Moduli and Hodge Theory *

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*Talk at the "Geometry at the Frontier" conference, Pucón, Chile (November 12, 2018), and based in part on joint work in progress with Mark Green, Radu Laza and Colleen Robles. Selected references to works quoted in or related to this talk are given at the end.

Outline

I. *Introduction*

Background and statement of the two main results

II. *Hodge theory*

Basic definitions; limiting mixed Hodge structures

III. *Moduli and period mappings*

The canonical minimal completion of the image of a period mapping

IV. *Use of Hodge theory to analyze the moduli space of I-surfaces*

Illustration of how Hodge theory guides the determination of the boundary structure of moduli of regular, general type surfaces X with $p_g(X) = 2$, $K_X^2 = 1$.[†]

[†]Based in significant part on the work of Marco Franciosi, Rita Pardini and Sönke Rollenske (FPR); cf. the references at the end.

I. Introduction

- The construction and study of moduli spaces is of central interest in algebraic geometry.
- Algebraic varieties are built out of three basic types:
  - rationally connected ($\kappa(X) = -\infty$; for curves $g = 0$)
  - Calabi-Yau's, abelian varieties, ... ($\kappa(X) = 0$; for curves $g = 1$)
  - general type ($\kappa(X) = \dim X$; for curves $g \geqq 2$).

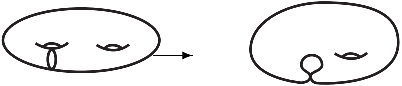
  Here

$$\dim H^0(mK_X) \sim m^{\kappa(X)} + \cdots$$

  measures the growth of the dimension of the space of global differential forms $f(z)(dz^1 \wedge \cdots \wedge dz^n)^m$.

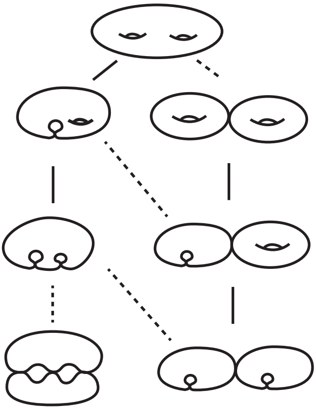
- The moduli spaces (if they exist) behave quite differently in the three cases — for X of general type and with a fixed Hilbert polynomial $\overset{m}{\oplus}(\chi(mK_X)$ Kollár-Shepherd-Barron- Alexeev (KSBA) proved the existence of $\mathcal{M}$ with a canonical projective completion $\overline{\mathcal{M}}$ — for surfaces we need only specify $q(X) = h^{1,0}$, $p_g(X) = h^{2,0}(X)$ and $K_X^2 = c_1^2(X)$. For $q(X) = 0$, $p_g(X) = 2$ and $K_X^2 = 1$ we have an *I-surface* with moduli space $\mathcal{M}_I$.

- For $\dim X = n = 1$ we have $\mathcal{M}_g$ with an essentially smooth $\overline{\mathcal{M}}_g$ having a canonical stratification of $\partial\mathcal{M}_g = \overline{\mathcal{M}}_g \backslash \mathcal{M}_g$ — $\overline{\mathcal{M}}_g$ is much studied and very beautiful — the analysis of the classical period matrix of degenerating curves



provided an early guide to understanding $\partial\mathcal{M}_g$.

- The picture of $\overline{\mathcal{M}}_2$ is

(∗) 

This gives the stratification of $\overline{\mathcal{M}}_2$ together with the incidence (degeneration) relations among the strata. (The solid and dotted arrows will be explained later.)

- For $n \geqq 2$ aside from the work of FPR I know of no example where a significant part of the structure of $\partial\mathcal{M}$ has been analyzed.
- Associated to a stable curve X as in $(*)$ is a Hodge structure (period matrix) in the case when X is smooth and a limiting mixed Hodge structure (LMHS) in the general case. There is a stratification on the space of Gr(LMHS)'s, and this stratification determines the one pictured in $(*)$. The objective of our work is to be able to use Hodge theory in a similar way to study the moduli space $\mathcal{M}$ for general type surfaces.

- The space of (equivalence classes of) LMHS's of a given type may be described using Lie theory. What is needed is

   (i) to connect $\overline{\mathcal{M}}$ to this space via an extended period mapping
   (ii) to then apply this to some interesting examples to determine $\overline{\mathcal{M}}$.

  Following a discussion in Part II of some definitions and properties from Hodge theory, carrying out (i) will be explained in Part III of this talk, and in Part IV we will apply this to the I-surface described above. The results will be
   - the picture $(*)$ seems to carry over very closely as in the curve case[‡]
   - there is the added benefit that whereas $\overline{\mathcal{M}}_g$ is smooth, $\overline{\mathcal{M}}_I$ is highly singular along the boundary and the *proof* of (i) suggests how one might desingularize it.

