# Isolated Hypersurface Singularities*

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# Outline

# I.A. Introduction

- These notes, largely written in outline form, present aspects of the classical study of isolated singularities of the local hypersurface $V_0 \subset \mathbb{C}^{n+1}$ defined by

$$f(x_1, x_2, \ldots, x_{n+1}) = 0$$

  where $f$ is an analytic function defined in a neighborhood $\mathcal{U}$ of the origin $\{0\}$ in $\mathbb{C}^{n+1}$. With the exceptions to be noted below, the material in the lectures generally follows the presentations in the monographs [AGLV], [K], [M] and [D], where extensive additional references to the literature up until about 2000 may be found.
- There are some differences in the presentation here and the treatments in the literature. The major one is that the basic structure theory of the map $f : \mathcal{U} \to \mathbb{C}$ and its generic perturbations are here based on two classical topics:

- One is the general theory of finite holomorphic mappings, applied here to the Gauss mapping of the non-singular hypersurface $X \subset \mathbb{C}^{n+2}$ given by the graph of the mapping $f(x) = t$. The basic observation is that

  > *the condition that this mapping have an isolated critical point at the origin is equivalent to the condition that Gauss mapping give a finite holomorphic mapping in a neighborhood of* $(\{0\}, f(0))$.

- With this observation in hand, the methods of Picard and Lefschetz, as given e.g. in the classic [L], may be adapted to the isolated hypersurface singularity situation. This leads directly to the basic results about the Milnor fibration and its generic perturbations, the Milnor number etc. appearing here as local versions of the global methods of Lefschetz.

- A second point, one that is more one of emphasis rather than one of substance, is that the approach taken here will be basically analytic Much of the early understanding of the topology of algebraic varieties was motivated by questions in analysis and many of the results were either proved, or heuristic arguments given for, by analytic methods. This classical approach evolved in modern algebraic geometry to the topic of Hodge theory. One motivation for the approach in these notes is that there is a local version of Hodge theory due to Steenbrink and others (cf. the history and literature given in the references) associated to the study of isolated (and other more general) singularities. This will be an integral part of some work currently in progress applying Hodge theory to moduli.

There wasn't sufficient time in the lectures to get into this aspect; an appendix is included with these notes which gives a summary of some of the Hodge theory associated to an isolated hypersurface singularity.

As just noted above the use of Hodge theory in the study of singularities, e.g. those that arise in varieties that lie over the boundary of moduli spaces of surfaces of general type, is a very active current area of research and the possibility of using additional geometric/analytic techniques arising from the theory of isolated hypersurface singularities provided a background motivation for the approach taken in these lectures.

- In the remainder of the introduction we will give a list of notations and informal statements of some of the basic results to be presented.

- In Lecture I, $\mathcal{U}$ will be an open neighborhood of the origin in $\mathbb{C}^m$ with coordinates $(z_1, \ldots, z_m)$ and $F = (F_1, \ldots, F_m) : \mathcal{U} \to \mathbb{C}^m$ will be a finite holomorphic mapping; $J_F(z) = \det \|F_{i,z_j}(z)\|$ denotes the Jacobian determinant; $\mathcal{O}_z = \mathcal{O}_{\mathcal{U},\{0\}}$ is the local ring at the origin with $I_F = \{F_1, \ldots, F_m\}$ the ideal in $\mathcal{O}_z$ generated by the $F_1, \ldots, F_m$. We will allow $\mathcal{U}$ to be shrunk as needed, so it really should be thought of as a germ of a neighborhood of the origin.

- In Lectures II and III we will use the notations
  - $f : \mathcal{U} \to B$ where $\mathcal{U}$ is a Stein neighborhood of the origin in $\mathbb{C}^{n+1}$ with coordinates $(x_1, \ldots, x_{n+1})$ and smooth boundary $\partial\mathcal{U}$; $B$ is a neighborhood of the origin in $\mathbb{C}$ with coordinate $t$. Again, $\mathcal{U}$ should be though of as a germ of a neighborhood of the origin.†
- We assume $f(x)$ has an isolated critical point at the origin and we set $\mathcal{O}_x = \mathcal{O}_{\mathcal{U},\{0\}}$, $J_f = \{f_{x_1}, \ldots, f_{x_{n+1}}\}$ the Jacobian ideal generated by $f_{i,x_j} = \partial f_i(x)/\partial x_j$ at $x = 0$, and
  - $Q_f = \mathcal{O}_x/J$ with $\dim Q_f = \mu$ being the Milnor number,
  - $\Omega_f = Q_f \otimes \Omega^{n+1}_{\mathcal{U},\{0\}}$;
  - $V = f^{-1}(t)$ for a general $t \in B' = B \backslash \{0\}$ (occasionally we shall write $V_t$ for $f^{-1}(t)$);

†We apologize for using the same letter $\mathcal{U}$ for an open set in $\mathbb{C}^m$ which is the domain of a finite holomorphic mapping, and for an open set in $\mathbb{C}^{n+1}$ which is the domain of an analytic function with an isolated critical point at the origin. As noted above, the first use of $\mathcal{U}$ will only be in Lecture I, while the second (and main) use of $\mathcal{U}$ will be in Lectures II and III.

- $V_0 = f^{-1}(0)$ has an isolated singular point $p$ at the origin;
- $\mathcal{U}' = f^{-1}(B')$ so that $f : \mathcal{U}' \to B'$ is a holomorphic fibration with smooth Stein fibres;
- $K = \partial V$ is a smooth, real $(2n-1)$-dimensional manifold, $K \cong \partial V_0$ as $C^\infty$ manifolds;
- $f : \partial\mathcal{U} \to B$ is a $C^\infty$ fibration; hence $\partial\mathcal{U}$ is diffeomorphic to $B \times K$.

That we may choose $\mathcal{U}$ and $B$ with these properties will be proved in the lectures.

- The classical example of an isolated singularity is when $p$ is an ordinary double point (ODP)

$$f(x) = x_1^2 + \cdots + x_{n+1}^2 + (\text{HOT} = \text{higher order terms})$$

(sometimes $f$ is said to have a Morse-type singularity).

- In this case the reduced homology group $\widetilde{H}_p(V_t, \mathbb{Z})$ is zero for $p \neq n$, and for $\mathrm{Im}\, t = 0$, $\mathrm{Re}\, t > 0$; setting

$$S_t^n = \{x : f(x) = t\}$$

and

$$\delta_t \text{ the homology class of } S_t^n$$

one has that

$$H_n(V_t, \mathbb{Z}) \cong \mathbb{Z} \cdot \delta_t.$$

The standard term (due to Lefschetz) is that $\delta_t$ is a *vanishing cycle* (since $S_t^n$ shrinks to a point as $t \to 0$) that generates the homology group $H_n(V_t, \mathbb{Z})$.

The main results to be proved in the notes are

- a generic perturbation $f_\epsilon : \mathcal{U}_\epsilon \to B_\epsilon$ of $f$ by a small linear function has $\mu$ Morse-type critical points;

- by taking the limit as $\epsilon \to 0$ we will see that $V$ is homotopy equivalent to the wedge $VS^n$ of $\mu$ $n$-spheres that arise as limits of the $\mu$ vanishing cycles in the nearby perturbations of $V$; thus the reduced homology group $\widetilde{H}_p(V, \mathbb{Z}) = 0$ for $p \neq n$ while $H_n(V, \mathbb{Z}) \cong \mathbb{Z}^\mu$;
- for the monodromy

  $$T : H^n(V, \mathbb{Z}) \to H^n(V, \mathbb{Z})$$

  associated to the fibration $f : \mathcal{U}' \to B'$ and with the Jordan decomposition

  $$T = T_s T_u, \quad T_s T_u = T_u T_s,$$

  the semi-simple part $T_s$ of $T$ has eigenvalues equal to roots of unity,[‡] and the unipotent part $T_u = e^N$ has Jordan blocks of length $\leqq n+1$ (i.e., $N^{n+2} = 0$).

[‡]Thus the characteristic polynomial of $T$ is a product of cyclotromic polynomials. The suitably labelled eigenvalues of $T_s$ will be part of the *spectrum* of $T$.

- Finally we define a bilinear form

$$( \ , \ ) : H_n(V, \mathbb{Z}) \otimes H_n(V, \mathbb{Z}) \to \mathbb{Z}$$

by combining
  - Poincaré-Lefschetz duality $H_n(V, \mathbb{Z}) \cong H^n(V, K; \mathbb{Z})$
  - $H^n(V, K; \mathbb{Z}) \to H^n(V, \mathbb{Z})$;
  - $H_n(V, \mathbb{Z}) \otimes H^n(V, K; \mathbb{Z}) \to H_n(V, \mathbb{Z}) \otimes H^n(V, \mathbb{Z}) \to \mathbb{Z}$.

