THE THEORY OF VARIFOLDS

A Variational Calculus in the Large for the k-Dimensional Area Integrand

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1. INTRODUCTION

This paper introduces and studies a space of measure theoretic surfaces called varifolds. Varifolds of dimension k are defined in R^n for all non-negative integers $k \leq n$ as certain non-linear real valued functions on the space of all continuous k-forms on R^n. Varifolds on a Riemannian manifold can be defined either intrinsically on the manifold or, as is done for convenience in this paper, using an isometric imbedding of the manifold in some Euclidean space.

Many objects of geometric interest can be considered as varifolds in a natural way; including, for example, differentiable varieties which are locally of finite area, singular chains satisfying a Lipschitz condition, flat chains of finite mass over a finite coefficient group, integral currents, and more general surfaces such as are approximated by soap films. Since most natural geometric constructions are possible in the space of varifolds, the language of varifolds is useful for an assortment of geometric problems. The existence of several geometrically important compact subsets in the space of varifolds and the continuity (in contrast with the usual lower semi-continuity) of the k-dimensional area function $\underline{\underline{W}}$ form the basis for the use of varifolds in the study of calculus of variations problems in the large for the k-dimensional area integrand.

The most important subset of the space of all varifolds is the space of integral varifolds. Except for arbitrarily small k-dimensional measure, each integral varifold of dimension k is a compact differentiable manifold of dimension k with positive integer multiplicities on its various components. A k-dimensional integral varifold is called stationary if and only if it has zero initial rate of change of k-dimensional area (measured counting multiplicities) for each differentiable deformation of the supporting space. Within the framework of integral varifolds we are able to give a solution to the existence portion of two problems of long standing.

First, as the most important results in this paper and as the initial justification for defining varifolds, we establish general topological conditions sufficient for the existence of k-dimensional minimal surfaces, i.e. stationary integral varifolds, on compact Riemannian manifolds (15.1). In particular, whenever $k \leq n$ are positive integers, we show that each n-dimensional compact Riemannian manifold M of class 3 supports at least one stationary integral varifold V of dimension k (15.2). V will have boundary only if M does, and, in that case, the boundary of V will lie on the boundary of M and itself be a regular integral varifold of dimension k-1. (The boundary of a stationary varifold is defined analytically and does not necessarily coincide with the topological boundary, (11.1(4)(5)). There are no curvature restrictions on M. As a step in proving this result we show that the homology groups of M, with a downward shift in dimension by k, are naturally a direct summand of the (appropriately defined) homotopy groups of the VZ-space consisting of pairs (V, T) where V is a k-dimensional varifold on M and T is flat k-chain on M related to V(13.5).

The proof of (15.1) requires the use of most of the other results in this paper.

Second, we show the existence of a "best possible solution" to Plateau's problem, i.e. the problem of least area. If N is a compact (k-1)-dimensional submanifold of R^n of class 3 without boundary, then the set of non-zero k-dimensional stationary integral varifolds V having N as (analytic) boundary is compact with a positive lower bound on k-area. Hence there is a stationary integral varifold of least area having N as boundary (11.5). The term "best possible solution" seems justified since there is always at least one non-zero stationary integral varifold having N as boundary and, in every other formulation of Plateau's problem known to this author, each solution surface (if one exists at all) is naturally a stationary integral varifold. In general, solutions by other methods will have strictly larger area. The examples of

(11.1) illustrate the necessity for admitting surfaces of the generality of integral varifolds in the study of least area problems.

In proving the above results one first obtains the desired surface as a (function valued) measure on R^n and then shows that this measure lies on a rectifiable subset of R^n with integer densities.

In addition to the two main results above, a number of other results of classical interest are obtained through the methods of varifold geometry. For example, we obtain an isoperimetric inequality for compact manifolds with bounded mean curvature (8.7), (8.9). Also we show that a k-dimensional manifold of bounded mean curvature, but arbitrary topological type, which lies near a k-disk must have k-area nearly equal an integral multiple of the area of the k-disk (9.10).

The author is indebted to H. Federer for his suggestion that results similar to those of M. Morse [MR] for the existence of geodesics on manifolds might be obtained for the existence of higher dimensional minimal surfaces using topological information about the integral cycle groups. Some of our results are indeed similar to those of Morse. In the present context, however, it does not seem possible to satisfy simultaneously the axioms of "upper semi-reducibility" and "accessibility" upon which the Morse theory is based.

The measure theoretic foundations of this paper are due largely to H. Federer, and the applications of these results to show the rectifiability properties of varifolds is similar to a technique used by W.H. Fleming. This paper is, in part, derived also from the work of A. S. Besicovitch, E. R. Reifenberg, H. Whitney, and L. C. Young. Throughout the preparation of this paper, the author has enjoyed stimulating and fruitful conversations with H. Federer, W.H. Fleming, D.C. Spencer, and H. Whitney, among others, and a number of the ideas generated by these conversations appear in this paper.

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2. DEFINITION AND BASIC PROPERTIES OF VARIFOLDS

2.1 DEFINITIONS. Preliminaries.

(1) R denotes the real number field. R_0^+ denotes the positive real numbers and R_0^- the negative real numbers. We set $R^+ = R_0^+ \cup \{0\}$ and $R^- = R_0^- \cup \{0\}$. For each non-negative integer n, R^n denotes Euclidean n-dimensional space with the usual inner product $\cdot$ and norm $|\ |$. For each $p \in R^n$ and $r \in R_0^+$ we define

$$\underline{\underline{D}}^n(p, r) = R^n \cap \{x : |x-p| \le r\},$$
$$\underline{\underline{D}}_0^n(p, r) = R^n \cap \{x : |x-p| < r\},$$
$$\partial \underline{\underline{D}}^n(p, r) = R^n \cap \{x : |x-p| = r\}, \text{ and}$$
$$S^{n-1} = \partial \underline{\underline{D}}^n(0, 1).$$

(2) For integers $0 \le k \le n$ we denote by $(n : k)$ the binomial coefficient $[k!\,(n-k)!]^{-1} n!$.

(3) For integers $0 \le k \le n$ we denote by $\underline{\underline{\Lambda}}_k(R^n)$ and $\underline{\underline{\Lambda}}^k(R^n)$ the dual vector spaces of k-vectors and k-covectors on R^n. For $\lambda \in \underline{\underline{\Lambda}}^k(R^n)$ and $\mu \in \underline{\underline{\Lambda}}_k(R^n)$ we write $\lambda(\mu) = \lambda \cdot \mu$. The direct sums

$$\underline{\underline{\Lambda}}_*(R^n) = \bigoplus_j \underline{\underline{\Lambda}}_j(R^n), \qquad \underline{\underline{\Lambda}}^*(R^n) = \bigoplus_j \underline{\underline{\Lambda}}^j(R^n)$$

are the contravariant and covariant Grassmann algebra of R^n with the exterior multiplication $\wedge$. The inner product $\cdot$ and norm $|\ |$ of R^n induce inner products and norms on $\underline{\underline{\Lambda}}_*(R^n)$ and $\underline{\underline{\Lambda}}^*(R^n)$ also denoted $\cdot$ and $|\ |$. We define

$$\underline{\underline{D\Lambda}}^k(R^n) = \underline{\underline{\Lambda}}^k(R^n) \cap \{\lambda : |\lambda| \le 1\} \text{ and}$$
$$\partial \underline{\underline{D\Lambda}}^k(R^n) = \underline{\underline{\Lambda}}^k(R^n) \cap \{\lambda : |\lambda| = 1\}.$$

If $\mu \in \underline{\underline{\Lambda}}_h(R^n)$ and $\lambda \in \underline{\underline{\Lambda}}^j(R^n)$ for integers $0 \le h \le j \le n$ we define $\mu \wedge \lambda \in \Lambda^{j-h}(R^n)$ by requiring for each $\nu \in \underline{\underline{\Lambda}}_{j-h}(R^n)$ that $(\mu \wedge \lambda) \cdot \nu = \lambda \cdot (\mu \wedge \nu)$. $\wedge = \lrcorner$ ([F] 1.5.1)

If $\{x^1, x^2, \ldots, x^n\}$ is an orthonormal basis for R^n, then, in the usual notation,

$$\{(\partial/\partial x^{i(1)}) \wedge (\partial/\partial x^{i(2)}) \wedge \ldots \wedge (\partial/\partial x^{i(k)}) : 1 \le i(1) < i(2) < \ldots < i(k) \le n\} \text{ and}$$
$$\{dx^{i(1)} \wedge dx^{i(2)} \wedge \ldots \wedge dx^{i(k)} : 1 \le i(1) < i(2) < \ldots < i(k) \le n\}$$

are dual orthonormal bases for $\Lambda_k(W)$ and $\Lambda^k(W)$ respectively where $W \cong R^n$ is the tangent space of R^n at some point in R^n. For simplicity of notation we will identify R^n with its various tangent spaces W throughout this paper.

For each integer $m \geq k$ a differentiable map $f : R^n \longrightarrow R^m$ has a differential $Df(p) : R^n \longrightarrow R^n$ at each $p \in R^n$ which in turn induces linear mappings $f_{\#}(p) : \Lambda_k(R^n) \longrightarrow \Lambda_k(R^m)$ and $f^{\#}(p) : \Lambda^k(R^m) \longrightarrow \Lambda^k(R^n)$. $f_{\#}(p) = \Lambda_k Df(p)$

$f^{\#}(p) = \Lambda^k Df(p)$ Let $0 \leq j \leq n$ and $0 \leq k \leq n$ be integers, $\lambda \in \Lambda^j(R^n)$, and $\mu \in \Lambda^k(R^n)$. We say that λ is <u>parallel with</u> μ or, equivalently, μ is <u>parallel with</u> λ if and only if for some $\omega \in \Lambda^*(R^n)$ either $\mu = \omega \wedge \lambda$ or $\lambda = \omega \wedge \mu$. A similar definition defines parallel j and k-vectors. If $\nu \in \Lambda_k(R^n)$ is dual to μ we say that λ is parallel with ν or, equivalently, μ is parallel with ν if and only if λ is parallel with μ. j-vectors and j-covectors are <u>parallel with</u> an affine subspace of R^n of dimension k if and only if they are parallel with each k-vector lying in that subspace.

(4) For integers $0 \leq k \leq n$ we denote by $C^k(R^n)$ the real vector space of all continuous differential k-forms on R^n. Each such form can be regarded as a continuous map $\varphi : R^n \longrightarrow \Lambda^k(R^n)$. We define a norm $|\ |$ on $C^k(R^n)$ by setting for each $\varphi \in C^k(R^n)$,

$$|\varphi| = \sup\{|\varphi(x)| : x \in R^n\} \in R^+ \cup \{\infty\}.$$

This defines the uniform or $|\ |$ topology on $C^k(R^n)$. For each $\varphi \in C^k(R^n)$, the <u>support of</u> φ, written $\operatorname{spt}(\varphi)$, is $\operatorname{clos}\{x : \varphi(x) \neq 0\} \subset R^n$. We set

$$C_0^k(R^n) = C^k(R^n) \cap \{\varphi : \operatorname{spt}(\varphi) \text{ is compact}\}.$$

and topologize $C_0^k(R^n)$ as the inductive limit of its subsets $C_0^k(R^n) \cap \{\varphi : \operatorname{spt}(\varphi) \subset D^n(0, j)\}$, $j = 1, 2, 3, \ldots$ each of which has the $|\ |$ topology. Note that if $\varphi, \varphi_1, \varphi_2, \varphi_3, \ldots \in C_0^k(R^n)$, then $\lim_i \varphi_i = \varphi$ if and only if $\lim_i |\varphi_i - \varphi| = 0$ and $\bigcup_i \operatorname{spt}(\varphi_i)$ is bounded.

The direct sums

$$C^*(R^n) = \bigoplus_j C^j(R^n) \quad \text{and} \quad C_0^* = \bigoplus_j C_0^j(R^n)$$

are graded algebras with exterior multiplication $\wedge$.

For $U \subset R^n$ we set $\underline{\underline{C}}^k(U) = \{\varphi|U : \varphi \in \underline{\underline{C}}^k(R^n)\}$ and $\underline{\underline{C}}_0^k(U) = \{\varphi|U : \varphi \in \underline{\underline{C}}_0^k(R^n)\}$. If U is bounded then $\underline{\underline{C}}^k(U) = \underline{\underline{C}}_0^k(U)$.

For each integer $m \geq k$ each continuously differentiable map $f : R^n \longrightarrow R^m$ induces a linear map $f^{\#} : \underline{\underline{C}}^k(R^m) \longrightarrow \underline{\underline{C}}^k(R^n)$ given for $\varphi \in \underline{\underline{C}}^k(R^m)$ and $x \in R^n$ by $f^{\#}(\varphi)(x) = f^{\#}(p)(\varphi(f(x)))$. If $f^{-1}(K)$ is compact whenever $K \subset R^m$ is compact then $f^{\#} : \underline{\underline{C}}_0^k(R^m) \longrightarrow \underline{\underline{C}}_0^k(R^n)$.

If $\mu : R^n \longrightarrow \underline{\underline{\Lambda}}_h(R^n)$ is continuous and $\varphi \in \underline{\underline{C}}^j(R^n)$ for integers $0 \leq h \leq j$ we define $\mu \wedge \varphi \in \underline{\underline{C}}^{j-h}(R^n)$ by setting $\mu \wedge \varphi(x) = \mu(x) \wedge \varphi(x)$ for each $x \in R^n$.

For each $\lambda \in \underline{\underline{\Lambda}}^k(R^n)$ we define $\omega(\lambda) \in \underline{\underline{C}}^k(R^n)$ by setting for each $x \in R^n$, $\omega(\lambda)(x) = \lambda$. $[\Lambda^k(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n)]$ // $G(n,k)$

(5) Let $0 \leq k \leq n$ be integers. We denote by $\underline{\underline{\Omega}}_k(R^n)$ the Grassmann manifold of all unoriented k-plane directions in R^n with the usual metric and measure. For each $\lambda \in \underline{\underline{\Lambda}}^k(R^n)$ we define the continuous function $\|\lambda\| : \underline{\underline{\Omega}}_k(R^n) \longrightarrow R^+$ by setting for each $P \in \underline{\underline{\Omega}}_k(R^n)$,

$$\|\lambda\|(P) = \|\lambda\| \cdot P = |\lambda \cdot \mu| = |\lambda(\mu)|$$

where $\mu \in \underline{\underline{\Lambda}}_k(R^n)$ is a simple unit vector parallel with any k-plane in R^n having direction P. The absolute value function in the definition eliminates the ambiguity in the choice of μ.

We identify also $\underline{\underline{\Omega}}_k(R^n)$ as the space of all orthogonal projections $R^n \longrightarrow \mathbb{R}^k$

(6) Let $0 \leq m \leq n$ be integers. By an (abstract) m-dimensional closed Riemannian manifold M we mean a closed submanifold of some (abstract) m-dimensional Riemannian manifold N (for example, the double of M). If $0 \leq k \leq m$ is an integer we define $\underline{\underline{C}}^k(N)$ and $\underline{\underline{C}}_0^k(N)$ in the obvious way. We define $\underline{\underline{C}}^k(M) = \{\varphi|M : \varphi \in \underline{\underline{C}}_0^k(N)\}$ and $\underline{\underline{C}}_0^k(M) = \underline{\underline{C}}_0^k(N) \cap \underline{\underline{C}}^k(M)$. Throughout this paper the word "manifold" always implies a differentiability structure of class at least 1. We denote by ∂M the boundary of manifold M. If r is a positive integer and M is a manifold of (differentiability) class r then, by definition, ∂M is also a manifold of class r. If N is a submanifold of manifold M, then, by definition, $\partial M \cap N = \phi$. dim(M) denotes the dimension of manifold M.

(7) Let $0 \leq k \leq n$ be integers. $\underline{\underline{H}}^k$ denotes k-dimensional Hausdorff measure on R^n [FF 8.2]. $\underline{\underline{H}}^n$ equals Lebesgue n-dimensional measure on R^n. We set $\alpha(n) = \underline{\underline{H}}^n(\underline{\underline{D}}^n(0, 1))$. We write $\alpha((n:k)) = \alpha(n:k)$. If μ is a Borel measure on R^n and $x \in R^n$ we define

$$\odot^k(\mu, x) = \lim_{r \to 0^+} \alpha(k)^{-1} r^{-k} \mu(\underline{\underline{D}}^n(x, r)) = \vartheta^k(\mu, x)$$

to be the k-dimensional density of μ at x, provided this limit exists; similarly the upper and lower densities

$$\overline{\odot}^k(\mu, x) \quad \text{and} \quad \underline{\odot}^k(\mu, x)$$

are defined as the corresponding lim sup and lim inf.

(8) If q is a positive integer and $a_1, a_2, \ldots, a_q \in R$, we define

$$\text{average}\{a_1, a_2, \ldots, a_q\} = q^{-1}(a_1 + a_2 + \ldots + a_q).$$

(9) If n is a positive integer and $A \subset R^n$ we denote by clos(A) the closure of A and by diam(A) the diameter of A.

(10) If A and B are metric spaces and $f : A \longrightarrow B$, we define $\text{Lip}(f) : A \longrightarrow R^+ \cup \{\infty\}$ by setting for $a \in A$,

$$\text{Lip}(f)(a) = \limsup_{r \to 0^+} \{[\text{dist}(a, c)]^{-1} \text{dist}(f(a), f(c)) : c \in A \text{ and } 0 < \text{dist}(a, c) < r\}.$$

We write also $\text{Lip}(f) = \sup\{\text{Lip}(f)(a) : a \in A\}$.

(11) If $f : A \longrightarrow R$ we set $f^+, f^- : A \longrightarrow R$ by setting for $a \in A$, $f^+(a) = \max\{f(a), 0\}$ and $f^-(a) = \min\{f(a), 0\}$. Note $f = f^+ - f^-$.

(12) If $f : A \times B \longrightarrow C$ we define for each $a \in A$ and $b \in B$, $f(\cdot, b) : A \longrightarrow C$ and $f(a, \cdot) : B \longrightarrow C$ by setting $f(\cdot, b)(a) = f(a, \cdot)(b) = f(a, b)$.

(13) If f is a function we denote by dmn(f) the domain of f.

(14) Several computations in this paper involve functions of more than one variables For notational simplicity we adopt the following conventions. Let $g : R \longrightarrow R$ an arbitrary function. $O_z(g)$ will denote any of several functions $f(z)$, depending only on the variable z, such that for some $c \in R^+$ $\liminf_{z \to 0^+} c|g(z)| - |f(z)| = 0$.

$o_z(g)$ will denote any of several functions $f(z)$, depending only on the variable z, for which $\lim_{z \to 0^+} g(z)^{-1} f(z) = 0$.

2.2 DEFINITIONS. Varifolds. This section includes the basic definitions for varifolds. For expositional reasons we have included some definitions at this point whose justification depends on later results. Let $0 \le k \le n$ be integers.

(1) A varifold of dimension k in R^n is a function $V : \underline{\underline{C}}_0^k(R^n) \longrightarrow R^+$ satisfying the following three axioms:

(i) $V(r\varphi) = |r| V(\varphi)$ for each $r \in R$ and $\varphi \in \underline{\underline{C}}_0^k(R^n)$;

(ii) $V(\varphi + \psi) \le V(\varphi) + V(\psi)$ for each $\varphi, \psi \in \underline{\underline{C}}_0^k(R^n)$; and

(iii) $V(f \wedge \varphi + g \wedge \varphi) = V(f \wedge \varphi) + V(g \wedge \varphi)$ for each $\varphi \in \underline{\underline{C}}_0^k(R^n)$ and $f, g \in \underline{\underline{C}}^0(R^n)$ with $f = f^+$ and $g = g^+$.

We denote by $\mathcal{V}_k(R^n)$ the space of all varifolds of dimension k in R^n. In case j is an integer and $j \notin \{0, 1, 2, \ldots, n\}$ we set $\mathcal{V}_j(R^n) = \{0\}$.

For each $V \in \mathcal{V}_k(R^n)$ we define the support of V, written $\operatorname{spt}(V)$, to be the smallest closed set K for which $V(\varphi) = 0$ whenever $\varphi \in \underline{\underline{C}}_0^k(R^n)$ and $\operatorname{spt}(\varphi) \cap K = 0$. We denote by $\underline{\underline{V}}_k(R^n)$ the subset of $\mathcal{V}_k(R^n)$ of varifolds having compact support. In case j is an integer and $j \notin \{0, 1, 2, \ldots, n\}$ we set $\underline{\underline{V}}_j(R^n) = \{0\}$.

If $0 \le j \le k$ is an integer, $\varphi \in \underline{\underline{C}}^j(R^n)$, and $V \in \mathcal{V}_k(R^n)$, we define $V \wedge \varphi \in \mathcal{V}_{k-j}(R^n)$ by setting for each $\psi \in \underline{\underline{C}}_0^{k-j}(R^n)$, $V \wedge \varphi(\psi) = V(\varphi \wedge \psi)$. If $\varphi \in \underline{\underline{C}}_0^j(R^n)$, then $V \wedge \varphi \in \underline{\underline{V}}_k(R^n)$.

If $0 \le h \le n-k$ is an integer, $\mu : R^n \longrightarrow \underline{\underline{\Lambda}}_h(R^n)$ is continuous, and $V \in \mathcal{V}_k(R^n)$ we define $V \wedge \mu \in \mathcal{V}_{k+j}(R^n)$ by setting for each $\varphi \in \underline{\underline{C}}_0^{k+h}(R^n)$, $V \wedge \mu(\varphi) = V(\mu \wedge \varphi)$. If μ has compact support then $V \wedge \mu \in \underline{\underline{V}}_{k+h}(R^n)$.

Each $V \in \underline{\underline{V}}_k(R^n)$ admits a unique continuous extension $V : \underline{\underline{C}}^k(R^n) \longrightarrow$ [illegible] for which (i), (ii), (iii) above are valid with $\varphi, \psi \in \underline{\underline{C}}^k(R^n)$. If $V_1, V_2, V_3, \ldots \in \underline{\underline{V}}_k(R^n)$ and $\lim_i V_i(\varphi) = V(\varphi)$ for each $\varphi \in \underline{\underline{C}}^k(R^n)$, then $V \in \underline{\underline{V}}_k(R^n)$ and R^n has a compact subset containing the supports of all the V_i.

For each $U \subset R^n$ we set

$$\mathcal{V}_k(U) = \mathcal{V}_k(R^n) \cap \{V : \mathrm{spt}(V) \subset U\} \text{ and}$$
$$\underline{\underline{V}}_k(U) = \underline{\underline{V}}_k(R^n) \cap \{V : \mathrm{spt}(V) \subset U\}.$$

(2) Let $k \leq m \leq n$ be integers and $A \subset R^n$ be an m-dimensional closed submanifold of R^n of class 1. $V \in \mathcal{V}_k(R^n)$ is said to <u>lie intrinsically on</u> A if and only if $V(\varphi) = 0$ for each $\varphi \in \underline{\underline{C}}_0^k(R^n)$ for which $\varphi(x) \cdot \lambda = 0$ whenever $x \in A -$ ∂ and $\lambda \in \underline{\underline{\Lambda}}^k(R^n)$ is parallel with the m-plane in R^n tangent to A at x. If V li intrinsically on A, then $V \in \mathcal{V}_k(A)$. If $B \subset R^n$ is another manifold and $W \in \mathcal{V}_k$ then (V, W) is said to <u>lie intrinsically on</u> (A, B) if and only if V lies intrinsical on A and W lies intrinsically on B.

It is clear how one defines the space $\mathcal{V}_k(M)$ of varifolds of dimension k on an m-dimensional closed Riemannian manifold M; namely as continuous functions $V : \underline{\underline{C}}_0^k(M) \longrightarrow R^+$ satisfying the axioms of (1) above. So defined, $\mathcal{V}_k(M)$ is naturally isomorphic with

$$\mathcal{V}_k(A) \cap \{V : V \text{ lies intrinsically on } A\}$$

whenever $A \subset R^n$ is a closed isometric imbedding of M into R^n. $\underline{\underline{V}}_k(M)$ has the obvious meaning.

(3) The <u>sum</u> $V + W$ of two varifolds $V, W \in \mathcal{V}_k(R^n)$ is again a varifold given for $\varphi \in \underline{\underline{C}}_0^k(R^n)$ by $(V + W)(\varphi) = V(\varphi) + W(\varphi)$.

(4) $\mathcal{V}_k(R^n)$ admits a natural partial ordering $\leq$ as follows: For $V, W \in \mathcal{V}_k(R^n)$ we say $V \leq W$ if and only if $V(\varphi) \leq W(\varphi)$ for each $\varphi \in \underline{\underline{C}}_0^k(R^n)$. If $V \leq W$ and $W \leq V$, then $V = W$.

(5) The <u>union</u> $V \cup W$ of two varifolds $V, W \in \mathcal{V}_k(R^n)$ is that unique smallest varifold in $\mathcal{V}_k(R^n)$ such that $V \leq V \cup W$ and $W \leq V \cup W$, i.e. if $U \in \mathcal{V}_k(R^n)$ and $V \leq U$ and $W \leq U$, then $V \cup W \leq U$. The existence of $V \cup W$ is established in 3.10. Clearly $V \cup W \leq V + W$.

(6) The <u>intersection</u> $V \cap W$ of two varifolds $V, W \in \mathcal{V}_k(R^n)$ is that unique largest varifold in $\mathcal{V}_k(R^n)$ such that $V \cap W \leq V$ and $V \cap W \leq W$, i.e. if $U \in \mathcal{V}_k($ and $U \leq V$ and $U \leq W$, then $U \leq V \cap W$. The existence of $V \cap W$ is establis

in 3.10. Often $V \cap W = 0$ when $V \neq 0$ and $W \neq 0$.

(7) If $A \subset R^n$ is a Borel set and $V \in \mathcal{V}_k(R^n)$, we denote by $V \cap A$ the <u>intersection of</u> V with A. The definition and proof of existence of $V \cap A \in \mathcal{V}_k(R^n)$ are given in 3.9.

(8) $\mathcal{V}_k(R^n)$ can be regarded as a subset of the cartesian product space $\prod \{R_\varphi : \varphi \in \underline{\underline{C}}_0^k(R^n)\}$ where R_φ denotes R indexed by φ. The relative topology on $\mathcal{V}_k(R^n)$ is called the <u>weak topology</u>. Note that if $V, V_1, V_2, V_3, \ldots \in \mathcal{V}_k(R^n)$, then $\lim_i V_i = V$ in the weak topology if and only if $\lim_i V_i(\varphi) = V(\varphi)$ for each $\varphi \in \underline{\underline{C}}_0^k(R^n)$.

(9) We define the $\underline{\underline{F}}$ metric on $\underline{\underline{V}}_k(R^n)$ by setting for each $V, W \in \underline{\underline{V}}_k(R^n)$,

$$\underline{\underline{F}}(V, W) = \sup\{|V(\varphi) - W(\varphi)| : \varphi \in \underline{\underline{C}}_0^k(R^n),\ |\varphi| \leq 1, \text{ and } \operatorname{Lip}(\varphi) \leq 1\}.$$

We write also $\underline{\underline{F}}(V) = \underline{\underline{F}}(V, 0)$.

(10) We define the $\underline{\underline{M}}$ metric on $\underline{\underline{V}}_k(R^n)$ by setting for each $V, W \in \underline{\underline{V}}_k(R^n)$,

$$\underline{\underline{M}}(V, W) = \sup\{V(\varphi) - W(\varphi) + W(\psi) - V(\psi) : \varphi, \psi \in \underline{\underline{C}}_0^k(R^n) \text{ and } |\varphi| = |\psi| = 1\}$$

We write also $\underline{\underline{M}}(V) = \underline{\underline{M}}(V, 0)$.

(11) We define the <u>weight metric</u> $\underline{\underline{W}}$ on $\underline{\underline{V}}_k(R^n)$ by setting for each $V, W \in \underline{\underline{V}}_k(R^n)$,

$$\underline{\underline{W}}(V, W) = \sup_f \Big\{ \gamma(n, k) \int_{\lambda \in \underline{\underline{D\Lambda}}^k(R^n)} \Big[V(f(\lambda, \cdot) \wedge \omega(\lambda)) - W(f(\lambda, \cdot) \wedge \omega(\lambda))$$

$$+ V([1 - f(\lambda, \cdot)] \wedge \omega(\lambda)) - W([1 - f(\lambda, \cdot)] \wedge \omega(\lambda)) \Big] \, d\underline{\underline{H}}^{(n:k)}\lambda \Big\}$$

where the supremum is taken over all continuous functions $f : \underline{\underline{\Lambda}}^k(R^n) \times R^n \longrightarrow \{t : 0 \leq t \leq 1\}$. Here

$$\gamma(n, k)^{-1} = \int_{\lambda \in \underline{\underline{D\Lambda}}^k(R^n)} |\lambda \cdot \mu| \, dH^{(n:k)}\lambda$$

for some $\mu \in \underline{\underline{\Lambda}}_k(R^n)$ with $|\mu| = 1$. We write also

$$\underline{\underline{W}}(V) = \underline{\underline{W}}(V, 0) = \gamma(n, k) \int_{\underline{\underline{D\Lambda}}^k(R^n)} V(\omega(\lambda)) \, dH^{(n:k)}\lambda.$$

Axiom (i) of (1) implies that, with a change of constants, the integrations above could be taken over $\partial \underline{\underline{D\Lambda}}^k(R^n)$ with respect to $H^{(n:k)-1} \cap \partial \underline{\underline{D\Lambda}}^k(R^n)$.

If $|A| \in \underline{\underline{V}}_k(R^n)$ is the varifold corresponding to a compact submanifold A o R^n of dimension k (5.3), then $\underline{\underline{W}}(|A|) = \underline{\underline{M}}(|A|) = \underline{\underline{H}}^k(A)$.

(13) For each $V \in \mathcal{V}_k(R^n)$ we define Radon measures $\underline{\underline{M}}V$ and $\underline{\underline{W}}V$ on R^n by setting $\underline{\underline{M}}V(\varphi) = \underline{\underline{M}}(V \wedge \varphi)$ and $\underline{\underline{W}}V(\varphi) = \underline{\underline{W}}(V \wedge \varphi)$ for each $\varphi \in \underline{\underline{C}}_0^0(R^n)$ with $\varphi = \varphi^+$.

(14) We denote by $\underline{\underline{U}}_k(R^n)$ the set of all continuous functions $F : \underline{\underline{\Lambda}}^k(R^n) \longrightarrow R^+$ for which

(i) $F(r\lambda) = |r|F(\lambda)$ for $r \in R$ and $\lambda \in \underline{\underline{\Lambda}}^k(R^n)$, and

(ii) $F(\lambda+\mu) \leq F(\lambda) + F(\mu)$ for $\lambda, \mu \in \underline{\underline{\Lambda}}^k(R^n)$.

Note that (i) and (ii) imply that F is convex. We define the $\underline{\underline{M}}$ and $\underline{\underline{W}}$ metrics on $\underline{\underline{U}}_k(R^n)$ by setting for each $F, G \in \underline{\underline{U}}_k(R^n)$,

$$\underline{\underline{M}}(F, G) = \sup\{F(\lambda) - G(\lambda) + G(\mu) - F(\mu) : \lambda, \mu \in \underline{\underline{D\Lambda}}^k(R^n)\}$$

$$\underline{\underline{W}}(F, G) = \sup_f \left\{ \gamma(n, k) \int_{\lambda \in \underline{\underline{D\Lambda}}^k(R^n)} \Big[F(f(\lambda) \wedge \lambda) - G(f(\lambda) \wedge \lambda) + G([1-f(\lambda)] \wedge \lambda) - F([1-f(\lambda)] \wedge \lambda) \Big] \, d\underline{\underline{H}}^{(n:k)}\lambda \right\}$$

where the supremum is taken over all continuous functions $f : \underline{\underline{\Lambda}}^k(R^n) \longrightarrow \{t : 0 \leq t \leq 1\}$. We write also

$$\underline{\underline{M}}(F) = \underline{\underline{M}}(F, 0) = \sup\{F(\lambda) : \lambda \in \underline{\underline{D\Lambda}}^k(R^n)\}, \text{ and}$$

$$\underline{\underline{W}}(F) = \underline{\underline{W}}(F, 0) = \gamma(n, k) \int_{\underline{\underline{D\Lambda}}^k(R^n)} F(\lambda) \, dH^{(n:k)}\lambda.$$

One verifies the existence of $c \in R_0^+$ for which

$$c^{-1}\underline{\underline{M}}(F) \leq \underline{\underline{W}}(F) \leq c\underline{\underline{M}}(F)$$

for each $F \in \underline{\underline{U}}_k(R^n)$.

Note that if $V \in \underline{\underline{V}}_k(R^n)$ then the function $V(\omega(\cdot)) : \underline{\underline{\Lambda}}^k(R^n) \longrightarrow R^+$ sendi $\lambda \in \underline{\underline{\Lambda}}^k(R^n)$ to $V(\omega(\lambda)) \in R^+$ is in $\underline{\underline{U}}_k(R^n)$ and, in particular, $\underline{\underline{M}}(V(\omega(\cdot))) \leq \underline{\underline{M}}(V)$ nd $\underline{\underline{W}}(V(\omega(\cdot))) = \underline{\underline{W}}(V)$. $V(\omega(\cdot))$ is the function $\underline{\underline{m}}V(\operatorname{spt}(V))$ of 3.6.

For each F, G $\underline{\underline{U}}_k(R^n)$ we define the <u>union</u> $F \cup G$ and <u>intersection</u> $F \cap G$ of F and G in $\underline{\underline{U}}_k(R^n)$ by setting for each $\lambda \in \underline{\underline{\Lambda}}^k(R^n)$

$$F \cup G(\lambda) = \max\{F(\lambda), G(\lambda)\}, \quad \text{and}$$

$$F \cap G(\lambda) = \sup\{H(\lambda) : H \in \underline{\underline{U}}_k(R^n) \text{ and for each } \mu \in \underline{\underline{\Lambda}}^k(R^n)$$
$$H(\mu) \leq \min\{F(\mu), G(\mu)\}\}.$$

(15) We define

$$\underline{\underline{v}} : R^n \times \underline{\underline{U}}_k(R^n) \longrightarrow \underline{\underline{V}}_k(R^n).$$

by setting for each $p \in R^n$, $F \in \underline{\underline{U}}_k(R^n)$, and $\varphi \in \underline{\underline{C}}_0^k(R^n)$, $\underline{\underline{v}}(p, F)(\varphi) = F(\varphi(p))$. Each such $\underline{\underline{v}}(p, F)$ is called an <u>elementary varifold of dimension</u> k in R^n. Note that $\underline{\underline{F}}(\underline{\underline{v}}(p, F)) = \underline{\underline{M}}(\underline{\underline{v}}(p, F)) = \underline{\underline{M}}(F)$. 3.6 and 3.7 imply that the space of finite sums of elementary varifolds of dimension k is dense in $\mathcal{V}_k(R^n)$ in the weak topology.

(16) We define

$$\underline{\underline{v}} : R^n \times \underline{\underline{\Lambda}}_k(R^n) \longrightarrow \underline{\underline{V}}_k(R^n)$$

by setting for each $p \in R^n$, $\mu \in \underline{\underline{\Lambda}}_k(R^n)$, and $\varphi \in \underline{\underline{C}}_0^k(R^n)$, $\underline{\underline{v}}(p, \mu)(\varphi) = |\varphi(p) \cdot \mu|$. Each such $\underline{\underline{v}}(p, \mu)$ is called an <u>elementary normal varifold of dimension</u> k <u>in</u> R^n. If μ is a simple k-vector, $\underline{\underline{v}}(p, \mu)$ is called an <u>elementary geometric varifold of dimension</u> k <u>in</u> R^n. Note that in either case

$$\underline{\underline{F}}(\underline{\underline{v}}(p, \mu)) = \underline{\underline{M}}(\underline{\underline{v}}(p, \mu)) = \underline{\underline{W}}(\underline{\underline{v}}(p, \mu)) = |\mu|.$$

$\underline{\underline{v}}$ does not preserve addition in $\underline{\underline{\Lambda}}_k(R^n)$ since, for example, if $\mu \in \underline{\underline{\Lambda}}_k(R^n) - \{0\}$, then

$$\underline{\underline{v}}(p, \mu + (-\mu)) = 0 < \underline{\underline{v}}(p, \mu) + \underline{\underline{v}}(p, -\mu) = 2\underline{\underline{v}}(p, \mu).$$

More subtly notice that in R^4,

$$\underline{\underline{v}}(p, (\partial/\partial x^1)\wedge(\partial/\partial x^2) + (\partial/\partial x^3)\wedge(\partial/\partial x^4)) < \underline{\underline{v}}(p, (\partial/\partial x^1)\wedge(\partial/\partial x^2))$$
$$+ \underline{\underline{v}}(p, (\partial/\partial x^3)\wedge(\partial/\partial x^4)).$$

We denote by $\mathcal{N}\mathcal{V}_k(R^n)$ the closure in $\mathcal{V}_k(R^n)$ in the weak topology of the space of finite sums of elementary normal varifolds of dimension k in R^n, and by $\mathcal{G}\mathcal{V}_k(R^n)$ the closure in $\mathcal{V}_k(R^n)$ in the weak topology of the space of finite sums of elementary normal varifolds of dimension k in R^n. In 5.3 we define $\mathcal{J}\mathcal{V}_k$

and $\mathcal{R}\mathcal{V}_k(R^n)$. The following inclusions hold and, in general, are proper:

$$\mathcal{I}\mathcal{V}_k(R^n) \subset \mathcal{R}\mathcal{V}_k(R^n) \subset \mathcal{G}\mathcal{V}_k(R^n) \subset \mathcal{N}\mathcal{V}_k(R^n) \subset \mathcal{V}_k(R^n).$$

Also, in the weak topology,

$$\operatorname{clos}(\mathcal{I}\mathcal{V}_k(R^n)) = \operatorname{clos}(\mathcal{R}\mathcal{V}_k(R^n)) = \mathcal{G}\mathcal{V}_k(R^n).$$

$\mathcal{V}_k(R^n)$ is itself the smallest closed subset of $\mathcal{V}_k(R^n)$ which contains $\mathcal{N}\mathcal{V}_k(R^n)$ and is closed under the formation of unions.

A varifold $V \in \mathcal{N}\mathcal{V}_k(R^n)$ (resp. $\mathcal{G}\mathcal{V}_k(R^n)$, $\mathcal{R}\mathcal{V}_k(R^n)$, $\mathcal{I}\mathcal{V}_k(R^n)$) is called a <u>normal</u> (resp. <u>geometric</u>, <u>real</u>, <u>integral</u>) <u>varifold of dimension</u> k <u>in</u> R^n.

One sets $\underline{\underline{NV}}_k(R^n) = \mathcal{N}\mathcal{V}_k(R^n) \cap \underline{\underline{V}}_k(R^n)$, $\underline{\underline{GV}}_k(R^n) = \mathcal{G}\mathcal{V}_k(R^n) \cap \underline{\underline{V}}_k(R^n)$, $\underline{\underline{RV}}_k(R^n) = \mathcal{R}\mathcal{V}_k(R^n) \cap \underline{\underline{V}}_k(R^n)$, and $\underline{\underline{IV}}_k(R^n) = \mathcal{I}\mathcal{V}_k(R^n) \cap \underline{\underline{V}}_k(R^n)$.

2.3 PROPOSITION. <u>Let</u> $0 \le k \le n$ <u>and</u> $q > 0$ be <u>integers and</u> $V \in \mathcal{V}_k(R^n)$. <u>Let</u> $\varphi_1, \varphi_2, \ldots, \varphi_q \in \underline{\underline{C}}^0(R^n)$ <u>with</u> $\{x : \varphi_i(x) \neq 0$ <u>and</u> $\varphi_j(x) \neq 0\} = \phi$ <u>whenever</u> $1 \le i < j \le q$. <u>Then</u> $V \wedge \sum_i \varphi_i = \sum_i V \wedge \varphi_i$. <u>In particular,</u> $V \wedge f = V \wedge f^+ + V \wedge f^-$ <u>for each</u> $f \in \underline{\underline{C}}^0_0(R^n)$.

PROOF.

<u>Part</u> I. If $\{\lambda^i\}_i$ is an orthonormal basis for $\underline{\underline{\Lambda}}^k(R^n)$ and $\varphi \in \underline{\underline{C}}^k(R^n)$ with $|\varphi| \le 1$, we can write uniquely $\varphi = \sum_i \varphi^i \wedge \omega(\lambda^i)$ for some $\varphi^i \in \underline{\underline{C}}^0_0(R^n)$ with $|\varphi^i| \le 1$ for each i. For $V \in \underline{\underline{V}}_k(R^n)$,

$$\begin{aligned} V(\varphi) &\le \textstyle\sum_i V(\varphi^i \wedge \omega(\lambda^i)) \\ &\le \textstyle\sum_i V([\varphi^i + 1]\omega(\lambda^i) - \omega(\lambda^i)) \\ &\le \textstyle\sum_i V(2\omega(\lambda^i)) + V(\omega(\lambda^i)) \\ &\le 3 \textstyle\sum_i V(\omega(\lambda^i)), \end{aligned}$$

which implies $\underline{\underline{M}}(V) \le 3 \sum_i V(\omega(\lambda^i)) < \infty$.

<u>Part</u> 2. It is sufficient to prove the proposition for $q = 2$. Let $\varphi_1, \varphi_2 \in \underline{\underline{C}}^0(R^n)$, $\psi \in \underline{\underline{C}}^k_0(R^n)$, and $V \in \mathcal{V}_k(R^n)$, and choose $\varphi \in \underline{\underline{C}}^0_0(R^n)$ for which $\varphi \wedge \psi = \psi$. Then $V \wedge \varphi \in \underline{\underline{V}}_k(R^n)$, and $V \wedge \varphi \wedge \varphi_1(\psi) = V \wedge \varphi_1(\psi)$, $V \wedge \varphi \wedge \varphi_2(\psi) = V \wedge \varphi_2(\psi)$, and $V \wedge \varphi \wedge (\varphi_1 + \varphi_2)(\psi) = V \wedge (\varphi_1 + \varphi_2)(\psi)$. In view of the obvious approximations

based on part 1 and the continuity of φ_1 and φ_2, it is sufficient to show the conclusion of the proposition under the stronger hypothesis that $\operatorname{spt}(\varphi_1) \cap \operatorname{spt}(\varphi_2) = 0$. With this hypothesis we choose $\alpha, \beta \in \underline{\underline{C}}^0(R^n)$ as a partition of unity for R^n such that $\operatorname{spt}(\varphi_1) \subset \alpha^{-1}(1)$ and $\operatorname{spt}(\varphi_2) \subset \beta^{-1}(1)$. Then

$$\begin{aligned} V \wedge (\varphi_1 + \varphi_2)(\psi) &= V((\varphi_1 + \varphi_2) \wedge \psi) \\ &= V(\alpha \wedge [(\varphi_1 + \varphi_2) \wedge \psi] + \beta \wedge [(\varphi_1 \wedge \varphi_2) \wedge \psi]) \\ &= V(\alpha \wedge (\varphi_1 + \varphi_2) \wedge \psi) + V(\beta \wedge (\varphi_1 \wedge \varphi_2) \wedge \psi) \\ &= V(\varphi_1 \wedge \psi) + V(\varphi_2 \wedge \psi). \end{aligned}$$

The proposition follows.

2.4 PROPOSITION. Let $0 \leq k \leq n$ be integers and $K \subset R^n$ be compact.

(1) The functions $\underline{\underline{F}}, \underline{\underline{M}}, \underline{\underline{W}} : \underline{\underline{V}}_k(R^n) \longrightarrow R^+$ are lower semi-continuous in the weak topology on $\underline{\underline{V}}_k(R^n)$. However, if $V, V_1, V_2, V_3, \ldots \in \underline{\underline{V}}_k(K)$ with $\lim_i V_i = V$ in the weak topology, then $\lim_i \underline{\underline{F}}(V_i) = \underline{\underline{F}}(V)$, $\lim_i \underline{\underline{W}}(V_i) = \underline{\underline{W}}(V)$, and

$$(n : k)^{-1} \limsup_i \underline{\underline{M}}(V_i) \leq \underline{\underline{M}}(V) \leq \liminf_i \underline{\underline{M}}(V_i) .$$

(2) There exists $c \in R_0^+$ such that for each $V \in \underline{\underline{V}}_k(R^n)$ and $\varphi \in \underline{\underline{C}}^k(R^n)$,

(a) $$V(\varphi) \leq \underline{\underline{M}}(V)|\varphi| \leq c\underline{\underline{W}}(V)|\varphi|,$$

and thus

(b) $$\underline{\underline{F}}(V) \leq \underline{\underline{M}}(V) \leq c\underline{\underline{W}}(V) \leq c\alpha(n : k)\gamma(n ; k)\underline{\underline{F}}(V).$$

Also, for each $V, W \in \underline{\underline{V}}_k(R^n)$,

(c) $$\underline{\underline{F}}(V, W) \leq \underline{\underline{M}}(V, W),$$

and

(d) $$|\underline{\underline{W}}(V) - \underline{\underline{W}}(W)| \leq \alpha(n : k)\gamma(n, k)\underline{\underline{F}}(V, W),$$

which means

(e) $$\operatorname{Lip}(\underline{\underline{W}}) \leq \alpha(n : k)\gamma(n, k)$$

with respect to the $\mathbf{F}$ metric on $\mathbf{V}_k(R^n)$.

(3) The $\mathbf{F}$ metric topology coincides with the weak topology on $\mathbf{W}$ bounded subsets of $\mathbf{V}_k(K)$. Furthermore, for each $b \in R^+$, $\mathbf{V}_k(K) \cap \{V : \mathbf{F}(V) \leq b\}$, $\mathbf{V}_k(K) \cap \{V : \mathbf{M}(V) \leq b\}$, and $\mathbf{V}_k(K) \cap \{V : \mathbf{W}(V) \leq b\}$ are each compact in the weak topology and hence in the $\mathbf{F}$ metric topology also.

(4) Let $f : R_0^+ \longrightarrow R^+$ be continuous and non-decreasing. Then

$$\mathcal{V}_k(R^n) \cap \{V : X(V \cap \mathbf{D}^n(0, r)) \leq f(r) \text{ for each } r \in R_0^+\}$$

is compact in the weak topology whenever $X = \mathbf{F}$, $\mathbf{M}$, or $\mathbf{W}$.

PROOF. Most of conclusion (1) is straightforward. Note that if $\{\lambda^i\}_i$ is as in 2.3 (part 1), then

$$\liminf_j \mathbf{M}(V_j) \leq \lim_j \Sigma_i \mathbf{M}(\omega(\lambda^i)) = \Sigma_i V(\omega(\lambda^i)) \leq (n : k)\mathbf{M}(V).$$

Note also that $|V(\varphi) - V(\psi)| \leq \mathbf{M}(V)|\varphi - \psi|$ whenever $V \in \mathbf{V}_k(R^n)$ and $\varphi, \psi \in \mathbf{C}^k(R^n)$.

Conclusions (2), (3), and (4) are left to the reader. Note that $\mathcal{V}_k(R^n)$ is separable in the weak topology.

2.5 PROPOSITION. Let $0 \leq k \leq n$ be integers, $f \in \mathbf{C}^0(R^n)$ with $\text{Lip}(f) \leq 1$, and $U_r = R^n \cap \{x : f(x) < r\}$ for each $r \in R$. Then for each $V, W \in \mathbf{V}_k(R^n)$ and $\delta \in R_0^+$,

$$\mathbf{F}(V \cap U_r, W \cap U_r) \leq (1 + \delta^{-1})\, \mathbf{F}(V, W) + c\, \mathbf{W}[(V + W) \cap (U_\delta - U_{-\delta})],$$

whenever $0 < r < \delta$. Here $c \in R_0^+$ is the constant of 2.4 (2a).

PROOF. Let f, V, W, δ be as above. Choose $g, h : R^n \longrightarrow \{t : 0 \leq t \leq 1\}$ such that $U_{-\delta} \subset g^{-1}(1)$, $(R^n - U_0) \subset g^{-1}(0)$, $\text{Lip}(g) \leq \delta^{-1}$, and $h = 1 - g$. For $0 < r < \delta$ one has

$$\mathbf{F}(V \cap U_r, W \cap U_r) = \mathbf{F}(V \wedge g + V \wedge h \cap U_r, W \wedge g + W \wedge h \cap U_r)$$

$$= \sup\{|V \wedge g(\varphi) + V \wedge h \cap U_r(\varphi) - W \wedge g(\varphi) - W \wedge h \cap U_r(\varphi)| : \varphi \in \mathbf{C}_0^k(R^n), |\varphi| \leq 1 \text{ and } \text{Lip}(\varphi) \leq 1\}$$

$$\leq \sup\{|V(g \wedge \varphi) - W(g \wedge \varphi)| + (V+W) \wedge h \cap U_r(\varphi) : \varphi \in \mathbf{C}_0^k(R^n), |\varphi| \leq 1, \text{ and } \text{Lip}(\varphi) \leq 1\}$$

$$\leq (1 + \delta^{-1})\mathbf{F}(V, W) + c\mathbf{W}[(V+W) \cap (U_\delta - U_{-\delta})],$$

because $\text{Lip}(g \wedge \varphi) \leq \text{Lip}(g) + \text{Lip}(\varphi) \leq \delta^{-1} + 1$.

