Positivity in Hodge theory and algebraic geometry

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References

I. Introduction

Let Y be a smooth, d-dimensional quasi-projective algebraic variety. Three interesting potential properties of Y have been and continue to be topics of active research.

Y is of log general type.

This means that there exists a smooth completion $\overline{Y}$ of Y with complement $Z = \overline{Y} \setminus Y$ a reduced normal crossing divisor and where

(A) $\Omega^d_{\overline{Y}}(\log Z)$ *is big.*

This property is independent of the smooth completion of Y.

Keeping these notations a second potential property is

(B) $\Omega^1_{\overline{Y}}(\log Z)$ *is big.*

Again this depends only on Y and not on the particular completion.

Finally there is the property

(C) *Y is hyperbolic.*

Here there are several notions of hyperbolicity; One is the little Picard version

a holomorphic mapping $f : \mathbb{C} \to Y$ *is constant.*

The big Picard version is: For A an algebraic variety any holomorphic mapping

$$f : A \to Y$$

is algebraic. This is the one we will be mainly concerned with.

There is also the notion of hyperbolicity modulo a proper subvariety. This is given by a proper algebraic subvariety $V \subset Y$ such that

any holomorphic mapping $f : \mathbb{C} \to Y$ *has image* $f(\mathbb{C}) \subset V$.

This will also be discussed.

There are relations among (A), (B), (C). Suitably interpreted, (B) tends to imply (A) and (C).

The purposes of this talk are

(1) to discuss these properties from a Hodge theoretic perspective, i.e., when Y parametrizes a variation of Hodge structure (VHS);
(2) to apply the results here to the geometric case, i.e., when there is a family
$$f : X \to Y$$
of smooth projective varieties $X_y = f^{-1}(y)$ and the VHS arises from the cohomology of the fibres;
(3) to extend these and other results to the case where the X_y are only assumed to be of general type.

There is extensive literature on all of these and at the end we shall give selected references emphasizing those works that are at least partly expository and can serve as a guide to the literature. Among these the paper [Z1] has been important for the works that followed it, and it also has been influential in the preparation of these notes.

An informal summary of the results is

(I) *If the differential of the period map is generically injective, then all three properties (A), (B), (C) hold.*

(II) *In the geometric case, if local Torelli holds for a general* $y \in Y$*, then properties (A), (B), (C) all hold.*

For (C) we only have big Picard modulo a proper subvariety unless we assume that the differential of the period map is everywhere 1-1.

It is basically clear that (I) $\implies$ (II). More subtle is

(III) *If the fibres X_y are of general type and* $\mathrm{Var}\, f = \dim Y$, *then properties (A), (B), (C) all hold.*

Here $\mathrm{Var}\, f = \dim Y$ means that for general $y \in Y$ the Kodaira-Spencer map

$$\rho_y : T_y Y \to H^1(X_y, T_{X_y})$$

is injective. And again we only have big Picard modulo a proper subvariety.

A further property in the geometric case is the Iitaka conjecture (now a theorem): Let $\bar{f} : \overline{X} \to \overline{Y}$ be a completion of $f : X \to Y$ where $\overline{X}$ and $\overline{Y}$ are smooth. Then

(D) $\kappa(\overline{X}) \geqq \kappa(X_y) + \kappa(\overline{Y})$.

(IV) *If X_y is of general type and* $\mathrm{Var}\, f = \dim Y$, *then (D) holds.*

This raises the question

> *How can Hodge theory be used to derive properties of algebraic varieties from their pluricanonical systems* $|mK_X|$*?*

Of course if local Torelli holds for the $H^0(K_X) \cong H^{n,0}(X)$ part of the Hodge structure on $H^n(X, \mathbb{Q})$, then the use of Hodge theory to study the moduli of X is classical. However even for varieties of general type local Torelli holds in a number of interesting examples but it is far from a general phenomenon.

This issue will be taken up when we discuss the Iitaka conjecture in Section V.

As remarked above there is a vast literature concerning all of these results. Although (A) and (B) of (III) and (IV) are purely algebro-geometric results, the original proofs used Hodge theory. In fact the curvature properties of bundles arising from Hodge theory continue to provide heuristic arguments suggesting algebro-geometric results. In proofs of these results the arguments generally require important and frequently complex technical issues that may be a combination of Lie theory (e.g., the singular behavior of the metrics, curvatures and Chern forms of Hodge bundles), complex analysis (e.g., the estimates needed to apply the L^2-$\overline{\partial}$ techniques), and of course the algebraic methods from birational geometry and singularity theory. The goal of this lecture is to isolate some of the central Hodge-theoretic aspects mentioning but not substantially delving into the technical issues.

II. Hodge theoretic preliminaries

- Notations
  - $\overline{Y}$ is a smooth projective variety;
  - $Z \subset \overline{Y}$ is a reduced normal crossing divisor with components Z_i; we sometimes set $Z = \sum Z_i$;
  - $Y = \overline{Y} \backslash Z$.
- We assume given a variation of weight n polarized Hodge structures $(\mathcal{V}, F^\bullet, \nabla)$ over Y. Here $\mathcal{V}$ is the sheaf of $\mathcal{O}_Y$-modules obtaining by tensoring $\mathcal{O}_Y$ with the local system underlying the VHS
  - $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_Y$ is the Gauss-Manin connection with $\mathcal{V}^\nabla = \ker \nabla$ the corresponding local system;
  - $F^\bullet = \{F^p\}$ is the system of Hodge bundles $F^n \subset F^{n-1} \subset \cdots \subset F^0 = \mathcal{V}$;
  - the polarization is given by a horizontal bilinear form $Q : \mathcal{V}^\nabla \otimes \mathcal{V}^\nabla \to \mathbb{C}_Y$ that induces on each fibre $(\mathcal{V}_b, F^\bullet_b)$ a polarized Hodge structure.

- The data $(\mathcal{V}, F^\bullet, \nabla)$ is equivalent to a period mapping

$$\Phi : Y \to \Gamma \backslash D \tag{II.1}$$

  where
  - D is the period domain parametrizing polarized Hodge structures $(V, F^\bullet, Q)$ where V is a $\mathbb{Q}$-vector space and $F^\bullet = \{F^p\}$ is a Hodge structure that is polarized by $Q : V \otimes V \to \mathbb{Q}$;*
  - Φ is a local liftable holomorphic mapping and $\Gamma \subset G := \mathrm{Aut}(V, Q)$ is a discrete subgroup that contains the monodromy group.

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*We will use the same notation F^p for the vector bundles that arise in a VHS and for the filtration on the vector space $V_{\mathbb{C}}$ that defines a PHS. Hopefully the context will make clear which use is intended.

