Global properties of period mappings on the boundary

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Outline

I. Introduction

The *global* study of variations of polarized, pure Hodge structures is an extensively studied and in many ways fairly well-developed subject (cf. [CM-SP]). Thus for a period mapping

$$\Phi : B \to \Gamma \backslash D$$

where B is a smooth quasi-projective variety, if we assume that the differential Φ_* is generically injective, then it is known that both $K_{\overline{B}}(\log Z)$ and $T^*_{\overline{B}}(\log Z)$ are nef and big (cf. [Z]). Here $\overline{B}$ is any completion of B with boundary $Z = \overline{B} \backslash B$ a normal crossing divisor. It is also known that B is hyperbolic modulo a proper subvariety (cf. [LSZ]). With no assumption on Φ_* it has been recently proved that for Γ arithmetic the image $\Phi(B) \subset \Gamma \backslash D$ is a quasi-projective algebraic variety over which the augmented Hodge bundle $L \to P$ is ample ([BBT]).*

*For background and the terminology from Hodge theory used in these notes we refer to [GGLR] and [GG].

On the other hand, assuming unipotent monodromies around the irreducible branches Z_i of Z, it is well known that the VHS over B extends canonically to $\overline{B}$ and on the boundary Z one has a variation of limiting mixed Hodge structures whose *local* structure has been extensively studied through the work of Cattani-Kaplan- Schmid and others (cf. [CKS]). From multiple perspectives it has recently become clear that what is now needed is a *global* study of the variation of *limiting* mixed Hodge structures along Z.[†] That is what will be undertaken in these notes.

[†] There is a global study of variations of regular mixed Hodge structures (cf. [PS] and the references there). As we will see the global study for limiting mixed Hodge structures is a somewhat different story.

We denote by $Z_I = \bigcap_{i \in I} Z_i$ the closed strata in Z with $Z_I^* \subset Z_I$ being the Zariski open smooth part of Z_I. Then taking the associated graded pure Hodge structures of the limiting mixed Hodge structures along Z_I^* gives period mappings

$$\Phi_I : Z_I^* \to \Gamma_I \backslash D_I \tag{I.1}$$

in the usual sense.

Thus the new information concerns the question

What happens globally along the fibres Y^* of (I.1)*?*

What is varying along these fibres is the *extension data* associated to the family of limiting mixed Hodge structures along Y^*. Here it is important to keep in mind is that these limiting mixed Hodge structures have locally constant associated graded pure Hodge structures. There will be global monodromy arising from the action of $\pi_1(Y^*)$ but it will preserve the weight filtration along each connected component of Y^*, and will act by a finite group on the graded quotients.

As a first step this invites the study of the geometry of the set $\mathcal{E}$ of all extension data associated to the set of mixed Hodge structures with a fixed associated graded pure Hodge structures. This is discussed in Section II, and there it is observed that the set $\mathcal{E}_k$ of at most k-fold extensions of pure Hodge structures, which we shall refer to as *extensions of level at most* k, fibres over $\mathcal{E}_{k-1}$ with typical fibre an $\mathrm{Ext}^1_{\mathrm{MHS}}(H^{m+k}, H^m)$ where H^i is a pure Hodge structure of weight i. We shall refer to these fibres as *extensions of level* k.

From [C] we have

> $\mathrm{Ext}^1_{\mathrm{MHS}}(H^{m+k}, H^m)$ *is the quotient of a complex Euclidean space by a discrete abelian subgroup.*

In fact, this Ext-group is an extension of a compact complex torus by a product of $\mathbb{C}^*$'s and $\mathbb{C}$'s, which with a somewhat abuse of language we shall call a *semi-abelian-torus*. In general other than this structure result there doesn't seem to be a lot more that one can say about the set $\mathcal{E}$ of extensions of mixed Hodge structures having a fixed associated graded.

However when we come to *limiting* mixed Hodge structures the story is richer. For such a LMHS $(V, Q, W(N), F)$ where N lies in the interior of a monodromy cone σ with dual open cone $\check{\sigma}$, elements A in $\check{\sigma} \otimes \mathbb{Z}$ canonically define line bundles $L_A \to J$ over the level 1 extension data J.

Now J is a compact complex torus that looks like an intermediate Jacobian of the type $H^{2m-1}_{\mathbb{C}}/F^m H^{2m-1}_{\mathbb{C}} + H^{2m-1}_{\mathbb{Z}}$, and one may define the abelian part $J_{ab} \subset J$ of J to be the largest sub-torus lying under $H^{m,m-1} \oplus H^{m-1,m}$. Then for $A \in \check{\sigma}$ the restriction

$$L_A \to J_{ab}$$

is positive. The construction of $L_A \to J$ involves the level 2 extension data and the structure of the LMHS. The point here is that the extension data for *limiting* mixed Hodge structures has an associated geometry not present for general mixed Hodge structures.

We now turn to the Abel-Jacobi type maps that arise on a subvariety $Y^* \subset Z_I^*$ along which the associated graded to the limiting mixed Hodge structures are locally constant. There are inductively defined maps, the first of which is[‡]

$$\Phi_1 : Y^* \to \{ \text{ level 1 extension data} \}.$$

Concerning Φ_1 there are three main results:

- Φ_1 extends to the closure $Y \subset Z_I$;
- $\Phi_1 : Y \to J_{ab}$ maps to the abelian part of J;

[‡]Actually the first should probably be though of as the map

$$\Phi_e : \overline{B} \to \overline{P}$$

where $\overline{P}$ is the canonical set-theoretic completion of the image $P = \Phi(B) \subset \Gamma \backslash D$ of the original period mapping. On the boundary Z the map $\Phi_e := \Phi_0$ associates to a limiting mixed Hodge structure the associated graded pure Hodge structure. Thus it is a minimal Satake-Baily-Borel (SSB) type completion of P (cf. [GGLR] and [GG]).

and we have the equation of the line bundles over Y

$$\text{(I.2)} \qquad \Phi_1^* L_A = -\sum_{i \in I} \langle A, N_i \rangle \, [Z_i] \big|_Y .$$

In (I.2) the pairing is between σ and the dual cone $\check{\sigma}$; the $N_i \in \sigma$ are the logarithms of monodromy around the branches Z_i of Z. We note that the formula holds when we sum over all the indices where we set $[Z_j]_Y = 0$ if $Y \cap Z_j = \emptyset$. This basic formula relates the variation of the extension data *along* $Y \subset Z$ to the normal bundle to Z in $\overline{B}$ which points *out of* Y into $\overline{B}$.

For example

If Y is contained in one Z_i and doesn't meet the other Z_j's, then

$$\Phi_1^* L_A = \langle A, N_i \rangle \, N^*_{Z_i/B}\Big|_Y .$$

In particular, if $\langle A, N_i \rangle > 0$ and the differential $\Phi_{1,*}$ is injective, then $N^*_{Z/\overline{B}}\Big|_Y$ is ample.

We note that the proof of the above results are Hodge theoretic. The first uses mixed Hodge theory, the second the infinitesimal period relation (IPR), and the third the detailed structure equations of a family of degenerating Hodge structures.

The next map

$$\Phi_2 : \Phi_1^{-1}(\text{point}) \to \{\text{level 2 extension data}\}$$

is defined on Zariski open sets S^* in the fibres S of Φ_1 of the fibration $\mathcal{E}_2 \to \mathcal{E}_1$. This mapping turns out to have some surprising features, properties that are of both a local and global nature. For the first, the level 2 extension data is an extension of a $\mathbb{C}^m/\Lambda$ where Λ is a partial lattice by a quotient M of a product of $\mathbb{C}^*$'s. Then as a result of the IPR we have

(1) $\Phi_2 : S^* \to M$.

The second is a global result which informally state is

(2) Φ_2 is determined up to a constant by discrete data.

This discrete data involves the monodromy arising from the minimal stratum Z_I^* that contains S^*, the monodromies Z_j where $j \notin I$ but Z_j intersects S, and the line bundle

$$N^*_{Z_i/\overline{B}}\big|_S.$$

The third is in response to the question

(3) *What object does all of S map to?*

The answer is that there is a toroidal variety $\overline{M}_J$ that is a partial completion of M and whose construction involves the j's where $j \notin I$ but $Z_j \cap S \neq \emptyset$.

A fourth feature is that if we think of attaching to $\overline{P}$ the Hodge-theoretically constructed objects P_1 and P_2 given by the image of Φ_1 and Φ_2, then one may think of all of the data as given a toroidal object lying over the minimal SBB completion $\overline{P}$ of the image of a period mapping. As will be illustrated by example, this construction suggests how one may at least partially desingularize moduli spaces of some general type surfaces.

For the next step we will state it informally and refer to Section IV(A) below for explanations of the terminology and a proof. The result is

> *the Abel-Jacobi maps $\Phi_k, k \geqq 3$, are determined up to constants by Φ_1 and Φ_2.*

In particular,

> *If Φ_1 and Φ_2 are constant, then the map to the extension data is constant.*

This is again a Hodge theoretic result whose proof uses the IPR.

In Section VI we will discuss what seem to be rather natural conditions for local Torelli to hold, including along the boundary. The main result is that if local Torelli holds, then there is a natural completion of the image P of a period mapping that captures the complete information in the limiting mixed Hodge structures along the boundary of P.

In Section VII we will give a fairly complete analysis of the global structure of variations of Hodge structure over complete algebraic surfaces.

Using the notations from [GGLR] in summary the results will include

(i) the image $\Phi_e(\overline{B})$ of the extended period mapping is either an algebraic curve or an algebraic surface;

(ii) the augmented Hodge line bundle $\Lambda_e \to \overline{B}$ is free and

$$\Phi_e(\overline{B}) = \mathrm{Proj}(\Lambda_e);$$

(iii) in case Φ_* is everywhere injective, the are integers $a_i \geqq 0$ and there is an m_0 such that for $m \geqq m_0$ the line bundle

$$m\Lambda_e - \sum a_i[Z_i]$$

is ample on $\overline{B}$; and

(iv) in general, if $\dim \Phi_e(B) = 2$, there is a structure result that will be explained in Section VII.

In the result (iii) among the Z_i there will be a subset Z_α such that

$$\dim \Phi_e(Z_\alpha) = 1.$$

For these Z_α we have $a_\alpha = 0$ in (iii).
As will be further discussed in Section VII, assuming for simplicity of explanation that all $\Phi_e(Z_i) =$ point, one may consider the two conditions:

(a) the intersection matrix $\|Z_i \cdot Z_j\| < 0$ is negative definite;

(b) for some positive integers a_i the line bundle $\sum_i a_i[Z_i]$ restricted to Z is negative.

These will be seen to be equivalent. The first condition will be established using Hodge theory. This is what relates to (ii) above. We note that (b) is *not* equivalent to the statement that $\sum_i [Z_i]\big|_Z$ is negative.

We may summarize the above as saying that when $\dim B = 2$ there is a fairly complete qualitative global description of a variation of Hodge structure, including its behavior along the boundary.

