THE GLOBAL ASYMPTOTIC STRUCTURE OF PERIOD MAPPINGS

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ABSTRACT. This work is part of a project to construct completions of period mappings $\Phi : B \to \Gamma\backslash D$. A proper topological SBB-esque completion $\Phi^0 : \overline{B} \to \overline{\wp}^0$ is constructed. The fibres of Φ^0 are projective varieties, and the image $\overline{\wp}^0$ is a union of quasi-projective varieties; one wants to endow the topological completion with a compatible algebraic structure. This raises questions about: (i) the global geometry of the Φ^0–fibres; and (ii) the existence of period matrix representations on neighborhoods of such fibres over which the restricted extension is still *proper*. The purpose of this paper is to investigate these questions.

1. Introduction

1.1. **Overview.** The motivation behind this work is to construct completions of period mappings, and to apply those completions to study moduli and their compactifications. The goal is to develop an analog of the Satake–Baily–Borel (SBB) compactification and Borel's extension theorem [BB66, Bor72] for arbitrary period mappings; see Conjecture 1.6 for a precise statement. The conjecture raises a number of questions about the *global asymptotic structure* of a period mapping (§1.4). The purpose of this work is to establish properties of this structure.

Remark 1.1. We distinguish the *global asymptotic structure* studied here from both *global properties* of the period mapping and the *local asymptotic structure*. The first concerns properties of a variation of Hodge structures over a quasi-projective base. This is a classical and much studied subject beginning with [Gri70], and with recent developments including [BKT20, BBT23, BBKT20, BBT20]. The second concerns local properties of degenerations of period mappings beginning with the nilpotent and SL(2) orbit theorems [Sch73, CKS86].[1] The orbit theorems describe the period mapping over a local coordinate chart at infinity. The period map will not (in general) be proper when restricted to this local coordinate

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[1]Significant applications include the Iitaka conjecture [Vie83a, Vie83b, Kol87] and the arithmetically of Hodge loci [CDK95].

chart. Very roughly, what we mean by the *global asymptotic structure* is properties over larger neighborhoods at infinity where both the period map and its extension are proper, cf. §1.5.

1.2. **The set-up.** We consider triples $(\overline{B}, Z; \Phi)$ consisting of a smooth projective variety $\overline{B}$ and a reduced normal crossing divisor Z whose complement

$$B \;=\; \overline{B}\backslash Z$$

has a variation of (pure) polarized Hodge structure

$$\begin{array}{c} \mathcal{F}^p \subset \mathcal{V} = \tilde{B} \times_{\pi_1(B)} V \\ \downarrow \\ B \end{array} \tag{1.2a}$$

inducing a period map

$$\Phi : B \;\to\; \Gamma\backslash D\,. \tag{1.2b}$$

Here $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a rational vector space with underlying lattice $V_{\mathbb{Z}}$; D is a period domain parameterizing pure, weight n, Q–polarized Hodge structures on the vector space V; and $\pi_1(B) \twoheadrightarrow \Gamma \subset \mathrm{Aut}(V, Q)$ is the monodromy representation. Without loss of generality the period map (1.2b) is proper [GS69]. The image

$$\wp \;=\; \Phi(B)$$

is a quasi-projective variety [BBT23]. The motivating goal behind this paper is to construct a projective compactification $\overline{\wp}$ of $\wp$ and an extension $\Phi^{\mathrm{e}} : \overline{B} \to \overline{\wp}$ of the period map.

The compactification should be obtained from Hodge theoretic data at infinity; that is, from the limiting mixed Hodge structures of the period map. Write

$$Z \;=\; Z_1 \cup Z_2 \cup \cdots \cup Z_\nu\,,$$

with smooth irreducible components Z_i. We denote by

$$Z_I \;=\; \bigcap_{i \in I} Z_i$$

the closed strata, and $Z_I^* \subset Z_I$ the Zariski open smooth locus. As we approach a point $b \in Z_I^*$ the period map Φ degenerates to a limiting mixed Hodge structure (W, F) that is polarized by nilpotent operators in the local monodromy cone σ_I. The Hodge filtration $F \in \check{D}$ will vary along Z_I^*, and is well-defined only up to the action of $\exp(\mathbb{C}\sigma_I)$ on the compact dual $\check{D}$. This induces a map

$$\Phi_I : Z_I^* \;\to\; (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I\,, \tag{1.3a}$$

cf. §A.2. Nonetheless, because $N(W_a) \subset W_{a-2}$ for all $N \in \sigma_I$, the induced Hodge filtration $F^p(\mathrm{Gr}^W_a)$ on the graded quotient $\mathrm{Gr}^W_a = W_a/W_{a-1}$ is well-defined. In this way we obtain a period map

$$\Phi^0_I : Z^*_I \;\to\; \Gamma_I \backslash D^0_I \tag{1.3b}$$

factoring through Φ_I.

Theorem† (Theorem 2.20)**.** *The maps* (1.3b) *can be patched together to define an extension*

$$\Phi^0 : \overline{B} \;\to\; \overline{\wp}^0 \tag{1.4}$$

of $\Phi : B \to \wp$. The image $\overline{\wp}^0$ is a Hausdorff topological space compactifying $\wp$, and is a finite union (not necessarily disjoint) of quasi-projective varieties. The map Φ^0 is continuous and proper, and the fibres are projective algebraic varieties.

(In the interest of conciseness, many the results discussed in this Introduction are stated imprecisely and/or incompletely; this is indicated by the superscript †. The reader will find the formal statements, with all necessary definitions, in the body of the paper.)

Remark 1.5. The completion $\Phi^0 : \overline{B} \to \overline{\wp}^0$ encodes the variations of limiting mixed Hodge structures *modulo extension data* along the strata. This is the sense in which $\overline{\wp}^0$ is a *minimal Hodge theoretic compactification.*

1.3. **The motivating conjecture.** Theorem 2.20 gives us a candidate for the generalization the Satake–Bailey–Borel compactification and Borel's extension theorem. More precisely, this paper is motivated by Conjecture 1.6.

Theorem† (Theorem 2.24)**.** *The Stein factorization*

$$B \xrightarrow{\;\hat\Phi\;} \hat\wp \longrightarrow \wp, \qquad \Phi : B \to \wp$$

of the period mapping extends to a "Stein factorization"

$$\overline{B} \xrightarrow{\;\hat\Phi^0\;} \hat\wp^0 \longrightarrow \overline{\wp}^0, \qquad \Phi^0 : \overline{B} \to \overline{\wp}^0$$

of (1.4)*.* The fibres of $\hat\Phi^0$ are connected, and the fibres of $\hat\wp^0 \to \overline{\wp}^0$ are finite. And, as in Theorem 2.20, the image $\hat\wp^0$ is a Hausdorff topological space compactifying $\hat\wp$, and a finite union of normal complex analytic varieties.

Conjecture 1.6 ([GGLR20])**.** *The image $\hat\Phi^0(\overline{B}) = \hat\wp^0$ is projective algebraic, and the extension $\hat\Phi^0 : \overline{B} \to \hat\wp^0$ is an algebraic morphism.*

Remark 1.7. In the classical case that D is hermitian and Γ is arithmetic, a stronger version of Conjecture 1.6 holds: $\overline{\wp}^0$ is the closure of $\wp$ in the Satake-Baily-Borel compactification of $\Gamma\backslash D$, and Φ^0 is Borel's extension [BB66, Bor72]. In particular, if D is hermitian, the extension (1.4) is algebraic.

Theorem 1.8 ([GGLR20])**.** *If* $\dim B = 2$*, then Conjecture 1.6 holds.*

For more on the $\dim B = 2$ case, see [GG22].

Remark 1.9. Bakker–Brunebarbe–Tsimerman have applied the o-minimal structures of model theory to prove the long standing conjecture that the image $\wp = \Phi(B)$ of the period map is quasi-projective [BBT23]. In particular, they show that the (*augmented*) *Hodge line bundle*

$$\Lambda \;=\; \det(\mathcal{F}^n) \otimes \det(\mathcal{F}^{n-1}) \otimes \cdots \otimes \det(\mathcal{F}^{\lceil (n+1)/2 \rceil})$$

is semi-ample over B, and that $\wp = \mathrm{Proj}\left(\oplus_d H^0(B, d\Lambda)\right)$. Note however that this result does not suffice to establish the existence of a *completion* of the period map. What is missing is to show that Deligne's extension Λ_{e} is semi-ample over $\overline{B}$. This is conjectured to be the case in [GGLR20], and proven there for $\dim \wp = 1$; and for $\dim B = 2$. What we can say in this direction is

Theorem† (Corollary 3.10)**.** *A power of the line bundle* Λ_{e} *descends to* $\hat{\wp}^0$*.*

Corollary 3.10 is a consequence of (i) the existence of a neighborhood $\overline{\mathcal{O}}^0 \subset \overline{B}$ of A^0 over which Φ^0 is proper and admits a period matrix representation (§1.5); and (ii) strong constraints on the monodromy Γ_{A^0} of the variation of Hodge structure over $B \cap \overline{\mathcal{O}}^0$ (§3.1). There is also an arithmetic component of the argument (Theorem 3.22, which asserts that the values of a certain character are roots of unity) requiring the construction of a rational representation $\Gamma_{A^0} \to \mathrm{Aut}(U)$ with certain properties (§3.5.1). The construction is intricate (because Γ_{A^0} need not commute with the local monodromy at infinity), and requires that we work with a more expansive notion of what it means to polarize a mixed Hodge structure than has previously been considered. (See §3.5.5 for an overview.)

It is an open question whether or not Λ_{e} descends to $\overline{\wp}^0$.

1.4. **The motivating questions.** The main challenge that arises when trying to prove Conjecture 1.6 is to show that $\hat{\wp}^0$ admits the structure of a complex analytic variety. It will then follow from [GGLR20] that Λ_{e} is ample over $\hat{\wp}^0$. This raises a number of questions, including two that are the focus of this paper. Let A^0 be a connected component of a Φ^0–fibre.

(i) *Does* A^0 *admit a neighborhood* $\overline{\mathcal{O}}^0 \subset \overline{B}$ *with the following properties? The restriction of* Φ^0 *to* $\overline{\mathcal{O}}^0$ *is proper, and the holomorphic functions on* $\overline{\mathcal{O}}^0$ *separate the fibres of* $\Phi^0|_{\overline{\mathcal{O}}^0}$*?*

(ii) *What is the global geometry A^0?*

Remark 1.10. Question (i) arises because this would establish the desired complex analytic structure, by either [BB66, Theorem 9.2] or [Gra83]. Question (ii) is relevant because we are conjecturing that there is a map contracting the fibre A^0, cf. [Gra62, Art70, Fuj75].

1.5. **Period matrix representations.** A partial answer to Question §1.4(i) is given by

Theorem† (Corollaries 2.25 & 3.7). *Given a connected component A^0 of a Φ^0–fibre, there exists a neighborhood $\overline{\mathcal{O}}^0 \subset \overline{B}$ of A^0 with the properties:*

(a) *the restriction of Φ^0 to $\overline{\mathcal{O}}^0$ is proper, and*

(b) *the restriction of Φ to $\mathcal{O}^0 = B \cap \overline{\mathcal{O}}^0$ admits a period matrix representation.*

What is meant by "admits a period matrix representation" is discussed in Definition 1.12. The theorem allows us to reduce Conjecture 1.6 to the problem of extending functions off $Z \cap \overline{\mathcal{O}}^0$.

Theorem† (Theorem 3.30). *Conjecture 1.6 holds if the holomorphic functions on $Z_I \cap \overline{\mathcal{O}}^0$ extend to holomorphic functions on $\overline{\mathcal{O}}^0$.*

This reduction of the conjecture is deduced from [BBT23] and the following theorem.

Let $\mathcal{F}^p_{\mathrm{e}}$ and Λ_{e} denote Deligne's extensions of the Hodge vector bundle and augmented Hodge line bundle to $\overline{B}$.

Theorem† (Theorem 3.8 and Corollary 3.9). *There exists $m_p > 0$ so that the power $\det(\mathcal{F}^p_{\mathrm{e}})^{m_p}$ is trivial over $\overline{\mathcal{O}}^0$. In particular, there exists $m > 0$ so that the power Λ^m_{e} is trivial over $\overline{\mathcal{O}}^0$.*

Remark 1.11. As noted in [GGLR20], the existence of theses trivializations implies that implies that $\det(\mathcal{F}^p_{\mathrm{e}})^{m_p}$ and $\Lambda_{\mathrm{e}}^{\otimes m}$ descend to the finite cover $\hat{\wp}^0$. Returning to Question 1.4(i), and the problem of separating fibres, it follows from this triviality and [BBT23], that: (i) the functions on $\overline{\mathcal{O}}^0$ separate the fibres of $\hat{\Phi}^0$ over $\mathcal{O}$; and (ii) functions on $Z_I \cap \overline{\mathcal{O}}^0$ separate the fibres of $\hat{\Phi}^0$ over $Z_I^* \cap \overline{\mathcal{O}}^0$. So what is left to establish an analytic structure on $\hat{\wp}^0$ is to show that functions on $Z_I \cap \overline{\mathcal{O}}^0$ can be extended to $\overline{\mathcal{O}}^0$.

Definition 1.12. Period matrix representations are closely related to Schubert cells (§A.5). The compact dual $\check{D} \supset D$ can be covered by Zariski open Schubert cells. Each such cell is biholomorphic to $\mathbb{C}^m$, with $m = \dim D$. (These are local coordinate charts on $\check{D}$.) We say that *the period mapping Φ can be represented by a period matrix over an open set $\mathcal{O} \subset B$* when the following two conditions hold:

(i) The lift $\tilde{\Phi} : \tilde{\mathcal{O}} \to D$ of Φ to the universal cover of $\mathcal{O}$ takes value in a Schubert cell $\mathcal{S} \subset \check{D}$.

(ii) The monodromy $\Gamma_{\mathcal{O}} \subset \Gamma$ of the variation over $\mathcal{O}$ preserves $D \cap \mathcal{S}$.

Under these conditions, the pullback of the coordinates on $\mathcal{S} \to \mathbb{C}^m$ yields the period matrix representation of $\Phi|_{\mathcal{O}}$. If $\Gamma_{\mathcal{O}}$ is nontrivial, then the entries of the period matrix may be multi-valued (cf. the logarithm in Example 1.17). Nonetheless, we may think of this as giving us a (possibly multi-valued) coordinate representation of $\Phi|_{\mathcal{O}}$.

Example 1.13. The fact that period maps are locally liftable implies that they can always be locally represented by period matrices. Schmid's nilpotent orbit theorem implies that this property also holds at infinity: points $b \in Z$ admit local coordinates $\overline{\mathcal{U}} \subset \overline{B}$ so that the restriction of Φ to $\mathcal{U} = B \cap \overline{\mathcal{U}}$ can be represented by a period matrix. The expression (1.18) is an example of one such representation.

1.6. **The geometry at infinity.** Question §1.4(ii) is answered by Corollaries 5.5 and 5.8, which may be summarized as follows. The variation of limiting mixed Hodge structures over Z_I^* defines a map $\Phi^1 : A^0 \cap Z_I^* \to \mathcal{J}_I$, with $\mathcal{J}_I$ an abelian variety. This map encodes the level one extension data in the variation of limiting mixed Hodge structure along $A^0 \cap Z_I^*$. It extends to the Zariski closure $A_I^0 = A^0 \cap Z_I$,

$$\Phi^1 : A_I^0 \;\to\; \mathcal{J}_I \,. \tag{1.14}$$

The abelian variety admits a family $\{\mathcal{L}_M\}$ of ample line bundles. (The latter was independently observed in [BBT20].)

Theorem† (Corollary 5.5)**.** *There exist integers $\kappa_i = \kappa_i(M)$ so that the line bundle $\mathcal{L}_M$ is related to the normal bundles $\mathcal{N}_{Z_i/\overline{B}}$ by*

$$(\Phi^1\big|_{A_I^0})^*(\mathcal{L}_M) \;=\; \sum \kappa_i [Z_i]|_{A_I^0} \;=\; \sum \kappa_i\, \mathcal{N}_{Z_i/\overline{B}}\Big|_{A_I^0} \,. \tag{1.15}$$

Remark 1.16. The expression (1.15) relates the geometry *along* A^0 to the geometry *normal* to $Z \subset \overline{B}$. Moreover, by Theorem 6.1, *this is the central geometric information that arises when considering the variation of limiting mixed Hodge structure along A_I^0.*

Theorem† (Corollary 5.8)**.** *If the differential of $\Phi^1\big|_{A_I^0}$ is injective and $M \in \mathbf{N}_I^{\mathfrak{sl}_2}$, then the line bundle $\sum \kappa_i \mathcal{N}^*_{Z_i/\overline{B}}\big|_{A_I^0}$ is ample.*

Example 1.17. Consider a weight $n = 1$ variation of Hodge structure with Hodge numbers $\mathbf{h} = (2,2)$. Suppose that $\dim B = 2$, and fix local coordinates $(t,w) \in \Delta^2 = \overline{\mathcal{U}}$ on $\overline{B}$ centered at a point $b \in Z$ so that $Z = \{t = 0\}$ locally. (Here $\Delta \subset \mathbb{C}$ is the unit disc.) Suppose the local nilpotent logarithm of monodromy about $t = 0$ has rank one. (This is the mildest possible non-trivial degeneration. Imagine a 2-parameter family of smooth genus two curves acquiring a node.) Then the restriction of Φ to $\Delta^* \times \Delta = \mathcal{U}$ may be represented by the period matrix

$$\widetilde{\Phi}(t,w) \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha(t,w) & \lambda(t,w) \\ \hat{\nu}(t,w) & \alpha(t,w) \end{bmatrix}, \tag{1.18}$$

with $\alpha(t,w)$, $\lambda(t,w)$, $\nu(t,w) = \hat{\nu}(t,w) - \log(t)/2\pi\mathbf{i}$ holomorphic functions on Δ^2.

We can choose the neighborhood $\overline{\mathcal{O}}^0$ so that the monodromy over $\mathcal{O}^0$ takes the form

$$\gamma \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathsf{a} & 1 & 0 & 0 \\ \mathsf{b} & 0 & 1 & 0 \\ \mathsf{c} & \mathsf{b} & -\mathsf{a} & 1 \end{bmatrix},$$

with $\mathsf{a},\mathsf{b},\mathsf{c} \in \mathbb{Z}$. Then the period matrix $\widetilde{\Phi}(t,w)$ transforms as

$$\gamma \cdot \widetilde{\Phi}(t,w) \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha(t,w) + \mathsf{b} - \mathsf{a}\lambda(t,w) & \lambda(t,w) \\ \hat{\nu}(t,w) + \mathsf{c} - \mathsf{ab} + 2\mathsf{a}\alpha(t,w) + \mathsf{a}^2\lambda(t,w) & \alpha(t,w) + \mathsf{b} - \mathsf{a}\lambda(t,w) \end{bmatrix}$$

Under this action, $\nu(t,w)$ transforms as

$$\nu(t,w) \;\mapsto\; \nu(t,w) + \mathsf{c} - \mathsf{ab} + 2\mathsf{a}\alpha(t,w) + \mathsf{a}^2\lambda(t,w)\,,$$

so that

$$\tau(t,w) \;= \exp(2\pi\mathbf{i}\,\hat{\nu}(t,w)) \;=\; t\exp(2\pi\mathbf{i}\,\nu(t,w))$$

transforms as

$$\begin{aligned} \tau(t,w) \;&\mapsto\; t\exp 2\pi\mathbf{i}\,(\varepsilon(t,w) + \mathsf{a}^2\lambda(t,w) - 2\mathsf{a}\alpha(t,w)) \\ &=\; \tau(t,w)\exp 2\pi\mathbf{i}\,(\mathsf{a}^2\lambda(t,w) - 2\mathsf{a}\alpha(t,w))\,. \end{aligned}$$

This is the functional equation for the classical theta function. We may normalize our choice of coordinates (t,w) so that $\nu(t,w) = 0$. Then, this computation implies that $t \cdot \vartheta$, with ϑ a section of the dual to the theta line bundle, is globally well-defined along the fibre.

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Organization of the paper. • In §2.1 we review the Lie theoretic structure of the extension data in the limiting mixed Hodge structures along Z_I^*. In §2.2 we use the reduced limit period map to establish a very close relationship between the restriction of Φ^0 to Z_I^* and the topological boundary of the period domain in the compact dual (Proposition 2.7).

• The topological completion $\Phi^0 : \overline{B} \to \overline{\wp}^0$ is constructed in §§2.3–2.4.

• In §§3.1–3.2 we show that A^0 admits a neighborhood $\overline{\mathcal{O}}^0 \subset \overline{B}$ with the property that the restriction Φ^0 to $\overline{\mathcal{O}}^0$ is still proper, and admits a period matrix representation.

• In §§3.3–3.4 we show that some multiple $\det(\mathcal{F}_{\mathrm{e}}^p)^{\otimes m_p} \to \overline{B}$ of the extended line bundles are trivial over $\overline{\mathcal{O}}^0$. From this it follows that a power $\Lambda_{\mathrm{e}}^{\otimes m}$ descends to the finite cover $\hat{\wp}^0 \to \overline{\wp}^0$ (Corollary 3.10).

• In §3.6 we reduce Conjecture 1.6 to an extension problem.

• In §4.1 we study the monodromy about $A_I^0 = A^0 \cap Z_I$. In §4.2 we construct explicit sections s_M of line bundles over a neighborhood $\overline{\mathcal{O}}_I^0 \supset A_I^0$. These sections will be used in §5 to establish (1.15), which relates the geometry of the Φ^0–fibre A^0 to the normal bundles $\mathcal{N}_{Z_i/\overline{B}}$.

• In §5 we study the level one extension data map $A_I^0 \to \mathcal{J}_I$. In §6 we show that, modulo a nilpotent orbit, the higher level extension data is locally constant on fibres of (1.14).

• We need to set notation and review the local behavior of period maps at infinity. Because this material is classical, we streamline the presentation by placing this this review (which also includes the proofs of a few technical lemmas) in §§A–B.

Contents

2. Period mappings at infinity

2.1. **A tower of maps: extension data.** To understand the geometry of the fibres of Φ^0, we note that what varies along the fibre A^0 is the extension data of the mixed Hodge structure (W, F). This suggests that we study of the geometry of the extension data. To that end we realize (1.3) as the extremal maps in a tower

$$
\begin{array}{ccc}
Z_I^* & \xrightarrow{\Phi_I} & (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I \\
 & \searrow^{\Phi_I^a} & \downarrow \\
 & & (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I^a \\
 & \searrow^{\Phi_I^2} & \downarrow \\
 & & (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I^2 \\
 & \searrow^{\Phi_I^1} & \downarrow \\
 & & \Gamma_I\backslash D_I^1 \\
 & \searrow^{\Phi_I^0} & \downarrow \\
 & & \Gamma_I\backslash D_I^0\,,
\end{array}
\tag{2.1}
$$

with $3 \le a \le 2n-1$, that is defined as follows.

