Extended Period Mappings*

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- Theme of the lecture is global properties of period mappings with applications to the geometry of completions of moduli spaces

I. Introduction
II. Construction and properties of extended period mappings
III. Geometry of extension data
IV. Basic formula
V. Applications to moduli of general type algebraic surfaces[†]
    A. Infinite local monodromy
    B. Finite local monodromy

[†]This section is based in part on joint work with Radu Laza and on the work of and discussion with Marco Franciosi, Rita Pardini and Sönke Rollenske.

I. Introduction

- Given $(\overline{B}, Z; \Phi)$ where
  - — $\overline{B}$ is a smooth projective variety, $Z = \cup Z_i$ is a normal crossing divisor in $\overline{B}$ and $B = \overline{B} \backslash Z$;
  - — $\Phi : B \to \Gamma \backslash D$ is a *period mapping* where $D = G_{\mathbb{R}}/H$, $\rho : \pi_1(B) \to \Gamma \subset G_{\mathbb{Z}}$ is monodromy.[‡]

[‡]WLOG we assume that the local monodromy around Z_i is of infinite order. Then Φ is proper and $P \subset \Gamma \backslash D$ is a closed analytic subvariety.

In the extensive literature there are

— *global results on B* (theorem of the fixed part, image $\Phi(B) = P \subset \Gamma\backslash D$ is an algebraic variety over which the Hodge line bundle $\overset{p}{\otimes} \det F^p := L \to P$ is ample, algebraicity of Hodge loci)

— *local results on neighborhoods* $\Delta^{*k} \times \Delta^{\ell}$ in B of points at infinity in Z. (Nilpotent and sl_2-orbit theorems, existence and properties of several variable limiting mixed Hodge structures, Chern forms of the extended Hodge bundles; these results are used in proving global results on B.)

This talk will be concerned with global results on $\overline{B}$; specifically

— extensions of Φ mapping to completions of P

$$\begin{tikzcd} & \overline{P}_T \arrow[d, "f"] \\ \overline{B} \arrow[ur, dashed, "\Phi_T"] \arrow[r, "\Phi_S"'] & \overline{P}_S \end{tikzcd}$$

— properties of $\overline{P}_T, \overline{P}_S$ (e.g., ample line bundles on them)
— geometry of the fibres of f (this is of particular interest)
— will mostly restrict to the case $\dim B = 2$ and will assume $\dim \Phi(B) = 2$.[§]
— traditionally one seeks completions of $\Gamma\backslash D$ to which $\overline{B}$ maps (cf. [KU]); the approach here is *relative* in that the completions $\overline{P}_T$ and $\overline{P}_S$ depend on $(\overline{B}, Z; \Phi)$.

[§]A fundamental invariant of any VHS is monodromy, which lives on a general 2-dimensional section of the parameter space.

Will also discuss some applications to moduli of general type algebraic surfaces, illustrated by one interesting surface. Main emphasis will be on the case of infinite local monodromy around the Z_i; will also briefly discuss the finite local monodromy case.

II. Construction and properties of extended period mappings[¶]

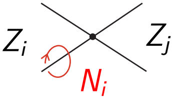
- Given V, Q
  - polarized Hodge structure (PHS) is $(V, F), F = \{F^p\}$ where the Hodge filtration F satisfies Hodge-Riemann I, II
  - mixed Hodge structure (MHS) is (V, W, F), where $W = \{W_k\}$ is the weight filtration
  - limiting mixed Hodge structure (LMHS) is $(V, W(N), F)$ where $N \in \mathrm{End}_Q(V)$ is a nilpotent operator with $N : F^p \to F^{p-1}$ and

$$\begin{cases} N : W_k(N) \to W_{k-2}(N) \\ N^k : W_{n+k}(N) \xrightarrow{\sim} W_{n-k}(N) \end{cases}$$

    uniquely characterizes the monodromy weight filtration $W(N)$.

