q, 0, \ldots 0) \cdot h(r)$$

where q and r run over dual spaces $(q \cdot = \frac{d}{dr})$, $(q, 0, \ldots 0) \in \Delta(a)$; thus

$$\int_{-\infty}^{+\infty} \ell(r)h(r)dr = [a_\varepsilon(q, 0, \ldots 0) \cdot h(r)]_{r=0}, \tag{15.7}$$

where $[\ldots]_{r=0}$ is the value of ... for $r = 0$. Formula (15.6) and (15.7) prove that the following theorem is true if a is replaced by $a_{\mathcal{E}}$. Therefore, upon application of (15.1), this theorem is true:

Theorem 15.1. If $a(p)$ is a function of $p_1^2 + \ldots + p_\ell^2$, such that $\Delta(a)$ is non-void, then

$$a(p) \cdot f(x) = [\frac{q}{\pi} a(q, 0, \ldots 0) \cdot h(r)]_{r=0},$$

where

$$(q, 0, \ldots 0) \in \Delta(a),$$

$$h(r) = 0 \text{ for } r < 0,$$

$$h(r) = (\frac{Q}{\pi})^{\frac{\ell-3}{2}} \cdot \int_{\Omega_R}\cdots\int f(y)\, \omega(y,dy) \text{ for } r > 0,\ Q > 0;$$

$(\frac{Q}{\pi})^{\frac{\ell-3}{2}} > 0$; q and Q are the dual variables of r and $R = r^2$; Ω_R and $\omega(y,dy)$ are defined by (15.4) and (15.5); $f(x)$ is an infinitely differentiable function with compact support.

Note. This theorem expresses the operator of several variables $a(p) \cdot = a(p_1, \ldots p_\ell)$. by means of the operator of one variable $a(p_1, 0, \ldots 0)$. : it is always easier to apply Theorem 6.2 to $a(p_1, 0, \ldots 0)$ than to $a(p_1, \ldots p_\ell)$; moreover, it happens often that Theorem 6.2 can be applied to $a(p_1, 0, \ldots 0)$ but not to $a(p_1 \ldots p_2)$. More precisely: if $\|f(x) \exp.(-x \cdot \xi)\|_2 < +\infty$ fpr $\xi \in \Delta(a)$, then the function $a(p) \cdot f(x)$ is defined, but only almost everywhere; we suppose $f(x)$ so regular that this function $a(p) \cdot f(x)$ is continuous and defined everywhere; the theorem expresses its value at each point by means of the operator $a(p_1, 0, \ldots 0)$.

16. Examples.

If $\ell = 3$, then $a(p) \cdot f(x) = [\frac{q}{\pi} a(q, 0, 0) \int_{\Omega_{r^2}} \cdots \int f(y)\, \omega(y, dy)]_{r=0}$

where $(q, 0, 0) \in \Delta(a)$.

If $a(p) = \dfrac{1}{p_1^2 + p_2^2 + \cdots + p_\ell^2 - c^2}$ and $c = $ const.,

then using Theorem 6.2, the relation $h(r) = 0$ for $r < 0$ and Cauchy's theory of residues, we obtain

$$[\frac{q}{\pi} a(q, 0, \ldots 0) \cdot h(r)]_{r=0} = -\frac{1}{2\pi} \int_0^{+\infty} \exp.(-cr) h(r) dr \text{ where } |q| < c$$

For $\ell = 3$ and $a(p) = \dfrac{1}{p_1^2 + p_2^2 + p_3^2 - c^2}$, we obtain in this way the classical integral, where $r^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2$;

$$a(p) \cdot f(x) = -\frac{1}{4\pi} \iiint_X \frac{\exp.(-cr)}{r} f(y)\, dy_1 dy_2 dy_3.$$

§3. Symbolic product by a function of $-p_1^2 + p_2^2 + \cdots + p_\ell^2$.

(§3 proves that the formula, which expresses the symbolic product by a function of $p_1^2 + p_2^2 + \cdots + p_\ell^2$, expresses also the symbolic product by a function of $-p_1^2 + p_2^2 + \cdots + p_\ell^2$. This formula is in particular convenient to solve Cauchy's problem for classical wave equations: see chapter V).

17. The relation between $\Delta(a)$ and $\delta(a)$.

Proposition 17.1. Let $a(\zeta)$ be a function of $-\zeta_1^2 + \zeta_2^2 + \cdots + \zeta_\ell^2$; $\Delta(a)$ is a domain

(17.1) $\sqrt{\xi_2^2 + \cdots + \xi_\ell^2 + \text{const.}} < \xi_1$ (where const. > 0)

<u>or a domain</u>

(17.2) $$\xi_1 < -\sqrt{\xi_2^2 + \dots + \xi_\ell^2} + \text{const.} \qquad (\text{const.} > 0).$$

<u>Suppose</u> (17.1) <u>holds</u>; <u>let</u> $\rho = \sqrt{\xi_1^2 - \xi_2^2 - \dots - \xi_\ell^2} > 0$; <u>then</u>

(17.3) $$\| a(\xi + i\eta)\|_\infty = \sup_{\rho \le \xi_0} \| a(\xi_0 + i\eta_0, 0, \dots 0)\|_\infty;$$

<u>therefore the intersection of</u> Δ(a) <u>by the axis</u> $\xi_2 = \dots = \xi_\ell = 0$ <u>is a connected component of</u> δ(a).

<u>Note</u>. Conversely the half line $\rho \le \xi_1 < +\infty$, $\xi_2 = 0, \dots \xi_\ell = 0$ belongs to a domain Δ(a) if and only if

(17.4) $$\sup_{\rho \le \xi_0} \| a(\xi_0 + i\eta_0, 0, \dots 0)\|_\infty < +\infty.$$

<u>Proof</u>. Δ(a) is a convex domain invariant under the linear mappings of Ξ which leave invariant $\xi_1^2 - \xi_2^2 - \dots - \xi_\ell^2$; therefore Δ(a) is necessarily of the type (17.1) or (17.2). Now the assumptions are (17.1) and

$$a(\zeta_1, \dots \zeta_\ell) = \phi(\zeta_1^2 - \zeta_2^2 - \dots - \zeta_\ell^2);$$

is particular

$$a(\xi_0 + i\eta_0, 0, \dots 0) = \phi(\xi_0^2 - \eta_0^2 + 2i\xi_0\eta_0)$$

$$a(\xi_1 + i\eta_1, i\eta_2, \dots i\eta_\ell) = \phi(\xi_1^2 - \eta_1^2 + \eta_2^2 + \dots + \eta_\ell^2 + 2i\xi_1\eta_1).$$

Let $P(\xi_0)$ be the parabola described by

$$z = \xi_0^2 - \eta_0^2 + 2i\xi_0\eta_0 \qquad (\xi_0 \text{ fixed, } \eta_0 \text{ variable});$$

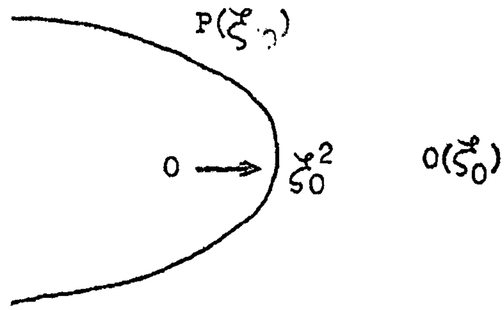
let $O(z_0)$ be the outside of this parabola; preceding formula give

$$\|a(\xi_0 + i\eta_0, 0, \ldots 0)\|_\infty = \operatorname*{Sup.}_{z \in P(\xi_0)} |\phi(z)|$$

$$\|a(\xi_1 + i\eta_1, i\eta_2, \ldots i\eta_\ell)\|_\infty = \operatorname*{Sup.}_{z \in O(\xi_1)} |\phi(z)|;$$

but obviously

$$O(\xi_1) = \bigcup_{\xi_0 \geq \xi_1} P(\xi_0);$$



therefore

$$\|a(\xi_1 + i\eta_1, i\eta_2, \ldots i\eta_\ell)\|_\infty = \operatorname*{Sup.}_{\xi_0 \geq \xi_1} \|a(\xi_0 + i\eta_0, 0, \ldots 0)\|_\infty;$$

this proves the note and the proposition, since $\|a(\xi + i\eta)\|_\infty$ is obviously a function of ρ.

18. <u>Properties</u> <u>of</u> $\zeta_1^2 - \zeta_2^2 - \ldots - \zeta_\ell^2$. (Purpose of n°18 is to find a <u>convergence</u> <u>factor</u> having the properties of the convergence factor exp. $[\varepsilon(\zeta_1^2 + \ldots + \zeta_\ell^2)]$ used in n°15, but depending now on $\zeta_1^2 - \zeta_2^2 - \ldots \zeta_\ell^2$.) The set of values taken by

$$z(\zeta) = \zeta_1^2 - \zeta_2^2 - \ldots - \zeta_\ell^2 \tag{18.1}$$

in the tube of basis $0 \neq \xi_1, \xi_2 = \ldots = \xi_\ell = 0$ is the set of complex numbers

$$\xi_1^2 - \eta_1^2 + \eta_2^2 + \ldots + \eta_\ell^2 + 2i\,\xi_1\eta_1 \qquad (\xi_1 \neq 0);$$

this set is $\bigcup_{0<\xi_1} P(\xi_1)$; that is the set of the non-negative complex numbers; now the linear real mappings of $\Xi + i\,\Xi$, leaving $z(\zeta)$ invariant, map this tube onto the tube with basis $\xi_2^2 + \dots + \xi_\ell^2 < \xi_1^2$; therefore the set of values taken by $z(\zeta)$ in the tube with basis $\xi_2^2 + \dots + \xi_\ell^2 < \xi_1^2$ is the set of the non-negative complex numbers. Thus we have, in the tube of basis

$$\Delta:\ \sqrt{\xi_2^2 + \dots + \xi_\ell^2} < \xi_1, \tag{18.2}$$

$$-\pi < \arg.\, z(\zeta) < \pi. \tag{18.3}$$

This inequality proves that the following function is uniform in the tube with basis Δ:

<u>Definition</u> 18.1.

$$b_\varepsilon(\zeta) = \exp.\,[-\,\varepsilon\, z^{\frac{1}{4}}(\zeta)], \qquad \text{where } \varepsilon > 0,\ \xi \in \Delta.$$

<u>Lemma</u> 18.1. $b_\varepsilon(\zeta)$ and each derivative of $b_\varepsilon(\zeta)$ is bounded by

$$\text{const.}\ \exp.\,[-\,\text{const.}\,|\zeta|^{\frac{1}{4}}],$$

in any tube whose basis is a compact subset of Δ. In particular

$$\|b_\varepsilon(\xi + i\eta)\|_\infty \leq 1, \qquad \|b_\varepsilon(\xi + i\eta)\|_1 < +\infty;$$

more precisely, $\lim_{\|\xi\|\to\infty} \|b_\varepsilon(\xi + i\eta)\|_1 = 0$ when ε is fixed and ξ describes inside Δ a half line, whose origin is 0.

<u>Proof</u>. (18.3) shows that

$$|\exp.\,[-\,\varepsilon\, z^{\frac{1}{4}}]| \leq \exp.\,[-\,\frac{\varepsilon}{\sqrt{2}}\,|z|^{\frac{1}{4}}];$$

thus the lemma follows from a convenient lower bound of $|z|$; the linear real mappings of $\Xi + i\Xi$ leaving $z(\zeta)$ invariant show that it is sufficient to obtain this lower bound when $\xi_2 = \ldots = \xi_\ell = 0$; in this case

$$|z(\zeta)|^2 = (\xi_1^2 - \eta_1^2 + \eta_2^2 + \ldots + \eta_\ell^2)^2 + 4\xi_1^2\eta_1^2$$
$$\geq \xi_1^4 + 2\xi_1^2(\eta_1^2 + \eta_2^2 + \ldots + \eta_\ell^2).$$

19. <u>Laplace transform of a function of</u> $\zeta_1^2 - \zeta_2^2 - \ldots - \zeta_\ell^2$.
Let $\phi(\zeta) = \phi(\rho)$ be a function of $\rho = \sqrt{\zeta_1^2 - \zeta_2^2 - \ldots - \zeta_\ell^2}$ with the following properties:

$\phi(\zeta)$ is analytic in a tube with basis

(19.1) $\Delta : \sqrt{\xi_2^2 + \ldots + \xi_\ell^2 + \text{const.}} < \xi_1$ (const. > 0);

$\phi(\zeta)$ and each derivative of $\phi(\zeta)$ is bounded by

(19.2) const. exp. $[- \text{const.} |\zeta|^{\frac{1}{4}}]$

in any tube whose basis is a compact subset of Δ;

(19.3) $\| \phi(\xi + i\eta)\|_1 \longrightarrow 0$ when ξ describes the intersection of Δ and any line containing 0.

Let

(19.4) $f(x) = \mathcal{L}^{-1}[\phi(\zeta)] = \int \underset{\Xi}{\ldots} \int \phi(\xi + i\eta) \exp. [2\pi x \cdot (\xi + i\eta)] d\eta_1 \ldots d\eta_\ell$.

Let $\|\xi\| \longrightarrow \infty$: according to (19.3) and (19.4), $f(x) = 0$, if there is some $\zeta \in \Delta$ such that $x \cdot \zeta < 0$; that is: $f(x) = 0$ outside the cone

$$(19.5) \qquad C: \sqrt{x_2^2 + \ldots + x_\ell^2} < x_1.$$

On the other hand $f(x)$ is invariant under the linear mappings of X contragrediant with those of Ξ which leave $\phi(\zeta)$ invariant; that is: $f(x)$ is in C a function $f(r)$ of

$$(19.6) \qquad r = \sqrt{x_1^2 - x_2^2 - \ldots - x_\ell^2};$$

we denote

$$(19.7) \qquad r_1^2 = x_1^2 - r^2 = x_2^2 + \ldots + x_\ell^2.$$

Now let us simplify the relation between the function $\phi(\rho)$ and $f(r)$; the value of $\phi(\zeta) = \mathcal{L}[f(x)]$ at the point $(\rho, 0, \ldots 0)$ is

$$\phi(\rho) = \int\!\!\ldots\!\!\int_C f(r) \exp.(-2\pi x_1 \rho)\, dx_1 \ldots dx_\ell$$

$$= \int_0^{+\infty} \exp.(-2\pi x_1 \rho)\, dx_1 \int\!\!\ldots\!\!\int_{r_1 < x_1} f(r)\, dx_2 \ldots dx_\ell$$

since the measure of the sphere with dimension $\ell - 2$ and radius r_1 is $\dfrac{2\pi^{\frac{\ell-1}{2}}}{\Gamma(\frac{\ell-1}{2})} r_1^{\ell-2}$ we have

$$\int\!\!\ldots\!\!\int_{r_1 < x_1} f(r)\, dx_2 \ldots dx_\ell = \frac{2\pi^{\frac{\ell-1}{2}}}{\Gamma(\frac{\ell-1}{2})} \int_0^{x_1} f(r) r_1^{\ell-2} dr_1;$$

thus

$$\phi(\rho) = \int_0^{+\infty} \exp.(-2\pi x_1 \rho)\, dx_1 \frac{\pi^{\frac{\ell-1}{2}}}{\Gamma(\frac{\ell-1}{2})} \int_0^{x_1} f(r)(x_1^2 - r^2)^{\frac{\ell-3}{2}} d(r^2);$$

replacing x_1 and r respectively by r and t we obtain

$$(19.8)\qquad \phi(\rho) = \int_0^{+\infty} \exp.\,(-\,2\pi r\rho)dr\, \frac{\pi^{\frac{\ell-1}{2}}}{\Gamma(\frac{\ell-1}{2})} \int_{0<t<r} (r^2 - t^2)^{\frac{\ell-3}{2}} f(t)d(t^2).$$

Let $\mathcal{L}^*$ be the Laplace transformation of functions of r into functions of ρ:

$$\mathcal{L}^*[g(r)] = \int_{-\infty}^{+\infty} \exp.\,(-\,2\pi r\rho)g(r)dr;$$

let $R = -\,r^2$, $T = -\,t^2$; (19.8) becomes

$$(19.9)\qquad \mathcal{L}^{*-1}[\phi(\rho)] = \frac{\pi^{\frac{\ell-1}{2}}}{\Gamma(\frac{\ell-1}{2})} \int_R^0 (T - R)^{\frac{\ell-3}{2}} f(t)dT.$$

Let us define

$$(19.10)\qquad \boxed{\begin{array}{ll} g(R) = 0 & \text{for } R > 0; \\ g(R) = \mathcal{L}^{*-1}[\phi(\rho)] & \text{for } R = -\,r^2,\ r > 0; \end{array}}$$

<u>let</u> Q <u>run over the space dual to the one over which</u> R <u>runs</u>; (19.9) becomes (see n°10)

$$(19.11)\qquad \boxed{f(r) = \left(-\,\frac{Q}{\pi}\right)^{\frac{\ell-1}{2}} \cdot g(R) \text{ for } R = -\,r^2,\ r > 0.}$$

20. <u>Calculation of</u> $a(p)\cdot f(x)$. Let $a(p)$ be a function of $-\,p_1^2 + p_2^2 + \ldots + p_\ell^2$ such that $\Delta(a)$ is non-void. Theorem 6.2 can not be directly applied; now let us use the <u>convergence factor</u> $b_\varepsilon(\zeta)$ [see Definition 18.1] and define

$$a_\varepsilon(\zeta) = a(\zeta)b_\varepsilon(\zeta), \text{ where } \varepsilon > 0,\ \varepsilon \longrightarrow 0.$$

On the one hand the assumption $\|a(\xi + i\eta)\|_\infty < +\infty$, the properties

$$\|b_\varepsilon(\xi + i\eta)\|_\infty < 1, \qquad \lim_{\varepsilon \to 0} b_\varepsilon(\xi + i\eta) = 1$$

and the definition (5.1) of the symbolic calculus show that

$$(20.1)\qquad \lim_{\mathcal{E}\to 0} a_{\mathcal{E}}(p)\cdot f(x) = a(p)\cdot f(x)$$

$$\lim_{\mathcal{E}\to 0} a_{\mathcal{E}}(p_1, 0, \dots 0) f(x) = a(p_1, 0, \dots 0)\cdot f(x)$$

at each point x, if $p \in \Delta(a)$ and if f(x) is for instance an infinitely differentiable function with compact support.

On the other hand Theorem 6.2 can be applied to $a_{\mathcal{E}}(p)$ and gives

$$a_{\mathcal{E}}(p)\cdot f(x) = \int_X\dots\int k(x-y)f(y)\,dy_1\dots dy_{\ell},$$

where

$$k(x) = \mathcal{L}^{-1}[a_{\mathcal{E}}(2\pi\zeta)];$$

moreover $a_{\mathcal{E}}(\zeta)$ verifies all the conditions which n°19 imposes on $\phi(\zeta)$: for instance, if ξ describes the intersection of $\Delta(a)$ and any line containing 0, then

$$\|a_{\mathcal{E}}(\xi + i\eta)\|_1 \leq \|a(\xi + i\eta)\|_{\infty}\,\|b_{\mathcal{E}}(\xi + i\eta)\|_1 \to 0$$

since $\|a(\xi + i\eta)\|_{\infty}$ is bounded according to (17.3) and $\|b_{\mathcal{E}}(\xi + i\eta)\|_1 \to 0$ according to Lemma 18.1. Thus, upon application of n°19, k(x) is

zero outside the domain $\sqrt{x_2^2 + \dots + x_{\ell}^2} < x_1$,

$$k(r) = \left(-\frac{Q}{\pi}\right)^{\frac{\ell-1}{2}}\cdot \mathcal{l}(R) \text{ inside,}$$

where:

$$\mathcal{l}(R) = 0 \text{ for } R > 0;$$

$$\mathcal{l}(R) = \frac{1}{i}\int_{\zeta_1-i\infty}^{\zeta_1+i\infty} a_{\mathcal{E}}(2\pi\zeta_1, 0, \dots 0)\exp.(2\pi r\zeta_1)\,d\zeta_1,$$

that is

$$(20.2)\quad \ell(R) = \frac{1}{2\pi i} \int_{\xi_1 - i\infty}^{\xi_1 + i\infty} a_{\mathcal{E}}(\zeta_1, 0, \dots 0) \exp.(r\zeta_1) d\zeta_1$$

for $R = -r^2$, $r > 0$, $(\zeta_1, 0, \dots 0) \in \Delta(a)$;

thus

$$(20.3)\qquad a_{\mathcal{E}}(p) \cdot f(x) = \int_{-\infty}^{0} [(-\frac{Q}{\pi})^{\frac{\ell-1}{2}} \cdot \ell(R)] g(R) dR$$

if

$$R = -r^2,\ g(R) = \int \dots \int_{\Omega_R} f(y)\, \omega(y, dy).$$

$\omega(y,dy)$ is an exterior differential form of $y_1, \dots y_\ell$ such that

$$(20.4)\ d[-(y_1 - x_1)^2 + (y_2 - x_2)^2 + \dots + (y_\ell - x_\ell)^2]\, \omega(y,dy) = dy_1 \dots dy_\ell;$$

Ω_R is the half hyperboloid

$$(20.5)\quad -(y_1 - x_1)^2 + (y_2 - x_2)^2 + \dots + (y_\ell - x_\ell)^2 = R < 0,\ x_1 - y_1 > 0$$

with that orientation which makes $\omega(y,dy) > 0$.

Suppose $f(x)$ is for instance an infinitely differentiable function with compact support; (20.3) can be written

$$a_{\mathcal{E}}(p) \cdot f(x) = \int_{-\infty}^{0} \ell(R) [(\frac{Q}{\pi})^{\frac{\ell-1}{2}} \cdot g(R)] dR$$

or

$$(20.6)\qquad a_{\mathcal{E}}(p) \cdot f(x) = \int_{-\infty}^{0} \ell(R) \frac{1}{\pi} \frac{d}{dr} [(\frac{Q}{\pi})^{\frac{\ell-3}{2}} \cdot g(R)] dr$$

where $R = -r^2$, $r < 0$ (when $r > 0$ up to that time).

Now the integral (20.2) is zero if $r < 0$, for (see Lemma 18.1)

$$\lim_{\xi_1 \to \infty} \| a_\varepsilon(\xi_1 \mp i\eta_1, 0, \ldots 0) \|_1 = 0;$$

this fact and Theorem 6.2 give

$$\int_0^{+\infty} \ell(-s^2) h(r-s)\,ds = a(q, 0, \ldots 0) \cdot h(r)$$

where q and r run over dual spaces $(q \cdot = \frac{d}{dr})$, $(q, 0, \ldots 0) \in \triangle(a)$; thus replacing r by 0 and s by $-r$,

(20.7) $\int_{-\infty}^{0} \ell(R) h(r)\,dr = [a(q, 0, \ldots 0) \cdot h(r)]_{r=0}$ where $R = -r^2$, $r < 0$.

Formulas (20.6) and (20.7) prove that the following theorem is true if a is replaced by a_ε. Therefore, upon application of (20.1), this theorem is true:

Theorem 20.1. *If* $a(p)$ *is a function of* $-p_1^2 + p_2^2 + \ldots + p_\ell^2$, *such that* $\triangle(a)$ *is non-void, then*

$$a(p) \cdot f(x) = [\tfrac{q}{\pi}\, a(q, 0, \ldots 0) \cdot h(r)]_{r=0}$$

where

$p \in \triangle(a)$, $(q, 0, \ldots 0) \in \triangle(a)$, $\triangle(a)$ *being of the type* (17.1),

$h(r) = (\frac{Q}{\pi})^{\frac{\ell-3}{2}} \cdot \int \ldots \int_{\Omega_R} f(y)\,\omega(y, dy)$ for $r < 0$, $R = -r^2$, $Q > 0$,

$(\frac{Q}{\pi})^{\frac{\ell-3}{2}} > 0$; q *and* Q *are the dual variables of* r *and* $R = -r^2 \cdot \Omega_R$ *and* $\omega(y, dy)$ *are defined by* (20.4) *and* (20.5); $f(x)$ *is an infinitely differentiable function with compact support.*

21. *Examples.* *Corollary* 21.1. If $\sqrt{p_2^2 + \ldots + p^2} < p_1$, then

$$(21.1)\quad \frac{1}{p_1^2 - p_2^2 - \dots - p_\ell^2} \cdot f(x) = \frac{1}{2\pi}\left[\left(\frac{Q}{\pi}\right)^{\frac{\ell-4}{2}} \cdot \int_{\Omega_R}\!\!\dots\!\int f(y)\,\omega(y,dy)\right]_{R=0}$$

where $Q > 0$, $\left(\frac{Q}{\pi}\right)^{\frac{\ell-4}{2}} > 0$; Q and R are dual variables; Ω_R and $\omega(y,dy)$ are defined by (20.4) and (20.5). For instance ($\ell = 4$):

$$(21.2)\quad \frac{1}{p_1^2 - p_2^2 - p_3^2 - p_4^2} \cdot f(x) = \frac{1}{2\pi} \int_{\Omega_0}\!\!\dots\!\int f(y)\,\omega(y,dy).$$

Proof. For $q > 0$

$$\left[\frac{q}{\pi}\, a(q, 0, \dots 0) \cdot h(r)\right]_{r=0} = \left[\frac{1}{\pi q} \cdot h(r)\right]_{r=0} = \frac{1}{\pi}\int_{-\infty}^{0} h(r)\,dr =$$

$$\frac{1}{2\pi}\int_{-\infty}^{0} \frac{1}{\sqrt{-R}}\, h(r)\; dR = \frac{1}{2\pi}\left[\left(\frac{Q}{\pi}\right)^{-\frac{1}{2}} \cdot h(r)\right]_{R=0}$$

History. (21.2) was given by Poisson in July, 1819; (21.1), for even, was given by Tedone in 1898 and J. Hadamard in 1904 - 1908: see [5], p. 336; (21.1), for ℓ even and odd, was given by M. Riesz in 1949: see [20], p. 93, (A); [M. Riesz gives a concise expression of his formula (A) only for ℓ even; however his formula, where α is to be replaced by 2, does not differ from our formula (21.1)].

CHAPTER III

SYMBOLIC PRODUCT BY $a^{\beta}(p)$, WHERE a IS A POLYNOMIAL AND β A COMPLEX NUMBER; THE CASE $\beta = -1$

(Chapter III uses and completes Gårding [4_1]; its results are expressed in §5; they are used in chapters VII and VIII to solve Cauchy's problem.)

§1. The real projection of the algebraic manifold $a(\zeta) = 0$ and the complement $\Delta(a)$ of the closure of this projection

($\Delta(a)$ is the inside of the set where $\|a^{\beta}(\xi + i\eta)\|_{\infty} < +\infty$ if $\mathcal{R}(\beta) < 0$; therefore the connected components $\Delta_{\alpha}(a)$ of $\Delta(a)$ are convex domains; some $\Delta_{\alpha}(a)$, the $\Delta^*(a)$, have a simpler definition. Let us recall that the symbolic product by the restriction of $a^{\beta}(p)$ to each $\Delta_{\alpha}(a)$ is defined and depends on α.)

22. The convex domains $\Delta_{\alpha}(a)$.

Notation. $a(\xi)$ is a polynomial defined on the vector space Ξ (dim. $\Xi < +\infty$); $h(\xi)$ is its principal part; W is the algebraic manifold

$$a(\zeta) = 0, \text{ where } \zeta = \xi + i\eta \in \Xi + i\Xi;$$

the real projection of the point $\zeta = \xi + i\eta$ is $\pi(\zeta) = \xi$; thus $\pi(W)$ is the set of the $\xi \in \Xi$ such that $a(\xi + i\eta) = 0$ for some $\eta \in \Xi$.

Definition 22.1. $\Delta(a)$ is the complement of $\overline{\pi(W)}$. ($\overline{\pi(W)}$ denotes the closure of $\pi(W)$).

Proposition 22.1. The connected components $\Delta_1(a), \ldots, \Delta_{\alpha}(a), \ldots$ of $\Delta(a)$ are convex domains.

Proof. $\Delta(a)$ is the basis of the greatest tube in which $a^{-1}(\zeta)$ is regular; Bochner's theorem (see n°4) is used.

Proposition 22.2. Let the polynomial $a(\xi)$ be real; $\pi(W)$ is obviously the set of the centers of the chords joining conjugate imaginary points of W; $\overline{\pi(W)}$ contains the real asymptotes of W (i.e. the real lines tangent to W at infinite points); thus $\overline{\pi(W)}$ is the union of $\pi(W)$ and of the real asymptotes of W, if W has not singular, real, infinite points.

Proof. Let the first axis be a real asymptote; Puiseux's theorem (E. Picard, Traité d'analyse, t. II, ch. XIII, §1) shows that a branch of the intersection of W by the plane containing the first and second axes is given by:

$$\zeta_2 = a_1 \zeta_1^{\gamma} + a_2 \zeta_2^{2\gamma} + \dots, \quad \zeta_3 = \dots = \zeta_\ell = 0$$

(ζ_1 near ∞; γ rational and < 0); let ξ_1 be fixed, $\eta_1 \to \infty$ and ζ be the point with coordinates

$$(\zeta_1 = \xi_1 + i\eta_1, \ \zeta_2 = a_1 \zeta_1^{\gamma} + \dots \to 0, \ \zeta_3 = 0, \ \dots \ \zeta_\ell = 0);$$

$\pi(\zeta)$ has the coordinates

$$(\xi_1, \ \xi_2 \to 0, \ 0, \ \dots \ 0)$$

thus $\overline{\pi(W)}$ contains the point $(\xi_1, 0, \dots 0)$ and therefore the first axis.

Let us now study $\| a^{\beta}(\xi + i\eta) \|_{\infty}$ in $\Delta(a)$ for $\mathcal{R}(\beta) < 0$.

($\mathcal{R} \dots$: real part of ... ; $\mathcal{I} \dots$: imaginary part of ...)

Lemma 22.1. Let $P(\lambda)$ be a polynomial of one variable λ; let $H(\lambda)$ be its principal part; if $P(\lambda) \neq 0$ for $-1 < \mathcal{R}(\lambda) < 1$, then $P(0) \geq H(1)$.

<u>Proof.</u> $P(\lambda) = H(1)(\lambda - \lambda_1) \ldots (\lambda - \lambda_n), |\mathcal{R}(\lambda_n)| \geq 1$; thus

$$|P(0)| = |H(1)\lambda_1 \ldots \lambda_n| \geq |H(1)|.$$

<u>Lemma</u> 22.2. If ζ and θ are two points of Ξ such that $a(\zeta + \lambda\theta + i\eta) \neq 0$ for $-1 < \lambda < 1$ and all $\eta \in \Xi$, then $|a(\zeta + i\eta)| > |h(\theta)|$ for all $\eta \in \Xi$.

<u>Proof.</u> According to the assumption,

$$a(\zeta + \lambda\theta + i\eta) \neq 0$$

for all $\eta \in \Xi$ and all complex numbers λ such that $-1 < \mathcal{R}(\lambda) < 1$; Lemma 22.1 is applied to $P(\lambda) = a(\zeta + \lambda\theta + i\eta)$.

<u>Lemma</u> 22.3. Let $P(\lambda)$ be a polynomial of one variable λ and of degree m; if $P(\lambda) \neq 0$ for $0 \leq \lambda \leq 1$, then

$$|\arg. P(1) - \arg. P(0)| < m\pi.$$

<u>Proof.</u> $P(\lambda)$ is the product of m polynomials of degree 1, for each of which this assertion is obvious.

<u>Lemma</u> 22.4. Let $a(\zeta)$ be a polynomial of degree m; in any convex domain where $a(\zeta) \neq 0$,

$$[\text{osc. arg. } a(\zeta)] < m\pi.$$

<u>Proof.</u> Let ζ' and ζ'' be two points of such a domain; let

$$P(\lambda) = a(\zeta' + \lambda(\zeta'' - \zeta'));$$

Lemma 22.3 gives

$$|\arg. a(\zeta') - \arg. a(\zeta'')| < m\pi.$$

<u>Proposition</u> 22.3. <u>Let</u> β <u>be a complex number such that</u> $\mathcal{R}(\beta) < 0$; <u>then</u>

1°. $\log \| a^{\beta}(\xi + i\eta) \|_{\infty}$ is a convex function of ξ;

2°. $\Delta(a)$ is the greatest open set where this function is finite;

3°. if the segment $(\xi - \theta, \xi + \theta)$ belongs to $\Delta(a)$, then one of the branches of $a^{\beta}(\zeta)$ (all its branches for β real) satisfies

$$\| a^{\beta}(\xi + i\eta) \|_{\infty} < |h(\theta)|^{\mathcal{R}(\beta)} \exp.[m\pi \mathcal{J}(\beta)]$$

where m is the degree of $a(\zeta)$.

Proof of 1°. Hardy's theorem (see n°4).

Proof of 3°. According to the assumption

$$a(\xi + \lambda\theta + i\eta) \neq 0 \text{ for } -1 < \lambda < 1, \quad \eta \in \Xi;$$

Lemmas 22.2 and 22.4 are used.

Proof of 2°. According to 3°. $\| a^{\beta}(\xi + i\eta) \|_{\infty}$ is finite when $\xi \in \Delta(a)$. If $\xi \notin \Delta(a)$, then the definition of $\Delta(a)$ and $\pi(W)$ show that ξ is a limit of points of $\pi(W)$, at which $\| a^{\beta}(\xi + i\eta) \|_{\infty} = +\infty$.

23. The domains $\Delta^{*}_{\alpha}(a)$. Now suppose the polynomial $a(\xi)$ is real and denote by V the real part of W, that is the set of ξ such that

$$a(\xi) = 0.$$

Let $\delta(\xi + i\eta)$ be the real line joining the non-real point $\xi + i\eta$ $(\eta \neq 0)$ of W and its conjugate. In the space of the real lines of Ξ, these lines constitute an open set Ω; the complement of Ω is the set of the lines cutting W only at real points; the boundary of Ω is the set of the tangents to V cutting W only at real points.

Let $\pi^{*}(W)$ be the set of all points of all lines belonging to Ω;

the definitions of Π and Π^* show that

(23.1) $$\Pi(W) \subset \Pi^*(W).$$

The properties of Ω give the following ones:

(23.2) $\Pi^*(W)$ is an open subset of Ξ;

(23.3) the boundary of $\Pi^*(W)$ is the set of those points of the tangents to V which are not in $\Pi^*(W)$.

But let ξ be a real point of a tangent to V other than the contact point; ξ belongs obviously to real lines cutting W at non-real points (use Puiseux): $\xi \in \Pi^*(W)$; therefore (23.3) can be improved as follows:

(23.4) the boundary of $\Pi^*(W)$ belongs to V.

Let us call $\Delta^*(a)$ the complement of $V \cup \Pi^*(W)$; that is

Definition 23.1. $\Delta^*(a)$ is the set of points $\xi \in \Xi$ and $\notin V$ such that each real line through ξ cuts W only at real points.

Proposition 23.1. Each connected component of Δ^* is a connected component of Δ, whose boundary belongs to V.

Proof. (23.1) and (23.4) give

$$\overline{\Pi(W)} \subset V \cup \Pi^*(W);$$

the complements of these sets are

$$\Delta^*(a) \subset \Delta(a).$$

According (23.4) $\Delta^*(a)$ is open and its boundary belongs to V, which is outside $\Delta(a)$; now proposition 23.1 is obvious.

Notation. The components $\Delta_\alpha(a)$ of $\Delta(a)$ belonging to Δ^* will be denoted by $\Delta^*_\alpha(a)$.

24. Projective properties of $\triangle^*(a)$. Now let us replace the vector space Ξ by the associated projective space; let $\triangle^*(a)$ be now the set of the points ξ of this projective space satisfying definition 23.1; the image of $\triangle^*(a)$ under any projective mapping of Ξ is obviously the $\triangle^*$-domain associated to the image of V. Thus proposition 23.1 proves the following proposition 24.1. 1°.:

Proposition 24.1. 1°. To each $\triangle_1^*$, whose director cone has an interior, is associated another $\triangle_\alpha^*$, having the opposite director cone.

2°. A convenient projective mapping maps $\triangle_1^*$ or, if $\triangle_1^*$ has an associated domain $\triangle_2^*$, maps $\triangle_1^* \cup \triangle_2^*$ into a bounded convex domain of the type.

(24.1) $\quad \Xi_1 + B_2$ (B_2: bounded and convex domain of Ξ_2),

where Ξ_1 and Ξ_2 are subspaces of the vector space Ξ such that

$$\Xi_1 + \Xi_2 = \Xi \text{ (direct sum).}$$

Proof of 2°. Suppose ~~at~~ first that $\triangle_1^*$ has an associated domain $\triangle_2^*$; there is a hyperplane separating these two convex domains (see: Seminar on convex sets, p. 73); let this hyperplane become the infinite hyperplane: $\triangle_1^* \cup \triangle_2^*$ becomes a domain $\triangle_1^*$ without associated domain. Therefore it is sufficient to treat the case where $\triangle_1^*$ is a non-bounded domain without associated domain.

If the director cone Γ_1 of $\triangle_1^*$ does not contain a line, there is obviously a hyperplane whose intersection by Γ_1 is the vertex of Γ_1; some parallel hyperplane does not meet the closure of $\triangle_1^*$; let this hyperplane become the infinite hyperplane; $\triangle_1^*$ becomes a bounded and convex domain. Therefore it is sufficient to treat the case where Γ_1 contains a line.

Then $a(\xi)$ depends on less than ℓ linear functions (see Proposition 27.1): W and therefore Δ_1^* are cylinders; it is sufficient to prove the proposition for their bases; thus an induction on ℓ achieves the proof.

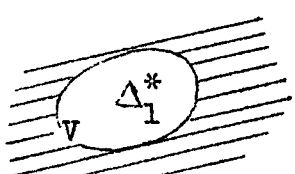
Proposition 24.2. *Suppose that the dimension of the set of the singular real points of* V *is* $< \ell - 2$. *Then there are at most two domains* Δ_ℓ^*, *and, if there are two such domains, they have opposite director cones.*

Proof. Assume this last assertion false; Proposition 24.1 enables us to replace this assumption by the following: there exist a domain Δ_1^* of the type (24.1) and a point $\xi \in \Delta^*$, but $\notin \overline{\Delta}_1^*$. The cone with vertex ξ circumscribing Δ_1^* has the dimension $\ell - 1$; but a line through ξ can not be tangent to the boundary of Δ_1^*, because this boundary belongs to V and $\xi \in \Delta^*$; thus this cone circumscribes Δ_1^* along a $(\ell - 2)$-dimensional manifold, all of whose points are singular points of V; therefore the assumption of Proposition 24.2 is false.

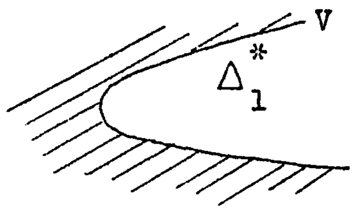
25. *Example.* $a(\xi)$ *is a real polynomial with degree* 2. Obviously: all chords with center ξ of a quadric surface W are in the hyperplane containing both ξ and the polar of the line joining ξ and the center of W; the intersection of W and this hyperplane is an ellipsoid with real points, if and only if $\xi \neq \pi(W)$ (see Proposition 22.2).

Therefore: *if* $a(\xi)$ *is a real polynomial of degree* 2, then $\Delta(a)$ *is empty except in the following cases:*

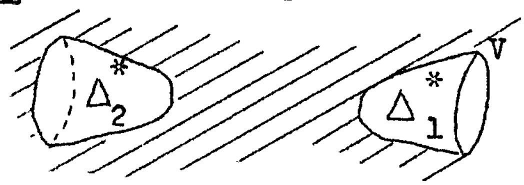
1°. V *is an ellipsoid containing real points.* (That is: the image of $\xi_1^2 + \ldots + \xi_\ell^2 = 1$ by some linear mapping of Ξ); V is the boundary of $\Delta = \Delta_1^*$.



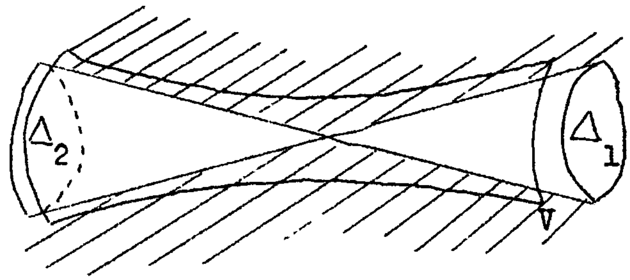
2°. V is an elliptical paraboloid; (the image of $2\xi_1 = \xi_2^2 + \dots + \xi^2$ by some linear mapping of Ξ); V is the boundary of $\Delta = \Delta_1^*$.



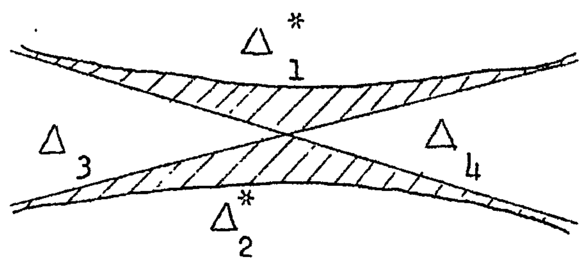
3°. V is a convex hyperboloid and $\ell > 2$; the image of $\xi_1^2 = \xi_2^2 + \dots + \xi^2 + 1$ by some linear mapping of Ξ); $\Delta = \Delta^* = \Delta_1^* \cup \Delta_2^*$; the boundary of Δ is V.



4°. V is a no convex hyperboloid with convex director cone and $\ell > 2$; (that is: the image of $\xi_1^2 = \xi_2^2 + \dots + \xi_\ell^2 - \text{const.}$, where const. ≥ 0); $\Delta = \Delta_1 \cup \Delta_2$; the boundary of Δ is the asymptotic cone of V.



5°. V is a hyperbola and $\ell = 2$; $\Delta = \Delta_1^* \cup \Delta_2^* \cup \Delta_3 \cup \Delta_4$; the boundary of Δ^* is the hyperbola; the boundary of $\Delta_3 \cup \Delta_4$ is the union of the two asymptotes.



6°. V is a cylinder whose basis is of the type 1°., 2°., 3°., 4°., 5°., or 6°.

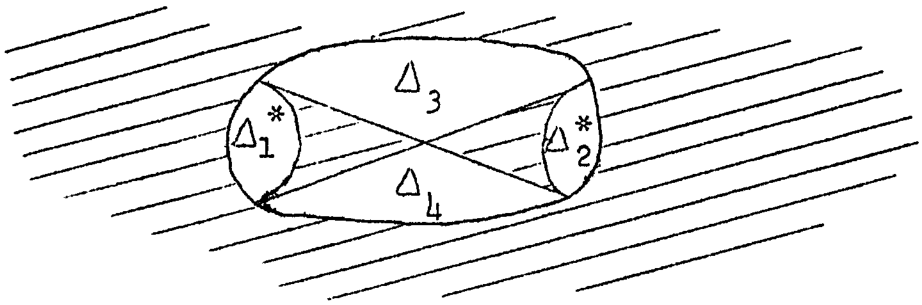
26. Further examples.

Proposition 26.1. $\Delta(a_1 a_2) = \Delta(a_1) \cap \Delta(a_2)$.

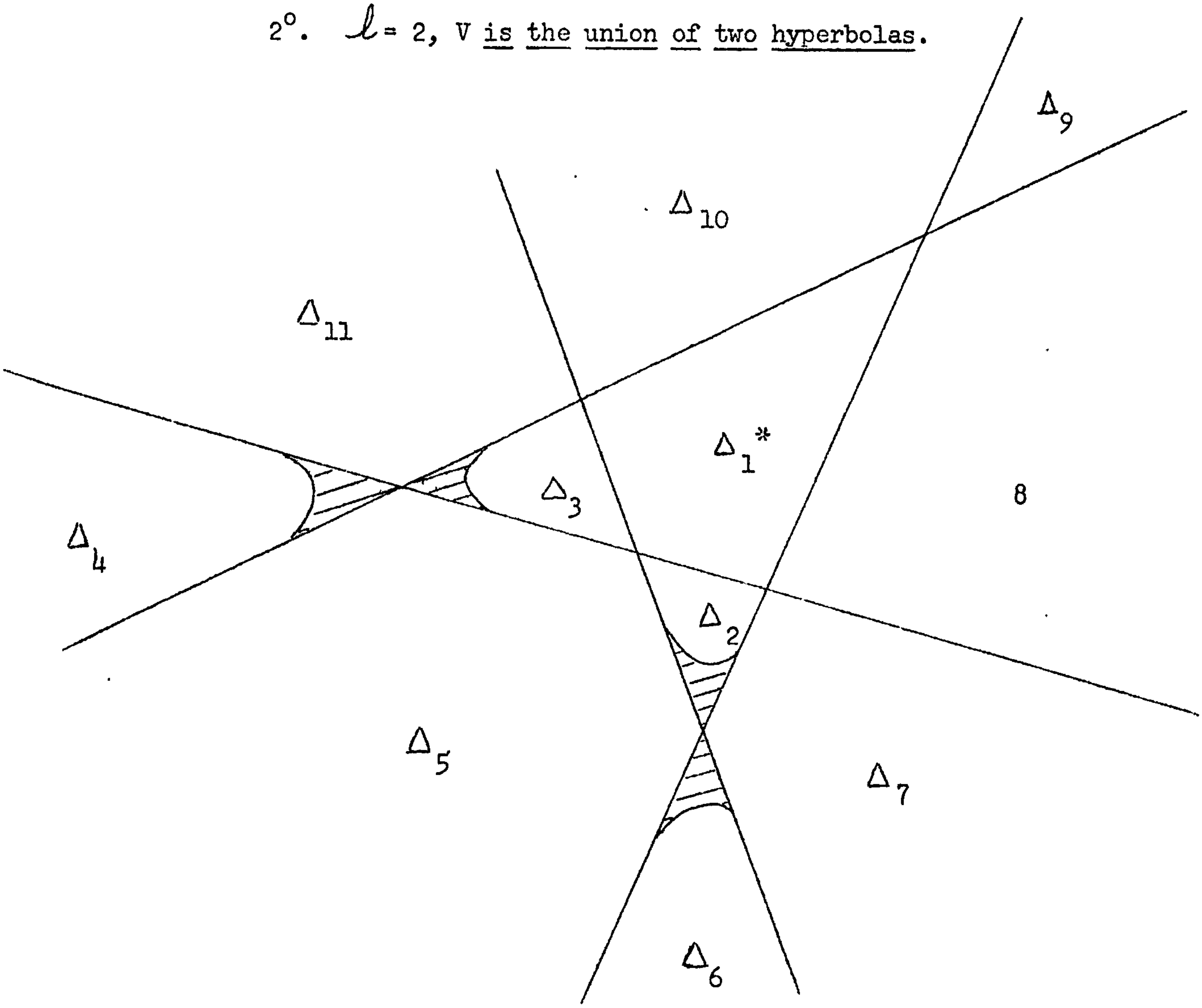
Proof. $\pi(V_1 \cup V_2) = \pi(V_1) \cup \pi(V_2)$; definition 22.1.

This proposition and n°25 enable us to treat the case where V is a union of quadric surfaces. For instance:

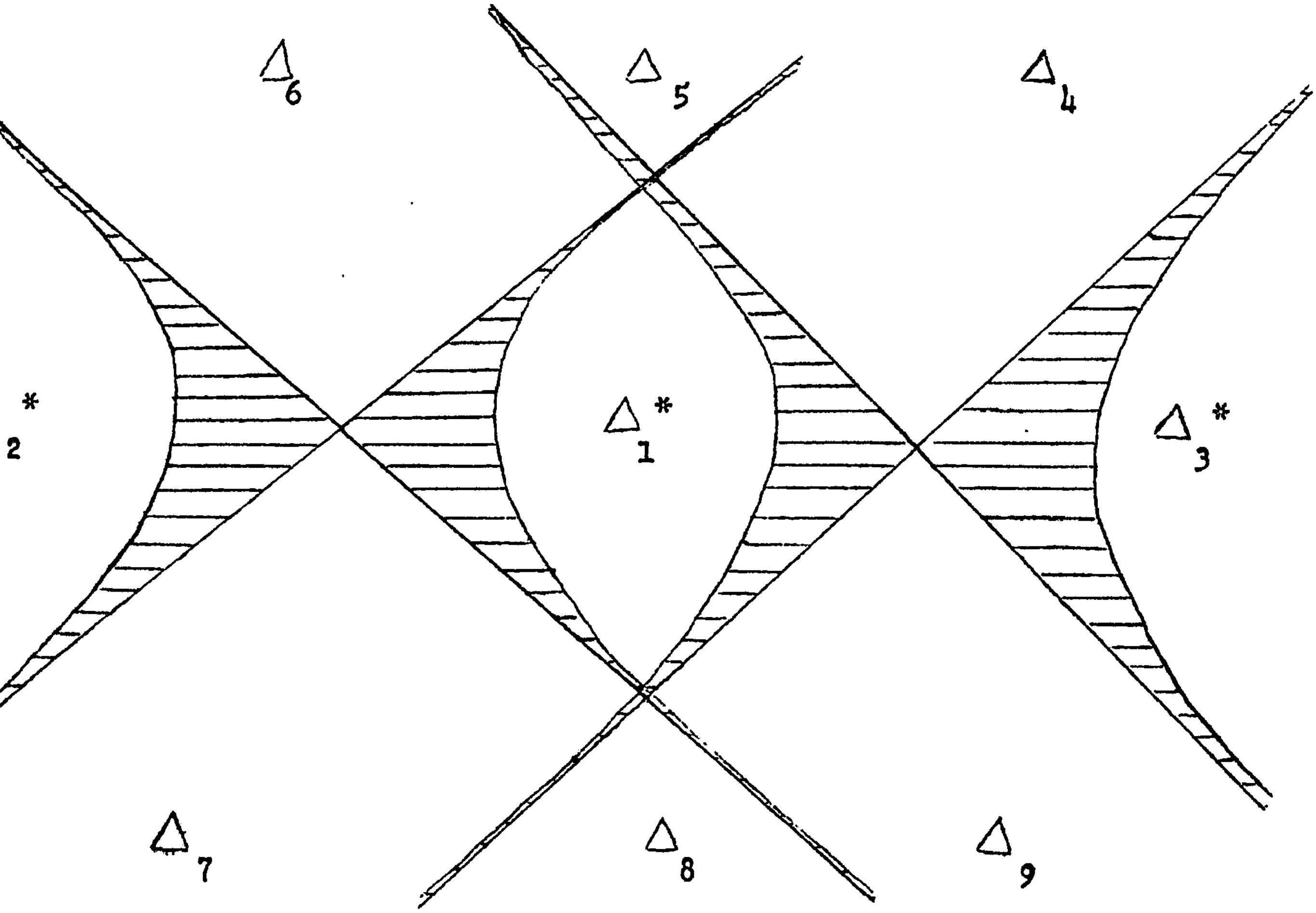
1°. $\ell = 2$, V is the union of an ellipse and a hyperbola.



2°. $\ell = 2$, V is the union of two hyperbolas.



3°. V <u>is the union of two hyperboles with parallel asymptotes</u>.



§2. <u>The director cone</u> $\Gamma(a)$ <u>of</u> $\Delta(a)$ <u>and its dual</u> $C(a)$.

(§2 gives Gårding's relation between $\Gamma(a)$ and $\Delta(h) = \Delta^*(h)$ and Gårding's theorem asserting that the dependence domain of $a^\beta(p)$ for $p \in \Delta_\alpha(a)$ is the cone $C_\alpha(a)$ dual to $\Gamma_\alpha(a)$.)