[‡]Some of this, together with an extension to H-surfaces $(q = 0, p_g = 2, K_X^2 = 2)$, is work in progress with GLR and FPR.

II. Hodge theory

- Associated to a smooth projective variety X is the *Hodge structure* (HS) of *weight m*

$$H^m(X,\mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}, \quad \overline{H}^{p,q} = H^{q,p}$$

  on its cohomology. Here

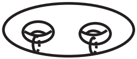
$$H^{m,0}(X) = H^0(\Omega_X^m).$$

Example: For $m = n = 1$ the HS is determined by the period matrix

$$\Omega = \left\| \int_{\gamma_i} \omega_\alpha \right\|$$

$\omega_\alpha \in H^0(\Omega_X^1)$ (dim $= g$)

$\gamma_i \in H_1(X,\mathbb{Z})$ $(\cong \mathbb{Z}^{2g})$



Example: For $m = n = 2$ it is determined by a similar period matrix where

$$\dim H^0(\Omega_X^2) = p_g(X), \quad H_2(X, \mathbb{Z})/\text{torsion} \cong \mathbb{Z}^{b_2(X)}.$$

- A *polarized Hodge structure* of weight m (PHS) is (V, Q, F)
  - $Q : V \otimes V \to \mathbb{Q}$, $Q(u, v) = (-1)^m Q(v, u)$
  - $\begin{cases} F^m \subset F^{m-1} \subset \cdots \subset F^0 = V_{\mathbb{C}} & \text{(Hodge filtration)} \\ F^p \oplus \overline{F}^{m-p+1} \xrightarrow{\sim} V_{\mathbb{C}}. \end{cases}$

  For $V^{p,q} = F^p \cap \overline{F}^q$ the second condition is the same as

$$\begin{cases} V_{\mathbb{C}} = \oplus V^{p,q}, & \overline{V}^{p,q} = V^{q,p} \\ F^p = \bigoplus_{p' \geqq p} V^{p',q} \end{cases}$$

  - $\begin{cases} Q(F^p, F^{m-p+1}) = 0 & \text{(HR I)} \\ i^{p-q} Q(V^{p,q}, \overline{V}^{p,q}) > 0 & \text{(HR II)} \end{cases}$

- $H^m(X) = \oplus$ PHS's — in the examples above Q is the intersection form and (HR I) and (HR II) result from

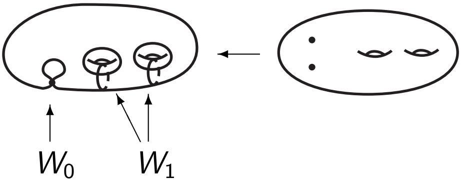
$$\begin{cases} \int_X \omega \wedge \omega' = 0 & \text{(because } \omega \wedge \omega' = 0\text{)} \\ c_n \int_X \omega \wedge \overline{\omega} > 0 & \text{(because } c_n\, \omega \wedge \overline{\omega} > 0\text{)} \end{cases}$$

  where $\omega, \omega' \in H^0(\Omega_X^n)$, $\dim X = n$, and c_n is a constant.
- *Mixed Hodge structure* (MHS) is $(V, W_\bullet, F^\bullet)$
  - $(0) \subset W_0 \subset \cdots \subset W_\ell = V$ (weight filtration)
  - $F^m \subset F^{m-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}$ (Hodge filtration)

  where
  - $F^\bullet$ induces a HS of weight k on

$$\mathrm{Gr}_k^W V = W_k / W_{k-1}.$$

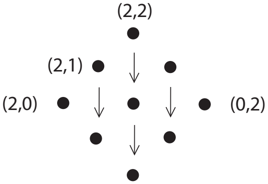
Example: $H^m(X)$ where $X =$ complete algebraic variety of dimension m and the weight filtration is $W_0 \subset \cdots \subset W_m$.



- *Limiting mixed Hodge structure* (LMHS)
  - $N : V \to V$ with $N^{m+1} = 0$

$$\begin{cases} W_0(N) \subset \cdots \subset W_{2m}(N) \quad \text{(monodromy weight filtration)} \\ \text{with } N : W_\ell(N) \to W_{\ell-2}(N) \text{ and} \\ N_k : \mathrm{Gr}^{W(N)}_{m+k} \xrightarrow{\sim} \mathrm{Gr}^{W(N)}_{m-k}\,. \end{cases}$$

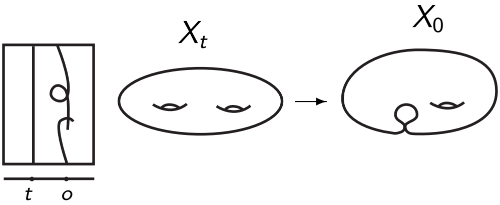
- $\begin{cases} (V, W(N), F^{\bullet}_{\lim}) \text{ is a MHS with} \\ N : F^{p}_{\lim} \to F^{p-1}_{\lim}. \end{cases}$
- There will also be a Q in the picture.
- Gr(LMHS)$\cong \bigoplus_{\ell=0}^{2m} H^{\ell}$ where H^{ℓ} is a HS of weight ℓ — picture is a Hodge diamond. Here $m = 2$ and N is the vertical arrows — the dots are the $H^{p,q}$'s



- we set $h^{p,q}$ = dimension of the (p, q) dot.