3. VARIFOLD MEASURES

3.1 DEFINITIONS. Let n be a non-negative integer.

(1) We set $\underline{\underline{C}} = \underline{\underline{C}}_0^0(R^n)$.

(2) We denote by $\mathcal{L}$ the set of all functions $L : \underline{\underline{C}} \longrightarrow R$ for which

(a) $L(r\varphi + s\psi) = r\,L(\varphi) + s\,L(\psi)$ for each $r, s \in R$ and $\varphi, \psi \in \underline{\underline{C}}$, and

(b) $L(\varphi) \geq 0$ whenever $\varphi \in \underline{\underline{C}}$ and $\varphi = \varphi^+$.

(3) We denote by $\mathcal{V}$ the set of all functions $V : \underline{\underline{C}} \longrightarrow R^+$ for which

(a) $V(r\varphi) = |r|V(\varphi)$ for each $r \in R$ and $\varphi \in \underline{\underline{C}}$,

(b) $V(\varphi+\psi) \leq V(\varphi) + V(\psi)$ for each $\varphi, \psi \in \underline{\underline{C}}$, and

(c) $V(\varphi+\psi) = V(\varphi) + V(\psi)$ whenever $\varphi, \psi \in \underline{\underline{C}}$ and $\varphi = \varphi^+$, $\psi = \psi^+$.

(4) We define $\sigma : \mathcal{L} \longrightarrow \mathcal{V}$ and $\tau : \mathcal{V} \longrightarrow \mathcal{L}$ by setting for each $L \in \mathcal{L}$, $V \in \mathcal{V}$, and $\varphi \in \underline{\underline{C}}$, $\sigma(L)(\varphi) = L(|\varphi|)$ and $\tau(V)(\varphi) = V(\varphi^+) - V(\varphi^-)$.

3.2 LEMMA. <u>Let</u> n <u>be a non-negative integer</u>, $\varphi, \psi \in \underline{\underline{C}}$, <u>and</u> $V \in \mathcal{V}$. <u>Then</u>

(1) $(\varphi^+ + \psi^+) - (\varphi+\psi)^+ = (\varphi^- + \psi^-) - (\varphi+\psi)^-$,

(2) $(\varphi^+ + \psi^+) \geq (\varphi+\psi)^+$ and $(\varphi^- + \psi^-) \geq (\varphi+\psi)^-$,

(3) $V((\varphi+\psi)^+) - V((\varphi+\psi)^-) = V(\varphi^+ + \psi^+) - V(\varphi^- + \psi^-)$, <u>and</u>

(4) σ <u>and</u> τ <u>are well defined mappings with</u> $\tau \circ \sigma$ <u>equal to the identity map on</u> $\mathcal{L}$ <u>and</u> $\sigma \circ \tau$ <u>equal to the identity map on</u> $\mathcal{V}$.

PROOF.

(1) Note that $\varphi + \psi = \varphi^+ + \psi^+ - \varphi^- - \psi^- = (\varphi+\psi)^+ - (\varphi+\psi)^-$.

(2) Obvious.

(3) Note that

$$V(\varphi^+ + \psi^+) = V((\varphi+\psi)^+ + [(\varphi^+ + \psi^+) - (\varphi+\psi)^+])$$
$$= V((\varphi+\psi)^+) + V((\varphi^+ + \psi^+) - (\varphi+\psi)^+)$$
$$V(\varphi^- + \psi^-) = V((\varphi+\psi)^-) + V((\varphi^- + \psi^-) - (\varphi+\psi)^-)$$

which implies

$$V((\varphi+\psi)^+) - V((\varphi+\psi)^-) = V(\varphi^+ + \psi^+) - V(\varphi^- + \psi^-).$$

(4) : (4) follows easily from (3).

3.3 PROPOSITION. Let n be a non-negative integer. Then corresponding to each $V \in \mathcal{V}$ there exists a unique Borel measure $\underline{\underline{m}}$ on R^n such that for each $\varphi \in \underline{\underline{C}}$,

$$V(\varphi) = \int_{R^n} |\varphi(x)| \, d\,\underline{\underline{m}}\, x .$$

PROOF. The measure $\underline{\underline{m}}$ is that unique Borel measure such that for each $\varphi \in \underline{\underline{C}}$,

$$\tau(V)(\varphi) = \int_{R^n} \varphi(x) \, d\,\underline{\underline{m}}\, x .$$

3.4 COROLLARY. Let $0 \leq k \leq n$ be integers, $V \in \mathcal{V}_k(R^n)$, and $\lambda \in \underline{\underline{\Lambda}}^k(R^n)$. Then there exists a unique Borel measure $\underline{\underline{m}}(V, \lambda)$ such that for each $\varphi \in \underline{\underline{C}}^0_0(R^n)$,

$$V(\varphi\, \omega(\lambda)) = \int_{R^n} |\varphi(x)| \, d\, \underline{\underline{m}}(V, \lambda)\, x .$$

3.5 DEFINITIONS. Integration of differential forms with respect to $\underline{\underline{U}}_k(R^n)$ valued measures. Let $0 \leq k \leq n$ be integers. By a Borel measure $\underline{\underline{m}}$ on R^n with values in $\underline{\underline{U}}_k(R^n)$ we mean a set function $\underline{\underline{m}}$ which assigns to each bounded Borel set $A \subset R^n$ a function $\underline{\underline{m}}(A) \in \underline{\underline{U}}_k(R^n)$ such that

(a) $\underline{\underline{m}}(\phi) = 0$, and

(b) $\underline{\underline{m}}(A) = \Sigma_i\, \underline{\underline{m}}(A_i)$ whenever $A_1, A_2, A_3, \ldots$ R^n comprise a pairwise disjoint sequence of bounded Borel sets in R^n with $A = \bigcup_i A_i$ bounded. The sum is to exist in the $\underline{\underline{M}}$ metric or, equivalently, in the $\underline{\underline{W}}$ metric on $\underline{\underline{U}}_k(R^n)$. Note that

$$\underline{\underline{W}}(\underline{\underline{m}}(A)) = \Sigma_i\, \underline{\underline{W}}(\underline{\underline{m}}(A_i)), \text{ and}$$

$$\underline{\underline{M}}(\underline{\underline{M}}(A)) \leq \Sigma_i\, \underline{\underline{M}}(\underline{\underline{m}}(A_i)) \leq c\, \underline{\underline{W}}(\underline{\underline{m}}(A)) ,$$

where c is the constant of 2.4 (2a).

If $\varphi \in \underline{\underline{C}}_0^k(R^n)$ and $\underline{\underline{m}}$ is a Borel measure on R^n with values in $\underline{\underline{U}}_k(R^n)$, we define the integral $\int_{R^n} \varphi \, d\, \underline{\underline{m}}$ by setting

$$\int_{R^n} \varphi \, d\, \underline{\underline{m}} = \lim_i \Sigma_j \, \underline{\underline{m}}(A(i,j))(\varphi[p(i,j)]),$$

where for each i=1, 2, 3, ..., $\{A(i,j)\}_j$ is a Borel partition of R^n such that $\operatorname{diam}(A(i,j)) \leq 2^{-i}$ and $p(i,j) \in A(i,j)$ for each j = 1, 2, 3, ... The existence of this limit and its independence of the choice of $\{A(i,j)\}_{i,j}$ follows in the usual way using the observation that for each bounded Borel set $B \subset R^n$ and each i = 1, 2, 3, ...,

$$\Sigma_j \, \underline{\underline{M}}(\underline{\underline{m}}[B \cap A(i,j)]) \leq c \, \underline{\underline{W}}(\underline{\underline{m}}(B)) < \infty .$$

3.6 DEFINITIONS. <u>The Borel measure $\underline{\underline{m}}V$ with values in $\underline{\underline{U}}_k(R^n)$ corresponding to a varifold</u> V. Let $0 \leq k \leq n$ and $V \in \mathcal{V}_k(R^n)$. Corresponding to V we define the Borel measure $\underline{\underline{m}}\,V$ on R^n by setting for each bounded Borel set $A \subset R^n$ and each $\lambda \in \underline{\underline{\Lambda}}^k(R^n)$, $\underline{\underline{m}}V(A)(\lambda) = \underline{\underline{m}}(V,\lambda)(A)$, where $\underline{\underline{m}}(V,\lambda)$ is the real valued Borel measure of 3.4. Note that if $\varphi \in \underline{\underline{C}}_0^0(R^n)$, $\lambda, \mu \in \underline{\underline{\Lambda}}^k(R^n)$, and $A \subset R^n$ is a bounded Borel set, then

$$\underline{\underline{m}}(V, \lambda)(\varphi) = \underline{\underline{m}}(V \wedge \varphi, \lambda)(\operatorname{spt}(\varphi)) = V(\varphi \wedge \omega(\lambda)),$$

and thus,

$$\underline{\underline{m}}(V, \lambda+\mu) \leq \underline{\underline{m}}(V, \lambda) + \underline{\underline{m}}(V, \mu),$$

which implies

$$\underline{\underline{m}}\,V(A)(\lambda+\mu) \leq \underline{\underline{m}}\,V(A)(\lambda) + \underline{\underline{m}}\,V(A)(\mu) .$$

The rest of the verification that $\underline{\underline{m}}V$ is well defined is left to the reader.

3.7. THEOREM. <u>Let</u> $0 \leq k \leq n$ <u>be integers</u>, $V \in \mathcal{V}_k(R^n)$, <u>and</u> $\varphi \in \underline{\underline{C}}_0^k(R^n)$. <u>Then</u> $V(\varphi) = \int_{R^n} \varphi \, d\, \underline{\underline{m}}V$.

3.8 THEOREM. Let $0 \leq k \leq n$ be integers and $V \in \mathcal{V}_k(R^n)$. Then (1) For $\underline{\underline{M}}V$ almost all $x \in R^n$ or, equivalently, for $\underline{\underline{W}}V$ almost all $x \in R^n$, there exist $\vec{\underline{\underline{M}}}V(x) \in \underline{\underline{U}}_k(R^n)$ and $\vec{\underline{\underline{W}}}V(x) \in \underline{\underline{U}}_k(R^n)$ characterized by the property that for each $\lambda \in \underline{\underline{\Lambda}}^k(R^n)$,

$$\vec{\underline{\underline{M}}}V(x)(\lambda) = \frac{d\underline{\underline{m}}(V, \lambda)}{d\underline{\underline{M}}V}(x) \quad \text{and} \quad \vec{\underline{\underline{W}}}V(x)(\lambda) = \frac{d\underline{\underline{m}}(V, \lambda)}{d\underline{\underline{W}}V}(x) \quad .$$

(2) $\underline{\underline{M}}(\vec{\underline{\underline{M}}}V(x)) \leq 1$ and $\underline{\underline{W}}(\vec{\underline{\underline{W}}}V(x)) \leq 1$ whenever $x \in R^n$ and $\vec{\underline{\underline{M}}}V(x)$ and $\vec{\underline{\underline{W}}}V(x)$ exist, and equality holds for $\underline{\underline{M}}V$ almost all $x \in R^n$ or, equivalently, for $\underline{\underline{W}}V$ almost all $x \in R^n$. Also for some $t \in R_0^+$ depending on $\vec{\underline{\underline{M}}}(x)$, $\vec{\underline{\underline{M}}}V(x) = t\,\vec{\underline{\underline{W}}}(x)$. In case $x \in R^n$, $\mu \in \underline{\underline{\Lambda}}_k(R^n)$, $\vec{\underline{\underline{M}}}(x)$ exists, and for each $\lambda \in \underline{\underline{\Lambda}}^k(R^n)$, $\vec{\underline{\underline{M}}}V(x)(\lambda) = |\lambda \cdot \mu|$, then $\vec{\underline{\underline{M}}}V(x) = \vec{\underline{\underline{W}}}V(x)$ and $\underline{\underline{M}}(\vec{\underline{\underline{M}}}V(x)) = \underline{\underline{W}}(\vec{\underline{\underline{W}}}V(x)) = |\mu|$.

(3) For each $\varphi \in \underline{\underline{C}}_0^k(R^n)$,

$$V(\varphi) = \int_{R^n} \vec{\underline{\underline{M}}}V(x)(\varphi(x)) \; d\,\underline{\underline{M}}V\,x = \int_{R^n} \vec{\underline{\underline{W}}}V(x)(\varphi(x)) \; d\,\underline{\underline{W}}V\,x.$$

PROOF. For a summary of relevant measure theoretic information, see [FF 8.7]. The uniqueness almost everywhere of $\vec{\underline{\underline{M}}}V(x)$ and $\vec{\underline{\underline{W}}}V(x)$ is clear and their existence can be shown by prescribing their values on a countable dense subset of $\underline{\underline{\Lambda}}^k(R^n)$ whenever all these derivates exist. The convexity of elements of $\underline{\underline{U}}_k(R^n)$ facilitates these arguments. The rest of the proof is left to the reader.

3.9 DEFINITION. The intersection of a varifold with a Borel set. Let $0 \leq k \leq n$ be integers, $V \in \mathcal{V}_k(R^n)$, and $A \subset R^n$ be a Borel set. We define the intersection of V with A, written $V \cap A$, to be that varifold in $\mathcal{V}_k(R^n)$ given for each $\varphi \in \underline{\underline{C}}_0^k(R^n)$ by, $V \cap A(\varphi) = \int_A \varphi \; d\,\underline{\underline{m}}V$.

3.10. DEFINITIONS. Union and intersection of varifolds. Let $0 \le k \le n$ be integers and $V, W \in \mathcal{V}_k(R^n)$. For each $i = 1, 2, 3, \ldots$ let $\{A(i, j)\}_j$ be a partition of R^n with $\operatorname{diam}(A(i, j)) \le 2^{-i}$ and $p(i, j) \in A(i, j)$ for each $j = 1, 2, 3, \ldots$. We define

$$V \cup W = \lim_i \Sigma_j \underline{\underline{v}}(p(i, j), \underline{\underline{m}}[V \cap A(i, j)] \cup \underline{\underline{m}}[W \cap A(i, j)]),$$

$$V \cap W = \lim_i \Sigma_j \underline{\underline{v}}(p(i, j), \underline{\underline{m}}[V \cap A(i, j)] \cap \underline{\underline{m}}[W \cap A(i, j)]).$$

Note that for $F, F', G, G' \in \underline{\underline{U}}_k(R^n)$,

$$(F + F') \cup (G + G') \le (F \cup G) + (F' \cup G'), \quad \text{and}$$
$$(F + F') \cap (G + G') \ge (F \cap G) + (F' \cap G').$$

The existence of the limits above follows. The characterization of $V \cup W$ and $V \cap W$ in 2.2(6) and 2.2(7) is immediate.

3.11. PROPOSITION. Let $0 < k \le n$ be integers. The smallest closed subset of $\mathcal{V}_k(R^n)$ which contains $\mathcal{N}\mathcal{V}_k(R^n)$ and is closed under the formation of unions is $\mathcal{V}_k(R^n)$ itself.

PROOF. For each $F \in \underline{\underline{U}}_k(R^n)$ and each $\varepsilon > 0$ there exists a positive integer q and and $\mu_1, \mu_2, \ldots, \mu_q \in \underline{\underline{\Lambda}}_k(R^n)$ and corresponding $F_1, F_2, \ldots, F_q \in \underline{\underline{U}}_k(R^n)$ with $F_i(\lambda) = |\lambda \cdot \mu|$ for each $\lambda \in \underline{\underline{\Lambda}}^k(R^n)$ and $i = 1, 2, \ldots, q$, such that $\underline{\underline{W}}(F, F_1 \cup F_2 \cup F_3 \cup \ldots \cup F_q) < \varepsilon$. The proposition follows by elementary arguments.

4. MAPPINGS OF VARIFOLDS

4.1 DEFINITIONS. The mapping of varifolds induced by a differentiable map. Let k, m, n, p be integers with $0 \le k \le \min\{m, n, p\}$.

(1) Let $U \subset R^m$ be open and $f: U \longrightarrow R^n$ be continuously differentiable. Then f induces a mapping

$$f_{\#} : (\underline{\underline{V}}_k(U), \underline{\underline{NV}}_k(U), \underline{\underline{GV}}_k(U), \underline{\underline{RV}}_k(U), \underline{\underline{IV}}_k(U))$$
$$\longrightarrow (\underline{\underline{V}}_k(R^n), \underline{\underline{NV}}_k(R^n), \underline{\underline{GV}}_k(R^n), \underline{\underline{RV}}_k(R^n), \underline{\underline{IV}}_k(R^n))$$

given for each $V \in \underline{\underline{V}}_k(R^m)$ and $\varphi \in \underline{\underline{C}}_0^k(R^n)$ by $f_{\#}(V)(\varphi) = V(f^{\#}(\varphi))$. If $K \subset U$ is compact and $V, V_1, V_2, V_3, \ldots \in \underline{\underline{V}}_k(K)$ with $\lim_i V_i = V$, then $\lim_i f_{\#}(V_i) = f_{\#}(V)$. In case $f^{-1}(C)$ is compact for each compact $C \subset R^n$, then $f_{\#}$ extends to give a mapping

$$f : (\mathcal{V}_k(U), \mathcal{N}\mathcal{V}_k(U), \mathcal{G}\mathcal{V}_k(U), \mathcal{R}\mathcal{V}_k(U), \mathcal{I}\mathcal{V}_k(U))$$
$$\longrightarrow (\mathcal{V}_k(R^n), \mathcal{N}\mathcal{V}_k(R^n), \mathcal{G}\mathcal{V}_k(R^n), \mathcal{R}\mathcal{V}_k(R^n), \mathcal{I}\mathcal{V}_k(R^n))$$

If $f, f^1, f^2, f^3, \ldots : U \longrightarrow R^n$ are continuously differentiable mappings with $\lim_i f^i = f$ and $\lim_i Df^i = Df$ uniformly on compact sets then $\lim_i f^i_{\#}(V) = f_{\#}(V)$ for each $V \in \underline{\underline{V}}_k(U)$.

(2) For each linear mapping $L : R^m \longrightarrow R^n$ we set

$$\mathcal{L}^k(L) = \sup\{ |f_{\#}(\mu)| : \mu \in \partial\underline{\underline{D\Lambda}}_k(R^m)\} .$$
$$\mathcal{L}_k(L) = \inf\{ |f_{\#}(\mu)| : \mu \in \partial\underline{\underline{D\Lambda}}_k(R^m)\} .$$

(3) Let $f : U \longrightarrow R^n$ be as in (1). We define continuous functions $\mathcal{L}^k(f)$, $\mathcal{L}_k(f) : U \longrightarrow R^+$ by setting for each $x \in U$, $\mathcal{L}^k(f)(x) = \mathcal{L}^k(Df(x))$ and $\mathcal{L}_k(f)(x) = \mathcal{L}_k(Df(x))$. If $f : R^m \longrightarrow R^n$ is a diffeomorphism, then $\mathcal{L}^k(f) = \mathcal{L}_k(f^{-1}) \circ f$ and $\mathcal{L}_k(f) = \mathcal{L}^k(f^{-1}) \circ f$. If $f : R^m \longrightarrow R^n$ and $g : R^n \longrightarrow R^p$ are both continuously differentiable, then $\mathcal{L}^k(g \circ f) \le (\mathcal{L}^k(g) \circ f)\, \mathcal{L}^k(f)$ and $\mathcal{L}_k(g \circ f) \ge (\mathcal{L}_k(g) \circ f)\, \mathcal{L}_k(f)$.

4.2 REMARK. Let n be a positive integer and $L : R^n \longrightarrow R^n$ be linear. Then there exists an orthonormal basis $e^1, e^2, \ldots, e^n$ for R^n, real numbers $r_1 \geq r_2 \geq \ldots \geq r_n \geq 0$, and an orthogonal transformation $\theta : R^n \longrightarrow R^n$ with $|L(e^i)|\theta(e^i) = L(e^i)$ for each i such that $L = \theta \circ E_n \circ E_{n-1} \circ \ldots \circ E_1$ where $E_i : R^n \longrightarrow R^n$ is given by $E_i(e^i) = r_i e^i$ and $E_i(e^j) = e^j$ for $j \neq i$. One verifies that $\mathcal{L}^k(L) = r_1 \cdot r_2 \cdot \ldots \cdot r_k$ and $\mathcal{L}_k(L) = r_{n-k+1} \cdot r_{n-k+2} \cdot \ldots \cdot r_n$.

4.3 LEMMA. Let $0 \leq k \leq m \leq n$ be integers and $f : R^m \longrightarrow R^n$ be given by $f(x^1, x^2, \ldots, x^m) = (x^1, x^2, \ldots, x^m, 0, 0, \ldots, 0)$ for each $(x^1, x^2, \ldots, x^m) \in R^m$. Then for each $V \in \underline{\underline{V}}_k(R^n)$, $\underline{\underline{M}}(f_{\#}(V)) = \underline{\underline{M}}(V)$ and $\underline{\underline{W}}(f_{\#}(V)) = \underline{\underline{W}}(V)$.

4.4 LEMMA. Let $0 \leq k \leq n$ be integers, $L : R^n \longrightarrow R^n$ be linear, and $V \in \underline{\underline{V}}_k(R^n)$. Then $\mathcal{L}_k(L)\, \underline{\underline{M}}(V) \leq \underline{\underline{M}}(L_{\#}(V)) \leq \mathcal{L}^k(L)\, \underline{\underline{M}}(V)$, and $\mathcal{L}_k(L)\, \underline{\underline{W}}(V) \leq \underline{\underline{W}}(L_{\#}(V)) \leq \mathcal{L}^k(L)\, \underline{\underline{W}}(V)$.

PROOF.

Part 1. Let q be a positive integer, $m : R^q \longrightarrow R$ be convex, and $0 \leq t \leq 1$. Then

$$\int_{x \in \underline{\underline{D}}^q} m(x^1, x^2, \ldots, x^q)\, d\,\underline{\underline{H}}^q x \geq \int_{x \in \underline{\underline{D}}^q(0,1)} m(tx^1, x^2, x^3, \ldots, x^q)\, d\,\underline{\underline{H}}^q x$$

Part 2. The first conclusion of the lemma is immediate. We verify the second. If L is singular then $\mathcal{L}_k(L) = 0$ and the left hand inequality is immediate. If L is not singular, the left hand inequality follows from the right hand inequality applied to L^{-1}. Write $L = \theta \circ E_n \circ E_{n-1} \circ \ldots \circ E_1$ as in 4.2. Similarly, write $L\,(0) = \Omega \circ F_{(n:k)} \circ F_{(n:k)-1} \circ \ldots \circ F_1$ where $\Omega : \underline{\underline{\Lambda}}^k(R^n) \longrightarrow \underline{\underline{\Lambda}}^k(R^n)$ is orthogonal and for some orthonormal basis $\{\lambda^i\}_i$ of $\underline{\underline{\Lambda}}^k(R^n)$ and real numbers $s_1 \geq s_2 \geq \ldots \geq s_{(n:k)} \geq 0$, $F_i(\lambda^i) = s_i \lambda^i$ and $F_i(\lambda^j) = \lambda^j$ for $i \neq j$ whenever $i, j = 1, 2, \ldots, (n:k)$. Note that $s_1 = r_1 \cdot r_2 \cdot \ldots \cdot r_k = \mathcal{L}^k(L)$. Thus

$$\underline{\underline{W}}(L_{\#}(V)) = \gamma(n, k) \int_{\underline{\underline{D}}\underline{\underline{\Lambda}}^k(R^n)} L_{\#}(V)\,(\omega(\nu))\, d\, \underline{\underline{H}}^{(n:\,k)}\nu$$

$$= \gamma(n, k) \int_{\underline{\underline{D}}\underline{\underline{\Lambda}}^k(R^n)} \underline{\underline{m}}\, V(\mathrm{spt}(V))\, (L^{\#}(O, \nu))\, d\, \underline{\underline{H}}^{(n:\,k)}\nu$$

$$= \gamma(n; k) \int_{\underline{\underline{D}}\underline{\underline{\Lambda}}^k(R^n)} \underline{\underline{m}}\, V(\mathrm{spt}(V))(\Omega \circ F_{(n:\,k)} \circ F_{(n:\,k)-1} \circ \cdots \circ F_1(\nu))\, d\, \underline{\underline{H}}^{(n:\,k)}\nu$$

$$= s_1\, \gamma(n, k) \int_{\underline{\underline{D}}\underline{\underline{\Lambda}}^k(R^n)} \underline{\underline{m}}\, V(\mathrm{spt}(V))(G_{(n:\,k)} \circ G_{(n:\,k)-1} \circ \cdots \circ G_1(\nu))\, d\, \underline{\underline{H}}^{(n:\,k)}\nu$$

$$\leq s_1\, \gamma(n, k) \int_{\underline{\underline{D}}\underline{\underline{\Lambda}}^k(R^n)} \underline{\underline{m}}\, V(\mathrm{spt}(V))(G_{(n:\,k)} \circ G_{(n:\,k)-1} \circ \cdots \circ G_2(\nu))\, d\, \underline{\underline{H}}^{(n:\,k)}\nu$$

$$\vdots$$

$$\leq s_1\, \gamma(n, k) \int_{\underline{\underline{D}}\underline{\underline{\Lambda}}^k(R^n)} \underline{\underline{m}}\, V(\mathrm{spt}(V))(G_{(n:\,k)}(\nu))\, d\, H^{(n:\,k)}\nu$$

$$\leq s_1\, \gamma(n, k) \int_{\underline{\underline{D}}\underline{\underline{\Lambda}}^k(R^n)} \underline{\underline{m}}\, V(\mathrm{spt}(V))(\nu)\, d\, \underline{\underline{H}}^{(n:\,k)}\nu$$

$$= \mathcal{L}^k(L)\, \underline{\underline{W}}(V)\ .$$

Here for $i = 1, 2, \ldots, (n:k)$, $G_i : \underline{\underline{\Lambda}}^k(R^n) \longrightarrow \underline{\underline{\Lambda}}^k(R^n)$ is given by $G_i(\lambda^i) = s_1^{-1} s_i \lambda^i$ and $G_i(\lambda^j) = \lambda^j$ for $i \neq j$. In the estimates above we used 2.2 (1(i), 1(ii)) and made repeated use of part 1.

4.5 THEOREM. Let k, m, n be integers with $0 \leq k \leq \min\{m, n\}$, $f: R^m \longrightarrow R^n$ be continuously differentiable, and $\underline{\underline{V}}_k(R^n)$. Then

$$(n:k)^{-1}\, \underline{\underline{M}}\, V(\mathcal{L}_k(f)) \leq \underline{\underline{M}}(f_{\#}(V)) \leq \underline{\underline{M}}\, V(\mathcal{L}^k(f)), \text{ and}$$

$$\underline{\underline{W}}\, V(\mathcal{L}_k(f)) \leq \underline{\underline{W}}(f_{\#}(V)) \leq \underline{\underline{W}}\, V(\mathcal{L}^k(f))\ .$$

PROOF. Approximate V by finite sums of elementary varifolds and use 2.4, 4.3, 4.4 and the continuity properties of $f_{\#}$ for varifolds with supports in a fixed compact set.

4.6 COROLLARY. Let f and V be as in 4.5. Then $\underline{\underline{M}}(f_{\#}(V)) \leq [\mathrm{Lip}(f)]^k \underline{\underline{M}}(V)$ and $\underline{\underline{W}}(f_{\#}(V)) \leq [\mathrm{Lip}(f)]^k \underline{\underline{W}}(V)$.

4.7 REMARK. If $0 \leq k \leq m$, $k \leq n$ are integers, if $A_1, A_2, A_3, \ldots \subset R^m$ are k-rectifiable sets (5.1) such that $V = \sum_{i=1}^{\infty} |A_i| \in \underline{\underline{IV}}_k(R^m)$ (5.3, 5.4), and if $f : R^m \longrightarrow R^n$ is continuously differentiable, then $f_{\#}(V) \geq \sum_{i=1}^{\infty} |f(A_i)|$, with equality in case f is one to one on spt(V).

5. INTEGRAL AND REAL VARIFOLDS AND THE VZ SPACES.

5.1 DEFINITIONS. Rectifiable sets .. Let $1 \leq k \leq n$ be integers and $A \subset R^n$. A is called a k-dimensional rectifiable subset of R^n, or, equivalently, k-rectifiable, if and only if A is a bounded Borel set and each of the following four equivalent statements is true [F1] [F2.3.5, 3.7, 4.3, 4.6, 4.7, 5.8, 8.7] [FF 8.9, 8.16] [W1].

(1) For each $\varepsilon > 0$ there exists a compact k-dimensional submanifold M of R^n of class 1 such that $\underline{\underline{H}}^k([A-M] \cup [M-A]) < \varepsilon$.

(2) For each $\varepsilon > 0$ there exists a bounded open set $U \subset R^k$ and a mapping $f: R^k \longrightarrow R^n$ of class 1 such that

$$\underline{\underline{H}}^k([A-f(U)] \cup [f(U)-A]) < \varepsilon .$$

(3) $\underline{\underline{H}}^k(A) < \infty$ and there exists a countable family of k-dimensional non-parametric manifolds of class 1 whose union contains $\underline{\underline{H}}^k$ almost all of A.

(4) $\underline{\underline{H}}^k(A) < \infty$ and A has no Borel subset B such that $\underline{\underline{H}}^k(B) > 0$ and $\underline{\underline{H}}^k(p(B)) = 0$ for almost all $p \in \underline{\underline{\Omega}}_k(R^n)$.

A is called a 0-dimensional rectifiable subset of R^n, or, equivalently, 0-rectifiable, if and only if A is finite.

A is called a (-1)-dimensional rectifiable subset of R^n, or, equivalently (-1)-rectifiable, if and only if $A = \phi$.

Note that for each $k = -1, 0, 1, 2, \ldots, n$ the finite union or finite intersection of k-rectifiable sets is again a k-rectifiable set.

5.2 PROPOSITION. Let $0 \leq k \leq n$ be integers and $A \subset R^n$ be a k-dimensional rectifiable subset of R^n. Then for $\underline{\underline{H}}^k$ almost all $x \in A$:

(1) There exists a k-dimensional non-parametric submanifold M of R^n of class 1 containing x as an interior point such that $\Theta^k(\underline{\underline{H}}^k \cap ([M-A] \cup [A-M]), x) = 0$.

(2) A has an approximate k-dimensional tangent plane at x, i.e. there exists $B \subset R^n$ with $\Theta^k(\underline{\underline{H}}^k \cap B, x) = 0$ such that the tangent cone (10.3) to A-B.

at x is a k-dimensional disk through x.

(3) The function $\vec{A} : A \longrightarrow \underline{\underline{\Omega}}_k(R^n)$ of 5.3(1) is $\underline{\underline{H}}^k \cap A$ measurable.

PROOF. [F5]

5.3 DEFINITIONS. Rectifiable varifolds. Let $0 \leq k \leq n$ be integers and $A \subset R^n$ be a k-dimensional rectifiable subset of R^n.

(1) For each $x \in A$ at which A has an approximate k-dimensional tangent plane we denote by $\vec{A}(x) \in \underline{\underline{\Omega}}_k(R^n)$ the unoriented k-plane direction in R^n of the approximate k-dimensional tangent plane of A at x.

(2) We denote by $|A|$ the varifold in $\underline{\underline{V}}_k(R^n)$ given for each $\varphi \in \underline{\underline{C}}_0^k(R^n)$ by

$$|A|(\varphi) = \int_{R^n} \|\varphi(x)\| \cdot \vec{A}(x) \quad d\,\underline{\underline{H}}^k \cap A\, x$$

noting that $\vec{A}(x)$ is well defined for $\underline{\underline{H}}^k \cap A$ almost all $x \in R^n$ and is $\underline{\underline{H}}^k \cap A$ measurable. A varifold $V \in \underline{\underline{V}}_k(R^n)$ is called a k-dimensional rectifiable varifold in R^n, or equivalently, k-rectifiable varifold, if and only if $V = |B|$ for some k-rectifiable set $B \subset R^n$. Note that the finite union and finite intersection of k-rectifiable varifolds is again a k-rectifiable varifold corresponding respectively to the finite union and finite intersection of the associated k-rectifiable sets.

5.4 DEFINITIONS. Integral and real varifolds. Let $0 \leq k \leq n$ be integers.

(1) A varifold $V \in \underline{\underline{V}}_k(R^n)$ is called a k-dimensional integral varifold in R^n if and only if each of the following three equivalent statements is true.

(a) For each $\varepsilon > 0$ there exists a finite number of pairwise disjoint compact k-dimensional submanifolds $A_1, A_2, \ldots, A_q$ of R^n of class 1 together with positive integers $a_1, a_2, \ldots, a_q$ such that $\underline{\underline{W}}(V, a_1|A_1| + a_2|A_2| + \ldots + a_q|A_q|) < \varepsilon$.

(b) There exists a sequence $A_1, A_2, A_3, \ldots$ of k-dimensional rectifiable subsets of R^n such that $\bigcup_i A_i$ is bounded, $\Sigma_i \underline{\underline{H}}^k(A_i) < \infty$, and $V = \Sigma_i |A_i|$.

(c) There exists a k-dimensional rectifiable subset A of R^n and a positive integer valued $\underline{\underline{H}}^k$ measurable function f on R^n such that for each $\varphi \in \underline{\underline{C}}_0^k(R^n)$,

$$V(\varphi) = \int_{R^n} (\|\varphi(x)\| \cdot \vec{A}(x))\, f(x) \quad d\, \underline{\underline{H}}^k \cap A\, x\ .$$

Here we write $V = |A| \wedge f$. Clearly for $\underline{\underline{H}}^k$ almost all $x \in A$, $f(x) = \odot^k(\underline{\underline{W}}V, x) = \odot^k(\underline{\underline{M}}V, x)$ and $\|\varphi(x)\| \cdot \vec{A}(x) = \vec{\underline{\underline{W}}}V(x)(\varphi(x)) = \vec{\underline{\underline{M}}}V(x)(\varphi(x))$.

We denote by $\underline{\underline{IV}}_k(R^n)$ the space of all k-dimensional integral varifolds in $\underline{\underline{V}}_k(R^n)$. A varifold $V \in \mathcal{V}_k(R^n)$ is called a k-dimensional integral varifold in R^n if and only if $V \cap U \in \underline{\underline{IV}}_k(R^n)$ for each bounded open set $U \subset R^n$. We denote by $\mathcal{I}\mathcal{V}_k(R^n)$ the space of all k-dimensional integral varifolds in $\mathcal{V}_k(R^n)$. For $U \subset R^n$ we write $\underline{\underline{IV}}_k(U) = \underline{\underline{IV}}_k(R^n) \cap \underline{\underline{V}}_k(U)$ and $\mathcal{I}\mathcal{V}_k(U) = \mathcal{I}\mathcal{V}_k(R^n) \cap \mathcal{V}_k(U)$. Note that the finite union and finite intersection of k-dimensional integral varifolds in R^n is again a k-dimensional integral varifold in R^n.

(2) A varifold $V \in \underline{\underline{V}}_k(R^n)$ is called a k-dimensional real varifold in R^n if and only if each of the following three equivalent statements is true.

(a) For each $\varepsilon > 0$ there exists a finite number of pairwise disjoint compact k-dimensional submanifolds $A_1, A_2, \ldots, A_q$ of R^n of class 1 together with positive real numbers $a_1, a_2, \ldots, a_q$ such that $\underline{\underline{W}}(V, a_1|A_1| + a_2|A_2| + \ldots + a_q|A_q|) < \varepsilon$.

(b) For each $\varepsilon > 0$ there exists a compact k-dimensional submanifold A of R^n of class 1 and $\varphi \in \underline{\underline{C}}_0^0(R^n)$ such that $\underline{\underline{W}}(V, |A| \wedge \varphi) < \varepsilon$.

(c) There exists a sequence $A_1, A_2, A_3, \ldots$ of pairwise disjoint k-dimensional rectifiable subsets of R^n and a sequence of $\underline{\underline{H}}^k$ measurable functions $f_1, f_2, f_3, \ldots : R^n \longrightarrow R^+$ such that $\bigcup_i A_i$ is bounded, $\lim_i \sup\{f_i(x) : x \in R^n\}$ ...

$\Sigma_i \underline{\underline{H}}^k(A_i) \sup\{f_i(x) : x \in R^n\} < \infty$, and for each $\varphi \in \underline{\underline{C}}_0^k(R^n)$,

$$V(\varphi) = \sum_i \int_{R^n} (\|\varphi(x)\| \cdot \vec{A}_i(x))\, f_i(x) \quad d\, \underline{\underline{H}}^k \cap A_i\, x\,.$$

Here we write $V = \Sigma_i |A_i| \wedge f_i$. Clearly for $\underline{\underline{H}}^k$ almost all $x \in A_i$, $f(x) = \odot^k(\underline{\underline{W}}V, x) = \odot^k(\underline{\underline{M}}V, x)$ and $\|\varphi(x)\| \cdot \vec{A}_i(x) = \vec{\underline{\underline{W}}}V(x)\,(\varphi(x)) = \vec{\underline{\underline{M}}}V(x)\,(\varphi(x))$. In case $\inf\{\odot^k(\underline{\underline{W}}V, x) : x \in \mathrm{spt}(V)\} > 0$ one can take $A_i = \phi$ for $i > 1$ and $V = A_1 \wedge f_1$.

We denote by $\underline{\underline{R}}\underline{\underline{V}}_k(R^n)$ the space of all k-dimensional real varifolds in $\underline{\underline{V}}_k(R^n)$. A varifold $V \in \mathcal{V}_k(R^n)$ is called a <u>k-dimensional real varifold in</u> R^n if and only if $V \cap U \in \underline{\underline{R}}\underline{\underline{V}}_k(R^n)$ for each bounded open set $U \subset R^n$. We denote by $\mathcal{R}\mathcal{V}_k(R^n)$ the space of all k-dimensional real varifolds in $\mathcal{V}_k(R^n)$. For $U \subset R^n$ we write $\underline{\underline{R}}\underline{\underline{V}}_k(U) = \underline{\underline{R}}\underline{\underline{V}}_k(R^n) \cap \underline{\underline{V}}_k(U)$ and $\mathcal{R}\mathcal{V}_k(U) = \mathcal{R}\mathcal{V}_k(R^n) \cap \mathcal{V}_k(U)$. Note that the finite union or the finite intersection of k-dimensional real varifolds in R^n is again a k-dimensional real varifold in R^n.

5.5 PROPOSITION. <u>Let</u> $0 \leq k \leq m \leq n$ <u>be integers and</u> $M \subset R^n$ <u>be a closed</u> m<u>-dimensional submanifold of</u> R^n <u>of class</u> 1. <u>Then each</u> $V \in \mathcal{R}\mathcal{V}_k(M)$ <u>lies intrinsically on</u> M <u>and the space of all</u> $V \in \mathcal{V}_k(M)$ <u>which lie intrinsically on</u> M <u>is closed in</u> $\mathcal{V}_k(R^n)$.

5.6 REMARK. Let $0 \leq k \leq n$ be integers and $f: \underline{\underline{R}}\underline{\underline{V}}_k(R^n) \longrightarrow R^+$ with

(a) $f(r\,V + s\,W) = r\,f(V) + s\,f(W)$ for each $r, s \in R^+$ and $V, W \in \underline{\underline{R}}\underline{\underline{V}}_k(R^n)$, and

(b) $\sup\{\underline{\underline{W}}(V)^{-1} f(V) : V \in \underline{\underline{R}}\underline{\underline{V}}_k(R^n) - \{0\}\} < \infty$.

Then assuming the continuum hypothesis and the well ordering principle, one can use 3.3 and the methods of [A3] to prove the existence of a differential k-form $\varphi: R^n \longrightarrow \underline{\underline{\Lambda}}^k(R^n)$ such that for each $V \in \underline{\underline{R}}\underline{\underline{V}}_k(R^n)$,

$$f(V) = \int_{R^n} \vec{\underline{\underline{W}}}V(x)\,(\varphi(x))\;\; d\,\underline{\underline{W}}V\,x\ .$$

5.7 DEFINITIONS. <u>Flat chains and their associated integral varifolds.</u> Let $0 \le k \le n$ be integers and $B \subset A \subset R^n$. We denote by G either the additative group of integers with the absolute value norm $|\;|$ or a finite abelian group with a translation invariant metric such that $|g| = \text{dist}(g, 0)$ is an integer for each $g \in G$. Such a group is called <u>admissible.</u> We denote by $\underline{\underline{Z}}_k(A, B; G)$ the abelian group of all flat k-chains T over G in R^n [FL3 1] such that the mass of T is finite, the mass of ∂T is finite, spt(T) is compact and contained in A, and spt(∂T) is compact and contained in B. In case G is the integers, $\underline{\underline{Z}}_k(A, B; G)$ is the abelian group of all k-dimensional integral currents in R^n[FF 3.7] with spt(T) $\subset$ A and spt(∂T) $\subset$ B.

For each $T \in \underline{\underline{Z}}_k(A, B; G)$ we denote by $\underline{\underline{M}}(T)$ the <u>mass</u> of T [FL3 3] [FF 2.4]. We define the $\underline{\underline{M}}$ metric on $\underline{\underline{Z}}_k(A, B; G)$ by setting for each $S, T \in \underline{\underline{Z}}_k(A, B; G)$, $\underline{\underline{M}}(S, T) = \underline{\underline{M}}(S-T)$. For each $T \in \underline{\underline{Z}}_k(A, B; G)$ we define

$$\underline{\underline{F}}(T) = \inf\{\underline{\underline{M}}(P) + \underline{\underline{M}}(Q) : T = P + \partial Q \quad \text{where} \quad P \in \underline{\underline{Z}}_k(A, B; G),\ Q \in \underline{\underline{Z}}_{k+1}(A, A; G)\}\ .$$

We define the $\underline{\underline{F}}$ metric on $\underline{\underline{Z}}_k(A, B; G)$ by setting for each $S, T \in \underline{\underline{Z}}_k(A, B; G)$, $\underline{\underline{F}}(S, T) = \underline{\underline{F}}(S-T)$. Unless otherwise indicated $\underline{\underline{Z}}_k(A, B; G)$ will have the $\underline{\underline{F}}$ metric topology.

To each $T \in \underline{\underline{Z}}_k(A, B; G)$ there is associated a Borel measure $\underline{\underline{M}}T$ on R^n (denoted μ_T in [FL3 4] and $\|T\|$ in [FF 2.4]) characterized by the property that for each open set $U \subset R^n$, $\underline{\underline{M}}T(U) = \underline{\underline{M}}(T \cap U)$.

For $T \in \underline{\underline{Z}}_k(A, B; G)$ let $A_T = R^n \cap \{x : \overline{\Theta}^k(\underline{\underline{M}}T, x) > 0\}$.

Since T is rectifiable [FL3 10.1] [FF 3.7], A_T will be k-rectifiable and $\Theta^k(\underline{\underline{M}}T, x)$ will exist for $\underline{\underline{H}}^k \cap A_T$ almost all $x \in R^n$, be a positive integer, and be $\underline{\underline{H}}^k \cap A_T$ measurable as a function of x.

Corresponding to each $T \in \underline{\underline{Z}}_k(A, B; G)$ we denote by $|T|$ the varifold in $\underline{\underline{IV}}_k(R^n)$ given for each $\varphi \in \underline{\underline{C}}_0^k(R^n)$ by

$$|T|(\varphi) = \int_{R^n} (\|\varphi(x)\| \cdot \vec{A}_T(x)) \odot^k(\underline{\underline{M}}T, x) \; d\, \underline{\underline{H}}^k \cap A_T\, x \,.$$

One verifies that the mapping $|\,|: \underline{\underline{Z}}_k(A, B; G) \longrightarrow \underline{\underline{IV}}_k(R^n)$ has the following properties:

(1) For each $T \in \underline{\underline{Z}}_k(A, B; G)$, $\underline{\underline{W}}(|T|) = \underline{\underline{M}}(|T|) = \underline{\underline{M}}(T)$.

(2) $|\,|$ is continuous in the $\underline{\underline{M}}$ metric topology on $\underline{\underline{Z}}_k(A, B; G)$ and both the $\underline{\underline{M}}$ metric topology and the $\underline{\underline{W}}$ metric topology on $\underline{\underline{IV}}_k(R^n)$.

(3) $|\,|$ is not, in general, one to one, nor is addition preserved; indeed if $T \in \underline{\underline{Z}}_k(A, B; G) - \{0\}$ then

$$|T + (-T)| = 0 \neq |T| + |-T| = 2|T| = |2T|$$

(4) $|\,|$ is not, in general, continuous in the $\underline{\underline{F}}$ metric topology on $\underline{\underline{Z}}_k(A, B; G)$ and the $\underline{\underline{F}}$ metric topology on $\underline{\underline{IV}}_k(R^n)$ since, in particular, $\underline{\underline{W}}$ is continuous on $\underline{\underline{IV}}_k(K)$ when $K \subset R^n$ is compact, while $\underline{\underline{M}}$ is only lower semi-continuous on $\underline{\underline{Z}}_k(A, B; G)$. However, if $T, T_1, T_2, T_3, \ldots \in \underline{\underline{Z}}_k(A, B; G)$, $\lim_i \underline{\underline{F}}(T, T_i) = 0$, and $\lim_i \underline{\underline{M}}(T_i) = \underline{\underline{M}}(T)$, then $\lim_i |T_i| = |T|$.

5.8 DEFINITIONS. Varifold tuples. Let $1 \leq k \leq n$ and $0 \leq j \leq k$ be integers and $A_j, A_{j+1}, A_{j+2}, \ldots, A_k \subset R^n$.

(1) We define

$$\underline{\underline{V}}_k(A_k, A_{k-1}, \ldots, A_j) = \underline{\underline{V}}_k(A_k) \times \underline{\underline{V}}_{k-1}(A_{k-1}) \times \ldots \times \underline{\underline{V}}_j(A_j)\,,$$

$$\underline{\underline{NV}}_k(A_k, A_{k-1}, \ldots, A_j) = \underline{\underline{NV}}_k(A_k) \times \underline{\underline{NV}}_k(A_{k-1}) \times \ldots \times \underline{\underline{NV}}_j(A_j).$$

$\underline{\underline{GV}}_k(A_k, A_{k-1}, \ldots, A_j)$, $\underline{\underline{RV}}_k(A_k, A_{k-1}, \ldots, A_j)$, and $\underline{\underline{IV}}_k(A_k, A_{k-1}, \ldots, A_j)$ have corresponding definitions.

(2) We define the $\underline{\underline{F}}$ metric and the $\underline{\underline{W}}$ metric on $\underline{\underline{V}}_k(A_k, A_{k-1}, \ldots, A_j)$ by setting for each $(V_k, V_{k-1}, \ldots, V_j), (W_k, W_{k-1}, \ldots, W_j) \in \underline{\underline{V}}_k(A_k, A_{k-1}, \ldots$

$$\underline{\underline{F}}[(V_k, V_{k-1}, \ldots, V_j), (W_k, W_{k-1}, \ldots, W_j)] = \sum_{i=j}^{k} \underline{\underline{F}}(V_i, W_i)$$

$$\underline{\underline{W}}[(V_k, V_{k-1}, \ldots, V_j), (W_k, W_{k-1}, \ldots, W_j)] = \sum_{i=j}^{k} \underline{\underline{W}}(V_i, W_i) .$$

We write also

$$\underline{\underline{F}}[(V_k, V_{k-1}, \ldots, V_j)] = \underline{\underline{F}}[(V_k, V_{k-1}, \ldots, V_j), (0, 0, \ldots, 0)],$$

$$\underline{\underline{W}}[(V_k, V_{k-1}, \ldots, V_j)] = \underline{\underline{W}}[(V_k, V_{k-1}, \ldots, V_j), (0, 0, \ldots, 0)].$$

(3) For $(V_k, V_{k-1}, \ldots, V_j), (W_k, W_{k-1}, \ldots, W_j) \in \underline{\underline{V}}_k(A_k, A_{k-1}, \ldots, A_j)$ we define

$$(V_k, V_{k-1}, \ldots, V_j) + (W_k, W_{k-1}, \ldots, W_j)$$
$$= (V_k + W_k, V_{k-1} + W_{k-1}, \ldots, V_j + W_j)$$
$$\in \underline{\underline{V}}_k(A_k, A_{k-1}, \ldots, A_j)$$

(4) If m is a non-negative integer, $f : R^n \longrightarrow R^m$ is of class 1, and $(V_k, V_{k-1}, \ldots, V_j) \in \underline{\underline{V}}_k(A_k, A_{k-1}, \ldots, A_j)$, we define

$$f_{\#}(V_k, V_{k-1}, \ldots, V_j) = (f_{\#}V_k, f_{\#}V_{k-1}, \ldots, f_{\#}V_j)$$
$$\in \underline{\underline{V}}_k(A_k, A_{k-1}, \ldots, A_j) .$$

(5) Definitions similar to those of (1) define

$$\mathcal{V}_k(A_k, A_{k-1}, \ldots, A_j), \ \mathcal{N}\mathcal{V}_k(A_k, A_{k-1}, \ldots, A_j), \ \mathcal{G}\mathcal{V}_k(A_k, A_{k-1}, \ldots, A_j),$$
$$\mathcal{R}\mathcal{V}_k(A_k, A_{k-1}, \ldots, A_j) \text{ and } \mathcal{I}\mathcal{V}_k(A_k, A_{k-1}, \ldots, A_j) .$$

5.9 DEFINITIONS. <u>The</u> VZ <u>spaces</u>. Let $1 \leq k \leq n$ be integers, $B \subset A \subset R^n$, and G be an admissible group.