Notations and terminology

- $\Phi : B \to \Gamma\backslash D$ denotes a period mapping from a smooth, quasi-projective variety B to the quotient of a period domain $D = G_{\mathbb{R}}/H$;
- this period mapping corresponds to a variation of weight n polarized Hodge structures over B; we denote by $F^p \to B$ the corresponding Hodge bundles;
- the *augmented Hodge line bundle* is defined to be

  $$\Lambda = \bigotimes_{p=0}^{[(n-1)/2]} \det(\mathrm{Gr}^{n-p} F)^{n_p} = \bigotimes_{p=0}^{[(n-1)/2]} \det F^{n-p}$$

  where $[n_p = (n-p+1)/2]$; it has a canonical Hodge metric with Chern form ω, and for $\xi \in T_b B$ we have

  $$\|\phi_*(\xi)\|^2 = \omega(\xi) \tag{I.3}$$

  where the left-hand side is given by the $G_{\mathbb{R}}$-invariant metric on TD;

- the image $P = \Phi(B) \subset \Gamma\backslash D$ is a locally closed analytic subvariety that has a canonical algebraic structure over which $L \to P$ is ample (cf. [BBT]);
- we assume that B has a smooth projective completion $\overline{B}$ such that the divisor at infinity $Z := \overline{B}\backslash B$ is a reduced normal crossing divisor $\sum_i Z_i$ where the smooth irreducible branches locally look like a set of hyperplanes in general position in $\mathbb{C}^d$ ($d = \dim B$);
- we assume that the local monodromies around the Z_i are unipotent with logarithm N_i, and we denote by $F_e^p \to \overline{B}$, $L_e \to \overline{B}$ etc. the canonical Deligne extensions of the bundles F^p, Λ_e;

- there is a canonical completion $\overline{P}$ as a compact Hausdorff space to which the period mapping extends to a proper continuous mapping

  $$\Phi_e : \overline{B} \to \overline{P};$$

  $\overline{P}$ is stratified by complex analytic subvarieties and Φ_e is holomorphic on the inverse images of these strata (cf. [GGLR]);
- it is conjectured that $L_e \to \overline{B}$ is free[§] and that $\overline{P} = \mathrm{Proj}(L_e)$; in Section VII we will prove that when $\dim B = 2$, L_e is free and that at the set-theoretic level $\overline{P} = \mathrm{Proj}(L_e)$;

[§] By the results of Satake-Baily-Borel this is true in the classical case when D is a Hermitian symmetric domain. In the non-classical case it holds when $\dim B = 2$, when $-K_{\overline{B}}$ is nef, and in a few other special cases.

- we will introduce inductively defined maps $\Phi_0, \Phi_1, \Phi_2, \ldots$ where
    - $\Phi_0 = \Phi_e$,
    - Φ_1 is defined on the fibres of Φ_0 and maps to level one extension data;
    - Φ_2 is defined on the fibres of Φ_0 and Φ_1 and maps to level two extension data of level $\leqq 2$.

      $\vdots$

  We will see that the Φ_k for $k \geqq 3$ are determined up to constants by Φ_0, Φ_1, and Φ_2.

II. Extension data for a mixed Hodge structure

In these notes we assume the existence of a lattice $V_{\mathbb{Z}}$ in the $\mathbb{Q}$-vector space V. In this section we will consider extension data for mixed Hodge structures $(V, W, F^{\bullet})$. The weight filtration is

$$\{0\} \subset W_0 \subset W_1 \subset \cdots \subset W_n = V$$

whose graded quotients are weight k pure Hodge structures

$$H^k = \mathrm{Gr}_k^W(V).$$

We will consider only those extensions for which the H^k are fixed Hodge structures, and $\mathcal{E}$ will denote the set of all such. There is a filtration

$$\mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$$

where $\mathcal{E}_m$ denotes the set of at most m-fold extensions in $\mathcal{E}$, which we will refer to as extension of level $\leq m$.

For example, $\mathcal{E}_1$ is the set of extensions

$$0 \to H^k \to W_{k+1}/W_{k-1} \to H^{k+1} \to 0,$$

$\mathcal{E}_2$ is the set of these extensions plus the extensions

$$0 \to H^{k-2} \to W_k/W_{k-3} \to W_k/W_{k-2} \to 0.$$

Equivalently, the filtration of $\mathcal{E}$ by levels is the one induced on the sets of extensions by the filtration $W_\bullet \operatorname{End}(V)$. Thus the level 1 information is in $\operatorname{Gr}^W_{-1} \operatorname{End}(V)$, the level 2 information is in the level 1 information together with $W_{-2} \operatorname{End}(V)/W_0 \operatorname{End}(V)$, and so forth. One might say that the level 2 information reflects extensions of extensions from level 1.

We note that the level 1 extension data is equivalently given by

$$\bigoplus_{k=1}^{n} \operatorname{Ext}^1_{\mathrm{MHS}}(H^k, H^{k-1}).\P \tag{II.1}$$

The main points concerning the structure of $\mathcal{E}$ are

(i) $\mathcal{E}$ is a complex manifold that is an iterated fibration

$$\mathcal{E}_{k+1} \to \mathcal{E}_k; \tag{II.2}$$

(ii) the fibres of (II.2) are connected, abelian complex Lie groups that are extensions of a compact complex torus by a product of C^*'s and $\mathbb{C}$'s; we will refer to these are semi-abelian-tori;

¶In contrast, one has for any mixed Hodge structures that

$$\operatorname{Ext}^i_{\mathrm{MHS}}(A, B) = 0, \qquad i \geqq 2.$$

For reference a proof of this well-known fact will be given in the appendix to this section.

(iii) $\mathcal{E}_1$ given by (II.1) is a sum of compact, complex tori

$$\begin{aligned} J_k &:= \mathrm{Ext}^1_{\mathrm{MHS}}(H^k, H^{k-1}) \\ &\cong \frac{\mathrm{Hom}_{\mathbb{C}}(H^k, H^{k-1})}{F^0 \,\mathrm{Hom}_{\mathbb{C}}(H^k, H^{k-1}) + \mathrm{Hom}_{\mathbb{Z}}(H^k, H^{k-1})}. \end{aligned}$$

Here $\mathrm{Hom}_{\mathbb{Z}}(H^k, H^{k-1}) := \mathrm{Hom}(H^k_{\mathbb{Z}}, H^{k-1}_{\mathbb{Z}})$ where we use the integral structures induced by $V_{\mathbb{Z}}$.

(iv) In J_k there is a compact sub-torus

$$J_{k,ab} \subset J_k$$

given by intersecting $\mathrm{Hom}_{\mathbb{Z}}(H^k, H^{k-1})$ with the $(0,-1) \oplus (-1,0)$ part of the weight -1 Hodge structure on $\mathrm{Hom}_{\mathbb{C}}(H^k, H^{k-1})$. We set

$$\begin{cases} J = \overset{k}{\oplus} J_k \\ J_{ab} = \oplus J_{k,ab}; \end{cases}$$

(v) the fibration $\mathcal{E}_2 \to \mathcal{E}_1$ is given by

$$\left\{ \begin{array}{l} \dfrac{\overset{k}{\oplus}\,\mathrm{Hom}_{\mathbb{C}}(H^k, H^{k-2})}{F^0\,\mathrm{Hom}_{\mathbb{C}} + \mathrm{Hom}_{\mathbb{Z}}} \longrightarrow \mathcal{E}_2 \\ \qquad\qquad\qquad\qquad\qquad\quad \Big\downarrow \\ \qquad\qquad\qquad\qquad \dfrac{\overset{k}{\oplus}\,\mathrm{Hom}_{\mathbb{C}}(H^k, H^{k-1})}{F^0\,\mathrm{Hom}_{\mathbb{C}} + \mathrm{Hom}_{\mathbb{Z}}}, \end{array} \right.$$

or equivalently by

(II.3)
$$\left\{ \begin{array}{l} \dfrac{\mathrm{Gr}^W_{-2}\,\mathrm{End}_{\mathbb{C}}(V)}{F^0\,\mathrm{Gr}^W_{-2}\,\mathrm{End}_{\mathbb{C}}(V) + \mathrm{Gr}^W_{-2}\,\mathrm{End}_{\mathbb{Z}}(V)} \longrightarrow \mathcal{E}_2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow \\ \qquad\qquad\qquad\qquad \dfrac{\mathrm{Gr}^W_{-1}\,\mathrm{End}(V)}{F^0\,\mathrm{Gr}^W_{-1}\,\mathrm{End}_{\mathbb{C}}(V) + \mathrm{Gr}^W_{-1}\,\mathrm{End}_{\mathbb{Z}}(V)} \end{array} \right.$$

where the F^0's in the denominator are induced by the Hodge filtration on the numerator.

(vi) the topological line bundles over J are uniquely specified by their Chern classes. Noting that $\mathrm{Gr}^W_{-1}\mathrm{End}_{\mathbb{Z}}(V)$ is the lattice that defines J as a quotient of a complex vector space we have

$$\left\{\begin{array}{c}\text{topological line}\\ \text{bundles over } J\end{array}\right\} \cong \wedge^2 \mathrm{Gr}^W_{-1}\mathrm{End}_{\mathbb{Z}}(V)^*.$$

Using the mapping

$$\mathrm{Gr}^W_{-1}\mathrm{End}(V) \otimes \mathrm{Gr}^W_{-1}\mathrm{End}(V) \to \mathrm{Gr}^W_{-2}\mathrm{End}(V)$$

given by composition, by dualizing we obtain

$$\text{(II.4)} \qquad \begin{array}{ccc} \mathrm{Gr}^W_{-2}\mathrm{End}_{\mathbb{Z}}(V)^* & \longrightarrow & \wedge^2 \mathrm{Gr}^W_{-1}\mathrm{End}_{\mathbb{Z}}(V)^* \\ & & \wr\| \\ & & H^2(J,\mathbb{Z}). \end{array}$$

The elements in $\mathrm{Gr}^W_{-2}\,\mathrm{End}_{\mathbb{Z}}(V)^*$ that map to (1,1) classes in $H^2(J,\mathbb{Z})$ give rise to holomorphic line bundles on J; they are well defined up to translation.
We next observe that $\mathrm{Gr}^W_{-2}\,\mathrm{End}_{\mathbb{Z}}(V)^*$ is naturally isomorphic to H^1(fibres of (II.3), $\mathbb{Z}$).The mapping (II.4) may be identified with the transgression mapping

$$H^0(H^1(\text{fibre})) \xrightarrow{d_2} H^2(\text{base})$$

in the Leray spectral sequence of the fibration (II.3).
Summarizing in words: *For a set of mixed Hodge structures with fixed associated graded Hodge structure, the level 1 extension data is a (direct sum of) compact complex tori. The level 2 extension data is a complex manifold that fibres holomorphically over the level 1 extension data with fibres consisting of semi-abelian-tori.*

Appendix to Section II

We will give a proof that for mixed Hodge structures A, B

$$\text{(A.1)} \qquad \mathrm{Ext}^2_{\mathrm{MHS}}(B, A) = 0.$$

The idea behind the argument can be used to show that for $k \geqq 2$ all of the $\mathrm{Ext}^k_{\mathrm{MHS}}(B, A) = 0$.

Recall that $\mathrm{Ext}^k_{\mathrm{MHS}}(B, A)$ is generated by exact sequences of mixed Hodge structures

$$0 \longrightarrow A \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_k \longrightarrow B \longrightarrow 0$$

modulo equivalences generated by commutative diagrams

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & A & \longrightarrow & E_1' & \longrightarrow & \cdots & \longrightarrow & E_k' & \longrightarrow & B & \longrightarrow & 0 \\
 & & \| & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_k & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

where the maps are morphisms of MHS's. The zero element is the class of

$$0 \longrightarrow A \xrightarrow{=} A \longrightarrow 0 \cdots 0 \longrightarrow B \xrightarrow{=} B \longrightarrow 0.$$

To establish (A.1) we will proceed in two steps.

