Using Hodge theory to detect the structure of a compactified moduli space*

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*This talk is based on joint work in progress with Mark Green, Radu Laza and Colleen Robles. The example is drawn from the work of Marco Franciosi, Rita Pardini and Sönke Rollenske and on discussions that we have had with them.

Abstract

For a moduli space $\mathcal{M}$ of smooth general type varieties X with a given Hilbert polynomial $\overset{m}{\oplus}\,\chi(mK_X)$, Kollár-Shepherd-Barron-Alexev proved the existence of a canonical completion $\overline{\mathcal{M}}$. For curves the structure (stratification) of $\overline{\mathcal{M}}$ is well known and may be described Hodge theoretically. For algebraic surfaces the picture is quite different. We will discuss (i) some general results about how Hodge theory may be used to study moduli of surfaces, and (ii) how these results go some distance towards determining the structure of $\overline{\mathcal{M}}$ for one very beautiful surface. One new ingredient is the definition and use of a cohomological expression for derivative of the period mapping at a singular surface.

Outline

I. I. Statement of the problem

II. Background and general results

III. I-surfaces

I. Statement of the problem

The cohomology groups $H^n(X)$ of a smooth general type algebraic variety carry polarized Hodge structures. Denoting by D the period domain of all PHS's (V, Q, F) of this type there is a period mapping

$$\Phi : \mathcal{M} \to \Gamma \backslash D$$

where $G = \mathrm{Aut}(V, Q)$ and $\Gamma \in G$ is a discrete group of automorphisms of D that contains the monodromy group associated to the family of varieties paramterized by $\mathcal{M}$. The first general result, not restricted to the case of surfaces, is that there is a canonical extension of the period mapping to

$$\Phi_e : \overline{\mathcal{M}} \to \overline{\mathcal{P}} \tag{1}$$

where $\overline{\mathcal{P}}$ is a compact Hausdorff space that has a Hodge-theoretically constructed stratification by complex analytic subvarieties.

It is conjectured, and proved in special cases, that $\overline{\mathcal{P}}$ has an analytic structure and that the Hodge line bundle $\Lambda_e \to \overline{\mathcal{P}}$ is ample.[†] The stratification of $\overline{\mathcal{P}}$ is known by Lie theory, and one would like to use that together with the extended period mapping (1) to provide information about $\overline{\mathcal{M}}$. Specifically we have the

Problem: *To what extent does the stratification of* $\overline{\mathcal{P}}$ *determine that of* $\overline{\mathcal{M}}$*?*

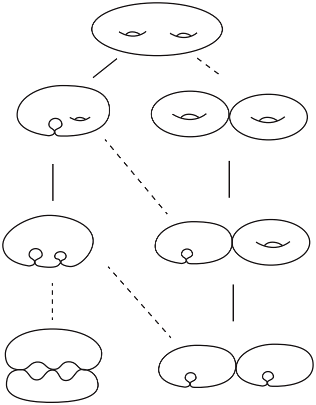
[†]cf. [GGLR] and especially [BBT] for a very interesting approach and results about this issue.

The stratification of $\overline{\mathcal{M}}$ is by deformation type. The singular surfaces X parametrized by the boundary of $\mathcal{M}$ admit desingularizations $\widetilde{X}$ and one wants to know which $\widetilde{X}$'s occur in the general classification of algebraic surfaces and how is X obtained from $\widetilde{X}$? One also wants to understand the dimensions and incidence relations among the strata; i.e., which degenerations $X \to X'$ occur? For the particular I-surface discussed later in this talk there is the work in [FPR] which gives the most complete analysis of this question that I am aware of in a particular example.

On the Hodge theory side there are two basic types of subvarieties of $\overline{\mathcal{P}}$ and then there is the amalgam of these. The first type is the stratification associated to the boundary components given by limiting mixed Hodge structures $(V, Q, W(N), F_{\mathrm{lim}})$ that occur when polarized Hodge structures degenerate. Roughly speaking one thinks of going in $\mathcal{P}$ to the boundary of $\Gamma\backslash D$. In classical terms the period matrices are polynomials in $\log t$ with analytic coefficients, and we let $t \to 0$. Lie theory provides a classification of how this may happen.

The other type of subvarieties of $\overline{\mathcal{P}}$ is that given by the Mumford-Tate sub-domains $D' \subset D$. Associated to a polarized Hodge structure (V, Q, F) is the algebra $T(F)$ of Hodge tensors in $\overset{k,\ell}{\oplus}(\otimes^k V) \otimes (\otimes^\ell V^*)$, and D' is the orbit in D of $T(F)$ under the Lie group associated to the subgroup $G' \subset G$ preserving that algebra. Geometrically, for algebraic surfaces in first approximation one thinks of those X's having additional Hodge classes in $H^2(X)$.

Example: For algebraic curves the structure of $\overline{\mathcal{M}}_g$ is a much studied and very beautiful subject. For the first case $g = 2$ the picture of the stratification is



Here the solid lines reflect degenerations of the first type (infinite order monodromy), while the dotted lines reflect those of the second type (finite order monodromy).

The results we shall discuss about algebraic surfaces are of the following two types.

1. General results valid for any KSBA moduli space of general type surfaces.
2. For I surfaces, defined to be smooth surfaces X with $q(X) = 0$, $p_g(X) = 2$ and K_X ample. Informally stated we shall see there are three results about the completed moduli space $\overline{\mathcal{M}}_I$:
   (a) for the part $\overline{\mathcal{M}}_I^G$ of Gorenstein degenerations there is an analogous picture to the solid line part of the one above for $g = 2$ curves; the stratification of $\overline{\mathcal{M}}^G$ is faithfully captured by the extended period mapping
   $$\Phi_e : \overline{\mathcal{M}}^G \to \overline{\mathcal{P}};$$
   (b) in a phenomenon not present in the curve case, Hodge theory provides a guide as to how to desingularize a general point of the boundary $\overline{\mathcal{M}}_I^G \backslash \mathcal{M}_I$;

(c) for the part $\overline{\mathcal{M}}_I^{NG}$ of normal, non-Gorenstein degenerations corresponding to the dotted lines in the $g = 2$ example there are partial results, a very interesting example, and a question/conjecture about what part of the general story might be.

Parts (a) and (b) above have been the subject of a number of previous talks for which [G1] and [G2] contain slides and text. These results are based in large part on the work of [FPR] and on discussions we have had with them.
In this talk we will focus more on part (c) as the geometric issues involved necessitate considerations not present in the curve case including the cohomological analysis of the derivative of a period mapping at a *singular* variety.