(1) We define $\underline{\underline{VZ}}_k(A, B; G) \subset \underline{\underline{V}}_k(A, B) \times \underline{\underline{Z}}_k(A, B; G) \times \underline{\underline{Z}}_{k-1}(B, \phi; G)$ to be the set of all quadruples $(V, W; T, \partial T)$ such that for some sequence $T_1, T_2, T_3, \ldots \in \underline{\underline{Z}}_k(A, B; G)$, $\lim_i |T_i| = V \in \underline{\underline{V}}_k(A)$, $\lim_i |\partial T_i| = W \in \underline{\underline{V}}_{k-1}(B)$, $\lim_i T_i = T \in \underline{\underline{Z}}_k(A, B; G)$, $\lim_i \partial T_i = \partial T$, and there exists a compact subset

of R^n containing the supports of all the T_i. Note that for $(V, W; T, \partial T)$ $\in \underline{\underline{V}}\underline{\underline{Z}}_k(A, B; G)$ it is not in general true that $V = |T|$ or that $W = |\partial T|$; one has $\underline{\underline{M}}(T) \leq \underline{\underline{W}}(V)$ and $\underline{\underline{M}}(\partial T) \leq \underline{\underline{W}}(W)$.

(2) We define the $\underline{\underline{F}}$ metric and the $\underline{\underline{W}}$ metric on $\underline{\underline{V}}\underline{\underline{Z}}_k(A, B; G)$ by setting for $(V, W; T, \partial T)$, $(V', W'; T', \partial T') \in \underline{\underline{V}}\underline{\underline{Z}}_k(A, B; G)$,

$$\underline{\underline{F}}[(V, W; T, \partial T), (V', W'; T', \partial T')] = \underline{\underline{F}}[(V, W), (V', W')] + \underline{\underline{F}}(T, T'),$$

$$\underline{\underline{W}}[(V, W; T, \partial T), (V', W'; T', \partial T')] = \underline{\underline{W}}[(V, W), (V', W')] .$$

We write also

$$\underline{\underline{F}}[(V, W; T, \partial T)] = \underline{\underline{F}}[(V, W; T, \partial T), (0, 0; 0, 0)] ,$$

$$\underline{\underline{W}}[(V, W; T, \partial T)] = \underline{\underline{W}}[(V, W; T, \partial T), (0, 0; 0, 0)] .$$

(3) For $(V, W; T, \partial T)$, $(V', W'; T', \partial T') \in \underline{\underline{V}}\underline{\underline{Z}}_k(A, B; G)$ we define

$$(V, W; T, \partial T) + (V', W'; T', \partial T') = (V + V', W + W'; T + T', \partial T + \partial T').$$

See 5.13 and 5.14.

(4) If m is a non-negative integer, $f : R^n \longrightarrow R^m$ is of class 1, and $(V, W, T, \partial T)$ $\in \underline{\underline{V}}\underline{\underline{Z}}_k(A, B; G)$, we define

$$f (V, W; T, \partial T) = (f_{\#}V, f_{\#}W; f_{\#}T, f_{\#}\partial T) .$$

See 5.12.

5.10 LEMMA. Let $0 \leq k \leq n-2$ be integers and $A \subset R^n$ be k-rectifiable. Then for almost all $p \in \underline{\underline{\Omega}}_{n-1}(R^n)$,

$$\underline{\underline{H}}^k(A \cap \{x : p^{-1}\{p(x)\} - \{x\} \neq \phi\} = 0 .$$

PROOF. In view of 5.1 we can assume without loss of generality that A is a compact k-dimensional submanifold of R^n of class 1. Let A have the induced metric, let $A \times A$ have the product metric, and let D denote the diagonal of $A \times A$. We define

$P : A \times A - D \longrightarrow \underline{\underline{\Omega}}_{n-1}(R^n)$ by requiring $P(x, y)(x) = P(x, y)(y)$

for $x, y \in A$, $x \neq y$. One verifies that P is locally Lipschitzian which implies by [F3 3.1, 3.2] that for almost all $p \in \underline{\underline{\Omega}}_{n-1}(R^n)$, $(A \times A - D) \cap \{(x, y) : P(x, y) = p\}$ will be the countable union of j-dimensional rectifiable subsets of $A \times A - D$ where $j = \max\{-1, 2k-n+1\} \leq k-1$, and, in particular, will be of $\underline{\underline{H}}^k$ measure zero. The lemma follows since projection does not increase Hausdorff measure and for each $p \in \underline{\underline{\Omega}}_{n-1}(R^n)$,

$$A \cap \{x\colon\ p^{-1}\{p(x)\} - \{x\} \neq \phi\} = A \cap \{x\colon \text{ for some } (x, y) \in A \times A - D,\ P(x, y) = p\}.$$

5.11 COROLLARY. *Let $0 \leq k \leq n-2$ be integers and $A \subset R^n$ be k-rectifiable. Then there exists a dense subset θ of $\underline{\underline{\Omega}}_{k+1}(R^n)$ such that for each $p \in \theta$*

$$\underline{\underline{H}}^k(A \cap \{x\colon\ p^{-1}\{p(x)\} - \{x\} \neq \phi\}) = 0.$$

5.12 THEOREM. *Let $0 \leq p \leq m$, $0 \leq q \leq n$, and k be integers with $0 \leq k \leq \min\{m, n-1\}$ and $k-1 \leq \min\{p, q-1\}$. Let G be an admissible group and $f\colon (R^m, R^p) \longrightarrow (R^n, R^q)$ be of class 1. Then for each $(V, W; T, \partial T) \in \underline{\underline{V}}\underline{\underline{Z}}_k(R^m, R^p; G)$, $f_{\#}(V, W; T, \partial T) \in \underline{\underline{V}}\underline{\underline{Z}}_k(R^n, R^q; G)$, and thus the mapping*

$$f_{\#}\colon\ \underline{\underline{V}}\underline{\underline{Z}}_k(R^m, R^p; G) \longrightarrow \underline{\underline{V}}\underline{\underline{Z}}_k(R^n, R^q, G)$$

is well defined. Furthermore $f_{\#} \big| \underline{\underline{V}}\underline{\underline{Z}}_k(R^m \cap K, R^p \cap K; G)$ is continuous in the $\underline{\underline{F}}$ metric topologies for each compact set $K \subset R^n$.

PROOF. Let $(V, W; T, \partial T) \in \underline{\underline{V}}\underline{\underline{Z}}_k(R^m, R^p; G)$ and $T_1, T_2, T_3, \ldots \in \underline{\underline{Z}}_k(R^m, R^p; G)$ such that $\lim_i |T_i| = V$, $\lim_i |\partial T_i| = W$, $\lim_i T_i = T$, and $\lim_i \partial T_i = \partial T$. To prove that $f_{\#}(V, W; T, \partial T) \in \underline{\underline{V}}\underline{\underline{Z}}_k(R^n, R^q; G)$ we must find $S_1, S_2, S_3, \ldots \in \underline{\underline{Z}}_k(R^n, R^q; G)$ such that $\lim_i |S_i| = f_{\#}(V)$, $\lim_i |\partial S_i| = f_{\#}(W)$, $\lim_i S_i = f_{\#}(T)$, and $\lim_i \partial S_i = f_{\#}(\partial T_i)$. Observe that f can be factored

$$\begin{tikzcd}
R^m \arrow[rrr, "f"] \arrow[d, "g"'] & & & R^n \\
R^m \times R^n \arrow[rr, "p"] & & R^m \times R^n \arrow[ur, "h"'] &
\end{tikzcd}$$

where for $x \in R^m$ and $y \in R^n$, $g(x) = (x, f(x))$, $p(x, y) = (0, y)$, and $h(x, y) = y$. Note that $p \in \underline{\underline{\Omega}}_n(R^m \times R^n)$ and that h maps all n-dimensional subspaces of $R^m \times R^n$ which are sufficiently close to $p(R^m \times R^n)$ linearly isomorphically onto R^n. 5.7(4), 5.11, and the fact that $\underline{\underline{M}}T$ is absolutely continuous with respect to $\underline{\underline{H}}^k$ imply the existence of $p_1, p_2, p_3, \ldots \in \underline{\underline{\Omega}}_n(R^m \times R^n)$ with $\lim_i p_i = p$ such that $|(h \circ p_i \circ g)_{\#}(T)| = (h \circ p_i \circ g)(|T|)$ for each i. A similar argument yields $q_1, q_2, q_3, \ldots : R^p \longrightarrow R^q$ converging together with first derivatives uniformly on compact sets to $f|R^p$ such that $|q_{i\#}(\partial T)| = q_{i\#}(|\partial T|)$ for each i. From [FL3 7.6] one concludes the existence of $Q_1, Q_2, Q_3, \ldots \in \underline{\underline{Z}}_k(R^n, R^n; G)$ with $\lim_i \underline{\underline{M}}(Q_i) = 0$ such that $\partial[(h \circ p_i \circ g)_{\#}(T) + Q_i] = q_{i\#}(\partial T)$. for each i. We set $S_i = (h \circ p_i \circ g)_{\#}(T) + Q_i$.

5.13 COROLLARY. Let $0 \le p \le m$ and $0 \le k$ be integers with $k \le m-1$ and $k-1 \le p-1$. Let G be an admissible group and $\mu, \nu \in \underline{\underline{V}}\underline{\underline{Z}}_k(R^m, R^p; G)$. Then $\mu + \nu \in \underline{\underline{V}}\underline{\underline{Z}}_k(R^m, R^p; G)$.

5.14 EXAMPLE. Let G be the additive group of integers with metric $|\,|$. Let $T \in \underline{\underline{Z}}_1(\underline{\underline{D}}^1(0, 1), \partial\, \underline{\underline{D}}^1(0, 1); G)$ correspond to the interval $\underline{\underline{D}}^1(0, 1)$ with its usual orientation. Then

$$(|T|, |\partial T|; T, \partial T), (|-T|, |-\partial T|; -T, -\partial T) \in \underline{\underline{V}}\underline{\underline{Z}}_1(\underline{\underline{D}}^1(0, 1), \partial \underline{\underline{D}}^1(0, 1); G)$$

but

$$(|T|, |\partial T|; T, \partial T) + (|-T|, |-\partial T|; -T, -\partial T)$$
$$= (2|T|, 2|\partial T|; 0, 0) \notin \underline{\underline{V}}\underline{\underline{Z}}_1(\underline{\underline{D}}^1(0, 1), \partial D^1(0, 1); G)$$

5.15 REMARK. Let $0 \leq k \leq m \leq n$ be integers and M be an (abstract) compact m-dimensional Riemannian manifold of class 1 having $A \subset R^n$ as its image under an isometric imbedding. One (abstractly) defines $\underline{\underline{V}}\underline{\underline{Z}}_k(M, M; G)$ in the obvious way for admissible groups G. Clearly $\underline{\underline{V}}\underline{\underline{Z}}_k(M, M; G)$ is naturally isomorphic with $\underline{\underline{V}}\underline{\underline{Z}}_k(A, A; G)$. One verifies results corresponding to 5.12 and 5.13 for (abstract and imbedded) manifolds, submanifolds, and differentiable maps.

6. REGULAR AND STATIONARY VARIFOLDS

6.1 THEOREM. Let $0 \leq k \leq n$ be integers. Let $f : \{(t, x) : t \in R, x \in R^n\} \longrightarrow R^n$ have as continuous partial derivatives $(\partial f/\partial t)$, $(\partial f/\partial x^i)$, $(\partial^2 f/\partial t \partial x^i)$ for each $i = 1, 2, \ldots, n$, with $f(0, x) = x$ for each $x \in R^n$. Define $S : \underline{\underline{V}}_k(R^n) \longrightarrow R$ by setting for each $V \in \underline{\underline{V}}_k(R^n)$,

$$S(V) = (d/dt)\underline{\underline{W}}(f(t, \cdot)_{\#}(V))\Big|_{t=0}$$

Then S is well defined and for each $V \in \underline{\underline{V}}_k(R^n)$, $S(V)$ equals

(a) $$\gamma(n, k) \int_{\lambda \in D\Lambda^k(R^n)} \int_{x \in R^n} D\vec{\underline{\underline{W}}}V(x)\left(\lambda, (d/dt)\left[f(t, \cdot)^{\#}(x, \lambda)\right]\Big|_{t=0}\right) d\underline{\underline{W}}Vx \; d\underline{\underline{H}}^{(n:k)}\lambda.$$

Also $S|\underline{\underline{V}}_k(K)$ is continuous for each compact $K \subset R^n$.

PROOF.

Part 1. Let $F : R^n \longrightarrow R$ be convex and suppose DF exists at $x \in R^n$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $G : R^n \longrightarrow R$ is convex, DG exists at x, and $\sup\{|F(y) - G(y)| : |y - x| < 1\} < \delta$, then $\sup\{|DF(x, v) - DG(x, v)| : v \in S^{n-1}\} < \varepsilon$.

Part 2. Denote by $\underline{\underline{U}}$ the set of all functions $F : R^n \longrightarrow R$ which are convex and satisfy a Lipschitz condition with constant $\eta < \infty$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ (depending on η and ε) such that if $F, G \in U$ with $\sup\{|F(x) - G(x)| : x \in R^n\}$ then

(1) For $\underline{\underline{H}}^n$ almost all $x \in R^n$, DF and DG both exist at x and $\sup\{|DF(x, v) - DG(x, v)| : v \in S^{n-1}\} \leq 2\eta$;

(2) $\underline{\underline{H}}^n(\underline{\underline{D}}^n(0, 1) \cap \{x : DF$ and DG both exist at x and $\sup\{|DF(x, v) - DG(x, v)| : v \in S^{n-1}\} > \varepsilon\}) < \varepsilon$.

Part 3. Let f be as in the hypothesis and $V \in \underline{\underline{V}}_k(R^n)$. Then there exists $\varepsilon > 0$ such that whenever $-\varepsilon < s < \varepsilon$, $(d/ds)\underline{\underline{W}}(f(t, \cdot)_{\#}(V))\Big|_{t=s}$ exists and equals (a) with, of course, "$t = 0$" replaced by "$t = s$").

Proof of part 3. We will abbreviate $\gamma = \gamma(n, k)$, $\underline{\underline{D}} = \underline{\underline{D\Lambda}}^k(R^n)$, $q = (n : k)$, $\lambda_t(x) = f(t, \cdot)^\#(x, \lambda)$, and $\dot{\lambda}_t(x) = (d/ds)f(s, \cdot)^\#(x, \lambda)\big|_{s=t}$. Note that $\lambda_t(x)$ and $\dot{\lambda}_t(x)$ are continuous as functions of t and x.

We compute

$$\begin{aligned}
\underline{\underline{W}}(f(t, \cdot)\ V) &= \gamma \int_{\underline{\underline{D}}} f(t, \cdot)_\#(V)(\omega(\lambda))\, d\underline{\underline{H}}^q\lambda \\
&= \gamma \int_{\underline{\underline{D}}} V(f(t, \cdot)^\#\omega(\lambda))\, d\underline{\underline{H}}^q\lambda \\
&= \gamma \int_{\underline{\underline{D}}} \int_{R^n} \vec{\underline{\underline{W}}}V(x)(\lambda_t(x))\, d\underline{\underline{W}}Vx\, d\underline{\underline{H}}^q\lambda \\
&= \gamma \int_{\underline{\underline{D}}} \int_{R^n} \vec{\underline{\underline{W}}}V(x)(\lambda_s(x) + (t-s)\dot{\lambda}_s(x) + [\lambda_t(x) - \lambda_s(x) - (t-s)\dot{\lambda}_s(x)] \\
&\qquad\qquad d\underline{\underline{W}}Vx\, d\underline{\underline{H}}^q\lambda
\end{aligned}$$

$$\begin{aligned}
\text{(b)}\qquad &= \gamma \int_{\underline{\underline{D}}} \int_{R^n} \vec{\underline{\underline{W}}}V(x)(\lambda_s(x))\, d\underline{\underline{W}}Vx\, d\underline{\underline{H}}^q\lambda \\
&+ (t-s)\gamma \int_{\underline{\underline{D}}} \int_{R^n} D\vec{\underline{\underline{W}}}V(x)(\lambda_s(x), \dot{\lambda}_s(x))\, d\underline{\underline{W}}Vx\, d\underline{\underline{H}}^q\lambda \\
&+ \gamma \int_{\underline{\underline{D}}} \int_{R^n} D\vec{\underline{\underline{W}}}V(x)(\lambda_s(x), \lambda_t(x) - \lambda_s(x) - (t-s)\dot{\lambda}_s(x))\, d\underline{\underline{W}}Vx\, d\underline{\underline{H}}^q\lambda \\
+ \gamma \int_{\underline{\underline{D}}} \int_{R^n} &\vec{\underline{\underline{W}}}V(x)(\lambda_t(x)) - \vec{\underline{\underline{W}}}V(x)(\lambda_s(x)) - D\vec{\underline{\underline{W}}}V(x)(\lambda_s(x), \lambda_t(x) - \lambda_s(x))\, d\underline{\underline{W}}Vx\, d\underline{\underline{H}}^q\lambda
\end{aligned}$$

provided these integrals are all well defined.

There exists a compact set $K \subset R^n$ and $\varepsilon > 0$ such that $f(t, \cdot)(\mathrm{spt}(V)) \subset K$ whenever $|t| < \varepsilon$ and such that for each $x \in K$ and $-\varepsilon < s < \varepsilon$ the mapping $\underline{\underline{D\Lambda}}^k(R^n) \longrightarrow \underline{\underline{\Lambda}}^k(R^n)$, sending $\lambda \in \underline{\underline{D\Lambda}}^k(R^n)$ to $\lambda_s(x)$, is a diffeomorphism onto its image. If $|s| < \varepsilon$ then the integral

$$\int_{\underline{\underline{D}}} \int_{R^n} D\vec{\underline{\underline{W}}}V(x)(\lambda_s(x), \varphi(x, \lambda))\, d\underline{\underline{W}}Vx\, d\underline{\underline{H}}^q\lambda$$

is well defined for each continuous mapping $\varphi: R^n \times \underline{\underline{D\Lambda}}^k(R^n) \longrightarrow \underline{\underline{\Lambda}}^k(R^n)$. Thus, in particular, the integrals above in (b) are well defined for $|s| < \varepsilon$. We will have proved part 3 if we can show that the t derivatives of the third and fourth summands of (b) are zero. Since

$$\lim_{t \to s} (t-s)^{-1}[\lambda_t(x) - \lambda_s(x) - (t-s)\dot{\lambda}_s(x)] = 0$$

uniformly on compact subsets of $\underline{\underline{\Lambda}}^k(R^n)$ and R^n one sees easily that the t derivative of the third summand of (b) is zero.

Note that for each $x \in R^n$, $\lambda, \mu \in \underline{\underline{\Lambda}}^k(R^n)$ for which $D\vec{\underline{\underline{W}}}V(\lambda, \cdot)$ exists we have

$$\vec{\underline{\underline{W}}}V(x)(\lambda+\mu) - \vec{\underline{\underline{W}}}V(x)(\lambda) - D\vec{\underline{\underline{W}}}V(x)(\lambda,\mu) \leq 2|\mu| \operatorname{Lip}(\vec{\underline{\underline{W}}}V(x))$$

and that the function

$$\psi(\lambda, s) = s^{-1} \sup\{\vec{\underline{\underline{W}}}V(x)(\lambda+\mu) - \vec{\underline{\underline{W}}}V(x)(\lambda) - D\vec{\underline{\underline{W}}}V(x)(\lambda,\mu) : |\mu| \leq s\}$$

is non-increasing as a function of s for $s \in R^+$ with

$$\lim_{s \to 0^+} \psi(\lambda, s) = 0.$$

This follows using the convexity and positive homogeneity of $\vec{\underline{\underline{W}}}(x)$. One now uses 3.8 and Lebesgue's theorem on bounded convergence to conclude that the t derivative of the fourth summand of (b) is zero.

<u>Part</u> 4. The function S is well defined and continuous.

<u>Proof of part</u> 4. That S is well defined follows from the formula (a) established in part 3. The continuity is straightforward using part 2 and Lebesgue's theorem on bounded convergence.

6.2 DEFINITION. <u>The cartesian product of a varifold with an interval.</u> Let $0 \leq k \leq n$ be integers and $a < b$ be real numbers. Let $R^{n+1} = R \times R^n$ have orthonormal coordinate functions $t, x^1, x^2, \ldots, x^n$ and let $\mu : R \times R^n \longrightarrow \{(\partial/\partial t)\} \subset \underline{\underline{\Lambda}}_1(R \times R^n)$. Set $[a, b] = |R \cap \{t : a \leq t \leq b\}| \in \underline{\underline{IV}}_1(R)$. For each $V \in \mathcal{V}_k(R^n)$ we define the cartesian product of $[a, b]$ with V to be that varifold $[a, b] \times V \in \mathcal{V}_{k+1}(R^{n+1})$ given for each $\varphi \in \underline{\underline{C}}_0^{k+1}(R^{n+1})$ by

$$([a, b] \times V)(\varphi) = \int_{R \times R^n} \vec{\underline{\underline{W}}}V(x)[(\mu \wedge \varphi)(t, x)]\, d(\underline{\underline{W}}[a, b] \times \underline{\underline{W}}V)(t, x).$$

One verifies that $[a, b] \times V \in \mathcal{V}_{k+1}(R^{n+1})$. If $V \in \mathcal{G}\mathcal{V}_k(R^n)$, then $[a, b] \times V \in \mathcal{G}\mathcal{V}_{k+1}(R^{n+1})$. Also the function $\mathcal{V}_k(R^n) \longrightarrow \mathcal{V}_{k+1}(R^n)$ sending $V \in \mathcal{V}_k(R^n)$ to $[a, b] \times V$ is continuous.

6.3 THEOREM. *Let* $0 \le k \le n$ *be integers and* $f : \{(t, x) : t \in R,\ x \in R^n\} \longrightarrow R^n$ *be of class* 1 *with* $f(0, x) = 0$ *for each* $x \in R^n$. *Define* $T : \underline{\underline{V}}_k(R^n) \longrightarrow R^+$ *by setting for each* $V \in \underline{\underline{V}}_k(R^n)$,

$$T(V) = \lim_{t \longrightarrow 0^+} t^{-1} \underline{\underline{W}}(f(t, \cdot)_{\#}([0, t] \times V)).$$

Then T *is well defined and for each* $V \in \underline{\underline{V}}_k(R^n)$, $T(V)$ *equals*

(a) $$\gamma(n, k+1) \int_{\lambda \in \underline{\underline{D\Lambda}}^{k+1}(R^n)} \int_{x \in R^n} \vec{\underline{\underline{W}}}V([\mu \wedge f^{\#}(\omega(\lambda))](0, x))\, d\underline{\underline{W}}Vx\, d\underline{\underline{H}}^{(n:k+1)}\lambda$$

where $\mu : R \times R^n \longrightarrow \{(\partial/\partial t)\} \in \underline{\underline{\Lambda}}_1(R \times R^n)$ *and we identify* $[\mu \wedge f^{\#}(\omega(\lambda))](0, x) \in \underline{\underline{\Lambda}}^k(R \times R^n)$ *as an element of* $\underline{\underline{\Lambda}}^k(R^n)$, which is possible because $[\mu \wedge f^{\#}(\omega(\lambda))](0, x)$ *is parallel with* $\{0\} \times R^n$ *for each* $x \in R^n$. *Also* $T | \underline{\underline{V}}_k(K)$ *is continuous for each compact* $K \subset R^n$.

PROOF. The proof is left to the reader.

6.4 DEFINITIONS. *Vector fields, deformations, and the functions* $\underline{\underline{S}}$, $\underline{\underline{T}}$, $\underline{\underline{P}}$, *and* $\underline{\underline{Q}}$. We assume throughout this section that $1 \le k \le n$ are integers.

(1) Let $A \subset R^n$ be a closed submanifold of R^n of class 3, perhaps with boundary. We say that a vector field $g : R^n \longrightarrow R^n$ is *tangent* to A if and only if (a) for each non-boundary point $x \in A$, $g(x)$ is a tangent vector to A at x, and (b) for each boundary point $x \in \partial A$, $g(x)$ lies in the tangent half plane to A at x.

(2) Let q be a positive integer and $A_1, A_2, \ldots, A_q \subset R^n$ be closed submanifolds of R^n of class 3, perhaps with boundary, and $C \subset A$ be closed. We denote by $\underline{\underline{X}}(R^n, A_1, A_2, \ldots, A_q; C)$ the real vector space of all vector fields $g : R^n \longrightarrow R^n$ of class 1 with $\mathrm{Lip}(Dg) < \infty$, having compact support disjoint from C, and tangent to A_i for each i. We write also $\underline{\underline{X}}(R^n) = \underline{\underline{X}}(R^n, \emptyset; \emptyset)$, $\underline{\underline{X}}(R^n, A_1, \ldots, A_q) = \underline{\underline{X}}(R^n, A_1, \ldots, A_q; \emptyset)$, and $\underline{\underline{X}}(R^n; C) = \underline{\underline{X}}(R^n, \emptyset; C)$.

(3) Let $\underline{\underline{X}}(R^n, A_1, \ldots, A_q; C)$ be as above and $g \in \underline{\underline{X}}(R^n, A_1, \ldots, A_q; C)$. associate with C two deformations $f_1, f_2 : R \times R^n \longrightarrow R^n$ which are of class and, in addition, twice continuously differentiable in the first variable t,

$f_1(t, x) = x + tg(x)$ for $t \in R$ and $x \in R^n$ and f_2 is characterized by the conditions (a) $f_2(0, x) = x$, and (b) $(\partial/\partial t)f_2(t, x) = g(f_2(t, x))$ for each $t \in R$ and $x \in R^n$. Note that $f_1(t, x) = f_2(t, x) = x$ for each $t \in R$ and $x \in C$, and that for each $V \in \underline{\underline{V}}_k(\cup_i A_i)$, $f_2(t, \cdot)_\#(V) \in \underline{\underline{V}}_k(\cup_i A_i)$ for each $t \in R^+$. We observe that for each $\lambda \in \underline{\underline{\Lambda}}^k(R^n)$,

$$(d/dt)f_1(t, \cdot)^\#(x, \lambda)\Big|_{t=0} = (d/dt)f_2(t, \cdot)^\#(x, \lambda)\Big|_{t=0}$$

and thus, by 6.1,

$$(d/dt)\underline{\underline{W}}(f_1(t, \cdot)_\#(V))\Big|_{t=0} = (d/dt)\underline{\underline{W}}(f_2(t, \cdot)_\#(V))\Big|_{t=0}$$

for each $V \in \underline{\underline{V}}_k(R^n)$. We define

$$\underline{\underline{S}} : \underline{\underline{V}}_k(R^n) \times \underline{\underline{X}}(R^n) \longrightarrow R$$

by setting for each $V \in V_k(R^n)$ and $g \in \underline{\underline{X}}(R^n)$,

$$\underline{\underline{S}}(V, g) = (d/dt)\underline{\underline{W}}(f_1(t, \cdot)_\#(V))\Big|_{t=0} = (d/dt)\underline{\underline{W}}(f_2(t, \cdot)_\#(V))\Big|_{t=0}.$$

For fixed $g \in \underline{\underline{X}}(R^n)$, 6.1 implies the continuity of $\underline{\underline{S}}(\cdot, g) : \underline{\underline{V}}_k(R^n) \longrightarrow R$ and for each $V, W \in \underline{\underline{V}}_k(R^n)$ and $r, s \in R^+$,

$$\underline{\underline{S}}(rV + sW, g) = r\underline{\underline{S}}(V, g) + s\underline{\underline{S}}(W, g).$$

For fixed $V \in \underline{\underline{V}}_k(R^n)$, 6.1 implies the linearity of $\underline{\underline{S}}(V, \cdot) : \underline{\underline{X}}(R^n) \longrightarrow R$.

(4) If $g \in \underline{\underline{X}}(R^n)$, $p \in R^n$, and $\mu_1, \mu_2, \ldots, \mu_k \in \underline{\underline{\Lambda}}_1(R^n)$ are an orthonormal set of vectors, then one computes

(a) $$\underline{\underline{S}}(\underline{\underline{v}}(p, \mu_1 \wedge \mu_2 \wedge \ldots \wedge \mu_k), g) = \sum_{i=1}^{k} Dg(p, \mu_i) \cdot \mu_i$$

[FL p. 18]. Writing $g = (g^1, g^2, \ldots, g^n)$ we have, in particular,

(b) $$\underline{\underline{S}}(\underline{\underline{v}}(p, (\partial/\partial x^1) \wedge (\partial/\partial x^2) \wedge \ldots \wedge (\partial/\partial x^k)), g) = \sum_{i=1}^{k} (\partial g^i/\partial x^i)(p)$$

If $V \in \underline{\underline{GV}}_k(R^n)$, i.e. V is the weak limit of finite sums of elementary geometric varifolds, then (b) together with the additivity and continuity properties of $\underline{\underline{S}}(\cdot, \cdot)$ implies

(c) $$|\underline{\underline{S}}(V, g)| \leq k \underline{\underline{W}}V(\mathrm{Lip}(g)(\cdot)),$$

or, in particular,

(d) $$|\underline{\underline{S}}(V, g)| \leq k \underline{\underline{W}}(V)\, \mathrm{Lip}(g)$$

for each $g \in \underline{\underline{X}}(R^n)$. The inequality of (c) is the best possible as one sees by choosing $g \in \underline{\underline{X}}(R^n)$ with $g(x) = x$ for x in some neighborhood of $\mathrm{spt}(V)$.

If $p, u \in R^n$, $h : R^n \longrightarrow R$ is of class 2 with compact support, and $\mu \in \underline{\underline{\Lambda}}_k(R^n)$ is simple, then one computes

(e) $$\underline{\underline{S}}(\underline{\underline{v}}(p, \mu), hu) = |\mu|\ Dh(p, u_*)$$

[FL p. 19] where u_* denotes the orthogonal projection of u onto the k-plane of μ.

(5) We define a norm $\| \ \| : \underline{\underline{X}}(R^n) \longrightarrow R^+$ by setting for each $g \in \underline{\underline{X}}(R^n)$,

$$\|g\| = \sup\{|g(x)| : x \in R^n\} + \sup\{|Dg(x, v)| : x \in R^n \text{ and } v \in S^{n-1}\}.$$

$\underline{\underline{X}}(R^n)$ is given the $\| \ \|$ topology. Note that if $g_0, g_1, g_2, \ldots \in \underline{\underline{X}}(R^n)$ with $\lim_i \|g_i - g_0\| = 0$, if $f_0, f_1, f_2, \ldots : R \times R^n \longrightarrow R^n$ are characterized by being of class 1 and satisfying (i) $f_i(0, x) = x$, and (ii) $(\partial/\partial t) f_i(t, x) = g_i(f_i(t, x))$ for each $(t, x) \in R \times R^n$, and if $t_0, t_1, t_2, \ldots \in R$ with $\lim_i t_i = t_0$, then, by standard theorems on differential equations,

$$\lim_i f_i(t_i, \cdot) = f_0(t_0, \cdot) \text{ and } \lim_i Df_i(t_i, \cdot) = Df_0(t_0, \cdot)$$

uniformly on compact sets, and hence for each $V \in \underline{\underline{V}}_k(R^n)$,

$$\lim_i f_i(t_i, \cdot)_{\#}(V) = f_0(t_0, \cdot)_{\#}(V).$$

(6) For each $V \in \underline{\underline{GV}}_k(R^n)$, 4(c) above implies that $\underline{\underline{S}}(V, \cdot) : \underline{\underline{X}}(R^n) \longrightarrow R$ is linear and continuous, and, in fact, satisfies a Lipschitz condition with constant $k\underline{\underline{W}}(V)$ in the $\| \ \|$ metric on $\underline{\underline{X}}(R^n)$. This implies then that $\underline{\underline{S}} : \underline{\underline{GV}}_k(K) \times \underline{\underline{X}}(R^n) \longrightarrow R$ is continuous in the product topology whenever $K \subset R^n$ is compact.

For $A_1, A_2, \ldots, A_q$, C as in (2) above and $0 \leq j \leq k$, we define

$$\underline{\underline{S}}(R^n, A_1, A_2, \ldots, A_q; C) : \underline{\underline{GV}}_k(R^n, R^n, \ldots, R^n) \longrightarrow R^+$$

by setting for each $(V_k, V_{k-1}, \ldots, V_j) \in \underline{\underline{GV}}_k(R^n, R^n, \ldots, R^n)$,

$$\underline{\underline{S}}(R^n, A_1, A_2, \ldots, A_q; C)(V_k, V_{k-1}, \ldots, V_j)$$
$$= \sup\{-\sum_{i=j}^{k} \underline{\underline{S}}(V_i, g) : g \in \underline{\underline{X}}(R^n, A_1, A_2, \ldots, A_q, C), \|g\| \le 1, \text{ and } \mathrm{Lip}(Dg) \le 1\}$$

The continuity of $\underline{\underline{S}}(R^n, A_1, A_2, \ldots, A_q; C)$ can be established by straightforward arguments.

(7) We define

$$\underline{\underline{T}} : \underline{\underline{V}}_k(R^n) \times \underline{\underline{X}}(R^n) \longrightarrow R^+$$

by setting for each $V \in \underline{\underline{V}}_k(R^n)$ and $g \in \underline{\underline{X}}(R^n)$,

$$\underline{\underline{T}}(V, g) = \lim_{t \to 0^+} t^{-1}\underline{\underline{W}}[f_{1\#}([0, t] \times V)]\Big|_{t=0} = \lim_{t \to 0^+} t^{-1}\underline{\underline{W}}[f_{2\#}([0, t] \times V)]\Big|_{t=0},$$

where f_1 and f_2 are the deformations associated with g as in (3) above. If $\mu \in \underline{\underline{\Lambda}}_k(R^n)$ is simple, $p \in R^n$, and $g \in \underline{\underline{X}}(R^n)$, then

(a) $$\underline{\underline{T}}(\underline{\underline{v}}(p, \mu), g) = |g(p) \wedge \mu| \le \underline{\underline{W}}\,\underline{\underline{v}}(p, \mu)(|g|).$$

For each $V \in \underline{\underline{GV}}_k(R^n)$, the additivity and continuity of $\underline{\underline{T}}(\cdot, g)$ for fixed $g \in \underline{\underline{X}}(R^n)$ imply

(b) $$\underline{\underline{T}}(V, g) \le \underline{\underline{W}}V(|g|).$$

In general this inequality is strict. Similarly, for $V \in \underline{\underline{GV}}_k(R^n)$ and $g \in \underline{\underline{X}}(R^n)$,

(c) $$\underline{\underline{T}}(V, g) = \underline{\underline{W}}(V \wedge g).$$

(8) We define

$$\underline{\underline{P}}^+, \underline{\underline{P}}^-, \underline{\underline{P}} : \underline{\underline{V}}_k(R^n, R^n) \times \underline{\underline{X}}(R^n) \cap \{(V, W; g) : \underline{\underline{T}}(V, g) > 0\} \longrightarrow R^+$$

by setting for each $(V, W; g) \in \mathrm{dmn}(\underline{\underline{P}})$,

$$\underline{\underline{P}}^+(V, W; g) = \max\{0, [\underline{\underline{T}}(V, g)]^{-1}[\underline{\underline{S}}(V, g) - \underline{\underline{T}}(W, g)]\},$$
$$\underline{\underline{P}}^-(V, W; g) = \max\{0, [\underline{\underline{T}}(V, g)]^{-1}[-\underline{\underline{S}}(V, g) - \underline{\underline{T}}(W, g)]\}, \text{ and}$$
$$\underline{\underline{P}}(V, W; g) = \max\{\underline{\underline{P}}^+(V, W; g), \underline{\underline{P}}^-(V, W; g)\}.$$

(9) We define

$$\underline{\underline{Q}}^+,\ \underline{\underline{Q}}^-,\ \underline{\underline{Q}} : \underline{\underline{V}}_k(R^n, R^n) \times \underline{\underline{X}}(R^n) \cap \{(V, W; g) : \underline{\underline{W}}V(|g|) > 0\} \longrightarrow R^+$$

by setting for each $(V, W; g) \in \mathrm{dmn}(Q)$,

$$\underline{\underline{Q}}^+(V, W; g) = \max\{0, [\underline{\underline{W}}V(|g|)]^{-1}[\underline{\underline{S}}(V, g) - \underline{\underline{T}}(W, g)]\}$$

$$\underline{\underline{Q}}^-(V, W; g) = \max\{0, [\underline{\underline{W}}V(|g|)]^{-1}[-\underline{\underline{S}}(V, g) - \underline{\underline{T}}(W, g)]\}$$

$$\underline{\underline{Q}}(V, W; g) = \max\{\underline{\underline{Q}}^+(V, W; g), \underline{\underline{Q}}^-(V, W; g)\}$$

(10) Let $A_1, A_2, \ldots, A_q, C \subset R^n$ be as in (2) above and $0 \le j < k$ be integers. We define functions

$$\underline{\underline{P}}^a(R^n, A_1, A_2, \ldots, A_q; C) : \underline{\underline{V}}_k(R^n, R^n, \ldots, R^n) \longrightarrow R^+ \cup \{\infty\}$$

$$\underline{\underline{Q}}^a(R^n, A_1, A_2, \ldots, A_q; C) : \underline{\underline{V}}_k(R^n, R^n, \ldots, R^n) \longrightarrow R^+ \cup \{\infty\}$$

where $\underline{\underline{P}}^a$ denotes $\underline{\underline{P}}^+$, $\underline{\underline{P}}^-$, or $\underline{\underline{P}}$ and $\underline{\underline{Q}}^a$ denotes $\underline{\underline{Q}}^+$, $\underline{\underline{Q}}^-$, or $\underline{\underline{Q}}$, by setting for each $(V_k, V_{k-1}, \ldots, V_j) \in \underline{\underline{V}}_k(R^n, R^n, \ldots, R^n)$,

$$\underline{\underline{P}}^a(R^n, A_1, \ldots, A_q; C)(V_k, \ldots, V_j)$$

$$= \sum_{i=j+1}^{k} \sup\{\underline{\underline{P}}^a(V_i, V_{i-1}; g) : g \in \underline{\underline{X}}(R^n, A_1, \ldots, A_q; C) \text{ and } \underline{\underline{T}}(V_i, g) > 0\}$$

$$\underline{\underline{Q}}^a(R^n, A_1, \ldots, A_q; C)(V_k, \ldots, V_j)$$

$$= \sum_{i=j+1}^{k} \sup\{\underline{\underline{Q}}^a(V_i, V_{i-1}; g) : g \in \underline{\underline{X}}(R^n, A_1, \ldots, A_q; C) \text{ and } \underline{\underline{W}}V_i(|g|) > 0\}$$

In view of the continuity of $\underline{\underline{S}}(\cdot, g)$, $\underline{\underline{T}}(\cdot, g)$, and $\underline{\underline{W}}(\cdot)(|g|)$ for fixed $g \in \underline{\underline{X}}(R^n)$, each of the functions above is lower semi-continuous in the $\underline{\underline{F}}$ metric topology on $\underline{\underline{V}}_k(K, K, \ldots, K)$ whenever $K \subset R^n$ is compact. Also, in case $A_1, A_2, \ldots, A_q$ are closed submanifolds of R^n of class 3 without boundary then the functions $\underline{\underline{P}}^a(R^n, A_1, \ldots, A_q; C)$ coincide and the functions $\underline{\underline{Q}}^a(R^n, A_1, \ldots, A_q; C)$ coincide. Note that if $(V, W) \in \underline{\underline{V}}_k(R^n, R^n)$, then $\underline{\underline{P}}(R^n; \mathrm{spt}(W))(V, 0) \le \underline{\underline{P}}(R^n)(V, W)$ and $\underline{\underline{Q}}(R^n; \mathrm{spt}(W))(V, 0) \le \underline{\underline{Q}}(R^n)(V, W)$.

We sometimes write $\underline{\underline{P}}^a(R^n, \cdot; \cdot)(V)$ for $\underline{\underline{P}}^a(R^n, \cdot; \cdot)(V, 0)$ and $\underline{\underline{Q}}^a(R^n, \cdot; \cdot)(V)$ for $\underline{\underline{Q}}^a(R^n, \cdot; \cdot)(V, 0)$.

6.5 DEFINITIONS. Stationary, $\underline{\underline{P}}$ regular, and $\underline{\underline{Q}}$ regular varifolds. Let $1 \le k \le n$ be integers and $A, B \subset R^n$ be closed submanifolds of R^n of class 3. Let $C \subset R^n$ be closed and $(V, W) \in \mathcal{V}_k(A, B)$.

(1) (V, W) is called stationary on (A, B) with respect to C if and only if for each $r \in R_0^+$,

$$\underline{\underline{P}}^-(R^n, A, B; C \cup [R^n - \underline{\underline{D}}_0^n(0, r)])(V \cap \underline{\underline{D}}^n(0, r), W \cap \underline{\underline{D}}^n(0, r))$$
$$= \underline{\underline{Q}}^-(R^n, A, B; C \cup [R^n - \underline{\underline{D}}_0^n(0, r)])(V \cap \underline{\underline{D}}^n(0, r), W \cap \underline{\underline{D}}^n(0, r))$$
$$= 0$$

(2) (V, W) is called $\underline{\underline{P}}$ regular on (A, B) with respect to C if and only if for each $r \in R_0^+$,

$$\underline{\underline{P}}^-(R^n, A, B; C \cup [R^n - \underline{\underline{D}}_0^n(0, r)])(V \cap \underline{\underline{D}}^n(0, r), W \cap \underline{\underline{D}}^n(0, r)) < \infty.$$

(3) (V, W) is called $\underline{\underline{Q}}$ regular on (A, B) with respect to C if and only if for each $r \in R_0^+$,

$$\underline{\underline{Q}}^-(R^n, A, B; C \cup [R^n - \underline{\underline{D}}_0^n(0, r)])(V \cap \underline{\underline{D}}^n(0, r), W \cap \underline{\underline{D}}^n(0, r)) < \infty.$$

(4) For each $c \in R^+$ we define the boundary of $\underline{\underline{P}}$ regularity c of (V, W) on (A, B), written $\underline{\underline{BP}}^c(R^n, A, B)(V, W)$, to be the intersection of all those closed sets $D \subset R^n$ having the property that for each $r \in R_0^+$,

$$\underline{\underline{P}}^-(R^n, A, B; D \cup [R^n - \underline{\underline{D}}_0^n(0, r)])(V \cap \underline{\underline{D}}^n(0, r), W \cap \underline{\underline{D}}^n(0, r)) \le c.$$

In a similar way one defines the boundary of $\underline{\underline{Q}}$ regularity c of (V, W) on (A, B), written $\underline{\underline{BQ}}^c(R^n, A, B)$. Clearly

$$\underline{\underline{P}}^-(R^n, A, B; \underline{\underline{BP}}^c(R^n, A, B)(V, W))(V, W) \le c,$$
$$\underline{\underline{Q}}^-(R^n, A, B; \underline{\underline{BQ}}^c(R^n, A, B)(V, W))(V, W) \le c,$$

and

$$\underline{\underline{BQ}}^c(R^n, A, B)(V, W) = \operatorname{clos} \underline{\underline{BQ}}^c(R^n, A, B)(V, W) \subset \underline{\underline{BP}}^c(R^n, A, B)(V, W)$$
$$= \operatorname{clos} \underline{\underline{BP}}^c(R^n, A, B)(V, W) \subset R^n.$$

6.6 PROPOSITION. Let $1 \le k \le n$ and $q \ge 1$ be integers and $f_1, f_2, \ldots, f_q : R^n \longrightarrow \{t : 0 \le t \le 1\}$ be a partition of unity on R^n of class [illegible] with $\operatorname{Lip}(Df_i) < \infty$ for each i. Let $g \in \underline{\underline{X}}(R^n)$ and $(V, W) \in \underline{\underline{V}}_k(R^n)$ with $\underline{\underline{I}}(V, f_i \cdot g) > 0$ for each i. Then

$$\underline{\underline{P}}(V, W; g) \leq \max\{\underline{\underline{P}}(V, W; f_i \cdot g) : i = 1, 2, \ldots, q\},$$

and

$$\underline{\underline{Q}}(V, W; g) \leq \max\{\underline{\underline{Q}}(V, W; f_i \cdot g) : i = 1, 2, \ldots, q\}.$$

PROOF. It is sufficient to consider the case $q = 2$. The first conclusion is immediate in case $\underline{\underline{P}}(V, W; g) = 0$. For $\underline{\underline{P}}(V, W; g) > 0$ we have

$$\begin{aligned}
\underline{\underline{P}}(V, W; g) &= [\underline{\underline{T}}(V, g)]^{-1}[|\underline{\underline{S}}(V, g)| - \underline{\underline{T}}(W, g)] \\
&= [\underline{\underline{T}}(V, f_1 g) + \underline{\underline{T}}(V, f_2 g)]^{-1} \cdot \\
&\quad [|\underline{\underline{S}}(V, f_1 g) + \underline{\underline{S}}(V, f_2 g)| - \underline{\underline{T}}(W, f_1 g) - \underline{\underline{T}}(W, f_2 g)] \\
&\leq [\underline{\underline{T}}(V, f_1 g) + \underline{\underline{T}}(V, f_2 g)]^{-1} \cdot \\
&\quad ([|\underline{\underline{S}}(V, f_1 g)| - \underline{\underline{T}}(W, f_1 g)] + [|\underline{\underline{S}}(V, f_2 g)| - \underline{\underline{T}}(W, f_2 g)]) \\
&\leq \max\{[\underline{\underline{T}}(V, f_1 g)]^{-1}[|\underline{\underline{S}}(V, f_1 g)| - \underline{\underline{T}}(W, f_1 g)], \\
&\quad [\underline{\underline{T}}(V, f_2 g)]^{-1}[|\underline{\underline{S}}(V, f_2 g)| - \underline{\underline{T}}(W, f_2 g)]\}
\end{aligned}$$

The second conclusion follows by a similar argument.

6.7 PROPOSITION. Let $1 \leq k \leq n$ be integers and $(V, W) \in \underline{\underline{GV}}_k(R^n, R^n)$ with $\underline{\underline{P}}(R^n)(V, W) < \infty$ (resp. $\underline{\underline{Q}}(R^n)(V, W) < \infty$). Then for each $\varepsilon > 0$ there exists a unit vector $u \in R^n$ and a class ∞ function $f : R^n \longrightarrow R^+$ with $\operatorname{diam}(\operatorname{spt}(f)) < \varepsilon$ such that $\underline{\underline{P}}(V, W; fu) \geq \underline{\underline{P}}(R^n)(V, W) - \varepsilon$ (resp. $\underline{\underline{Q}}(V, W; fu) \geq \underline{\underline{Q}}(R^n)(V, W) - \varepsilon$).

PROOF.

Suppose $\underline{\underline{P}}(R^n)(V, W) < \infty$ and $\varepsilon > 0$. We will show the existence of u and f as above so that $\underline{\underline{P}}(V, W; fu) \geq \underline{\underline{P}}(R^n)(V, W) - \varepsilon$. In case $\underline{\underline{P}}(R^n)(V, W) = 0$ this result is immediate. We assume therefore that $0 < \underline{\underline{P}}(R^n)(V, W) < \infty$ and choose $g \in \underline{\underline{X}}(R^n)$ such that $\operatorname{Lip}(g) \leq 1$ and $\underline{\underline{P}}(V, W; g) \geq \underline{\underline{P}}(R^n)(V, W) - (1/3)\varepsilon$.

For each $\delta = 1^{-1}, 2^{-1}, 3^{-1}, \ldots$ let $\{f(\delta, i) : i = 1, 2, 3, \ldots\}$ be a class ∞ partition of unity on R^n such that $\operatorname{diam}(\operatorname{spt}[f(\delta, i)]) < \delta$ for each $i = 1, 2, 3, \ldots$. Now for each $\delta = 1^{-1}, 2^{-1}, 3^{-1}, \ldots$ and $i = 1, 2, 3, \ldots$, choose $p(\delta, i) \in \operatorname{spt}(f(\delta, i))$ and define $g(\delta, i) = f(\delta, i)g(p(\delta, i)) \in \underline{\underline{X}}(R^n)$.