Example:

$$\begin{tikzcd} \mathcal{X}^* \arrow[d] & \subset & \mathcal{X} \arrow[d] \\ \Delta^* & \subset & \Delta \end{tikzcd}$$



- monodromy $T : H^m(X_t) \to H^m(X_t)$

$$\begin{cases} T = T_s T_u & \text{(Jordan decomposition)} \\ T_s^k = I, & T_u = e^N \ \text{ with } \ N^{m+1} = 0 \end{cases}$$

  thus (i) eigenvalues are roots of unity, and (ii) length of Jordan blocks is $\leqq m$.

- the solid lines in the diagram in the introduction represent degenerations with $N \neq 0$.

Theorem (Schmid)

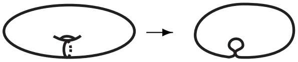
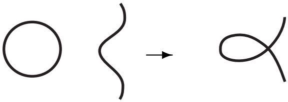
Given $\mathcal{X} \to \Delta$ as above

$$\lim_{t \to 0} H^m(X_t) = \mathrm{LMHS}.$$

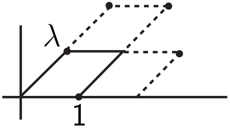
Proof is a combination of

- Lie theory
- complex analysis
- differential geometry

Example:

-  topological picture
-  $y^2 = x(x-1)(x-t)$ algebraic picture

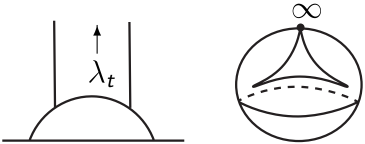
- $X = \mathbb{C}/\Lambda$, $\Lambda = \{1, \lambda\}$



analytic picture

λ determined up to $\lambda \to \frac{a\lambda+b}{c\lambda+d}$ where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

- $\mathcal{M}_1 \cong \mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}$, $\mathcal{H} = \{\lambda : \mathrm{Im}\,\lambda > 0\}$
- in above example $\lambda_t = \frac{\log t}{2\pi i}$



$$\begin{cases} \text{space of PHS's is } \mathcal{H} \subset \mathbb{P}^1,\ V = \left(\begin{smallmatrix} * \\ * \end{smallmatrix}\right),\ Q = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right),\ T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \\ F^1 = \left[\begin{smallmatrix} \lambda \\ 1 \end{smallmatrix}\right] \in \mathbb{P}^1,\ \text{HR II} \iff \mathrm{Im}\,\lambda > 0 \\ \text{as } \lambda \to i\infty \text{ we have } F^1 \to \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = F^1_{\lim}. \end{cases}$$

How does Lie theory enter?

- *Period domain*
  $D = \{F^\bullet = \text{ flag in } V_{\mathbb{C}} : (V, Q, F^\bullet) = \text{PHS}\}$ where a flag is $\{F^m \subset \cdots \subset F^0 = V_{\mathbb{C}}\}$
- *compact dual*
  $\check{D} = \{F^\bullet \text{ is a flag with } Q(F^p, F^{m-p+1}) = 0\}$
- $G = \mathrm{Aut}(V, Q) = \mathbb{Q}$-algebraic group
- $G_{\mathbb{R}}$ acts transitively on D so that

$$\begin{array}{l} D = G_{\mathbb{R}}/H \text{ with } H \text{ compact} \\ \cap \\ \check{D} = G_{\mathbb{C}}/P \text{ with } P \text{ parabolic } \begin{pmatrix} * & * & * & * \\ \circ & * & * & * \\ \circ & \circ & * & * \\ \circ & \circ & \circ & * \end{pmatrix} \end{array}$$

Then $D =$ open $G_{\mathbb{R}}$-orbit in $\check{D}$.