27. <u>The director cone</u> $\Gamma_\alpha(a)$ <u>of</u> $\Delta_\alpha(a)$. Let $\Gamma_\alpha(a)$ be the director cone with vertex 0 of a non-bounded $\Delta_\alpha(a)$; $\Gamma_\alpha(a)$ is closed; let $\mathring{\Gamma}_\alpha(a)$ be its interior; let $\Gamma(a)$ be the union of the $\Gamma_\alpha(a)$. Obviously

$$\Delta_\alpha(a) + \Gamma_\alpha(a) \subset \Delta_\alpha(a) \,. \tag{27.1}$$

<u>Proposition</u> 27.1. <u>If</u> $\Gamma_\alpha(a)$ <u>contains a line, then</u> $a(\xi)$ <u>depends on less than</u> ℓ <u>linear functions of</u> ξ.

<u>Proof</u>. Assume $\pm\gamma \in \Gamma_\alpha$ and $\delta \in \Delta_\alpha$; then, according to (27.1)

$$\delta + \lambda\gamma \in \Delta_\alpha \text{ for all real number } \lambda;$$

therefore

$$a(\delta + \lambda\gamma) \neq 0 \text{ for all complex numbers } \lambda;$$

thus

$$a(\delta + \lambda\gamma) = a(\delta) \text{ for all } \delta.$$

<u>Proposition</u> 27.2. <u>Let</u> $h(\xi)$ <u>be the principal part of</u> $a(\xi)$; <u>let</u> $V(h)$ <u>be the cone</u> $h(\xi) = 0$; <u>then</u>

1°. $\Delta(h) = \Delta^*(h)$;

2°. $\Gamma(a) \subset \Delta(h) \cup V(h)$; <u>thus</u> $\mathring{\Gamma}(a) \subset \Delta(h)$;

3°. $(a) = \quad (h)$ <u>if</u> $V(h)$ <u>has no real singular generator</u>.

<u>Remark</u>. It can happen that

$$\mathring{\Gamma}(a) \neq \Delta(h);$$

indeed, if $V(a)$ is a parabola, then $\overset{\circ}{\Gamma}(a)$ is void, but $\Delta(h) = \Xi$.

Proof of 1°. Let $\xi \notin \Delta^*(h)$; then

$$h(\xi + \lambda\eta) = 0 \text{ for some } \eta \in \Xi \text{ and some non-real number } \lambda ;$$

therefore

$$h((\mu + i\nu)\xi + i\eta) = 0, \text{ where } \mu + i\nu = i\lambda^{-1}, \ \mu \neq 0;$$

thus

$$h(\xi + i\eta') = 0, \text{ where } \eta' = \frac{\nu\xi + \eta}{\mu} \in \Xi ; \tag{27.2}$$

thus

$$\xi \notin \Delta(h) ;$$

that is

$$\Delta(h) \subset \Delta^*(h) .$$

Proposition 23.1 achieves the proof.

Proof of 2°. Let $\xi \notin \Delta(h) \cup V(h)$; according to (27.2) there is an $\eta \in \Xi$ such that

$$h(\xi + i\eta) = 0 , \quad h(\xi) \neq 0 ;$$

let $\delta \in \Delta_\alpha(a)$ and $\rho \longrightarrow +\infty$; the equation with unknown λ

$$a(\delta + \lambda\xi + i\rho\eta) = 0$$

has a root $\lambda = \sigma + i\tau$ such that $\lambda/\rho \longrightarrow 1$; the equation

$$a(\delta + \sigma\xi + i(\tau\xi + \rho\eta)) = 0$$

shows that $\delta + \sigma\xi \notin \Delta(a)$, where $\sigma \longrightarrow +\infty$; thus, according to (27.1),

$$\xi \notin \Gamma_\alpha(a) .$$

Proof of 3°. Let A be the union of the real asymptotes of $W(a)$, I be the set of the infinite points of $W(a)$, N be a neighborhood of I in $W(a)$ and M be the complement of N in $W(a)$;

$$\pi(W) = \pi(M) \cup \pi(N) .$$

$\pi(M)$ is bounded; $\pi(N) \subset \pi^*(N)$; $\pi^*(N)$ is arbitrarily near the closed set $A \cup \pi^*(I)$. Let ξ be any infinite point outside this closed set: ξ has in the projective space Ξ a neighborhood outside $\pi(W)$; that is

$$\xi \in \mathring{\Gamma}(a).$$

But ξ is any infinite point of $\Delta(h)$; thus $\Delta(h) \subset \mathring{\Gamma}(a)$.

<u>Proposition</u> 27.3. <u>If the cone</u> $V(h)$ <u>has no real singular generator and</u> $\ell > 2$, <u>then either all</u> $\mathring{\Gamma}_\alpha(a)$ <u>are void or there are two non-void and opposite</u> $\mathring{\Gamma}_\alpha(a)$.

<u>Proof</u>. Proposition 24.2 and 27.3°.

28. <u>The cone dual to</u> $\Gamma(a)$. Let us define <u>the cone dual to the cone</u> $\Gamma_\alpha(a)$; it is the set of $x \in X$ such that

$$x \cdot \xi \geq 0 \text{ for all } \xi \in \Gamma_\alpha(a); \tag{28.1}$$

this cone is convex and closed. Its <u>interior</u> is the set of the $x \in X$ such that

$$x \cdot \xi > 0 \text{ for all } \xi \in \Gamma_\alpha(a). \tag{28.2}$$

The cone (28.1) is the closure of its interior, if it has an interior (because it is convex); it has an interior if and only if $\Gamma_\alpha(a)$ does not contain any line.

Let β be a complex number such that $\mathcal{R}(\beta) \leq 0$; let $D_{\alpha\beta}$ be the set of the $x \in X$ such that

$$\operatorname*{Inf.}_{\xi \in \Delta_\alpha} [x \cdot \xi + \log \| a^\beta(\xi + i\eta) \|_\infty] > -\infty; \tag{28.3}$$

that is the definition (7.2) where D, Δ and $a(\zeta)$ are replaced by $D_{\alpha\beta}$, Δ_α and $a^\beta(\zeta)$. Let us likewise define $C_{\alpha\beta}$ by replacing Δ and $a(\zeta)$

by Δ_α and $a^\beta(\zeta)$ in (7.1).

Lemma 28.1. $D_{\alpha\beta}$ is contained in the cone (28.1) and contains the cone (28.2).

Proof that (28.3) implies (28.1). Let x be a point of X such that (28.1) does not hold: there is a $\gamma \in \Gamma_\alpha$ such that $x \cdot \gamma < 0$. Let $\delta \in \Delta_\alpha$; there is a $\theta \in \Xi$ such that $\delta \pm \theta \in \Delta_\alpha$, $h(\theta) \neq 0$. Let λ be a real number $\to +\infty$;

$$\delta + \lambda\gamma \in \Delta_\alpha\,;\ \delta \pm \theta + \lambda\gamma \in \Delta_\alpha\,;$$

Proposition 22.3, 3° shows that

$$\operatorname{Sup.}_\lambda \| a^\beta(\delta + \lambda\gamma + i\eta)\|_\infty < +\infty;$$

therefore

$$x \cdot (\delta + \lambda\gamma) + \log\| a^\beta(\delta + \lambda\gamma + i\eta)\|_\infty \to -\infty;$$

thus (28.3) does not hold.

Proof that (28.2) implies (28.3). Let x be a point of X such that (28.2) holds: there is a constant $\varepsilon > 0$ such that

$$x \cdot \xi > \varepsilon\|\xi\| \text{ for } \xi \in \Delta_\alpha,\ \|\xi\| \text{ sufficiently large.}$$

But

$$\log\|a^\beta(\xi + i\eta)\|_\infty \geq \log|a^\beta(\xi)| \geq \text{const.} + \text{const.}\log\|\xi\|.$$

Thus

$$x \cdot \xi + \log\|a^\beta(\xi + i\eta)\|_\infty \to +\infty \text{ for } \xi \in \Delta_\alpha,\ \|\xi\| \to +\infty;$$

therefore (28.3) holds.

Lemma 28.2. If $\Gamma_\alpha(a)$ does not contain any line, then

$$C_{\alpha\beta} = \overline{D}_{\alpha\beta} = \text{dual of } \Gamma_\alpha(a).$$

Proof. The open cone (28.2) is non-void; its closure is the cone (28.1); Lemmas 28.1 and 7.5 are used.

Proposition 28.1. $C_{\alpha\beta} = \overline{D}_{\alpha\beta}$ = dual of Γ_α.

Proof. Lemma 28.2 and Proposition 27.1 show that it is sufficient to treat the case where $a(\xi)$ is independent of ξ_1. Then by replacing X by its hyperplane $x_1 = 0$ we do not change either $C_{\alpha\beta}$, $D_{\alpha\beta}$ (see Proposition 7.2) or the cone dual to Γ_α [see its definition (28.1) where $(\xi_1, 0, \ldots, 0) \in \Gamma_\alpha(a)$]. Therefore an induction on ℓ proves the proposition.

§3. The convex domain $\Delta_\alpha(a, \beta, b)$ such that the operator $b(p)\, a^\beta(p)$ is bounded for $p \in \Delta_\alpha(a, \beta, b)$, where $b(p)$ is a polynomial.

(It is useful to complete Gårding's results by the following: some particular cases of which, for instance, have already been proved and used in Chapter II to study the convergence factor of §3).

29. Lower bounds of $|a(\xi + i\eta)|$. Lemma 29.1. Let $P(\lambda)$ be the real polynomial

$$P(\lambda) = a_0 + a_1\lambda + \dots + a_m\lambda^m ,$$

where $a_0 \neq 0$: $P(\lambda)$ has m roots, always $\neq 0$, but allowed to be infinite. When λ is real, then

$$|P(i\lambda)|^2 = (a_0 - a_2\lambda^2 + \dots)^2 + (a_1\lambda - a_3\lambda^3 + \dots)^2$$

$$= A_0 + \dots + A_\mu\lambda^{2\mu} + \dots + A_m\lambda^{2m} ,$$

where $A_\mu(a_0, a_1, \dots, a_m)$ is a quadratic form:

$$A_0 = a_0^2, \quad A_1 = a_1^2 - 2a_0a_2, \quad \dots, \quad A_{m-1} = a_{m-1}^2 - 2a_ma_{m-2}, \quad A_m = a_m^2 .$$

1°. $a_\mu = a_{\mu+1} = \dots = a_m = 0$ implies $A_\mu = 0$.

2°. If all roots of $P(\lambda)$ are real, then:

$$A_\mu(a_0, a_1, \dots, a_m) \geqq 0 ;$$

$$A_\mu(a_0, a_1, \dots, a_m) = 0 \text{ implies } a_\mu = a_{\mu+1} = \dots = a_m = 0 .$$

Proof of 1°. If $a_\mu = a_{\mu+1} = \dots = a_m = 0$, then $P(\lambda)$ has a degree $< \mu$ and therefore $|P(i\lambda)|^2$ has a degree $< 2\mu$.

Proof of 2°. The roots of $P(\lambda)$ are m infinite or real numbers $\neq 0$: $\lambda_1, \dots, \lambda_m$. The polynomial

$$A_o + \dots + A_\mu \lambda^{2\mu} + \dots + A_m \lambda^{2m},$$

being $|P(i\lambda)|^2$ for λ real, has the roots

$$\pm i\lambda_1, \dots, \pm i\lambda_m ;$$

thus the polynomial of Λ

$$A_o + \dots + (-1)^\mu A_\mu \Lambda^\mu + \dots + (-1)^m A_m \Lambda^m$$

has the roots $\lambda_1^2, \dots, \lambda_m^2$; therefore

$$\sum \frac{1}{\lambda_{\beta_1}^2 \cdots \lambda_{\beta_\mu}^2} = \frac{A_\mu}{A_o}, \text{ where } A_o = a_o^2 > 0 ;$$

now the assertion 2° is obvious.

Lemma 29.2. Let $a(\xi)$ be a real polynomial of degree m; let $h(\xi)$ be its principal part; let us decompose $|a(\xi + i\eta)|^2$ into components homogeneous in η:

$$|a(\xi + i\eta)|^2 = A_o(\xi) + \dots + A_\mu(\xi, \eta) + \dots A_m(\eta)$$

where $A_\mu(\xi, \eta)$ is homogeneous and has the degree 2μ in η;

$$A_o(\xi) = a^2(\xi) ; \quad A_m(\eta) = h^2(\eta) .$$

1°. If the line $\xi + \lambda\eta$ $(-\infty < \lambda < +\infty)$ cuts V at infinity more than $(m - \mu)$-times, then $A_\mu(\xi, \eta) = 0, \dots A_m(\eta) = 0$.

2°. Let $\xi \in \triangle^*(a)$; then

$$A_\mu(\xi, \eta) \geq 0 \text{ for all } \mu ;$$

if $A_\mu(\xi, \eta) = 0$, then the infinite point in the direction of η is a point of V whose order is $> m - \mu$.

<u>Proof of 1°.</u> Let

$$P(\lambda) = a(\xi + \lambda\eta) = a_o + a_1\lambda + \dots + a_m\lambda^m ;$$

if the line $\xi + \lambda\eta(-\infty < \lambda < +\infty)$ cuts V at infinity more than $(m-\mu)$-times, then $a_\mu = a_{\mu+1} = \dots = a_m = 0$; Lemma 29.1.1° is used.

<u>Proof of 2°.</u> Let $\xi \in \triangle^*(a)$: all roots of $P(\xi + \lambda\eta)$ are real; Lemma 29.1.2° gives: $A_\mu \geq 0$; if $A_\mu = 0$, then the line $\xi + \lambda\eta(-\infty < \lambda < +\infty)$ cuts V at infinity more than $(m-\mu)$-times. But this line is not an asymptote of V (see Proposition 22.2 and Definition 22.1).

<u>Lemma 29.3.</u> Assume that V(a) has no real, infinite point of order $> \mu$ and that ξ belongs to some compact part of $\triangle^*(a)$; then

$$|a(\xi + i\eta)| > \text{const.}\ (1 + \|\eta\|)^{m-\mu} \qquad (\text{const.} > 0)$$

<u>Proof.</u> Lemma 29.2.2° gives

$$|a(\xi + i\eta)|^2 \geq a^2(\xi) + A_{m-\mu}(\xi, \eta), \quad a(\xi) \neq 0, \quad A_{m-\mu}(\xi, \eta) > 0$$

for $\xi \in \triangle^*(a)$, $\eta \neq 0$; $A_{m-\mu}$ is homogeneous and has the degree $2(m - \mu)$ in η.

<u>Lemma 29.4.</u> Assume that the cone V(h) has no real singular point except its vertex, and that ξ belongs to some closed cone $\subset \mathring{\Gamma}_\alpha(a)$; then for $\|\xi\|$ sufficiently large,

$$|a(\xi + i\eta)| \geq \text{const.} \|\xi\| [\|\xi\| + \|\eta\|]^{m-1} .$$

<u>Proof.</u> We have

$$|a(\xi + i\eta)|^2 = a^2(\xi) + \dots + A_\mu(\xi, \eta) + \dots + h^2(\eta)$$

and similarly

$$|h(\xi + i\eta)|^2 = h^2(\xi) + \dots + H_\mu(\xi, \eta) + \dots + h^2(\eta) ;$$

obviously $H_\mu(\xi, \eta)$ is the principal part of $A_\mu(\xi, \eta)$; but the Proposition 27.2. (1° and 2°) and the Lemma 29.2.2° show that

$$H_\mu(\xi, \eta) > 0 \quad \text{for} \quad \eta \neq 0 , \qquad 1 < \mu < m ;$$

therefore

$$A_\mu(\xi, \eta) > 0 \quad \text{for} \quad \eta \neq 0, \quad 1 < \mu < m , \quad \|\xi\| > \text{bound};$$

therefore, when $\|\xi\| >$ bound,

$$|a(\xi + i\eta)|^2 \geq a^2(\xi) + A_{m-1}(\xi, \eta)$$

$$> \text{const.}\ [\|\xi\|^{2m} + \|\xi\|^2 \|\eta\|^{2(m-1)}] .$$

The following lemma completes the statement of Lemma 29.3:

Lemma 29.5. Assume that $V(h)$ is void and that ξ belongs to some compact part of $\triangle(a)$; then

$$|a(\xi + i\eta)| > \text{const.}\ [1 + \|\eta\|]^m .$$

Proof. $a(\xi + i\eta) \neq 0$ and its principal part is $i^m h(\eta) \neq 0$ for $\eta \neq 0$.

30. The domains $\triangle_\alpha(a, \beta, b)$. Notation. $a(\xi)$ and $b(\xi)$ are two polynomials of degrees m and n, β is a complex number, whose real part is $\Re(\beta) < 0$.

Definition. $\triangle_\alpha(a, \beta, b)$ is the set of the points $\xi \in \triangle_\alpha(a)$ such that

$$\| b(\xi + i\eta)\, a^\beta(\xi + i\eta) \|_\infty < + \infty ;$$

$\triangle_\alpha(a, \beta, b)$ is denoted by $\triangle^*_\alpha(a, \beta, b)$ when $\triangle_\alpha(a)$ is denoted by $\triangle^*_\alpha(a)$.

Proposition 30.1. $\triangle_\alpha(a, \beta, b)$ is either void or a convex domain.

<u>Proof.</u> It is sufficient to show that, if ξ, ξ', and $\xi'' \in \Delta(a)$ and have the coordinates

$$(\xi_1, 0, \ldots, 0), \ (\xi'_1, 0, \ldots, 0) \text{ and } (\xi''_1, 0, \ldots, 0), \text{ where } \xi'_1 \leq \xi_1 \leq \xi''_1,$$

then

$$\| b(\xi + i\eta) \, a^{\beta}(\xi + i\eta) \|_{\infty}$$

is less than the larger of the numbers

$$\| b(\xi' + i\eta) \, a^{\beta}(\xi' + i\eta) \|_{\infty} \quad \text{and} \quad \| b(\xi'' + i\eta) \, a^{\beta}(\xi'' + i\eta) \|_{\infty} \ ;$$

obviously it is sufficient to prove this when $b(\xi)$ and $a(\xi)$ depend only on ξ_1; but in this case, since $\| a^{\beta}(\xi + i\eta) \|_{\infty} < +\infty$ for $\xi \in \Delta_{\alpha}(a)$, it follows from the Phragmen-Lindelöf principle (see: G. Julia, Principes géométriques d'analyse, t. II, I, n°11, p.14; Paris 1932).

<u>Proposition 30.2.</u> <u>Assume that</u> $a(\xi)$ <u>is real and that</u> $V(h)$ <u>is void;</u> <u>then</u>

$$\Delta_{\alpha}(a, \beta, b) = \Delta_{\alpha}(a) \quad \underline{\text{for}} \quad n + m\mathcal{R}(\beta) \leq 0.$$

<u>Proof.</u> Lemmas 22.4 and 29.5.

<u>Proposition 30.3.</u> <u>Assume that</u> $a(\xi)$ <u>is real and that</u> $V(a)$ <u>has no infinite point of order</u> $> \mu$; <u>then</u>

$$\Delta^*_{\alpha}(a, \beta, b) = \Delta^*_{\alpha}(a) \quad \text{if} \quad n + (m - \mu)\mathcal{R}(\beta) \leq 0.$$

<u>Proof.</u> Lemmas 22.4. and 29.3.

<u>Proposition 30.4.</u> <u>Assume that</u> $a(\xi)$ <u>is real, that</u> $V(h)$ <u>has no real singular point except its vertex, that</u> $\overset{\circ}{\Gamma}_{\alpha}(a)$ <u>is non-void and that</u> $n+(m-1)\mathcal{R}(\beta) \leq 0$; <u>then</u> $\Delta_{\alpha}(a, \beta, b)$ <u>is non-void; its director cone is</u> $\Gamma_{\alpha}(a)$; <u>on any closed cone</u> $\subset \overset{\circ}{\Gamma}(a)$,

$$\| b(\xi + i\eta) \, a^{\beta}(\xi + i\eta) \|_{\infty} \longrightarrow 0 \ \underline{\text{for}} \ \|\xi\| \longrightarrow +\infty \ .$$

<u>Proof.</u> Lemmas 22.4 and 29.4.

§4. The elementary solution

(Theorem 6.2 asserted that the symbolic product by $a^{\beta}(p)$ is the convolution by a distribution; in important cases this distribution is a function defined by an ℓ-tuple integral; if $\beta = -1$, Cauchy's residue formula replaces this integral by an $(\ell-1)$-tuple one; this result is classical in Heaviside's case: $\ell = 1$; it is important in solving Cauchy's problem.)

31. Definition. Let us call the elementary solution of $a(p)$ corresponding to $\Delta_\alpha(a)$ the distribution

(31.1) $$k_x = \mathcal{L}^{-1}[a^{-1}(2\pi\xi)], \text{ where } \xi \in \Delta_\alpha(a);$$

obviously

(31.2) $$a(p) \cdot k_x = \text{Dirac's measure (see [21])};$$

according to theorem 6.2

(31.3) $$a^{-1}(p) \cdot f(x) = k_x * f(x) \text{ for } p \in \Delta_\alpha(a).$$

Note. (31.2) is Schwartz's definition of the elementary solution; Hadamard and Bureau use another, not equivalent definition.

32. The $(\ell-1)$-tuple integral giving $k(x)$. Let us suppose

(32.1) $$||a^{-1}(\xi + i\eta)||_1 < +\infty \text{ for } \xi \in \text{some open subset of } \Delta_1(a);$$

(lemmas 29.3, 29.4, and 29.5 give cases where this happens); then, according to theorem 6.2, k_x is the function

$$k(x) = \frac{1}{2(\pi i)^{\ell}} \int \cdots \int_{\xi+i\Xi} a^{-1}(\zeta) \exp(x \cdot \zeta) d\zeta_1 \ldots d\zeta_\ell,$$

Cauchy's residue formula gives for $x_1 > 0$

$$\frac{1}{2\pi i}\int_{\xi_1 - i\infty}^{\xi_1 + i\infty} \frac{\exp.(x \cdot \zeta)}{a(\zeta)}\, d\zeta_1 = \sum_{\zeta^*} \frac{\exp.(x \cdot \zeta^*)}{a_1'(\zeta^*)},$$

where:

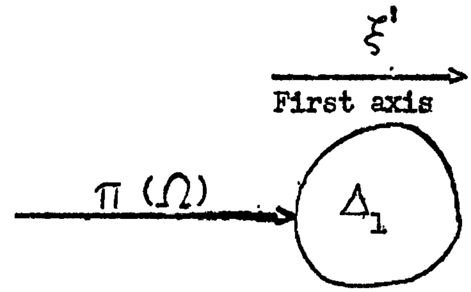
$$a_1'(\zeta) = \frac{\partial a(\zeta_1, \dots \zeta_\ell)}{\partial \zeta_1}, \quad a(\zeta^*) = 0;$$

$$\zeta^*, = (\zeta_1^*, \zeta_2, \dots \zeta_\ell), \ \mathcal{R}(\zeta_1^*) = \xi_1^* < \xi_1.$$

Hence

$$(32.2) \quad k(x) = \frac{1}{(2\pi i)^{\ell - 1}} \int \dots \int_\Omega \frac{\exp.(x \cdot \zeta)}{a_1'(\zeta)}\, d\zeta_2 \dots d\zeta_\ell$$

where Ω is the part of W whose real projection $\pi(\Omega)$ belongs to a half line parallel to the first axis and ending in $\Delta_1(a)$; the orientation of Ω is such that $d\eta_2 \dots d\eta_\ell > 0$ if $x_1 > 0$; the integral is absolutely convergent after *summing* the terms corresponding to the same values of $\zeta_2, \dots \zeta_\ell$: this summing is to be done at first; if $a_1'(\zeta) = 0$, then $\frac{\exp.(x \cdot \zeta)}{a_1'(\zeta)}$ *has to be replaced by the residue of the function* $\frac{\exp.(x \cdot \zeta)}{a(\zeta)}$ *of* ζ_1.



This result can be expressed as follows, the first coordinate axis being replaced by a vector ξ':

Proposition 32.1. *Suppose* (32.1) *holds. Let* $\omega(\zeta, d\zeta)$ *be an differential form such that*

$$(32.3) \qquad da(\zeta) \cdot \omega(\zeta, d\zeta) = d\zeta_1 \dots d\zeta_\ell;$$

$\omega(\zeta, d\zeta)$ is defined mod. da, so that its restriction to W is defined without ambiguity. Let ξ' be some po... of Ξ such that

(32.4) $$x \cdot \xi' > 0.$$

Let Ω be the part of W, whose real projection belongs to some half line parallel to ξ' and ending in $\Delta_1(a)$; the orientation of Ω to be used is such that

(32.5) $$\frac{1}{i^{\ell-1}}\left[\xi_1' \frac{\partial a(\zeta)}{\partial \zeta_1} + \dots + \xi_\ell' \frac{\partial a(\zeta)}{\partial \zeta_\ell}\right]\omega(\zeta, d\zeta) > 0.$$

Then the elementary solution of a(p) corresponding to $\Delta_1(a)$ is

(32.6) $$k(x) = \frac{1}{(2\pi i)^{\ell-1}} \int \cdots \int_{\Omega} \exp.(x \cdot \zeta)\,\omega(\zeta, d\zeta);$$

the elements of this integral corresponding to the points of Ω belonging to a complex line parallel to ξ' have to be summed: the integral thus becomes absolutely convergent. Let us recall (theorem 6.2) that

(32.7) $$\| k(x) \exp.(-x \cdot \xi) \|_q < +\infty \text{ for } 2 \leq q \leq +\infty.$$

33. The case where $\overset{\circ}{\Gamma}_1(a)$ is non-void. Lemma 7.1 shows that $k(x) = 0$ outside $C_1(a)$; suppose

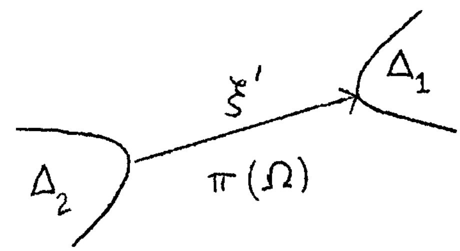
$$x \in C_1(a) = \text{dual of } \Gamma_1(a) \text{ (see proposition 28.1);}$$

definition (28.1) enables us to choose

$$\xi' \in \overset{\circ}{\Gamma}_1(a);$$

suppose that there is a $\Delta_2(a)$ with director cone opposite to $\Gamma_1(a)$; the half line parallel to ξ', containing $\pi(\Omega)$ and ending in $\Delta_1(a)$ starts from $\Delta_2(a)$. Thus Ω is the part of W whose real projection belongs to a segment joining $\Delta_2(a)$ to $\Delta_1(a)$; but a Cauchy-Poincaré theorem shows that integral (32.6) does not depend on the choice of such a segment. Therefore:

Proposition 33.1. *Suppose* $\Delta_1(a)$ *has a non-void director cone and* $\Delta_2(a)$ *has the opposite director cone* (*see Proposition* 27.3); *then Theorem* 32.1 *holds for* $x \notin C_2$ *with following definitions*: Ω *is the part of* W *whose real projection belongs to some vector* ξ' *joining* $\Delta_2(a)$ *to* $\Delta_1(a)$.



34. *The general case.* Let $\xi \in \Delta_1(a)$; there are polynomials $b(\zeta)$ such that

(34.1) $$\| a^{-1}(\xi + i\eta) b^{-1}(\xi + i\eta) \|_1 < +\infty .$$

On the one hand

$$a^{-1}(p) \cdot f(x) = b(p) a^{-1}(p) b^{-1}(p) \cdot f(x);$$

on the other hand $a^{-1}(p)b^{-1}(p) \cdot f(x)$ is given by the Proposition 32.1; thus, for $p \in \Delta_1(a)$,

$$a^{-1}(p) \cdot f(x) = k_x * f(x), \text{ where } k_x = \frac{1}{(2\pi i)^{\ell - 1}} b(p) \int \cdots \int_\Omega \exp. (x \cdot \zeta) \bar{\omega}(\zeta, d\zeta);$$

Ω is the part of $W(a) \cup W(b)$ whose real projection belongs to the half line parallel to ξ' and ending in ξ; $\bar{\omega}(\zeta, d\zeta)$ is such that

$$d[a(\zeta)b(\zeta)] \cdot \bar{\omega}(\zeta, d\zeta) = d\zeta_1 \ldots d\zeta_\ell .$$

According to n°25 and Lemma 29.3 we can choose $b(\zeta)$ such that $\Delta(b)$ contains the preceding half line; then $\Omega \subset W(a)$; now, ω being the same as in the Proposition 32.1, we have on W(a)

$$b(\zeta)\varpi(\zeta, d\zeta) = \omega(\zeta, d\zeta).$$

Therefore

Proposition 34.1. The elementary solution of a(p) corresponding to $\Delta_1(a)$ is

(34.2) $k_x = \frac{1}{(2\pi i)^{\ell-1}} b(p) \int_{\Omega}\cdots\int b^{-1}(\zeta) \exp(x \cdot \zeta)\omega(\zeta, d\zeta)$ when $x \cdot \zeta' > 0$;

ω and Ω are the same as in the proposition 32.1; $b(\zeta)$ is a polynomial satisfying (34.1) and such that the half line containing $\pi(\Omega)$ and ending in $\Delta_1(a)$ belongs to $\Delta(b)$.

This result can be completed, when W(a) has no singular point even at infinity. Let $g(\zeta)$ be a polynomial such that:

Ω and its infinite boundary points are outside W(g);

degree $g(\zeta) > \ell$ - degree $a(\zeta)$.

Let us choose $b(\zeta)$ satisfying these conditions. Then, since $\exp(x \cdot \zeta)$ is bounded on Ω, the uniform convergences justifying the following derivations hold:

$$g(p)b(p) \int_{\Omega}\cdots\int g^{-1}(\zeta)b^{-1}(\zeta) \exp(x \cdot \zeta)\omega(\zeta, d\zeta)$$

$$= b(p) \int_{\Omega}\cdots\int b^{-1}(\zeta) \exp(x \cdot \zeta)\omega(\zeta, d\zeta)$$

$$= g(p) \int_{\Omega}\cdots\int g^{-1}(\zeta) \exp(x \cdot \zeta)\omega(\zeta, d\zeta).$$

Therefore

Proposition 34.2. If W(a) has no singular points (finite or infinite), then the elementary solution of a(p) corresponding to $\Delta_1(a)$ is

(34.3) $k_x = \frac{1}{(2\pi i)^{\ell-1}} g(p) \int_{\Omega}\cdots\int g^{-1}(\zeta) \exp(x \cdot \zeta)\omega(\zeta, d\zeta)$;

$g(\zeta)$ is a polynomial with degree $> \ell$ -degree $a(\zeta)$ such that Ω and its infinite boundary points are outside $W(g)$; ω and Ω are the same as in the Proposition 32.1 (then $x \cdot \xi' > 0$) or 33.1 (then $x \notin C_2$).

§5 Conclusions

Properties of $a^{\beta}(p)$. Let us sum up preceding results:

Theorem 34.1. Let $a(\xi)$ be a polynomial, m its degree and β a complex number:

1°. The symbolic product by $a^{\beta}(p)$ is defined for $p \in \Delta_{\alpha}(a)$ and depends on α; (the properties of $\Delta_{\alpha}(a)$ are given by the propositions of $n^{\circ}22$, 23, 24).

2°. Let $C_{\alpha}(a)$ be the cone dual to the director cone $\Gamma_{\alpha}(a)$ of $\Delta_{\alpha}(a)$; if $p \in \Delta_{\alpha}(a)$ and if G is an open part of X, then the datum of f_x in $G - C_{\alpha}(a)$ determines $a^{\beta}(p) \cdot f_x$ in G; (the properties of $\Gamma_{\alpha}(a)$ are given by the propositions of $n^{\circ}27$).

3°. Let $a(p)$ be real; let $b(p)$ be a second polynomial and n its degree; the symbolic product by $b(p)a^{\beta}(p)$ is a bounded operator for the norm $\|f(x) \exp. (- x \cdot \xi)\|_2$ if $\xi \in \Delta_{\alpha}(a, \beta, b)$; (the properties of $\Delta_{\alpha}(a, \beta, b)$ are given by the propositions of $n^{\circ}30$).

4°. $a^{-1}(p) \cdot f_x = k_x * f_x$ for $x \in \Delta_1(a)$, k_x being the elementary solution of $a(p)$ corresponding to $\Delta_1(a)$; (the Propositions 32.1, 33.1, 34.1, 34.2 give expressions of k_x).

CHAPTER IV

SYMBOLIC PRODUCT BY $a^{-1}(p)$, WHEN $a(p)$ IS AN HOMOGENEOUS POLYNOMIAL

(A calculation, which Herglotz began [7] and Petrowsky pursued [9], expresses the elementary solution of a(p) by means of periods of abelian integrals. We improve and achieve their calculation: as it was pointed out by Fl. Bureau, they have uncautiously permuted non-absolutely convergent integrals (Herglotz, second paper, p. 290); their assumption that the cone $a(\xi) = 0$ has no singular points is removed; an invariant expression is given to their results; using Schwartz's distributions we define the elementary solution everywhere and not only there where it is a function.)

§1. The exterior differential calculus

35. The rules of the exterior differential calculus. (See the E. Cartan and Kähler papers and the de Rham - Kodaira Seminar.) The calculus of the exterior differential forms

$$\omega(\xi, d\xi) = \sum_{\alpha_1,\dots,\alpha_q} a_{\alpha_1,\dots,\alpha_q}(\xi) d\xi_{\alpha_1} \cdots d\xi_{\alpha_q}$$

$$(q = \text{degree of } \omega)$$

is defined by the product rules

$$d\xi_\alpha \cdot d\xi_\beta = -d\xi_\beta \cdot d\xi_\alpha, \quad (d\xi_\alpha)^2 = 0$$

and by the differentiation rule

$$d\omega(\xi, d\xi) = \sum_{\alpha_1,\dots,\alpha_q} da_{\alpha_1,\dots,\alpha_q}(\xi) d\xi_{\alpha_1} \cdots d\xi_{\alpha_q};$$

hence

(35.1) $$\omega \cdot \overline{\omega} = (-1)^{qr}\, \overline{\omega} \cdot \omega \quad (q = \text{degree of } \omega,\ r = \text{degree of } \overline{\omega})$$

(35.2) $$d^2\omega = 0,\ d(\omega \cdot \overline{\omega}) = (d\omega) \cdot \overline{\omega} + (-1)^q \omega \cdot d\overline{\omega}.$$

A Poincaré's theorem asserts that ω is a differential if and only if $d\omega = 0$.

Note. More generally the exterior differential calculus can be defined on a differentiable manifold and Poincaré's theorem is a special case of the Poincaré - E. Cartan - de Rham definition of the cohomology ring: the exterior forms which are zero outside compact subsets of a manifold constitute a ring $\mathcal{A}$; let $\mathcal{C}$ be the set of the elements of $\mathcal{A}$ whose differentials are 0; let $\mathcal{D}$ be the set of the differentials of elements of $\mathcal{A}$: $\mathcal{C}$ is a subring of $\mathcal{A}$; $\mathcal{D}$ is an ideal of $\mathcal{C}$; $\mathcal{C}/\mathcal{D}$ is the cohomology ring (with real coefficients) of the manifold.

Note. Let V be an orientable manifold of dimension q; let βV be its boundary; Stokes's formula can be written

(35.3) $$\int_{\beta V} \omega(\xi, d\xi) = \int_V d\omega(\xi, d\xi).$$

36. The hypersurface Ω_a and the differential form ω_a. Let $a(\xi)$ be a differentiable function on the vector space Ξ; let us denote by Ω_a the hypersurface $a(\xi) = 0$ and by $\omega_a(\xi, d\xi)$ the exterior differential form such that

(36.1) $$da(\xi) \cdot \omega_a(\xi, d\xi) = d\xi_1 \ldots d\xi_\ell;$$

$\omega_a(\xi, d\xi)$ is defined mod. da; thus its restriction to Ω_a is defined without ambiguity and $\int\!\!\ldots\!\!\int_{\Omega_a} \omega_a(\xi, d\xi)$ has a meaning when Ω_a has been oriented.

(36.2) $$\omega_a(\xi, d\xi) = (-1)^{i-1} \frac{d\xi_1 \ldots d\xi_{i-1} d\xi_{i+1} \ldots d\xi_\ell}{a_i'(\xi)} \quad \text{mod. da;}$$

let us choose the axes such that $a_2'(\xi) = \dots = a_\ell'(\xi) = 0$: (36.2) shows that

$$|\overrightarrow{\text{grad.}}\, a(\xi)|\, |\omega_a(\xi, d\xi)|$$

is the measure of the element of the hypersurface Ω_a, if Ξ has a euclidean metric.

On the other hand, if $f(\xi)$ is a function, if w is a parameter and if Ω_{a-w} is so oriented that $\omega_a(\xi, d\xi) > 0$, then

$$\int_u^v dw \int_{\Omega_{a-w}} f(\xi)\,\omega_a(\xi, d\xi) = \int_{u < a(\xi) < v} f(\xi)\,d\xi_1 \dots d\xi_\ell;$$

(proof: replace in the first member ω_a by $\dfrac{d\xi_2 \dots d\xi_\ell}{a_1'(\xi)}$ and in the second member use $w = a(\xi)$, $\xi_2, \dots \xi_\ell$ as independent variables).

The preceding formula gives

$$(36.3)\qquad \int_{\Omega_a} f(\xi)\,\omega_a(\xi, d\xi) = \Big[\frac{d}{dv}\int_{a(\xi) < v} f(\xi)\,d\xi_1 \dots d\xi_\ell\Big]_{v=0},$$

if $\omega_a(\xi, d\xi) > 0$ on Ω_a.

37. The manifold $\Omega_{a,b}$ and the differential form $\omega_{a,b}$. The definitions of n°36 can easily be extended: let $a(\xi)$ and $b(\xi)$ be two functions; let $\Omega_{a,b}$ be the manifold $a(\xi) = b(\xi) = 0$; let $\omega_{a,b}(\xi, d\xi)$ be the differential exterior form such that

$$(37.1)\qquad da \cdot db \cdot \omega_{a,b}(\xi, d\xi) = d\xi_1 \dots d\xi_\ell;$$

$\omega_{a,b}(\xi, d\xi)$ is defined mod. (da, db); thus its restriction to $\Omega_{a,b}$ is defined without ambiguity;

$$(37.2)\qquad db \cdot \omega_{a,b}(\xi, d\xi) = \omega_a(\xi, d\xi) \qquad \text{mod. } da$$

$$(37.3)\qquad da \cdot \omega_{a,b}(\xi, d\xi) = -\,\omega_b(\xi, d\xi) \qquad \text{mod. } db$$

$$(37.4)\qquad \omega_{a,b}(\xi, d\xi) = (-1)^{i+j-1}\frac{d\xi_1 \ldots d\xi_{i-1}d\xi_{i+1}d\xi_{j-1}d\xi_{j+1}d\xi_\ell}{\dfrac{D(a,b)}{D(\xi_i, \xi_j)}}.$$

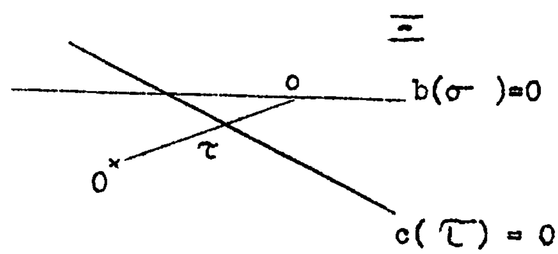
$|\overrightarrow{\text{grd}}\, a|\,|\overrightarrow{\text{grd}}\, b|\,|\sin(\overrightarrow{\text{grd}}\, a, \overrightarrow{\text{grd}}\, b)|\,|\omega_{a,b}(\xi, d\xi)|$ is the measure of the element of the manifold $\Omega_{a,b}$, if Ξ has a euclidean metric. Finally, if $\Omega_b(a < v)$ is the part of Ω_b where $a(\xi) < v$, then

$$(37.5)\qquad \int_{\Omega_{a,b}} f(\xi)\,\omega_{a,b}(\xi, d\xi) = \Big[\frac{d}{dv}\int_{\Omega_b(a<v)} f(\xi)\,\omega_b(\xi, d\xi)\Big]_{v=0}$$

where $\omega_{a,b}(\xi, d\xi) > 0$ on $\Omega_{a,b}$ and $\omega_a(\xi, d\xi) > 0$ on Ω_a.

38. Use of differential exterior forms in a projective space. The following proposition, where Ξ is an ℓ-dimensional vector space, affords differential forms of the $(\ell-1)$-dimensional projective space.

Lemma 38.1. Let $a(\xi)$ and $h(\xi)$ be homogeneous functions of degrees m and $n = m - \ell$; let $b(\xi)$ and $c(\xi)$ be linear functions such that $b(0) = c(0) \neq 0$; let σ run over $\Omega_{a,b}$ and τ run over $\Omega_{a,c}$ in such a way that σ and τ are proportional: thus $\Omega_{a,b}$ is mapped one-one onto $\Omega_{a,c}$. This mapping is such that



$$(38.1)\qquad h(\sigma)\,\omega_{a,b}(\sigma, d\sigma) = h(\tau)\,\omega_{a,c}(\tau, d\tau).$$

Proof. The definition (37.1) of $\omega_{a,b}$ shows that this assertion is not modified by a linear mapping of Ξ; this enables us to assume that

$$b(\xi) = \xi_1 - 1, \; c(\xi) = \xi_2 - 1.$$

Thus

$$(38.2) \qquad \frac{1}{\tau_1} = \frac{\sigma_2}{1} = \frac{\sigma_3}{\tau_3} = \dots = \frac{\sigma_\ell}{\tau_\ell}$$

$$(38.3) \qquad \omega_{a,b}(\sigma, d\sigma) = (-1)^{\ell-1} \frac{d\sigma_2 \dots d\sigma_{\ell-1}}{a'_\ell(\sigma)};$$

$$\omega_{a,c}(\tau, d\tau) = (-1)^{\ell} \frac{d\tau_1 d\tau_3 \dots d\tau_{\ell-1}}{a'_\ell(\tau)}.$$

But (38.2) gives

$$a'_\ell(\sigma) = \tau_1^{-(m-1)} a'_\ell(\tau); \; d\sigma_2 \dots d\sigma_{\ell-1} = -\tau_1^{-(\ell-1)} d\tau_1 d\tau_3 \dots d\tau_{\ell-1};$$

hence, according to (38.3),

$$\omega_{a,b}(\sigma, d\sigma) = \tau_1^{n} \omega_{a,c}(\tau, d\tau);$$

but, upon application of (38.2)

$$h(\sigma) = \tau_1^{-n} h(\tau);$$

(38.1) follows from the two preceding formulas.

From lemma 38.1 follows the

Proposition 38.1. *If $a(\xi)$ and $h(\xi)$ are homogeneous functions of degrees m and $n = m - \ell$ and if $c(\xi)$ is a linear function such that $c(0) = -1$, then the differential form of*

$$\frac{\xi_2}{\xi_1}, \dots \frac{\xi_\ell}{\xi_1}, \; d\left(\frac{\xi_2}{\xi_1}\right), \dots d\left(\frac{\xi_\ell}{\xi_1}\right)$$

which is defined on the cone Ω_a and is equal to $h(\xi)\,\omega_{a,c}(\xi, d\xi)$

on $\Omega_{a,c}$ does not depend on the choice of $c(\xi)$.

Note. $h(\xi)\,\omega_{a,b,c}(\xi, d\xi)$ has the same property on the cone $\Omega_{a,b}$ if $a(\xi)$, $b(\xi)$, $h(\xi)$ are homogeneous and if the degree of $\dfrac{a(\xi)b(\xi)}{h(\xi)}$ is ℓ.

Note. Let $h(\xi)$ have the degree $n = \ell - m$ and $\Omega_{a,c}(x \cdot \xi > 0)$ be the part of $\Omega_{a,c}$ where $x \cdot \xi > 0$; then

$$(38.4)\qquad \int_{\Omega_{a,x\cdot\xi,c}} \xi_1 h(\xi)\,\omega_{a,x\cdot\xi,c}(\xi, d\xi) = \frac{\partial}{\partial x_1}\int_{\Omega_{a,c}(x\cdot\xi>0)} h(\xi)\,\omega_{a,c}(\xi, d\xi);$$

where $\omega_{a,x\cdot\xi,c} > 0$ on $\Omega_{a,x\cdot\xi,c}$ and $\omega_{a,c} > 0$ on $\Omega_{a,c}$. (Proof: choose $c(\xi) = \xi_1 - 1$ and apply (37.5).)

§2. Herglotz's formula

39. A first expression of the elementary solution $k(x)$ of the homogeneous polynomial $a(p)$. Let $a(\xi)$ be a real homogeneous polynomial such that $\Delta(a)$ is non-void; proposition 27.1 shows that $\Delta(a) \neq \Xi$, if $a(\xi) \neq$ const.; proposition 27.2.1° and 23.1 prove that the components of $\Delta(a) = \Delta^*(a)$ are convex cones, whose boundaries belong to $a(\xi) = 0$. Let $\Delta_1(a)$ be one of these convex cones and $\Delta_2(a)$ be the opposite cone, which is also a component of $\Delta(a)$ [see proposition 24.1, 1°].

Assume

$$(39.1)\qquad \|a^{-1}(\xi + i\eta)\|_1 < +\infty \text{ for } \xi \in \Delta_1(a);$$

lemma 29.3 shows that this happens if any real point of the cone $a(\xi) = 0$

(<u>except its vertex</u>) <u>has an order</u> $< n = m - \ell$; we assume

(39.2) $$n = m - \ell > 0.$$

Assume that the first axis is contained in $\Delta_1(a)$; let us apply proposition 33.1, ξ' having the direction of this axis, but being reduced to the origin: the elementary solution of $a(p)$ corresponding to $\Delta_1(a)$ is

$$k(x) = \frac{1}{(2\pi i)^{\ell - 1}} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \sum_{\zeta} \frac{\exp.(x.\zeta)}{a_1'(\zeta)} d\zeta_2 \dots d\zeta_\ell,$$

where $\zeta = (i\eta_1, i\eta_2, \dots i\eta_\ell)$, $a(\zeta) = 0$, $x \in C_1(a)$;
$\eta_1, \eta_2, \dots \eta_\ell$ are real, since $\pi(\Omega)$ is the origin.

This result can be expressed thus:

(39.3) $$k(x) = \frac{1}{(2\pi)^{\ell - 1}} \frac{1}{i^{m-1}} I,$$

(39.4) $$I = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \sum_{\xi} \frac{\exp.(ix.\xi)}{a_1'(\xi)} d\xi_2 \dots d\xi_\ell ;$$

$\xi = (\xi_1, \xi_2, \dots \xi_\ell)$ runs over the set of points satisfying

$$a(\xi) = 0; \quad \xi_2, \dots \xi_\ell : \text{ given and real;}$$

all this m points are real, since the infinite point of the first axis is in $\Delta^*(a)$.

<u>Note</u>. (39.4) is <u>an absolutely convergent integral</u>, which becomes divergent by permutation of $\int \dots \int$ and $\sum$.

40. <u>The expression of I after a projective mapping of the cone</u> $a(\xi) = 0$ <u>into a cylinder</u> $t(\tau) = 0$. Let us define:

$$\frac{\xi_1}{\tau_1} = \frac{\xi_2}{\tau_2} = \dots = \frac{\xi_{\ell - 1}}{\tau_{\ell - 1}} = \frac{\xi_\ell}{1} = \rho,$$

$$x \cdot \tau = x_1\tau_1 + \dots + x_{\ell-1}\tau_{\ell-1} + x_\ell,$$

$$t(\tau) = a(\tau_1, \tau_2, \dots \tau_{\ell-1}, 1).$$

We have:

$$\frac{D(\xi_2, \dots \xi_{\ell-1}, \xi_\ell)}{D(\tau_2, \dots \tau_{\ell-1}, \rho)} = \rho^{\ell-2},$$

$$x \cdot \xi = \rho(x \cdot \tau),$$

$$a_1'(\xi) = \rho^{m-1} t_1'(\tau).$$

Hence, using the definition (39.4),

$$\text{(40.1)} \qquad I = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} J(\tau_2, \dots \tau_{\ell-1}) d\tau_2 \dots d\tau_{\ell-1}$$

where

$$\text{(40.2)} \quad J(\tau_2, \dots \tau_{\ell-1}) = \int_{-\infty}^{+\infty} \operatorname{sgn}(\rho^\ell) \sum_\tau \frac{\exp.(i\rho(x \cdot \tau))}{t_1'(\tau)} \frac{d\rho}{\rho^{n+1}};$$

τ runs over the set of the m points satisfying

$t(\tau) = 0$; $\tau_2, \dots \tau_\ell$ are prescribed and real:

all these points are real; (40.1) and (40.2) are absolutely convergent integrals; but in (40.2) $\int$ and $\sum$ cannot be permuted.

Let us define

$$\text{(40.3)} \qquad b = \frac{1}{t_1'(\tau)}, \quad c = x \cdot \tau$$

Since the sum of the residues of the function of τ_1

$$\frac{\tau_1^q}{t(\tau)}$$

is zero for $0 \leq q < m - 1$,

(40.4) $$\sum_{\tau} bc^q = 0 \text{ for } 0 \leq q < m - 1.$$

Let us express (40.2) and (40.4) as follows:

(40.5) $$J = \int_{-\infty}^{+\infty} \operatorname{sgn}(\rho^{\ell}) \sum_{\alpha=1}^{m} b_\alpha \exp.(ic_\alpha \rho) \frac{d\rho}{\rho^{n+1}}$$

(40.6) $$\sum_{\alpha=1}^{m} b_\alpha c_\alpha^q = 0 \text{ for } 0 \leq q \leq n;$$

b_α and c_α are real.