- We assume that the local monodromies $T_i = \exp N_i$ around the irreducible components Z_i of Z are unipotent.

  We will also use the notations

  - $\check{D}$ is the compact dual of D consisting of all filtrations $F^\bullet$ on $V_{\mathbb{C}}$ that satisfy the 1^{st} Hodge-Riemann bilinear relation $Q(F^p, F^{n-p+1}) = 0$.
  - $\mathcal{V}_e, F_e^p$ will denote the canonical Deligne extensions of $\mathcal{V}, F^p$ for $\overline{Y}$; then
  - $\nabla : F_e^p \to F_e^{p-1} \otimes \Omega^1_{\overline{Y}}(\log Z)$ has residue $\operatorname{Res}_{Z_i} \nabla = N_i$;
  - at each point of $Z_I := \cap_{i \in I} Z_i$ there is a monodromy cone σ_I and a limiting mixed Hodge structure $(V, W\{\sigma_I\}, F^\bullet_{\text{lim}})$;
  - by the results of [CKS], any $N \in \sigma_I$ defines the same monodromy weight filtration $W(N) = W(\sigma_I)$, and each particular $N \in \sigma_I$ defines a polarization on the limiting mixed Hodge structure.

- The period mapping (II.1) may be extended across any branches Z_i of Z where $N_i = 0$, and assuming that all $N_i \neq 0$ the mapping Φ is proper and the image

$$\Phi(Y) := P \subset \Gamma \backslash D \tag{II.2}$$

is a closed analytic subvariety. When Γ is arithmetic the recent theorem in [BBT2] gives that P is an algebraic variety over which the characteristic Hodge bundle

$$L := (\det F^n)^{\otimes n} \otimes \cdots \otimes (\det F^{(n+1)/2}) \to P$$

is ample.

- The 1st and 2nd Hodge-Riemann bilinear relations define Hermitian metrics in the Hodge bundles $F^p \to Y$ and on the associated graded bundles

$$E^p := F^p / F^{p+1}.$$

There are then canonically associated Chern connections with associated curvatures

$$\Theta_p \in A^{1,1}(Y, \mathrm{End}(E^p)).$$

From Θ_p one constructs the Chern forms that represent the Chern classes in cohomology.

- The singularity properties of these metrics, curvatures and Chern forms have been analyzed in the fundamental work [CKS] with amplifications by others including [K] (cf. [GG] for a recent account). There it is shown that the Chern forms on Y extend to $\overline{Y}$ as differential forms whose coefficients are in L^1_{loc}. As such they define currents and in [CKS] it is proved that these currents are closed and define in $H^*(\overline{Y})$ the Chern classes of the canonically extended bundles $E^p_e := F^p_e / F^{p+1}_e$.

Of particular importance for this talk will be the Chern form ω_e for the canonically extended characteristic Hodge line bundle $L_e \to \overline{Y}$. This form is extensively discussed in [GG] and there it is noted that

- the powers $\omega_e^k = \overbrace{\omega_e \wedge \cdots \wedge \omega_e}^{k}$ obtained by formally multiplying the ω_e exist as differential forms whose coefficients are in L^1_{loc} and the closed currents they define represent $c_1(L_e)^k$;
- these forms ω_e^k may be restricted to the strata Z_I and there they represent the cohomology classes $c_1(L_e\big|_{Z_I})^k$ associated to the restrictions $L_e\big|_{Z_I} \to Z_I$;
- finally, for $\omega = \omega_e\big|_Y$ the Chern form of $L \to Y$ we have the positivity property
  (II.3)
  $$\omega \geqq 0 \text{ and for } \xi \in T_y Y \;\; \omega(\xi) = 0 \iff \Phi_*(\xi) = 0.$$

This positivity will be extended below to ω_e.

- By [GGLR] P has a canonical compactification $\overline{P}$ as a stratified compact Hausdorff space with complex analytic strata and where (II.1) extends to a proper continuous mapping

$$\Phi_e : \overline{Y} \to \overline{P}$$

which is complex analytic on the inverse images of the strata in $\overline{P}$. It is conjectured that $\overline{P}$ has a complex analytic structure and that the characteristic Hodge line bundle on P extends to an ample line bundle $L_e \to \overline{P}$. This would plausibly follow from the

Conjecture A: *The canonically extended characteristic Hodge line bundle $L_e \to \overline{Y}$ is semi-ample.*

If the conjecture is true, then $\mathrm{Proj}(L_e)$ exists as a projective variety and, as discussed in [GG], at least at the set-theoretic level we have

$$\overline{P} = \mathrm{Proj}(L_e).$$

- Part of the heuristic evidence for this conjecture is provided by the properties of the extended Chern form ω_e. Even though it is not known that $\overline{P}$ has a complex structure, for $Z_I^* := Z_I \setminus \cup_{j \notin I} Z_{I \cup \{j\}}$ the open strata

  $$P_I := \Phi_e(Z_I^*)$$

  have complex structures and setting $\Phi_I := \Phi_e \big|_{Z_I^*}$

  $$\Phi_I : Z_I^* \to P_I$$

  is a holomorphic mapping.

Moreover, even though the coefficients of ω_e are only in L^1_{loc}, by the restriction property mentioned above for $y \in Z_I$ and $\xi \in T_y\overline{B}$ the condition

$$\omega_e(\xi) = 0$$

is well defined and is equivalent to $\Phi_{I,*}(\xi) = 0$. This may be expressed as saying that although $\Phi_e : \overline{Y} \to \overline{P}$ has not been proved to be a holomorphic map, the tangent spaces to the fibres of Φ_e are well defined holomorphically varying and fiberwise linear subspaces of $T\overline{Y}$ defined by the condition

$$\omega_e(\xi) = 0.$$

The structure sheaf $\mathcal{O}_{\overline{P}}$ may be defined to be given by the sheaf of continuous functions that are holomorphic when restricted to the complex analytic strata in $\overline{P}$. Proving that $\mathcal{O}_{\overline{P}}$ has enough functions is a global issue along the compact fibres of Φ_e.