This bilinear form satisfies

$$(\alpha, \beta) = (-1)^n(\beta, \alpha)$$

and is preserved by monodromy.

For the vanishing cycle $\delta \in H_n(V, \mathbb{Z})$ in the ODP case we have

$$(\delta, \delta) = \begin{cases} 0 & n \equiv 1 \pmod 2 \\ 2 & n \equiv 0 \pmod 4 \\ -2 & n \equiv 2 \pmod 4. \end{cases}$$

In general for $\gamma \in H_n(V, \mathbb{Z})$ there is the Picard-Lefschetz formula

$$T\gamma = \gamma + \epsilon_n(\gamma, \delta)\delta \text{ where } \epsilon_n = (-1)^{n(n-1)/2}.$$

# I.B. *Finite holomorphic mappings*[§]

- With

$$\begin{array}{c} \mathbb{C}^m \\ \cup \\ F : \mathcal{U} \to \mathbb{C}^m \end{array}$$

giving a holomorphic mapping

$$(*) \qquad w = F(z)$$

where $z = (z_1, \ldots, z_m)$, $w = (w_1, \ldots, w_m)$, $F = (F_1, \ldots, F_m)$, and where we we assume that set-theoretically

$$F^{-1}(0) = \{0\},$$

we have a finite holomorphic mapping.

[§]This material is taken from Chapter V in [GH])

- The basic results about such maps are
  - $F$ is an open mapping (not like $(u,v) \to (u,uv)$); choosing $W$ to be a small ball around the origin in $\mathbb{C}^m$ and $\mathcal{U} = F^{-1}(W)$ it follows that $F : \mathcal{U} \to W$ is a finite branched covering with branch locus a divisor $D \subset W$;
  - $F^{-1}(w) = \sum n_\alpha z_\alpha(w)$, where the $F(z_\alpha(w)) = w$ are the finitely many solutions to $(*)$ in $\mathcal{U}$; $n_\alpha$ is the *multiplicity* of $z_\alpha(w)$ as a solution to $(*)$.
  - $\det \|F_{i,z_j}(z)\| \not\equiv 0$ and if it is non-zero at $z = \{0\}$ the multiplicity at the origin is equal to 1; we will see that for $w \in W \backslash D$ all $n_\nu = 1$ for $F^{-1}(w) = \sum_\nu n_\nu z_\nu(w)$.
  - $d = \sum \nu_\alpha$ is independent of $w$; this is the *degree* of $F$; we will use residues to prove this.

- denoting by $\mathcal{O}_z, \mathcal{O}_w$ the local rings given by the germs of holomorphic functions at the origins, for $u \in \mathcal{O}_z$

$$\begin{aligned} H(z) =: \prod_{\nu=1}^{d} \big(u(z) - u(z_\nu(F(z)))\big) \\ &= u(z)^d + a_1(w)u(z)^{d-1} + \cdots + a_d(w), \quad a_i \in \mathcal{O}_w \\ &\equiv 0.^{\P} \end{aligned}$$

Thus

$$\begin{cases} [\mathcal{O}_z : \mathcal{O}_w] = d \\ u \in \mathfrak{m}_z \implies u^d \in (F_1, \ldots, F_m) = I_F. \end{cases}$$

The proof of this last statement (which is the nullstellensatz in this case) goes as follows:

$$u \in \mathfrak{m}_z \implies a_i(w) \to 0 \text{ as } z \to 0 \implies a_i(F(z)) \in I_F.$$

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¶ Thus $a_\nu(w)$ is the $\nu^{\mathrm{th}}$ elementary symmetric function of the values of $u$ at the $d$-points lying over $w$.

**Residues:** For $G \in \mathcal{O}(\mathcal{U})$ we set

$$\omega_G = \frac{G(z)dF_1 \wedge \cdots \wedge dF_m}{F_1(z) \cdots F_m(z)}$$

$$\mathrm{Res}_{\{0\}} \omega_G = \left(\frac{1}{2\pi i}\right)^m \int_\Gamma \omega_G$$

where $\Gamma = \{|F_1| = \epsilon_1, \ldots, |F_m| = \epsilon_m\}$ oriented by $d(\arg F_1) \wedge \cdots \wedge d(\arg F_m) \geqq 0$. For $G = 1$ we set $\omega_G = \omega$

- $\mathrm{Res}_{\{0\}} \omega_G$ is linear in $G$, alternating in the $F_i$ and $\mathrm{Res}_{\{0\}} \omega_G = 0$ if $G \in I_F$.

- Transformation law: If $F = \{F_1, \ldots, F_m\}$ and $G = \{G_1, \ldots, G_m\}$ with set-theoretic equality $F^{-1}(0) = \{0\} = G^{-1}(0)$ and

$$I_G \subseteq I_F \iff G_i(z) = \sum A_{ij}(z) F_j(z),$$

then for $H \in \mathcal{O}(\mathcal{U})$

$$\mathrm{Res}_{\{0\}}\left(\frac{H\, dz_1 \wedge \cdots \wedge dz_m}{F_1 \cdots F_m}\right) = \mathrm{Res}_{\{0\}}\left(\frac{H \det A\, dz_1 \wedge \cdots \wedge dz_m}{G_1 \cdots G_m}\right).$$

- The pairing
$$\mathcal{O}_z/I_F \otimes \mathcal{O}_z/I_F \to \mathbb{C}$$
given by
$$H_1 \otimes H_2 \to \operatorname{Res}_{\{0\}} \omega_{H_1 H_2}$$
is non-degenerate; as will be explained below, setting $\Omega^m_{F,z} = \mathcal{O}_z/I_F \otimes \Omega^m_{\mathbb{C}^m,z}$ *local duality* is expressed by saying that the pairing
$$\mathcal{O}_z/I_F \otimes \operatorname{Ext}^m_{\mathcal{O}_z}(\mathcal{O}_z/I_F, \Omega^m_{F,z}) \to \mathbb{C}$$
is non-degenerate.[‖]

[‖]Thus in coordinates $\operatorname{Ext}^m_{\mathcal{O}_z}(\mathcal{O}_z/I_F, \Omega^m_{F,z}) \cong \mathcal{O}_z/I_F$; by the transformation law the pairing with the Ext in it is canonical.

- Two of the main kernels in complex analysis are the Cauchy and Bergman kernels given respectively by

$$K(z) = \left(\frac{1}{2\pi i}\right) \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_m}{z_m}$$

corresponding to the polycylinder $\Delta^m$ and

$$B(z, \bar{z}) = C_m \frac{\sum (-1)^{i-1} \bar{z}_i d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_i} \wedge \cdots \wedge d\bar{z}_m \wedge dz_1 \wedge \cdots \wedge dz_m}{\|z\|^{2m}}$$

corresponding to the ball $B^m$.

For $f(z)$ holomorphic

$$\int_{\partial_S \Delta^m} f(z)K(z) = \int_{\partial B^m} f(z)B(z,\bar{z}) = f(0)$$

where $\partial_S \Delta^m = \{|z_i| = \epsilon\}$ is the Silov boundary of $\Delta^m$.

- $\mathrm{Res}_{\{0\}}\, \omega_G = \int_{\partial B^m} G(z)B(z, \overline{F(z)})$.

The last formula may be used to express $\deg F$ as an integral over $\partial B^m = S^{2m-1}$. It is also an integer. Now use the Bergman kernel $B(z, \overline{F(z) - w})$ on $B^m \times B^m$ to obtain $n(z_\nu(w))$ for $F^{-1}(w)$. This is a continuously varying integer, hence is constant.

**Summary:** For a holomorphic mapping $\mathcal{U} \to W$ we have

- $F$ is finite $\iff$ $F_1, \ldots, F_m$ is a regular sequence,[**] and in this case for the degree $d$ we have the descriptions

algebraic

- $\deg[\mathcal{O}_z, \mathcal{O}_w] = d$
- $\dim \mathcal{O}_z / I_F = d$ [††]

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[**] This means that $F_i$ is not a 0-divisor in $\mathcal{O}_z/(F_1, \ldots, F_{i-1})$ for $i = 1, \ldots, m$ (cf. III.A for further discussion).

