Hodge theory: What is it? How can it be used in algebraic geometry?

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I. Origins

- Integrals of algebraic functions

$$\int \frac{p(x,y)}{q(x,y)}\,dx, \qquad f(x,y(x)) = 0.$$

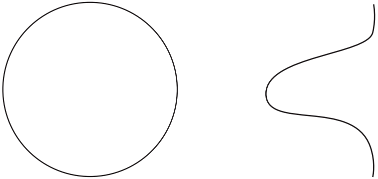
For example

$$\begin{cases} \int \frac{q(x)\,dx}{\sqrt{p(x)}} = \int \frac{q(x)\,dx}{y(x)} = \int \omega \\ y(x)^2 = p(x). \end{cases}$$

For deg $p(x) \leqq 2$ can be evaluated using "elementary functions," $r(x), \log x, \arcsin x, \ldots$.

For deg $p(x) \geqq 3$ this is not the case; e.g.,

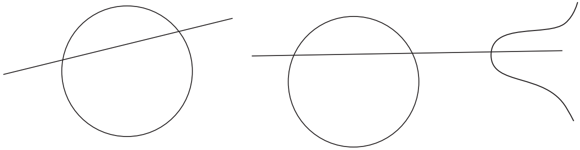
$$\int \frac{dx}{y}, \qquad y^2 = p_3(x)$$



- Abel — two ideas

Abelian sum: $I(t) = \sum \int^{(x_i(t),y_i(t))} \omega$ where

$$(x_i(t), y_i(t)) = \begin{cases} \text{solutions of} \\ \begin{cases} f(x,y) = 0 \\ g(x,y,t) = 0. \end{cases} \end{cases}$$



$$x^2 + y^2 = 1 \qquad\qquad y^2 = p_3(x)$$

Abel's theorem: $I(t) =$ elementary function.

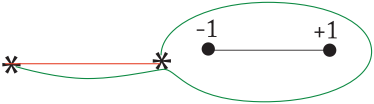
Second idea:

Inversion: Define $x(u), y(u)$ by

$$u = \int^{(x(u),y(u))} \omega$$

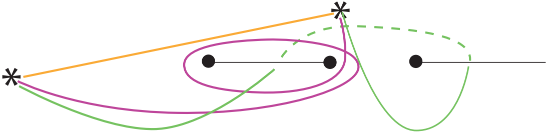
Ex: $u = \int^{(\sin u, \cos u)} \frac{dx}{\sqrt{1-x^2}}$.

Periodic



Ex: $$u = \int^{(p(u),q(u))} \frac{dx}{\sqrt{p_3(x)}}.$$

Doubly periodic

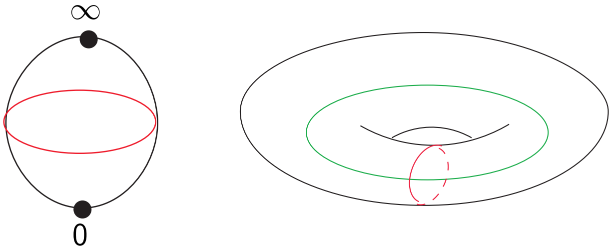


Abel's theorem $\implies$ addition theorem for

$$\begin{cases} \sin x : \sin(x+x') = \sin x \cos x' + \sin x' \cos x \\ p(x) = \begin{Bmatrix} \text{Weierstrass} \\ p\text{-function} \end{Bmatrix} : \begin{matrix} \text{taking } x' = x \text{ gives} \\ p(2x) = R(p(x), p'(x)) \end{matrix} \end{cases}$$

In both cases above since the integrand is $\frac{dx}{y}$ we have $y(u) = x'(u)$.

- Reimann —- concept of *Riemann surface* X



— X = 1-dimensional complex manifold, local holomorphic coordinate z

— ω = meromorphic 1-form, $\omega = f(z)dz$

$$\omega = \frac{q(x)dx}{\sqrt{p(x)}}, \quad x = \frac{1}{x'} \text{ and } dx = -\frac{dx'}{x'^2}$$

gives ω holomorphic on $X \iff \int \omega < \infty \iff \deg q \leqq \lceil \deg p/2 - 2 \rceil$; these are *integrals of first kind*; space of these is $H^0(\Omega^1_X)$.