2.1.1. *Mixed Hodge structures.* Given a MHS (W, F_0), define Hodge numbers $f_\ell^p := \dim F_0^p(\mathrm{Gr}_\ell^W)$, and set

$$D_W \ = \ \{F \in \check{D} \mid (W,F) \text{ is a MHS, } \dim F^p(\mathrm{Gr}_\ell^W) = f_\ell^p\}\,.$$

Set

$$G \ = \ \mathrm{Aut}(V, Q)\,,$$

and let $P_W \subset G$ be the $\mathbb{Q}$–algebraic group stabilizing the weight filtration. (See §A.1.1 for further discussion of group notation.) Given any $g \in P_W$, there is an induced action on the quotients $W_\ell / W_{\ell-a}$. The normal subgroups

$$P_W^{-a} \ = \ \{g \in P_W \mid g \text{ acts trivially on } W_\ell/W_{\ell-a} \ \forall\, \ell\}$$

define a filtration $P_W = P_W^0 \supset P_W^{-1} \supset \cdots$. The group

$$G_W \ = \ (P_{W,\mathbb{R}}/P_{W,\mathbb{R}}^{-1}) \ltimes P_{W,\mathbb{C}}^{-1} \tag{2.2}$$

acts transitively D_W, [KP16].

2.1.2. *Limiting mixed Hodge structures.* Now suppose that the MHS (W, F_0) is polarized by a nilpotent cone

$$\sigma_I \ = \ \mathrm{span}_{\mathbb{R}_{>0}}\{N_i \mid i \in I\} \ \subset \ \mathrm{End}(V_\mathbb{R}, Q)$$

of commuting logarithms of monodromy. (Here $\exp(N_i)$ is a local monodromy operator about Z_i^*.) Define

$$D_I \ = \ \{F \in D_W \mid (W,F) \text{ is polarized by } \sigma_I\}\,.$$

Then $W = W(\sigma_I)$ implies that the $\mathbb{Q}$–algebraic group $C_I \subset \mathrm{Aut}(V,Q)$ centralizing the cone σ_I is a subgroup of P_W. Note that this centralizer also admits a filtration $C_I = C_I^0 \supset C_I^{-1} \supset \cdots$ by normal subgroups

$$C_I^{-a} \;=\; C_I \cap P_W^{-a}\,.$$

The group

$$G_I \;=\; (C_{I,\mathbb{R}}/C_{I,\mathbb{R}}^{-1}) \ltimes C_{I,\mathbb{C}}^{-1}$$

acts transitively D_I, [KP16].

2.1.3. *Definition of the tower.* Let

$$\Gamma_I \;=\; \Gamma \cap C_{I,\mathbb{Q}}\,.$$

The variation of limiting mixed Hodge structures along Z_I^* in §1.2 induces the map Φ_I of (2.1), cf. §A.2.4. The maps Φ_I^a are defined by passing to the quotient spaces $D_I^a = C_{I,\mathbb{C}}^{-a-1}\backslash D_I$. Define

$$\wp_I^a \;=\; \Phi_I^a(Z_I^*)\,.$$

We have natural surjections $\wp_I^{a+1} \twoheadrightarrow \wp_I^a$. Theorem 6.1(c) implies

Theorem 2.3. *The maps $\wp_I^{a+1} \twoheadrightarrow \wp_I^a$ are finite to one for all $a \geq 2$.*

We have tower of fibre bundles

$$D_I \;\twoheadrightarrow \cdots \twoheadrightarrow\; D^{a+1} \;\twoheadrightarrow\; D_I^a \;\twoheadrightarrow \cdots \twoheadrightarrow\; D_I^0\,.$$

We say that the quotient D_I^a has automorphism group $G_I^a = G_I/C_{I,\mathbb{C}}^{-a-1}$ to indicate that G_I acts on D_I^a, with the normal subgroup C_I^{-a-1} acting trivially. The base space D_I^0 is a Mumford–Tate domain with Mumford–Tate group G_I^0.

Definition 2.4 (Extension data of LMHS). If $\delta_I = \delta_{I,F} = \delta_{W,F} \cap D_I$ is the fibre of the surjection $D_I \to D_I^0$ and $\Gamma_I^{-1} = \Gamma \cap C_{I,\mathbb{Q}}^{-1}$, then $\Gamma_I^{-1}\backslash\delta_{I,F}$ is the *(polarized) extension data of the limiting mixed Hodge structure* (W,F). The image $\delta_I^a = \delta_{I,F}^a$ of δ_I under the projection $D_I \to D_I^a$ is also a fibre of $D_I^a \twoheadrightarrow D_I^0$, and we say that $\Gamma_I^{-1}\backslash\delta_{I,F}^a$ is the *(polarized) extension data of level $\leq a$.*

Remark 2.5. Theorem 2.3 asserts that the level ≤ 2 extension data map Φ_I^2 determines the full extension data map Φ_I up to constants of integration. Additionally the level 2 extension data is discrete. (The data not given by constants of integration is given by sections of line bundles with fixed divisor, Theorem 6.1(c) and Remark 6.2.) So it is not surprising that we will see that the answer to Question §1.4(ii) is to be found in studying the level one extension data (which is given by the map Φ_I^1) along the the Φ_I^0–fibre $A^0 \cap Z_I^*$. This restriction takes value in some $\Gamma_I\backslash\delta_{I,F}^1$. The spaces $\Gamma_I\backslash\delta_{I,F}^1$ and $\Gamma\backslash\delta_{W,F}^1$ of level one extension data carry rich geometric structure. As observed by Carlson, these spaces are tori, and $\Gamma_I\backslash\delta_{I,F}^1$ is an abelian subvariety when $F^p(\mathrm{Gr}^W_{-1})$ defines a level one Hodge structure [Car87]. To this we add Theorem 5.3, and Corollary (5.5); the later encodes the central geometric information that arises when considering the VLMHS along A^0_I.

2.2. **Reduced limit period map.** The purpose of this section is to describe an important relationship between the period map $\Phi^0_I : Z^*_I \to \Gamma_I \backslash D^0_I$ and the topological boundary ∂D of the period domain in the compact dual $\check{D}$. In general, the limit Hodge filtration F associated with a point $b \in Z^*_I$ (cf. §A.2.4) will not lie in the boundary. However, there is a "naïve", or *reduced limit* $F_\infty(b)$, that does lie in ∂D (§2.2.1). Each of these limits takes value in a $C_{I,\mathbb{R}}$–orbit $\mathcal{O}_I \subset \partial D$, and there is an induced map

$$\Phi^\infty_I : Z^*_I \ \to \ \Gamma_I \backslash \mathcal{O}_I \,. \tag{2.6}$$

Let

$$\wp^\infty_I \ = \ \Phi^\infty_I(Z^*_I) \ \subset \ \Gamma_I \backslash \mathcal{O}_I$$

denote the image.

Proposition 2.7. *The period map Φ^0_I factors through the reduced limit period map Φ^∞_I. Moreover, the map Φ^∞_I is locally constant on Φ^0_I–fibres. In particular, the map $\pi_I : \wp^\infty_I \to \wp_I$ is finite.*

The proposition is proved in §§2.2.2–2.2.4. It imposes an additional constraint on the monodromy over a neighborhood $\mathcal{O}^0$ of a A^0 (Lemma 3.3). This constraint makes it possible for us to show that $\Phi|_{\mathcal{O}^0}$ admits a period matrix representation (Corollary 3.7).

2.2.1. *Definition.* Fix a local lift $\widetilde{\Phi}(t,w)$, and let (W,F,σ) be the associated limiting mixed Hodge structure (§§A.2.2–A.2.4). The *reduced limit period*

$$F_\infty(w) \ = \ \lim_{y\to\infty} \widetilde{\Phi}(z,w) \ = \ \lim_{y\to\infty} \exp(\mathbf{i}yN)\tilde{g}(0,w)\cdot F \ \in \ \overline{D}$$

is independent of our choice of $N \in \sigma$, [GGK13, KP14, GGR17]. (The limit is understood to be taken with x bounded.) The two filtrations F and $F_\infty(0)$ are related by the Deligne splitting (§A.3)

$$F^p \ = \ \bigoplus_{a\ge p} V^{a,b}_{W,F} \quad \text{and} \quad F^p_\infty(0) \ = \ \bigoplus_{b\le n-p} V^{a,b}_{W,F} \,. \tag{2.8}$$

In particular, the Lie algebra $\mathfrak{f}_\infty$ of the stabilizer $\mathrm{Stab}_{G_\mathbb{C}}(F_\infty(0))$ is

$$\mathfrak{f}_\infty \ = \ \bigoplus_{q\le 0} \mathfrak{g}^{p,q}_{W,F} \,. \tag{2.9}$$

Recalling that the map $\tilde{g}(0,w)$ takes value in $C_{I,\mathbb{C}}$ (§A.2.4), we see that

$$F_\infty(w) \ = \ \tilde{g}(0,w)\cdot F_\infty(0) \,. \tag{2.10}$$

In particular, the map $F_\infty : \{0\} \times \Delta^r \to \check{D}$ is holomorphic, and takes value in the $C_{I,\mathbb{C}}$–orbit of $F_\infty(0)$. What is less obvious is that: (i) The holomorphic $F_\infty(0,w)$ takes value in the real orbit

$$\mathcal{O}_I \;=\; C_{I,\mathbb{R}} \cdot F_\infty(0) \;\subset\; \check{D}\,.$$

(ii) The real orbit $\mathcal{O}_I$ is open in the (complex) orbit $C_{I,\mathbb{C}} \cdot F_\infty(0)$, and so is a complex submanifold of $\check{D}$. See [GG15, KP14] for details.

The reduced limit F_∞ is independent of the local coordinates (t,w) expressing $\widetilde{\Phi}$. So the reduced period limit induces a well-defined holomorphic map (2.6).

2.2.2. *Proof: period map factors through reduced limit.* Observe that there is a natural identification

$$D_I^0 \;\simeq\; C_{I,\mathbb{R}}^{-1} \backslash \mathcal{O}_I\,.$$

This identification induces

$$\pi_I : \Gamma_I \backslash \mathcal{O}_I \;\to\; \Gamma_I \backslash D_I^0\,. \tag{2.11}$$

We have

$$\Phi_I^0 \;=\; \pi_I \circ \Phi_I^\infty\,. \tag{2.12}$$

In particular, $\pi_I : \wp_I^\infty \twoheadrightarrow \wp_I$.

Remark 2.13. When D is hermitian the map (2.11) is an isomorphism and $\Phi_I^0 = \Phi_I^\infty$.

2.2.3. *Proof of finiteness: formulation of the argument.* It is enough to show that $F_\infty(w)$ is constant along the Φ^0–fibres in $\{0\} \times \Delta^r$. This is a consequence of the infinitesimal period relation. The essential point is that the map

$$w \;\mapsto\; \tilde{g}(0,w) \cdot F \quad \text{is horizontal.} \tag{2.14}$$

Recall that $\tilde{g}(t,w)$ takes value in $\exp(\mathfrak{f}^\perp)$, and $\tilde{g}(0,w)$ takes value in $\exp(\mathfrak{c}_{I,\mathbb{C}})$, cf. §A.3 and §A.2.4. We have

$$\mathfrak{f}^\perp \,\cap\, \mathfrak{c}_{I,\mathbb{C}} \;=\; \bigoplus_{\substack{p<0\\ p+q\le 0}} \mathfrak{c}_{I,F}^{p,q}\,.$$

Note that

$$\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}} \cap \mathfrak{f}_\infty \;=\; \bigoplus_{\substack{p<0\\ q\le 0}} \mathfrak{c}_{I,F}^{p,q}\,,$$

and consider the decomposition

$$\mathfrak{f}^\perp \,\cap\, \mathfrak{c}_{I,\mathbb{C}} \;=\; \mathfrak{d} \;\oplus\; \mathfrak{e} \;\oplus\; (\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}} \cap \mathfrak{f}_\infty)$$

defined by

$$\mathfrak{d} \;=\; \bigoplus_{\substack{p<0\\ p+q=0}} \mathfrak{c}^{p,q}_{I,F} \quad\text{and}\quad \mathfrak{e} \;=\; \bigoplus_{\substack{p<0<q\\ p+q<0}} \mathfrak{c}^{p,q}_{I,F}\,.$$

Each of these three summands is a Lie subalgebra of $\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}}$.

Since $\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}}$ is nilpotent, the function $\tilde g(0,w)$ may be uniquely decomposed as

$$\tilde g(0,w) \;=\; e(w) f(w) s(w)$$

with $f(w) \in \exp(\mathfrak{d})$, $e(w) \in \exp(\mathfrak{e})$ and $s(w) \in \exp(\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}} \cap \mathfrak{f}_\infty)$. Since $\tilde g(0,w) = e(w)\, f(w) s(w) f(w)^{-1}\, f(w)$, and both $e(w)$ and $f(w)s(w)f(w)^{-1}$ take value in the unipotent radical $C^{-1}_{I,\mathbb{C}}$, we may

$$\text{identify } \Phi^0_I(0,w) \text{ with } f(w).$$

Furthermore, since $\mathfrak{f}_\infty$ is the stabilizer of $F_\infty(0)$ in $\mathfrak{f}$, (2.10) implies we may

$$\text{identify } F_\infty(w) \text{ with } e(w)f(w).$$

So to prove the lemma, it suffices to show that

$$e(w) \text{ is locally constant along } f\text{–fibres.}$$

So we assume

$$\mathrm{d}f = 0\,, \tag{2.15a}$$

and will show that $\mathrm{d}e = 0$; equivalently,

$$e^{-1}\mathrm{d}e \;=\; 0\,. \tag{2.15b}$$

2.2.4. *Proof of finiteness: horizontality.* Horizontality is the condition

$$(\xi^{-1}\mathrm{d}\xi)^{p,q} \;=\; 0\,, \quad \forall\, p \le -2\,, \tag{2.16}$$

with $(\xi^{-1}\mathrm{d}\xi)^{p,q}$ the component of the $\mathfrak{f}^\perp$–valued $\xi^{-1}\mathrm{d}\xi$ taking value in $\mathfrak{g}^{p,q}_{W,F}$, cf. §A.5 and §A.4. At $(0,w)$ we have

$$\begin{aligned} \xi^{-1}\mathrm{d}\xi \;&=\; (efs)^{-1}\mathrm{d}(efs)\\ &=\; \mathrm{Ad}^{-1}_{fs}(e^{-1}\mathrm{d}e) \;+\; \mathrm{Ad}^{-1}_{s}(f^{-1}\mathrm{d}f) \;+\; s^{-1}\mathrm{d}s\\ &\overset{(2.15a)}{=}\; \mathrm{Ad}^{-1}_{fs}(e^{-1}\mathrm{d}e) \;+\; s^{-1}\mathrm{d}s\,. \end{aligned} \tag{2.17}$$

Note that $e^{-1}\mathrm{d}e$ and $s^{-1}\mathrm{d}s$ take value in $\mathfrak{e}$ and $\mathfrak{f}_\infty$, respectively. Furthermore, (A.6d) and $fs \in \exp(\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}})$ imply that

$$e^{-1}\mathrm{d}e \;=\; 0 \quad\text{if and only if}\quad \left(\mathrm{Ad}^{-1}_{fs}(e^{-1}\mathrm{d}e)\right)^{p,q} \;=\; 0$$

for all $q > 0$ and $p+q<0$. At the same time (A.6d), (2.16) and (2.17) imply that

$$0 \;=\; (\xi^{-1}\mathrm{d}\xi)^{p,q} \;=\; \left(\mathrm{Ad}^{-1}_{fs}(e^{-1}\mathrm{d}e)\right)^{p,q}$$

for all $q > 0$ and $p + q < 0$. The desired (2.15b) now follows, completing the proof of Proposition 2.7.

2.3. **Extension to proper maps.** The period map $\Phi_I^0 : Z_I^* \to \Gamma_I \backslash D_I^0$ may not be proper, however it has a proper extension [GS69]. Let Π be the finest possible partition of the power set $\{I\}$ of $\{1, \ldots, \nu\}$ that satisfies the following property: given $\pi \in \Pi$ and $I \in \pi$, if $I \subset J$ and Φ_I extends to Z_J^* then $J \in \pi$. The Γ–conjugacy class $[W]$ of the weight filtration is well-defined along

$$Z_\pi \;=\; \bigcup_{I \in \pi} Z_I^* \,.$$

The intersection $Z_I \cap Z_\pi$ is the *weight-closure* of Z_I^*. The maps Φ_I^0 and Φ_I^1 in the tower (2.1) extend to the weight-closure (Lemma B.1), as follows. Given $I, J \in \pi$ with $I \subset J$ we have $Z_J^* \subset Z_I \cap Z_\pi$ and $\Gamma_J \subset \Gamma_I$. It is also the case that $D_J \subset D_I$ (§B.4). This induces maps $\Gamma_J \backslash D_J^a \to \Gamma_I \backslash D_I^a$, and a commutative diagram

$$\begin{array}{ccccc} Z_J^* & \xrightarrow{\Phi_J^1} & \Gamma_J \backslash D_J^1 & \longrightarrow & \Gamma_I \backslash D_I^1 \\ & \searrow^{\Phi_J^0} & \downarrow & & \downarrow \\ & & \Gamma_J \backslash D_J^0 & \longrightarrow & \Gamma_I \backslash D_I^0 \,. \end{array}$$

The resulting

$$\begin{array}{ccc} Z_I \cap Z_\pi & \xrightarrow{\Phi_I^1} & \Gamma_I \backslash D_I^1 \\ & \searrow^{\Phi_I^0} & \downarrow \\ & & \Gamma_I \backslash D_I^0 \end{array} \tag{2.18}$$

are proper extensions of the maps Φ_I^0 and Φ_I^1 in (2.1). The proper mapping theorem implies that the images

$$\wp_I^0 \;=\; \Phi_I^0(Z_I \cap Z_\pi) \quad \text{and} \quad \wp_I^1 \;=\; \Phi_I^1(Z_I \cap Z_\pi)$$

are complex analytic spaces. We reiterate that the image $\wp_I^0$ of Φ_I^0 parameterizes σ_I–polarized Hodge structures (a.k.a., level 0 extension data) along $Z_I \cap Z_\pi$, and the image $\wp_I^1$ of Φ_I^1 parameterizes the extension data of level ≤ 1 along $Z_I \cap Z_\pi$.[2]

[2]If we restrict to local lifts at infinity we can say more: it is a corollary of Lemma B.20 that the maps (2.18) patch together nicely to locally define analytic maps along Z_π.

2.4. **Two topological completions.** Consider the disjoint unions

$$\tilde{\wp}^0_\pi \ = \ \bigcup_{I\in\pi} \wp^0_I \quad \text{and} \quad \tilde{\wp}^1_\pi \ = \ \bigcup_{I\in\pi} \wp^1_I \,.$$

Following the discussion of §2.3, if $b \in Z_I \cap Z_{I'} \cap Z_\pi$ then we identify the two points $x = \Phi^0_I(b) \in \wp^0_I$ and $x' = \Phi^0_{I'} \in \wp^0_{I'}$. Let $\wp^0_\pi$ be the quotient of $\tilde{\wp}^0_\pi$ by the equivalence relation generated by this identification. Define $\wp^1_\pi$ analogously. Set

$$\overline{\wp}^0 \ = \ \bigcup \wp^0_\pi \quad \text{and} \quad \overline{\wp}^1 \ = \ \bigcup \wp^1_\pi \,.$$

Note that we have injections

$$\wp^0_I \ \hookrightarrow \ \wp^0_\pi \ \hookrightarrow \ \overline{\wp}^0 \quad \text{and} \quad \wp^1_I \ \hookrightarrow \ \wp^1_\pi \ \hookrightarrow \ \overline{\wp}^1 \,.$$

Define maps

$$\overline{B} \xrightarrow{\ \Phi^1\ } \overline{\wp}^1 \twoheadrightarrow \overline{\wp}^0 \,, \qquad \Phi^0 : \overline{B} \to \overline{\wp}^0 \tag{2.19}$$

by specifying $\Phi^0\big|_{Z_I\cap Z_\pi} = \Phi^0_I$ and $\Phi^1\big|_{Z_I\cap Z_\pi} = \Phi^1_I$.

Let $\epsilon = 0, 1$. Fix a Riemannian metric on on $\overline{B}$. Since the fibres of Φ^ϵ are compact, there is an induced metric on $\overline{\wp}^\epsilon$. Endow $\overline{\wp}^\epsilon$ with the metric topology.

Theorem 2.20. *The topology on $\overline{\wp}^\epsilon$ is Hausdorff. The induced subspace topology on $\wp^\epsilon_I$ coincides with the natural topology on $\wp^\epsilon_I$ as a complex analytic space. The map $\Phi^\epsilon : \overline{B} \to \overline{\wp}^\epsilon$ is continuous and proper, and $\overline{\wp}^\epsilon$ is a topological compactification of $\wp$.*

Proof. It is clear that that the induced subspace topology coincides with the natural topology on $\wp^\epsilon_I$. The topology on $\overline{\wp}^\epsilon$ is Hausdorff if and only if the map Φ^ϵ is continuous. In this case, the map is necessarily proper. So it suffices to establish the continuity of Φ^ϵ.

Suppose that $b_i \in \overline{B}$ is a sequence of points converging to $b_\infty \in \overline{B}$. Let A_i and A_∞ be the fibres of Φ^ϵ through b_i and b_∞, respectively. Let $b'_i \in A_i$. Since $\overline{B}$ is compact, $\{b'_i\}$ contains a convergent subsequence; abusing notation, let $\{b'_i\}$ denote that convergent subsequence with limit b'_∞. The essential point is to prove that

$$b'_\infty \ = \ \lim_{i\to\infty} b'_i \ \in \ A_\infty \,. \tag{2.21}$$

Informally this says

$$\lim_{i\to\infty} A_i \ \subset \ A_\infty \,.$$

The local analog of this assertion is Lemma B.5. The "globalization" will follow from a certain finiteness result for Siegel domains.