II. Background and general results

This section is divided into three subsections:

A. Background from moduli theory
B. Background from Hodge theory
C. Some general results.

A. Moduli theory. We will give an informal account of what we will use from moduli theory, restricting here to the cases of curves and surfaces. The two main points are these:

(i) Given a family $\mathcal{X}^* \xrightarrow{f} \Delta^*$ of minimal smooth varieties $X_t = f^{-1}(t)$ of general type, one wants to define a *unique* limit X_0.[‡]

[‡]Here it is understood that the total space $\mathcal{X}^*$ is smooth. This is not the most general formulation of the issue. One wants to also allow the X_t to have canonical singularities. For curves the X_t should be smooth but for surfaces they could have -2 curves; i.e., smooth rational curves $C \subset X_t$ with $C^2 = -2$. By adjunction this implies that $K_{X_t} \cdot C = 0$ so that these C's are contracted by the pluricanonical mappings.

The result is that after possibly allowing a base change there is a unique completion of $\mathcal{X}^* \to \Delta^*$ to a family

$$\mathcal{X} \xrightarrow{f} \Delta$$

such that $\mathcal{X}$ has canonical singularities and the dualizing sheaf $\omega_{\mathcal{X}/\Delta}$ is relatively ample.

Operationally this means that X_0 should be reduced and the Weil canonical sheaf K_{X_0} should have the properties

- mK_{X_0} is a line bundle for some $m > 0$; and
- K_{X_0} is ample.

For curves the condition that the surface $\mathcal{X}$ have canonical singularities implies that X_0 has only nodes and the first condition is vacuous. In general an X_0 satisfying these conditions is said to be *stable*. The main result is (again here informally stated)

A moduli space $\mathcal{M}$ has a unique completion $\overline{\mathcal{M}}$ where all varieties corresponding to points of $\overline{\mathcal{M}}$ are stable.

There are a number of quite non-trivial technical issues required to properly formulate much less prove this result. For a discussion of these we refer to [K] and the references cited there.

(ii) The second point is that for curves and surfaces the singularities of a stable X_0 have been classified.

For simplicity of notation we shall simply use X instead of X_0. For curves as mentioned above, the singularities of X consist of nodes. For surfaces a rough organization of the singularity type is given by the table

X	normal singularities	non-normal singularities
K_X	G	NG

where G stands for Gorenstein and NG stands for non-Gorenstein.

In the K_X-NG spot, by definition there is smallest integer, the *index* $m \geqq 2$ of X, such that mK_X is a line bundle.[§] The first row means that the singularities of X could be isolated (i.e., points), or could occur along curves. In the K_X-G spot, $K_X = \omega_X$ is the dualizing sheaf and is a line bundle.
In this talk we shall be especially interested in the case when X has normal singularities; we shall denote by (X, p) the pair given by a stable surface X and a normal (and hence isolated) unique singular point p. Then the classification breaks into 2-types.

[§] A significant issue is to give a good bound on the index. Here we refer to [RU] for very interesting recent work.

$\mathbf{K_X}$-G: These are the *canonical singularities*, concerning which there is a rich and vast literature (e.g., Chapter 4 in [R]). They are also referred to as Du Val or ADE singularities and are locally analytically equivalent to isolated hypersurface singularities $f(x_1, x_2, x_3) = 0$ in $\mathcal{U} \subset \mathbb{C}^3$. For example, A_n is given by

$$x_1^2 + x_2^2 + x_3^{n+1} = 0.$$

For $n = 1$ there is the standard resolution $(\widetilde{X}, C) \to (X, p)$ where $C \cong \mathbb{P}^1$ is a -2 curve (i.e., $C^2 = -2$). In general the C is a configuration of -2 rational curves corresponding to the nodes in a Dynkin diagram.

For the next type we shall use the singularity theorists' notation

$$\frac{1}{n}(1, r), \qquad \text{gcd }(n, r) = 1$$

for the quotient $\mathbb{C}^2 / \left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta^r \end{smallmatrix}\right)$ where $\zeta = e^{2\pi i/n}$ is a primitive n^{th} root of unity.

$\mathbf{K_X}$-NG: These are required to be $\mathbb{Q}$-Gorenstein smoothable, meaning that there should be a local smoothing whose relative dualizing sheaf is $\mathbb{Q}$-Cartier. Then there are two types of such singularities:

(i) the $\frac{1}{dn^2}(1, dna-1)$ singularities; for $d=1$ these are called *Wahl singularities*. Again for these there is an extensive literature (cf. [H1] and the references cited therein);

(ii) the $\mathbb{Z}_2$-quotients of simple elliptic or cusp singularities (cf. (3.24)(c) in [K]).

The non-isolated KSBA singularities are given by pairs (X, C) where C is a (possibly reducible) double curve having isolated pinch points and nodes. Typically there is a resolution

$$(\widetilde{X}, \widetilde{C}) \to (X, C)$$

where $\widetilde{X}$ is smooth, $\widetilde{C} \subset \widetilde{X}$ is a possibly reducible nodal curve with an involution

$$\tau : \widetilde{C} \to \widetilde{C},$$

and (X, C) is the quotient of $(\widetilde{X}, \widetilde{C})$ by the involution τ where we identify $p \in C$ with $\tau(p) \in C$. References for and examples of these are [K], [FPR] and the various lecture notes in [G1] and [G2].

B. Hodge theory:

We shall not give an extensive discussion here but refer to [GGLR], [GG], and the sets of notes [G1] and [G2] for the definitions and statements of the results from Hodge theory that will be used in this talk. Here we shall review the terminology and establish notations.

The main objects of Hodge theory are

- *Polarized Hodge structure* (PHS) of weight n is given by the data (V, Q, F) where
  - $F = \{F^p\}$ is a decreasing *Hodge filtration* on $V_{\mathbb{C}}$ satisfying $F^p \oplus \overline{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}}$;
  - for $V^{p,q} = F^p \cap \overline{F}^q$ this is equivalent to a *Hodge decomposition* $V_{\mathbb{C}} = \overset{p+q=n}{\oplus} V^{p,q}$, $\overline{V}^{p,q} = V^{q,p}$;
  - $Q : V \otimes V \to \mathbb{Q}$ is non-degenerate with $Q(u, v) = (-1)^n Q(v, u)$ and which satisfies the Hodge-Riemann I, II bilinear relations.