[¶] A general reference for Hodge theory is [CM-SP]. For limits of Hodge structures see [CK] and the references cited therein.

The Q will be understood for MHS's and LMHS's. Will also consider $(V, W(\sigma), F)$ where $\sigma = \mathrm{span}_{\mathbb{Q}^+}\{N_1, \dots, N_k\}$ is a monodromy cone. When $\dim B = 2$ we have



$$\sigma = \begin{cases} \sigma_i \\ \sigma_{ij} \end{cases}$$

— equivalence class $[V, W(\sigma), F] := \mathcal{L}$ where

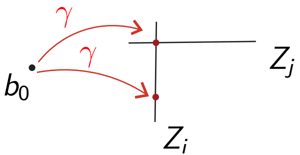
$$F \sim \exp(\lambda N)F, \quad \lambda \in \mathbb{C} \text{ and } N \in \sigma$$

— assuming T_i unipotent‖ there are canonical extensions $F_e^p \to \overline{B}$

— extended Hodge line bundle $L_e = \overset{p}{\otimes} \det F_e^p$.

‖With slightly more elaborate statements this assumption may be dropped.

Definition: $\overline{P}_T$ = quotient by Γ of the set $\{(\gamma, [V, W(\sigma), F_b)]\} = \{\gamma, \mathcal{L}_b\}$ where $\gamma = \overline{b_o b}$



- Given a MHS (V, W, F) the associated graded is a direct sum of PHS's. For $\mathcal{L} = [V, W(\sigma), F]$, $\mathrm{Gr}(\mathcal{L})$ is well-defined.

Definition: $\overline{P}_S$ = quotient by Γ of the set $\{\gamma, \mathrm{Gr}(\mathcal{L}_b)\}$.

- We have

$$\begin{tikzcd} & \overline{P}_T \arrow[d] \\ \overline{B} \arrow[ur, dashed] \arrow[r] & \overline{P}_S \end{tikzcd}$$

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In the following we assume $\dim B = 2$.

Theorem:** (i) $\overline{P}_S$ *is a compact analytic surface.*
(ii) *The Hodge line bundle descends to an ample line bundle on* $\overline{P}_S$.

— Regarding (i) essential case is $Z = \cup Z_i$ where $L_e \cdot Z_i = 0$. Then

$$\begin{aligned} \dim B = 2 &\implies \|Z_i \cdot Z_j\| < 0 \qquad \text{(Hodge index theorem)} \\ &\implies Z \text{ contracts to a normal singular point (Grauert).} \end{aligned}$$

**Part (i) is in [GGLR] and part (ii) follows from there and [GGR] for the descent result.

- Regarding (ii), even if we know that $L_e\big|_Z \cong \mathcal{O}_Z$ there are generally non-trivial obstructions to trivialize L_e in a neighborhood of Z. Proof involves
  - new ingredient in Hodge theory (semi-global representation of Φ by period matrices)
  - observation that $L =$ pullback of $\mathcal{O}(1)$ under the Plücker embedding

    $$D_i \subset \prod^{p} \mathbb{P}(\wedge^{h_p} F^p), \quad h_p = \text{rank } F_p,$$

    applied to the maps
    $\mathrm{Gr}(\mathcal{L}) \to \{\text{Mumford-Tate domains } D_i\}$.
- Result is conjectured to hold in general. True when the original period domain D is Hermitian symmetric and Γ is arithmetic (Baily-Borel).

Theorem: (i) $\overline{P}_T$ *is a compact analytic variety.*[†] (ii) *Assuming* $\Phi : B \to P$ *does not contract any curve,*[‡] *there exists* m_0 *and* $a_i > 0$ *such that*

$$L_{e,a} := mL_e - \sum_i a_i Z_i$$

is ample for $m \geqq m_0$.

[†] This result is also conjectured to be the case for general dim B. When Γ is arithmetic and has a fan it follows from the theory developed in [KU]. In the non-classical case the existence of a fan seems to be quite restrictive.