— DEFINITION (ABEL): The genus $g(X)$ is dim(integrals of first kind);
$g(x) = \dim H^0(\Omega^1_X) = \dim H^{1,0}(X)$.

— THEOREM (RIEMANN): $\dim H^1(X) = 2g$

$$H^1_{\mathrm{DR}}(X, \mathbb{C}) \cong H^{1,0}(X) \oplus H^{0,1}(X), H^{0,1}(X) = \overline{H^{1,0}(X)}.$$

— also bilinear form

$$\begin{cases} \int_X \omega \wedge \omega = 0, & dz \wedge dz = 0, \\ \left(\frac{i}{2}\right) \int_X \omega \wedge \bar{\omega} > 0, & \left(\frac{i}{2}\right) dz \wedge d\bar{z} > 0. \end{cases}$$

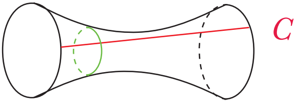
- $H^1(X)$ gives a point in the space of Hodge structures $\left[\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right]$, $\mathrm{Im}(z_0/z_1) > 0$ of weight 1, dim $= 2$.
- Picard-Poincaré — single and double integrals

$$\begin{cases} \int p(x,y,z)dx + q(x,y,z)dy & f(x,y,z(x,y)) = 0, \\ \iint r(x,y,z) dx \wedge dy. \end{cases}$$

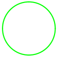
— definition of integrals of first kind as before; gives $H^0(\Omega^1_X) = H^{1,0}(X), H^0(\Omega^2_X) = H^{2,0}(X)$ and

$$\begin{cases} H^1(X) \cong H^{1,0}(X) \oplus H^{0,1}(X), & H^{0,1}(X) = \overline{H^{1,0}(X)}, \\ H^2(X) \cong H^{2,0}(X) \oplus \, ? \, \oplus H^{0,2}(X), & H^{0,2}(X) = \overline{H^{2,0}(X)}. \end{cases}$$

— class $[C]$ of algebraic curve in $X \implies [C]$ belongs to ?; locally $C = g(z, w) = 0$,
$\omega = a(z, w) dz \wedge dw \implies \omega\big|_C = 0$



$$x^2 + y^2 + z^2 = 1$$

$[C] \neq 0$ because $C \cdot \mathfrak{h} = 1$ ($C =$ ) and $\mathfrak{h} =$ 

— Picard: $h \in H_2(X)$ is class of hyperplane section; then

$$H_3(X) \xrightarrow[\sim]{h} H_1(X)$$

- Lefschetz: $\dim X = n$
  - — $b_{2p+1}(X) \equiv 0 \pmod 2$
  - — $b_{2p}(X) \geqq 1$

$$\text{(HL)} \quad h^k : H_{n+k}(X) \xrightarrow{\sim} H_{n-k}(X) \qquad \text{(Hard Lefschetz)}$$

- Proof of (HL) was incomplete (Picard's proof of $n = 2$ case was OK — method didn't extend)

II. Topology of algebraic varieties

- $H^k(X,\mathbb{C}) \cong H_{\mathrm{DR}}(X) = \dfrac{\{\omega \in A^k(X) : d\omega = 0\}}{\{\omega = d\eta, \eta \in A^{k-1}(X)\}}$

$A^k(X) = \overset{p+q=k}{\oplus} A^{p,q}(X)$ where

$$A^{p,q}(X) = \left\{ \omega = \sum_{\substack{|I|=p \\ |J|=q}} f_I dz^I \wedge d\bar{z}^J \right\} = \overline{A^{q,p}(X)} \text{ leads to}$$

Hodge:

$$H^k(X,\mathbb{C}) \cong \overset{p+q=k}{\oplus} H^{p,q}(X), \qquad H^{q,p}(X) = \overline{H^{p,q}(X)},$$

$$H^{p,q}(X) \cong H^q(\Omega_X^p) \qquad (? = H^1(\Omega_X^1))$$

$$H^0(\Omega_X^k) \hookrightarrow H^k(X)$$ (holomorphic forms are closed and not exact).

