New Geometric Invariants Arising from Hodge Theory*

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*Talk given at the conference "Algebraic Geometry in Mexico" taking place at Puerta Escondido, December 2–7, 2018. Partly based on work in progress with Mark Green, Radu Laza and Colleen Robles. Some selected references are given at the end of these slides.

Abstract

The use of the Hodge line bundle, especially its positivity properties, in algebraic geometry is classical. The Hodge line bundle only sees the associated graded to the limiting mixed Hodge structures (LMHS's) that arise from the singular members in a family of algebraic varieties. This talk will introduce a new type of geometric object associated to the extension data in LMHS's (and not present for general MHS's) whose positivity properties play a central role in the application of Hodge theory to moduli. Their construction will be discussed and illustrated.

Outline

I. *Introduction* (including a challenge to and a question for algebraic geometers)

II. *Formulation of the problem of constructing analytic functions*

   A. The case of Hodge structures (HS's)

   B. The case of limiting mixed Hodge structures (LMHS's)

III. *Extension data*

   A. Generalities

   B. The LMHS case

IV. *Construction in weights $n = 1, 2$*

   A. Case $n = 1$; appearance of generalized theta functions

   B. Case $n = 2$; theta functions and functions constructed from iterated integrals using the infinitesimal period relation (IPR)

V. *Application to moduli*

   The Satake-Baily-Borel (SBB) completion of the image of a period mapping

I. Introduction

- Given

$$\begin{array}{ccc} \mathcal{X} & \subset & \overline{\mathcal{X}} \\ \downarrow & & \downarrow f \\ B & \subset & \overline{B} \end{array}$$

  - $X_b = f^{-1}(b)$ smooth algebraic surface for $b \in B^{\dagger}$
  - $\omega_{\overline{\mathcal{X}}/\overline{B}}$ Cartier

Theorem: $\det(f_* \omega_{\overline{\mathcal{X}}/\overline{B}})$ *is free.*

  - purely algebraic result whose existing proof uses pretty much what is known about variations of Hodge structure (VHS)

    Lie theory, differential geometry, complex analysis

†The result stated below can be formulated for X_b's of any dimension.

plus a new aspect of Hodge theory

geometry of extension data in LMHS's (not just MHS's)

- will explain how this new aspect arises and how it is used
- *Challenge:* Give a purely algebraic proof of the theorem.
- *Problem:* If a general X_b is of general type, then is $\det(f_*\omega_{\overline{\mathcal{X}}/B})$ free for $m \gg 0$?

- the analytic proof of the above theorem has an interesting[‡] application to moduli
  - $\overline{\mathcal{M}}_g$ is essentially smooth
  - completed KSBA moduli spaces $\overline{\mathcal{M}}$ for general type surfaces typically have very singular boundaries $\partial\mathcal{M} = \overline{\mathcal{M}} \backslash \mathcal{M}$.

[‡]The assumptions needed to apply the standard base point free theorem in birational geometry are pretty much tantamount to the statement of the theorem.

Problem: *How to construct a "natural" desingularization of $\overline{\mathcal{M}}$?*

- the *proof* of the theorem suggests how to
  (i) use Lie theory to organize the Gr(LMHS's) along $\partial\mathcal{M}$
  (ii) use the geometry of the extension data associated to a fixed Gr(LMHS) to suggest what surfaces should be in the fibres of a desingularization.
- this has been largely carried out for I-surfaces ($q(X) = 0$, $p_g(X) = 2$, $K_X^2 = 1$) and gives a very beautiful answer — if time permits will explain this.

- The current proof of the above theorem and method for addressing the question require an *existence* result in *algebraic* geometry — this will be discussed now.

II. Formulation of the problem of constructing analytic functions

A. The case of HS's

- Given VHS $(\mathbb{V}, \mathbb{F}^\bullet)$ over B
  - $\mathbb{V}$ = local system of Q-vector spaces
  - $\mathbb{F}^\bullet$ = filtration of $\mathcal{O}_B(\mathbb{V}_\mathbb{C})$ such that $\mathbb{F}^\bullet_b$ induces a Hodge filtration on $\mathbb{V}_{\mathbb{C},b}$
  - assume HS's polarized by $Q : \mathbb{V} \otimes \mathbb{V} \to \mathbb{Q}$
- $\mathcal{F} \subset B$ is a complete irreducible subvariety such that the VHS is (locally) constant * — where $\mathcal{U}$ is a small neighborhood of $\mathcal{F}$ in B.