[††] Cf. [GH]; will be proved below in the case we will use.

topological

- $F : \partial\mathfrak{U} \to \partial W$ has degree $d$
- $\deg F^{-1}(w) = \sum n_\nu = d$

analytic

- $\mathrm{Res}_{\{0\}}\,\omega = d$
- $\int_{\partial W} B(z, \overline{F(z)}) = d$

Briefly: In all essential aspects finite holomorphic mappings share the properties of those in 1-variable, and the proofs can be similarly done using residues.

# II.A. Gauss mapping and basic structure results

- $\mathbb{P}^{n+2}$ is projective space with dual projective space $\check{\mathbb{P}}^{n+2} = \{\text{set of linear hyperplanes in } \mathbb{P}^{n+2}\}$, we will usually work in the standard affine open set $\mathbb{C}^{n+2} \subset \mathbb{P}^{n+2}$ and with the corresponding part of $\check{\mathbb{P}}^{n+2}$ consisting of affine linear hyperplanes in $\mathbb{C}^{n+2}$.
- $X^{n+1} \subset \mathbb{P}^{n+2}$ is a smooth complex hypersurface; in practice $X$ will be given in $\mathbb{C}^{n+2}$ by an equation $\varphi(z_1, \ldots, z_{n+2}) = 0$.
- The *Gauss map*

$$G : X \to \check{\mathbb{P}}^{n+2}$$

is given for $z \in X$

$$G(z) = \text{tangent hypeplane to } X \text{ at } z.$$

It is the affine linear space through $z$ of points $\xi = (\xi_1, \dots, \xi_{n+2})$ defined by

$$\sum_{i=1}^{n+2} \varphi_{z_i}(z)\xi_i = 0.$$

- The image $\check{X} \subset \check{\mathbb{P}}^{n+2}$ of the Gauss mapping is the *dual variety* to $X$; it is a set of hyperplanes $H$ such that $H \cap X$ is singular.
- The smooth points $\check{X}_{\mathrm{reg}}$ of $\check{X}$ correspond to hyperplanes $H$ such that the intersection $H \cap X$ has an ordinary double point (ODP) at the point of tangency.

If $z$ is the origin with $\varphi(0) = 0$ and $z_{n+2} = 0$ is the hyperplane tangent to $X$ at the origin, then ODP means that

$$\varphi(z) = \sum_{i,j=1}^{n+1} a_{ij} z_i z_j + \mathrm{HOT}$$

where $a_{ij} = a_{ji}$ and $\det \|a_{ij}\| \neq 0$.

- For $\mathcal{U} \subset \mathbb{C}^{n+1}$ with coordinates $x = (x_1, \ldots, x_{n+1})$ we consider a holomorphic mapping

$$f : \mathcal{U} \to \mathbb{C}$$

given by $f(x) = t$ with $f(0) = 0$ and which has an isolated singular point at that point; thus the gradient mapping

$$\nabla f = (f_{x_1}, \ldots, f_{x_{n+1}}) : \mathcal{U} \to \mathbb{C}^{n+1}$$

has $(\nabla f)^{-1}(0) = \{0\}$.

- Define

$$X \subset \mathbb{C}^{n+2}$$

to be the *graph* of $f$; thus $X = \{(x,t) : f(x) = t\}$.[‡‡] and There is the obvious map $\mathcal{U} \to X$ given by $x \to (x, f(x)) \in \mathbb{C}^{n+2}$. Composing this with the Gauss map to affine hyperplanes in $\mathbb{C}^{n+2}$ and then translating the image hyperplane to the origin in $\mathbb{C}^{n+2}$ gives a map

$$F : \mathcal{U} \to \mathbb{P}^{n+1}$$

where here $\mathbb{P}^{n+1}$ = hyperplanes through the origin in $\mathbb{C}^{n+2}$.

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[‡‡]In coordinates

$$\begin{aligned} z_i &= x_i \qquad 1 \leqq i \leqq n+1 \\ z_{n+2} &= t \end{aligned}$$

$X$ is given by

$$\varphi(z_1, \dots, z_{n+2}) = f(x_1, \dots, x_{n+1}) - t = 0.$$

**Main observation:** *Shrinking $\mathcal{U}$ as necessary, the origin is an isolated critical point of $f$ if, and only if, $F$ is a finite holomorphic mapping of degree $\mu =$ Milnor number.*

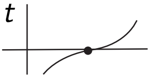
## Proof.

Unwinding the definitions, $F$ maps to an affine $\mathbb{C}^{n+1} \subset \mathbb{P}^{n+1}$ and it is given in coordinates by

$$(x_1, \ldots, x_{n+1}) \to (f_{x_1}(x), \ldots, f_{x_{n+1}}(x)).$$

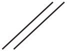
Since $F$ is constructed from the Gauss mapping on the graph of $f$, the result follows. The critical points of $f$ are the zeroes of $\nabla f$, and these are isolated if, and only if, $F$ does not have a positive dimensional fibre through the origin. □

The pictures are (here the vertical axis is the $t$ coordinate)




has tangent line $t = 0$ (counted twice)

Nearby tangent lines are

 giving  which gives that the inverse image

of the corresponding line through the origin are the two tangent lines to the graph.

- We may now apply the general theory of finite holomorphic mappings to the mapping $F$ and interpret those results in the context of the holomorphic mapping

$$\begin{array}{ccc} \mathbb{C}^{n+1} & & \mathbb{C} \\ \cup & & \cup \\ f : \mathcal{U} & \to & B \end{array}$$

which has the origin as an isolated critical point with $f(0) = \{0\}$.
  - $\mathcal{U}$ is a contractible Stein manifold with smooth boundary $\partial\mathcal{U} = f^{-1}(\partial B)$;
  - setting $B' = B\backslash\{0\}$ and $\mathcal{U}' = f^{-1}(B')$, the restriction

    $$f : \mathcal{U}' \to B'$$

    is a smooth holomorphic fibration with general fibre $V$, which is a complex $n$-dimensional Stein manifold with smooth boundary a real $(2n-1)$-dimensional manifold $K$;

- $V_0 =: f^{-1}(0)$ is a Stein variety with an isolated singular point $p$ $(= \{0\})$ and smooth boundary $K_0 \cong K$;
- $f : \partial\mathfrak{U} \to \partial B$ is a topologically trivial fibre bundle with typical fibre $K$.

In terms of the above graph construction, the fibres $V_t = f^{-1}(t)$ are the intersections of the graph $X$ with the hyperplanes $t =$ constant. This is a line $L$ in the dual projective space $\check{\mathbb{P}}^{n+2}$, and locally the line $L$ meets the dual variety $\check{X}$ in a single point $P \in \check{X}_{\mathrm{sing}}$.

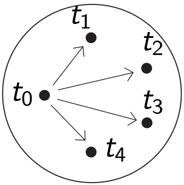
- We now perturb $L$ generically to a line $L_\epsilon$ in $\check{\mathbb{P}}^{n+2}$ that meets $\check{X}$ only at regular points (this is possible since $\operatorname{codim} \check{X}_{\text{sing}} = 2$ and therefore a generic line in $\check{\mathbb{P}}^{n+2}$ will not meet $\check{X}_{\text{sing}}$). This corresponds to generically perturbing $f$ to $f(x) + \sum \epsilon_i x_i$ giving rise to

$$f_\epsilon : \mathcal{U}_\epsilon \to B_\epsilon$$

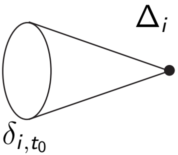
which will have exactly $\mu$ non-degenerate critical points corresponding to $L_\epsilon \cap X_{\text{reg}}$. The general fibre $V_\epsilon$ of $f_\epsilon$ is topologically the same as $V$, from which we may draw the

**Conclusion:** *$V$ is topologically a wedge of $\mu$ $n$-spheres.*

Proof: The argument is essentially the same as in the classic book [L] by Lefschetz. We picture $B_\epsilon$ as a disc with a base point $t_0$ and paths drawn to the critical values $t_1, \ldots, t_\mu$ of $f_\epsilon$




As $t$ traverses the path $\overline{t_0 t_i}$ there is a vanishing cycle $\delta_{i,t} \in H^n(V_t, \mathbb{Z})$; $\delta_{i,t} \cong S^n$, and along the path the locus of the $\delta_{i,t}$ traces out an $(n+1)$-cell $\Delta_i$



Now on the one hand

- $\mathcal{U}_\epsilon$ retracts onto $f^{-1}$ (union of the cuts $\overline{t_0 t_i}$), and thus $\mathcal{U}_\epsilon$ is homotopy equivalent to $V_{t_0}$ with $(n+1)$-cells $\Delta_i$ attached at the $\mu$ $n$-spheres $\delta_i$,

while on the other hand

- $\mathcal{U}_\epsilon$ is contractible.