- For X a smooth algebraic variety, $H^k(X) := V$ has a functorial *Hodge structure* (V, F) of weight k.

  Here $F^p = \bigoplus_{p' \geqq p} H^{p',q}$ is the decreasing *Hodge filtration*

$$H^{p,q} = F^p \cap \overline{F}^q, \qquad F^p \subset F^{p-1}.$$

- Hodge structures on cohomology have strong implication — e.g., $b_{2k+1}(X) \equiv 0(2)$;
- $X \xrightarrow{f} Y$ holomorphic fibration $\implies$ Leray spectral sequence degenerates at E_2; e.g., $\pi_1(Y)$ acts trivially on $H^q(X_y) \implies$ additively $H^k(X) \cong \bigoplus^{p+q=k} H^p(X_y) \otimes H^q(Y)$.
- X simply connected $\implies \pi_*(X) \otimes \mathbb{Q}$ is determined by $H^*(X)$ (Massey products $= 0$, etc.); also results on $\widehat{\pi_1(X)}_{\mathbb{Q}}$.

- For a smooth $X^n \subset \mathbb{P}^N$, $L \in H^2(X) =$ class of a hyperplane section, Hodge proved (HL)

$$H^{n-k}(X) \xrightarrow{L^k} H^{n+k}(X)$$

$\implies$ there is an sl_2-action on $H^*(X)$ where $L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\implies$ $H^*(X) = \oplus L^k H^{*-k}(X)_{\mathrm{prim}}$.

- For $V = H^{n-k}(X)_{\mathrm{prim}}$ there is a *polarized Hodge structure* (V, Q, F) where

$$Q(\varphi, \omega) = \int_X L \wedge \varphi \wedge \omega.$$

Deligne: For X a complex algebraic variety, $H^k(X)$ has a functorial *mixed Hodge structure* (V, W, F) where $W_\ell \subset W_{\ell+1}$ is the increasing *weight filtration*

$$\mathrm{Gr}_k^W(V) = W_k/W_{k-1} \text{ has a Hodge structure of weight } k$$
$$F^p\,\mathrm{Gr}_k^W(V) = F^p \cap W_k/W_{k-1}.$$

- Relative cohomology groups $H^k(X, Y)$, homology groups $H_k(X), \ldots$ all have MHS's and the usual exact sequences, dualities,...are all morphism of MHS's.
- Morphisms $\varphi : (V, W, F) \to (V', W, F')$ of mixed Hodge structures are *strict*

$$\begin{cases} \varphi(V) \cap W'_k = \varphi(W_k) \\ \varphi(V_{\mathbb{C}}) \cap F'^p = \varphi(F^p) \end{cases}$$

This simple linear algebra property has very strong implications on topology of algebraic varieties.

Ex: For $X \xrightarrow{f} Y$ with $\pi_1(Y)$ acting trivially on $H^*(X_y)$

$$H^p(X_y) \otimes H^q(Y) \xrightarrow{d_2} H^{p+2}(X_y) \otimes H^{q-1}(Y)$$

is a morphism between Hodge structures of different weights $\implies d_2 = 0$.

III. Uses of Hodge theory (A)

- Hodge theory has many aspects
  - — topology of algebraic varieties
  - — analysis, both complex analysis and PDE's
  - — geometry of algebraic varieties; e.g., classification and moduli.
- Usual applications are to restrict cohomological properties of varieties and morphisms between them. Here we will take a different perspective and will discuss how Hodge may be used to *construct* something.

By way of an illustration I will discuss what is the strongest result to date on a famous problem, the Shafarevich conjecture. This approach to this result was initiated by Katzarkov, and he is one of the co-authors of the final result.