First assume that both sequences $\{b_i\}$ and $\{b'_i\}$ are contained in B. Fix two coordinate charts $\overline{\mathfrak{U}}$ and $\overline{\mathfrak{U}}'$ centered at b_∞ and b'_∞ respectively, and local lifts $\widetilde{\Phi}(t,w)$ and $\widetilde{\Phi}'(t,w)$. Without loss of generality, $b_i \in \mathfrak{U}$ and $b'_i \in \mathfrak{U}'$. Since $b_i, b'_i \in A_i$, there exists $\gamma_i \in \Gamma$ so that

$\widetilde{\Phi}'(b_i') = \gamma_i \cdot \widetilde{\Phi}(b_i)$. Shrinking $\overline{\mathfrak{U}}$ if necessary, there exists a finite union $\mathfrak{D} \subset D$ of Siegel sets so that $\widetilde{\Phi}(\widetilde{\mathfrak{U}}) \subset \mathfrak{D}$. (In the case of one-variable degenerations this is a corollary of Schmid's SL(2) orbit theorem [Sch73, (5.26)]. In the general case, this is [**?**, Theorem 1.5], and is key to the Bakker–Klingler–Tsimerman result that period maps are $\mathbb{R}_{\mathrm{an,exp}}$–definable.) Likewise, we have a finite union $\mathfrak{D}' \subset D$ of Siegel sets so that $\widetilde{\Phi}'(\widetilde{\mathfrak{U}}') \subset \mathfrak{D}'$. It follows that there are only finitely many distinct γ_i. Restricting to a subsequence with all $\gamma_i = \gamma$ equal, we have $\widetilde{\Phi}'(b_i') = \gamma \cdot \widetilde{\Phi}(b_i)$. Since we may replace the local lift $\widetilde{\Phi}'$ with $\gamma^{-1}\widetilde{\Phi}'$, this forces b_∞ and b'_∞ to lie in the same Φ^ϵ–fibre. This establishes the desired (2.21) in the case that $\{b_i\}$ and $\{b_i'\}$ are contained in B.

For the general case, we may assume without loss of generality that $\{b_i\} \subset Z_I^*$ and $\{b_i'\} \subset Z_{I'}^*$ with $W^I = W^{I'}$. We leave it an exercise for the reader to verify that Lemma B.5 allows us to modify the argument above to treat the general case. □

2.5. **A "Stein factorization" of Φ^ϵ.** Since the period mapping $\Phi : B \to \Gamma\backslash D$ is proper, we may consider the Stein factorization

$$B \xrightarrow{\hat{\Phi}} \hat{\wp} \longrightarrow \wp\,, \qquad \Phi = (\hat{\wp} \to \wp) \circ \hat{\Phi} \tag{2.22a}$$

The fibres of $\hat{\Phi}$ are connected, the fibres of $\hat{\wp} \to \wp$ are finite, and $\hat{\wp}$ is a normal complex analytic space. Likewise, we have Stein factorizations

$$\begin{array}{ccccc} & \overset{\hat{\Phi}^1}{\nearrow} & \hat{\wp}^1_I & \longrightarrow & \wp^1_I \\ Z_I \cap Z_\pi & & \downarrow & & \downarrow \\ & \underset{\hat{\Phi}^0}{\searrow} & \hat{\wp}^0_I & \longrightarrow & \wp^0_I \end{array} \tag{2.22b}$$

of the maps (2.18). Again, the fibres of $\hat{\Phi}^\epsilon_I$ are connected, the fibres of $\hat{\wp}^\epsilon_I \to \wp^\epsilon_I$ are finite, and $\hat{\wp}^\epsilon_I$ is normal.

The construction in §2.4 applies here to define $\hat{\wp}^\epsilon$ analogously to $\overline{\wp}^\epsilon$. Again, we have injections $\hat{\wp}^\epsilon_I \hookrightarrow \hat{\wp}^\epsilon$. The "Stein factorization"

$$\overline{B} \xrightarrow{\hat{\Phi}^\epsilon} \hat{\wp}^\epsilon \longrightarrow \overline{\wp}^\epsilon, \qquad \Phi^\epsilon = (\hat{\wp}^\epsilon \to \overline{\wp}^\epsilon) \circ \hat{\Phi}^\epsilon \tag{2.23}$$

of the map Φ^ϵ in (2.19) is given by specifying that the restrictions to $Z_I \cap Z_\pi$ coincide with (2.22). Again, the fibres of $\hat{\Phi}^\epsilon$ are connected, and the fibres of $\hat{\wp}^\epsilon \to \overline{\wp}^\epsilon$ are finite. The obvious analog of Theorem 2.20 holds by essentially the same argument.

Theorem 2.24. *The topology on $\hat{\wp}^\epsilon$ is Hausdorff. The induced subspace topology on $\hat{\wp}^\epsilon_I$ coincides with the natural topology on $\hat{\wp}^\epsilon_I$ as a normal complex analytic space. The maps of* (2.23) *are continuous and proper, and $\hat{\wp}^\epsilon$ is a topological compactification of $\hat{\wp}$.*

Corollary 2.25. *Let $\hat{A} \subset \overline{B}$ be a fibre of $\hat{\Phi}^\epsilon$.* (Equivalently, $\hat{A}$ is a connected component of a Φ^ϵ–fibre.) *Fix a neighborhood $\hat{\mathcal{O}} \subset \hat{\wp}^\epsilon$ of $\hat{\Phi}^\epsilon(\hat{A}) \in \hat{\wp}$. Then $\overline{\mathcal{O}} = (\hat{\Phi})^{-1}(\hat{\mathcal{O}}) \subset \overline{B}$ is a neighborhood of $\hat{A}$ with the property that the maps $\Phi|_{B\cap\overline{\mathcal{O}}}$, $\Phi^\epsilon|_{\overline{\mathcal{O}}}$ and $\hat{\Phi}^\epsilon\big|_{\overline{\mathcal{O}}}$ are all proper.*

3. Neighborhood of a $\hat{\Phi}^0$–fibre

Let A^0 be the $\hat{\Phi}^0$–fibre $\hat{A}$ of Corollary 2.25.

3.1. **Monodromy about the fibre.** The restriction of the variation of Hodge structure (1.2a) to $\mathcal{O}^0 = \overline{\mathcal{O}}^0 \cap B$ induces a period map

$$\Phi_{A^0} : \mathcal{O}^0 \ \to \ \Gamma_{A^0}\backslash D \tag{3.1}$$

with monodromy $\Gamma_{A^0} \subset \Gamma$.

Lemma 3.2. *We may choose the neighborhood $\overline{\mathcal{O}}^0$ of Corollary 2.25 so that*

$$\Gamma_{A^0} \ \subset \ P_{W,\mathbb{Q}}\,,$$

and the induced action

$$\mathrm{Gr}^W(\Gamma_{A^0}) \ = \ \Gamma_{A^0}/(\Gamma_{A^0} \cap P_W^{-1}) \ \subset \ \mathrm{Aut}(\mathrm{Gr}^W) \ = \ \oplus_\ell \, \mathrm{Aut}(\mathrm{Gr}^W_\ell)$$

of Γ_{A^0} on Gr^W stabilizes the Hodge filtration $F(\mathrm{Gr}^W)$.

Proof. The fibre A^0 is contained in a weight strata Z_π (§2.3). Along Z_π we have a variation of mixed Hodge structures (W,F). Here the weight filtration is constant, and it is the Hodge filtration $F \in \check{D}$ that varies. These filtrations lie in a G_W–orbit D_W (§2.1.1). So we may choose the neighborhood $\overline{\mathcal{O}}^0$ so that $\Gamma_{A^0} \subset P_{W,\mathbb{Q}}$.

When we restrict to A^0 the variation of mixed Hodge structures has the property that the Hodge decomposition $F^p(\mathrm{Gr}^W_\ell)$ is *constant*. So we may further assume that Γ_{A^0} fixes $F(\mathrm{Gr}^W) \in D^0_W$; equivalently, the discrete quotient $\mathrm{Gr}^W(\Gamma_{A^0})$ stabilizes $F(\mathrm{Gr}^W)$. □

Lemma 3.2 can be further strengthened. Recall the reduced limit period filtration $F_\infty \in \check{D}$ of §2.2.1.

Lemma 3.3. *We may choose the neighborhood $\overline{\mathcal{O}}^0$ so that $\Gamma_{A^0} \subset \mathrm{Stab}_{G_\mathbb{C}}(F_\infty)$.*

Proof. Proposition 2.7 implies that the the reduced period limit is constant on the connected A^0. □

Since Γ is real, it follows that

$$\Gamma_{A^0} \;\subset\; S \;=\; \mathrm{Stab}_{G_{\mathbb{C}}}(F_\infty) \cap \mathrm{Stab}_{G_{\mathbb{C}}}(\overline{F_\infty})\,. \tag{3.4a}$$

And (2.9) implies

$$S \;\subset\; P_{W,\mathbb{C}}\,. \tag{3.4b}$$

3.2. **Period matrix representation.** Consider the Schubert cell (§A.5)

$$\mathcal{S} \;=\; \exp(\mathfrak{f}^\perp)\cdot F \;=\; \left\{ \tilde F \in \check D \mid \dim(\tilde F^a \cap \overline{F^b_\infty}) = \dim(F^a \cap \overline{F^b_\infty})\,,\ \forall\, a, b \right\}\,. \tag{3.5}$$

Lemma 3.6. *The action of Γ_{A^0} on $\check D$ preserves the cell $\mathcal{S} \subset \check D$.*

Corollary 3.7 (Period matrix representation)**.** *Every local lift of Φ_{A^0} over a chart $\overline{\mathcal{U}}$ centered at a point $b \in A^0$ takes value in $\mathcal{S}$. In particular, the lift of Φ_{A^0} to the universal cover $\tilde{\mathcal{O}}^0 \to \mathcal{O}^0$ takes value in the Schubert cell:*

$$\begin{array}{ccc} \tilde{\mathcal{O}}^0 & \xrightarrow{\;\tilde\Phi_{A^0}\;} & \mathcal{S}\cap D \\ \big\downarrow & & \big\downarrow \\ \mathcal{O}^0 & \xrightarrow{\;\Phi_{A^0}\;} & \Gamma_{A^0}\backslash(\mathcal{S}\cap D)\,. \end{array}$$

Proof of Lemma 3.6. Since $\mathcal{S}$ is by definition those filtrations $\tilde F \in \check D$ intersecting $\overline{F_\infty}$ generically, it follows from (3.4a) that $\mathcal{S}$ is preserved by Γ_{A^0}. □

3.3. **Trivializations about the fibre.** Let $\mathcal{F}^r_{\mathrm{e}} \to \overline{B}$ denote Deligne's extension of the Hodge vector bundle [Del97].

Theorem 3.8. *There exists $m_p > 0$ so that the power $\det(\mathcal{F}^r_{\mathrm{e}})^{m_p}$ is trivial over $\overline{\mathcal{O}}^0$. If Γ is neat, then we make take $m_p = 1$.*

The theorem is proved in §3.4.

Let

$$\Lambda_{\mathrm{e}} \;=\; \det(\mathcal{F}^n_{\mathrm{e}}) \otimes \det(\mathcal{F}^{n-1}_{\mathrm{e}}) \otimes \cdots \otimes \det(\mathcal{F}^{\lceil (n+1)/2 \rceil}_{\mathrm{e}})\,.$$

be the extended (augmented) Hodge line bundle.

Corollary 3.9. *There exists $m > 0$ so that the power Λ^m_{e} is trivial over $\overline{\mathcal{O}}^0$. If Γ is neat, then we may take $m = 1$.*

As noted in [GGLR20], Theorem 3.8 and Corollary 3.9 yield

Corollary 3.10. *There exist integers $m, m_p > 0$ so that the line bundles $\det(\mathcal{F}^r_{\mathrm{e}})^{m_p}$ and Λ^m_{e} descend to $\hat\wp^0$.*

Proof. Recall that $\overline{\mathcal{O}}^0$ is the $\hat\Phi^0$ pre-image of an open set $\hat{\mathcal{O}}^0 \subset \hat\wp^0$ (Corollary 2.25). It follows from Theorem 3.8 that $\det(\mathcal{F}^r_{\mathrm{e}})^{m_p}$ descends to a line bundle on $\hat\wp^0$ that is trivial over $\hat{\mathcal{O}}^0$. □

3.4. **Proof of Theorem 3.8.** The map $\exp(\mathfrak{f}^\perp) \to \exp(\mathfrak{f}^\perp) \cdot F = \mathcal{S}$ is a biholomorphism (§A.5). So Corollary 3.7 implies that there is a uniquely determined holomorphic map

$$g : \tilde{\mathcal{O}}^0 \ \to\ \exp(\mathfrak{f}^\perp) \ \subset\ G_{\mathbb{C}} \tag{3.11a}$$

so that

$$\tilde{\Phi}(\zeta) \ =\ g(\zeta) \cdot F\,. \tag{3.11b}$$

Set $d_p = \dim_{\mathbb{C}} F^p$, fix a nonzero λ in the line $\det(F^p) \subset \bigwedge^{d_p} V_{\mathbb{C}}$, let m_p be any positive integer and define

$$f : \tilde{\mathcal{O}}^0 \ \to\ (\textstyle\bigwedge^{d_p} V_{\mathbb{C}})^{\otimes m_p}$$

by

$$f(\zeta) \ =\ g(\zeta) \cdot \lambda^{m_p}\,.$$

Remark 3.12. If

$$f(\zeta \cdot \gamma) \ =\ \gamma^{-1} \cdot f(\zeta)\,, \tag{3.13}$$

then f defines a section of $\det(\mathcal{F}^p)^{m_p} \to \mathcal{O}^0$. This section will define a trivialization of $\det(\mathcal{F}^p_{\mathrm{e}})^{m_p}$ over $\overline{\mathcal{O}}^0$, by essentially the same arguments as in [Del97]. So to prove the theorem, it suffices to establish (3.13).

3.4.1. *Decompositions of monodromy.* The proof of the theorem will make use of two decompositions (3.15a) and (3.16) of the group S defined in (3.4a). Both are induced by natural decompositions of parabolic groups containing S, [ČS09, Theorem 3.1.3]. To begin we note that it follows from (2.8) that the Lie algebra of S is

$$\mathfrak{s} \ =\ \bigoplus_{p,q\le 0} \mathfrak{g}^{p,q}_{W,F}\,. \tag{3.14}$$

The first decomposition

$$S \ =\ S^{-1}_W \rtimes S^0_W\,. \tag{3.15a}$$

is induced by the parabolic subgroup $P_{W,\mathbb{C}} \supset S$. The unipotent radical of S is

$$S^{-1}_W \ =\ S \cap P^{-1}_{W,\mathbb{C}} \ =\ \{g \in S \mid g \text{ acts trivially on } \mathrm{Gr}^W_\ell \ \forall\, \ell\}\,, \tag{3.15b}$$

and has Lie algebra

$$\mathfrak{s}^{-1}_W \ =\ \bigoplus_{\substack{p,q \le 0\\ (p,q)\neq(0,0)}} \mathfrak{g}^{p,q}_{W,F}\,.$$

The reductive subgroup

$$S^0_W \ =\ \{g \in S \mid g \text{ preserves } V^{p,q}_{W,F} \ \forall\, p,q\} \tag{3.15c}$$

has Lie algebra $\mathfrak{g}^{0,0}_{W,F}$, and is a Levi factor of S.

The second decomposition

$$S \;=\; S_\infty^{-1} \rtimes S_\infty^0 \tag{3.16}$$

is induced by the parabolic subgroup $\mathrm{Stab}_{G_\mathbb{C}}(\overline{F}_\infty) \supset S$. Specifically

$$S_\infty^{-1} \;=\; \{g \in S \mid g \text{ acts trivially on } \overline{F}{}^q_\infty/\overline{F}{}^{q+1}_\infty \;\forall\, q\} \;=\; \exp(\mathfrak{f}^\perp)$$

has Lie algebra

$$\mathfrak{s}_\infty^{-1} \;=\; \mathfrak{s}_W^{-1} \cap \mathfrak{f}^\perp \;=\; \bigoplus_{\substack{p<0\\ q\le 0}} \mathfrak{g}^{p,q}_{W,F}$$

and

$$S^0_\infty \;=\; S \cap \mathrm{Stab}_{G_\mathbb{C}}(F)$$

has Lie algebra

$$\mathfrak{s}^0_\infty \;=\; \bigoplus_{b\le 0} \mathfrak{g}^{0,b}_{W,F} \;\supset\; \mathfrak{s}^0_W\,.$$

3.4.2. *Monodromy action.* We have

$$\tilde\Phi(\zeta\cdot\gamma) \;=\; \gamma^{-1}\cdot\tilde\Phi(\zeta)\,;$$

equivalently,

$$g(\zeta\cdot\gamma)\cdot F \;=\; \gamma^{-1}g(\zeta)\cdot F\,.$$

However, while γ^{-1} preserves the Schubert cell $\mathcal{S}$, it need not be an element of $\exp(\mathfrak{f}^\perp)$. So we can not assert that $g(\zeta\cdot\gamma) = \gamma^{-1}g(\zeta)$.

In order to determine $g(\zeta\cdot\gamma)$ we factor the monodromy using (3.16). Write

$$\gamma^{-1} \;=\; \alpha\,\beta\,,$$

with $\beta \in S^0_\infty$ and $\alpha \in S^{-1}_\infty$. Then the action of γ^{-1} on $g\cdot F \in \mathcal{S}$ is given by

$$\gamma^{-1}g\cdot F \;=\; \alpha\beta g\cdot F \;=\; \alpha\,\beta g\beta^{-1}\,\beta\cdot F \;=\; \alpha\,(\beta g\beta^{-1})\cdot F\,. \tag{3.17}$$

Noting that $\beta g\beta^{-1} \in \exp(\mathfrak{f}^\perp) = S^{-1}_\infty$, this implies that

$$g(\zeta\cdot\gamma) \;=\; \alpha\,\beta g(\zeta)\beta^{-1}\,.$$

Since β^{-1} preserves F, it stabilizes the line $\det(F^p)$. So there is a group homomorphism

$$\chi : S^0_\infty \;\to\; \mathbb{C}\backslash\{0\}$$

such that

$$\beta^{-1}(\lambda) \;=\; \chi(\beta^{-1})\,\lambda\,.$$

Remark 3.18. If every $\chi(\beta^{-1})$ is an m_p–th root of unity, then (3.13) will hold. We will show that there is a group homomorphism $\chi_\infty : \Gamma_{A^0} \to S^1 \subset \mathbb{C}$ such that $\overline{\chi_\infty(\gamma)} = \chi(\beta)$, and each $\chi_\infty(\gamma)$ is a root of unity (Lemmas 3.19 and 3.20, and Theorem 3.22). Since Γ_{A^0} is contained in an arithmetic group, and every arithmetic group contains a neat subgroup of finite index, it follows that there exists a choice of m_p so that the elements of $\chi_\infty(\Gamma_{A^0}) \subset S^1$ are all m_p–th roots of unity. This will establish Theorem 3.8 (Remark 3.12).

3.5. **The character χ_∞.** Because Γ_{A^0} stabilizes F_∞, the line

$$\det(F^p_\infty) \ \subset \ \textstyle\bigwedge^{d_p} V_{\mathbb{C}}$$

is an eigenline of Γ_{A^0}. Let

$$\chi_\infty : \Gamma_{A^0} \ \to \ \mathbb{C}\backslash\{0\}$$

be the associated character. Note that $\chi_\infty(\gamma) = \det\{\gamma : F^p_\infty \to F^p_\infty\}$.

Lemma 3.19. *We have* $\chi(\beta) = \overline{\chi_\infty(\gamma)}$.

Proof. The polarization Q induces a nondegenerate bilinear form on $\bigwedge^{d_p} V$, also denoted by Q, with the property that

$$\textstyle\bigwedge^{d_p} V_{\mathbb{C}} \ = \ \det(F^p)^\perp \ \oplus \ \det(\overline{F^p_\infty}) \ = \ \det(\overline{F^p_\infty})^\perp \ \oplus \ \det(F^p)\,.$$

In particular, given $0 \neq \mu \in \det(F^p_\infty)$, the pairing $Q(\lambda\,,\,\bar\mu)$ is nonzero. We have

$$Q(\gamma^{-1}\cdot\lambda\,,\,\bar\mu) \ = \ Q(\lambda\,,\,\gamma\cdot\bar\mu) \ = \ \overline{\chi_\infty(\gamma)}\,Q(\lambda\,,\,\bar\mu)\,.$$

On the other hand, since $\alpha \in S^{-1}_\infty$, it acts trivially on $\det(\overline{F^p_\infty})$ and we have $Q(\alpha\cdot\lambda,\bar\mu) = Q(\lambda,\alpha^{-1}\cdot\bar\mu) = Q(\lambda,\bar\mu)$, so that

$$Q(\gamma^{-1}\cdot\lambda\,,\,\bar\mu) \ = \ Q(\alpha\beta\cdot\lambda\,,\,\bar\mu) \ = \ \chi(\beta)\,Q(\alpha\cdot\lambda\,,\,\bar\mu) \ = \ \chi(\beta)\,Q(\lambda\,,\,\bar\mu)\,.$$

Thus $\chi(\beta) = \overline{\chi_\infty(\gamma)}$. □

Lemma 3.20. *The character χ_∞ takes value in the unit circle $S^1 \subset \mathbb{C}$.*

Proof. Let $N \in \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{g}^{-1,-1}_{W,F}$ be a nilpotent operator polarizing (W,F). There exists k, independent of the choice of N, so that $N^k(\det(F^p)) = \det(F^p_\infty)$ and $N^{k+1}(\det(F^p)) = 0$.

It follows from $N \in \mathfrak{g}^{-1,-1}_{W,F}$, (2.8) and the decomposition (3.15a) that

$$\gamma\,N^k(\lambda) \ = \ N^k\,\gamma(\lambda)\,. \tag{3.21}$$

In particular,

$$\chi_\infty(\gamma)\,N^k(\lambda) \ = \ \gamma\,N^k(\lambda) \ = \ N^k\,\gamma(\lambda)\,.$$

Without loss of generality, $\mu = N^k(\lambda)$. Then

$$\begin{aligned} 0 \ &\neq \ \chi_\infty(\gamma)\,Q(\mu\,,\,\bar\lambda) \ = \ Q(N^k\,\gamma(\lambda)\,,\,\bar\lambda) \ = \ (-1)^k\,Q(\gamma(\lambda)\,,\,\bar\mu) \\ &= \ (-1)^k\,\overline{\chi_\infty(\gamma^{-1})}\,Q(\lambda\,,\,\bar\mu)\,. \end{aligned}$$

Thus $\chi_\infty(\gamma)^{-1} = \pm\,\overline{\chi_\infty(\gamma)}$. □

Theorem 3.22. *Given $\gamma \in \Gamma_{A^0}$, the eigenvalue $\chi_\infty(\gamma) = \det\{\gamma : F^p_\infty \to F^p_\infty\}$ is a root of unity.*

This will establish Theorem 3.8 (Remark 3.18). The remainder of §3.5 is occupied with the proof of Theorem 3.22.