- Finally, the result in [BBT2] is proved by first showing that P has the structure of a complex algebraic variety (cf. [BBT1]) and then proving that $L \to P$ is ample. A consequence of their argument is that $L \to Y$ is semi-ample. Denoting by $H^0(\overline{Y}, L^m(*Z))$ the global sections of $L^{\otimes m} \to \overline{Y}$ that vanish to some unspecified order on Z, they prove that for $m \gg 0$ the sections in $H^0(\overline{Y}, L^m(*Z))$ generate the fibres of $L^{\otimes m} \to Y$ and induce a projective embedding $P \hookrightarrow \mathbb{P}^N$. A strengthened version of [BBT2], which does not assume that Γ is arithmetic, would result from

Conjecture B: *There exist non-negative integers k_i and on m_0 such that for $m \gg m_0$ the line bundle $mL - \sum_i k_i[Z_i]$ is semi-ample.*

The $k_i = 0$ if $N_i = 0$. The conjecture is true when $\dim B = 2$ and Z_i are contracted by Φ_e. In this case the k_i are determined from the smallest eigenvalue of the matrix $\|Z_i \cdot Z_j\|$.
If we assume that Φ_* is everywhere injective, then the above line bundle is ample.

- We will say that a VHS $(V, F^\bullet, Y)$ *arises from geometry* if there is a smooth family

$$f : X \to Y$$

of projective varieties $X_y = f^{-1}(y)$, $y \in Y$, whose associated variation of the Hodge structure on the cohomology in the degree n primitive cohomology is $(\mathcal{V}, F^\bullet, Y)$.

In this case there will be a completion

$$\begin{array}{ccc} X & \subset & \overline{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ Y & \subset & \overline{Y} \end{array}$$

where $Y = \overline{Y} \backslash Z$ and over Z the mapping $\bar{f}$ has the Abramovich-Karu form of semi-stable reduction (cf. [AK]).

For our purposes nothing essential will be lost if we think of the singular fibres X_y, $y \in Z$, as being locally a product of normal crossing varieties.

If we have $f : X \to Y$ as above, then $Var\, f = dim\ Y$ means that the Kodaira-Spencer maps $\rho_y : T_y Y \to H^1(T_{X_y})$ are generically 1-1, and we shall say that

> *Local Torelli (LT) holds generically if the differential of the period mapping associated to the corresponding VHS is generically 1-1.*

III. General type

We will state the result and then discuss some of the main points in Zuo's proof ([Z1]) using the positivity of the curvature forms of sub-bundles of Hodge bundles that are defined as kernels of iterated Kodaira-Spencer maps.

Theorem

If the differential of the period mapping arising from a VHS is generically 1-1, then $(\overline{Y}, Z)$ is of log general type.

- Following Simspon ([Sim]) one associates to a VHS $(\mathcal{V}, F^\bullet, \nabla)$ over Y a *Higgs bundle* (E, θ) where (here we set $F^{-1} = 0$),
  - $E = \bigoplus_{p=0}^{n} E^p, \quad E^p = F^p / F^{p+1};$
  - $\theta = \bigoplus_{p=1}^{n}$ where $\theta_p : E^p \to E^{p-1} \otimes \Omega^1_Y$ is induced by ∇;

- the E^p have Hermitian metrics; and

(III.1) - the integrability condition $\theta \wedge \theta$ is satisfied.

The last condition is a consequence of $\nabla^2 = 0$.

- As noted above, associated to the Hodge metrics in the E^p there are canonical Chern connections

$$D_p : A^0(E^p) \to A^1(E^p)$$

which are compatible with the metrics and where $D_p'' = \overline{\partial}$. The curvatures

$$\Theta_p \in A^{1,1}(\mathrm{End}(E^p))$$

of the Higgs bundle associated to a VHS are given by

$$\Theta_p + \theta_{p+1} \wedge \theta_p^* + \theta_{p-1}^* \wedge \theta_p = 0$$

where θ_p^* is the adjoint of θ_p relative to the Hodge metrics in the E^p's.

We note that if $e \in E^p_y$ and $\xi \in T_y Y$, the curvature form

$$\text{(III.2)} \qquad \Theta_p(e)(\xi) = \| \langle \theta_p(e), \xi \rangle \|^2 - \| \left\langle \theta^*_p(e), \xi \right\rangle \|^2$$

is a *difference* of non-negative terms. Hence only for $p = n, 0$ does it have a sign. The first basic result is given by the

Observation ([Z1]): Assume that $K^p := \ker \theta_p$ is a sub-bundle.† Then the curvature form for the induced metric on K^p is $\leqq 0$. Moreover, if it is equal to zero, then K^p is D'_p-invariant, and in fact K^p is a flat sub-bundle of E^p.

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†In general K is a subsheaf; the technical issues needed to deal with this are given in [Z1].

Proof.

The $\leqq 0$ is clear from (III.2). We then recall that the curvature form for K^p is given by the restriction of the curvature form of E^p restricted to K^p *minus* the norm of the 2^{nd} fundamental form of K^p in E^p (in Hermitian differential geometry curvatures decrease on sub-bundles). Since the 2^{nd} fundamental form is the *difference* of the restriction $D_p\big|_{K^p}$ and the Chern connection for the induced metric on K^p, the result follows. □

Note: We will further review these differential geometric points when we discuss the Iitaka conjecture in the last part of these notes.

- Given a Higgs bundle (E, θ) any standard linear algebra construction may be applied to produce another Higgs bundle. We apply this to obtain a Higgs bundle $(\mathrm{Hom}(E, E), \theta^{\mathrm{end}})$. Over the locus where the map

$$TY \to \oplus \, \mathrm{Hom}(E^p, E^{p-1})$$

induced by θ is 1-1 the image $K = \bigoplus_{p=1}^{n} \theta^p(TY)$ is a sub-bundle, and we have the second basic

Observation ([Z1]): $K \subseteq \ker \theta^{\mathrm{end}}$.

This has the following

Consequence: Let ω_0 be the Kähler metric induced from the Chern form of the characteristic Hodge line bundle on the open set $Y_0 \subset Y$ where Φ_* is 1-1. Then the holomorphic bi-sectional curvatures $R(\xi, \eta)$ and holomorphic sectional curvatures $R(\xi) = R(\xi, \xi)$ of ω_0 satisfy

(i) $R(\xi, \eta) \leqq 0$ and there exists $\xi \in T_y Z_0$ such that $R(\xi, \eta) < 0$ for all $\eta \in T_y Y_0$;

(ii) $R(\xi) \leqq -c$ for a constant $c > 0$ depending only on D (here $\|\xi\| = \|\eta\| = 1$).

Only the first part of (i) has been proved here. We note that it is a general result in Kähler geometry that (ii) $\Longrightarrow$ (i) ([BKT]). A full proof of (ii) from a Lie theory perspective is given in [CM-SP]. An alternative linear algebra argument is given in [GG]. Our main objective here is to draw attention to the above application of the general result that on bundles given by the kernels of Kodaira-Spencer maps with the induced Hodge metrics the corresponding curvature forms have signs.