Problem: Construct functions $f \in \mathcal{O}_B(\mathcal{U})$ such that the level sets f = constant define the subvarieties of $\mathcal{U}$ along which the VHS is (locally) constant.

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*Locally constant means finite monodromy over $\mathcal{F} \iff \nabla(\mathbb{F}^p) = 0$ where ∇ is the Gauss-Manin connection.

Solution: VHS equivalent to period mapping

$$\Phi : B \to \Gamma \backslash D$$

- $D =$ period domain for PHS's $(V, Q, F^\bullet) = G_{\mathbb{R}}/H$
- Φ holomorphic with $\Phi_* : \pi_1(B) \to \Gamma \subset G_{\mathbb{Z}}$
- may assume $Z = \cup Z_i$ is a normal crossing divisor with monodromy T_i around Z_i
- $T_i = T_{i,s} e^{N_i}$, $T_{i,s}^{m_i} = \mathrm{Id}$ †
- $N_i = 0 \implies$ may extend Φ across Z_i †
- resulting Φ is proper $\implies \Phi(B) = \mathcal{H} \subset \Gamma \backslash D$ † is a closed analytic subvariety
- $\Phi(\mathcal{F}) = p \in \mathcal{H}$, $\Phi(\mathcal{U}) \subset$ neighborhood W of p

$$\Downarrow$$

$$\Phi^{-1}(\mathcal{O}_{\mathcal{H}}(W)) \cap \mathcal{O}_B(\mathcal{U}) \text{ solves the problem.}$$

† These steps marked with a † have Lie theory, differential geometry (curvature), complex analysis (Ahlfors-Schwarz lemma, proper mapping theorem) behind them.

B. The case of LMHS's (work in progress):

- Following the works of Schmid, Cattani-Kaplan-Schmid, and others the VHS over B extends to a VLMHS $(\mathbb{V}_e(\log Z), \mathbb{F}_e^\bullet)$ over $\overline{B}$
- over each open stratum Z_I^* there is $\mathrm{VLMHS}_\mathrm{I}$ with weight filtrations $W(N_I)$ where $N_I = \sum_{i \in I} N_i$
- passing to $\mathrm{Gr}(\mathrm{LMHS}_I)$ we have period mappings

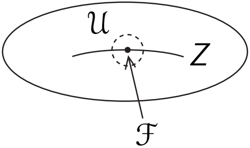
$$\Phi_I : Z_I^* \to \Gamma_I \backslash D_I$$

with images $\mathcal{H}_I$, and we set

$$\begin{cases} \overline{\mathcal{H}} = \mathcal{H} \cup \big(\bigcup_I \mathcal{H}_I\big) \\ \Phi_e : \overline{B} \to \overline{\mathcal{H}}. \end{cases}$$

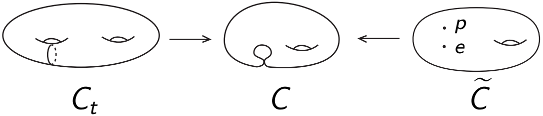
Theorem: $\overline{\mathcal{H}}$ *has the structure of a complete algebraic variety to which the Hodge line bundle* $\det \mathbb{F}_e^n$ *extends. When* $n = 2$ *it is ample.*[‡]

- the issue is to construct local analytic functions on $\overline{\mathcal{H}}$ — an instructive special case is



$\dim B = 3$
$\dim Z = 2$
$\dim \mathcal{F} = 1$

Example: $\overline{\mathcal{M}}_2$



[‡]For general n we use the augmented Hodge line bundle $\bigotimes_{p \leqq [\frac{n-1}{2}]} (\det \mathbb{F}^{n-p})$.

- Here $\mathcal{F}$ = curves C where $j(\widetilde{C})$ = constant
- C is described by $(\widetilde{C}, \mathrm{AJ}_{\widetilde{C}}(p-e))$
- the extension data in the MHS on $H^1(C)$ is $\mathrm{AJ}_{\widetilde{C}}(p-e) = p$ — the issue is: *how to use this to find analytic functions in $\mathcal{U}$?* — as will be illustrated below the point is to look at the extension data in the $\underline{\text{L}}$MHS
- If we have $f \in \mathcal{O}_{\overline{B}}(\mathcal{U})$ with $f\big|_{\mathcal{F}} = 0$,

$$df \in H^0(N^*_{\mathcal{F}/\mathcal{U}}).$$

Thus the existence of functions implies existence of sections of $N^*_{\mathcal{F}/\mathcal{U}}$ — *Why should this bundle have positivity?*