3.5.1. *Basic idea of the proof.* We will construct a rational representation $\Gamma_{A^0} \to \mathrm{Aut}(U)$ with the property that the γ–eigenvalues of U include $\overline{\chi_\infty(\gamma)}^{-1}$, and all have absolute value one (Lemma 3.29). Since these eigenvalues are the roots of a rational polynomial (the characteristic polynomial of $\gamma : U \to U$), it will then follow from Kronecker's Theorem that these eigenvalues are roots of unity. The essential observation underlying the construction of U is that the mixed Hodge structure (W,F) is *polarized* by *rational* $N \in \mathfrak{g}_\mathbb{Q} \cap \mathfrak{g}^{-1,-1}_{W,F}$.

In the special case that there exist Γ_{A^0}–invariant N polarizing (W,F), standard constructions yield the rational representation $\Gamma_{A^0} \to \mathrm{Aut}(U)$ (§3.5.4). In the general case, a new perspective is needed to deal with this lack of invariance; see §3.5.5 for an outline of issues involved and how they are addressed.

3.5.2. *Induced mixed Hodge structure.* The mixed Hodge structure (W,F) on V induces one on $\bigwedge^{d_p} V$. It follows from (2.8) that $\det(F^p_\infty) = (\bigwedge^{d_p} V)^{\mathsf{a},\mathsf{b}}_{W,F}$ is a summand of the Deligne splitting; and that

$$(3.23)\qquad \begin{aligned} (\textstyle\bigwedge^{d_p} V)^{r,s}_{W,F} \;&=\; 0\,, \quad \text{for all} \;\; s < \mathsf{b}\,,\\ (\textstyle\bigwedge^{d_p} V)^{r,\mathsf{b}} = 0 \;&=\; 0\,, \quad \text{for all} \;\; r \neq \mathsf{a}\,. \end{aligned}$$

(This is pictured in Figure 3.5.2.) Setting $m = nd_p$ and $k = m - \mathsf{a} - \mathsf{b}$, we likewise have $\det(F^p) = (\bigwedge^{d_p} V)^{\mathsf{a}+k,\mathsf{b}+k}_{W,F}$. And $(\bigwedge^{d_p} V)^{r,s}_{W,F} = 0$ for all $r > \mathsf{a}+k$ and $(\bigwedge^{d_p} V)^{\mathsf{a}+k,s}_{W,F}$ for all $s \neq \mathsf{b}+k$.

3.5.3. *The weight-graded quotient.* The action of Γ_{A^0} on V induces an action on $\bigwedge^d V$. This action preserves the weight filtration (3.4), and so descends to an action on $H_\ell = \mathrm{Gr}^W_\ell(\bigwedge^{d_p} V)$. This gives H_ℓ the structure of a rational Γ_{A^0}–representation.

It follows from (3.15) that the action preserves the Hodge decomposition $H_\ell \otimes \mathbb{C} = \oplus_{r+s=\ell} H^{r,s}$ induced by F. We have natural identifications $H^{r,s} \simeq (\bigwedge^{d_p} V)^{r,s}_{W,F}$, with respect to which

$$(3.24)\qquad \det(F^p) \;\simeq\; H^{\mathsf{a}+k,\mathsf{b}+k} \quad\text{and}\quad \det(F^p_\infty) \;\simeq\; H^{\mathsf{a},\mathsf{b}}\,.$$

Given $\gamma \in \Gamma_{A^0}$, let $\gamma^{r,s} : H^{r,s} \to H^{r,s}$ be the induced action. Then we have

$$(3.25\text{a})\qquad \chi_\infty(\gamma) \;=\; \det\{\gamma : F^p_\infty \to F^p_\infty\} \;=\; \det(\gamma^{\mathsf{a},\mathsf{b}})\,.$$

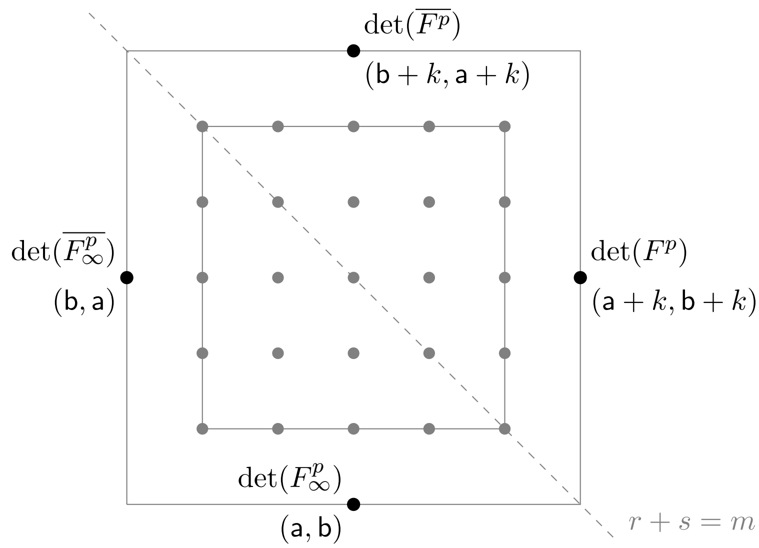

FIGURE 1. Hodge diamond of the MHS on $\bigwedge^{d_p} V$.

And more generally

$$\det(\gamma^{r,s}) \;=\; \overline{\det(\gamma^{s,r})} \;=\; \det(\gamma^{m-r,m-s})^{-1}\,. \tag{3.25b}$$

3.5.4. *Proof of Theorem 3.22: special case.* Suppose that (W,F) can be polarized by an element $N \in \mathfrak{g}_\mathbb{Q} \cap \mathfrak{g}^{-1,-1}_{W,F}$ that is *invariant* under Γ_{A^0}. In this case, both $P_N = \ker\{N^{k+1} : H_{m+k} \to H_{m-k-2}\}$ and $Q_N(u,v) = Q(u, N^k v)$ are Γ_{A^0}–invariant. Then the facts that Q_N polarizes the weight $m+k$ Hodge structure on P_N, and Γ_{A^0} preserves the Hodge decomposition, imply that the Γ_{A^0}–eigenvalues of P_N all have absolute value one. Kronecker's Theorem implies these eigenvalues are all roots of unity (§3.5.1). Finally, we observe that $\det(F^p) \in P_N$, and therefore the eigenvalue $\overline{\chi_\infty(\gamma)}^{-1}$ is a root of unity.

3.5.5. *Proof of Theorem 3.22: outline for the general case.* The main difficulty that must be handled is the fact that there may be no Γ_{A^0}–invariant N that polarize the mixed Hodge structure. This leads to the introduction of

$$\mathcal{N} \;=\; \{\mathrm{Ad}_\gamma(N) \mid \gamma \in \Gamma_{A^0}\,,\; N \in \mathfrak{g}_\mathbb{Q} \cap \mathfrak{g}^{-1,-1}_{W,F} \text{ polarizes the MHS}\} \;\subset\; \mathfrak{g}_\mathbb{Q}\,.$$

By (3.4a) and (3.14),

$$\mathcal{N} \;\subset\; \bigoplus_{p,q\le -1} \mathfrak{g}^{p,q}_{W,F}\,. \tag{3.26}$$

In general, $\mathcal{N} \not\subset \mathfrak{g}^{-1,-1}_{W,F}$, and so elements of $\mathcal{N}$ will not "polarize" the mixed Hodge structure (W,F) in the sense of [CKS86, (2.6)]. Nonetheless, we still have a well-defined isomorphism

$M^k : H_{m+k} \to H_{m-k}$ with the properties that

$$M^k(H^{r+k,s+k}) \ = \ H^{r,s}\,, \tag{3.27}$$

and $Q_M(u,v) = Q(u, M^k\overline{u})$ polarizes the induced Hodge structure on $\ker\{M^{k+1} : H_{m+k} \to H_{m-k-2}\}$. (The map $M^k : H_{m+k} \to H_{m-k}$ is pictured in Figure 3.5.6.) And while P_N will not be Γ_{A^0}–invariant in general, the intersection

$$P \ = \ \bigcap_{M\in\mathcal{N}} \ker\{M^{k+1} : H_{m+k} \to H_{m-k-2}\} \ \subset \ H_{m+k}$$

is a rational Γ_{A^0}–submodule. Note that P is nontrivial as (3.23) and (3.26) imply

$$\det(F^p) \ \subset \ P \otimes \mathbb{C}\,, \tag{3.28}$$

under the identification (3.24). The intersection P inherits the Hodge decomposition, and every Q_M, $M \in \mathcal{N}$, polarizes this Hodge structure.

We still have to deal with the fact that Q_M need not be Γ_{A^0}–invariant. To that end we introduce a decomposition $Q_M = Q_{M,1} + Q'_M$ with both summands rational and $Q_{M,1}$ the Γ_{A^0}–invariant part of Q_M (§3.5.6). If $Q_{M,1}$ polarized the Hodge structure on P, then we would be done; but it need not. So we introduce a rational Γ_{A^0}–submodule $U \subset P$, which inherits the Hodge decomposition, and with the property that the Hodge structure on U is polarized by the $Q_{M,1}$. It then follows (as in §3.5.4) that the γ–eigenvalues of U are roots of unity. It remains to observe that $\det(F^p) \subset U \otimes \mathbb{C}$.

3.5.6. *Proof of Theorem 3.22.* The polarization Q identifies $H_{m-k} \simeq H^*_{m+k}$, and the bilinear form Q_M is an element of the rational Γ_{A^0}–submodule

$$S \ = \ \begin{cases} \mathrm{Sym}^2 H_{m-k} \simeq \mathrm{Sym}^2 H^*_{m+k}\,, & m-k \text{ is even,} \\ \bigwedge^2 H_{m-k} \simeq \bigwedge^2 H^*_{m+k}\,, & m-k \text{ is odd.} \end{cases}$$

Let $\tilde{S}_\nu \subset S \otimes \mathbb{C}$ be the (generalized) eigenspaces, as ν runs over the *distinct* eigenvalues of $\gamma : S \to S$. Both $\tilde{S}_1$ and $\oplus_{\nu\neq 1}\tilde{S}_\nu$ are defined over $\mathbb{Q}$, and we will be interested in the associated rational Γ_{A^0}–module decomposition $S = S_1 \oplus S'$. Let $Q_M = Q_{M,1} + Q'_M$ be the corresponding decomposition. Then

$$U \ = \ \bigcap_{M\in\mathcal{N}} \{u \in P \mid Q'_M(u,P) \ = \ 0\}$$

is a rational Γ_{A^0}–submodule.

Lemma 3.29. *The eigenvalues of U include $\overline{\chi_\infty(\gamma)}^{-1}$ and all have absolute value one.*

As discussed in §3.5.1, this establishes Theorem 3.22, and by extension Theorem 3.8 (Remark 3.18).

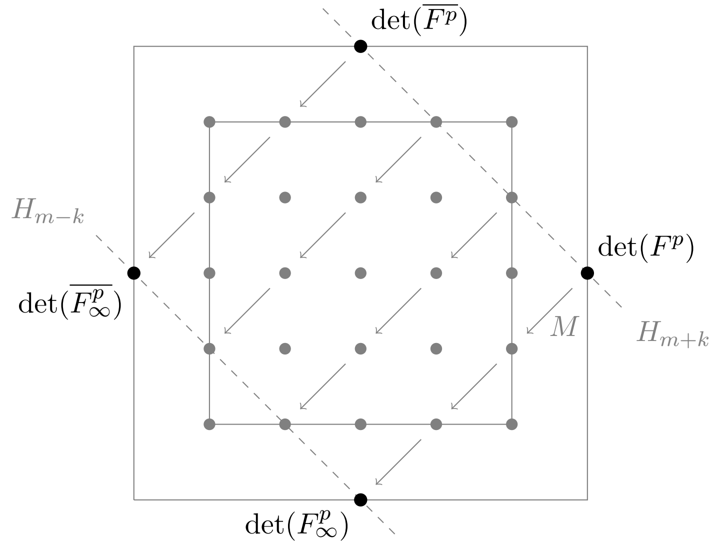

FIGURE 2. The map $M^k : H_{m+k} \to H_{m-k}$.

Proof. Let $\Lambda(H_{m-k})$ denote the γ–eigenvalues of H_{m-k} listed with (algebraic) multiplicity. We may fix a basis $\{e_\nu\}_{\nu\in\Lambda(H_{m-k})}$ of H_{m-k} so that each e_ν is a (generalized) γ–eigenvector with eigenvalue ν, and each e_ν is contained in some $H^{r,s}$. Without loss of generality we may assume that the basis is closed under complex conjugation, $\overline{e_\nu} = e_{\overline{\nu}}$. Let $\{f_\nu\}_{\nu\in\Lambda(H_{m-k})}$ be the dual basis of H_{m+k} defined by $Q(e_\mu, f_\nu) = \delta_{\mu\nu}$. Each f_ν is a (generalized) eigenvector with eigenvalue ν^{-1}, and is contained in $H^{m-r,m-s}$. Then

$$Q_M \;=\; \sum_{\mu,\nu} q_{M\mu\nu}\, e_\mu \otimes e_\nu$$

defines $q_{M\mu\nu}$; and we have

$$Q_{M,1} \;=\; \sum_{\mu\nu=1} q_{M\mu\nu}\, e_\mu \otimes e_\nu \quad \text{and} \quad Q'_M \;=\; \sum_{\mu\nu\neq 1} q_{M\mu\nu}\, e_\mu \otimes e_\nu\,.$$

Note $\det(F^p)$ is a γ–eigenline of $P\otimes\mathbb{C}$ with eigenvalue $\overline{\chi_\infty(\gamma)}^{-1}$: this follows from (3.25) and (3.28). And (3.27) implies $M^k(\det(F^p)) = \det(F^p_\infty)$, under the identifications (3.24). It then follows from Lemma 3.20 that $\det(F^p) \subset U\otimes\mathbb{C}$, and $\overline{\chi_\infty(\gamma)}^{-1}$ is an eigenvalue of U.

It follows from (3.27) that $U \subset P \subset H_{m+k}$ inherit the Hodge decomposition. So we may assume without loss of generality that the basis $\{e_\nu\}$ was chosen so that $\{f_\nu\}_{\nu\in\Lambda(U)^*}$ spans U, and $\{f_\nu\}_{\nu\in\Lambda(P)^*}$ spans P, for some subsets $\Lambda(U)^* \subset \Lambda(P)^* \subset \Lambda(H_{m-k})$. Now suppose that $f_\nu \in P^{r,s}$. We necessarily have $\mathbf{i}^{r-s} Q_M(f_\nu, \overline{f_\nu}) = \mathbf{i}^{r-s} Q_M(f_\nu, f_{\overline{\nu}}) > 0$, as discussed in §3.5.5. So if $f_\nu \in U^{r,s}$, then $q_{M\nu\overline{\nu}} = Q_{M,1}(f_\nu, f_{\overline{\nu}}) \neq 0$. It follows that $|\nu|^2$ is an eigenvalue of S_1; that is, $|\nu|^2 = 1$. $\square$

3.6. **Discussion of Conjecture 1.6.** The results of §3 reduce Conjecture 1.6 to an extension problem.

Theorem 3.30. *Conjecture 1.6 holds if the holomorphic functions on $Z_I \cap \overline{\mathcal{O}}^0$ extend to holomorphic functions on $\overline{\mathcal{O}}^0$.*

3.6.1. *Outline of proof.* Recall that $\Lambda_{\rm e}$ descends to $\hat{\wp}^0$ (Corollary 3.10). Suppose we can show that $\hat{\wp}^0$ is a normal Moishezon variety containing $\hat{\wp}$ as a Zariski open subset, and that the extension $\hat{\Phi}^0$ of $\hat{\Phi}$ is a morphism of algebraic spaces. It will then follow from [GGLR20] that $\Lambda_{\rm e}$ is ample over $\hat{\wp}^0$, establishing Conjecture 1.6.

The set

$$\hat{X} \;=\; \{(b_1, b_2) \in \overline{B} \times \overline{B} \mid \hat{\Phi}^0(b_1) = \hat{\Phi}^0(b_2)\}$$

defines an equivalence relation on $\overline{B}$ with the property that $\hat{\Phi}^0 : \overline{B} \to \hat{\wp}^0$ is the quotient map. Suppose that $\hat{X}$ defines a proper, holomorphic equivalence relation on $\overline{B}$. Then [Gra83, §3, Theorem 2] asserts that the quotient $\hat{\wp}^0$ is a compact, complex analytic variety, and the quotient map $\hat{\Phi}^0$ is a proper holomorphic completion of $\hat{\Phi}$. Since $\overline{B}$ is projective (and therefore Moishezon) it follows that $\hat{\wp}^0$ is Moishezon [AT82, §5, Corollary 11]. As Moishezon spaces are algebraic, Serre's GAGA implies $\hat{\Phi}^0$ is a morphism, [Art70, §7].

So the essential problem is to show that $\hat{X}$ is defines a proper, holomorphic equivalence relation. Since $\Phi^0\big|_{\mathcal{O}^0}$ is proper (Corollary 2.25), it suffices to show that the holomorphic functions on $\overline{\mathcal{O}}^0$ separate the fibres of $\Phi^0\big|_{\overline{\mathcal{O}}^0}$.

3.6.2. *Separation of fibres.* Bakker–Brunebarbe–Tsimerman [BBT23] have shown that $\wp = \Phi(B)$ is projectively embedded by sections of $\Lambda_{\rm e}^{\otimes m} \to \overline{B}$ that vanish along Z. We have seen that a multiple of $\Lambda_{\rm e}$ is trivial over $\overline{\mathcal{O}}^0$ (Corollary 3.9). It follows that the holomorphic functions on $\overline{\mathcal{O}}^0$ will separate any fibre of $\Phi^0\big|_{\mathcal{O}^0}$ from any other fibre of $\Phi^0\big|_{\overline{\mathcal{O}}^0}$.

By the same argument, $\wp^0_I = \Phi^0_I(Z_I \cap Z_\pi)$ is projectively embedded by sections of $\Lambda_{\rm e}^{\otimes m} \to Z_I$ that vanish along $Z_I \backslash Z_\pi$, and holomorphic functions on Z_I will separate any fibre of $\Phi^0\big|_{Z_I \cap Z_\pi \cap \overline{\mathcal{O}}^0}$ from any other fibre of $\Phi^0\big|_{Z_I \cap \overline{\mathcal{O}}^0}$.

So to prove Conjecture 1.6 it remains to show that holomorphic functions on $Z_I \cap \overline{\mathcal{O}}^0$ extend to all of $\overline{\mathcal{O}}^0$. This completes the proof of Theorem 3.30.

4. Neighborhood of a $\hat{\Phi}^0_I$–fibre

Next we restrict out attention to a fibre

$$A^0_I \;=\; A^0 \cap Z_I$$

of the map $\hat{\Phi}^0_I$ of (2.22b).

4.1. **Monodromy about the fibre.** Along $Z_I \cap Z_\pi$ we have a variation of limiting mixed Hodge structures (W, F, σ_I). The Hodge filtrations lie in a G_I–orbit D_I (§2.1.2). Arguing as in the proof of Lemma 3.2, we may choose a neighborhood $\overline{\mathcal{O}}_I^0 \subset \overline{\mathcal{O}}^0$ of A_I^0 so that the monodromy $\Gamma_{A_I^0}$ of the restriction of the variation of Hodge structure (1.2a) to $\mathcal{O}_I^0 = \overline{\mathcal{O}}_I^0 \cap B$ takes value in the centralizer $C_{I,\mathbb{Q}}$ of σ_I

$$\Gamma_{A_I^0} \ \subset\ \Gamma_{A^0} \,\cap\, C_{I,\mathbb{Q}}\,, \tag{4.1}$$

and the induced action on Gr^W stabilizes $F(\mathrm{Gr}^W)$. Let

$$\Phi_{A_I^0} : \mathcal{O}_I^0 \ \to\ \Gamma_{A_I^0} \backslash (D \cap \mathcal{S}) \tag{4.2}$$

be the induced period map.

Remark 4.3. Unlike (3.1), the period map (4.2) will not in general be proper.

Because the induced action of $\Gamma_{A_I^0}$ on Gr^W_ℓ preserves both the Hodge filtration $F^p(\mathrm{Gr}^W_\ell)$ and the polarization by $N \in \sigma_I$, it necessarily takes value in a compact subgroup of $\mathrm{Aut}(\mathrm{Gr}^W)$. Since Γ is discrete, this forces the image $\mathrm{Gr}^W(\Gamma_{A_I^0})$ of $\Gamma_{A_I^0} \to \mathrm{Aut}(\mathrm{Gr}^W)$ to be finite. If we assume that Γ is neat, then this finite group is trivial, and the monodromy

$$\Gamma_{A_I^0} \ \subset\ \Gamma_{A^0} \,\cap\, C^{-1}_{I,\mathbb{Q}} \ \subset\ S^{-1}_W \tag{4.4}$$

is unipotent.

4.2. **Divisors at infinity.** The purpose of this section is to to construct, from the period map (4.2), explicit sections $s_M \in H^0(\overline{\mathcal{O}}_I^0, L_M)$ of certain line bundles $L_M \to \overline{\mathcal{O}}_I^0$. We will see that the sections have divisor

$$(s_M) \ =\ \sum \kappa(M, N_i)\, (Z_i \cap \overline{\mathcal{O}}_I^0) \tag{4.5a}$$

for some integers $\kappa(M, N_i)$. In particular,

$$L_M \ =\ \sum \kappa(M, N_i)\ [Z_i]|_{\overline{\mathcal{O}}_I^0}\ . \tag{4.5b}$$

4.2.1. *Line bundles over* $\Gamma_{A_I^0}\backslash\mathcal{S}$. Recall the Schubert cell $\mathcal{S}$ of (3.5) and Lemma 3.6. We will construct line bundles over $\Gamma_{A_I^0}\backslash\mathcal{S}$ from the data:

- The left-action of $\Gamma_{A_I^0}$ on $\mathcal{S}$ induces a right-action on the functions $f : \mathcal{S} \to \mathbb{C}$ by the prescription $(f \cdot \gamma)(\xi) = f(\gamma \cdot \xi)$.
- Let
$$\mathfrak{f}^1 \ =\ F^1(\mathfrak{g}_\mathbb{C}) \ =\ \bigoplus_{p \ge 1} \mathfrak{g}^{p,q}_{W,F}$$
be the nilpotent radical of the Lie algebra $\mathfrak{f}$ stabilizing F. The relation (A.6c) implies that the bilinear pairing
$$\kappa : \mathfrak{f}^1 \times \mathfrak{f}^\perp \ \to\ \mathbb{C}$$

is nondegenerate.