Example: $H^n(X, \mathbb{Q})$ *where* X *is a smooth projective variety.* Without the polarization condition we have a *Hodge structure of weight* n. In practice we usually have a lattice $V_{\mathbb{Z}} \subset V$.

- *Mixed Hodge structure* (MHS) is given by the data (V, W, F) where $W = \{W_k\}$ is an increasing filtration on V such that the filtration induced by $\{F^p\}$ on $\mathrm{Gr}_k^W V = W_k / W_{k-1}$ is a Hodge structure of weight k.

Example: $H^n(X, \mathbb{Q})$ where X is a complete algebraic variety and the weight filtration is $W_0 \subset \cdots \subset W_n = V$.

Given a nilpotent endomorphism $N \in \mathrm{End}(V, Q)$, $N^{m+1} = 0$ there is a unique filtration $W(N)$ with

- $W_0(N) \subset \cdots \subset W_{2m}(N) = V$;
- $N : W_k(N) \to W_{k-2}(N)$;
- $N^k : \mathrm{Gr}_{m+k}^{W(N)} V \xrightarrow{\sim} \mathrm{Gr}_{m-k}^{W(N)}(V)$.

- *Limiting mixed Hodge structure (LMHS)* $(V, W(N), F)$ is given by a MHS with weight filtration $W(N)$ where the condition

$$N : F^p \to F^{p-1}$$

is satisfied.

One frequently writes $F_{\lim}$ for F. Below we will give an example of a LMHS.

- *Period domain* D is given by the set of PHS's (V, Q, F) with given Hodge numbers $h^{p,q} = \dim V^{p,q}$.

It is a homogeneous complex manifold $G_{\mathbb{R}}/H$ where $G_{\mathbb{R}}$ is the real Lie group asssociated to the $\mathbb{Q}$-algebraic group $G := \mathrm{Aut}(V, Q)$. The isotropy group H is compact. The *classical case* is when D is a Hermitian symmetric domain; equivalently $H = K$ is the maximal compact subgroup of $G_{\mathbb{R}}$. Otherwise we are in the *non-classical case.*

Example: When $n = 1$ and $h^{1,0} = g$,

$$D = \mathrm{Sp}(2g, \mathbb{R})/\mathcal{U}(g) = \mathcal{H}g.$$

When $n = 2$ and $h^{2,0} = a$, $h^{1,1} = b$,

$$D = 0(2a, b)/\mathcal{U}(a) \times O(b).$$

D is non-classical if, and only if, $a \geqq 2$.

- *The compact dual* $\check{D}$ is the set of filtrations F satisfying HRI. It is a rational homogeneous variety

$$\check{D} = G_{\mathbb{C}}/P$$

where P is a parabolic subgroup of $G_{\mathbb{C}}$. The period omain is an open $G_{\mathbb{R}}$-orbit $D \subset \check{D}$.

Example: $D = \mathcal{H}$ and $\check{D} = \mathbb{P}^1$.

We recall that there is a canonical $G_{\mathbb{C}}$-invariant inclusion

$$T\check{D} \subset \oplus \operatorname{Hom}(F^p, V_{\mathbb{C}}/F^p).$$

- *The infinitesimal period relation* (IPR) is the $G_{\mathbb{C}}$-invariant sub-bundle $I \subset T\check{D}$ defined by

$$I = \{\theta \in T\check{D} \subset \oplus \operatorname{Hom}(F^p, V_{\mathbb{C}}/F^p) : \theta(F^p) \subset F^{p-1}\}.$$

- A *variation of Hodge structure* (VHS) is given by

$$\Phi : B \to \Gamma\backslash D$$

  where B is a complex manifold, Φ is a locally liftable holomorphic mapping satisfying

$$\Phi_* : TB \to I,$$

  and where $\Gamma \subset G_{\mathbb{Z}}$ is a discrete group containing the image of

$$\Phi_* : \pi_1(B) \to G_{\mathbb{Z}}.$$

We frequently refer to Γ as the monodromy group associated to the VHS.

Example: Let $\mathcal{X} \xrightarrow{\pi} B$ be a smooth family of projective algebraic varieties $X_b = f^{-1}(b)$. Setting $V = H^n(X_{b_0})$ for some base point $b_0 \in B$, a VHS is given by

$$\Phi(b) = \text{PHS on } H^n(X_b), \quad b \in B.$$

Here the $H^n(X_b)$ are a monodromy invariant direct sum of PHS's.

- An *infinitesimal variation of Hodge structure* (IVHS) (V, Q, F, T, θ) is given by a point $F \in D$, a vector space T and a map

$$\theta : T \to I_F \subset \oplus \operatorname{Hom}(F^p, F^{p-1}/F^p)$$

which satisfies

$$\theta \wedge \theta = 0.$$

Example: Given a VHS as above, for a point $b \in B$, $T = T_b$ and $\theta = \Phi_*$ define an IVHS.
Given an IVHS as above we set

$$\mathrm{Gr}_F\, V_{\mathbb{C}} = \oplus F^p / F^{p+1}.$$

Then there is an induced map

$$\theta \in \mathrm{Hom}^{-1}(\mathrm{Gr}_F\, V_{\mathbb{C}}) = \oplus \mathrm{Hom}(F^p/F^{p+1}, F^{p-1}/F^p).$$

The integrability condition $\theta \wedge \theta = 0$ then gives

$$\mathrm{Gr}_F\, V_{\mathbb{C}} \textit{ is an } ST\textit{-module.}$$

Here $ST = \oplus \mathrm{Sym}^k\, T$ is the symmetric algebra on T.

Let $B = \Delta^*$ be the unit disc $\{t \in \mathbb{C} : 0 < |t| < 1\}$ and

$$\Phi : \Delta^* \to \Gamma_T \backslash D$$

a VHS over Δ^*. We denote by $T \in \mathrm{Aut}(V_{\mathbb{Z}}, Q)$ the canonical generator of $\Phi_*(\pi_1(\Delta^*))$, and by $\Gamma_T = \{T^k : k \in \mathbb{Z}\}$ the subgroup of $G_{\mathbb{Z}}$ generated by T. Letting $T = T_s T_u$ denote the Jordan decomposition of T into semi-simple and unipotent factors, the *monodromy theorem* gives

- $T_s^\ell = I$ for some ℓ;
- $T_u = e^N$ where $N^{m+1} = 0$ for some $m \leq n$.