It follows from the exact homotopy sequence of the pair $(\mathcal{U}_\epsilon, V_{t_0})$ that $V_{t_0}$ is homotopy equivalent to a wedge of $\mu$ $n$-spheres. □

# II.B de Rham cohomology

- Recall our basic setup

$$(*) \qquad \begin{array}{ccccccccl} V & \subset & \mathcal{U}' & \subset & \mathcal{U} & \subset & \mathbb{C}^{n+1} & & \text{with coordinates } x = (x_1, \ldots, x_{n+1}) \\ \downarrow & & \downarrow f & & \downarrow f & & & & \\ \{t\} & \subset & B' & \subset & B & \subset & \mathbb{C} & & \text{with coordinate } t,\ B' = \{t \neq 0\} \end{array}$$

given by $f(x) = t$ where $f$ has an isolated critical point at the origin with $f(0) = 0$;

- de Rham cohomology provides a basic method to study topology using analysis; we will use use this in the situation $(*)$; differential forms will always be holomorphic (expressions using $dx_i$'s and no $d\bar{x}_i$'s and with holomorphic functions as coefficients); in practice they will be algebraic ($f(x)$ is a polynomial, etc.);
- Will use elementary sheaf theory; a sheaf over a space gives for each open set a group; sheaf cohomology provides a means of passing from local to global; standard notations are
  - $\mathcal{O}_{\mathcal{U}} =$ sheaf of holomorphic functions (sections over an open set are the holomorphic functions)
  - $\Omega^p_{\mathcal{U}} =$ sheaf of holomorphic $p$-forms ($\Omega^0_{\mathcal{U}} = \mathcal{O}_{\mathcal{U}}$, similar to $\mathcal{O}_{\mathcal{U}}$ but using holomorphic forms)

- $\Omega^p_{\mathcal{U}/B} = \Omega^p_{\mathcal{U}}/f^*\Omega^1_B \wedge \Omega^{p-1}_{\mathcal{U}}$ is the sheaf of relative differential forms*
- $\mathbb{C}$ = constant sheaf (sections over a connected open set are just constant functions)
- for a sheaf $\mathcal{S}$ over $\mathcal{U}$, $R^q_f\mathcal{S}$ is the sheaf associated to

$$W \to H^q(f^{-1}W, \mathcal{S}).^\dagger$$

- The basic steps are
  - the de Rham theorem for a single Stein manifold $M$ such as $\mathcal{U}$ or $V$;
  - the relative de Rham for a smooth fibration such as $f : \mathcal{U}' \to B'$ where the fibres are Stein manifolds; and most interestingly
  - the correction to this last step created by the isolated singular point on $V$.

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*In a local product situation $(u, v) \to v$ where differential forms involve $du_i$'s and $dv_\alpha$'s, with holomorphic functions $h(u, v)$ as coefficients, passing to relative differential forms means "set $dv_\alpha = 0$."

†This means to each open set $W$ we associate the cohomology group as indicated.

## Fundamental exact sequence (notation to be explained)

$$(**)\quad \boxed{0 \to \underbrace{R_f^n\mathbb{C}\otimes \mathcal{O}_B}_{\text{topological}} \to \underbrace{\mathcal{H}^n_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}/B})}_{\text{analytic}} \to \underbrace{\Omega_f}_{\substack{\text{invariant of}\\ \text{the singularity}}} \to 0.}$$

- Step one: For a Stein manifold $M$ we have the de Rham complex of sheaves

$$0 \to \mathbb{C} \to \Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m_M \to 0, \quad d^2 = 0$$

  $H^p_{\mathrm{DR}}(\Omega^\bullet_M) = p^{\mathrm{th}}$ cohomology of the complex of global sections

$$\to H^0(M,\Omega^p_M) \to H^0(M,\Omega^{p+1}_M \cdots) \to$$

  Then we have the *de Rham theorem* for $M$

$$H^p(M,\mathbb{C}) \cong H^p_{\mathrm{DR}}(\Omega^\bullet_M).$$

The basic ingredients in the proof are

- local; $\mathcal{H}^p(\Omega^\bullet_M) = 0$ for $p > 0$ (Poincaré lemma)[‡]
- global; $H^q(\Omega^\bullet_M) = 0$ for $q > 0$ (Stein)

The second is because in the Stein situation the higher cohomology of any coherent sheaf is zero.

[‡] $\mathcal{H}^p(\Omega^\bullet_M)$ are cohomology sheaves of the above complex of sheaves.

- Step two: For $f : \mathcal{U} \to B$ we have

$$\begin{cases} \Omega^\bullet_{\mathcal{U}/B} = \Omega^\bullet_{\mathcal{U}}/df \wedge \Omega^{\bullet-1}_{\mathcal{U}}, \\ d : \Omega^p_{\mathcal{U}/B} \to \Omega^{p+1}_{\mathcal{U}/B} \text{ gives rise to the relative} \\ \qquad \text{de Rham sheaf complex } (\Omega^\bullet_{\mathcal{U}/B}, d) \end{cases}$$

$$\mathcal{H}^p_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}/B}) =: \text{ cohomology of}$$

$$\{\cdots \to f_*\Omega^p_{\mathcal{U}/B} \xrightarrow{d} f_*\Omega^{p+1}_{\mathcal{U}/B} \to \cdots\}.^{\S}$$

Then we have the *relative de Rham theorem* for $f : \mathcal{U}' \to B'$

$$R^p_f\mathbb{C} \otimes \mathcal{O}_{B'} \cong \mathcal{H}^p_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}'/B'}).$$

[§] We will think of this as a holomorphic vector bundle whose fibres are the global, holomorphic de Rham cohomology group along the fibre. Over $\mathcal{U}'$ this is just the relative de Rham theorem for a holomorphic family of Stein manifolds.

As before there are two ingredients in the proof:

- local; $\mathcal{H}^q(\Omega^p_{\mathcal{U}'/B'}) = 0$ for $q > 0$ (Poincaré lemma with dependence on parameters)[¶]
- global; fibres are Stein, which gives $R^q_f \Omega^p_{\mathcal{U}/B} = 0$ for $q > 0$.

**Note:** We will see that the first step breaks down exactly at the isolated singular point of $V_0 \subset \mathcal{U}$, and this is what leads to the right-hand term in $(**)$.

[¶]Locally, $\mathcal{U}' \to B'$ is holomorphically a product $(z^1, \ldots, z^n; t)$ and relative differentials have only $dz^i$'s with holomorphic coefficients $g(z, t)$.

# III.A. Koszul complexes

- The Koszul complex emanates from the following linear algebra construction: Given a $d$-dimensional complex vector space $E$ and non-zero vector $e^* \in E^*$, denoting by $i(e^*)$ the contraction operator, the sequence

$$0 \to \Lambda^d E \xrightarrow{i(e^*)} \Lambda^{d-1} E \to \cdots \to \Lambda^2 E \xrightarrow{i(e^*)} E \xrightarrow{i(e^*)} \mathbb{C} \to 0$$

  is exact.[‖] Dually the sequence

$$0 \to \mathbb{C} \xrightarrow{e^*} E^* \xrightarrow{\wedge e^*} \wedge^2 E^* \to \cdots \xrightarrow{\wedge e^*} \wedge^{d-1} E^* \xrightarrow{\wedge e^*}\to \wedge^d E^* \to 0$$

  is exact.

---

[‖] Here $\mathbb{C} = \wedge^\circ E$. In the dual sequence we likewise identify $\mathbb{C}$ with $\wedge^\circ E^*$.

- Let $R$ be a ring and $r_1, \ldots, r_d$ elements of $R$ that generate an ideal $I = \{r_1, \ldots, r_d\}$. For $e_1, \ldots, e_d$ the standard basis of $\mathbb{C}^d$, set

$$E_k = R \otimes_{\mathbb{C}} \wedge^k \mathbb{C}^d$$
$$e_J = e_{j_1} \wedge \cdots \wedge e_{j_k} \text{ where } J = (j_1, \ldots, j_k)$$

and define the Koszul complex $(E_\bullet, \partial)$ where

$$\partial : E_k \to E_{k-1}$$

is given by the usual boundary formula

$$\partial(e_J) = \sum_{\nu=1}^{k} (-1)^{\nu-1} r_{j_\nu} e_1 \wedge \cdots \wedge \hat{e}_{j_\nu} \wedge \cdots \wedge e_{j_k}.$$

Thus $\partial = i(r_1, \ldots, r_d)$. The homology of $(E_\bullet, \partial)$ is denoted by $H_*(E_\bullet, \partial)$.