[‡] This assumption can be removed with a slightly more elaborate statement of the result.

Regarding the proof of (ii), the a_i are chosen so that for each j

$$Z_j \cdot \sum_i a_i Z_i > 0.$$

That this is possible is a property of negative definite symmetric matrices. The a_i reflect the nature of the singularity to which Z contracts.

In summary

- $\overline{P}_S = \mathrm{Proj}\,(L_e)$
- $\overline{P}_T = \mathrm{Proj}\,(L_{e,a})$

III. Geometry of extension data

Still assuming that $\dim B = \dim \Phi(B) = 2$, in the diagram

$$\begin{tikzcd}
 & \overline{P}_T \arrow[d, "f"] \\
\overline{B} \arrow[ur, "\Phi_T"] \arrow[r, "\Phi_S"'] & \overline{P}_S
\end{tikzcd}$$

- Z_i not a fibre of $\Phi_S \implies \Phi_T\big|_{Z_i^*}$ is like a usual period mapping
- Z_i is a fibre of $\Phi_S \iff \mathcal{L}\big|_{Z_i}$ has locally constant $\mathrm{Gr}(\mathcal{L})$.

Taking the second case, for simplicity assume along Z_i^* we have VLMHS $\mathcal{L}$ where $\mathrm{Gr}(\mathcal{L}) = \{H^0, \dots, H^m\}$ is constant.*
The fibres of f are compact analytic subvarieties of a complex manifold that is an iterated fibration of $\mathrm{Ext}^1_{\mathrm{MHS}}(\bullet, \bullet)$'s that give extension data of levels $1, 2, \dots$ †

- $(V, F), (V', F')$ Hodge structures of weights $k > k'$

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathrm{MHS}}(V, V') & = & \frac{\mathrm{Hom}_{\mathbb{C}}(V,V')}{F^0\,\mathrm{Hom}_{\mathbb{C}}(V,V') + \mathrm{Hom}_{\mathbb{Z}}(V,V')} \\ \| & & \\ E & \cong & \mathbb{C}^m/\Lambda, \quad \Lambda \text{ discrete} \end{array}$$

*The $\mathcal{L}\big|_{Z_i}$ are related to the unipotent variations of mixed Hodge structure studied in [HZ]. The fact that we have *limiting* MHS's provides rich additional structure.

†$\mathrm{Ext}^q_{\mathrm{MHS}}(\bullet, \bullet) = 0$ for $q \geqq 2$.

Tangent spaces to variations of the extension data of levels $1, 2, \ldots$ lie over the red.

- $k' = k - 1$ gives for $\mathrm{Hom}_{\mathbb{C}}(V, V')$

$$\underbrace{(k-1,-k)\oplus\cdots\oplus(0,-1)}_{F^0}\oplus\underbrace{\underbrace{(-1,0)}\oplus\cdots\oplus(-k,k-1)}_{T_eE}$$

$E =$ compact complex torus with $E \supset E_{ab}$ where $T_eE_{ab} \subset \underbrace{(-1,0)}$

- $k' = k - 2$ gives

$$\underbrace{(k-2,-k)\oplus\cdots\oplus(0,-2)}_{F^0}$$

$$\oplus\underbrace{\underbrace{(-1,-1)}\oplus(-2,0)\oplus\cdots\oplus(-k,k-2)}_{T_eE}$$

connected analytic subgroup S whose T_eS is over $\underbrace{\qquad}$ is a $\mathbb{C}^{*k}$.

- $k' = k - 3$ gives

$$\underbrace{(k-3,-k) \oplus \cdots \oplus}_{F^0}$$

$$\underbrace{\underbrace{(-1,-2) \oplus (-2,-1)} \oplus \cdots \oplus (-k, k-3)}_{T_e E}.$$

This will imply that there is no VLMHS with non-trivial tangent space to level 3 extension data. Thus up to integration constants the fibres of f are given by extension data of levels 1 and 2.