Example: Associated to the above VHS over Δ^* there is a LMHS $H^n_{\lim}$; in the precise sense given by [S] one has

$$\lim_{t\to 0} \Phi(t) = H^n_{\lim}$$

which by suggestive abuse of notation is sometimes written as

$$\lim_{t\to 0} H^n(X_t) = H^n_{\lim}.$$

Given the data

- A complex manifold $\overline{B}$, a reduced normal crossing divisor $Z \subset \overline{B}$ with complement $B = \overline{B} \backslash Z$ and a VHS

$$\Phi : B \to \Gamma \backslash D$$

  with image $\mathcal{P} = \Phi(B)$, one seeks to complete $\mathcal{P}$ to a $\overline{\mathcal{P}}$ by adding some of the information in the LMHS's that arise from the limits $\lim_{b\to Z} \Phi(b)$. In this talk we will be mainly concerned with the case $\overline{B} = \Delta$ and $B = \Delta^*$.

- There are basically two types of information in the LMHS's. One is the associated graded to the LMHS's. This leads to Satake-Baily-Borel (SBB) type completions, which may be thought of as minimal ones; these are the main ones used in the construction of $\overline{\mathcal{P}}$. The other is the full LMHS's, which may be thought of as adding the extension data to the information in the SBB completion (cf. [KU] for the general theory here). As will be illustrated below, these toroidal type objects may be used as a guide to desingularizing moduli spaces of surfaces.

- Very roughly speaking there are two types of boundary components; viz. over $\mathbb{Q}$ and over $\mathbb{Z}$. There is yet to be a formal definition of the latter, which in this talk this will be taken to be the conjugacy class of T_s (which is closely related to the *spectrum* in the case of isolated hypersurface singularities). For the former we use the conjugacy class of N. For $n = 1$ since $N^2 = 0$ this is determined by rank N. For $n = 2$ one has the classification
  - $N^2 = 0$; then we have rank N.
  - $N^2 \neq 0$; then we have rank N and rank N^2.

  One may picture the $\mathbb{Q}$-boundary structure by a diagram in which the conjugacy classes and possible degenerations are represented. For $n = 1$ and $h^{1,0} = g$ the diagram is

$$\mathrm{I}_0 \text{ ------ } \mathrm{I}_1 \text{ ------ } \cdots \text{ ------ } \mathrm{I}_g .$$

For $n = 2$ and $h^{2,0} = 2$ the diagram is

$$
\begin{tikzcd}
 & & \mathrm{III} \arrow[dr, dash] & & \\
\mathrm{I}_0 \arrow[r, dash] & \mathrm{I} \arrow[ur, dash] \arrow[dr, dash] & & \mathrm{IV} \arrow[r, dash] & \mathrm{V} \\
 & & \mathrm{II} \arrow[ur, dash] & &
\end{tikzcd}
$$

References to these diagrams are given in the talks [G1], [G2].

- The final Hodge theoretic part of the story for this talk is to describe a cohomological expression for the derivative of the period mapping at a singular variety. We will formulate the general question and then give the answer in the very special case to be used in the examples below.

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Let
$$\mathcal{X}^* \to \Delta^*$$
be a family of smooth projective varieties giving a period mapping
$$\Phi : \Delta^* \to \Gamma_T \backslash D.$$
We assume that the monodromy $T = T_s$ is of finite order. Then Φ extends across the origin and one wants a cohomological expression for the derivative of Φ at $t = 0$. The first issue is to define the derivative. For this we note that $\Phi(0) \in \Gamma_T \backslash D$ is an orbifold singularity of the analytic variety $\Gamma_T \backslash D$, and we make a base change $\widetilde{t} = t^m$ to have a diagram

$$\begin{tikzcd} \widetilde{\Delta} \arrow[r, "\widetilde{\Phi}"] \arrow[d] & D \arrow[d] \\ \Delta \arrow[r, "\Phi"] & \Gamma_T \backslash D. \end{tikzcd}$$

Then we *define* the derivative to be $\widetilde{\Phi}_*$ at $\tilde{t} = 0$.

To give a cohomological expression for the differential the general procedure is to do semi-stable reduction (SSR) to obtain

$$\begin{tikzcd} \widetilde{\mathcal{X}} \arrow[r] \arrow[d, "\tilde{f}"'] & \mathcal{X} \arrow[d, "f"] \\ \widetilde{\Delta} \arrow[r] & \Delta \end{tikzcd}$$

where $\widetilde{\mathcal{X}}$ is smooth and the fibre $\widetilde{X}_0 = \tilde{f}^{-1}(0)$ is a reduced normal crossing divisor. There is then a recipe for computing $\lim_{\tilde{t}\to 0} H^n(\widetilde{X}_t)$. Here with our applications in mind we change notation and assume given a family of surfaces

$$\mathcal{X} \xrightarrow{f} \Delta$$

when $\mathcal{X}$ is smooth, which is a smooth fibration over Δ^* and where

$$X_0 = X_1 \cup_C X_2 \tag{2}$$

with X_1, X_2 being smooth surfaces glued along a smooth rational curve C in each X_i to give X_0.

This implies that the monodromy $T = \mathrm{Id}$; then we have

$$\Phi : \Delta \to D.$$

We want a cohomological expression for

$$\Phi_*(d/dt)\big|_{t=0} := d/dt.$$

What could d/dt be? First

$$H^2_{\lim} = (V, Q, F_{\lim})$$

is a pure Hodge structure of weight 2 associated to the X_0 above. The obvious weight 2 Hodge structures are

$$H^2(X_1),\ H^2(X_2),\ H^0(C)(-1).$$

Secondly, by the IPR

$$d/dt \in \mathrm{Hom}(F^2_{\mathrm{lim}}, F^1_{\mathrm{lim}}/F^2_{\mathrm{lim}}).$$

The condition on normal bundles

$$N_{C/X_1} \cong N^*_{C/X_2} \tag{3}$$

should enter reflecting the 1^{st} order information that $\mathcal{X} \to \Delta$ should give a smoothing deformation. The result is the

Theorem

There is a natural map of direct sums

$$H^0(\Omega^2_{X_1}) \oplus H^0(\Omega^2_{X_2}) \to \big(H^1(\Omega^1_{X_2})/[C]\big) \oplus \big(H^1(\Omega^1_{X_1})/[C]\big)$$

that gives d/dt at $t = 0$.[¶]

[¶] The important point is the intertwining of X_1 and X_2 which will be seen to be a reflection of (3).