Recall the biholomorphism $X : \mathcal{S} \xrightarrow{\simeq} \mathfrak{f}^\perp$ of (A.9). Given $M \in \mathfrak{f}^1$, define

$$f_M : \mathcal{S} \ \to\ \mathbb{C} \quad \text{by} \quad f_M = \exp 2\pi \mathbf{i}\, \kappa(M, X)\,.$$

Given $\gamma \in \Gamma_{A_I^0}$, define a holomorphic function $e^M_\gamma : \mathcal{S} \to \mathbb{C}^*$ by

$$e^M_\gamma \ = \ \frac{f_M \cdot \gamma}{f_M} \ = \ \frac{\exp 2\pi\mathbf{i}\,\kappa(M, X\cdot\gamma)}{\exp 2\pi\mathbf{i}\,\kappa(M,X)}\,. \tag{4.6}$$

Then

$$e^M_{\gamma_1\gamma_2}(\xi) \ = \ e^M_{\gamma_1}(\gamma_2\cdot\xi)\, e^M_{\gamma_2}(\xi)\,.$$

so that

$$\gamma\cdot(z,\xi) \ = \ (z e^M_\gamma(\xi), \gamma\cdot\xi)$$

defines a left action of $\Gamma_{A_I^0}$ on $\mathbb{C}\times\mathcal{S}$. Let

$$\begin{array}{l} \mathcal{L}_M = (\mathbb{C}\times\mathcal{S})/\sim \\ \ \downarrow \\ \Gamma_{A_I^0}\backslash\mathcal{S} \end{array}$$

be the associated line bundle over the quotient. Then f_M induces a section s_M

$$\begin{array}{c} \mathcal{L}_M \\ {\scriptstyle s_M}\uparrow\ \downarrow \\ \Gamma_{A_I^0}\backslash\mathcal{S}\,. \end{array}$$

4.2.2. *Line bundles over $\mathcal{O}_I^0$.* Pull the line bundle $\mathcal{L}_M$ back to the (punctured) neighborhood $\mathcal{O}_I^0$

$$\begin{array}{ccc} (\Phi_{A_I^0})^*\mathcal{L}_M & & \mathcal{L}_M \\ {\scriptstyle \Phi^*_{A_I^0}(s_M)}\uparrow\ \downarrow & & \downarrow\ \uparrow{\scriptstyle s_M} \\ \mathcal{O}_I^0 & \xrightarrow{\ \Phi_{A^0}\ } & \Gamma_{A_I^0}\backslash\mathcal{S}\,. \end{array}$$

Recall the local lift (A.2), and note that the local expression for the pulled-back section $\Phi^*_{A_I^0}(s_M)$ is

$$\tau_M(t,w) \ = \ f_M\circ\widetilde{\Phi}(t,w) \ = \ \exp 2\pi\mathbf{i}\kappa(M, X\circ\widetilde{\Phi}(t,w))\,. \tag{4.7}$$

If $M \in \mathfrak{g}^{1,\bullet}_{W,F}$ and $\kappa(M,N_i)\in\mathbb{Z}$ for all $i \in I_\pi$, then (A.10) implies

$$\tau_M(t,w) \ = \ \exp 2\pi\mathbf{i}\kappa(M,\tilde{X}(t,w)) \prod t_i^{\kappa(M,N_i)} \tag{4.8}$$

is a well-defined holomorphic function on $\mathcal{U}$. If in addition $0 \le \kappa(M,N_i)\in\mathbb{Z}$ for all $i\in I_\pi$, then $\tau_M(t,w)$ is holomorphic on $\overline{\mathcal{U}}$.

4.2.3. *Extension to $\overline{\mathcal{O}}{}^0_I$.* Define

$$\mathbf{N}^* \;=\; \{M \in \mathfrak{g}^{1,\le 1}_{W,F} \mid \kappa(M,N_i) \in \mathbb{Z}\,,\ \forall\, i \in I_\pi\}\,. \tag{4.9}$$

Lemma 4.10. *If $M \in \mathbf{N}^*$, then the line bundle $(\Phi_{A^0_I})^*\mathcal{L}_M$ is the restriction to $\mathcal{O}^0_I$ of a holomorphic vector bundle $L_M \to \overline{\mathcal{O}}{}^0_I$. And $(\Phi_{A^0})^* s_M$ extends to a section of L_M* (which, in a minor abuse of notation, we also denote s_M)

$$\begin{array}{ccccc}
L_M & & (\Phi_{A^0_I})^*\mathcal{L}_M & & \mathcal{L}_M \\
{\scriptstyle s_M}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle (\Phi_{A^0})^* s_M} & & \downarrow\uparrow{\scriptstyle s_M} \\
\overline{\mathcal{O}}{}^0_I & \longleftrightarrow & \tilde{\mathcal{O}}^0_I & \xrightarrow[\Phi_{A^0}]{} & \Gamma_{A^0_I}\backslash\mathcal{S}\,.
\end{array}$$

The desired (4.5) now follows from (4.7) and (4.8).

Proof. Set

$$\tilde X_\gamma(t,w) \;=\; (X\cdot\gamma)\circ\widetilde\Phi(t,w) \;-\; \textstyle\sum \ell(t_i)\,N_i$$

Again, the key point is that it follows from (A.6d), (A.10), (3.4a) and (4.1) that the component $\tilde X_\gamma^{-1,q}(t,w)$ taking value in $\mathfrak{g}^{-1,q}_{W,F}$ is a well-defined holomorphic function on $\overline{\mathcal{U}}$, so long as $q \ge -1$. So $\kappa(M,\tilde X_\gamma(t,w))$ is a holomorphic function on $\overline{\mathcal{U}}$, so long as $M \in \mathbf{N}^*$. Then

$$\begin{array}{rcl}
(\widetilde\Phi)^*(f_M\cdot\gamma)(t,w) & = & (f_M\cdot\gamma)\circ\widetilde\Phi(t,w) \\
 & = & \exp 2\pi\mathbf{i}\kappa(M,\tilde X_\gamma(t,w)) \prod t_i^{\kappa(M,N_i)}\,,
\end{array}$$

and

$$\begin{array}{rcl}
(\widetilde\Phi)^*(e^M_\gamma)(t,w) & = & \dfrac{(\widetilde\Phi)^*(f_M\cdot\gamma)(t,w)}{(\widetilde\Phi)^*(f_M)(t,w)} \\
 & = & \dfrac{\exp 2\pi\mathbf{i}\,\kappa(M,\tilde X_\gamma(t,w))}{\exp 2\pi\mathbf{i}\,\kappa(M,\tilde X(t,w))}
\end{array} \tag{4.11}$$

is a well-defined holomorphic function on $\overline{\mathcal{U}}$. □

5. Level one extension data

Along A^0_I we have a variation of limiting mixed Hodge structures (W,F), with the property that the Hodge decomposition $F^p(\mathrm{Gr}^W_\ell)$ is *constant*, and all of which are polarized by a fixed cone σ_I. In this section we study the σ_I–polarized, level one extension data (Definition 2.4) of these limiting mixed Hodge structures. This extension data is encoded by the restriction

$$\Phi^1_I : A^0_I \;\to\; \Gamma_{A^0_I}\backslash\delta^1_I \tag{5.1}$$

to A^0_I of the map $\Phi^1_I : Z_I \cap Z_\pi \to \Gamma_I\backslash D^1_I$ introduced in (2.18).

Note that δ_I is a subset of the Schubert cell $\mathcal{S}$ of (3.5). It follows that the quotient inherits the line bundles

$$\begin{array}{c} \mathcal{L}_M \\ \downarrow \\ \Gamma_{A_I^0} \backslash \delta_I \end{array} \tag{5.2}$$

of Lemma 4.10.

Theorem 5.3. *Assume that the monodromy* Γ *is neat* (cf. §4.1)*.*

(a) *The bundle* $\pi_I^1 : \Gamma_{A_I^0} \backslash D_I^1 \twoheadrightarrow D_I^0$ *admits a subbundle*

$$\begin{array}{ccccc} \mathfrak{J}_I & \hookrightarrow & \mathcal{J}_I & \subset & \Gamma_{A_I^0}\backslash D_I^1 \\ & & \big\downarrow \pi_I^1 & & \\ & & D_I^0 & & \end{array}$$

that is fibered by abelian varieties $\mathfrak{J}_I$*. The restriction* $\Phi^1\big|_{A_I^0}$ *takes value in a translate of* $\mathfrak{J}_I$*.*

(b) *If* $M \in \mathfrak{g}^{1,1}_{W,F}$*, then the line bundle* $\mathcal{L}_M$ *of* (5.2) *descends to* $\Gamma_{A_I^0} \backslash \delta_I^1$*. In the case that* $M \in \mathbf{N}^* \cap \mathfrak{g}^{1,1}_{W,F}$*, we have*

$$L_M|_{A_I^0} \;=\; (\Phi_I^1|_{A_I^0})^*(\mathcal{L}_M)\,. \tag{5.4}$$

(c) *There is a nonempty subset* $\mathbf{N}_I^{\mathfrak{sl}_2} \subset \mathbf{N}^* \cap \mathfrak{g}^{1,1}_{W,F}$ *with the property that the abelian variety* $\mathfrak{J}_I$ *is polarized by the* $\mathcal{L}_M^*$ *with* $M \in \mathbf{N}_I^{\mathfrak{sl}_2}$*.*

(d) *The set* $\mathbf{N}_I^{\mathfrak{sl}_2,+} \;=\; \{M \in \mathbf{N}_I^{\mathfrak{sl}_2} \mid \kappa(M,N_i) > 0\,,\ \forall\ i \in I\}$ *is nonempty. Indeed the dimension of the real span is* $\dim \sigma_I$*.*

Theorem 5.3 and (4.5) yield

Corollary 5.5. *The line bundles* $\mathcal{L}_M \to \mathfrak{J}_I$ *are related to the normal bundles by*

$$(\Phi_I^1|_{A_I^0})^*(\mathcal{L}_M) \;=\; \sum_{i\in I} \kappa(M,N_i)[Z_i]|_{A_I^0} \;=\; \sum_{i \in I} \kappa(M,N_i)\, \mathcal{N}_{Z_i/\overline{B}}\Big|_{A_I^0}\,. \tag{5.6}$$

It follows from Theorem 6.1 that (5.6) *is the central geometric information that arises when considering the variation of limiting mixed Hodge structure along* A^0.

Example 5.7. Suppose that $A^0 \subset Z_i^*$ and $N_i \neq 0$. Taking $I = \{i\}$, we may choose $M \in \mathbf{N}_i^{\mathfrak{sl}_2,+}$, so that $\mathcal{L}_M^* \to \mathfrak{J}_i$ is ample and $\kappa(M,N_i) > 0$. Then $\mathcal{N}^*_{Z_i/\overline{B}}\Big|_{A^0}$ is ample if the differential of $\Phi^1\big|_{A^0}$ is injective.

More generally, we have

Corollary 5.8. *Suppose the differential of* $\Phi^1\big|_{A_I^0}$ *is injective and* $M \in \mathbf{N}_I^{\mathfrak{sl}_2}$*. Then the line bundle* $\sum \kappa(M,N_i)\mathcal{N}^*_{Z_i/\overline{B}}\big|_{A_I^0}$ *is ample.*

Remark 5.9. The sum in Corollary 5.8 is over those j with $Z_j \cap A_I^0$ nonempty. Theorem 5.3(d) asserts that we may choose M so that the integers $\kappa(M,N_j)$ are positive when $j \in I$; we are not able to say the same when $j \notin I$.

The remainder of §5 is occupied with the proof of Theorem 5.3. In outline, the argument is as follows:

- In §5.1 it is shown that the level one extension data takes value in the translate of a compact torus $\mathcal{J}_I$. This is a consequence of a Lie theoretic description of the extension data, and the infinitesimal period relation.
- The action of $\Gamma_{A_I^0}$ on $\delta_I \subset \mathcal{S}$ is analyzed in §5.2.
- The line bundle $\mathcal{L}_M \to \Gamma_{A^0}\backslash\delta_I$ descends to $\Gamma_{A_I^0}\backslash\delta_I^1$ if and only if the functions e_γ^M of (4.6) are constant on the fibres of $\delta_I \twoheadrightarrow \delta_I^1$. In §5.2.4 it is shown that the bundles parameterized by $M \in \mathfrak{g}^{1,1}_{W,F}$ have this property. If, in addition, $M \in \mathbf{N}^* \cap \mathfrak{g}^{1,1}_{W,F}$ then we also have a line bundle $L_M|_{A_I^0}$ (Lemma 4.10). In order to see that (5.4) holds, we show that the associated systems of multipliers coincide.
- We then restrict to a subset $\mathbf{N}^1 \subset \mathfrak{g}^{1,1}_{W,F} \cap \mathbf{N}^*$ (which may be thought as imposing an integrality condition on M) and compute the Chern forms ω_M in §5.3.
- We restrict to a final subset $\mathbf{N}_I^{\mathfrak{sl}_2} \subset \mathbf{N}^1$ (which may be thought of as a positivity condition) and confirm that $-\omega_M$ is positive on $\mathcal{J}_I$. It then follows that the line bundle $\mathcal{L}_M^* \to \mathcal{J}_I$ is ample and $\mathcal{J}_I$ is an abelian variety.

5.1. **Extension data and tori.**

5.1.1. *Lie theoretic description.* The level one extension data

$$\Gamma_{A_I^0}\backslash\delta_I^1 \;=\; (\Gamma_{A_I^0}\cdot C_I^{-2})\backslash(C_I^{-1}\cdot F)$$

is defined in Definition 2.4, a Lie theoretic description of $\delta_I^1 = \delta_{I,F}^1$ is given in §A.6. With that as our starting point, we note that the biholomorphism $\exp : \mathfrak{c}_{I,\mathbb{C}}^{-1} \to C_{I,\mathbb{C}}^{-1}$ yields a canonical identification

$$C_{I,\mathbb{C}}^{-1}/C_{I,\mathbb{C}}^{-2} \;\simeq\; \bigoplus_{p+q=-1} \mathfrak{c}_{I,F}^{p,q}\,.$$

Setting

$$\mathbb{L}_I \;=\; \bigoplus_{\substack{p+q=-1\\ p<0}} \mathfrak{c}_{I,F}^{p,q}\,,$$

we have

$$C_{I,\mathbb{C}}^{-1}/C_{I,\mathbb{C}}^{-2} \;\simeq\; \mathbb{L}_I \oplus \overline{\mathbb{L}}_I\,.$$

Additionally $\mathfrak{c}^{-a}_{I,\mathbb{C}} = (\mathfrak{f}\cap\mathfrak{c}^{-a}_{I,\mathbb{C}})\oplus(\mathfrak{f}^\perp\cap\mathfrak{c}^{-a}_{I,\mathbb{C}})$, and the map $\mathfrak{f}\cap\mathfrak{c}^{-1}_{I,\mathbb{C}} \to \delta_I$ given by $x\mapsto \exp(x)\cdot F$ is a biholomorphism. It follows that we have a canonical identification

$$C^{-2}_{I,\mathbb{C}}\backslash(C^{-1}_{I,\mathbb{C}}\cdot F) \;=\; \mathbb{L}_I\,.$$

Taking Λ_I to be the discrete image of $\Gamma_{A^0_I}$ under the projection $C^{-1}_{I,\mathbb{C}}\to\mathbb{L}_I$, we obtain

$$\Gamma_{A^0_I}\backslash\delta^1_I \;=\; \Lambda_I\backslash\mathbb{L}_I \;=\; \mathbb{C}^{d_{I,1}}\;\times\;(\mathbb{C}^*)^{d_{I,2}}\;\times\;\mathcal{J}_I\,, \tag{5.10}$$

with $(\mathbb{C}^*)^{d_{I,2}}\;\times\;\mathcal{J}_I$ a complex torus having compact factor $\mathcal{J}_I$.

5.1.2. *The infinitesimal period relation along fibres.* Let $\omega = g^{-1}\mathrm{d}g$ be the pull-back on the Maurer-Cartan form on $\exp(\mathfrak{f}^\perp)\subset G_\mathbb{C}$ under the map (3.11). The infinitesimal period relation (§A.4) implies ω takes value in $\mathfrak{f}^{-1,\bullet}$.

The local lift (A.2) implies that ω may be regarded as a multi-valued logarithmic 1-form on $\overline{\mathcal{O}}^0$ with poles along $Z\cap\overline{\mathcal{O}}^0$. More precisely, let $\omega^{-1,q}$ be the component of ω taking value in $\mathfrak{g}^{-1,q}_{W,F}$. If $q\neq -1$, then $\omega^{-1,q}$ is a holomorphic (but a priori multi-valued) 1-form on $\overline{\mathcal{O}}^0$. The component $\omega^{-1,-1}$ is a logarithmic 1-form with poles along $Z\cap\overline{\mathcal{O}}^0$.

The restriction $\omega_{A^0_I}$ to A^0_I takes value in $\oplus_{q\le 0}\,\mathfrak{c}^{-1,q}_{I,F}$. The hypothesis that Γ is neat (§4.1) implies that the component $\omega^{-1,0}_{A^0_I}$ taking value in $\mathfrak{c}^{-1,0}_{I,F}$ is well-defined (single-valued).

Write $g=\exp(X)$, with $X:\tilde{\mathcal{O}}^0\to\mathfrak{f}^\perp$. Then $\omega^{-1,q}=\mathrm{d}X^{-1,q}$. And the infinitesimal period relation implies that the restriction of $\mathrm{d}X^{p,q}$ to A^0_I is zero for all $p+q\ge -1$ with $p\le -2$. (Alternatively, this is (2.15).)

5.1.3. *Compact torus.* It follows from (3.14) that

$$\Lambda_I\;\subset\;\mathfrak{c}^{-1,0}_{I,F}\;\subset\;\mathbb{L}_I\,.$$

In particular, the torus factor $(\mathbb{C}^*)^{d_{I,2}}\;\times\;\mathcal{J}_I$ of (5.10) is contained in the image of $\mathfrak{c}^{-1,0}_{I,F}\to\Lambda_I\backslash\mathbb{L}_I$. It follows from the infinitesimal period relation (§5.1.2) and the compactness of A^0_I that the image of $\Phi^1: A^0_I\to\Gamma_{A^0}\backslash\delta^1_I$ is contained in a translate of the compact torus $\mathcal{J}_I$.

We will show that $\mathcal{J}_I$ is abelian by exhibiting ample Lie bundles $\mathcal{L}_M\to\mathcal{J}_I$.

5.2. **Action on LMHS of the fibre.** Before describing the action, we first need to work out some formula.

5.2.1. *Logarithms of monodromy.* To begin, we note that (3.15a) and (4.4) imply that any element $\gamma\in\Gamma_{A^0_I}$ may be expressed as

$$\gamma\;=\;e^c$$

for a unique

$$c\;\in\;\mathfrak{s}^{-1}_W\;\cap\;\mathfrak{c}^{-1}_{I,\mathbb{C}}\,.$$

Applying the decomposition (3.16) to factor

$$\gamma \;=\; \alpha\,\beta$$

with $\alpha \in S^{-1}_\infty \cap C^{-1}_{I,\mathbb{C}}$ and $\beta \in S^0_\infty \cap C^{-1}_{I,\mathbb{C}}$, we again have unique

$$a \;\in\; \mathfrak{s}^{-1}_\infty \,\cap\, \mathfrak{c}^{-1}_{I,\mathbb{C}} \quad\text{and}\quad b \;\in\; \mathfrak{s}^{0}_\infty \,\cap\, \mathfrak{c}^{-1}_{I,\mathbb{C}}$$

so that

$$\alpha \;=\; e^a \quad\text{and}\quad \beta \;=\; e^b\,.$$

Given any $x \in \mathfrak{g}_\mathbb{C}$, the Deligne splitting (§A.3) yields unique $x^{p,q} \in \mathfrak{g}^{p,q}_{W,F}$ so that

$$x \;=\; \sum x^{p,q}\,.$$

One may verify that the logarithms satisfy

$$\begin{aligned} c^{-1,0} &\;=\; a^{-1,0} \\ c^{0,-1} &\;=\; b^{0,-1} \\ c^{-1,-1} &\;=\; a^{-1,-1} \,+\, \tfrac{1}{2}[a^{-1,0}, b^{0,-1}]\,. \end{aligned}$$

5.2.2. *Action on the fibre.* When restricted to $\delta_I \subset \mathcal{S}$, the map $X : \mathcal{S} \to \mathfrak{f}^\perp$ of §4.2.1 takes value in

$$X : \delta_I \;\to\; \mathfrak{c}^{-1}_{I,\mathbb{C}} \cap \mathfrak{f}^\perp\,.$$

Set $\xi = \exp(X)$. The action of γ on $\xi = \exp(X)\cdot F \in \delta_I$ satisfies

$$(\log \alpha\,\beta\xi\beta^{-1})^{-1,0} \;=\; X^{-1,0} \,+\, a^{-1,0} \tag{5.11a}$$

$$(\log \alpha\,\beta\xi\beta^{-1})^{-1,-1} \;=\; X^{-1,-1} \,+\, a^{-1,-1} \,+\, [b^{0,-1}, X^{-1,0}]\,. \tag{5.11b}$$

The containment (4.4) implies

$$(\log \alpha\,\beta\xi\beta^{-1})^{p,q} \;=\; X^{p,q}\,, \quad \forall\; p+q = -1 > p\,. \tag{5.11c}$$

Let λ be the image of γ under the map $\Gamma_{A^0_I} \to \Lambda_I$. Under the identifications of §5.1.1 we have

$$\lambda \;=\; a^{-1,0} \quad\text{and}\quad \overline{\lambda} \;=\; b^{0,-1}\,,$$

and $(X^{p,-1-p})_{p\le -1} = X^{-1,0} + X^{-2,1} + X^{-3,2} + \cdots$ parameterizes a point in $\mathbb{L}_I$. So (5.11) is describing the action of Λ_I on $\mathbb{L}_I$.