- We recall that $r_1, \ldots, r_d$ is a *regular sequence* if for each $k$ with $1 \leqq k \leqq d$

$$r_k \text{ is not a 0-divisor in } R/\{r_1, \ldots, r_{k-1}\}$$

(here we set $r_0 = 0$). A standard result is
*If* $\{r_1, \ldots, r_d\}$ *is a regular sequence, then*

$$\begin{cases} H_q(E_\bullet, \partial) = 0 \text{ for } q > 0 \\ H_0(E_\bullet, \partial) \cong R/I. \end{cases}$$

- Dually we set $\mathfrak{r} = (r_1, \ldots, r_d)$ and have the complex $(E^*, \partial^*)$

$$\mathbb{C} \xrightarrow{\partial^*} E^* \xrightarrow{\partial^*} \wedge^2 E^* \to \cdots \xrightarrow{\partial^*} \wedge^{d-1} E^* \xrightarrow{\partial^*} \wedge^d E^*$$

where $\partial^*$ is the wedge product with $\mathfrak{r}$.

For a regular sequence the cohomology groups

$$(*) \qquad \begin{cases} H^q(E^*, \partial^*) = 0 \ \text{ for } \ q < d \\ H^d(E^*, \partial^*) \cong R \otimes \wedge^d \mathbb{C}^d. \end{cases}$$

## Examples:

- For $\mathcal{U} \subset \mathbb{C}^m$ and $F = (F_1, \ldots, F_m) : \mathcal{U} \to \mathbb{C}^m$ a holomorphic mapping, for $z \in \mathcal{U}$ and $\mathcal{O}_z = \mathcal{O}_{\mathcal{U},z}$, $I_F = \{F_1, \ldots, F_m\} \subset \mathcal{O}_z$

  $F$ *is finite* $\iff$ $F_1, \ldots, F_m \in \mathcal{O}_z$ *is a regular sequence* .

  In this case
  $$\deg F = \dim \mathcal{O}_z / I_F.$$

- For $\mathcal{U} \subset \mathbb{C}^{n+1}$ and $f : \mathcal{U} \to \mathbb{C}$ a holomorphic function $f(x)$ with an isolated critical point at the origin,

  $f_{x_1}, \ldots, f_{x_{n+1}}$ *is a regular sequence in* $\mathcal{O}_x = \mathcal{O}_{\mathcal{U},0}$.

In this case, setting $Q_f = \mathcal{O}_{\mathcal{U},0}/J_f$ where $J_f = \{f_{x_1}, \ldots, f_{x_{n+1}}\}$, the fundamental exact sequence $(*)$ above implies that the sequence
$(**)$

$$0 \to \Omega^0_{\mathcal{U}} \xrightarrow{df} \Omega^1_{\mathcal{U}} \xrightarrow{\wedge df} \cdots \xrightarrow{df} \Omega^n_{\mathcal{U}} \xrightarrow{df} \Omega^{n+1}_{\mathcal{U}} \to \Omega_f \to 0$$

is exact. Using that the fibres of $f : \mathcal{U} \to B$ are Stein, from this exact sequence we may infer that for $W \subset B$ an open set and $\varphi \in H^0(f^{-1}W, \Omega^p_{\mathcal{U}})$ a holomorphic $p$-form with $0 < p \leqq n$ satisfying $\varphi \wedge df = 0$, there exists a holomorphic $(p-1)$-form $\psi$ in $f^{-1}W$ with

$$(***) \qquad \varphi = \psi \wedge df.$$

By $(**)$ this is true locally along the fibres of $f^{-1}W \to W$, and then by the Stein property it is also true globally.
For $p = n+1$ this breaks down locally around the critical point of $f$.

# III.B. Gauss-Manin connection; Picard-Fuchs equation

- The sheaf $R^q_f\mathbb{C}$ arises from $W \to H^q(f^{-1}W, \mathbb{C})$; over $B'$ where $f : \mathcal{U}' \to B'$ is locally topologically a product this sheaf is locally constant; it is called a *local system*; one may think of it as a vector bundle with constant transition functions which then gives rise to the *Gauss-Manin connection*

$$\nabla : R^q_f\mathbb{C} \otimes \mathcal{O}_{B'} \to R^q_f\mathbb{C} \otimes \Omega^1_{B'}, \qquad \nabla^2 = 0$$

  where $\nabla(\xi \otimes g) = \xi \otimes dg$ for $\xi \in R^q_f\mathbb{C}, g \in \mathcal{O}_{B'})$.
- We have for $f : \mathcal{U} \to B$

$$(R^q_f\mathbb{C})_t = \begin{cases} 0 & q \neq 0, n \\ \mathbb{C}^\mu & q = n \text{ and } t \neq 0 \\ 0 & q = n \text{ and } t = 0. \end{cases}$$

The third is because a neighborhood of $V_0$ in $\mathcal{U}$ retracts onto $V_0$ which is homeomorphic to the cone over $K_0 = \partial V_0$ and therefore is a contractable space.

- One may ask: For a section $\varphi$ of $\mathcal{H}^p_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}'/B'})$ over an open set $W \subset B'$, how does one calculate $\langle \nabla\varphi, \partial/\partial t\rangle = \nabla_t\varphi$? The answer is: $\varphi$ is represented by a holomorphic $q$-form $\Phi$ defined in $f^{-1}W$. The condition that $d\Phi\big|_{V_t} = 0$ for $t \in W$ is equivalent to

$$d\Phi = \psi \wedge df.^{**}$$

** Here we are using that $\pi^{-1}W$ is a Stein manifold to infer that if we have a holomorphic form $\eta$ defined in $\pi^{-1}W$ and satisfying $\eta \wedge df = 0$, so that locally in $\pi^{-1}W$ we have $\eta = \sigma \wedge df$, then we may globally find a $\sigma$ in all of $\pi^{-1}W$ such that $\eta = \sigma \wedge df$.

Then

$$0 = d^2\Phi = d\psi \wedge df \implies \psi\Big|_{V_t} \text{ is a closed } p\text{-form},$$

and one may check that

$$\nabla_t \varphi \textit{ is represented by } \psi.$$

- A more analytic way of arriving at this answer (and the way it was done historically) uses the isomorphism

$$\mathcal{H}^p_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}'/B'}) \cong R^p_f \mathbb{C} \otimes \mathcal{O}_{B'}.$$

We use that $(R^q_f \mathbb{C})_t$ is dual to $H_q(V_t, \mathbb{C}) \cong H_q(V, \mathbb{Z}) \otimes \mathbb{C}$, and then for a family $\delta_t \in H_p(V_t, \mathbb{Z})$ of geometric $p$-cycles in the same homology class (this uses the homeomorphism $\pi^{-1}W \cong W \times V_t$ for some $t \in W$), it is clear that the *period*

$$\pi_{\delta_t}(\varphi) = \int_{\delta_t} \varphi$$

is a holomorphic function of $t$.

We claim that

$$\frac{d}{dt}(\pi_{\delta_t}(\psi)) = \int_{\delta_t} \psi$$

where $\psi$ is defined above. To see this, for $\epsilon > 0$, let $\Delta_{t_0,\epsilon}$ be the locus of the $\delta_t$ from $t_0$ to $t_0 + \epsilon$. Then $\partial\Delta_{t_0,\epsilon} = \delta_{t_0+\epsilon} - \delta_{t_0}$, and by Stokes' theorem

$$\begin{aligned}\frac{1}{\epsilon}\left(\int_{\delta_{t_0+\epsilon}} \varphi - \int_{\delta_{t_0}} \varphi\right) &= \frac{1}{\epsilon}\left(\int_{\Delta_{t_0,\epsilon}} d\varphi\right) \\ &= \frac{1}{\epsilon}\left(\int_{\Delta_{t_0,\epsilon}} \psi \wedge dt\right),\end{aligned}$$

and the limit as $\epsilon \to 0$ is given by $\int_{\delta_{t_0}} \psi$. □

- We next show that the inclusion of sheaves

$$0 \to R^q_f \mathbb{C} \otimes \mathcal{O}_{B'} \to \mathcal{H}^p_{\mathrm{DR}} \left( \Omega^\bullet_{\mathcal{U}'/B'} \right)$$

  extends across $t = 0$ to all of $B$. From the above description of $R^p_f \mathbb{C}$ what has to be checked is
    - $\mathcal{H}^p_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}/B})_0 = 0$ for $0 < p < n$;
    - the limit as $t \to 0$ of the image of $(R^n_f \mathbb{C})_t \to \mathcal{H}^n_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}/B})_t$ is equal to zero.
- For $0 \leqq p \leqq n$ we will use $(***)$ in the preceding section to show that for $0 \leqq p < n$ the connection $\nabla$ in $\mathcal{H}^p_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}'/B'})$ extends to one in $\mathcal{H}^p_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}/B})$. The point is that a section $\varphi$ over $W \subset B$ of $\mathcal{H}^p_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}/B})$ is given by a $p$-form defined in $f^{-1}W$ and satisfying $d\varphi = \psi \wedge df$. To show that setting $\nabla_t \varphi = \psi$ is well defined we need to know that if we have a $p$-form $\eta$ in $\pi^{-1}W$ with $\eta \wedge df = 0$, then $\eta = \lambda \wedge df$ for some $\lambda$ defined in $\pi^{-1}W$. This is what $(***)$ gives.