Remark: For the period matrices of a VMHS of Hodge-Tate type (the $H^{2p} = \oplus \mathbb{Q}(-p)$'s) the

- level 1 extension data is trivial
- level 2 extension data is given by $\log t_i$'s
- level 3 extension data is given by $Li_2 t_i$'s
- d (level 3) $\in$ level 2 $\implies$ ODE expressing $Li_2 t_i$ in terms of $\log t_i$'s, etc.

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- Along Z_i^* the level 1 the extension data gives

$$\Phi_1 : Z_i^* \to E_{ab} \subset E \qquad \text{(actually } E_{ab} + c\text{)}$$

$$\begin{tikzcd}
\rightsquigarrow Z_i \arrow[rr] \arrow[dr] & & E_{ab} \\
& \mathrm{Alb}\, Z_i \arrow[ur] &
\end{tikzcd}$$

- Φ_1 (locally) constant $\rightsquigarrow \Phi_2 : Z_i^* \to \mathbb{C}^{*k}$.
- If Φ_1, Φ_2 are both constant along Z_i^*, then $\Phi_3 = \Phi_4 = \cdots =$ constant along Z_i.

Theorem: $\dim \Phi(B) = 2 \implies$ *the map to extension data is non-constant* on any Z_i.

Corollary: $\dim \Phi(B) = 2 \implies \Phi_T$ *contracts no curves in Z.*

- At a point of $Z_i \cap Z_j$ if N_i, N_j are linearly independent, then Φ_T extends by filling in the origin to some of the $\mathbb{C}^*$'s; essentially $\Delta^* \times \Delta^*$ completes to $\Delta \times \Delta$.
- If N_i, N_j are linearly dependent, then $\Delta^* \times \Delta^*$ fills in to $\Delta \times \Delta$ with the axes contracted to points.[‡]

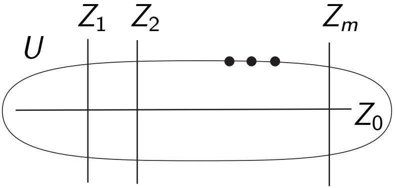
[‡]The general version of this case involves a somewhat subtle analysis of the relations among the N_i in a nilpotent orbit

$$\exp\left(\sum_i \left(\frac{\log t_i}{2\pi\sqrt{-1}}\right) N_i\right) \cdot F;$$

also in general Φ_T will be a meromorphic mapping.

IV. Basic formula

- Relates the geometry *along* a Z_i to the geometry *normal* to it



$U^* = U \backslash (Z \cap U)$

- Assume $\Phi_S(Z_0^*) =$ point, thus $\mathcal{L}$ locally constant along $Z_0^* \implies \pi_1(U^*)$ acts as a finite group on $\mathrm{Gr}(\mathcal{L})$; assume this group is trivial; then
  - — $W(N_0) = W(N_i)$ and $\mathrm{Gr}^W(V)$ is a fixed vector space;
  - — $N_0, N_i \in \mathrm{Gr}^W_{-2}\,\mathrm{End}(V)$, gives a cone $\sigma \subset \mathrm{Gr}^W_{-2}\,\mathrm{End}(V)$;
  - — $\mathrm{Gr}^W_{+2}\,\mathrm{End}(V) \cong \mathrm{Gr}^W_{-2}\,\mathrm{End}(V)^*$ (uses Q);
  - — $M \in \mathrm{Gr}^W_{+2}\,\mathrm{End}(V)$ gives $L_M \to E$ and $M \in \check{\sigma} \implies L_M \to E_{\mathrm{ab}}$ ample.