Step one: Given

$$0 \longrightarrow A \xrightarrow{i} E_1 \xrightarrow{f} E_2 \xrightarrow{\pi} B \longrightarrow 0$$

then for $C = \mathrm{im}\, f$ we have the solid arrows in

(A.2)
$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & B \arrow[u, two heads, "c"'] \arrow[r, equal] & B \arrow[u, two heads] & \\
0 \arrow[r, dashed] & A \arrow[r, dashed, "a"] & G \arrow[r, dashed, "b"] \arrow[u, dashed, two heads] & E_2 \arrow[r, dashed] \arrow[u, two heads, "\pi"'] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[u, equal] & E_1 \arrow[r, "h"] \arrow[u, dashed, "d"] & C \arrow[r] \arrow[u, "g"'] & 0 \\
 & & 0 \arrow[u, two heads] & 0 \arrow[u, dashed, two heads] &
\end{tikzcd}$$

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We claim that if we can find a mixed Hodge structure G so that the dotted arrows can be filled in to give a commutative diagram, then we will have (A.1).
Given (A.2) we have

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & E_1 \arrow[r, "g\circ h"] \arrow[d, "d"'] & E_2 \arrow[r, "\pi"] \arrow[d, "\mathrm{id}\oplus\pi"'] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "a"'] & G \arrow[r, "b\oplus c"'] & E_2 \oplus B \arrow[r, "\pi\oplus\mathrm{id}"'] & B \arrow[r] & 0. \end{tikzcd}$$

One checks commutativity so that this diagram gives an equivalence of extensions.
Next we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "="] \arrow[d, equal] & A \arrow[r, "0"] \arrow[d, "a"] & B \arrow[r, "="] \arrow[d, "0\oplus\mathrm{id}"] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "a"'] & G \arrow[r, "b\oplus c"'] & E_2 \oplus B \arrow[r, "\pi\oplus\mathrm{id}"'] & B \arrow[r] & 0 \end{tikzcd}$$

which also gives an equivalence of extensions. Combining these gives (A.1).

Step two: We note that the desired G has a filtration

$$A \subset E_1 \subset G$$

with graded pieces A, C, B. To construct it we need

$$e_1 \in \mathrm{Ext}^1_{\mathrm{MHS}}(B, E_1);$$

i.e.,

$$e_1 \in \frac{\mathrm{Hom}_{\mathbb{C}}(B, E_1)}{F^0 \mathrm{Hom}_{\mathbb{C}}(B, E_1) + \mathrm{Hom}_{\mathbb{Z}}(B, E_1)} \xrightarrow{\quad h \quad} \frac{\mathrm{Hom}_{\mathbb{C}}(B, C)}{F^0 \mathrm{Hom}_{\mathbb{C}}(B, C) + \mathrm{Hom}_{\mathbb{Z}}(B, C)}$$

gives the extension class of E_2. But

$$h : \mathrm{Hom}(B, E_1) \to \mathrm{Hom}(B, C)$$

is surjective since both $E \twoheadrightarrow C$ and $B \oplus E_1 \twoheadrightarrow B \oplus C$ are surjective.

If G is the extension defined by e_1, then we have

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
 & & B \arrow[u] \arrow[r, equal] & B \arrow[u] & \\
0 \arrow[r] & A \arrow[uu] \arrow[r] & G \arrow[u] \arrow[r] & E_2 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[u, equal] \arrow[r] & E_1 \arrow[u] \arrow[r] & C \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

and we are done. □

The key is given A, B, C where $0 \to A \to E_1 \to C \to 0$ to be able to construct G with these graded pieces. This is possible because there are a lot of mixed Hodge structures.

III. Extension data for limiting mixed Hodge structures

We assume given a limiting mixed Hodge structure $(V, Q, W(N), F^\bullet)$. With the standard notations ([GGLR]) we have

- $N : W_k(N) \to W_{k-2}(N)$ and
$$N^k : \mathrm{Gr}^{W(N)}_{n+k}(V) \xrightarrow{\sim} \mathrm{Gr}^{W(N)}_{n-k}(V);$$
- $Q : V \otimes V \to \mathbb{Q}$ and $N \in W_{-2}(N)\,\mathrm{End}(V)$ preserves Q.

The Q will always be present but we shall omit it in the notation; thus, e.g., it is understood that

$$\mathrm{End}(V) = \mathrm{End}(V, Q);$$

- From an equivalent but alternative perspective, there is a non-degenerate pairing

$$\mathrm{Gr}^{W(N)}_{n+k} V \otimes \mathrm{Gr}^{W(N)}_{n-k} \to \mathbb{Q}$$

given by

$$u \otimes v \to Q(u, v).$$

This gives an isomorphism

$$H^i(-(n-i))^* \cong H^i$$

and then

$$N^{n-i} \in H^i(-(n-i))^* \otimes H^i \cong H^i \otimes H^i.$$

In fact, N^{n-i} lies in the symmetric part

$$N^{n-i} \in S^2 H^i \cong S^2 H^i(-(n-i))^*.$$

We shall use these identifications without further comment.

- N can uniquely be completed to an $\mathrm{sl}_2\{N, H, N^+\}$ acting on $\mathrm{Gr}_\bullet^{W(N)}(V)$ and preserving Q;
- we decompose $\mathrm{Gr}_\bullet^{W(N)}(V)$ into a direct sum of irreducible sl_2-modules; the resulting summands will be called N-strings;
- under this decomposition the $\mathrm{Gr}_k^{W(N)}(V)$'s are direct sums of polarizable Hodge structures; by abuse of language we shall simply refer to $\mathrm{Gr}_k^{W(N)}(V)$ as a polarized Hodge structure.

Now we come to the main point:

(III.1) *Over the level one extension data there is a canonical line bundle $L_N \to J$ such that*

$$L_N \to J_{ab}$$

is positive (ample).

Proof: This is a consequence of the following diagram in which it is to be understood that we are restricting to the lattices $V_{\mathbb{Z}} \subset V$ and that $N : V_{\mathbb{Z}} \to V_{\mathbb{Z}}$ (we may have to replace N by a multiple to achieve this), and where the top horizontal arrow is defined by commutativity of the diagram:

$$
\begin{tikzcd}
\mathrm{Gr}^{W(N)}_{-1}\mathrm{End}_{\mathbb{Z}}(V)\otimes \mathrm{Gr}^{W(N)}_{-1}\mathrm{End}_{\mathbb{Z}}(V) \arrow[r] \arrow[d] & \mathbb{Z} \\
\mathrm{Gr}^{W(N)}_{-2}\mathrm{End}_{\mathbb{Z}}(V) \arrow[d, "N^+"] & \\
\mathrm{Gr}^{W(N)}_{0}\mathrm{End}_{\mathbb{Z}}(V); \arrow[uur, "\mathrm{Tr}"'] &
\end{tikzcd} \tag{III.2}
$$

here the top vertical arrow is composition of endomorphisms and Tr is the trace. Referring to (II.4) one may check that

- the element of $H^2(J,\mathbb{Z})$ defined by (II.4) via (III.2) is of Hodge type (1,1);
- as a consequence of the 2^{nd} Hodge-Riemann bilinear relation for polarized Hodge structures this class is represented by a positive (1,1) form.

- In general the Hodge classes in $\mathrm{Gr}_{-2}^{W(N)} \mathrm{End}_{\mathbb{Z}}(V)$ are dual to the Hodge classes in $\mathrm{Gr}_{+2}^{W(N)} \mathrm{End}_{\mathbb{Z}}(V)$, which in particular are $(1,1)$ classes. This is the reason that N^+ appears, here using that

$$Q(N, N^+) \text{ is a positive integer.}$$

In general if we have a several variable limiting mixed Hodge structure that defines a monodromy cone

$$\sigma \subset \mathrm{Gr}_{-2}^{W(N)} \mathrm{End}_{\mathbb{Z}}(V)$$

consisting of Hodge classes, there is a dual cone

$$\begin{aligned} \check{\sigma} \subset \mathrm{Gr}_{-2}^{W(N)} \mathrm{End}_{\mathbb{Z}}(V)^* &\cong \mathrm{Gr}_{2}^{W(N)} \mathrm{End}_{\mathbb{Z}}(V) \\ &\cong H^2(J, \mathbb{Z}). \end{aligned}$$

Here we are using that any $N \in \sigma$ defines the same weight filtration. Although $W(\sigma)$ would probably be better notation, we will continue to use $W(N)$, keeping in mind that the sl_2 and resulting polarizations on the $\mathrm{Gr}_k^{W(N)}(V)$ depend on the particular N.

For any $A \in \check{\sigma} \otimes \mathbb{Z}$ the above identification defines a line bundle $L_A \to J$ with the property that

$$L_A \to J_{ab} \text{ is positive if } A \in \check{\sigma}.$$

In words,

> *The integral elements $A \in \check{\sigma} \otimes \mathbb{Z}$ define line bundles $L_A \to J$ over the level 1 extension data. For $A \in \check{\sigma}$ the line bundle $L_A \to J_{ab}$ is ample over the Hodge part J_{ab} of J.*

We note that to define L_A with the above properties we must have the structure of level 2 extension data plus the properties of LMHS's.

The above raises two natural questions. The first is that for a limiting mixed Hodge structure the level 1 extension data is a direct sum of compact tori that using

$$H^i(-(n-i))^* \cong H^i$$

occur in dual pairs, with the understanding that when n is even the middle term is self dual. One may ask if the Poincaré line bundle is a part of the picture, and the answer is that it doesn't seem to be.

More interesting is the following question: There is a bijection

$$\left\{\begin{array}{l}\text{equivalence classes of}\\ \text{limiting mixed Hodge}\\ \text{structures}\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\text{equivalence classes}\\ \text{of nilpotent orbits}\end{array}\right\}$$

What, if any, is the relation between the natural line bundles over the level 1 extension data on the left and the 1^{st} order "smoothing" variation of LMHS's that arise on the right?$^{\|}$ The interesting and somewhat subtle answer to this question will be taken up in the next section.

$^{\|}$In the geometric case where we have a family of smooth varieties X_b parametrized by B and singular varieties lying over Z, then for each point $b_0 \in Z$ there is a limiting mixed Hodge structure $\lim_{b \to b_0} H^n(X_b)$. Thus in general taking the limiting mixed Hodge structure at b_0 and moving it out into B in a normal direction to Z may be thought of as "smoothing" the LMHS at b_0.

IV. Period mappings to extension data (A)

With the setup and notation explained in Section I, we denote by B_p a connected component of a fibre of the extended period mapping Φ_e. We recall the following general facts:

- B_p is complete and is contained in the closure of $\overline{Z}_I$ of a unique minimal stratum $Z_I = \bigcap_{i \in I} Z_i$;
- $B_p^* := B_p \backslash$(the union of B_p intersect the strata $Z_{I \cup \{j\}}$ for $j \notin I$) is a Zariski open $B_p \cap Z_I^*$ in B_p;
- along B_p^* we have a variation of limiting mixed Hodge structures with locally constant associated graded pure Hodge structures; the behavior of the variation of limiting mixed Hodge structures along Z_I^* at the intersection with other strata is developed in [CKS];

- we denote by $\mathcal{E}$ the set of extension data, as described in sections II and III above, for the locally constant associated graded to the LMHS's along B_p^*;
- Γ_I will denote the action of the monodromy on $\mathcal{E}$ induced by the monodromy action of $\pi_1(B_p^*)$ on the family of limiting mixed Hodge structures along B_p^*.

