# Completions of Period Mappings

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## Outline

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I. Introduction Given the data $(\overline{B}, Z; \Phi)$ where

- $(\overline{B}, Z)$ is a pair consisting of a smooth projective variety $\overline{B}$ and $Z = \cup Z_i$ is a simple normal crossing subvariety of $\overline{B}$ with complement $B = \overline{B} \backslash Z$; and
- $\Phi$ is given equivalently by a period mapping

$$\Phi : B \to P \subset \Gamma \backslash D \tag{I.1}$$

with image $P$, or by a variation of polarized Hodge structure $(\mathcal{V}, \mathcal{F}, \nabla; B)$. We want to define and give some properties of natural completions

$$\overline{\Phi} : \overline{B} \to \overline{P} \tag{I.2}$$

of (I.1).[†]

[†]These notations will be explained below.

Informally stated among the results are

(i) There are two natural completions, a maximal one $\overline{P}^T$ and a minimal one $\overline{P}^S$. The latter is analogous to the Satake-Baily-Borel compactification $\overline{\Gamma\backslash D}^{\mathrm{SBB}}$ in the classical case when $D$ is a Hermitian symmetric domain and $\Gamma$ is an arithmetic group. The former is in some ways analogous to a toroidal compactification $\overline{\Gamma\backslash D}^{\mathrm{Tor}}$; it provides a potential candidate for the general definition of a toroidal completion $\overline{P}^T$ of the image $P$ of a period mapping (I.1).‡

‡In [CCK79], [Cat84] for the classical case of algebraic curves there is a harbinger of some of the material in this talk.

(ii) Among the applications of the methods used to construct the completions of $\overline{P}$ are properties of line bundles associated to $(\overline{B}, Z; \Phi)$. There are many results, both classical and recent, concerning nefness and bigness of such line bundles. Frequently more subtle are those involving freeness. For these it is natural to make generic local Torelli assumptions as the general case can usually be reduced to this one. Still even more subtle are ampleness results; for these it is natural to make local Torelli assumptions that hold everywhere. Under a probably unnecessary mild technical assumption two sample applications are

- If generic local Torelli holds, then for sufficiently small $\epsilon > 0$ the line bundle

$$K_{\overline{B}} + (1+\epsilon)Z$$

is free; and

- If local Torelli holds, then $K_{\overline{B}} + Z$ is free, and there are necessary and sufficient geometrically expressed conditions that it be ample.

(iii) Other applications that we shall have time to only briefly illustrate are to the boundary structure of moduli spaces of general type algebraic varieties. The Hodge theoretic boundary strata of the minimal completion provides a guide to how one may organize the stratification of the algebro-geometric boundary; the mapping $\overline{P}^T \to \overline{P}^S$ suggests how one might desingularize it.

We emphasize that in (i) the constructions

$$(\overline{B}, Z; \Phi) \rightsquigarrow \overline{P}^T, \overline{P}^S$$

are *relative*; given the data $(\overline{B}, Z; \Phi)$ completions are produced. This is in contrast to the usual process where for arithmetic $\Gamma$ various completions $\overline{\Gamma\backslash D}$ are constructed and, under certain conditions, a period mapping $\Phi : B \to \Gamma\backslash D$ is proved to extend to $\overline{\Phi} : \overline{B} \to \overline{\Gamma\backslash D}$. In the classical case there is [AMRTS10] and the references cited there.) In the non-classical case there is the work [KU09]. The somewhat different methods used here to construct $\overline{P}$'s and to prove results about them might be summarized as *the global study of the period mapping at infinity*.

The *global* study of period mapping (I.1) on the open quasi-projective variety $B$ is a long standing subject, one of current active research (cf. [BB20], [BKT13], [D20], [P20], [PLSZ19], [LSZ19], [Zou00] for a sample of recent work and extensive bibliographies). Typical results are that under local Torelli assumptions

- $(B, \overline{Z})$ is of log general type;
- $\Omega^1_{\overline{B}}(\log Z)$ is nef and big;
- hyperbolicity properties of $B$.

The *local* properties around a point of $Z$ is also classical ([CKS86], partially summarized in [GGLR20]). Of interest in their own right they also enter into the proofs of the results just mentioned. However it has become apparent that to obtain more refined conclusions about the data $(\overline{B}, Z; \Phi)$ one needs *global* information about the variation of the limiting mixed Hodge structures on $Z$, specifically the *global* behavior along the compact subvarieties of $Z$ where the associated graded pure Hodge structures remain locally constant. The key geometric result (Theorem C below) relates the variation of the extension data of the limiting mixed Hodge structure along these subvarieties to the geometry of the co-normal bundle $N^*_{Z/\overline{B}}$ restricted to them.

Finally we note that in [BBT20] there are techniques related to some of those discussed below.