Example:

$m=1$: $D=\mathrm{Sp}(2g,\mathbb{R})/\mathcal{U}(g)=\mathcal{H}_g$ where $g=h^{1,0}$

$m=2$: $D=\mathrm{SO}(2k,\ell)/\mathcal{U}(k)\times\mathrm{SO}(\ell)$ where $k=h^{2,0}$, $\ell=h^{1,1}$

- *Classical case:*

$$\begin{array}{l} D = \text{ Hermitian symmetric domain (HSD)} \\ \; \| \\ G_R/K,\ K = \text{ maximal compact.} \end{array}$$

Two classical cases are

$m = 1$ (curves, abelian varieties)

$m = 2$ is HSD $\iff$ $k = 1$ (K3's)

thus $h^{2,0} \geqq 2$ is non-classical

- $$\left\{\begin{array}{ccc} T_{F^\bullet}\check{D} & \subset & \overset{p}{\oplus}\operatorname{Hom}(F^p, V_{\mathbb{C}}/F^p) \\ \cup & & \\ \xi & \longrightarrow & \overset{p}{\oplus}\,\xi \cdot F^p/F^p \end{array}\right.$$

  (think of $F_t^p \to dF_t^p/dt\big|_{t=0} \bmod F_0^p$)
- *Infinitesimal period relation* (IPR)

$$\{\xi : \xi \cdot F^p \subseteq F^{p-1}\} = I \subset T\check{D}$$

$I = T\check{D} \iff D$ is HSD (classical case)

Example: $m = 2$, $D = \mathrm{SO}(2k, \ell)/\mathcal{U}(k) \times \mathrm{SO}(\ell)$ — first non-classical case when $k = 2$

- dim $D = 2\ell + 1$
- $I =$ contact structure

- Period domains have sub-domains corresponding to PHS's with additional structure; e.g.,

$$\begin{array}{ccc} D' & \subset & D \\ \| & & \\ \left\{ \begin{smallmatrix} \text{reducible PHS's} \\ \text{that are } \oplus\text{'s} \end{smallmatrix} \right\} & & \end{array}$$

  This is what the dotted lines represent in the diagram in the introduction for $\overline{\mathcal{M}}_2$.

- In general the D's are Mumford-Tate domains defined to be those PHS's with a given algebra of Hodge tensors.

- *period mapping* (next topic) will arise from holomorphic mappings
$$\begin{cases} \Phi : & M \to D \\ \Phi_* : & TM \to I \subset TD \end{cases}$$
the differential constraint given by I is the primary feature of the non-classical case
- differential geometry enters via holomorphic vector bundles
$$\mathbb{F}^p \to M$$
which have canonical *Hermitian metrics* due to HR II — these then have curvatures which turn out to have signs.

III. Moduli and period mappings

- Variety Y has *canonical singularities* if for any desingularization $\widetilde{Y} \xrightarrow{f} Y$ we have

$$f^* K_Y = K_{\widetilde{Y}}.$$

  Equivalently, if Y is normal, then for $Y^* = Y \backslash Y_{\mathrm{sing}}$ any $\omega \in H^0(K_{Y^*})$ has

$$\int_{Y^*} \omega \wedge \overline{\omega} < \infty.$$

- $\mathcal{M}$ = moduli space for varieties that are smooth or have canonical singularities.

Question: *What varieties X do we add to obtain $\overline{\mathcal{M}}$?*

- Use valuative criterion: Given $\mathcal{X}^* \to \Delta^*$ what X do we use to uniquely fill in over the origin to have

$$\begin{array}{ccc} \mathcal{X}^* & \subset & \mathcal{X} \\ \downarrow & & \downarrow \\ \Delta^* & \subset & \Delta \end{array}$$

- **Answer (KSBA):** There are two equivalent criterion:
  - X should
    (a) have semi-log-canonical (slc) singularities (local)
    (b) K_X should be ample (global)
  - $\mathcal{X}$ should
    (a′) have canonical singularities (local)
    (b′) $\omega_{\mathcal{X}/\Delta}$ should be relatively ample (global)

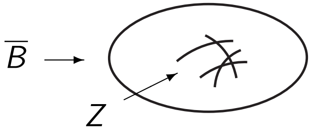
For curves

$$\begin{cases} (\mathrm{a}) = (\mathrm{a}') \iff X \text{ is nodal} \\ (\mathrm{a}) + (\mathrm{b}) = (\mathrm{a}') + (\mathrm{b}') \iff X \text{ is stable.} \end{cases}$$

For surfaces there is a list of slc singularities

- normal singularities (the Gorenstein ones are simple elliptic and cusps)
- non-normal (double curve with pinch points and nodes satisfying conditions with respect to the involution)

- Let B = smooth quasi-projective variety with a smooth, projective completion $\overline{B}$ with $B = \overline{B} \backslash Z$ where $Z = \bigcup Z_i$ is a reduced normal crossing divisor



- *Period mapping* is

$$\Phi : B \to \Gamma \backslash D, \qquad \Gamma \subset \mathrm{Aut}(V_{\mathbb{Z}}, Q)$$

  that satisfies
  - Φ locally liftable and holomorphic
  - $\Phi_* : TB \to I \subset T(\Gamma\backslash D) \qquad (\dot{F}_b^p \subset F_b^{p-1})$

  Then local monodromies around Z_i are quasi-unipotent.
- **Example:** $\mathcal{X} \xrightarrow{\pi} B$ projective family with $\pi^{-1}(b) = X_b$ smooth gives a period mapping where
  - $\Phi(b) =$ PHS on $H^m(X_b)$
  - $\Phi_* : \pi_1(B) \to \Gamma \subset \mathrm{Aut}(X_b)$ is global monodromy.
- *Hodge line bundle* $\Lambda = \det \mathbb{F}^n$ when $m = n$.