One can write, of course, for each δ

$$g = \sum_i g(\delta, i) + [g - \sum_i g(\delta, i)].$$

By the continuity of g we have

$$\lim_{\delta \to 0^+} \sup\{|g(x) - \sum_i g(\delta, i)(x)| : x \in R^n\} = 0$$

which implies

$$\lim_{\delta \to 0^+} \underline{\underline{T}}(V, g - \sum_i g(\delta, i)) = \lim_{\delta \to 0^+} \underline{\underline{T}}(W, g - \sum_i g(\delta, i)) = 0$$

which, since $\underline{\underline{P}}(R^n)(V, W) < \infty$, implies

$$\lim_{\delta \to 0^+} \underline{\underline{S}}(V, g - \sum_i g(\delta, i)) = 0.$$

Thus for all sufficiently small values of δ,

$$\underline{\underline{P}}(V, W; \sum_i g(\delta, i)) \geq \underline{\underline{P}}(R^n)(V, W) - (2/3)\varepsilon.$$

One observes that for each $\eta = 1^{-1}, 2^{-1}, 3^{-1}, \ldots,$ each $i = 1, 2, 3, \ldots,$ and each $x \in R^n$,

$$|g(\delta, i)(x) - f(\delta, i)(x) \qquad {}_i g(\delta, i)(x)| \leq 2\delta f(\delta, i)(x)$$

because $\mathrm{Lip}(g) \leq 1$. Hence

$$\lim_{\delta \to 0^+} \sum_i \underline{\underline{T}}(V, g(\delta, i)) = \lim_{\delta \to 0^+} \underline{\underline{T}}(V, \sum_i g(\delta, i)),$$

$$\lim_{\delta \to 0^+} \sum_i \underline{\underline{T}}(W, g(\delta, i)) = \lim_{\delta \to 0^+} \underline{\underline{T}}(W, \sum_i g(\delta, i)).$$

Thus for all sufficiently small δ, using the linearity of $\underline{\underline{S}}(V, \cdot)$, we have $[\sum_i \underline{\underline{T}}(V, g(\delta, i))]^{-1}[\sum_i \underline{\underline{S}}(V, g(\delta, i)) - \sum_i \underline{\underline{T}}(W, g(\delta, i)] \geq \underline{\underline{P}}(R^n)(V, W) - \varepsilon$. Thus we can choose $\delta_0 < \varepsilon$ and i_0 such that $[\underline{\underline{T}}(V, g(\delta_0, i_0))]^{-1}[\underline{\underline{S}}(V, g(\delta_0, i_0) - \underline{\underline{T}}(W, g(\delta_0, i_0))] \geq \underline{\underline{P}}(R^n)(V, W) - \varepsilon$. One, of course, takes $f = f(\delta_0, i_0)$ and $u = |g(p(\delta_0, i_0))|^{-1} g(p(\delta_0, i_0))$ which gives

$$\underline{\underline{P}}(V, W; fu) \geq \underline{\underline{P}}(R^n)(V, W) - \varepsilon$$

which was to be shown.

The proof of the other conclusion of the proposition follows by virtually identical arguments.

6.8 PROPOSITION. Let $1 \leq k \leq n$ be integers and $A \subset R^n$ be a closed submanifold of R^n with boundary B. Let C be a closed submanifold of A of class 3 with boundary D. Let $(V, W) \in \underline{\underline{GV}}_k(C, C)$ with $\underline{\underline{Q}}(R^n, A, C)(V, W) < \infty$. Then V lies intrinsically on C and $V \cap D$ lies intrinsically on D.

PROOF. We will prove the more difficult assertion; namely, that $V \cap D$ lies intrinsically on D. Choose $g \in \underline{\underline{X}}(R^n)$ so that for each x in some neighborhood U of D, $g(x) = x - N(x)$ where $N(x)$ is the nearest point on D to x. If $V \cap D$ does not lie intrinsically on D then $\underline{\underline{S}}(V \cap D, g) > 0$. We now modify g to obtain $g_0 \in \underline{\underline{X}}(R^n, A, C)$ for which $g_0(x) = g(x) = 0$ and $Dg_0(x, \cdot) = Dg(x, \cdot)$ for each $x \in D$, which implies $\underline{\underline{S}}(V \cap D, g_0) = \underline{\underline{S}}(V \cap D, g)$. It is not difficult to establish the existence of a sequence $g_1, g_2, g_3, \ldots \in \underline{\underline{X}}(R^n, A, C)$ such that

(1) $|g_i(x)| \leq 1$ and $\sup\{Dg_i(x, v) : v \in S^{n-1}\} \leq 2$ for each $i = 1, 2, 3, \ldots$;

(2) $g_i | U_i = g_0 | U_i$ for some neighborhood U_i of D for each $i = 1, 2, 3, \ldots$;

(3) $\lim_i g_i(x) = 0$ for each $x \in R^n - D$; and

(4) $\lim_i Dg_i(x, \cdot) = 0$ for each $x \in R^n - D$.

By 6.4 (4a) and Lebesgue's Theorem on bounded convergence we have that

$$\lim_i \underline{\underline{S}}(V \cap (R^n - D), g_i) = \lim_i \underline{\underline{T}}(W, g_i) = \lim_i \underline{\underline{W}}V(|g_i|) = 0$$

If $V \cap D$ does not lie intrinsically on D, then

$$\lim_i \underline{\underline{S}}(V \cap D, g_i) = \underline{\underline{S}}(V \cap D, g_0) > 0$$

and hence $\underline{\underline{Q}}(R^n, A, C)(V, W) = \infty$ contradicting our hypothesis.

6.9 LEMMA. Let $1 \leq k \leq n$ be integers, $(V, W) \in \underline{\underline{GV}}_k(R^n, R^n)$, and $g, h \in \underline{\underline{X}}(R^n)$ with $\underline{\underline{T}}(V, g) > 0$, $\underline{\underline{T}}(V, h) > 0$, and $\underline{\underline{T}}(V, g+h) > 0$. Then

(1) $\underline{\underline{T}}(V; g+h) \le \underline{\underline{T}}(V, g) + \underline{\underline{T}}(V, h)$;

(2) $\underline{\underline{W}}V(|g+h|) \le \underline{\underline{W}}V(|g|) + \underline{\underline{W}}V(|h|)$;

(3) $$[\underline{\underline{T}}(V, g+h)]^{-1}[\underline{\underline{S}}(V, g+h) - \underline{\underline{T}}(W, g+h)]$$
$$\ge [\underline{\underline{T}}(V, g) + \underline{\underline{T}}(V, h)]^{-1}[\underline{\underline{S}}(V, g) - \underline{\underline{T}}(W, g) + \underline{\underline{S}}(V, h) - \underline{\underline{T}}(W, h)]$$
$$\ge [\underline{\underline{T}}(V, f)]^{-1}[\underline{\underline{S}}(V, f) - \underline{\underline{T}}(W, f)]$$
<u>for some</u> $f \in \{g, h\}$;

(4) $$[\underline{\underline{W}}V(|g+h|)]^{-1}[\underline{\underline{S}}(V, g+h) - \underline{\underline{T}}(W, g+h)]$$
$$\ge [\underline{\underline{W}}V(|f|)]^{-1}[\underline{\underline{S}}(V, f) - \underline{\underline{T}}(W, f)]$$
<u>for some</u> $f \in \{g, h\}$;

(5) $[\underline{\underline{T}}(V, g)]^{-1}|\underline{\underline{S}}(V, g) - \underline{\underline{T}}(W, g)| \ge [\underline{\underline{W}}V(|g|)]^{-1}|\underline{\underline{S}}(V, g) - \underline{\underline{T}}(W, g)|$;

(6) $\underline{\underline{P}}^a(V, W; g+h) > 0$ <u>whenever both</u> $\underline{\underline{P}}^a(V, W; g) > 0$ <u>and</u> $\underline{\underline{P}}^a(V, W; h) > 0$. <u>Here</u> $\underline{\underline{P}}^a$ <u>denotes either</u> $\underline{\underline{P}}^+$ <u>or</u> $\underline{\underline{P}}^-$; <u>and</u>

(7) $\underline{\underline{Q}}^a(V, W; g+h) > 0$ <u>whenever both</u> $\underline{\underline{Q}}^a(V, W; g) > 0$ <u>and</u> $\underline{\underline{Q}}^a(V, W; h) > 0$. <u>Here</u> $\underline{\underline{Q}}^a$ <u>denotes either</u> $\underline{\underline{Q}}^+$ or $\underline{\underline{Q}}^-$.

6.10 LEMMA. <u>Let</u> $1 \le k \le m \le n$ <u>be integers and</u> $f^{m+1}, f^{m+2}, \ldots, f^n : R^m \to R$ <u>be of class</u> 3 <u>with</u> $f^i(0) = 0$ <u>and</u> $(\partial f^i/\partial x^j)(0) = 0$ <u>for each</u> $i = m+1, m+2, \ldots, n$ <u>and each</u> $j = 1, 2, \ldots, m$. <u>Let</u>

$$A = R^n \cap \{x : x^i = f^i(x^1, x^2, \ldots, x^m) \text{ for each } i = m+1, m+2, \ldots, n\}$$

<u>and</u> $g \in \underline{\underline{X}}(R^n)$ <u>such that</u> $g(0) = (0, 0, \ldots, 0, 1)$ <u>and for each</u> $x \in A$, $g(x)$ <u>is perpendicular to the</u> m-<u>plane tangent to</u> A <u>at</u> x. <u>Then</u>

(1) $$\underline{\underline{S}}(\underline{\underline{v}}(0, (\partial/\partial x^1)\wedge(\partial/\partial x^2)\wedge\ldots\wedge(\partial/\partial x^k)), g) = -|g(0)| \sum_{i=1}^{k} (\partial^2 f^n/(\partial x^i)^2)(0),$$

(2) $$\underline{\underline{S}}(\underline{\underline{v}}(0, (\partial/\partial x^1)\wedge(\partial/\partial x^2)\wedge\ldots\wedge(\partial/\partial x^{k-1})\wedge(a(\partial/\partial x^k) + b(\partial/\partial x^n)), g)$$
$$= -|g(0)| \sum_{i=1}^{k-1} (\partial^2 f^n/(\partial x^i)^2)(0) - a^2|g(0)|(\partial^2 f^n/(\partial x^k)^2)(0)$$
$$+ b^2(\partial g^n/\partial x^n)(0) + ab[(\partial g^k/\partial x^n)(0) + (\partial g^n/\partial x^k)(0)]$$

whenever $a, b \in R^+$ with $a^2 + b^2 = 1$.

PROOF. For $i = 1, 2, \ldots, m$ and $g = (g^1, g^2, \ldots, g^n)$ we define $\theta_i : R^n \longrightarrow R$ by setting for each $x \in R^n$,

$$\theta_i(x) = [(\partial/\partial x^i) + \sum_{j=m+1}^{n} (\partial f^j/\partial x^i)(x)(\partial/\partial x^j)]\cdot[\sum_{j=1}^{n} g^j(x)(\partial/\partial x^j)]$$

$$= g^i(x) + \sum_{j=m+1}^{n} g^j(x)(\partial f^j/\partial x^i)(x).$$

Since for each $x \in A$, $(\partial/\partial x^i) + \sum_{j=m+1}^{n} (\partial f^i/\partial x^j)(\partial/\partial x^j)$ is tangent to A at x, and $g(x) = \sum_{j=1}^{n} g^j(x)(\partial/\partial x^j)$ is perpendicular to A at x, $\theta_i | A \equiv 0$. Since for each $i = 1, 2, \ldots, m$, $(\partial/\partial x^i)$ is tangent to A at 0,

$$0 = (\partial\theta_i/\partial x^i)(0)$$

$$= (\partial g^i/\partial x^i)(0) + \sum_{j=m+1}^{m} (\partial g^j/\partial x^i)(0)(\partial f^j/\partial x^i)(0) + g^j(0)(\partial^2 f^j/\partial x^i)^2)(0)$$

$$= (\partial g^i/\partial x^i)(0) + g^n(0)(\partial^2 f^n/(\partial x^i)^2)(0),$$

which implies that

$$(\partial g^i/\partial x^i)(0) = -g^n(0)(\partial^2 f^n/(\partial x^i)^2)(0) = -|g(0)|(\partial^2 f^n/(\partial x^i)^2)(0).$$

6.4 (4b) implies

$$\underline{\underline{S}}(\underline{\underline{v}}(0, (\partial/\partial x^1)\wedge(\partial/\partial x^2)\wedge\ldots\wedge(\partial/\partial x^k)), g) = \sum_{i=1}^{k} (\partial g^i/\partial x^i)(0)$$

$$= -|g(0)| \sum_{i=1}^{k} (\partial^2 f^n/(\partial x^i)^2)(0)$$

which is (1). (2) follows by similar arguments based on 6.4 (4b).

6.11 DEFINITIONS. Let $0 \leq q \leq p \leq n$ be integers, $A \subset R^n$ be a closed p-dimensional submanifold of R^n of class 3 with boundary B, and $C \subset A$ be a closed q-dimensional submanifold of A of class 3 with boundary D.

Let $y \in B$ [resp., let $z \in C-D$]. Then for some set of orthonormal coordinate functions $(x^1, x^2, \ldots, x^n)$ on R^n the manifold $B - y = \{(x-y) : x \in B\}$ [resp., the manifold $C-Z = \{x-z : x \in C\}$] can be represented in a neighborhood of 0 as a non-parametric surface.

(*) $(x^1, \ldots, x^{p-1}) \longrightarrow (x^1, \ldots, x^{p-1}, f^p(x^1, \ldots, x^{p-1}), \ldots, f^n(x^1, \ldots, x^{p-1}))$

[resp. (**) $(x^1, \ldots, x^q) \longrightarrow (x^1, \ldots, x^q, g^{q+1}(x^1, \ldots, x^q), \ldots, g^n(x^1, \ldots, x^q))$]

where $f^p, \ldots, f^n : R^{p-1} \longrightarrow R$ [resp. $g^{q+1}, \ldots, g^n : R^q \longrightarrow R$] are functions of class 3 such that the vector $e_n = (0, \ldots, 0, 1)$ is contained in the tangent half-space of $A - y$ at 0 [resp., in the tangent space of $A - z$ at 0] and $f^i(0) = 0$ and $(\partial f^i / \partial x^j)(0) = 0$ for each $i = p, p+1, \ldots, n$ and $j = 1, 2, \ldots, p-1$ [resp. $g^i(0) = 0$ and $(\partial g^i / \partial x^j)(0) = 0$ for each $i = q+1, q+2, \ldots, n$ and $j = 1, 2, \ldots, q$].

We define for each integer k, $1 \le k \le p-1$ [resp. $1 \le k \le q$],

$$\Delta_k(B, A; y, (x^1, \ldots, x^n)) = -\sum_{i=1}^{k} \frac{\partial^2 f^n}{(\partial x^i)^2}(0)$$

[resp. $\Delta_k(C, A; z, (x^1, \ldots, x^n)) = -\sum_{i=1}^{k} \frac{\partial^2 g^n}{(\partial x^i)^2}(0)$]

$\Delta_k^+(B, A; y) = -\inf\{\Delta_k(B, A; y, (x^1, \ldots, x^n)) : (x^1, \ldots, x^n)$ is an orthonormal set of coordinate functions for R^n for which $B - y$ can be represented as a non-parametric surface as in (*) in some neighborhood of 0 with e_n in the tangent half space of $A - y$ at 0.$\}$

$\Delta_k^-(B, A; y) = \sup\{\Delta_k(B, A; y, (x^1, \ldots, x^n)) : (x^1, \ldots, x^n)$ is as in the preceding definition$\}$

[resp. $\Delta_k(C, A; z) = \sup\{\Delta_k(C, A; y, (x^1, \ldots, x^n)) : (x^1, \ldots, x^n)$ is an orthonormal set of coordinate functions for R^n for which $C - z$ can be represented as a non-parametric surface as in (**) in some neighborhood of 0 with e_n in the tangent space of $A - z$ at 0$\}$]

$$\Delta_k^+(B, A) = \max\{\sup\{\Delta_k^+(B, A; y) : y \in B\}, 0\}$$

$$\Delta_k^-(B, A) = \max\{\sup\{\Delta_k^-(B, A; y) : y \in B\}, 0\}$$

$$\Delta_k(B, A) = \max\{\Delta_k^+(B, A), \Delta_k^-(B, A)\}$$

[resp. $\Delta_k(C, A) = \sup\{\Delta_k(C, A; z) : z \in C-D\}$]

For $k > p-1$ [resp. $k > q$] we define

$$\Delta_k(B, A) = \Delta_k^+(B, A) = \Delta_k^-(B, A) = 0$$

[resp. $\Delta_k(C, A) = 0$].

6.12 THEOREM. *Let* $1 \le k \le n$ *be integers and suppose*

(i) $A \subset R^n$ *is a closed submanifold of* R^n *of class* 3 *with boundary* B;

(ii) $C \subset A$ *is a closed submanifold of* A *of class* 3 *with boundary* D;

(iii) $E \subset C$ *is a closed submanifold of* C *of class* 3 *without boundary*;

(iv) $F \subset R^n$ *is a closed subset of* R^n; *and*

(v) $(V, W) \in \mathbf{GV}_k(C, C)$.

Then

(1) $$\mathbf{P}(R^n, A)(V, W) \le \mathbf{P}(R^n, A, C)(V, W) + \Delta_k(C, A);$$

(2) $$\mathbf{P}(R^n, A)(V, W) \le \mathbf{P}^-(R^n, A, C)(V, W) + \Delta_k^-(D, C) + \Delta_k(C, A)$$

provided W *lies intrinsically on* C;

(3) $$\mathbf{P}(R^n, A)(V, W) \le \mathbf{P}^+(R^n, A, C)(V, W) + \Delta_k^+(D, C) + \Delta_k(C, A)$$

provided W *lies intrinsically on* C;

(4) $$\mathbf{P}(R^n, C)(V, W) \le \mathbf{P}(R^n, C, D)(V, W) + \Delta_k^+(D, C)$$

provided W *lies intrinsically on* C;

(5) $$\mathbf{P}(R^n, C)(V, W) \le \mathbf{P}(R^n, C, E)(V, W) + \Delta_k(E, C)$$

provided W *lies intrinsically on* C *and* $W \cap E$ *lies intrinsically on* E.

(6) *Statements corresponding to* (1), (2), (3), (4), (5) *are true with* $\mathbf{Q}$ *repl*[illegible] *and without the additional hypothesis on* W.

(7) Statements corresponding to (1), (2), (3), (4), (5), (6) are true with $\underline{\underline{P}}(R^n, \cdot, \cdot; F)$ replacing $\underline{\underline{P}}(R^n, \cdot, \cdot)$ and $\underline{\underline{Q}}(R^n, \cdot, \cdot; F)$ replacing $\underline{\underline{P}}(R^n, \cdot, \cdot)$, etc.

PROOF. We assume without loss of generality that $\Delta_k(C, A) < \infty$, $\Delta_k(D, C) < \infty$, and $\Delta_k(E, A) < \infty$ since all our arguments will take place in a bounded neighborhood of $\mathrm{spt}(V) \cup \mathrm{spt}(W)$.

Part 1. Let $g \in \underline{\underline{X}}(R^n, A)$. Then in some neighborhood of C we can write $g = g' + g''$ where for each $x \in C-D$, $g'(x)$ is perpendicular to C at x and $g''(x)$ is parallel with C at x. If V lies intrinsically on C, we have by 6.4 (4b), 6.10, and the continuity of $\underline{\underline{S}}$ and $\underline{\underline{T}}$ that

$$\underline{\underline{S}}(V, g') \leq \Delta_k(C, A) \cdot \underline{\underline{T}}(V, g')$$

Thus

$$\begin{aligned}\underline{\underline{P}}(V, W; g) &= [\underline{\underline{T}}(V, g)]^{-1}[|\underline{\underline{S}}(V, g)| - \underline{\underline{T}}(W, g)] \\ &\leq [\underline{\underline{T}}(V, g'')]^{-1} \cdot [|\underline{\underline{S}}(V, g'')| - \underline{\underline{T}}(W, g'')] + \Delta_k(C, A)\end{aligned}$$

since $\underline{\underline{T}}(W, g'') \leq \underline{\underline{T}}(W, g)$, $\underline{\underline{T}}(V, g') \leq \underline{\underline{T}}(V, g)$, and $\underline{\underline{T}}(V, g'') \leq \underline{\underline{T}}(V, g)$ provided V and W lie intrinsically on C. To establish conclusion (1) we must now show that

$$\underline{\underline{P}}(V, W, g'') \leq \underline{\underline{P}}(R^n, A, C)(V, W)$$

Observe that if $\underline{\underline{P}}(R^n, A, C)(V, W) = \infty$, conclusion (1) is immediate. We therefore assume that $\underline{\underline{P}}(R^n, A, C)(V, W) < \infty$ which, of course, implies that V lies intrinsically on C.

Part 2. Let $g \in \underline{\underline{X}}(R^n, A)$ be chosen so that for each $x \in C-D$, $g(x)$ is parallel to C at x, $1 \leq \underline{\underline{T}}(V, g) \leq 2$, and $\underline{\underline{P}}(V, W; g) > 0$. We write $g = g' + g''$, $g' \in \underline{\underline{X}}(R^n, A)$, $g'' \in \underline{\underline{X}}(R^n, A, C)$ such that $g'(x)$ is perpendicular to D for each $x \in D$. There are several, not necessarily disjoint, possibilities:

Possibility (a). $V \cap C-D = 0$ and $\underline{\underline{T}}(V, g') = 0$. Since V lies intrinsically on D and $\underline{\underline{T}}(V, g') = 0$, we have that $\underline{\underline{S}}(V, g') = 0$. Thus $\underline{\underline{P}}(V, W; g) \leq \underline{\underline{P}}(V, W; g'') \leq \underline{\underline{P}}(R^n, A, C)(V, W)$.

Possibility (b). $V \cap C-D \neq 0$. Here we can assume without loss of generality that $\underline{\underline{T}}(V \cap [C-D], g) > 0$, by making arbitrarily small modifications of g if necessary.

Possibility (c). $\underline{\underline{T}}(V, g') > 0$. Here we can assume without loss of generality that

$$V \cap B \cap \{x : g'(x) \text{ points into } C\} \neq 0$$

by replacing g with $-g$ if necessary.

Let $\varepsilon > 0$ and choose $h \in \underline{\underline{X}}(R^n, A, C)$ with the following properties:

(i) For each $x \in D \cap \operatorname{spt}(g)$, $|h(x)| > 0$ and $h(x)$ is perpendicular to D at x and points into A;

(ii) $\underline{\underline{W}}V(|h|) \leq \min\{2^{-1}\varepsilon \underline{\underline{T}}(V \cap C-D, g'), \varepsilon \underline{\underline{W}}(V \cap D \cap \{x, g'(x) \text{ points into } C\})\}$;

(iii) $|\underline{\underline{S}}(V, h)| + \underline{\underline{W}}W(|h|) \leq \varepsilon^2[|\underline{\underline{S}}(V, g)| - \underline{\underline{T}}(W, g)]$.

One now chooses functions $\alpha, \beta : R^n \longrightarrow \{t : 0 \leq t \leq 1\}$ of class ∞ so that

(i) For each $x \in R^n$, $\alpha(x) + \beta(x) = 1$;

(ii) $\alpha(x) = 1$ for each $x \in D$ for which $g(x) + h(x)$ points out of C or is tangent to D;

(iii) $\beta(x) = 1$ for each $x \in D$ for which $g(x)$ points into C or is tangent to D; and

(iv) If possibility (b) occurs, then

$$\underline{\underline{T}}(V \cap (C-D) \cap \{x : \beta(x) = 1\}, g) \geq 2^{-1}\underline{\underline{T}}(V \cap (C-D), g).$$

By 6.6 we have

$$\underline{\underline{P}}(V, W; g) \leq \max\{\underline{\underline{P}}(V, W; \alpha \wedge g), \underline{\underline{P}}(V, W; \beta \wedge g)\}$$

We now examine several cases:

Case 1. Possibility (a) holds. Here we already know that $\underline{\underline{P}}(V, W; g) \leq \underline{\underline{P}}(R^n, A, C)(V, W)$ and there is nothing more to prove.

Case 2. Possibility (b) or possibility (c) holds and $\underline{\underline{P}}(V, W; g) \leq \underline{\underline{P}}(V, W; \alpha \wedge g)$. Here $-(\alpha \wedge g) \in \underline{\underline{X}}(R^n, A, C)$, giving

$$\underline{\underline{P}}(V, W; g) \leq \underline{\underline{P}}(V, W; \alpha \wedge g) = \underline{\underline{P}}(V, W; -\alpha \wedge g) \leq \underline{\underline{P}}(R^n, A, C)(V, W)$$

and we are done.

Case 3. Possibility (b) or possibility (c) holds, $\mathbf{P}(V, W; g) \leq \mathbf{P}(V, W; \beta \wedge g)$, and $|\mathbf{S}(V, \beta \wedge g)| - \mathbf{T}(W, \beta \wedge g) < \varepsilon[|\mathbf{S}(V, g)| - \mathbf{T}(W, g)]$. In this case we have

$$
\begin{aligned}
&(|\mathbf{S}(V, g)| - \mathbf{T}(W, g)) - (|\mathbf{S}(V, \alpha \wedge g)| - \mathbf{T}(W, \alpha \wedge g)) \\
&\quad = |\mathbf{S}(V, \alpha\wedge g) + \mathbf{S}(V, \beta\wedge g)| - |\mathbf{S}(V, \alpha\wedge g)| - |\mathbf{S}(V, \beta\wedge g)| + (|\mathbf{S}(V, \beta\wedge g)| - \mathbf{T}(V, \beta\wedge g)) \\
&\quad \leq \varepsilon(|\mathbf{S}(V, g)| - \mathbf{T}(W, g)),
\end{aligned}
$$

which implies that

$$\mathbf{P}(V, W; g) \leq (1-\varepsilon)^{-1}\mathbf{P}(V, W; -\alpha \wedge g) \leq \mathbf{P}(R^n, A, C)(V, W).$$

Case 4. Possibility (b) or possibility (c) holds, $\mathbf{P}(V, W; g) \leq \mathbf{P}(V, W; \beta \wedge g)$, and $|\mathbf{S}(V, \beta \wedge g)| \geq \mathbf{T}(W, \beta \wedge g) \geq \varepsilon[|\mathbf{S}(V, g)| - \mathbf{T}(W, g)]$. In this case we have

$$
\begin{aligned}
|\mathbf{S}(V, \beta\wedge g+h)| - \mathbf{T}(W, \beta \wedge g+h) &\geq |(\mathbf{S}(V, \beta\wedge g)| - \mathbf{T}(W, \beta\wedge g)) - (|\mathbf{S}(V, h)| + \mathbf{T}(W, h)) \\
&\geq (1-\varepsilon)(|\mathbf{S}(V, \beta \wedge g)| - \mathbf{T}(W, \beta \wedge g))
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{T}(V, \beta \wedge g+h) &\leq \mathbf{T}(V, \beta \wedge g) + \mathbf{W}V(|h|) \\
&\leq \mathbf{T}(V, \beta \wedge g) + \varepsilon \mathbf{T}(V, \beta \wedge g) \\
&= (1+\varepsilon)\mathbf{T}(V, \beta \wedge g) .
\end{aligned}
$$

Thus

$$\mathbf{P}(V, W; \beta\wedge g) \leq (1-\varepsilon)^{-1}(1+\varepsilon)\mathbf{P}(V, W; \beta\wedge g+h) \leq (1-\varepsilon)^{-1}(1+\varepsilon)\mathbf{P}(R^n, A, C)(V, W).$$

The proof of conclusion (1) is complete since the choice of ε and g was arbitrary.

Part 3. We now will prove conclusion (2). From part (1) we have

$$\mathbf{P}(R^n, A)(V, W) \leq \mathbf{P}(R^n, A, C)(V, W) + \Delta_k(C, A)$$

We will have verified conclusion (2) if we can show that

$$\mathbf{P}(R^n, A, C)(V, W) \leq \mathbf{P}^-(R^n, A, C)(V, W) + \Delta_k^-(D, C)$$

Let $g \in \mathbf{X}(R^n, A, C)$. In case $\mathbf{S}(V, g) \leq 0$, then

$$\mathbf{P}(V, W; g) = \mathbf{P}^-(V, W; g) \leq \mathbf{P}^-(R^n, A, C)(V, W).$$

We will be done if we can show that if $\underline{\underline{S}}(V, g) > 0$ then

$$\underline{\underline{P}}(V, W; g) \leq \underline{\underline{P}}^{-}(R^n, A, C)(V, W) + \Delta_k^{-}(D, C)$$

If $V \cap (C-D) = \emptyset$, then this result follows from the definitions and 6.9. Suppose then $\underline{\underline{S}}(V, g) > 0$. We can suppose without loss of generality that $\underline{\underline{T}}(V \cap C-D; g) > 0$, making an arbitrarily small modification of g if necessary. For each sufficiently small $\delta > 0$ let $f_\delta : R^n \longrightarrow \{t : 0 \leq t \leq 2\}$ of class 2 have the following properties:

(i) $f_\delta(x) = f_\delta(y)$ whenever $\operatorname{dist}(x, D) = \operatorname{dist}(y, D)$;

(ii) $f_\delta(x) = 1$ if $x \in D$;

(iii) $f_\delta(x) = 0$ if $\operatorname{dist}(x, D) \geq \delta$;

(iv) $|\operatorname{grad}(f_\delta(x))| \geq (2\delta)^{-1}$ for each $x \in C$ with $\operatorname{dist}(x, D) \leq 2^{-1}\delta$;

and

(v) $\underline{\underline{W}}V\{x : f_\delta(x) > 0$ and $|\operatorname{grad}(f_\delta(x))| < (2\delta)^{-1}\} < \delta \underline{\underline{W}}V\{x : f_\delta(x) > 0\}$.

We write $g = g' + g''$, $g' \in \underline{\underline{X}}(R^n, A)$, $g'' \in \underline{\underline{X}}(R^n, A, C)$, where $g'(x)$ is perpendicular to D at x and $g''(x)$ is parallel with D at x for each $x \in D$. Let $g''' \in \underline{\underline{X}}(R^n, A)$ have the following properties:

(i) $g'''(x) = g'(x)$ for each $x \in D$;

(ii) there exists a neighborhood U of $D \cap \operatorname{spt}(g)$ so that for each $x \in U$, $g'''(x)$ is perpendicular to the manifold $C \cap \{y : \operatorname{dist}(y, D) = \operatorname{dist}(x, D)\}$.

Observe that

$$\limsup_{\delta \longrightarrow 0^+} \underline{\underline{S}}(V, f_\delta \wedge g''') \leq \Delta_k^{-}(D, C) \cdot \underline{\underline{T}}(V \cap D, g''')$$

which is a consequence of straightforward continuity arguments based on 6.10 (2).

Let $\varepsilon > 0$ and choose $\delta > 0$ so that

$$\underline{\underline{S}}(V, f_\delta \wedge g''') \leq \Delta_k^{-}(D, C) \cdot \underline{\underline{T}}(V \cap D, g''') + \varepsilon,$$

$$\underline{\underline{T}}(W, f_\delta \wedge g''') - \underline{\underline{T}}(W, g) \leq \varepsilon, \text{ and}$$

$$0 < \underline{\underline{T}}(V, f_\delta \wedge g''' - g) \leq \underline{\underline{T}}(V, g) + \varepsilon.$$

Observe that $f_\delta \wedge g''' - g \in \underline{\underline{X}}(R^n, A, C)$ and that

$$[\mathbf{T}(V, f_\delta \wedge g''' - g)]^{-1}[-\mathbf{S}(V, f_\delta \wedge g''' - g) - \mathbf{T}(W, f_\delta \wedge g''' - g)]$$

$$\geq [\mathbf{T}(V, g) - \varepsilon]^{-1}[\mathbf{S}(V, g) - \Delta_k^-(D, C) \cdot \mathbf{T}(V \cap D, g''') - \mathbf{T}(W, g) - 2\varepsilon]$$

$$\geq [\mathbf{T}(V, g) - \varepsilon]^{-1}[\mathbf{S}(V, g) - \mathbf{T}(W, g) - \Delta_k^-(D, C)\mathbf{T}(V, g) - 2\varepsilon]$$

Since ε is arbitrary we have

$$\mathbf{P}^-(R^n, A, C) \geq \mathbf{P}(V, W; g) - \Delta_k^-(D, C)$$

which establishes conclusion (2). Conclusion (3) is established by essentially identical arguments. Conclusions (4), (5), (6) are left to the reader.

6.13 COROLLARY. Let $1 \leq k \leq m \leq n$ be integers and $A \subset R^n$ be a compact m-dimensional submanifold of R^n of class 3 with boundary B. Let $(V, W) \in \mathbf{GV}_k(A, A)$. Then

(1) The following statements are equivalent:

(a) $\mathbf{P}(R^n)(V, W) < \infty$,

(b) $\mathbf{P}(R^n, A)(V, W) < \infty$,

(c) $\mathbf{P}(R^n, A, C)(V, W) < \infty$,

(d) $\mathbf{P}^+(R^n)(V, W) < \infty$,

(e) $\mathbf{P}^-(R^n)(V, W) < \infty$,

etc.; and

(2) The following statements are equivalent:

(a) $\mathbf{Q}(R^n)(V, W) < \infty$,

(b) $\mathbf{Q}(R^n, A)(V, W) < \infty$,

etc.

6.14 PROPOSITION. Let $1 \leq k \leq m \leq n$ be integers, $A \subset R^n$ be a compact submanifold of R^n of dimension m and class 3, and $C \subset A$ be a compact submanifold of A of dimension k and class 3 with boundary D. Then $(|C|, |D|) \in \mathbf{IV}_k(A, A)$ and

(1) $\underline{\underline{P}}(R^n, A)(|C|, |D|) = \underline{\underline{Q}}(R^n, A)(|C|, |D|) = \Delta_k(C, A)$;
(2) $\underline{\underline{P}}^+(R^n, C)(|D|, 0) = \underline{\underline{Q}}^+(R^n, C)(|D|, 0) = \Delta_k^+(D, C)$; and
(3) $\underline{\underline{P}}^-(R^n, C)(|D|, 0) = \underline{\underline{Q}}^-(R^n, C)(|D|, 0) = \Delta_k^-(D, C)$.

PROOF. Let $g \in \underline{\underline{X}}(R^n, A)$. One can write $g = g' + g''$ where for each x in some neighborhood of C, $g'(x)$ is perpendicular to C at x and $g''(x)$ is parallel with C at x. One verifies that

$$|\underline{\underline{S}}(|C|, g'')| \leq \underline{\underline{T}}(|D|, g'') \leq \underline{\underline{W}}|D|(|g''|),$$

and 6.9 implies that

$$|\underline{\underline{S}}(|C|, g')| \leq \underline{\underline{T}}(|C|, g'')\cdot\Delta_k(C, A) = \underline{\underline{W}}|C|(|g''|)\cdot\Delta_k(C, A).$$

This with 6.9 establishes that $\underline{\underline{Q}}(R^n, A)(|C|, |D|) \geq \underline{\underline{P}}(R^n, A)(|C|, |D|)$. 6.9 (5) implies the opposite inequality. Conclusions (2) and (3) follow by similar arguments.

6.15 COROLLARY. Let $1 \leq k \leq n$ be integers and $A \subset R^n$ be a compact submanifold of R^n of dimension k and class 2 with boundary B. The following statements are equivalent:

(1) $(|A|, |B|)$ is stationary on (R^n, R^n);
(2) $(|A|, 0)$ is stationary on (R^n, R^n) with respect to B;
(3) $\underline{\underline{S}}(R^n; B)(|A|) = 0$;
(4) $\underline{\underline{P}}(R^n; B)(|A|) = 0$;
(5) $\underline{\underline{Q}}(R^n; B)(|A|) = 0$;
(6) The mean curvature of A is zero at each non-boundary point of A;
(7) $A-B$ is a real analytic submanifold of R^n satisfying the minimal surface equation;
(8) The coordinate functions of R^n restricted to A are harmonic in the Riemannian metric on A induced by its imbedding.

6.16 PROPOSITION. Let $1 \leq k \leq n$ be integers and $(V, W) \in \underline{\underline{RV}}_k(R^n, R^n)$. If $\underline{\underline{P}}(R^n)(V, W) < \infty$ then $\underline{\underline{P}}(R^n)(V, W) = \underline{\underline{Q}}(R^n)(V, W)$.

PROOF. Suppose $0 < \mathbf{P}(R^n)(V, W) < \infty$, $\eta > 0$, and $g \in \mathbf{X}(R^n)$ with $\mathbf{T}(V, g) > 0$ and $\mathbf{P}(V, W; g) \geq \mathbf{P}(R^n)(V, W) - \eta$. Let $\varepsilon > 0$. Since $(V, W) \in \mathbf{RV}_k(R^n)$ there exist $g', g'' \in \mathbf{X}(R^n)$ with $g = g' + g''$ and

$$\mathbf{T}(V, g') \leq \varepsilon \mathbf{T}(V, g), \qquad \mathbf{T}(W, g') \leq \varepsilon \mathbf{T}(W, g)$$
$$\mathbf{T}(V, g'') \geq (1-\varepsilon)\mathbf{W}V(|g''|)$$

Now

$$[\mathbf{T}(V, g')]^{-1}[|\mathbf{S}(V, g')| - \mathbf{T}(W, g')] \leq \mathbf{P}(R^n)(V, W) < \infty$$

which implies

$$|\mathbf{S}(V, g')| \leq \varepsilon[\mathbf{T}(V, g)\cdot\mathbf{P}(R^n)(V, W) + \mathbf{T}(W, g)] = \varepsilon\cdot C$$

and hence

$$\begin{aligned} |\mathbf{S}(V, g'')| &= |\mathbf{S}(V, g) - \mathbf{S}(V, g')| \\ &\geq |\mathbf{S}(V, g)| - \varepsilon C \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{Q}(V, W; g'') &= [\mathbf{W}V(|g''|)]^{-1}[|\mathbf{S}(V, g'')| - \mathbf{T}(W, g'')] \\ &\geq [(1-\varepsilon)^{-1}\mathbf{T}(V, g'')][|\mathbf{S}(V, g)| - \varepsilon C - \mathbf{T}(W, g'')] \\ &\geq [(1-\varepsilon)^{-2}\mathbf{T}(V, g)][|\mathbf{S}(V, g)| - \varepsilon C - (1-\varepsilon)^{-2}\mathbf{T}(W, g)] \\ &\geq \mathbf{P}(V, W; g) - o_\varepsilon(1) \\ &\geq \mathbf{P}(R^n)(V, W) - o_\varepsilon(1) - \eta \end{aligned}$$

The proposition follows since ε and η are arbitrary and $\mathbf{P}(R^n)(V, W) \geq \mathbf{Q}(R^n)(V, W)$.

6.17 PROPOSITION. Let $1 \leq k \leq n$ be integers and $(V, W) \in \mathbf{GV}_k(R^n, R^n)$. Let $f : R^n \longrightarrow R_0^+$ be of class 1. Then

$$\mathbf{Q}(R^n)(V \wedge f, W \wedge f) \leq \mathbf{Q}(R^n)(V, W) + \sup\{f(x)^{-1} : x \in \operatorname{spt}(V)\}\, \sup\{\operatorname{Lip}(f)(x) : x \in \operatorname{spt}(V)\}$$

PROOF.

Part 1. Let $1 \leq k \leq n$ be integers and $V \in \mathbf{GV}_k(R^n)$. Then for each $\varepsilon > 0$ there exists a positive integer q, functions $\varphi_1, \varphi_2, \ldots, \varphi_q : R^n \longrightarrow R^+$ of class ∞ with compact pairwise disjoint supports, and simple unit k-vectors $\mu_1, \mu_2, \ldots, \mu_q \in \mathbf{\Lambda}_k(R^n)$ such that $\mathbf{F}(V, \sum_i \varphi_i \mu_i) < \varepsilon$ and

$\bigcup_i \operatorname{spt}(\varphi_i) \subset \{x : \operatorname{dist}(x, \operatorname{spt}(V)) < \varepsilon\}$. Here for each $\psi \in \underline{\underline{C}}_0^k(R^n)$, $[\sum_i \varphi_i \mu_i](\psi) = \sum_i \int_{R^n} \varphi_i(x)|\psi(x) \cdot \mu_i| \, d\underline{\underline{H}}^n x$.

Part 2. Let $1 \leq k \leq n$ be integers, $V \in \underline{\underline{GV}}_k(R^n)$, $f : R^n \longrightarrow R^+$ be of class 1, $h : R^n \longrightarrow R$ be of class ∞ with compact support, and $u \in R^n$. Then

$$\underline{\underline{S}}(V \wedge f, hu) \leq \sup\{|\operatorname{grad}(f)(x)| : x \in \operatorname{spt}(V)\} \underline{\underline{W}}V(|h|) + \underline{\underline{S}}(V, fhu).$$

Proof of part 2. Let $\varepsilon > 0$ and choose $\{\varphi_i, \mu_i\}_i$ to approximate V as in part 1. We define $u_* : R^n \longrightarrow R^n$ by setting $u_*(x)$ equal to $\varphi_i(x)$ times the orthogonal projection of u onto the k-plane of μ_i if $x \in \operatorname{spt}(\varphi_i)$ for some i and $u_*(x) = 0$ otherwise. u_* is of class ∞ and 6.1 and 6.4 (4e) imply

$$\underline{\underline{S}}(\textstyle\sum_i \varphi_i \mu_i, hu) = \displaystyle\int_{R^n} f(x)(\operatorname{grad}(h)(x) \cdot u_*(x)) dx^1 dx^2 \ldots dx^n$$

$$= \sum_{i=1}^n \int_{R^n} f(x)(\partial h/\partial x^i)(x) u_*^i(x) \, dx^1 dx^2 \ldots dx^n$$

$$= \sum_{i=1}^n \int_{R^{n-1}} f(x)h(x)u_*^i(x) \Big|_{x^i=-\infty}^{x^i=\infty} - \int_R h(x)(\partial/\partial x^i)(fu_*^i)(x) \; dx^1 \ldots dx^{i-1} dx^{i+1} \ldots dx^n$$

$$= -\int_{R^n} \sum_{i=1}^n (\partial f/\partial x^i)(x) u_*^i(x) h(x) \, dx^1 dx^2 \ldots dx^n - \sum_{i=1}^n \int_{R^n} f(x)h(x)(\partial u_*^i/\partial x^i)(x) dx^1 dx^2 \ldots dx^n$$

$$\leq \sup\{|\operatorname{grad}(f)(x)| : x \in \textstyle\bigcup_i \operatorname{spt}(\varphi_i)\} \displaystyle\int_{R^n} \sum_j \varphi_j(x) h(x) dx^1 dx^2 \ldots dx^n$$

$$- \sum_{i=1}^n \int_{R^{n-1}} f(x)h(x)u_*^i(x) \Big|_{x^i=-\infty}^{x^i=\infty} dx^1 \ldots dx^{i-1} dx^{i+1} \ldots dx^n$$

$$+ \sum_{i=1}^n \int_{R^n} (\partial/\partial x^i)(fh)(x) u_*^i(x) dx^1 dx^2 \ldots dx^n.$$

$$\leq \sup\{|\operatorname{grad}(f)(x)| : \operatorname{dist}(x, \operatorname{spt}(V)) < \varepsilon\} \underline{\underline{W}} \textstyle\sum_i \varphi_i \mu_i(|h|) + \underline{\underline{S}}(\sum_i \varphi_i \mu_i, fhu).$$

The continuity of $\underline{\underline{S}}(\cdot, fhu)$ and $W[\cdot](|h|)$ imply part 2 since ε is arbitrary.

Part 3. Let V, W, f be as in the hypothesis of the proposition and h, u as in part 2. Assume $\underline{\underline{W}}V(|hu|) > 0$. Then

$$\begin{aligned}\underline{\underline{Q}}(V\wedge f,\ W\wedge f;\ hu) &= [\underline{\underline{W}}V\wedge f(|hu|)]^{-1}[|\underline{\underline{S}}(V\wedge f,\ hu)| - \underline{\underline{T}}(W\wedge f,\ hu)] \\ &\le [\underline{\underline{W}}V\wedge f(|hu|)]^{-1}\sup\{|\operatorname{grad}(f)(u)| : x\in \operatorname{spt}(V)\}\underline{\underline{W}}V(|h|) \\ &\quad + [\underline{\underline{W}}V(|fhu|)]^{-1}[|\underline{\underline{S}}(V,\ fhu)| - \underline{\underline{T}}(W,\ fhu)] \\ &\le \sup\{|\operatorname{grad}(f)(x)| : x\in\operatorname{spt}(V)\}\sup\{f^{-1}(x) : x\in\operatorname{spt}(V)\} \\ &\quad + \underline{\underline{Q}}(R^n)(V,\ W).\end{aligned}$$

The proposition follows with the use of 6.7.

6.18 EXAMPLE. Let $1 \le k \le n$ be integers and set

$$A = \underline{\underline{D}}^k(0,\ 1)\times\{0\}\subset R^k\times R^{n-k} = R^n,$$

$$B = \partial\underline{\underline{D}}^k(0,\ 1)\times\{0\}\subset R^k\times R^{n-k} = R^n.$$

Let $f : R^n \longrightarrow R_0^+$ be of class 1. Then

(1) $\underline{\underline{Q}}(R^n)(|A|\wedge f,\ |B|\wedge f) = \underline{\underline{Q}}(R^n;\ B)(|A|\wedge f,\ 0) = \sup\{[f(x)]^{-1}\operatorname{Lip}(f|A)(x) : x\in A\}$

(2) $\underline{\underline{P}}(R^n)(|A|\wedge f,\ |B|\wedge f) = \underline{\underline{P}}(R^n;\ B)(|A|\wedge f,\ 0) = \begin{cases}\infty & \text{if } \operatorname{Lip}(f|A) > 0\ ;\\ 0 & \text{if } f|A \text{ is constant.}\end{cases}$

6.19 PROPOSITION. Let $1 \le k \le n$ be integers and $(V,\ W)\in \underline{\underline{V}}_k(R^n,\ R^n)$. Let $r\in R$ and $f : R^n \longrightarrow R^n$ be given by $f(x) = rx$ for each $x\in R^n$. Then $|r|\underline{\underline{P}}(R^n)(f_{\#}(V,\ W)) = \underline{\underline{P}}(R^n)(V,\ W)$ and $|r|\underline{\underline{Q}}(R^n)(f_{\#}(V,\ W)) = \underline{\underline{Q}}(R^n)(V,\ W)$.

PROOF. Note that for each $g\in\underline{\underline{X}}(R^n)$,

$|r|^k\underline{\underline{S}}(V,\ g) = \underline{\underline{S}}(f_{\#}(V),\ f_{\#}(g))$, $|r|^k\underline{\underline{T}}(W,\ g) = \underline{\underline{T}}(f_{\#}(W),\ f_{\#}(g))$,

$|r|^{k+1}\underline{\underline{T}}(V,\ g) = \underline{\underline{T}}(f_{\#}(V),\ f_{\#}(g))$, and $|r|^{k+1}\underline{\underline{W}}V(|g|) = \underline{\underline{W}}f_{\#}(V)(|f_{\#}(g)|)$.

The proposition follows.

6.20 REMARK. Most mappings do not preserve $\underline{\underline{P}}$ and $\underline{\underline{Q}}$ regularity. Let $1 \le k < n$ be integers and suppose $f : R^n \longrightarrow R^n$ is a diffeomorphism of class 3. Then one can prove:

(1) $f_{\#}$ maps $\underline{\underline{V}}_k(R^n) \cap \{(V, W) : \underline{\underline{P}}(R^n)(V, W) = 0\}$ onto itself if and only if f is the composition of an isometry with a uniform expansion or contraction (as in 6.19).

(2) $f_{\#}$ maps $\underline{\underline{V}}_k(R^n) \cap \{(V, W) : \underline{\underline{P}}(R^n)(V, W) < \infty\}$ onto itself if and only if f is conformal.

(3) (1) and (2) above are true with $\underline{\underline{Q}}$ replacing $\underline{\underline{P}}$.

Observe that conformal mappings are precisely those preserving tangent cones (10.3).

6.21 DEFINITIONS. Higher variations.

(1) Let $1 \leq k \leq n$ and $2 \leq q$ be integers and suppose $f : R \times R^n \longrightarrow R^n$ is of class $q+1$ with $f(0, x) = x$ for each $x \in R^n$. It is not difficult to verify that the function $\underline{\underline{S}}^q(\cdot, f) : \underline{\underline{RV}}_k(R^n) \longrightarrow R$ given for $V \in \underline{\underline{RV}}_k(R^n)$ by

$$\underline{\underline{S}}^q(V, f) = (d^q/dt^q)\underline{\underline{W}}(f(t, \cdot)_{\#}(V))\Big|_{t=0}$$

is well defined and continuous on $\underline{\underline{RV}}_k(K)$ for each compact $K \subset R^n$.