- To complete the proof of the theorem stated above several further steps are necessary. Before listing these we note that if $Y = \overline{Y}$ and Φ_* is everywhere 1-1, then the induced metric in $K_{\overline{Y}}$ has a positive curvature form and so $K_{\overline{Y}}$ is ample. Thus one has to
  - deal with the degeneracies of ω on Y;
  - deal with the singularities of ω_e on Z; and
  - show that when these are taken care of then $\Omega^d_{\overline{Y}}(\log Z)$ is big.

  Since these issues (and more) are dealt with in [Z1], we want to formulate a question/conjecture that we think could lead to a in some ways more natural proof of the theorem.

- The basic point is that general type questions is where the canonical bundle, i.e., differential forms of the top degree, are involved and this invites the consideration of volume forms and their associated Ricci forms. This is in contrast to hyperbolicity which involves questions of 1-forms where curvature of the full cotangent bundle (essentially holomorphic bi-sectional curvatures) enter in. We will discuss the canonical bundle case now.

- A singular volume form Ω on a d-dimensional complex manifold M is given locally by

$$\Omega = h \bigwedge_{j=1}^{d} \left(\frac{i}{2}\right) dz_j \wedge d\overline{z}_j$$

  where h is a real and non-negative function. The h's we shall consider will be locally products of
  - a positive C^∞ function h_0;
  - $\|f\|^2$ where $f(z)$ is a $\mathbb{C}^m$-valued holomorphic function; and
  - $R(\log(g_1(z)), \ldots, \log(g_\ell(z))$ where R is a rational function and the $g_\alpha(z)$ are holomorphic functions.

In our case h will be in L^1_{loc} so that $\int \Omega < \infty$. Since h is in L^1_{loc}, the distributional derivative

$$\left(\frac{i}{2\pi}\right) \partial\overline{\partial} \log h := \mathrm{Ric}\,\Omega$$

is defined and will be a closed $(1,1)$ current that represents $c_1(K_M)$ in cohomology.

- Given a VHS where $\dim Y = d$ and Φ_* is generically 1-1 we set
  - $\Omega = \omega^d$;
  - $\Omega_e = \omega_e^d = \Omega\big|_Y$.

  From the above discussion about the curvature of the Kähler metric ω_0 on Y_0, since the usual Ricci form $R_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is the *negative* of $\mathrm{Ric}\,\Omega_0$, it follows that $\mathrm{Ric}\,\Omega_0 > 0$. Thus if $Y_0 = Y = \overline{Y}$, we again have that $K_{\overline{Y}}$ is positive. In general we have the

Conjecture C: *The singular differential form* $\mathrm{Ric}\,\Omega_0$ *defines on* $\overline{Y}$ *a* $(1,1)$ *current whose coefficients are in* L^1_{loc}*, and we have the equation of currents*

$$\mathrm{Ric}\,\Omega = \mathrm{Ric}\,\Omega_0 - Z + R$$

where R *is an effective divisor on* $\overline{Y}$*.*

It follows that $c_1(\Omega^d_{\overline{Y}}(\log Z))$ is represented by $\mathrm{Ric}\,\Omega_0 + R$, from which the above theorem is an immediate consequence.

A central aspect of the conjecture is that as we approach the boundary Z the singularities of the curvature of the Kähler metric ω_0 on Y_0 should be mild in a somewhat, but not entirely, similar way as is the case for the singularities of the curvature of the canonically extended Hodge bundles. Recall that for the Hodge bundles the Chern forms on Y extend to $\overline{Y}$ as differential forms whose coefficients are in L^1_{loc}, and which then define currents on $\overline{Y}$ that turn out to be closed and in cohomology define the Chern classes of the canonically extended Hodge bundles. In particular, the Lelong number of the corresponding currents are all equal to zero.

The difference here is that although $\mathrm{Ric}\,\Omega_0$ extends to $\overline{Y}$ as an L^1_{loc} differential form we must add δ-function type terms to get $c_1(K_{\overline{Y}})$. The R arises from the degeneracies, rather than the singularities, of Ω; one may think of it as a Riemann-Hurwitz correction term. The Z arises from the singularities of ω_e along Z. These are illustrated by the following

Toy example: On the disc $\Delta = \{0 \leqq |t| < 1\}$ the Poincaré form

$$\pi = \frac{dt \wedge d\bar{t}}{|t|^2(-\log|t|)^2}$$

is integrable on $\Delta_\epsilon = \{0 \leqq |t| \leqq \epsilon\}$ where $\epsilon < 1$. Denoting by π^* the restriction of π to $\Delta^*_\epsilon = \Delta_\epsilon \cap \Delta^*$, we have

$$\mathrm{Ric}\,\pi^* = \pi^*.$$

But on Δ_ϵ

$$\mathrm{Ric}\,\pi = \pi + \delta_{\{0\}}\left(\frac{i}{2}\right) dt \wedge \overline{dt}.$$

Here in order to highlight the essential points we are omitting some inessential positive constants.

- As previously noted, in Hodge theory the known properties of the curvatures of Hodge bundles and the corresponding Chern forms provide the tools to be able to show that certain bundles have positivity properties such as bigness and nefness. For applications to algebraic geometry, e.g. in moduli questions, one would like to know that certain bundles are *semi-ample* or *free*.‡

‡We shall use these terms interchangeably. For line bundles $L \to X$ they mean that all fibres of some $L^m \to X$ are globally generated; i.e., for all $x \in X$ we have $H^0(L^m) \to L^m_x \to 0$. For vector bundles $E \to X$ we shall use as definitions of big, nef, and free the corresponding property for the line bundle $\mathcal{O}_{\mathbb{P}E}(1)$. We note that $L \to X$ is semi-ample $\iff \overset{m}{\oplus} H^0(X, L^m)$ is finitely generated. Thus only for such bundles do we have an associated Proj.

Even better but much harder is to have that a bundle is actually ample. Here we will formulate a question/conjecture related to the above theorem.

Using the setting described above, we define the differential of the canonically extended VHS to be the map

$$\Phi_{e,*} : T_{\overline{Y}}(-\log Z) \to \mathrm{End}(E_e)$$

induced by θ_e. We say that the canonically extended VHS has the *local Torelli* (LT) property if $\Phi_{e,*}$ is everywhere injective.

Question/Conjecture D: *Does the local Torelli property imply that $\Omega^d_{\overline{Y}}(\log Z)$ is ample?*[§]

[§] More precisely, is $K_{\overline{B}} + \sum(\mathrm{rank}\, N_i)[Z_i]$ an ample line bundle? The heuristic evidence is that near Z_i the Chern form ω_e has principal part $(\mathrm{rank}\, N_i)\pi_i +$ (lower order terms) where π_i is the Poincaré form in the normal to Z_i.