We will see that this monodromy action is at most a finite group, one that we shall at least initially ignore, acting on the complex torus J given by the level 1 extension data. Thus in the obvious way we may define an Abel-Jacobi type mapping

$$\text{(IV.1)} \qquad \Phi_1 : B_p^* \to J$$

by assigning to each point $b \in B_p^*$ the level 1 extension data in the limiting mixed Hodge structure $\Phi_e(b)$. The main result is the

Theorem (IV.2): *The mapping* (IV.1) *extends to a mapping on all of B_p. There it maps to a translate of the abelian subvariety J_{ab} of the complex torus J and we have*

$$\boxed{\text{(IV.3)} \qquad \Phi_1^* L_A = -\sum_i \langle A, N_i \rangle \, [Z_i]\big|_{B_p}.}$$

Here $L_A \to J$ is the line bundle defined for $A \in \check{\sigma}_Y \otimes \mathbb{Z}$ in Section III above. The sum is over all the line bundles $[Z_i]\big|_{B_p}$, noting that this bundle is trivial if $Z_i \cap B_p = \emptyset$ is empty.

Corollary (IV.4): *If $B_p \subset Z_i$ does not meet other strata and $\langle A, N_i \rangle > 0$, then $\Phi_1^* L_A \to B_p$ is a positive line bundle.*

Proof of (IV.3): We begin by explaining the central idea behind the first two assertions with some of the details given below and the rest to be provided elsewhere. For the first statement the mapping (IV.1) induces the top row in

$$\begin{tikzcd} H_1(B_p^*, \mathbb{Z}) \arrow[rr, "\Phi_{1,*}"] \arrow[dr] & & H_1(J, \mathbb{Z}) \\ & H_1(B_p, \mathbb{Z}) \arrow[ur, dashed] & \end{tikzcd}$$

This top row is a morphism of mixed Hodge structures. By a weight argument the kernel of the slanted solid arrow is of strictly lower weight than the weight of the pure Hodge structure on $H_1(J, \mathbb{Z})$. Therefore the mapping $\Phi_{1,*}$ factors in the way indicated by the dotted arrow in the diagram. Thus there is an induced morphism of Hodge structures

(IV.5) $$H_1(\mathrm{Alb}\, B_p, \mathbb{Z}) \to H_1(J, \mathbb{Z})$$

where $\mathrm{Alb}\, B_p$ is the Albanese variety of (any desingularization of) the complete variety B_p. From this it follows that Φ_1 in (IV.1) extends to give the composed mapping in

$$B_p \to \mathrm{Alb}\, B_p \to J$$

where the second arrow is induced by the mapping in (IV.5).

For the second part of the theorem, $\mathrm{Gr}_{-1}^{W(N)} \mathrm{End}(E)$ is a pure Hodge structure of weight -1. The Hodge decomposition of this pure Hodge structure looks like

$$(m-1,-m)+\cdots+\underbrace{(0,-1)+(-1,0)}+\cdots+(-m,m-1).$$

The induced morphism of Hodge structures

$$H_1(\mathrm{Alb}\, B_p) \to H_1(J)$$

has image in a Tate twist of the term over the brackets in the above direct sum. It also lands in a subgroup of $H_1(\mathrm{Gr}_{-1}^{W(N)} \mathrm{End}(E), \mathbb{Z})$, which implies that $\Phi_1(B_p)$ lands in a translate of J_{ab}.

In somewhat more detail and from a slightly different perspective, suppose we inductively define maps $\Phi_1, \Phi_2, \dots$ by

- Φ_1 maps B_p to level 1 extension data;
- Φ_2 maps the fibres of Φ_1 to level 2 extension data

and so forth. Let $C \subset B_p$ be an irreducible curve that is a fibre of $\Phi_1, \dots, \Phi_{k-1}$. Then there is a Zariski open set $C \backslash D$ in C and a map

$$\Phi_k : C\backslash D \longrightarrow \frac{\mathrm{Gr}_{-k}^{W(N)} \mathrm{End}_{\mathbb{C}}(V)}{F^0 \,\mathrm{Gr}_{-k}^{W(N)} \mathrm{End}_{\mathbb{C}}(V) + \mathrm{Gr}_{-k}^{W(N)} \mathrm{End}_{\mathbb{Z}}(V)}$$

when the RHS is a semi-abelian-torus. This maps factors through

$$C\backslash D \longrightarrow \frac{H^1(C, D; \mathbb{C})}{F^1 H^1(C, D; \mathbb{C}) + H^1(C, D; \mathbb{Z})}.$$

The numerator is the mixed Hodge structure $H^1(C,D)$ with associated graded described by (here using $\mathbb{Q}$-coefficients)

$$0 \longrightarrow \frac{H^0(D)}{H^0(C)} \longrightarrow H^1(C,D) \longrightarrow H^1(C) \longrightarrow 0.$$

We then have

$$\begin{tikzcd} C\backslash D \arrow[d] \arrow[dr] & \\ \frac{H^1(C,D;\mathbb{C})}{F^1H^1(C,D;\mathbb{C})+H^1(C,D;\mathbb{Z})} \arrow[r] & \frac{\mathrm{Gr}^{W(N)}_{-k}\mathrm{End}_{\mathbb{C}}(V)}{F^0\,\mathrm{Gr}^{W(N)}_{-k}\mathrm{End}_{\mathbb{C}}(V)+\mathrm{Gr}^{W(N)}_{-k}\mathrm{End}_{\mathbb{Z}}(V)} \end{tikzcd}.$$

Then $\Phi_{k,*}$ lands in $F^{-1}\,\mathrm{Gr}_{-k}^{W(N)}\,\mathrm{End}_{\mathbb{C}}(V)$, which has Hodge type

$$(-1,-(k-1))+(0,-k)+\cdots+(-(k-1),-1).$$

It follows that (with the hopefully obvious notation)

$$H^1(C,D;\mathbb{Z})\to F^{-1}\,\mathrm{Gr}_k\cap\overline{F^{-1}\,\mathrm{Gr}_k}.$$

For

$$\begin{array}{ll} k=1 & \text{this is } (-1,0)+(0,-1) \\ k=2 & \text{this is } (-1,-1) \\ k\geqq 2 & \text{this is empty.} \end{array}$$

As a consequence

(IV.6) *For $k\geqq 3$ the maps Φ_k are determined up to constants by Φ_1,Φ_2.*

This is a rather remarkable geometric fact. It says that if in the fibres of Φ_e we consider the successive maps to extension data of increasingly higher levels, then up to constants these maps are already determined by what happens at the first two levels.

In the classical case this is perhaps not so surprising. For $n = 1$ there are only two levels. Any Hermitian symmetric domain may be equivariantly embedded in the period domain for polarized Hodge structures of weight $n = 1$.

In the non-classical case it is consistent with the general philosophy that even when the period domain is not Hermitian symmetric, period mappings behave in much the same way as in the classical case.

We now turn to a discussion of the main formula (IV.3). A formal proof of this will be given elsewhere. Here we will present some special cases that illustrate why the result should be true. We begin with the simplest non-trivial case:

- $n = 1$ and $g = 2$ (here g is $h^{1,0}$ for a weight 1 PHS (V, Q, F));
- $\dim B = 2$ and there are local coordinates (t, w) where Z is given by $t = 0$;
- the normalized period matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & \lambda \\ \beta & \alpha \end{pmatrix}$$

where $\alpha(t, w), \lambda(w)$ with $\operatorname{Im} \lambda > 0$ are holomorphic and

$$\beta(t, w) = \ell(t) + b(t, w)$$

where $\ell(t) = \log t/2\pi i$ and $b(t, w)$ is holomorphic.

The fibres of Φ_e are given by $\lambda =$ constant. In this case the Q, N and weight filtration are given by

$$Q = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\left.\begin{array}{l} \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix} \begin{array}{l} \\ \\ \\ \} W_0 \end{array} \quad \left.\begin{array}{l} \\ \\ \end{array}\right\} W_1 \end{array}\right\} W_2 = V$$

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The level 1 extension data is given by $\alpha(t, w)$ and the level 2 extension data by $\beta(t, w)$. An element A in the dual cone $\check{\sigma}$ is given by a positive integer a, and the pairing that defines the line bundle L_A works out to give that

$$\exp(2^{\mathrm{nd}} \text{ level extension data}) \\ = \exp(2\pi i a \beta(t, w)) = t^a \exp(2\pi i b(t, w))$$

is a nowhere vanishing section of L_{-A}. On the other hand, the LHS gives the transition functions for the line bundle L_{-A}. From this the result follows.

In more detail, the points in the argument are as follows:

- For a compact complex torus $T = \mathbb{C}^m/\Lambda$ there is the identification
$$H^2(T, \mathbb{Z}) \cong \wedge^2(\Lambda^*);$$
- For a $(1,1)$ class A in $\wedge^2(\Lambda^*)$ there is a holomorphic line bundle $L_A \to T$, defined up to translation and whose Chern class is A;

- The pullback to $\mathbb{C}^m$ of this bundle is trivial, and it is defined by "transition functions" that are exponentials of linear functions on $\mathbb{C}^m$ whose coefficients are linear in the entries of A;
- Thus the sections of $L_A \to T$ are given by holomorphic functions $\theta(z)$ satisfying $\theta(z+\lambda) = exp\,\langle \ell(A,\lambda), z\rangle$ where $\ell(A,\lambda)$ is linear in A and satisfies a cocycle rule in $\lambda \in \Lambda$. In the above special case the period matrix has been normalized to be of the form $\binom{I}{\Omega}$. The quotient of $\mathbb{C}^4$ by I-part of the lattice gives transition functions that are trivial; i.e., the line bundle L_A descends to $\mathbb{C}^* \times \mathbb{C}^*$. The transition functions for the second part of the lattice are given by exponentials of the Ω-part of the period matrix. If we cover Z by the open sets $\mathcal{U}_k$ where Z is defined by $t_k = 0$, then in $\mathcal{U}_k \cap \mathcal{U}_\ell$ we have $t_k = f_{k\ell} t_\ell$ where $f_{k\ell}\big|_Z$ give the transition functions for $N^*_{Z/\overline{B}}$. This then identifies the transition functions for $N^*_{Z/\overline{B}}$ with those of $\Phi_1^* L_{-A}$.

- In Section III above we have explained how level 2 extension data gives $(1,1)$ classes on the compact, complex torus J given by level 1 extension data.

The above calculation carries out this general procedure in the special case described there. The normal bundle to Z appears because the level 2 extension data has an $\ell(t)N$ term in the matrix representing it and exponentiating $\ell(t)N+$ (holomorphic term) produces a "t" that is a local section of $N^*_{Z/\overline{B}}$. For $n \geqq 2$ the period matrix will generally contain $(\ell(t)N)^m$ terms where $m \geqq 2$. The general rule is

- the fibres of the level k extension data over the level $k-1$ extension data are contained in

$$\mathrm{Ext}^2_{\mathrm{MHS}}(H^{m+k}, H^m)\text{'s;} \tag{IV.7}$$

- only for $k = 2m$, $m \geqq 1$ do we get $(\ell(t)N)^m$-terms appearing in the $W_{-k}\,\mathrm{End}(V)$'s;
- thus no matter what the weight n of the original polarized Hodge structure is, for level 2 extensions corresponding to $W_{-2}\,\mathrm{End}(V)$ (thus $m = 1, k = 2$) we get only $\ell(t)N$'s in the group (IV.7)

A consequence of (IV.6) is

(IV.8) *Let B_p be a connected fibre of Φ_e and*

$$\Phi : B_p \to \Gamma_I \backslash \mathcal{E}$$

the map to the extension data of the limiting mixed Hodge structures along B_p. Then this map is constant if, and only if, the maps Φ_1 and Φ_2 to extension data of levels 1 and 2 are constant.

We conclude this section with a result that although the hypotheses are quite restrictive is a harbinger of what one would like to prove and also indicates that the analysis given above will have interesting consequences.

Suppose that we have a variation of Hodge structure over B with an extension to $\overline{B}$ as described in Section I. The general question/conjecture is to show that *under local Torelli type assumptions there are non-negative integers a_i and an m_0 such that the line bundle*

$$L_m = mL - \sum a_i[Z_i] \tag{IV.9}$$

is ample for $m \geqq m_0$.