We will not prove this but will show where the natural maps come from. There are two exact sequences that intertwine the roles of X_1, X_2 and whose cohomology sequences lead to the map. They are

$$
\begin{tikzcd}
0 \arrow[r] & \Omega^2_{X_1}(-C) \arrow[r] & \Omega^2_{X_1} \arrow[r] & \Omega^1_C \otimes N^*_{C/X_1} \arrow[r] \arrow[d, phantom, "\wr\|"] & 0 \\
& & & \Omega^1_C \otimes N_{C/X_2}. &
\end{tikzcd}
\tag{4}
$$

Here the last term in the top sequence uses adjunction $K_C \cong K_{X_1} \otimes_{\mathcal{O}_C} N_{C/X_1}$; the vertical isomorphism uses (3) above and is *the key step*. The next sequence is

$$
0 \longrightarrow \Omega^1_{X_2}(\log C) \longrightarrow \Omega^1_{X_2}(C) \longrightarrow \Omega^1_C \otimes N_{C/X_2} \longrightarrow 0. \tag{5}
$$

Here the right-hand term uses $\mathcal{O}_C(C) \cong N_{C/X_2}$. The final sequence is

$$(6) \qquad 0 \longrightarrow \Omega^1_{X_2} \longrightarrow \Omega^1_{X_2}(\log C) \xrightarrow{\mathrm{Res}} \mathcal{O}_C \longrightarrow 0,$$

where the last map is the usual Poincaré residue map. Putting together the cohomology sequences arising from (4), (5), (6) gives

$$\begin{tikzcd}
 & & & & H^0(\mathcal{O}_C)(-1) \arrow[d] \\
0 \arrow[r] & H^0(K_{X_1}(-C)) \arrow[r] & H^0(K_{X_1}) \arrow[r] & H^0(K_C \otimes N^*_{C/X_1}) \arrow[d, phantom, "\wr\|"] & H^1(\Omega^1_{X_2}) \arrow[d] \\
 & & & H^0(K_C \otimes N_{C/X_2}) \arrow[r] \arrow[ur, dashed] & H^1(\Omega^1_{X_2}(\log C)) \arrow[d] \\
 & & & & H^1(\mathcal{O}_C)(-1).
\end{tikzcd}$$

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Since C is rational, we have $H^1(\mathcal{O}_C)(-1) = 0$, so that there is a factorization as indicated by the dotted arrow. This then defines the map

$$\Phi_*(d/dt) : H^0(K_{X_1}) \to H^1(\Omega^1_{X_2})$$

in the theorem. The $[C]$ comes from the image of the Gysin map $H^0(\mathcal{O}_C)(-1) \to H^1(\Omega^1_{X_2})$.

C. General results

(i) Let $\mathcal{M}$ be a KSBA moduli space for a class of surfaces of general type and with canonical completion $\overline{\mathcal{M}}$. Our first general result concerns mappings of $\mathcal{M}$ and $\overline{\mathcal{M}}$. It is known that the structure of $\overline{\mathcal{M}}$ may be arbitrarily nasty and we have not worked out the exact technical conditions under which the following results will hold. We do assume that each component of $\overline{\mathcal{M}}$ is generically reduced and that a general point corresponds to a smooth surface. Then there is a holomorphic period mapping

$$\Phi : \mathcal{M} \to \mathcal{P} \subset \Gamma \backslash D$$

whose image $\mathcal{P}$ is a locally closed analytic subvariety.

From [BBT] it follows that the closure of $\mathcal{P}$ in $\Gamma\backslash D$ is a quasi-projective algebraic variety over which the Hodge line bundle is ample.[‖]
The first main "result" is that the above period mapping extends to

$$\Phi_e : \overline{\mathcal{M}} \to \overline{\mathcal{P}} \tag{7}$$

and that over the extended Hodge line bundle is ample. This "result" has been established only in special cases. What is known [GGLR] is that $\overline{\mathcal{P}}$ exists as a compact Hausdorff space with a stratification by complex analytic subvarieties and that Φ_e is defined and is a continuous mapping.

[‖] The interesting work [BBT] uses *o*-minimal structures (arising initially from model theory) to put an algebraic structure on $\mathcal{P}$. The techniques introduced there and in the references to that work seem certain to have further applications to Hodge theory.

As a set $\overline{\mathcal{P}}$ consists of the associated graded PHS's to the equivalence classes of LMHS's obtained from families $\mathcal{X}^* \to \Delta^*$ of smooth surfaces parametrized by discs $g : \Delta^* \to \mathcal{M}$. The essential geometric content of the statement is that

$$\mathrm{Gr}\left(\lim_{t\to 0} H^2(X_t)\right)$$

depends only on the limit surface X_o and not on the disc $\overline{g} : \Delta \to \overline{\mathcal{M}}$ extending g above with $\overline{g}(0)$ corresponding to X_0. Informally stated, one does not have to blow up $\overline{\mathcal{M}}$ along $\partial\mathcal{M}$ to extend the period mapping. In the examples of I-surfaces discussed below one may see this directly.

(ii) For the next general result we denote by

$$\mathcal{M}_f \subset \overline{\mathcal{M}}$$

the subvariety of $\overline{\mathcal{M}}$ parametrizing singular surfaces X such that there exists a smoothing $\mathcal{X} \to \Delta$ of $X = X_0$ with finite monodromy; using the above notation $T = T_s$ and $N = 0$. A general result is

The period mapping extends to $\Phi : \mathcal{M}_f \to \Gamma\backslash D$,

and in the situation at hand we have

$$\overline{\mathcal{M}}^{\mathrm{NG}} \subset \mathcal{M}_f. \tag{8}$$

Here, $\overline{\mathcal{M}}^{\mathrm{NG}}$ denotes the subvariety of $\overline{\mathcal{M}}$ parametrizing *normal* surfaces X having non-Gorenstein singularities. Informally stated, to a normal and smoothable surface X having non-Gorenstein semi-log-canonical singularities (slc) one may associate a polarized Hodge structure $H^2_{\lim}(X)$.

This result is really more of an observation than a theorem: it is a consequence of the statements

- normal surfaces with rational singularities are parametrized by a subvariety of $\mathcal{M}_f$ (i.e., they have finite monodromy); and
- normal, non-Gorenstein slc singularities are rational.

Canonical singularities are also rational so they are again parametrized by a subvariety in $\mathcal{M}_f$. For I-surfaces thus far there is no example other than the above of singular surfaces with finite monodromy.