# II. Preliminaries

A variation of Hodge structure $(\mathcal{V}, \mathcal{F}, \nabla; B)$ is given by

- a holomorphic vector bundle $\mathcal{V} \to B$ with integrable Gauss-Manin connection
$$\nabla : \mathcal{V} \to \Omega^1_B \otimes \mathcal{V};$$
- the kernel of $\nabla$ is a local system $\mathbb{V}_{\mathbb{C}} = \mathbb{V}_{\mathbb{Z}} \otimes \mathbb{C}$ whose monodromy preserves $\mathbb{V}_{\mathbb{Z}}$;
- the holomorphic vector bundle $\mathcal{F}$ has a filtration $\mathcal{F}^n \subset \mathcal{F}^{n-1} \subset \cdots \subset \mathcal{F}^0 = \mathcal{V}$ where
$$\nabla(\mathcal{F}^p) \subseteq \Omega^1_B \otimes \mathcal{F}^{p-1};$$
- not included in the notation is the existence of a bilinear form
$$Q : \mathbb{V}_{\mathbb{Z}} \otimes \mathbb{V}_{\mathbb{Z}} \to \mathbb{Z}$$
such that each $(\mathcal{V}_b, \mathcal{F}_b, Q)$ defines a polarized Hodge structure.

As usual, without essential loss of generality we may assume that the monodromy around the irreducible components $Z_i$ of $Z$ is unipotent. There are then canonical extensions, recalled in [GGLR20], $\mathcal{V}_e, \mathcal{F}_e$ of $\mathcal{V}, \mathcal{F}$ such that

$$\nabla : \mathcal{F}_e^p \to \Omega^1_{\overline{B}}(\log Z) \otimes \mathcal{F}_e^{p-1}.$$

Dualizing gives a bundle map

$$T_{\overline{B}}(-\log Z) \to F^{-1}\,\mathrm{End}(\mathcal{V}_e).$$

It is convenient to use the Higgs formalism where we set $\mathcal{E} = \mathrm{Gr}^{\mathcal{F}}\,\mathcal{V}$, $\mathcal{E}_e = \mathrm{Gr}^{\mathcal{F}_e}(\mathcal{V}_e)$ and then the above map induces

$$T_{\overline{B}}(-\log Z) \xrightarrow{\delta} F^{-1}\,\mathrm{End}(\mathcal{E}_e). \tag{II.1}$$

In the period mapping formulation we are given $(V_{\mathbb{Z}}, Q)$ where $V_{\mathbb{Z}}$ is a lattice and $Q : V_{\mathbb{Z}} \otimes V_{\mathbb{Z}} \to \mathbb{Z}$ a non-degenerate bilinear form. Setting $V = V_{\mathbb{Z}} \otimes \mathbb{Q}$ and given a set of Hodge numbers $h^{p,q}$ we denote by $D$ the period domain of polarized Hodge structures $(V, F)$ where $F$ is a filtration

$$F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}} \tag{II.2}$$

that together with $Q$ define a polarized Hodge structure of weight $n$ and where $h^{p,q} = \dim(F^p \cap \overline{F}^q)$. Setting $G = \mathrm{Aut}(V, Q)$ with $G_{\mathbb{R}}$ the corresponding real Lie group, it is known that $D$ is a homogeneous complex manifold that is an open $G_{\mathbb{R}}$-orbit in the compact dual $\check{D}$, which is the rational homogeneous projective variety consisting of all $F$'s in (II.2) satisfying the first Hodge-Riemann bilinear relation.

Upon choice of a reference point $b_0 \in B$, giving $(\mathcal{V}, \mathcal{F}, \nabla; B)$ is equivalent to giving a period mapping (I.1) where the monodromy group $\Gamma \subset G_{\mathbb{Z}}$ is the image of the monodromy representation

$$\pi_1(B, b_0) \to \mathrm{Aut}(V_{\mathbb{Z}}).$$

Using the period domain formulation allows one to apply Lie theory to the variation of Hodge structure. Before explaining how these methods apply to study the period mapping at infinity we recall a couple of further definitions.

A mixed Hodge structure $(V, W, F)$ is given by $(V, F)$ together with a weight filtration

$$W_0 \subset W_1 \subset \cdots \subset W_m = V$$

such that $F$ induces a Hodge structure on each $\mathrm{Gr}_k^W V = W_k / W_{k-1} := H^k$. The set of mixed Hodge structures with fixed $H^k$'s will be denoted by $E$.

We have maps

$$E = E_m \to E_{m-1} \to \cdots \to E_2 \to E_1$$

where $E_k$ are the $k$-fold extensions in $E$. Thus $E_1$ is the set of extensions

$$0 \to H^k \to W_k/W_{k-2} \to H^{k-1} \to 0,$$

and $E_2$ consists of these extensions together with the pairs of extensions

$$\begin{cases} 0 \to W_{k-1}/W_{k-3} \to W_k/W_{k-3} \to H^k \to 0 \\ 0 \to H^{k-2} \to W_k/W_{k-3} \to W_k/W_{k-2} \to 0. \end{cases}$$

We note that

$$E_1 \cong \overset{k}{\oplus} \mathrm{Ext}^1_{\mathrm{MHS}}(H^k, H^{k-1}) := J$$

where $J$ is a compact complex torus with tangent space a Hodge structure of weight $-1$ and whose $k^{\mathrm{th}}$ summand in the above direct sum decomposition has Hodge type

$$(k-1,-k)+\cdots+\underbrace{(0,-1)+(-1,0)}+\cdots+(-k,k-1).^{\S}$$

A fibre of $E_\ell \to E_{\ell-1}$ is $\overset{k}{\oplus} \mathrm{Ext}^1_{\mathrm{MHS}}(H^{k+\ell}, H^k)$; it is an extension of a compact complex torus by a product of $\mathbb{C}^*$'s. The tangent space is a Hodge structure of weight $-\ell$ that for $\ell \geqq 2$ has no summands of Hodge type $(a,b)$ with $|a-b| \leqq 1$.