III. Extension data

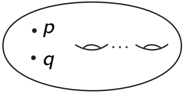
A. Generalities

- Given HS's A, C with weights a, c where $a < c$ the set of equivalence classes of exact sequences of MHS's

$$0 \to A \to B \to C \to 0$$

is the connected abelian complex Lie group

$$\mathrm{Ext}^1_{\mathrm{MHS}}(C, A) \cong \frac{\mathrm{Hom}_{\mathbb{C}}(C, A)}{F^0 \,\mathrm{Hom}_{\mathbb{C}}(C, A) + \mathrm{Hom}_{\mathbb{Z}}(C, A)}.$$

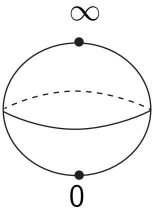
Example: 

$\widetilde{C}$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Q} \arrow[r] \arrow[d, equal] & H^1(\widetilde{C}, p+q) \arrow[r] & H^1(\widetilde{C}) \arrow[r] & 0 \\ & \mathbb{Q}([p]-[q]) & & & \end{tikzcd}$$

$$\Longrightarrow \begin{cases} \mathrm{Ext}^1_{\mathrm{MHS}}(H^1(\widetilde{C}), \mathbb{Q}) \cong J(\widetilde{C}) \\ \text{particular extension is } AJ_{\widetilde{C}}(p-q) \end{cases}$$

Example: $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(-1), \mathbb{Q}) \cong \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$



$$
\begin{array}{ccccccc}
0 \longrightarrow & H^2(\mathbb{P}^1) & \longrightarrow & H^2(\mathbb{P}^1 \backslash \{0,\infty\}) & \xrightarrow{\mathrm{Res}} & \mathbb{Q}(-1) & \longrightarrow 0 \\
& \| & & & & \cap & \\
& \mathbb{Q} & & & & H^0(\{0,\infty\})(-1) &
\end{array}
$$

Example: If $A = \mathbb{Q}(-1)$ and C is a weight $n = 3$ HS, then

$$\mathrm{Ext}^1_{\mathrm{MHS}}(C, A) := J$$

is a compact torus that contains a maximal abelian sub-variety $J_{\mathrm{ab}}(-1)$ (think here of the intermediate Jacobian of a smooth 3-fold).

More generally, if $A =$ Hodge-Tate HS of weight 2 and C as above, then

$$\mathrm{Ext}^1_{\mathrm{MHS}}(C, A) \cong A_{\mathbb{Z}} \otimes J$$

where J is a compact complex torus that contains a maximal abelian sub-variety $J_{\mathrm{ab}}(-1)$.

B. The limiting mixed Hodge structure case

- Given a nilpotent $N = \mathrm{End}_Q(V)$ with $N^{m+1} = 0$ there is a weight filtration

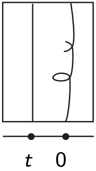
$$0 \subset W_0(N) \subset \cdots \subset W_{2m}(N) = V$$

characterized by

$$\begin{cases} N : W_k(N) \to W_{k-2}(N) \\ N^\ell : \mathrm{Gr}^{W(N)}_{m+\ell}(V) \xrightarrow{\sim} \mathrm{Gr}^{W(N)}_{m-\ell}(V). \end{cases}$$

- A *limiting mixed Hodge structure* (LMHS) is a MHS $(V, W(N), F^\bullet)$ with

$$N : F^p \to F^{p-1}.$$

Example: Given  with monodromy

$$T : H^m(X_t) \to H^m(X_t)$$

we have

$$T = T_s T_u = T_s e^N \quad \begin{cases} T_s^k = 0 \\ N^{m+1} = 0 \end{cases}$$