(iii) The normal surfaces X with infinite monodromy are those with either simple elliptic singularities or cusps. To state the following result for simplicity we shall assume that a general smooth surface is regular and that the singular surface has either e simple elliptic singularities or c cusps but does not have some of each.** The result is then

$$\begin{cases} \operatorname{rank} N \leqq e \leqq p_g + 1 \\ \operatorname{rank} N^2 \leqq c \leqq p_g + 1. \end{cases} \tag{9}$$

**There is a general result without these assumptions, but it is more complicated to formulate and the special case given here captures the essential geometric content of the result.

(iv) Finally we will explain a general result one would like to hold and that can be established in a couple of cases. It deals with the question posed in the title of this talk. Recall the notation $\mathcal{M}_f \subset \overline{\mathcal{M}}$ for the subvariety of a complete KSBA moduli space parametrizing smoothable surfaces X around which the local monodromy of a smoothing deformation is finite.[††] The period mapping then extends from $\mathcal{M}$ to give

$$\Phi : \mathcal{M}_f \to \Gamma \backslash D.$$

One may show that the image B is a *closed* analytic subvariety. As noted above it follows from the results in [BBT] that $\mathcal{P}$ is quasi-projective; indeed, the Hodge line bundle $\Lambda \to \mathcal{P}$ is ample.

[††] For I-surfaces discussed below, it seems to be the case that all such surfaces X are normal; i.e., if X is not normal, then any smoothing deformation has infinite order monodromy. We do not know how general one might expect this phenomenon to be.

Let $M \subset \mathcal{M}_f$ be an irreducible component of $\mathcal{M}_f$. One would like to show that

There exists a Γ-invariant Mumford-Tate subdomain $D' \subset D$ with Γ' the discrete group of automorphisms of D' induced by Γ such that

$$M = \Phi^{-1}(\mathcal{P} \cap \left(\Gamma' \backslash D'\right)).$$

Informally this means that these components of moduli can be detected Hodge theoretically.
How might one prove this, at least in special cases such as the two discussed below? For those M such that the surfaces X parametrized by M are normal with either canonical or non-Gorenstein singularities, such singularities are rational and the resolution

$$(\widetilde{X}, D) \to (X, p)$$

of a particular one has for D a configuration of $\mathbb{P}^1$'s.

Recalling that X gives a PHS denoted here by $\Phi(X) \in D$, the $\mathbb{P}^1$'s give Hodge classes not present on a general point in D, and then D' could be the Mumford-Tate domain defined by PHS's having these additional Hodge classes. One might then use a variational argument to show that in $T\,\mathrm{Def}\,X$ the condition to retain these Hodge classes defines the tangent space to $M \subset \overline{\mathcal{M}}$. As an illustration this argument will be carried out in the two cases

- an A_1-singularity that is not a base point of $|K_X|$;
- the $\frac{1}{4}(1,1)$-singularity on the general I-surface having that type of singularity.

Both arguments will use the differential of the period mapping at a singular surface that was discussed above.

III. I-surfaces

This section is divided into three sub-sections:

A. Generalities on I-surfaces

B. Hodge theory and moduli: the non-Gorenstein case

C. Hodge theory and moduli: the Gorenstein case

A. Generalities on I-surfaces

Definition: An I-surface is a connected, reduced surface X that is of general type and that satisfies

- $K_X^2 = 1$
- $\chi(\mathcal{O}_X) = 3$
- $h^1(\mathcal{O}_X) = 0$.

We note that if X is Gorenstein, i.e., the Weil divisor K_X is a line bundle, then $K_X^2 = 1$ implies that X is irreducible. There is essentially one known example of a reducible I-surface ([FPR]). Henceforth we will assume that X is irreducible.
If X is Gorenstein, then the pluricanonical ring

$$R(X) = \overset{m}{\oplus} H^0(mK_X)$$

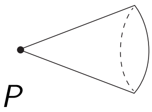
has the *postulated form*; i.e., generators and relations are added only when they are forced to by the formulas

$$\begin{aligned} h^0(K_X) &= 2 \\ h^0(mK_X) &= \frac{m(m-1)}{2} + 2, \qquad m \geqq 2. \end{aligned}$$

It follows that

- the canonical map $\varphi_{K_X} : X \dashrightarrow \mathbb{P}^1$ has one base point and the reduced fibres are curves C with arithmetic genus $p_a(C) = 2$;
- the bi-canonical map gives a morphism
$$\varphi_{2K_X} : X \to \mathbb{P}(1,1,2),$$
which is a 2:1 covering branched over a quintic $V \in |\mathcal{O}_{\mathbb{P}(1,1,2)}(5)|$ not passing through the singular point $P = (0,0,1)$ of $\mathbb{P}(1,1,2)$.

Remark: $\mathbb{P}(1,1,2)$ is realized as a singular quadric



in $\mathbb{P}^3$; the curves $C \in |K_X|$ are branched over P and the remaining 5 points of the intersection of V with the rulings of that quadric.

- the 5-canonical map

$$\varphi_{5K_X} : X \hookrightarrow \mathbb{P}(1,1,2,5)$$

is an embedding that realizes X by an equation

$$z^2 = a_0 y^5 + a_1 y^4 + \cdots + a_5$$

where $a_i(x_0, x_1)$ is homogeneous of degree $2i$ and $a_0 \neq 0$;
- the *local Torelli property* (LT), i.e., that the differential of the period mapping is 1-1, holds for any smooth X (cf. [PZ], [CT] and [G1], [G2]);
- it is suspected that generic global Torelli holds in the sense that the period mapping

$$\Phi : \mathcal{M} \to \mathcal{P} \subset \Gamma \backslash D$$

has degree 1, where $\Gamma = \Phi_*(\pi_1(\mathcal{M}))$ is the global monodromy group;
- Γ is an arithmetic group (it is plausible that $\Gamma = G_{\mathbb{Z}}$, but this has not been proved).

Remark: For generic global Torelli there are two heuristic arguments using known generic global properties for the surfaces parametrized by boundary components of $\overline{\mathcal{M}}_I$; one of these will be discussed below.

- the period domain has dimension 57 and the IPR is a contact system; thus $\mathcal{P}$ is a contact submanifold of $\Gamma\backslash D$.

Finally we note that there are three known divisors in $\overline{\mathcal{M}}_I$

- that given by the Hodge line bundle $\Lambda_e \to \overline{\mathcal{M}}_I$;
- the locus $\mathcal{M}_{\mathfrak{n}}$ of I-surfaces X having a node (A_1-singularity); and
- the locus $\mathcal{M}_W$ of I-surfaces having a Wahl $\frac{1}{4}(1,1)$ singularity.