§cf. [Car87]

Given a nilpotent endomorphism $N \in \mathrm{End}(V, Q)$ where $N^{m+1} = 0$ there is a unique weight filtration $W(N)$ satisfying

$$\begin{cases} N : W_k(N) \to W_{k-2}(N) \\ N^k : W_{m+k}(N) \xrightarrow{\sim} W_{m-k}(N) \end{cases} .$$

A limiting mixed Hodge structure is a mixed Hodge structure $(V, W(N), F)$ where

$$N : F^p \to F^{p-1}.$$

Using the $Q$ one shows that the associated graded is a direct sum of polarized Hodge structures.

More generally, giving commuting nilpotents $N_1, \dots, N_r$ that generate a cone $\sigma = \mathrm{span}_{\mathbb{Z}^+}\{N_1, \dots, N_r\}$ such that each $N \in \sigma$ gives a limiting mixed Hodge structure, the weight filtration $W(N)$ is independent of $N$ and will be denoted $W(\sigma)$. The induced polarizations on $\mathrm{Gr}^{W(\sigma)} V$ depend on the particular $N \in \sigma$.

The extension data associated to a limiting mixed Hodge structure has a geometric structure only partially present for general graded polarized mixed Hodge structures. Denoting by $\check{\sigma}$ the dual cone to $\sigma$ there is an inclusion

$$\check{\sigma} \hookrightarrow \mathrm{Pic} J.$$

Consequently for each $M \in \check{\sigma}$ we have a line bundle $L_M \to J$.

# III. Basic results

Given the data $(\overline{B}, Z; \Phi)$ from [CKS] it is known that to each point $b \in \overline{B}$ there is associated an equivalence class of limiting mixed Hodge structures $(V, W(\sigma_I), F_b)$. If $b \in B$, we take the polarized Hodge structure corresponding to $\Phi(b)$. If $b \in Z_I^*$, we define two limiting mixed Hodge structures $(W, W(\sigma_I), F_b)$ and $(V, W(\sigma_{I'}), F_b')$ to be equivalent if there exists $\gamma \in \Gamma$ and $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r$ such that

- $\sigma_{I'} = \gamma \sigma_I \gamma^{-1}$;
- $F_b' = \gamma \exp(\sum \lambda_i N_i) F_b$.[¶]

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[¶]This is the definition one expects. Only up to $\gamma$ can we identify the fibres of $\mathcal{V}$ with $V_{\mathbb{C}}$, and the Hodge filtration for the limiting mixed Hodge structure is only defined up to scaling by conormal vectors

$$\lambda \in N^*_{Z_I^*/\overline{B}}$$

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Definition: $\overline{P}^T$ is the set of equivalence classes of limiting mixed Hodge structure associated to $(\overline{B}, Z; \Phi)$, and

(III.1) $$\Phi^T : \overline{B} \to \overline{P}^T$$

is the obvious map.

Conjecture: *$\overline{P}^T$ is a projective algebraic variety.*

The normalization $\widehat{\overline{P}}^T$ can be defined and using Lie theory one may prove the

Theorem A: *$\widehat{\overline{P}}$ is a compact complex analytic variety*.

The set $\overline{P}^T$ is constructed to retain the maximal amount of Hodge-theoretic information associated to $(\overline{B}, Z; \Phi)$. At the opposite extreme we denote by $D_I$ the period domain (actually a Mumford-Tate domain) parametrizing Hodge structures of the type $(\mathrm{Gr}^{W(\sigma_I)} V, F)$ where $\sigma_I$ is a polarizing cone.

There are period mappings

$$\Phi_I : Z_I^* \to P_I \subset \Gamma_I \backslash D_I,$$

where, for $b \in Z_I^*$, $\Phi_I(b)$ is the associated graded to $(V, W(\sigma_I), F_b)$. Set

$$\overline{P}^S = \bigcup_I P_I$$

and define

$$\Phi^S : \overline{B} \to \overline{P}^S \tag{III.2}$$

by $\Phi^S\big|_{Z_I^*} = \Phi_I$. It is conjectured and proved in special cases (cf. [GGLR20]) that $\overline{P}^S$ is a projective algebraic variety over which the Hodge line bundle

$$L := (\det \mathcal{F}_e^n) \otimes (\det \mathcal{F}_e^{n-1}) \otimes \cdots \otimes \det \left(\mathcal{F}_e^{[(n+1)/2]}\right)$$

is ample.‖

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‖In this regard see [BBT20] and the related papers [BBT18] and [BKT18].