(2) Let $1 \leq k \leq n$ be integers; $g, h \in \underline{\underline{X}}(R^n)$; and $f : R \times R^n \longrightarrow R^n$ given for $t \in R$ and $x \in R^n$ by

$$f(t, x) = x + t g(x) + 2^{-1}t^2 h(x) + 0_t(t^3).$$

One computes explicitly:

$$\begin{aligned}
&(d^2/dt^2)\underline{\underline{W}}[f(t, \cdot)_{\#}\underline{\underline{v}}(0, (\partial/\partial x^1)\wedge(\partial/\partial x^2)\wedge \cdots \wedge(\partial/\partial x^j))]\Big|_{t=0} \\
&\quad = \sum_{i=1}^{k} \sum_{j=k+1}^{n} [(\partial g^j/\partial x^i)(0)]^2 \\
&\quad + 2 \sum_{1 \leq i < j \leq k} (\partial g^i/\partial x^i)(0)\cdot(\partial g^j/\partial x^j)(0) + (\partial g^j/\partial x^i)(0)\cdot(\partial g^i/\partial x^j)(0) \\
&\quad + \sum_{i=1}^{k} (\partial h^i/\partial x^i)(0).
\end{aligned}$$

(3) Let $1 \leq k \leq n$ be integers, $A \subset R^n$ be a closed submanifold of R^n, $C \subset$ [illegible] be closed, and $V \in \mathcal{RV}_k(A)$. We say that V is stable on A with respect to C

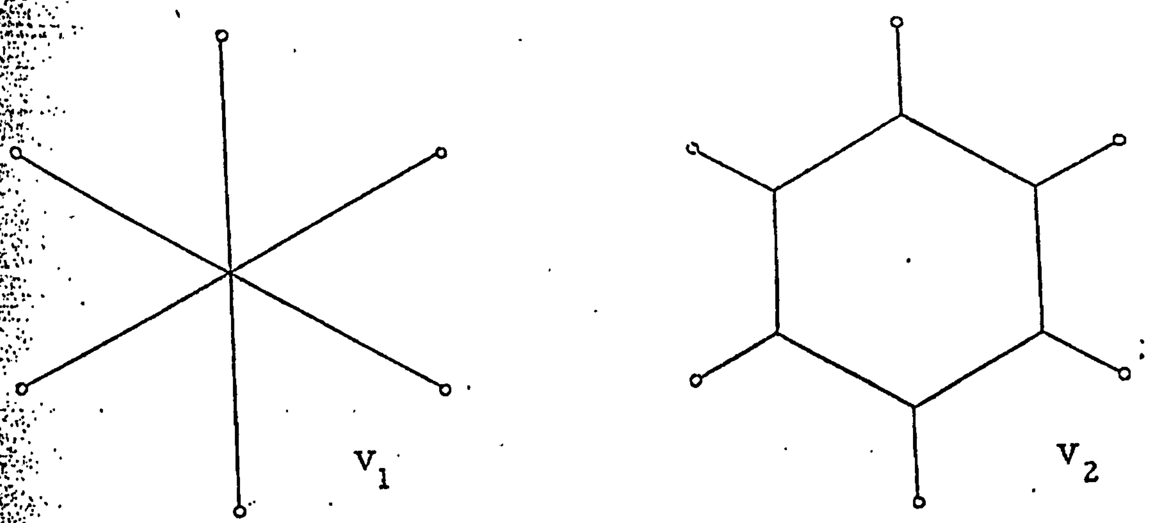
and only if $\underline{\underline{BP}}^0(R^n, A, A)(V, 0) \subset C$ and $\underline{\underline{S}}^2(V \cap \underline{\underline{D}}^n(0, r), f) \geq 0$ for each $r \in R_0^+$ and each $f: R \times R^n \longrightarrow R^n$ of class 3 for which $f(0, x) = x$ for each $x \in R^n$, $f(t, \cdot)(A) \subset A$ for each $t \in R$, and

$$(\partial/\partial t)f(t, \cdot)\Big|_{t=0} \in \underline{\underline{X}}(R^n, A; C \cup [R^n - \underline{\underline{D}}^n(0, r)]).$$

(1) and (2) above imply that if $V \in \underline{\underline{RV}}_1(R^n)$ and $\underline{\underline{Q}}(R^n; C)(V) = 0$, then V is stable in R^n with respect to C.

6.22 EXAMPLES.

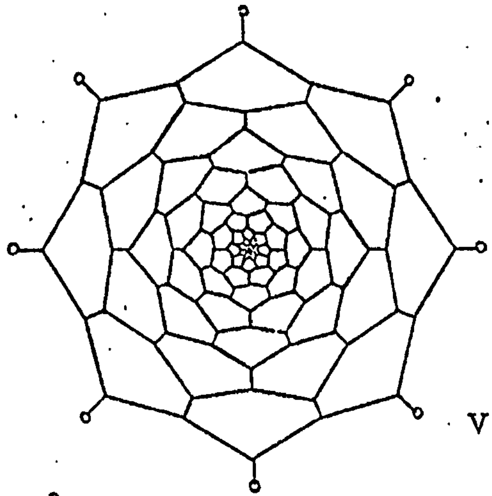
(1) Let $W \in \underline{\underline{IV}}_0(R^2)$ consist of six points equally spaced around the unit circle. The two distinct varifolds $V_1, V_2 \in \underline{\underline{IV}}_1(R^n)$ sketched below have $\underline{\underline{P}}(R^2)(V_1, W) = \underline{\underline{P}}(R^2)(V_2, W) = 0$ and agree in a neighborhood of $\operatorname{spt}(W)$.



V_1 and V_2 are two members of a continuum of 1 dimensional integral varifolds in R^2 which are stable with respect to $\operatorname{spt}(W)$, the other members of this continuum being similar to V_2 but with larger or smaller central hexagons.

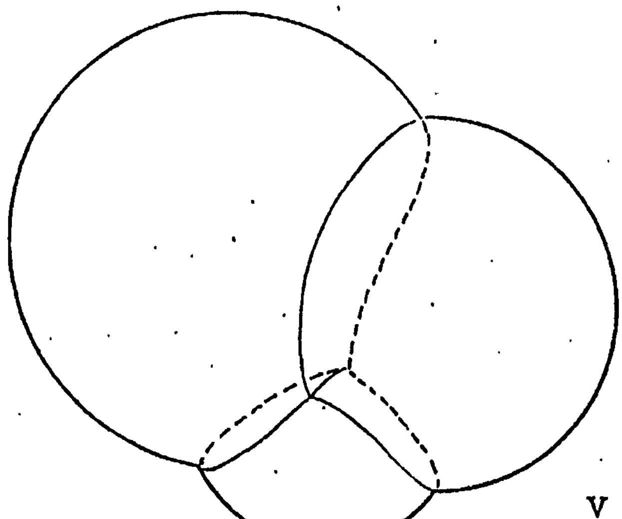
(2) Let $W \in \underline{\underline{IV}}_0(R^2)$ consist of eight points equally spaced around the unit circle. A varifold $V \in RV_1(R^2)$ with $\underline{\underline{P}}(R^2)(V, W) = 0$ is sketched below. The

outermost rays can be taken to have density 1. One can then compute the density of the other segments using the force diagram method of elementary physics. One verifies, either by computation or 10.6, that $\odot^1(\underline{\underline{W}}V; 0) = 0$.



V

(3) The varifold $V \in \underline{\underline{IV}}_2(R^3)$ sketched below provides a mathematical model for the familiar soap bubble. V is $\underline{\underline{P}}$ regular, i.e. $\underline{\underline{P}}(R^3)(V, 0) < \infty$.



V

7. VARIFOLD SLICES.

7.1. DEFINITIONS. Slicing a varifold with a differentiable function.

(1) Let $1 \leq k \leq n$ be integers, $U \subset R^n$ be open, $V \in \mathbf{V}_k(R^n)$, and $f: U \longrightarrow R$ be of class 1. One sets $[\mathrm{grad}(f)]^* \in \mathbf{C}^1(U)$ as the covector field dual to the vector field $\mathrm{grad}(f) : U \longrightarrow \mathbf{\Lambda}_1(R^n)$. We define the slice of V by f at $r \in R$, denoted $\mathbf{B}(V, f, r)$, to be that element of $\mathbf{V}_{k-1}(R^n)$ given for $\varphi \in \mathbf{C}_0^{k-1}(U)$ by

$$(*) \quad \mathbf{B}(V, f, r)(\varphi) = \lim_{\substack{s \to 0^+ \\ t \to 0^+}} (s+t)^{-1} V \cap \{x : r-s < f(x) < r+t\}([\mathrm{grad}(f)]^* \wedge \varphi)$$

whenever

(i) the limit of $(*)$ exists for each $\varphi \in \mathbf{C}_0^{k-1}(U)$,

(ii) $\limsup_{t \to 0^+} t^{-1} \mathbf{W}(V \cap \{x : r-t < f(x) < r+t\}) < \infty$, and

(iii) $\mathbf{B}(V, f, r)$ defined by $(*)$ is in $\mathbf{V}_{k-1}(R^n)$ (this is, in fact, implied by (i) and (ii)).

One verifies immediately that if $\mathbf{B}(V, f, r)$ exists, then $\mathbf{B}(V, f, r)$ lies intrinsically on $f^{-1}(r)$.

(2) Let $1 \leq k \leq n$ be integers, $U \subset R^n$ be open, $V \in \mathbf{V}_k(R^n)$, and $f, g : R^n \longrightarrow R$ be of class 1 with $|\mathrm{grad}(f)(x)| > 0$ for each $x \in U$. We define

$$\mathbf{b}(\cdot, f) : \mathbf{V}_k(U) \longrightarrow \mathbf{V}_k(U)$$

$$\mathbf{b}(V, f) = V \wedge ([\mathrm{grad}(f)]^{-1}[\mathrm{grad}(f)]^*) \wedge ([\mathrm{grad}(f)]^{-1}\mathrm{grad}(f))$$

for each $V \in \mathbf{V}_k(R^n)$. One verifies that

(i) $\mathbf{b}(\cdot, f) | \mathbf{V}_k(K)$ is continuous for each compact $K \subset U$,

(ii) $\mathbf{B}(V, f, r) = \mathbf{B}(\mathbf{b}(V, f), f, r)$ if $V \in \mathbf{NV}_k(U)$, $r \in R$, and either side exists,

(iii) $\mathbf{W}(\mathbf{b}(V, f)) \leq \mathbf{W}(V)$ if $V \in \mathbf{NV}_k(U)$,

(iv) if $V \in \mathbf{NV}_k(U)$ and $\mathbf{W}(\mathbf{b}(V, f)) = \mathbf{W}(V)$, then $\mathbf{b}(V, f) = V$,

(v) if f and g are of class 2, $r \in R$, and $f^{-1}(r) = g^{-1}(r)$, then $\mathbf{B}(V, f, r) = \mathbf{B}(V, g, r)$ whenever either side exists.

7.2. THEOREM. Let $1 \leq k \leq n$ be integers, $U \subset R^n$ be open, $V \in \underline{\underline{GV}}_k(U, U)$, and $f : U \longrightarrow R$ be of class 3 with $|\mathrm{grad}(f)(x)| > 0$ for each $x \in U$. If $r \in R$ and $\underline{\underline{B}}(V, f, r)$ exists, then

$$\underline{\underline{P}}(R^n)(V \cap \{x : f(x) < r\},\ W \cap \{x : f(x) < r\} + \underline{\underline{B}}(V, f, r)) \leq \underline{\underline{P}}(R^n)(V, W) ,$$

and

$$\underline{\underline{Q}}(R^n)(V \cap \{x : f(x) < r\},\ W \cap \{x : f(x) < r\} + \underline{\underline{B}}(V, f, r)) \leq \underline{\underline{Q}}(R^n)(V, W) .$$

PROOF. We will prove the first conclusion. This is immediate if $\underline{\underline{P}}(R^n)(V, W) = \infty$. We assume therefore that $\underline{\underline{P}}(R^n)(V, W) < \infty$, and we assume also $\underline{\underline{B}}(V, f, 0)$ exists. Let $g \in \underline{\underline{X}}(R^n)$ with $\underline{\underline{T}}(V \cap \{x : g(x) < 0\}, g) > 0$ and for each sufficiently small $\varepsilon > 0$, choose $\varphi_\varepsilon : U \longrightarrow \{t : 0 \leq t \leq 1\}$ of class 2 such that

(1) $\varphi_\varepsilon^{-1}(1) \supset \{x : f(x) \geq -\delta\}$ for some $\delta > 0$;
(2) $\varphi_\varepsilon(x) = \varphi_\varepsilon(y)$ if $f(x) < 0$, $f(y) < 0$, and $\mathrm{dist}(x, f^{-1}(0)) = \mathrm{dist}(y, f^{-1}(0))$;
(3) $\varphi_\varepsilon^{-1}(0) \supset \{x : f(x) \leq -\varepsilon\}$; and
(4) $\mathrm{Lip}(\varphi_\varepsilon)(x) \leq \varepsilon^{-1}|\mathrm{grad}(f)(x)| + o_\varepsilon(1)$ for each $x \in U$.

Let $\varepsilon > 0$, $p \in U$ with $-\varepsilon < f(p) < 0$, $a, b \in R^+$ with $a^2 + b^2 = 1$, and $x^1, x^2, \ldots, x^n$ be orthonormal coordinates for R^n such that $\mathrm{grad}(f)(p)$ is parallel with the x^n direction. 6.10(2) implies

$$\underline{\underline{S}}(\underline{v}(p, (\partial/\partial x^1) \wedge (\partial/\partial x^2) \wedge \ldots \wedge (\partial/\partial x^{k-1}) \wedge (a(\partial/\partial x^k) + b(\partial/\partial x^n)), \varphi_\varepsilon g)$$
$$= \sum_{i=1}^{k-1} (\partial/\partial x^i)(\varphi_\varepsilon g^i)(p) + a^2(\partial/\partial x^k)(\varphi_\varepsilon g^k)(p)$$
$$+ ab[(\partial/\partial x^k)(\varphi_\varepsilon g^n)(p) + (\partial/\partial x^n)(\varphi_\varepsilon g^k)(p) + b^2(\partial/\partial x^n)(\varphi_\varepsilon g^n)(p)$$
$$= \sum_{i=1}^{k-1} \varphi_\varepsilon(p)(\partial/\partial x^i)g^i(p) + a^2\varphi_\varepsilon(p)(\partial/\partial x^k)g^k(p)$$
$$+ ab\,\varphi_\varepsilon(p)[(\partial/\partial x^k)g^n(p) + (\partial/\partial x^n)g^k(p)]$$
$$+ b^2\varphi_\varepsilon(p)(\partial/\partial x^n)g^n(p) + ab\, g^k(p)(\partial/\partial x^n)\varphi_\varepsilon(p) + b^2 g^n(p)(\partial/\partial x^n)\varphi_\varepsilon(p)$$
$$\leq K + \varepsilon^{-1}|\mathrm{grad}(f)(p)|(ab\, g^k(p) + b^2 g^n(p)) + o_\varepsilon(1)$$

$$\leq K + \varepsilon^{-1} |\mathrm{grad}(f)(p)| (g^k(p)^2 + g^n(p)^2)^{1/2} + o_\varepsilon(1)$$

for some constant K.

6.4(7c) implies

$$\underline{\underline{T}}(\underline{\underline{B}}(V, f, 0), \varphi_\varepsilon g) = \underline{\underline{W}}(\underline{\underline{B}}(V, f, 0) \wedge g)$$

$$= \gamma(n, k) \int_{\underline{\underline{D\Lambda}}^k(R^n)} \underline{\underline{B}}(V, f, 0) \wedge g(\omega(\lambda)) \, d\, \underline{\underline{H}}^{(n:k)}\lambda$$

$$= \gamma(n, k) \int_{\underline{\underline{D\Lambda}}^k(R^n)} (d/dr)[V \cap \{x : f(x) < r\}([\mathrm{grad}(f)]^* \wedge g \wedge \omega(\lambda))]\Big|_{r=0} \, d\, \underline{\underline{H}}^{(n:k)}\lambda$$

$$= \gamma(n, k)(d/dr) \int_{\underline{\underline{D\Lambda}}^k(R^n)} V \cap \{x : f(x) < r\}([\mathrm{grad}(f)]^* \wedge g \wedge \omega(\lambda)) \, d\, \underline{\underline{H}}^{(n:k)}\lambda \Big|_{r=0}$$

$$= o_\varepsilon(1) + \varepsilon^{-1}\gamma(n, k) \int_{\underline{\underline{D\Lambda}}^k(R^n)} V \cap \{x : -\varepsilon < f(x) < 0\}([\mathrm{grad}(f)]^* \wedge g \wedge \omega(\lambda)) \, d\, H^{(n:k)}\lambda .$$

Observe that

$$\gamma(n, k) \int_{\underline{\underline{D\Lambda}}^k(R^n)} \underline{\underline{v}}(p, (\partial/\partial x^1) \wedge (\partial/\partial x^2) \wedge \ldots \wedge (\partial/\partial x^{k-1}) \wedge [a(\partial/\partial x^k) + b(\partial/\partial x^n)])$$
$$([\mathrm{grad}(f)]^* \wedge [g \wedge \omega(\lambda)]) \, d\, H^{(n:k)}\lambda$$

equals

$$b|\mathrm{grad}(f)(p)| \, \underline{\underline{W}}(\underline{\underline{v}}(p, (\partial/\partial x^1) \wedge (\partial/\partial x^2) \wedge \ldots \wedge (\partial/\partial x^{k-1})) \wedge g)$$

which is greater than or equal to

$$b'|\mathrm{grad}(f)(p)| (g^k(p)^2 + g^n(p)^2)^{1/2} .$$

From 6.1 and the fact that $V \in \underline{\underline{GV}}_k(R^n)$, one concludes that

$$\underline{\underline{S}}(V \cap \{x : f(x) < 0\}, \varphi_\varepsilon g) - \underline{\underline{T}}(\underline{\underline{B}}(V, f, 0), \varphi_\varepsilon g) \leq o_\varepsilon(1) .$$

Using the fact that $\underline{\underline{B}}(V, f, 0)$ exists and $\underline{\underline{W}}W(f^{-1}(0)) = 0$, we have

$\underline{\underline{T}}(V \cap \{x : f(x) < 0\},\ g)]^{-1}[\underline{\underline{S}}(V \cap \{x : f(x) < 0\}) - \underline{\underline{T}}(W \cap \{x : f(x) < 0\} + \underline{\underline{B}}(V,\ f,\ 0),\ g)]$

$[\underline{\underline{T}}(V,\ (1 - \varphi_\varepsilon)g) + o_\varepsilon(1)]^{-1}[\underline{\underline{S}}(V,\ (1 - \varphi_\varepsilon)g) - \underline{\underline{T}}(W,\ (1 - \varphi_\varepsilon)g) - o_\varepsilon(1)]$

$[\underline{\underline{T}}(V \cap \{x : f(x) < 0\},\ g)]^{-1}[\underline{\underline{S}}(V \cap \{x : f(x) < 0\},\ \varphi_\varepsilon g) - \underline{\underline{T}}(\underline{\underline{B}}(V,\ f,\ r),\ \varphi_\varepsilon g)]$

$\underline{\underline{P}}(V,\ W;\ (1 - \varphi_\varepsilon)g) + o_\varepsilon(1)$

$\underline{\underline{P}}(R^n)(V,\ W) + o_\varepsilon(1)$.

Conclusion (1) follows since ε is arbitrary and g is arbitrary. Conclusion (2) follows by similar arguments.

7.3. EXAMPLE. Let G be the additive group of integers and $T \in \underline{\underline{Z}}_1(R^2,\ R^2;\ G)$ be the integral current given as follows. ∂T consists of four points: $(-1,\ -1)$ with coefficient -1, $(-1,\ 1)$ with coefficient $+1$, $(1,\ -1)$ with coefficient -1, and $(1,\ 1)$ with coefficient $+1$. T consists of the line segment joining $(-1,\ 1)$ to $(1,\ 1)$ and the line segment joining $(-1,\ 1)$ to $(1,\ -1)$, each with the appropriate orientation. Let $f : R^2 \longrightarrow R$, $f(x^1,\ x^2) = x^1$ for each $(x^1, x^2) \in R^2$. Then

(1) $\underline{\underline{P}}(R^2)(|T|,\ |\partial T|) = \underline{\underline{Q}}(R^2)(|T|,\ |\partial T|) = 0$;

(2) $\underline{\underline{B}}(|T|,\ f,\ 0)$ is defined and equals the point $(0,\ 0)$ with multiplicity 2 ;

(3) $\partial(T \cap \{x : f(x) < 0\}) - \partial T \cap \{x : f(x) \leq \infty\}$ is defined [FF 3.9] and equals 0 ;

(4) $\underline{\underline{P}}(R^2)(|T| \cap \{x : f(x) < 0\},\ |\partial T| \cap \{x : f(x) < 0\} + \underline{\underline{B}}(|T|,\ f,\ 0))$
$= \underline{\underline{Q}}(R^2)(|T| \cap \{x : f(x) < 0\},\ |\partial T| \cap \{x : f(x) < 0\} + \underline{\underline{B}}(|T|,\ f,\ 0)) = 0$;

(5) $\underline{\underline{P}}(R^2)(|T \cap \{x : f(x) < 0\}|,\ |\partial(T \cap \{x : f(x) < 0\})|)$
$= \underline{\underline{Q}}(R^2)(|T \cap \{x : f(x) < 0\}|,\ |\partial(T \cap \{x : f(x) < 0\})|) = \infty$.

This example illustrates one difference between the definition of boundary in the chain sense and the varifold definition of boundary.

7.4. THEOREM. Let $1 \leq k \leq n$ be integers, $U \subset R^n$ be open, $V \in \underline{\underline{NV}}_k(U)$, and $f : U \longrightarrow R$ be of class 1 with $0 < |\mathrm{grad}(f)(x)| \leq \xi < \infty$ for each $x \in U$. Then for $\underline{\underline{H}}^1$ almost all $r \in R$,

(1) $\underline{\underline{B}}(V,\ f,\ r)$ exists;

(2) $(d/ds)\underline{\underline{W}}(V \cap \{x : f(x) < s\} \wedge |\mathrm{grad}(f)|)\Big|_{s=r}$ exists;

(3) $(d/ds)\underline{\underline{W}}(\underline{\underline{b}}(V, f) \cap \{x : f(x) < s\} \wedge |\mathrm{grad}(f)|)\Big|_{s=r}$ <u>exists</u>;

(4) $(d/ds)\underline{\underline{W}}(\underline{\underline{b}}(V, f) \cap \{x : f(x) < s\})\Big|_{s=r}$ <u>exists</u>;

(5) $$\begin{aligned} \underline{\underline{W}}(\underline{\underline{B}}(V, f, r)) &= (d/ds)\underline{\underline{W}}(\underline{\underline{b}}(V, f) \cap \{x : f(x) < s\} \wedge |\mathrm{grad}(f)|)\Big|_{s=r} \\ &\leq (d/ds)\underline{\underline{W}}(V \cap \{x : f(x) < s\} \wedge |\mathrm{grad}(f)|)\Big|_{s=r} \\ &\leq \xi(d/ds)\underline{\underline{W}}(V \cap \{x : f(x) < s\})\Big|_{s=r}; \quad \underline{\text{and}} \end{aligned}$$

(6) $\int_a^b \underline{\underline{W}}(\underline{\underline{B}}(V, f, r))dr \leq \xi\underline{\underline{W}}(V \cap \{x : a < f(x) < b\})$

<u>for each</u> $-\infty \leq a < b \leq \infty$.

PROOF. Choose $\varphi_1, \varphi_2, \varphi_3, \ldots \in \underline{\underline{C}}_0^{k-1}(R^n)$ so that $\{\varphi_i | U\}_i$ is dense in $\underline{\underline{C}}_0^{k-1}(U)$ in the $|\;|$ topology. Considered as a function of r each of the expressions $V \cap \{x : f(x) < r\}([\mathrm{grad}(f)]^* \wedge \varphi_i)$ $(i = 1, 2, 3, \ldots)$, $\underline{\underline{W}}(\underline{\underline{b}}(V, f) \cap \{x : f(x) < r\} \wedge |\mathrm{grad}(f)|)$, $\underline{\underline{W}}(V \cap \{x : f(x) < r\} \wedge |\mathrm{grad}(f)|)$, and $\underline{\underline{W}}(V \cap \{x : f(x) < r\})$ is non-decreasing for $r \in R$ and hence differentiable for $\underline{\underline{H}}^1$ almost all $r \in R$. Since these functions are only countable in number they are simultaneously differentiable for $\underline{\underline{H}}^1$ almost all $r \in R$. For each such r the conclusions of the theorem are true. To see this one observes that for each $\varphi \in \underline{\underline{C}}_0^k(U)$,

$$\begin{aligned} &\limsup_{\substack{s \to 0^+ \\ t \to 0^+}} (s+t)^{-1} V \cap \{x : r-s < f(x) < r+t\}([\mathrm{grad}(f)]^* \wedge \varphi) \\ &\qquad \leq c\,|[\mathrm{grad}(f)]^* \wedge \varphi|\,(d/ds)\underline{\underline{W}}(V \cap \{x : f(x) < s\})\Big|_{s=r} \\ &\qquad \leq c\xi|\varphi|\,(d/ds)\underline{\underline{W}}(V \cap \{x : f(x) < r\}) \end{aligned}$$

where c is the constant of 2.4(2a). Conclusion (1) follows by elementary arguments. We already know (2), (3), and (4) to hold. We examine the equality of (5).

For each $\varepsilon > 0$,

$$\begin{aligned} &(d/ds)\underline{\underline{W}}(\underline{\underline{b}}(V, f) \cap \{x : f(x) < s\} \wedge |\mathrm{grad}(f)|)\Big|_{s=r} \\ &\quad = o_\varepsilon(1) + \varepsilon^{-1}\underline{\underline{W}}(\underline{\underline{b}}(V, f) \cap \{x : r-\varepsilon < f(x) < r\} \wedge |\mathrm{grad}(f)|) \\ &\quad = o_\varepsilon(1) + \varepsilon^{-1}\gamma(n, k) \int_{D\Lambda^k(R^n)} \underline{\underline{b}}(V, f) \cap \{x : r-\varepsilon < f(x) < r\} \wedge |\mathrm{grad}(f)|\;(\ldots)\;\underline{\underline{\mu}}^{(n:k)}, \end{aligned}$$

$$(a) = o_\varepsilon(1) + \varepsilon^{-1}\gamma(n,k) \int_{\underline{\underline{D}}\Lambda^k(R^n)} V \cap \{x : v-\varepsilon < f(x) < r\} \wedge |\mathrm{grad}(f)| \wedge [|\mathrm{grad}(f)|^{-1}[\mathrm{grad}(f)]^*]\,(\omega(\lambda))\, d\, \underline{\underline{H}}^{(n:k)}\lambda \,.$$

Also

$$\underline{\underline{W}}(\underline{\underline{B}}(V, f, r))$$

$$(b) = o_\varepsilon(1) + \varepsilon^{-1}\gamma(n, k-1) \int_{\underline{\underline{D}}\Lambda^k(R^n)} V \cap \{x : r-\varepsilon < f(x) < r\} \wedge \mathrm{grad}(f)\ (\omega(\eta))\, d\, \underline{\underline{H}}^{(n:k-1)}\eta \,.$$

One verifies by explicit computation that expressions (a) and (b) differ only by $o_\varepsilon(1)$ whenever we set $V \cap \{x : r-\varepsilon < f(x) < r\} = \underline{v}(p, \mu)$ for any $p \in U$ with $r-\varepsilon < f(p) < r$ and any $\mu \in \underline{\underline{D}}\Lambda_k(R^n)$. Since $V \in \underline{\underline{NV}}_k(U)$ by hypothesis, (a) and (b) differ only by $o(1)$. The equality of (5) follows since ε is arbitrary. 7.1(2c) implies the first inequality, and the second inequality is obvious. Conclusion (6) is immediate.

7.5. LEMMA. <u>Let</u> $1 \le k \le n$ <u>be integers,</u> $U \subset R^n$ <u>be open</u> $V, V_1, V_2, V_3, \ldots \in \underline{\underline{NV}}_k(U)$ <u>with</u> $\lim_i \underline{\underline{M}}(V_i, V) = 0$, <u>and</u> $f : U \longrightarrow R$ <u>be of class</u> 1. <u>Then for</u> $\underline{\underline{H}}^1$ <u>almost all</u> $r \in R$,

(1) $\underline{\underline{B}}(V, f, r)$ <u>exists;</u>

(2) $\underline{\underline{B}}(V_i, f, r)$ <u>exists for each</u> $i = 1, 2, 3, \ldots$; <u>and</u>

(3) $\lim_{j} \underline{\underline{M}}(\underline{\underline{B}}(V_{i(j)}, f, r), \underline{\underline{B}}(V, f, r)) = 0$ <u>for some subsequence</u> $i(1), i(2), i(3), \ldots$ of $1, 2, 3, \ldots$.

PROOF. 7.3 implies (1) and (2). We prove (3). Let $V, W \in \underline{\underline{NV}}_k(U)$ and for each $\varepsilon > 0$ let $L(\varepsilon)$ denote all those $r \in \mathrm{range}(f)$ for which both $\underline{\underline{B}}(V, f, r)$ and $\underline{\underline{B}}(W, f, r)$ exist and $\underline{\underline{M}}(\underline{\underline{B}}(V, f, r), \underline{\underline{B}}(W, f, r)) \ge \varepsilon$. For each $r \in L(\varepsilon)$ there exists an open interval $I(r) \subset R$ and $\varphi(r) \in \underline{\underline{C}}_0^{k-1}(U)$ with $|\varphi(r)| \le 1$ such that

$$|V \cap \{x : f(x) \in J(r)\}([\mathrm{grad}(f)]^* \wedge \varphi(r)) - W \cap \{x : f(x) \in J(r)\}([\mathrm{grad}(f)]^* \wedge \varphi(r))| \ge 2^{-1}\varepsilon\, \underline{\underline{H}}^1(J(r))$$

whenever $J(r) \subset I(r)$ is an interval containing the midpoint of $I(r)$. Interchanging V and W if necessary, one can choose a finite set of pairwise disjoint intervals

$J_1, J_2, \ldots, J_q$ and differential forms $\varphi_1, \varphi_2, \ldots, \varphi_q$ such that

$$\sum_i \underline{\underline{H}}^1(J_i) \geq \min\{1,\ 3^{-1}\underline{\underline{H}}^1(L(\varepsilon))\}\ ,$$

and for each i,

$$V \cap \{x : f(x) \in J_i\}([\mathrm{grad}(f)]^* \wedge \varphi_i) - W \cap \{x : f(x) \in J_i\}([\mathrm{grad}(f)]^* \wedge \varphi_i) \geq 3^{-1}\varepsilon \underline{\underline{H}}^1(J_i)\ .$$

One now chooses $\psi_1, \psi_2, \psi_3, \ldots \in \underline{\underline{C}}_0^k(R^n)$ such that if
$\xi = \sup\{|\mathrm{grad}(f)(x)| : x \in \mathrm{spt}(V) \cup \mathrm{spt}(W)\}$, then
(1) $|\psi_i| \leq 2\xi$ for each i;
(2) $\lim_i \psi_i(x) = [\mathrm{grad}(f)]^* \wedge \varphi_j$ for each $x \in U$, and $f(x) \in J_j$ for some $j = 1, 2, \ldots, q$;
and
(3) $\lim_i \psi_i(x) = 0$ whenever $x \notin U$ or $x \in U$ and $f(x) \notin \bigcup_j J_j$.

Lebesgue's theorem on bounded convergence implies that for all sufficiently large values of i,

$$V(\psi_i) - W(\psi_i) \geq 4^{-1}\ \varepsilon \min\{1,\ 3^{-1}\underline{\underline{H}}^1(L(\varepsilon))\}\ .$$

Hence

$$\underline{\underline{M}}(V,\ W) \geq (8\xi)^{-1}\ \varepsilon \min\{1,\ 3^{-1}\underline{\underline{H}}^1(L(\varepsilon))\}$$

or $\underline{\underline{H}}^1(L(\varepsilon)) \leq 24\ \varepsilon^{-1}\xi \underline{\underline{M}}(V,\ W)$ whenever $\underline{\underline{M}}(V,\ W) < (8\xi)^{-1}\varepsilon$. One now chooses a subsequence $i(1), i(2), i(3), \ldots$ of $1, 2, 3, \ldots$ such that $\sum_j \underline{\underline{M}}(V_{i(j)},\ V) < \infty$. Then for each $\varepsilon > 0$,

$$0 = \underline{\underline{H}}^1(R \cap \{r : \underline{\underline{B}}(V,\ f,\ r) \text{ exists, } \underline{\underline{B}}(V_{i(j)},\ f,\ r) \text{ exists for each } j,$$
$$\text{and } \underline{\underline{M}}(\underline{\underline{B}}(V_{i(j)},\ f,\ r),\ \underline{\underline{B}}(V,\ f,\ r)) \geq \varepsilon \text{ for infinitely many}$$
$$\text{positive integers } j\})\ .$$

The lemma follows since ε is arbitrary.

7.6. THEOREM. Let $1 \leq k \leq n$ be integers, $U \subset R^n$ be open, $V \in \underline{\underline{IV}}_k(U)$, $W \in \underline{\underline{RV}}_k(U)$, and $f : R^n \longrightarrow R$ be of class 1. Then for $\underline{\underline{H}}^1$ almost

$r \in R$, $\mathbf{B}(V, f, r) \in \mathbf{IV}_{k-1}(U)$ and $\mathbf{B}(W, f, r) \in \mathbf{RV}_k(U)$.

PROOF. In view of 5.4(1) and 7.5 it is sufficient to prove the theorem in case $V = W = |A|$ where $A \subset R^n$ is a compact submanifold of R^n of dimension k and class 1. One uses [F3 3.1, 3.2] to verify that for $\mathbf{H}^1$ almost all $r \in R$, $f^{-1}(r) \cap A$ is a k-rectifiable subset of R^n, and $\mathbf{B}(|A|, f, r) = |A \cap f^{-1}(r)|$. The theorem follows.

7.7. REMARK. Let $1 \leq k \leq n$ be integers and $(V, W) \in \mathbf{V}_k(R^n, R^n)$ with $\mathbf{P}(R^n)(V, W) < \infty$. It is not difficult to verify the existence of $Y \in \mathbf{V}_{k-1}(R^n)$ such that $\mathbf{P}(R^n)(V, Y) \leq \mathbf{P}(R^n)(V, W)$ and $\mathbf{W}(Y) = \inf\{\mathbf{W}(U) : U \in \mathbf{V}_{k-1}(R^n)$ and $\mathbf{P}(R^n)(V, U) \leq \mathbf{P}(R^n)(V, W)\}$. Also if $W \in \mathbf{RV}_{k-1}(R^n)$, there exists a unique $Z \in \mathbf{V}_{k-1}(R^n)$ such that $Z \leq W$ (which implies $Z \in \mathbf{RV}_{k-1}(R^n)$) and $\mathbf{W}(Z) = \inf\{\mathbf{W}(U) : U \in \mathbf{V}_{k-1}(R^n)$, $U \leq W$, and $\mathbf{P}(R^n)(V, U) \leq \mathbf{P}(R^n)(V, W)\}$. The uniqueness of Z follows from the observation that if $W, Z_1, Z_2 \in \mathbf{RV}_k(R^n)$ with $Z_1 \leq W$ and $Z_2 \leq W$, then $Z_1 \cap Z_2 \leq W$ and

$$\mathbf{P}(R^n)(V, Z_1 \cap Z_2) \leq \max\{\mathbf{P}(R^n)(V, Z_1), \mathbf{P}(R^n)(V, Z_2)\} .$$

The above statements remain valid if $\mathbf{Q}$ replaces $\mathbf{P}$ above.

8. INEQUALITIES FOR REGULAR VARIFOLDS.

8.1. THEOREM. Let $0 \le m < k \le n$ be integers, $c \in R^+$, and $A \subset R^n$ a closed m-dimensional submanifold of R^n of class 3 without boundary. Let $(V, W) \in \underline{\underline{GV}}_k(R^n, R^n)$ with

$$\liminf_{t \to 0^+} t^{-1} \underline{\underline{W}}(V \cap \{x : 0 < \operatorname{dist}(x, \underline{\underline{BQ}}^c(R^n, R^n)(V, W)) < t\}) < \infty .$$

Then there exists $r_0 \in R_0^+ \cup \{\infty\}$ such that

(1) for $\underline{\underline{H}}^1$ almost all r, $0 < r < r_0$, $\underline{\underline{\ell}}'(r)$ and $\underline{\underline{m}}'(r)$ exist and

$$\begin{aligned} 0 &\le -(k-m)\underline{\underline{m}}(r) + r\underline{\underline{\ell}}'(r) + r\underline{\underline{n}}_1(r) + kr\underline{\underline{n}}_2(r) \\ &\qquad + mr(\Delta_1^{-1} - r)^{-1}\underline{\underline{m}}(r) + cr\underline{\underline{m}}(r) \\ &\le -(k-m)\underline{\underline{m}}(r) + r\underline{\underline{m}}'(r) + r\underline{\underline{n}}_1(r) + kr\underline{\underline{n}}_2(r) \\ &\qquad + mr(\Delta_1^{-1} - r)^{-1}\underline{\underline{m}}(r) + cr\underline{\underline{m}}(r) . \end{aligned}$$

(2) $[r^{k-m}(\Delta_1^{-1} - r)^m]^{-1}\underline{\underline{m}}(r) \cdot \exp\left[cr + \int_0^r [\underline{\underline{m}}(t)]^{-1}[\underline{\underline{n}}_1(t) + k\underline{\underline{n}}_2(t)]dt\right]$

is non-decreasing as a function of r for $0 < r < r_0$ provided the integral exists.

(3) Conclusions (1) and (2) remain valid if $\underline{\underline{BQ}}^c$ is replaced with $\underline{\underline{BP}}^c$ throughout.

Here for $0 < r < r_0$,

$\underline{\underline{\ell}}(r) = \underline{\underline{W}}(\underline{\underline{b}}(V, \operatorname{dist}(\cdot, A)) \cap [\{x : 0 < \operatorname{dist}(x, A) < r\} - \underline{\underline{BQ}}^c(R^n, R^n)(V, W)])$

(which has the obvious meaning)

$$\underline{\underline{m}}(r) = \underline{\underline{W}}(V \cap [\{x : 0 < \operatorname{dist}(x, A) < r\} - \underline{\underline{BQ}}^c(R^n, R^n)(V, W)])$$

$$\begin{aligned} \underline{\underline{n}}_1(r) &= r^{-1}\underline{\underline{T}}(W \cap [\{x : \operatorname{dist}(x, A) < r\} - \underline{\underline{BQ}}^c(R^n, R^n)(V, W)], g) \\ &\le \underline{\underline{W}}(W \cap [\{x : 0 < \operatorname{dist}(x, A) < r\} - \underline{\underline{BQ}}^c(R^n, R^n)(V, W)]) \end{aligned}$$

where $g(x) = \operatorname{dist}(x, A) \cdot \operatorname{grad}(\operatorname{dist}(\cdot, A))$ is well defined for $0 < \operatorname{dist}(x, A) < r_0$.

$\underline{\underline{n}}_2(r) = \liminf_{t \to 0^+} t^{-1}\underline{\underline{W}}(V \cap \{x : \operatorname{dist}(x, \underline{\underline{BQ}}^c(R^n, R^n)(V, W) - A) < t$ and $\operatorname{dist}(x, A) < r\})$

$\Delta_1 = \sup\{\Delta_1(A, R^n; x) : x \in A \cap \{x : \operatorname{dist}(x, \operatorname{spt}(V)) < r_0\}\}$

whenever $m \geq 1$. In case either $m = 0$ or $m \geq 1$ and $\Delta_1 = 0$, we take $[r(\Delta_1^{-1} - r)^{-1} \underline{\underline{m}}(r)] = 0$ in (1) and $[(\Delta_1^{-1} - r)^m] = 1$ in (2).

PROOF.

Part 1. Let $0 \leq m < n < \infty$ and $f^{m+1}, f^{m+2}, \ldots, f^n : R^n \longrightarrow R$ be of class 3 with $f^j(0) = 0$ and $(\partial f^j / \partial x^i)(0) = 0$ for each $j = m+1, m+2, \ldots, n$ and each $i = 1, 2, \ldots, m$. Suppose also $a_1, a_2, \ldots, a_m \in R$ and

$$f^n(x^1, x^2, \ldots, x^m) = \sum_{i=1}^m a_i (x^i)^2 + 0_{(x^1, x^2, \ldots, x^m)}([\sum_{i=1}^m (x^i)^2]^{3/2}) .$$

Define

$$A = R^n \cap \{x : x^j = f^j(x^1, x^2, \ldots, x^m) \text{ for each } j = m+1, m+2, \ldots, n\} ,$$

and $\underline{\underline{r}} : R^n \longrightarrow R^+$, $\underline{\underline{r}}(x) = \text{dist}(x, A)$ for $x \in R^n$, and $g : R^n \longrightarrow R^n$ given for $x \in R^n$ by $g(x) = \underline{\underline{r}}(x)\text{grad}(\underline{\underline{r}})(x)$ whenever $\text{grad}(\underline{\underline{r}})(x)$ exists and $g(x) = 0$ in case $\text{grad}(\underline{\underline{r}})(x)$ does not exist. One verifies that in some neighborhood U of $0 \in R^n$, $g|U$ is of class 2.

Let $x_0^n \in R^+$ with $x_0^n < \inf\{a_i^{-1} : a_i > 0 \text{ and } i = 1, 2, \ldots m\}$ and $p = (0, 0, \ldots, x_0^n) \in U$. We compute

$$\begin{aligned} & g(x^1, x^2, \ldots, x^n) \\ &= ((a_1 x_0^n - 1)^{-1} a_1 x^1 x_0^n + 0_{x-p}(|x-p|^2) , \\ &\qquad (a_2 x_0^n - 1)^{-1} a_2 x^2 x_0^n + 0_{x-p}(|x-p|^2), \ldots, \\ &\qquad (a_m x_0^n - 1)^{-1} a_m x^m x_0^n + 0_{x-p}(|x-p|^2), x^{m+1}, x^{m+2}, \ldots x^n) . \end{aligned}$$

Let $m < k \leq n$ and $1 \leq i_1 < i_2 < \ldots < i_h \leq m < i_{h+1} < \ldots < i_k \leq n$. Then

$$\begin{aligned} & \underline{\underline{S}}(\underline{\underline{v}}(p, (\partial/\partial x^{i_1}) \wedge (\partial/\partial x^{i_2}) \wedge \ldots \wedge (\partial/\partial x^{i_k})), fg) \\ &\qquad = \sum_{j=1}^k (a_{i_j} x_0^n - 1)^{-1} a_{i_j} x_0^n + k-h \\ &\qquad \geq (k-h) - h x_0^m [\inf\{a_i^{-1} : a_i > 0 \text{ and } i = 1, 2, \ldots, m\} - x_0^n]^{-1} \\ &\qquad \geq (k-m) - m\underline{\underline{r}}(p)[\inf\{a_i^{-1} : a_i > 0 \text{ and } i = 1, 2, \ldots, m\} - \underline{\underline{r}}(p)]^{-1} \end{aligned}$$

whenever $f \in \mathbf{C}_0^0(\mathbf{R}^n)$ is identically equal to 1 in some neighborhood of p and $g \in \mathbf{X}(\mathbf{R}^n)$.

Part 2. Let m, k, n, A, V, W be as in the hypothesis of the theorem and $\mathbf{r}, g$ as in part 1. Then there exists a largest $r_0 \in \mathbf{R}_0^+ \cup \{\infty\}$ such that $\mathbf{r}|\{x : \mathbf{r}(x) < r_0$ and $\operatorname{dist}(x, \operatorname{spt}(V)) < r_0\}$ and $g|\{x : \mathbf{r}(x) < r_0$ and $\operatorname{dist}(x, \operatorname{spt}(V)) < r_0\}$ are of class 2. We conclude from part 1 that, in case $V \in \mathbf{GV}_k(\{x : \mathbf{r}(x) < r_1\} \cap U)$, where $0 < r_1 < r_0$, then

$$\mathbf{S}(V, fg) \geq [(k-m) - mr_1(\Delta_1^{-1} - r_1)^{-1}]\mathbf{W}(V)$$

whenever $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ of class ∞ satisfies $f|\operatorname{spt}(V) = 1$ and $f|\{x : \mathbf{r}(x) \geq r_0\} = 0$.

Part 3. Let $m, k, n, A, V, W, \mathbf{r}, g, r_0$ be as in part 2 and $0 < r < r_0$, for which $\mathbf{m}'(r)$ exists. For each ε, $0 < \varepsilon < 3^{-1}r$ we choose $f_\varepsilon : \mathbf{R}^n \longrightarrow \{t : 0 \leq t \leq 1\}$ of class ∞ having compact support such that for x in some neighborhood of $\operatorname{spt}(V) \cup \operatorname{spt}(W)$

(i) $f_\varepsilon(x) = 1$ if $\varepsilon \leq \mathbf{r}(x) \leq r-\varepsilon$ and $\operatorname{dist}(x, \operatorname{clos}[\mathbf{BQ}^c(\mathbf{R}^n, \mathbf{R}^n)(V, W) - A]) \geq \varepsilon$

(ii) $f_\varepsilon(x) = 0$ if $\mathbf{r}(x) \geq r$ or if $\operatorname{dist}(x, A \cup \mathbf{BQ}^c(\mathbf{R}^n, \mathbf{R}^n)(V, W)) \leq \varepsilon^2$

(iii) $\operatorname{Lip}(f_\varepsilon) \leq [\varepsilon - 2\varepsilon^2]^{-1}$.

By the definition of $\mathbf{BQ}^c(\mathbf{R}^n, \mathbf{R}^n)(V, W)$, we have that

$$[\mathbf{W}V(|f_\varepsilon g|)]^{-1}[|\mathbf{S}(V, f_\varepsilon g)| - \mathbf{T}(|f_\varepsilon g|)] \leq c$$

which when evaluated using 6.4(4b, 4c), 6.10(2), and part 2, one obtains for arbitrarily small values of $\varepsilon > 0$:

$$(k-m)\mathbf{m}(r) - mr(\Delta_1^{-1} - r)^{-1}\mathbf{m}(r) - (\varepsilon - 2\varepsilon^2)^{-1}r[\boldsymbol{\ell}(r) - \boldsymbol{\ell}(r-\varepsilon)]$$

$$- rk(\varepsilon - 2\varepsilon^2)^{-1}\mathbf{W}(V \cap \{x : \operatorname{dist}(x, \mathbf{BQ}^c(\mathbf{R}^n, \mathbf{R}^n)(V, W) - A) < \varepsilon \text{ and } \operatorname{dist}(x, A) < r\})$$

$$- r\mathbf{n}_1(r) + o_\varepsilon(1) \leq cr\mathbf{m}(r) .$$

Taking the lower limit as $\varepsilon \longrightarrow 0^+$ and noting that for $\mathbf{H}^1$ almost all r, $\boldsymbol{\ell}'(r) \leq \ldots(r)$.

obtains conclusion (1). $\underline{m}$, $\underline{n}_1$, and $\underline{n}_2$ are clearly non-decreasing in s, $0 < s < r_0$, and hence differentiable for $\underline{H}^1$ almost all r, $0 < r < r_0$. Integration of conclusion (1) after dividing through by $r\underline{m}(r)$, yields

$$0 \le (d/ds)[-(k-m)\log s + \log \underline{m}(s) + \int [\underline{m}(t)]^{-1}[\underline{n}_1(t) + k\underline{n}_2(t)]dt - m\log(\Delta_1^{-1} - r) + cs]\Big|_{s=r}.$$

Conclusion (2) follows since the logarithm function is non-decreasing. Conclusion (3) follows by virtually identical arguments.

8.2. COROLLARY. Let $1 \le k \le n$ be integers, $C \subset R^n$ be closed, $(V, W) \in \underline{GV}_k(R^n, R^n)$, and $p \in R^n - [C \cup \mathrm{spt}(W)]$.

(1) If $\underline{P}(R^n; C)(V, W) < \infty$, then

$$r^{-k}\underline{W}(V \cap \underline{D}_0^n(p, r))\exp[r\underline{P}(R^n; C)(V, W)]$$

is non-decreasing as a function of r for $0 < r < \mathrm{dist}(p, C \cup \mathrm{spt}(W))$.

(2) If $\underline{Q}(R^n; C)(V, W) < \infty$, then

$$r^{-k}\underline{W}(V \cap \underline{D}_0^n(p, r))\exp[r\underline{Q}(R^n; C)(V, W)]$$

is non-decreasing as a function of r for $0 < r < \mathrm{dist}(p, C \cup \mathrm{spt}(W))$.

(3) If $\underline{Q}(R^n; C)(V, W) = 0$, then $r^{-k}\underline{W}(V \cap \underline{D}_0^n(0, r))$ is non-decreasing as a function of r for $0 < r < \mathrm{dist}(p, C \cup \mathrm{spt}(W))$.

8.3. COROLLARY. Let $1 \le k \le n$ be integers, $C \subset R^n$ be closed, and $W \in \underline{GV}_{k-1}(R^n)$. Let $V, V_1, V_2, V_3, \ldots \in \underline{GV}_k(R^n)$ with $\lim_i V_i = V$ and $\sup_i \underline{Q}(R^n; C)(V_i, W) < \infty$, and $p, p_1, p_2, p_3, \ldots \in R^n - (C \cup \mathrm{spt}(W))$ with $\lim_i p_i = p$. Then

(1) $\Theta^k(\underline{W}V, p)$ exists;

(2) $\Theta^k(\underline{W}V, \cdot) : R^n - (C \cup \mathrm{spt}(W)) \longrightarrow R^+$ is upper semi-continuous;

(3) $\Theta^k(\underline{W}V, p) \ge \limsup_i \Theta^k(\underline{W}V_i, p_i)$;

(4) $\Theta^k(\underline{W}V, p) \le \alpha(k)^{-1}\mathrm{dist}(p, C \cup \mathrm{spt}(W))^{-k}\underline{W}(V) \cdot \exp(\mathrm{dist}(p, C \cup \mathrm{spt}(W)) \cdot \underline{Q}(R^n; C)(V, \ldots$

8.4. COROLLARY. Let $1 \le k \le n$ be integers and $V \in \underline{GV}_k(R^n)$ with $\underline{Q}(R^n)(V) < \infty$. Then

(1) $\operatorname{spt}(V) \cap [R^n - \mathbf{D}_0^n(p, r)] \neq \emptyset$ whenever $p \in R^n$ and $r \in R_0^+$ with $r\mathbf{Q}(R^n)(V) < k$; and

(2) $(\operatorname{diam}[\operatorname{spt}(V)]) \cdot \mathbf{Q}(R^n)(V) \geq k$.