We shall show that this result holds under the following assumptions:

(i) the differential of Φ_e is 1-1 except along the fibres B_p;

(ii) along the fibres B_p the differential of Φ_1 is 1-1;

(iii) Z has one component; and

(iv) the cone $\mathrm{Eff}^1(\overline{B})$ of effective 1-cycles on $\overline{B}$ is finitely generated.

Then under these assumptions there is an m_0 such that

$$mL - [Z] \text{ is ample for } m \geqq m_0.$$

Proof: Given an irreducible curve $C \subset \overline{B}$ we have to show that

$$(mL - [Z]) \cdot C := \deg\left((mL - [Z])\big|_C \right) > 0 \tag{IV.10}$$

for $m \gg 0$.

If C is not a fibre B_p of Φ_e, then using assumption (i) this is a consequence of

$$L \cdot C = \deg(L\big|_C) > 0.$$

If $C \subset B_p$, then by the basic formula (IV.3) using assumption (ii) we have

$$\text{(IV.3)} = \deg\left(N^*_{Z/\overline{B}}\big|_C\right) > 0.$$

Assumption (iii) is used in that C does not meet any other strata of Z, and (iv) is used for it to be sufficient to show (IV.10) for any fixed curve C. □

In general, from the construction proposed in [GGLR] for the Satake-Baily-Borel completion $\overline{P}$ of the image $P = \Phi(B) = \Gamma\backslash D$ of a period mapping, it is expected that for fibres $B_p = \Phi_e^{-1}(p)$, $p \in \overline{P}$ we will have positivity of the bundles

$$N^*_{Z/\overline{B}}\big|_{B_p} \to B_p. \tag{IV.11}$$

Modulo issues of the finite generation of the effective cone of curves on a variety (which will be taken up elsewhere), for smooth B_p's the ampleness of the line bundle (IV.11) is equivalent to

$$\deg\left(N^*_{Z/\overline{B}}\big|_C\right) > 0 \tag{IV.12}$$

for text curves $C \subset B_p$. (IV.9) above is a result in this direction and we shall now give another one extending that result.

Given C there will be a smallest stratum Z_I with

$$C \subset Z_I$$

(thus I is a maximal index set with $C \subset Z_i$, $i \in I$).

We will show

(IV.13) *If* $\Phi_1\big|_C$ *is non-constant and the* N_i*,* $i \in I$*, are linearly independent, then* (IV.12) *holds.*

Proof: The cone σ_I is a face of $\sigma_{I\cup J}$, and for $A \in \check{\sigma}_I$ we have

$$\begin{cases} \langle A, N_i\rangle > 0, & i \in I \\ \langle A, N_j\rangle \geq 0, & j \in I. \end{cases}$$

For such A from (IV.3) and setting $d_i = \deg\left(N^*_{Z_i/\overline{B}}\big|_C\right)$ we have

$$0 < \deg\left(\Phi_1^* L_A\big|_C\right) = \sum_{i\in I}\langle A, N_i\rangle\, d_i - \sum_{j\in J}\langle A, N_j\rangle\,(Z_j \cdot C).$$

This gives

$$0 \leq \sum_{j \in J} \langle A, N_j \rangle \, (Z_j \cdot) < \sum_{i \in I} \langle A, N_i \rangle \, d_i .$$

Using the assumed linear independence of the N_i, $i \in I$ and letting A vary over $\check{\sigma}_I$ we may conclude (IV.12). $\square$

Discussion of the proof of (IV.3)

We will give two arguments for (IV.3). The first will be an elaboration and extension of the above period matrix one. This is included because period matrix collaborations were how many aspects of Hodge theory were first understood. The second will be the Lie theoretic, a method that (the very general cohomological/D-module techniques not withstanding) remains an extremely powerful approach to Hodge theory.

> *What do we man by a period matrix associated to a limiting mixed Hodge structure?*

We will illustrate this when $n = 2$ and $N^2 \neq 0$. Then the associated graded to the LMHS is a direct sum

$$H^0 \oplus H^1 \oplus \underbrace{H^0(-1) \oplus H^2} \oplus H^1(-1) \oplus H^0(-2)$$

where H^i is a Hodge structure of weight i and the arrows are the action of N (N-strings). We choose a basis for $V_{\mathbb{C}}$ adapted to $W(N)$ and shall use H to stand for a generic holomorphic term in (t, w). The period matrix is then

(IV.14)

$$
\begin{array}{c c}
 & \begin{matrix} \quad H^0 \quad & \quad F^1H^1 \quad & \quad F^2H^2 \quad \end{matrix} \\
\begin{matrix} H^0(-2) \\[6pt] H^1(-1) \\[6pt] H^0(-1) \oplus H^3 \\ H^1 \\ H^0 \end{matrix} &
\begin{pmatrix}
I & 0 & 0 \\[6pt]
\boxed{H} & \begin{matrix} I \\ \Lambda \end{matrix} & 0 \\[6pt]
\ell(t)N + H & \boxed{H} & I \\
H & \ell(t)N + H & \boxed{H} \\
(\ell(t)N)^2 + \ell(t)N + H & \ell(t)N + H & H
\end{pmatrix}
\end{array}
$$

The way to read this is to imagine a family of surfaces X_t degenerating to a surface $X = \cup X_i$ that has normal crossings with smooth components X_i, a double curve $D = \bigcup_{i<j} X_i \cap X_j$ and triple points $T = \bigcup_{i<j<k} X_i \cap X_j \cap X_k$.

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We denote by $\Omega^2_X(\log D)$ the sheaf of meromorphic 2-forms on X having log poles on D and with appropriate residues on a component $X_i \cap X_j$ of D, and by $\Omega^1_D(\log T)$ the meromorphic 1-forms on D with log poles at the triple point p with a relation among the three residues from the branches of D through p. Then

$$\lim_{t\to 0} H^0\left(\Omega^2_{X_t}\right) = H^0\left(\Omega^2_X(\log D)\right)$$

where the RHS here is filtered by

(IV.15)

$$\begin{array}{ccccc}
\begin{pmatrix}\text{double and single}\\ \text{residues vanish}\end{pmatrix} & \subset & \begin{pmatrix}\text{double residues}\\ \text{vanish}\end{pmatrix} & \subset & H^0\left(\Omega^2_X(\log T)\right)\\
\| & & \| & & \\
\begin{pmatrix}\text{kernel of the map}\\ \text{of } H^0\left(\Omega^2_X(\log D)\right) \text{ to}\\ H^0(T) \text{ and } H^0\left(\Omega^1_D(\log t)\right)\end{pmatrix} & & \begin{pmatrix}\text{kernel of the}\\ \text{map to } H^0(T)\end{pmatrix} & &
\end{array}$$

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The entries in the matrix (IV.14) are obtained by taking the rows to correspond to the associated graded to the weight filtration and the columns to the associated graded to the filtration (IV.15). The entries in the matrix (IV.14) represent $\mathrm{End}_{\mathbb{C}}(V)$ with the zero blocks in the upper right corresponding to the modding out by $F^0\,\mathrm{End}_{\mathbb{C}}(V)$.

Now we come to the punch line.

- the terms in the boxes $\square$ represent $\mathrm{Gr}_{-1}\,\mathrm{End}_{\mathbb{C}}(V)$;
- the terms in the circles $\bigcirc$ represent $\mathrm{Gr}_{-2}\,\mathrm{End}_{\mathbb{C}}(V)$;
- the terms in squiggles represent $\mathrm{Gr}_{-4}\,\mathrm{End}_{\mathbb{C}}(V)$.

This is consistent with the observation that in a period matrix only the terms in $\mathrm{Gr}_{-2m}\,\mathrm{End}_{\mathbb{C}}(V)$ have logarithmic entries and there the leading term is $\ell(t)^m$.

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Recalling that the compact torus J is a quotient of its tangent space $\widetilde{J} := \mathrm{Gr}^W_{-1}\,\mathrm{End}_{\mathbb{C}}(V)/F^0\,\mathrm{Gr}^W_{-1}\,\mathrm{End}_{\mathbb{C}}(V)$ by a lattice Γ, and that line bundles on $J = \widetilde{J}/\Gamma$ arise from Hodge classes in $\mathrm{Gr}^W_{-2}\,\mathrm{End}_{\mathbb{C}}(V)$ and they are constructed by taking quotients of the trivial bundle on $\widetilde{J}$ by cocycles of the form $\exp(2\pi i \times \text{linear function on } \widetilde{V}, \gamma)$ where $\gamma \in \Gamma$, we see why only $\exp \ell(t)$'s and not $\exp(\ell(t)^m)$'s for $m \geqq 2$ enter. Now $\exp \ell(t) = t$ is a local section of $N^*_{Z/\overline{B}}$, and this is what is behind the mechanism that relates information along a fibre B_p to information normal to fibres in $\overline{B}$.

Discussion of the proof of (IV.6)

We will illustrate by a period matrix calculation in the case $n = 2$ where the limiting mixed Hodge structure is Hodge-Tate. Then the period matrix is

$$\begin{array}{c} H^0(-2) \\ \big\downarrow N \\ H^0(-1) \\ \big\downarrow N \\ H^0 \end{array} \qquad \begin{pmatrix} I \\ \\ A = \ell(t)I + A_0 \\ \\ B = \frac{\ell(t)^2}{2}I + \ell(t)B_1 + B_0 \end{pmatrix}$$

where A_0, B_1, B_0 are holomorphic. We may take

$$Q = \begin{pmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{pmatrix}.$$

Then the 1$^{\mathrm{st}}$ Hodge-Riemann bilinear relation is

$$B + {}^tB = {}^tAA.$$

The symmetric part of B

$$B_s = \frac{1}{2}{}^tAA$$

is determined by A.
The IPR is

$$dB = {}^tA\,dA. \tag{IV.16}$$

The level 2 extension data is given by A and the level 4 extension data by B. From (IV.16) we see quite explicitly how in this case the extension data of higher level is determined up to constants by the extension data of levels 1,2.

V. Period mappings to extension data (B)

Reviewing briefly, for the complete varieties

$$B_p = \Phi_e^{-1}(p), \quad p \in \overline{P}$$

we have defined a complex torus J_{ab}, holomorphic line bundles $L_A \to J_{ab}$ and an Abel-Jacobi mapping $\Phi_1 : B_p^* \to J_{ab}$ where B_p^* is the Zariski open obtained by removing from B_p the lower dimensional intersections with other strata. We then showed that the above mapping extends to

$$\Phi_1 : B_p \to J_{ab},$$

and from (IV.3) we have

$$\Phi_1^* L_A = -\sum_i \langle A, N_i \rangle \, N^*_{Z_i/\overline{B}}\Big|_{B_p}.$$

On the fibres of Φ_1 there is a mapping
(V.1)
$\Phi_2 : \{\text{Zariski open in } \Phi_1^{-1} \text{ points}\} \to \{\text{level 2 extension data}\}.$

The purpose of this section is to analyze this mapping. We denote by S a typical fibre of (V.1) and assume for simplicity that S is irreducible. Then there will be a maximal index set I such that

$$S \subset Z_I.$$

There will be another index set $J = \{j \notin I : S \cap Z_j \neq \emptyset\}$. We let

$$S^* = S \backslash \left(\bigcup_{j \in J} S \cap Z_j \right)$$

be the Zariski open obtained by removing from S the lower dimensional intersections with the other strata of Z.