We remark that Conjectures A and D seem to be related through the work of Zuo. In fact, in Corollary 0.3 in [Z1] there is a relation of $\mathbb{Q}$-divisors

$$L = rK_{\overline{Y}}(\log Z) - P + N$$

where r is a positive rational number, P is an effective divisor and N is semi-negative. Understanding the precise geometric meaning of P and N may provide a way to connect these two conjectures.

The issues here are also related to those in Conjecture B above. They center around trying to understand the geometry of the fibres of

$$\Phi_e : Z \to \partial P := \overline{P} \backslash P.$$

Recalling that Φ_e associates to a limiting mixed Hodge structure the corresponding associated graded polarized Hodge structure, along a fibre

$$Y_p := \Phi_e^{-1}(p), \qquad p \in \partial D$$

the extension data in the LMHS is varying. This extension data seems to have a richer structure than the extension data for a general mixed Hodge structure,[¶] and the issue is how to convert this information along the fibre to information about the restriction of the normal bundle to Z in $\overline{Y}$ along Y_p. Specifically, without labeling it a formal conjecture there are indications that a local Torelli assumption will imply that

$$N^*_{Z/Y}\Big|_{Y_P} \textit{ is ample.}$$

[¶]cf. [PP] for a discussion of some differential geometric aspects of MHS's.

This is closely related to Conjecture B and would imply it under local Torelli assumptions such as the injectivity of the map $\Phi_{e,*}$ above.

- An obvious geometric application of the above theorem is the following:
  Let $f : X \to Y$ be as above and assume that
  - $\mathrm{Var}\, f = \dim Y$;
  - *local Torelli holds for a general fibres $X_y = f^{-1}(y)$.*

  Then $(\overline{Y}, Z)$ is of log general type.

There are many results of this type in the literature; a few references to some of these such as [Be], [Fu], [Ka1], [Ka2], [Ka3], [V1], [V2] are given at the end of these notes.

IV. Symmetric differentials and hyperbolicity*

- For a holomorphic vector bundle $E \to M$ over a compact complex manifold there are two notions of *big*. The first is that there is on m_0 such that for $m \geqq m_0$ the evaluation maps

$$H^0(S^m E) \to E_p$$

are surjective for all $p \in M$.

There is some evidence that for studying holomorphic curves in varieties of general type (or of pairs of varieties of log general type) jet differentials rather than symmetric powers of the cotangent (or log-cotangent) bundles may be more useful (cf. [De2]). However, in Hodge theory because of the negativity properties of the holomorphic bi-sectional curvatures rather than just the holomorphic sectional curvatures, symmetric differentials seem to be important (cf. [BKT], [LSZ] and [Z1]).

For the second we recall that $\mathbb{P}E \to M$ is defined to be the bundle of hyperplanes in the fibres of $E \to M$. Thus for $p \in M$ the fibre

$$(\mathbb{P}E)_p = \mathbb{P}E_p^*.$$

Over $\mathbb{P}E$ there is the tautological bundle $\mathcal{O}_{\mathbb{P}E}(1)$ whose fibre at $(p, [\lambda]) \in \mathbb{P}E$ is

$$E_p/\mathbb{C}\lambda$$

where $\lambda \in E_p^*$ is thought of as a hyperplane in E_p and $[\lambda]$ is the corresponding point in $\mathbb{P}E_p^*$.

Definition: *$E \to M$ is big if $\mathcal{O}_{\mathbb{P}E}(1)$ is big in the first sense above.*

Suppose now that $E \to M$ has an Hermitian metric whose curvature form is semi-positive; i.e.,

$$\Theta_E(e, \xi) \geqq 0 \text{ for all } p \in M,\ e \in E_p, \xi \in T_pM.$$

Lemma *Suppose that there is an $e \in E_p$ such that $\Theta_E(e, \xi) > 0$ for all $p \in M$ and all $\xi \in T_pM$. Then $E \to M$ is big.*†

†We recall that the curvature form for $p \in M$, $e \in E_p$, $\xi \in T_pM$ is defined by

$$\Theta_E(e, \xi) = \left\langle (\Theta_E(e), e), \xi \wedge \bar{\xi} \right\rangle.$$

This will be further discussed in the last part of this talk.

Proof Let ω be the Chern form of $\mathcal{O}_{\mathbb{P}E}(1)$ defined by the curvature form for that bundle with the induced metric. Then

$$\omega \geqq 0 \ \text{and}\ \omega > 0 \ \text{at}\ (p, [e^{\perp}]) \in \mathbb{P}E_{(p,[e^{\perp}])}.$$

Here $e \in E_p$ and $e^{\perp} \in E_p$ is the corresponding hyperplane defined using the Hermitian inner product. It follows that $\omega^{\dim \mathbb{P}E} > 0$ on a Zariski open set and the lemma follows from results of Siu and Demailly (cf. [De1]). □

We now want to use a similar lemma in the situation where M is replaced by $\overline{Y}$ and TM by $T_{\overline{Y}}(-\log Z)$. The issue is that assuming that the differential

$$T_{\overline{Y}}(-\log Z) \to \mathrm{End}(E_e)$$

is injective, the metric on $T_{\overline{Y}}(-\log Z)$ induced by the Hodge metric in E is singular. As previously noted it is known that the Chern forms expressed in terms of the singular curvature in $\mathrm{End}(E_e)$ are given by differential forms in L^1_{loc} that defined closed currents whose de Rham cohomology classes are the Chern classes of $\mathrm{End}(E_e)$.

It follows from the discussion in [Z1] that a similar property holds in the present situation. One philosophical reason why this should be so is this: The deviation of the curvature of $T_{\overline{Y}}(-\log Z)$ with the induced metric from the restriction of the curvature in $\mathrm{End}(E_e)$ to that sub-bundle is of the form

$$ {}^{t}\overline{A} \wedge A $$

where A is a matrix of (1,0) forms. In particular the curvature of $T_{\overline{Y}}(-\log Z)$ with the induced metric is more negative than that induced from

$$ \mathrm{End}(E_e)\Big|_{T_{\overline{Y}}(-\log Z)}. $$

The signs then go the right way to increase the bigness of $T_{\overline{Y}}(-\log Z)$ with A.

We may now apply the above lemma to obtain the results (A) and (B) in the introduction.

Hyperbolicity

- Two of the main principles in this subject are
  - the Ahlfors-Schwarz lemma; and
  - curvatures decrease on holomorphic sub-bundles.