PROOF. If $\operatorname{spt}(V) \subset \mathbf{D}^n(0, t)$ for some $t \in R_0^+$, then $r^{-k}\mathbf{W}(V)\exp(r\mathbf{Q}(R^n)(V))$ is non-decreasing as a function of r for $r > t$, which implies

$$0 \leq (d/ds)[s^{-k}\mathbf{W}(V)\exp(s\mathbf{Q}(R^n)(V))|_{s=r}$$

$$= \mathbf{W}(V)r^{-k}\exp(r\mathbf{Q}(R^n)(V))(Q(R^n)(V) - r^{-1}k) \quad ,$$

from which it follows that $r\mathbf{Q}(R^n)(V) \geq k$ and hence $t\mathbf{Q}(R^n)(V) \geq k$ which implies conclusion (1). Conclusion (2) is immediate.

8.5. COROLLARY. Let $1 \leq k \leq n$ be integers and $A \subset R^n$ be a compact $(k-1)$-dimensional submanifold of R^n of class 3 without boundary. For each $c < \infty$ and $d > 0$ there exists $b > 0$ such that if $V \in \mathbf{GV}_k(R^n)$ with

(1) either $\mathbf{Q}(R^n; A)(V) \leq c$ or $\mathbf{Q}(R^n)(V, |A|) \leq c$,

(2) $\operatorname{spt}(V) \cap (R^n - A) \neq \emptyset$, and

(3) $\inf\{\Theta^k(\mathbf{W}V, p) : p \in \operatorname{spt}(V) - A\} \geq d$, then $\mathbf{W}(V) \geq b$.

PROOF. One shows first the existence of a positive lower bound on the maximum distance from points in $\operatorname{spt}(V)$ to A by arguments using 8.1 in a similar way to its use in 8.4. The conclusion now follows from 8.3(4) and the lower bound on the density.

8.6. LEMMA. To each non-negative integer n there corresponds a positive integer $N(n)$ with the following properties:

(1) If $B_1, B_2, B_3, \ldots \subset R^n$ comprise a sequence of bounded open balls in R^n such that $\operatorname{center}(B_i) \notin B_j$ for $i \neq j$, then each point $p \in R^n$ is contained within at most $N(n)$ distinct balls of the sequence;

(2) If $A \subset R^n$ is compact and $f : A \longrightarrow R_0^+$ is upper semi-continuous, then there exists a finite number of points $p_1, p_2, \ldots, p_q \in A$ such that the open balls $B_i = \mathbf{D}_0^n(p_i, f(p_i))$, $i = 1, 2, \ldots, q$, cover A and each point $p \in R^n$ is contained within at most $N(n)$ distinct balls.

PROOF. The proof of (1) follows by obvious arguments from the proof of lemma 1 in [B I]. To prove (2) one chooses $p_1 \in A$ such that $f(p_1) = \max\{f(x) : x \in A\}$, chooses $p_2 \in A - \underline{\underline{D}}^n_0(p_1, f(p_1))$ such that $f(p_2) = \max\{f(x) : x \in A - \underline{\underline{D}}^n_0(p_1, f(p_1))\}$, etc.

8.7. THEOREM. Let $1 \le k \le n$ be integers, $C \subset R^n$ be closed, and $(V, W) \in \underline{\underline{GV}}_k(R^n, R^n)$ with $\underline{\underline{W}}V(\operatorname{spt}(W) \cup C) = 0$. If

$$[\underline{\underline{Q}}(R^n; C)(V, W)]^k \underline{\underline{W}}(V) \le a(k) d(V; W, C) ,$$

then

(1) $$d(V; W, C)^{1/(k-1)} \underline{\underline{W}}(V) \le b(k, n)[\underline{\underline{W}}(W) + \underline{\underline{W}}(C)]^{k/(k-1)}$$

if $k \ge 2$, and

(2) $$d(V; W, C) \le b(1, n)[\underline{\underline{W}}(W) + \underline{\underline{W}}(C)] \text{ if } k = 1.$$

Here

$$a(k) = 2^{-k-1} \alpha(k)[\log(2)]^k ,$$

$$b(1, n) = 2[\log(2)]^{-1} N(n) ,$$

$$b(k, n) = 2^{-1} \alpha(k)^{-1/(k-1)} [\log(2)]^{-k/(k-1)} [4N(n)]^{k/(k-1)} \text{ for } k \ge 2 ,$$

$$d(V; W, C) = \inf\{ \odot^k(\underline{\underline{W}}V, x) : x \in \operatorname{spt}(V) - (\operatorname{spt}(W) \cup C)\}, \text{ and}$$

$$\underline{\underline{W}}(C) = k \liminf_{t \to 0^+} t^{-1} \underline{\underline{W}}(V \cap \{x : \operatorname{dist}(x, C) < t\}) .$$

PROOF.

Part 1. In view of the mappings $\underline{\underline{GV}}_k(R^n) \longrightarrow \underline{\underline{GV}}_k(R^n)$ which (a) multiply varifolds by a fixed positive real number changing the densities accordingly, and (b) are induced by elementary expansions or contractions (as in 6.19), we can assume without loss of generality that $\underline{\underline{W}}(V) = 2^{-1}$, $\underline{\underline{Q}}(R^n; C)(V, W) < \infty$, and

$$\inf\{ \odot^k(\underline{\underline{W}}V, p) : p \in \operatorname{spt}(V) - (\operatorname{spt}(W) \cup C)\} = \alpha(k)^{-1} ,$$

since if any of these conditions is impossible to achieve, the theorem holds trivially.

Let $p \in \mathrm{spt}(V) - (\mathrm{spt}(W) \cup C)$ and define for each $r \in R_0^+$,

$\underline{\underline{m}}(r) = \underline{\underline{W}}(V \cap \underline{\underline{D}}_0^n(p, r))$, $\underline{\underline{n}}_1(r) = \underline{\underline{W}}(W \cap \underline{\underline{D}}_0^n(p, r))$, and

$$\underline{\underline{n}}_2(r) = \liminf_{t \to 0^+} t^{-1} \underline{\underline{W}}(V \cap \underline{\underline{D}}_0^n(p, r) \cap \{x : \mathrm{dist}(x, C) < t\} .$$

Taking $A = \{p\}$ in 8.1, we have by 8.1 and 8.3 that

$$1^{-k}\underline{\underline{m}}(1) \exp\left[\underline{\underline{Q}}(R^n; C)(V, W) + \int_0^1 [\underline{\underline{m}}(r)]^{-1}[\underline{\underline{n}}_1(r) + k\underline{\underline{n}}_2(r)]dr\right]$$

$$\geq \lim_{r \to 0^+} r^{-k}\underline{\underline{m}}(r) \exp\left[r\underline{\underline{Q}}(R^n; C)(V, W) + \int_0^1 [\underline{\underline{m}}(r)]^{-1}[\underline{\underline{n}}_1(r) + k\underline{\underline{n}}_2(r)]dr\right]$$

$$\geq \alpha(k) \inf\{ \Theta^k(\underline{\underline{W}}V, x) : x \in \mathrm{spt}(V) - (\mathrm{spt}(W) \cup C)\}$$

$$= 1 .$$

Our assumption that $\underline{\underline{m}}(1) \leq \underline{\underline{W}}(V) = 2^{-1}$ implies that

$$2^{-1} \exp\left[\underline{\underline{Q}}(R^n; C)(V, W) + \int_0^1 [\underline{\underline{m}}(r)]^{-1}[\underline{\underline{n}}_1(r) + k\underline{\underline{n}}_2(r)]dr\right] \geq 1 ,$$

or, equivalently,

$$\int_0^1 [\underline{\underline{m}}(r)]^{-1}[\underline{\underline{n}}_1(r) + k\underline{\underline{n}}_2(r)]dr \geq \log(2) - \underline{\underline{Q}}(R^n; C)(V, W) .$$

Our hypothesis that

$$2^{k+1}[\underline{\underline{Q}}(R^n; C)(V, W)]^k \underline{\underline{W}}(V) \leq [\log(2)]^k \alpha(k) \inf\{ \Theta^k(\underline{\underline{W}}V, x) : x \in \mathrm{spt}(V) - (\mathrm{spt}(W) \cup C)\}$$

implies $\underline{\underline{Q}}(R^n; C)(V, W) \leq 2^{-1}\log(2)$, giving

$$\int_0^1 [\underline{\underline{m}}(r)]^{-1}[\underline{\underline{n}}_1(r) + k\underline{\underline{n}}_2(r)]dr \geq 2^{-1}\log(2) .$$

Therefore

$$\{t : 0 \leq t \leq 1 \text{ and } \underline{\underline{n}}_1(t) + k\underline{\underline{n}}_2(t) \geq 2^{-1}\log(2)\underline{\underline{m}}(t)\}$$

has positive $\underline{\underline{H}}^1$ measure and is closed from the left.

Part 2. Let $f_1, f_2, f_3, \ldots : R^n \longrightarrow R^+$ form a non-decreasing sequence of continuous functions converging pointwise to the set function f of $\underline{\underline{D}}_0^n(0, 1)$. For each $x \in R^n$, $r \in R_0^+$, and $i = 1, 2, 3, \ldots$ we define

$$\mu_i(x, r) = \int_{R^n} f_i(r^{-1}(y-x)) d\underline{\underline{W}} Vy \text{ and } \nu_i(x, r) = \int_{R^n} f_i(r^{-1}(y-x)) d\underline{\underline{W}} Wy ,$$

noting that each μ_i and each ν_i is continuous on the domain $R^n \times R_0^+$ and that the sequences $\{\mu_i\}_i$ and $\{\nu_i\}_i$ are non-decreasing. One observes that the functions μ and ν^1 given for $(x, r) \in R^n \times R_0^+$ by

$$\mu(x, r) = \lim_i \mu_i(x, r) = \int_{R^n} f(r^{-1}(y-x)) d\underline{\underline{W}} Vy = \underline{\underline{W}}(V \cap \underline{\underline{D}}_0^n(x, r)) ,$$

$$\nu^1(x, r) = \lim_i \nu_i(x, r) = \int_{R^n} f(r^{-1}(y-x)) d\underline{\underline{W}} Wy = \underline{\underline{W}}(W \cap \underline{\underline{D}}_0^n(x, r)) ,$$

are Baire functions on $R^n \times R_0^+$.

Let $\varepsilon > 0$ and $g_1^\varepsilon, g_2^\varepsilon, g_3^\varepsilon, \ldots : R^n \longrightarrow R^+$ form a non-decreasing sequence of continuous functions converging pointwise to the set functions g^ε of $\{x : \operatorname{dist}(x, C) < \varepsilon\}$. For each $x \in R^n$, $r \in R_0^+$, and $i = 1, 2, 3, \ldots$ we define

$$\nu_i^2(x, r, \varepsilon) = \int_{R^n} f_i(r^{-1}(y-x)) g_i^\varepsilon(y) d\underline{\underline{W}} Vy ,$$

noting that each $\nu_i^2(\cdot, \cdot, \varepsilon)$ is continuous on the domain $R^n \times R_0^+$. One observes then that the function $\nu^2(\cdot, \cdot, \varepsilon)$ given for $(x, r) \in R^n \times R_0^+$ by

$$\nu^2(x, r, \varepsilon) = \lim_i \nu_i^2(x, r, \varepsilon) = \int_{R^n} f_i(r^{-1}(y-x)) g^\varepsilon(y) d\underline{\underline{W}} Vy$$

$$= \underline{\underline{W}}(V \cap \underline{\underline{D}}_0^n(x, r) \cap \{y : \operatorname{dist}(y, C) < \varepsilon\} .$$

is a Baire function on $R^n \times R_0^+$.

It is not difficult to verify the existence of a decreasing sequence $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \in R_0^+$ with $\lim_i \varepsilon_i = 0$ such that if $\varphi : R_0^+ \longrightarrow R_0^+$ is non-dec[illegible] then

$$\liminf_{\delta\to 0^+} \delta^{-1}\varphi(\delta) = \liminf_{i\to\infty} \varepsilon_i^{-1}\varphi(\varepsilon_i) = \lim_{i\to\infty} \inf\{\varepsilon_j^{-1}\varphi(\varepsilon_j) : j > i\} ,$$

noting that $\{\inf\{\varepsilon_j^{-1}\varphi(\varepsilon_j) : j > i\}\}_i$ is a non-decreasing sequence. Letting $\{\varepsilon_i\}_i$ be as above, we define ${}_j\nu^2 : R^n \times R_0^+ \longrightarrow R^+$ for each $j = 1, 2, 3, \dots$ by setting for each $(x, r) \in R^n \times R_0^+$,

$${}_j\nu^2(x, r) = \inf\{\nu^2(x, r, \varepsilon_i) : i \geq j\} .$$

Each ${}_j\nu^2$ is a Baire function since it is the infimum of a sequence of Baire functions. We define $\nu^2 : R^n \times R_0^+ \longrightarrow R^+$ by setting for each $(x, r) \in R^n \times R_0^+$,

$$\nu^2(x, r) = \lim_j {}_j\nu^2(x, r) .$$

ν^2 is a Baire function since it is the pointwise limit of a sequence of Baire functions. Note that in the notation of part 1, $\underline{\underline{m}}(r) = \mu(p, r)$, $\underline{\underline{n}}_1(r) = \nu^1(p, r)$, and $\underline{\underline{n}}_2(r) = \nu^2(p, r)$. Since μ, ν^1, and ν^2 are Baire functions, so is

$$\eta = \nu^1 + k\nu^2 - 2^{-1}\log(2)\mu .$$

We set

$$Y = R^n \times R_0^+ \cap \{(x, r) : \eta(x, r) \geq 0 \text{ and } 0 < r \leq 1\} .$$

Clearly Y is a Borel set, and part 1 implies that for $\underline{\underline{W}}V$ almost all $x \in R^n$, $(x, r) \in Y$ for some r. We define $h : R^n \longrightarrow \{r : 0 \leq r \leq 1\}$ by setting for each $x \in R^n$, $h(x) = \sup\{r : (x, r) \in Y\}$ if $(x, r) \in Y$ for some r, and $h(x) = 0$ otherwise. Observe that for each $t \in R_0^+$,

$$R^n \cap \{x : h(x) \geq t\} = q(Y \cap \{(x, r) : r \geq t\}) .$$

where $q : R^n \times R_0^+ \longrightarrow R^n$, $q(x, r) = x$ for $(x, r) \in R^n \times R_0^+$. Since $Y \cap \{(x, r) : r \geq t\}$ is a Borel set, $q(Y \cap \{(x, r) : r \geq t\})$ is a Suslin (analytic) set which makes $q(Y \cap \{(x, r) : r \geq t\})$ $\underline{\underline{W}}V$ measurable (since $\underline{\underline{W}}V$ measures all closed sets, it measures Suslin sets). Hence h is $\underline{\underline{W}}V$ measurable.

For each $\delta > 0$ Luzin's theorem guarantees the existence of a compact set $H \subset R^n$ such that $h|H$ is continuous, $h(x) > 0$ for each $x \in H$, and $\underline{\underline{W}}V(H) \geq \underline{\underline{W}}V(R^n) - \delta$. Using 8.6, we cover H by a finite number of open balls $B_1, B_2, \ldots, B_m$ such that center$(B_i) \notin B_j$ for $i \neq j$ and each point $x \in R^n$ is contained within no more than $N(n)$ distinct balls. One concludes, therefore, that

$$
\begin{aligned}
& N(n)[\underline{\underline{W}}(W) + k \liminf_{\varepsilon \to 0^+} \varepsilon^{-1} \underline{\underline{W}}(V \cap \{x : \operatorname{dist}(x, C) < \varepsilon\})] \\
& \geq \sum_{i=1}^{m} \underline{\underline{W}}W(B_i) + k \liminf_{\varepsilon \to 0^+} \varepsilon^{-1} W(V \cap B_i \cap \{x : \operatorname{dist}(x, C) < \varepsilon\}) \\
& \geq \sum_{i=1}^{m} 2^{-1} \log(2) \underline{\underline{W}}V(B_i) \\
& \geq 2^{-1} \log(2) \underline{\underline{W}}V(\cup_i B_i) \\
& \geq 2^{-1} \log(2) \underline{\underline{W}}V(H) \\
& \geq 2^{-1} \log(2)(\underline{\underline{W}}(V) - \delta) \\
& \geq 4^{-1} \log(2) - 2^{-1}\delta \log(2)
\end{aligned}
$$

which implies, since δ is arbitrarily small, that for $k \geq 2$,

$$
\begin{aligned}
& [\underline{\underline{W}}(W) + k \liminf_{\varepsilon \to 0^+} \varepsilon^{-1} \underline{\underline{W}}(V \cap \{x : \operatorname{dist}(x, C) < \varepsilon\})]^{k/(k-1)} \\
& \geq N(n)^{-k/(k-1)} 2^{-2k/(k-1)} \cdot \log(2)^{k/(k-1)} ,
\end{aligned}
$$

and

$$
\begin{aligned}
& [\underline{\underline{W}}(W) + k \liminf_{\varepsilon \to 0^+} \varepsilon^{-1} \underline{\underline{W}}(V \cap \{x : \operatorname{dist}(x, C) < \varepsilon\}]^{k/(k-1)} \\
& \geq N(n)^{-k/(k-1)} 2^{-2k/(k-1)} \log(2)^{k/(k-1)} \cdot 2\underline{\underline{W}}(V) \cdot \alpha(k) \cdot \inf\{ \odot^k(\underline{\underline{W}}V, x) : x \in \operatorname{spt}(V) \\
& \qquad - (\operatorname{spt}(W) \cup C)\} .
\end{aligned}
$$

The theorem follows if $k \geq 2$. For $k = 1$ the theorem is immediate.

8.8. COROLLARY. Let $1 \leq k \leq n$ be integers and $V \in \underline{\underline{GV}}_k(R^n)$. Then $[\underline{Q}(R^n)(V)]^k \underline{\underline{W}}(V) > a(k) \sup\{ \odot^k(\underline{\underline{W}}V, x) : x \in \operatorname{spt}(V)\}$ where $a(k)$ is as in 8.7.

8.9. COROLLARY. Let $2 \le k \le n$ be integers and $a(k)$ and $b(k, n)$ as in 8.7. Let A be a compact k-dimensional manifold of class 3 with boundary B, and let $f : A \longrightarrow R^n$ be a class 3 immersion of A into R^n. If M denotes the maximum mean curvature of $f(A)$ in R^n and M^k (k-area of $f|A) \le a(k)$, then (k-area of $f|A) \le b(k, n)((k-1)$-area of $f|B)^{k/(k-1)}$. In particular, if f satisfies the minimal surface equation, then, without additional hypotheses,

$$(\text{k-area of } f|A) \le b(k, n)((\text{k-1})\text{-area of } f|B)^{k/(k-1)} .$$

8.10. REMARK. Let $2 \le k \le n$ be integers and $b(k, n)$ be as in 8.7. Let G denote the additive group of integers and $T \in \underline{\underline{Z}}_k(R^n, R^n; G)$ be minimal in the weak sense [FL2, p. 17]. The example below shows that it is not true in general that $\underline{\underline{M}}(T) \le b(k, n)\underline{\underline{M}}(\partial T)^{k/(k-1)}$. However, if for some $\varepsilon > 0$, $\underline{\underline{M}}(S) \ge \underline{\underline{M}}(T)$ whenever $S \in \underline{\underline{Z}}_k(R^n, R^n; G)$, $\partial S = \partial T$, and $\underline{\underline{F}}(S, T) < \varepsilon$, then $\underline{\underline{M}}(T) \le b(k, n)\underline{\underline{M}}(\partial T)^{k/(k-1)}$.

Example. Let $C = \partial \underline{\underline{D}}^2(0, 1) \times \{0\} \subset R^3$ be given an orientation and regarded as an element of $\underline{\underline{Z}}_1(R^3, \emptyset; G)$ where G is the group of integers. Let $\varepsilon > 0$ and D_ε be an oriented smooth simple closed curve of length no greater than ε, which coincides with C along an arc of length at least $3^{-1}\varepsilon$, and which does not lie entirely in $R^2 \times \{0\} \subset R^3$. We regard D_ε as an element of $\underline{\underline{Z}}_1(R^3, \emptyset; G)$ also. Let $S, S_\varepsilon \in \underline{\underline{Z}}_1(R^3, \emptyset; G)$ be of least area, where $\partial S = C$ and $\partial S_\varepsilon = C - D_\varepsilon$. [FL2, Thm. 2] implies that $\underline{\underline{M}}(S - S_\varepsilon) = \underline{\underline{M}}(S) + \underline{\underline{M}}(S_\varepsilon)$. For each (large) positive integer m, set $S(m, \varepsilon) = (m+1)S - mS_\varepsilon$ noting that $\partial S(m, \varepsilon) = C + mD_\varepsilon$. Hence

$$\lim_{\substack{m \to \infty \\ m\varepsilon \to 0}} \underline{\underline{M}}(S(m, \varepsilon))^{-1}\underline{\underline{M}}(\partial S(m, \varepsilon))^2 = 0 .$$

8.11. REMARK. [A2] provides a partial converse to 8.7.

9. VARIFOLDS LYING NEAR A DISK.

9.1. PROPOSITION. *Let* $1 \leq k \leq n$ *be integers and* $V \in \mathbf{GV}_k(\mathbf{D}^k(0, 1) \times \{0\}) \subset \mathbf{GV}_k(R^k \times R^{n-k})$.

(1) *If* $\mathbf{P}(R^n; \partial \mathbf{D}^k(0, 1) \times \{0\})(V, 0) < \infty$, *then there exists* $r \in R^+$ *such that*

$$V \cap (\mathbf{D}_0^k(0, 1) \times \{0\}) = r\,|\mathbf{D}^k(0, 1) \times \{0\}| \in \mathbf{RV}_k(R^n) .$$

(2) *If* $\mathbf{Q}(R^n; \partial \mathbf{D}^k(0, 1) \times \{0\})(V, 0) < \infty$, *then either*

$$V \cap [\mathbf{D}_0^k(0, 1) \times \{0\}] = 0$$

or there exists $f : R^n \longrightarrow R_0^+$ *such that*

$$\sup\{f(x)^{-1}\mathrm{Lip}(f)(x) : x \in R^n\} \leq \mathbf{Q}(R^n; \partial \mathbf{D}^k(0, 1) \times \{0\})(V, 0)$$

and

$$V \cap (\mathbf{D}_0^k(0, 1) \times \{0\}) = |\mathbf{D}^k(0, 1) \times \{0\}| \wedge f \in \mathbf{RV}_k(R^n) .$$

(3) *If* $\mathbf{P}(R^n; \partial \mathbf{D}^k(0, 1) \times \{0\})(V, |\mathbf{D}^{k-1}(0, 1) \times \{0\}|) < \infty$, *then there exist* $r, s \in R^+$ *with* $|r - s| \leq 1$ *such that*

$$V \cap [\mathbf{D}_0^k(0, 1) \times \{0\}] = r\,|[\mathbf{D}^k(0, 1) \cap \{x : x^k \geq 0\}] \times \{0\}|$$
$$+ s\,|[\mathbf{D}^k(0, 1) \cap \{x : x^k \leq 0\}] \times \{0\}| \in \mathbf{RV}_k(R^n) .$$

(4) *If* $\mathbf{Q}(R^n; \partial \mathbf{D}^k(0, 1) \times \{0\})(V, |\mathbf{D}^{k-1}(0, 1) \times \{0\}|) < \infty$, *then there exist* *Lipschitz functions* $f, g : R^n \longrightarrow R^+$ *such that*

(i) $\sup\{|f(x) - g(x)| : x \in \mathbf{D}^{k-1}(0, 1) \times \{0\}\} \leq 1$.

(ii) *If* f *is not identically* 0, *then* $f : R^n \longrightarrow R_0^+$ *and*

$$\sup\{f(x)^{-1}\mathrm{Lip}(f)(x) : x \in R^n\} \leq \mathbf{Q}(R^n; \partial \mathbf{D}^k(0, 1) \times \{0\})(V, |\mathbf{D}^{k-1}(0, 1)$$

(iii) *If* g *is not identically* 0, *then* $g : R^n \longrightarrow R_0^+$ *and*

$$\sup\{g(x)^{-1}\mathrm{Lip}(g)(x) : x \in R^n\} \leq \mathbf{Q}(R^n; \partial \mathbf{D}^n(0, 1) \times \{0\})(V, |\mathbf{D}^{k-1}(0,$$

(iv) $V \cap [\underset{=}{D}_0^k(0, 1) \times \{0\}] = |[\underset{=}{D}^k(0, 1) \cap \{x : x^k \geq 0\}] \times \{0\}| \wedge f$

$$+ |[\underset{=}{D}^k(0, 1) \cap \{x : x^k \leq 0\}] \times \{0\}| \wedge g \ .$$

PROOF. The proof, while not trivial, is straightforward and is left to the reader. A more sophisticated version of the deformations required appears in the proof of 10.4. See 6.17, 6.18, 8.2, and 8.3.

9.2. COROLLARY. Let $1 \leq k \leq n$ be integers and $c \in R^+$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $V \in \underline{\underline{GV}}_k(\underset{=}{D}^k(0, 1) \times \underset{=}{D}^{n-k}(0, \delta))$ with $\underset{=}{W}(V) \leq c$, and if $W \in \underline{\underline{GV}}_{k-1}(R^n)$ with $\underset{=}{F}(W, |\underset{=}{D}^{k-1}(0, 1) \times \{0\}|) \leq \delta$, and if

(1) $\underset{=}{P}(R^n; \partial \underset{=}{D}^k(0, 1) \times \underset{=}{D}^{n-k}(0, \delta))(V) \leq c$,

or (2) $\underset{=}{Q}(R^n; \partial \underset{=}{D}^k(0, 1) \times \underset{=}{D}^{n-k}(0, \delta))(V) \leq c$,

or (3) $\underset{=}{P}(R^n; \partial \underset{=}{D}^k(0, 1) \times \underset{=}{D}^{n-k}(0, \delta))(V, W) \leq c$,

or (4) $\underset{=}{Q}(R^n; \partial \underset{=}{D}^k(0, 1) \times \underset{=}{D}^{n-k}(0, \delta))(V, W) \leq c$,

then, respectively,

(1) There exists $r \in R^+$ such that

$$\underset{=}{F}(V \cap [\underset{=}{D}^{n-k}(0, 1-\varepsilon) \times R^{n-k}], r|\underset{=}{D}^k(0, 1-\varepsilon) \times \{0\}|) < \varepsilon ,$$

or (2) There exists $f : R^n \longrightarrow R_0^+$ with

$$\sup\{f(x)^{-1} \cdot \mathrm{Lip}(f)(x) : x \in R^n\} \leq c$$

such that

$$\underset{=}{F}(V \cap [\underset{=}{D}^{n-k}(0, 1-\varepsilon) \times R^{n-k}], |\underset{=}{D}^k(0, 1-\varepsilon) \times \{0\}| \wedge f) < \varepsilon ,$$

or (3) There exist $r, s \in R^+$ with $|r-s| \leq 1$ such that

$$\underset{=}{F}(V \cap [\underset{=}{D}^{n-k}(0, 1-\varepsilon) \times R^{n-k}], r|[\underset{=}{D}^k(0, 1) \cap \{x : x^k \geq 0\}] \times \{0\}|$$
$$+ s|[\underset{=}{D}^k(0, 1) \cap \{x : x^k \leq 0\}] \times \{0\}|) < \varepsilon ,$$

or (4) There exist $f, g : R^n \longrightarrow R_0^+$ such that

$$\sup\{f(x)^{-1}\cdot \operatorname{Lip}(f)(x) : x\in R^n\} \le c \ ,$$

$$\sup\{g(x)^{-1}\cdot \operatorname{Lip}(f)(x) : x\in R^n\} \le c \ ,$$

$$\sup\{|f(x) - g(x)| : x\in \underline{\underline{D}}^{k-1}(0,\ 1)\times\{0\}\} \le 1 \ ,$$

and

$$\underline{\underline{F}}(V\cap[\underline{\underline{D}}^k(0,\ 1-\varepsilon)\times R^{n-k}],\ |[\underline{\underline{D}}^k(0,\ 1)\cap\{x : x^k \ge 0\}]\times\{0\}|\wedge f$$
$$+\ |[\underline{\underline{D}}^k(0,\ 1)\cap\{x : x^k \le 0\}]\times\{0\}|\wedge g) < \varepsilon$$

9.3. COROLLARY. *Let* $1 \le h < k \le n$ *be integers* $c\in R^+$, *and* $d\in R^+_0$. *For each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if*

$$(V,\ W)\in \underline{\underline{RV}}_k(R^n,\ R^n\cap\{x : \operatorname{dist}(x,\ \partial\underline{\underline{D}}^h(0,\ 1)\times\{0\}) < \delta\})$$

with $\underline{\underline{Q}}(R^n)(V,\ W) \le c$ *and*

$$\inf\{\odot^k(\underline{\underline{W}}V,\ p) : p\in \operatorname{spt}(V) - \operatorname{spt}(W)\} \ge d \ ,$$

then

$$\operatorname{spt}(V)\cap[\underline{\underline{D}}^h(0,\ 1-\varepsilon)\times R^{n-h}] = \emptyset \ .$$

PROOF. Observe that

$$R^n\cap\{x : \operatorname{dist}(x,\ \partial\underline{\underline{D}}^h(0,\ 1)\times\{0\}) < \delta\}\subset R^n\cap\{x : \operatorname{dist}(x,\ \partial\underline{\underline{D}}^k(0,\ 1)\times\{0\}) < \delta\} \ .$$

9.4. COROLLARY. *Let* $2 \le k \le n$ *be integers. There exists* $L\in R^+_0$ *such that if* $V\in \underline{\underline{GV}}_k(R^n)$ *and* $W_1,\ W_2\in \underline{\underline{GV}}_{k-1}(R^n)$ *with* $\underline{\underline{Q}}(R^n)(V,\ W_1 + W_2) = 0$, $\operatorname{diam}(\operatorname{spt}(W_1)) \le 1$, $\operatorname{diam}(\operatorname{spt}(W_2)) \le 1$, *and* $\operatorname{dist}(\operatorname{spt}(W_1),\ \operatorname{spt}(W_2)) \ge L$, *then one can find* $V_1,\ V_2$ $\underline{\underline{GV}}_k(R^n)$ *such that* $V = V_1 + V_2$ *and* $\underline{\underline{Q}}(R^n)(V_1,\ W_1) = \underline{\underline{Q}}(R^n)(V_2,\ W_2) = 0$.

PROOF. Apply 9.3 with $h = 0$.

9.5. DEFINITIONS. Let n be a positive integer.

(1) For each $i = 1,\ 2,\ \ldots,\ n$, we define $e^i : R^n \longrightarrow R$ by setting $e^i(x) = x^i$ for $x\in R^n$.

For each $k = 1, 2, \ldots, n$, we define $\pi^k : R^n \longrightarrow R^k$ by setting
$x) = (x^1, x^2, \ldots, x^k)$ for $x = (x^1, x^2, \ldots, x^n) \in R^n$.
or each $x \in R^n$, each $r, \eta \in R_0^+$, and each $k = 1, 2, \ldots, n$ we define

$$, r, \eta) = [\mathbf{D}^k(\pi^k(x), r) \times R^{n-k}] \cap \{y : \sum_{i=k+1}^{n} (y^i - x^i)^2 \le \eta^2 \sum_{i=1}^{k} (y^i - x^i)^2\}.$$

9.6. PROPOSITION. *Let* $0 \le k \le n$ *be integers and* $V \in \mathbf{GV}_k(R^n)$ *such that*
$(R^n; \partial \mathbf{D}^k(0, 1) \times R^{n-k})(V, 0) = c < \infty$;
af$\{\Theta^k(\mathbf{W}V, x) : x \in \operatorname{spt}(V) - (\partial \mathbf{D}^k(0, 1) \times R^{n-k})\} = d > 0$; *and*
$(V, e^i) = 0$ *for each* $i = k+1, k+2, \ldots, n$.
there exist (at most $[d\alpha(k)]^{-1}\mathbf{W}V$*) distinct points* $p_1, p_2, \ldots, p_q \in R^{n-k}$ *and*
ions $f_1, f_2, \ldots, f_q : R^n \longrightarrow \{t : t \ge d\}$ *with*

$$\sup\{f_i(x)^{-1} \operatorname{Lip}(f)(x) : x \in R^n,\ i = 1, 2, \ldots, q\} \le c$$

that

$$V \cap (\mathbf{D}_0^k(0, 1) \times R^{n-k}) = \sum_{i=1}^{q} |\mathbf{D}^k(0, 1) \times \{p_i\}| \wedge f_i \ .$$

use $\mathbf{P}(R^n; \partial \mathbf{D}^k(0, 1) \times R^{n-k})(V, 0) < \infty$, *one can choose* f_i *to be constant*
ach $i = 1, 2, \ldots, q$.

PROOF. 7.2 and 7.4(5) imply that for $\mathbf{H}^1$ almost all r R,

$$\mathbf{Q}(R^n; \partial \mathbf{D}^k(0, 1) \times R^{n-k})(V \cap \{x : e^i(x) < r\}) \le c \ .$$

repeated use of such partitions and 8.2(2), the theorem reduces to 9.1.

9.7. LEMMA. *Let* k *be a positive integer and* $A \subset \mathbf{D}_0^k(0, 4)$ *be open*
$\mathbf{D}^k(0, 1) - A \ne \emptyset$. *Then there exists a finite set of points*
$p_2, \ldots, p_q \in \mathbf{D}^k(0, 1) \cap A$ *and numbers* $r_1, r_2, \ldots, r_q \in R^+$ *such that*
ist$(p_i, \mathbf{D}_0^k(0, 4) - A) = r_i$ *for each* $i = 1, 2, \ldots, q$;
$\mathbf{D}^k(p_i, 2r_i) \cap \mathbf{D}^k(p_j, 2r_j) = \emptyset$ *whenever* $i \ne j$; *and*
$\sum_{i=1}^{q} \mathbf{H}^k(\mathbf{D}^k(p_i, r_i)) \ge 2^{-1} 5^{-k} \mathbf{H}^k(A \cap \mathbf{D}^k(0, 1))$.

PROOF. One chooses the p_i and r_i as follows: Let $\mathbf{r} : A \longrightarrow R_0^+$,
$= \operatorname{dist}(x, R^k - A)$. Choose $p_1 \in \mathbf{D}^k(0, 1)$ as a maximum for $\mathbf{r} | A \cap \mathbf{D}^k(0, 1)$

and set $r_1 = \mathbf{r}(p_1)$. Choose $p_2 \in \mathbf{D}^k(0, 1)$ as a maximum for $\mathbf{r}|[A \cap \mathbf{D}^k(0, 1)] - \mathbf{D}_0^k(p_1, 4r_1)$ (provided $[A \cap \mathbf{D}^k(0, 1)] - \mathbf{D}_0^k(p_1, 4r_1) \neq \emptyset$; if this is not the case, we are done) and set $r_2 = \mathbf{r}(p_2)$. Proceeding in this manner, one chooses $p_j \in \mathbf{D}^k(0, 1)$ as a maximum for $\mathbf{r}|[A \quad \mathbf{D}^k(0, 1)] - \bigcup_{i<j} \mathbf{D}_0^k(p_i, 4r_i)$ for $j = 3, 4, 5, \ldots$ so long as these choices are possible and sets $r_j = \mathbf{r}(p_j)$. One verifies that $A \cap \mathbf{D}^k(0, 1) \subset \bigcup_i \mathbf{D}^k(p_i, 5r_i)$. The lemma follows.

9.8. THEOREM. Let $1 \leq k \leq n$ be integers, $d \in \mathbf{R}_0^+$, and $N \in \mathbf{R}^+$. For each $\varepsilon > 0$ there exists $\delta > 0$ (depending on k, n, d, N, ε) such that if $V \in \mathbf{GV}_k(\mathbf{R}^n)$ with

(1) $\operatorname{spt}(V) \subset \mathbf{D}^k(0, 4) \times \mathbf{D}^{n-k}(0, \delta)$,

(2) $\mathbf{W}(V) \leq N$,

(3) $\mathbf{Q}(\mathbf{R}^n; \partial \mathbf{D}^k(0, 4) \times \mathbf{R}^{n-k})(V) \leq N$, and

(4) $\inf\{ \Theta^k(\mathbf{W}V, x) : x \in \operatorname{spt}(V) - (\partial \mathbf{D}^k(0, 4) \times \mathbf{R}^{n-k})\} \geq d$,

then

$$\mathbf{H}^k(\pi^k(\operatorname{spt}(V) \quad \mathbf{D}^k(0, 1) \times \mathbf{R}^{n-k})) \geq (1-\varepsilon)\mathbf{H}^k(\mathbf{D}^k(0, 1)) .$$

PROOF.

Part 1. Let $k, n, \quad N$ be as above. Then for each $\varepsilon > 0$ there exists $\delta > 0$ (depending on k, n, N, ε) such that if $V \in \mathbf{GV}_k(\mathbf{R}^n)$ with

(1) $\operatorname{spt}(V) \subset \mathbf{D}^k(0, 1+\varepsilon) \times \mathbf{D}^{n-k}(0, \delta)$, and

(2) $\mathbf{Q}(\mathbf{R}^n; \partial \bar{\mathbf{D}}^k(0, 1+\varepsilon))(V) \leq N$,

then

$$\mathbf{W}(\mathbf{b}(V, e^j) \cap \mathbf{D}^k(0, 1) \times \mathbf{R}^{n-k}) \leq \varepsilon \mathbf{W}(V)$$

for each $i = k+1, k+2, \ldots, n$.

Proof of part 1. Part 1 follows from the continuity of $\mathbf{b}(\bullet, e^j)$ and 9.1.

Part 2. Let k, n, d, N be as above. There exists $c \in \mathbf{R}_0^+$ (depending on k, n, d, N) with the following property: Let $V \in \mathbf{GV}_k(\mathbf{R}^n)$ such that

(1) $\operatorname{spt}(V) \subset \mathbf{D}^k(0, 4) \times \mathbf{R}^{n-k}$,

(2) $\mathbf{W}(V) \leq N$,

(3) $\underline{\underline{Q}}(R^n;\ \partial \underline{\underline{D}}^k(0,\ 4) \times R^{n-k})(V) \leq N$, and
(4) $\inf\{\ \Theta^k(\underline{\underline{W}}V,\ x) : x \in \operatorname{spt}(V) - (\partial \underline{\underline{D}}^k(0,\ 4) \times R^{n-k})\} \geq d$;
and suppose there exists $r \in R_0^+$ such that
(1) $0 < r \leq 2$,
(2) $\operatorname{spt}(V) \cap \underline{\underline{D}}_0^k(0,\ r) \times R^{n-k} = \emptyset$, and
(3) $\operatorname{spt}(V) \cap \partial \underline{\underline{D}}^k(0,\ r) \times R^{n-k} \neq \emptyset$.
Then

$$\underline{\underline{W}}(\underline{\underline{b}}(V,\ e^j)) \geq c\ \underline{\underline{H}}^k(\underline{\underline{D}}^k(0,\ r))$$

for some $j = k+1,\ k+2,\ \ldots,\ n$.

Proof of part 2. 9.6 implies that one can find c such that

$$\underline{\underline{W}}(\underline{\underline{b}}(V,\ e^j) \cap \underline{\underline{D}}^k(0,\ 2r) \times R^{n-k}) \geq c\ \underline{\underline{H}}^k(\underline{\underline{D}}^k(0,\ r))$$

for some $j = k+1,\ k+2,\ \ldots,\ n$.

Part 3. Let k, n, d, N, and V be as in the hypotheses and set

$$A = \underline{\underline{D}}_0^k(0,\ 4) \cap \{x : (\{x\} \times R^{n-k}) \cap \operatorname{spt}(V) = \emptyset\}\ .$$

Choose $p_1,\ p_2,\ \ldots,\ p_q \in \underline{\underline{D}}^k(0,\ 1)$ and $r_1,\ r_2,\ \ldots,\ r_q \in R_0^+$ as in 9.7. Then for some $j = k+1,\ k+2, \ldots,\ n$

$$\begin{aligned}\underline{\underline{W}}(\underline{\underline{b}}(V,\ e^j)) &\geq \sum_{i=1}^{q} \underline{\underline{W}}(\underline{\underline{b}}(V \cap \underline{\underline{D}}^k(p_i,\ r_i) \times R^{n-k},\ e^j) \\ &\geq (n-k)^{-1} \sum_{i=1}^{q} c\ \underline{\underline{H}}^k(\underline{\underline{D}}^k(p_i,\ r_i)) \\ &\geq (n-k)^{-1}\ c\ 2^{-1} \cdot 5^{-k}\ \underline{\underline{H}}^k(A \cap \underline{\underline{D}}^k(0,\ 1))\end{aligned}$$

where c is the constant of part 2. The theorem now follows from part 1. Note that 8.2 and the hypotheses that $\operatorname{spt}(V) \cap (\underline{\underline{D}}^k(0,\ 1) \times R^{n-k}) \neq \emptyset$ and $\inf\{\ \Theta^k(\underline{\underline{W}}V,\ x) : x \in \operatorname{spt}(V) - (\partial \underline{\underline{D}}^n(0,\ 4) \times R^{n-k})\} > 0$ imply a lower bound for $\underline{\underline{W}}(V)$.

9.9. THEOREM. Let $1 \leq k \leq n$ and $Z \geq 0$ be integers and $\alpha \in R_0^+$. For each $\varepsilon > 0$ there exists $\delta > 0$ (depending on k, n, Z, and α) such that if

$V \in \underline{\underline{IV}}_k(R^n)$ with $\operatorname{spt}(V) \subset \underline{\underline{D}}^k(0, 2+\alpha) \times \underline{\underline{D}}^{n-k}(0, \delta)$ and
$(R^n; \partial \underline{\underline{D}}^k(0, 2+\alpha) \times R^{n-k})(V, 0) \leq \delta$, then

$$\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 2))^{-1} \underline{\underline{W}}(V \cap (\underline{\underline{D}}^k(0, 2) \times R^{n-k})) \notin \{z : Z + \varepsilon \leq z \leq Z + 1 - \varepsilon\} .$$

PROOF. The proof is in four parts. The crucial idea appears in part 2.

Part 1. Corresponding to each $c \in R^+$ and $\varepsilon > 0$, there exists $\delta > 0$ ch that if $V \in \underline{\underline{GV}}_k(R^n)$ with

$\operatorname{spt}(V) \subset \underline{\underline{D}}^k(0, 1) \times \underline{\underline{D}}^{n-k}(0, \delta)$,

$\underline{\underline{W}}(V) \leq c$, and

$\underline{\underline{Q}}(R^n, \partial \underline{\underline{D}}^k(0, 1) \times R^{n-k})(V, 0) \leq \delta$,

en

$$|\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 1-\varepsilon))^{-1} \underline{\underline{W}}(V \cap \underline{\underline{D}}^k(0, 1-\varepsilon) \times R^{n-k}) - \underline{\underline{H}}^k(\underline{\underline{D}}^k(p, r))^{-1} \underline{\underline{W}}(V \cap \underline{\underline{D}}^k(p, r) \times R^{n-k})| \leq \varepsilon$$

enever $r \geq \varepsilon$ and $\underline{\underline{D}}^k(p, r) \subset \underline{\underline{D}}^k(0, 1-\varepsilon)$.

Proof of part 1. Part 1 follows from 9.1, 9.2 and the continuity of $\underline{\underline{W}}$.

Part 2. Suppose Z_0 is a non-negative integer and that the theorem (9.9) s been proved for each non-negative integer $Z < Z_0$. Then for each $\varepsilon > 0$ ere exists $\delta > 0$ such that if $V \in \underline{\underline{IV}}_k(R^n)$ satisfies

$\operatorname{spt}(V) \subset \underline{\underline{D}}^k(0, 1) \times \underline{\underline{D}}^{n-k}(0, 1)$,

$\underline{\underline{Q}}(R^n; \partial \underline{\underline{D}}^k(0, 1) \times R^{n-k})(V, 0) \leq \delta$,

$\underline{\underline{W}}(\underline{\underline{b}}(V, e^j)) \leq \delta$ for each $j = k+1, k+2, \ldots, n$,

$\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 1))^{-1} \underline{\underline{W}}(V) \leq Z_0 + 1 - \varepsilon$,

there exist $y, z \in R^{n-k}$ with $|y - z| \geq \varepsilon$ such that

$$\operatorname{spt}(V) \cap [\underline{\underline{D}}^k(0, 1-\varepsilon) \times \{y\}] \neq \emptyset$$

$$\operatorname{spt}(V) \cap [\underline{\underline{D}}^k(0, 1-\varepsilon) \times \{z\}] \neq \emptyset ,$$

en

$$\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 1-\varepsilon))^{-1} \underline{\underline{W}}(V \cap [\underline{\underline{D}}^k(0, 1-\varepsilon) \times R^{n-k}]) \leq Z_0 + \varepsilon .$$

Proof of part 2. It is clear that we can fix $y, z \in R^{n-k}$. For each ...

small $\gamma > 0$ one sees by 8.2 and 9.6 that there exists $\delta_1 > 0$ (depending on γ) such that if $\delta < \delta_1$ then one can find $p_1, p_2, \ldots, p_q \in R^{n-k}$ with $|p_i - p_j| \geq 3\gamma$ whenever $i \neq j$ such that

$$\operatorname{spt}(V) \cap [\mathbf{D}^k(0, 1-\varepsilon) \times R^{n-k}] \subset \bigcup_{i=1}^{q} \mathbf{D}^k(0, 1-\varepsilon) \times \mathbf{D}^{n-k}(p_i, \gamma) .$$

Clearly $q \geq 2$ and we can assume without loss of generality that

$$\operatorname{spt}(V) \cap \mathbf{D}^k(1-\varepsilon) \times \mathbf{D}^{n-k}(p_i, \gamma) \neq \emptyset$$

for each $i = 1, 2, \ldots, q$. Part 1 guarantees the existence of $\delta_2 > 0$ (depending on γ) such that if $\delta < \delta_2$, then

$$\mathbf{H}^k(\mathbf{D}^k(0, 1-\varepsilon))^{-1} \mathbf{W}(V \cap [\mathbf{D}^k(0, 1-\varepsilon) \times \mathbf{D}^{n-k}(p_i, \gamma)]) \geq 1 - \gamma$$

for each $i = 1, 2, \ldots, q$. We apply our hypothesis to each

$$V \cap \mathbf{D}^k(0, 1-\varepsilon) \times \mathbf{D}^{n-k}(p_i, \gamma)$$

to conclude that

$$\mathbf{W}(V \cap [\mathbf{D}^k(0, 1-\varepsilon) \times \mathbf{D}^{n-k}(p_i, \gamma)])$$

is very nearly an integer multiple of $\mathbf{H}^k(\mathbf{D}^k(0, 1-\varepsilon))$. Part 2 follows from the observation that the sum of q numbers each of which is very nearly an integer is itself very nearly an integer. The details are left to the reader.

Part 3. We will prove the theorem by induction on non-negative integ... Suppose $Z = 0$. In case δ is sufficiently small it follows from 8.2(2) that ei... $\operatorname{spt}(V) \cap \mathbf{D}^k(0, 2) \times R^{n-k}$ is empty, in which case the theorem holds trivially, ... there exist points p in $\operatorname{spt}(V)$ close to $\{0\} \times R^{n-k}$. By 8.3, $\Theta^k(\mathbf{W}V, p)$ ... The theorem then follows by 8.2(2) if δ is sufficiently small. The theore... thus valid for $Z = 0$.

Now let Z_0 be a positive integer and assume that the theorem h... proved for all non-negative integers $Z < Z_0$. Let $\alpha, \varepsilon_0 \in R_0^+$ be fixed.

$\underline{\underline{U}}_k(Z_0, \varepsilon_0)$ denote the set of all varifolds $V \in \underline{\underline{IV}}_k(\underline{\underline{D}}^k(0, 2+a) \times R^{n-k})$ for which

$$Z_0 + \varepsilon_0 \le \underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 2+2^{-1}a))\underline{\underline{W}}(V \cap [\underline{\underline{D}}^k(0, 2+2^{-1}a) \times R^{n-k}]) \le Z_0 + 1 - \varepsilon .$$

It is now necessary to make careful selection of several small numbers

$\delta_1, \delta_2, \delta_3, \eta_1, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \beta_1, \beta_2, \beta_3, \delta_0$.