41. The explicit expression of J. Let us calculate the absolutely convergent integral

(41.1) $$K = \int_0^{+\infty} \sum_{\alpha=1}^{m} b_\alpha \exp.(ic_\alpha \rho) \frac{d\rho}{\rho^{n+1}}.$$

Let $\mathcal{E}$ be a positive number tending to zero:

$$K = \lim_{\mathcal{E} \to 0} \sum_\alpha \int_{\mathcal{E}}^{+\infty} b_\alpha \exp.(ic_\alpha s) \frac{ds}{s^{n+1}};$$

$$= \lim_{\mathcal{E} \to 0} \sum_\alpha \int_{\mathcal{E} c_\alpha}^{\operatorname{sgn}(c_\alpha)\infty} b_\alpha c_\alpha^n \exp.(is) \frac{ds}{s^{n+1}};$$

but $\int_{-\infty}^{+\infty} \exp.(is) \frac{ds}{s^{n+1}} = 0$ if s remains in the half plane $\mathcal{J}(s) > 0$; on the other hand $\sum_\alpha b_\alpha c_\alpha^n = 0$; hence

$$K = \lim_{\mathcal{E} \to 0} \sum_\alpha \int_{\mathcal{E} c_\alpha}^{\mathcal{E} c_1} b_\alpha c_\alpha^n \exp.(is) \frac{ds}{s^{n+1}}, \text{ where } \mathcal{J}(s) > 0,$$

$$= \lim_{\mathcal{E} \to 0} \frac{1}{\mathcal{E}^n} \sum_\alpha \int_{c_\alpha}^{c_1} b_\alpha c_\alpha^n \exp.(i \mathcal{E} s) \frac{ds}{s^{n+1}}$$

$$= \lim_{\varepsilon \to 0} \sum_{\alpha} \int_{c_\alpha}^{c_1} b_\alpha c_\alpha^n \left[\frac{1}{\varepsilon^n s^{n+1}} + \frac{i}{\varepsilon^{n-1} s^n} + \dots + \frac{i^n}{n!s} + \varepsilon \dots\right] ds$$

but

$$\sum_{\alpha} \int_{c_\alpha}^{c_1} b_\alpha c_\alpha^n \frac{ds}{s^{q+1}} = \frac{1}{q}\left(\sum_{\alpha} b_\alpha c_\alpha^{n-q} - \sum_{\alpha} b_\alpha c_\alpha^n c_1^{-q}\right) = 0 \text{ for } 0 < q \leq n$$

$$\sum_{\alpha} \int_{c_\alpha}^{c_1} b_\alpha c_\alpha^n \frac{ds}{s} = - \sum_{\alpha} b_\alpha c_\alpha^n \log c_\alpha ,$$

where, since $\mathcal{J}(s) > 0$,

(41.2) $$\mathcal{J}(\log c_\alpha) = 0 \text{ for } c_\alpha > 0, = \pi \text{ for } c_\alpha < 0.$$

Hence

(41.3) $$K = - \frac{i^n}{n!} \sum_{\alpha} b_\alpha c_\alpha^n \log c_\alpha .$$

Therefore

$$\begin{aligned} L &= \int_{-\infty}^{0} \sum_{\alpha} b_\alpha \exp.(ic_\alpha \rho) \frac{d\rho}{\rho^{n+1}} \\ &= (-1)^{n+1} \int_{0}^{+\infty} \sum_{\alpha} b_\alpha \exp.(-ic_\alpha \rho) \frac{d\rho}{\rho^{n+1}} \\ &= (-1)^{n+1} \overline{K} = \frac{i^n}{n!} \sum_{\alpha} b_\alpha c_\alpha^n \overline{\log c_\alpha}. \end{aligned}$$

Hence

$$J = K + (-1)^{\ell} L = - \frac{i^n}{n!} \sum_{\alpha} b_\alpha c_\alpha^n [\log c_\alpha + (-1)^{\ell-1} \overline{\log c_\alpha}];$$

that is

(41.4) $$J = - \frac{2i^{m-1}}{n!} \mathcal{R}\left[\frac{1}{i^{\ell-1}} \sum_{\alpha} b_\alpha c_\alpha^n \log c_\alpha\right].$$

42. Herglotz's formula. (39.3), (40.1), (40.3) and (41.4) prove that the elementary solution is the absolutely convergent integral

$$(42.1)\quad k(x) = -\mathcal{R}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\frac{2}{(2\pi i)^{\ell-1}}\sum_{\tau}\frac{(x\cdot\tau)^n}{n!}\log(x\cdot\tau)\frac{1}{t_1'(\tau)}\,d\tau_2\cdots d\tau_{\ell-1}$$

where $\tau = (\tau_1 \dots \tau_{\ell-1})$ runs over the set of the m points satisfying

$$t(\tau) = 0;\ \tau_2, \dots \tau_{\ell-1} \text{ given and real;}$$

all these points are real;

$$(42.2)\qquad \sum_{\tau}(x\cdot\tau)^q\frac{1}{t_1'(\tau)} = 0 \text{ for } 0 \leq q \leq n.$$

§3. The case: ℓ even, $m - \ell > 1$ (Herglotz)

43. The invariant expression of $k(x)$. Formulas (42.1), (42.2) and (41.2) give

$$k(x) = \frac{1}{2(2\pi i)^{\ell-2}}\int\cdots\int_{t(\tau)=0}\frac{(x\cdot\tau)^n}{n!}\operatorname{sgn}(x\cdot\tau)\frac{d\tau_2\cdots d\tau_{\ell-1}}{t_1'(\tau)}\quad(\tau:\ \text{real})$$

$$= \frac{1}{2(2\pi i)^{\ell-2}}\int\cdots\int_{a(\xi)=0,\xi_\ell=1}\frac{(x\cdot\xi)^n}{n!}\operatorname{sgn}(x\cdot\xi)\frac{d\xi_2\cdots d\xi_{\ell-1}}{\dfrac{D(a,\xi_\ell)}{D(\xi_1,\xi_\ell)}}\quad(\xi:\ \text{real})$$

where the integral elements corresponding to the same $\xi_2, \dots \xi_\ell$ and different ξ_ℓ have first to be summed; that is

$$(43.1)\qquad k(x) = \frac{1}{2(2\pi i)^{\ell-2}}\int_{\Omega_{a,b}}\frac{(x\cdot\xi)^n}{n!}\,\omega_{a,b}(\xi, d\xi),$$

where $b(\xi) = \xi_{\ell-1}$; $\Omega_{a,b}$ is the real part of the variety $a(\xi) = .b(\xi) = 0$, with the orientation such that

$$(x\cdot\xi)\,a_1'(\xi)\,\omega_{a,b}(\xi, d\xi) > 0;$$

thus this orientation changes along the intersection of $\Omega_{a,b}$ with the hyperplane $x \cdot \xi = 0$; since $m-n = \ell$ is even, the preceding formula can be written

$$\frac{a_1'(\xi)}{(x\cdot\xi)^{m-1}} \cdot (x\cdot\xi)^n\,\omega_{a,b}(\xi, d\xi) > 0;$$

this formula, (43.1) and proposition 38.1 show that $b(\xi)$ can be replaced by any linear function of ξ such that $b(0) = -1$. On the other hand, since the direction of a real tangent to $a(\xi) = 0$ can not belong to $\Delta^*(a)$,

$$\xi_1^* a_1'(\xi) + \dots + \xi_\ell^* a_\ell'(\xi) \neq 0 \text{ for } a(\xi) = 0,\ \xi^* \in \Delta_1(a);$$

therefore, since the first axis is in $\Delta_1(a)$, $a_1'(\xi)$ has the sign of

$$\xi_1^* a_1'(\xi) + \dots + \xi_\ell^* a_\ell'(\xi);$$

thus the last inequality can be written

$$(43.2)\quad [\xi_1^* a_1'(\xi) + \dots + \xi_\ell^* a_\ell'(\xi)](x\cdot\xi)\,\omega_{a,b}(\xi, d\xi) > 0.$$

Finally,

Theorem 43.1. Suppose ℓ even. Let $a(\xi)$ be a homogeneous polynomial of degree m such that any real point of the cone $a(\xi) = 0$ (except its vertex) has an order $< n = m - \ell$. Let $b(\xi)$ be an arbitrary linear function of ξ such that $b(0) = -1$; let ξ^* be an arbitrary point such that

$$b(\xi^*) = 0,\quad \xi^* \in \Delta_1(a).$$

Let $\omega_{a,b}(\xi, d\xi)$ be the differential form such that

$$da(\xi) \cdot db(\xi) \cdot \omega_{a,b}(\xi, d\xi) = d\xi_1 \cdot d\xi_2 \dots d\xi_\ell;$$

$\omega_{a,b}(\xi, d\xi)$ is defined without ambiguity on the variety

$$a(\xi) = b(\xi) = 0;$$

the real part of this variety is non-orientable; let $\Omega_{a,b}$ be this real part oriented outside the hyperplane $x \cdot \xi = 0$ in such a way that

(43.2) $[\xi_1^* a_1'(\xi) + \dots + \xi_\ell^* a_\ell'(\xi)](x \cdot \xi)\,\omega_{a,b}(\xi, d\xi) > 0$:

the boundary of $\Omega_{a,b}$ is twice the real part of the variety

$$a(\xi) = b(\xi) = x \cdot \xi = 0.$$

Then the elementary solution of $a(p)$ for $p \in \Delta_1(a)$ is the $(\ell-2)$-tuple integral $[x \notin -C_1(a)]$

(43.1) $$k(x) = \frac{1}{2(2\pi i)^{\ell-2}} \int_{\Omega_{a,b}} \frac{(x \cdot \xi)^n}{n!}\,\omega_{a,b}(\xi, d\xi)$$

A line of the hyperplane $b(\xi) = 0$ running through ξ^* cuts $\Omega_{a,b}$ at m points; the corresponding elements of the integral (43.1) have first to be summed: thus this integral becomes absolutely convergent.

44. The $(n+1)$th derivatives of $k(x)$ are periods of abelian integrals. Let $h_q(p)$ be a homogeneous polynomial of degree q; (43.1) gives

$$h_n(p) \cdot k(x) = \frac{1}{2(2\pi i)^{\ell-2}} \int_{\Omega_{a,b}} h_n(\xi)\,\omega_{a,b}(\xi, d\xi);$$

hence, upon application of (38.4) and since the orientation of $\Omega_{a,b}$ changes along $\Omega_{a,x\cdot\xi,b}$, we have

$$h_{n+1}(p) \cdot k(x) = \frac{1}{(2\pi i)^{\ell-2}} \int_{\Omega_{a,x\cdot\xi,b}} h_{n+1}(\xi)\,\omega_{a,x\cdot\xi,b}(\xi, d\xi)$$

where $\Omega_{a,x\cdot\xi,b}$ has the orientation such that

$$[\xi_1^* a_1'(\xi) + \dots + \xi_\ell^* a_\ell'(\xi)]\,\omega_{a,x\cdot\xi,b}(\xi, d\xi) > 0.$$

To justify these derivations, it is sufficient to assume that the last integral in uniformly convergent; hence

Theorem 44.1. Let us preserve the assumptions of theorem 43.1. Let x be a point such that the hyperplane $x \cdot \xi = 0$ does not touch the cone $a(\xi) = 0$, nor contain real singular points of this cone, except its vertex. Let $\Omega_{a,x.\xi,b}$ be the real part of the variety

$$a(\xi) = x \cdot \xi = b(\xi) = 0$$

with the orientation such that

$$[\xi_1^* a_1'(\xi) + \dots + \xi_\ell^* a_\ell'(\xi)]\, \omega_{a,x.\xi,b}(\xi, d\xi) > 0.$$

The (n+1)th derivatives of $k(x)$ are given by the $(\ell-3)$-tuple integral

$$h(p) \cdot k(x) = \frac{1}{(2\pi i)^{\ell-2}} \int_{\Omega_{a,x.\xi,b}} h(\xi)\, \omega_{a,x.\xi,b}(\xi, d\xi)$$

where $h(p)$ is a homogeneous polynomial of degree $n+1$, and $x \notin -C_1(a)$.

§4. The case: ℓ odd, $m - \ell > 1$ (Petrowsky)

45. A modification of Herglotz's formula. When ℓ is even, Herglotz's formula (42.1) gives the expression for $k(x)$ directly by means of the integral of a rational function (see n°43); when ℓ is odd, such an expression still exists, though Herglotz's formula gives the formula containing a logarithm:

$$k(x) = -\frac{2}{(2\pi i)^{\ell-1}} \int_{-\infty}^{+\infty}\!\!\dots\!\int_{-\infty}^{+\infty} \sum_{\tau} \frac{(x.\tau)^n}{n!} \log|x \cdot \tau| \frac{1}{t_1'(\tau)}\, d\tau_2 \dots d\tau_{\ell-1},$$

indeed Petrowsky succeeded in suppressing this logarithm.

Let us suppose that all the axes of Ξ, except the first one, are orthogonal to the given vector x:

(45.1) $$x_1 > 0,\ x_2 = 0,\ \dots\ x_\ell = 0;$$

thus $x \cdot \tau = x_1 \tau_1$ and, since $\sum_\tau \dfrac{1}{t_1'(\tau)} = 0$,

$$(45.2) \quad k(x) = - \frac{2}{(2\pi i)^{\ell-1}} \frac{x_1^{\,n}}{n!} \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \sum_{\tau} \frac{\tau_1^{\,n}}{t_1'(\tau)} \log|\, t_1| \, d\tau_2 \ldots d\tau_{\ell-1}$$

where τ_1 runs over the set of the roots of the polynomial $t(\tau_1) = t(\tau_1, \tau_2, \ldots \tau_{\ell-1})$; all these roots are real.

Let us replace $\sum_{\tau_1} \frac{\tau_1^{\,n}}{t'(\tau_1)} \log|\tau_1|$ by an integral in the plane of the complex variable τ_1; let us cut this τ_1-plane along the line $(-i\infty, 0)$, the two edges of the cut being called L_1 and L_2; let us use the branch of $\log \tau_1$ which is uniform in this cut plane and is real for $\tau_1 > 0$; let L_3 be the half line $(0, +i\infty)$; according to the residue formula

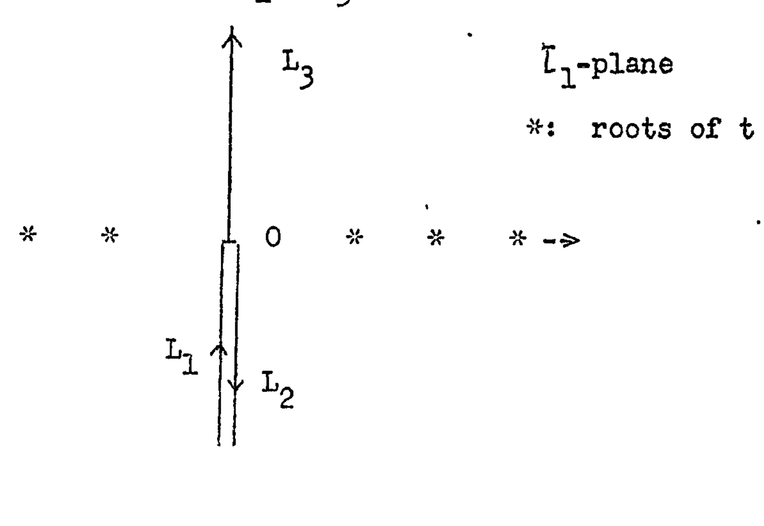
$$- 2 \sum_{\tau_1} \frac{\tau_1^{\,n}}{t'(\tau_1)} \log |\tau_1| =$$

$$- 2 \sum_{\tau_1} \frac{\tau_1^{\,n}}{t'(\tau_1)} \log \tau_1 + 2 i \sum_{\tau_1<0} \frac{\tau_1^{\,n}}{t'(\tau_1)} =$$

$$- \frac{1}{\pi i} \int_{L_1+L_2} \frac{\tau_1^{\,n}}{t(\tau_1)} \log \tau_1 d\tau_1 + \int_{L_1+L_3} \frac{\tau_1^{\,n}}{t(\tau_1)} d\tau_1 =$$

$$- 2 \int_{L_1} \frac{\tau_1^{\,n}}{t(\tau_1)} d\tau_1 + \int_{L_1+L_3} \frac{\tau_1^{\,n}}{t(\tau_1)} d\tau_1 = \int_L \operatorname{sgn} \mathcal{J}(\tau_1) \frac{\tau_1^{\,n}}{t(\tau_1)} d\tau_1,$$

where $L = L_1 + L_3$.



Consequently (45.4) and (45.5) give

$$M(\tau_3, \dots \tau_{\ell-1}) = \int_L \operatorname{sgn} \mathcal{J}(\tau_1) d\tau_1 \int_{-\infty}^{+\infty} \frac{(x\cdot\tau)^n}{n!} \frac{d\tau_2}{t(\tau)}$$

$$= \pi i \int_L \sum_{\tau} \frac{(x\cdot\tau)^n}{n!} \operatorname{sgn} [\mathcal{J}(\tau_1)\, \mathcal{J}(\tau_2)] \frac{d\tau_1}{t_2'(\tau)}$$

hence, upon application of (45.3)

$$(45.7)\quad k(x) = \frac{1}{2(2\pi i)^{\ell-2}} \int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty} d\tau_3 \dots d\tau_{\ell-1} \int_L \sum_{\tau} \frac{(x\cdot\tau)^n}{n!} \operatorname{sgn} [\mathcal{J}(x\cdot\tau)\, \mathcal{J}(\tau_2)] \frac{d\tau_1}{t_2'(\tau)}$$

where τ_2 runs over the set of those points for which $t(\tau) = 0$, $\tau_1, \tau_3, \dots \tau_{\ell-1}$ being prescribed; L is a line from $-i\infty$ to $i\infty$ through 0; (45.1) is assumed.

46. <u>The invariant expression of</u> k(x). The preceding formula can be written as follows

$$(46.1)\qquad k(x) = \frac{1}{2(2\pi i)^{\ell-2}} \int_{\Omega_{a,b}} \frac{(x\cdot\zeta)^n}{n!}\, \omega_{a,b}(\zeta, d\zeta)$$

where $b(\zeta) = \zeta_{\ell}-1$; $\Omega_{a,b}$ is the part of the variety

$$a(\zeta) = b(\zeta) = 0$$

whose real projection belongs to the hyperplane $x\cdot\zeta = 0$ and whose imaginary projection belongs to the 2-dimensional plane containing two vectors $i\xi^*$ and $i\xi^{**}$ chosen such that

$$b(\xi^*) = b(\xi^{**}) = -1,\ \xi^* \in \Delta_1(a),\ x\circ\xi^{**} = 0;$$

$$(\xi^* \neq \xi^{**} \text{ since } x \in \text{dual of } \Delta_1(a));$$

if $\zeta \in \Omega_{a,b}$, let us write

(46.2) $$\zeta = \xi + i\eta, \quad \eta = \sigma_1\xi^* + \sigma_2\xi^{**}$$

($\sigma_1 \neq 0$ for $\sigma_2 \neq 0$, since ξ^* is outside the real projection of $a(\zeta) = 0$) and define

(46.3) $$\mathcal{E}(\zeta) = \operatorname{sgn}(\sigma_1\sigma_2) = \operatorname{sgn}\sigma_2 \cdot \operatorname{sgn}\mathcal{J}(x \cdot \zeta)$$

the orientation of $\Omega_{a,b}$ is such that

(46.4) $$\frac{1}{i}[\xi_1^{**}a_1'(\zeta) + \dots + \xi_\ell^{**}a_\ell'(\zeta)]\,\mathcal{E}(\zeta)\,\omega_{a,b}(\zeta, d\zeta) > 0.$$

The function $\mathcal{E}(\zeta)$ is not defined if $x \cdot \zeta = 0$, but is continuous at each real point of $\Omega_{a,b}$ for which $x \cdot \zeta = 0$ (if $a_2'(\zeta) \neq 0$): thus the orientation of $\Omega_{a,b}$ changes along the imaginary part of its intersection by the hyperplane $x \cdot \zeta = 0$. Hence, according to proposition 38.1:

<u>Theorem 46.1. Suppose ℓ odd. Let $a(\xi)$ be a homogeneous polynomial of degree m, such that any real point of the cone $a(\xi) = 0$ (except its vertex) has an order $< n = m - \ell$. Let ξ^* and ξ^{**} be arbitrary real vectors such that</u>

$$\xi^* \in \Delta_1(a), \quad x \cdot \xi^{**} = 0,$$

<u>let $b(\xi)$ be an arbitrary linear real function of ξ such that</u>

$$b(0) = b(\xi^*) = b(\xi^{**}) = -1.$$

<u>Let $\omega_{a,b}(\zeta, d\zeta)$ be the differential form such that</u>

$$da(\zeta) \cdot db(\zeta) \cdot \omega_{a,b}(\zeta, d\zeta) = d\zeta_1 d\zeta_2 \dots d\zeta_\ell;$$

<u>$\omega_{a,b}(\zeta, d\zeta)$ is defined without ambiguity on the variety</u>

$$a(\zeta) = b(\zeta) = 0.$$

<u>Let $\Omega_{a,b}$ be the part of this variety whose real projection belongs to the hyperplane $x \cdot \zeta = 0$ and whose imaginary projection belongs to the 2-dimensional plane containing the vectors $i\xi^*$ and $i\xi^{**}$; $\Omega_{a,b}$ is non-orientable; let</u>

us oriente it outside the hyperplane $x \cdot \zeta = 0$ in such a way that (46.4) holds: the boundary of $\Omega_{a,b}$ is twice the closure of the imaginary part of its intersection by this hyperplane. Then the elementary solution of $a(p)$ for $p \in \Delta_1(a)$ is the $(\ell-2)$-tuple integral $[x \notin - C_1(a)]$

$$(46.3) \qquad k(x) = \frac{1}{2(2\pi i)^{\ell-2}} \int_{\Omega_{a,b}} \frac{(x\cdot\zeta)^n}{n!}\,\omega_{a,b}(\zeta, d\zeta).$$

outside $-C_1(a)$; $k(x) = 0$ on $-C_1(a)$. The elements of this integral corresponding to points on a parallel to ξ^{**} have first to be summed; then ζ has first to run parallel to $i\xi^*$.

47. The $(n + 1)$th derivatives of $k(x)$ are periods of abelian integrals. The preceding theorem is similar to the theorem 43.1; theorem 44.1 follows from theorem 43.1; in the same way, and using in addition Stokes's formula and the obvious fact that $d[x \cdot \zeta)^q \omega_{a,b}(\zeta, d\zeta)] = 0$ on the variety $a(\zeta) = b(\zeta) = 0$, we obtain the following

Theorem 47.1. Let us preserve the assumptions of theorem 46.1. Let x be a point such that the hyperplane $x \cdot \zeta = 0$ does not touch the cone $a(\zeta) = 0$, and does not contain non-real, singular points of this cone. Let $\Omega_{a,x\cdot\zeta,b}$ be the closure of the part of the variety

$$a(\zeta) = x \cdot \zeta = b(\zeta) = 0$$

where

$$\zeta = \xi + i\eta, \quad \eta = \sigma\,\xi^{**} \quad (\sigma: \text{ real number} \neq 0);$$

$\Omega_{a,x\cdot\zeta,b}$ is orientable; let it have the orientation such that

$$(47.1) \quad \sigma \cdot [\xi_1^{**}a_1'(\zeta) + \dots + \xi_\ell^{**}a_\ell'(\zeta)] \cdot \omega_{a,x\cdot\zeta,b}(\zeta, d\zeta) > 0.$$

The $(n + 1)$th derivatives of $k(x)$ are given by the $(\ell-3)$-tuple integral

$$(47.2) \qquad h(p) \cdot k(x) = \frac{1}{(2\pi i)^{\ell-2}} \int_{\Omega_{a,x\cdot\zeta,b}} h(\zeta)\,\omega_{a,x\cdot\zeta,b}(\zeta, d\zeta)$$

where $x \notin - C_1(a)$.

48. A complement of the theorems 46.1 and 47.1. The proposition 38.1 enables us to replace in these theorems $b(\zeta)$ by any real linear function $c(\zeta)$ such that $c(0) = -1$; the definition of $\Omega_{a,c}$ and its orientation are not easy; we define explicitly only $\Omega_{a,x\cdot\zeta,c}$ and its orientation.

Let us multiply ζ^{**} by a real number such that

$$c(\zeta^{**}) = 0;$$

$\Omega_{a,x\cdot\zeta,c}$, being the projection of $\Omega_{a,x\cdot\zeta,b}$ from 0 into $x \cdot \zeta = 0$, is the closure of the set of the non real points $\tilde{\zeta}$ of the variety

$$a(\zeta) = x \cdot \zeta = c(\zeta) = 0$$

such that the line joining ζ^{**} and $\tilde{\zeta}$ is real.

Now let us define the orientation of $\Omega_{a,x\cdot\zeta,c}$;

$\tilde{\zeta} = \zeta^{**} + (i + \tau)\,\tilde{\eta}$ (τ: real number; $\tilde{\eta} \in \Xi$); the corresponding point of $\Omega_{a,x\cdot\zeta,b}$ is

$$\zeta = \rho\,\tilde{\zeta},$$

ρ being the complex number such that

$$b(\zeta) = 0;$$

since $b(\zeta) + 1$ is homogeneous and $b(\zeta^{**}) + 1 = 0$,

$$\rho = \frac{1}{(\tau + i)[b(\tilde{\eta}) + 1]};$$

that is

$$\zeta = \sigma(i - \tau)\,\tilde{\zeta}, \quad \eta = \sigma\,\zeta^{**} \quad (\sigma: \text{ real number});$$

therefore, upon application of the proposition 38.1,

$$h(\zeta)[\xi_1^{**}a_1'(\zeta) + \ldots]\omega_{a,x\cdot\zeta,b}(\zeta, d\zeta) = h(\tilde{\zeta})[\xi_1^{**}a_1'(\tilde{\zeta}) + \ldots]\omega_{a,x\cdot\zeta,c}(\tilde{\zeta}, d\tilde{\zeta})$$

where $h(\zeta)$ is homogeneous of degree $2-\ell$; thus (47.1) becomes, since ℓ is odd,

$$(i-\tau)^{\ell-2}[\xi_1^{**}a_1'(\tilde{\zeta}) + \ldots]\omega_{a,x\cdot\zeta,c}(\tilde{\zeta}, d\tilde{\zeta}) > 0.$$

Hence:

Theorem 48.1. The theorem 47.1 remains true when $b(\zeta)$, ξ^{**} and $\Omega_{a,x\cdot\zeta,b}$ are defined as follows:

$b(\zeta)$ is a linear real function such that $b(0) = -1$;

ξ^{**} is a real vector of Ξ such that $x\cdot\xi^{**} = b(\xi^{**}) = 0$;

$\Omega_{a,x\cdot\zeta,b}$ is the closure of the set of the points

$\zeta = \xi^{**} + (i+\tau)\eta$ (τ: real number; $\eta\in\Xi$; $\eta\neq 0$)

belonging to the variety

$$a(\zeta) = x\cdot\zeta = b(\zeta) = 0.$$

The orientation of $\Omega_{a,x\cdot\zeta,b}$ is such that

$$(i+\tau)^{2-\ell}[\xi_1^{**}a_1'(\zeta) + \ldots + \xi_\ell^{**}a_\ell'(\zeta)]\omega_{a,x\cdot\zeta,b}(\zeta, d\zeta) < 0.$$

§5. The general case

(For $n = m-\ell < 0$, the elementary solutions k_x of $a(p)$ are distributions, which Petrowsky determines only outside the variety dual of $W(a)$, that is where these distributions are functions. We give their general expression. Our method differs essentially from Petrowsky's not explicit derivations of integrals: [9] p. 322-323.)

48. <u>Preliminary</u>. Let $a_1(\zeta)$, $a_2(\zeta)$, ... $a_r(\zeta)$ be homogeneous polynomials of degrees m, 2, ... 2, such that the varieties $W_j = W(a_j)$ are without singular point and without contact and such that there is a domain Δ belonging to all $\Delta(a_j)$. Let k and k_j be the elementary solutions of $a_1(p) \dots a_r(p)$ and of $a_j(p)$ for $p \in \Delta$; they are distributions such that

(48.1) $$k_1 = a_2(p) \dots a_r(p)k,$$

since

$$a_1^{-1}(p) = a_2(p) \dots a_r(p)[a_1(p) \dots a_r(p)]^{-1}$$

$\Omega_j = \Omega_{a_j,b}$ and $\omega_j = \omega_{a_j,b}$ are defined as in theorem 43.1 if ℓ is even, as in theorem 46.1 if ℓ is odd; these theorems and (48.1) give, r being such that

$$r < n = m_1 + 2(r-1) - \ell,$$

$$2(2\pi i)^{\ell-2}k_1 = a_2(p)\dots a_r(p)\Big[\int_{\Omega_1}\frac{(x\cdot\zeta)^n\omega_1(\zeta,d\zeta)}{n!a_2(\zeta)\dots a_r(\zeta)} + \int_{\Omega_2}\frac{(x\cdot\zeta)^n\omega_2(\zeta,d\zeta)}{n!a_1(\zeta)a_3(\zeta)\dots a_r(\zeta)} + \dots\Big];$$

let us suppose Ω_i outside W_j for $i \neq j$, the integrals are absolutely convergent and

$$a_2(p)\int_{\Omega_2}\frac{(x\cdot\zeta)^n\omega_2(\zeta,d\zeta)}{n!a_1(\zeta)a_3(\zeta)\dots a_r(\zeta)} = \int_{\Omega_2}\frac{a_2(\zeta)(x\cdot\zeta)^{n-2}\omega_2(\zeta,d\zeta)}{(n-2)!a_1(\zeta)a_3(\zeta)\dots a_r(\zeta)} = 0$$

since $\Omega_2 \subset W_2$; therefore

(48.3) $$2(2\pi i)^{\ell-2}k_x = a_2(p) \dots a_r(p)\int_{\Omega_1}\frac{(x\cdot\zeta)^n\omega_1(\zeta,d\zeta)}{n!a_2(\zeta)\dots a_r(\zeta)}.$$

<u>This formula gives the elementary solutions k_1 of any homogeneous polynomial</u> $a_1(p)$; indeed

<u>Lemma</u> 48.1. The polynomial $a_1(\zeta)$, the integer r and the two points $\zeta^* \in \Delta(a_1)$, $x \in X$ being given such that $x \cdot \zeta^* > 0$, we can find r - 1

homogeneous polynomials $a_2(\zeta)$, ... $a_r(\zeta)$ of degree 2 with the following properties: $\zeta^* \in \Delta(a_j)$; W_j has no singular points $(j = 2, \dots r)$; W_j does neither touch W_i nor cut $\Omega_i (i \neq j;\ i, j = i, \dots r)$.

By induction upon r, lemma 48.1 follows immediately from

Lemma 48.2. The polynomial $a_1(\zeta)$ and the two points $\zeta^* \in \Delta(a_1)$, $x \in X$ being given such that $x \cdot \zeta^* > 0$, we can find a polynomial $a_2(\zeta)$ of degree 2 with the following properties: $\zeta^* \in \Delta(a_2)$; W_2 has no singular points and does not touch W_1; W_2 is outside Ω_1 and W_1 is outside Ω_2.

Proof for ℓ even. Let the first axis contain ζ^* and

$$a_2(\zeta) = -\ \varepsilon\,\zeta_1^{\ 2} + \zeta_2^{\ 2} + \dots + \zeta_\ell^2;$$

ε is small and > 0; Ω_i is the real part V_i of W_i (theorem 43.1); V_2 is near the first axis and therefore outside V_1.

Proof for ℓ odd. Let the firxt axis contain ζ^*, the others be orthogonal to x, the second be outside W_1 and

$$a_2(\zeta) = -\ \zeta_1^{\ 2} + \varepsilon\,\zeta_2^{\ 2} + \zeta_3^{\ 2} + \dots + \zeta_\ell^2;$$

ε is small and > 0. In the theorem 46.1, let us choose ζ^{**} on the second axis: Ω_j is the set of the points $\zeta \in W_j$ such that

$$\zeta_1 = i\eta_1,\ \zeta_2 = \xi_2 + i\eta_2,\ \zeta_3 = \xi_3, \dots \zeta_\ell = \xi_\ell$$
$$(\eta_1, \xi_2, \eta_2, \xi_3, \dots \xi_\ell \in \Xi).$$

Therefore

$$\Omega_1 \cap W_2 = \Omega_2 \cap W_1.$$

On Ω_2,

$$\varepsilon\,\zeta_2^{\ 2} + \eta_1^{\ 2} + \xi_3^{\ 2} + \dots + \xi_\ell^2 = 0;$$

thus Ω_2 is near the second axis and $\Omega_2 \cap W_1$ is void.

49. A particular expression of the elementary solutions.

Proposition 49.1. Let $a(\zeta)$ be a homogeneous polynomial of degree m, such that W(a) has no singular points; let $n = m - \ell$; let $\Delta_1(a)$ be one of the components of $\Delta(a)$; let $\Omega_{a,b}$ and $\omega_{a,b}$ be defined as in the theorem 43.1 for ℓ even, as in the theorem 46.1 for ℓ odd; let $x \notin - C_1(a)$. There are homogeneous polynomials $g(\zeta)$ of degree $q \geq -n$ such that, for this point x, $\Omega_{a,b}$ and its infinite points are outside W(g); in the neighborhood of x, the elementary solution a(p), for $p \in \Delta_1(a)$, is

$$(49.1)\quad k_x = \frac{1}{2(2\pi i)^{\ell-2}} g(p) \int_{\Omega_{a,b}} g^{-1}(\zeta) \frac{(x\cdot\zeta)^{n+q}}{(n+q)!} \omega_{a,b}(\zeta, d\zeta).$$

Note. The following n°50 shows how k_x is given by an integration on any cycle homologous with $\Omega_{a,b}$ or $\Omega_{a,x\cdot\zeta,b}$.

Proof. According to n°48, the formula (49.1) holds when $g(\zeta)$ is some product $h(\zeta)$ of polynomials of degree 2. Now, if s is the degree of $h(\zeta)$,

$$(49.2)\quad h(p) \int_{\Omega_{a,b}} h^{-1}(\zeta) \frac{(x\cdot\zeta)^{n+s}}{(n+s)!} \omega_{a,b}(\zeta, d\zeta) = g(p)h(p) \int_{\Omega_{a,b}} g^{-1}(\zeta)h^{-1}(\zeta)$$
$$\frac{(x\cdot\zeta)^{n+q+s}}{(n+q+s)!} \omega_{a,b}(\zeta, d\zeta) = g(p) \int_{\Omega_{a,b}} g^{-1}(\zeta) \frac{(x\cdot\zeta)^{n+q}}{(n+q)!} \omega_{a,b}(\zeta, d\zeta).$$

50. The general expression of the elementary solutions. Let $a(\zeta)$ be a homogeneous polynomial of degree m, such that the cone W(a) is without singularity; let $b(\zeta)$ a linear function of ζ such that $b(0) = -1$; let W(a, b) and $W(a, x\cdot\zeta, b)$ be the projective algebraic varieties

$$a(\zeta) = b(\zeta) = 0,\ a(\zeta) = x\cdot\zeta = b(\zeta) = 0;$$

<u>let</u> $\Gamma(x)$ <u>be</u> <u>a</u> $(\ell-2)$-<u>dimensional</u> <u>chain</u>[1] <u>of</u> $W(a, b)$ <u>depending</u> <u>continuously</u> <u>on</u> $x \in X$, <u>in</u> <u>such</u> <u>a</u> <u>way</u> <u>that</u> <u>its</u> <u>boundary</u> $\beta\Gamma$ <u>is</u> <u>independent</u> <u>of</u> x <u>outside</u> <u>the</u> <u>neighborhood</u> <u>of</u> $W(a, x\cdot\zeta, b)$. Let $n = m - \ell$ and $f(\zeta)$ be a homogeneous polynomial of degree $q \geq -n$ such that $\Gamma \cap W(f)$ is void for a point x; then, in a neighborhood of this point,

$$\int_\Gamma \frac{(x\cdot\zeta)^{n+q}}{(n+q)!f(\zeta)}\,\omega_{a,b}(\zeta, d\zeta)$$

is a continuous function of x; it does not change when we replace $b(\zeta)$ by another linear function $c(\zeta)$ such that $c(0) = -1$ and Γ by its projection from 0 into $W(a, c)$ (proposition 38.1); we have for any homogeneous polynomial $g(\zeta)$ of degree $r \leq n + q$

$$(50.1)\quad g(p)\cdot\int_\Gamma \frac{(x\cdot\zeta)^{n+q}}{(n+q)!f(\zeta)}\,\omega_{a,b}(\zeta, d\zeta) = \int_\Gamma \frac{g(\zeta)}{f(\zeta)}\,\frac{(x\cdot\zeta)^{n+q-r}}{(n+q-r)!}\,\omega_{a,b}(\zeta, d\zeta);$$

thus formula (49.2) holds when $\Omega_{a,b}$ is replaced by Γ. Therefore the distribution

$$(50.2)\quad k_x(\Gamma) = f(p)\cdot\int_\Gamma \frac{(x\cdot\zeta)^{n+q}}{(n+q)!f(\zeta)}\,\omega_{a,b}(\zeta, d\zeta)$$

$$(\Gamma \cap W(f) \text{ void; degree } f = q \geq -n)$$

depends neither on f nor on b and satisfies [choose $g = a$ in (50.1)]

$$(50.3)\qquad a(p)\cdot k_x(\Gamma) = 0.$$

$k_x(\Gamma)$ is defined for small Γ. If $k(\Gamma)$ is defined and if

$$\Gamma = \Gamma_1 + \dots + \Gamma_s$$

is a subdivision of Γ (support of $\Gamma_i \subset$ support of Γ), then obviously

$$k_x(\Gamma) = k_x(\Gamma_1) + \dots + k_x(\Gamma_s);$$

therefore, if

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[1] In the meaning of the algebraic topology.

$$\Gamma = \Gamma_1 + \dots + \Gamma_s = \Gamma_{s+1} + \dots \Gamma_t$$

are two subdivisions of any Γ and if they are so fine that $k_x(\Gamma_1), \dots k_x(\Gamma_t)$ are defined, then the use of a common subdivision gives

$$k_x(\Gamma_1) + \dots + k_x(\Gamma_s) = k_x(\Gamma_{s+1}) + \dots + k_x(\Gamma_t):$$

this allows us to define

$$k_x(\Gamma) = k_x(\Gamma_1) + \dots + k_x(\Gamma_s).$$

Let us sum up:

Definition 50.1. $k_x(\Gamma)$ *is a homomorphism of the group of the chains* $\Gamma(x)$ *into the group of the distributions* k_x *which are homogeneous of degree* n *and annul* a(p); $k_x(\Gamma)$ *is defined by* (50.2) *for small* Γ.

If Γ belongs to some algebraic subvariety of W(a, b), in particular to W(a, x $\cdot$ ζ, b), then

(50.4) $$k_x(\Gamma) = 0,$$

for the differential form $\omega_{a,b}(\zeta, d\zeta)$ is 0 on any algebraic variety of complex dimension < $\ell - 2$.

If $\Gamma = \beta\Delta$, we can find a subdivision of Δ

$$\Delta = \Delta_1 + \dots + \Delta_s$$

so fine that we are allowed to write

$$k_x(\Gamma) = \sum_j k_x(\beta\Delta_j) = \sum_j f_j(p) \int_{\beta\Delta_j} \frac{(x\cdot\zeta)^{n+q}}{(n+q)!f_j(\zeta)} \omega_{a,b}(\zeta, d\zeta)$$

$$= \sum_j f_j(p) \int_{\Delta_j} d\Big[\frac{(x\cdot\zeta)^{n+q}}{(n+q)!f_j(\zeta)} \omega_{a,b}(\zeta, d\zeta)\Big];$$

hence

(50.5) $$k_x(\Gamma) = 0,$$

for a differential form of ζ, $d\zeta$ is 0 on W(a, b) if its degree > $\ell - 2$.

(50.4) and (50.5) give

(50.6) $$k_x(\Gamma) = 0 \text{ if } \Gamma \sim 0 \text{ mod } W(a, x \cdot \zeta, b)$$

($\sim$: homologous in the meaning of thé algebraic topology).

Let h be the homology class of the cycle Γ of W(a, b) mod W(a, x $\cdot$ ζ, b); let βh be the homology class of the cycle $\beta\Gamma$ of W(a, x $\cdot$ ζ, b); (50.6) allowed us to define

(50.7) $$k_x(h) = k_x(\Gamma).$$

(50.2) proves that $k_x(\Gamma)$ is a polynomial when Γ is independent of x; therefore $k_x(h)$ is a polynomial homogeneous of degree n (is 0 if n < 0) when h belongs to the image of the homology group of W(a, b), that is contains cycles Γ of W(a, b).

Now βh determines h mod. this image; thus βh determines $k_x(h)$ mod. the polynomials of degree n. In fact, if $g(\zeta)$ is any homogeneous polynomial of degree $n + 1 \geq 0$, (50.2) gives

(50.8) $$g(p) \cdot k_x(h) = \int_B g(\zeta)\, \omega_{a,x\cdot\zeta,b}(\zeta, d\zeta), \text{ where } B \in \beta h;$$

if $n + 1 \leq 0$, suppose there are a cycle $B(x) \in \beta h$ and a homogeneous polynomial $g(\zeta)$ of degree $-(n+1)$, such that $W(g) \cap B(x)$ is void for a point x; then in the neighborhood of this point

(50.9) $$k_x(h) = g(p) \int_B g^{-1}(\zeta)\, \omega_{a,x\cdot\zeta,b}(\zeta, d\zeta)$$

Let us sum up:

Definition 50.2. *Let* h(x) *be a* (ℓ-2)-*dimensional homology class of* W(a, b) mod W(a, x $\cdot$ ζ, b); *suppose that* h(x) *depends continuously on* x *and define*

$$k_x(h) = k_x(\Gamma), \text{ where } \Gamma \in h;$$

$k_x(h)$ is a homomorphism of the group of the $h(x)$ into the group of the distributions k_x homogeneous of degree n annuling $a(p)$. Let βh the homology class of $W(a, x \cdot \zeta, b)$ whose elements are the $\beta\Gamma(\Gamma \in h)$; βh defines $k_x(h)$ mod. the homogeneous polynomials of degree n: let $g(\zeta)$ be a polynomial of degree $\pm(n+1)$ and $B \in \beta h$;

(50.8) $$g(p) \cdot k_x(h) = \int_B g(\zeta)\omega_{a,x\cdot\zeta,b}(\zeta, d\zeta) \quad \text{for } n+1 \geq 0;$$

(50.9) $$k_x(h) = g(p) \int_B g^{-1}(\zeta)\omega_{a,x\cdot\zeta,b}(\zeta, d\zeta) \quad \text{for } n+1 \leq 0$$

in the neighborhood of x if $W(g) \cap B(x)$ is void.

The definition of $k_x(h)$ allows us to express as follows the proposition 49.1, that is all the results obtained in this chapter when $W(a)$ has no singularity:

Theorem 50.1. If the cone $W(a)$ has no singularity, then, outside $-C_1(a)$, the elementary solution of $a(p)$ for $p \in \Delta_1(a)$ is

$$\frac{1}{(2\pi i)^{\ell-2}} k_x(h)$$

where h is the homology class of $W(a, b)$ mod. $W(a, x \cdot \zeta, b)$ which contains the cycle $\frac{1}{2}\Omega_{a,b}$ defined by Theorem 43.1 for ℓ even and by Theorem 46,1 for ℓ odd; βh contains the cycles $\Omega_{a,x\cdot\zeta,b}$ defined by Theorem 44.1 for ℓ even and by Theorem 47.1, 48.1 for ℓ odd.

Note. n is arbitrary; $k_x = 0$ outside $C_1(a)$.

51. Petrowsky's paper [9] contains results which we now sum up. It would be important to clarify their proof. He assumes x outside the dual of $W(a)$: $W(x \cdot \zeta)$ does not touch $W(a)$; $W(a, x \cdot \zeta, b)$ is an algebraic variety without singularity; he asserts (p. 320) that the $(\ell-3)$th homology group of a such variety of complex dimension $\ell - 3$ is the direct sum of two subgroups, the subgroup of the <<algebraic>> homology classes and the subgroup of the

<<*finite*>> classes: a class is << algebraic>> when it contains <<algebraic>> cycles, which are algebraic subvarieties; a class is <<finite>> when its intersection by the homology class of the hyperplane sections of the variety is 0; a <<finite>> class contains <<finite>> cycles, which do not meet an arbitrarily chosen hyperplane, the <<infinite>> hyperplane. Therefore the intersection of βh by the class of the hyperplane sections contains an algebraic cycle; like βh, this cycle is a boundary in $W(a, b)$; but 0 is the only boundary which is an algebraic cycle (Lefschetz); thus

$\beta h(x)$ *is a finite class, for* $x \notin$ *dual of* $W(a)$.

Thus, *for these points* x, (50.9) *can be used*, B *being a* <<*finite*>> *cycle*, $W(g)$ *being the* <<*infinite*>> *hyperplane*: this is Petrowsky's expression of $k_x(h)$; *it holds on the complement of the dual of* $W(a)$.

Petrowsky shows that $k_x(h)$ is an analytic function of x in each connected component of this complement; he calls *lacunas* the components where $k_x(h) = 0$; by a *stable lacuna* he understands a lacuna which is not destroyed by any sufficiently small variation of the coefficients of $a(\zeta)$; his conclusion can be expressed as follows:

$x \in$ *stable lacuna*

is equivalent to: $\beta h(x) = 0$, that is: $h(x)$ *is the image of a homology class of* $W(a, b)$, *if* $n < 0$; $h(x)$ *is the image of an algebraic homology class of* $W(a, b)$, *if* $n \geq 0$.

Note. This statement differs from Petrowsky's statement: he uses only βh, without remarking that βh is finite and that βh is a boundary in $W(a, b)$; he defines $h(x)$ indirectly and only when $\beta h = 0$.

§6. Example: the waves equation

(To show how the theorem 50.1 can be actually used, we deduce formula (21.1) from this theorem)

Let us find the elementary solution k_x of

$$a(p) = p_1^2 - p_2^2 - \dots - p_\ell^2 \text{ for } p_1 > p_2^2 + \dots + p_\ell^2$$

52. A first expression of k_x. We choose

$$b(\zeta) = \zeta_1 - 1,\ g(\zeta) = \zeta_1^n,\ B(x) = \Omega_{a,x\cdot\zeta,b}$$

and apply (50.9); the condition

$$W(g) \cap B(x) \text{ is void}$$

becomes

$$\Omega_{a,x\cdot\zeta,b} \text{ is bounded;}$$

it will be satisfied. Thus (Theorem 50.1)

$$k_x = \frac{1}{(2\pi i)^{\ell-2}}\, p_1^{\ell-3} \int_{\Omega_{a,x\cdot\zeta,b}} \omega_{a,x\cdot\zeta,b}(\zeta, d\zeta) \tag{52.1}$$

for

$$x_1 > -r_1 = -\sqrt{x_2^2 + \dots + x_\ell^2}.$$

$W(a, x\cdot\zeta, b)$ is the sphere

$$\zeta_1 = 1,\ \zeta_2^2 + \dots + \zeta_\ell^2 = 1,\ x_2\zeta_2 + \dots + x_\ell\zeta_\ell + x_1 = 0;$$

let ζ^{**} be its center (for ℓ odd; Theorem 48.1): the preceding integral is a function of x_1 and r_1; this allows us to choose

$$x_2 = r_1,\ x_3 = \dots = x_\ell = 0;$$

hence $W(a, x\cdot\zeta, b)$ is the sphere

(52.2) $$\zeta_1 = 1,\ \zeta_2 = -\frac{x_1}{r_1},\ \zeta_3^2 + \dots + \zeta_\ell^2 = 1 - \frac{x_1^2}{r_1^2};$$

(52.3) $$\omega_{a,x\cdot\zeta,b} = -\frac{d\zeta_4 \dots d\zeta_\ell}{2r_1\zeta_3}.$$

The case: ℓ even. $\Omega_{a,x\cdot\zeta.b}$ is the real part of the sphere (52.2) with the orientation $\omega_{a,x\cdot\zeta,b} > 0$ (Theorem 44.1); $\dfrac{\sqrt{r_1^2 - x_1^2}}{}\,\dfrac{d\zeta_4 \dots d\zeta_\ell}{\zeta_3}$ is the elementary measure of this sphere; hence, for $x_1 > -r_1$,

(52.4) $$k_x = \frac{1}{2^{\ell-2}\pi^{\frac{\ell-2}{2}}}\, p_1^{\ell-3}\left[\frac{(x_1^2 - r_1^2)^{\frac{\ell-4}{2}}}{\Gamma(\frac{\ell-2}{2})r_1^{\ell-3}}\, Y\left(\frac{x_1^2}{r_1^2} - 1\right)\right]$$

where $Y(t)$ is Heavidise's function:

$$Y(t) = 0 \text{ for } t < 0, = 1 \text{ for } t > 0.$$

The case: ℓ odd. ξ^{**} is the point $(1, -\frac{x_1}{r_1}, 0, \dots 0)$; according to Theorem 48.1, $\zeta \in \Omega_{a,x\cdot\zeta,b}$ if

$$\zeta = \xi^{**} + i\eta,\quad \eta = (0, 0, \eta_3, \dots \eta_\ell),\quad \eta_3^2 + \dots + \eta_\ell^2 = \frac{x_1^2}{r_1^2} - 1;$$

$$\omega_{a,x\cdot\zeta,b} = i^{\ell-2}\frac{d\eta_4 \dots d\eta_\ell}{2r_1\eta_3};$$

the orientation of $\Omega_{a,x\cdot\zeta,b}$ is such that $\dfrac{d\eta_4 \dots d\eta_\ell}{\eta_3} > 0$: finally (52.4) still holds.

53. A second expression of k_x. Let

$$\delta(t) = \text{Dirac's measure} = \frac{dY}{dt};$$

$$X_j = x_j^2,\ P_j = \frac{\partial}{\partial X_j};$$

according to (10.1), (52.4) expresses that

$$k_x = \frac{1}{2^{\ell-2}\pi^{\frac{\ell-2}{2}}} p_1^{\ell-3} P_1^{-\frac{\ell-2}{2}} X_1^{\frac{3-\ell}{2}} \delta(X_1 - X_2 - \dots - X_\ell);$$

hence, upon application of (53.2),

$$(53.1) \qquad k_{\mathfrak{X}} = \frac{1}{2\pi} \left(\frac{P_1}{\pi}\right)^{\frac{\ell-4}{2}} \delta(X_1 - X_2 - \dots - X_\ell),$$

<u>Proposition</u> 53.1. <u>Let</u> x <u>be</u> <u>a</u> <u>real</u> <u>number</u> <u>and</u>

$$X = x^2,\ p = \frac{d}{dx},\ P = \frac{d}{dX};$$

<u>then</u>, <u>for</u> $X > 0$ <u>and</u> <u>any</u> <u>integer</u> $\alpha > 0$,

$$(53.2) \qquad p^\alpha f_X = 2^\alpha P^{\frac{\alpha-1}{2}} x^\alpha P^{\frac{\alpha+1}{2}} f_X.$$

<u>Proof</u>. (10.1) easily gives

$$(53.3) \qquad P^\beta X g_X = X P^\beta g_X + \beta P^{\beta-1} g_X.$$

Hence

$$\frac{1}{4} p^2 P^{-\frac{1-\alpha}{2}} x^{-\alpha} f_X = xPxP^{\frac{1-\alpha}{2}} x^{-\alpha} f_X = x^2 P^{\frac{3-\alpha}{2}} x^{-\alpha} f_X + \frac{1}{2} P^{\frac{1-\alpha}{2}} x^{-\alpha} f(x) =$$

$$P^{\frac{3-\alpha}{2}} x^{2-\alpha} f_X - \frac{2-\alpha}{2} P^{\frac{1-\alpha}{2}} x^{-\alpha} f_X = P^{\frac{1-\alpha}{2}} [P x^{2-\alpha} f_X - \frac{2-\alpha}{2} x^{-\alpha} f_X] = P^{\frac{1-\alpha}{2}} x^{2-\alpha} P f_X.$$

Replacing f_X by $P^{\frac{1-\alpha}{2}} f_X$, we obtain

$$\frac{1}{2^\alpha} p^\alpha P^{\frac{-1-\alpha}{2}} x^{-\alpha} P^{\frac{1-\alpha}{2}} f_X = \frac{1}{2^{\alpha-2}} p^{\alpha-2} P^{\frac{1-\alpha}{2}} x^{2-\alpha} P^{\frac{3-\alpha}{2}} f_X;$$

in other words the operator

$$\frac{1}{2^{\alpha}} p^{\alpha} P^{\frac{-1-\alpha}{2}} x^{-\alpha} P^{\frac{1-\alpha}{2}}$$

does not change when α is replaced by $\alpha - 2$; but it is obviously the identity for $\alpha = 0$ or 1; thus it is the identity for any integer $\alpha > 0$.

54. The invariant expression of k_x. The change of variables in the distributions theory is very easy, but has not yet been described:

Definition. Let φ_s be a distribution of one variable s and $u(x)$ be a numerical, infinitely derivable function on X; $\varphi_{u(x)}$ denotes the distribution defined on X by the following extension of (36.3):

$$(54.1) \quad \int_X \varphi_{u(x)} f(x)\, dx_1 \ldots dx_\ell = \int_{-\infty}^{+\infty} \varphi_s ds \int_{u(x)=s} f(x)\, \omega_u(x, dx);$$

ω_u is a differential form such that

$$du(x) \cdot \omega_u(x, dx) = dx_1 \ldots dx_\ell;$$

the manifold $u(x) = s$ has the orientation such that $\omega_u(x, dx) > 0$.