Theorem (Schmid): $\lim_{t \to 0} H^m(X)$ *is a LMHS.*

- **Main point:** *The set of extension data for a LMHS has a rich geometric structure, one not present for the case of just a MHS*
- *What do we mean by the set $\mathcal{E}$ of extension data for a LMHS?* Given a LMHS $(V, W(N), F_0)$ with

$$\mathrm{Gr}(V, W(N), F_0) \cong \bigoplus_{p=0}^{2m} \mathrm{Gr}_{0,p}$$

the set of LMHS's $(V, W(N), F)$ with
  - $\mathrm{Gr}_p \cong \mathrm{Gr}_{0,p}$
  - $\begin{tikzcd} \mathrm{Gr}_{m+\ell} \arrow[r, "\sim"] \arrow[d, "N^\ell"'] & \mathrm{Gr}_{0,m+\ell} \arrow[d, "N_\ell"] \\ \mathrm{Gr}_{m-\ell} \arrow[r, "\sim"] & \mathrm{Gr}_{0,m-\ell} \end{tikzcd}$

is a complex manifold $\mathcal{E}$ that is an iterated fibration

$$E_{2m} \to E_{2m-1} \to \cdots \to E_1.$$

- If $\mathrm{Gr}_p = H^p$, $p \leqq m$ then for $q \geqq 0$, $\mathrm{Gr}_{m+q} = H^{m-q}(-q)$

$\Longrightarrow$ ingredients in $\mathcal{E}$ are

  - $\mathrm{Ext}^1_{\mathrm{MHS}}(H^q, H^p)$, $q > p$
  - $\mathrm{Ext}^1_{\mathrm{MHS}}(H^{m-q}(-q), H^p)$

and their duals using $H^{m+q} \cong H^{m-q}(-q)^*, \quad q \geqq 0$

Example: $m = 1$. Write $\mathrm{Gr}_{0,p} = H^p$ and

$$\mathrm{Gr}_0 = H^0 \oplus H^1 \oplus H^0(-1)$$

where $N : H^0(-1) \xrightarrow{\sim} H^0$. Then

- $\mathrm{Ext}^1_{\mathrm{MHS}}(H^1, H^0) \cong H^0_{\mathbb{Z}} \otimes J$
- $\mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-1), H^1) \cong H^{0*}_{\mathbb{Z}} \otimes J$ (by duality)
- $$\begin{tikzcd} \mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-1), H^0) \arrow[r] & \mathcal{E} \arrow[d, "\pi"] \\ & \mathrm{Ext}^1_{\mathrm{MHS}}(H^1, H^0) = H^0_{\mathbb{Z}} \otimes J. \end{tikzcd}$$

- The fibre is a set of line bundles — using

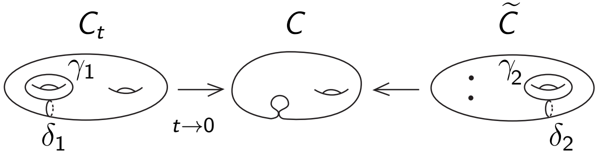
$$\begin{cases} Q_0 : H^0_{\mathbb{Z}}(-1) \otimes H^0_{\mathbb{Z}}(-1) \to \mathbb{Z} \\ Q_0(u, v) = Q(Nu, v) \end{cases}$$

defines a cone $\sigma \subset \mathrm{Pic}(J)$ of ample line bundles.
- Construction for $n \geqq 2$, even $n = 2$ (algebraic surfaces) has very interesting new features — will discuss below.

IV. Construction for weights $n = 1, 2$

- Will take an example and illustrate how the geometric structures arising from the extension data may be used to construction functions.

Example:



$$\lambda = \int_{\gamma_2} \omega_2$$

The period matrix is

$$\begin{array}{c} F^1 \cap W_1 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & \lambda \\ b & a \end{pmatrix} \\ \text{periods of } \omega_1 \quad \text{periods of } \omega_2 \end{array} \quad \begin{array}{l} e_1 \\ e_2 \\ e_3 \\ e_4 \end{array}$$

$$Q(e_1, e_4) = -Q(e_4, e_1) = 1$$
$$Q(e_2, e_3) = -Q(e_3, e_2) = 1$$

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

- fiber is $\lambda = \lambda_0$, $t = 0$ $\qquad (\mathrm{Gr}_1 \cong H^1(\widetilde{C}, \mathbb{Z}))$
- *problem:* t is not well defined — need functions $f = f(a, \lambda, b)$ that define $\mathcal{F}$.