For the first, a pseudo-metric on the disc $\Delta = \{z : |z| < 1\}$ is given by a (1,1) form

$$\omega = \left(\frac{i}{2}\right) h\, dz \wedge d\bar{z}$$

where h is a non-negative C^∞ function on Δ that locally is

$$(z - z_0)^\alpha h_0, \quad h_0 > 0.$$

The Ricci form is

$$\mathrm{Ric}\,\omega = \left(\frac{i}{2\pi}\right) \partial\overline{\partial} \log h.$$

Up to a positive constant

$$\operatorname{Ric}\omega = -K \cdot \omega$$

where K is the Gauss curvature of the pseudo-metric. At points where $\alpha > 0$ we set $K = -\infty$.
The Poincaré metric is

$$\pi = \left(\frac{i}{2}\right) \frac{dz \wedge d\bar{z}}{|z|^2(-\log|z|)^2}.$$

The fact that it has constant negative Gaussian curvature translates into

$$\operatorname{Ric}\pi = \pi.$$

The Ahlfors lemma is

$$\operatorname{Ric}\omega \geqq \omega \implies \omega \leqq \pi.$$

For the little Picard theorem one uses that if ω is defined on $\Delta(r) = \{|z| < r\}$ and satisfies $\mathrm{Ric}\,\omega \geqq \omega$, then by a scaling argument it follows that $r \leqq 1$. In particular there is no metric in $\mathbb{C}$ with Gauss curvature $K \leqq -c$ where $c > 0$.

For the proof of the Ahlfors lemma we consider the function

$$g = \omega/\pi.$$

If g has a maximum at $z_0 \in \Delta$, then at z_0

$$0 \geqq \left(\frac{i}{2}\right) \partial\overline{\partial} \log g = \mathrm{Ric}\,\omega - \mathrm{Ric}\,\pi \geqq g - \pi,$$

which gives $g(z_0) \leqq 1$. If there is no interior maximum, then one uses the same argument for ω/π_r on the smaller disc $\Delta(r)$ for $r < 1$ where π_r is the Poincaré metric with constant Gauss curvature -1 there to have $\omega \leqq \pi_r$ and one then lets $r \to 1$.

Corollary

If M is a complex manifold with a Hermitian metric whose holomorphic sectional curvatures are $\leqq -c$ where $c > 0$, then any holomorphic mapping

$$f : \mathbb{C} \to M$$

is constant.

Application: Assume there is a VHS over Y whose corresponding period mapping has injective differential. Then any holomorphic mapping $f : \mathbb{C} \to Y$ is constant.

If the differential is only injective on a Zariski open set in Y, then the argument gives that the image of any such holomorphic mapping lies in a proper subvariety.

Regarding the relation between symmetric differentials and hyperbolicity, if we have a compact, complex manifold M and sections $\varphi_1, \ldots, \varphi_k \in H^0(Sym^n \Omega^1_M)$, then we may define a Finsler pseudo-metric in TM by taking for $\xi \in T_p$

$$\|\xi\|^2 = \left(\sum_{i=1}^{k} |\varphi_i(\xi^m)|^2 \right)^{1/m} .$$

These metrics tend to have semi-negative curvature. If T^*M is big, then using these metrics and the Ahlfors lemma leads to hyperbolicity modulo a proper subvariety (cf. [Pa] and the references cited there). There is also a logarithmic version which may be applied to $\Omega^1_B(\log Z)$ in the Hodge theoretic setting of these notes. Significantly refined versions of this general approach are given in [LSZ] and [P3].

Turning to big Picard theorems in the formulation that for A an algebraic variety any holomorphic mapping

$$f : A \to Y$$

is algebraic there is both a classical and a very active recent literature of such results (cf. [De2], [D1], [D2], [LSZ], [PS1] and the references given in these works). Basically in one form or another one has to show that any holomorphic mapping

$$f : \Delta^* \to Y$$

extends to a holomorphic mapping

$$\bar{f} : \Delta \to \overline{Y}.$$

Here $\overline{Y}$ can be any completion of Y. If one had information on the curvature of the Kähler metric given by the Chern form of the characteristic Hodge line bundle, e.g., Poincaré metric type behavior locally along the normal crossing divisor $Z = \overline{Y} \backslash Y$, then the result would follow. But such information is currently lacking, and there are ingenious arguments in the above references that lead to the algebraicity results under assumptions given there. Of particular interest to me is the use of Nevanlinna theory given in [LSZ] as that theory is, quoting Herman Weyl, a jewel in 20^{th} century mathematics and may provide as good an understanding of hyperbolicity as any method.

V. Iitaka Conjecture

- In this part of the talk all varieties will be complete (we drop the $\overline{}$ on $\overline{X}$ and just use X). Let

$$f : X \to Y$$

be a morphism between smooth projective varieties and $X_y = f^{-1}(y)$ a general fibre. Then the conjecture, now a theorem, is

$$\kappa(X) \geqq \kappa(X_y) + \kappa(Y). \tag{V.1}$$

We shall discuss (V.1) in what turns out to be an essential case for establishing the general conjecture, namely when

(A) $\kappa(X_y) = \dim X_y$ (X_y is of general type)

(B) $\mathrm{Var}\, f = \dim Y$

(cf. [V1], [V2], [K]).

Here the second assumption intuitively means that the map from the parameter space Y to the moduli space $\mathcal{M}$ of a general fibre X_y is generically locally 1-1. In technical terms we recall that it means that for a general point $y \in Y$ the Kodaira-Spencer map

$$\rho_y : T_y Y \to H^1(X_y, TX_y)$$

is injective. The result (V.1) may be deduced from the following result (cf. [V1], [V2], [VZ1], [K], [Pa], [PS2] and [Sc]): *Under the assumptions (A) and (B), for* $m \gg 0$

(V.2) $$\det(f_* \omega^m_{X/Y}) \text{ is big and nef.}$$

This implies that for $m \gg 0$ the line bundle $\det(f_* \omega^m_{X/Y})$ has lots of sections, and an elementary argument can be used to explain how this leads to sufficiently many sections of $K_X^{\otimes \ell} \to X$, $\ell \gg 0$, to give the result.
In fact to better understand the original intuition, under additional assumptions we shall first prove a stronger result than (V.2). This argument will illustrate the essence of (a) above, i.e., how Hodge theory enters the picture. The additional assumptions we make are

$$\text{(V.3)} \qquad \begin{cases} \bullet \ \textit{the fibres } X_y \textit{ are all smooth;} \\ \bullet \ \textit{a suitable version of local Torelli holds.} \end{cases}$$

Here the second assumption means the following: Using

$$H^0(X_y, K_{X_y}) = H^{n,0}(X_y) \subset H^n(X_y, \mathbb{C})$$

we obtain a local mapping from Y to the Grassmannian of $h^{n,0}(X_y)$ planes in the locally constant vector space $H^n(X_y, \mathbb{C})$, and local Torelli here means that this mapping should have injective differential.[‡]

Theorem

Under the assumption (V.3) *above*

$$\det(f_*\omega_{X/Y}) \text{ is ample.}$$

[‡]The usual meaning of local Torelli is that locally in moduli the way in which the Hodge decomposition on $H^n(X_y)$ varies determines X_y. Here we only consider the end piece $H^{n,0}(X_y) \cong H^0(\Omega^n_{X_y})$ of the Hodge decomposition.