In general the level 2 extension data is a direct sum of

$$\mathrm{Ext}^1_{\mathrm{MHS}}\left(H^{k+2}, H^k\right)\text{'s.} \tag{V.2}$$

These are quotients of Hodge structures of weight -2. The integral classes of type $(-1,-1)$ then project to a subgroup, denoted by $\mathrm{Hg}\otimes\mathbb{C}^*$, of the direct sum of the terms (V.2). We thus have

$$0 \to \mathrm{Hg}\otimes\mathbb{C}^* \to \begin{pmatrix}\text{extension data}\\ \text{of level 2}\end{pmatrix} \to T \to 0 \tag{V.3}$$

where T is a quotient of a $\mathbb{C}^m$ by a discrete group, which in general is not a full lattice. The notation is chosen because $\mathrm{Hg}\otimes\mathbb{C}^*$ is constructed from a product of $\mathbb{C}^*$'s.

In (V.2) the H^p's are graded quotients of a filtration on the fixed vector space V. When we vary along S^* we have to take into account the monodromy action. We will denote by M, to be described explicitly later, the resulting quotient.

Proposition V.4: *The set-theoretic mapping* (V.1) *induces a holomorphic mapping*

$$\Phi_2 : S^* \to M.$$

Proof: The point here is that the differential of Φ_2 maps to zero in the quotient T in (V.3). This is essentially the argument just above (IV.6) in the case $k = 2$ there. We will give the details here since contrary to the case of Φ_1 monodromy along S^* enters the picture. The map (V.1) is induced by passing to a quotient of locally defined mapping

$$S^* \to \frac{\mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{C}}(V)}{F^0 \,\mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{C}}(V) + \mathrm{End}_{\mathbb{Z}}(V)}. \tag{V.5}$$

Now

$$\mathrm{Gr}_{-2}^{W(\sigma)}\mathrm{End}_{\mathbb{C}}(V) = \bigoplus_{p+q=-2} \mathrm{Gr}_{-2}^{W(\sigma)}\mathrm{End}_{\mathbb{C}}(V)^{p,q}$$

$$\frac{\mathrm{Gr}_{-2}^{W(\sigma)}\mathrm{End}_{\mathbb{C}}(V)}{F^0\,\mathrm{Gr}_{-2}^{W(\sigma)}\mathrm{End}_{\mathbb{C}}(V)} \cong (-1,-1)\oplus(-2,0)\oplus(-3,1)\oplus\cdots\,.$$

We let

$$\mathrm{Hg}\left(\mathrm{Gr}_{-2}^{W(\sigma)}\mathrm{End}_{\mathbb{Z}}(V)\right) := \left((-1,-1)\ \text{summand}\ \ \cap \mathrm{Gr}_{-2}^{W(\sigma)}\mathrm{End}_{\mathbb{Z}}(V)\right).$$

Then

$$\frac{\mathrm{Hg}\left(\mathrm{Gr}_{-2}^{W(\sigma)}\mathrm{End}_{\mathbb{Z}}(V)\right)\otimes\mathbb{C}}{\mathrm{Hg}\left(\mathrm{Gr}_{-2}^{W(\sigma)}\mathrm{End}_{\mathbb{Z}}(V)\right)} \hookrightarrow \frac{\mathrm{Gr}_{-2}^{W(\sigma)}\mathrm{End}_{\mathbb{C}}(V)}{F^0\,\mathrm{Gr}_{-2}^{W(\sigma)}\mathrm{End}_{\mathbb{C}}(V)+\mathrm{End}_{\mathbb{Z}}(V)}.$$

This gives

$$\mathrm{Hg}\left(\mathrm{Gr}_{-2}^{W(\sigma)}\mathrm{End}_{\mathbb{Z}}(V)\right)\otimes\mathbb{C}^* \hookrightarrow \frac{\mathrm{Gr}_{-2}^{W(\sigma)}\mathrm{End}_{\mathbb{C}}(V)}{F^0\,\mathrm{Gr}_{-2}^{W(\sigma)}\mathrm{End}_{\mathbb{C}}(V)+\mathrm{End}_{\mathbb{Z}}(V)}.$$

As noted above, the quotient is generally a non-compact $\mathbb{C}^m/\Lambda$.

By the argument just above (V.6) the derivative of the map (V.5) lands in the $(-1,-1)$ part of the Hodge decomposition of the tangent space. On the other hand, the derivative is the complex linear map induced by

$$H_1(S^*, \mathbb{Z}) \to \mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{Z}}(V)/\sigma_{\mathbb{Z}}$$

where the $\sigma_{\mathbb{Z}} = \mathrm{span}_{\mathbb{Z}}\{N_i : i \in I\}$ reflects the action of monodromy. It follows that (V.5) is a map

$$S^* \to \mathrm{Hg}\left(\mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{Z}}(V)/\sigma_{\mathbb{Z}}\right) \otimes \mathbb{C}^* \tag{V.6}$$

where the RHS defines the M in the statement of Proposition V.4.

Remark: We have an exact sequence of mixed Hodge structures

$$0 \to H_0(S\backslash S^*, \mathbb{Z})(-1) \to H_1(S^*, \mathbb{Z}) \to H_1(S, \mathbb{Z})$$

and by a weight argument any map of mixed Hodge structures

$$H_1(S, \mathbb{Z}) \to \mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{Z}}(V)$$

is zero. This suggests that the mapping (V.6) should in some sense be determined by the $H_0(S\backslash S^*, \mathbb{Z})$ part of $H_1(S^*, \mathbb{Z})$. As will now be explained, this is indeed the case.
For simplicity we assume that S is an irreducible curve.

Proposition V.7: *The mapping Φ_2 in* (V.6) *is determined up to a constant by the discrete data*

$$N_k \text{ for } k \in I \cup J; \ \deg N^*_{Z_-/\overline{B}}\Big|_S \text{ for } i \in I, \ Z_j \cdot S \text{ for } j \in J.$$

Proof: A formal proof will be given later. Here we will give a period matrix argument as in the proof of (IV.3). The period matrix is

$$\begin{pmatrix} I & 0 \\ 0 & I \\ \alpha & \lambda \\ \beta & {}^t\alpha \end{pmatrix}$$

where the entries are now block matrices. Using coordinates $t_i, i \in I$ and $t_j, j \in J$ we have

$$\beta = \sum_{i \in I} \ell(t_i) N_i + \sum_{j \in J} \ell(t_j) N_j + H$$

where H is holomorphic.

Moreover β is a coordinate representation of the level 2 extension information. For a symmetric matrix A we have

$$e^{2\pi i\langle A,\beta\rangle} = \prod_{i\in I} t_i^{\langle A,N_i\rangle} \prod_{j\in j} t_j^{\left\langle A,N_j\right\rangle} e^{2\pi iH}.$$

Now we use that S lies in a fibre of Φ_2. The line bundle $\Phi_1^* L_{-A}$ is then a trivial bundle on S. As in the argument for (IV.3), $e^{2\pi i\langle A,\beta\rangle}$ is the local coordinate representation of a *global* holomorphic section of this trivial bundle. Thus

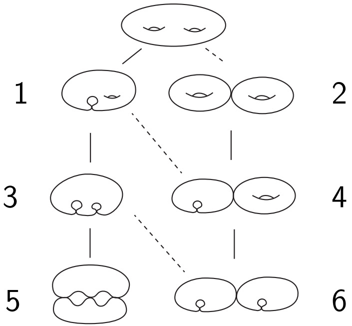
$$e^{2\pi i\langle A,\beta\rangle} = \text{constant}$$

(this is the constant in the statement of V.7). Continuing to follow as in the argument for (IV.3), $e^{-2\pi iH}$ is a section over S of the line bundle $\prod_{i\in I} N^{*\otimes\langle A,N_i\rangle}_{Z_i/\overline{B}}$ that vanishes to order $\langle A,N_j\rangle\,(Z_j\cdot S)$ at the points of $S\cap Z_j$, $j\in J$. $\square$

Appendix to Sections IV and V: Examples

We shall present some examples that illustrate the constructions in these notes.

Example 1: This is $\overline{\mathcal{M}}_2$, the Deligne-Mumford compactification of genus 2 curves. Although it is the simplest example it illustrates many of the basic constructions. The stratification of $\overline{\mathcal{M}}_2$ may be pictured as



The solid lines represent degenerations with infinite oder monodromy ($N \neq 0$). The dotted ones are degenerations with trivial monodromy ($N = 0$). Although $\overline{\mathcal{M}}_2$ is not smooth the singularities are simple rational quotient ones and we shall ignore them. For $(\overline{B}, Z)$ we shall use the successive blow ups of $\overline{\mathcal{M}}_2$ along the strata pictured above, beginning with the most singular one. This is just a convenience to fit the following discussion into the general framework of these notes.

The stratum Z_3 whose general member is a curve of type 2

We begin with this one as it is the simplest and also illustrates a general principal we shall use repeatedly without further comment. Since the monodromy is trivial we can extend the original period mapping (here $D = \mathcal{H}_2$ and $\Gamma = \mathrm{Sp}(4, \mathbb{Z})$)

$$\mathcal{M}_2 \xrightarrow{\Phi} \Gamma \backslash D$$

across it. Thus in the $(\overline{B}, Z)$ setting we need not include it in Z.

The stratum Z_1 whose general member is a curve of type 1

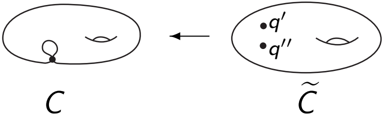
The image $\overline{P}$ of the extended period mapping Φ_e is smooth on $P = \Phi(B)$ and everywhere singular along the boundary; i.e.,

$$\overline{P}_{|sing} = \overline{P} \backslash P.$$

The image $\Phi_e(Z_1) := \overline{P}_1$ is a curve that in an open set in $\overline{P}$ near a curve of type 1 we have a 1-parameter family of simple elliptic surface singularities. A general fibre B_p of the map

$$\Phi_e : Z_1 \to \overline{P}_1$$

consists of curves C whose normalization $\widetilde{C}$ is a fixed elliptic curve

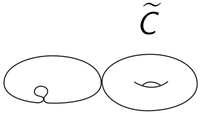


The abelian variety J_{ab} is just the Jacobian $J(\widetilde{C})$ and the mapping

$$\Phi_1 : B_p \to J(\widetilde{C})$$

is $\mathrm{AJ}_{\widetilde{C}}(q' - q'')$ (we are being sloppy about double coverings here). The line bundle L_{N^+} turns out to be 2Θ, twice the theta line bundle on an elliptic curve.

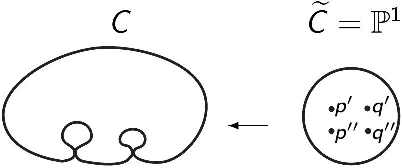
In general we shall let Z_k denote the component of Z obtained by blowing up the stratum in $\overline{\mathcal{M}}_2$ consisting of curves corresponding to k in the diagram. The general fibre of Φ_e/Z_1 meets the stratum Z_4 in points corresponding to a curve

$C_0 =$ 

The vanishing cycle as $C \to C_0$ is homologically trivial and Φ_1 obviously extends. The intersection $Z_1 \cap Z_4$ occurs when $q' = q''$ in the above picture.
In this case Φ_1 is locally 1-1 and there is no need to consider Φ_2.

The stratum Z_2 whose general member is a curve of type 3

A general point on this stratum is on the blow up of curves of type 3 in the above picture. The picture of the corresponding curve is



Since the limiting mixed Hodge structure is of Hodge-Tate type,

$$\Phi_e(Z_2) = \text{point and } \Phi_1 \text{ is trivial.}$$

Thus the interesting map is

$$\Phi_2 : Z_2^* \to \mathbb{C}^*$$

given by the cross-ratio of $(p', p''; q', q'')$.