- monodromy is

$$W_1 \left\{ \begin{array}{r} e_1 \to e_1 + me_2 + ne_3 + pe_4 \\ e_2 \to e_2 + qe_4 \\ e_3 \to e_3 + re_1 \\ W_0\{ \; e_4 \to e_4 \end{array} \right.$$

- to preserve Q

$$q = n$$
$$r = -m$$

and the new period matrix is

$$\begin{pmatrix} 1 & 0 \\ m & 1 \\ a+n & \lambda \\ b+p & a+n-m\lambda \end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a+n-m\lambda & \lambda \\ b+p-mn+m^2\lambda-ma & a+n-m\lambda \end{pmatrix}$$

- when $a \to a+n-m\lambda$ we have

$$b \to b+p-nm+m^2\lambda-ma$$
$$e^{2\pi ib} \to e^{2\pi ib}e^{-2\pi ima+2\pi im^2\lambda}$$

- now $a \in \widetilde{C} = \mathbb{C}/\{1,\lambda\}$ — taking
  - $a \to a+\lambda$ gives
  - $e^{2\pi ib} \to e^{2\pi ib}e^{2\pi ia+2\pi i\lambda}$
  - $\theta(a+\lambda) = \theta(a)e^{-2\pi ia-\pi i\lambda}$
  - $e^{-2\pi i(a+\lambda)} = e^{-2\pi ia}e^{-2\pi i\lambda}$

$\Longrightarrow$ $\left(e^{2\pi ib}\right)^2\theta(a)^2e^{-2\pi ia}$ is a function on $\overline{\mathcal{H}}$

- $b = \frac{\log t}{2\pi i} + b'$ gives the function $t^2 \left(e^{2\pi i b'}\right)^2 \theta(a)^2 e^{-2\pi i a}$

Conclusion: *The functions that define $\mathcal{F}$ are*

$$\begin{cases} \lambda - \lambda_0 \\ (e^{2\pi i b})^2 \theta(a)^2 e^{-2\pi i a} \end{cases}$$

Example: $m = 2$ and the LMHS is Hodge-Tate

- This has a much different character and introduces some new elements into the story.

- The period matrices are

$$\begin{array}{cc} F^2 & F^2/F^1 \\ \end{array}$$
$$\begin{pmatrix} I & 0 \\ A & I \\ B & {}^tA \end{pmatrix} \;\}W_0 \;\}W_2 \;\}W_4$$

$$N = \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 0 & I \\ 0 & -I & 0 \\ I & 0 & 0 \end{pmatrix}$$

- The basic constraints are
  - Hodge-Riemann I $\implies B + {}^tB = {}^tAA$
  - Infinitesimal period relation (IPR) $\implies dB = {}^tA\,dA \implies d\,{}^tA \wedge dA = 0$

  The second of these is a new ingredient not present in the classical case.

- The extension data map is $\mathcal{F} \to \mathcal{E}$ where

$$\begin{tikzcd} \mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-2), H^0) \arrow[r] & \mathcal{E} \arrow[d] \\ & \mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-1), H^0) \end{tikzcd}.$$

- $\mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-1), H^0) \cong \mathrm{Hom}_{\mathbb{Z}}(H^0(-1), H^0) \otimes \mathbb{C}^*$, and for $A = \|a_{ij}\|$ as above the $e^{2\pi i a_{ij}}$ are well-defined functions — indeed, monodromy acts by

$$\begin{pmatrix} I & 0 & 0 \\ M & I & 0 \\ L & {}^tM & I \end{pmatrix}$$

where M, L are integral matrices — then our claim follows from

$$A \to A + M.$$

- The interesting new point is that

$$B \to B + {}^tAM + L$$
$$\implies e^{2\pi i b_{ij}} \text{ is \textit{not} well defined.}$$

- Here the IPR comes to the rescue — if we let γ be a path in $\mathcal{F}$ from x to x_0, then

$$B(x) = \int_{x_0}^{x} dB = \int_{x_0}^{x} {}^tA\, dA \text{ mod periods}$$

  and from $d({}^tA\, dA) = 0$ we see that the integral depends on the homotopy class of γ.
- Thus the functions

$$e^{2\pi i b_{ij}(x)} = e^{2\pi i b_{ij}(x_0)} e^{2\pi i \int ({}^tA\, dA)_{ij}}$$

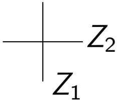
  are well defined and together with the $e^{2\pi i a_{ij}}$ give the desired functions.