- As an introduction I will consider the question

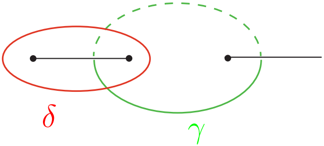
  Can one use Hodge theory to construct interesting analytic or meromorphic functions?

— For the elliptic curve in Weierstrass form

$$y^2 = x^3 + ax + b$$

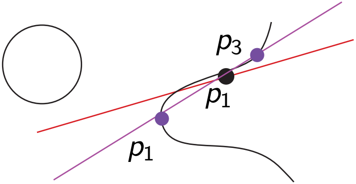
the inversion of the elliptic integral

$$u = \int^{(x(u),y(u))} \frac{dx}{y}$$



defines local analytic functions $x(u), y(u)$ with $y(u) = x'(u)$.

— Abel's theorem



$$u_1 + u_2 + u_3 = \text{ constant}$$

is used to show that $x(u), y(u)$ defined in $|u| < \epsilon$ may be extended to $|u| < 2\epsilon$ etc., leading to entire meromorhic functions $x(u) = p(u)$, $y(u) = p'(u)$.

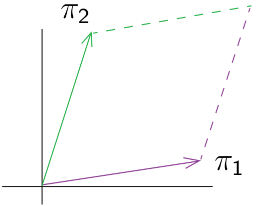
If

$$\pi_1 = \int_\gamma \frac{dx}{y}, \quad \pi_2 = \int_\delta \frac{dx}{y}$$

are the periods of $\int \frac{dx}{y}$ then a calculation gives

$$\left(\frac{i}{2}\right) \iint \frac{dx}{y} \wedge \overline{\frac{dx}{y}} > 0 \implies \mathrm{Im}(\pi_1/\pi_2) > 0$$

giving the parametrization of C by elliptic functions



$$= \mathbb{C}/\Lambda \xrightarrow{(p(u),p'(u))} \{C =: y^2 = x^3 + ax + b)\}$$

- A consequence is that

$$z := \int \frac{dx}{y}$$

gives a holomorphic function on the universal covering $\widetilde{C}$ of C. A far reaching generalization is

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Theorem (Katzarkov): *$X =$ smooth projective variety with $\pi_1(X)$ is nilpotent. Then for $\omega_1, \ldots, \omega_q$ giving a basis for $H^0(\Omega^1_X)$ the map $(\int \omega_1, \ldots, \int \omega_q) : \widetilde{X} \to \mathbb{C}^q$ has image a Stein variety. It contracts the compact subvarieties given by the inverse image in $\widetilde{X}$ of the set*

$(*)$

$$\{\text{subvarieties } Y \subset X \text{ with trivial map } H_1(Y, \mathbb{Q}) \to H_1(X, \mathbb{Q})\}$$

- A general question is: *How does one construct functions on the universal cover $\widetilde{X}$ of a smooth algebraic variety X?*

The above gives one answer when $\pi_1(X)$ nilpotent.

The other extreme is when there is a linear representation

$$\rho : \pi_1(X) \to \mathrm{GL}(N)$$

whose image is semi-simple and "big"?* We will illustrate by giving a classical method (due to Picard) when $X = \{\mathbb{P}^1 \backslash \{0, 1, t\}$.

- The algebraic curve

$$C_t = \{y^2 = x(x-1)(x-t)\}$$

  is smooth for $t \neq 0, 1, \infty$. Then

$$w(t) = \int_\gamma \omega / \int_\delta \omega$$

  is a locally defined analytic function on X. Analytic continuation gives a holomorphic mapping

$$\widetilde{X} \to \mathcal{H} = \{\mathrm{im}\, w > 0\}.$$

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*We may use any semi-simple reductive subgroup $G \subset \mathrm{GL}(N)$. "Big" means that the image $\Gamma = \rho(\pi_1(X))$ is Zariski dense.

In this way we have the bounded analytic function

$$\zeta = \frac{w-i}{w+i} \in \Delta = \{|\zeta| < 1\}$$

on $\widetilde{X}$.