Theorem (basic formula): *$\Phi_1 : Z_0 \to E_{ab}$ and in $\mathrm{Pic}(Z_0)$*

$$(*) \qquad -\Phi_1^*(L_M) = \left\{ \sum_{i=0}^{m} \langle M, N_i \rangle [Z_i] \right\} \Bigg|_{Z_0} .$$

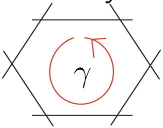
Corollary: $-\deg \Phi_1^*(L_M) = \langle M, N_0 \rangle Z_0^2 + \sum_{i=1}^m \langle M, N_i \rangle$.

- RHS is $\left\langle M, \begin{smallmatrix} N_i\text{'s corresponding to the row having } Z_0^2 \\ \text{on the diagonal in the intersection matrix} \end{smallmatrix} \right\rangle$; $(*)$ tells us how negative that row is in terms of the variation of the level 1 extension data.

Corollary: $\Phi_T \big|_{Z_0} \neq$ *constant.*

This uses $\dim \Phi(B) = 2$ and $N_0 \neq 0$.

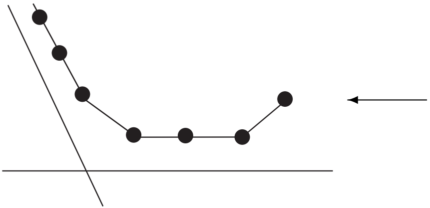
- Special case: Z is a *cycle*



$$Z_i^* = \mathbb{C}^*$$

$$-\langle M, Z_i \rangle Z_i^2 = \langle M, Z_{i-1} \rangle + \langle M, Z_{i+1} \rangle .$$

- monodromy $\overset{\gamma}{\circlearrowleft}$ gives a *circuit* — then
  — γ acting on the N_i spans a 2-plane in $\mathrm{Gr}^W_{-2}\,\mathrm{End}(V)$, and in this plane there is a sector such that the $\gamma^k N_i$ give in the sector a convex figure where $\gamma =$ translation by m



$$\left\{\begin{array}{c}\text{isomorphic to}\\ \text{dual graph of}\\ \text{the universal}\\ \text{covering of } Z\end{array}\right\}$$

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— from the basic formula $(*)$

$$\begin{cases} \text{straight line at } Z_i \leftrightarrow Z_i^2 = -2 \\ \text{bend at } Z_i \leftrightarrow Z_i^2 \leqq -3. \end{cases}$$

— Hodge-Riemann II $\implies$ convexity. Hilbert modular surface picture is general.

— In this case the basic formula gives a proof that $\|Z_i \cdot Z_j\| < 0$.

V. Application to moduli of general type surfaces

A. Infinite monodromy

We begin with the question

- *What are the singularities of $\overline{P}_T$ and $\overline{P}_S$?*
  - — the singularities of $P = \Phi(B)$ seem to be arbitrary
  - — we will illustrate the general principle that the LMHS along a Z_i helps determine the singularity type of $\Phi_S(Z_i)$.

Example 1:[§] Weight $n = 2m$

— $Z =$ smooth curve and $\Phi_S(Z) = p \in \overline{P}_S$;

— $N^2 = 0$, rank $N = 2$;

— $\mathrm{Gr}\,\mathcal{L} = \{H^{2m-1}, H^{2m}, H^{2m-1}(-1)\}$, with dim $H^{2m-1} = 2$ and $N : H^{2m-1}(-1) \xrightarrow{\sim} H^{2m-1}$, then $H^{2m-1} = H^1(C)(-(m-1))$ for an elliptic curve C;

[§] These are maximal dimension LMHS's.

— for simplicity assume rank $\mathrm{Hg}^m = 1$ and C is general;

$\implies E_{ab} = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathrm{Hg}^m, H^{2m-1}) \cong H^1(C)$;

— $* \implies \Phi_1 : Z \to C$ is a finite morphism and for $U =$ neighborhood of Z in $\overline{B}$

$$\begin{array}{ccc} U & \xrightarrow{\Phi_S} & \overline{P}_S \\ \cup & & \Cup \\ Z & \xrightarrow{\Phi_1} & \{p\} \end{array}$$

gives a resolution of an elliptic singularity.