An interesting question is whether or not we have

$$\mathrm{Pic}(\overline{\mathcal{M}}_I) \otimes \mathbb{Q} \text{ has } \Lambda_e, \mathcal{M}_{\mathfrak{n}}, \mathcal{M}_W \text{ as a basis.}$$

B. Hodge theory and moduli; the non-Gorenstein case

One may formulate a desired result, putting in precise terms the statement

> *In a KSBA moduli space the locus of normal surfaces having finite monodromy may be detected Hodge theoretically. This locus consists of exactly surfaces with canonical singularities and non-Gorenstein isolated singularities.*

There are two cases where this result is known and we shall discuss those here.

$\mathbf{A_1}$-singularities: Let (X, p) be a surface with an ordinary double point p and $(\widetilde{X}, C) \to (X, p)$ the standard desingularization. Locally analytically the picture is

- the singularity is analytically equivalent to the origin in the surface $x^2 + y^2 + z^2 = 0$;
- we desingularize by blowing up the origin which then gives a $C \cong \mathbb{P}^1$ with $C^2 = -2$;
- we take a smooth quadric $Q \subset \mathbb{P}^3$ with hyperplane section isomorphic to C;
- taking $X_1 = \widetilde{X}$ and $X_2 = Q$, the surface

$$X_1 \cup_C X_2$$

satisfies the condition (3) above.

The map

$$d/dt : H^0(K_{\widetilde{X}}) \to H^1(\Omega^1_Q)/[C] \cong H^2(Q)_{\mathrm{prim}}$$

is non-zero if, and only if, there is an $\widetilde{\omega} \in H^0(K_{\widetilde{X}})$ with $\widetilde{\omega}\big|_C \neq 0$. We note that

$$K_{\widetilde{X}}\big|_C \cong K_C \otimes N^*_{C/\widetilde{X}} \cong \mathcal{O}_C.$$

Using $H^0(K_{\widetilde{X}}) \cong H^0(K_X)$, with $\omega \in H^0(K_X)$ corresponding to $\widetilde{\omega}$ the above condition is

$$\omega(p) \neq 0;$$

i.e., p is not a base point of the canonical series.

This gives the following result, which is of interest not for the statement but for the method of proof.

> *If $|K_X|$ is base-point-free, then the condition to have a node is a smooth divisor in moduli.*

Note: At the risk of being pedantic the precise formulation is this: Let (X, p) be a surface that is smooth except for one A_1-singularity p that is not a base point of $|K_X|$. Assume that the local deformation space $\mathrm{Def}\, X$ is irreducible and contains a smoothing deformation. Then

- the derivative of the period mapping $\mathrm{Def}\, X \to \Gamma_T \backslash D$ is non-zero in the smoothing direction;
- the locus in $\mathrm{Def}\, X$ of nodal surfaces is a reduced divisor.

The proof of the second statement requires two steps:

(i) the locus in $\mathrm{Def}\,\widetilde{X}$ where the line bundle $L = [C]$ deforms is a reduced divisor;

(ii) if L deforms with $\widetilde{X}$, then the section $C \in H^0(\widetilde{X}, L)$ deforms with $(\widetilde{X}, L)$; and

(iii) if C deforms with $(\widetilde{X}, L)$, then the deformed curve may be contracted to a node.

These can be checked for the A_1-singularity; in fact, (iii) follows automatically from (ii). In the next example this will no longer be the case.
Using the simultaneous resolution of ADE singularities [A], it seems likely that the above can be extended to general such surfaces; this is a work in progress.

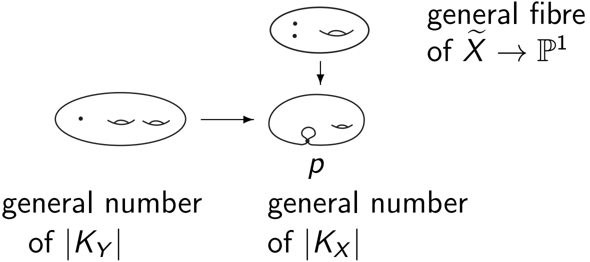
Wahl singularity on an I-surface

The only known non-Gorenstein singularity on a normal I-surface X is the $\frac{1}{4}(1,1)$ singularity described in [FPR] and [H2]. We shall give a geometric description of the surface, referring to [H2] for the equations of X and of its $\mathbb{Q}$-Gorenstein smoothing. We recall that the bi-canonical model of a smooth I-surface Y gives a 2-sheeted crossing

$$Y \to \mathbb{P}(1,1,2)$$

branched over a quintic $V_Y \in |\mathcal{O}_{\mathbb{P}(1,1,2)}(5)|$ not passing through the vertex P of the singular quadric $\mathbb{P}(1,1,2) \hookrightarrow \mathbb{P}^3$. The I-surface X with a Wahl $\frac{1}{4}(1,1)$ singularity arises by allowing V_Y to pass through P but otherwise be general. To desingularize X we first consider the desingularization $\mathbb{F}_2 \to \mathbb{P}(1,1,2)$ where $\mathbb{F}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ has a section E with $E^2 = -2$ and that contracts to P.

The blow up $\widetilde{Y}$ of Y at the base point of $|K_Y|$ then gives a morphism $\widetilde{Y} \to \mathbb{P}^1$ whose fibres are the $p_a = 2$ curves in $|K_X|$, and a 2:1 morphism $\widetilde{Y} \to \mathbb{F}_2$ with branch curve $E + V_Y$. The curve E defines a section of $\widetilde{Y} \to \mathbb{P}^1$ that meets a general fibre in a Weierstrass point of that fibre.
In the limit as V_Y tends to V_X the surface $\widetilde{Y}$ tends to the desingularization $\widetilde{X}$ of X. The picture of the limit of general fibres of $|K_Y|$ and $|K_X|$ is



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It follows that $\widetilde{X} \to \mathbb{P}^1$ is a regular elliptic surface with $p_g(\widetilde{X}) = 2$ and having a bi-section $\widetilde{C}$ with $\widetilde{C}^2 = -4$. This section contracts to $p \in X$.