From (54.1) follows

$$(54.2) \qquad \varphi_{u(x)} * f(x) = \int_{-\infty}^{+\infty} \varphi_s ds \int_{u(x-y)=s} f(y)\, \omega_u(x-y, dy),$$

where

$$\omega_u(x - y, dy) > 0 \text{ on } u(x - y) = s.$$

Likewise, let $\varphi_{s,t}$ be a distributibn of two variables and $u(x)$, $v(r)$ be two numerical functions:

$$\int_X \varphi_{u(x),v(x)} f(x)\, dx_1 \ldots dx_\ell = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi_{s,t}\, dsdt \int_{\substack{u(x)=s \\ v(x)=t}} f(x)\, \omega_{u,v}(x, dx),$$

where

$$du(x) \cdot dv(x)\, \omega_{u,v}(x, dx) = dx_1 \dots dx_\ell$$

$$\omega_{u,v}(x, dx) > 0 \text{ on the manifold } u(x) - s = v(x) - t = 0.$$

$$\varphi_{u(x),\, v(x)} * f(x) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi_{s,t}\, ds\, dt \int_{\substack{u(x-y)=s \\ v(x-y)=t}} f(y)\, \omega_{u,v}(x - y, dy)$$

where $\omega_{u,v}(x - y, dy) > 0$ on $u(x - y) - s = v(x - y) - t = 0$.

Two successive changes of variables are equivalent with the composed change.

This enables us to apply the Theorem 6.1 to (53.1): P_1 can be replaced by the derivationwithrespect to $X_1 - X_2 - \dots - X_\ell$; finally we obtain the following expression of k_x, which is invariant under the linear mappings of X leaving $x_1^2 - x_2^2 - \dots - x_\ell^2$ invariant:

<u>Theorem</u> 54.1. <u>The elementary solution of</u>

$$p_1^2 - p_2^2 - \dots - p_\ell^2 \ \underline{\text{for}}\ p_1 > \sqrt{p_2^2 + \dots + p_\ell^2}$$

<u>is</u>, <u>when</u> $x_1 > -\sqrt{x_2^2 + \dots + x_\ell^2}$,

$$(54.3) \quad k_x = \frac{1}{2\pi} \left(\frac{Q}{\pi}\right)^{\frac{\ell-4}{2}} \cdot \delta(R) \ \underline{\text{for}}\ Q > 0,\ Q^{\frac{1}{2}} > 0;\ Q = \frac{d}{dR},\ R = x_1^2 - x_2^2 - \dots - x_\ell^2.$$

<u>Note</u>. (21.1) immediately follows from (54.3), (54.2) and (6.12).

Second Part

Linear Hyperbolic Equations with Variable Coefficients

Introduction

Let

(1) $$a(x, p)u(x) = v(x)$$

be a linear equation of order m:

$x = (x_1, \ldots, x\ ) \in X$, real vector space;

$p = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_\ell}) \in \Xi$, vector space dual to X;

$v(x)$ is a given function on X;

$a(x, p)$ is a polynomial in p; its coefficients are functions defined on X; its degree is m;

$u(x)$ is the unknown function.

Cauchy's problem consists of finding $u(x)$ when $u(x)$ and its derivatives of order $< m$ are given on the boundary of a domain of X; the solution is said to be <u>global</u> when it is defined on the whole domain and <u>local</u> when it is defined on a neighborhood of the boundary.

I. <u>History</u>. For the analytic case Cauchy and Kowalewski proved the existence of local solutions of Cauchy's problem [32]; in his well known book [33], J. Hadamard emphasized the uselessness for physics of the local solutions and proved the existence of global solutions for the hyperbolic equation of second order; he used Riemann's geometry and Green's formula, which transforms the search for regular solutions into the search for a solution having a convenient singularity; at first he had to deal with the analytic case. In 1935 J. Schauder [36] gave an easier method; the classic energy relation enables him to extend the local Cauchy-Kowalewski's solution into a global one for the analytic and finally non analytic equations of second order. Two years later I. Petrowsky [34] defined the hyperbolic equations of order m ($m > 2$) and, using Schauder's process, proved the existence of the global solutions of Cauchy's problem. Recently M. Riesz [35] published a deep study of the equation of second order; and Y. Fourès-Bruhat [31] gave explicit solutions for systems of second order equations.

I. Petrowsky's paper needs to be completed; its main part is the a priori limitation of the solution of Cauchy's problem; this limitation consists in seventeen pages of inequalities without comments; Petrowsky's first step is surprising: he defines a transformation involving both the Fourier transformation and the use in the (ℓ-1)-dimensional space of a variable frame depending only on its first vector; he assumes[1] that this frame depends continuously on its first vector; this assumption does not differ from the assumption that the (ℓ-2)-sphere is parallelisable, that is: there exists on this sphere a frame depending continuously on its origin. He uses this assumption in order to extend to the case $\ell > 2$ the important particularities [39] which occur for $\ell = 2$ and which are closely related to the properties of the complex numbers and the analytic functions. Now obviously a continuous vector field cannot be drawn on the 2-sphere, which therefore is not parallelisable; the same holds [40] for any even dimensional sphere; recently N. E. Steenrod and J. H. C. Whitehead [41] gave a deep theorem asserting that a sphere whose dimension is not $2^k - 1$ is not parallelisable; as for the $(2^k - 1)$-spheres,

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[1] Pp. 821-822: << die Functionen k_{ij} besitzen...stetige partielle Abteilungen beliebig hoher Ordnung nach α'_k >>; more explicitly, p. 861: << Die Axen Ox'_2, ... $Ox'_{\ell-1}$ wählen wir derart, dass ihre Richtungtskosinusen in bezug auf die alten Koordinatenachsen Ox_1, ... $Ox_{\ell-1}$ Functionen von den Richtungskosinusen α'_1, ... $\alpha'_{\ell-1}$ von Ox'_1 seien, die stetige Abteilungen erster Ordnung besitzen.>>

we know [42] only what happens for $k = 1, 2$, and 3: the complex numbers, the quaternions and the Cayley's numbers define easily parallelisms on the 1, 3 and 7 spheres. Thus we can use Petrowsky's proof only for $\ell = 3$, 5 and 9; moreover, Hadamard's descent method immediately extends Petrowsky's statement to all $\ell \leq 9$.

Our first purpose is a complete proof of Petrowsky's assertion that Cauchy's problem has global solutions when the equation is hyperbolic; our proof has no connection either with the parallelism of the sphere or with the Cayley's numbers.

But, however interesting may local Cauchy's problem be (see: E. Cartan, Les systèmes différentiels extérieurs et leurs applications géométriques, Hermann, 1945), the global Cauchy problem is a secondary problem: let $T_1 \subset T$ be two domains of X; let $u_1(x)$ be the global solution in T_1 of Cauchy's problem; let $u(x)$ be any function such that $u(x) = u_1(x)$ in T_1; obviously $u(x)$ is the global solution in T of a Cauchy problem: if the global Cauchy problem is solved in T, then it is solved in T_1. A more complete study shows that the fundamental problem is the problem which could be called <<the Cauchy problem with Cauchy's data zero at infinity>>; its solution is given by the inverse operator of $a(x, p)$, which was denoted $a^{-1}(p)$ when $a(x, p)$ was independent of x (First Part). The definition and the study of this inverse operator is our essential purpose.

II. Summary. Chapter V states inequalities applicable to the local Cauchy-Kowalewski solution of (1), and, using these inequalities, extends these solutions to the whole space: Petrowsky's assertion is proved. Now these first inequalities give an imprecise information about the global behaviour of the solutions defined on the whole space; Chapter VI gives for such solutions new and simpler inequalities; moreover, it weakens extremely the assumptions about the coefficients of (1), which are finally assumed to be bounded and lipschitzan. Chapter VII extends these results to a manifold, defines the elementary solutions and solves Cauchy's problem. Chapter VIII studies similarly hyperbolic systems.

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CHAPTER V

THE EXISTENCE OF GLOBAL SOLUTIONS ON A VECTOR SPACE

Introduction to Chapter V

(The precise work begins in §1; the Introduction is a motivating discussion.)

I. Let us consider the linear differential equation of order m (see p. 104):

(1) $$a(x, p)u(x) = v(x).$$

Denote x_1 by t; suppose Cauchy's data given for $t = 0$ and the equation to be analytic: Cauchy-Kowalewski's theorem gives a local solution, defined for $0 \leq t \leq \tau$. A convenient a priori limitation of the local solutions of (1) can allow us to extend this local resolution to non analytic Cauchy's data, to non analytic v(x) and then successively to the intervals $0 \leq t \leq 2\tau$, $0 \leq t \leq 3\tau$, and so on: we obtain global solutions. Summing up: <u>the existence of global solutions of Cauchy's problem can result from an a priori bound of its local solutions</u>.

II. In order to obtain a relation similar to the classic energy equality of the mechanics (see (4)) let us replace (1) by the equivalent system

(2) $$\frac{dU(t)}{dt} = A \cdot U(t) + V(t)$$

where U(t) is the unknown vector $(u, \frac{\partial u}{\partial t}, \ldots, \frac{\partial^{m-1} u}{\partial t^{m-1}})$;

V(t) is the given vector $(0, 0, \ldots, 0, v)$;

A is the matrix

$$(3) \qquad A = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdot & -\alpha_{m-2} & -\alpha_{m-1} \end{pmatrix}$$

whose element $-\alpha_\mu(x_1, \dots x_\ell; p_2, \dots p_\ell)$ is a differential operator independent of p_1 and of order $m - \mu$. Let us consider that, for any value of t, $U(t)$ and $V(t)$ are real points of some Hilbert's space, whose norm and scalar product are denoted by $||U||$, (U, V); A has an adjoint operator A^*; (2) gives

$$(4) \qquad \frac{d}{dt}||U(t)||^2 = ((A + A^*)U(t), U(t)) + 2(U(t), V(t)).$$

Since

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*),$$

$\frac{1}{2}(A + A^*)$ and $\frac{1}{2i}(A - A^*)$ being obviously hermitian operators,

$\frac{1}{2}(A + A^*)$ shall be called <u>the hermitian part of</u> A. Suppose this hermitian part has a finite upper bound γ; (see [37] Defin. 2.13, 2.14, p. 56); (4) gives

$$(5) \qquad \frac{d}{dt}[\,||U(t)||\ \exp(-\gamma t)] \leq ||V(t)||\ \exp(-\gamma t).$$

An a priori limitation is obtained.

Now A is a non bounded operator, whose hermitian part will be bounded only if the Hilbert's norm is judiciously chosen; and the Hilbert's norm used for U will be a trivial one. Thus we have to find some hermitian positive operator B such that the hermitian part of A is bounded for the norm $\sqrt{(BU, U)}$, whose scalar product is (BU, V) = (U, BV) (U, V: real). For this norm the adjoint of A is not its trivial adjoint A^* but the operator C such that

$$(BCU, V) = (BU, AV);$$

that is such that: $BC = A^*B$; thus for the norm defined by B the hermitian part of A is $\frac{1}{2}[A + B^{-1}A^*B]$: *the assumption that the hermitian part of A is bounded for the norm defined by B is the assumption that*

$$BA + (BA)^* < \text{const. } B.$$

If such an operator B exists, then the local solutions of (1) can be a priori bounded.

III. Now, since A is a matrix whose element $a_{\lambda\mu}(x_1, \dots x_\ell; p_2 \dots p_\ell)$ is a differential operator of order $\lambda - \mu + 1$, let us choose for B a matrix whose element $b_{\lambda\mu}(x_1, \dots x_\ell; p_2, \dots p_\ell)$ is a differential operator of order $2n - \lambda - \mu$, n being a fixed integer. Thus the difference of the orders of the corresponding elements of BA and B is at most one. Therefore, when the bounded operators are neglected, we can deal with the x_λ and p_λ as if they were commutative variables; and *the search for an operator B, such that the hermitian part of the operator A is bounded for the norm defined by B, is reduced to the purely algebraic problem stated in* n°iv.

IV. Let A and B now be matrices of rank m, whose elements are real numbers; let us say that A *is symmetric for the norm defined by* B when B is symmetric positive and BA symmetric, which fact requires that the characteristic roots of A are real.

An algebraic problem. Let $x_1, \dots x_\ell, \xi_2, \dots \xi_\ell$ be real commutative variables; let $A(x, \xi)$ be the matrix (3), where $\alpha_\mu(x_1, \dots x_\ell; p_2 \dots p_\ell)$ is replaced by the principal part of $\alpha_\mu(x_1, \dots x_\ell; \xi_2, \dots \xi_\ell)$: thus the element $a_{\lambda\mu}(x, \xi)$ of $A(x, \xi)$ is a polynomial in $(\xi_2, \dots \xi_\ell)$ homogeneous of degree $\lambda - \mu + 1$; we ask for a matrix $B(x, \xi)$ such that $A(x, \xi)$ is symmetric for the norm defined by $B(x, \xi)$ and such that the element $b_{\lambda\mu}(x_1, \dots, x_\ell; \xi_2, \dots \xi_\ell)$ of $B(x, \xi)$ is a polynomial in $(\xi_2, \dots \xi_\ell)$ homogeneous of degree $2n - \lambda - \mu$.

This algebraic problem is possible only if all characteristic roots of $A(x, \xi)$ are real; when they are real and distinct, this problem has solutions, even if $A(x, \xi)$ does not belong to the type (3). But it has a much simpler solution if it belongs to this type (3): we shall consider only this simpler solution.

Obviously a complete proof of these assertions must follow the reverse order of the preceding rough sketch.

The condition that the characteristic roots are real and distinct is the condition that the differential equation (1) is <u>hyperbolic</u>.

§1. The matrices B defining norms for which a given matrix A is hermitian

61. Notations.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot \\ a_{21} & a_{22} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \quad \text{and } B = \begin{pmatrix} b_{11} & b_{12} & \cdot \\ b_{21} & b_{22} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

are matrices with m rows and m columns; their elements are real or complex numbers. The identity matrix is

$$I = \begin{pmatrix} 1 & 0 & \cdot \\ 0 & 1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}.$$

The elements of $A \cdot B$ are $\sum_{\nu=1}^{m} a_{\lambda\nu} b_{\nu\mu}$.

det A means the determinant of A. The characteristic polynomial of A is

$$\det (rI - A);$$

its roots $r_1, \ldots, r_m$ are the characteristic roots; A and $C^{-1} \cdot A \cdot C$ have the same characteristic roots.

The transpose ${}^tA$ of A is the matrix whose elements are

$${}^ta_{\lambda\mu} = a_{\mu\lambda}; \qquad {}^t(A \cdot B) = {}^tB \cdot {}^tA.$$

A is symmetric when ${}^tA = A$; A is symmetric positive and we write

$$A > 0,$$

when $\sum_{\lambda,\mu} a_{\lambda\mu}\xi_\lambda\xi_\mu > 0$ for $\xi_1, \ldots, \xi_\ell$ real, $\xi_1^2 + \ldots + \xi_\ell^2 \neq 0$. If A is symmetric, then ${}^tC \cdot A \cdot C$ is also symmetric. If $A > 0$, C real and det $C \neq 0$, then ${}^tC \cdot A \cdot C > 0$; in particular $A^{-1} > 0$.

The adjoint A^* of A is the matrix such that

$$a^*_{\lambda\mu} = \bar{a}_{\mu\lambda} \text{ (that is: } a^*_{\lambda\mu} \text{ and } a_{\mu\lambda} \text{ are conjugate complex numbers);}$$

$$(A \cdot B)^* = B^* \cdot A^*.$$

A is hermitian when $A^* = A$; A is hermitian positive and we write

$$A > 0$$

when $\sum_{\lambda,\mu} a_{\lambda\mu}\zeta_\lambda\bar{\zeta}_\mu > 0$ for $\zeta_1, \ldots, \zeta_\ell$ complex, $|\zeta_1|^2 + \ldots + |\zeta_\ell|^2 \neq 0$. If A is hermitian, then A^* is also hermitian. If $A > 0$ and det $C \neq 0$, then $C^* \cdot A \cdot C > 0$; in particular $A^{-1} > 0$. A real symmetric matrix is hermitian; it is hermitian positive if and only if it is symmetric positive.

N° IV of Introduction to Chapter V gave the reason of the following terminology: A <u>is said to be hermitian for the norm defined by</u> B <u>when</u> B

is hermitian positive and $B \cdot A$ *hermitian;* when A and B are moreover real, A *is said to be symmetric for the norm defined by* B.

62. *The characteristic roots of* A *have to be real.*

Proposition 62.1. *If* A *is hermitian for the norm defined by* B, *then the characteristic roots of* A *are real.*

Proof. There is a matrix C such that both $C^* \cdot B \cdot C$ and $C^* \cdot B \cdot A \cdot C$ are real diagonal matrices (see for instance [7], §113, Hermitesche Formen). Thus

$$C^{-1} \cdot A \cdot C = (C^* \cdot B \cdot C)^{-1}(C^* \cdot B \cdot A \cdot C)$$

is a real diagonal matrix. Its diagonal elements are the characteristic roots of A.

63. *A matrix* B *such that* A *is symmetric for the norm defined by* B. (Suppose the roots of A real and moreover distinct; then there are matrices B defining norms for which A is hermitian; these matrices constitute a convex m-dimensional set. And, when A is real, that set contains a matrix B, whose elements are polynomials in the elements of A. But we shall neither use nor prove that general assertions: we have to consider the following special case.)

Let us suppose

$$(63.1) \qquad A = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdot & -\alpha_{m-2} & -\alpha_{m-1} \end{pmatrix}.$$

Its characteristic polynomial is

$$P(r) = \alpha_0 + \alpha_1 r + \dots + \alpha_{m-1} r^{m-1} + r^m.$$

Let $r_1, \dots r_m$ be the characteristic roots; let

$$R = \begin{pmatrix} 1 & 1 & 1 & \vdots & 1 \\ r_1 & r_2 & r_3 & \vdots & r_m \\ r_1^2 & r_2^2 & r_3^2 & \vdots & r_m^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ r_1^{m-1} & r_2^{m-1} & r_3^{m-1} & \vdots & r_m^{m-1} \end{pmatrix}, \qquad s_\lambda = \sum_{\mu=1}^{m} r_\mu^\lambda,$$

$$S = R \cdot {}^tR = \begin{pmatrix} s_0 & s_1 & s_2 & \vdots & s_{m-1} \\ s_1 & s_2 & s_3 & \vdots & s_m \\ s_2 & s_3 & s_4 & \vdots & s_{m+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{m-1} & s_m & s_{m+1} & \vdots & s_{2m-2} \end{pmatrix};$$

hence, if the r_μ are real, $S > 0$. We have (Van der Monde's determinant)

$$\text{det. } R = \prod_{\lambda>\mu} (r_\lambda - r_\mu); \text{ hence: } \text{det. } S = \prod_{\lambda>\mu} (r_\lambda - r_\mu)^2.$$

The s_λ are polynomials in α_μ defined by Newton's formulas

$$s_0 = m$$

$$s_1 + \alpha_{m-1} = 0$$

$$s_2 + \alpha_{m-1} s_1 + 2\alpha_{m-2} = 0$$

$$s_{m-1} + \alpha_{m-1} s_{m-2} + \dots + \alpha_2 s_1 + (m-1)\alpha_1 = 0$$

$$s_m + \alpha_{m-1} s_{m-1} + \dots + \alpha_1 s_1 + \alpha_0 s_0 = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$s_{m+\lambda} + \alpha_{m-1} s_{m-1+\lambda} + \dots + \alpha_1 s_{1+\lambda} + \alpha_0 s_\lambda = 0 \qquad (\lambda \geq 0).$$

hence

$$A \cdot S = \begin{pmatrix} s_1 & s_2 & s_3 & \vdots & s_m \\ s_2 & s_3 & s_4 & \vdots & s_{m+1} \\ s_3 & s_4 & s_5 & \vdots & s_{m+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_m & s_{m+1} & s_{m+2} & \vdots & s_{2m-1} \end{pmatrix}$$

is symmetric; and, since S is symmetric,

$$S^{-1} \cdot (A \cdot S) \cdot S^{-1} = S^{-1} \cdot A$$

is also symmetric. Finally define

$$(63.2) \qquad B = S^{-1} \cdot \det S.$$

B is a matrix whose elements are real polynomials of the elements of A; B and $B \cdot A$ are symmetric; if the characteristic roots of A are real and distinct, then $B > 0$ (for $S > 0$, $\det S > 0$) and thus A is symmetric for the norm defined by B.

Note 63. $B > 0$ means: strictly positive; that is $(BF, F) > 0$ for $F \neq 0$.

64. A real matrix $B(\xi)$ such that $A(i\xi)$ is hermitian for the norm defined by $B(i\xi)$. Now, using the matrix B defined by (63.2), we can define the matrix that §2 requires.

Definition. Let $\xi = (\xi_1, \ldots, \xi_\ell)$ be ℓ real variables. Suppose α_μ is a real homogeneous polynomial in ξ of degree $m - \mu$. Then A and the matrix B defined by (63.2) are denoted $A(\xi)$ and $C(\xi)$.

Properties of $A(\xi)$. The element $a_{\lambda\mu}(\xi)$ of $A(\xi)$ is a real homogeneous polynomial of degree $\lambda - \mu + 1$.

Properties of $C(\xi)$.

1°. The element $c_{\lambda\mu}(\xi)$ of $C(\xi)$ is a real homogeneous polynomial of degree $2n - \lambda - \mu$, where

$$2n = m(m-1) + 2.$$

2°. $C(\xi)$ and $C(\xi) \cdot A(\xi)$ are symmetric; $C(\xi) > 0$ when the characteristic roots of $A(\xi)$ are real and distinct, and ξ is real.

Proof of 1°: $r_\lambda(\xi)$ is a homogeneous function of degree 1; $s_\lambda(\xi)$ is a homogeneous polynomial of degree λ.

Proof of 2°: See $n^\circ 63$.

Let d be a real or complex number and $\bar{d}$ its conjugate; let

$$D = \begin{pmatrix} d^{m-1} & 0 & \cdot & 0 & 0 \\ 0 & d^{m-2} & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & d & 0 \\ 0 & 0 & \vdots & 0 & 1 \end{pmatrix}; \qquad D^* = \begin{pmatrix} \bar{d}^{m-1} & 0 & \cdot & 0 & 0 \\ 0 & \bar{d}^{m-2} & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & \bar{d} & 0 \\ 0 & 0 & \vdots & 0 & 1 \end{pmatrix};$$

the degrees of the homogeneous $a_{\lambda\mu}(\xi)$ and $b_{\lambda\mu}(\xi)$ show that

$$A(d\xi) = dD^{-1} \cdot A(\xi) \cdot D \tag{64.1}$$

$$C(d\xi) = d^{2(n-m)} D \cdot C(\xi) \cdot D; \tag{64.2}$$

hence

$$C(d\xi) \cdot A(d\xi) = d^{2(n-m)+1} D \cdot C(\xi) \cdot A(\xi) \cdot D. \tag{64.3}$$

Define

$$C(d, \xi) = d^{2(m-n)} D^* \cdot D^{-1} \cdot C(\xi). \tag{64.4}$$

Properties of $C(d, \xi)$.

1°. The element $c_{\lambda\mu}(d, \xi)$ of $C(d, \xi)$ is a homogeneous polynomial in ξ of degree $2n - \lambda - \mu$.

2°. $C(d, \xi)$ is real if id and ξ are real.

3°. $C(d, d\xi)$ and $d^{-1}C(d, d\xi) \cdot A(d\xi)$ are hermitian if ξ is real; if moreover the characteristic roots of $A(\xi)$ are real and distinct, then $C(d, d\xi) > 0$.

Proof of 2°. If id and ξ are real, $d^{2(m-n)}$ and $D^* \cdot D^{-1}$ are real.

Proof of 3°. (64.2) and (64.4) give

$$C(d, d\xi) = D^* \cdot C(\xi) \cdot D,$$

which is obviously hermitian or hermitian > 0, according as $B(\xi)$ is hermitian or hermitian > 0. On the other hand (64.3) and (64.4) give

$$d^{-1}C(d, d\xi) \cdot A(d\xi) = D^* \cdot C(\xi) \cdot A(\xi) \cdot D,$$

which is hermitian because $C(\xi) \cdot A(\xi)$ is hermitian.

Now let

$$n > \frac{m(m-1)}{2} \qquad (n:\ \text{integer}); \tag{64.5}$$

define

$$B(\xi) = (-\xi_1^2 - \dots - \xi_\ell^2)^{n-1-m(m-1)/2} C(i, \xi); \tag{64.6}$$

the properties of $C(d, \xi)$ give the following conclusion:

Proposition 64.1. *Let $A(\xi)$ be a given real matrix belonging to the type (63.1), where α_μ is a homogeneous polynomial in $\xi = (\xi_1, \dots \xi_\ell)$ of degree $m - \mu$: thus the element $a_{\lambda\mu}$ of $A(\xi)$ is a homogeneous polynomial in ξ of degree $\lambda - \mu + 1$; suppose the characteristic roots of $A(\xi)$ to be real and distinct. To such a matrix $A(\xi)$ and any integer $n > \frac{m(m-1)}{2}$ is associated another real matrix $B(\xi)$ with the following properties: its element $b_{\lambda\mu}$ is a homogeneous polynomial in ξ of degree $2n - \lambda - \mu$; $iA(i\xi)$ is hermitian for the norm defined by $B(i\xi)$.*

Note. This last assertion means

$$B(i\xi) \text{ and } iB(i\xi) \cdot A(i\xi) \text{ are hermitian matrices;} \tag{64.7}$$

(64.8) $$B(i\xi) > 0.$$

(64.7) is equivalent to

(64.9) $$B(i\xi) = B^*(i\xi), \qquad B(i\xi)\cdot A(i\xi) + [B(i\xi)\cdot A(i\xi)]^* = 0$$

or, since $B(\xi)$ is real, to

(64.10) $$B(\xi) = {}^tB(-\xi), \qquad B(\xi)\cdot A(\xi) + {}^t[B(-\xi)\cdot A(-\xi)] = 0.$$

<u>Note</u>. The coefficients of the polynomials $b_{\lambda\mu}(\xi)$ are themselves polynomials in the coefficients of the polynomials $\alpha_\mu(\xi)$.

65. <u>Example</u>. Let $m = 2$,

$$A(\xi) = \begin{pmatrix} 0 & 1 \\ \alpha_0(\xi) & \alpha_1(\xi) \end{pmatrix}$$

where $\alpha_0(\xi)$ and $\alpha_1(\xi)$ are homogeneous of degrees 2 and 1 and such that $4\alpha_0(\xi) < \alpha_1^2(\xi)$; let $n = 2$;

for ξ real: $$B(\xi) = \begin{pmatrix} 2\alpha_0(\xi) - \alpha_1^2(\xi) & -\alpha_1(\xi) \\ \alpha_1(\xi) & 2 \end{pmatrix} \text{ is real,}$$

$$B(i\xi) = \begin{pmatrix} 2\alpha_0(\xi) + \alpha_1^2(\xi) & -i\alpha_1(\xi) \\ i\alpha_1(\xi) & 2 \end{pmatrix} \text{ is hermitian} > 0,$$

$$iB(i\xi)A(i\xi) = -\begin{pmatrix} \alpha_0(\xi)\alpha_1(\xi) & -2i\alpha_0(\xi) \\ 2i\alpha_0(\xi) & \alpha_1(\xi) \end{pmatrix} \text{ is hermitian.}$$

§2. <u>The operators</u> B <u>defining norms for which the hermitian part of a given operator</u> A <u>is bounded</u>.

The n°66 and 67 extend an inequality which L. Gårding [43] used for solving Dirichlet's problem; Višik had just [44] announced its solution but did not indicate his method.

66. <u>Preliminaries.</u> X and Ξ are dual vector spaces of dimension ℓ; $p = (p_1, \dots, p_\ell) = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x})$ has to be considered as a point of Ξ; all the functions of x to be used are infinitely derivable. $F(x) = (f_1(x), \dots, f_m(x))$ is a mapping of X into an m-dimensional vector space Y; $A(x, p)$ is a matrix with m rows and m columns whose element $a_{\lambda\mu}(x, p)$ is a polynomial in p;

$$a_{\lambda\mu}(x, p) = \sum_{n_1, \dots, n_\ell} c^{\lambda\mu}_{n_1, \dots, n_\ell}(x) \frac{\partial^{n_1 + \dots + n_\ell}}{\partial x_1^{n_1} \dots \partial x_\ell^{n_\ell}};$$

$A(x, p)F(x)$ is also a mapping of X into Y.

We use the Hilbert norm

$$||f|| = [\int \underset{X}{\dots} \int \sum_\lambda |f_\lambda(x)|^2 dx_1 \dots dx_\ell]^{1/2}$$

and the corresponding scalar product (F, G); the <u>adjoint</u> of the operator $A(x, p)$ is the operator $B(p, x)$ whose element $b_{\mu\lambda}(p, x)$ has the following definition:

$$b_{\mu\lambda}(p, x)f_\lambda(x) = \sum_{n_1, \dots, n_\ell} (-1)^{n_1 + \dots + n_\ell} \frac{\partial^{n_1 + \dots + n_\ell}}{\partial x_1^{n_1} \dots \partial x_\ell^{n_\ell}} [\overline{c^{\lambda\mu}_{n_1, \dots, n_\ell}(x)} f_\lambda(x)];$$

that is (the components $x_1 \dots x_\ell$, $\xi_1 \dots \xi_\ell$, $\eta_1 \dots \eta_\ell$ of $x \in X$, $\xi \in \Xi$ and $\eta \in \Xi$ being <u>commutative variables</u>):

$$b_{\lambda\mu}(\xi, x) = \overline{a_{\mu\lambda}(x, -\xi)};$$

or

$$b_{\lambda\mu}(\zeta, x) = \overline{a_{\mu\lambda}(x, -\overline{\zeta})} \text{ for } \zeta = \xi + i\eta, \overline{\zeta} = \xi - i\eta, \xi \text{ and } \eta \in \Xi;$$

or

$$b_{\lambda\mu}(i\xi, x) = \overline{a_{\mu\lambda}(x, i\xi)} \text{ for } \xi \in \Xi;$$

this means that the matrices $A(x, i\xi)$ and $B(i\xi, x)$, whose elements are complex numbers, are adjoint; hence:

Lemma 66.1. The operators $A(x, p)$ and $B(p, x)$ are adjoint if and only if the matrices $A(x, i\xi)$ and $B(i\xi, x)$ are adjoint.

A hermitian operator $A(x, p)$ is said to be ≥ 0 when $(A(x, p)F(x), F(x)) \geq 0$ (F and its derivatives square integrable). This definition gives an ordering relation for the hermitian operators $A(x, p)$; similarly an ordering relation exists for the hermitian matrices. Obviously

Lemma 66.2. If A and B are hermitian and if

$$A \leq B,$$

then

$$C^* \cdot A \cdot C \leq C^* \cdot B \cdot C \text{ for any } C.$$

Lemma 66.3. If C has the bound γ, if α and β are positive numbers such that $\gamma^2 \leq \alpha\beta$, then

$$A \cdot C \cdot B + (A \cdot C \cdot B)^* \leq \alpha A \cdot A^* + \beta B^* \cdot B.$$

Proof.

$$([A \cdot C \cdot B + (A \cdot C \cdot B)^*]F, F) = 2\mathcal{R}(A \cdot C \cdot B \cdot F, F) \leq 2|(C \cdot B \cdot F, A^*F)|$$

$$\leq 2\gamma\|B \cdot F\| \cdot \|A^* \cdot F\| \leq \alpha\|A^*F\|^2 + \beta\|BF\|^2 = ([\alpha A \cdot A^* + \beta B^* \cdot B]F, F).$$

According to Lemma 66.1, $A(p)$ is hermitian if and only if $A(i\xi)$ is hermitian; an easy application of Fourier's transformation gives (see Theorem 6.3, (6.11^{bis})).

Lemma 66.4. $A(p)$ is hermitian ≥ 0 if and only if $A(i\xi)$ is hermitian ≥ 0.

In order to extend this lemma to the operators $A(x, p)$ we have to consider the highest orders terms of $A(x, p)$; with this view define two vector spaces and a homomorphism:

<u>Definition</u> 66.1. $\mathcal{A}$ is the vector space of the operators $A(x, p)$ such that: $a_{\lambda\mu}(x, p)$ has an order $\leq 2n - \lambda - \mu$, where $n > m$ is given; the coefficients of $a_{\lambda\mu}(x, p)$ and their derivatives are bounded.

$\mathcal{H}$ is the vector space of the matrices $H(x, i\xi)$ such that: $h_{\lambda\mu}(x, i\xi)$ is homogeneous of degree $2n - \lambda - \mu$; the coefficients of the polynomials $h_{\lambda\mu}(x, i\xi)$ and their derivatives are bounded.

Φ is the homomorphism

$$\Phi : \ A(x, p) \longrightarrow H(x, i\xi)$$

of $\mathcal{A}$ onto $\mathcal{H}$ such that $h_{\lambda\mu}(x, i\xi)$ is the sum of the terms of $a_{\lambda\mu}(x, i\xi)$ which are homogeneous of degree $2n - \lambda - \mu$.

According to Lemmas 66.1 and 66.4:

<u>Φ maps the subspace $\mathcal{A}'$ of the hermitian $A(x, p)$ onto the subspace $\mathcal{H}'$ of the hermitian $H(x, i\xi)$; Φ maps the subspace of the $A(p)$ hermitian > 0 onto the subspace of the $H(i\xi)$ hermitian > 0.</u>

We need an extension of this last assertion to the $A(x, p)$ and $H(x, i\xi)$.

67. <u>Gårding's lemma.</u> Define

$$X(\xi) = -\xi_1^2 - \xi_2^2 - \dots - \xi_\ell^2$$

$$Q_1(\xi) = \begin{pmatrix} X(\xi) & 0 & 0 & \vdots \\ 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & \vdots \\ \dots & \dots & \dots & \vdots \end{pmatrix}, \quad Q_2(\xi) = \begin{pmatrix} 0 & 0 & 0 & \vdots \\ 0 & X(\xi) & 0 & \vdots \\ 0 & 0 & 0 & \vdots \\ \dots & \dots & \dots & \vdots \end{pmatrix}, \dots$$

$$Q(\xi) = Q_1^{n-1}(\xi) + Q_2^{n-2}(\xi) + \dots + Q_m^{n-m}(\xi).$$

According to Lemma 66.1: $Q_\lambda(p)$ is hermitian;

(67.1) $$0 \leq Q_\lambda(p)$$

(67.2) $$rQ_\lambda^{r-1}(p) \leq (r-1)\,\varepsilon\, Q_\lambda^r(p) + \varepsilon^{1-r}\, I$$

(r: integer > 1; ε : any number > 0; I: identity matrix).

$$(67.3)\qquad \pm\begin{pmatrix} a(p) & 0 & 0 & \vdots \\ 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & \vdots \\ \cdots & \cdots & \cdots & \cdot \end{pmatrix} \le Q_1^r(p) \text{ if } a(p) = p_1^{r_1}\cdots p_\ell^{r_\ell} \text{ and } r_1+\cdots+r_\ell = 2r.$$

Lemma 67.1. If $u = a(p)c(x)b(p)$,

$a(p) = p_1^{r_1}\cdots p_\ell^{r_\ell}$, $b(p) = p_1^{s_1}\cdots p_\ell^{s_\ell}$, $r_1+\cdots+r_\ell = r$, $s_1+\cdots+s_\ell = s$.

$\operatorname{Sup}|c(x)| = \gamma$, then for any positive numbers α and β such that $\gamma^2 \le \alpha\beta$,

$$(67.4)\qquad \begin{pmatrix} u+u^* & 0 & 0 & \vdots \\ 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & \vdots \\ \cdots & \cdots & \cdots & \cdot \end{pmatrix} \le \alpha Q_1^r(p) + \beta Q_1^s(p)$$

$$(67.5)\qquad \begin{pmatrix} 0 & u & 0 & \vdots \\ u^* & 0 & 0 & \vdots \\ 0 & 0 & 0 & \vdots \\ \cdots & \cdots & \cdots & \cdot \end{pmatrix} \le \alpha Q_1^r(p) + \beta Q_2^s(p).$$

Proof of (67.4). Lemma 66.2 is applied with

$$A = \begin{pmatrix} a(p) & 0 & \vdots \\ 0 & 0 & \vdots \\ \cdots & \cdots & \cdot \end{pmatrix},\ B = \begin{pmatrix} b(p) & 0 & \vdots \\ 0 & 0 & \vdots \\ \cdots & \cdots & \cdot \end{pmatrix},\ C = \begin{pmatrix} c(x) & 0 & \vdots \\ 0 & 0 & \vdots \\ \cdots & \cdots & \cdot \end{pmatrix};$$

(67.3) is used.

Proof of (67.5). Lemma 66.2 is applied with

$$A = \begin{pmatrix} a(p) & 0 & \vdots \\ 0 & 0 & \vdots \\ \cdots & \cdots & \cdot \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & \vdots \\ 0 & b(p) & \vdots \\ \cdots & \cdots & \cdot \end{pmatrix},\ C = \begin{pmatrix} 0 & c(x) & \vdots \\ 0 & 0 & \vdots \\ \cdots & \cdots & \cdot \end{pmatrix}.$$

Notation. $A(x, p) \in \mathcal{A}'$ (see Definition 66.1); the coefficients of the polynomials $a_{\lambda\mu}(x, \xi)$ and all their derivatives are bounded;

$$H(x, i\xi) = \Phi A;$$

the $c_\gamma(x)$ are the coefficients of the polynomials $h_{\lambda\mu}(x, \xi)$. We ask for two numbers γ and γ' such that

$$A(x, p) \leq \gamma Q(p) + \gamma' I. \tag{67.6}$$

Lemma 67.2. (67.6) holds for any

$$\gamma > \sum_{\gamma} \operatorname{Sup}_{x \in X} |c_\gamma(x)|. \tag{67.7}$$

Proof. (67.4) and (67.5) show that

$$A(x, p) \leq \gamma Q(p) + \gamma'' \sum_{0 \leq \mu < n - \lambda} Q_\lambda^\mu(p);$$

(67.2) achieves the proof.

Definition. σ is the smallest number such that

$$H(x, i\xi) \leq \sigma Q(i\xi) \text{ for all } x \in X.$$

Justification. If p and $A(x, p)$ are replaced by $i\xi$ and $H(x, i\xi)$ in the proofs of Lemmas 67.1 and 67.2, it appears that

$$H(x, i\xi) \leq \gamma Q(i\xi) \text{ for any } \gamma \text{ satisfying (67.7);}$$

therefore σ exists.

Let us improve Lemma 67.2:

Lemma 67.3. (67.6) holds for

$$\gamma > \sigma + \omega$$

where $\omega = \sum_{\gamma} \operatorname{Oscill.}_{x \in X} c_\gamma(x)$.

Proof: Let $H(\xi)$ be the value of $H(x, \xi)$ at some point x of X; according to Lemmas 66.4 and 67.2:

$$H(p) \leq \sigma Q(p); \; A(x, p) - H(p) \leq \omega' Q(p) + \gamma', \text{ if } \omega' > \omega.$$

Lemma 67.4. (67.6) holds for $\gamma > \sigma$.

Proof. Let η be a number > 0 and z any point of X whose coordinates are multiples of η; let k(x) be a real function such that

$$\sum_z k^2(x + z) = 1, \qquad k(x) = 0 \qquad \text{outside the cube } C: \ |x_\lambda| < 2\eta.$$

Lemma 67.3 and 66.2 give:

$$(67.8) \quad k(x + z)A(x, p)k(x + z) \leq \gamma k(x + z)Q(p)k(x + z) + \gamma' k^2(x + z)I$$

for any z and $\gamma > \sigma + \omega$; now the first member of (67.8) depends only on the restriction of A(x, p) to the cube z + C; thus ω can be replaced by the number

$$\omega = \sum_\nu \operatorname*{Oscill.}_{z + C} c_\nu(x),$$

which is small when η is small: (67.8) holds for any z and

$$(67.9) \qquad \gamma > \sigma$$

when η is sufficiently small. Now (see Definition 66.1 of Φ)

$$\Phi[A(x, p) - \sum_z k(x + z)A(x, p)k(x + z)] = 0, \ \Phi[Q(p) - \sum_z k(x + z)Q(p)k(x + z)] = 0;$$

thus, according to Lemma 67.2,

$$(67.10) \qquad A(x, p) \leq \sum_z k(x + z)A(x, p)k(x + z) + \varepsilon Q + \gamma'' I$$

$$(67.11) \qquad \sum_z k(x + z)Q(p)k(x + z) \leq (1 + \varepsilon)Q(p) + \gamma''' I,$$

where ε is arbitrarily small. The lemma results from (67.8), (67.9), (67.10) and (67.11).

Lemma 67.5. Suppose $H(x, i\xi) < 0$ and $\lim_{\|x\| \to \infty} H(x, i\xi) < 0$ (see Note 63.1); (67.6) holds for some $\gamma < 0$.

Proof. $\sigma < 0$.

Replacing in Lemma 67.5 A(x, p) by $-$ B(x, p) and using Lemma 67.2, we obtain the following variant of Gårding's lemma [43]:

Proposition 67.1. Let $K(x, i\xi)$ be a hermitian matrix $\in \mathcal{H}$; suppose $K(x, i\xi) > 0$ and $\lim_{\|x\| \to \infty} K(x, i\xi) > 0$ (see: Note 63.1, Definition 66.1). There is a hermitian operator $B(x, p) \in \mathcal{A}$ such that

(67.11) $$\Phi B = K;$$

there is a number β such that any $C(x, p) \in \mathcal{A}$ is dominated by some multiple of $B(x, p) + \beta I$:

(67.12) $$C(x, p) \leq \gamma[B(x, p) + \beta I] \qquad \text{for some number } \gamma.$$

68. An operator B defining a norm for which the hermitian part of A is bounded.

Proposition 68.1. Let $A(x, p)$ be an operator with the following properties:

it belongs to the type (63.1);

its element $a_{\lambda\mu}(x, p)$ is a differential operator of order $\lambda - \mu + 1$;

the coefficients of $a_{\lambda\mu}(x, p)$ and all their derivatives are bounded;

if $H(x, \xi)$ is the matrix whose element $h_{\lambda\mu}(x, \xi)$ is the homogeneous part of degree $\lambda - \mu + 1$ of $a_{\lambda\mu}(x, \xi)$, then the characteristic roots of the matrix $H(x, \xi)$ (and of its limits for $\|x\| \to +\infty$) are real and distinct.

To such an operator $A(x, p)$ and any integer $n > \frac{m(m-1)}{2}$ can be associated a real hermitian operator $B(x, p) \in \mathcal{A}$ (Definition 66.1) with the following properties:

the hermitian part of $A(x, p)$ is bounded for the norm defined by $B(x, p)$;

any hermitian $C(x, p) \in \mathcal{A}$ is dominated by some multiple of $B(x, p)$.

Note. These assertions mean that

(68.1) $$-\beta B(x, p) \leq B(x, p)A(x, p) + [B(x, p)A(x, p)]^* \leq \beta B(x, p)$$

for some number β;

(68.2) $$C(x, p) \leq \gamma B(x, p) \text{ for some number } \gamma.$$

Proof. Proposition 64.1 (see (64.8) and (64.9)) gives a real matrix $K(x, \xi)$ such that $K(x, i\xi)$ is hermitian > 0, $\varepsilon\, \mathcal{H}$ and satisfies

(68.3) $$K(x, i\xi) \cdot H(x, i\xi) + [K(x, i\xi) \cdot H(x, i\xi)]^* = 0.$$

Proposition 67.1 gives a real hermitian > 0 operator $B(x, p) \varepsilon\, \mathcal{H}$ such that

(68.4) $$\Phi B = K(x, i\xi)$$

and (68.2) holds.

Now (68.3), (68.4) and Lemma 66.1 show that

$$B(x, p) \cdot A(x, p) + [B(x, p) \cdot A(x, p)]^* \varepsilon\, \mathcal{A};$$

hence (68.1) results from (68.2).

§3. A priori bound for the local solutions of the hyperbolic equation.

69. Definition of a regularly hyperbolic equation on a vector space.

Let

(69.1) $$a(x, p)u(x) = v(x)$$

be a differential equation of order m; $u(x)$ is the unknown function; $v(x)$ is a given function; $a(x, \xi)$ is a given real polynomial in ξ of degree m. Let $h(x, \xi)$ be the sum of its homogeneous terms of degree m; let $V_x(h)$ be the cone defined in Ξ by $h(x, \xi) = 0$; $a(x, p)$ is said to be hyperbolic at the point x if Ξ contains points $\xi \varepsilon \Xi$ such that any real line through ξ cuts the cone $V_x(h)$ at m real and distinct points; these points ξ constitute the interior of two opposite convex and closed half cones $\Gamma_x(a)$ and $-\Gamma_x(a)$, whose boundaries belong to $V_x(h)$. (Proposition 27.3. For $\ell = 2$ these points ξ constitute several such pairs of half cones; $\Gamma_x(a)$ is any of these half cones). $V_x(h)$ has no singular generators. The cones with vertex x dual to

$\Gamma_x(a)$ and $-\Gamma_x(a)$ are denoted by $C_x(a)$ and $-C_x(a)$.

Assume that: $a(x, p)$ is hyperbolic at each point x of X;

$$\Gamma_X = \bigcap_{x \in X} \Gamma_x \text{ has a \underline{not} \underline{void} interior } \overset{0}{\Gamma}_X;$$

no limit of $h(x, \xi)$ for $||x|| \to \infty$ is 0; no limit of the cones $V_x(h)$ for $||x|| \to \infty$ has singular generator: then $a(x, p)$ is said to be regularly hyperbolic on X.

70. A priori bound of a local solution.

Notation. We denote by

$$(70.1) \qquad ||u||_t = \left[\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u^2(x)\,dx_2 \ldots dx_\ell\right]^{\frac{1}{2}}, \text{ where } x_1 = t,$$

the norm of the restriction of $u(x)$ to the hyperplane $x_1 = t$; $(u, v)_t$ is the corresponding scalar product; $u(x)$ is said to be uniformly square integrable on the hyperplanes $x_1 = t$ when $||u||_t$ is bounded on any finite interval $\tau_1 \leq t \leq \tau_2$.

Proposition 70.1. Assumption about $a(x, p)$: $a(x, p)$ is regularly hyperbolic on X; the first axis of Ξ belongs to $\overset{0}{\Gamma}_X$; the derivatives[1] of $a(x, p)$ are bounded. Assumptions about $v(x)$: The derivatives of $v(x)$ are uniformly square integrable (and therefore uniformly bounded) functions on the hyperplanes $x_1 = t$.

Assumption about $u(x)$: $u(x)$ is defined for $0 \leq x_1 \leq \tau$; the derivatives[1] of $u(x)$ are uniformly square integrable functions on the hyperplanes $x_1 = t$ $(0 \leq t \leq \tau)$; $u(x)$ is a solution of the Cauchy problem defined by the equation (69.1) and the Cauchy data;

$$(70.2) \qquad u(x) = \ldots = p_1^{m-1}u(x) = 0 \qquad \text{for } x_1 = 0.$$

Statement: There are positive numbers $\beta, \beta', \beta'', \beta'''$, γ depending only on $a(x, p)$ and n and such that:

[1] By the derivatives of a function is meant the function itself and its derivatives of any order.

$$(70.3)\quad ||u||_t + ||p^{n-1}u||_t \underset{t}{\leq} \beta[||v||_t + ||p^{n-1}v||_t] + \beta'[||v||_t + ||p^{n-1}v||_t]_{t=0}$$
$$\exp(\gamma t) + \beta'' \int_0^t [||v||_s + ||p^{n-1}v||_s] \exp \gamma(t - s)ds$$

(*for* $n > \frac{m(m-1)}{2}$, $0 \leq t \leq \tau$);

$$(70.4)\quad |u| + |p^r u| \leq \beta'''[||v||_t + ||p^{n-1}v||_t]_t +$$
$$\beta'''[||v||_t + ||p^{n-1}v||_t]_{t=0} \exp(\gamma t) + \beta''' \int_0^t [||v||_s + ||p^{n-1}v||_s] \exp(t - s)ds$$

(*for any* r; n *depends on* r);

by $||p^r u||_t$ and $|p^r u|$ are meant

$$||p^r u||_t = \sum_{r_1+\dots+r_\ell=r} ||p_1^{r_1} \dots p_\ell^{r_\ell} u||_t, \quad |p^r u| = \sum_{r_1+\dots+r_\ell=r} |p_1^{r_1} \dots p_\ell^{r_\ell} u|.$$

Note. β, β', β'', β''' and γ remain bounded when $a(x, p)$ varies so that the assumptions about $a(x, p)$ are uniformly satisfied.

Note. Chapter IX, §1 improves this proposition.

Proof. Let

$$(70.5)\qquad a(x, p) = p_1^m + \alpha_{m-1}(x, p)p_1^{m-1} + \dots + \alpha_o(x, p)$$

where $\alpha_\mu(x, p)$ is independent of p_1; let $A(x, p)$ be the matrix (63.1) and U and V be the vectors

$$U = (u, p_1 u, \dots, p_1^{m-1}u), \qquad V = (0, 0, \dots, 0, v);$$

(69.1) becomes

$$(70.6)\qquad p_1 U = A(x, p)U + V, \quad \text{where } A(x, p) \text{ is independent of } p_1.$$

Let $H(x, \xi)$ be the matrix whose element $h_{\lambda\mu}(x, \xi)$ is the homogeneous part of degree $\lambda - \mu + 1$ of the element $a_{\lambda\mu}(x, \xi)$ of $A(x, \xi)$; according to n°63 the characteristic roots of the matrix $H(x, \xi)$ are the numbers ξ_1

such that $h(x, \xi) = 0$; by assumption they are real and distinct; for any $n > \frac{m(m-1)}{2}$ Proposition 68.1 gives some real hermitian $B(x, p)$ independent of p_1, satisfying (68.1) and (6812) for any value of x_1. Now, applying the process by which the energy relation is obtained in the mechanics, let us calculate

$$\frac{d}{dt}(B \cdot U, U)_t = (\{B \cdot A + [B \cdot A]^*\} U, U)_t + (B_t' U, U)_t + 2(BU, V)_t$$

(B_t' is the derivative of B with respect to $x_1 = t$); hence, according to (68.1) and (68.2) and using Schwarz's inequality

$$\frac{1}{2}\frac{d}{dt}(BU, U)_t \leq \gamma(BU, U)_t + \sqrt{(BU, U)_t}\sqrt{(BV, V)_t};$$

that is

$$\frac{d}{dt}\left[\sqrt{(BU, U)_t} \exp.(-\gamma t)\right] \leq \sqrt{(BV, V)_t} \exp.(-\gamma t);$$

hence

(70.7) $$\sqrt{(BU, U)_t} < \int_0^t \sqrt{(BV, V)_s} \exp. \gamma(t-s)ds + \sqrt{(BU, U)_0} \exp. \gamma t.$$

According to (68.2) and Lemma 67.2, $\sqrt{(BU, U)_t}$ is a norm equivalent to

$$\sum_{0 \leq m_1 < m} \|p_1^{m_1} u\|_t + \sum_{\substack{m_1 + \dots + m_\ell = n-1 \\ 0 \leq m_1 < m}} \|p_1^{m_1} \dots p_\ell^{m_\ell} u\|_t;$$

now (69.1) and (70.5) show that $p_1^{n_1} \dots p_\ell^{n_\ell} u$, for $n_1 \geq m$, $n_1 + \dots + n_\ell = n - 1$, is a linear function of the $p_1^{m_1} p_2^{r_2} \dots p_\ell^{r_\ell} u$ and $p_1^{s_1} \dots p_\ell^{s_\ell} v$ such that

$$m_1 < m, \ m_1 + r_2 + \dots + r_\ell < n, \qquad s_1 + \dots + s_\ell < n;$$

moreover Fourier's transformation shows that $\|p_1^{m_1} p_2^{r_2} \dots p_\ell^{r_\ell} u\|_t$ is dominated by a linear function of $\|p_1^{m_1} u\|_t$ and $\|p_1^{m_1} p_2^{m_2} \dots p_\ell^{m_\ell} u\|_t$ when

$$r_2 \leq m_2, \; \ldots \; r_\ell \leq m_\ell;$$

thus

(70.8) $\quad \|u\|_t + \|p^{n-1}u\|_t \leq \text{const.} \; \sqrt{(BU, U)_t} + \text{const.} \; [\|v\|_t + \|p^{n-1}v\|_t].$

Similarly, $\|p_1^{m_1}v\|_s$ is dominated by a linear function of $\|p_1^{n-1}v\|_s$ and $\|v\|_s$ for $m_1 < n$; thus

(70.9) $$\sqrt{(BV, V)_s} \leq \text{const.} \; [\|v\|_s + \|p^{n-1}v\|_s].$$

And

(70.10) $$\sqrt{(BU, U)_0} \leq \text{const.} \; [\|v\|_0 + \|p^{n-1}v\|_0].$$

(70.3) follows from (70.7), (70.8), (70.9) and 70.10).