Example 2:¶ $n = 2m$

— $Z =$ cycle;

— $N^2 \neq 0$, $N^3 = 0$ and rank $N = 1$.

¶These are the maximal dimension LMHS's with $N^2 \neq 0$

Then a similar analysis to the elliptic singularity case might be used to show that $\Phi_S(Z) =$ cusp singularity (work in progress).

$$\text{———}<\circ>\text{———}$$

Moduli‖

Will assume the structure theorems from Section II hold in general.

- $\mathcal{M} =$ irreducible and reduced KSBA moduli space whose general point corresponds to a smooth general type surface.
- $\overline{\mathcal{M}} =$ canonical completion whose boundary points correspond to surfaces X_0 having slc-singularities.

Even if $\mathcal{M}$ is irreducible and almost smooth (e.g., looks locally like the moduli space of an ADE singularity), in contrast to $\overline{\mathcal{M}}_g$ the boundary may be quite singular. There are geometric and Hodge theoretic reasons why this should be so.

‖[K] is a reference for moduli.

Question: *How can Hodge theory help understand the geometry of $\mathcal{M}$ near $\partial\mathcal{M}$?*

- Basic idea: Using Lie theory LMHS's have been classified; use this to infer results about $\partial\mathcal{M}$.
- If a point of $\partial\mathcal{M}$ corresponds to a normal surface X_0 having a singular point p and where $N \neq 0$ for a general smoothing of X_0, then from the list in [K]
  p is either a simple elliptic singularity or a cusp.**

**Interestingly if p is non-Gorenstein, then it is a rational singularity and consequently $N = 0$.

- a general result, here stated informally, is that for a singular surface X_0 corresponding to a point x_0 of $\partial\mathcal{M}$, the associated graded to the LMHS $\mathcal{L}$ for *any* smoothing of X_0 the $\mathrm{Gr}(\mathcal{L})$ is the same.[††]
- Suggests that the map

$$\overline{\mathcal{M}} \dashrightarrow \overline{P}_T$$

may provide a guide to a resolution of the singularities of $\overline{\mathcal{M}}$.

[††] More precisely the smoothings of X_0 may have several components and the $\mathrm{Gr}(\mathcal{L})$ depends only on the particular component. This result suggests why $\partial\mathcal{M}$ should be singular along components where $N \neq 0$. Specifically, if $\dim \partial\mathcal{M}$ is much less than $\dim \mathcal{M}$ at p, then since $\mathcal{L}$ does not depend on the direction of approach then $\overline{\mathcal{M}}$ should be singular at p. We will see below that we can also obtain divisors in $\partial\mathcal{M} \subset \overline{\mathcal{M}}$ along certain components where $N = 0$.

Example ([FPR]): The “first” general type surface with $p_g \geqq 2$ is an I-surface X

$$p_g(X) = 2,\ q(X) = 0,\ K_X^2 = 1;$$

- well known classically, it is on the Noether line $p_g = [K_X^2/2 + 2]$;
- $\mathcal{M}_I$ is almost smooth, $\dim \mathcal{M}_I = 28$;[†]
- $D = \mathrm{SO}(4, 28)/U(2) \times SO(28)$, $\dim D = 57$;
- IPR is a contact system and $\Phi(\mathcal{M}_I)$ is a contact subvariety;

[†]More precisely, if X has at most a canonical singularity, then the Kuranishi space at X is smooth.