Recalling that $\dim \overline{\mathcal{M}}_I = 28$ one may show that

> $\#$ moduli $\widetilde{X} = 32$
>
> $\#$ moduli $(\widetilde{X}, L) = 30$ (thus $p_g(\widetilde{X})$ imposes independent conditions to deforming L along with $\widetilde{X}$)
>
> $\#$ moduli $(\widetilde{X}, L, \widetilde{C}) = 27$ (thus there is one condition that $\widetilde{C}$ deform along with $(\widetilde{X}, \widetilde{C})$).

The computation of d/dt is then used to show that

> *I-surfaces* (X, p) *with a Wahl* $\frac{1}{4}(1,1)$ *singularity form a reduced divisor in* $\overline{\mathcal{M}}_I$ *that may be detected Hodge-theoretically.*

C. Hodge theory and moduli; the Gorenstein case

The result here is

> *The irreducible components of $\overline{\mathcal{M}}_I^G$, together with the incidence relations (degenerations) among them, map 1-1 to the Hodge-theoretically defined strata in $\overline{\mathcal{P}}$.*

The proof is done using the classification of the strata in $\overline{\mathcal{M}}_I^G$ from [FPR] together with an analysis of the LMHS's in the various cases.

- Rather than display the whole table the following is just the part for simple elliptic singularities (types I_k and III_k in the earlier diagram of types of LMHS's). These degenerations have $N^2 = 0$ since for the semi-stable-reduction (SSR) only double curves (and no triple points) occur. All of the other types occur if we include cusp singularities.

In the following

- $k = \#$ elliptic singularities; in general, as previously noted using Hodge theory one may show that $k \leqq p_g + 1$.
- $d_i =$ degree of elliptic singularity.
- $\widetilde{X} =$ minimal desingularization of X. In a SSR given by $\widetilde{\mathfrak{X}} \to \widetilde{\Delta}$ the surface $\widetilde{X}$ will appear as one component of the fibre over the origin.

In the following table, in the 1^{st} column subscripts denote the degrees of the elliptic singularities, which are uniquely determined by the $[T_s]$'s; we will explain the $\sum(9 - d_i)$ column below.

stratum	dimension	minimal resolution $\widetilde{X}$	$\sum_{i=1}^{k}(9-d_i)$	k	codim in $\overline{\mathcal{M}}_I$
I_0	28	canonical singularities	0	0	0
I_2	20	blow up of a K3-surface	7	1	8
I_1	19	minimal elliptic surface with $\chi(\widetilde{X})=2$	8	1	9
$\mathrm{III}_{2,2}$	12	rational surface	14	2	16
$\mathrm{III}_{1,2}$	11	rational surface	15	2	17
$\mathrm{III}_{1,1,R}$	10	rational surface	16	2	18
$\mathrm{III}_{1,1,E}$	10	blow up of an Enriques surface	16	2	18
$\mathrm{III}_{1,1,2}$	2	ruled surface with $\chi(\widetilde{X})=0$	23	3	26
$\mathrm{III}_{1,1,1}$	1	ruled surface with $\chi(\widetilde{X})=0$	24	3	27

Note that the last column is the sum of the two columns preceding it.

Example: For I_2 the picture is

$$
\begin{tikzcd}
 & (\widetilde{X}, \widetilde{C}) \arrow[dl] \arrow[dr] & \\
(X_{\min}, C) & & (X, p)
\end{tikzcd}
$$

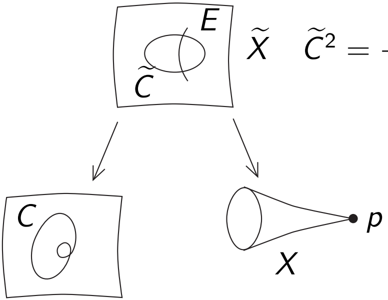
Here, $p =$ isolated normal singular point on X, $\widetilde{C} =$ curve on $\widetilde{X}$ with $\widetilde{C}^2 = -2$ and that contracts to p. From Hodge theory

$$2 = p_g(\widetilde{X}) + g(\widetilde{C}) \text{ and } p_g(\widetilde{X}) = 1$$

we see that $g(\widetilde{C}) = 1$ (simple elliptic singularity)

- It may be shown that $\mathrm{Hg}^1(\widetilde{X})$ has a $\mathbb{Z}^2$ summand with intersection form
$$\begin{pmatrix} -2 & 2 \\ 2 & -1 \end{pmatrix};$$
and that the basis classes are effective.

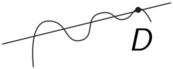
- Hodge theory then suggests the picture



$\widetilde{C}^2 = -2,\ E^2 = -1$

$$\begin{cases} X_{\min} = K3 \\ C^2 = 2 \end{cases}$$

$\implies X_{\min} \xrightarrow{2:1} \mathbb{P}^2$ branched over D



- LMHS has
  - $\mathrm{Gr}_2 \cong H^2(X_{\mathsf{min}})_{\mathrm{prim}}$
  - $\mathrm{Gr}_3 \cong H^1(\widetilde{C})(-1)$
- $\#$ of PHS's of type $\mathrm{Gr}_3 \oplus \mathrm{Gr}_2 = 19 + 1 = 20$ which suggests that for the boundary component of $\overline{\mathcal{M}}_I$ we have codim $= 8$.
- How to get this number? First approximation to the fibre over the origin in a SSR is blowing up p in $\mathcal{X}$ to have

$$\widetilde{X} \cup_{\widetilde{C}} (m\mathbb{P}^2)$$

where $\widetilde{C} \in |\mathcal{O}_{\mathbb{P}^2}(3)|$ and m is the multiplicity of p. Next one does base change and normalization to arrive at a SSR. Rather than proceed this way we just take $\widetilde{X} \cup_{\widetilde{C}} \mathbb{P}^2$ and ask what we need to do to smooth this surface.

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For this have to blow up $9 - (-\widetilde{C}^2) = 7$ points on $\widetilde{C}$ to obtain triviality of the infinitesimal normal bundle as a necessary condition for smoothability. This suggests that

- The extension data for the LMHS contains a factor

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathrm{Hg}^1(\widetilde{\mathbb{P}}^2), H^1(\widetilde{C})(-1)) \cong \oplus J(\widetilde{C})$$

in which the seven points appear.[‡‡]

Fibre over origin in a several parameter SSR is given by blowing up seven points on $\widetilde{C}$; this is a del Pezzo surface.

- Hodge theory suggests where to look — the seven parameters arise from the possible extension data for GR(LMHS) — and following FPR one may go back and prove things algebraically.

[‡‡]Considerations of this type first appear in [F].

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