**Basic idea behind the study of the period mapping at infinity:** The period mapping at infinity is essentially the restriction of $\Phi^T$ to $Z$. We then have

$$\begin{array}{ccc} \Phi^T\big|_Z & \longrightarrow & \overline{P}^T \\ \big\downarrow & \overset{\Phi_Z}{\searrow} & \big\downarrow \\ \Phi^S\big|_Z & \longrightarrow & \overline{P}^S. \end{array}$$

With two major differences the composed mapping $\Phi_Z : Z \to \prod_I \Gamma_I \backslash D_I$ is like an ordinary period mapping (I.1). The first major difference is that along the fibres there is a non-trivial Hodge-theoretic information given by the variation of the extension data defined by limiting mixed Hodge structures. The second is that, as mentioned above and will be discussed in detail below, *the* key point is to relate this geometry along the fibre of $\Phi_Z$ to the geometry normal to those fibres in $Z$.

Our next step is to introduce spaces $\overline{P}^k$ that interpolate between $\overline{P}^T$ and $\overline{P}^S$. As a set

$$\overline{P}_k = \left\{ \begin{matrix} \text{set of extension data of level} \leqq k \text{ in the equivalence} \\ \text{classes of limiting mixed Hodge structures in } \overline{P}^T. \end{matrix} \right\}$$

There is a tower of maps

(III.3)

$$\begin{array}{ccccc}
 & & \overline{P}^{2n} = \overline{P}^T & & \Phi^{2n} = \Phi^T \\
 & \overset{\Phi^{2n}}{\nearrow} & \downarrow & & \\
 & & \overline{P}^{2n-1} & & \\
 & \nearrow & \downarrow & & \\
\overline{B} & & \vdots & & \\
 & \underset{\Phi^{1}}{\searrow} & \downarrow & & \\
 & & \overline{P}^{1} & & \\
 & \underset{\Phi^{0}}{\searrow} & \downarrow & & \\
 & & \overline{P}^{0} = \overline{P}^S & & \Phi^{0} = \Phi^S
\end{array}$$

We emphasize that these are maps of sets, not of varieties (except conjecturally the composite map $\overline{P}^T \to \overline{P}^S$ should be a map between algebraic varieties). To study the period mapping at infinity, i.e., the maps $\Phi^k$ along $Z$, we set

$$F(\Phi^k) = \text{typical fibre of } \Phi^k \text{ in } Z.$$

It is known ([GGLR]) that

(III.4) $\qquad F(\Phi^0)$ is a compact subvariety of $Z$.

The proof of this uses the Cattani-Kaplan-Schmid local analysis of the canonically extended variation of Hodge structure together with the global fact that ordinary period mappings extend across subvarieties around which there is finite monodromy.

In the following result we assume that a fibre $F(\Phi^0)$ of $\Phi^0$ is irreducible. It then has a Zariski open set in a minimal stratum $Z_I^*$.

**Theorem B:** (i) *The level 1 extension data map*

$$\Phi^1 : F(\Phi^0) \cap Z_I^* \to J$$

*extends to* $F(\Phi^0)$*.* (ii) *The image lies in a translate of an abelian subvariety* $J_{\mathrm{ab}} \subset J$*.* (iii) *For* $M \in \check{\sigma}_I$ *the line bundle* $L_M \to J$ *is positive on the image of* $F(\Phi^0)$*.*

The first statement is a special case of the general result that a holomorphic mapping $\alpha^0 : X^0 \to T$ from a Zariski open set $X^0$ of an irreducible algebraic variety $X$ to a compact complex torus $T$ extends to a holomorphic mapping $\alpha : X \to T$. The reason is that in the diagram

$$\begin{array}{ccc} H_1(X^0,\mathbb{Z}) & \longrightarrow & H_1(X,\mathbb{Z}) \\ & {\scriptstyle \alpha^0_*} \searrow \qquad \swarrow {\scriptstyle \alpha_*} & \\ & H_1(T,\mathbb{Z}). & \end{array}$$


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the dotted arrow fills in by a weight argument in mixed Hodge theory. The morphism $\alpha_*$ of mixed Hodge structures is then induced by a holomorphic mapping filling in the dotted arrow in the diagram

$$\begin{array}{ccc} X^0 \subset X & \longrightarrow & \mathrm{Alb}(X) \\ & \searrow \qquad \swarrow_{\alpha} & \\ & T. & \end{array}$$

The statement (ii) follows from the infinitesimal period relation that gives that the differential of the holomorphic mapping $F(\Phi^0) \to J$ lies in the $(0,-1) \oplus (-1,0)$ part of the tangent space to $J$ (this is the part over the bracket in the diagram above). The result (iii) is a consequence of the ampleness of $L_M \to J_{\mathrm{ab}}$ for $M \in \check{\sigma}_I$.