Now Fourier's transformation shows that the first member of (70.4) is dominated by a sum of first members of (70.3): the proof is complete.

71. Example. $m = 2$, $n \geq 2$; $a(x, p) = p_1^2 + \alpha_1(x, p)p_1 + \alpha_0(x, p)$;

$$(B(x, p)U, U)_t = 2(\alpha_0(x, p)q^{n-2}u, u)_t - (\alpha_1(x, p)\alpha_1(x, p)q^{n-2}u, u)_t$$
$$- 2(\alpha_1(x, p)q^{n-2}p_1u, u)_t + 2(q^{n-2}p_1u, p_1u)_t + \beta(u, u)_t + \beta(p_1u, p_1u)_t,$$

where $q = -p_2^2 - \ldots - p_\ell^2$, β = large number.

§4. Existence theorems.

72. Local Cauchy's problem for analytic data. (Cauchy-Kowalewski's theorem: see [32]; Schauder's complement: [36], p. 229; [34], p. 840.)

Lemma 72.1. Assumption about $a(x, p)$: The coefficients of

$$a(x, p) = p_1^m + \alpha_0(x, p)p_1^{m-1} + \ldots + \alpha_{m-1}(x, p)$$

are holomorphic and their modulus have a bound $M < +\infty$ for

$$x = x' + ix'', \qquad x' \text{ and } x'' \in X, \qquad \|x''\| \leq \mathcal{E};$$

$\mathcal{E}$ is a given positive number.

Assumption about $v(x)$: $v(x)$ is an entire function;

$\sup_{\|x^*\| < \mathcal{E}} |v(x + x^*)|$, where $x^* \in X + iX$, is a bounded square integrable function of $x \in X$ on the hyperplane $x_1 = 0$.

Cauchy's data:

$$u(x) = \ldots = p_1^{m-1}u(x) = 0 \qquad \text{for } x_1 = 0. \tag{72.1}$$

Statement: The Cauchy problem defined by (69.1) and (72.1) has a unique holomorphic solution $u(x)$ defined for $|x_1| < \tau$; the positive number τ is a function of $\mathcal{E}$ and M; τ is independent of $v(x)$; the derivatives of $u(x)$ are uniformly square integrable and bounded functions on the hyperplanes $x_1 = t$ $(0 \leq t \leq \tau)$.

Proof. We can suppose $\mathcal{E} = \ell$. Taylor's expansion of $u(x)$ at the point $x = 0$ can be immediately deduced from (69.1) and (72.1); its coefficients are polynomials in the coefficients of the Taylor's expansions of the given functions:

$$v(x),\ -\alpha_0(x, \xi),\ \ldots,\ -\alpha_m(x, \xi);$$

these polynomials have possitive coefficients. Thus $u(x)$ exists and has the majorant (see [2]) $w(x)$ if $w(x)$ has the following properties;

(72.2) at $x = 0$, $w(x)$ has a Taylor expansion with positive coefficients;

(72.3) $w(x)$ is the solution of a problem whose data are majorants of the given data.

Now any function with modulus $\leq M$ for $|x_1| < 1, \ldots |x_\ell| < 1$ has the majorant

$\frac{M}{(1 - x_1) \dots (1 - x_\ell)}$ and therefore the majorant $\frac{M}{1 - \rho x_1 - x_2 - \dots x_\ell}$ for $\rho > 1$, $\rho |x_1| + \dots + |x_\ell| < 1$; thus (72.3) can be replaced by

(72.4) $p_1^m w(x) =$

$$\frac{1}{1 - \rho x_1 - x_2 - \dots - x_\ell} [b(p_1, p_2, \dots p_\ell) w(x) + c(p_1 \dots p_\ell) w(x) + d]$$

where $b(\xi_1, \dots \xi_\ell)$ is a homogeneous polynomial of degree m such that $b(1, 0, \dots 0) = 0$, $c(\xi_1, \dots \xi_\ell)$ is a polynomial of degree $m - 1$ and d is a constant; $\rho > 1$; b and c depend on $a(x, p)$ and d on $v(x)$.

We choose $w(x) = w(s)$, where $s = \rho x_1 + x_2 + \dots + x_\ell$; (72.2) and (72.4) become

(72.5) $w(s)$ has a positive expansion at $s = 0$;

(72.6) $$\frac{d^m w(s)}{ds^m} = \frac{1}{\rho^m - b(\rho) - s\rho^m} [c(\rho \frac{d}{ds}, \frac{d}{ds}, \dots \frac{d}{ds}) w(x) + d]$$

where $b(\rho) = b(\rho, 1, 1, \dots 1)$ has the degree $m - 1$ and c the order $m - 1$.

We choose ρ so large that $b(\rho) < \rho^m$ and $w(s)$ such that

(72.7) $$w(0) = (\frac{dw}{ds})_0 = \dots = (\frac{d^{m-1}w}{ds^{m-1}})_0 = 0:$$

(72.6) and (72.7) define a function $w(s)$ satisfying (72.5); its Taylor's expansion converges for $|s| < 1 - b(\rho)\rho^{-m}$ (see [2], Ch. XIX, Systèmes d'équations linéaires); $w(s)$ is proportional to d.

Hence, replacing the origin by any point x such that $x_1 = 0$ the given Cauchy problem has a unique holomorphic solution $u(x)$ inside a sphere whose center is on $x_1 = 0$ and whose radius is τ; τ is independent of x and $v(x)$; inside this sphere we have

$$|p_1^{n_1} \dots p_\ell^{n_\ell} u(x)| < \beta \operatorname*{Sup.}_{\|x^*\| < \varepsilon} |v(x + x^*)|$$

where $x^* \in X + iX$; β is independent of x and $v(x)$.

73. Global Cauchy's problem for analytic $a(x, p)$.

Lemma 73.1. Assumption about $a(x, p)$: the assumptions of Proposition 70.1 and Lemma 72.1

Assumption about $v(x)$: the assumptions of Proposition 70.1.

Cauchy's data: $u(x), \dots p_1^{m-1}u(x)$ are given functions for $x_1 = 0$; these functions and their derivatives are square integrable.

Statement. The Cauchy problem defined by (69.1) and these data has a unique solution $u(x)$ defined for $x_1 > 0$ and such that $u(x)$ and any derivative of $u(x)$ is bounded and uniformly square integrable on the hyperplanes $x_1 = t$.

Proof for a $v(x)$ with compact support, Cauchy's data zero and $0 \leq x_1 \leq \tau$. Let η be a positive number tending to zero and

$$\overline{v(x)} = (2\pi\eta)^{-\frac{\ell}{2}} \exp. \left(-\frac{x^2}{2\eta}\right) * v(x) \qquad (\text{see } n^o 9); \tag{73.1}$$

$\overline{v(x)}$ satisfies the assumption of Lemma 72.1; therefore, if $v(x)$ is replaced by $\overline{v(x)}$, Cauchy's problem has a solution $u_\eta(x)$ for $0 \leq x_1 \leq \tau$; τ is the number independent of η and $v(x)$ used in Lemma 72.1. Now $\overline{v(x)} - v(x)$ and $\| \overline{v(x)} - v(x) \|_t$ tend uniformly to 0. Thus according to Proposition 70.1 $u_\eta(x)$ tends to a limit $u(x)$; $u(x)$ satisfies the assertion of Lemma 73.1 for $0 \leq x_1 \leq \tau$.

Proof for Cauchy's data 0 and $0 \leq x_1 \leq \tau$. Let $v(x)$ be a function satisfying the assumptions of Lemma 73.1; there is a sequence of $v^*(x)$ such that their supports are compact, these assumptions are satisfied and $\| v^*(x) - v(x) \|_t$ tends uniformly to 0. According to Proposition 70.1 the solutions $u^*(x)$ of the corresponding Cauchy problem have a limit satisfying the Lemma 73.1 for $0 \leq x_1 \leq \tau$.

<u>Proof for</u> $0 \leq x_1 \leq \tau$. Any Cauchy's problem is equivalent to a Cauchy's problem with data 0: replace u(x) by u(x) - w(x), w(x) satisfying Cauchy's data.

<u>Proof for</u> $0 \leq x_1$. The preceding result enables us to solve the given global Cauchy problem by solving successively local problems for $0 \leq x_1 \leq \tau$, $\tau \leq x_1 \leq 2\tau$, $2\tau \leq x_1 \leq 3\tau$, etc.

<u>The uniqueness</u> of the solution follows from Proposition 70.1.

74. <u>Global Cauchy's problem for an infinitely differentiable</u> a(x, p).

<u>Proposition</u> 74.1. <u>Consider a Cauchy problem such that</u> a(x, p) <u>and</u> v(x) <u>satisfy the assumptions of Proposition</u> 70.1. <u>This problem has a unique global solution</u> u(x) <u>satisfying the assumptions of Proposition</u> 70.1.

<u>Note</u>. This Proposition 70.1 gives some information about the behaviour of u(x).

<u>Proof</u>. Define $\bar{a}(x, p)$ by means of (73.1); when η is small, $\bar{a}(x, p)$ satisfies the assumptions of Lemma 73.1. Then the equation

$$\bar{a}(x, p)u^*(x) = v(x)$$

has a solution $u^*(x)$ satisfying the Cauchy data and the assumption about u(x) of Proposition 70.1. This proposition shows that $u^*(x)$ and any of its derivatives is equicontinuous for $\eta \longrightarrow 0$; therefore (see Ascoli's theorem: Bourbaki, Topologie génerale, Ch. X, Expaces fonctionnels; or easier: Courant Hilbert, Methoden der Math. Phys. I, Ch. II, §2, Häufungsprinzip für Funktionen) we can find a sequence of numbers $\eta \longrightarrow 0$ such that $u^*(x)$ (and its derivatives) tend to a limit u(x) (and its derivatives); u(x) is a solution of the given Cauchy problem; u(x) satisfies the assumptions of Proposition 70.1 about u(x). Proposition 70.1 shows the uniqueness of the solution of this problem.

75. The inverse operators of $a(x, p)$. The Proposition 75.1 completes the Chap. III as follows:

Proposition 75.1. Assumptions about $a(x, p)$: $a(x, p)$ is regularly hyperbolic on X; the coefficients of $a(x, p)$ and their derivatives of any order are bounded functions of x. When $\|x\|$ is large, $a(x, p)$ is independent of x.

Notation: When x is large $a(x, p)$ and Γ_x are denoted $a(\infty, p)$ and Γ_∞. The components of the complement of the closure of the real projection of the algebraic variety $a(\infty, \zeta) = 0$ are convex domains; one of them has the director cone Γ_∞ (Propositions 24.2 and 27.2.3°) and is denoted by Δ_∞.

Assumption about $v(x)$: $v(x) \exp. (- x \cdot \xi)$ and its derivatives of any order are square integrable on X for $\xi \in \Delta_\infty$.

Statement: There is a solution $u(x)$ of the equation

$$a(x, p)u(x) = v(x) \tag{75.1}$$

such that $u(x) \exp. (- x \cdot \xi)$ and its derivatives of any order are square integrable on X when $\xi \in \Delta_\infty$; this solution is unique.

Proof. Suppose the first axis of Ξ in $\overset{0}{\Gamma}_X$. According to Theorem 6.3 and Proposition 22.3.2° the equation

$$a(\infty, p)w(x) = v(x) \tag{75.2}$$

has a solution $w(x)$ such that $w(x) \exp. (- x \cdot \xi)$ and its derivatives are square integrable on X when $\xi \in \Delta_\infty$. Therefore $w(x) \exp. (- x \cdot \xi)$, $v(x) \exp. (- x \cdot \xi)$ and their derivatives of any order are bounded and uniformly square integrable on the hyperplanes $x_1 = t$. Suppose

$$a(x, p) = a(\infty, p) \text{ for } x_1 \leq \tau;$$

define:

for $x_1 \leq \tau$, $\quad u(x) = w(x)$;

for $\tau \leq x_1$, $\quad u(x) - w(x)$ is the solution of the equation

$$(75.3) \qquad a(x, p)[u(x) - w(x)] = v(x) - a(x, p)w(x)$$

with Cauchy's data 0 for $x_1 = \tau$ (see Proposition 74.1). This function $u(x)$ is the only one which can satisfy the statement.

In fact this function $u(x)$ satisfies (75.1) and according to Proposition 70.1 $u(x)$ exp. $(- x \cdot \xi)$ and its derivatives of order $\leq m$ are square integrable when ξ belongs to the first axis of Ξ and is sufficiently large. Now

$$(75.4) \qquad a(\infty, p)u(x) = f(x)$$

where

$$(75.5) \qquad f(x) = v(x) + [a(\infty, p) - a(x, p)]u(x);$$

since $a(x, p) = a(\infty, p)$ when $\| x \|$ is large, $f(x)$ exp. $(- x \cdot \xi)$ and its derivatives of any order are square integrable for any $\xi \in \Delta_\infty$; thus, since $u(x)$ is a solution of (75.4) and since $u(x)$ exp. $(- x \cdot \xi)$ is square integrable for <u>some</u> $\xi \in \Delta_\infty$ the Theorem 6.1 shows that $u(x)$ exp. $(- x \cdot \xi)$ and its derivatives are square integrable for <u>any</u> $\xi \in \Delta_\infty$: the proof is complete.

CHAPTER VI

THE INVERSES OF A HYPERBOLIC OPERATOR ON A VECTOR SPACE

Introduction to Chapter VI

The results of Chapter V followed from the a priori bound (70.3) of the local solutions of the hyperbolic equation

$$a(x, p)u(x) = v(x);$$

from this bound can be deduced an a priori bound of the norm

$$\|u(x) \exp. (- x \cdot \xi)\|_2 \qquad (\xi \in \bigcap_{x\in X} \Gamma_x, \qquad \|\xi\| \text{ large})$$

of a global solution of this equation; but the so obtained bound is much less precise that the bound obtained in Chapter III when $a(x, p)$ is independent of x; now the study of its explicit calculation[1] shows that this way is not the most direct way leading to bounds of global solutions and suggests the use of an essentially different process. This process is used in the present chapter; it works only for global solutions and thus can not be used in the proof of the first existence theorem (Chapter V, §4); it uses symmetrically all the coordinates, whereas $x_1 = t$ played a special role in Chapter V; it needs only a weak assumptions about the data; its results are as precise as the results obtained in Chapter III when $a(x, p)$ is independent of x; these results enable us to define inverse operators of $a(x, p)$ and to state their main properties (Theorems 89, 93, 97); Chapter VII extends these properties.

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[1] $S^{-1}U$ is explicitly calculated, when $U = (1, \tau, \ldots, \tau^{m-1})$; S denotes the matrix of n°63.

§1. The cones whose sheets separate the sheets of a given cone.

76. Polynomial whose roots separate the roots of another polynomial.

Definition. Let $P(\lambda)$ and $Q(\lambda)$ be two polynomials of degree m and $m - 1$ in one variable λ; we say that the roots of $Q(\lambda)$ separate the roots of $P(\lambda)$ when $P(\lambda)$ and $Q(\lambda)$ have real distinct roots

$$\lambda_1 < \lambda_2 < \dots < \lambda_m \text{ and } \lambda_1' < \lambda_2' < \dots < \lambda_{m-1}'$$

such that

$$\lambda_1 < \lambda_1' < \lambda_2 < \lambda_2' < \dots < \lambda_{m-1} < \lambda_{m-1}' < \lambda_m.$$

Example 76. If $P(\lambda)$ has real and distinct roots, they are separated by the roots of its derivative $P'(\lambda)$.

Lemma 76.1. Let $P(\lambda)$ and $Q(\lambda)$ be two real polynomials in the complex variable λ;

1°. $$\rho = \operatorname{Inf.}_{\lambda} \frac{\mathcal{J}[Q(\overline{\lambda})P(\lambda)]}{(1 + |\lambda|^2)^{m-1}\mathcal{J}(\lambda)} \text{ and } \sigma = \operatorname{Sup.}_{\lambda} \frac{\mathcal{J}[Q(\overline{\lambda})P(\lambda)]}{(1 + |\lambda|^2)^{m-1}\mathcal{J}(\lambda)}$$

are continuous functions of the coefficients of $P(\lambda)$ and $Q(\lambda)$;

2°. When the roots of $Q(\lambda)$ separate the roots of $P(\lambda)$, then

$$\rho\sigma > 0.$$

Note. This last assertion is essentially the classic statement: $Q^{-1}(\lambda)P(\lambda)$ maps the half-plane $\mathcal{J}(\lambda) > 0$ into itself or into its complement (see: G. Julia, Principes géométriques d'analyse, Vol. 1, Chap. 3, §1, n°3, p. 127).

Proof of 1°. $\mathcal{J}[Q(\overline{\lambda})P(\lambda)]\mathcal{J}^{-1}(\lambda)$ is a polynomial in $\mathcal{R}(\lambda)$ and $\mathcal{J}(\lambda)$ whose principal part is const. $|\lambda|^{2(m-1)}$; thus the function $\frac{\mathcal{J}[Q(\overline{\lambda})P(\lambda)]}{(1 + |\lambda|^2)^{m-1}\mathcal{J}(\lambda)}$ of λ and of the coefficients of P and Q is continuous not only for λ finite but also for λ infinite.

Proof of 2°. $\frac{Q(\lambda)}{P(\lambda)} = \sum_\mu \frac{Q(\lambda_\mu)}{P'(\lambda_\mu)} \frac{1}{\lambda - \lambda_\mu}$;

hence

$$Q(\overline{\lambda})P(\lambda) = |P(\lambda)|^2 \sum_\mu \frac{Q(\lambda_\mu)}{P'(\lambda_\mu)} \frac{\lambda - \lambda_\mu}{|\lambda - \lambda_\mu|^2}$$

and

$$\frac{\mathcal{J}[Q(\overline{\lambda})P(\lambda)]}{\mathcal{J}(\lambda)} = \sum_\mu \frac{Q(\lambda_\mu)}{P'(\lambda_\mu)} \left|\frac{P(\lambda)}{\lambda - \lambda_\mu}\right|^2. \tag{76.1}$$

Now, since the roots of $Q(\lambda)$ and also the roots of $P'(\lambda)$ separate the roots of $P(\lambda)$,

$$Q(\lambda_\mu)Q(\lambda_{\mu+1}) < 0, \qquad P'(\lambda_\mu)P'(\lambda_{\mu+1}) < 0;$$

thus

$$\text{sign } Q(\lambda_\mu)P'(\lambda_\mu)$$

is independent of μ and according to (76.1)

$$\rho\sigma > 0.$$

77. <u>The cones whose sheets separate the sheets of a given cone.</u> The notations of Chapter III are used: Ξ is a vector space; $\xi \in \Xi$; $h(\xi)$ is a real homogeneous polynomial of degree m; $V(h)$ is the cone $h(\xi) = 0$; $\Gamma(h)$ is the set of the points ξ such that any line through ξ cuts $V(h)$ at real points; the connected components $\Gamma_1(h)$, $\Gamma_2(h)$, ... of $\Gamma(h)$ are closed convex half-cones, whose boundaries belong to $V(h)$.

<u>Definition</u> 77.1. Assume that $V(h)$ has no singular generator and that the interior $\overset{\circ}{\Gamma}(h)$ of $\Gamma(h)$ is not void; (for dim. $\Xi > 2$ the cone $\Gamma(h)$ has two connected components, $\Gamma_1(h)$ and $\Gamma_2(h)$; they are opposite half cones: see Proposition 27.3). We say that the sheets of a cone $V(k)$ of degree $m - 1$ separate the sheets of the cone $V(h)$ when any line parallel to any $\xi \in \overset{\circ}{\Gamma}_1(h)$ cuts $V(k)$ at $m - 1$ real points separating the m real

distinct points where this line cuts $V(h)$. This happens when it happens for any line parallel to some $\xi \in \Gamma_1(h)$. Obviously:

$$\Gamma_1(h) \subset \Gamma(k);$$

$$k(\xi)h(\xi) \neq 0 \text{ in } \overset{\circ}{\Gamma}_1(h);$$

$V(k)$ has no real singular generator.

For dim. $\Xi = 2$, the meaning of Definition 77.1 can depend on the choice of $\Gamma_1(h)$.

Proposition 77.1. Assumption: $V(h)$ has no singular generator; $\overset{\circ}{\Gamma}(h)$ is not void.

Definition:

$$k(\xi) = \xi'_1 \frac{\partial h}{\partial \xi_1} + \dots + \xi'_\ell \frac{\partial h}{\partial \xi_\ell}, \text{ where } \xi' \in \overset{\circ}{\Gamma}_1(h).$$

Statement: The sheets of $V(k)$ separate the sheets of $V(h)$; $k(\xi)h(\xi) > 0$ in $\overset{\circ}{\Gamma}_1(h)$.

Proof. Example 76.

Proposition 77.2. Assumption: $h(\xi)$ and $k(\xi)$ are two real homogeneous polynomials in ξ of degrees m and $m - 1$.

Definition:

$$\rho(\xi) = \operatorname*{Inf.}_{\eta \in \Xi} \frac{\mathcal{R}[k(\overline{\zeta})h(\zeta)]}{\|\zeta\|^{2m-2}}, \qquad \sigma(\xi) = \operatorname*{Sup.}_{\eta \in \Xi} \frac{\mathcal{R}[k(\overline{\zeta})h(\zeta)]}{\|\zeta\|^{2m-2}}$$

where $\zeta = \xi + i\eta$, $\overline{\zeta} = \xi - i\eta$.

Statement:

1°. $\rho(\xi)$ and $\sigma(\xi)$ are positively homogeneous functions of degree 1 in ξ; $\rho(\xi)$ and $\sigma(\xi)$ are continuous functions of ξ and of the coefficients of $h(\xi)$ and $k(\xi)$.

2°. <u>If the sheets of</u> $V(k)$ <u>separate the sheets of</u> $V(h)$ <u>and if</u> $k(\xi)h(\xi) > 0$ <u>in</u> $\overset{\circ}{\Gamma}_1(h)$, <u>then</u>

$$\rho(\xi) > 0 \text{ in } \overset{\circ}{\Gamma}_1(h).$$

<u>Proof</u>. Assume $\xi = (\xi_1, 0, 0, \ldots 0)$; then

$$\frac{\mathcal{R}[k(\overline{\zeta})h(\zeta)]}{\|\xi\|^{2m-2}} = \frac{\mathcal{R}[k(\xi_1 - i\eta_1, -i\eta_2, \ldots, -i\eta_\ell)h(\xi_1 + i\eta_1, i\eta_2, \ldots i\eta_\ell)]}{(\xi_1^2 + \eta_1^2 + \theta^2)^{m-1}}$$

$$= -\frac{\mathcal{J}[k(\eta_1 + i\xi_1, \eta_2, \ldots \eta_\ell)h(\eta_1 - i\xi_1, \eta_2, \ldots \eta_\ell)]}{(\xi_1^2 + \eta_1^2 + \theta^2)^{m-1}}$$

that is

$$\text{(77.1)} \qquad \frac{\mathcal{R}[k(\overline{\zeta})h(\zeta)]}{\|\zeta\|^{2m-2}} = \xi_1 \frac{\mathcal{J}[Q(\overline{\lambda})P(\lambda)]}{(1 + |\lambda|^2)^{m-1}\mathcal{J}(\lambda)}$$

where

$$\theta = \sqrt{\eta_2^2 + \ldots + \eta_\ell^2}, \qquad \lambda = \frac{\eta_1 - i\xi_1}{\theta},$$

$$P(\lambda) = h(\lambda, \frac{\eta_2}{\theta}, \ldots, \frac{\eta_\ell}{\theta}, \qquad Q(\lambda) = k(\lambda, \frac{\eta_2}{\theta}, \ldots, \frac{\eta_\ell}{\theta});$$

according to Lemma 76.1.1°.

$$\text{(77.2)} \quad \operatorname*{Inf.}_{\lambda} \frac{\mathcal{J}[Q(\overline{\lambda})P(\lambda)]}{(1 + |\lambda|^2)^{m-1}\mathcal{J}(\lambda)} \text{ and } \operatorname*{Sup.}_{\lambda} \frac{\mathcal{J}[Q(\overline{\lambda})P(\lambda)]}{(1 + |\lambda|^2)^{m-1}\mathcal{J}(\lambda)}$$

are continuous functions of the coefficients of $h(\xi)$ and $k(\xi)$ and of the point $(\frac{\eta_2}{\theta}, \ldots, \frac{\eta_\ell}{\theta})$, which runs on a sphere; therefore $\rho(\xi)$ and $\sigma(\xi)$ are continuous functions of ξ and of the coefficients of $h(\xi)$ and $k(\xi)$.

When $\xi \in \overset{\circ}{\Gamma}_1(h)$, then the roots of $Q(\lambda)$ separate the roots of $P(\lambda)$; the Lemma 76.1.2°., the formula (77.1) and the continuity of the functions (77.2) show that

$$\rho(\xi) \cdot \sigma(\xi) > 0.$$

Now $\sigma(\xi) > 0$ since $k(\xi) \cdot h(\xi) > 0$. Hence $\rho(\xi) > 0$.

78. A first example. $m = 1$; $h(\xi) = \xi_1$; we take $\Gamma_1(h)$ as

$$\xi_1 > 0.$$

According to Proposition 77.1 choose

$$k(\xi) = 1.$$

Then

$$\rho(\xi) = \xi_1 > 0 \text{ in } \Gamma_1(h),$$

as Proposition 77.2 asserts.

$$\sigma(\xi) = \xi_1.$$

A second example. $m = 2$; $h(\xi) = \xi_1^2 - \xi_2^2 - \xi_3^2$; we take $\Gamma_1(h)$ as

$$\sqrt{\xi_2^2 + \xi_3^2} < \xi_1.$$

According to Proposition 77.1 choose

$$k(\xi) = \xi_1.$$

Then:

$$\rho(\xi) = \xi_1 - \sqrt{\xi_2^2 + \xi_3^2} > 0 \text{ in } \Gamma_1(h) \tag{78.1}$$

as Proposition 77.2 asserts.

$$\sigma(\xi) = \xi_1 + \sqrt{\xi_2^2 + \xi_3^2}. \tag{78.2}$$

§2. The hyperbolic operators of order m - 1 whose product by a given hyperbolic operator of order m has a positive hermitian part

[We give an extension of Gårding's lemma: see n°.66 and 67].

79. <u>Preliminaries</u>. X and Ξ are dual spaces of dimension ℓ; $x \in X$; ξ and $p = (\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_\ell}) \in \Xi$. We use for the functions f(x) defined on X the Hilbert norm depending on $\xi \in \Xi$

$$(79.1) \qquad \|f\| = [\int_X \dots \int |f(x)|^2 \exp. (-2x \cdot \xi) dx_1 \dots dx_\ell]^{\frac{1}{2}};$$

(f, g) is the corresponding scalar product; a^* is the adjoint operator of a for this norm;

$$(79.2) \qquad p^*_\lambda = 2\xi_\lambda - p_\lambda .$$

Let $a(\xi^*, x, \xi)$ be a real polynomial in $\xi^* \in \Xi$ and $\xi \in \Xi$; let n^* and n be its degrees in ξ^* and in ξ. We study <u>the operator of order</u> (n^*, n):

$$a(p^*, x, p) = a(2\xi - p, x, p);$$

its meaning is clear: for instance

$$p_1^* c(x) p_1 f(x) = (2\xi_1 - \frac{\partial}{\partial x_1})[c(x) \frac{\partial f}{\partial x_1}];$$

$$(p_1^* c(x) p_1 f(x), g(x)) =$$

$$(c(x) \frac{\partial f}{\partial x_1}, \frac{\partial g}{\partial x_1}) = \int\int_X c(x) \frac{\partial f}{\partial x_1} \frac{\partial \bar{g}}{\partial x_1} \exp. (-2x \cdot \xi) dx_1 \dots dx_\ell .$$

More generally $(a(p^*, x, p)f(x), g(x))$ is defined when f, g and their derivatives of orders n and n^* have finite norms, if the coefficients of the polynomial $a(\xi^*, x, \xi)$ are bounded functions of x.

<u>Lemma</u> 79.1. Let us denote $a(\xi^*, x, \xi)$ by $b(\xi, x, \xi^*)$; then for the norm (79.1) the adjoint of the operator $a(p^*, x, p)$ of order (n^*, n) is

$$[a(p^*, x, p)]^* = b(p^*, x, p),$$

whose order is (n, n^*).

An operator $a(p^*, x, p)$ hermitian for any ξ is said to be ≥ 0 when $(a(p^*, x, p)f(x), f(x)) \geq 0$ for any ξ and any $f(x)$. The three following lemmas are similar to the Lemmas 66.2, 66.3 and 66.4.

Lemma 79.2. If a and b are hermitian and if

$$a \leq b,$$

then

$$c^*ac \leq c^*bc$$

for any c.

Lemma 79.3. If $\gamma = \operatorname{Sup.}_X |c(x)|$, if $\gamma^2 \leq \alpha\beta$, $\alpha > 0$, $\beta > 0$, then

$$a(p^*)c(x)b(p) + [a(p^*)c(x)b(p)]^* \leq \alpha\, a(p^*)a(p) + \beta\, b(p^*)b(p).$$

Lemma 79.4. $a(p^*, p)$ is hermitian > 0 if and only if $a(\overline{\zeta}, \zeta)$ is real > 0.

Lemma 79.5. If $a(p^*, x, p)$ has the order $(m - 2, m)$, then

$$a = \sum_{\lambda=1}^{\ell} \xi_\lambda b_\lambda(p^*, x, p) + b'(p^*, x, p)$$

where b_λ and b' have respectively the orders $(m - 2, m - 1)$ and $(m - 1, m - 1)$. The coefficients of b_λ (of b') belong to the vector space generated by the coefficients of a (by these coefficients and their first derivatives).

Proof. We can suppose

$a = g(p^*)c(x)p_\lambda h(p)$, g of order $m - 2$, h homogeneous of order $m - 1$;

according to (79.2)

$$a = 2\xi_\lambda g(p^*)c(x)h(p) - g(p^*)p^*_\lambda c(x)h(p) - g(p^*)c'_\lambda(x)h(p)$$

where $c'_\lambda(x) = \dfrac{\partial c(x)}{\partial x_\lambda}$.

Lemma 79.6. If $a(p^*, x, p)$ has the order $(m - 1, m)$, then

$$a + a^* = \sum_{\lambda=1}^{\ell} \xi_\lambda \, b_\lambda(p^*, x, p) + b'(p^*, x, p)$$

where b_λ and b' are hermitian and have the order $(m - 1, m - 1)$. The coefficients of b_λ (of b') belong to the vector space generated by the coefficients of a (by these coefficients and their first derivatives).

Proof. Let B be the vector space whose elements are the operators

$$\sum_{\lambda=1}^{\ell} \xi_\lambda \, b_\lambda(p^*, x, p) + b'(p^*, x, p)$$

such that b_λ and b' have the order $(m - 1, m - 1)$ and have coefficients in these vector spaces; according to Lemma 79.5 we have to prove that

$$a + a^* \in B \text{ when } a = p^*_{\lambda_1} \cdots p^*_{\lambda_{m-1}} c(x) p_{\mu_1} \cdots p_{\mu_m}.$$

Now, mod. B,

$$\begin{aligned} a &= - p^*_{\lambda_1} \cdots p^*_{\lambda_{m-1}} p^*_{\mu_1} c(x) p_{\mu_2} \cdots p_{\mu_m} \\ &= p^*_{\lambda_2} \cdots p^*_{\lambda_{m-1}} p^*_{\mu_1} c(x) p_{\lambda_1} p_{\mu_2} \cdots p_{\mu_m} = \cdots \\ &= - p^*_{\mu_1} \cdots p^*_{\mu_m} c(x) p_{\lambda_1} \cdots p_{\lambda_m} = - a^*. \end{aligned}$$

80. An extension of Gårding's lemma. Define

(80.1) $$q = p^*_1 p_1 + \cdots + p^*_\ell p_\ell ;$$

according to Lemma (79.4)

(80.2) $$\|\xi\|^{2r} q^s \leq q^{r+s} \qquad (r, s: \text{ positive integers})$$

$$(80.3) \qquad p^*_{\lambda_1} \cdots p^*_{\lambda_r} p_{\lambda_1} \cdots p_{\lambda_r} \leq q^r$$

$$(80.4) \quad rq^{r-1} \leq (r-1)\,\varepsilon\, q^r + \varepsilon^{1-r} \quad (r:\ \text{integer} > 1;\ \varepsilon:\ \text{any positive number}).$$

Notation. $a(p^*, x, p)$ has the order $(m-1, m)$; we ask for a positively homogeneous function $\gamma(\xi)$ of degree 1 and for two numbers γ', γ'' such that

$$(80.5) \qquad a + a^* \leq \gamma(\xi)q^{m-1} + \gamma' q^{m-1} + \gamma''.$$

We denote by $c'_\mu(x)$ the coefficients of the polynomial $a(\xi^*, x, \xi)$, by $c_\nu(x)$ the coefficients of its principal part $h(\xi^*, x, \xi)$, which is homogeneous of degree $m-1$ in ξ^* and m in ξ.

Lemma 80.1. If $a(p^*, x, p)$ has the order $(m-1, m-1)$ and if its coefficients are bounded, then (80.5) holds for $\gamma = 0$, $\gamma' = n \cdot \operatorname{Sup}_{\mu, x} |c'_\mu(x)|$, where n is a function of ℓ and m.

Proof. Lemma 79.3 and (80.3) give

$$a + a^* \leq n \operatorname{Sup}_{\mu,x} |c'_\mu(x)|(q^{m-1} + q^{m-2} + \ldots + q + 1);$$

(80.4) achieves the proof.

Lemma 80.2. If $a(p^*, x, p)$ has the order $(m-2, m-1)$ and if its coefficients are bounded, then (80.5) holds for $\gamma = 0$ and any $\gamma' > 0$.

Proof. Lemma 79.3 and (80.3) give

$$a + a^* \leq \text{const.}\ (\varepsilon q^{m-1} + (\varepsilon^{-1} + 1)q^{m-2} + q^{m-3} + \ldots + 1);$$

(80.4) achieves the proof.

Henceforward the coefficients of $a(p^*, x, p)$ and their first derivatives are assumed to be bounded.

Lemma 80. 3. If $a(p^*, x, p)$ has the order $(m-2, m)$, then (80.5) holds for $\gamma(\xi) = \varepsilon \|\xi\|$ if $\varepsilon > 0$.

<u>Proof.</u> Lemmas 79.5, 80.1 and 80.2.

<u>Lemma</u> 80.4. (80.5) holds for

$$\gamma(\xi) = [n \cdot \operatorname*{Sup.}_{\nu, x} |c_\nu(x)| + \varepsilon] \, \|\xi\|$$

if $\varepsilon > 0$; n is a function of ℓ and m.

<u>Proof.</u> $a(p^*, x, p) = h(p^*, x, p) + a_1(p^*, x, p) + a_2(p^*, x, p)$ where h is the principal part of a, a_1 has the order $(m - 2, m)$ and a_2 the order $(m - 1, m - 1)$; the coefficients of h, a_1 and a_2 are coefficients of a; Lemmas 79.6 and 80.1 are applied to h, Lemma 80.3 to a_1 and Lemma 80.1 to a_2.

<u>Definition.</u>

$$\sigma(\xi) = \operatorname*{Sup}_{x, \eta} \cdot \frac{\mathcal{R} h(\overline{\zeta}, x, \zeta)}{\|\zeta\|^{2m-2}}, \text{ where } \zeta = \xi + i\eta;$$

$\sigma(\xi)$ is positively homogeneous; $\|\xi\|^{-1}\sigma(\xi)$ is bounded (see: Proposition 77.2.1°.).

<u>Lemma</u> 80.5. (80.5) holds for

$$\gamma(\xi) = 2\sigma(\xi) + n\omega\|\xi\|,$$

if n is a function of (ℓ, m) and $\omega > \sum_\nu \operatorname*{Oscill.}_{x \in X} c_\nu(x)$.

<u>Proof.</u> $h(\xi^*, x, \xi)$ denotes the principal part of $a(\xi^*, x, \xi)$; let $h(\xi^*, \xi)$ be its value at some point of X. According to Lemmas 79.4 and 80.4

$$h(p^*, p) + [h(p^*, p)]^* \leq 2\sigma(\xi)q^{m-1}$$

$$a(p^*, x, p) - h(p^*, p) + [a(p^*, x, p) - h(p^*, p)]^* \leq n\,\omega\|\xi\|q^{m-1} + \gamma' q^{m-1} + \gamma''.$$

<u>Lemma</u> 80.6. (80.5) holds for

$$\gamma(\xi) = 2\sigma(\xi) + \varepsilon\|\xi\| \quad \text{if} \quad \varepsilon > 0.$$

<u>Proof.</u> Let δ be a number > 0 and z any point of X whose coordinates are multiples of δ; let $k(x)$ be a real infinitely differentiable function such that

(80.6) $$\sum_z k^2(x + z) = 1,$$

$k(x) = 0$ outside the cube C: $|x_\lambda| < 2\delta$.

Lemmas 80.5 and 79.2 give

(80.7) $$k(x + z)\,[a + a^*]\,k(x + z) \leq [\gamma(\xi) + \gamma']k(x + z)q^{m-1}k(x + z) + \gamma' k^2(x+z)$$

if

(80.8) $$\gamma(\xi) = 2\sigma(\xi) + n\omega\|\xi\|$$

(80.9) $$\omega > \sum_\nu \operatorname*{Oscill.}_{x \in X} c_\nu(x);$$

now the first member of (80.7) depends only on the restriction of $a(\xi^*, x, \xi')$ to the cube $z + C$; therefore we can replace (80.9) by

$$\omega > \sum_\nu \operatorname*{Oscill.}_{z + C} c_\nu(x);$$

thus ω can be chosen small when δ is chosen small: if ε is any positive number, then (80.7) holds for any z and

(80.10) $$\gamma(\xi) = 2\sigma(\xi) + \varepsilon\|\xi\|,$$

if δ is sufficiently small.

Now (80.6) shows that

$$a(p^*, x, p) - \sum_z k(x + z)a(p^*, x, p)k(x + z)$$

is the sum of an operator of order $(m - 2, m)$ and an operator of order $(m - 1, m - 1)$; thus, according to Lemmas 80.1 and 80.3

(80.11) $$a + a^* \leq \sum_z k(x+z)\,[a + a^*]\,k(x+z) + \varepsilon\|\xi\|q^{m-1} + \gamma' q^{m-1} + \gamma''.$$

Similarly

$$q^{m-1} - \sum_z k(x+z)q^{m-1}k(x+z)$$

is the sum of two operators of orders $(m-2, m-1)$ and $(m-1, m-2)$; thus, according to Lemma 80.2

$$(80.12) \qquad \sum_z k(x+z)q^{m-1}k(x+z) \leq (1+\mathcal{E})q^{m-1} + \gamma''.$$

The lemma results from (80.6), (80.7), (80.10), (80.11) and (80.12).

Proposition 80.1. Assumption: $a(p^*, x, p)$ has the order $(m-1, m)$; its coefficients and their first derivatives are bounded functions of x.

Definition: The hermitian part of $a(p^*, x, p)$ is

$$(80.13) \qquad c(p^*, x, p) = \tfrac{1}{2}(a + a^*);$$

$\alpha(\xi)$ and $\beta(\xi)$ are the largest and smallest numbers such that

$$(80.14) \qquad \alpha(\xi)(q+1)^{m-1} \leq c(p^*, x, p) \leq \beta(\xi)(q+1)^{m-1};$$

$h(\overline{\zeta}, x, \zeta)$ is the principal part of $a(\overline{\zeta}, x, \zeta)$; $\zeta = \xi + i\eta$, $\overline{\zeta} = \xi - i\eta$;

$$(80.15) \qquad \rho(\xi) = \operatorname*{Inf.}_{x,\eta} \frac{\mathcal{R}h(\overline{\zeta}, x, \zeta)}{\|\zeta\|^{2m-2}}, \qquad \sigma(\xi) = \operatorname*{Sup.}_{x,\eta} \frac{\mathcal{R}h(\overline{\zeta}, x, \zeta)}{\|\zeta\|^{2m-2}}$$

are positively homogeneous functions of ξ of degree 1; we know that $\|\xi\|^{-1}\rho(\xi)$ and $\|\xi\|^{-1}\sigma(\xi)$ are bounded.

Assertion: $\alpha(\xi)$ and $\beta(\xi)$ are finite; for $\|\xi\| \longrightarrow \infty$

$$(80.16) \qquad 0 \leq \underline{\lim}.\ \frac{\alpha(\xi) - \rho(\xi)}{\|\xi\|}; \qquad \overline{\lim}.\ \frac{\beta(\xi) - \sigma(\xi)}{\|\xi\|} \leq 0.$$

Proof. Lemma 80.6 asserts that

$$(80.17) \qquad 2c \leq [2\sigma(\xi) + \varepsilon\|\xi\| + \gamma']q^{m-1} + \gamma''$$

where $\mathcal{E}$ is any positive number; γ' and γ'' are functions of $\mathcal{E}$; therefore

$\beta(\xi)$ is finite. Now (80.2) and (80.17) give

$$2c \leq [2\sigma(\xi) + \varepsilon\|\xi\| + \gamma' + \gamma''\|\xi\|^{2(1-m)}]q^{m-1};$$

(80.2) gives moreover

$$\text{(80.18)} \qquad \left(\frac{\|\xi\|^2}{\|\xi\|^2+1}\right)^{m-1}(q+1)^{m-1} \leq q^{m-1} \leq (q+1)^{m-1};$$

hence:

$$2\beta(\xi) \leq 2\sigma(\xi) + \varepsilon\|\xi\| + \gamma' + \gamma''\|\xi\|^{1-m}$$

when the second member is positive; when it is negative

$$2\beta(\xi) \leq [2\sigma(\xi) + \varepsilon\|\xi\| + \gamma' + \gamma''\|\xi\|^{1-m}]\left(\frac{\|\xi\|^2}{1+\|\xi\|^2}\right)^{m-1};$$

thus the second inequality (80.16) is proved; replacing a by - a it becomes the first one.

81. <u>A hyperbolic operator of order</u> m - 1 <u>whose product by a given hyperbolic operator of order</u> m <u>has a positive hermitian part.</u>

We use the notations of n° 69.

<u>Proposition</u> 81.1. <u>Assumption</u>: $a(x, p)$ <u>is regularly hyperbolic on</u> X; <u>the coefficients of</u> $a(x, \xi)$ <u>and their first derivatives are bounded functions of</u> x.

<u>Definition</u>:

$$\text{(81.1)} \quad b(x, \xi) = \xi'_1 \frac{\partial a(x, \xi)}{\partial \xi_1} + \dots + \xi'_\ell \frac{\partial a(x, \xi)}{\partial \xi_\ell}, \text{ where } \xi' \in \overset{\circ}{\Gamma}_X(a);$$

<u>the hermitian part of</u> $[b(x, p)]^* a(x, p)$ <u>is</u>

$$\text{(81.2)} \qquad c(p^*, x, p) = \tfrac{1}{2}[b(x, p)]^* a(x, p) + \tfrac{1}{2}[a(x, p)]^* b(x, p);$$

$\alpha(\xi)$ <u>and</u> $\beta(\xi)$ <u>are the largest and smallest numbers such that</u>

$$\text{(81.3)} \qquad \alpha(\xi)(q+1)^{m-1} \leq c(p^*, x, p) \leq \beta(\xi)(q+1)^{m-1}.$$

Assertion: $\alpha(\xi)$ and $\beta(\xi)$ are finite; on any closed cone $\subset \overset{\circ}{\Gamma}_X(a)$,

(81.4) $\alpha(\xi) = O(||\xi||)$ and $\beta(\xi) = O(||\xi||)$ for $||\xi|| \longrightarrow \infty$.

Note 81.1. $\alpha(\xi) = O(||\xi||)$ means that there are positive constants such that

$$\text{const.}\ ||\xi|| \leq \alpha(\xi) \leq \text{const.}\ ||\xi|| \text{ for } ||\xi|| > \text{const.};$$

These constants can be chosen independent of $a(x, p)$ when $a(x, p)$ varies and satisfies uniformly the assumptions.

Note 81.2. (81.3) means

$$c(p^*, x, p) = O(||\xi||q^{m-1}) \quad \text{for } ||\xi|| \longrightarrow \infty,\ \xi \in \overset{\circ}{\Gamma}_X(a).$$

Note 81.3. $\Gamma_X(a) \subset \Gamma_X(b)$.

Note 81.4. Choices of $b(x, \xi)$ other than (81.1) are possible; in particular there are choices of $b(x, \xi)$ such that the coefficients of $b(x, \xi)$ are infinitely differentiable functions of x.

Proof. Propositions 80.1, 77.1 and 77.2, where

$$\Gamma_1(h) = \Gamma_x(a) \supset \Gamma_X(a).$$

82. A first example. $m = 1$, $a(x, p) = p_1$. Then $\Gamma_X(a)$ is either $\xi_1 \geq 0$ or $\xi_1 \leq 0$; we take $\Gamma_X(a)$ as

$$\xi_1 \geq 0.$$

Then

$$b(x, p) = 1,$$

$$c(p^*, x, p) = \xi_1$$

$$\alpha(\xi) = \beta(\xi) = \frac{\xi_1}{2} = O(||\xi||) \text{ inside } \overset{\circ}{\Gamma}_X(a).$$

A second example. $m = 2$, $a(x, p) = p_1^2 - p_2^2 - p_3^2$. We take $\Gamma_X(a)$ as

$$\sqrt{\xi_2^2 + \xi_3^2} \leq \xi_1.$$

Then

$$b(x, p) = p_1;$$

the Laplace transform of (78.1) and (78.2) is

(82.1) $$(\xi_1 - \sqrt{\xi_2^2 + \xi_3^2})q \leq c(p^*, x, p) \leq (\xi_1 + \sqrt{\xi_2^2 + \xi_3^2})q;$$

the statement of the Proposition 81.1 is an obvious consequence of (82.1) and (80.18).

Note. For $\xi_2 = \xi_3 = 0$ (82.1) becomes

(82.2) $$c(p^*, x, p) = \xi_1 q$$

which means that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\partial u}{\partial x_1}\left[\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2}\right] \exp.\,(-2x_1\xi_1)dx_1dx_2dx_3 =$$

$$\xi_1 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2\right] \exp.\,(-2x_1\xi_1)dx_1dx_2dx_3;$$

this relation is the Laplace transform of the following one

(82.3) $$\frac{1}{2}\frac{\partial}{\partial x_1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2\right]dx_2dx_3 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v\,\frac{\partial u}{\partial x_1}\,dx_2dx_3$$

where

(82.4) $$\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} = v.$$

Now (82.4) is the waves equation; (82.3) is the classic energy relation for this equation: (82.3) follows easily from (82.4). Schauder's process (see the Introduction, p. 104) is based on this energy relation.

Note. The fundamental inequalities (68.1) and (81.3) of the Chapters V and VI are essentially different for $m > 2$; but they are closely related for $m = 2$.

§3. The inverses of a regularly hyperbolic operator

83. The norm $||r^s f||$. We use the same norm as in n°79:

$$||f|| = [\int_X \cdots \int |f(x)|^2 \exp(-2x.\xi)\, dx_1 \ldots dx_\ell\,]^{\frac{1}{2}}, \text{ where } \xi \in \Xi ;$$

(f, g) is the corresponding scalar product; a* is the adjoint operator of a; for instance

$$p^*_\lambda = 2\xi_\lambda - p_\lambda .$$

As in n°80

$$q = p_1^* p_1 + \ldots + p^*_\ell p_\ell ;$$

let r be the positive and self-adjoint operator

$$r = \sqrt{1 + q} ;$$

we use also the norm

$$||r^s f|| = \sqrt{(f,\ (1+q)^s f)} ,$$

where s is a positive or negative integer: for $s < 0$ the definition of this norm requires the symbolic calculus (Ch. I); for $s \geq 0$ this norm has an elementary definition and is equivalent to

$$\sum_{s_1+\ldots+s_\ell=0}^{s} ||p_1^{s_1} \ldots p_\ell^{s_\ell} f||.$$

Obviously

$$(83.1) \qquad ||r^{s+1} f||^2 = ||r^s f||^2 + \sum_\lambda ||r^s p_\lambda f||^2.$$

Hence

$$||rf||^2 = ||f||^2 + \sum_\lambda ||p_\lambda f||^2;$$

(83.2)

$$||r^s f||^2 = ||f||^2 + \sum_{s_1+\dots+s_\ell=0}^{s} ||p_1^{s_1}\dots p_\ell^{s_\ell} f||^2.$$

Besides, when ξ is given and when $v(x)$ is any function such that $||r^{-s}v|| < \infty$, then

$$||r^s u|| = \operatorname*{Sup}_v \frac{(u, v)}{||r^{-s}v||}. \tag{83.3}$$

Proof. $(u, v) = (r^s u, r^{-s}v)$; Schwarz's inequality is applied to $(r^s u, r^{-s}v)$.

Lemma 83.1. If a and a^* are adjoint operators and if there is a constant γ such that

$$||r^t au|| < \gamma ||r^s u||,$$

then

$$||r^{-s}a^*u|| < \gamma\, ||r^{-t}u||.$$

Proof. According to (83.3)

$$||r^{-s}a^*u|| = \operatorname*{Sup}_v \frac{(a^*u, v)}{||r^s v||} = \operatorname*{Sup}_v \frac{(u, av)}{||r^s v||} \leq ||r^{-t}u|| \operatorname*{Sup}_v \frac{||r^t av||}{||r^s v||}.$$

Let us denote for a moment (u, v) and $||r^s u||$ by $(u, v)_\xi$ and $||r^s u||_\xi$.

Lemma 83.2. $||r^s u(x)||_\xi$ and $||r^s[u(x)\exp(-x.\xi)]||_0$ are equivalent norms:

$$1 \leq \frac{||r^s u(x)||_\xi}{||r^s[u(x)\exp(-x.\xi)]||_0} \leq (1 + \sum_\lambda \xi_\lambda^2)^{\frac{s}{2}} \quad \text{for } 0 \leq s; \tag{83.4}$$

the opposite inequalities hold for $s \leq 0$.

Proof. According to (6.7)

$$||r^s[u(x)\exp(-x.\xi)]||_0^2 =$$

$$(u(x)\ \exp(-x.\xi),\ (1 - \sum_\lambda p_\lambda^2)^s[u(x)\exp(-x.\xi)])_o =$$

$$(u(x)\ \exp(-2x.\xi),\ [1 - \sum_\lambda (p_\lambda - \xi_\lambda)^2]^s u(x))_o =$$

$$(u,\ (1 + q - \sum_\lambda \xi_\lambda^2)^s u)_\xi\ ;$$

thus the inequalities to be proved are:

$$\frac{1}{(1 + \sum_\lambda \xi_\lambda^2)^s}(1 + q)^s \leq (1 + q - \sum_\lambda \xi_\lambda^2)^s \leq (1 + q)^s \text{ for } s \geq 0$$

and the opposite inequalities for $s \leq 0$; that is (Lemma 79.4):

$$\frac{1}{(1 + \sum_\lambda \xi_\lambda^2)^s}(1 + \sum_\lambda |\zeta_\lambda|^2)^s \leq (1 + \sum_\lambda \eta_\lambda^2)^s \leq (1 + \sum_\lambda |\zeta_\lambda|^2)^s$$

for $\zeta_\lambda = \xi_\lambda + i\eta_\lambda$, $s \geq 0$; which is obvious.