  Example: For $\mathcal{X} \xrightarrow{f} B$

$$\Lambda = \det(f_* \omega_{\mathcal{X}/B})$$

- may extend Φ across Z_i where $N_i = 0$ and

$$\Phi : B \to \mathcal{H} \subset \Gamma \backslash D \text{ proper, holomorphic mapping.}$$

Theorem A1: *There exists a canonical minimal completion $\overline{\mathcal{H}}$ of $\mathcal{H}$ to which the augmented Hodge line bundle extends as an ample line bundle $\Lambda_e \to \overline{\mathcal{H}}$.*[§] *Moreover there is an extension of the period mapping to*

$$\Phi_e : \overline{B} \to \overline{\mathcal{H}}.$$

- *What is the boundary $\partial\mathcal{H} = \overline{\mathcal{H}} \backslash \mathcal{H}$?* For $b_0 \in Z$

$$\Phi_e(b_0) = \mathrm{Gr}\left\{ \lim_{b \to b_0} H^m(X_b) \right\}.$$

[§] The augmented Hodge line bundle is $\bigotimes_{p=0}^{[m+1/2]} \det \mathbb{F}^p$. We shall mainly be concerned with the cases $m = 1, 2$.

Example:

$$X_b = \text{ curve } \implies \Phi_e(b_0) = \{H^0(X_{b_0}), H^1(\widetilde{X}_{b_0})\}$$

where $\widetilde{X}_{b_0}$ = normalization of X_{b_0}.

Example: *I*-surface example to be discussed below.

- Regarding the of proof of Theorem A1: Line bundle $L \to Y$ over a compact analytic variety is *free*[¶] if some $L^m \to Y$ is globally generated

$$\Updownarrow$$

$$\begin{cases} \varphi : Y \to \mathbb{P}^N, \\ \varphi^* \mathcal{O}_{\mathbb{P}^N}(1) = L^m \end{cases}$$

$\implies \mathrm{Proj}\, L$ exists as a projective variety (use Spec of $\oplus H^0(L^k)$).

[¶]Also called *semi-ample*.

$(**)$ **Theorem:** *$\Lambda_e \to \overline{B}$ is free.*

Definition: $\overline{\mathcal{H}} = \operatorname{Proj} \Lambda_e$
(depends only on $\mathcal{H}$ and not on $B, \overline{B}$)

- Proof of $(**)$ uses pretty much what is known about VHS's together with some new aspects involving the *geometry* of extension data.
- **Definition:** $\overline{\mathcal{H}}$ is the *Satake-Baily-Borel* (SBB) completion of $\mathcal{H}$.
- Challenge to algebraic geometers: Given a family $\overline{\mathcal{X}} \xrightarrow{f} \overline{B}$ where a general $X_b = f^{-1}(b)$ is smooth and $\omega_{\overline{\mathcal{X}}/\overline{B}}$ is Cartier, Theorem $(**)$ implies that $\det(f_* \omega_{\overline{\mathcal{X}}/\overline{B}})$ is free. I do not know of an algebraic proof of this result.

We note that this is a *relative* construction; it depends on $\Phi : B \to \Gamma \backslash D$, in contrast to the classical case where there is a $\overline{\Gamma \backslash D}^{\mathrm{SBB}}$ where Φ extends to $\Phi_e : \overline{B} \to \overline{\Gamma \backslash D}^{\mathrm{SBB}}$ and $\overline{\mathcal{H}}$ is the image.

- $\mathcal{M}$ = KSBA moduli space, $\overline{B} = \overline{\mathcal{M}}$ is a desingularization.

Theorem A2:[‖] *There is a factorization*

$$\begin{tikzcd} B \arrow[d] & \subset & \overline{B} \arrow[d] \arrow[dr, "\Phi_e"] & \\ \mathcal{M} & \subset & \overline{\mathcal{M}} \arrow[r, dashed] & \overline{\mathcal{H}} \end{tikzcd}$$

Briefly this says

- the period mapping $\mathcal{M} \to \mathcal{H} \subset \Gamma \backslash D$ extends to $\Phi_e : \overline{\mathcal{M}} \to \overline{\mathcal{H}}$; i.e., to a surface corresponding to a boundary point of $\mathcal{M}$ we can *uniquely* associate the associated graded to the LMHS;
- the extended Hodge line bundle on $\overline{B}$ descends to $\overline{\mathcal{M}}$ and there it is free.

[‖]The detailed statement and proof of this result are a work in progress.