We now come to the

**Theorem C (main formula):** *Denoting by $\Phi^1_0$ the restriction of $\Phi^1$ to a fibre $F(\Phi^0)$, for $M \in \check{\sigma}$ we have*

$$(\text{III.5}) \qquad (\Phi^1_0)^* L_M = -\sum \langle M, N_k \rangle \, [Z_k]\Big|_{F(\Phi^0)}.$$

If as above we have a minimal inclusion $F(\Phi^0) \subset Z_I$, then for $J = \{j \notin I : Z_I \cap Z_j \neq \emptyset\}$, the sum in (III.5) is only over the $k \in I \cup J$.

To get a feeling for the formula, if $F(\Phi^0) \subset Z_I^*$ and the $N_i$, $i \in I$, are linearly independent, then letting $M$ range over the $N_i$ we conclude that

(III.6) If $\Phi^1\big|_{F(\Phi^0)}$ *is a finite-to-one mapping, then* $N_{Z_i/\overline{B}}\big|_{F(\Phi^0)}$ *is a negative line bundle.*

At the other extreme, in a fibre $F(\Phi^1)$ we have that as $\mathbb{Q}$-line bundles

$$N_{Z_i/\overline{B}}\Big|_{F(\Phi^1)} \cong \mathcal{O}_{F(\Phi^1)}.$$

In general the terms in (III.5) with $i \in I$ have opposite signs to those with $j \in J$. This subtle interplay is a key phenomenon. The point is that the main formula (III.5) relates the geometry along the fibres of the maps in (III.3) to the geometry normal to those fibres in $\overline{B}$.

Turning to higher level extension data we have

Theorem D: *$\Phi^2$ maps a $F(\Phi^1)$ to product of $\mathbb{C}^*$'s.*

The point here is again that using the infinitesimal period relation and the fact that the differential of the map $F(\Phi^1) \to$ (level 2 extension data) is a mapping between real vector spaces, the image of $TF(\Phi^1)$ lies in the Hodge classes part of the weight $-2$ Hodge structure $T(\overset{k}{\oplus} \mathrm{Ext}^1_{\mathrm{MHS}}(H^{k+2}, H^k))$.

A further use of the infinitesimal period relation is given by the

**Theorem E:** *For $k \geqq 3$ the fibres in* (III.3) *of* $\overline{P}^k \to \overline{P}^{k-1}$ *are finite.*

In this case the differential of the map $F(\Phi^{k-1}) \to$ (level $k$ extension data) is zero.

In the classical case when $D$ is Hermitian symmetric one may show that $n = 2$ (any such $D$ may be equivalently embedded in a Siegel generalized upper-half-plane).

In the non-classical case in coordinates the fibres of $\overline{P}^{2n} \to \overline{P}^2$ may be expressed as periods of integrals of polylogarithmic type and give constants that resemble those that arise in the expressions for multi-zeta functions. This is perhaps not so surprising as such numbers arise as periods of mixed Tate motives that express the extension data of particular mixed Hodge structures of Hodge-Tate type in terms of special values of multi-zeta functions. There may well be a very nice story here. In particular the variations of graded polarized mixed Hodge structures of Hodge Tate that arise in moduli will be variations of *limiting* mixed Hodge structures and using Theorem C these will have what is turning out to be rich additional structure.

The general picture that emerges of $\overline{P}^T$ is

- there is a stratification of $\overline{P}^T$ derived from the stratification of $Z$ and that maps to a Hodge-theoretically defined stratification of $\overline{P}^S$;

- the open strata in this stratification of $\overline{P}^T$ are families of subvarieties of semi-abelian varieties, and in the non-classical case at a point of each strata there is a finite set of additional information expressed in coordinates by iterated integrals of polylogarithmic type;**

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**The intrinsic explanation for this is that using Lie theory the period matrices of the extended variation of Hodge structure around a point of $Z_I$ may be put in block form whose entries are polynomials in the log $t_i$ and with holomorphic coefficients. Using the infinitesimal period relation the derivatives of the higher degree polynomials in the log $t_i$'s may be expressed in terms of the ones that are linear in them. Since one half of the level of the extension data in the limiting mixed Hodge structure corresponds to the degree of these polynomials (this can be made precise) the extension data of levels $\geqq 3$ can be obtained by iterated integrals of level $\leqq 2$ extension data in which the log $t_i$'s appear linearly.

# IV. Applications

We shall give two types of results that use the global analysis of the period mapping at infinity. One of these concerns the finer properties of freeness and ampleness of line bundles associated to $(\overline{B}, Z; \Phi)$. We shall state these under a simplifying technical assumption that helps to isolate the essential geometric content of the results. The second will be a very brief discussion of an application to moduli of general type surfaces.