Lemma 83.3. If $||r^s u||_{\xi'} < \infty$, $||r^{-s} v||_{\xi''} < \infty$ and $2\xi = \xi' + \xi''$, then $(u, v)_\xi < \infty$.

Proof. The preceding Lemma shows that the assumptions

$$||r^s u||_{\xi'} < \infty, \quad ||r^{-s} v||_{\xi''} < \infty$$

mean

$$||r^s[u(x)\ \exp(-x.\xi')]||_o < \infty,\ ||r^{-s}[v(x)\ \exp(-x.\xi'')]||_o < \infty;$$

hence, according to (83.3),

$$(u(x)\ \exp(-x.\xi'),\ v(x)\ \exp(-x.\xi''))_o < \infty,$$

that is

$$(u,\ v)_\xi < \infty.$$

Lemma 83.4. There is a constant $\gamma(\ell)$ such that

$$||c(x)u(x)|| \leq \gamma\left[\int \cdots \int_X c^2(x)dx_1 \ldots dx_\ell\right]^{\frac{1}{2}} \cdot ||r^{\ell'} u||, \tag{83.5}$$

where ℓ' is the smallest integer $> \frac{\ell}{2}$.

Note. Obviously

(83.6) $$||c(x)u(x)|| < \operatorname{Sup}_X |c(x)| \cdot ||u||.$$

Proof. S. Sobolev [48] improved as follows a Lemma of Schauder ([36], Hilfsatz I): there is a constant $\gamma(\ell')$ such that

(83.7) $$\operatorname{Sup}_X |u(x)| < \gamma ||r^{\ell'} u||_0;$$

hence, according to (83.4),

$$\operatorname{Sup}_X |u(x) \exp(-x. \xi)| < \text{const.} ||r^{\ell'}[u(x) \exp(-x. \xi)]||_0$$
$$< \text{const.} ||r^{\ell'} u||_\xi .$$

Now

$$||c(x)u(x)|| < [\int_X \cdots \int c^2(x)dx_1 \ldots, dx_\ell]^{\frac{1}{2}} \cdot \operatorname{Sup}_X |u(x) \exp(-x. \xi)|.$$

Lemma 83.5. Let $b(p)$ be an operator of order t and $a(x, p)$ or $a(p, x)$ be another operator of order $m = s-t+1$. Its coefficients are bounded; either they have bounded derivatives of order $\leq M$ for some M satisfying $1 \leq M \leq \ell'$; or they have bounded derivatives of order $\leq \ell'$ and square integrable derivatives of order $> \ell'$ and $\leq M$ for some $M > \ell'$. There are constants γ such that:

(83.8) $$||[a(x, p)b(p) - b(p)a(x, p)]u|| < \gamma ||r^s u|| \quad \text{if } 0 \leq t \leq M;$$

(83.9) $$||r^{1-m}[a(p, x)b(p) - b(p)a(p, x)]u|| < \gamma ||r^t u|| \text{ if } 0 \leq t < M;$$

(83.10) $$||r^t au|| < \gamma ||r^{s+1} u||$$

if $|t| \leq M$, $a = a(x, p)$ and if $|s + 1| \leq M$, $a = a(p, x)$.

Note. γ depends on a, b, s, t, but is independent of u, ξ; the assumption that a is hyperbolic is useless.

Proof of (83.8).

$$a(x, p)b(p) - b(p)a(x, p) = c_1(x, p) + c_2(x, p)$$

where the coefficients of c_1 [or c_2] are derivatives of order $\leq \ell'$ [or $> \ell'$] of the coefficients of $a(x, p)$: they are bounded [or square integrable]; the order of c_1 [or c_2] is s [or $s - \ell'$]; (83.6) and (83.5) give respectively

$$||c_1(x, p)u|| < \gamma ||r^s u||, \qquad ||c_2(x, p)u|| < \gamma ||r^s u||.$$

<u>Proof of</u> (83.9).

$$a(p, x)b(p) - b(p)a(p, x) = c_1(p, x, p) + c_2(p, x, p),$$

where: $c_1(\xi, x, \xi')$ and $c_2(\xi, x, \xi')$ are polynomials in ξ and ξ', whose degree in ξ is $m - 1$; the coefficients of c_1 [or c_2] are derivatives of order $\leq \ell'$ [or $> \ell'$] of the coefficients of $a(p, x)$: they are bounded [or square integrable]; the degree in ξ' of c_1 [or c_2] is t [or $t - \ell'$]; (83.1), (83.6) and (83.5) give

$$||r^{1-m} c_1(p, x, p)u|| < \gamma ||r^t u||; \quad ||r^{1-m} c_2(p, x, p)u|| < \gamma ||r^t u||.$$

<u>Proof of (83.10) for $0 \leq t \leq M$, $a(x, p)$ and for $0 \leq s < M$, $a(p, x)$.</u>

$$b(p)a = c_1(x, p) + c_2(x, p)$$

where the coefficients of c_1 [or c_2] are derivatives of order $\leq \ell'$ [or $> \ell'$] of the coefficients of a: they are bounded or square integrable; the order of c_1 [or c_2] is $s + 1$ [or $s - \ell'$]; (83.6) or (83.5) give

$$||c_1(x, p)u|| < \gamma ||r^{s+1} u||, \quad ||c_2(x, p)|| < \gamma ||r^{s+1} u||.$$

<u>Proof of (83.10) for $-M \leq t < 0$, $a(x, p)$ and for $-M - 1 \leq s < 0$, $a(p, x)$.</u>

Assume: $0 \leq t \leq M$, $a = a(x, p)$; or $0 \leq s < M$, $a = a(p, x)$; (83.10) holds; hence, according to Lemma 83.1,

$$||r^{-s-1} a^* u|| < \gamma ||r^{-t} u||,$$

which is the assertion to be proved.

84. <u>Notations.</u> <u>1°</u> The definitions of n°69 are used: a(x, p) or a(p, x) denotes a <u>regularly hyperbolic</u> operator of order m.

<u>2°</u> Its coefficients are bounded; either they have bounded derivatives of order $\leq M$, for some M satisfying $1 \leq M \leq \ell'$; or they have bounded derivatives of order $\leq \ell'$ and square integrable derivatives of order $> \ell'$ and $\leq M$, for some $M > \ell'$; ℓ' is the smallest integer $> \frac{\ell}{2}$. s and t are integers (≥ 0 or < 0) such that

$$(84.1) \qquad m = s - t + 1.$$

<u>3°</u> Δ denotes any domain of Ξ whose director cone is

$$\Gamma_X(a) = \bigcap_{x \subset X} \Gamma_x(a);$$

$\mu(\xi)$ denotes any positive function defined on Δ and such that on any closed cone interior to $\Gamma_X(a)$

$$\mu(\xi) = O(||\xi||^{-1}) \quad \text{for } ||\xi|| \longrightarrow \infty.$$

<u>4°</u> $D^s = D^s_+$ (or D^s_-) denotes the <u>vector space</u> whose elements are the distributions f_x such that

$$(84.2) \qquad ||r^s f|| < \infty \text{ for any } \xi \in \Delta \text{ (or } \xi \in -\Delta).$$

(D^0 was denoted by $F(\Delta)$ in n°3).

$D^s \subset D^t$ if $t < s$. Moreover D^s is dense in D^t: if $f \subset D^t$ and if $\overline{f}$ is defined by (73.1), where $\eta \longrightarrow 0$, then $\overline{f} \subset D^s$ and $||r^t(\overline{f} - f)|| \longrightarrow 0$ for any $\xi \in \Delta$ [see n°9 and (6.11)].

<u>Lemma</u> 84. . If $u \subset D^s$ and $v \subset D_-^{-s}$, then (u, v) is defined for any $\xi \in \Xi$.

<u>Proof.</u> Lemma (83.3), where ξ' is chosen in Δ and ξ'' in $-\Delta$ such that $\xi' + \xi'' = 2\xi$.

85. <u>The mapping</u> T is helpful in n°87, 88 and 89.

<u>Definition.</u> $Tu(x) = u(-x) \exp(2x.\xi)$.

Properties. (85.1) $T^2 = 1$

(85.2) $T = T^*$

(85.3) $Ta(p)u(x) = a(p^*)Tu(x)$

(85.4) $||Tu|| = ||u||;\quad ||r^s Tu|| = ||r^s u||$

(85.5) $Ta(p,\ x)u(x) = a(p^*,\ -x)Tu(x);\quad Ta(x,\ p)u(x) = a(-x,\ p^*)Tu(x).$

Proof. (85.1) is obvious.

$$(Tu,\ v) = \int_X\!\cdots\!\int u(-x)v(x)dx_1\cdots dx_\ell = \int_X\!\cdots\!\int u(x)v(-x)dx_1\cdots dx_\ell = (u,\ Tv) = (T^*u,\ v).$$

According to (6.5) and (6.7)

$$Ta(p)u(x) = \exp(2x,\xi)a(-p)u(-x) = a(2\xi - p)[u(-x)\exp(2x,\xi)] = a(p^*)Tu(x).$$

$$||Tu||^2 = (Tu,\ Tu) = (T^2u,\ u) = ||u||^2.$$

$$||r^s Tu|| = ||Tr^s u|| = ||r^s u||.$$

(85.5) follows from (85.3).

86. Existence theorem for the equation

(86.1) $a(x,\ p)u(x) = v(x).$

Lemma 86.1. There are a domain $\triangle$ and a function $\mu(\xi)$ independent of v such that, if $\xi \in \triangle$ and

(86.2) $||r^m u|| < \infty,$

then

(86.3) $||r^{m-1}u|| < \mu(\xi)||v||.$

Proof. Assuming at first that

(86.4) $||r^{2m}u|| < \infty,$

apply Proposition 81.1:

$$(bu, v) = (u, b^*au) = (a^*bu, u) = (u, cu) \geq \alpha(\xi)||r^{m-1}u||^2;$$

now according to (83.2) and Schwarz's inequality

$$(bu, v) \leq \text{const.}\ ||r^{m-1}u||\cdot||v||;$$

thus (86.3) holds if u satisfies (86.4). Now the functions satisfying (86.4) are dense on the space of the functions satisfying (86.2); and on this space both members of (86.3) are continuous functions of u.

<u>Lemma</u> 86.2. There are a domain $\triangle$ and a function $\mu(\xi)$ independent of v such that, if

$$||r^{s+1}u|| < \infty,\ \xi \in \triangle,\ 0 \leq t \leq M,$$

then

(86.5) $$||r^s u|| < \mu(\xi)||r^t v||.$$

<u>Proof.</u> If c(p) has the order t, then according to (83.8)

$$||a(x, p)c(p)u - c(p)v|| < \gamma||r^s u||$$

hence

$$||a(x, p)c(p)u|| < ||c(p)v|| + \gamma||r^s u||$$

and, upon application of the preceding Lemma,

$$||r^{m-1}c(p)u|| < \mu(\xi)[||c(p)v|| + \gamma||r^s u||];$$

hence [see (83.2)]

(86.6) $$||r^s u|| < \mu(\xi)[||r^t v|| + \gamma||r^s u||];$$

since $\mu(\xi) = O(||\xi||^{-1})$ inside $\triangle$, the subset of $\triangle$ where $\gamma\mu(\xi) < 1$ is another domain $\triangle$, where (86.6) means that (86.5) holds.

<u>Lemma</u> 86.3. There are a domain $\triangle$ and a function $\mu(\xi)$ such that, if for

any or some $\xi \in \Delta$ $||r^t v|| < \infty$ and if $0 \leq t \leq M$, then (86.1) has at least one solution $u(x)$ satisfying

$$(86.5) \qquad ||r^s u|| < \mu(\xi)||r^t v||.$$

<u>Note.</u> Δ and $\mu(\xi)$ are independent of v; they depend on $a(x, p)$, s and t, which satisfy (84.1).

<u>Note.</u> If $t = 0$, $s = m - 1$: the first member of (86.1) is not a function, but a distribution.

<u>Proof.</u> Let η be a positive number $\rightarrow 0$; define $\bar{v}(x)$ and $\bar{a}(x, p)$ by means of (73.1); according to n°9 and (6.11) $\bar{v}(x)$ and the coefficients of $\bar{a}(x, p)$ are infinitely differentiable functions of x and are arbitrarily near $v(x)$ and $a(x, p)$; define

$$a_\eta(x, p) = \bar{a}(\varphi(\eta||x||)x, p)$$

where $\varphi(\tau)$ is some infinitely differentiable function of τ such that

$$\varphi(\tau) = 1 \text{ for } 0 \leq t \leq 1, \qquad \varphi(\tau) = 0 \text{ for } 2 \leq \tau.$$

Obviously a_η is regularly hyperbolic:

$$\Gamma_X(\bar{a}) \subset \Gamma_X(a_\eta).$$

According to Proposition 75.1 the equation

$$a_\eta(x, p)u_\eta = \bar{v}$$

has a unique solution u_η such that

$$||r^\sigma u_\eta|| < \infty \text{ for any } \sigma > 0 \text{ and any } \xi \in \Delta_\infty;$$

moreover Lemma 86.2 and Note 81.1 give a domain Δ and a function $\mu(\xi)$ having the following properties:

Δ and $\mu(\xi)$ are independent of η;

$\Delta \subset \lim \Delta_\infty(a_\eta)$;

(86.5) holds for u_η, $\bar{v}$, $\xi \in \Delta$, η small.

Since $||r^s u_\eta||$ has a bound independent of η when $\xi \in \Delta$, there is a sequence of $\eta \longrightarrow 0$ such that $u_\eta(x)$ and its derivatives of orders $\leq s$ converge weakly to a function u(x) and its derivatives (F. Riesz, Math. Ann., 1910, vol. 69); u(x) satisfies (86.1) and (86.5).

87. Existence theorem for the equation

(87.1) $$a(p, x)u(x) = v(x).$$

Note. An operator a(p, x) is not necessarily equal to some operator a(x, p) when $M < m$; moreover n°87 extends the preceding assumptions about s and t: the foregoing condition $0 \leq t$ is more strict than the condition here that $0 \leq s$.

Lemma 87.1. There are a domain Δ and a function $\mu(\xi)$ indpependent of v such that, if $\xi \in \Delta$ and

(87.1) $$||ru|| < \infty,$$

then

(87.2) $$||u|| < \mu(\xi)||r^{1-m}v||.$$

Proof. Assume at first $||r^{m+1}u|| < \infty$. The mapping T (see n°85) transforms the assertion to be proved into the following one:

if

$$[a(x, p)]^* u(x) = v(x), \quad ||r^{m+1}u|| < \infty,$$

then

$$||u|| < \mu(\xi)||r^{1-m}v||.$$

Define b(x, p) by means of Proposition 81.1 and Note 81.4; according to the existence theorem of n°86 (Lemma 86.3) there is a function w(x) such that

$$b(x, p)w(x) = u(x), \quad ||r^{2m-1}w|| < \infty;$$

we have

$$||r^{m-1}w||\cdot||r^{1-m}v|| > (w,\ v) = (w,\ a^*bw) = (b^*aw,\ w) = (w,\ cw) \geq \alpha(\xi)||r^{m-1}w||^2;$$

hence (87.2). Thus (87.2) holds when $||r^{m+1}u|| < \infty$. Now the functions u(x) satisfying this condition are dense in the space of the functions u(x) satisfying (87.1); and in this space both members of (87.2) are continuous functions of u (see (83.1)).

Lemma 87.2. There are a domain Δ and a function $\mu(\xi)$ independent of v such that, if

$$||r^{s+1}u|| < \infty,\ \xi \in \Delta,\ 0 \leq s < M$$

then

(87.4) $$||r^s u|| < \mu(\xi)||r^t v||.$$

Proof. If c(p) has the order s, then according to (83.9)

$$||r^{1-m}[a(p,\ x)c(p)u(x) - c(p)v(x)]|| < \gamma||r^s u||;$$

hence, upon application of (83.1)

$$||r^{1-m}a(p,\ x)c(p)u(x)|| < \gamma||r^t v|| + \gamma||r^s u||$$

and, upon application of the preceding Lemma,

$$||c(p)u(x)|| < \mu(\xi)[\gamma||r^t v|| + \gamma||r^s u||]$$

hence (see (83.2))

(87.5) $$||r^s u|| < \mu(\xi)[\gamma||r^t v|| + \gamma||r^s u||].$$

Since $\mu(\xi) = O(||\xi||^{-1})$ inside Δ, the subset of Δ where $\gamma\mu(\xi) < 1$ is another domain Δ, where (87.5) means that (87.4) holds.

Lemma 87.3. There are a domain Δ and a function $\mu(\xi)$ such that, if for any or for some $\xi \in \Delta$

$$||r^t v|| < \infty$$

and if $0 \leq s < M$, then (87.1) has at least one solution u(x) satisfying

(87.4) $$||r^s u|| < \mu(\xi)||r^t v||.$$

Note. If $s < m$ both members of (87.1) are distributions.

Proof: Similar to the proof of Lemma 86.3.

88. **Uniqueness theorems.** The preceding existence theorem for $a(p, x)$ gives by duality a uniqueness theorem for $a(x, p)$:

Lemma 88.1. Consider anew the equation

(86.1) $$a(x, p)u(x) = v(x).$$

There are a domain Δ and a function $\mu(\xi)$ independent of v such that, if

$$||r^{s+1}u|| < \infty, \quad \xi \in \Delta, \quad -M < t \leq 0,$$

then

(88.1) $$||r^s u|| < \mu(\xi)||r^t v||.$$

Proof: Choose Δ and $\mu(\xi)$ such that Lemma 87.3 holds for a^*; transform this Lemma by T (n°85) and replace in this Lemma s and t by $-t$ and $-s$; this does not alter (84.1): its assumption becomes $-M < t \leq 0$; its assertion shows that, for a given $\xi \in \Delta$:

the set of the functions a^*w such that

$$||r^{-t}w|| < \mu(\xi)||r^{-s}a^*w|| < \infty$$

is the space of the functions v' such that $||r^{-s}v'|| < \infty$. Hence, according to (83.3),

$$||r^s u|| = \operatorname{Sup}_w \frac{(u, a^*w)}{||r^{-s}a^*w||} = \operatorname{Sup}_w \frac{(v, w)}{||r^{-s}a^*w||} \leq ||r^t v|| \operatorname{Sup}_w \frac{||r^{-t}w||}{||r^{-s}a^*w||}$$

$$< \mu(\xi)||r^t v||.$$

The use of the formula $(u, a^*w) = (v, w)$ needs the assumption $||r^{s+1}u|| < \infty$.

Similarly Lemma 86.3 gives:

Lemma 88.2. Consider the equation (87.1). There are a domain $\triangle$ and a function $\mu(\xi)$ independent of v such that, if

$$||r^{s+1}u|| < \infty, \ \xi \in \triangle, \ -M \leq s \leq 0,$$

then

$$||r^s u|| < \mu(\xi)||r^t v||.$$

Summing up:

Proposition 88. Let X be an ℓ-dimensional vector space. Let ℓ' be the smallest integer $> \frac{\ell}{2}$; let a be a regularly hyperbolic operator on X (n^o69); its order is m; its coefficients are bounded; either they have bounded derivatives of order M ($1 \leq M \leq \ell'$); or they have bounded derivatives of order $\leq \ell'$ and square integrable derivatives of order $> \ell'$ and $\leq M$ ($\ell' < M$). Let s and t be two positive or negative integers such that:

$$m = s - t + 1; \ -M < t \leq M \text{ if } a = a(x, p); \ -M \leq s < M \text{ if } a = a(p, x).$$

There are a domain $\triangle$ and a function $\mu(\xi)$ dependent on a, s, t, but independent of v, with the following properties:

1) The equation au = v has at most one solution u(x) such that

$$||r^{s+1}u|| < \infty \text{ for any } \xi \in \triangle.$$

2) The equation au = v has at least one solution u(x) such that

$$||r^s u|| < \mu(\xi)||r^t v|| \text{ for any } \xi \in \triangle, \text{ if } ||r^t v|| < \infty. \tag{88.1}$$

Note. There is a constant γ dependent on a, s, t such that

$$||r^t v|| < \gamma||r^{s+1}u||, \quad ||r^{t-1}v|| < \gamma||r^s u|| \text{ for any } \xi \in \Xi. \tag{88.2}$$

Note. There is another domain $\triangle_-$ with the same properties for $\xi \in \triangle_-$ and with the director cone $-\Gamma_X(a)$.

Proof of 1) Lemma 88.1 or 88.2.

Proof of (88.2)': (83.10).

Proof of 2). Assume $a = a(x, p)$. Lemmas 86.3 and 88.1 show that the equation $au = v$ defines a mapping

$$a^{-1}: \; v \longrightarrow u,$$

which maps continuously D^t into D^s, when $0 \leq t \leq M$; (88.1) enables us to extend this mapping to the case: $-M < t < 0$. We have

$$aa^{-1} = 1$$

on D^t for $0 \leq t \leq M$, hence on D^0, that is on a dense subset of D^t when $-M < t < 0$ ($n^o 84.4$); now (88.1) and (88.2) show that aa^{-1} is a continuous mapping of D^t into D^{t-1}; hence

$$aa^{-1} = 1 \text{ on } D^t:$$

thus $u = a^{-1}v$ satisfies $au = v$, if $v \in D^t$.

For $a = a(p, x)$ the proof is the same, Lemmas 87.3 and 88.2 replacing Lemmas 86.3 and 88.1.

89. The inverses of a regularly hyperbolic operator and of its adjoint.

Proposition 88, where s and t are replaced by $-t$ and $-s$ when $a = a(p, x)$, gives immediately the first and second part of

Theorem 89. *On an ℓ-dimensional vector space X let $a(x, p)$ be a regularly hyperbolic operator ($n^o 69$); its order is m; its adjoint is $a^*(p, x)$; its coefficients are bounded; either they have bounded derivatives of order $\leq M$ ($1 \leq M \leq \ell'$) or they have bounded derivatives of order $\leq \ell'$ and square integrable derivatives of order $> \ell'$ and $\leq M$ ($\ell' < M$). (ℓ' is the smallest integer $> \ell/2$). Then*

$a(x, p)$ has two inverses $a_+^{-1} = a^{-1}$ and a_-^{-1};

$a^*(p, x)$ has two inverses $a_+^{*-1} = a^{*-1}$ and a_-^{*-1}.

1) Let s and t be two integers such that

$$m = s - t + 1; \quad -M < t \leq M.$$

There are functional spaces (see n°84) D^s, D^t, D^{-s}, D^{-t} such that

$$a^{-1}: \ D^t \longrightarrow D^s; \quad a^{*-1}: \ D^{-s} \longrightarrow D^{-t}.$$

We have

$$a: \ D^s \longrightarrow D^{t-1}; \quad a^*: \ D^{-t} \longrightarrow D^{-s-1};$$

$$a: \ D^{s+1} \longrightarrow D^t; \ a^*: \ D^{-t+1} \longrightarrow D^{-s};$$

$$aa^{-1} = 1 \text{ on } D^t; \qquad a^*a^{*-1} = 1 \text{ on } D^{-s};$$

$$a^{-1}a = 1 \text{ on } D^{s+1}; \ a^{*-1}a^* = 1 \text{ on } D^{-t+1}.$$

2) The same statement holds when a^{-1}, a^{*-1}, D are replaced by a_-^{-1}, a_-^{*-1}, D_-.

3) The adjoint of a^{-1} is a_-^{*-1}; that is

$$(a^{-1}v, u) = (v, a_-^{*-1}u) \text{ for } v \in D^t, \ u \in D_-^{-s}.$$

Note. (88.1) and (88.2) give bounds of a and a*, and arbitrarily small bounds of their inverses, which is essential in the theory of systems (n°116) and in the theory of the non-linear equations and systems (n°123).

Proof of 3). Assume either $v \in D^{t+1}$ if $t < M$, or $u \in D_-^{-s+1}$ if $1 - M < t$: thus either $a^{-1}v \in D^{s+1}$, $a_-^{*-1}u \in D^{-t}$, or $a^{-1}v \in D^s$, $a_-^{*-1}u \in D_-^{-t+1}$; hence, since $m = s - t + 1$, $(a^{-1}v, u) = (a^{-1}v, a^*a_-^{*-1}u) = (aa^{-1}v, a_-^{*-1}u) = (v, a_-^{*-1}u)$. Now D^{t+1} is dense in D^t, D_-^{-s+1} is dense in D_-^{-s} (n°84), $(a^{-1}v, u)$ and $(v, a_-^{*-1}u)$ are continuous for $v \in D^t$, $u \in D_-^{-s}$; hence

$$(a^{-1}v, u) = (v, a_-^{*-1}u) \text{ for } v \in D^t, \ u \in D_-^{-s}.$$

§4. Emission and dependence domain

90. The emission.

Let us now introduce a notion that S. C. Zaremba [47] and A. Marchaud [46] studied independently.

Definition 90.1. A path is said to be timelike when its positive semi-tangents at x belong to C_x; the emission of a subset Y of X is the union of the timelike paths originating from Y; it is denoted by $\mathcal{E}(Y)$ or $\mathcal{E}_+(Y)$; $\mathcal{E}_-(Y)$ denotes the union of the timelike paths ending on Y: $\mathcal{E}(Y)$ becomes $\mathcal{E}_-(Y)$ when C_x is replaced by $-C_x$.

$\mathcal{E}(Y)$ can be constructed by solving ordinary differential equations: see §5.

Properties of the emission. 1) When K is compact, then $\mathcal{E}(K)$ is closed and depends continuously on K and C_x (Marchaud, Zaremba).

2) When D is a domain, then $\mathcal{E}(D)$ is a domain (assumption 94.1 is made).

3) Let S, T, K be a closed and two compact subsets of X such that:

$$S \subset C_X + K;$$

at any point y of S exterior to T, S has at least a semi-tangent belonging to $-C_y$; then (Marchaud):

$$S \subset \mathcal{E}(T).$$

Proof of 2). Let $z \in \mathcal{E}(D)$: there is a timelike path $\widehat{yz}$ such that $y \in D$; according to Lemma 97.4, since y is not on the boundary of $\mathcal{E}(D)$, z cannot be on this boundary.

91. The functional spaces $\mathcal{D}^s(K)$.

Definition 91.1. For $0 \leq s$, $\tilde{\mathcal{D}}^s$ is the vector space whose elements are the functions u(x) with locally square integrable derivatives of order $\leq$ s: u(x) is defined on X; u(x) and its derivatives of order $\leq$ s are square integrable on any compact subset of X. The topology of $\breve{\mathcal{D}}^s$ is defined by the following fundamental system of neighborhoods $V_{K,\varepsilon}$ of 0: K is a compact subset of X; ε is a number > 0; $u(x) \in V_{K,\varepsilon}$ if

$$\int_K \cdots \int |u(x)|^2 dx_1 \ldots dx_\ell < \varepsilon, \ \ldots, \ \int_K \cdots \int |p_{\lambda_1} \ldots p_{\lambda_s} u(x)|^2 dx_1 \ldots dx_\ell < \varepsilon .$$

For $s < 0$, $\breve{\mathcal{D}}^s$ is the vector space whose elements are <u>sums of derivatives of order ≤ -s of locally square integrable functions</u>. The topology of $\breve{\mathcal{D}}^s$ is defined by the following fundamental system of neighborhoods $V_{K,\varepsilon}$ of 0 (K compact, $\varepsilon > 0$): $u_x \in V_{K,\varepsilon}$ if u_x is the sum of derivatives of order $\leq -s$ of functions $u_\alpha(x)$ such that

$$\sum_\alpha \int_K \cdots \int |u_\alpha(x)|^2 dx_1 \ldots dx_\ell < \varepsilon .$$

<u>Definition</u> 91.2. Let K be a subset of X; $\mathcal{D}^s(K) = \mathcal{D}^s_+(K)$ and $\mathcal{D}^s_-(K)$ are subsets of $\breve{\mathcal{D}}^s$:

$u \in \mathcal{D}^s(K)$ if $u \in \breve{\mathcal{D}}^s$ and $S(u) \subset \mathcal{E}(K)$;

$u \in \mathcal{D}^s_-(K)$ if $u \in \breve{\mathcal{D}}^s$ and $S(u) \subset \mathcal{E}_-(K)$.

$S(u)$ denotes the <u>support</u> of the distribution u_x, that is, the smallest closed subset of X outside which $u_x = 0$.

92. <u>The dependence domain</u>. The notations of n°69 are used. a denotes either $a(x, p)$ or $a(p, x)$. We assume that $S(v)$ is compact and that u is a solution, given by Proposition 88 of the equation

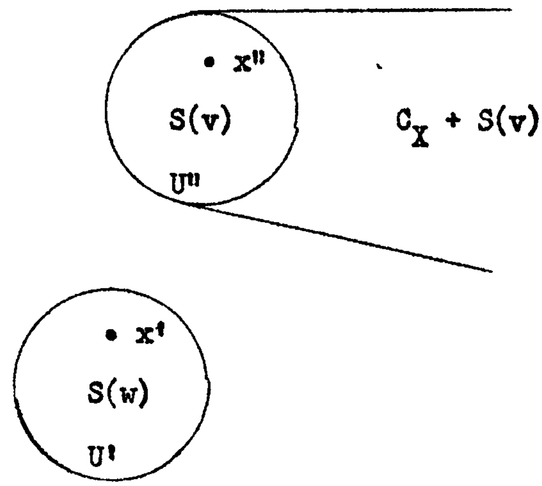
$$au = v.$$

<u>Lemma</u> 92.1. $S(u) \subset C_X + S(v)$

where C_X denotes the cone with vertex 0 dual to Γ_X.

Proof. Let x' and x'' be two points of X such that

(92.1) $\qquad x' - x'' \notin C_X;$



we have to show that, if $S(v)$ is near x'', then

$u(x) = 0 \quad$ near x'.

The assumption (92.1) means that there is a point $\xi \in \Gamma_X$ such that

(92.2) $\qquad (x' - x'')\cdot \xi < 0;$

Γ_X being convex and having an interior is the closure of its interior; therefore (92.2) holds for some

$$\xi \in \mathring{\Gamma}_X.$$

Let U' and U'' be neighborhoods of x' and x'' such that

(92.3) $\qquad (u' - u'')\cdot \xi < 0$ for $u' \subset U'$ and $u'' \subset U''$.

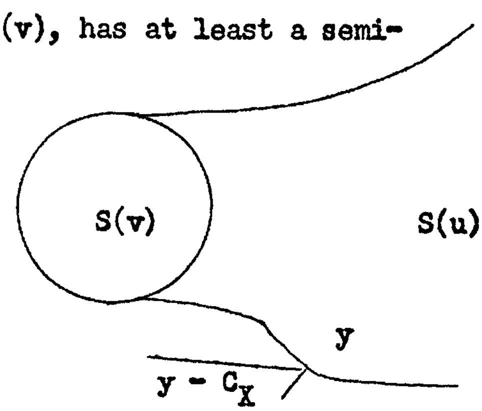
Suppose $S(v) \subset U''$ and denote by $w(x)$ the restriction of $u(x)$ to U'; (88.1) (where we choose either $s \geq 0$, $t = 0$ or $s = 0$, $t \leq 0$), (83.1) and the mean value theorem give

(92.4) $\qquad ||w||_2 \leq \mu(\xi)||v||_2 \exp(u' - u'')\cdot \xi ,$

where u' and u'' are the points of $S(w)$ and $S(v)$ where $u\cdot\xi$ is maximum and minimum respectively. Let ξ run on a half-line belonging to $\mathring{\Gamma}_X$; (92.3) and (92.4) show that $w = 0$.

Lemma 92.2. $S(u)$, at its points exterior to $S(v)$, has at least a semitangent belonging to $-C_X$.

Proof. Assume this assertion to be false: there are a point y and a number $\varepsilon > 0$ with the following properties:



$$y \in S(u); \quad y \notin S(v);$$

any x such that

(92.5) $$x \in S(u),\ y - x \in C_X,\ x \neq y$$

satisfies

(92.6) $$||y - x|| > 2\varepsilon.$$

Let $\varphi(x)$ be an infinitely differentiable function such that

(92.7) $$\varphi(x) = 1 \text{ for } ||x - y|| \leq \varepsilon, \quad \varphi(x) = 0 \text{ for } ||x - y|| \geq 2\varepsilon.$$

On the first hand

(92.8) $$y \in S(\varphi u).$$

On the second hand we have

$a\varphi u = 0$ near y because $\varphi = 1$, $au = 0$ near y;

$a\varphi u = 0$ near any x satisfying $y - x \subset C_X$, $x \neq y$, because

$\varphi u = 0$ near x according to (92.5), (92.6), (92.7);

thus

(92.9) $S(a\varphi u)$ is outside $y - C_X$.

Hence, according to (92.8)

$$S(\varphi u) \not\subset C_X + S(a\varphi u)$$

which is contradictory to Lemma 92.1.

Lemma 92.3. $S(u)$, at a point y exterior to $S(v)$, has at least a semi-tangent belonging to $-C_y$.

Proof. Let y be a point of $S(u)$ exterior to $S(v)$; let us apply the preceding lemma to $u(x)$ and the hyperbolic differential operator

$$a(y + \varphi(\varepsilon^{-1}||x - y||)(x - y),\ p),$$

where ε is a positive number $\longrightarrow 0$ and $\varphi(t)$ a function such that

$$0 \leq \varphi(t) \leq 1; \qquad \varphi(t) = 1 \text{ for } 0 \leq t \leq 1/2; \qquad \varphi(t) = 0 \text{ for } 1 \leq t;$$

the lemma asserts that $S(u)$ has at the point y at least a semi-tangent belonging to the cone dual to $-\bigcap_{||x-y||\leq\varepsilon} \Gamma_x$; therefore $S(u)$ has at the point y at least a semi-tangent belonging to the limit of this cone for $\varepsilon \longrightarrow 0$; this limit is the cone $-C_y$ dual to $-\Gamma_y$.

<u>Proposition</u> 92. 1) <u>If</u> $S(v)$ <u>is compact, then</u>

$$S(a^{-1}v) \subset \mathcal{E}(Sv), \quad S(a^{*-1}v) \subset \mathcal{E}(Sv).$$

(a^{-1}, a^{*-1}, $\mathcal{E}$ can be replaced by a_-^{-1}, a_-^{*-1}, $\mathcal{E}_-$).

2) <u>The restriction of</u> $a^{-1}v$ <u>to a domain</u> D <u>of</u> X <u>depends only on the restrictions of</u> a <u>and</u> v <u>to the domain</u> $\mathcal{E}_-(D)$. (Assumption 94.1 is made.)

<u>Proof of</u> 1): Property 3) of the emission ($n^o 90$); Lemmas 92.1 and 92.3.

<u>Proof of</u> 2): Obvious consequence of 1) and of the properties 1) and 2) of the emission ($n^o 90$).

93. <u>The inverses of a hyperbolic operator.</u> If $S(v)$ is compact, then the conditions $v \in D^s$ and $v \in \tilde{\mathcal{B}}^s$ are obviously equivalent. Thus Theorem 89.1 defines $a^{-1}v$ when

$$v \in \mathcal{B}^t(K), \quad S(v) \text{ is compact};$$

Proposition 92 shows that

$$a^{-1}v \in \mathcal{B}^s(K),$$

and that the restriction of $a^{-1}v$ to a domain D depends only on the restriction of v to $\mathcal{E}_-(D)$. Now this restriction of v to $\mathcal{E}_-(D)$ is the restriction of a function whose support is compact when

$v \in \mathcal{D}^t(K)$, $D \subset K'$, K and K' being compact.

Hence the definition of $a^{-1}v$ for any $v \in \mathcal{D}^t(K)$.

Similarly a^{*-1} can be defined on $\mathcal{D}^{-s}(K)$.

Hence the following Theorem, under the <u>assumptions of Theorem</u> 89; K and K' denote compact subsets of X.

<u>Theorem</u> 93. <u>The hyperbolic operator</u> $a(x, p)$ <u>has two inverses</u> $a_+^{-1} = a^{-1}$ <u>and</u> a_-^{-1}; <u>its adjoint</u> $a^*(p, x)$ <u>has also two inverses</u> $a_+^{*-1} = a^{*-1}$ <u>and</u> a_-^{*-1}.

1) <u>Let</u> s <u>and</u> t <u>be two integers such that</u>

$$s - t + 1 = m; \quad -M < t \leq M.$$

<u>For any</u> K

$$a^{-1}: \mathcal{D}^t(K) \longrightarrow \mathcal{D}^s(K); \quad a^{*-1}: \mathcal{D}^{-s}(K) \longrightarrow \mathcal{D}^{-t}(K);$$
$$a: \mathcal{D}^s(K) \longrightarrow \mathcal{D}^{t-1}(K); \quad a^*: \mathcal{D}^{-t}(K) \longrightarrow \mathcal{D}^{-s-1}(K);$$
$$a: \mathcal{D}^{s+1}(K) \longrightarrow \mathcal{D}^t(K); \quad a^*: \mathcal{D}^{-t+1}(K) \longrightarrow \mathcal{D}^{-s}(K).$$

<u>These mappings are continuous</u>.

$$aa^{-1} = 1 \text{ on } \mathcal{D}^t(K); \quad a^*a^{*-1} = 1 \text{ on } \mathcal{D}^{-s}(K);$$
$$a^{-1}a = 1 \text{ on } \mathcal{D}^{s+1}(K); \quad a^{*-1}a^* = 1 \text{ on } \mathcal{D}^{-t+1}(K).$$

2) <u>The same statement holds when</u> a^{-1}, a^{*-1}, $\mathcal{D}$ <u>are replaced by</u> a_-^{-1}, a_-^{*-1}, $\mathcal{D}_-$.

3) <u>The adjoint of</u> a^{-1} <u>is</u> a_-^{*-1}; <u>that is</u>

$$(a^{-1}v, u) = (v, a_-^{*-1}u) \text{ for } v \in \mathcal{D}^t(K),\ u \in \mathcal{D}_-^{-s}(K').$$

<u>Note</u> 93.1. Chapter VII shows that this theorem holds on a manifold for any hyperbolic operator and any K "compact toward the past".

In particular Proposition 92.2 shows immediately that:

<u>Note</u> 93.2. Theorem 93. holds when we replace the vector space X by $\mathcal{E}_-(D)$, D being a domain of X (assumption 94.1 is made).

§5. The emission front is a locus of trajectories

Using more precise assumptions than S. C. Zaremba and A. Marchaud, we prove what A. Marchaud proved partially: the emission front is a characteristic; in other words: the emission front is a locus of trajectories defined by Jacobi's, Lagrange's or Hamilton's equations or by the extremal principle of the particle mechanics (about these notions see E. Cartan's fundamental treatise: «Invariants intégraux» [45]). Our proof of Theorem 97 follows a way indicated by E. Cartan [45], chapter XIX, in particular n°195-196; unfortunately his n°196 is wrong from p. 198 «On voit immédiatement que ...».

94. Emission front, characteristics and bicharacteristics. (Notation: see n°69).

Definition 94.1. The emission front $\mathcal{F}(Y)$ is the part of the boundary $\dot{\mathcal{E}}(Y)$ of $\mathcal{E}(Y)$ exterior to Y.

Definition 94.2. The characteristics of a(p, x) [or a(p, x)] are the hyper-surfaces w(x) = 0 satisfying the condition

$$pw \in \pm \dot{\Gamma}_x \qquad (\dot{\Gamma}_x = \text{boundary of } \Gamma_x);$$

they satisfy the homogeneous first order equation

(94.1) $$h(x, pw) = 0 \qquad \text{(Jacobi's equation).}$$

Assumption 94.1. We assume that $h(x, \xi)$ has continuous first order derivatives and that the total curvature of $\dot{\Gamma}_x$ is > 0.

We introduce the equation g(x, x') of $\dot{C}_x$, that is the tangential equation of $\dot{\Gamma}_x$:

Definition 94.3. We denote by g(x, x') the function obtained by eliminating ξ from

(94.2) $$x'_\lambda = \frac{\partial h(x, \xi)}{\partial \xi_\lambda}, \quad g(x, x') = \sum_\lambda \xi_\lambda \frac{\partial h(x, \xi)}{\partial \xi_\lambda} - h(x, \xi) = (m-1)h(x, \xi)$$

g(x, x') is defined at least when x' is near $\dot{C}_x$, since the functions

$\frac{\partial h(x, \xi)}{\partial \xi_\lambda}$ are independent on the hypersurface $\dot{\Gamma}_\xi$ whose curvature is > 0.

We have

$$(94.3)\quad \xi_\lambda = \frac{\partial g(x, x')}{\partial x'_\lambda}, \quad h(x, \xi) = \sum_\lambda x'_\lambda \frac{\partial g(x, x')}{\partial x'_\lambda} - g(x, x') = \frac{1}{m-1} g(x, x')$$

$$(94.4)\qquad \frac{\partial h(x, \xi)}{\partial x_\lambda} = - \frac{\partial g(x, x')}{\partial x_\lambda};$$

$g(x, x')$ is homogeneous of degree $\frac{m}{m-1}$ with respect to x'.

__Proof:__ (94.2) gives

$$dg = \sum_\lambda \xi_\lambda \, dx'_\lambda - \sum_\lambda \frac{\partial h(x, \xi)}{\partial x_\lambda} dx_\lambda;$$

hence (94.3) and (94.4).

__Definition__ 94.4. The bicharacteristics of $a(x, p)$ [or $a(p, x)$] have the four following definitions, whose equivalence results obviously from (94.2), (94.3), (94.4):

1) they are __the characteristics of the Jacobi equation__ (94.1)

2) they are the solutions of the __Hamilton equations__

$$(94.5)\qquad \frac{dx_\lambda}{\frac{\partial h(x, \xi)}{\partial \xi_\lambda}} = - \frac{d\xi_\mu}{\frac{\partial h(x, \xi)}{\partial x_\mu}}, \quad h(x, \xi) = 0 \quad (\lambda, \mu = 1, 2, \ldots, \ell)$$

3) they are the solutions of the __Lagrange equations__

$$(94.6)\qquad \frac{d}{ds} \frac{\partial g(x, x')}{\partial x'_\lambda} - \frac{\partial g(x, x')}{\partial x_\lambda} = 0, \quad \frac{d}{ds} x_\lambda = x'_\lambda, \quad g(x, x') = 0;$$

4) they are the __extremals__ of $\int g(x, dx)^{(m-1)/m}$ satisfying $g(x, dx) = 0$.

95. __A choice of the coordinates.__ We choose the first axis of Ξ interior to $\Gamma_X = \bigcap_{x \in X} \Gamma_x$: any line parallel to this axis cuts $\dot{\Gamma}_x$ at real distinct points; upon application of (28.1)

$x_1 > 0$ on C_x except at its vertex.

This choice was the choice of the hyperplanes x_1 = const.. We choose now for the other coordinates $x_2, \dots, x_\ell$ such functions of x that the direction

$$dx_1 > 0, \quad dx_2 = \dots = dx_\ell = 0 \text{ belongs to } C_x;$$

hence

$$\xi_1 > 0 \text{ on } \Gamma_x \text{ except at its vertex.}$$

Assuming

(95.1) $$h(x, \xi) \geq 0 \text{ for } \xi \in \Gamma_x$$

we obtain thus

(95.2) $$h(x, \xi) = 0,\ \xi_1 > 0,\ \frac{\partial h(x, \xi)}{\partial \xi_1} > 0 \text{ for } \xi \in \dot{\Gamma}_x,\ \xi \neq 0;$$

hence, upon application of (94.2) and (94.3)

(95.3) $$g(x, x') = 0,\ x'_1 > 0,\ \frac{\partial g(x, x')}{\partial x'_1} > 0 \text{ if } x' \subset \dot{C}_x,\ x' \neq 0;$$

therefore

(95.4) $$g(x, x') \geq 0 \text{ means that } x' \in C_x.$$

$g(x, x')$ has been defined when x' is near $\mathring{C}_x$; we extend its definition to any x' in such a way that:

(95.4) holds;

$g(x, x')$ is positively homogeneous of degree $\frac{m}{m-1}$ with respect to x'.

(95.5) $$\frac{\partial g(x, x')}{\partial x'_1} > 0 \text{ if } x' \subset C_x,\ x' \neq 0.$$

96. Timelike paths neighboring a given path.

Lemma 96.1. Along a timelike path $\widehat{yz}$ a parameter s can be used such that the first Lagrange equation holds:

$$\frac{d}{ds}\frac{\partial g(x, x')}{\partial x_1'} - \frac{\partial g(x, x')}{\partial x_1} = 0 , \tag{96.1}$$

and moreover

$$0 < \operatorname{Inf} \frac{d}{ds} x_1 , \quad \operatorname{Sup} \left| \frac{d}{ds} x_\lambda \right| < \infty, \tag{96.2}$$

$$x(0) = y, \quad x(1) = z. \tag{96.3}$$

<u>Proof</u>: x_1 increases along the path, which can be defined by the obviously lipschitzian functions:

$$x_2(x_1), \ldots, x_\ell (x_1);$$

thus the derivatives $x^*_\lambda = \frac{d}{dx_1} x_\lambda$ exist almost everywhere and are bounded. According to (95.5), $\operatorname{Inf} \frac{\partial g(x, x^*)}{\partial x^*_1} > 0$; thus the differential equation

$$\frac{d}{dx_1} \left(\varphi \frac{\partial g(x, x^*)}{\partial x^*_1}\right) - \varphi \frac{\partial g(x, x^*)}{\partial x_1} = 0$$

has solutions $\varphi(x_1)$ such that

$$0 < \operatorname{Inf} \varphi \leq \operatorname{Sup} \varphi < \infty.$$

Now $g(x, x^*)$ is homogeneous of degree $\frac{m}{m-1}$ in x^*; thus if we introduce the parameter s such that

$$ds = \varphi^{1-m} dx_1$$

and if we define $x'_\lambda = \frac{d}{ds} x_\lambda$, then the relations (96.1) and (96.2) hold. They remain true when we replace s by any linear function of s: (96.3) can be satisfied.

<u>Lemma</u> 96.2. Let x(s) be the previous parametric representation of the timelike path $\widehat{yz}$; $x(s) + \delta x(s)$ is also timelike if

$$\beta(||\delta x|| + ||\delta x'||) \leq \sum_\lambda \frac{\partial g(x, x')}{\partial x_\lambda} \delta x_\lambda + \sum_\lambda \frac{\partial g(x, x')}{\partial x'_\lambda} \delta x'_\lambda ; \tag{96.4}$$

where

$$||\delta x||^2 = \sum_\lambda (\delta x_\lambda)^2; \qquad \delta x' = \frac{d\delta x}{ds};$$

$\beta(\varepsilon)$ is an increasing function of $\varepsilon \geq 0$ such that

$$\varepsilon^{-1}\beta(\varepsilon) \longrightarrow 0 \text{ when } \varepsilon \longrightarrow 0;$$

$\beta(\varepsilon)$ depends only on $x(s)$.

<u>Proof.</u> $g(x, x') \geq 0$; therefore

$$g(x + \delta x, x' + \delta x') \geq \sum_\lambda \frac{\partial g}{\partial x_\lambda}(x + \theta\delta x, x' + \theta\delta x')\delta x_\lambda + \frac{\partial g}{\partial x'}(x + \theta\delta x, x' + \theta\delta x')\delta x'$$

$$\geq \sum_\lambda \frac{\partial g}{\partial x_\lambda}(x, x')\delta x_\lambda + \frac{\partial g}{\partial x'_\lambda}(x, x')\delta x'_\lambda - \beta(||\delta x|| + ||\delta x'||),$$

where $\beta(\varepsilon)$ has the above-said properties, since $\frac{\partial g(x, x')}{\partial x_\lambda}$ and $\frac{\partial g(x, x')}{\partial x'_\lambda}$ are continuous.

<u>Lemma</u> 96.3. Let $\widehat{yz}$ be a timelike path; let δy and δz be vectors of X.

1) If $\widehat{yz}$ is a bicharacteristic, there is a timelike path from $y + \delta y$ to $z + \delta z$ when

(96.5) $$\beta(||\delta z|| + ||\delta y||) \leq \sum_\lambda \frac{\partial g(z, z')}{\partial z'_\lambda}\delta z_\lambda - \frac{\partial g(y, y')}{\partial y'_\lambda}\delta y_\lambda;$$

$\beta(\varepsilon)$ is a function of ε such that

$$\varepsilon^{-1}\beta(\varepsilon) \longrightarrow 0 \text{ when } \varepsilon \longrightarrow 0;$$

$\beta(\varepsilon)$ depends only on the path $\widehat{yz}$.

2) If $\widehat{yz}$ is not a bicharacteristic, then there is a timelike path from $y + \delta y$ to $z + \delta z$ when $||\delta z|| + ||\delta y||$ is small.

<u>Proof.</u> - <u>Notation.</u> Let $x(s)$ be the parametric representation of $\widehat{yz}$ defined by Lemma 96.1; define $f_\lambda(s)$ as follows:

(96.6) $$\frac{d}{ds} f_\lambda(s) = \frac{\partial g(x, x')}{\partial x_\lambda}; \quad \int_0^1 [f_\lambda(s) - \frac{\partial g(x, x')}{\partial x'_\lambda}]ds = 0;$$

in particular, according to (96.1) and (95.5)

(96.7) $$f_1(s) = \frac{\partial g(x, x')}{\partial x'_1} > 0.$$

Let $x(s) + \delta x(s)$ be a path from $y + \delta y$ to $z + \delta z$ defined by the data

(96.8) $$\delta x_2(s), \ldots, \delta x_\ell(s), \quad \text{such that} \quad \delta x_\lambda(0) = \delta y_\lambda, \quad \delta x_\lambda(1) = \delta z_\lambda \quad (\lambda = 2, \ldots, \ell)$$

and by the condition

$$\sum_\lambda \frac{\partial g(x, x')}{\partial x_\lambda} \delta x_\lambda + \frac{\partial g(x, x')}{\partial x'_\lambda} \delta x'_\lambda = \rho, \qquad (\rho = \text{const.});$$

thus, upon application of (96.6), δx_1 is given by the relation

$$\sum_\lambda [f_\lambda(s)\delta x_\lambda - f_\lambda(0)\delta y_\lambda] + \int_0^s \sum_{\lambda>1} [\frac{\partial g(x, x')}{\partial x'_\lambda} - f_\lambda(s)]\delta x'_\lambda \, ds = \rho s;$$

the condition $\delta x_1(0) = \delta y_1$ is satisfied; but we have to satisfy the condition

$$\delta x_1(1) = \delta z_1,$$

that is: ρ has to be chosen as follows

(96.9) $$\rho = \sum_\lambda [f_\lambda(1)\delta z_\lambda - f_\lambda(0)\delta y_\lambda] + \int_0^1 \sum_{\lambda>1} [\frac{\partial g(x, x')}{\partial x'_\lambda} - f_\lambda(s)]\delta x'_\lambda \, ds .$$

This definition of the path $x(s) + \delta x(s)$ transforms (96.4) into

$$\beta(||\delta x|| + ||\delta x'||) \leq \rho ;$$

therefore Lemma 96.2 shows that <u>there is a timelike path from</u> $y + \delta y$ <u>to</u> $z + \delta z$ <u>if the data</u> (96.8) <u>and the number</u> ρ <u>defined by</u> (96.9) <u>satisfy</u>

(96.10) $$\beta\,(\text{const.}\,|\rho| + \text{const.} \sum_{\lambda>1} |\delta x_\lambda| + |\delta x'_\lambda|) \leq \rho.$$

<u>Proof of</u> 1) If $x(s)$ is a bicharacteristic, then, upon application of (94.6), (96.6) becomes

$$f_\lambda = \frac{\partial g(x, x')}{\partial x'_\lambda} ;$$

hence (96.9) becomes

$$\rho = \sum_{\lambda} \left(\frac{\partial g(z, z')}{\partial z'_{\lambda}} \delta z_{\lambda} - \frac{\partial g(y, y')}{\partial y'_{\lambda}} \delta y_{\lambda}\right);$$

let us choose for $\lambda > 1$

$$\delta x_{\lambda}(s) = (1 - s)\,\delta y_{\lambda} + s\,\delta z_{\lambda};$$

thus

$$|\rho| < \text{const.}(||\delta y|| + ||\delta z||);$$

therefore, if

$$\beta\,(\text{const.}||\delta y|| + \text{const.}||\delta z||) \leq \sum_{\lambda} \left(\frac{\partial g(z, z')}{\partial z'_{\lambda}} \delta z_{\lambda} - \frac{\partial g(y, y')}{\partial y'_{\lambda}} \delta y_{\lambda}\right),$$

then (96.10) is satisfied and there is a timelike path from $y + \delta y$ to $z + \delta z$.