As an informal argument for the second, for $t \neq 0$ the homology $H_n(V_t, \mathbb{Z})$ is generated by the vanishing cycles $\delta_t$ described above as the limit of the ODP vanishing cycles for a generic perturbation of $V_t$, and for $\varphi$ a holomorphic $n$-form defined in a neighborhood of $V_0$ in $\mathcal{U}$ it is then intuitively pretty clear that we have

$$\lim_{t \to 0} \int_{\delta_t} \varphi = 0.$$

- It may be shown that by shrinking $\mathcal{U}$ to a sufficiently small neighborhood of $V_0$ we may choose a holomorphic $n$-form $\varphi$ defined in $\mathcal{U}$ and such that for all $t \in B'$

  $$\varphi, \nabla_t \varphi, \ldots, \nabla_t^{\mu-1} \varphi \text{ gives basis for } \mathcal{H}^n_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}'/B'})_t.$$

  There is then a relation

  $$\nabla_t^\mu \varphi + a_1(t) \nabla_t^{\mu-1} \varphi + \cdots + a_\mu(t) = 0.$$

Equivalently, for every $\delta_t \in H_n(V_t, \mathbb{Z})$ the periods

$$\pi(t) = \int_{\delta_t} \varphi$$

satisfy a *Picard-Fuchs equation*

$$(*) \qquad \pi^{(\mu)} + a_1(t)\pi^{(\mu-1)} + \cdots + a_\mu(t)\pi = 0.$$

Moreover, it is known that the Picard-Fuchs equation has *regular singular points*, meaning that

$$(**) \qquad a_j(t) = \frac{b_j(t)}{t^j}$$

where $b_j(t)$ is holomorphic across $t = 0$. Equivalently any solution $\pi(t)$ to the Picard-Fuchs equation has moderate growth in the sense that

$$|\pi(t)| = O(|t|^{-N})$$

as $t \to 0$.

- If $\pi_1(t), \ldots, \pi_\mu(t)$ are a fundamental set of solutions to $(*)$, then under analytic constriction around $t = 0$ the $\pi_j(t)$ are transformed by the dual of the previously defined monodromy matrix $T$.
- If $\lambda_1, \ldots, \lambda_k$ are the roots of the characteristic equation $\det|T - \lambda I| = 0$ of multiplicities $m_1, \ldots, m_k$, then assuming the condition $(**)$ on regular singular points we may choose the $\pi_i(t)$ to be of the form

$$t^{\alpha_j}\varphi_{j_1}(t), t^{\alpha_j}\log t\varphi_{j_2}(t), \ldots, t^{\alpha_j}(\log t)^{m_j-1}\varphi_{jm_j}(t)$$

where $\alpha_j = \left(\frac{1}{2\pi i}\right)\log\lambda_j$ and where the coefficients in the linear combinations and the $\varphi_{j_\alpha}(t)$ are holomorphic in $|t| < \epsilon$.

# III.C. Monodromy; examples

- It is standard that the linear homogeneous ODE $(*)$ above can be transformed into a linear ODE system

$$(\natural) \qquad X' = A(t)X$$

  where $A(t)$ is holomorphic in $\Delta^* =: 0 < |t| < \epsilon$. The system $(\sharp)$ may be considered as a bundle with connection $\nabla$.

- Assuming that the regular singular point condition $(**)$ is satisfied, the matrix $A(t)$ will be mermorphic in $\Delta = \{t : |t| < 1\}$ having at most a $1^{\mathrm{st}}$ order pole at $t = 0$. Moreover, we may make a holomorphic change of frame that transforms the system $(\sharp)$ to a new one

$$(\sharp\sharp) \qquad Y' = B(t)Y$$

where

$$B(t) = \left(\frac{1}{2\pi i}\right) \frac{N}{t} + \text{(holomorphic matrix)}.$$

A subtle point (cf. (7.76) in [K]) is that $e^N$ will be the same as a conjugate of the monodromy matrix $T$ except that the eigenvalues $\lambda_j$ can be shifted by integers. By abuse of language we shall refer to $N$ as the logarithm of monodromy, written suggestively but incorrectly as

$$\mathrm{Res}_{\{0\}} \nabla = \log T$$

and expressed by *the residue of the Gauss-Manin connection is the logarithm of monodromy*.

- We now sketch the proof, due to Brieskorn, that

  *the eigenvalues of $\lambda_j$ of monodromy are roots of unity.*

The essential points are

- the eigenvalues of $\exp(\mathrm{Res}_{\{0\}})$ are roots of an equation with integer coefficients;
- in case $f(x)$ is a polynomial the cohomology bundle $\mathcal{H}_{\mathrm{DR}}(\Omega^{\bullet}_{\mathcal{U}/B})$ and the Gauss-Manin connection $\nabla$ are constructed algebraically;
- it follows that if $\sigma$ is any automorphism of $\mathbb{C}$ and $\lambda$ is an eigenvalue of $T$ with $\lambda = e^{2\pi i a}$, then if we apply $\sigma$ to the construction of $\mathcal{H}^{n}_{\mathrm{DR}}(\Omega^{\bullet}_{\mathcal{U}/B})$, we obtain $\sigma(\lambda) = e^{2\pi i \sigma(a)}$. Now $\sigma(\lambda)$ is an algebraic number, and if $\sigma(a)$ were transcendental it could be chosen to be any transcendental number. But the $\sigma(\lambda)$'s are countable, hence $\sigma(a)$ must be algebraic.
- Finally the theorem of Gelfond-Schneider stating that if $a$ and $e^{2\pi i a}$ are both algebraic, then $a \in \mathbb{Q}$ is rational gives the result.

Note: This argument is the tip of a deep and not yet understood iceberg. Namely, suppose that $f(x) \in \overline{\mathbb{Q}}[x_1, \dots, x_{n+1}]$. Then in the complex vector space $H^n(V, \mathbb{C})$ we have two lattices:

- $H^n(V, \mathbb{Q})$;
- $H^n_{\mathrm{DR}}(\Omega^\bullet_{V(\overline{\mathbb{Q}})})$

and an isomorphism given by periods

$$H^n(V, \mathbb{Z}) \otimes \mathbb{C} \cong H^n_{\mathrm{DR}}(\Omega^\bullet_{V(\overline{\mathbb{Q}})}) \otimes \mathbb{C}.$$

The arithmetic relation between the two is a very deep issue.

Example: For weights $w_1, \dots, w_{n+1}$ where the $w_i$ are positive integers with $\gcd(w_1, \dots, w_{n+1}) = 1$, we define a $\mathbb{C}^*$ action on $\mathbb{C}^{n+1} \backslash \{0\}$ by

$$\lambda \cdot x = (\lambda^{w_1} x_1, \dots, \lambda^{w_{n+1}} x_{n+1}).$$

Let $f(x)$ be a *weighted homogeneous polynomial* of degree $d$; this means that

$$f(\lambda \cdot x) = \lambda^d f(x).$$

Then we have the *global affine Milnor fibration*

$$(\sharp\sharp\sharp) \qquad \mathbb{C}^{n+1} \backslash f^{-1}(0) \xrightarrow{f} \mathbb{C}^*,$$

which can be shown to be topologically equivalent to the standard Milnor fibration

$$\mathcal{U}' \xrightarrow{f} B'.$$

Denoting by $V_1$ the fibre of $(\sharp\sharp\sharp)$ over $t = 1$, from

$$f(e^{2\pi it/d} \cdot x) = e^{2\pi it} f(x)$$

we see that as $t$ turns around the unit circle the resulting action of the circle lifts to the fibration $(\sharp\sharp\sharp)$. In particular, the monodromy $T$ is finite of order $\text{l.c.m}(w_1, \ldots, w_{n+1})$ and its eigenvalues with their multiplicities can be computed from the weights $w_i$ (cf. [AGLV] and [K]).