(1) $\delta_1 = \delta_1(Z_0, \varepsilon_0)$. $\delta_1 > 0$ is chosen sufficiently small so that if $V \in \underline{\underline{U}}_k(Z_0, \varepsilon_0)$ with

$$\operatorname{spt}(V) \subset \underline{\underline{D}}^k(0, 2+a) \times \underline{\underline{D}}^{n-k}(0, \delta_1)$$

and

$$\underline{\underline{Q}}(R^n; \partial\underline{\underline{D}}^k(0, 2+a) \times R^{n-k})(V, 0) \le \delta_1 ,$$

then

$$8^{-1}\varepsilon_0 \underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 1)) \ge \sup\{\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, r))^{-1}[\underline{\underline{W}}(V \cap \underline{\underline{D}}^k(0, r) \times R^{n-k}) - \underline{\underline{W}}(\Pi^k_{\#}V \cap \underline{\underline{D}}^k(0, r))] : 1 \le r \le 2\} .$$

(2) $\eta_1 = \eta_1(Z_0, \varepsilon_0)$. $\eta_1 > 0$ is chosen sufficiently small so that if $V \in \underline{\underline{U}}_k(Z_0, \varepsilon_0)$ with

$$\underline{\underline{Q}}(R^n, \partial\underline{\underline{D}}^k(0, 2+a) \times R^{n-k})(V, 0) \le \eta_1$$

and if $x \in \underline{\underline{D}}^k(0, 2) \times R^{n-k}$, $r \in R^+_0$ such that

$$0 < r \le 2 + 2^{-1}a - |\Pi^k(x)|$$

and

$$\operatorname{spt}(V) \cap [\underline{\underline{D}}^k_0(0, 2+a) - \underline{\underline{D}}^k(\Pi^k(x), r)] \times R^{n-k} \subset \underline{\underline{E}}^k(x, 6, \eta_1) \cap [\underline{\underline{D}}^k(0, 2+a) \times \underline{\underline{D}}^{n-k}(0, \cdot) ,$$

then

$$\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, r))^{-1}\underline{\underline{W}}(V \cap [\underline{\underline{D}}^k(\Pi^k(x), r) \times R^{n-k}]) \le Z_0 + 1 - \varepsilon_0 + 8^{-1}\varepsilon_0 .$$

The existence of such a choice follows from 8.2(2), part 1, and elementary geometric arguments relating cones and balls which are left to the reader.

(3) $\varepsilon_1 = \varepsilon_1(\varepsilon_0)$. Choose $\varepsilon_1 = 8^{-1}\varepsilon_0 \underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 1))$.

(4) $\varepsilon_2 = \varepsilon_2(Z_0, \varepsilon_0)$. Choose $\varepsilon_2 = [8Z_0 N(k)]^{-1}\varepsilon_0$ where $N(k)$ is as defined in 8.6.
(5) $\varepsilon_3 = \varepsilon_3(Z_0, \varepsilon_2)$. $0 < \varepsilon_3 < 4^{-1}$ is chosen so that if $V \in \underline{\underline{V}}_k(\underline{\underline{D}}^k(0, 2) \times R^{n-k})$ and $\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 2))^{-1}\underline{\underline{W}}(V) \leq Z_0 + 1$, then for some r, $3/2 < r < 2$,

$$\underline{\underline{H}}^k[\underline{\underline{D}}^k(0, r) - \underline{\underline{D}}^k(0, (1-3\varepsilon_3)r)]^{-1}\underline{\underline{W}}(V \cap ([\underline{\underline{D}}_0^k(0, r) - \underline{\underline{D}}^k(0, (1-3\varepsilon_3)r)] \times R^{n-k})) \leq 2(Z_0 +$$

and

$$Z_0^{-1}\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, r))^{-1}\underline{\underline{W}}(V \cap ([\underline{\underline{D}}_0^k(0, r) - \underline{\underline{D}}^k(0, (1-3\varepsilon_3)r)] \times R^{n-k})) \leq \varepsilon_2 .$$

(6) $\beta_1 = \beta_1(Z_0, \varepsilon_0, \eta_1, \varepsilon_3)$. $\beta_1 > 0$ is chosen sufficiently small so that if $V \in \underline{\underline{IV}}_k(\underline{\underline{D}}^k(0, 1) \times R^{n-k})$ with

$$\underline{\underline{Q}}(R^n; \partial\underline{\underline{D}}^k(0, 1) \times R^{n-k})(V) \leq \beta_1 ,$$

$$\underline{\underline{W}}(\underline{\underline{b}}(V, e^j)) \leq \beta_1 \text{ for each } j = k+1, k+2, \ldots, n ,$$

$$\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 1))^{-1}\underline{\underline{W}}(V) \leq Z_0 + 1 - \varepsilon_0 + 8^{-1}\varepsilon_0 ,$$

and if there exist $p, q \in \operatorname{spt}(V) \cap [\underline{\underline{D}}^k(0, 2/3) \times R^{n-k}]$ such that

$$\sum_{i=k+1}^{n} (p^i - q^i)^2 \geq (4^{-1}\eta_1)^2 ,$$

then

$$\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 1-\varepsilon_3))^{-1}\underline{\underline{W}}(V \cap [\underline{\underline{D}}^k(0, 1-\varepsilon_3) \times R^{n-k}]) \leq Z_0 + 8^{-1}\varepsilon_0 .$$

Part 2 justifies such a choice.
(7) $\varepsilon_4 = \varepsilon_4(\varepsilon_0)$. Choose $\varepsilon_4 = 8^{-1}\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 1))\varepsilon_0$.
(8) $\varepsilon_5 = \varepsilon_5(Z_0, \varepsilon_0, \varepsilon_4)$. Choose $0 < \varepsilon_5 < 1$ sufficiently small so that if $V \in \underline{\underline{U}}_k(Z_0, \varepsilon_0)$, then for some r, $1 < r < 2$,

$$\underline{\underline{W}}(V \cap ([\underline{\underline{D}}^k(0, r) - \underline{\underline{D}}^k(0, r - \varepsilon_5)] \times R^{n-k})) \leq \varepsilon_4 .$$

(9) $\delta_2 = \delta_2(Z_0, \varepsilon_0, \varepsilon_5, \eta_1)$. $\delta_2 > 0$ is chosen sufficiently small so that if $V \in \underline{\underline{U}}^k(Z_0, \varepsilon_0)$ with $\operatorname{spt}(V) \subset \underline{\underline{D}}^k(0, 2+\alpha) \times \underline{\underline{D}}^{n-k}(0, \delta_2)$, then for each $p \in \operatorname{spt}(V) \cap \underline{\underline{D}}^k(0, 2) \times R^{n-k}$,

$\inf\{r : r > 0 \text{ and } \operatorname{spt}(V) \cap [\underline{\underline{D}}^k(0, 2+\alpha) - \underline{\underline{D}}^k(p, r)] \times R^{n-k} \subset \underline{\underline{E}}^k(p, r, \eta_1)\} \leq 2^{-1}\varepsilon_5$.

(10) $\varepsilon_6 = \varepsilon_6(\varepsilon_3)$. Choose $\varepsilon_6 = k^{-1}\varepsilon_3$.

(11) $\beta_2 = \beta_2(Z_0, \varepsilon_0, \varepsilon_3, \varepsilon_6)$. Choose $\beta_2 > 0$ sufficiently small so that if $V \in \underline{\underline{GV}}_k(\underline{\underline{D}}^k(0, 1) \times R^{n-k})$ with

$$\underline{\underline{Q}}(R^n; \partial \underline{\underline{D}}^k(0, 1) \times R^{n-k})(V) \leq \beta_2 ,$$

$$\underline{\underline{W}}(\underline{\underline{b}}(V, e^j)) \leq \beta_2 \underline{\underline{W}}(V) \text{ for each } j = k+1, k+2, \ldots, n, \text{ and}$$

$$\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 1-\varepsilon_3))^{-1}\underline{\underline{W}}(V \cap \underline{\underline{D}}^k(0, 1-\varepsilon_3) \times R^{n-k}) \leq Z_0 + 8^{-1}\varepsilon_0 ,$$

then

$$\underline{\underline{H}}^k(K)^{-1}\underline{\underline{W}}(V \cap [K \times R^{n-k}]) \leq Z_0 + 4^{-1}\varepsilon_0$$

for each k-cube $K \subset \underline{\underline{D}}^k(0, 1-2\varepsilon_3)$ of side length ε_6.

(12) $\beta_3 = \beta_3(Z_0, \varepsilon_0, \varepsilon_1, \beta_2)$. Let $\beta_3 = [N(k)\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 2))]^{-1}\beta_2 , \varepsilon_1$.

(13) $\delta_3 = \delta_3(Z_0, \varepsilon_0, \varepsilon_1, \beta_2, \beta_3)$. Choose $\delta_3 > 0$ sufficiently small so that if $V \in \underline{\underline{U}}_k(Z_0, \varepsilon_0)$ with

$$\operatorname{spt}(V) \subset \underline{\underline{D}}^k(0, 2+\alpha) \times \underline{\underline{D}}^{n-k}(0, \delta_3)$$

and

$$\underline{\underline{Q}}(R^n, \partial \underline{\underline{D}}^k(0, 2+\alpha) \times R^{n-k})(V) \leq 1 ,$$

then

$$\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 2))^{-1}\underline{\underline{W}}(\underline{\underline{b}}(V, e^j) \cap [\underline{\underline{D}}^k(0, 2) \times R^{n-k}]) \leq \beta_3$$

for each $j = k+1, k+2, \ldots, n$.

(14) $\delta_4 = \delta_4(Z_0, \varepsilon_0)$. Choose $\delta_4 > 0$ sufficiently small so that if $V \in \underline{\underline{U}}_k(Z_0, \varepsilon_0)$ with $\operatorname{spt}(V) \subset \underline{\underline{D}}^k(0, 2+\alpha) \times \underline{\underline{D}}^{n-k}(0, \delta_4)$, then for each r, $1 < r < 2$,

$$\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, r))^{-1}\underline{\underline{W}}(\pi^k_{\#}(V) \cap \underline{\underline{D}}^k(0, r)) \geq Z_0 + \varepsilon_0 - 8^{-1}\varepsilon_0 .$$

(15) Choose $\delta_0 = \min\{\delta_1, \delta_2, \delta_3, \delta_4, \eta_1, \beta_1, \beta_2\}$.

Part 4. The covering construction. Let $V \in \underline{\underline{IV}}_k(\underline{\underline{D}}^k(0, 2+\alpha) \times \underline{\underline{D}}^{n-k}(0, \ldots$ with $\underline{\underline{Q}}(R^n; \partial \underline{\underline{D}}^k(0, 2+\alpha) \times R^{n-k})(V) \leq \delta_0$. We will prove $V \notin \underline{\underline{U}}_k(Z_0, \varepsilon_0)$. To accomplish this we now assume $V \in \underline{\underline{U}}_k(Z_0, \varepsilon_0)$ and demonstrate a contradiction ...

We define for each positive integer Z,

$$\Delta(Z) = R^k \cap \{x : \overline{\Theta}^k(\underline{\underline{W}}\ \Pi^k_{\#}(V),\ x) \geq Z\} \ ,$$

noting that $\Delta(Z)$ is $\underline{\underline{H}}^k$ measurable for each Z.

(1) The outer annulus A_0. Choose r_0, $1 < r_0 < 2$, such that

$$\underline{\underline{W}}(V \cap [A_0 \times R^{n-k}]) \leq \varepsilon_4$$

where
$$A_0 = \underline{\underline{D}}^k(0,\ r_0) - \underline{\underline{D}}^k(0,\ r_0 - \varepsilon_5) \subset R^k \ .$$

(2) The function ρ.

(a) For each $x \in \Delta(Z_0+1) \cap \underline{\underline{D}}^k(0,\ r_0 - \varepsilon_5)$, let $\rho_1(x)$ be the smallest non-negative number r such that for some $p \in \operatorname{spt}(V)$, $\Pi^k(p) = x$, and

$$\operatorname{spt}(V) \cap ([\underline{\underline{D}}^k(0,\ 2+a) - \underline{\underline{D}}^k(x,\ r)] \times R^{n-k}) \subset \underline{\underline{E}}^k(p,\ 6,\ \eta_1)$$

(b) For each $x \in \Delta(Z_0+1) \cap \underline{\underline{D}}^k(0,\ r_0 - \varepsilon_5)$, let $\rho_2(x)$,

$$(3/2)\rho_1(x) \leq \rho_2(x) \leq 2\rho_1(x) \ ,$$

be chosen so that if

$$A'(x) = \underline{\underline{D}}^k_0(x,\ \rho_2(x)) - \underline{\underline{D}}^k(x,\ (1-3\varepsilon_3)\rho_2(x)) \ ,$$

then

$$\underline{\underline{W}}(V \cap [A'(x) \times R^{n-k}]) \leq 2(Z_0+1)\underline{\underline{H}}^k(A'(x))$$

and

$$\underline{\underline{W}}(V \cap [A'(x) \times R^{n-k}]) \leq \varepsilon_2 Z_0 \underline{\underline{H}}^k(\underline{\underline{D}}^k(0,\ \rho_2(x))) \ .$$

This is possible by the selection of ε_3. Note that the choice of η_1 guarantees that

$$\underline{\underline{W}}(V \cap \underline{\underline{D}}^k(x,\ \rho_2(x)) \times R^{n-k}) \leq (Z_0 + 1 - \varepsilon_0 + 8^{-1}\varepsilon_0)\underline{\underline{H}}^k(\underline{\underline{D}}^k(x,\ \rho_2(x)))$$

since, with $\rho_2(x) > \rho_1(x)$,

$$\operatorname{spt}(V) \cap [\underline{\underline{D}}^k(0,\ 2+a) - \underline{\underline{D}}^k(x,\ \rho_2(x))] \times R^{n-k} \subset \underline{\underline{E}}^k(p,\ 6,\ \eta_1) \ .$$

for some $p \in R^n$ with $\Pi^k(p) = x$.

(c) Define

$$\rho : \operatorname{clos}[\Delta(Z_0 + 1)] \longrightarrow R^+$$

by setting for each $x \in \operatorname{dmn}(\rho)$,

$$\rho(x) = \lim_{\varepsilon \to 0^+} \sup\{\rho_2(y) : y \in \Delta(Z_0 + 1),\ |y-x| < \varepsilon\} .$$

One verifies that ρ has the following properties:

(i) $\rho(x) \leq \varepsilon_5 \leq 1$ for each $x \in \operatorname{dmn}(\rho)$, by the choice of ε_5,

(ii) ρ is upper semi-continuous with a compact set as domain,

(iii) For each $x \in \operatorname{clos}(\Delta(Z_0 + 1))$, set

$$A(x) = \underline{\underline{D}}_0^k(x,\ \rho(x)) - \underline{\underline{D}}^k(x,\ (1-3\varepsilon_3)\rho(x)) \subset R^k .$$

By Fatou's lemma, for each $x \in \operatorname{clos}(\Delta(Z_0 + 1))$,

$$\underline{\underline{W}}(V \cap [A(x) \times R^{n-k}]) \leq \varepsilon_2 Z_0 \underline{\underline{H}}^k(\underline{\underline{D}}^k(0,\ \rho(x))) .$$

(iv) $\rho(x) > 0$ for each $x \in \Delta(Z_0 + 1)$. If this were not the case, it would follow from our assumptions about η_1 that

$$\overline{\odot}^k(\underline{\underline{W}}\, \Pi^k_{\#}(V),\ x) \leq Z_0 + 1 - \varepsilon_0 + 8^{-1}\varepsilon_0$$

contradicting $x \in \Delta(Z_0 + 1)$.

(v) For each $x \in \operatorname{dmn}(\rho)$ for which $\rho(x) > 0$, there exist $p,\ q \in \operatorname{spt}(V) \cap [\underline{\underline{D}}^k(x,\ (2/3)\rho(x)) \times R^{n-k}]$ such that

$$\left[\sum_{i=k+1}^{n} (p_i - q_i)^2\right]^{1/2} \geq 4^{-1}\eta_1 \rho(x) .$$

This is the crucial property which enables us to use the results of part 2 in our covering construction. This property of ρ follows from the geometry of $\underline{\underline{E}}^k(p,\ 6,\ \eta_1)$, the definition of ρ, and the compactness of $\operatorname{spt}(V)$.

(3) <u>Selection of the balls</u> B_i', B_i <u>and the small annuli</u> A_i. We now choose a

sequence of points $p_1, p_2, p_3, \ldots \in \underline{\underline{D}}^k(0, r_0-\varepsilon_5) \cap \operatorname{clos}(\Delta(Z_0+1))$. For each $i = 1, 2, 3, \ldots$ we then set $B_i^! = \underline{\underline{D}}_0^k(p_i, \rho(p_i))$.

(a) $p_1 \in \underline{\underline{D}}^k(0, r_0-\varepsilon_5) \cap \operatorname{clos}(\Delta(Z_0+1))$ is chosen so that $\rho(p_1)$ is a maximum for $\rho|\underline{\underline{D}}^k(0, r_0-\varepsilon_5) \cap \operatorname{clos}(\Delta(Z_0+1))$.

(b) $p_2 \in \underline{\underline{D}}^k(0, r_0-\varepsilon_5) \cap \operatorname{clos}(\Delta(Z_0+1)) - B_1^!$ is chosen so that $\rho(p_2)$ is a maximum for $\rho|\underline{\underline{D}}^k(0, r_0-\varepsilon_5) \cap \operatorname{clos}(\Delta(Z_0+1)) - B_1^!$.

(c) Assuming $p_1, p_2, \ldots, p_{q-1}$ have been chosen, choose $p_q \in \underline{\underline{D}}^k(0, r_0-\varepsilon_5) \cap \operatorname{clos}(\Delta(Z_0+1)) - \bigcup_{i=1}^{q-1} B_i^!$ so that $\rho(p_q)$ is a maximum for $\rho|\underline{\underline{D}}^k(0, r_0-\varepsilon_5) \cap \operatorname{clos}(\Delta(Z_0+1)) - \bigcup_{i=1}^{q-1} B_i^!$.

This defines the p_i and $B_i^!$ for $i = 1, 2, 3, \ldots$. Note that $\rho(p_i) \geq \rho(p_{i+1})$ for each $i = 1, 2, 3, \ldots$ and, in particular, $\operatorname{center}(B_i^!) \notin B_j^!$ whenever $i \neq j$. We now define for each $i = 1, 2, 3, \ldots$,

$$B_i = \underline{\underline{D}}^k(p_i, (1-3\varepsilon_3)\rho(p_i)) \text{ and } A_i = B_i^! - B_i = A(p_i) .$$

It is a consequence of the remark above and 8.6 that each point $x \in \underline{\underline{D}}^k(0, r_0)$ is contained in no more than $N(k)$ distinct annuli A_i.

Now define

$$\Theta = \{1, 2, 3, \ldots\} \cap \{i : \underline{\underline{W}}(\underline{\underline{b}}(V \cap [B_i^! \times R^{n-k}], e^j) \leq \beta_2 \underline{\underline{W}}(V \cap [B_i^! \times R^{n-k}]) \text{ for each } j = k+1, k+2, \ldots, n\} .$$

By the choice of δ_3 we have that for each $j = k+1, k+2, \ldots, n$,

$$\begin{aligned} \underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 2))\beta_3 &\geq \underline{\underline{W}}(\underline{\underline{b}}(V \cap [\underline{\underline{D}}^k(0, 2) \times R^{n-k}], e^j)) \\ &\geq N(k)^{-1} \sum_{i \notin \Theta} \underline{\underline{W}}(\underline{\underline{b}}(V \cap [B_i^! \times R^{n-k}], e^j)) \\ &\geq N(k)^{-1}\beta_2 \sum_{i \notin \Theta} \underline{\underline{W}}(V \cap [B_i^! \times R^{n-k}]) \end{aligned}$$

so that, in particular,

$$\sum_{i \notin \Theta} \underline{\underline{W}}(V \cap [B_i' \times R^{n-k}]) \leq \beta_2^{-1}\beta_3 N(k)\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 2)) \leq \varepsilon_1 .$$

(4) Selection of cubes. Let $\Theta = \{i_1, i_2, i_3, \ldots\}$, $i_1 < i_2 < i_3 < \ldots$.

(a) We choose a collection of closed rectilinear k-cubes, each of side length $\varepsilon_6\rho(p_{i_1})$, which cover B_{i_1} so that the interiors of each pair of distinct cubes are disjoint and such that each cube has a non-empty intersection with B_{i_1}.

(b) We now cover that part of B_{i_2} which is not covered by $A_{i_1} \cup B_{i_1}$ by a collection of closed rectilinear cubes, each of side length $\varepsilon_6\rho(p_{i_2})$, such that the interiors of each pair of distinct cubes are disjoint and such that each cube has a non-empty intersection with $B_{i_2} - (A_{i_1} \cup B_{i_1})$. This construction guarantees that no cube touching B_{i_2} touches any cube touching B_{i_1} by the definition of ε_6, i.e. recall that the diameter of the unit k-cube is $k^{1/2} \leq k$.

(c) We now continue in the manner initiated above to cover all the B_i, $i \in \Theta$, by small rectilinear cubes with pairwise disjoint interiors.

(5) Covering observations. We observe the following:

(a) $$\underline{\underline{D}}^k(0, r_0-\varepsilon_5) \cap \Delta(Z_0 + 1) \subset \bigcup_{i=1}^{\infty} A_i \cup B_i .$$

(b) The union of all the rectilinear cubes together with $\bigcup_{i \in \Theta} A_i$ covers $\bigcup_{i \in \Theta} B_i'$, and, furthermore,

$$\underline{\underline{W}}V([\Delta(Z_0 + 1) \cap \underline{\underline{D}}^k(0, r_0-\varepsilon_5) - \bigcup_{i \in \Theta} B_i'] \times R^{n-k}) \leq \varepsilon_1$$

(c) $$\underline{\underline{W}}\, \Pi^k_{\#}(V)(\underline{\underline{D}}^k(0, r_0-\varepsilon_5) \cap \{x : \overline{\Theta}^k(\underline{\underline{W}}\, \Pi^k_{\#}(V), x) \leq Z_0\})$$

$$\leq \underline{\underline{W}}\, \Pi^k_{\#}(V)(\underline{\underline{D}}^k(0, r_0-\varepsilon_5) - \Delta(Z_0 + 1))$$

(d) $\underline{\underline{W}}\, \Pi^k_{\#}(V)(\underline{\underline{D}}^k(0, r_0))$ is no more than the sum of the following terms:

(i) $$\underline{\underline{W}}V(A_0 \times R^{n-k}) \leq \varepsilon_4 = 8^{-1}\varepsilon_0 \underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 1))$$

$$\leq 8^{-1}\varepsilon_0 \underline{\underline{H}}^k(\underline{\underline{D}}^k(0, r_0)) .$$

(ii) $\sum\{\underline{\underline{W}}V(K \times R^{n-k}) : K$ is a rectilinear cube in R^n chosen in (4) above$\}$

$$\leq (Z_0 + 4^{-1}\varepsilon_0)\underline{\underline{H}}^k(\bigcup\{K : K \text{ is a rectilinear cube as above}\})$$

$$\leq (Z_0 + 4^{-1}\varepsilon_0)\underline{\underline{H}}^k(\bigcup_{i\in\Theta} B_i') .$$

(iii) $\sum_{i\in\Theta} \underline{\underline{W}}V(A_i \times R^{n-k}) \leq Z_0\varepsilon_2 \sum_{i\in\Theta} \underline{\underline{H}}^k(B_i')$

$$\leq Z_0\varepsilon_2 N(k)\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, r_0))$$

$$\leq 8^{-1}\varepsilon_0\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, r_0)) .$$

(iv) $\underline{\underline{W}}V([\Delta(Z_0 + 1) - \bigcup_{i\in\Theta} A_i \cup B_i] \times R^{n-k})$

$$\leq \varepsilon_1 = 8^{-1}\varepsilon_0\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, 1)) \leq 8^{-1}\varepsilon_0\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, r_0)) .$$

(v) $\underline{\underline{W}}\,\Pi^k_{\#}(V)(\underline{\underline{D}}^k(0, r_0-\varepsilon_5) - \bigcup_{i=1}^{\infty} B_i')$

$$= \underline{\underline{W}}\,\Pi^k_{\#}(V)(\underline{\underline{D}}^k(0, r_0-\varepsilon_5) - \bigcup_{i=1}^{\infty} B_i' \cap \{x : \overline{\odot}^k(\underline{\underline{W}}\,\Pi^k_{\#}(V), x) \leq Z_0\})$$

$$\leq Z_0\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, r_0-\varepsilon_5) - \bigcup_{i=1}^{\infty} B_i') .$$

Here we are using the fact that $\Pi^k_{\#}(V) \in \underline{\underline{IV}}_k(R^k)$ and hence corresponds to a $\underline{\underline{H}}^k$ measurable integer valued function whose value at $x \in \underline{\underline{R}}^k$ is $\overline{\odot}^k(\underline{\underline{W}}\,\Pi^k_{\#}(V), x)$ for $\underline{\underline{H}}^k$ almost all $x \in R^k$.

(e) Summation of the terms listed in (d) yields:

$$\underline{\underline{W}}\,\Pi^k_{\#}(V)(\underline{\underline{D}}^k(0, r_0))$$

$$\leq (3/8)\varepsilon_0\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, r_0))$$

$$+ [Z_0 + (2/8)\varepsilon_0][\underline{\underline{H}}^k(\bigcup_{i\in\Theta} B_i') + \underline{\underline{H}}^k(\underline{\underline{D}}^k(0, r_0-\varepsilon_5) - \bigcup_{i=1}^{\infty} B_i')]$$

$$\leq [Z_0 + (5/8)\varepsilon_0]\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, r_0)) .$$

(f) By our choice of δ_1,

$$\underline{\underline{W}}V(\underline{\underline{D}}^k(0, r_0) \times R^{n-k}) \leq [Z_0 + (6/8)\varepsilon_0]\underline{\underline{H}}^k(\underline{\underline{D}}^k(0, r_0)) .$$

From the definition of δ_4 we must have

$$\underline{\underline{W}}V(\underline{\underline{D}}^k(0,\ r_0)\times R^{n-k})\geq[Z_0+(7/8)\varepsilon_0]\underline{\underline{H}}^k(\underline{\underline{D}}^k(0,\ r_0)) .$$

The contradiction is apparent. We conclude $V\notin\underline{\underline{U}}_k(Z_0,\ \varepsilon_0)$ and the theorem is proved.

9.10. THEOREM. Let $0\leq k\leq n$ be integers. There exists a function

$$\varphi(n,\ k) : R^+\times R_0^+\longrightarrow R_0^+$$

with the following property: If

(1) $(m,\ \varepsilon)\in R^+\times R_0^+$;

(2) $V\in\underline{\underline{IV}}_k(\underline{\underline{D}}^k(0,\ 1+\ \varphi(n,\ k)(m,\ \varepsilon))\times\underline{\underline{D}}^{n-k}(0,\ \varphi(n,\ k)(m,\ \varepsilon)))$ with $\underline{\underline{W}}(V)\leq m$,

(3) $W\in\underline{\underline{GV}}_k([\underline{\underline{D}}^k(0,\ 1+\ \varphi(n,\ k)(m,\ k))-\underline{\underline{D}}^k(0,\ 1)]\times\underline{\underline{D}}^{n-k}(0,\ \varphi(n,\ k)(m,\ \varepsilon)))$ with $\underline{\underline{W}}(W)\leq m$, and

(4) $\underline{\underline{Q}}(R^n)(V,\ W)\leq m$,

then there exists some non-negative integer Z such that

$Z-\varepsilon\leq\underline{\underline{H}}^k(\underline{\underline{D}}^k(0,\ 1))^{-1}\underline{\underline{W}}(V)\leq Z+\varepsilon$.

PROOF. 8.1(1) together with 9.2, 7.2, and 7.4 imply that $\underline{\underline{W}}V$ near $\partial\underline{\underline{D}}^k(0,\ 1)\times\{0\}$ can be made small by choosing $\varphi(n,\ k)(m,\ \varepsilon)$ small. 6.19 together with 9.9 imply that we can assume $\underline{\underline{Q}}(R;\ \partial\underline{\underline{D}}^k(0,\ 1)\times R^{n-k})(V)$ small by choosing $\varphi(n,\ k)(m,\ \varepsilon)$ small and applying an elementary expansion of R^n. The theorem then follows using 7.2 and 9.9.

9.11. EXAMPLE. Let $V\in\mathcal{V}_2(R^4)$ be the stationary integral varifold corresponding to the complex algebraic variety $C^2\cap\{(z,\ w) : w=cz^2\}$ (11.1(2)). Here C^2 denotes the complex plane and $c\in R^+$. If c is large then $V\cap\underline{\underline{D}}^4(0,\ 1)$ will lie near the unit disk in $C^2\cap\{(z,\ w) : z=0\}$ and have 2-dimensional area nearly equal twice $\alpha(2)$. V is a counterexample to a number of plausible conjectures concerning the behavior of a minimal surface which lies near a disk.

10. COMPACTNESS THEOREMS FOR INTEGRAL AND REAL VARIFOLDS

10.1 DEFINITION. Cones. Let $0 \leq k \leq n$ be integers, $p \in R^n$, and $V \in \mathbf{V}_k(R^n)$. We define the cone pV over V with center at p by $pV = f_{\#}([0, 1] \times V) \in \mathbf{V}_{k+1}(R^n)$ where $f : R \times R^n \longrightarrow R^n$, $f(t, x) = p + t(x-p)$ for $t \in R$, $x \in R^n$. If $V \in \mathbf{GV}_k(R^n)$ and $r = \sup\{|x-p| : x \in \operatorname{spt}(V)\}$ one verifies that $\mathbf{W}(pV) \leq k^{-1} r\, \mathbf{W}(V)$. Note that if $V \in \mathbf{GV}_k(R^n)$ then $pV \in \mathbf{GV}_{k+1}(R^n)$; if $V \in \mathbf{RV}_k(R^n)$ then $pV \in \mathbf{RV}_{k+1}(R^n)$; and if $V \in \mathbf{IV}_k(R^n)$ then $pV \in \mathbf{IV}_{k+1}(R^n)$.

10.2 PROPOSITION. Let $2 \leq k < n$ be integers and $V \in \mathbf{GV}_k(R^n)$ lie intrinsically on S^{n-1}. Then $\mathbf{S}(R^n, S^{n-1})(V) = 0$ if and only if $\mathbf{S}(R^n, S^{n-1})(0V) = 0$.

PROOF.

(1) Suppose $\mathbf{S}(R^n, S^{n-1})(V) = 0$. Then if $\mathbf{S}(R^n, S^{n-1})(0V) \neq 0$, there exists $g \in \mathbf{X}(R^n, S^{n-1})$ such that $\mathbf{S}(0V, g) > 0$. For each $\varepsilon > 0$ let $f_\varepsilon : R^n \longrightarrow \{t : 0 \leq t \leq 1\}$ be chosen of class ∞ so that $f_\varepsilon(x) = 0$ whenever $|x| \leq \varepsilon$, $f_\varepsilon(x) = 1$ whenever $|x| \geq 2\varepsilon$, and $\operatorname{Lip}(f_\varepsilon) \leq 2^{-1}\varepsilon$. One notes that, since V lies intrinsically on S^{n-1}, $\mathbf{W}(0V \cap \mathbf{D}^n(0, r)) = k^{-1} r^k\, \mathbf{W}(V)$ whenever $0 < r \leq 1$, and concludes from 6.4(4d) and the linearity of $\mathbf{S}(V, \cdot)$ that $\lim_{\varepsilon \to 0+} \mathbf{S}(0V, f_\varepsilon g) = \mathbf{S}(0V, g)$. Hence there exists $\varepsilon > 0$ such that $\mathbf{S}(0V, f_\varepsilon g) > 0$. We now write uniquely $f_\varepsilon g = h_1 + h_2$ where $h_1, h_2 \in \mathbf{X}(R^n, S^{n-1})$ satisfy $|h_1(x) \cdot x| = |h_1(x)|\,|x|$ and $h_2(x) \cdot x = 0$ for each $x \in R^n$. It is clear that $\mathbf{S}(0V, h_1) = 0$. Hence $\mathbf{S}(0V, h_2) > 0$. An easy calculation using 6.10 (1) shows this implies $\mathbf{S}(R^n, S^{n-1})(V) > 0$.

(2) Suppose $\mathbf{S}(R^n, S^{n-1})(0V) = 0$. The argument to show $\mathbf{S}(R^n, S^{n-1})(V) = 0$ is not difficult. See, for example [FL2 p. 20].

10.3 DEFINITIONS. Tangent cones. Let $0 \leq k \leq n$ be integers, $C \subset R^n$ be closed, and $V \in \mathbf{V}_k(R^n)$ with $\mathbf{Q}(R^n; C)(V) < \infty$. For each

$p \in \mathrm{spt}(V) - C$ we define the set of tangent cones to V at p, denoted $\underline{\underline{K}}(V, p)$, to consist of all varifolds $K \in \underline{\underline{V}}_k(\underline{\underline{D}}^n(p, 1))$ such that for each $\varepsilon > 0$ there exist arbitrarily small values of $r > 0$ for which $\underline{\underline{F}}(K, f(p, r)_{\#}(V) \cap \underline{\underline{D}}^n(p, 1)) < \varepsilon$. Here $f(p, r) : R^n \longrightarrow R^n$, $f(p, r)(x) = p + r^{-1}(x-p)$ for $x \in R^n$. One verifies easily that for each $p \in \mathrm{spt}(V) - C$, $\underline{\underline{K}}(V, p)$ is not empty, $\underline{\underline{K}}(V, p)$ is compact, and $\underline{\underline{W}}(K) = \Theta^k(\underline{\underline{W}}V, p)\, \alpha(k)$ for each $K \in \underline{\underline{K}}(V, p)$.

If $A \subset R^n$ is a set and $p \in A$, the tangent cone to A at p consists of all points $x \in \underline{\underline{D}}^n(p, 1)$ such that for each $\varepsilon > 0$ there exist arbitrarily small values of $r > 0$ for which $\mathrm{dist}(x, f(p, r)(A)) < \varepsilon$.

10.4 THEOREM. Let $0 \le k \le n$ be integers, $C \subset R^n$ be closed, $d > 0$ and $V \in \underline{\underline{R}}\,\underline{\underline{V}}_k(R^n)$ with $\underline{\underline{Q}}(R^n; C)(V) < \infty$ and $\Theta^k(\underline{\underline{W}}V, p) \ge d$ for each $p \in \mathrm{spt}(V) - C$. Then

(1) $\mathrm{spt}(V) - C$ is a k-dimensional rectifiable subset of R^n and $\underline{\underline{W}}V(A) \ge d\, \underline{\underline{H}}^k(A)$ for every Borel set $A \subset \mathrm{spt}(V) - C$.

(2) For $\underline{\underline{H}}^k$ almost all $p \in \mathrm{spt}(V) - C$, there exists a simple k-vector $\mu \in \underline{\underline{\Lambda}}_k(R^n)$ with $|\mu| \ge d$, $\vec{\underline{\underline{M}}}V(p)$ exists, $\vec{\underline{\underline{W}}}V(p)$ exists, and for each $\lambda \in \underline{\underline{\Lambda}}^k(R^n)$,

$$\vec{\underline{\underline{M}}}V(p)(\lambda) = \vec{\underline{\underline{W}}}V(p)(\lambda) = |\lambda \cdot \mu| .$$

Furthermore the tangent cone of $\mathrm{spt}(V)$ at p is contained in the k-plane of μ, and $\underline{\underline{K}}(V, p)$ consists of a single varifold D corresponding to a unit k-disk centered at p and contained in the k-plane of μ having constant density no less than d.

PROOF. The theorem follows from 5.2, 5.4(2C), 6.19, 8.2, 9.1, 9.2, and the arguments in the proof of [FF 9.14].

10.5 LEMMA. Let $0 \le k \le n$ be integers, $C \subset R^n$ be closed, and $V \in \underline{\underline{G}}\,\underline{\underline{V}}_k(R^n)$ with $\underline{\underline{Q}}(R^n; C)(V) < \infty$. If $\mathrm{spt}(V)$ is a k-dimensional rectifiable subset of R^n, then $V \cap (R^n - C) \in \underline{\underline{R}}\underline{\underline{V}}_k(R^n)$.

PROOF. From 5.2(1) and 9.1 one concludes that for $\underline{\underline{H}}^k$ almost all $x \in \operatorname{spt}(V) - C$ there exists some $\mu \in \underline{\underline{\Lambda}}_k(R^n)$ parallel with the approximate tangent plane to A at x such that $\vec{\underline{\underline{W}}}V(x)$ exists and for each $\lambda \in \underline{\underline{\Lambda}}^k(R^n)$, $\underline{\underline{W}}V(x)(\lambda) = |\lambda \cdot \mu|$. The lemma follows from 5.4 (2c).

10.6 LEMMA. Let $1 \le k \le n$ be integers, $C \subset R^n$ be closed, and $V \in \underline{\underline{G}}\underline{\underline{V}}_k(R^n)$ with $\underline{\underline{Q}}(R^n; C)(V) < \infty$. Let $p \in \operatorname{spt}(V) - C$, $K \in \underline{\underline{K}}(V, p)$ and $e : R^n \longrightarrow R^+$, $e(x) = |x-p|$ for $x \in R^n$. Then

(1) $\underline{\underline{S}}(R^n, \partial\underline{\underline{D}}^n(p, 1))(K) = 0$;

(2) $\underline{\underline{b}}(K, e) = K$;

(3) For $\underline{\underline{H}}^1$ almost all r, $0 \le r < 1$, $\underline{\underline{B}}(K, e, r)$ exists, $\underline{\underline{S}}(R^n; \partial\underline{\underline{D}}^n(p, r))(\underline{\underline{B}}(K, e, r)) = 0$, and $\underline{\underline{W}}K \cap \underline{\underline{D}}^n(p, r) = \underline{\underline{W}}p\underline{\underline{B}}(K, e, r)$;

(4) If $0 < r < 1$, $\underline{\underline{B}}(K, e, r)$ exists, and $\underline{\underline{B}}(K, e, r) \in \underline{\underline{R}}\underline{\underline{V}}_{k-1}(R^n)$, then $p\underline{\underline{B}}(K, e, r) \in \underline{\underline{R}}\underline{\underline{V}}_k(R^n)$ and $K \cap \underline{\underline{D}}^n(p, r) = p\underline{\underline{B}}(K, e, r)$;

(5) If $\inf\{\underline{\underline{\Theta}}^k(\underline{\underline{W}}K, x) : x \in \operatorname{spt}(K) \cap \underline{\underline{D}}^n_0(p, 1)\} > 0$ then $\inf\{\underline{\underline{\Theta}}^{k-1}(\underline{\underline{W}}\,\underline{\underline{B}}(K, e, r), x) : x \in \operatorname{spt}(\underline{\underline{B}}(K, e, r))\} > 0$ whenever $0 < r < 1$ and $\underline{\underline{B}}(K, e, r)$ exists.

PROOF

Part 1. Conclusion (1) follows from 6.19. The first part of conclusion (3) follows from 7.4(1).

Part 2, Abbreviate $Q = \underline{\underline{Q}}(R^n; C)(V)$ and define for $0 < r < \operatorname{dist}(p, C)$, $\underline{\underline{\ell}}(r) = \underline{\underline{W}}(\underline{\underline{b}}(V, e) \cap \underline{\underline{D}}^n(p, r))$ and $\underline{\underline{m}}(r) = \underline{\underline{W}}(V \cap \underline{\underline{D}}^n(p, r))$. Since $\underline{\underline{\ell}}$ and $\underline{\underline{m}}$ are non-decreasing as functions of r, $\underline{\underline{\ell}}'(r)$ and $\underline{\underline{m}}'(r)$ exist for $\underline{\underline{H}}^1$ almost all r, $0 < r < \operatorname{dist}(p, C)$, and 7.1(2iii)) and 8.1(1) imply

$$\begin{aligned}\int_0^r \underline{\underline{\ell}}'(s) \le \underline{\underline{\ell}}(r) \le \underline{\underline{m}}(r) &\le k^{-1}\, r\, \underline{\underline{\ell}}'(r) + k^{-1} Q r\, \underline{\underline{m}}(r) \\ &\le k^{-1}\, r\underline{\underline{m}}'(r) + k^{-1} Q r\, \underline{\underline{m}}(r) .\end{aligned}$$

and hence, as in 8.2, $r^{-k}\underline{\underline{m}}(r)\cdot\exp(Qr)$ is non-decreasing for $0 < r < \operatorname{dist}(p, C)$.
If $\varepsilon > 0$ and $0 < r_0 < \operatorname{dist}(p, C)$ and $\underline{\underline{m}}(r) \leq (1+\varepsilon)\, r^k\, \Theta^k(\underline{\underline{W}}V, p)\alpha(k)$ for $0 < r < r_0$ then

$$\Theta^k(\underline{\underline{W}}V, p)\alpha(k) r^k \leq k^{-1}\, r\underline{\underline{\ell}}'(r) + k^{-1}(1+\varepsilon)\, Q\, \Theta^k(\underline{\underline{W}}V, p)\alpha(k)\, r^{k+1}$$

whenever $0 < r < r_0$ and hence

$$\liminf_{r \to 0+} r^{-k+1}\, \underline{\underline{\ell}}'(r) \geq k\, \Theta^k(\underline{\underline{W}}V, p)\alpha(k),$$

$$\lim_{r \to 0+} \underline{\underline{\ell}}(r)^{-1}\, \underline{\underline{m}}(r) = 1,$$

and $r^{-k}\ell(r)\exp(cr)$ is non-decreasing for $0 < r < r_0$ and suitable $c \in R^+$.
The continuity of $\underline{\underline{b}}(\cdot, e)$ and 7.1(2(iv)) imply conclusion (2).

Part 3. Let $q \in R^n - \{p\}$. For each $x \in R^n - \{p\}$ we define $\theta_q(x) \in R^+$ to be the angle between px and pq. Let $K \in \underline{\underline{K}}(V, p)$. For each ω, $0 < \omega \leq \pi$ we define

$$\underline{\underline{n}}(\theta) = \underline{\underline{W}}(V \cap \{x : x \in R^n - \{p\} \text{ and } \theta_q(x) < \theta\}).$$

Since $\underline{\underline{n}}$ is a non-decreasing function of θ for $0 < \theta \leq \pi$, $\underline{\underline{n}}'(\theta)$ exists for $\underline{\underline{H}}^1$ almost all θ, $0 < \theta < \pi$. Let $0 < \theta_0 < \pi$ such that $\underline{\underline{n}}'(\theta_0)$ exists. For each δ, $0 < \delta < \theta_0$, let $f_\delta : R^n - \{p\} \longrightarrow R^+$ be chosen of class ∞ so that for each $x, y \in R^n - \{p\}$

(i) $f_\delta(x) = 0$ if $\theta_q(x) > \theta_0$,
(ii) $f_\delta(x) = 1$ if $\theta_q(x) < \theta_0 - \delta$,
(iii) $f_\delta(p + t(x-p)) = f_\delta(x)$ for each $t \in R_0^+$,
(iv) $f_\delta(x) = f_\delta(y)$ if $\theta_q(x) = \theta_q(y)$, and
(v) $\operatorname{Lip}(f_\delta \mid \partial\underline{\underline{D}}^n(p, 1)) \leq 2\delta^{-1}$.

For each $\varepsilon > 0$ and $0 < r_1 < r_2 < 1$, let $g_\varepsilon : R^n \longrightarrow R^+$ be chosen of class ∞ such that for each $x, y \in R^n$

(i) $g_\varepsilon(r_1, r_2)(x) = 0$ if $|x-p| \leq r_1$ or $|x-p| \geq r_2$
(ii) $g_\varepsilon(r_1, r_2)(x) = 1$ if $r_1 + \varepsilon \leq |x-p| \leq r_2 - \varepsilon$

(iii) $\mathrm{Lip}(g_\varepsilon) \leq \varepsilon^{-1} + \varepsilon$

For each ε, δ, r_1, r_2 as above, define $h(r_1, r_2; \varepsilon, \delta) \in \underline{\underline{X}}(R^n, \partial \underline{\underline{D}}^n(p, 1))$ by setting for $x \in R^n$, $h(r_1, r_2; \varepsilon, \delta)(x) = f_\delta(x) \cdot g_\varepsilon(r_1, r_2)(x) \cdot (x-p)$.

From the continuity of $\underline{\underline{b}}(\cdot, e)$ and the fact that $V \in \underline{\underline{GV}}_k(R^n)$, we see that K is the weak limit of finite sums of elementary geometric varifolds $\underline{\underline{v}}$ for which $\underline{\underline{b}}(\underline{\underline{v}}, e) = 0$. One uses 6.1(4b) and conclusion (1) above to verify that if $0 < r_1 < r_2 < 1$ and

$$(d/dr)\ \underline{\underline{W}}(K \cap \{x\colon 0 < |x-p| < r \text{ and } \theta_q(x) \leq \theta_0\})$$

exists for $s = r_1, r_2$, then

$$\begin{aligned} 0 &= \lim_{\substack{\varepsilon \to 0+ \\ \varepsilon^{-1}\delta \to 0+}} \underline{\underline{S}}(K, h(r_1, r_2; \varepsilon, \delta)) \\ &= k\ \underline{\underline{W}}(K \cap \{x\colon r_1 \leq |x-p| \leq r_2 \text{ and } \theta_q(x) \leq \theta_0\}) \\ &\quad + r_1 (d/dr)\ \underline{\underline{W}}(K \cap \{x\colon 0 < |x-p| < r \text{ and } \theta_q(x) \leq \theta_0\}) \Big|_{r = r_1} \\ &\quad + r_2 (d/dr)\ \underline{\underline{W}}(K \cap \{x\colon 0 < |x-p| < r \text{ and } \theta_q(x) \leq \theta_0\}) \Big|_{r = r_2} \end{aligned}$$

for fixed q and θ_0 let

$$\underline{\underline{u}}(r) = \underline{\underline{W}}(V \cap \{x\colon 0 < |x-p| < r \text{ and } \theta_q(x) \leq \theta_0\}) .$$

Then for $\underline{\underline{H}}^1$ almost all r_1 and r_2 with $0 < r_1 < r_2 < 1$,

$$k\ \underline{\underline{u}}(r_2) - ku(r_1) = r_2\ \underline{\underline{u}}'(r_2) - r_1\ \underline{\underline{u}}'(r_1) .$$

and hence

$$\begin{aligned} k\ \underline{\underline{u}}(r) &= r\ \underline{\underline{u}}'(r) \ ; \\ \underline{\underline{u}}(r) &= c\ r^k , \\ \underline{\underline{u}}'(r) &= c\ k\ r^{k-1} , \end{aligned}$$

for some constant c, $0 \leq c \leq \Theta^k(\underline{\underline{W}}V, p)\alpha(k)$. Since q, θ_0, r_1, r_2 are arbitrary one concludes that $\underline{\underline{W}}K \cap \underline{\underline{D}}^n(p, r) = \underline{\underline{W}}\, p\underline{\underline{B}}(K, e, r)$ whenever $\underline{\underline{B}}(K, e, r)$ exists, and by 7.4 $\underline{\underline{B}}(K, e, r)$ exists for $\underline{\underline{H}}^1$ almost all r, $0 < r < 1$. Conclusion 5 is an immediate consequence.

Part 4. Let $0 < r < 1$ such that $\underline{\underline{B}}(K, e, r)$ exists and suppose $g \in \underline{\underline{X}}(R^n, \partial \underline{\underline{D}}^n(p, r))$ such that $\underline{\underline{S}}(\underline{\underline{B}}(K, e, r), g) > 0$. For each $\varepsilon > 0$ choose $f_\varepsilon : R_0^+ \longrightarrow \{t: 0 \leq t \leq 1\}$ of class ∞ such that

(i) $f_\varepsilon(x) = 0$ whenever $||x-p|-r| \geq \varepsilon$,

(ii) $f_\varepsilon(x) = 1$ whenever $||x-p|-r| \leq 2^{-1}\varepsilon$,

(iii) $\text{Lip}(f_\varepsilon) \leq 4\varepsilon^{-1}$.

Using 6.1, conclusion (2), and the assumption that $\underline{\underline{B}}(K, e, r)$ exists, one verifies the existence of $\varepsilon_0 > 0$ such that $\underline{\underline{S}}(K, h_\varepsilon) > 0$ whenever $0 < \varepsilon < \varepsilon_0$. Here $h_\varepsilon \in \underline{\underline{X}}(R^n, \partial\underline{\underline{D}}^n(p, r))$ is given for $x \in R^n - \{p\}$ by $h_\varepsilon(x) = f_\varepsilon(|x-p|) \cdot g(x)$ and $h_\varepsilon(p) = 0$. Hence the existence of g contradicts conclusion (1). Conclusion (3) is now established.

Conclusion (4) is verified using 5.4(2a), 9.2, and conclusions (1), (2), (3).

10.7 THEOREM. Let $0 \leq k \leq n$ be integers, $C \subset R^n$ be closed, and $V \in \underline{\underline{GV}}_k(R^n)$ with $\underline{\underline{Q}}(R^n; C)(V) < \infty$ and $\inf\{\Theta^k(\underline{\underline{W}}V, p) : p \in \text{spt}(V) - C\} > 0$. Then $V \cap (R^n - C) \in \underline{\underline{RV}}_k(R^n)$.