- Remark that the a_{ij} and b_{ij} will have logarithmic singularities at intersections of $\mathcal{F}$ with other strata — from

$$e^{2\pi i\left(\frac{\log t}{2\pi i}\right)} = t$$

we see that our functions will vanish at these intersections

— this reflects the fact that for $Z_1 = \{x = 0\}$,



$$d(xy) = x\,dy + y\,dx$$

$$d(xy)\big|_{Z_1} = y\,dx \in N^*_{Z_1/\mathcal{U}} \text{ has a zero at } Z_1 \cap Z_2$$

V. Application to moduli

A. The Satake-Baily-Borel (SBB) completion of the image of a period mapping

- Given $\Phi : B \to \mathcal{H} \subset \Gamma\backslash D$ with $B = \overline{B}\backslash Z$ where $Z = \cup Z_i$ is a normal crossing divisor with Z_i irreducible with $N_i \neq 0$ one defines
  - $\Phi_{I,e} : Z^*_{I,e} \to \mathcal{H}_I \subset \Gamma_I\backslash D_I$,
  - $\overline{\mathcal{H}} = \mathcal{H} \cup \left(\underset{I}{\cup} \mathcal{H}_I\right)$, where there are identifications arising from non-empty intersections $Z^*_{I,e} \cap Z^*_{I,e'}$,

  and then one shows
  - $\overline{\mathcal{H}}$ is a compact Hausdorff space and $\Phi_e : \overline{B} \to \overline{\mathcal{H}}$ is a proper continuous mapping with fibres $\mathcal{F}$ that are compact complex analytic subvarieties of $\overline{B}$.

- If we define

$$\mathcal{O}_{\overline{\mathcal{H}}} = \left\{ \begin{array}{c} \text{sheaf of continuous functions for } \overline{\mathcal{H}} \\ \text{such that } f \circ \Phi_e \text{ is holomorphic} \end{array} \right\}$$

  the issue is to show that $\mathcal{O}_{\overline{\mathcal{H}}}$ defines the structure of a complex analytic variety
  - this is an *existence* result — one has to produce analytic functions on a neighborhood $\mathcal{U}$ of $\mathcal{F}$ that are constant on the fibres of Φ_e
  - the issue is *global* along the compact analytic subvarieties $\mathcal{F}$.
- One may think of $\overline{\mathcal{H}}$ as the quotient of $\overline{B}$ by the equivalence relation

$$b \sim b' \iff \left\{ \begin{array}{l} \text{the (limiting mixed) Hodge} \\ \text{structures corresponding to} \\ b, b' \text{ are equivalent} \end{array} \right\}$$

  (cf. the references BBT and BKT at the end.

- One next shows that the augmented Hodge line bundle lives on $\mathcal{H}$; i.e., we have

$$\Lambda_e \to \overline{\mathcal{H}}$$

with $\Phi_e^*(\Lambda_e) =$ augmented Hodge line bundle over $\overline{B}$ and where
  - there is a (singular) metric in $\Lambda_e \to \overline{\mathcal{H}}$ whose Chern form ω_e may be defined as a current whose wave front set has a very special structure
  - $\omega_e > 0$, and an extension of the proof of the classical Kodaira theorem shows that Λ_e is ample — in particular, it is free
  - in the weight $n = 2$ geometric case this implies the theorem stated in the introduction.[§]

[§]This is what gives the challenge to algebraic geometers given at the beginning.

Definition: $\overline{\mathcal{H}}$ is the SBB completion of the image $\mathcal{H}$ of the period mapping $\Phi : B \to \Gamma\backslash D$.

Remark: In the classical case and for Γ arithmetic one has

$$\Gamma\backslash D \subset \overline{(\Gamma\backslash D)}^{\mathrm{SBB}}$$

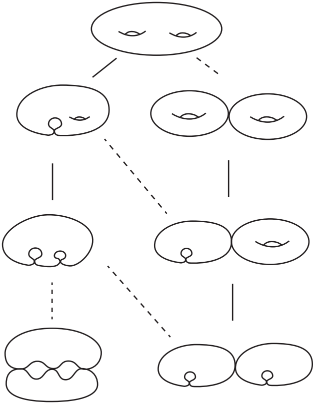
and $\Phi : B \to \Gamma\backslash D$ extends to a holomorphic mapping $\Phi_e : \overline{B} \to \overline{(\Gamma\backslash D)}^{\mathrm{SBB}}$ and $\overline{\mathcal{H}} = \Phi_e(\overline{B})$. The ampleness of the Hodge line bundle is proved by the *global* construction of automorphic forms.

In the non-classical case the situation is quite different — e.g., Γ may be a thin matrix group.