- As noted we may think of $\mathcal{H} \subset \mathbb{P}^1$ as the set of Hodge structures of weight 1 and dimension 2

$$[\omega] \in \mathcal{H} \text{ and}$$
$$[\omega] \oplus [\bar{\omega}] \in H^{1,0} \oplus H^{0,1}.$$

Analytic continuation of the period ratio $w(t)$ around a closed curve in $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ gives

$$\begin{pmatrix} \gamma \\ \delta \end{pmatrix} \to \begin{pmatrix} a\,\gamma + b\,\delta \\ c\,\gamma + d\,\delta \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. This gives

$$(**) \qquad \begin{array}{ccc} \widetilde{X} & \longrightarrow & \mathcal{H} \quad = \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}(2) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H} \end{array}$$

- COROLLARY (PICARD): *An entire meromorphic function $f(z)$ with $f(z) \neq 0, 1, \infty$ is constant.*

- What is important here is the diagram $(**)$, meaning a family of Hodge structures parametrized by X. In general the Hodge structures of a fixed type form a homogeneous complex manifold

$$D = G_{\mathbb{R}}/H$$

and one may try to construct variations of Hodge structure

$$(***) \qquad \Phi : X \to \Gamma \backslash D$$

where

$$\rho : \pi_1(X) \to \Gamma \subset G_{\mathbb{Z}}$$

is the monodromy representation. If we have a family of smooth algebraic varieties parametrized by X, we get such a Φ. In general this may or may not be possible, and an existence theorem is needed to in general produce a Φ. Stated informally the result here is

Theorem (Simpson): *For X compact, if $\pi_1(X)$ has a faithful linear representation, then there is a variation of Hodge structure $(***)$.*

This is a beautiful existence theorem whose proof involves solving a non-linear Yang-Mills type equation.†

- That's the good news. The bad news is that in general there are no non-constant analytic functions on D. However D has a compact dual $\check{D}$ $(= \mathbb{P}^1$ above) and the ratio of the invariant volume forms, which in the $\mathbb{P}^1$ case is

$$f = \frac{(1+|\zeta|^2)^2}{(1-|\zeta|^2)^2},$$

† There is an existence theorem due to Mochizuki when X is a Zariski open in a complete algebraic variety (like $\mathbb{P}^1 \backslash \{0, 1, \infty\}$). I don't know the exact statement well enough to discuss it here.

gives an exhaustion function $f : D \to \mathbb{R}$ such that when we have a VHS $(***)$ we have that for $F = \Phi \circ f$

$$(i/2)\partial\overline{\partial} \log F \geqq 0$$

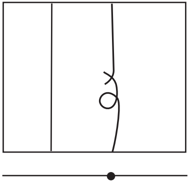
with > 0 where Φ_* is 1-1.

One knows that pluri-sub-harmonic functions lead to the construction of analytic functions. The upshot is the result due to Eyssidieux, Katzarkov, Pantev and Ramachandran saying that quotienting X by subvarieties satisfying a condition similar to $(*)$ the resulting space has universal covering a Stein variety.[‡]

[‡]Their proof also uses Hodge theory but in a more elaborate fashion than as done here. An open question is whether $\widetilde{X}$ may be realized as a *bounded* analytic subvariety in a $\mathbb{C}^m$. What is known is that $\widetilde{X}$ is hyperbolic; i.e., any holomorphic map $\mathbb{C} \to \widetilde{X}$ is constant.