5.2.3. *Logarithms of products.* For later use we consider $\gamma_i = \alpha_i\beta_i \in \Gamma_{A^0}$, with $\gamma_i = e^{c_i}$, $\alpha_i = e^{a_i}$ and $\beta_i = e^{b_i}$, as above. Suppose that $\gamma = \gamma_1\,\gamma_2$. Then one may verify that

$$\begin{aligned} a^{-1,0} &= a_1^{-1,0} + a_2^{-1,0} \\ b^{0,-1} &= b_1^{0,-1} + b_2^{0,-1} \\ c^{-1,-1} &= c_1^{-1,-1} + c_2^{-1,-1} + \tfrac{1}{2}[a_1^{-1,0}\,,\, b_2^{0,-1}] + \tfrac{1}{2}[b_1^{0,-1}\,,\, a_2^{-1,0}] \\ a^{-1,-1} &= a_1^{-1,-1} + a_2^{-1,-1} + [b_1^{0,-1}, a_2^{-1,0}]\,. \end{aligned}$$

5.2.4. *Proof of Theorem 5.3(b).* The line bundle $\mathcal{L}_M \to \Gamma_{A_I^0}\backslash\delta_I$ descends to $\Gamma_{A_I^0}\backslash\delta_I^1$ if and only if the functions e_γ^M of (4.6) are constant on the fibres of $\delta_I \twoheadrightarrow \delta_I^1$. If $M \in \mathfrak{g}_{W,F}^{1,1}$, then (3.17), (4.6), (5.11) and (A.6c) yield

$$e_\gamma^M(X) \;=\; \exp 2\pi\mathbf{i}\kappa(M, a^{-1,-1} + [b^{0,-1}, X^{-1,0}]) \tag{5.12}$$

on δ_I. These functions are constant on the fibres of $\delta_I \twoheadrightarrow \delta_I^1$, and so descend to well-defined on functions on δ_I^1. There they induce a line bundle, (also denoted)

$$\begin{array}{c} \mathcal{L}_M \\ \downarrow \\ \Gamma_{A_I^0}\backslash\delta_I^1\,, \end{array}$$

over the level one extension data. Additionally, if $M \in \mathbf{N}^* \cap \mathfrak{g}_{W,F}^{1,1}$, then (4.11), (5.11) and (5.12) yield

$$(\widetilde{\Phi})^*(e_\gamma^M)\Big|_{A_I^0} \;=\; (\Phi_I^1)^* e_\gamma^M(X)\,;$$

establishing (5.4).

5.3. **Chern classes.** We now wish to compute the first Chern class $c_1(\mathcal{L}_M)$ of $\mathcal{L}_M \to \Gamma_{A_I^0}\backslash\delta_I^1 = \Lambda_I\backslash\mathbb{L}_I$ for $M \in \mathfrak{g}_{W,F}^{1,1}$. We have [*Below* $\mathbb{L}_I$ *should be replaced with* $\mathrm{span}_{\mathbb{C}}\Lambda_I \subset \mathbb{L}_I$*. Thank Haohua Deng. 22.07.29*]

$$H^1(\Lambda_I\backslash\mathbb{L}_I, \mathbb{C}) \;=\; (\mathbb{L}_I \oplus \overline{\mathbb{L}}_I)^* \;\simeq\; \bigoplus_{p+q=-1} \mathfrak{c}_{I,F}^{p,q}\,,$$

and

$$\begin{aligned} H^2(\Lambda_I\backslash\mathbb{L}_I, \mathbb{C}) &= \textstyle\bigwedge^2 H^1(\Lambda_I\backslash\mathbb{L}_I, \mathbb{C}) \;=\; \bigwedge^2(\mathbb{L}_I \oplus \overline{\mathbb{L}}_I)^*\,, \\ H^{1,1}(\Lambda_I\backslash\mathbb{L}_I) &= \mathbb{L}_I^* \otimes \overline{\mathbb{L}}_I^*\,. \end{aligned}$$

Define a map

$$\omega : \mathfrak{g}_{W,F}^{1,1} \;\hookrightarrow\; \mathbb{L}_I^* \otimes \overline{\mathbb{L}}_I^* \;\simeq\; H^{1,1}(\Lambda_I\backslash\mathbb{L}_I)\,,$$

that sends $M \in \mathfrak{g}_{W,F}^{1,1}$ to the form $\omega_M \in H^{1,1}(\Lambda_I\backslash\mathbb{L}_I)$ defined by

$$\omega_M(u, \bar{v}) \;:=\; \kappa(M, [u, \bar{v}]) \;=\; -\kappa(u, \mathrm{ad}_M(\bar{v}))$$

with $u, v \in \mathbb{L}_I$.

Recall the definition of $\mathbf{N}^*$ in (4.9) and consider the subset

$$\mathbf{N}^1 \;=\; \left\{ M \in \mathfrak{g}^{1,1}_{W,F} \;\middle|\; \begin{array}{l} \kappa(M\,,\,[a^{-1,0}, b^{0,-1}]) \in \mathbb{Z}\,,\ \forall\, \gamma \in \Gamma_{A_I^0}\,; \\ \kappa(M, N_j) \in \mathbb{Z}\,,\ \forall\, j \text{ s.t. } Z_j \cap (A_I^0) \neq \emptyset \end{array} \right\}.$$

(If $j \notin I$, then N_j is not necessarily well-defined. However, $\kappa(M, N_j)$ is.)

Remark 5.13. (i) When $\gamma = \exp(N_i)$, we have $a^{-1,-1} = N$ and $a^{-1,0}, b^{0,-1} = 0$.

(ii) The fact that κ is defined over $\mathbb{Q}$ implies that $\mathbf{N}^1$ is non-empty; in fact, $\mathbf{N}^1$ spans $\mathfrak{g}^{1,1}_{W,F}$.

Lemma 5.14. *If $M \in \mathbf{N}^1$, then the form ω_M represents the Chern class $c_1(\mathcal{L}_M)$.*

Proof. Define a smooth function $h_M : \mathbb{L}_I \to \mathbb{R}$ by

$$h_M(z) \;:=\; \exp 2\pi \mathbf{i}\, \kappa\, (M, [z, \bar z])\ .$$

With the formula of §5.2, is straightforward to confirm

$$h_M(z + \lambda) \;=\; |e^M_\gamma(z)|^{-2}\, h_M(z)\,.$$

So h_M defines a metric on $\mathcal{L}_M \to \Lambda_I \backslash \mathbb{L}_I$ with curvature form $-\partial\bar\partial \log h_M$, cf. [GH94, p. 310–311]. It follows that the Chern form of $\mathcal{L}_M$ is

$$c_1(\mathcal{L}_M) \;=\; -\frac{\mathbf{i}}{2\pi}\partial\bar\partial \log h_M \;=\; \partial\bar\partial\kappa\,(M, [z, \bar z]) \;=\; \kappa\,(M, [\mathrm{d}z, \mathrm{d}\bar z]) \;=\; \omega_M\,.$$

□

5.4. **$\mathfrak{sl}_2$–triples.** The ample line bundles $\mathcal{L}_M \to \mathcal{J}_I$ are constructed from $\mathfrak{sl}_2$–triples $\{M, Y, N\}$ constructed from the data of a LMHS (W, F, N), $N \in \sigma_I$. Here we briefly review this well-known construction (see, for example, [CM93] or [Sch73]), and discuss those properties that we will use later.

Define $Y \in \mathrm{End}(\mathfrak{g}_\mathbb{C})$ by specifying that Y acts on $\mathfrak{g}^{p,q}_{W,F}$ by the eigenvalue $(p+q)$. Then $Y \in \mathfrak{g}^{0,0}_{W,F} \cap \mathfrak{g}_\mathbb{R}$, and

$$\mathrm{ad}_Y(N) \;=\; [Y, N] \;=\; -2\, N\,.$$

Notice that Y depends only on (W, F); in particular Y is independent of N. The pair $\{Y, N\}$ may be uniquely completed to a triple $\{M, Y, N\} \subset \mathfrak{g}_\mathbb{R}$ with the properties that

$$[M, N] \;=\; Y \quad\text{and}\quad [Y, M] \;=\; 2\, M\,; \tag{5.15}$$

In particular, $\{M, Y, N\}$ spans a subalgebra of $\mathfrak{g}_\mathbb{R}$ that is isomorphic to $\mathfrak{sl}_2\mathbb{R}$. We have

$$M \;\in\; \mathfrak{g}^{1,1}_{W,F} \cap \mathfrak{g}_\mathbb{R}\,.$$

From $[M, N] = Y$ and $\kappa(Y, Y) > 0$ it follows that

$$0 \;<\; \kappa(Y, Y) \;=\; \kappa([M, N], Y) \;=\; \kappa(M, [N, Y]) \;=\; 2\kappa(M, N)\,. \tag{5.16}$$

We regard (W, F), and hence Y, as fixed. And consider $M = M(N)$ as a function of $N \in \sigma_I$.

5.5. **Ample line bundles.** Define

$$\mathbf{N}_I^{\mathfrak{sl}_2} \;=\; \{M \in \mathbf{N}^1 \mid M = M(N) \text{ for some } N \in \sigma_I\}\,.$$

The fact that both σ_I and κ are defined over $\mathbb{Q}$ implies that $\mathbf{N}_I^{\mathfrak{sl}_2}$ is nonempty.

We have $NMu = u$ for all $u \in \mathfrak{c}_{I,F}^{p,q}$ with $p + q = -1$. The fact that that $N \in \sigma_I$ polarizes the MHS (W, F) on $(\mathfrak{g}, -\kappa)$ implies that

$$\begin{aligned} -\mathbf{i}\,\omega_M(u,\bar u) &= -\mathbf{i}\,\kappa(M,[u,\bar u]) \;=\; \mathbf{i}\,\kappa(u,\mathrm{ad}_M\bar u) \\ &= \mathbf{i}\,\kappa(\mathrm{ad}_N\mathrm{ad}_M u\,,\,\mathrm{ad}_M\bar u) \;=\; -\mathbf{i}\,\kappa(\mathrm{ad}_M u\,,\,\mathrm{ad}_N\mathrm{ad}_M\bar u) \;<\; 0 \end{aligned}$$

for all $0 \neq u \in \mathfrak{c}_{I,F}^{-1,0} \subset \mathbb{L}_I$. It follows that the line bundle $\mathcal{L}_M^* \to \Gamma_{A_I^0}\backslash\delta_I^1$ has positive Chern form $-\omega_M$ for every $M \in \mathbf{N}_I^{\mathfrak{sl}_2}$ (Lemma 5.14). Thus $\mathcal{L}_M^* \to \mathcal{J}_I$ is ample.

5.6. **Positivity.** It remains to establish Theorem 5.3(d); this is a consequence of Remark 5.17 and Lemma 5.23.

Remark 5.17. The map $N \mapsto M(N)$ is the restriction to σ_I of a diffeomorphism $M : \mathcal{N} \to \mathcal{M}$ from an open cone $\mathcal{N} \subset \mathfrak{g}_{W,F}^{-1,-1}$ onto an open cone $\mathcal{M} \subset \mathfrak{g}_{W,F}^{1,1}$. This is a well-known and classical result in the theory of nilpotent elements of semisimple Lie algebras, cf. [CM93] and the references therein, and is discussed in the context of Hodge theory and polarized mixed Hodge structures in [BPR17, §3.2]. In general the map is not linear; in particular, while the image $M(\sigma_I)$ is a cone, it need not be convex.

Notice that the first equation of (5.15) implies that

$$M(\lambda N) \;=\; \tfrac{1}{\lambda} M(N)\,, \tag{5.18}$$

for all $\lambda > 0$. We claim that

$$\mathrm{ad}_N^2(\mathrm{d}M) \;=\; 2\mathrm{d}N\,. \tag{5.19}$$

To see this note that the fact that $Y = [M, N]$ is constant implies

$$[N, \mathrm{d}M] \;=\; [M, \mathrm{d}N]\,.$$

Since elements of the vector subspace $\mathrm{span}_{\mathbb{R}}\,\sigma_I \subset \mathfrak{g}_{W,F}^{-1,-1} \cap \mathfrak{g}_{\mathbb{R}}$ commute, we also have

$$[N, \mathrm{d}N] \;=\; 0\,. \tag{5.20}$$

Thus

$$\mathrm{ad}_N^2(\mathrm{d}M) \;=\; [N\,,\,[M\,\mathrm{d}N]] \;=\; [\mathrm{d}N\,,\,[M,N]] \;=\; 2\,\mathrm{d}N\,.$$

In particular, the differential $\mathrm{d}M$ of $N \mapsto M(N)$ is injective.

Notice that (5.19) and (5.20) imply that

$$\mathrm{ad}_N^3(\mathrm{d}M) \;=\; 0\,.$$

Since $N \in \sigma_I$ polarizes the MHS (W,F) on $(\mathfrak{g},-\kappa)$, we have

$$0 \;\le\; -\tfrac{1}{2}\kappa(\mathrm{d}M, \mathrm{ad}_N^2(\mathrm{d}M)) \;=\; -\kappa(\mathrm{d}M,\mathrm{d}N)\,, \tag{5.21}$$

with equality if and only if $\mathrm{d}N = 0$.

Lemma 5.22. *Fix $0 \neq N' \in \mathrm{span}_{\mathbb{R}}\,\sigma_I$. The set*

$$\sigma'_0 \;=\; \{N \in \sigma_I \mid \kappa(M(N),N') = 0\}$$

is contained in the closure of

$$\sigma'_+ \;=\; \{N \in \sigma_I \mid \kappa(M(N),N') > 0\}\,.$$

Proof. Suppose that $N \in \sigma'_0$. Fix a smooth curve $\nu(t)$ in σ_I with the property that $\nu(0) = N$ and $\nu'(0) = -N'$. Set $\mu(t) = M(\nu(t))$. Then (5.21) implies

$$0 \;<\; \kappa(\mu'(0),N')\,.$$

In particular, $\nu(t) \in \sigma'_+$ for small $t > 0$. □

Lemma 5.23. *The cone*

$$\sigma_I^+ \;=\; \{N \in \sigma_I \mid \kappa(M(N),N_i) > 0\,,\ \forall\, i \in I\}$$

is open and nonempty.

Proof. In the case that $\dim\sigma_I = 1$, (5.18) and (5.24) yield $\sigma_I^+ = \sigma_I$.

For the general case $\dim\sigma_I \ge 1$, with $I = \{1,\ldots,k\}$, set

$$\mathbb{R}^k_+ \;=\; \{y = (y^1,\ldots,y^k) \in \mathbb{R}^k \mid y^i > 0\}$$

so that

$$\sigma_I \;=\; \{N(y) = y^i N_i \mid y \in \mathbb{R}^k_+\}\,.$$

Set $M(y) = M(N(y))$ and $\kappa_i(y) = \kappa(M(y),N_i)$. Then it suffices to show that the cone

$$S^+ \;=\; \{y = (y^1,\ldots,y^k) \in \mathbb{R}^k_+ \mid \ \kappa_i(y) > 0\}$$

is open. From (5.15) and (5.16) we see that

$$0 \;<\; \kappa(M(y),N(y)) \;=\; y^i\,\kappa_i(y)\,. \tag{5.24}$$

Since the y^i are all positive, this forces some $\kappa_i(y)$ to be positive (with i depending on y).

Decompose

$$\mathbb{R}^k_+ \;=\; S \cap S' \cap S''$$

with

$$\begin{aligned} S_1 &= \left\{ y \in \mathbb{R}^k_+ \;\Big|\; \kappa_1(y) \geq 0\,,\ \textstyle\sum_{i=2}^k y^i \kappa_i(y) \geq 0 \right\} \\ S_1' &= \left\{ y \in \mathbb{R}^k_+ \ | \ \kappa_1(y) < 0 \right\} \\ S_1'' &= \left\{ y \in \mathbb{R}^k_+ \;\Big|\; \textstyle\sum_{i=2}^k y^i \kappa_i(y) < 0 \right\} . \end{aligned}$$

The inequality (5.24) forces the open sets S_1' and S_1'' to be disjoint. Since $\mathbb{R}^k_+$ is open and connected, this in turn forces S to be nonempty. Then Lemma 5.22 implies that the cone

$$S_1^+ \;=\; \left\{ y \in \mathbb{R}^k_+ \;\Big|\; \kappa_1(y) > 0\,,\ \textstyle\sum_{i=2}^k y^i \kappa_i(y) > 0 \right\} \;\subset\; S$$

is nonempty and open in $\mathbb{R}^k_+$. This proves Theorem 5.3(d) in the case that $|I| \leq 2$.

For the general case $|I| = k$ we induct. Assume that the cone

$$S_a^+ \;=\; \left\{ y \in \mathbb{R}^k_+ \;\Big|\; \kappa_i(y) > 0\,,\ 1 \leq i \leq a\,; \quad \textstyle\sum_{i=a+1}^k y^i \kappa_i(y) > 0 \right\}$$

is nonempty (and therefore open) for some $1 \leq a \leq k-1$. Define a decomposition

$$S_a^+ \;=\; S_{a+1} \cup S_{a+1}' \cup S_{a+1}''$$

by

$$\begin{aligned} S_{a+1} &= \left\{ y \in S_a^+ \;\Big|\; \kappa_{a+1}(y) \geq 0\,,\ \textstyle\sum_{i=a+2}^k y^i \kappa_i(y) \geq 0 \right\} \\ S_{a+1}' &= \left\{ y \in S_a^+ \ | \ \kappa_{a+1}(y) < 0 \right\} \\ S_{a+1}'' &= \left\{ y \in S_a^+ \;\Big|\; \textstyle\sum_{i=a+2}^k y^i \kappa_i(y) < 0 \right\} . \end{aligned}$$

The definition of S_a^+ forces the open sets S_{a+1}' and S_{a+1}'' to be disjoint. Since S_a^+ is open, every connected component of S_a^+ must have nonempty intersection with S_{a+1}. Then Lemma 5.22 implies that the cone S_{a+1}^+ is nonempty and open in $\mathbb{R}^k_+$. This completes the inductive step. □

6. Higher level extension data

Let A_I^1 be a connected component of a fibre of the level one extension data map $\Phi_I^1 : A_I^0 \to \Gamma_{A_I^0} \backslash \delta_I$ introduced in (5.1). Along A_I^1 we have a fixed cone σ_I and weight filtration W, and a variation of limiting mixed Hodge structures (W, F, σ_I) with the property that both the Hodge decomposition $F^p(\mathrm{Gr}_\ell^W)$ and the level one extension data are *constant*. The goal here is to study the higher level extension data. Over $A_I^1 \cap Z_I^*$ this extension data is encoded by the map Φ_I of (2.1). We will see that, after modding out by some level two extension data (which can be recovered from the sections s_M of §4.2), the map Φ_I is constant along $A_I^1 \cap Z_I^*$ and extends to $A_I^1 \cap Z_I$. Very roughly, this days that the full extension data is determined by the level ≤ 2 extension data (Remark 6.2).

We assume that the monodromy Γ is neat. Arguing as in §3.1 and §4.1, we may choose a neighborhood $\overline{\mathcal{O}}^1_I \subset \overline{\mathcal{O}}^0_I$ of A^1_I so that the monodromy $\Gamma_{A^1_I} \subset \Gamma_{A^0_I}$ of the restriction of the variation of Hodge structures on $\mathcal{O}^1_I = \overline{\mathcal{O}}^1_I \cap B$ takes value in $C^{-2}_{I,\mathbb{Q}} \cap S$. Set

$$J \;=\; \{j \mid Z_j \cap A^1_I \neq \emptyset\} \;\supset\; I\,.$$

While the cone σ_I is well-defined (because the monodromy $\Gamma_{A^1_I}$ centralizes the cone), the larger cone

$$\sigma_J \;=\; \mathrm{span}_{\mathbb{R}_{>0}}\{N_j \mid j \in J\} \;\supset\; \sigma_I$$

is, a priori, defined only up to the action of $\Gamma_{A^1_I}$.

Theorem 6.1. *Assume that Γ is neat.*

(a) *The neighborhood $\overline{\mathcal{O}}^1_I$ may be chosen so that σ_J is well-defined, and the monodromy $\Gamma_{A^1_I} \subset \exp(\mathbb{C}\sigma_J)$.*

(b) *There is a well-defined holomorphic map*

$$\Psi_I : Z_I \cap \overline{\mathcal{O}}^1_I \;\to\; (\exp(\mathbb{C}\sigma_J)\Gamma_{A^1_I})\backslash D_I\,,$$

that "extends" the map Φ_I of (2.1) *in the sense that we have a commutative diagram*

$$\begin{array}{ccc} Z_I^* \cap \overline{\mathcal{O}}^1_I & \xrightarrow{\Phi_I} & (\exp(\mathbb{C}\sigma_I)\Gamma_{A^1_I})\backslash D_I \\ \downarrow & & \downarrow \\ Z_I \cap \overline{\mathcal{O}}^1_I & \xrightarrow{\Psi_I} & (\exp(\mathbb{C}\sigma_J)\Gamma_{A^1_I})\backslash D_I\,. \end{array}$$

(c) *The map Ψ_I is locally constant on the fibres of the level ≤ 1 extension data map $\Phi^1_I : Z_I \cap Z_\pi \cap \overline{\mathcal{O}}^1_I \to \Gamma_{A^1_I}\backslash D^1_I$ introduced in* (2.18).

Remark 6.2. The information contained in $\exp(\mathbb{C}\sigma_J)$ is level two extension data. So the content of Theorem 6.1(c) is that *the full extension data is determined by the level ≤ 2 extension data*, up to constants of integration.[3] The level 2 extension data contained in $\exp(\mathbb{C}\sigma_J)$ is not lost; it is encoded in the sections $s_M \in H^0(\overline{\mathcal{O}}^0, L_M)$, with $M \in \mathfrak{g}^{1,1}_{W,F}$, of §4.2. These sections are essentially discrete data as their restriction to the Φ^0–fibres is determined up to a constant factor by (4.5).

[3]In the case that D is hermitian, all extension data is level ≤ 2; that is, $D_I = D^2_I$. So here we find here another example of the ansatz that horizontality (the IPR) forces period maps and their images to behave "as if they were hermitian".

6.1. **Outline of the proof.** Theorem 6.1 is proved by an inductive analysis of the higher level extension data along $A^1_I \subset A^0_I$. Over A^{0*}_I the extension data is encoded by the map Φ_I of (2.1). We have seen that the projection Φ^1_I to level ≤ 1 extension data extends to A^0_I (§2.3). In general, the higher level maps will not extend. However we do obtain local "extensions" after quotienting out by the larger monodromy cone σ_J. These extensions are constructed level by level. We begin with the level ≤ 2 extension data. First note that $\Gamma_{A^1_I} \subset C^{-2}_{I,\mathbb{Q}}$ implies that σ_J is well-defined modulo $\mathfrak{c}^{-4}_{I,\mathbb{R}} \subset \mathfrak{c}^{-3}_{I,\mathbb{R}}$, so that $\exp(\mathbb{C}\sigma_J)$ is well-defined modulo $C^{-3}_{I,\mathbb{C}}$. So (2.1) and Lemma B.20 yield a well-defined map

$$\Psi^2_I : Z_I \cap \overline{\mathcal{O}}^1_I \ \to \ (\exp(\mathbb{C}\sigma_J)\Gamma_{A^1_I})\backslash D^2_I$$

and a commutative diagram

$$\begin{array}{ccc} Z^*_I \cap \overline{\mathcal{O}}^1_I & \xrightarrow{\Phi^2_I} & (\exp(\mathbb{C}\sigma_I)\Gamma_{A^1_I})\backslash D^2_I \\ \downarrow & & \downarrow \\ Z_I \cap \overline{\mathcal{O}}^1_I & \xrightarrow{\Psi^2_I} & (\exp(\mathbb{C}\sigma_J)\Gamma_{A^1_I})\backslash D^2_I \,. \end{array} \tag{6.3}$$

This says that we may "extend" the map Φ^2_I from $Z^*_I \cap \overline{\mathcal{O}}^1_I$ to $Z_I \cap \overline{\mathcal{O}}^1_I$ if we quotient by the larger $\exp(\mathbb{C}\sigma_J)$. The map Ψ^2_I encodes level two extension data modulo the cone $\sigma_J \supset \sigma_I$. The following lemma asserts that this data is (locally) constant whenever the level one extension data is fixed.