For the first, among the properties that a line bundle $L \to X$ over a compact complex manifold may have are

$$\left.\begin{array}{l} L \to X \text{ is nef} \\ L \to X \text{ is big} \end{array}\right\}$$

$$\left.\begin{array}{l} L \to X \text{ if free} \\ L \to X \text{ is ample} \end{array}\right\}.$$

The first pair are of a numerical character; the second are geometric. For the data $(\overline{B}, Z; \Phi)$, arising from the sign and singularity properties of the Chern forms of the canonically extended Hodge bundles there are numerous results of the first type (some sample references were given above). For example, assuming generic local Torelli that $\delta$ in (II.1) is injective at a general point

(IV.1)

- $K_{\overline{B}}$ is of log general type;
- $\Omega^1_{\overline{B}}(\log Z)$ is big (and there are related hyperbolicity results);
- the Hodge line bundle $L \to \overline{B}$ is nef and big.

The proofs of these results use the *global* geometry of the period mapping (I.1) on $B$ and the *local* analysis of the variation of Hodge structure around the points of $Z$. One might hope that having an understanding of the *global* geometry of the period mapping at infinity, i.e., on all of $\overline{B}$, could lead to more refined results of the second type above.

The technical assumption we shall make is that the mappings

$$\Phi_I : Z_I^* \to P_I \subset \Gamma_I \backslash D_I \tag{IV.2}$$

are fibrations. In general one probably needs to further stratify $Z_I$ so that this fibration property holds along the strata and that the pieces fit together properly.

The next property is a conjecture about which we can prove special cases, and where the main formula (Theorem C) may be used to reduce the desired result to a combinatorial property of monodromy cones.

Conjecture F: *Assume that generic local Torelli holds and that the restriction of $\Phi_1$ to the fibres of $\Phi_0$ is an immersion. Then there are $a_i \in \mathbb{Q}^{\geqq 0}$ such that the $\mathbb{Q}$ line bundle $L_e - \sum a_i Z_i$ is free.*

We remark that in general we cannot take the $a_i$ to be equal. This will be discussed below.

Without the local Torelli assumption (which is convenient but not essential) and the assumption about $\Phi_1$ on the fibres of $\Phi_0$, but with the assumption that $\Gamma$ is arithmetic, it is proved in [BBT] that

(IV.3) *$P$ has an algebraic structure and $L \to P$ is ample.*[††]

Denoting by $L_e(*Z)$ the sheaf of holomorphic sections of $L_e \to \overline{B}$ that vanish to some order on $Z$, (IV.3) proves that $L_e(*Z)$ is free in the sense that there is a finite dimensional space of sections in $H^0(\overline{B}, L_e^{\otimes m}(*Z))$ that projectively embed $P = \Phi(B)$. Thus Conjecture F would give a stronger form of the result in [BBT20].

[††] The paper [BBT20] is especially noteworthy in that it makes use of the very interesting new technique of o-minimal structures (cf. [BBT18] and [BKT18]).

To explain the combinatorial statement that would give a proof of Conjecture F, using the fibration assumption and the base point free theorem from birational geometry the result is reduced to showing that for an irreducible curve C

$$C \cdot L = 0 \implies C \cdot (\Sigma a_i Z_i) < o.$$

Denoting by $\sigma$ the cone $\mathrm{span}_{\mathbb{Q}^+}\{N_i\}$ and by $\mathrm{Eff}^1_L(\overline{B})$ the cone of effective 1-cycles $X$ satisfying $X \cdot L = 0$, there is a pairing

$$\Psi : \check{\sigma} \otimes \mathrm{Eff}^1_L(\overline{B}) \to \mathbb{Q}$$

given by

$$\Psi(M \otimes C) = -\sum_i \langle M, Z_i \rangle \, (C \cdot Z_i).$$

Then using Theorem C and the base-point-free theorem one needs to show

*There exists $M \in \check{\sigma}$ such that $\Psi(M, C) > 0$ for all curves $C$ with $C \cdot L = 0$.*

The first non-trivial case of this is when there exists a connected fibre $Z = Z_1 + Z_2$ of $\Phi_0$ where $Z_1 \cap Z_2 \neq \emptyset$, and there exist curves $C_1 \subset Z_1$, $C_2 \subset Z_2$ with $C_1 \cdot Z_2 \neq 0$, $C_2 \cdot Z_1 \neq 0$. Setting

$$e_{ij} = C_i \cdot Z_j$$

an argument using Theorem C gives $e_{11} < 0$, $e_{22} < 0$; our assumption gives $e_{12} > 0$, $e_{21} > 0$. Elementary reasoning then shows that we may choose $a_1, a_2 > 0$ such that

$$C \cdot (a_1 Z_1 + a_2 Z_2) < 0.$$

However it is in general not possible to choose $a_1 = a_2$.

We note in passing that Hodge theory as in Theorem C gives

$$e_{11}e_{22} - e_{12}e_{21} > 0;$$

this Hodge index-like property suggests an indication of the nature of the conjecture.

For the next result, keeping the assumption (IV.2) we have

**Theorem G:** *$K_{\overline{B}} + Z$ is nef, and for rational $\epsilon > 0$ sufficiently small the $\mathbb{Q}$-line bundle*

$$K_{\overline{B}} + (1+\epsilon)Z$$

*is free.*

We suspect that the assumption (IV.2) is not necessary and that that the result holds when $\epsilon = 0$.