Proof (sketch): $f_*\omega_{X/Y}$ is a vector bundle with fibre $H^0(X_y, \Omega^n_{X_y})$ the space of global holomorphic n-forms on X_y. This Hodge bundle has a metric given by the inner product

$$(\varphi, \psi) = c_n \int_{X_y} \varphi \wedge \overline{\psi}$$

where $\varphi, \psi \in H^0(X_y, \Omega^n_{X_y})$ and c_n is a suitable constant. The metric then induces one in the Hodge line bundle

$$\Lambda := \det f_*\omega_{X/Y}.$$

The theorem will follow from Kodaira's theorem if we show that

(V.4) *the curvature form of $\Lambda \to Y$ is positive.*

To prove (V.4) we let $\mathcal{U} \subset Y$ be a neighborhood of a point and

$$\Psi : \mathcal{U} \to \mathrm{Grass}(h^{n,0}, \mathbb{C}^b)$$

the map described above where all $H^n(X_y, \mathbb{C})$ for $y \in \mathcal{U}$ are identified with a fixed $\mathbb{C}^b$. Then the crucial curvature computation is that for $\xi \in T_y Y$

$$\left(\frac{i}{2}\right) \Theta_\wedge(\xi) = \|\Psi_*(\xi)\|^2.$$

We note that the curvature of the Hodge line bundle is a *first order* (not second as is usually the case for curvatures) invariant that measures the size of the first order variation of $H^{n,0}(X_y)$ in the locally constant vector space $H^n(X_y, \mathbb{C})$. With the above local Torelli assumption we may then conclude the result.

The assumption that local Torelli holds is too strong; this is frequently but not always the case (the general question of finding conditions where it holds is one of interest). A better algebro-geometric assumption is that the X_y are canonically embedded, so we now assume (cf. [VZ1])

$$\text{(V.5)} \quad \begin{cases} \bullet \quad \text{the } X_y \text{ are smooth and } \rho_y \text{ is everywhere 1-1} \\ \bullet \quad \text{the canonical maps } \varphi_{I_{X_y}} : X_y \to \mathbb{P}H^0(X, K_{X_y})^* \\ \quad\ \ \text{are embeddings.} \end{cases}$$

Theorem

Under the assumptions (V.5)*, for* $m \gg 0$

$$\det(f_*\omega_{X/Y}^{\otimes m}) \text{ is ample.}$$

If we only assume that ρ_y is generically 1-1, i.e., that $\mathrm{Var}\, f = \dim Y$, then the result becomes

$$\det\left(f_*\omega_{X/Y}^{\otimes m}\right) \text{ is big and nef,}$$

which with only the assumption that X_y is of general type is basically the result that seems to be the one most frequently appearing in the literature. There appears to be little known about the stable base locus of the linear systems $|\ell \det(f_*\omega_{X/Y}^{\otimes m})|$ for $\ell, m \gg 0$. For example there is the question

Q: *Are there natural assumptions which imply that* $\det\left(f_*\omega_{X/Y}^{\otimes m}\right)$ *is free for* $m \gg 0$*?*

Proof of the theorem: The argument is particularly interesting due to the blend of differential-geometric and algebro-geometric inputs that go into it. We shall use the exact sequence

$$\text{(V.6)} \qquad 0 \to R_m \to S^m(f_*\omega_{X/Y}) \xrightarrow{\mu} f_*\omega_{X/Y}^{\otimes m}.$$

The mapping μ is given by the pointwise multiplication

$$S^m H^0(X_y, K_{X_y}) \to H^0(X_y, K_{X_y}^{\otimes m})$$

of global sections of $K_{X_y} \to X_y$. For $m \gg 0$ the kernel R_m of μ is a vector bundle given by the degree m defining equations of the canonical model of the X_y's. We denote by C_m the image of μ so that for the Hodge vector bundle

$$H = f_*\omega_{X/Y}$$

the sequence (V.6) becomes

$$\text{(V.7)} \qquad \begin{cases} 0 \to R_m \to S^m H \to C_m \to 0 \\ 0 \to C_m \to f_*(\omega_{X/Y}^{\otimes m}) \to D_m \to 0. \end{cases}$$

The ideas are the following:

- H is semi-positive, and thus so is $S^m H$ (semi-positivity is a general property of globally generated vector bundles);
- since positivity increases on quotient bundles (another general property), C_m is semi-positive;
- $f_*(\omega_{X/Y}^{\otimes m})$ is semi-positive (discussed below), and as a consequence we may infer that

$$\det(f_*\omega_{X/Y}^{\otimes m}) = \det C_m \otimes \det D_m$$

  is semi-positive and is *strictly* positive if $\det C_m$ is positive;
- the positivity of C_m is increased from that induced from $S^m H$ by an amount that reflects the "twisting" or variation (suitably defined) of the subspaces $R_{m,y} \subset S^m H_y$;

and finally

- since for $m \gg 0$ the subspace $R_{m,y} \subset S^m H_y$ determines X_y, we will see that the assumption that the Kodaira-Spencer maps are 1-1 will imply there is sufficient twisting to give that $\det C_m$ is positive.

In order to give the details we recall the following

Differential-geometric preliminaries (cf. [GG] and [De1]): For a Hermitian vector bundle $E \to M$ the curvature form is defined for each $x \in M$, $e \in E_x$ and $\xi \in T_x M$ by

$$\Theta_E(e, \xi) = \left\langle (\Theta_E(e), e), \xi \wedge \bar{\xi} \right\rangle . \tag{V.8}$$

The bundle is *semi-positive* if $\Theta_E(e, \xi) \geqq 0$, *positive* if strict inequality holds (for e, ξ non-zero). Semi-positivity is a fairly common property of vector bundles, strict positivity much less so. We will abbreviate (V.8) by $\Theta_E \geqq 0$.