B. An illustration of how SBB is used

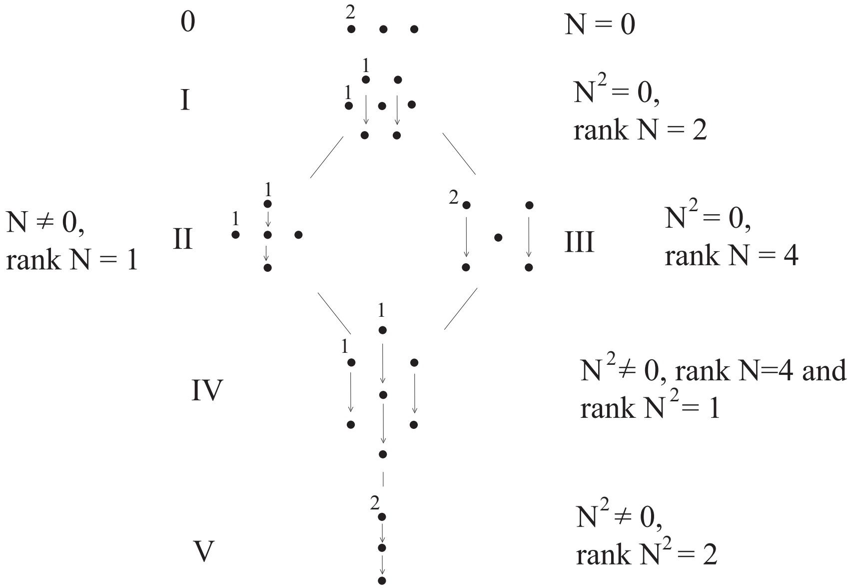
- LMHS's and Mumford-Tate sub-domains of period domains can be analyzed by Lie theory
- this leads to a stratification of $\overline{\mathcal{H}}$
- using this as a guide helps to organize the boundary $\partial\mathcal{M} = \overline{\mathcal{M}} \backslash \mathcal{M}$ of a KSBA moduli space
- for $\overline{\mathcal{M}}_g$ this gives the stratification structure of $\overline{\mathcal{M}}_g$

- The picture of $\overline{\mathcal{M}}_2$ is



This gives the stratification of $\overline{\mathcal{M}}_2$ together with the incidence (degeneration) relations among the strata.

- I-surface X is a surface
  - of general type
  - $q(X) = 0$ and $p_g(X) = 2$
  - $K_X^2 = 1$ (first non-classical case)
- following is the classification over Q of LMHS's with $N \neq 0$


0
N = 0
I
N^2 = 0,
rank N = 2
N ≠ 0,
rank N = 1
II
III
N^2 = 0,
rank N = 4
IV
N^2 ≠ 0, rank N=4 and
rank N^2 = 1
V
N^2 ≠ 0,
rank N^2 = 2

- one refines to LMHS's over $\mathbb{Z}$ by using the conjugacy class of T_s
- drawing on the beautiful work in FPR this is then used to classify the strata of $\partial\mathcal{M}_I$ having normal, Gorenstein slc singularities — the non-normal case has a similar table

stratum	dimension	minimal resolution $\widetilde{X}$	$\sum_{i=1}^{k}(9-d_i)$	k	codim in $\overline{\mathcal{M}}_I$
I_0	28	canonical singularities	0	0	0
I_2	20	blow up of a K3-surface	7	1	8
I_1	19	minimal elliptic surface with $\chi(\widetilde{X})=2$	8	1	9
$\mathrm{III}_{2,2}$	12	rational surface	14	2	16
$\mathrm{III}_{1,2}$	11	rational surface	15	2	17
$\mathrm{III}_{1,1,R}$	10	rational surface	16	2	18
$\mathrm{III}_{1,1,E}$	10	blow up of an Enriques surface	16	2	18
$\mathrm{III}_{1,1,2}$	2	ruled surface with $\chi(\widetilde{X})=0$	23	3	26
$\mathrm{III}_{1,1,1}$	1	ruled surface with $\chi(\widetilde{X})=0$	24	3	27

- an extra benefit is this:
  - $\overline{\mathcal{M}}_g$ is essentially smooth
  - however, $\overline{\mathcal{M}}_I$ is very singular along $\partial\mathcal{M}_I$, and the *proof* of the above theorem using the extension data in the LMHS's can be used to suggest a natural resolution of singularities of $\overline{\mathcal{M}}_I$

Thank you

References

FPR

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