For $\overline{B} = \overline{\Gamma\backslash D}^{\mathrm{Tor}}$ a toroidal compactification where $D$ is Hermitian symmetric and $\Gamma$ is arithmetic, there is a canonical ample "automorphic form" line bundle

$$L_e \to \overline{\Gamma\backslash D}^{\mathrm{SBB}}$$

over the Satake-Baily-Borel compactification of $\Gamma\backslash D$ ($L_e$ is the extended Hodge line bundle).

For $\overline{\Gamma\backslash D}^{\mathrm{Tor}} \xrightarrow{\pi} \overline{\Gamma\backslash D}^{\mathrm{SBB}}$ from [Mu77] we have

$$K_{\overline{B}} + Z = \pi^* \mathcal{O}(1). \tag{IV.4}$$

Thus $K_{\overline{B}} + Z$ is free but not ample; its $\mathrm{Proj}$ is $\overline{\Gamma\backslash D}^{\mathrm{SBB}}$.

For the next result, on the fibres $F(\Phi^0)$ the level 1 extension data gives a morphism

$$\Phi^1 : F(\Phi^0) \to J_{\mathrm{ab}}$$

from $F(\Phi^0)$ to an abelian variety. We assume that this mapping is an immersion and denote by $G(\Phi^1)$ the associated Gauss mapping to the Grasmannian of $d = \dim F(\Phi^0)$-planes in the tangent space $T_e J_{\mathrm{ab}}$.

Theorem H: *Assume local Torelli in the form that the mapping* (II.1) *is injective. Then* $K_{\overline{B}} + Z$ *is free, and it is ample if, and only if,* $G(\Phi^1)$ *is an immersion.*

The example (IV.4) when $B = \mathcal{A}_g$ shows that the assumption about the Gauss mapping is essential.

Assuming local Torelli, the Chern form $\omega$ of the Hodge line bundle gives a complete Kähler metric $h_\omega$ on $B$. It is known that the holomorphic sectional curvatures of $h_\omega$ are $\leqq -c$ for some constant $c > 0$, and also that the holomorphic bi-sectional curvatures are $\leqq 0$ and strictly negative on particular open sets (cf. [BKT13] and [GG20]). If $\dim B = d$ and $\Omega = \omega^d$ is the volume form associated to $\omega$, then the Ricci form $\mathrm{Ric}\,\Omega$ is strictly positive on $B$.

The proof of Theorem H requires some understanding of the singularities of $\mathrm{Ric}\,\Omega$ on $Z$, especially along the fibres of $\Phi^0$. This involves the second fundamental form of the sub-bundle

$$T_{\overline{B}}(-\log Z) \hookrightarrow F^{-1}\,\mathrm{End}(\mathcal{E}_e).$$

A second type of application is to moduli spaces $\mathcal{M}$ for varieties of general type $X$ with a given Hilbert polynomial $\overset{m}{\oplus}\,\chi(mK_X)$. For algebraic curves $\mathcal{M} = \mathcal{M}_g$ and the canonical compactification $\overline{\mathcal{M}}_g$ is a very beautiful and much studied space. It is essentially smooth, meaning that the Kuranishi space of a stable curve is reduced and of dimension $h^1(TX)$. Thus we may take our $\overline{B} = \overline{\mathcal{M}}_g$. The period mapping (I.1) is

$$\Phi : \mathcal{M}_g \to P \subset \Gamma\backslash D$$

where $D = \mathcal{H}_g$ and $\Gamma = \mathrm{Sp}(2g,\mathbb{Z})$.

There are various toroidal compactifications and for some, but *not* all, the period mapping extends to

$$\overline{\Phi} : \overline{\mathcal{M}}_g \to \overline{\Gamma\backslash D}^{\mathrm{Tor}}$$

(cf. [Nam80]). A reasonable supposition that we have not verified is that for the second Voronoi toroidal compactification of $\Gamma\backslash D$ the image $\overline{\Phi}(\overline{\mathcal{M}}_g)$ is the $\overline{P}^T$ of Theorem A (cf. [Cat84]).

When we come to higher dimensions, e.g., $X$ a smooth algebraic surface, the situation is very different. The moduli space $\mathcal{M}_X$ having a canonical projective completion $\overline{\mathcal{M}}_X$ parametrizing stable surfaces with the given Hilbert polynomial has been constructed by Kollár, Shepherd-Barron, Alexeev (cf. [CCK79] and [Kol13]). However the situation is quite unlike the curve case. Even when $\mathcal{M}_X$ may be essentially smooth, i.e., the Kuranishi space has dimension $h^1(T_X)$, $\overline{\mathcal{M}}_X$ may be quite singular along the boundary $\partial\mathcal{M}_X = \overline{\mathcal{M}}_X \backslash \mathcal{M}_X$.