Here we should quotient $\mathbb{C}^*$ by the action of monodromy but we shall ignore this. There are two degenerations of the above curve C. One is to a curve of type 6. For the same reason as when a curve of type 1 degenerates to one of type 4, the mapping Φ_2 extends across this locus and continues to map to $\mathbb{C}^*$.

However when C degenerate to a curve of type 5, i.e., when there is a non-trivial vanishing cycle and the monodromy cone goes from dimension 2 to dimension 3, we must add to point to $\mathbb{C}^*$ to receive the image of Φ_2. This is a partial toroidal completion of the type to be discussed in a continuation of these notes.

Example: This example illustrates how lifting the mapping

$$\Phi_0 : \overline{B} \to \overline{P}_{\mathrm{SBB}}{}^{**}$$

to the (set-theoretic) mapping

$$\Phi_2 : \overline{B} \to \overline{P}_T$$

may be used to suggest how one may at least partially desingularize completed moduli spaces for general type surfaces. This is in contrast to the case of curves where $\overline{\mathcal{M}}_g$ is essentially smooth and maps to a suitable toroidal completion $(\overline{\Gamma\backslash D})^{\mathrm{tor}}$ where $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$ and $D = \mathcal{H}_g$. References to the background and details of the following discussion may be found in [G1] and [G2].

$^{**}\overline{P}_{\mathrm{SBB}}$ denotes $\overline{P}$ as used elsewhere in these notes.

In summary form the basic points needed for this example are the following (here stated informally):

- For a given type of smooth general type surface X, Kollár-Shepherd-Barron-Alexeev have defined a moduli space $\mathcal{M}$ with a canonical projective completion $\overline{\mathcal{M}}$;
- the surfaces X_0 that are added to compactify the moduli space have $\mathbb{Q}$-Gorenstein canonical divisor class K_{X_0} and semi-log-canonical (slc) singularities;
- for surfaces there is a classification of slc singularities; in the case when X_0 is normal with a singular point p
  - (i) if X_0 is Gorenstein, then p is either simple elliptic or a cusp;
  - (ii) if X_0 is non-Gorenstein, then p is a rational singularity;

- still in the case of surfaces the period mapping $\Phi : \mathcal{M} \to \Gamma\backslash D$ extends to

$$\Phi_0 : \overline{\mathcal{M}} \to \overline{P}_{\mathrm{SBB}} \tag{A.V.1}$$

Remark: The period mapping Φ extends across the locus in $\overline{\mathcal{M}}$ of normal surfaces X_0 of type (ii) (the monodromy around X_0 is finite); for type (i) surfaces in general there is a non-trivial limiting mixed Hodge structure associated to a degeneration $X \to X_0$; it is this case that we shall be concerned with here.

- In contrast to the case of curves, $\overline{\mathcal{M}}$ is singular along the boundary $\partial M = \overline{\mathcal{M}}\backslash\mathcal{M}$; moreover, the mapping (A.V.1) does *not* lift to give the dotted arrow in

$$\begin{tikzcd} & \overline{P}_T \arrow[d] \\ \Phi_e : \overline{\mathcal{M}} \arrow[ur, dashed] \arrow[r] & \overline{P}_{\mathrm{SBB}} \end{tikzcd}$$

This suggests that, at least in cases where one has a local Torelli property, one might blow up $\overline{\mathcal{M}}$ to make the dotted arrow well defined and use this to partially desingularize $\overline{\mathcal{M}}$. This turns out to be the case for a very interesting class of surfaces. Referring to [FPR], [G1], [G2], [G3] for more details the example is the following:

- An *I-surface* is a smooth,[††] minimal general type surface X with
  - $K_X^2 = 1$;
  - $p_g(X) = 2$ and $q(X) = 0$.

  It is known that the moduli space $\mathcal{M}_I$ of I-surfaces is essentially smooth of dimension 28:

$$h^1(T_X) = 28 \text{ and } h^0(T_X) = h^2(T_X) = 0.$$

[††] Everything that follows works if we assume X has canonical (ADE or DuVal) singularities.

- The period domain D is of dimension 57, the IPR is a contact structure and the Torelli (or period) mapping

$$\Phi : \mathcal{M}_I \to \Gamma \backslash D$$

is locally 1-1;[‡‡] the image $\Phi(\mathcal{M}_I) = P \subset \Gamma\backslash D$ is a contact manifold;
- There is a 20-dimensional boundary component $\mathcal{N}_2 \subset \overline{\mathcal{M}_I}$ whose general point corresponds to a singular I-surface X_0 with the property
X_0 is normal and Gorenstein with a simple elliptic singularity of degree 2.

[‡‡]It is known that the monodromy group Γ is of finite index in the full arithmetic group $G_{\mathbb{Z}}$. Since $K_X^2 = 1$, the intersection form is unimodular on the primitive cohomology $H^2(X, \mathbb{Z})_{\mathrm{prim}} = c_1(K_X)^{\perp}$. The ideal situation would be that $\Gamma = G_{\mathbb{Z}}$ and that global Torelli (i.e., Φ is 1-1) holds, but this is not known.

The resolution of this singularity is $(\widetilde{X}, C) \to (X_0, p)$ where $\widetilde{X}$ is a smooth surface whose minimal model X_{min} is a K3 surface and $\widetilde{C} \subset \widetilde{X}$ is an elliptic curve with $\widetilde{C}^2 = -2$. The map $\widetilde{X} \to X$ is given by contradicting a -1 curve E with $E \cdot \widetilde{C} = 2$; it follows that the image $C \subset X$ of $\widetilde{C}$ is a curve C with $C^2 = 2$ and one node. From this it follows that X_{min} is a 2:1 cover of $\mathbb{P}^2$ branched over a sextic curve B and that C is a double cover of a tangent line ℓ to B;

- the limiting mixed Hodge structure corresponding to X_0 has associated graded

$$\Phi_e(X_0) = H^2(X_{\text{min}})_{\text{prim}} \oplus H^1(\widetilde{C});$$

it depends on 20 parameters and up to finite data determines the pair (X_{min}, C). In other words we have local Torelli for the boundary component.

What about the extension data in the limiting mixed Hodge structure?

To desingularize $\overline{\mathcal{M}}_I$ along $\mathcal{N}_2$ we need to blow up at a general point corresponding to a surface X_0 as described above. This means that we consider a 1-parameter degeneration $X_t \to X_0$ and do a semi-stable reduction to have a smooth total space with a normal crossing divisor $\widetilde{X}_0$ over the origin. From the Clemens-Schmid exact sequence one may guess that $\widetilde{X}_0$ has a double curve isomorphic to $\widetilde{C}$; i.e.,

$$\widetilde{X}_0 = \widetilde{X} \bigcup_{\widetilde{C}} Y$$

where Y is a smooth surface containing the curve $\widetilde{C}$. If $\widetilde{X}_0$ is the central fibre in a smooth family, then

$$N^*_{\widetilde{C}/\widetilde{X}} \cong N_{\widetilde{C}/Y}.$$

The line bundle on the left has degree 2, and if we think of Y as obtained from a smooth cubic in $\mathbb{P}^2$ by blowing up points q_i, then there must be 7 q_i's in order to have $\deg N_{\widetilde{C}/Y} = 2$.

Now the 1^{st} order extension data for the limiting mixed Hodge structure is $J = J(\widetilde{C})$ and the 7-parameters in the extension data corresponds to the points in $J(\widetilde{C})$ given by the to the q_i's. Of course there are important details required to make this precise, but this at least illustrates the point that the extension data in the limiting mixed Hodge structure may serve as a guide on how to desingularize some moduli spaces of surfaces.

VI. Local Torelli conditions

Given a variation of Hodge structure over a smooth quasi-projective variety B there is a corresponding period mapping

$$\Phi : B \to \Gamma \backslash D. \tag{VI.1}$$

There are two equivalent conditions that the *local Torelli* (LT) property should hold. One is that the differential of the mapping Φ should be injective.

Recall that by definition Φ should be locally liftable and this means that the differential of one, and hence any, local lifting should be injective.[†]

The other condition is given by considering the Higgs data (E, θ) given by the VHS. Here

- $E = \oplus E^p$ where $E^p = F^p / F^{p+1}$;
- $\theta = \oplus \theta^p$ where $\theta^p : E^p \to E^{p-1} \otimes \Omega^1_B$ is induced by the Gauss-Manin connection.

[†] There is a subtlety here. In moduli problems where the period mapping of a moduli space $\mathcal{M}$ maps a subvariety of $\mathcal{M}$ to a singular points of $\Gamma\backslash D$ corresponding to fixed points of Γ acting on D, the differential in the above sense may not be injective whereas in framework of moduli spaces or stacks it should be considered as being injective. The classical example here is the hyperelliptic locus in the moduli space $\mathcal{M}_g$ of smooth curves of genus $g \geqq 3$.

This gives a map

$$\text{(VI.2)} \qquad \theta : TB \to \mathrm{End}(E) \otimes \Omega^1_B$$

and in the Higgs setting local Torelli the condition is that this mapping should be an injective mapping of vector bundles. The observation then is that

These two local Torelli conditions are equivalent.

This is not entirely trivial. The usual expression for the differential of Φ is given by

$$\Phi_* : TB \to \oplus \operatorname{Hom}(F^p, V_C/F^p).$$

By the IPR this mapping is induced by

$$\begin{array}{rcl} \Phi_* : TB & \longrightarrow & \oplus \operatorname{Hom}(F^p, F^{p-1}/F^p) \\ & & \| \\ & & \oplus \operatorname{Hom}(F^p, E^{p-1}). \end{array}$$

Again by the IPR we have a factorization

$$\begin{tikzcd} TB \arrow[rr, "\Phi_*"] \arrow[dr, "\theta"'] & & \mathrm{Hom}(F^p, E^{p-1}) \\ & \mathrm{Hom}(E^p, E^{p-1}) \arrow[ur, "j"'] & \end{tikzcd}$$

where j is an inclusion of vector bundles. Thus

$$\ker \Phi_* = \ker \theta,$$

which implies the equivalence of the two LT conditions. Assuming unipotent local monodromies across the irreducible components Z_i of Z, the canonical extension $F_e^p \to \overline{B}$ of the Hodge bundles and the extension of the Gauss-Manin connection induces

(VI.3) $$\theta_e : T_{\overline{B}}(-\log Z) \to \mathrm{End}(E_e).$$

Definition (VI.4): *The local Torelli (LT) condition is satisfied on $\overline{B}$ if the map* (VI.3) *of vector bundles is injective.*

One may ask what the geometric meaning of this condition is? Let ω_e be the Chern form of the canonically extended augmented Hodge line bundle Λ_e. On B from the relation (I.3)

$$\omega(\xi) = \|\Phi_*(\xi)\|^2, \qquad \xi \in T_b B$$

we see that local Torelli is equivalent to the positivity $\omega > 0$ of the Chern form. On $\overline{B}$ the Chern form extends to a closed (1,1) current ω_e whose coefficients are in L^1_{loc} and which defines the Chern class of $\Lambda_e \to \overline{B}$. Moreover, along the divisor $Z = \overline{B} \backslash B$ even though ω_e is not smooth the condition

$$\text{(VI.5)} \qquad \omega_e(\xi) = 0, \qquad \xi \in T_b \overline{B}$$

is well defined.

Moreover, the fibres of Φ_e are defined by the exterior differential system $\omega_e = 0$. Since in general the limiting mixed Hodge structure is varying non-trivially along the B_p, $\omega_e(\xi) \neq 0$ does not give the right local Torelli condition to capture the full VLMHS.