IV. Use of Hodge theory to analyze the moduli space of I-surfaces

A. I-surfaces and their period mappings

- Murphy's law (Vakil) — whatever nasty property a scheme can have already occurs for the moduli spaces of general type surfaces — thus unlike curves should select "particular" surfaces to study — in geometry extremal cases are frequently interesting — Noether's inequality

$$p_g(X) \leqq \frac{K_X^2}{2} + 2$$

suggests studying surfaces close to extremal — the 1^{st} case is

Definition: An *I-surface* X is a regular $(q(X) = 0)$ general type surface that satisfies

$$p_g(X) = 2, K_X^2 = 1.$$

- One studies general type surfaces via their pluri-canonical maps

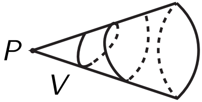
$$(***) \qquad \varphi_{mK_X} : X \dashrightarrow \mathbb{P}H^0(mK_X)^* \cong \mathbb{P}^{P_m - 1}$$

  and pluricanonical rings $R(X) = \oplus H^0(mK_X)$.
- Instead of $(***)$ frequently better to use weighted projective spaces corresponding to when we add new generators to $R(X)$ — from

$$P_m(X) = m(m-1)/2 + 3, \qquad m \geqq 2$$

  and Kodaira-Kawamata-Viehweg vanishing one has for the I-surface

$$\begin{aligned} \varphi_{K_X} &: X \dashrightarrow \mathbb{P}^1, \quad |K_X| = \text{ pencil of hyperelliptic curves} \\ \varphi_{2K_X} &: X \to \mathbb{P}(1,1,2) \hookrightarrow \mathbb{P}^3 \\ \varphi_{5K_X} &: X \hookrightarrow \mathbb{P}(1,1,2,5) \hookrightarrow \mathbb{P}^{12}. \end{aligned}$$

- Picture/equations
  - 

    $\mathbb{P}(1,1,2) \hookrightarrow \mathbb{P}^3$given by
    $(t_0, t_1) \hookrightarrow [t_0^2, t_0 t_1, t_1^2, y]$

    $X =$ 2:1 map branched over P and $V \in |\mathcal{O}_{\mathbb{P}^3}(5)|$
  - $z^2 = F_5(t_0, t_1, y)z + F_{10}(t_0, t_1, y)$ (weighted complete intersection) in $\mathbb{P}(1,1,2,5)$
- $\mathcal{M}_I$ is smooth and
  - $\dim \mathcal{M}_I = h^1(T_X) = 28$
  - $\dim D_I = 57 = 2 \dim \mathcal{M}_X + 1$
- $\Phi = \mathcal{M}_I \to \Gamma_I \backslash D_I$ has Φ_* injective (local Torelli)

$$\Downarrow$$

$$\Phi(\mathcal{M}_I) = \text{contact submanifold } \mathcal{H} \hookrightarrow \Gamma_I \backslash D_I$$

- Γ_I is arithmetic — not known are

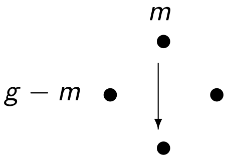
$$\begin{cases} \Gamma_I = G_{\mathbb{Z}} \\ \text{global Torelli} \end{cases} \quad ?$$

B. Stratification of the space of Gr(LMHS)'s

- For curves with $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$ we have for LMHS's

$$\begin{tikzcd} {} \arrow[r, dash] & I_0 \arrow[r, dash] \arrow[d, leftrightarrow] & I_1 \arrow[r, dashed, dash] \arrow[d, leftrightarrow] & I_2 \arrow[r, dash] \arrow[d, leftrightarrow] & \cdots \arrow[r, dash] & I_g \arrow[d, leftrightarrow] \\ & \mathcal{H}_g & \mathcal{H}_{g-1} & \mathcal{H}_{g-2} & & \mathcal{H}_0 \end{tikzcd}$$

  - note that I_{g-m} corresponds to $[N]$ with $N^2 = 0$, rank $N = m$.