# Appendix: Mixed Hodge structure (MHS) on the vanishing cohomology associated to an isolated hypersurface singularity (IHS)

For the reasons discussed in the introduction, and also because it is of interest in its own right (especially in its application to the classification of IHS's, one wishes to have a canonical MHS on $H^n(V)$. There is an extensive literature here (cf. the references to these lectures); here we shall just present an informal summary, largely following [K] but in the setting and notations of these lectures. A MHS has a *weight filtration* $W_\bullet$ and a *Hodge filtration* $F^\bullet$; we denote the object to be described by $H^n_{\lim}(V_0) = (H^n(V), W_\bullet, F^\bullet)$.

The basic ideas are

(i) to embed the family $\{V_t\}_{t\in B}$ in a family of projective varieties $\{X_t\}_{t\in B}$ where
  - $X_t$ is smooth for $t \neq 0$;
  - $V_t \subset X_t$ is an open subset and the complements $X_t \backslash V_t$ are smooth *for all* $t \in B$; in practice this will allow us to localize the interesting part of the cohomology $H^n(X_t)$ to what happens in $\mathcal{U}$;

(ii) the variations of Hodge structure given by the $H^m(X_t)$ for $t \neq 0$ give rise to limiting mixed Hodge structures $H^m_{\lim}(X_t)$ (cf. [CKS], [GGLR], [GG] and [PS]), and from the exact sequences associated to the cohomologies of the pairs $(X_t, V_t)$ and $(X_t, X_t \backslash V_t)$, by taking the limit as $t \to 0$ we will be able to dig out the desired $H^n_{\lim}(V_0)$;

(iii) the weight filtration on $H^n_{\lim}(V_0)$ will be constructed out of the monodromy weight filtrations on $H^n_{\lim}(X_t)$ and $H^{n+1}_{\lim}(X_t)$;

(iv) a similar statement will hold for the Hodge filtration on $H^n_{\lim}(V_0)$; However this approach doesn't really tell one how to *compute* $F^\bullet$, e.g., in terms of differential forms, and in the next subsection we shall discuss an alternative approach due to Varchenko and others that will describe $H^n_{\lim}(V_0)$ in terms of $\mathcal{H}^n_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}/B})$.

In the background is the LMHS on the cohomology of a family $\mathcal{X}^* \xrightarrow{\pi} \Delta^*$ of smooth projective varieties $X_t = \pi^{-1}(t)$ over a punctured disc. Instead of thinking of a "general" fibre $X_t$ and monodromy $T : H^n(X_t) \to H^n(X_t)$, even though this may be more intuitive (we may picture what happens in the limit as $t \to 0$ where a bunch of cycles vanish), from a more formal perspective (and one that is better adapted to standard cohomological techniques) is to use the *canonical fibre* $X_\infty$.

To define $X_\infty$ we complete the family $\mathcal{X}' \to B'$ to $\mathcal{X} \xrightarrow{\pi} B$ where

- $\mathcal{X}$ is smooth, and
- $\overline{X}_0 =: \pi^{-1}(0)$ is a not-necessarily reduced normal crossing divisor (NCD).[††]

The pullback of $\mathcal{X}' \to B'$ to the universal covering $\widetilde{B}' = \mathbb{H}$ (the upper half-plane in $\mathbb{C}$) is topologically a product, and one sets

$$X_\infty = \mathcal{X} \times_{B'} \mathbb{H}.$$

One may roughly think of $X_\infty$ as the ringed space $(X, \mathcal{O}_{X_0}[\log t])$.

[††] We use the notation $\overline{X}_0$ to distinguish from the singular fibre $X_0$ in the family $\{X_t\}_{t\in B}$. In practice we will have that $X_0$ is irreducible and

$$\overline{X}_0 = \widetilde{X}_0 + \Sigma n_i E_i$$

where $\widetilde{X}_0$ is a desingularization of $X_0$ and the $E_i$ are exceptional divisors.

The action of monodromy is induced from $s \to s+1$ where $s \in \mathbb{H}$ and $e^{2\pi is} = t$. This gives

$$T : H^m(X_\infty) \to H^m(X_\infty).$$

From the monodromy theorem

$$T = T_s T_u$$

where $T_s$ is semi-simple with eigenvalues roots of unity and $T_u$ is unipotent with logarithm $N = \log T_u$ that satisfies $N^{m+1} = 0$. From $N$ there is defined the *monodromy weight filtration*

$$W_0 \subset W_1 \subset \cdots \subset W_{2m-1} \subset W_m$$

on $H^m(X_\infty)$. The limiting Hodge filtration $F^\bullet_\infty$ is defined by the procedure of Schmid (cf. [K]). Of importance for this discussion is that

*Both $W_\bullet$ and $F^\bullet_\infty$ are invariant under the action of $T_s$.*

The basic result is the

Theorem (Schmid): *$(H^m(X_\infty), W_\bullet, F^\bullet_\infty)$ defines a mixed Hodge structure.*

This MHS has the additional properties

$$\begin{cases} N : W_k \to W_{k-2} \\ N : F^p \to F^{p-1}, \\ N^\ell : \mathrm{Gr}_{m+\ell} \xrightarrow{\sim} \mathrm{Gr}_{m-\ell} \quad \text{(Hard Lefschetz)} \end{cases}$$

which define it to be the *monodromy weight filtration* which together with $F^\bullet_\infty$ give a *limiting mixed Hodge structure* (LMHS).

# Weight filtration on vanishing cohomology

In light of the above discussion we now change notation and set

$$V_\infty = \mathcal{U} \times_{B'} \mathbb{H},$$

so that we want to define a MHS on $H^n(V_\infty)$. For this one constructs a diagram

$$\begin{array}{ccccc} \overline{X}_0 \subset \overline{\mathcal{X}} & \longleftarrow & \mathcal{X} & \supset & \pi^{-1}(0) = \widetilde{X}_0 + \Sigma n_i E_i \\ & \searrow^{\pi} & \downarrow \pi & & \\ & & B & & \end{array}$$

where the terms have the following meaning:

- we have

$$\begin{array}{ccc} \overline{\mathfrak{X}} & \supset & \mathfrak{U} \\ & {\scriptstyle \bar{\pi}} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & B & \end{array}$$

  where for $t \neq 0$ the $\overline{X}_t = X_t = \pi^{-1}(t)$ are smooth projective varieties containing $V_t = (X_t) \cap \mathfrak{U}$ as open sets;
- the $X_t \backslash V_t$ are smooth for *all* $t \in B$, so that the only singularities on $\bar{\pi}^{-1}(0) = \overline{X}_0$ are the isolated singular point on $V_0 \subset \overline{X}_0$; thus $\overline{\mathfrak{X}} \backslash \mathfrak{U} \to B$ is a topologically trivial fibration;
- $\mathfrak{X} \to \overline{\mathfrak{X}}$ is a resolution of singularities, and $\pi^{-1}(0)$ is a NCD one component of which is a resolution of the isolated singular point on $V_0$.

The exact cohomology sequences of the pair $(\overline{\mathcal{X}}, \mathcal{U})$ and $(X_t, V_t)$ then lead to the exact sequence

$$0 \to H^n(\overline{X}_0) \to H^n(X_\infty) \to H^n(V_\infty) \to 0$$

where the first two terms are MHS's and therefore there is an induced MHS on the third term. On the other hand the Clemens-Schmid sequence gives the local invariant cycle theorem expressed by the exact sequence

$$H^n(\overline{X}_0) \to H^n(X_\infty) \xrightarrow{N} H^n(X_\infty)$$

of MHS's. The image of the first map is equal to

$$\ker N = \ker(T_u - I) = \ker N \cap H^n(X_\infty)_1$$

there the subscript "1" refers to the eigenspace $\lambda = 1$ of $T_s$.

Comparing the two above exact sequences we have

$$\bigoplus_{\lambda \neq 1} H^n(V_\infty)_\lambda = \bigoplus_{\lambda \neq 1} H^n(X_\infty)_\lambda$$

and the weight filtration is centered at $n$, and

$$H^n(V_\infty)_1 = H^n(X_\infty)_1 / \ker(N)_1$$

and here from Clemens-Schmid the weight filtration is centered at $n+1$.