IV. Uses of Hodge theory (B)

- Although Hodge theory began as a tool to study the topology of smooth varieties, many of its applications are about the topology of singular ones. Of particular importance is the situation of a family X_t of varieties where for $t \neq 0$, X_t is smooth but where X_0 is singular



$$\begin{array}{c} \mathcal{X} \\ \downarrow{\scriptstyle \pi} \\ \Delta \end{array}$$

Over the punctured disc $\Delta^* = \{0 < |t| < 0\}$ we have a topological fibre bundle and for any such the basic invariant is monodromy

$$T : H^n(X_{t_0}, \mathbb{Z}) \to H^n(X_{t_0}, \mathbb{Z}).$$

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For just a topological fibration over S^1, T can be any automorphism of $H^n(X_{t_0}, \mathbb{Z})$ induced by a gluing homeomorphism $f : X_{t_0} \xrightarrow{\sim} X_{t_0}$ ($T = f_*$). However in algebraic geometry there are strong restrictions on T. Using the Jordan normal form to write

$$T = T_s T_u$$

where T_s is semi-simple

$$T_s = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_m \end{pmatrix}$$

and T_u is unipotent

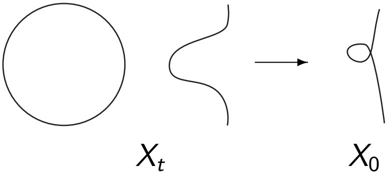
$$\log T_u = N = \begin{pmatrix} 0 & * & * \\ & \ddots & * \\ 0 & & 0 \end{pmatrix}$$

there is the

Monodromy theorem: (i) $\mu_\alpha = e^{2\pi i \lambda_\alpha}$, $\lambda_\alpha \in \mathbb{Q}$; (ii) $N^k = 0$ *for some* $k \leqq n+1$.

- As mentioned, Hodge theory involves complex analysis, differential geometry, and Lie theory. I will illustrate how these may be combined to give a generalizable argument for the monodromy theorem for a family of cubic curves

$f(x, y, t) = 0$



Schwarz lemma: $h : \Delta \to \Delta$, $h(0) = 0 \implies |h(t)| \leqq |t|$.

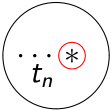
For the invariant Poincaré metric $\omega = \frac{dz d\bar{z}}{(1-|z|^2)^2}$ the Schwarz lemma gives

$$(\natural) \qquad h^*(\omega) \leqq \omega.$$

- If $\varphi = g(z)dz, d\bar{z}$ is any metric on Δ whose Gauss curvature $K = (i/2)\partial\bar{\partial}\log g(z)$ has $K \leqq -1$ Ahlfors extended (♮) to
$$h^*(\varphi) \leqq \omega$$
(h is distant decreasing).
- From the above we have

$$(\natural\natural) \qquad \begin{array}{ccc} \mathcal{H} & \xrightarrow{\widetilde{\Phi}} & \mathcal{H} \\ \downarrow & & \downarrow \\ \Delta^* & \xrightarrow{\Phi} & \{T^k\}\backslash\mathcal{H} \end{array} \qquad \Phi(t) = \int_{\gamma}\omega / \int_{\delta}\omega$$

— metric on Δ^* is $\frac{dt\, d\bar{t}}{|t|^2(\log t)^2} = \frac{dr\, d\theta}{r(\log r)^2}$. Length $|t| = r$ is $1/(\log r)^2 \to 0$ as $r \to 0$.

— 

and $e^{2\pi i} t_n :=$ circle through t_n

$d(t_n, e^{2\pi i} t_n) \to 0$ as $t_n \to 0$

— $d(\Phi(t_n), \Phi(e^{2\pi i} t_n)) = d(\Phi(t_n),\ T\Phi(t_n)) \to 0$

— $\Phi(t_n) = g_n \cdot w_0,\ g_n \in \mathrm{SL}_2(\mathbb{R})$

$$d(g_n w_0, T g_n w_0) \to 0$$

$$\|$$

$$d(w_0, g_n^{-1} T g_n w_0)$$

$\implies g_n^{-1} T g_n \to \mathrm{SO}(2)$ as $n \to \infty$

$\implies |\mu_\alpha| = 1$

$\implies$ (Kronecker) $\mu_\alpha = e^{2\pi i \lambda_\alpha},\ \lambda_\alpha = p_\alpha / q_\alpha \in \mathbb{C}$.