For each boundary component we have the stratification

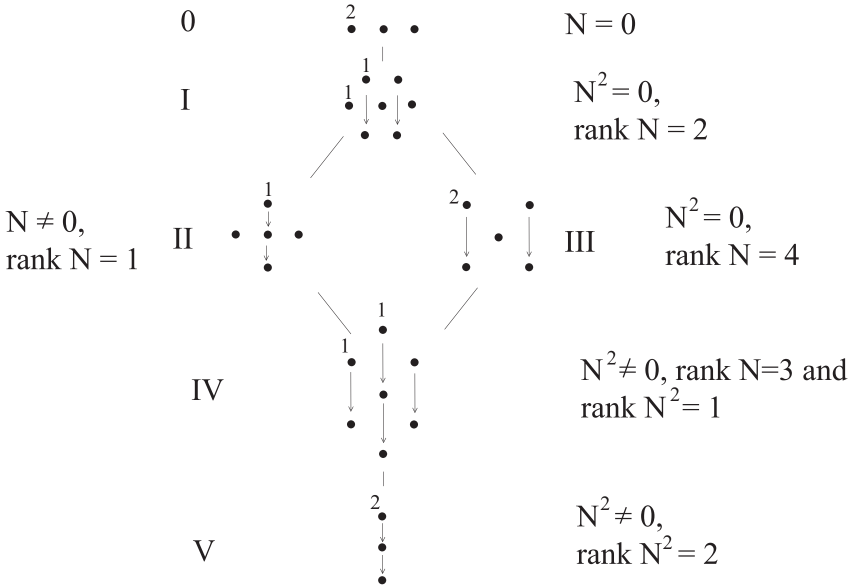
$$H^1 = \oplus H_i^1.$$

The composite of these induces a stratification of $\overline{\mathcal{M}}_g$ by

$$\{\# \text{ nodes}, \ \# \text{ components}\}.$$

Of course this is just the beginning of the story of $\overline{\mathcal{M}}_g$.

- For surfaces with $p_g = 2$ the classification of Gr(LMHS)'s/$\mathbb{Q}$ is


0
$N = 0$
I
$N^2 = 0$, rank $N = 2$
II
$N \neq 0$, rank $N = 1$
III
$N^2 = 0$, rank $N = 4$
IV
$N^2 \neq 0$, rank $N=3$ and rank $N^2 = 1$
V
$N^2 \neq 0$, rank $N^2 = 2$

- $\begin{cases} \text{For the refined Hodge-theoretic stratification of} \\ \text{Gr(LHHS}/\mathbb{Z})\text{'s we use } T_s \to \{\text{conjugacy class } [T_s] \\ \text{of } T_s \text{ in } \Gamma\}. \text{ Within each of these strata we use} \\ \text{Mumford-Tate sub-domains appearing} \\ \text{in Gr(LMHS)'s in } \overline{\mathcal{M}}_I. \end{cases}$
- We begin by considering the Gorenstein part $\overline{\mathcal{M}}_I^{\mathrm{Gor}} \subset \overline{\mathcal{M}}_I$ — one reason for this is the result

  if $X_t \to X$ is a KSBA degeneration of a surface where all the singularities of X are non-Gorenstein, then $N = 0$.

  Hence only Gorenstein singularities can non-trivially contribute to the LMHS/$\mathbb{Q}$.

The following results from coupling the classification in FPR with the analysis of the LMHS's in the various cases.

Theorem B

The Hodge theoretic stratification of $\overline{\mathcal{M}}$ given by the above diagram uniquely determines the stratification of $\overline{\mathcal{M}}_I^{\mathrm{Gor}}$.

- Rather than display the whole table the following is just the part for simple elliptic singularities (types I_k and III_k) — they have $N^2 = 0$ since for the semi-stable-reduction (SSR) of a degeneration only double curves (and no triple points) occur — all of the other types occur if we include cusp singularities.

- In the following
  - X is irreducible (since $K_X^2 = 1$ and any component of X will have positive K^2)
  - $d_i =$ degree of elliptic singularity
  - $k = \#$ elliptic singularities — by Hodge theory one shows in general that $k \leqq p_g + 1$
  - $\widetilde{X} =$ minimal desingularization of X — in a SSR given by $\widetilde{\mathfrak{X}} \to \Delta$ the surface $\widetilde{X}$ will appear as one component of the fibre over the origin.

In the following table, in the 1^{st} column subscripts denote $[T_s]$'s — will explain the $\sum(9 - d_i)$ column below.

stratum	dimension	minimal resolution $\widetilde{X}$	$\sum_{i=1}^{k}(9-d_i)$	k	codim in $\overline{\mathcal{M}}_I$
I_0	28	canonical singularities	0	0	0
I_2	20	blow up of a K3-surface	7	1	8
I_1	19	minimal elliptic surface with $\chi(\widetilde{X})=2$	8	1	9
$\mathrm{III}_{2,2}$	12	rational surface	14	2	16
$\mathrm{III}_{1,2}$	11	rational surface	15	2	17
$\mathrm{III}_{1,1,R}$	10	rational surface	16	2	18
$\mathrm{III}_{1,1,E}$	10	blow up of an Enriques surface	16	2	18
$\mathrm{III}_{1,1,2}$	2	ruled surface with $\chi(\widetilde{X})=0$	23	3	26
$\mathrm{III}_{1,1,1}$	1	ruled surface with $\chi(\widetilde{X})=0$	24	3	27

Example: For I_2 the picture is

$$\begin{tikzcd}[column sep=small] & (\widetilde{X}, \widetilde{C}) \arrow[dl] \arrow[dr] & \\ (X_{\min}, C) & & (X, p) \end{tikzcd}$$