In this way, by decomposing $H^n(V_\infty)$ into a direct sum of eigenspaces of $T_s$ we have defined the weight filtration on $H^n(V_\infty)$.

Summary: The weight filtration on $H^n(V_\infty) = H^n_{\lim}(V)$ is derived from the monodromy weight filtrations on the $H^m_{\lim}(X) = \lim H^m(X_t)$ for the family of projective varieties $X_t$ where $X_t \backslash V_t$ is smooth for *all* $t \in B$ and where $m = n, n+1$.

Alternatively, we may describe $W_\bullet$ by saying what $\lambda^{W_\bullet}$ is on the eigenspaces $H^n(V_\infty)_\lambda$ of $T_s$. If $\lambda = 1$ we shift the monodromy weight filtration of $N|_{H^n(V_\infty)_\lambda}$ up by one; if $\lambda \neq 1$ we just use the monodromy weight filtration of this restriction.

# Geometric sections of the de Rham cohomology bundle

For $\mathcal{U} \xrightarrow{f} B$ we have seen that $\mathcal{H}^n_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}/B}) =: H^n(f_*\Omega^\bullet_{\mathcal{U}/B})$ is a vector bundle whose fibres for $t \neq 0$ are isomorphic to $H^n(V_t, \mathbb{C})$, while for $t = 0$ the fibre is isomorphic to $\Omega_f = Q_f \otimes \Omega^{n+1}_{\mathcal{U},\{0\}}$ (cf. the fundamental exact sequence). There are distinguished *geometric sections* of $\mathcal{H}^m_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}/B})$, possibly with a pole at $t = 0$, defined as follows: For $\omega \in H^0(\Omega^{n+1}_{\mathcal{U}})$ a holomorphic $(n+1)$-form defined on $\mathcal{U}$, there exists an $n$-form $\psi$ that is holomorphic on $\mathcal{U}'$ but possibly with a pole along $V_0$, such that

$$\omega = \psi \wedge df\,.$$

It follows from the discussion above that for $t \neq 0$ the restrictions

$$\psi_t = \psi\big|_{V_t}$$

are well defined and holomorphic. We will see below that

$$\psi_0 = \psi\big|_{V_0}$$

is holomorphic on $V_0 \backslash \{p\}$. These $w$'s are the geometric sections of $\mathcal{H}^n_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}/B})$.
An alternate definition is that

$$\psi_t = \text{Poincaré residue of } \omega/f - t.$$

Here $f(x) - t$ is a holomorphic function on $\mathcal{U}$ so that $\omega/f - t$ is a meromorphic $(n+1)$-form with a $1^{\mathrm{st}}$ pole along $V_t$. With this description it is clear that $\psi_0$ has the properties mentioned above.

# Hodge filtration on the vanishing cohomology

We have seen above that the MHS on the vanishing cohomology $H^n(V_\infty)$ is described by embedding the family $\{V_t\}_{t\in B}$ in a family $\{X_t\}_{t\in B}$ of projective varieties where

- $X_t$ is smooth for $t \neq 0$;
- $X_0$ has an isolated singular point $p$ at the isolated singular point of $V_0 \subset X_0$.

Then the "action" on the LMHS on the $H^m_{\lim}(X_t) = H^m(X_\infty)$ due to the singularity of $X_0$ is concentrated around $p$ and so the MHS on $H^n(V_\infty)$ may be obtained by localizing using the exact sequences of $H^m(X_t, V_t)$ for $m = n, m+1$ and Clemens-Schmid.

Above we have described the weight filtration on $H^n(V_\infty)$, and the Hodge filtrations on the $H^m_{\mathrm{lim}}(X_t)$. Although this is a holomorphic description, it is *global* only in the sense that hypercohomology is global.
On the other hand the cohomology

$$H^n(V_t, \mathbb{C}) = \mathcal{H}^n_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}/B})_t$$

is for $t \neq 0$ defined *globally* holomorphically. Moreover there is the space of distinguished geometric holomorphic sections of $\mathcal{H}^n_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}/B})$ over all of $B$. A natural question is

> *Can the Hodge filtration on $H^n(V_\infty)$ be described globally holomorphically in terms of geometric sections of $\mathcal{H}^n_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}/B})$?*

In other words, can the Hodge filtration be described in terms of the asymptotic behavior of the periods

$$\int_{\delta_t} \mathrm{Res}_{V_t} \left( \frac{g(x) dx_1 \wedge \cdots \wedge dx_{n+1}}{f(x) - t} \right)$$

where $g(x) = \mathcal{O}(\mathcal{U})$? This question has a very nice positive answer due to Varchenko (cf. [AGLV] and [K]) which goes as follows:

Denote by $\omega$ the integrand in the above period integral, and form the vector $[\omega]$ of such periods when the vanishing cycles run over a basis for $H_n(V_t, \mathbb{Z})$. By the theorem on regular points there is an expansion

$$[\omega] = \sum_k t^\alpha (\ln t)^k A^\omega_{k,\alpha}/k!$$

where $\alpha = (1/2\pi i) \log \lambda$ with $\lambda$ running over the eigenvalues of the semi-singular part $T_s$ of monodromy.

Define the *order* $\alpha(\omega)$ of the geometric section to be the smallest value of $\alpha$ for which one of $A^{\omega}_{0,\alpha}, A^{\omega}_{1,\alpha}, \ldots$ is non-zero, and define the *principle part* $[\omega]_{\max}$ of $[\omega]$ by

$$[\omega]_{\max} = \sum_k t^{\alpha(\omega)} (\ln t)^k A^{\omega}_{k,\alpha(\omega)} / k!$$

Then the sub-bundle $F^p \mathcal{H}^n_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}/B})$ is described by

> $F^p \mathcal{H}^n_{\mathrm{DR}}(\Omega^\bullet_{\mathcal{U}/B})$ *is represented by the principal parts of the geometric sections of order* $\alpha(\omega) \leqq n - p$.

**Final question:** In the theory of moduli of general type one arrives at the situtaion

$$\mathcal{X} \xrightarrow{\pi} B$$

where for $t \neq 0$ the $X_t = \pi^{-1}(t)$ are smooth general type varieties while $X_0$ is normal and has an isolated semi-log-canonical (slc) singular point $p$. The total space $\mathcal{X}$ will then be smooth except that at $p$ it may have a canonical singularity at $p$. Localizing around $p$ one arrives at

$$\mathcal{U} \xrightarrow{f} B$$

as in the above notes but where $\mathcal{U}$ will not be smooth but will have a canonical singularity at $p \in V_0$. *One may ask about the extent to which the above theory carries over, perhaps with modifications, to this situation?*

Example: The central fibre $X_0$ has a simple elliptic singularity $p$ of degree $d$ with $1 \leqq d \leqq 9$. For $d \leqq 3$ we have an isolated hypersurface singularity as in these notes, while for $d \geqq 4$, $\mathcal{X}$ will be singular. In a semi-stable reduction of $\mathcal{X} \to B$ the central fibre may be taken to be $\widetilde{X}_0 \subset Y$ where $\widetilde{X}_0 \to X_0$ is a desingularization, $Y$ is a del Pezzo, and $\widetilde{X}_0 \cap Y = E$ is an elliptic curve with $E^2 = -d$ that contracts to $p$.[‡‡]

[‡‡] For $d = 4$ we have a complete intersection isolated singular point, a case to which much of the above theory extends (cf. [D]).

How does the assumption that $\mathcal{X}$, and therefore also $\mathcal{U}$, have a canonical singularity at $p$ enter? One way is that assuming $K_{\mathcal{U}}$ is a line bundle, a holomorphic section $\omega \in H^0(\mathcal{U}\backslash\{p\}, K_{\mathcal{U}})$ will induce a holomorphic section $\widetilde{\omega} \in H^0(\Omega^{n+1}_{\widetilde{\mathcal{U}}})$ for any resolution $\widetilde{\mathcal{U}} \to \mathcal{U}$ of the singularity $p \in \mathcal{U}$. Thus we may define the geometric sections of $f_*\Omega^{n+1}_{\mathcal{U}/B}$ as before by taking Poincaré residues along the $V_t$ of $\omega \in H^0(\mathcal{U}\backslash\{p\}, \Omega^{n+1}_{\mathcal{U}})$. This then leads as above to a possible definition of the spectrum and one may ask the extent to which this determines the singularity type.

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