- Using the Ahlfor's lemma this argument may be carried over verbatim in the general case once we know that the holomorphic sectional curvatures in $\Phi_*(T\Delta) \subset TD$ are $\leqq -c, c > 0$.
- What about the $N^{n+1} = 0$ part of the monodromy theorem?
  Replacing t by t^m we may assume $T = e^N$.
  Referring to ($\natural\natural$) set

$$\widetilde{\Psi}(w) = \exp(-wN)\widetilde{\Phi}(w).$$

  Then

$$\widetilde{\Phi}(w+1) = T\widetilde{\Phi}(w) \implies \widetilde{\Psi}(w+1) = \widetilde{\Psi}(w)$$
$$\implies \Psi : \Delta^* \to \mathbb{P}^1 \quad (= \check{D} \text{ in this case})$$

  Then Ψ extends to $\Psi : \Delta \to \mathbb{P}^1$.

Idea: $d_{\Delta^*}(t_0, t) \sim -\log|t|$.
Comparing metrics in Δ and $\mathbb{P}^1$ as above leads to no essential singularity $\implies \Psi(t) =$ meromorphic function, $\Psi(0) \in \mathbb{P}^1$

$$\Phi(t) \sim \exp(\log tN)\Psi(0).$$

Thus we may approximate any VHS by a nilpotent orbit (Schmid). This in effect moves algebraic geometry into the realm of Lie theory.

- Lie theoretic analysis of the nilpotent orbit leads (after some work) to the $N^{n+1} = 0$ result.

V. Reprise: What lies ahead for Hodge theory?

We have discussed how Hodge theory lies at the confluence of topology, analysis and geometry. One may ask: *What are the major questions that Hodge theory is seeking to address?*[§] I will not attempt to answer this question but will illustrate it with one further aspect of Hodge theory, namely the arithmetic side of the subject.

- We consider an integral

$$I(\xi) = \int_0^\xi f(x, y(x))dx$$

  where $f(x, y) \in \mathbb{Q}[x, y]$. The question is: For $\xi \in \mathbb{Q}$, what sort of number is $I(\xi)$?

[§] Other than the Hodge conjecture, which is the central problem in the subject (and which is one of the Clay millennium problems).

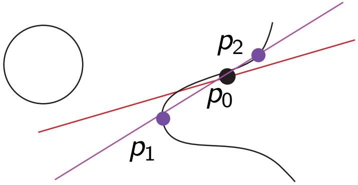
Very informally stated, an old result of Siegel is that, assuming $I(\xi)$ is not an algebraic function of ξ,

$$I(\xi) \text{ is a transcendental number.}$$

A related conjecture, due to Kontsevich-Zagier and here informally stated, is

> *For $\xi_1, \ldots, \xi_m$ with $\xi_i \in \overline{\mathbb{Q}}$, $|\xi_i| < \epsilon$, any algebraic relation among the $I(\xi_i)$'s defined over $\overline{\mathbb{Q}}$ will arise from geometry.*

Ex: For the cubic curve



Here is a flex-tangent and is a nearby line. Abel's theorem gives for $\omega = dx/y(x)$

$$\int^{p_1} \omega + \int^{p_2} \omega + \int^{p_3} \omega = \text{ constant}$$

or

$$I(\xi_1) + I(\xi_2) + I(\xi_3) = \text{ constant.}$$

- In general if X is an algebraic variety defined over $\mathbb{Q}$ and ω is an algebraic differential form on X that is also defined over $\mathbb{Q}$, then for a topological cycle $\Gamma \in H_m(X, \mathbb{Z})$ the period

$$\int_\Gamma \omega$$

is a complex number. There is then the general period conjecture, as formulated by Grothendieck and again very informally stated is

> *Any algebraic relation among the periods is due to the presence of an algebraic cycle in $X \times X$.*

This is a very deep existence question: it says that as a consequence of an *arithmetic* assumption on the periods of algebraic integrals a *geometric* construction is possible.

- I mention the above in part because even though the usual statement of the Hodge conjecture makes no mention of the arithmetic aspect of Hodge theory, its validity would have arithmetic consequences. In my view it is the better understanding of this aspect of Hodge theory that is necessary to make progress on the Hodge conjecture itself.