<u>Proof of</u> 2) <u>when</u> $\int_0^1 \sum_{\lambda} \left[\frac{\partial g(x, x')}{\partial x'_{\lambda}} - f_{\lambda}(s)\right]^2 ds > 0$.

Choose for $\lambda > 1$

$$\delta x_{\lambda} = (1 - s)\,\delta y_{\lambda} + s\,\delta z_{\lambda} + \gamma[||\delta y|| + ||\delta z||] \int_0^s \left[\frac{\partial g(x, x')}{\partial x'_{\lambda}} - f_{\lambda}(s)\right] ds,$$

γ = const.; according to (96.9) we can choose γ so large that

$$0 < \text{const.}[||\delta y|| + ||\delta z||] < \rho < \text{const.}[||\delta y|| + ||\delta z||];$$

then, if $||\delta y|| + ||\delta z||$ is small, (96.10) is satisfied and there is a timelike path from $y + \delta y$ to $z + \delta z$.

<u>Proof of</u> 2) <u>when</u> $\frac{\partial g(x, x')}{\partial x'_{\lambda}} = f_{\lambda}(s)$ <u>almost everywhere</u>.

From this assumption follows that the functions $x_{\lambda}(s)$ have bounded second derivatives, since the functions $f_{\lambda}(s)$ have bounded first derivatives; thus $x(s)$ satisfies

$$\frac{d}{ds} \frac{g(x, x')}{\partial x'_{\lambda}} = \frac{\partial g(x, x')}{\partial x_{\lambda}};$$

hence

$$\frac{d}{ds} g(x, x') = \sum_{\lambda} \left(x'_{\lambda} \frac{d}{ds} \frac{\partial g}{\partial x'_{\lambda}} + x''_{\lambda} \frac{\partial g}{\partial x'_{\lambda}}\right) = \frac{d}{ds} \sum_{\lambda} x'_{\lambda} \frac{\partial g}{\partial x'_{\lambda}} = \frac{m}{m-1} \frac{dg}{ds};$$

hence

$$\frac{d}{ds} g(x, x') = 0.$$

Now $x(s)$ is assumed to be timelike and not to be a bicharacteristic; thus

$$g(x, x') = \text{const.} > 0;$$

therefore $x(s) + \delta x(s)$ is timelike if δx and $\delta x'$ are small.

97. The properties of the emission front. Notation: Y is a subset of X; $\dot{\mathcal{E}}(Y)$ is the boundary of $\mathcal{E}(Y)$; $\complement\,\mathcal{E}(Y)$ is the complement of $\mathcal{E}(Y)$; x', y', z' are the tangents of the path $\widehat{yz}$ at x, y, z.

Lemma 97. Let $\widehat{yz}$ be a timelike path such that

$$y \in \mathcal{E}(Y), \quad z \in \hat{\mathcal{E}}(Y);$$

let x be a point of $\widehat{yz}$ different from y and z. Then

1) $\widehat{yz}$ is a bicharacteristic;

2) the semi-tangents δz of $\complement\,\mathcal{E}(Y)$ at z satisfy

$$\sum_\lambda \frac{\partial g(z, z')}{\partial z'_\lambda} \delta z_\lambda \leq 0;$$

3) the semi-tangents δy of $\mathcal{E}(Y)$ at y satisfy

$$\sum_\lambda \frac{\partial g(y, y')}{\partial y'_\lambda} \delta y_\lambda \geq 0;$$

4) $\widehat{yz}$ is on $\dot{\mathcal{E}}(Y)$.

5) $\dot{\mathcal{E}}(Y)$ has at x the tangent hyperplane

$$\sum_\lambda \frac{\partial g(x, x')}{\partial x'_\lambda} \delta x_\lambda = 0.$$

Proof of 1). Assume $y \in \mathcal{E}(Y)$, $\widehat{yz}$ timelike and not bicharacteristic; Lemma 96.3.2 shows that z is interior to $\mathcal{E}(Y)$.

Proof of 2). Choose $\delta y = 0$ in Lemma 96.3.1; we see that

$$z + \delta z \in \mathcal{E}(Y) \text{ when } \beta(||\delta z||) \leq \sum \frac{\partial g(z, z')}{\partial z'_\lambda} \delta z_\lambda .$$

Proof of 3). Assume this assertion to be false: there is a δy such that

$$y + \delta y \in \mathcal{E}(Y), \quad \sum_\lambda \frac{\partial g(y, y')}{\partial y'_\lambda} \delta y_\lambda < -\beta(||\delta y||);$$

thus (96.5) holds when δz is small: z is interior to $\mathcal{E}(Y)$.

Proof of 4). According to 3), y cannot be interior to $\mathcal{E}(Y)$:

$$y \in \mathring{\mathcal{E}}(Y).$$

We can replace y by any point of $\widehat{yz}$.

Proof of 5). x is both the origin and the end of arcs of a bicharacteristic belonging to $\mathcal{E}(Y)$ and to $\mathring{\mathcal{E}}(Y)$; thus according to 2)

$$\sum_\lambda \frac{\partial g(x, x')}{\partial x'_\lambda} \delta x_\lambda \leq 0$$

and according to 3)

$$\sum_\lambda \frac{\partial g(x, x')}{\partial x'_\lambda} \delta x_\lambda \geq 0.$$

Hence, any semi-tangent of $\dot{\mathcal{E}}(Y)$ at x satisfies

$$\sum_\lambda \frac{\partial g(x, x')}{\partial x'_\lambda} \delta x_\lambda = 0;$$

the converse is true since in X any line parallel to the first axis cuts $\dot{\mathcal{E}}(Y)$.

Theorem 97. *K is a compact subset of X. Assumption 94.1 is made.*

1) *$\mathcal{F}(K)$ is generated by arcs of bicharacteristics satisfying (94.5); the origin of such an arc is a point x of K where the inequality*

$$\delta x \circ \xi \geq 0$$

holds for any $\delta x \in C_x$ and also for any semi-tangent δx of K.

2) *Along such an arc, $\mathcal{F}(K)$ has a tangent hyperplane*

$$\delta x \circ \xi = 0 \quad [\delta x\text{: semi-tangent of } \mathcal{F}(K) \text{ at } x]$$

and thus is a characteristic.

3) *At the end of such an arc the inequality*

$$\delta x \circ \xi \leq 0$$

holds for any semi-tangent δx of $\mathcal{F}(K)$ at x.

Proof. Since K is compact, $\mathcal{E}(K)$ is closed ($n^{o}90$): any point z of $\mathcal{F}(K)$ is the end of a timelike path $\widehat{yz}$ such that $y \subset K$. We apply Lemma 97.

Chapter VII

The Inverses of a Hyperbolic Operator on a Manifold

Chapter VII states the main theorems about the linear hyperbolic differential equation (n^o99, 100, 103, 106, 108, 109). §1 defines a hyperbolic operator on a manifold X; for instance the operator $\frac{d}{dx}$ has inverses and is hyperbolic on a line, but has no inverses and is not hyperbolic on a circle: we assume that a timelike path is never closed and, more generally, that the set of the timelike paths from x to y is compact or void for any x and y $\in$ X. §2 extends Theorem 93 to manifolds (such an extension of the preliminary Theorem 89 is obviously impossible). §3 defines the elementary solution; §4 deals with Cauchy's problem, which has now less interest than it had for the local solutions of analytic equations (n^o72).

§1. Hyperbolic operators and emission

98. Definition of a hyperbolic operator on a manifold. Let X be an ℓ-dimensional (m+M)-smooth manifold; X is not necessarily complete. We use the definition given in n^o69 of an operator a(x, p) or a(p, x) hyperbolic at the point x; C_x is now a convex half cone belonging to the tangent hyperplane of X at x; its dual half cone Γ_x is in the dual hyperplane Ξ;

$$h(x, \xi) = 0 \text{ on } \Gamma_x \qquad (\xi \in \Xi),$$

h(x, ξ) being the principal part of a(x, ξ). m denotes the order of a(x, p) or a(p, x); m > 1.

A path is said to be timelike when its semi-tangents at x belong to C_x. Definition 90.1 of the emission $\mathcal{E}$ and Definition 91.2 of the functional spaces $\mathcal{D}^s(K)$ and $\mathcal{D}^s_-(K)$ are used ($dx_1 \ldots dx_\ell$ being now the measure of an element of X); they are also denoted by $\mathcal{D}^s(X, K)$ and $\mathcal{D}^s_-(X, K)$.

X and thus (Fréchet) the set of its paths are metric spaces.

Now a(x, p) [or a(p, x)] is said to be hyperbolic on X when the following conditions hold:

1) a(x, p) is hyperbolic at any point x of X.

2) the set of the timelike paths from y to z is compact or void for any y and z $\in$ X.

3) either the coefficients of a(x, p) have locally bounded (that is: bounded on any compact subset of X) derivatives of order M $(1 \leq M \leq \ell')$ or they have locally bounded derivatives of order $\leq \ell'$ and locally square integrable derivatives of order $> \ell'$ and $\leq$ M $(\ell \leq M)$. (ℓ' is the smallest integer $> \frac{\ell}{2}$).

4) the total curvature of $\dot{\Gamma}_x$ is > 0; if M = 1, then the first derivatives of the coefficients of h(x, ξ) are continuous.

Note 98.1. If X is complete and if the lengths of the paths from y to z are bounded, then Condition 3) holds.

Note 98.2. Condition 4) is required by Lemma 97, Theorem 97 and Theorem 99.1.1), which is used in §2. Now §2 holds when Condition 4) is not satisfied, provided that Theorem 99.1.1) holds.

Note 98.3. Let D be a domain of X; $\mathcal{E}(D)$, $\mathcal{E}_-(D)$, $\mathcal{E}(D) \cap \mathcal{E}_-(D)$ are manifolds (Theorem 99.1.1)) on which a(x, p) is still hyperbolic.

99. The properties of the emission.

Lemma 99.1. Let y and z be two points of X; they have compact neighborhoods V(y) and V(z) such that the set of the timelike paths from V(y) to V(z) is compact or void.

Proof: The cone Γ_x cannot be a cylinder, since it has no singular generators and is not a hyperplane; thus its dual cone C_x has an interior. Hence: there are y* near y and z* near z such that $\mathcal{E}(y^*)$ and $\mathcal{E}_-(z^*)$ are respectively neighborhoods of y and z. Now any path from $\mathcal{E}(y^*)$ to $\mathcal{E}_-(z^*)$ is an arc of a path from y^* to z^*; the set of the paths from y^* to z^* is assumed to be compact. The Lemma is now obvious. Hence

Lemma 99.2. Let Y and Z be two compact subsets of X. The set of the timelike paths from Y to Z is compact or void.

Lemma 97 holds on a manifold.

Proof: The proof given by n°97 holds on a manifold, when $\widehat{yz}$ is small: therefore if $y \in \mathcal{E}(Y)$ and $z \in \dot{\mathcal{E}}(Y)$, then $\widehat{yz}$ is on $\dot{\mathcal{E}}(Y)$ near z. Now this involves that the whole path $\widehat{yz}$ is on $\dot{\mathcal{E}}(Y)$ and satisfies Lemma 97.

Theorem 99.1. (Emission of open and compact subsets). 1) If D is a domain of X, then $\mathcal{E}(D)$ is a domain.

2) If Y and Z are compact subsets of X, then $\mathcal{E}(Y)$ and $\mathcal{E}_-(Z)$ are closed; $\mathcal{E}(Y) \cap \mathcal{E}_-(Z)$ is compact.

Proof of 1): Let $z \in \mathcal{E}(D)$: there is a timelike path $\widehat{yz}$ such that $y \in D$; hence $y \notin \dot{\mathcal{E}}(D)$ and $z \notin \dot{\mathcal{E}}(D)$, according to Lemma 97.4.

Proof of 2): Lemma 99.2 (and an easy process used by Zaremba and Marchaud) show that $\mathcal{E}(Y) \cap \mathcal{E}_-(Z)$ is compact. Therefore $\mathcal{E}(Y)$ and $\mathcal{E}_-(Z)$ are necessarily closed.

Theorem 99.2. (Semi-continuity of the emission). Assume that Y and C_x are the limits of Y_α and $C_{x,\alpha}$ when $\alpha \longrightarrow 0$ and that

$$Y \subset Y_\alpha, \qquad C_x \subset C_{x,\alpha}, \quad (Y,\ Y_\alpha \text{ compact});$$

let $\mathcal{E}_\alpha$ be the emission defined by $C_{x,\alpha}$; then

$$\mathcal{E}(Y) \text{ is the limit of } \mathcal{E}_\alpha(Y_\alpha).$$

Proof: The same as by Marchaud and Zaremba, Lemma 99.2 being used.

Theorem 97 (The emission front is a characteristic, that is a locus of trajectories) holds on a manifold.

Its proof holds, since Lemma 97 holds and $\mathcal{E}(K)$ is closed.

100. Compactness toward the past.

Definition 100. A subset K of X is said to be compact toward the past when $K \cap \mathcal{E}_-(x)$ is compact or void for any $x \in X$.

Theorem 100.1. Any compact subset of X is compact toward the past.

Proof: $\mathcal{E}_-(x)$ is closed.

Theorem 100.2. If K is compact towards the past, then

1) K is closed;

2) any closed subset of K is compact toward the past;

3) $\mathcal{E}(K)$ is also compact towards the past.

4) $K \cap \mathcal{E}_-(K')$ and $\mathcal{E}(K) \cap \mathcal{E}_-(K')$ are compact when K' is compact.

Proof of 1). Any $z \in X$ is interior to some $\mathcal{E}_-(z^*)$.

2) is obvious.

Proof of 3). The union of the timelike paths from K to x is

$$\mathcal{E}(K) \cap \mathcal{E}_-(x) = \mathcal{E}(K') \cap \mathcal{E}_-(x)$$

where $K' = K \cap \mathcal{E}_-(x)$ is compact; now $\mathcal{E}(K') \cap \mathcal{E}_-(x)$ is compact according to Theorem 99.1.2.

Proof of 4). There is z^* near $z \subset X$ such that $\mathcal{E}_-(z^*)$ is a neighborhood of z. Let us cover K' by a finite number of $\mathcal{E}_-(z^*_\alpha)$:

$$K' \subset \bigcup_\alpha \mathcal{E}_-(z^*_\alpha);$$

hence

$$\mathcal{E}_-(K') \subset \bigcup_\alpha \mathcal{E}_-(z^*_\alpha)$$

$$K \cap \mathcal{E}_-(K') \subset \bigcup_\alpha K \cap \mathcal{E}_-(z^*_\alpha),$$

which is compact. Moreover K and $\mathcal{E}_-(K')$ are closed. Thus $K \cap \mathcal{E}_-(K')$ is compact. $\mathcal{E}(K) \cap \mathcal{E}_-(K')$ is also compact, since $\mathcal{E}(K)$ is compact toward the past.

Theorem 100.3. Assume that: S is compact toward the past; $T \subset S$; T is closed; at any point x of S exterior to T, S has at least a semi-tangent belonging to $-C_x$. Then $S \subset \mathcal{E}(T)$.

We do not use this Theorem, which follows easily from the property 3) of the emission (n°90). We need the easier

Theorem 100.4. Let S be compact toward the past; let T be the set of the points x such that

$$S \cap \mathcal{E}_-(x) = x.$$

Then: $S \subset \mathcal{E}(T)$.

Proof: Let $y \in S$; consider the timelike paths $\widehat{xy}$ such that $x \in S$; such a path is maximal (that is: does not belong to another such path) if $x \in T$; there are maximal paths, since $S \cap \mathcal{E}_-(y)$ is compact; hence $y \in \mathcal{E}(T)$.

§2. The inverses of a hyperbolic operator

§2 extends Theorem 93 to manifolds.

101. The open subsets of X where a^{-1} exists. Notation. X is a manifold; $a(x, p)$ is hyperbolic on X ($n^{o}98$); K is a subset of X compact towards the past; D is an open subset of X such that $D = \mathcal{E}_{-}(D)$.

Thus: $a(x, p)$ is hyperbolic on D; $D \cap K$ is compact towards the past on D.

We denote $\mathcal{D}^{s}(D, D \cap K)$ (see $n^{o}98$) by $\mathcal{D}^{s}(D, K)$; we say that a^{-1} exists for (D, K) if there is a mapping a^{-1} such that, for any (s, t) satisfying

$$s - t + 1 = m, \qquad -M < t \leq M, \text{ we have:}$$

$$a^{-1}: \mathcal{D}^{t}(D, K) \longrightarrow \mathcal{D}^{s}(D, K) \quad \text{(continuously);}$$

$$aa^{-1} = 1 \text{ on } \mathcal{D}^{t}(D, K); \qquad a^{-1}a = 1 \text{ on } \mathcal{D}^{s+1}(D, K);$$

$$S(a^{-1}v) \subset \mathcal{E}\, S(v).$$

Hence: the restriction of $a^{-1}v$ to a domain $D' \subset D$ depends only on the restriction of v to the domain $\mathcal{E}_{-}(D')$.

Let us give some easy consequences of this definition of a^{-1}:

Lemma 101.1. If a^{-1} exists for (D, K), then a^{-1} is unique.

Proof: Let b be a mapping such that

$$b: \mathcal{D}^{t}(D, K) \longrightarrow \mathcal{D}^{s}(D, K) \quad \text{(continuously),}$$

$$ab = 1 \text{ on } \mathcal{D}^{t}(D, K).$$

If $1 - M < t \leq M$, then $b = a^{-1}ab = a^{-1}$ on $\mathcal{D}^{t}(D, K)$.

Now $\mathcal{D}^{t}(D, K)$ is dense on $\mathcal{D}^{t-1}(D, K)$, where b and a^{-1} are continuous; hence $b = a^{-1}$ on $\mathcal{D}^{t-1}(D, K)$.

Lemma 101.2. If a^{-1} exists for (D, K), then a^{-1} exists for (D', K'), where $D' = \mathcal{E}_{-}(D') \subset D$, $K' \subset K$.

<u>Proof</u>. Let v' be the restriction to D' of a function v defined on D; we define

$$a^{-1}v' = (a^{-1}v)'.$$

<u>Lemma</u> 101.3. a^{-1} exists for (D, K) when D is the complement of $\mathcal{E}(K)$.

<u>Proof</u>: D is open, $D = \mathcal{E}_-(D)$; $\mathcal{D}^t(D, K)$ contains only the function $v = 0$; $a^{-1}v = 0$.

<u>Lemma</u> 101.4. K being given, there is a <u>maximal</u> D such that a^{-1} exists for (D, K) (that is: if a^{-1} exists for (D', K), then $D' \subset D$).

<u>Proof</u>: Let D be the union of the D' such that a^{-1} exists for (D', K); obviously $D = \mathcal{E}_-(D)$; if $v \in \mathcal{D}^t(D, K)$, define $a^{-1}v$ as follows:

$$(a^{-1}v)' = a^{-1}v' \quad (v': \text{restriction of } v \text{ to } D').$$

This definition is coherent: indeed $a^{-1}v'$ and $a^{-1}v''$ have the same restriction to $D' \cap D''$ (Lemmas 101.1 and 101.2).

<u>Lemma</u> 101.5. If the function φ is defined on D and has locally bounded derivatives of order $m + M$, if $S(1 - \varphi)$ is compact towards the past, if a^{-1} exists for $(D, S(1 - \varphi))$ and for (D', K) where $D \supset D' = \mathcal{E}_-(D') \supset S(\varphi)$, then a^{-1} <u>exists for</u> (D, K).

<u>Proof</u>: Let $v \in \mathcal{D}^t(D, K)$; let v' be its restriction to D'; $a^{-1}v' \subset \mathcal{D}^s(D', K)$; define a mapping b as follows:

$$bv = \varphi a^{-1}v' \text{ on } D'; \; bv = 0 \text{ on } D \text{ outside } S(\varphi).$$

This mapping has the following properties:

(1) $\qquad b: \mathcal{D}^t(D, K) \longrightarrow \mathcal{D}^s(D, K);$

(2) $\qquad bau = \varphi u$ if $u \in \mathcal{D}^{s+1}(D, K)$;

$\qquad abv - \varphi v = c(x, p)a^{-1}v'$

$c(x, p)$ having the order $s - t$; its coefficients have derivatives of order M and are $= 0$ outside $S(1 - \varphi)$; hence

(3) $1 - ab\colon \mathcal{D}^t(D, K) \longrightarrow \mathcal{D}^t(D, K')$ where $K' = S(1 - \varphi) \cap \mathcal{E}(K)$.

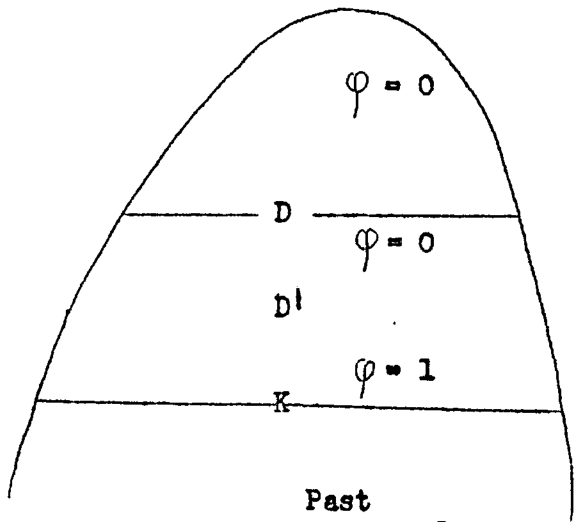
Now a^{-1} exists for (D, K'); let it be denoted by c:

(4) $$c\colon \mathcal{D}^t(D, K') \longrightarrow \mathcal{D}^s(D, K');$$

(5) $$ca = 1 \text{ on } \mathcal{D}^{s+1}(D, K');$$

(6) $$ac = 1 \text{ on } \mathcal{D}^t(D, K').$$

Let us show that a^{-1} exists for (D, K) and is the continuous mapping defined by



(7) $$a^{-1} = c(1 - ab) + b.$$

(1), (3) and (4) show that

$$a^{-1}\colon \mathcal{D}^t(D, K) \longrightarrow \mathcal{D}^s(D, K);$$

obviously

$$S(a^{-1}v) \subset \mathcal{E}(Sv);$$

(2) and (5) show that

$$a^{-1}a = c(a - aba) + ba = ca(1 - \varphi) + \varphi = (1 - \varphi) + \varphi = 1 \text{ on } \mathcal{D}^{s+1}(D, K);$$

finally (3) and (6) show that

$$aa^{-1} = ac(1 - ab) + ab = (1 - ab) + ab = 1 \text{ on } \mathcal{D}^t(D, K).$$

102. <u>The existence of a^{-1} on X.</u> Note 93.2 shows that a^{-1} exists for (D, K), when X is a vector space, $a(x, p)$ is regularly hyperbolic and $D = \mathcal{E}_-(D)$. Now any function $u \subset \mathcal{D}^s(D, K)$ is defined on D and $= 0$ outside $\mathcal{E}(K)$; thus a^{-1} still exists when we change X, D, K, $a(x, p)$ outside $D \cap \mathcal{E}(K)$. Hence:

<u>Lemma</u>: Let x be a point of the manifold X; let $a(x, p)$ be hyperbolic on X; if $D = \mathcal{E}_-(D)$ and if $D \cap \mathcal{E}(K)$ belongs to some neighborhood of x, then a^{-1} exists

for (D, K).

Now $\mathcal{E}_-$ and $\mathcal{E}$ are semi-continuous (Theorem 99.2); $\mathcal{E}_-(x) \cap \mathcal{E}(x) = x$; hence $\mathcal{E}_-(V) \cap \mathcal{E}(K)$ belongs to a given neighborhood of x when V and K are sufficiently near x. Remember that $\mathcal{E}_-(V)$ is open when V is open (Theorem 99.1.1). Hence:

<u>Lemma</u> 102.1. Let x be a point of the manifold X; if the open subset V and the compact subset K of X are sufficiently near x, then a^{-1} exists for $(\mathcal{E}_-(V), K)$.

<u>Lemma</u> 102.2. If a^{-1} exists for (D, K) and $D \neq X$, there is an open subset D' of X such that:

$D' \not\subset D$; a^{-1} exists for (D', K).

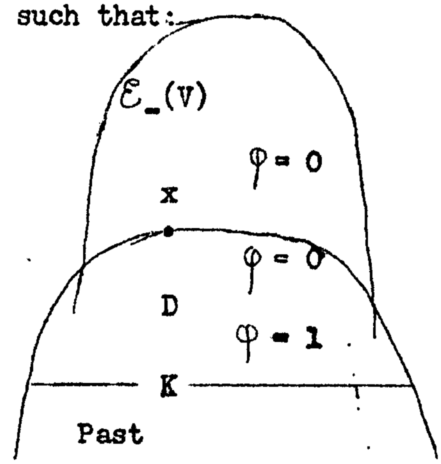
<u>Proof</u>: The Lemma is obvious, if D does not contain the complement of $\mathcal{E}(K)$: see Lemma 101.3. Assume that D contains this complement and apply Theorem 100.4 to the complement S of D (S being a closed subset of $\mathcal{E}(K)$ is compact towards the past: see Theorem 100.2.2): there is a point x such that

$$\mathcal{E}_-(x) \subset D \cup x, \qquad x \notin D.$$

Now $\mathcal{E}_-$ is semi-continuous; hence: if W is a given neighborhood of x, there is a neighborhood V of x such that

$$\mathcal{E}_-(V) \subset D \cup W.$$

Therefore, according to Lemma 102.1, there are two open neighborhoods V, W of x and a (m+M)-times differentiable function φ defined on X such that:



$S(1 - \varphi)$ is a compact neighborhood of x;

(1) a^{-1} exists for $(\mathcal{E}_-(V), S(1 - \varphi))$;

(2) $\mathcal{E}_-(V) \subset D \cup W$; $\varphi = 0$ on W.

Since a^{-1} exists for (D, K), Lemma 101.2 shows that:

(3) a^{-1} exists for $(D \cap \mathcal{E}_-(V), K)$;

moreover (2) shows that:

(4) $D \cap \mathcal{E}_-(V)$ contains the support of the restriction of φ to $\mathcal{E}_-(V)$.

Now (1), (3), (4) mean that the assumptions of Lemma 101.5 are satisfied, if in this Lemma D and D' are replaced by $\mathcal{E}_-(V)$ and $D \cap \mathcal{E}_-(V)$; hence: a^{-1} exists for $(\mathcal{E}_-(V), K)$. Now $\mathcal{E}_-(V) \not\subset D$ since $x \in V$, $x \notin D$.

Proposition 102. a^{-1} exists for (X, K), if K is compact towards the past.

Proof. The preceding Lemma shows that the maximal D defined by Lemma 101.4 is the manifold X itself.

103. Theorem 93 holds when X is a manifold, a(x, p) is hyperbolic on X (n°98) and K is compact towards the past.

The scalar product to be used on K is

$$(u, v) = \int \cdots \int_X u(x) v(x) \mu(x) \, dx_1 \ldots dx_\ell$$

$\mu(x) dx_1 \ldots dx_\ell$ being the measure of the element of X: Theorem 100.2 shows that (u, v) is defined when

$$u \in \mathcal{L}^s(K), \quad v \in \mathcal{D}^{-s}(K'),$$

K is compact towards the past and K' is compact (or K is compact and K' is compact towards the future).

The adjoint of a(x, p) is

$$a^*(p, x) = \frac{1}{\mu(x)} b(p, x) \mu(x)$$

where

$$b(-\xi, x) = a(x, \xi) \text{ when } x_\lambda \text{ and } \xi_\mu \text{ commute.}$$

Proof of Theorem 93.1: Proposition 102 gives the existence of a^{-1}. A similar Proposition holds when a, s, t, are replaced by a^*, $-t$, $-s$.

Proof of Theorem 93.2. Replace $\mathcal{E}$, a^{-1}, a^{*-1} by $\mathcal{E}_-$, a_-^{-1}, a_-^{*-1}, in the preceding proof.

Proof of Theorem 93.3. Assume either $v \in \mathcal{D}^{t+1}(K)$ if $t < M$ or $u \in \mathcal{D}_-^{-s+1}(K')$ if $1 - M < t$; thus either

$$a^{-1}v \in \mathcal{D}^{s+1}(K), \qquad a_-^{*-1}u \in \mathcal{D}_-^{-t}(K')$$

or

$$a^{-1}v \in \mathcal{D}^{s}(K), \qquad a_-^{*-1}u \in \mathcal{D}_-^{-t+1}(K');$$

hence, since $s - t + 1 =$ order of a and $\mathcal{E}(K) \cap \mathcal{E}_-(K')$ is compact:

$$(a^{-1}v,\ u) = (a^{-1}v,\ a^*a_-^{*-1}u) = (aa^{-1}v,\ a_-^{*-1}u) = (v,\ a_-^{*-1}u).$$

Now $\mathcal{D}^{t+1}(K)$ is dense in $\mathcal{D}^{t}(K)$ and $\mathcal{D}_-^{-s+1}(K')$ in $\mathcal{D}_-^{-s}(K')$; $(a^{-1}v,\ u)$ and $(v,\ a_-^{*-1}u)$ are continuous for $v \in \mathcal{D}^{t}(K)$, $u \in \mathcal{D}_-^{-s}(K')$; hence $(a^{-1}v,\ u) = (v,\ a_-^{*-1}u)$ for $v \in \mathcal{D}^{t}(K)$, $u \in \mathcal{D}_-^{-s}(K')$.

§3. The elementary solutions

(In the first part, Ch. III, §4, p. 66, the elementary solution and Dirac's measure were distributions k_x and δ_x on X; now they are distributions $k_{x,y}$ and $\delta_{x,y}$ on the product $X \times X$ of the manifold by itself: if X is a vector space, and $a(x, p) = a(p)$, then

$$k_{x,y} = k_{x-y}, \qquad \delta_{x,y} = \delta_{x-y}.)$$

104. Distributions on $X \times X$. Let us complete Ch. IV of Schwartz' treatise [21]. We use Y. Fourès-Bruhat's definitions; the manifold X and the functions $u(x)$, $v(x)$, $u(x, y)$ are (m+M)-time differentiable; these functions have compact supports; dx denotes the measure of the element of X.

Let $k_{x,y}$ be a distribution on $X \times X$; there is a distribution w_x on X such that

$$\int_X w_x u(x)\,dx = \int_{X\times X} k_{x,y} u(x) v(y)\,dx\,dy \text{ for any } u(x);$$

we write

(104.1) $$w_x = \int_X k_{x,y} v(y)\,dy;$$

thus the formula of the repeated integration holds:

(104.2) $$\int_X u(x)\,dx \int_X k_{x,y} v(y)\,dy = \int_{X\times X} k_{x,y} u(x) v(y)\,dx\,dy = \int_X v(y)\,dy \int_X k_{x,y} u(x)\,dx.$$

Assume now w_x to be a continuous function $w(x)$ of x for any (m+M)-times differentiable $v(y)$: x being given, there is on X a distribution h_y such that

$$w(x) = \int_X h_y v(y)\,dy$$

h_y is said to be the restriction of $k_{x,y}$ to $x \times X$; practically it is also denoted by $k_{x,y}$.

105. <u>Dirac's distribution</u> $\delta_{x,y}$ is defined on $X \times X$ by the formula $\int_{X\times X} \delta_{x,y} u(x,y)\,dx\,dy = \int_X u(x,x)dx$. Its support is the diagonal of $X \times X$.

$$u(x) = \int_X \delta_{x,y}\, u(y)dy = \int_X \delta_{y,x} u(y)dy;$$

thus $\delta_{x,y}$ has restrictions to $x \times X$ and $X \times x$.

If $a^*(p, x)$ is the adjoint of $a(x, p)$ and if $q = (\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_\ell})$ then

(105) $$a(x, p)\,\delta_{x,y} = a^*(q, y)\,\delta_{x,y}.$$

<u>Proof of</u> (105).

$$\int_{X\times X} u(x)v(y)[a(x, p)\,\delta_{xy}]dx\,dy = \int_{X\times X} v(y)\,\delta_{x,y}[a^*(p, x)u(x)]dx\,dy$$

$$= \int_X v(x)[a^*(p, x)u(x)]dx = \int_X u(x)[a(x, p)v(x)]dx$$

$$= \int_{X\times X} u(x)\,\delta_{x,y}[a(y, q)v(y)]dx\,dy = \int_{X\times X} u(x)v(y)[a^*(q, y)\,\delta_{x,y}]dx\,dy.$$

106. <u>The two elementary solutions of the adjoint hyperbolic operators</u> $a(x, p)$ <u>and</u> $a^*(p, x)$.

<u>Theorem</u> 106. <u>To</u> $\mathcal{E}$ <u>is associated an</u> « <u>elementary solution</u> » $k_{x,y}$ <u>of</u> a <u>and</u> a^*; $k_{x,y}$ <u>is a distribution on</u> $X \times X$ <u>having the following properties</u>:

1) <u>If</u> $(x, y) \in S(k_{x,y})$, <u>then there is a timelike path from</u> y <u>to</u> x.

2) <u>Let</u> K <u>be compact towards the past and</u> K' <u>be compact towards the future</u>:

$$a^{-1}v(x) = \int_X k_{x,y} v(y)dy \text{ if } v \in \mathcal{E}^{1-M}(K);$$

$$\underline{a^*}^{-1}u(y) = \int_X k_{x,y} u(x)dx \text{ if } u \in \underline{\mathcal{E}}^{1-m-M}(K').$$

3) $a(x, p)k_{x,y} = \delta_{x,y}$; $a^*(q, y)k_{x,y} = \delta_{x,y}$ $\quad (q = \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_\ell}))$

[hence another proof of (105)].

4) <u>If</u> $\ell - m < M$, <u>then</u> $k_{x,y}$ <u>has the following restriction to</u> $x \times X$:

$$k_{x,y} = \underline{a}^{*-1} \delta_{x,y}$$

<u>where</u> $\underline{a}^{*-1}$ <u>is the inverse of</u> $a^*(q, y)$, $q = (\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_\ell})$.

5) <u>If</u> $\ell < M$, <u>then</u> $k_{x,y}$ <u>has the following restriction to</u> $X \times y$:

$$k_{x,y} = a^{-1} \delta_{x,y}$$

<u>where</u> a^{-1} <u>is the inverse of</u> $a(x, p)$.

<u>Proof of</u> 2). According to Theorem 93.3

$$\int_X u(x) a^{-1} v(x) dx = \int_X v(y) \underline{a}^{*-1} u(y) dy$$

is a linear and continuous function of

$$u \in \mathcal{D}_{-}^{-s}(K') \text{ and } v \in \mathcal{D}^t(K),$$

that is of

$$w(x, y) = u(x) v(y)$$

($s - t + 1 = m$, $-M < t \leq M$, K or K' compact).

Obviously the definition of this linear function of w can be extended to any (m+M)-times differentiable w whose support is compact: there is a distribution $k_{x,y}$ on $X \times X$ such that

$$(106) \qquad \int_{X \times X} k_{x,y} u(x) v(y) dx\, dy = \int_X u(x) a^{-1} v(x) dx = \int_X v(y) \underline{a}^{*-1} u(y) dy.$$

Hence 2).

<u>Proof of</u> 1). Assume $S(u)$ and $S(v)$ to be compact; (106) and Theorem 93.1 show that

$$\int_{X \times X} k_{x,y} u(x) v(y) dx\, dy = 0$$

when $S(u)$ is outside $\mathcal{E} S(v)$.

<u>Proof of</u> 3). Replace in (106) v by av or u by a*u; apply Theorem 93.

<u>Proof of</u> 4). If $v \in \mathcal{D}^M(K)$, then $a^{-1} v \in \mathcal{D}^{m+M-1}(K) \in \mathcal{D}^\ell(K)$

and therefore $w(x) = a^{-1}v(x)$ is continuous; thus $k_{x,y}$ has a restriction to $x \times X$. Let

$$u \in \mathcal{D}_{-}^{-\ell}(K'), \qquad K' \text{ compact;}$$

then

$$\int_X u(y)w(y)\,dy = \int_X v(y)\,\underline{a}^{*-1}u(y)\,dx.$$

Now $\delta_{x,y} \in \mathcal{D}_{-}^{-\ell}(K')$ if $x \in K'$; let $u(y)$ tend to $\delta_{x,y}$ in $\mathcal{D}_{-}^{-\ell}(K')$; $\int_X u(y)w(y)\,dy$ tends to $w(x)$ and the preceding equation becomes

$$w(x) = \int_X h_y v(y)\,dy, \text{ where } h_y = \underline{a}^{*-1}\,\delta_{x,y};$$

that shows that $\underline{a}^{*-1}\,\delta_{x,y}$ is the restriction of $k_{x,y}$ to $x \times X$.

<u>Proof of</u> 5). Similar to the proof of 4).

§4. Cauchy's problem

(a^{-1} enables us to solve Cauchy's problem and to generalize it: the hypersurface carrying the data can be characteristic.)

107. Solution of $a(x, p)u(x) = 0$ with a given singularity. Notation. A hyperbolic operator $a(x, p)$ is given on the manifold X; $m \leq M$. A subset K of X is given; meas. $K = 0$; K is compact towards the past. A function $w(x)$ is given on a neighborhood of K; $w(x)$ is locally square integrable; its derivatives of order $\leq m$ are locally square integrable outside K. We denote by $[aw]$ the almost everywhere defined function such that

$$(107.1) \qquad [aw] = a(x, p)w(x) \text{ outside } K;$$

we denote by $\{aw\}$ the distribution

$$(107.2) \qquad \{aw\} = aw - [aw],$$

whose support belongs to K. We assume that $[aw]$ and its first derivatives are locally square integrable on a neighborhood of K.

The problem of finding a solution of $au = 0$ with the same singularity as w on K is the following.

Problem 107. We ask for a function $u(x)$ such that

$u(x)$ is defined on X;

$$S(u) \subset \mathcal{E}(K);$$

outside K, $u(x)$ and its derivatives of order $\leq m$ are locally square integrable and satisfy

$$a(x, p)u(x) = 0;$$

in the neighborhood of K, $u(x) - w(x)$ and its derivatives of order $\leq m$ are locally square integrable.

Proposition 107. The preceding problem has the unique solution

(107.3) $$u(x) = a^{-1}\{aw\}.$$

<u>Proof.</u> Let V be a neighborhood of K. We can assume that:

V is compact towards the past; $S(w) \subset V$; $w(x)$ is defined on X.

The function

$$v(x) = u(x) - w(x)$$

and its derivatives of order $\leq m$ are locally square integrable on X;

$$av = -[aw], \qquad S(v) \subset \mathcal{E}(V).$$

Hence (theorem 93)

$$v = -a^{-1}[aw];$$

that means: the problem has no other solution than

$$u = w - a^{-1}[aw] = a^{-1}\{aw\}.$$

Now this function u satisfies $S(u) \subset \mathcal{E}(K)$ and is actually a solution of the problem.

108. <u>Generalized Cauchy's problem.</u> <u>Note</u> 108. If $u(x)$ and its first derivatives are locally square integrable, then $u(x)$ has a locally square integrable restriction to any lipschitzian hypersurface K.

<u>Proof.</u> Let $x_1 = f(x_2, \ldots, x_\ell)$ be the equation of K;

$$u(x_1, x_2, \ldots, x_\ell) - \int_{f(x_2,\ldots,x_\ell)}^{x_1} \frac{\partial u(x_1,x_2,\ldots,x_\ell)}{\partial x_1}\, dx_1$$

is a locally square integrable function of $(x_2, \ldots, x_\ell)$, which is the restriction of $u(x)$ to K.

<u>Notation.</u> A hyperbolic operator $a(x, p)$ is given on the manifold X; $m \leq M$. A subset K of X is given; K is compact towards the past; K is the boundary of $\mathcal{E}(K)$. Hence, K is a lipschitzian hypersurface. [The boundary of any $\mathcal{E}(K')$, K' being compact towards the past, is such a subset K.] A function $w(x)$ is given on $\mathcal{E}(K)$;

$w(x)$ and its derivatives of order $\leq m$ are locally square integrable; the first derivatives of $a(x, p)w(x)$ are also square integrable;

$$a(x, p)w(x) = 0 \text{ on } K.$$

Generalized Cauchy's problem is the following

Problem 108. We ask for a function $u(x)$ defined on $\mathcal{E}(K)$ and such that: $u(x)$ and its derivatives of order $\leq m$ are locally square integrable;

$$a(x, p)u(x) = 0;$$

$u(x) - w(x)$ and its derivatives of order $< m$ are $= 0$ on K.

Theorem 108. The preceding problem has the unique solution

$$u(x) = a^{-1}\{aw\}$$

$\{aw\}$ and $[aw]$ being a function and a distribution defined on X by (107.1), (107.2), where

$$w(x) = 0 \text{ outside } \mathcal{E}(K).$$

Proof. Define $u(x) = 0$, $w(x) = 0$ outside $\mathcal{E}(K)$: Problem 108 becomes equivalent to problem 107.

109. Cauchy's problem is the problem of finding on $\mathcal{E}(K)$ a solution $u(x)$ of the equation

$$a(x, p)u(x) = 0,$$

with given values and derivatives of order $< m$ on the given hypersurface K, which satisfies the preceding conditions (n°108) and moreover the following one:

(109.1) K verifies nowhere the equation of the characteristics.

Theorem 109. Cauchy's problem has always a unique solution, if the data are so regular that (109.2) defines a function $w(x)$ whose derivatives of order $\leq m+1$ are locally square integrable.

<u>Proof.</u> Let $f(x) = 0$ be the equation of K; the assumption (109.1) is

$$h(x, pf) \neq 0.$$

Let $v(x)$ be a function with the given values and derivatives on K. According to Theorem 108 we have to find $w(x)$ such that:

$w(x) - v(x)$ and its derivatives of order $< m$ are $= 0$ on K;

$a(x, p)w(x) = 0$ on K.

We choose

$$w(x) = v(x) - \frac{f^m(x)}{m!\,h(x, pf)}\, a(x, p)v(x). \tag{109.2}$$

Chapter VIII

Hyperbolic systems

Chapter VIII explains briefly how the preceding properties of a hyperbolic equation can be extended to a hyperbolic system.

§1. Notation and results

110. Definition of a hyperbolic matrix. X is a manifold; $x \in X$; $(x_1, \ldots, x_\ell)$ are local coordinates of x; $p = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_\ell})$; $U(x) = (u_1(x), \ldots, u_n(x))$ and $V(x) = (v_1(x), \ldots, v_n(x))$ are vectors whose components are functions or distributions defined on X; $A(x, p)$ is a matrix whose element

$$a_{\tau\sigma}(x, p) \qquad (1 \leq \sigma \leq n,\ 1 \leq \tau \leq n)$$

is a linear differential operator of order

$$s(\sigma) - t(\tau) + 1;$$

$s(\sigma)$ and $t(\tau)$ are positive or negative integers depending on σ and τ; of course $s(\sigma)$ and $t(\tau)$ can be replaced by $s(\sigma)$ + const. and $t(\tau)$ + const.;

$$a_{\tau\sigma}(x, p) = 0 \text{ when } s(\sigma) - t(\tau) + 1 < 0.$$

$h_{\tau\sigma}(x, \xi)$ is the principal part of $a_{\tau\sigma}(x, \xi)$; the matrix whose element is $h_{\tau\sigma}(x, \xi)$ is denoted by

$$H(x, \xi) = \begin{pmatrix} H_1(x, \xi) & 0 & \cdot \\ 0 & H_2(x, \xi) & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}; \tag{110}$$

$$h_\lambda(x, \xi) = \det H_\lambda(x, \xi) \qquad (x \text{ and } \xi \text{ commute});$$

$$h(x, \xi) = \det H(x, \xi) = h_1(x, \xi) h_2(x, \xi) \cdots .$$

The matrix $A(x, p)$ is said to be hyperbolic at the point x when:

1) $h_1(x, p)$, $h_2(x, p)$, ... are hyperbolic at x;

2) the convex half cone $\Gamma_x(A) = \bigcap_\lambda \Gamma_x(h_\lambda)$ has an interior.

Of course $h(x, \xi) = 0$ on the boundaries of the cones $\Gamma_x(A)$ and $\Gamma_x(h_\lambda)$.

C_x denotes the cone dual to $\Gamma_x(A)$: C_x belongs to the vector space tangent to X at x.

A path is said to be timelike when its positive semi-tangents at x belong to C_x.

The matrix $A(x, p)$ is said to be hyperbolic on the manifold X when:

1) $A(x, p)$ is hyperbolic at any point $x \in X$;

2) the set of the timelike paths from y to z is compact or void for any y and $z \subset X$;

3) the total curvature of the boundary of $\Gamma_x(A)$ is > 0.

The definition 90.1 of the emission is used; its properties (n°99, 100) hold.

Let $s(\sigma)$ be a function which is defined for $\sigma = 1, \ldots, m$ and whose value is an integer (positive or negative); we denote by $\mathcal{B}^s(K)$ the vector space whose elements are the vectors

$$V(x) = (v_1(x), \ldots, v_m(x))$$

such that (see Definition 91.2)

$$v_\sigma \in \mathcal{B}^{s(\sigma)}(K).$$

Note. The coefficients of $A(x, p)$ are assumed to be locally bounded and to have locally square integrable derivatives of order $\leq M$. M has to be sufficiently large (in n°111 so large that the inequalities (115.1), (115.2), (115.3) hold).

111. The inverses of a hyperbolic matrix. Theorem 93 can be extended as follows:

A matrix $A(x, p)$, which is hyperbolic on a manifold X, has two inverses A^{-1} and A_-^{-1}. For any K compact towards the past

$$A^{-1}: \ \mathcal{D}^t(K) \longrightarrow \mathcal{D}^s(K);$$

$$A_-^{-1}: \ \mathcal{D}_-^t(K) \longrightarrow \mathcal{D}_-^s(K).$$

Let $A^*(p, x)$ be the adjoint of $A(x, p)$: when x and ξ commute

$$a^*_{\sigma\tau}(\xi, x) = a_{\tau\sigma}(x, -\xi).$$

The inverse A_-^{*-1} of A^* is the adjoint of the inverse A^{-1} of A.

112. The two elementary solutions. Let us denote by $\Delta_{x,y}$ Dirac's matrix of order m

$$\begin{pmatrix} \delta_{x,y} & 0 & \vdots \\ 0 & \delta_{x,y} & \vdots \\ \cdots & \cdots & \cdot \end{pmatrix}$$

where $\delta_{x,y}$ is Dirac's distribution ($n^o 105$). Theorem 106 can be extended as follows: To $\mathcal{E}$ is associated an elementary solution $K_{x,y}$ of $A(x, p)$ and $A^*(p, x)$; $K_{x,y}$ is a matrix; its rank is m; its elements are distributions defined on $X \times X$.

If $(x, y) \in S(K_{x,y})$, there is a timelike path from y to x.

$$A^{-1}V(x) = \int_X K_{x,y}\, V(y)\,dy$$

$$A_-^{*-1}U(y) = \int_X {}^tK_{x,y}\, U(x)\,dx \qquad ({}^tK = \text{transpose of } K)$$

$$A(x, p)K_{x,y} = \Delta_{x,y}; \quad A^*(q, y)\ {}^tK_{x,y} = \Delta_{x,y}, \text{ where}$$

$$q = \left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_\ell}\right);$$

$$K_{x,y} = A^{-1}\Delta_{x,y} = {}^t(A_-^{*-1}\Delta_{x,y}),$$

A^{-1} and A_-^{*-1} being the inverses of $A(x, p)$ and $A^*(q, y)$.

113. Cauchy's problem. Assume that:

$$t(\tau) \geq 0;$$

K is compact towards the past; K is the boundary of $\mathcal{E}(K)$.

Generalized Cauchy's problem. A vector

$$W(x) = (w_1(x), \ldots, w_n(x))$$

is given on $\mathcal{E}(K)$; $w_\sigma(x)$ and its derivatives of order $\leq s(\sigma) + 1$ are locally square integrable; $\sum_\sigma a_{\tau\sigma}(x, p)w_\sigma$ has locally square integrable derivatives of order $\leq t(\tau) + 1$; $\sum_\sigma a_{\tau\sigma}(x, p)w_\sigma$ and its derivatives of order $\leq t(\tau)$ are $= 0$ on K. We ask for a vector

$$U(x) = (u_1(x), \ldots, u_n(x))$$

defined on $\mathcal{E}(K)$ and such that: $u_\sigma(x)$ and its derivatives of order $\leq s(\sigma) + 1$ are locally square integrable;

$$A(x, p)U(x) = 0 \text{ on } \mathcal{E}(K);$$

$u_\sigma(x) - w_\sigma(x)$ and its derivatives of order $\leq s(\sigma)$ are $= 0$ on K.

Theorem 108 can be extended as follows: <u>The preceding problem has the unique solution</u>

$$U(x) = A^{-1}\{AW\}$$

<u>where $\{AW\}$ has the following definition:</u>

$W(x) = 0$ outside $\mathcal{E}(K)$; $[AW]$ is a vector, whose components are almost everywhere defined <u>functions</u> such that

$$[AW] = A(x, p)W(x) \text{ outside } K;$$

$$\{AW\} = AW - [AW] \text{ is a vector whose components are } \underline{\text{distributions}}$$

and whose support belongs to K.

<u>Cauchy's problem</u> is the problem of finding on $\mathcal{E}(K)$ a solution

$$U(x) = (u_1(x), \ldots, u_m(x)) \text{ of the system}$$
$$A(x, p)U(x) = 0$$

such that $u_\sigma(x)$ and its derivatives of order $\leq s(\sigma)$ have given values on K; K verifies nowhere the equation of the characteristic; moreover the system belongs to <u>Cauchy-Kowalewski's</u> type:

$$t(\tau) = 0.$$

Theorem 109 can be extended as follows:

<u>Cauchy's problem has always a unique solution</u>, given by the preceding theorem, if the data are sufficiently regular.

§2. The proof of the preceding statements

The process (n^o89-109) by means of which Theorems 93, 106, 108 and 109 were deduced from Proposition 88 shows obviously that the preceding extensions of these Theorems result from the extension of Proposition 88 to a hyperbolic system. Let us state this extension (Propositions 115) and give its proof (n^o116-119).

114. Definition of a regularly hyperbolic system on a vector space. X is a vector space; $A(x, p)$ is said to be regularly hyperbolic when

$h_\lambda(x, p)$ is regularly hyperbolic (see n^o69);

$\Gamma_X(A) = \bigcap_{x \in X} \Gamma_x(A) = \bigcap_\lambda \Gamma_X(h_\lambda)$ has an interior.

We study the system

$$A(x, p)U(x) = V(x)$$

where $U(x) = (u_1(x), \ldots, u_n(x))$ is unknown

$V(x) = (v_1(x), \ldots, v_n(x))$ is given.