A particular example here are the $I$-surfaces studied by Franciosi-Pardini-Rollenske (cf. [FPR15a], [FPR15b], [FPR17]). These are regular surfaces with $K_X^2 = 1$, $p_g(X) = 2$. The moduli space $\mathcal{M}_I$ has dimension $h^1(T_X) = 28$, and taking $B = \mathcal{M}_X$ the period mapping (I.1) is locally 1-1. However at least for normal Gorenstein $I$-surfaces $X_0$ corresponding to boundary points in $\overline{\mathcal{M}}_I \backslash \mathcal{M}_I$ the limit surface does not see the extension data in the limiting mixed Hodge structure in a specialization $X \to X_0$. This is explained in [Gri20], and in a joint project with FPR and GGLR we hope to explore the extent to which $\overline{P}^T$ desingularizes $\overline{\mathcal{M}}_X$. Informally stated, what is suggested is that the potential toroidal completion $\overline{P}^T$ may in some cases serve as a guide to how one desingularizes completed KSBA moduli spaces.

# Appendix: Elaboration on this example

Let $\mathcal{M}$ be a KSBA moduli space of surfaces of general type and with canonical completion $\overline{\mathcal{M}}$ where the boundary $\partial\mathcal{M} = \overline{\mathcal{M}} \backslash \mathcal{M}$ parametrizes surfaces $X$ with semi-log-canonical singularities (cf. [Kol13]). A part $\mathcal{N}_e$ of the boundary corresponds to normal Gorenstein $X$'s with simple elliptic singularities. In contrast to the case of algebraic curves where $\overline{\mathcal{M}}_g$ is essentially smooth, $\mathcal{N}_e \subset \partial\mathcal{M}$ is generally highly singular and consideration of the extension data in the LMHS suggests a natural desingularization of $\overline{\mathcal{M}}$ along $\mathcal{N}_e$. This phenomenon also extends to non-Gorenstein isolated singularities and to non-normal $X$'s as well.

Assume for simplicity that $\mathcal{M}$ is smooth and that a general point of $\mathcal{M}$ corresponds to a smooth regular surface. One may ask

- can Hodge theory suggest what surfaces $X$ appear on the boundary $\partial\mathcal{M} = \overline{\mathcal{M}}\backslash\mathcal{M}$?
- Can Hodge theory suggest how one might construct a desingularization $\widetilde{\overline{\mathcal{M}}}$ of $\overline{\mathcal{M}}$?

As noted above the answer to both questions is positive. For the first, a general limiting mixed Hodge structure has

$$N^2 = 0, \quad \mathrm{rank}\, N = 2.$$

Thus the LMHS has associated graded $(H^1, H^2, H^1(-1))$ where $H^1 = H^1(C)$ for a smooth elliptic curve $C$. Given a KSBA degeneration $\mathfrak{X} \to \Delta$ where $X_t$ is smooth for $t \neq 0$ and $X_0$ is a normal surface corresponding to a point of $\partial\mathcal{M}$, what is suggested is that $X_0 = (X, p)$ where $p$ is a simple elliptic singularity of a surface $X$ and where the resolution of that singularity is $(\widetilde{X}, C) \to (X, p)$ where $\widetilde{X}$ is smooth and $C \subset \widetilde{X}$ is an elliptic curve.

For the desingularization of $\overline{\mathcal{M}}$ one needs to do a semi-stable reduction

$$\widetilde{\mathcal{X}} \to \widetilde{\Delta}$$

of the family $\mathcal{X} \to \Delta$. Since $N^2 = 0$, Clemens-Schmid suggests that the central fibre $\widetilde{X}_0$ should have a double curve $C$ and no triple points. The simplest possibility is that

$$\widetilde{X}_0 = \widetilde{X} \bigcup_C Y$$

where $\widetilde{X}$ is as above and $Y$ is a smooth surface containing $C$. Since $C$ is a smooth elliptic curve we might try a smooth cubic $C \subset \mathbb{P}^2$. The normal bundle $N_{C/\widetilde{X}}$ has degree $d = -C^2$ while $N_{C/\mathbb{P}^2} \cong \mathcal{O}_C(3)$. To achieve the necessary condition

$$N_{C/\widetilde{X}} \cong \check{N}_{C/Y}$$

for smoothability, we must blow up $9-d$ points $p_i$ on $C$. Since for a smoothable elliptic singularity we have

$$1 \leqq d \leqq 9$$

so that $Y$ is a del Pezzo surface. Moreover from $\widetilde{X}_0$ as above we can construct the potential limiting mixed Hodge structure, and a standard computation gives that

$$\mathrm{Ext}^1_{\mathrm{MHS}}(H^1(-1), H^2)$$

contains a factor constructed from the subspace $\mathrm{Hg}^1(Y,\mathbb{Z})$ in $H^2$. It then follows that the information contained in the level 1 extension data in the LMHS is essentially the

$$\mathrm{AJ}_C(p_i - p_j) \in J(C).$$

This tells us which points $p_i$ on $C \subset \mathbb{P}^2$ to blow up to construct $Y$.

Of course the above is heuristic, but hopefully it does suggest the tight interplay between Hodge theory and geometry and illustrates how a geometric construction from a lMHS may be used to study moduli.

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