Lemma 6.4. *The map Ψ^2_I is locally constant on the fibres of the level ≤ 1 extension data map $\Phi^1_I : Z_I \cap Z_\pi \to \Gamma_{A^0_I}\backslash D^1_I$ introduced in* (2.18)*.*

A straightforward modification of the arguments in §3.1 and §4.1 establishes

Corollary 6.5. *We may choose the neighborhood $\overline{\mathcal{O}}^1_I$ so that $\Gamma_{A^1_I} \subset (\exp(\mathbb{C}\sigma_J)C^{-3}_{I,\mathbb{Q}}) \cap S$.*

As above, (2.1), Corollary 6.5 and Lemma B.20 yield a commutative diagram

$$\begin{array}{ccc} Z^*_I \cap \overline{\mathcal{O}}^1_I & \xrightarrow{\Phi^3_I} & (\exp(\mathbb{C}\sigma_I)\Gamma_{A^1_I})\backslash D^3_I \\ \downarrow & & \downarrow \\ Z_I \cap \overline{\mathcal{O}}^1_I & \xrightarrow{\Psi^3_I} & (\exp(\mathbb{C}\sigma_J)\Gamma_{A^1_I})\backslash D^3_I \,. \end{array}$$

The inductive step is

Lemma 6.6. *Fix $a \ge 2$. If the monodromy $\Gamma_{A^1_I}$ about A^1_I takes value in $\exp(\mathbb{C}\sigma_J)C^{-a}_I$, then there is a commutative diagram*

$$\begin{array}{ccc} Z^*_I \cap \overline{\mathcal{O}}^1_I & \xrightarrow{\Phi^a_I} & (\exp(\mathbb{C}\sigma_I)\Gamma_{A^1_I})\backslash D^a_I \\ \downarrow & & \downarrow \\ Z_I \cap \overline{\mathcal{O}}^1_I & \xrightarrow{\Psi^a_I} & (\exp(\mathbb{C}\sigma_J)\Gamma_{A^1_I})\backslash D^a_I \,. \end{array} \tag{6.7}$$

The map Ψ_I^a *is locally constant on the fibres of the level* ≤ 1 *extension data map* $\Phi_I^1 : Z_I \cap Z_\pi \to \Gamma_I \backslash D_I^1$ *introduced in* (2.18)*. And we may choose* $\overline{\mathcal{O}}_I^1$ *so that* $\Gamma_{A_I^1} \subset \exp(\mathbb{C}\sigma_J) C_{I,\mathbb{Q}}^{-a-1}$).

Note that Theorem 6.1 follows directly from Lemma 6.6. The remainder of §6 is occupied with the proof of Lemma 6.6 (which subsumes Lemma 6.4 and Corollary 6.5).

The existence of (6.7) follows from the monodromy assumption by an argument identical to that establishing (6.3). Assuming the constancy of Ψ_I^a, the existence of a neighborhood $\overline{\mathcal{O}}_I^1$ such that $\Gamma_{A_I^1} \subset \exp(\mathbb{C}\sigma_J) C_{I,\mathbb{Q}}^{-a-1}$ follows from an argument identical to those in §3.1. So the crux of the proof is to show that Ψ_I^a is constant on Φ_I^1 fibres. The argument (§6.4) makes use of a Lie theoretic description of the level a extension data (§6.2) and the infinitesimal period relation (§6.3).

6.2. **Lie theoretic description.** Fix $a \geq 2$. The fibres of $\Gamma_{A_I^1} \backslash \delta_I^a \twoheadrightarrow \Gamma_{A_I^1} \backslash \delta_I^{a-1}$ are the *level a extension data* (Definition 2.4). We begin by observing that these fibres are biholomorphic to the quotient $\Lambda_I^a \backslash \mathbb{L}_I^a$ of a vector space $\mathbb{L}_I^a$ by a discrete subgroup $\Lambda_I^a \subset \mathbb{L}_I^a$. To see this, first note that the fibre is

$$\begin{array}{ccc} \dfrac{C_{I,\mathbb{C}}^{-a} \cdot F}{(\Gamma_{A_I^1} \cap C_{I,\mathbb{C}}^{-a}) \cdot C_{I,\mathbb{C}}^{-a-1}} & \hookrightarrow & \Gamma_{A_I^1} \backslash \delta_I^a \\ & & \downarrow\!\!\downarrow \\ & & \Gamma_{A_I^1} \backslash \delta_I^{a-1} . \end{array}$$

We have

$$\begin{aligned} C_{I,\mathbb{C}}^{-a-1} \backslash C_{I,\mathbb{C}}^{-a} & \simeq \bigoplus_{p+q=-a} \mathfrak{c}_{I,F}^{p,q} \\ C_{I,\mathbb{C}}^{-a-1} \backslash (C_{I,\mathbb{C}}^{-a} \cdot F) & \simeq \bigoplus_{\substack{p+q=-a \\ p<0}} \mathfrak{c}_{I,F}^{p,q} \ = \ \mathbb{L}_I^a . \end{aligned}$$

The latter is an abelian group, with discrete subgroup

$$\Lambda_I^a \ = \ \frac{C_{I,\mathbb{C}}^{-a} \cap \Gamma_{A_I^1}}{C_{I,\mathbb{C}}^{-a-1} \cap \Gamma_{A_I^1}} .$$

We now see that the level a extension data of (W, F) is biholomorphic to the product

$$\Lambda_I^a \backslash \mathbb{L}_I^a \ \simeq \ \mathbb{C}^{d_1} \times (\mathbb{C}^*)^{d_2} \times \mathbb{T}^{d_3} \tag{6.8}$$

of an affine space $\mathbb{C}^{d_1}$ with a complex torus $(\mathbb{C}^*)^{d_2} \times \mathbb{T}^{d_3}$ having compact factor $\mathbb{T}^{d_3}$.

Since $\sigma_J \subset \mathfrak{c}_{I,F}^{-1,-1}$, it then follows that the fibres of

$$(\exp(\mathbb{C}\sigma_J)\Gamma_{A_I^1}) \backslash \delta_I^a \ \twoheadrightarrow \ (\exp(\mathbb{C}\sigma_J)\Gamma_{A_I^1}) \backslash \delta_I^{a-1}$$

are, for $a=2$:

$$\begin{array}{ccc} (\Lambda_I^2\cdot\sigma_J)\backslash\mathbb{L}_I^2 & \hookrightarrow & (\exp(\mathbb{C}\sigma_J)\Gamma_{A_I^1})\backslash\delta_I^2 \\ & & \downarrow \\ & & \Gamma_{A_I^1}\backslash\delta_I^1\,, \end{array}$$

and, for $a\geq 3$:

$$\begin{array}{ccc} \Lambda_I^a\backslash\mathbb{L}_I^a & \longhookrightarrow & (\exp(\mathbb{C}\sigma_J)\Gamma_{A_I^1})\backslash\delta_I^a \\ & & \downarrow \\ & & (\exp(\mathbb{C}\sigma_J)\Gamma_{A_I^1})\backslash\delta_I^{a-1}\,. \end{array}$$

Note that $(\Lambda_I^2\cdot\sigma_J)\backslash\mathbb{L}_I^2$ inherits (6.8) in the sense that it is also biholomorphic to a product

$$(\Lambda_I^2\cdot\sigma_J)\backslash\mathbb{L}_I^2 \;\simeq\; \mathbb{C}^{d_1}\,\times\,(\mathbb{C}^*)^{d_2}\,\times\,\mathbb{T}^{d_3} \tag{6.9}$$

of an affine space $\mathbb{C}^{d_1}$ with a complex torus $(\mathbb{C}^*)^{d_2}\times\mathbb{T}^{d_3}$ having compact factor $\mathbb{T}^{d_3}$. (We abuse notation by continuing to denote the dimensions by d_i.)

6.3. **The infinitesimal period relation along fibres.** Recall the Maurer-Cartan form ω of §5.1.2. Recall that the component $\omega^{-1,-1}$ taking value in $\mathfrak{g}_{W,F}^{-1,-1}$ has poles along $Z\cap\overline{\mathcal{O}}$. Let $\omega_{\overline{\mathcal{O}}_I^1}$ denote the restriction to $\overline{\mathcal{O}}_I^1$. We can finesse the poles in $\omega_{\overline{\mathcal{O}}_I^1}$ as follows. The assumption that $\Gamma_{A_I^1}$ takes value in $\exp(\mathbb{C}\sigma_J)C_I^{-1}$ implies that $\mathbb{C}\sigma_J+\mathfrak{c}_{I,\mathbb{C}}^{-a-2}$ is a well-defined subspace of $\mathfrak{g}_\mathbb{C}$. The 1-form $\omega_{\overline{\mathcal{O}}_I^1}$ naturally induces a 1-form η that takes value in $(\mathfrak{g}_{W,F}^{-1,\bullet}+\mathfrak{c}_{I,\mathbb{C}}^{-a-2})/(\mathbb{C}\sigma_J+\mathfrak{c}_{I,\mathbb{C}}^{-a-2})$. This 1-form is holomorphic (but possibly multi-valued). (Informally, we say that the poles of $\omega_{\overline{\mathcal{O}}_I^1}$ "live" in $\mathbb{C}\sigma_J$.) If we let $\eta^{-1,q}$ denote the component taking value in $(\mathfrak{g}_{W,F}^{-1,q}+\mathfrak{c}_{I,\mathbb{C}}^{-a-2})/(\mathfrak{g}^{-1,q}\cap\mathbb{C}\sigma_J+\mathfrak{c}_{I,\mathbb{C}}^{-a-2})$, then we have natural identifications $\omega_{\overline{\mathcal{O}}_I^1}^{-1,q}=\eta^{-1,q}$ for all $q\neq -1$. And we may informally think of $\eta^{-1,-1}$ as the quotient of $\omega_{\overline{\mathcal{O}}_I^1}^{-1,-1}$ by $\mathbb{C}\sigma_J$.

The restriction $\omega_{A_I^1}$ to A_I^1 takes value in $\oplus_{q\leq -1}\,\mathfrak{c}_{I,F}^{-1,q}$; equivalently, $\eta_{A_I^1}^{-1,q}=0$ for all $q\geq 0$. If $a\geq 2$, then Ψ^a is locally constant on the fibres of the level ≤ 1 extension data map if and only if $\eta_{A_I^1}^{-1,q}=0$ for all $q\geq 1-a$. The hypothesis that Γ is neat (§4.1) implies that the component $\eta_{A_I^1}^{-1,-a}$ is well-defined (single-valued). Also the infinitesimal period relation implies that the restriction of $\mathrm{d}X^{p,q}$ to A_I^0 is zero for all $p+q=-a-1$ with $p\leq -2$.

6.4. **Proof of Lemma 6.6.** The argument is inductive. Assume that $a\geq 1$ and that we have a well-defined

$$\Psi_I^{a+1}: Z_I\,\cap\,\overline{\mathcal{O}}_I^1\;\to\;(\exp(\mathbb{C}\sigma_J)\Gamma_{A^1})\backslash D_I^{a+1}\,.$$

We will show that Ψ_I^{a+1} is constant along A_I^1.

Recall §6.3. Fixing a point $z_0 \in A^1_I$ we may define a holomorphic map

$$A^1_I \;\to\; \begin{cases} (\Lambda^2_I \cdot \sigma_J)\backslash \mathbb{L}^2_I\,, & a = 1\,,\\ \Lambda^{a+1}_I \backslash \mathbb{L}^{a+1}_I\,, & a \geq 2\,, \end{cases} \tag{6.10a}$$

by integration

$$z \;\mapsto\; \int_{z_0}^{z} \eta^{-1,-a} \tag{6.10b}$$

along a curve $\alpha : [0,1] \to A^1$ joining $z_0 = \alpha(0)$ and $z = \alpha(1)$.

Because

$$\overline{\mathfrak{g}^{-1,-b}_{W,F}} \;=\; \mathfrak{g}^{-b,-1}_{W,F}\,,$$

the image of (6.10) necessarily lies in the noncompact factors $\mathbb{C}^{d_1} \times (\mathbb{C}^*)^{d_2}$ of (6.8) and (6.9). Since A^1 is compact, this map must be locally constant. This forces $\eta^{-1,-a} = 0$. This is precisely the statement that Ψ^{a+1}_I is locally constant along A^1_I. □

Appendix A. Asymptotics of period maps: review of local properties

Here we set notation and review well-known properties of period maps and their local behavior at infinity. Good references for this material include [CMSP17, CKS86, GGK12, GS69, PS08, Sch73].

A.1. **Notation.**

A.1.1. *Groups.* Given a $\mathbb{Q}$–algebraic group G, the Lie groups of real and complex points will be denoted by $G_\mathbb{R}$ and $G_\mathbb{C}$, respectively. The associated Lie algebras are denoted $\mathfrak{g}_\mathbb{R}$ and $\mathfrak{g}_\mathbb{C}$, respectively.

Let $V = V_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$ is a rational vector space, with underlying lattice $V_\mathbb{Z}$. Let $\mathrm{End}(V) = V \otimes V^*$ denote the Lie algebra of linear maps $V \to V$, and let $\mathrm{Aut}(V) \subset \mathrm{End}(V)$ denote the $\mathbb{Q}$–algebraic group of invertible linear maps.

Fix $n \in \mathbb{Z}$, and suppose that , and $Q : V \times V \to \mathbb{Q}$ is a nondegenerate (skew-)symmetric bilinear form satisfying

$$Q(u,v) \;=\; (-1)^n Q(v,u)\,, \quad \text{for all} \quad u,v \in V\,.$$

From this point on, G will denote the $\mathbb{Q}$–algebraic group

$$G \;=\; \mathrm{Aut}(V,Q) \;=\; \{g \in \mathrm{Aut}(V) \mid Q(gu,gv) = Q(u,v)\,,\ \forall\, u,v \in V\}\,.$$

with Lie algebra

$$\mathfrak{g} \;=\; \mathrm{End}(V,Q) \;=\; \{X \in \mathrm{End}(V) \mid 0 = Q(Xu,v) + Q(u,Xv)\,,\ \forall\, u,v \in V\}\,.$$

A.1.2. *Period domains.* Let $D = G_{\mathbb{R}}/K^0$ be the period domain parameterizing effective weight $n > 0$, Q–polarized Hodge structures on V with Hodge numbers $\mathbf{h} = (h^{n,0}, \ldots, h^{0,n})$. Given $\varphi \in D$, let

$$V_{\mathbb{C}} \;=\; \bigoplus_{p+q=n} V_{\varphi}^{p,q}$$

be the Hodge decomposition; let

$$F_{\varphi}^{n} \;\subset\; F_{\varphi}^{n-1} \;\subset \cdots \subset\; F_{\varphi}^{1} \;\subset\; F_{\varphi}^{0} \;=\; V_{\mathbb{C}}$$

be the Hodge filtration. The weight zero Hodge decomposition

$$\mathfrak{g}_{\mathbb{C}} \;=\; \oplus\, \mathfrak{g}_{\varphi}^{p,-p} \tag{A.1}$$

induced by φ, is polarized by $-\kappa$, where $\kappa \in \mathrm{Sym}^2 \mathfrak{g}_{\mathbb{C}}^*$ is the Killing form. The isotropy group $K^0 = \mathrm{Stab}_G(\varphi)$ stabilizing $\varphi \in D$ is compact, with complexified Lie algebra

$$\mathfrak{k}_{\mathbb{C}}^0 \;=\; \mathfrak{k}_{\mathbb{R}}^0 \otimes \mathbb{C} \;=\; \mathfrak{g}_{\varphi}^{0,0}\,.$$

Let $\check{D} = G_{\mathbb{C}}/P_{\varphi}$ denote the compact dual of D. Here P_{φ} is the complex parabolic stabilizer of the Hodge filtration F_{φ}, and has Lie algebra $\mathfrak{p}_{\varphi} = \oplus_{p \geq 0}\, \mathfrak{g}_{\varphi}^{p,-p}$.

A.2. **Period maps at infinity.**

A.2.1. Unit disc

$$\Delta \;=\; \{t \in \mathbb{C} \mid |t| < 1\}$$

and punctured unit disc

$$\Delta^* \;=\; \{t \in \mathbb{C} \mid 0 < |t| < 1\}\,.$$

Upper half plane

$$\mathcal{H} \;=\; \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$$

and covering map

$$\mathcal{H} \;\to\; \Delta^* \quad \text{sending} \quad z \;\mapsto\; t = e^{2\pi \mathbf{i} z}\,.$$

Multivalued inverse

$$\ell(t) \;=\; \frac{\log t}{2\pi \mathbf{i}}\,,$$

and (well-defined) differential $\mathrm{d}\ell \;=\; \dfrac{\mathrm{d}t}{2\pi \mathbf{i}\, t}$.

A.2.2. Fix a point $b \in Z_I^* \subset \overline{B}$. Choose a coordinate chart

$$(t,w) : \overline{\mathcal{U}} \subset \overline{B} \xrightarrow{\simeq} \Delta^{k+r}$$

centered at a point b with

$$(t,w) : \mathcal{U} = B \cap \overline{\mathcal{U}} \xrightarrow{\simeq} (\Delta^*)^k \times \Delta^r .$$

Reindexing the Z_i if necessary, we may assume that

$$\overline{\mathcal{U}} \cap Z_i = \{t_i = 0\} , \quad \text{for all} \quad 1 \le i \le k ,$$

and $\overline{\mathcal{U}} \cap Z_\mu = \emptyset$ for all $k+1 \le \mu \le \nu$. (We are assuming, as we may by shrining $\overline{\mathcal{U}}$ if necessary, that $\overline{\mathcal{U}} \cap Z_I = \overline{\mathcal{U}} \cap Z_I^*$.)

A.2.3. The counter-clockwise generator $\alpha_i \in \pi_1(\Delta^*) \hookrightarrow \pi_1((\Delta^*)^k) = \pi_1(\mathcal{U})$ induces a quasi-unipotent monodromy operator $\gamma_i \in \mathrm{Aut}(V,Q)$, $1 \le i \le k$ [Sch73]. Passing to a finite cover of B if necessary, we may assume without loss of generality that γ_i is unipotent; let

$$N_i = \log \gamma_i \in \mathfrak{g}$$

be the nilpotent logarithm of monodromy, and

$$\sigma_I = \mathrm{span}_{\mathbb{R}_{>0}}\{N_1, \ldots, N_k\} \subset \mathfrak{g}_\mathbb{R} = \mathrm{Aut}(V_\mathbb{R}, Q) ,$$

the *monodromy cone (for the coordinate chart centered at b)*.

A.2.4. The universal cover of $\mathcal{U}$ is

$$\widetilde{\mathcal{U}} = \mathcal{H}^k \times \Delta^r .$$

The local lift

$$\widetilde{\Phi} : \widetilde{\mathcal{U}} \to D$$

of $\Phi|_\mathcal{U}$ is of the form

$$\widetilde{\Phi}(t,w) = \exp(\textstyle\sum \ell(t_i) N_i) \tilde{g}(t,w) \cdot F . \tag{A.2}$$

Here, $F \in \check{D}$,

$$\tilde{g} : \overline{\mathcal{U}} \to G_\mathbb{C} \tag{A.3}$$

is a holomorphic map, and we abuse notation by regarding the multi-valued $\ell(t_i)$ as giving coordinates on $\mathcal{H}$. Additionally, if $F(w) = \tilde{g}(0,w) \cdot F$, then $(W, F(w))$, is a mixed Hodge structure (MHS) polarized by the local monodromy cone σ_I. We say (W, F, σ_I) is a *limiting mixed Hodge structure* (LMHS).

The infinitesimal period relation implies that the restriction $\tilde{g}_I = \tilde{g}|_{\overline{\mathcal{U}} \cap Z_I^*}$ takes value in the centralizer

$$C_{I,\mathbb{C}} = \{ g \in G_\mathbb{C} \mid \mathrm{Ad}_g N = N \, , \ \forall \ N \in \sigma_I \}$$

of the nilpotent cone σ_I. The map

$$F_I : Z_I^* \cap \overline{\mathcal{U}} \ \to\ D_I\,, \quad w \mapsto F_I(w) = \tilde{g}(0,w)\cdot F \tag{A.4}$$

defines a variation of limiting mixed Hodge structure $(W, F_I(w), \sigma_I)$ over $Z_I^* \cap \overline{\mathcal{U}}$. The map (A.4) is not well-defined; it depends on our choice of coordinates. What is well-defined is the composition

$$Z_I^* \cap \overline{\mathcal{U}} \xrightarrow{F_I} D_I \twoheadrightarrow \exp(\mathbb{C}\sigma_I)\backslash D_I\,.$$

(That is, it is the nilpotent orbit that is well-defined.) This yields the map Φ_I of §1.2 and (2.1).

The fact that $\exp(\mathbb{C}\sigma_I) \subset P_{W,\mathbb{C}}^{-2}$ implies that $(\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I^\epsilon = \Gamma_I\backslash D_I^\epsilon$ for $\epsilon = 0,1$. So (A.4) *does* induce well-defined maps

$$F_I^\epsilon : Z_I^* \cap \overline{\mathcal{U}} \ \to\ D_I^\epsilon\,. \tag{A.5}$$

The maps (A.5) are local lifts of the maps Φ_I^ϵ of (2.1).