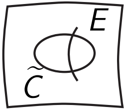
Here, $p =$ isolated normal singular point on X, $\widetilde{C} =$ curve on $\widetilde{X}$ that contracts to p — from Hodge theory

$$2 = p_g(\widetilde{X}) + g(\widetilde{C}) \text{ and } p_g(\widetilde{X}) = 1$$

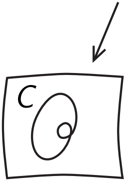
we see that $g(\widetilde{C}) = 1$ (simple elliptic singularity)

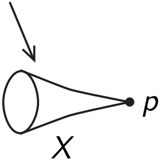
- $\mathrm{Gr}(\mathrm{LMHS})/\mathbb{Z}$ suggests that $\mathrm{Hg}^1(\widetilde{X})$ has a $\mathbb{Z}^2$ with intersection form
$$\begin{pmatrix} -2 & 2 \\ 2 & -1 \end{pmatrix}$$
for heuristic reasoning assume basis classes are effective.

- Hodge theory now suggests the picture



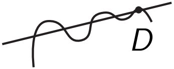
$\widetilde{X}$ $\quad \widetilde{C}^2 = -2,\ E^2 = -1$





$$\begin{cases} X_{\text{min}} = K3 \\ C^2 = 2 \end{cases}$$

$\implies X_{\text{min}} \xrightarrow{2:1} \mathbb{P}^2$ branched over D



- LMHS has $\begin{cases} \mathrm{Gr}_2 \cong H^2(X_{\mathsf{min}})_{\mathrm{prim}} \\ \mathrm{Gr}_3 \cong H^1(\widetilde{C})(-1) \end{cases}$
- $\#$ of PHS's of type $\mathrm{Gr}_3 \oplus \mathrm{Gr}_2 = 19 + 1 = 20$ which suggests
  - codim $= 8$
- How to get this number? First approximation to fibre over origin in a SSR is blowing up p in $\mathcal{X}$ to have

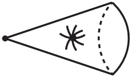
$$\widetilde{X} \cup_{\widetilde{C}} \mathbb{P}^2$$

  where $\widetilde{C} \in |\mathcal{O}_{\mathbb{P}^2}(3)|$
- Now have to blow up $9 - (-\widetilde{C}^2) = 7$ points on $\widetilde{C}$ to obtain triviality of the infinitesimal normal bundle as a necessary condition for smoothability. Thus

Fibre over origin in Δ^2 is given by blowing up seven points on $\widetilde{C}$ — this is a del Pezzo.

- Hodge theory suggests where to look and following FPR we can go back and prove things algebraically as follows:
  - X has one elliptic singularity P of degree 2
  - given in $\mathbb{P}(1,1,2,5) = \mathbb{P}$ by

$$\begin{cases} z^2 = f_{10}(x_1, x_2, y) \\ f \in (x_2^4, x_2^3 y, \dots, y^4) = (x_2, y)^4 \text{ is generic} \end{cases}$$

  - $X \xrightarrow{2:1} Q_0$ P  branched over $P + V$ where

    $V \in |\mathcal{O}_{Q_0}(5)|$ has ordinary quadruple point giving p.
  - smoothing $z^2 = f + \epsilon g$
  - $\dim \mathrm{I}_2 = 20 = \dim \left\{ \frac{g \in H^0(\mathcal{O}_{\mathbb{P}}(10))}{(x_2, y)^4 + (x_1 f_{x_2} + x_1^2 f_y)} \right\}$

- the $\sum_{i=1}^{k}(9-d_i)$ column translates into: To desingularize $\overline{\mathcal{M}}_I$ near the normal locus you do SSR using

$$\widetilde{X} \cup \Big(\bigcup_i Y_i\Big) \cup \Big(\bigcup_{\alpha=1}^{9-d_i} Z_{\alpha i}\Big).$$

- Finally, *what about the non-Gorenstein singularities?* From the list of normal slc singularities of surfaces these typically are quotient singularities. For those for which the local monodromy is a non-trivial quotient of the finite group that gives the singularity, one might say that they are detected Hodge-theoretically.

However there is one notable exception to this, namely the *Wahl singularity* ($T = \mathrm{Id}$). If

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \Delta^m \arrow[r] & \Delta \end{tikzcd}$$

is the minimal SSR, then for I-surfaces it turns out to be the case that

- the period mapping gives $\Phi : \Delta \to D$ (there is no need to quotient by a Γ);
- the point $\Phi(o) \in D$ is a PHS with an extra Hodge class arising from $\mathrm{Hg}^1(\widetilde{X})$, where $\widetilde{X} \to X$ is the minimal desingularization of X.

(Some details remain to be checked here.)

Conclusion

The SBB completion $\overline{\mathcal{H}}$ *of the image of moduli under the period mapping gives an invariant that has a rich structure and that provides an important and possibly complete guide to the boundary structure of the moduli space.*

Thank you

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FPR

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