- FPR have determined a stratification of $\overline{\mathcal{M}}_I^{\mathrm{Gor}}$, and have determined the divisors in $\partial\mathcal{M}_I$ ($\overline{\mathcal{M}}_I$ is difficult because one has to bound the index in the non-Gorenstein case);
- part of their table for elliptic singularities is

stratum	dimension	minimal resolution $\widetilde{X}$	$\sum_{i=1}^{k}(9-d_i)$	$k^{\ddagger}$	codim in $\overline{\mathcal{M}}_I$
I_0	28	canonical singularities	0	0	0
I_2	20	blow up of a K3-surface	7	1	8
I_1	19	minimal elliptic surface with $\chi(\widetilde{X})=2$	8	1	9
$\mathrm{II}_{2,2}$	12	rational surface	14	2	16
$\mathrm{II}_{1,2}$	11	rational surface	15	2	17
$\mathrm{II}_{1,1,R}$	10	rational surface	16	2	18
$\mathrm{II}_{1,1,E}$	10	blow up of an Enriques surface	16	2	18
$\mathrm{III}_{1,1,2}$	2	ruled surface with $\chi(\widetilde{X})=0$	23	3	26
$\mathrm{III}_{1,1,1}$	1	ruled surface with $\chi(\widetilde{X})=0$	24	3	27

‡Hodge theory gives that for a smoothable surface X_0 that is irreducible, regular and normal with k elliptic singularities implies $k \leqq p_g + 1$. Here $\widetilde{X} \to X$ contracts k elliptic curves $\widetilde{C}_i$ with $\widetilde{C}_i^2 = -d_i$.

I, II, III means 1,2,3 elliptic singularities. The subscripts are their degrees, which may also be bounded by Hodge theory.

- How can Hodge theory help understand the desingularization of $\overline{\mathcal{M}}_I$ along the boundary components?

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{C}) \arrow[dl] \arrow[dr] & \\ (X_{\min}, C) & & (X, p) \end{tikzcd}$$

Theorem (informal statement): *For I_1, I_2 the extended period mapping Φ_T gives a desingularization of a general point on the boundary.*

- Parameter counts in the table suggest that this result may be true for all cases of elliptic singularities.
- Cusp case has yet to be studied.

Example: For I_2 the picture is

$$
\begin{tikzcd}
 & (\widetilde{X}, \widetilde{C}) \arrow[dl] \arrow[dr] & \\
(X_{\mathrm{min}}, C) & & (X, p)
\end{tikzcd}
$$

Here, $p =$ isolated normal singular point on X, $\widetilde{C} =$ curve on $\widetilde{X}$ that contracts to p

$$
\text{LMHS has} \begin{cases} \mathrm{Gr}_2 \cong H^2(X_{\mathrm{min}})_{\mathrm{prim}} \\ \mathrm{Gr}_3 \cong H^1(\widetilde{C})(-1) \end{cases}
$$

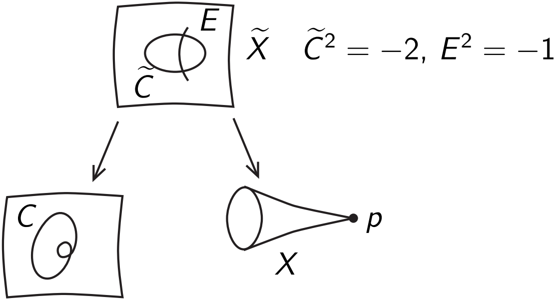
This will give for a general boundary point that Gr_2 isomorphic to the primitive cohomology of a polarized K3 surface and $g(\widetilde{C}) = 1$ (simple elliptic singularity).

- Gr(LMHS)$/\mathbb{Z}$ suggests that $\mathrm{Hg}^1(\widetilde{X})$ has a $\mathbb{Z}^2$ with intersection form

$$\begin{pmatrix} -2 & 2 \\ 2 & -1 \end{pmatrix}$$

for the purpose of heuristic reasoning we will assume classes are effective.