A.3. **Deligne bigrading.** Given a mixed Hodge structure (W,F) on (V,Q), we have a Deligne splitting

$$V_\mathbb{C} \ = \ \oplus\, V_{W,F}^{p,q}$$

satisfying

$$W_\ell \ = \ \bigoplus_{p+q\le\ell} V_{W,F}^{p,q} \quad \text{and} \quad F^k \ = \ \bigoplus_{p\ge k} V_{W,F}^{p,q}\,.$$

The induced splitting

$$\mathfrak{g}_\mathbb{C} \ = \ \oplus\, \mathfrak{g}_{W,F}^{p,q}\,, \tag{A.6a}$$

of the Lie algebra $\mathfrak{g}_\mathbb{C}$ is defined by

$$\mathfrak{g}_{W,F}^{p,q} \ = \ \{x \in \mathfrak{g}_\mathbb{C} \mid x(V_{W,F}^{r,s}) \subset V_{W,F}^{p+r,q+s}\,,\ \forall\, r,s\}\,, \tag{A.6b}$$

satisfies

$$\kappa(\mathfrak{g}_{W,F}^{p,q}\,,\, \mathfrak{g}_{W,F}^{r,s}) \ = \ 0 \quad \text{if} \quad (p,q)+(r,s) \ne (0,0)\,, \tag{A.6c}$$

and is compatible with the Lie bracket in the sense that

$$[\mathfrak{g}_{W,F}^{p,q}\,,\, \mathfrak{g}_{W,F}^{r,s}] \ \subset\ \mathfrak{g}_{W,F}^{p+r,q+s}\,. \tag{A.6d}$$

It follows that $\mathfrak{g}_\mathbb{C} = \mathfrak{f}\oplus\mathfrak{f}^\perp$ with

$$\mathfrak{f} \ = \ \oplus_{p\ge0}\, \mathfrak{g}_{W,F}^{p,q}$$

the parabolic Lie algebra of the stabilizer $\mathrm{Stab}_{G_\mathbb{C}}(F)$ of F, and

$$\mathfrak{f}^\perp \ = \ \oplus_{p<0}\, \mathfrak{g}_{W,F}^{p,q} \tag{A.7}$$

a nilpotent subalgebra of $\mathfrak{g}_\mathbb{C}$. The map (A.3) is determined by the property

$$\tilde g(t,w) \ \in \ \exp(\mathfrak{f}^\perp)\,.$$

Remark A.8. Without loss of generality, we may assume that (W,F) is $\mathbb{R}$–split

$$\overline{V^{p,q}_{W,F}} \ = \ V^{q,p}_{W,F}\,,$$

which implies

$$\overline{\mathfrak{g}^{p,q}_{W,F}} \ = \ \mathfrak{g}^{q,p}_{W,F}\,.$$

Then $\tilde g(0,0) \in P^{-2}_{W,\mathbb{C}}$.

A.4. **Infinitesimal period relation.** The pull-back of the Maurer-Cartan form on $\exp(\mathfrak{f}^\perp) \subset G_\mathbb{C}$ under the map (A.3) is $\tilde\omega = \tilde g^{-1}\mathrm{d}\tilde g$. The infinitesimal period relation asserts that $\tilde\omega$ takes value in $\mathfrak{f}^{-1,\bullet}$. And when restricted to $Z_I \cap \overline{\mathcal{U}}$, the form takes value in $\mathfrak{c}^{-1,\bullet}_{I,F}$.

A.5. **Period matrices and Schubert cells.** Since the *period matrix*

$$\exp(\textstyle\sum \ell(t_i)N_i)\tilde g(t,w)$$

of the local lift (A.2) takes value in $\exp(\mathfrak{f}^\perp)\cdot F$, the local lift $\widetilde\Phi(t,w)$ takes value in the open Schubert cell $\mathcal{S}$

$$\mathcal{S} \ = \ \exp(\mathfrak{f}^\perp)\cdot F \ = \ \left\{ E \in \check D \mid \dim\left(E^a \cap \overline{F^b_\infty}\right) = \dim\left(F^a \cap \overline{F^b_\infty}\right),\ \forall\ a,b \right\}\,,$$

defined by

$$\overline{F^b_\infty} \ = \ \bigoplus_{c\le n-b} V^{c,a}_{W,F}\,.$$

The map $\mathfrak{f}^\perp \to \mathcal{S}$ sending $X \mapsto \exp(X)\cdot F$ is a biholomorphism. Let

$$X : \mathcal{S} \ \xrightarrow{\simeq}\ \mathfrak{f}^\perp\,. \tag{A.9}$$

denote the inverse. The obvious analogs of (A.6) hold with $\mathrm{End}(V_\mathbb{C})$ in place of $\mathfrak{g}_\mathbb{C}$. Given $X \in \mathrm{End}(V_\mathbb{C})$, let $X^{p,q}$ denote the component taking value in $\mathrm{End}(V_\mathbb{C})^{p,q}_{W,F}$. Recalling the notation of §A.5, we have

$$(\log\tilde g(t,w))^{-1,q} \ = \ \tilde g(t,w)^{-1,q}\,,$$

and

$$\begin{aligned}(X\circ\widetilde\Phi)(t,w)^{-1,-1} \ &= \ \textstyle\sum_{i=1}^k \ell(t_i)N_i \ + \ \tilde g(t,w)^{-1,-1}\\ (X\circ\widetilde\Phi)(t,w)^{-1,q} \ &= \ \tilde g(t,w)^{-1,q}\,, \quad q \ne -1\,.\end{aligned}$$

We say

$$(X\circ\widetilde\Phi)^{-1,\bullet} \ = \ \textstyle\sum\,(X\circ\widetilde\Phi)^{-1,q}$$

is the *horizontal component of the* (logarithm of the) *period matrix*.

In general, the function $\tilde{X} : \tilde{\mathcal{U}} \to \mathfrak{f}^\perp$ defined by

$$\tilde{X}(t,w) \;=\; X \circ \widetilde{\Phi}(t,w) \;-\; \textstyle\sum \ell(t_i)\, N_i$$

is well-defined on $\widetilde{\mathcal{U}}$, but multi-valued over $\mathcal{U}$. But the discussion above implies

$$\tilde{X}^{-1,\bullet}(t,w) \;\in\; \mathcal{O}(\overline{\mathcal{U}})\,. \tag{A.10}$$

A.6. **Extension data.** The fibre $\delta_I = \delta_{I,F}$ of $D_I \twoheadrightarrow D_I^0$ through $F \in D_I$ is the set of $\tilde{F} \in D_I$ inducing the same pure, weight ℓ Hodge filtrations on the $H^{n-a}(-a)$ as F. It is a complex affine space. To see this, first note that $\delta^1_{I,F} = C^{-1}_{I,\mathbb{C}} \cdot F$. As a unipotent group $C^{-1}_{I,\mathbb{C}} = \exp(\mathfrak{c}^{-1}_{I,\mathbb{C}})$ is biholomorphic to its Lie algebra $\mathfrak{c}^{-1}_{I,\mathbb{C}}$. The Lie algebra of $C^{-a}_{I,\mathbb{C}}$ is

$$\mathfrak{c}^{-a}_{I,\mathbb{C}} \;=\; \bigoplus_{p+q\le -a} \mathfrak{c}^{p,q}_{I,F}\,. \tag{A.11}$$

Since

$$\mathfrak{c}^{-1}_{I,\mathbb{C}} \;=\; \left(\mathfrak{c}^{-1}_{I,\mathbb{C}} \cap \mathfrak{f}\right) \;\oplus\; \left(\mathfrak{c}^{-1}_{I,\mathbb{C}} \cap \mathfrak{f}^\perp\right)$$

with

$$\mathfrak{c}^{-1}_{I,\mathbb{C}} \cap \mathfrak{f} \;=\; \bigoplus_{\substack{p\ge 0\\ p+q\le -1}} \mathfrak{c}^{p,q}_{I,F}\,.$$

the stabilizer F in $\mathfrak{c}^{-1}_{I,\mathbb{C}}$ and

$$\mathfrak{c}^{-1}_{I,\mathbb{C}} \cap \mathfrak{f}^\perp \;=\; \bigoplus_{\substack{p< 0\\ p+q\le -1}} \mathfrak{c}^{p,q}_{I,F}\,,$$

we see that

$$\delta^1_{I,F} \;=\; \exp(\mathfrak{c}^{-1}_{\sigma,\mathbb{C}} \cap \mathfrak{f}^\perp) \cdot F\,,$$

and the map $\mathfrak{c}^{-1}_{\sigma,\mathbb{C}} \cap \mathfrak{f}^\perp \to \delta^1_{I,F}$ is a biholomorphism.

Likewise, $\mathbb{C}\sigma_I \subset \mathfrak{g}^{-1,-1}_{W,F}$ is an abelian ideal of the nilpotent algebra $\mathfrak{c}^{-1}_{I,\mathbb{C}} \cap \mathfrak{f}^\perp$, and we have a well-defined induced biholomorphism

$$\frac{\mathfrak{c}^{-1}_{I,\mathbb{C}} \cap \mathfrak{f}^\perp}{\mathbb{C}\sigma_I} \;\xrightarrow{\simeq}\; \exp(\mathbb{C}\sigma_I)\backslash \delta_{I,F}\,.$$

Appendix B. Compatibility of weight closures

The purpose of this section is to review compatibility properties between the weight filtrations $W^I = W(\sigma_I)$, and discuss some of the implications for local lifts of period maps. These local results will have global consequences, including the following corollary of Lemma B.20.

Lemma B.1. *The maps*

$$\begin{array}{ccc} Z_I^* & \xrightarrow{\Phi_I^1} & \Gamma_I\backslash D_I^1 \\ & \searrow^{\Phi_I^0} & \downarrow \\ & & \Gamma_I\backslash D_I^0 \end{array}$$

of (2.1) *extend to proper holomorphic maps on $Z_I \cap Z_\pi$. These extensions are compatible with the Φ_J^0 and Φ_J^1 on $Z_J^* \subset Z_I \cap Z_\pi$ in the sense that the we have a commutative diagram*

$$\tag{B.2} \begin{array}{ccc} Z_J^* & \hookrightarrow & Z_I \cap Z_\pi \\ \downarrow \Phi_J^1 & & \Phi_I^1 \downarrow \\ \Gamma_J\backslash D_J^1 & \longrightarrow & \Gamma_I\backslash D_I^1 \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ \Gamma_J\backslash D_J^0 & \longrightarrow & \Gamma_I\backslash D_I^0 \,. \end{array} \qquad (\Phi_J^0 : Z_J^* \to \Gamma_J\backslash D_J^0,\ \Phi_I^0 : Z_I \cap Z_\pi \to \Gamma_I\backslash D_I^0)$$

Lemma B.1 is a corollary of Lemma B.16.

B.1. **The commuting $\mathfrak{sl}(2)$'s.** Our constructions are defined over the open strata Z_I^*. We will need to see that these strata-wise constructions satisfying certain compatibility conditions in order to obtain the properties asserted in the lemmas above. The key technical result here is the SL(2) orbit theorem [CKS86]. We briefly review the theorem, and then discuss consequences.

Suppose that $Z_J \subset Z_I$; equivalently, $I \subset J$. To begin we assume that we have a local coordinate chart centered at $b \in Z_J^*$ with local monodromy cone $\sigma = \sigma_J$ generated by $N_1, \ldots, N_k$ as in §A.2. Given $I \subset J = \{1, \ldots, k\}$, let σ_I be the face of σ_J generated by the N_i, with $i \in I$. Define

$$N_I \ = \ \sum_{i \in I} N_i \quad \text{and} \quad N_J \ = \ \sum_{j \in J} N_j \,.$$

Given this data, the SL(2) orbit theorem [CKS86] produces "commuting $\mathfrak{sl}_2$–pairs

$$N_I, Y_I \,;\ \hat{N}_J, \hat{Y}_J \ \in \ \mathfrak{g}_\mathbb{R} \,.$$

These pairs have following properties: N_I and Y_I commute with $\hat{N}_J$ and $\hat{Y}_J$; and there is a $(Y_I, \hat{Y}_J)$–eigenspace decomposition $\mathfrak{g}_\mathbb{C} = \oplus\, \mathfrak{g}_{a,b}$,

$$\mathfrak{g}_{a,b} \ = \ \{\xi \in \mathfrak{g}_\mathbb{C} \mid [Y_I, \xi] = a\xi \,,\ [\hat{Y}_J, \xi] = b\xi\} \,,$$

with integer eigenvalues a, b that splits the weight filtrations

$$\tag{B.3} W_\ell^I(\mathfrak{g}_\mathbb{C}) \ = \ \bigoplus_{a \le \ell} \mathfrak{g}_{a,b} \quad \text{and} \quad W_\ell^J(\mathfrak{g}_\mathbb{C}) \ = \ \bigoplus_{a+b \le \ell} \mathfrak{g}_{a,b} \,.$$

We have

$$N_I \ \in \ \mathfrak{g}_{-2,0}$$

and

$$N_J \;\in\; \bigoplus_{a\le 0} \mathfrak{g}_{a,-a-2}\,.$$

If we write

(B.4a) $$N_J \;=\; \sum_{a\le 0} N_{J,a}\,,$$

with $N_{J,a} \in \mathfrak{g}_{a,-a-2}$, then

(B.4b) $$N_{J,0} \;=\; \hat{N}_J\,.$$

B.2. **Consequences for the local lifts of Φ^ϵ.** Recall the two maps $F_I^\epsilon : Z_I^* \cap \overline{\mathcal{U}} \to D_I^\epsilon$ of (A.5). Since $I \subset J$, we have $Z_J^* \subset Z_I$. Fix a coordinate neighborhood $(t,w) \in \overline{\mathcal{U}} \subset \overline{B}$ so that $Z_J^* \cap \overline{\mathcal{U}} = \{t = 0\}$.

Lemma B.5. *Suppose that (t_m, w_m) and (t'_m, w'_m) are two sequences in $Z_I^* \cap \overline{\mathcal{U}}$ converging to points $(0, w_\infty)$ and $(0, w'_\infty) \in Z_J^* \cap \overline{\mathcal{U}}$, respectively. If $F_I^\epsilon(t_m, w_m) = F_I^\epsilon(t'_m, w'_m)$ for all m, then $F_J^\epsilon(0,w) = F_J^\epsilon(0,w')$.*

This lemma is the analog of Theorem 2.20 for the "local lift" of Φ^ϵ, and implies that this lift is continuous.

Proof. Given $(t,w) \in Z_I^* \cap \overline{\mathcal{U}}$, recall that $F_I^\epsilon(t,w)$ is the composition of $F_I(t,w) = \tilde{g}(t,w)\cdot F$ with the projection $D_I \twoheadrightarrow D_I^\epsilon = C_{I,\mathbb{C}}^{-\epsilon-1}\backslash D_I$ (§A.2.4). Moreover, $\tilde{g}(t,w)$ is holomorphic (and therefore continuous) on Δ^{k+r}, and takes value in $C_{I,\mathbb{C}}$ when restricted to $Z_I^* \cap \overline{\mathcal{U}}$. So to prove the lemma, it suffices to show that

(B.6) $$W_\ell^I(\mathfrak{c}_J) \;\subset\; W_\ell^J(\mathfrak{c}_J)\,.$$

It is a general fact that the centralizers satisfy

$$\mathfrak{c}_J \;\subset\; W_0^J(\mathfrak{g}_\mathbb{C}) \quad\text{and}\quad \mathfrak{c}_J \;\subset \mathfrak{c}_I \;\subset\; W_0^I(\mathfrak{g}_\mathbb{C})\,.$$

So (B.3) implies

(B.7) $$\mathfrak{c}_J \;\subset\; \bigoplus_{\substack{a\,\le\,0\\ a+b\,\le\,0}} \mathfrak{g}_{a,b}\,.$$

Note that (B.7) implies the desired (B.6) for $\ell \ge 0$.

Suppose that $X \in W_\ell^I(\mathfrak{c}_J)$ for some $\ell < 0$. Then (B.3) and (B.7) imply that there exists unique $X_{a,b} \in \mathfrak{g}_\mathbb{C}$ so that

$$X \;=\; \sum_{\substack{a\,\le\,\ell\\ a+b\,\le\,0}} X_{a,b}\,.$$

In order to establish (B.6), we need to show

(B.8) $$X_{a,b} \;=\; 0 \quad\text{for all}\;\; a+b > \ell\,.$$

From $N_J(X) = 0$ and (B.4) we see that $\hat{N}_J(X_{\ell,b}) = 0$. Since $\{\hat{N}_J, \hat{Y}_J\}$ is an $\mathfrak{sl}_2$–pair, the centralizer $\mathfrak{c}(\hat{N}_J)$ of $\hat{N}_J$ satisfies

$$\mathfrak{c}(\hat{N}_J) \ \subset \ \bigoplus_{b\le 0} \mathfrak{g}_{a,b} \,. \tag{B.9}$$

This forces $X_{\ell,b} = 0$ for all $b > 0$, and yields the desired (B.8) for $a = \ell$.

Working inductively, fix $m < \ell < 0$ and assume that (B.8) holds for all $m < a \le \ell$. Again, $N_J(X) = 0$ and (B.4) implies $\hat{N}_J(X_{m,b}) = 0$ for all $m + b > \ell$. Since, $b > \ell - m > 0$, (B.9) implies $X_{m,b} = 0$ for all $m + b > \ell$. This establishes the desired (B.8) for $a = m$ and completes the induction. □

B.3. **When weight filtrations coincide.** The properties (B.3) and (B.4b) yield

Lemma B.10. *Suppose that $I \subset J$. The following are equivalent:*

(i) *The weight filtrations coincide $W^I = W^J$.*
(ii) *We have $\hat{Y}_J = 0$.*
(iii) *We have $\hat{N}_J = 0$.*
(iv) *The cone $\sigma_J \subset \mathfrak{c}_I^{-1}$.*

Corollary B.11. (a) *If $I \subset I' \subset J$ and $W^I = W^J$, then $W^I = W^{I'} = W^J$.*
(b) *If $W^{I_1} = W^{I_2}$, then $W^{I_i} = W^{I_1 \cup I_2}$.*
(c) *The union*

$$I_W \ = \ \bigcup_{W^I = W} I$$

is the unique maximal set I_W such that $W = W^{I_W}$.

If $W^I = W^J$, then $\mathfrak{g}_{a,\bullet} = \mathfrak{g}_{a,0}$ implies

$$\mathfrak{c}_J^{-a} \ \subset \ \mathfrak{c}_I^{-a} \,, \tag{B.12a}$$

and

$$\frac{\mathfrak{c}_J^{-a}}{\mathfrak{c}_J^{-a-1}} \ \hookrightarrow \ \frac{\mathfrak{c}_I^{-a}}{\mathfrak{c}_I^{-a-1}} \,. \tag{B.12b}$$

In the case $a = 1$, the inclusion (B.12a) yields the striking implication (known to the experts)

Lemma B.13. *If $\sigma_J \subset \mathfrak{c}_I^{-1}$, then $\sigma_J \subset \mathfrak{c}_I^{-2}$.*

Corollary B.14. *We have* $\exp(\mathbb{C}\sigma_{I_W}) \subset C_{I,\mathbb{C}}^{-2}$.

B.4. **Consequences for LMHS.** Note that $Z_J^* \subset Z_I$ if and only if $I \subset J$. In this case, $\Gamma_J \subset \Gamma_I$. We will also see that $D_J \subset D_I$, cf. (B.19). In particular, we have an induced $\Gamma_J \backslash D_J \to \Gamma_I \backslash D_I$. When $W^I = W^J$, then this map descends to $\Gamma_J \backslash D_J^a \to \Gamma_I \backslash D_I^a$.

Recall (§A.2.4) that the local lift of $\Phi_I : Z_I \to (\Gamma_I \exp(\mathbb{C}\sigma_I))\backslash D_I$ is

$$\nu_I \circ F_I : Z_I^* \cap \overline{\mathcal{U}} \to \exp(\mathbb{C}\sigma_I)\backslash D_I \,. \tag{B.15}$$

Define

$$Z_W \;=\; \bigcup_{W^I = W} Z_I^* \,.$$

Lemma B.16. *There is a well-defined holomorphic map*

$$\widetilde{\Phi}_I : Z_I \cap Z_W \cap \overline{\mathcal{U}} \;\to\; \exp(\mathbb{C}\sigma_{I_W})\backslash D_I \tag{B.17}$$

that, when restricted to $Z_J^* \subset Z_I \cap Z_W$, coincides with the composition $\nu_{I_W} \circ F_J$.

Proof of Lemma B.1. Given $\epsilon = 0, 1$, Corollary B.14 implies that

$$(\exp(\mathbb{C}\sigma_{I_W}) C_{I,\mathbb{C}}^{-\epsilon-1})\backslash D_I \;=\; C_{I,\mathbb{C}}^{-\epsilon-1}\backslash D_I \;=\; D_I^\epsilon \,.$$

So the composition

$$Z_I \cap Z_W \cap \overline{\mathcal{U}} \xrightarrow{\widetilde{\Phi}_I} \exp(\mathbb{C}\sigma_{I_W})\backslash D_I \longrightarrow\!\!\!\!\to (\exp(\mathbb{C}\sigma_{I_W}) C_{I,\mathbb{C}}^{-\epsilon-1})\backslash D_I \;=\; D_I^\epsilon$$

is the local coordinate expression for the extension $\Phi_I^\epsilon : Z_I \cap Z_W \to \Gamma_I \backslash D_I^\epsilon$ of (B.2). Thus Lemma B.1 follows directly from Lemma B.16. □

Proof of Lemma B.16. Suppose that $I \subset J$ and $W^I = W^J$. Consider a local lift $\widetilde{\Phi}(t, w)$ over a chart $\overline{\mathcal{U}}$ centered at $b \in Z_J^*$ (as in §A.2). Along

$$Z_J \cap \overline{\mathcal{U}} \;=\; \{t_j = 0 \;\forall\; j \in J\} \;=\; \{0\} \times \Delta^r \;\ni\; (0, w)$$

we have the map $F_J : Z_J^* \cap \overline{\mathcal{U}} \to D_J$ of (A.4)

$$F_J(w) \;=\; \tilde{g}(0, w) \cdot F \,. \tag{B.18a}$$

Along $Z_I^* \cap \overline{\mathcal{U}} = \{t_i = 0 \text{ iff } i \in I\}$ we may choose a well-defined branch of $\ell(t_j)$ for all $j \in J\backslash I$. Then the map $F_I : Z_I^* \cap \overline{\mathcal{U}} \to D_I$ is given by

$$F_I(t, w) \;=\; \exp\left(\sum_{j \in J\backslash I} \ell(t_j) N_j \right) \tilde{g}(t, w) \cdot F \,. \tag{B.18b}$$

Comparing the expressions (B.18) for F_J and F_I, and keeping $C_J \subset C_I$ and (B.12a) in mind, we see that

$$F \;\in\; D_J \;\subset\; D_I \tag{B.19}$$

and F_J takes value in D_I. (Note that the containment $F \in D_I$ is nontrivial, as F arises from the LMHS along Z_J^*.) It follows from (B.18) and (B.19) that

$$\nu_J \circ F_J : Z_J^* \cap \overline{\mathcal{U}} \ \to \ \exp(\mathbb{C}\sigma_J)\backslash D_I$$

also takes value in (a quotient of) D_I. The lemma now follows from (B.18). □

If follows from Corollary B.11(c) and (B.19) that the orbit

$$D_W \ = \ G_W \cdot F \ \supset \ D_I$$

is independent of our choice of D_I and $F \in D_I$ so long as $W^I = W$. It is straightforward to verify

Lemma B.20. *There is a well-defined holomorphic map*

$$\widetilde{\Phi}_W : Z_W \cap \overline{\mathcal{U}} \ \to \ \exp(\mathbb{C}\sigma_{I_W})\backslash D_W \tag{B.21}$$

that, when restricted to Z_I^*, coincides with $\nu_{I_W} \circ F_I$.

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