Notation. 1) As in n^o84, Δ denotes a domain of Ξ whose director cone is $\Gamma_X(A)$; $\mu(\xi)$ denotes a positive function defined on Δ and such that on any closed cone interior to $\Gamma_X(A)$

$$\mu(\xi) = O(||\xi||^{-1}) \text{ for } ||\xi|| \longrightarrow \infty.$$

2) The norm $||r^s U||$ is defined as follows:

$$||r^s U||^2 = \sum_\sigma ||r^{s(\sigma)} u_\sigma||^2.$$

3) If the element $h_{\tau\sigma}(x, \xi)$ of the matrix $H(x, \xi)$ belongs to $H_\lambda(x, \xi)$ (see (110)), then the degree of $h_\lambda(x, \xi)$ is denoted by

$$m(\sigma) = m(\tau).$$

Obviously:

$$m(\sigma) = \sum_\nu [s(\nu) - t(\nu) + 1] \qquad (\nu \text{ such that } h_{\nu\nu} \subset H_\lambda);$$

σ being given, there are elements $h_{\tau\sigma}(x, \xi)$ of H_λ such that

$$s(\sigma) - t(\tau) + 1 \leq m(\sigma);$$

hence

(114) $$s(\sigma) - m(\sigma) + 1 \leq \operatorname{Sup}_\tau t(\tau).$$

115. <u>The assertion to be proved</u> is the following extension of Proposition 88:

<u>Proposition</u> 115. 1) <u>There are a domain</u> Δ <u>and a function</u> $\mu(\xi)$ <u>dependent on</u> A, $s(\sigma)$, $t(\tau)$, <u>but independent of</u> V(x), <u>with the following properties:</u>

<u>The equation</u> $A(x, p)U(x) = V(x)$ <u>has at most one solution such that</u> $||r^{s+1}U|| < \infty$ (for any $\xi \in \Delta$); <u>if</u> $||r^tV|| < \infty$ <u>this equation has at least one solution such that</u>

$$||r^sU|| < \mu(\xi)||r^tV||.$$

2) <u>The assumptions about</u> M <u>are that for any</u> σ <u>and</u> τ

(115.1) $$t(\tau) \leq M$$

(115.2) $$t(\tau) - 2[s(\sigma) - m(\sigma) + 1] < M$$

(115.3) $$t(\tau) - [s(\sigma) - m(\sigma) + 1] + 2\ell' \leq M.$$

<u>Note.</u> (114) shows that (115.3) involves

(115.4) $$2\ell' \leq M.$$

<u>Note.</u> When the system reduces to an equation, the assumptions (115.1), (115.2) (115.3) become $-M < t \leq M$, $2\ell' \leq M$; this last one is then superfluous.

At first we prove Proposition 115.1 when M is <u>sufficiently large</u> (n^o116-117); then we prove Proposition 115.2 (n^o119).

116. <u>Special cases. Proof of Proposition</u> 115.1 <u>when</u> A(x, p) <u>is a diagonal matrix.</u> Proposition 115 does not differ from Proposition 88.

<u>Proof of Proposition</u> 115.1 <u>when</u> $a_{\tau\sigma}(x, p)$ <u>has an order</u> $\leq s(\sigma) - t(\tau)$ <u>for</u> $\sigma \neq \tau$. Let A(x, p) be a regularly hyperbolic diagonal matrix; let K(x, p) be a matrix whose element $k_{\tau\sigma}(x, p)$ has an order $\leq s(\sigma) - t(\tau)$. The assertions to be proved are:

1) $A(x, p)U(x) = K(x, p)U(x)$, $||r^s U|| < \infty$ for $\xi \in \Delta$ involve $U(x) = 0$.

2) The equation

$$A(x, p)U(x) = K(x, p)U(x) + V(x),$$

where $||r^t V|| < \infty$ for $\xi \in \Delta$, has a solution $U(x)$ such that

$$||r^s U|| < \mu(\xi)||r^t V||.$$

<u>Proof of</u> 1). Proposition 115.1 can be applied to the diagonal matrix $A(x, p)$; (83.10) can be applied to $K(x, p)U(x)$; hence

$$||r^s U|| < \mu(\xi)||r^t K(x, p)U|| < \text{const.}\ \mu(\xi)||r^s U||.$$

Now ξ can be chosen such that const. $\mu(\xi) < 1$; hence $U = 0$.

<u>Proof of</u> 2). Applying Proposition 115 to the diagonal matrix $A(x, p)$ and (83.10) to $K(x, p)U_\alpha(x)$, we define $U_1(x), \ldots, U_\alpha(x), \ldots$ such that

$$A(x, p)U_1 = V, \qquad ||r^s U_1|| < \mu(\xi)||r^t V||,$$
$$A(x, p)U_{\alpha+1} = K(x, p)U_\alpha, \qquad ||r^s U_{\alpha+1}|| < \mu(\xi)||r^s U_\alpha||.$$

Since $\mu(\xi) = O(||\xi||^{-1})$, we can assume $\mu(\xi) < \frac{1}{2}$ on Δ; $U(x) = U_1(x) + \ldots + U_\alpha(x) + \ldots$ converges and satisfies

$$A(x, p)U(x) = K(x, p)U(x) + V(x), \qquad ||r^s U|| < 2\mu(\xi)||r^t V||.$$

117. <u>Proof of Proposition</u> 115.1. <u>Notation.</u> Assume that x and ξ commute and define

$$B(x, \xi) = H^{-1}(x, \xi)\cdot \det H(x, \xi):$$

$$B = \begin{pmatrix} B_1 & 0 & \cdot \\ 0 & B_2 & \cdot \\ & & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

where $B_\lambda(x, \xi) = H_\lambda^{-1}(x, \xi)\cdot \det H_\lambda(x, \xi)$. The element $b_{\sigma\tau}(x, \xi)$ of $B_\lambda(x, \xi)$ is a polynomial in ξ, whose degree is

$$m(\sigma) - [s(\sigma) - t(\tau) + 1] = m(\tau) - [s(\sigma) - t(\tau) + 1].$$

Define

$$B(x, p)A(x, p) = C(x, p), \qquad A(x, p)B(x, p) = D(x, p).$$

The element $c_{\tau\sigma}(x, p)$ of $C(x, p)$ has the order

$$s(\sigma) - s(\tau) + m(\tau) - 1 \text{ for } \sigma \neq \tau, \quad m(\sigma) \text{ for } \sigma = \tau.$$

The element $d_{\tau\sigma}(x, p)$ of $D(x, p)$ has the order

$$m(\sigma) + t(\sigma) - t(\tau) - 1 \text{ for } \sigma \neq \tau, \qquad m(\sigma) \text{ for } \sigma = \tau.$$

$C(x, p)$ and $D(x, p)$ are regularly hyperbolic; according to n°116, Proposition 115.1 holds for $C(x, p)$ and for $D(x, p)$.

<u>Proof of Proposition</u> 115.1. There is a domain Δ such that

$$C(x, p)\, U(x) = 0 \text{ and } ||r^s U|| = 0 \text{ for } \xi \in \Delta$$

imply $U(x) = 0$; hence:

$$A(x, p)U(x) = 0 \text{ and } ||r^s U|| = 0 \text{ for } \xi \in \Delta \text{ imply } \quad U = 0.$$

There are a domain Δ and a function $\mu(\xi)$ such that the equation

$$D(x, p)W(x) = V(x),$$

where $||r^t V|| < \infty$ for $\xi \in \Delta$, has a solution W satisfying

$$||r^{m+t-1} W|| < \mu(\xi)||r^t V||.$$

Hence, upon application of (83.10),

$$U(x) = B(x, p)W(x)$$

satisfies

$$A(x, p)U(x) = V(x), \quad ||r^s U|| < \mu(\xi)||r^t V||.$$

118. <u>Schauder's functional ring</u> is used in the proof of Proposition 115.2 (n°119) and in the study of the non-linear hyperbolic equations (Third Part).

Remember that ℓ' denotes the smallest integer $> \frac{\ell}{2}$.

<u>Definition</u>. Let M be an integer such that

(118.1) $$M \geq 2\ell';$$

that is: $M > \ell + 1$ (or $M > \ell$ when ℓ is odd). The elements of Schauder's ring Sch(M) are the functions $c(x)$ defined over X and such that:

$c(x)$ is bounded;

its derivatives of order > 0 and $\leq M$ are square integrable.

Sch(M) has the following properties:

Lemma 118.1. An element of Sch(M) has bounded derivatives of order $\leq M - \ell'$ (in particular of order $\leq \ell'$).

Proof. Sobolev's inequality (83.7).

Lemma 118.2. 1) Sch(M) is a ring.

2) The functions $u(x)$ such that $||r^M u|| < \infty$ for $\xi \in \Delta$ constitute a vector space over Sch(M); if $0 \in \Delta$, they constitute an ideal of Sch(M).

Proof of 1). Let u_1 and $u_2 \in$ Sch(M); the derivatives of $u_1(x)u_2(x)$ of order $\leq M$ are sums of products

$$v_1(x)v_2(x),$$

where $v_\alpha(x)$ is a derivative of $u_\alpha(x)$ of order s_α;

$$s_1 + s_2 \leq M.$$

Assume $s_2 \leq s_1$; hence

$$2s_2 \leq M \leq 2M - 2\ell'$$

that is

$$s_2 \leq M - \ell';$$

thus according to the preceding Lemma v_2 is bounded; now v_1 is square integrable; hence $v_1(x)v_2(x)$ is square integrable.

Proof of 2). Let $u_1 \in$ Sch(M), $||r^M u_2|| < \infty$. We have to show that $||v_1(x)v_2(x)|| < \infty$ if $v_1(x)$ and $v_2(x)$ are derivatives of order s_1 and s_2 of u_1 and u_2 such that $s_1 + s_2 \leq M$. We have just shown that:

either $s_1 \leq M - \ell'$, then $\operatorname{Sup} |v_1| < \infty$ and a bound of $||v_1 v_2||$ is given by (83.6); or $s_2 \leq M - \ell' < s_1$ and a bound of $||v_1 v_2||$ is given by (83.5).

The Third Part uses the

Lemma 118.3. Assume that

$$u_1(x),\ u_2(x),\ \dots,\ \in \mathrm{Sch}(M);$$

and that for any constant γ :

$$\sup_{x \in X, |w_1| < \gamma} |F(x, w_1, w_2, \dots)| < \infty,$$

$$\sup_{|w_1| < \gamma} \left| \frac{\partial^{\alpha+\beta_1+\beta_2+\dots} F(x, w_1, w_2, \dots)}{\partial x^{\alpha} \partial w_1^{\beta_1} \partial w_2^{\beta_2} \dots} \right| \quad \text{is square integrable}$$

over X when $\alpha + \beta_1 + \beta_2 + \dots \leq M$. Then

$$F(x, u_1(x), u_2(x), \dots) \in \mathrm{Sch}(M).$$

Proof. A derivative of order $\leq M$ of $F(x, u_1(x), \dots)$ is a sum of terms

$$\frac{\partial^{\alpha+\beta_1+\dots} F(x, u_1(x), \dots)}{\partial x^{\alpha} \partial w_1^{\beta_1} \dots} v_1(x) \dots v_{\beta}(x) \tag{118.2}$$

where $v_1(x), \dots, v_\beta(x)$ are derivatives of $u_1(x), \dots$ whose orders $s_1, \dots, s_\beta$ are such that

$$\beta_1 + \beta_2 + \dots = \beta \tag{118.3}$$

$$\alpha + s_1 + s_2 + \dots + s_\beta \leq M. \tag{118.4}$$

We can assume

$$1 \leq s_\beta \leq \dots \leq s_2 \leq s_1. \tag{118.5}$$

Hence, according to (118.5), (118.4) and (118.1),

$$2s_2 \leq s_1 + s_2 \leq M \leq 2M - 2\ell';$$

that is

$$s_2 \leq M - \ell';$$

hence upon application of Lemma 118.1, $v_\beta(x)\ldots v_2(x)$ are bounded functions. If $s_1 \leq M - \ell'$, then $v_1(x)$ is also bounded and (118.2) is obviously square integrable. Assume now

$$M - \ell' < s_1;$$

hence, replacing in (118.4) s_1 by $M - \ell' + 1$ and $s_2, \ldots, s_\beta$ by 1

$$\alpha + \beta + M - \ell' \leq M;$$

hence, according to (118.1)

$$\alpha + \beta \leq M - \ell',$$

and using Sobolev's inequality (83.7)

$$\sup_{x \in X, |w_i| < \gamma} \left| \frac{\partial^{\alpha + \beta_1 + \cdots} F(x, w_1, \ldots)}{\partial x^\alpha \partial w_1^{\beta_1} \cdots} \right| < \infty;$$

hence, since $v_1(x)$ is square integrable, (118.2) is square integrable.

119. Proof of Proposition 115.2. N°116 and 117 showed that Proposition 115.1 holds when M is sufficiently large; we have to show that the assumptions (115.1), (115.2), (115.3) are sufficient.

Lemma 119.1. If $A(x, p)U(x) = V(x)$ and $||r^{s+1}U|| < \infty$, then

(119.1) $$||r^s U|| < \mu(\xi)||r^t V||.$$

Proof. $A(x, p)U(x) = V(x)$

gives

$$C(x, p)U(x) = B(x, p)V(x)$$

that is

(119.2) $$c_{\rho\rho}(x, p)u_\rho(x) = w_\rho(x)$$

(119.3) $$w_\rho(x) = \sum_\tau b_{\rho\tau}(x, p)v_\tau(x) - \sum_{\sigma \neq \rho} c_{\rho\sigma}(x, p)u_\sigma(x).$$

The coefficients of $A(x, p)$ belong to $\mathrm{Sch}(M)$ (see (115.4)); hence, since $\mathrm{Sch}(M)$ is a ring, the coefficients of $B(x, p)$ belong also to $\mathrm{Sch}(M)$; and, since

$$c_{\rho\sigma}(x, p) = \sum_{\tau} b_{\rho\tau}(x, p)a_{\tau\sigma}(x, p),$$

the coefficients of $c_{\rho\sigma}(x, p)$ belong to the ring (see (115.3))

$$\mathrm{Sch}(M + s(\rho) - m(\rho) + 1 - \operatorname*{Sup}_{\tau} t(\tau))$$

which contains $\mathrm{Sch}(M)$ (see (114)); the elements of this ring have bounded derivatives of order $\leq \ell'$ (Lemma 118.1). Hence:

(119.2) and Proposition 88 give

(119.4) $$||r^s U|| < \mu(\xi)||r^{s-m+1}W||;$$

(119.3) and (83.10) give

(119.5) $$||r^{s-m+1}W|| < \text{const.}\ ||r^t V|| + \text{const.}\ ||r^s U||;$$

we have to assume that the assumptions $-M < t \leq M$ of Proposition 88 and of formula (83.10) are satisfied:

M and t have to be replaced by

$$M + s(\rho) - m(\rho) + 1 - \operatorname*{Sup}_{\tau} t(\tau) \text{ and } s(\rho) - m(\rho) + 1;$$

we obtain the assumptions (115.1) and (115.2).

Now (119.1) follows from (119.4) and (119.5).

<u>Proof of Proposition</u> 115.2. The preceding Lemma shows that the solution of

$$A(x, p)U(x) = V(x)$$

is unique if $||r^{s+1}U|| < \infty$. Let us show that it exists if $||r^t V|| < \infty$. $A(x, p)$ is the limit of a sequence of $A^*(x, p)$ such that: the coefficients of the matrices $A^*(x, p)$ are bounded; their derivatives of any order are square integrable, this condition being uniformly satisfied by the derivatives of order $\leq M$.

According to Proposition 115.1 and to the preceding Lemma

$$A^*(x, p)U^*(x) = V(x)$$

has a solution $U^*(x)$ such that

$$||r^s U^*|| < \mu(\xi)||r^t V||,$$

$\mu(\xi)$ being the <u>same</u> for all the functions U^*; thus these functions U^* have at least one limit U, which satisfies

$$||r^s U|| < \mu(\xi)||r^t V||.$$

Third Part

Non-linear Equations and Systems

Introduction

A merely local existence theorem ($n^o 127$) can be proved. It shows that for hyperbolic equations the existence in the large of solutions depends upon the obtaining of a priori bounds for their derivatives of order $m + \ell + 3$ ($m + \ell + 2$ if ℓ is odd); except for equations which are linear outside a small domain, there are no examples where such a priori bounds are known. (For ordinary differential equations on the contrary this order is m and there are many types of equations whose solutions exist in the large: see for instance the mechanics of material systems depending on a finite number of parameters.)

Petrowsky [34] proved this local existence theorem by a very complicated process; thanks to the weakness of the assumptions about the coefficients in the Second Part ($t \leq M$), we can simply use the method of successive approximations, that J. Schauder [36] applied to equations of second order. We state the actual assumptions to be made. We prove furthermore a uniqueness theorem in the large ($n^o 127$).

We do not solve the " mixed boundary value problem " ; this problem has been solved for the second order by J. Schauder and M. Krzyżański (Studia mathematica, vol. 6, 1936, p. 162-189, p. 190-198).

§1. Preliminary: Quasi-linear equations and systems

120. Notation. Let X be a vector space; let $(x_1, \ldots, x_\ell)$ be the coordinates of $x \in X$; let Y be the strip

$$0 \leq x_1 < \sigma.$$

Let u(x) be a function defined on Y; we define

$$||u||^2 = \int_Y \cdots \int |u(x)|^2 \exp(-2x.\xi)dx_1 \ldots dx_\ell$$

$$||ru||^2 = ||u||^2 + \sum_\lambda ||p_\lambda u||^2; \quad ||r^{s+1}u||^2 = ||r^s u||^2 + \sum_\lambda ||r^s p_\lambda u||^2 \quad (s > 0).$$

When $\xi = 0$, then $||u||$ and $||r^s u||$ are denoted by $||u||_o$ and $||r^s u||_o$. Thus

$$||u|| = ||\exp(-x. \xi)u(x)||_o$$

$$||ru||^2 = ||\exp(-x. \xi)u(x)||_o^2 + \sum_\lambda ||\exp(-x. \xi)p_\lambda u||_o^2 \quad \text{etc.};$$

hence, when $\xi = (\xi_1, 0, \ldots, 0)$, $\xi_1 > 0$

(120) $$||r^s u||_o \exp(-\sigma \xi_1) \leq ||r^s u||_\xi \leq ||r^s u||_o.$$

Sobolev's inequality (83.7) and its consequences (83.5), (83.8) hold.

Note 120.1. Assume that in (83.7) and (83.5)

$$u = \ldots = p_1^{\ell' -1}u = 0 \text{ for } x_1 = 0; \text{ that in (83.8)}$$

$$u = \ldots = p_1^{s-1}u = 0 \text{ for } x_1 = 0.$$

Defining $u(x) = 0$ for $x_1 = 0$ and applying Sobolev's inequality to the half space $x_1 < \sigma$, we see that the constants γ used in (83.7), (83.5) and (83.8) are independent of σ.

The elements of the ring Sch(M) are the functions $c(x)$ such that:

$c(x)$ is defined on Γ;

$c(x)$ is bounded;

its derivatives of order > 0 and $\leq M$ are square integrable.

Lemmas 118.1, 118.2 and 118.3 hold.

Note 120.2. In Lemma 118.3 assume that

$$||r^M u_1||_o < \infty, \quad u_i = p_1 u_i = \ldots = p_1^{M-1}u_i = 0 \text{ for } x_1 = 0;$$

then $F(x, u_1(x), \ldots)$ and its derivatives of order $\leq M - \ell'$ and the integrals over Y of its derivatives of order > 0 and $\leq M$ are bounded by functions of $||r^M u_1||_0$ which are independent of σ.

121. Linear hyperbolic equations on a strip. Notation. $a(x, p)$ is a linear hyperbolic operator defined on the strip Y; it is regularly hyperbolic (n°69); it satisfies the assumptions 2° of n°84; the first axis of Ξ is inside $\Gamma_x(a)$. The order of $a(x, p)$ is m; s and t are two integers such that

$$m = s - t + 1; \qquad 0 \leq t \leq M.$$

Proposition 121. Let $v(x)$ be a function defined on Y and such that:

$v(x)$ and its first derivatives are locally square integrable; $v(x) = 0$ for $x_1 = 0$.

1) The equation

$$a(x, p)u(x) = v(x)$$

has a unique solution $u(x)$ defined on Y and such that: its derivatives of order $\leq m$ are locally square integrable;

$$u(x) = p_1 u(x) = \dots = p_1^{m-1} u(x) = 0 \text{ for } x_1 = 0.$$

2) Assume that $||r^t v|| < \infty$ and, if $t > 1$, that

$$v(x) = p_1 v(x) = \dots = p_1^{t-1} v(x) = 0 \text{ on } x_1 = 0;$$

then there are a domain Δ and a function $\mu(\xi)$ (which depend continuously on $a(x, p)$ and t but are independent of $v(x)$ and σ) such that

$$(121.1) \qquad ||r^s u|| < \mu(\xi)||r^t v|| \text{ for } \xi \in \Delta,$$

$$u(x) = p_1 u(x) = \dots = p_1^{s-1} u(x) = 0 \text{ for } x_1 = 0.$$

Hence there is a function $\nu(\sigma)$ (which depends only on $a(x, p)$, t, σ) such that:

(121.2) $$||r^s u||_0 < \gamma(\sigma)||r^t v||_0,$$

(121.3) $$\gamma(\sigma) = O(\sigma)$$ (That is: $\gamma(\sigma) <$ const. σ for $\sigma \to 0$).

<u>Proof of</u> 1. Extend $a(x, p)$ and $v(x)$ as follows to X:

$$a(-x_1, x_2, \ldots, x_\ell, p) = a(x_1, x_2, \ldots, x_\ell, p) \text{ for } 0 \leq x_1 \leq \sigma ;$$
$$a(x_1 + 2\sigma, x_2, \ldots, x_\ell, p) = a(x_1, x_2, \ldots, x_\ell, p);$$
$$v(x) = 0 \text{ for } x_1 < 0 \text{ and } \sigma < x_1.$$

Choose $u(x) = 0$ for $x_1 < 0$. Apply Theorem 93 to the half space $x_1 < \sigma$, according to n°103.

<u>Proof of</u> (121.1) <u>when</u> $t = 0$. $u(x)$ is the restriction to Y of the function $u(x)$ defined on X by Proposition 88; (121.1) follows from (88.1).

<u>Proof of</u> (121.1) <u>in a special case</u> $(1 \leq t)$. Define $u(x) = v(x) = 0$ for $x_1 < 0$; assume that $a(x, p)$ and $v(x)$ have extensions to X satisfying the assumptions of Proposition 88. Then $u(x)$ is the restriction to Y of the function $u(x)$ defined on X by this Proposition. However (88.1) does no more show that (121.1) holds; indeed the domain Δ and the function $\mu(\xi)$ used by (88.1) depend on the bounds of the extensions of $a(x, p)$, $v(x)$; now these bounds do no more depend only on the given bounds of $a(x, p)$, $v(x)$ on Y; thus (88.1) shows only that

$$||r^s u|| < \infty .$$

But the proof of (88.1), that is of Lemma 86.2, can be used on Y, Lemma 86.1 being replaced by the inequality (121.1) where $t = 0$. Hence (121.1) holds for $1 \leq t < M$. The note 120.1 shows that Δ, $\mu(\xi)$ can be chosen independent of σ.

<u>Proof of</u> (121.1). We approach $a(x, p)$ and $v(x)$ by approximations satisfying the preceding assumption, where M is replaced by $M + 1$; we obtain approximations of $u(x)$, all of them satisfying the <u>same</u> inequality (121.1) where $0 \leq t \leq M$. Hence $u(x)$ also

satisfies this inequality. These approximations and therefore $u(x)$ also satisfy

$$u(x) = p_1 u(x) = \dots = p_1^{s-1} u(x) = 0 \text{ on } x_1 = 0.$$

<u>Proof of</u> (121.2). According to (120) the inequality (121.1) shows that

$$||r^s u||_0 < \mu(\xi) \exp(\sigma \xi_1) \, ||r^t v||_0 ,$$

where $\xi = (\xi_1, 0, \dots, 0)$, ξ_1 being so large that $\xi \in \Delta$. When σ is sufficiently small we choose $\sigma = \xi_1^{-1}$; hence

$$\mu(\xi) \exp(\sigma \xi_1) = O(\sigma) \text{ for } \sigma \longrightarrow 0.$$

122. <u>A non-linear mapping defined by a linear hyperbolic equation.</u> <u>Notation.</u> Let

(122.1) $$a(x, v, p)u = b(x, v)$$

be a linear hyperbolic equation of order m, defined on the strip Y; $u(x)$ is the unknown function; $b(x, v)$ and the coefficients $c(x, v)$ of $a(x, v, u)$ are functions of x, of a given function $v(x)$ and of its derivatives of order $< m$. Denote by $b(x, W)$ and $c(x, W)$ the functions obtained by replacing in $b(x, v)$ and $c(x, v)$ the function v and its derivatives of order $< m$ by the components w_i of a vector W; we assume that, when W is sufficiently small:

$b(x, W)$ and $c(x, W)$ are defined;

$a(x, W, p)$ is regularly hyperbolic;

the first axis of the space Ξ is inside $\Gamma_x(a)$;

(122.2) $$\begin{cases} \operatorname*{Sup}_W \left| \dfrac{\partial^{\alpha+\beta_1+\dots} b(x, W)}{\partial x^{\alpha} \partial w_1^{\beta_1} \dots} \right| \text{ is square integrable over Y for } 0 \leq \alpha + \beta_1 + \dots \leq M; \\ \operatorname*{Sup}_W \; c(x, W) \text{ is bounded on Y;} \\ \operatorname*{Sup}_W \left| \dfrac{\partial^{\alpha+\beta_1+\dots} c(x, W)}{\partial x^{\alpha} \partial w_1^{\beta_1} \dots} \right| \text{ is square integrable over Y for } 0 \leq \alpha + \beta_1 + \dots \leq M. \end{cases}$$

$M \geq 2\ell'$ (that is $M \geq \ell + 2$ or $\geq \ell + 1$ when ℓ is odd);

s and t are integers such that

$$m = s - t + 1, \qquad 2\ell' \leq t \leq M.$$

We assume moreover that:

$$b(x, 0) = p_1 b(x, 0) = \dots = p_1^{t-1} b(x, 0) = 0 \text{ for } x_1 = 0;$$

$||r^s v||_o$ is sufficiently small;

$$v(x) = p_1 v(x) = \dots = p_1^{s-1} v(x) = 0 \text{ for } x_1 = 0.$$

Lemma 122. $\gamma_1, \gamma_2, \dots,$ denote positive numbers dependent on the functions $b(x, W)$, $c(x, W)$, on $||r^s v||_o$ and $||r^s v^*||_o$, but independent of σ (assumed to be bounded).

1) The equation (122.1) has a unique solution $u(x)$ defined on Y and such that

$$||r^m u||_o < \infty, \ u(x) = p_1 u(x) = \dots = p_1^{m-1} u(x) = 0 \text{ for } x_1 = 0;$$

it satisfies the conditions

$$(122.3) \qquad ||r^s u||_o < \gamma_1 \sigma, \quad u(x) = p_1 u(x) = \dots = p_1^{s-1} u(x) = 0 \text{ for } x_1 = 0.$$

2) Replace v by v*: u becomes u*; we have

$$(122.4) \qquad ||r^{s-1}(u - u^*)||_o < \gamma_2 \sigma ||r^{s-1}(v - v^*)||_o.$$

Proof of 1^o. Lemma 118.3 shows that $b(x, v)$ and $c(x, v)$ belong to Sch(t); $b(x, v)$ is obviously square integrable; thus the assumptions 2^o of $n^o 84$, where M is replaced by t, are satisfied; hence Proposition 121 can be applied: (122.3) follows from (121.2); Note 120.2 shows that γ_1 can be chosen independent of σ.

Proof of 2^o. Obviously

$$(122.5) \qquad a(x, v^*, p)(u^* - u) = b(x, v^*) - b(x, v) + a(x, v, p)u - a(x, v^*, p)u.$$

An easy extension of Lemma 118.3 gives

$$||r^{t-1}[b(x, v^*) - b(x, v)]||_o < \gamma_3||r^{s-1}(v^* - v)||_o$$

$$||r^{t-1}[a(x, v, p)u - a(x, v^*, p)u]||_o < \gamma_4||r^{s-1}(v^* - v)||_o$$

where γ_4 depends on the functions $c(x, W)$, on $||r^s u||_o$, $||r^{s-1}v||_o$ and $||r^{s-1}v^*||_o$; according to (122.3) we can choose a coefficient γ_4 depending on $b(x, W)$, $c(x, W)$, $||r^s v||_o$, $||r^{s-1}v^*||_o$. If now we apply Proposition 121 (where t and s are replaced by $t-1$ and $s-1$) to (122.5), we obtain (122.4); Note 120.2 shows that γ_2 can be chosen independent of σ.

123. The quasi-linear equations on a strip. We preserve the preceding notation; in Lemma 122 we assume now that

$$||r^s v||_o < \gamma_o, \qquad ||r^s v^*||_o < \gamma_o \qquad (\gamma_o > 0).$$

We lessen the width σ of the strip Y so that (122.3) and (122.4) become

(123.1) $\quad ||r^s u||_o < \gamma_o, \quad u(x) = p_1 u(x) = \dots = p_1^{s-1}u(x) = 0$ for $x_1 = 0$;

(123.2) $\quad ||r^{s-1}(u - u^*)||_o < \frac{1}{2}\,||r^{s-1}(v - v^*)||_o.$

The total curvature of $\hat{\Gamma}_x$ is assumed to be > 0.

Successive approximations give the following existence theorem:

Lemma 123.1. The quasi-linear equation

(123.3) $\qquad a(x, u, p)u = b(x, u)$

has a unique solution $u(x)$ defined on the strip Y and satisfying (123.1).

Proof. Uniqueness of $u(x)$: Let $u(x)$ and $u^*(x)$ be two solutions of (123.3) satisfying (123.1); (123.2) gives

$$||r^{s-1}(u - u^*)||_o < \frac{1}{2}\,||r^{s-1}(u - u^*)||, \text{ that is } u = u^*.$$

<u>Existence</u> <u>of</u> $u(x)$: Let $u_1(x)$ be a function satisfying (123.1); Lemma 122 gives a sequence of functions $u_\lambda(x)$ ($\lambda = 1, 2, \ldots$) satisfying (123.1) and

$$a(x, u_\lambda, p)u_{\lambda+1} = b(x, u_\lambda);$$

moreover, according to (123.2)

$$||r^{s-1}(u_{\lambda+1} - u_\lambda)||_o < \gamma_o 2^{2-\lambda};$$

hence $u_\lambda(x)$ has a limit $u(x)$ satisfying (123.1) and (123.3).

Let us state a <u>uniqueness</u> <u>theorem</u>:

<u>Lemma</u> 123.2. Assume that (123.3) has the solution $u(x) = 0$ on Y. Then

$$u^*(x) = 0 \text{ on } D$$

if the following assumptions hold:

$u^*(x)$ is defined on the domain D of Y;

$u^*(x)$ has locally square integrable derivatives of order $\leq s + 1$;

$u^*(x) = \ldots p_1^s u^*(x) = 0$ on D for $x_1 = 0$;

$a(x, u^*, p)u^* = b(x, u^*)$ on D;

$D = \mathcal{E}^*_-(D)$, where $\mathcal{E}^*_-$ is the emission defined by $a(x, u^*, p)$.

<u>Proof.</u> Assuming that $u^*(x)$ is defined on Y and that

$$u^*(x) = \ldots p_1^{s-1}u^*(x) = 0 \text{ on Y for } x_1 = 0,$$

$$a(x, u^*, p)u^* = b(x, u^*) \text{ on D},$$

let us prove successively the following assertions:

1^o If $||r^s u^*||_o < \gamma_o$, then $u^*(x) = 0$ on D;

2^o If $||r^s u^*||_o < \infty$, there exists $\sigma' > 0$ such that

$$u^*(x) = 0 \text{ on D for } 0 < x_1 < \sigma';$$

3^o If $||r^{s+1}u^*||_o < \infty$ and $u^*(x) = 0$ on D for $0 < x_1 < \sigma'$, there exists $\sigma'' > \sigma'$ such that

$$u^*(x) = 0 \text{ on } D \text{ for } 0 < x_1 < \sigma'';$$

4^o If $||r^{s+1}u^*||_o < \infty$, then $u^*(x) = 0$ on D.

The Lemma is an obvious consequence of 4^o: in order to prove that $u^*(y) = 0$ if $y \subset D$ when $u^*(x)$ satisfies the assumptions of the Lemma, we replace $u^*(x)$ by $\varphi(x)u^*(x)$, where $\varphi(x) = 1$ near $\mathcal{E}^*_-(y)$, $\varphi(x) = 0$ near the boundary of D; we define $u^*(x) = 0$ outside D and we replace D by a neighborhood of $\mathcal{E}^*_-(y)$ where $\varphi(x) = 1$: the assumptions of 4^o are now satisfied.

<u>Proof of</u> 1^o. In the proof of Lemma 123.1 (Existence), choose $u_1(x) = u^*(x)$; according to Proposition 92.2, $u_\lambda(x) = u^*(x)$ on D; now $u_\lambda(x)$ tends to $u(x) = 0$; hence $u^*(x) = 0$ on D.

<u>Proof of</u> 2^o. Apply 1^o to the strip $0 < x_1 < \sigma'$, σ' being so small that on this strip $||r^s u^*||_o < \gamma_o$.

<u>Proof of</u> 3^o. Let $u'(x)$ be a function such that: $u'(x)$ is defined on Y; $u'(x) = u^*(x)$ for $\frac{\sigma'}{2} < x_1 < \sigma'$; $u'(x) = 0$ on D; $||r^s u'||_o < \infty$.

Replacing in 2^o $u^*(x)$ by $u^*(x) - u'(x)$, 0 by σ', σ' by σ'' we obtain the conclusion to be proved:

$$u^*(x) = 0 \text{ on } D \text{ for } \sigma' < x_1 < \sigma''.$$

Such a function $u'(x)$ can be obtained as follows: let $b(x, p)$ be a regularly hyperbolic operator of order $s + 1$ such that

$$C_x(b) \subset C_x(a(x, u^*, p)):$$

for instance:

$$b(x, \xi) = b_1(x, \xi)b_2(x, \xi)\dots,$$

$b_\alpha(x, \xi) > 0$ being a convex cone of order 2 (or 1 if $\alpha = 1$, s even) containing

$b_{\alpha+1}(x, \xi) \geq 0$ and $\Gamma_x(a)$. Choose for $u'(x)$ the solution of

$$b(x, p)u'(x) = v'(x)$$

such that $u'(x) = u^*(x)$ for $\frac{\sigma'}{2} < x_1 < \sigma'$, $v'(x)$ being:

$$v'(x) = b(x, p)u^*(x) \text{ for } \frac{\sigma'}{2} < x_1 < \sigma'$$
$$v'(x) = 0 \qquad \text{for } \sigma' < x_1 < \sigma.$$

<u>Proof of</u> 4°. Let σ^* be the largest number $\leq \sigma$ such that

$$u^*(x) = 0 \text{ on } D \text{ for } 0 < x_1 < \sigma^*;$$

2° and 3° show that $\sigma^* = \sigma$.

124. <u>Local properties of special quasi-linear systems on a manifold.</u> <u>Notation.</u> Let X be a manifold. Let

(124.1) $$A(x, U, p)U = B(x, U)$$

be a quasi-linear system;

$$A(x, U, p) = \begin{pmatrix} a_1(x, U, p) & \vdots & 0 \\ \cdots\cdots\cdots & \cdot & \cdots\cdots\cdots \\ 0 & \vdots & a_n(x, U, p) \end{pmatrix}$$

is a diagonal matrix whose element $a_\tau(x, U, p)$ is a differential operator defined on S; its order is $m(\tau)$;

$$B(x, U) = (b_1(x, U), \ldots, b_n(x, U)) \ldots), \quad U(x) = (u_1(x), \ldots, u_n(x))$$

are vectors whose components are functions defined on X; $b_\tau(x, U)$ and the coefficients of $a_\tau(x, U, p)$ are functions of the functions $u_\sigma(x)$ and of their derivatives of order $\leq s(\sigma) - t(\tau)$;

$$m(\sigma) = s(\sigma) - t(\sigma) + 1;$$

of course $s(\sigma)$ and $t(\tau)$ can be replaced by $s(\sigma)$ + const., $t(\tau)$ + const. If we replace in $b_\tau(x, U)$ or in a coefficient of $a_\tau(x, U, p)$ the functions $u_\sigma(x)$ and their derivatives of order $\leq s(\sigma) - t(\tau)$ by the components w_i of a vector W we obtain functions $c_\tau(x, W)$ satisfying the following assumption :

(124.2) $$\operatorname*{Sup}_W \left| \frac{\partial^{M(\tau)} c_\tau(x, W)}{\partial x^\alpha \partial w_1^{\beta_1} \cdots} \right| \text{ is a locally square integrable function of } x.$$

We assume $t(\tau)$ such that

(124.3) $$2\ell \leq t(\tau) \leq M(\tau).$$

We assume X to be

$$\operatorname*{Sup}_{\sigma,\tau} [m(\tau) + M(\tau),\ s(\sigma) - t(\tau) + M(\tau)] \text{ - smooth;}$$

let K be a hypersurface of X as smooth as X itself. Let

$$V(x) = (v_1(x), \ldots, v_n(x))$$

be a vector whose elements are functions defined near K and such that

$v_\sigma(x)$ has locally square integrable derivatives of order $\leq s(\sigma) + 1$;

$a_\tau(x, V, p)v_\tau - b_\tau(x, V)$ and its derivatives of order $< t(\tau)$ are $= 0$ on K;

$A(x, V, p)$ is hyperbolic (n°110);

K is space like (i.e.: its tangents are outside C_x.)

<u>Cauchy's problem</u> asks for a solution $U(x)$ of (124.1) such that:

$u_\sigma(x)$ has locally square integrable derivatives of order $\leq s(\sigma)$;

$u_\sigma(x) - v_\sigma(x)$ and its derivatives of order $< s(\sigma)$ are $= 0$ on K;

<u>Proposition</u> 124. 1) <u>Let</u> $k \in K$; <u>Cauchy's problem has at least one solution near</u> k

2) <u>Two solutions</u> U <u>and</u> U^* <u>of Cauchy's problem are equal near</u> K <u>if</u> u_σ <u>and</u> u^*_σ <u>have locally square integrable derivatives of order</u> $\leq s(\sigma) + 1$.

3) <u>If two solutions</u> U <u>and</u> U^* <u>are defined near the point</u> y <u>and if</u> u_σ <u>and</u> u^*_σ <u>have</u>

<u>locally square integrable derivatives of order</u> $\leq s(\sigma) + 1$, <u>then</u> y <u>cannot be an isolated point of</u> $\mathcal{E}_{-}(y) \cap S(U - U^*)$.

[$\mathcal{E}_{-}$ denotes the emission of $A(x, U, p)$; S the support.]

<u>Proof of</u> 1^o <u>when</u> $n = 1$. $U = u_1$ is denoted by u, A by a, B by b. Replacing u by $u - v$ we reduce the proof to the case $v = 0$. We do not change the local property to be proved if we change X, $a(x, u, p)$, $b(x, u)$, K outside a neighborhood of k so that X becomes a strip, K a hyperplane, $a(x, W, p)$ a regularly hyperbolic operator. Hence the property to be proved follows from Lemma 123.1.

<u>Proof of</u> 2^o <u>and</u> 3^o <u>when</u> $n = 1$. A similar change of the data transforms X into a strip, K into a hyperplane, u into 0, u^* into a function satisfying

$$a(x, u^*, p)u^* = b(x, u^*), \; u^*(x) = \dots = p_1^s u^*(x) = 0 \text{ on } K,$$

inside a domain D such that $D = \mathcal{E}_{-}^{*}(D)$, $\mathcal{E}_{-}^{*}$ being the emission of $a(x, u^*, p)$. Then Lemma 123.2 is applied.

<u>The proof</u> in the general case is similar: indeed $n^o 123$ can be easily extended to the quasi-linear systems which we are studying.

§2. Non-linear equations

125. Notation. X is an ℓ-dimensional $(m + M)$-smooth manifold; $u(x)$ is an unknown function; $a(x, u)$ is a given function of x, of $u(x)$ and of its derivatives of order $\leq m$. If we replace u and these derivatives by the components w_i of a vector W, we obtain a function $a(x, W)$ satisfying the following assumptions:

$a(x, W)$ is defined on an open set;

$\sup_W \left| \frac{\partial^M a(x, W)}{\partial x^\alpha \partial w_1^{\beta_1} \cdots} \right|$ is a locally square integrable function of x.

We define

$$h(x, u, p) = \sum_{1 \leq \lambda_1 \leq \cdots \lambda_m \leq \ell} \frac{\partial a(x, u)}{\partial (p_{\lambda_1} \cdots p_{\lambda_m} u)} p_{\lambda_1} \cdots p_{\lambda_m},$$

K is a given open orientated $(m + M)$-smooth hypersurface of X; $v(x)$ is a given function defined near K; $h(x, v, p)$ is hyperbolic on K, which is space-like: the cone C_x defined by $h(x, v, p)$ at $x \in K$ does not contain any direction tangent to K; we choose C_x directed towards the positive side of K. $v(x)$ has locally square integrable derivatives of order $\leq s + 1$; $a(x, v)$ and its derivatives of order $< s - m$ are $= 0$ on K; s satisfies

$$(125.1) \qquad m + \ell + 2 < s < m + M \qquad (\ldots \leq s < \ldots \text{ if } \ell \text{ is odd});$$

thus

$$\ell + 3 < M \qquad (\ldots \leq \ldots \text{ if } \ell \text{ is odd}).$$

Cauchy's problem asks for a domain D and a function $u(x)$ such that:

$D \subset X$; K belongs to the boundary of D; D is on the positive side of K; $u(x)$ is defined on D; $u(x)$ has locally square integrable derivatives of order $\leq s$; $u(x) - v(x)$ and its derivatives of order $< s$ are $= 0$ on K.

$$a(x, u) = 0 \text{ on } D.$$

126. Local properties. Lemma 126.1. (Existence) Any point k of K has a neighborhood where Cauchy's problem has at least one solution.

Proof. Let b(p) be a linear operator of order 2 such that at k the operator h(x, V, p)b(p) is hyperbolic and the manifold K is space-like: its tangent hyperplane is exterior to the cone $C_x(b)$, which contains $C_x(h)$. The condition:

$$w(x) = 0 \text{ near } k$$

is equivalent to the conditions:

b(p)w(x) = 0 near k,

w(x) and its first derivatives are = 0 on K near k.

Thus Cauchy's problem does not change when we replace the non-linear equation

$$a(x, u) = 0$$

by the quasi-linear equation

$$b(p)a(x, u) = 0.$$

Hence our assertion follows from Proposition 124.1. In its assumptions (124.3) we have to replace $m(\tau)$, $M(\tau)$, $s(\sigma)$ by m + 2, M - 2, s;

$$t(\sigma) = s(\sigma) + 1 - m(\sigma) = s - m - 1; \qquad 2\ell' = \ell + 2 \quad \text{or } \ell + 1;$$

hence (124.3) becomes (125.1).

Note. The proof of the following Lemma could not use the preceding quasi-linear equation b(p)a(x, u) = 0: its emission is the emission of h(x, u, p)b(p), which differs from the emission of h(x, u, p).

Lemma 126.2. (Uniqueness) 1) Two solutions of Cauchy's problem are equal near K.

2) If u and u* are solutions of Cauchy's problem defined near $y \in X$, then y cannot be an isolated point of $\mathcal{E}_-(y) \cap S(u - u^*)$; $\mathcal{E}_-$ denotes the emission of

$h(x, u, p)$ or $h(x, u^*, p)$.

<u>Proof.</u> The vector $U(x) = (p_1 u, \ldots, p_\ell u)$ satisfies the <u>quasi-linear</u> system

$$p_1 a(x, u) = 0, \ldots, p_\ell a(x, u) = \Theta,$$

which belongs to the type studied in n°124; that system contains derivatives of order $\leq m$ of the components of $U(x)$ and also $u(x)$, which has to be considered as a derivative of order -1 of one of the components of U. Its emission is the emission defined by $h(x, u, p)$. Thus our assertion follows from Proposition 124.2°, 3°.

In its assumption (124.3) and in its assertion we have to replace $m(\tau)$, $M(\tau)$, $s(\sigma) + 1$ by m, $M - 1$, $s - 1$;

$$t(\sigma) = s(\sigma) + 1 - m(\sigma) = s - m - 1; \qquad 2\ell' = \ell + 2 \text{ or } \ell + 1;$$

thus (124.3) becomes

$$m + \ell + 2 < s \leq m + M \quad (\ldots \leq s \leq \ldots \text{ if } \ell \text{ is odd}).$$

127. <u>Properties in the large.</u>

<u>Theorem</u> 127.1. (Existence) <u>Cauchy's problem has solutions.</u>

<u>Proof.</u> According to Lemma 126.1 there are closed subsets K_α of K such that

1) Cauchy's problem has a solution $u_\alpha(x)$ near K_α;

2) $K = \bigcup_\alpha K_\alpha$; a point of K belongs to a finite number of K_α.

Let N_α be a neighborhood of K_α in X so small that:

1) $u_\alpha(x)$ is defined on the part of N_α belonging to the positive side of K;

2) $u_\alpha(x) = u_\beta(x) = \ldots = u_\gamma(x)$ on $N_\alpha \cap N_\beta \cap \ldots \cap N_\gamma$.

(Lemma 126.2.1 proves the existence of such N_α.)

Define $u(x) = u_\alpha(x)$ on N_α; $u(x)$ is defined on the part of $\bigcup_\alpha N_\alpha$ belonging to the positive side of K; $u(x)$ is a solution of Cauchy's problem.

<u>Theorem</u> 127.2. (Uniqueness) <u>Let</u> u, D <u>be a solution of Cauchy's problem.</u> <u>We</u>

can obviously lessen D so that:

$$(127.1)\quad \begin{cases} h(x, u, p) \text{ is hyperbolic on } D \cup K \text{ (see n°98)}; \\ \text{on } D \text{ any maximal}^{(1)} \text{ timelike}^{(2)} \text{ path originates from } K. \end{cases}$$

Then u is the only solution of Cauchy's problem defined on D.

Proof. Let $u^*(x)$ be another solution of Cauchy's problem defined on D. Assume $S(u - u^*)$ non void; choose $x \in S(u - u^*)$; let $\widehat{yx}$ be a maximal timelike(2) path originating from $y \in S(u - u^*)$. According to Lemma 126.2.1 $y \notin K$; thus u and u* are defined near the point y, which is an isolated point of $\mathcal{E}_-(y) \cap S(u - u^*)$; that is impossible, according to Lemma 126.2.2.

Notation. Let $\hat{\Gamma}_x = \bigcap_W \Gamma_x(h(x, W, p))$; let $\hat{C}_x$ be the cone dual to $\hat{\Gamma}_x$; (when $\hat{\Gamma}_x$ is void, then $\hat{C}_x$ is the whole hyperplane tangent to X at x). We shall consider domains D such that

$$(127.2)\quad \begin{cases} \text{on } D \text{ any path whose length is finite and whose semi-tangents at } x \text{ belong} \\ \text{to } \hat{C}_x \text{ originates from } D \cup K. \end{cases}$$

Corollary 127.1. Let u, D and u*, D* be two solutions of Cauchy's problem such that:

u, D satisfies (127.1); D* satisfies (127.2).

Then

$$u = u^* \text{ on } D \cap D^*.$$

Proof. u, $D \cap D^*$ satisfy (127.1) since $C_x \subset \hat{C}_x$.

Hence:

Corollary 127.2. Assume that the tangents of K at x are outside $\hat{C}_x$. There is a unique solution $\hat{u}$, $\hat{D}$ of Cauchy's problem such that:

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(1) Not belonging to another such path.

(2) With respect to h(x, u, p).

$\hat{u}$, $\hat{D}$ <u>satisfies</u> (127.1); $\hat{D}$ <u>satisfies</u> (127.2); $\hat{D}$ <u>is maximal</u>.

Obviously, if u, D is any solution such that (127.1) holds and that D is maximal, then $\hat{D} \subset D$, $\hat{u} = u$ on $\hat{D}$.

§3. Non-linear systems

128. Notation and results. The notation is similar to the notation defined by n °125;

$$U(x) = (u_1(x), \ldots, u_n(x))$$

is an unknown vector whose components $u_\sigma(x)$ are functions defined on D;

$$A(x, U) = (a_1(x, U), \ldots, a_n(x, U))$$

is a vector whose component $a_\tau(x, U)$ is a function of x, of $u_\sigma(x)$ and of its derivatives of order $\leq s(\sigma) - t(\tau) + 1$; $\operatorname{Sup}_W \left| \dfrac{\partial^M a_\tau(x, W)}{\partial x^\alpha \partial w_1^{\beta_1} \ldots} \right|$ is a locally square integrable function of x. H(x, U, p) is a matrix whose elements are

$$h_{\tau\sigma}(x, U, p) = \sum_{\lambda_1 \leq \lambda_2 \leq \ldots} \frac{\partial a_\tau(x, U)}{\partial(p_{\lambda_1} \ldots p_{\lambda_\mu} u_\sigma)} p_{\lambda_1} \ldots p_{\lambda_\mu} \quad (\mu = s(\sigma) - t(\tau) + 1);$$

we define $m(\sigma)$ as in n°114; we assume an inequality more strict than (114):

(128.1) $$s(\sigma) - t(\tau) + 1 < m(\sigma) \text{ when: } h_{\tau\sigma} \in H_\lambda, \text{ order } H_\lambda > 1.$$

$$V(x) = (v_1(x), \ldots, v_n(x))$$

is a given vector defined near K; H(x, V, p) is hyperbolic on K, which is space-like; $v_\sigma(x)$ has locally square integrable derivatives of order $\leq s(\sigma) + 1$; $a_\tau(x, V)$ and its derivatives of order $< t(\tau) - 1$ are $= 0$ on K;

(128.2) $$m(\sigma) + \ell + 2 < s(\sigma), \quad t(\tau) < M + 1.$$

Cauchy's problem asks for D ($K \subset \mathring{D}$) and U(x) such that: $u_\sigma(x)$ has locally square integrable derivatives of order $\leq s(\sigma)$; $u_\sigma(x) - v_\sigma(x)$ and its derivatives of order $< s(\sigma)$ are $= 0$ on K; $A(x, U) = 0$ on D.

Theorems 127.1 and 127.2, Corollaries 127.1, 127.2 hold (a, u, h being replaced by A, U, H).

Proof when $H(x, W, p)$ is a diagonal matrix. The system $A(x, U) = 0$ can be studied by the method applied to the equation $a(x, u) = 0$ in §2. Then:

Note 127. In the assumption (128.2) M can be replaced by $M(\tau)$.

Proof. Define $B(x, U, p)$ as in n°117; upon application of (128.1), $b_{\rho\tau}(x, U, p)$ contains derivatives of u_σ of order $\leq s(\sigma) - t(\tau') + 1 < s(\sigma) - s(\rho) + m(\rho)$; thus $b_{\rho\tau}(x, U, p)a_\tau(x, U)$ contains derivatives of u_σ of

$$\begin{aligned} \text{order} &< s(\sigma) - s(\rho) + m(\rho) && \text{if } \rho \neq \sigma \\ &= m(\rho) && \text{if } \rho = \sigma. \end{aligned}$$

Thus the matrix H related to the equation(1)

$$B(x, U, p)A(x, U) = 0 \tag{128.3}$$

is diagonal: existence and uniqueness theorem hold for this equation. Hence the uniqueness ~~theorems/for~~ hold the equation $A(x, U) = 0$ under assumptions which turn out to be (128.2). Hence also the existence theorem holds if the uniqueness theorem holds for the hyperbolic matrix $H(x, U, p)B(x, U, p)$ (U given); that is; if U is sufficiently regular; that is: under assumptions stronger than (128.2). If we weaken these assumptions, then this existence theorem remains true so long as a priori bounds exist for the solution of (128.3) (as in n°119, Proof of Proposition 115.2). Hence finally, using Note 127, the existence theorem under the assumptions (118.2).

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(1) This equation is not necessarily quasi-linear if some of the H_λ have the order 1