- Hodge theory now suggests the picture



$$\begin{cases} X_{\text{min}} = K3 \\ C^2 = 2 \end{cases}$$

$\implies X_{\text{min}} \xrightarrow{2:1} \mathbb{P}^2$ branched over a sextic C with a node

- $\#$ of PHS's of type $\mathrm{Gr}_3 \oplus \mathrm{Gr}_2 = 1 + 19 = 20 = \dim I_2$ which suggests local Torelli on I_2
- How to establish the theorem? The fibre over origin in a SSR is blowing up p in $\mathcal{X}$ to have

$$\widetilde{X} \cup_{\widetilde{C}} \mathbb{P}^2$$

  where $\widetilde{C} \in |\mathcal{O}_{\mathbb{P}^2}(3)|$
- Have to blow up $9 - (-\widetilde{C}^2) = 7$ points on $\widetilde{C}$ to obtain triviality of the infinitesimal normal bundle as a necessary condition for smoothability. Thus

Fibre over origin in a SSR is given by blowing up seven points on $\widetilde{C}$; thus is a del Pezzo.

- $\dim_p \partial\mathcal{M}_I + \underset{\substack{\| \\ 7}}{\dim(\textit{level 1 extension data})} = \dim \mathcal{M}_I - 1.$

B. Finite monodromy

- for $\Phi : \Delta^* \to \{T^k\}\backslash D$ where $T = T_s$ is of finite order

$$\begin{tikzcd} \widetilde{\Delta}^* \arrow[r, "\widetilde{\Phi}"] \arrow[d] & D \arrow[d] \\ \Delta^* \arrow[r, "\Phi"] & \{T^k\}\backslash \Delta \end{tikzcd} \rightsquigarrow \widetilde{\Phi} : \widetilde{\Delta} \to D \text{ extends.}$$

- In geometric case X_0 will be singular and the LMHS is a PHS (but $\neq H^n(\widetilde{X}_0)$).
- Generally $\widetilde{\Phi}_* : T_{\{0\}}\widetilde{\Delta} \to TD$ is zero but can define $\delta\Phi$ that has geometric information.
- For KSBA moduli of surfaces on $\partial\mathcal{M}_f$ we have X_0 where
  - X_0 is non-Gorenstein
  - singularity is a quotient singularity of type $\frac{1}{dn^2}(1, dna-1), (a,n) = 1$
  - rational $\implies N = 0$ (resolution is a tree of $\mathbb{P}^1$'s).

- Extension of Φ from $\mathcal{M}$ to $\mathcal{M}_f$ gives

$$\Phi_f : \mathcal{M}_f \to \Gamma \backslash D.$$

- In contrast to the $N \neq 0$ singularity the presence of an $N = 0$ singularity may define a divisor in $\overline{\mathcal{M}}$. This happens for the Wahl singularity $\frac{1}{4}(1,1)$; of particular interest as the monodromy $T = \mathrm{Id}$.

Example ([FPR]): For $\overline{\mathcal{M}}_I$ there are two divisors in $\partial \mathcal{M}_I$: I-surfaces (X_0, p) with a $\frac{1}{4}(1,1)$ or $\frac{1}{18}(1,5)$ singularity; denote first by $\mathcal{M}_{I,W}$.

- resolution of Wahl singularity is $(\widetilde{X}, E) \to (X, p)$ where $\widetilde{X}$ = elliptic surface with a bisection E, $E^2 = -4$;

- semi-stable-reduction has $\widetilde{X} \cup_E S$ where $S =$ Veronese surface ((X, p) looks locally like a hyperplane section through the vertex of a cone over S);
- $\widetilde{\Phi}(0) =$ HS computed from $\widetilde{X} \cup_E S$.

Theorem: *$\mathcal{M}_{I,W} =$ component of $\Phi_f^{-1}(\Gamma' \backslash D')$ where $D' \subset D$ is a Mumford-Tate domain.*

- Proof uses computation of $\delta\Phi$ in $T\,\mathrm{Def}(X)$.

Thus the presence of a Wahl singularity in $\overline{\mathcal{M}}_I$ is given by Hodge-theoretic conditions.

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