This raises the question *What is the geometric meaning of (VI.4)?*

Theorem (IV.6) : *The local Torelli condition (VI.4) is equivalent to*

- $\omega_e(\xi) \neq 0$ *for* ξ *not tangent to a fibre* B_p*;*
- $\Phi_{1,*}(\xi) \neq 0$ *and if* $\Phi_{1,*}(\xi) = 0$*, then* $\Phi_{2,*}(\xi) \neq 0$ *for* $\xi \in TB_p$*.*

Geometrically, we have the set-theoretic maps

$$\text{(VI.7)} \qquad \begin{cases} \Phi_e : \overline{B} \to \overline{P} \\ \Phi_1 : (\text{fibres } B_p \text{ of } \Phi_e) \to \mathcal{E}_1 \\ \Phi_2 : (\text{fibres of } \Phi_e \text{ and } \Phi_1) \to \mathcal{E}_2. \end{cases}$$

For each of these maps the kernel of the differential can be defined analytically, and the condition (VI.4) is equivalent to the kernels of all these maps should be zero. In words

> *Local Torelli means geometrically that the limiting mixed Hodge structures together with their extension data should to* 1^{st} *order be faithfully captured by the family of Hodge structures and limiting mixed Hodge structures parametrized by* $\overline{B}$.

One can imagine that there should be complete algebraic varieties $\overline{P} = P_0, P_I, P_2$ and maps

$$\text{(VI.8)}\qquad \begin{cases} \Phi_0 : \ \overline{B} \longrightarrow P_0 & \text{(equal to } \Phi_e : \overline{B} \to \overline{P}) \\ \Phi_1 : \ \overline{B} \dashrightarrow P_1 & \\ \Phi_2 : \ \overline{B} \dashrightarrow P_2. & \end{cases}$$

where outside of the fibres B_p of Φ_0 the map Φ_1 is equal to Φ_0 and along the B_p it gives the 1^{st} order extension data, and Φ_2 is similarly defined using Φ_1 in place of Φ_0 and it maps to level 2 extension data. One may imagine that there is a blowing up $\overline{P}_\infty$ of $\overline{P}$ along $\partial P = \overline{P} \backslash P$ that has the full set of extension data for the LMHS's over ∂P. The maps Φ_k are then quotients of the map to $\overline{P}_\infty$.

An interesting point here is that no matter what the weight n of the original family of Hodge structures over B is, we need only go to level 2 to capture the full limiting mixed Hodge structure along the boundary Z. Put differently, if we go ahead and inductively define $\Phi_3, \ldots, \Phi_n$ as above, then

$$\text{fibres of } \Phi_n = \text{fibres of } \Phi_2.$$

This suggests the

Possible definition: $\overline{P} = P_0$ is the *Satake-Baily-Borel completion* of the image $P \subset \Gamma \backslash D$ of the original period mapping, and P_2 is the *minimal toroidal completion of P.*

A Lie theoretic proof of (VI.6) is given in [R].

VII. The case when $\dim B = 2$

We consider the case where $\dim B = 2$ and make the following assumptions:

(VII.1) *The differential of Φ is everywhere injective;*

(VII.2) *The group $Eff^1(\overline{B})$ of effective 1-cycles on $\overline{B}$ is finitely generated;*

Theorem (VII.3): *Under these assumptions there are $a_i \geq 0$ such that for $c \gg 0$ the line bundle*

$$L = c\Lambda_e - \sum_i a_i [Z_i]$$

is ample.

The assumption (VII.1) is a local Torelli one on B; it is a reasonable one to have the result. If one weakens it to the assumption that Φ_* is just generically 1-1, i.e., the image $\Phi(B) = P \subset \Gamma \backslash D$ is a surface, and there is still a result that will be discussed elsewhere.

The assumption (VII.2) is not necessary and the proof that this is so will be also given elsewhere. The objective here is to give a simple clear statement whose proof illustrates the essential ideas behind a more general result.

Proof: Using (VII.2) there will be a finite set of irreducible "test curves" C that generate $\mathit{Eff}^1(\overline{B})$. We have to show that there exists c and the a_i such that for L in the statement of the theorem

$$L \cdot C = \deg L\big|_C > 0$$

for all such test curves. For this we separate the Z_i into

(1) the Z_i where $\Phi_e(Z_i)$ is a curve; call these Z_α's;

(2) the Z_i where $\Phi_e(Z_i)$ is a point; we continue to call these Z_i's.

Part of the reason for the notation is that for a Z_α in the first group we will have $a_\alpha = 0$.

Let C be a test curve. There are three possibilities:

(a) $C \cap B$ is a Zariski open set in C (i.e., $C \cap Z$ is a finite set of points on C);

(b) C is a curve Z_α from group (1) above;

(c) C is a curve Z_i from group (2) above.

For C in either group (a) or group (b) we have

$$\Lambda_e \cdot C > 0.$$

We will see below that the a_i are determined by the intersection matrix $\|Z_i \cdot Z_j\|$. Then for large enough c we will have

$$\left(c\Lambda_e - \sum_i a_i Z_i \right) \cdot C > 0.$$

For C in group (c) we have $\Lambda_e \cdot C = 0$. By the Hodge index theorem the intersection matrix $M =: \|Z_i \cdot Z_j\|$ is negative definite. There is then a linear algebra result whose proof will be given below.

Lemma: *Let M be an integral negative definite symmetric matrix where all the off-diagonal entries are $\geqq 0$. Then M has an eigenvector $a = {}^t(a_1, \ldots, a_m)$ with all $a_i > 0$.*

If μ is the eigenvalue, then

$$Ma = \mu a$$

where $\mu < 0$ since M is negative definite.

Applying this to the case at hand gives

$$\sum_j (a_j Z_j) \cdot Z_i < 0$$

for each Z_i. Thus

$$\left(c\Lambda_e - \sum_j a_j [Z_j] \right) \cdot Z_i > 0$$

and we are done. □

Note on Grauert's contactability theorem in the surface case (cf. [BS]):

Theorem

Let X be a normal complex analytic variety, $Z \subset X$, a compact local complete extension subvariety and assume that the normal bundle

$$N^*_{Z/X} \to Z$$

is ample. Then there exist a complex variety Y, a proper holomorphic mapping $f : X \to Y$ and $p \in Y$ such that

$$\begin{cases} f(Z) = p \ \textit{and} \\ f : X \backslash Z \xrightarrow{\sim} Y \backslash \{p\} \ \textit{is biholomorphic.} \end{cases}$$

Suppose now that $\dim X = 2$ and $Z = Z_1 \cup \cdots \cup Z_m$ are smooth curves forming a normal crossing divisor. Then

$$[Z] = \sum_{i=1}^{m} [Z_i],$$

and

$$N_{Z/X} = [Z]\big|_Z.$$

The condition that $N^*_{Z/X} \to Z$ be ample is

$$\left(\sum_{i=1}^{m} Z_i\right) \cdot Z_j < 0 \ \text{ for all } \ j.$$

We will show this implies that the intersection matrix

$$M := \|Z_i \cdot Z_j\| < 0$$

is negative definite.[‡]

For the proof using the above lemma we assume that M has an eigenvector a with maximal eigenvalue μ. Then all $a_i > 0$. We want to show that $\mu < 0$. Now

$$\left(\sum_i a_i Z_i\right) \cdot Z_j = \mu a_j \text{ for all } j.$$

Suppose that $\mu \geq 0$ and renumber so that we have $a_1 \geqq a_i$ for all i.

[‡] The converse is not true. The matrix $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ satisfies the first condition $\iff a + b < 0$ and $b + c < 0$, while the second condition is $a < 0, c < 0$ and $ac - b^2 < 0$. These are not equivalent conditions (e.g., take $\left(\begin{smallmatrix} -5 & 2 \\ 2 & -1 \end{smallmatrix}\right)$).

Then

$$a_1 Z_1 + \cdots + a_m Z_m = a_1(Z_1 + \cdots + Z_m) - \sum_{i>1}(a_1 - a_i) Z_i$$

$$\left(\sum_i a_i Z_i\right) \cdot Z_1 = a_1 \left(\sum_i Z_i\right) \cdot Z_1 - \left(\sum_{i>1}(a_1 - a_i) Z_i\right) \cdot Z_1.$$

By the ampleness assumption the first term is negative, and since $Z_i \cdot Z_1 \geqq 0$ for $i \neq 1$,

$$-\sum (a_1 - a_i) Z_i \cdot Z_i \leqq 0. \qquad \square$$

The point here is that the argument applies to our Hodge-theoretic situation if we take X to be a tubular neighborhood of Z in $\overline{B}$. If we know that $N^*_{Z/X}$ is ample, then we don't have to use the global Hodge index theorem to conclude that $\|Z_i \cdot Z_j\| < 0$.

This gives the

Conclusion (VII.4): *If Φ_1 is non-constant on each component Z_i of Z, then $N^*_{Z/X}$ is ample and $\|Z_i \cdot Z_j\| < 0$.*

This is a purely Hodge theoretic argument using only the behavior of Φ_e in a neighborhood of Z. This conclusion raises the question

> *Without assuming $N^*_{Z/\overline{B}}$ is ample, can we have $\Phi_1 =$ constant on Z?*

To give a partial answer to this we will show

(VII.5) Assuming that $\Phi(B)$ is a surface, Φ_1 *is non-constant on some component Z_i of Z.*

The proof is in two steps. We recall our notation $\lambda \in H^2(\overline{B})$ for the Chern class of the canonically extended augmented Hodge line bundle $\Lambda_e \to \overline{B}$.

Step one: In this argument we denote by $[Z_i] \in H^2(\overline{B})$ the Chern class of the line bundle $[Z_i]$. If $\lambda^2 > 0$ and $\lambda \cdot Z_i = 0$ for all i, then we claim that $[Z_1], \ldots, [Z_m]$ are linearly independent.

Proof: If $\sum_i a_i[Z_i]$ is a primitive relation, let

$$K_+ = \{i : a_i > 0\}, \quad K_- = \{i : a_i < 0\}.$$

Assume $K_+ \neq \emptyset$ and $K_- \neq \emptyset$. We have

$$\sum_{i \in K_+} a_i[Z_i] = \sum_{j \in K_-} (-a_j)[Z_j].$$

By primitivity of the relation $\sum_{i \in K_+} a_i[Z_i] = 0$ so

$$\left(\sum_{i \in K_+} a_i[Z_i]\right)^2 < 0.$$

But since $Z_i \cdot Z_j \geqq 0$ for $i \in K_+$, $j \in K_-$,

$$\left(\sum_{i \in K_+} a_i [Z_i]\right)^2 = \left(\sum_{i \in K_+} a_i [Z_i]\right)\left(\sum_{j \in K_-} (-a_j)[Z_j]\right) \geqq 0,$$

which is a contradiction. Thus either K_+ or K_- is empty; say $K_- = \emptyset$. Then we have a relation among the $[Z_i]$ with positive coefficients that cannot happen on a Kähler manifold.

Step two: Using the basic formula

$$\Phi_1^* L_A = -\sum_i \langle A, N_i \rangle [Z_i],$$

if Φ_1 is constant on each Z_i, we have

$$\sum_i \langle A, N_i \rangle Z_i \cdot Z_j = 0$$

for all j. Then

$$\sum \langle A, N_i \rangle [Z_i] \in \mathrm{span}\left([Z_1], \ldots, [Z_m]\right) \cap \mathrm{span}\left([Z_1], \ldots [Z_m]\right)^{\perp},$$

which is zero by the Hodge index theorem. Varying A over $\check{\sigma}$ we may conclude that all $N_i = 0$. $\square$

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