We note that $\Theta_E \geqq 0$ implies $\Theta_{\det E} \geqq 0$. Moreover, if M is a curve and everywhere along the curve we have $\Theta_E \geqq 0$ but do *not* have $\Theta_{\det E} > 0$ at some point, then $E \to M$ is flat (a positive semi-definite Hermitian matrix whose trace is zero is itself equal to zero).
Next let

$$0 \to S \to E \to Q \to 0$$

be an exact sequence of holomorphic vector bundles. Then there is a *second fundamental form* of $S \subset E$ that measures how much the Chern connection on E fails to leave S invariant. This leads to a C^∞ operator

$$A \in A^{1,0}(M, \mathrm{Hom}(Q, E))$$

such that for $e \in Q_x \cong S_x^\perp \subset E_x$ and $\xi \in T_x M$ we have

$$\text{(V.9)} \qquad \Theta_Q(e, \xi) = \Theta_E(e, \xi) + \left\langle (Ae, Ae), \xi \wedge \overline{\xi} \right\rangle .$$

In words, the amount by which the curvature increases in quotient bundles is measured by the size of the second fundamental form of S in E.
From this and the above we conclude that if $\Theta_E \geqq 0$ and if we do not have $\Theta_{\det Q} > 0$, then there is a curve $C \subset M$ such that

- $E\big|_C$ is flat;
- $S\big|_C \subset E\big|_C$ is a flat sub-bundle.

We now apply these general results to the situation at hand. As previously noted we have an Hermitian metric in H with $\Theta_H \geqq 0$. This gives $\Theta_{S^m H} \geqq 0$, and then $\Theta_{C_m} \geqq 0$. If we do *not* have $\Theta_{\det C_m} > 0$, then there is a curve $C \subset Y$ along which the $H^0(X_y, K_{X_y})$ are locally constant vector spaces and the $R_{m,y} \subset S^m H^0(X_y, K_{X_y})$ are locally constant subspaces.

This implies that the canonical models of the X_y do not vary along C, which contradicts our assumption about injectivity of Kodaira-Spencer maps. □

In general for

$$f : X \to Y$$

where we only assume

- X_y is general type for $y \in Y$;
- $\mathrm{Var}\, f = \dim Y$

then the canonical bundle $K_y \to X_y$ may well not have sections. The assumption is that for general $y \in Y$ the pluricanonical map

$$\varphi_{mK_{X_y}} : X_y \to \mathbb{P}H^0(X_y, K_{X_y}^{\otimes m})^*$$

is a rational map with image a birational model of X_y, and that as y varies these images modulo projective transformations also vary.

The basic idea, which is due to Iitaka and Kawamata (cf. [Fu], [Ka1], [Ka2], [Ka3]) and has been extensively developed and used by Viehweg [V1], [V2], and many others is in brief the following:

- For a smooth general type variety V assuming that K_V is ample, for $m \gg 0$ we may choose a smooth $R \in |mK_V|$ defined by $\zeta \in H^0(K_V^{\otimes m})$ and construct the cyclic covering

$$W \to V$$

  corresponding to $\zeta^{1/m}$.
- When this is done there is an inclusion

$$\text{(V.10)} \qquad H^0(K_V^{\otimes m}) \hookrightarrow H^0(K_W) \cong H^{n,0}(W)$$

  together with a canonical element $\zeta^{1/m} \in H^{n,0}(W)$.

- It is of interest to note that the Hodge metric induces a Finsler metric on $H^0(mK_V)$ by setting

$$\|\zeta\|^2 = \left(\frac{i}{2}\right)^{n^2} \int_V (\zeta \wedge \overline{\zeta})^{1/m}.$$

This norm is continuous on all of $H^0(mK_V)$ but it is smooth only on the open set where the $R \in |mK_V|$ are smooth. We note that this Finsler metric is equal to the Hodge metric of the corresponding element in $H^{n,0}(W)$. Thus one may anticipate positivity properties of the curvature of the Finsler metric (cf. [Pa] for a comprehensive discussion of this).

We note that when R becomes singular so that W does also, the norm does *not* become logarithmically infinite as in the usual case of the Hodge bundles over a family of degenerating algebraic varieties.

- If we scale ζ by setting $\zeta' = \lambda\zeta$ for $\lambda \in \mathbb{C}^*$, then with the hopefully obvious notation we have

$$W \text{ is biholomorphic to } W'.$$

Thus the natural parameter space is the projective space $\mathbb{P}H^0(mK_V) = |mK_V|$.

We now will apply this construction with dependence on parameters where V is a fibre X_y of $f : X \to Y$. Finessing many significant technical issues we choose $m \gg 0$ so that

- $\omega_{X/Y}^{\otimes m}$ is relatively very ample, and
- $f_*(\omega_{X/Y}^{\otimes m})$ is big.

The idea now is to take a family of cyclic coverings arising from sections in the $H^0(X_y, \omega_{X_y}^{\otimes m})$. There are two ways one might do this. A natural geometric construction would be to take

$$\mathbb{P} := \mathbb{P}\left(f_*(\omega_{X/Y}^{\otimes m})^*\right) \to Y$$

whose fibres are the projective spaces $|m\omega_{X_y}|$.
For each point $(y, [\zeta]) \in \mathbb{P}$ there is an m-sheeted branched covering $W_{(y,[\zeta])} \to X_y$ which gives a diagram

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & \overset{g}{\searrow} & \downarrow f \\ \mathbb{P} & \longrightarrow & Y \end{array}$$

and by the above construction with dependence on parameters

$$f_*(\omega_{X/Y}^{\otimes m}) \text{ is related to } g_*(\omega_{W/Y}).$$

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The problem is that due to the non-linearity of the map $\zeta \to \zeta^{1/m}$, there is no apparent relation between $g_*(\omega_{W/Y})$, which has positivity, and the bundle $f_*(\omega_{X/Y})$ that we are interested in.[§]

The approach taken by Viehweg, Zuo-Viehweg and others is to take a general section $\zeta \in H^0\left(Y, f_*(\omega_{X/Y}^{\otimes m})\right)$ and to construct the corresponding family $W \xrightarrow{\bar{g}} Y$ of cyclic branched coverings. In considering the VHS associated to $\bar{g}_*(\omega_{W/Y})$ the singular locus in Y will be increased for the reasons discussed above in that the divisors in the linear system $|m\omega_{X_y}|$ may become singular. These singularities are less significant and in [VZ1] and [Z] this distinction

[§]This approach was tried by Viehweg-Zuo (personal communication). It is also discussed in [GG]. The problem is that the positivity of $f_*(\omega_{X/Y}^{\otimes m})$ translates into positivity of the usual $\mathcal{O}_{\mathbb{P}f_*(\omega_{X/Y}^{\otimes m})}(1)$. But the geometry suggests the use of $\mathbb{P}(f_*(\omega_{X/Y}^{\otimes m})^*)$ which leads to some negativity that must be offset by positivity of $f_*\omega_{X/Y}^{\otimes m}$.

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