PROOF. We prove the theorem by induction on k. Clearly the theorem holds for $k = 0$. Suppose then $0 < h < n$ and the theorem holds for $0 \leq k \leq h-1 < n$. We will prove the theorem for $k = h$. Let then $V \in \underline{\underline{GV}}_h(R^n)$ with $\underline{\underline{Q}}(R^n; C)(V) = Q < \infty$ and $\inf\{\Theta^h(\underline{\underline{W}}V, p) : p \in \text{spt}(V) - C\} = d > 0$. By 10.5 it is sufficient to prove that $\text{spt}(V) - C$ is a h-dimensional rectifiable subset of R^n. $\text{spt}(V) - C$ is clearly a Borel set and by [F2 3.1] $d \cdot \underline{\underline{H}}^h \cap (\text{spt}(V) - C) \leq \underline{\underline{W}}V \cap (R^n - C) < \infty$. Let $B \subset \text{spt}(V) - C$ be a Borel set with $\underline{\underline{H}}^h(B) > 0$ and $p \in B$ such that $\Theta^h(\underline{\underline{W}}V \cap (R^n - B), x) = 0$. Let $K \in \underline{\underline{K}}(V, p)$ and $0 < r < 1$ such that $\underline{\underline{B}}(K, e, r)$ exists for e as in 10.6. By 10.6(5) $\inf\{\Theta^{h-1}(\underline{\underline{W}}\,\underline{\underline{B}}(K, e, r), x) : x \in \text{spt}(\underline{\underline{B}}(K, e, r))\} > 0$ and

$S(R^n, \partial D^n(p, r))(B(K, e, r)) = 0$. By 6.13 $Q(R^n)(B(K, e, r)) < \infty$. By our induction hypothesis $B(K, e, r) \in RV_{h-1}(R^n)$ and hence $K \in RV_h(R^n)$. By 10.6(1) and 6.13 $Q(R^n; \partial D^n(p, 1))(K) = 0$. One concludes from 10.4(2) and 9.8 that there exists $\varepsilon > 0$, $c > 0$ and an open set $U \subset \Omega_h(R^n)$ such that for each $\pi \in U$, $H^h(\pi(\mathrm{spt}(J))) \geq c$. whenever $J \in GV_h(R^n)$ with $Q(R^n; \partial D^n(p, 1))(J) \leq Q$, $\inf\{\Theta^h(WJ, x) : x \in \mathrm{spt}(J) - \partial D^n(p, 1)\} \geq d$, $W(J) \leq 2\,\Theta^h(WV, p)\alpha(h)$ and $F(K, J) \leq \varepsilon$. Let $0 < r_0 < 1$ be chosen so that $r^{-h} H^h([\mathrm{spt}(V) \cap D^n(p, r)] - B) \leq 2^{-1}c$ whenever $0 < r < r_0$. Now choose $0 < r_1 < r_0$ such that $F(K, f(p, r_1)_{\#}(V) \cap D^n(p, 1)) < \varepsilon$ and $r_1^{-h} W(V \cap D^n(p, r_1)) \leq 2\Theta^h(WV, p)\alpha(h)$ where $f(p, r_1)$ is as in 10.3. Clearly then $H^h(\pi[\mathrm{spt}(f(p, r_1)_{\#}(V))]) \geq c$ for each $\pi \in U$. Hence $H^h(\pi[\mathrm{spt}(V \cap D^n(p, r_1))]) \geq r^h c$ for each $\pi \in U$. Hence, $H^h(\pi[\mathrm{spt}(V \cap D^n(p, r_1)) \cap B]) \geq 2^{-1} r^h c$ for each $\pi \in U$. Since B is an arbitrary Borel subset of $\mathrm{spt}(V) - C$, 5.1(4) implies that $\mathrm{spt}(V) - C$ is an h-dimensional rectifiable subset of R^n. The theorem follows.

10.8 THEOREM.

Hypotheses:

(1) Let $0 \leq k \leq n$ be integers.

(2) Let $A_0, A_1, A_2, \ldots, A_k \subset R^n$ be closed subsets of R^n.

(3) Let $C_1 \quad C_2 \quad C_3 \quad \ldots \quad C_k \quad R^n$ be closed subsets of R^n.

(4) For each $i = 0, 1, 2, \ldots, k$ let $d_i, w_i : R^+ \longrightarrow R^+$ be continuous and non-decreasing.

(5) For each $i = 1, 2, \ldots, k$ let $q_i : R^+ \longrightarrow R^+$ be continuous and non-decreasing.

Conclusion: The following subset of $\mathcal{GV}_k(R^n, R^n, \ldots, R^n)$ is compact in the weak topology:

$$\mathcal{GV}_k(A_k, A_{k-1}, \ldots, A_0) \cap$$

$$\{(V_k, V_{k-1}, \ldots, V_0) : \text{(a) } W(V_i \cap D_0^n(0, r)) \leq w_i(r) \text{ for each } r \in R^+ \text{ and } i = 0, 1, 2, \ldots, k;$$

(b) $\Theta^k(\underline{\underline{W}}V_i, x) \geq d_i(r)$ <u>for each</u> $r \in R^+$, <u>each</u> $x \in [\operatorname{spt}(V) \cap \underline{\underline{D}}_0^n(0, r)] - C_i$, <u>and</u> $i = 0, 1, 2, \ldots, k$;

(c) $\underline{\underline{L}}(R^n; C_i \cup [R^n - \underline{\underline{D}}_0^n(0, r)])(V_i \cap \underline{\underline{D}}^n(0, r), V_{i-1} \cap \underline{\underline{D}}^n(0, r)$
$\leq q_i(r)$
<u>for each</u> $r \in R^+$ <u>and each</u> $i = 1, 2, \ldots, k$;

and (d) $V_i \cap (R^n - C) \in \mathcal{X}\mathcal{V}_i(R^n)$ <u>for each</u> $i = 0, 1, 2, \ldots, k\}$

<u>Here</u> $\underline{\underline{L}}$ <u>denotes either</u> $\underline{\underline{P}}$ <u>or</u> $\underline{\underline{Q}}$, <u>and</u> $\mathcal{X}\mathcal{V}_k(R^n)$ <u>denotes either</u> $\mathcal{I}\mathcal{V}_k(R^n)$ <u>or</u> $\mathcal{R}\mathcal{V}_k(R^n)$

PROOF.

<u>Part</u> 1. Let $1 \leq j \leq k$ be an integer and $(V, W) \in \underline{\underline{G}}\underline{\underline{V}}_j(R^n, R^n)$ with $\underline{\underline{Q}}(R^n, C_j)(V_i) < \infty$. For each $p \in R^n - C_j$, <u>either</u> (1) $\Theta^j(\underline{\underline{W}}V, p)$ exists and is finite, <u>or</u> (2) For each $\varepsilon > 0$ there exist arbitrarily small values of $r > 0$ for which $\underline{\underline{W}}(V \cap \underline{\underline{D}}^n(p, r)) \leq \varepsilon\, \underline{\underline{W}}(W \cap \underline{\underline{D}}^n(p, r))$.

<u>Proof of Part</u> 1. Let V, W, p be as above and for each $r \in R^+$ set $\underline{\underline{m}}(r) = \underline{\underline{W}}(V \cap \underline{\underline{D}}^n(p, r))$ and $\underline{\underline{n}}(r) = \underline{\underline{W}}(W \cap \underline{\underline{D}}^n(p, r))$. Both $\underline{\underline{m}}$ and $\underline{\underline{n}}$ are non-decreasing and thus $\underline{\underline{H}}^1$ measurable and differentiable $\underline{\underline{H}}^1$ almost everywhere. We consider two cases.

<u>Case</u> (a). Suppose $\int_0^1 [\underline{\underline{m}}(r)]^{-1} \underline{\underline{n}}(r)\, dr < \infty$. In this case 8.1 implies that

$$r^{-j}\, \underline{\underline{m}}(r) \exp[\, r\underline{\underline{Q}}(R^n; C)(V, W) + \int_0^r [\underline{\underline{m}}(s)]^{-1} \underline{\underline{n}}(s)\, ds\,]$$

is non-decreasing for $0 < r < \operatorname{dist}(p, C)$. Hence $\Theta^j(\underline{\underline{W}}V, p)$ exists and is finite.

<u>Case</u> (b). Suppose $\int_0^1 [\underline{\underline{m}}(r)]^{-1} \underline{\underline{n}}(r)\, dr = \infty$. Clearly for each $\varepsilon > 0$ there are arbitrarily small values of $r > 0$ for which $\underline{\underline{m}}(r) \leq \varepsilon\, \underline{\underline{n}}(r)$

Part 2. Let $0 \leq j \leq n$ be an integer and $A \subset R^n$ be a compact j-dimensional rectifiable subset of R^n. Let μ and ν be Radon measures on R^n such that for each $p \in A$ either $\Theta^j(\mu, p) = 0$ or for each $\varepsilon > 0$ there exist arbitrarily small values of $r > 0$ for which $\mu(\underline{\underline{D}}^n(p, r)) \leq \varepsilon\, \nu(\underline{\underline{D}}^n(p, r))$. Then $\mu(A) = 0$.

Proof of Part 2. Let $\varepsilon > 0$ and choose $\delta > 0$ so that $\nu(R^n \cap \{x: \operatorname{dist}(x, A) < \delta\}) < \varepsilon$. Set $B = A - \{x: \Theta^j(\mu, x) = 0\}$. Use [B][M0][FF 8.7] to cover μ almost all of B by a countable union of disjoint balls $\underline{\underline{D}}^n(p_i, r_i)$, $i=1, 2, 3, \ldots$, where $p_i \in B$, $0 < r_i < \delta$, and $\mu(\underline{\underline{D}}^n(p_i, r_i)) \leq \varepsilon\, \nu(\underline{\underline{D}}^n(p_i, r_i))$ for each i. We have

$$\begin{aligned}\mu(B) &\leq \Sigma_i\, \mu(\underline{\underline{D}}^n(p_i, r_i)) \\ &\leq \Sigma_i\, \varepsilon\, \nu(\underline{\underline{D}}^n(p_i, r_i)) \\ &\leq \varepsilon\, \nu(R^n \cap \{x: \operatorname{dist}(x, A) < \delta\}) \\ &\leq \varepsilon\, \nu(A) + \varepsilon\end{aligned}$$

for $p \in A-B$, $\Theta^j(\mu, p) = 0$. Since $\underline{\underline{H}}^j(A-B) < \infty$, [F2 3.6] implies $\mu(A-B) = 0$. Part 2 follows since ε is arbitrary.

Part 3. Let $2 \leq j \leq k$ be an integer and $(V_j, V_{j-1}, V_{j-2}) \in \underline{\underline{GV}}_k(R^n, R^n, R^n)$ such that

(1) $\underline{\underline{Q}}(R^n; C_j)(V_j, V_{j-1}) < \infty$

(2) $\underline{\underline{Q}}(R^n; C_{j-1})(V_{j-1}, V_{j-2}) < \infty$

(3) $\inf\{\Theta^j(\underline{\underline{W}}V_j, x) : x \in \operatorname{spt}(V_j) - [\operatorname{spt}(V_{j-1}) \cup C_j]\} > 0$

(4) $\inf\{\Theta^{j-1}(\underline{\underline{W}}V_{j-1}, x) : x \in \operatorname{spt}(V_{j-1}) - [\operatorname{spt}(V_{j-2}) \cup C_{j-1}]\} > 0$

(5) $\operatorname{spt}(V_{j-2}) - C_{j-2}$ is a (j-1) dimensional rectifiable subset of R^n with $\operatorname{clos}[\operatorname{spt}(V_{j-2})] \subset \operatorname{spt}(V_{j-2}) \cup C$. In case $j = 2$ one takes $C_0 = \phi$.

Then $\underline{\underline{W}}V_j(\operatorname{spt}(V_{j-1}) - C_j) = 0$ and $V \cap (R^n - C_j) \in \underline{\underline{RV}}_j(R^n)$.

Proof of part 3. By 10.4(1) $\operatorname{spt}(V_j) - (\operatorname{spt}(V_{j-1}) \cup C_j)$ is a j-dimensional rectifiable subset of R^n and $\operatorname{spt}(V_{j-1}) - C_{j-1}$ is a (j-1) dimensional rectifiable subset of R^n. Note that if $p \in \operatorname{spt}(V_{j-1}) - C_j$ such that $\Theta^j(\underline{\underline{W}}V_j, p)$ exists and is finite, then $\Theta^{j-1}(\underline{\underline{W}}V_j, p) = 0$. Since $C_{j-1} \subset C_j$, parts 1 and 2 imply that $\underline{\underline{W}}_j(\operatorname{spt}(V_{j-1}) - C_j) = 0$. 10.5 implies that $V_j - C \in \underline{\underline{RV}}_j(R^n)$.

Part 4. Let $0 < j \leq k$ be an integer, $d > 0$, $q < \infty$, and $V_1, V_2, V_3, \ldots \in \underline{\underline{RV}}_j(R^n)$ with $\lim_i V_i = V \in \underline{\underline{V}}_k(R^n)$, $\underline{\underline{Q}}(R^n; C_j)(V_i) \leq q$, and $\Theta^j(\underline{\underline{W}}V_j, p) \geq d$ for each $p \in \operatorname{spt}(V_i) - C_j$, for each $i = 1, 2, 3, \ldots$. Then $V \in \underline{\underline{RV}}_j(R^n)$, $\underline{\underline{Q}}(R^n; C_j)(V) \leq q$, and $\Theta^j(\underline{\underline{W}}V, p) \geq d$ for each $p \in \operatorname{spt}(V) - C_j$. In case $V_1, V_2, V_3, \ldots \in \underline{\underline{IV}}_j(R^n)$, then $V \in \underline{\underline{IV}}_j(R^n)$.

Proof of part 4. The first conclusion of part 4 follows from 8.3, 10.7, and the lower semi-continuity of $\underline{\underline{Q}}(R^n; C_j)(\cdot)$. The second conclusion follows from 5.4(c), 6.19, 8.2, 9.10, and 10.4.

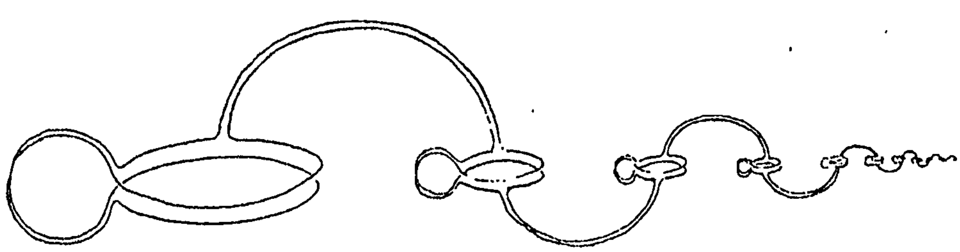
Part 5. 2.4(3), parts 3 and 4, and the Cantor diagonal process show the subsets of $\mathcal{G}\mathcal{V}_k(R^n, R^n, \ldots, R^n)$ of the conclusion to be sequentially compact. The fact that $\underline{\underline{C}}_0^j(R^n)$ is separable in the $||$ topology for each $j = 0, 1, 2, \ldots, k$ implies that $\mathcal{G}\mathcal{V}_k(R^n, R^n, \ldots, R^n)$ satisfies the first axiom of countability. $\mathcal{G}\mathcal{V}_j(R^n)$, $j = 0, 1, 2, \ldots, k$, contains a countable weakly dense subset, namely the space of all finite sums of elementary geometric varifolds $\underline{\underline{v}}(x, \lambda)$ where $x \in R^n$ has rational coordinates and $\lambda \in \underline{\underline{\Lambda}}_j(R^n)$ has rational coordinates. Thus $\mathcal{G}\mathcal{V}_k(R^n, R^n, \ldots, R^n)$ is separable. The compactness of the subsets above follows from their sequential compactness.

11. PLATEAU'S PROBLEM

11.1 Some phenomena of least area problems. J. Plateau (1801 - 1883) was a Belgian physicist and professor among whose accomplishments was a study of the geometric properties of soap bubbles and soap films [P]. The forces of surface tension (incidentally a soap film has two sides on which surface tension acts) are such that, to a good approximation, the "area" of a soap film having a given wire frame as boundary, will not exceed the area of surfaces which are obtained by small deformations, obtained, say, by blowing on the soap film [BO]. It is appropriate that the various mathematical problems having to do with least area should be called collectively Plateau's problem.

We now consider several examples which arise in the study of Plateau's problem and which illustrate the need for admitting surfaces of the generality of integral varifolds in its solution.

Example 1. The striking results obtained by J. Douglas [D] tell us that among all mappings of the 2-disk $\underline{\underline{D}}^2(0, 1)$ into R^3 which map the boundary circle $\partial\underline{\underline{D}}^2(0, 1)$ homeomorphically onto some given simple closed curve C in R^3 there will exist some mapping whose area is least. Let $A_0(C) > 0$ denote this least area. Douglas showed similarly for each $n = 1, 2, 3, \ldots$ there exists a map of least area among all mappings of the "2-disk with n-handles" into R^3 which send the boundary circle homeomorphically onto C. Let $A_n(C)$ denote this least area for $n = 1, 2, 3, \ldots$. Now consider a simple closed (unknotted) curve C in R^3 which looks like:



C

For such a curve W. H. Fleming [Fl] has proved the strict inequalities

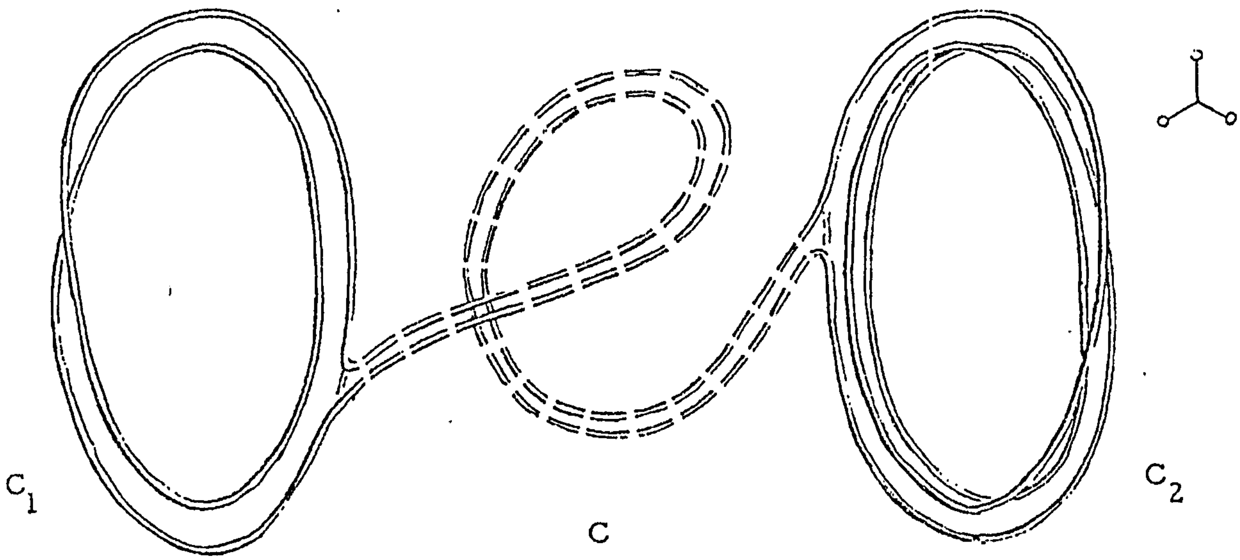
$$A_0(C) > A_1(C) > A_2(C) > \ldots > \inf_n A_n(C) > 0 .$$

One concludes then that if there should exist a manifold S having C as boundary and having least area (in some ultimate and, as yet, unspecified sense) then S must be of infinite topological type. One can in fact solve this least area problem in the space of oriented manifolds [FL2] and in the space of unoriented manifolds [FL3] [R3]. Such solutions will be real analytic submanifolds of R^3 at all non-boundary points and satisfy the minimal surface equation. The boundary behavior is not thoroughly understood. For a boundary similar to that of C above S. Scheinberg has observed that the values of area for manifolds having that boundary and satisfying the minimal surface equation can include all real numbers in an interval.

Example 2. H. Federer has shown that each k-dimensional complex algebraic variety in complex n-space is a surface of least oriented area whenever $1 \leq k \leq n$ are integers [F4 4.3]. In particular if V is such a variety and S is any oriented surface without boundary -- say, for example, a locally integral current -- which agrees with V outside a compact set, then unless S = V, the 2k-dimensional area of that portion of S not on V will be strictly greater than the 2k-dimensional area of that portion of V not on S. This is true whether or not V has singularities. Thus the class of singularities which arise naturally in the study of least area problems contains, at least, all singularities of complex algebraic varieties. Clearly $|V| \in \mathcal{IV}_{2k}(R^{2n})$ is stationary.

Example 3. R. Thom has given an example of a 14-dimensional real analytic manifold having a 7-dimensional integral homology class which cannot be represented by a 7-dimensional compact submanifold [T]. However, this homology class can be represented by a 7-dimensional oriented surface T, i. e. an integral current, of least area among all oriented surfaces representing this class [FF 9.6]. Thus T, of topological necessity, must have some singularity.

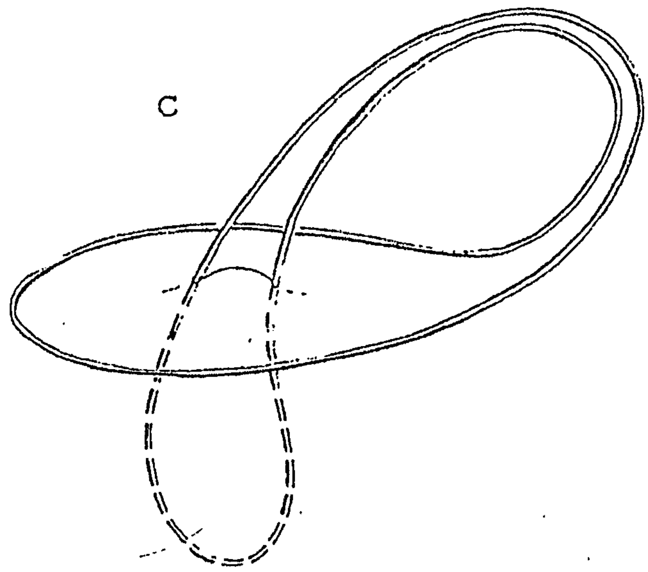
Example 4. Consider the curves C_1, C_2, and C below.



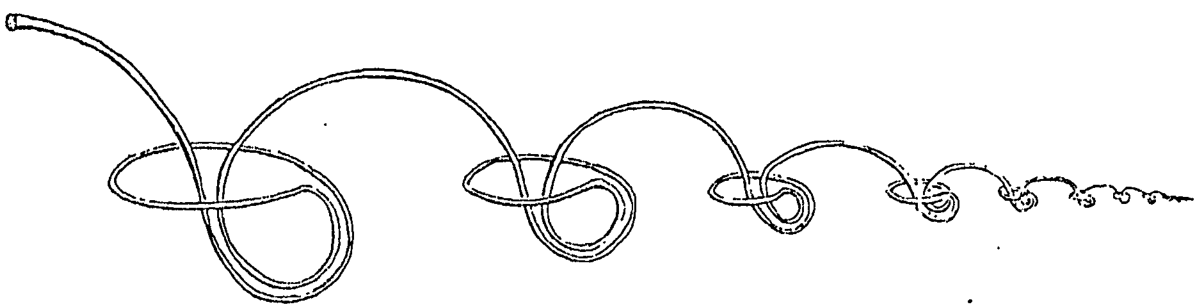
For the curve C_1 one would expect a surface S_1 of least area to resemble a Möbius band. Such a surface can in fact be obtained by solving the least area problem for boundary C_1 in the space of flat chains with the integers modulo 2 as coefficient group. Similarly, for the curve C_2 one would expect a surface S_2 of least area to resemble the so called triple Möbius band. (a section of which is indicated), and one can obtain such a surface by solving the least area problem for boundary C_2 in the space of flat chains with the integers modulo 3 as coefficient group. For the curve C (obtained by removing a small arc from both C_1 and C_2 and adding a bridge as indicated in the dashed lines) one would expect a surface of least area S to resemble S_1 (near C_1) attached to S_2 (near C_2) by a thin ribbon of surface filling in the bridge. In no topological sense does S have the boundary C for J. F. Adams has observed the existence of a retraction $(S, C) \longrightarrow (C, C)$ [Rl p. 80]. However, there is such a surface S for which

$\underset{=}{P}(R^n)(|S|, |C|) = 0$. The proof of this fact is beyond the scope of this paper.

Example 5. The sketch below is of a curved wire C and soap film S which actually forms on that wire



Note that the wire C (indicated in solid lines) does not form a closed loop! To achieve this film one makes a closed loop (indicated in solid and dashed lines), punctures the lower part of the soap film (within the dashed lines), and cuts off the obsolete wire (indicated in dashed lines). Tangent cone arguments show that to approximate this particular soap film by a stationary 2-dimensional integral varifold V in $\underset{=}{IV}_2(R^3)$ one cannot take as boundary an (infinitely thin) curve C_0 approximating C. However, it is plausible that if C_1 is a small tubular neighborhood of C_0 then there will exist V approximating S for which $\underset{=}{P}(R^n; C_1)(V) = 0$. Also, using a tubular neighborhood of tapering thickness, one would expect a closed set looking like



to contain the $\mathbf{BP}^0(R^3)$ boundary of 2-dimensional stationary integral varifolds of arbitrarily small area.

11.2 LEMMA. Let $1 \le k \le n$ be integers and $A \subset R^n$ be a compact $(k-1)$-dimensional submanifold of R^n of class 2 with or without boundary. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $p \in A$ and $0 < r \le \delta$,

$$\mathbf{W}(\,0[\,V \cap \mathbf{D}^n(p,\, r)]\,) \le \varepsilon\, \alpha(k)\, r^k .$$

PROOF. Straightforward calculation.

11.3 LEMMA. Let $1 \le k \le n$ be integers and $A \subset R^n$ be a compact $(k-1)$-dimensional submanifold of R^n of class 3 with or without boundary. Then for each $d > 0$ and $c < \infty$ there exists $\varepsilon > 0$ such that if $V \in \mathbf{RV}_k(R^n)$ with $\mathbf{W}(V) \le c$ and $\Theta^k(\mathbf{W}V;\, x) \ge d$ for each $x \in \mathrm{spt}(V) - A$, if $0 < r < \varepsilon$, if $p \in \mathrm{spt}(V) \cap A$, and if

$$\mathrm{spt}(V) \cap \mathbf{D}^n(p,\, r) \subset \{x\colon \mathrm{dist}(x,\, A) < \varepsilon\, r\},$$

then

$$\mathbf{Q}(R^n)\,(V,\, |A|) \ge c.$$

PROOF.

Part 1. Let $q \in A$ and $g \in \underline{\underline{X}}(R^n)$, $g(x) = x-q$ for $x \in R^n$. For each $r \in R_0^+$ note that

$$\underline{\underline{T}}(|A| \cap \underline{\underline{D}}^n(p, r), g) = \underline{\underline{W}}(q[|A| \cap \underline{\underline{D}}^n(p, r)]).$$

Let $d > 0$ and choose $\delta > 0$ as in 11.2 corresponding to $\varepsilon = 2^{-k-1}d$. For $r \in R_0^+$ we set

$$\underline{\underline{m}}(r) = \underline{\underline{W}}(V \cap \underline{\underline{D}}^n(p, r)),$$

$$\underline{\underline{m}}_1(r) = \underline{\underline{W}}(q[|A| \cap \underline{\underline{D}}^n(q, \delta)]) \cap \underline{\underline{D}}^n(q, r),$$

$$\underline{\underline{m}}_2(r) = \underline{\underline{m}}(r) - \underline{\underline{m}}_1(r),$$

$$\underline{\underline{n}}(r) = r^{-1}\, \underline{\underline{T}}(|A| \cap \underline{\underline{D}}^n(p, r), g).$$

Assume $\underline{\underline{Q}}(R^n)(V, |A|) < \infty$. By 8.1 we have that for $\underline{\underline{H}}^1$ almost all $R \in R_0^+$

$$0 \le -k\, \underline{\underline{m}}(r) + r\, \underline{\underline{m}}'(r) + r\, \underline{\underline{n}}(r) + r\, \underline{\underline{Q}}(R^n)(V, |A|)\, \underline{\underline{m}}(r)$$

which for $0 < r < \delta$ becomes

$$0 \le -k\, \underline{\underline{m}}_2(r) + r\, \underline{\underline{m}}_2'(r) + r\, \underline{\underline{Q}}(R^n)(V, |A|)\, \underline{\underline{m}}(r),$$

because for $\underline{\underline{H}}^1$ almost all $r \in R_0^+$,

$$\underline{\underline{m}}_1(r) = k^{-1}\, r\, \underline{\underline{m}}_1'(r) + r\, \underline{\underline{n}}(r)$$

If $2\underline{\underline{m}}_2(r) \ge \underline{\underline{m}}(r)$ for $0 < r < \delta$ integration shows that

$$r^{-k}\, \underline{\underline{m}}_2(r) \exp(2r\, \underline{\underline{Q}}(R^n)(V, |A|))$$

is non-decreasing as a function of r for $0 < r < \delta$.

Part 2. Let $c \in R^+$, $d \in R_0^+$, and $V \in \underline{\underline{RV}}_k(R^n)$ with $\underline{\underline{W}}(V) \leq c$, $\underline{\underline{Q}}(R^n)(V, |A|) < c$, and $\underline{\underline{\Theta}}^k(\underline{\underline{W}}V, x) \geq d$ for each $x \in \operatorname{spt}(V) - A$. Let $p \in \operatorname{spt}(V) \cap A$. 8.1 implies that $p \in \operatorname{clos}(\operatorname{spt}(V) - A)$. Therefore for each $\varepsilon > 0$ choose $q \in A$ with $|p-q| < \varepsilon$ and $z \in \operatorname{spt}(V)$ with $\operatorname{dist}(z, A) = |z-q| \leq \varepsilon$. For sufficiently small $\varepsilon > 0$, 8.2 implies that $\underline{\underline{W}}(V \cap \underline{\underline{D}}^n(z, |z-q|)) \geq (1-\varepsilon)\, d\, \alpha(k)\, |z-q|^k$ and thus $\underline{\underline{W}}(V \cap \underline{\underline{D}}^n(q, 2|z-q|) \geq (1-\varepsilon)\, d \cdot \alpha(k)\, |z-q|^k$. For δ chosen as in part 1, part 1 implies that for $\varepsilon < r < \delta$,

$$\underline{\underline{W}}(V \cap \underline{\underline{D}}^n(q, r)) \geq [\, 2^{-k}(1-\varepsilon) - 2^{-k-1}]\; d\, \alpha(k)\, r^k \exp(2\, r\, c).$$

Since ε is arbitrary, we have

$$\underline{\underline{W}}(V \cap \underline{\underline{D}}^n(p, r) \geq 2^{-k-1}\, d\, \alpha(k)\, r^k \exp(2\, r\, c)$$

whenever $0 < r < \delta$. The lemma now is an easy consequence of 8.7 and 9.2.

11.4 PROPOSITION. Let $1 \leq k \leq n$ be integers, $d > 0$, $c < \infty$, $A \subset R^n$ be a compact $(k-1)$-dimensional submanifold of R^n of class 3 with or without boundary, and $K \subset R^n$ be compact. Then the subset of $\underline{\underline{RV}}_k(K)$ consisting of all varifolds V for which $A \subset \operatorname{clos}(\operatorname{spt}(V) - A)$, $\underline{\underline{W}}(V) \leq c$, $\underline{\underline{Q}}(R^n)(V, |A|) \leq c$, and $\underline{\underline{\Theta}}^k(\underline{\underline{W}}V, x) \geq d$ for each $x \in \operatorname{spt}(V) - A$, is compact in the weak topology.

11.5 THEOREM. Let $1 \leq k \leq n$ and $q \geq 0$ be integers; A and B be compact $(k-1)$-dimensional submanifolds of R^n of class 3 with or without boundary; $M_1, M_2, \ldots, M_q$ be compact submanifolds of R^n of class 3 with or without boundary; $C \subset R^n$ be compact; and $W \in \underline{\underline{GV}}_{k-1}(R^n)$. Then

(1) There exists $V \in \underline{\underline{GV}}_k(R^n)$ such that

(a) $\underline{\underline{P}}(R^n, M_1, M_2, \ldots, M_q; A\;\; C)(V, |B|) = 0$

(b) $V \cap [\mathbf{R}^n - (\operatorname{spt}(W) \cup M_1 \cup M_2 \cup \ldots \cup M_q)] \in \underline{\underline{IV}}_k(\mathbf{R}^n)$

(c) $\underline{\underline{W}}(V) = \inf\{\underline{\underline{W}}(S) : S \in \underline{\underline{GV}}_k(\mathbf{R}^n) - \{0\}$ and satisfies (a) and (b) above with S replacing V$\}$

whenever there is at least one $S \in \underline{\underline{GV}}_k(\mathbf{R}^n)$ which satisfies (a) and (b) above with S replacing V. In case $\underline{\underline{Q}}(\mathbf{R}^n)(W) < \infty$ and

$$\inf\{\Theta^{k-1}(\underline{\underline{W}}W, x) : x \in \operatorname{spt}(W) - (A \cup C \cup M_1 \cup M_2 \cup \ldots \cup M_q)\} > 0,$$

then $V \cap [\mathbf{R}^n - (M_1 \cup M_2 \cup \ldots \cup M_q)] \in \underline{\underline{IV}}_k(\mathbf{R}^n)$. Also $\underline{\underline{W}}(V) > 0$ in case either

(i) $A \neq \phi$, $\partial A = B = M_1 = M_2 = \ldots = M_q = \phi$, $C = \phi$, and $W = 0$, or

(ii) $B \neq \phi$, $\partial B = A = M_1 = M_2 = \ldots = M_q = \phi$, $C = \phi$, and $W = 0$.

(2) There exists $V \in \underline{\underline{IV}}_k(\mathbf{R}^n) - \{0\}$ such that

(a) $\underline{\underline{P}}(\mathbf{R}^n; A)(V) = 0$,

(b) $A \subset \operatorname{clos}(\operatorname{spt}(V) - A)$, and

(c) $\underline{\underline{W}}(V) = \inf\{\underline{\underline{W}}(S) : S \in \underline{\underline{IV}}_k(\mathbf{R}^n) - \{0\}$ and satisfies (a) and (b) above with S replacing V$\}$,

provided either $A \neq \phi$ and $\partial A = \phi$ or there exists at least one $S \in \underline{\underline{IV}}_k(\mathbf{R}^n) - \{0\}$ which satisfies (a) and (b) above with S replacing V.

(3) There exists $V \in \underline{\underline{IV}}_k(\mathbf{R}^n) - \{0\}$ such that

(a) $\underline{\underline{P}}(\mathbf{R}^n)(V, |B|) = 0$,

(b) $B \subset \operatorname{clos}(\operatorname{spt}(V) - B)$, and

(c) $\underline{\underline{W}}(V) = \inf\{\underline{\underline{W}}(S) : S \in \underline{\underline{IV}}_k(\mathbf{R}^n) - \{0\}$ satisfies (a) and (b) above with S replacing V$\}$,

provided either $B \neq \phi$ and $\partial B = \phi$ or there exists at least one $S \in \underline{\underline{IV}}_k(\mathbf{R}^n) - \{0\}$
$S \in \underline{\underline{IV}}_k(\mathbf{R}^n) - \{0\}$ which satisfies (a) and (b) above with S replacing V.

PROOF. Most of the theorem is implied by 10.8 and 11.4. To obtain a stationary integral varifold having boundary A or B whenever there are manifolds without boundary one can, for example, solve the least area problem in the space of flat chains with coefficients in the integers modulo 2 for this boundary and then take the associated varifold.

11.6 REMARK. The preceding theorem gives a solution, in the context of integral varifolds, to a quite general formulation of Plateau's problem -- including the free boundary case. In every other formulation of the problem of least area known to this author, each solution surface (if one exists at all) is naturally a stationary integral varifold. The space in which one minimizes area in 11.5 is then, intuitively, the space of all solutions to different formulations of Plateau's problem. It seems justified then to call a minimizing varifold of 11.5 a best possible solution to Plateau's problem. The (weighted) area of the stationary integral varifold so obtained will in general be strictly less than the least area obtained by other formulations.

12. HOMOTOPY CLASSES OF MAPPINGS

12.1 DEFINITIONS. Cell complexes, # subdivisions, and the function $\mathcal{C}$.

(1) Let $\mathcal{J}^{\#}(1)$ denote the cell complex of the interval $\{x : 0 \leq x \leq 3\}$ whose 1-cell is $[0, 3]$ and whose 0-cells are the endpoints $[0]$ and $[3]$.

(2) For each positive integer m let

$$\mathcal{J}^{\#}(m) = \mathcal{J}^{\#}(1) \otimes \mathcal{J}^{\#}(1) \otimes \cdots \otimes \mathcal{J}^{\#}(1) \qquad \text{(m times)}$$

be the cell complex of $R^m \cap \{x : 0 \leq x^i \leq 3,\ i = 1, 2, \ldots, m\}$.

(3) If $\mathcal{J}$ is any cell complex we denote by $\mathcal{J}_j$ the set of cells of dimension j for $j = 0, 1, 2, \ldots$. If a is a cell of $\mathcal{J}$, $\dim(a)$ denotes the dimension of a. A 0-cell is called a vertex.

(4) We define the 0-th # subdivision of $\mathcal{J}^{\#}(1)$, denoted $\mathcal{J}^{\#}(1, 0)$ to be the cell complex given by

$$\mathcal{J}^{\#}(1, 0)_1 = \{[0, 1], [1, 2], [2, 3]\}$$
$$\mathcal{J}^{\#}(1, 0)_0 = \{[0], [1], [2]\}$$

(5) For each positive integer n we define the n-th # subdivision of $\mathcal{J}^{\#}(1)$, denoted $\mathcal{J}^{\#}(1, n)$, to be the cell complex given by

$$\begin{aligned}\mathcal{J}^{\#}(1, n)_1 = \{&[0, 1\cdot 2^{-n}], [1\cdot 2^{-n}, 2\cdot 2^{-n}], [2\cdot 2^{-n}, 3\cdot 2^{-n}], \ldots \\ &\ldots, [(2^n-1)\cdot 2^{-n}, 1], [1, 2], [2, 2+1\cdot 2^{-n}], \\ &[2+1\cdot 2^{-n}, 2+2\cdot 2^{-n}], \ldots, [2+(2^n-1)\cdot 2^{-n}, 3]\}\end{aligned}$$

$$\begin{aligned}\mathcal{J}^{\#}(1, n)_0 = \{&[0], [1\cdot 2^{-n}], [2\cdot 2^{-n}], \ldots, [(2^n-1)\cdot 2^{-n}], [1], [2], \\ &[2+1\cdot 2^{-n}], [2+2\cdot 2^{-n}], \ldots, [2+(2^n-1)2^{-n}], [3]\}.\end{aligned}$$

Note, in particular, that these cells are not evenly spaced, and that to each vertex $a \in \mathcal{J}^{\#}(1, n)_0$ there is a unique vertex $\eta(a) \in \mathcal{J}^{\#}(1)_0$ which is nearest to a, i.e. separated from a by the smallest number of 1-cells in $\mathcal{J}^{\#}(1, n)_1$.

(6) For each pair m and n of positive integers we define the n-th # subdivision of $\mathcal{J}^{\#}(m)$, denoted $\mathcal{J}^{\#}(m, n)$, by

$$\mathcal{J}^{\#}(m, n) = \mathcal{J}^{\#}(1, n) \otimes \mathcal{J}^{\#}(1, n) \otimes \ldots \otimes \mathcal{J}^{\#}(1, n) \qquad \text{(m times).}$$

Corresponding to each vertex $a \in \mathcal{J}^{\#}(m, n)_0$ there is a unique vertex $\eta(a) \in \mathcal{J}^{\#}(m)_0$ which is nearest to a, i.e. a can be joined to $\eta(a)$ by a path of 1-cells of $\mathcal{J}^{\#}(m, n)_1$ which contains fewer 1-cells than any other such path connecting a to any other vertex of $\mathcal{J}^{\#}(m)_0$.

(7) We define the boundary homomorphism $d : \mathcal{J}^{\#}(1, n) \longrightarrow \mathcal{J}^{\#}(1, n)$ by setting $d([a, b]) = [a] - [b]$ for each 1-cell $[a, b] \in \mathcal{J}^{\#}(1, n)_1$ and $d([c]) = 0$ for each 0-cell $[c] \in \mathcal{J}^{\#}(1, n)_0$. We define $d : \mathcal{J}^{\#}(m, n) \longrightarrow \mathcal{J}^{\#}(m, n)$ by setting for each cell $a = (a^1 \otimes a^2 \otimes \ldots \otimes a^m) \in \mathcal{J}^{\#}(m, n)$

$$d(a) = \sum_{i=1}^{m} (-1)^{\sigma(i)} a^1 \otimes \ldots \otimes a^{i-1} \otimes da^i \otimes a^{i+1} \otimes \ldots \otimes a^m$$

where $\sigma(i) = \sum_{j<i} \dim(i)$.

(8) Let m and n be positive integers. We will define

$$\mathcal{C}^1 : \mathcal{J}^{\#}(m, n+1)_0 \times \bigcup_j \mathcal{J}^{\#}(m)_j \longrightarrow \{0, 1, 2, \ldots, 2^n\}$$

to have the following properties:

(i) $| \mathcal{C}^1(a, \tau) - \mathcal{C}^1(\beta, \tau) | \leq 1$ whenever $a, \beta \in \mathcal{J}^{\#}(m, n+1)_0$ are endpoints of some 1-cell in $\mathcal{J}^{\#}(m, n+1)$ and τ is a cell in $\mathcal{J}^{\#}(m)$;

(ii) $\mathcal{C}^1(a, \tau) = 0$ whenever $a \in \mathcal{J}^{\#}(m, n+1)_0$ lies in the (n+1)-st $\#$ subdivision of some cell σ of $\mathcal{J}^{\#}(m)$ for which $\dim(\sigma) < \dim(\tau)$ where τ is a cell in $\mathcal{J}^{\#}(m)$;

(iii) For each $a \in \mathcal{J}^{\#}(m, n)_0$ there exists at least one cell $\tau \in \mathcal{J}^{\#}(m)$ for which $\mathcal{C}^1(a, \tau) = 2^n$.

We define $\mathcal{C}^1$ as follows:

(a) Define $\zeta_0 : \mathcal{J}^{\#}(m, 0)_0 \longrightarrow \mathcal{J}^{\#}(m)$ and, for $p = 1, 2, \ldots, n$, $\zeta_p : \mathcal{J}^{\#}(m, n)_0 \longrightarrow \mathcal{J}^{\#}(m, p-1)$ by setting for each $a \in \mathcal{J}^{\#}(m, 0)_0$ (resp. $a \in \mathcal{J}^{\#}(m, n)_0$), $\zeta_0(a)$ (resp. $\zeta_p(a)$) equal to that unique cell of least dimension in $\mathcal{J}^{\#}(m)$ (resp. in $\mathcal{J}^{\#}(m, p-1)$) whose subdivision contains a.

(b) For $a \in \mathcal{J}^{\#}(m, 0)_0$ and $(a, \tau) \in \mathrm{dmn}(\mathcal{C}^1)$ we set $\mathcal{C}^1(a, \tau) = 2^n$ [illegible] and $\mathcal{C}^1(\alpha, \tau) = 0$ otherwise. Note that for each $\alpha \in \mathcal{J}^{\#}(m, 0)$ there is some cell $\tau \in \mathcal{J}^{\#}(m)$ for which $(\alpha, \tau) \in \text{dmn}(\mathcal{C}^1)$ and $\mathcal{C}^1(\alpha, \tau) = 2^n$. Also $\mathcal{C}^1(\alpha, \tau) = 0$ whenever $\alpha \in \mathcal{J}^{\#}(m, 0)$ lies in the 0-th $\#$ subdivision of some cell σ of $\mathcal{J}^{\#}(m)$ where $\dim(\sigma) < \dim(\tau)$, τ a cell in $\mathcal{J}^{\#}(m)$.

(c) For $\alpha \in \mathcal{J}^{\#}(m, 1)_0$ and $(\alpha, \tau) \in \text{dmn}(\mathcal{C}^1)$ we set

$$\mathcal{C}^1(\alpha, \tau) = \max\{\mathcal{C}^1(\beta, \tau) : \beta \in \mathcal{J}^{\#}(m, 0)_0 \text{ is a vertex of } \zeta_1(\alpha)\} \in \{0, 2^n\}$$

Note that if $\alpha, \beta \in \mathcal{J}^{\#}(m, 1)_0$ are the endpoints of some 1-cell in $\mathcal{J}^{\#}(m, 1)_0$ and $(\alpha, \tau), (\beta, \tau) \in \text{dmn}(\mathcal{C}^1)$ then $|\mathcal{C}^1(\alpha, \tau) - \mathcal{C}^1(\beta, \tau)| \leq 2^n$. Observe also that $\mathcal{C}^1(\alpha, \tau) = 0$ whenever α $\mathcal{J}^{\#}(m, 1)_0$ lies in the 1-st $\#$ subdivision of some cell τ of $\mathcal{J}^{\#}(m)$ for which $\dim(\sigma) < \dim(\tau)$, τ a cell of $\mathcal{J}^{\#}(m)$. This is clear since each $\beta \in \mathcal{J}^{\#}(m, 0)_0$ which is a vertex of $\zeta_1(\alpha)$ will also be a vertex of σ, giving $\mathcal{C}^1(\beta, \tau) = 0$.

Note also that for each m-cell $\gamma \in \mathcal{J}^{\#}(m, 1)_m$ there exists some $\varepsilon \in \mathcal{J}^{\#}(m, 0)_0$ which is a vertex of γ. We assert that $\mathcal{C}^1(\alpha, \zeta_0(\varepsilon)) = 2^n$ for each vertex α of γ. To see this, observe that for such α, $\dim(\zeta_1(\alpha))$ equals the smallest number of 1-cells of $\mathcal{J}^{\#}(m, 1)_1$ necessary to form a path connecting α to some $\beta \in \mathcal{J}^{\#}(m, 0)_0$ which is also a vertex of γ. One concludes the existence of $\varepsilon \in \mathcal{J}^{\#}(m, 0)_0$ which is both a vertex of γ and a vertex of $\zeta_1(\alpha)$. To establish our assertion it is sufficient to verify that $\zeta_0(\varepsilon) = \zeta_0(\varepsilon')$ whenever $\varepsilon, \varepsilon' \in \mathcal{J}^{\#}(m, 0)_0$ are both vertices of γ. If τ_m denotes the unique m-cell of $\mathcal{J}^{\#}(m)_m$ and γ contains some vertex $\varepsilon \in \mathcal{J}^{\#}(m, 0)_0$ for which $\zeta_0(\varepsilon) = \tau_m$, then the definitions imply immediately that $\zeta_0(\varepsilon') = \tau_m$ whenever $\varepsilon' \in \mathcal{J}^{\#}(m, n)_0$ is another vertex of γ. Suppose then that for no vertex $\varepsilon \in \mathcal{J}^{\#}(m, 0)_0$ of γ does $\dim(\zeta_0(\varepsilon)) = m$. In that case each vertex $\varepsilon \in \mathcal{J}^{\#}(m, 0)_0$ of γ must lie in the 1-st $\#$ subdivision of some (m-1)-cell of $\mathcal{J}^{\#}(m)_{m-1}$ -- not necessarily the same cell for each vertex. If, however, for some vertex $\varepsilon \in \mathcal{J}^{\#}(m, 0)_0$ of γ, $\zeta_0(\varepsilon) = \tau_{m-1} \in \mathcal{J}^{\#}(m)_{m-1}$, then the definitions imply immediately that $\zeta_0(\varepsilon') = \tau_{m-1}$ whenever $\varepsilon' \in \mathcal{J}^{\#}(m, 0)_0$ is another vertex of γ. The proof of the assertion is obtained from arguments similar to those above applied to successively lower dimensional cells $\tau_{m-2}, \tau_{m-3}, \ldots, \tau_0$ of $\mathcal{J}^{\#}(m)$.

(d) For $\alpha \in \mathcal{J}^{\#}(m, 2)_0$ and $(\alpha, \tau) \in \operatorname{dmn}(\mathcal{C}^1)$ we define

$$\mathcal{C}^1_0(\alpha, \tau) = \operatorname{average}\{\mathcal{C}^1(\beta, \tau) : \beta \in \mathcal{J}^{\#}(m, 1)_0 \text{ is a vertex of } \zeta_2(\alpha)\}.$$

We assert that $|\mathcal{C}^1_0(\alpha, \tau) - \mathcal{C}^1_0(\beta, \tau)| \leq 2^{n-1}$ whenever $\alpha, \tau \in \mathcal{J}^{\#}(m, 2)_0 - \mathcal{J}^{\#}(m, 1)_0$ are the endpoints of some 1-cell in $\mathcal{J}^{\#}(m, 2)_1$ and $(\alpha, \tau), (\beta, \tau)$ $\operatorname{dmn}(\mathcal{C}^1)$. We see this as follows. Without loss of generality we can suppose $\dim(\zeta_2(\alpha)) = \dim(\zeta_2(\beta)) + 1$. There then exists a unique $\gamma \in \mathcal{J}^{\#}(m, 2)_0$ for which the vertices of $\zeta_2(\alpha)$ equal the union of the vertices of $\zeta_2(\beta)$ with the vertices of $\zeta_2(\gamma)$. Furthermore there is a natural one-to-one correspondence between the vertices of $\zeta_2(\beta)$ and the vertices of $\zeta_2(\gamma)$; namely vertex $\varepsilon(\beta)$ of $\zeta_2(\beta)$ corresponds to vertex $\varepsilon(\gamma)$ of $\zeta_2(\gamma)$ if and only if these vertices are the endpoints of some 1-cell in $\mathcal{J}^{\#}(m, 2)_1$. Since $|\mathcal{C}^1(\varepsilon(\beta), \tau) - \mathcal{C}^1(\varepsilon(\gamma), \tau)| \leq 2^n$ for each pair of vertices $\varepsilon(\beta), \varepsilon(\gamma)$ corresponding as above, we have that $|\mathcal{C}^1_0(\beta, \tau) - \mathcal{C}^1_0(\gamma, \tau)| \leq 2^n$. Since

$$\mathcal{C}^1_0(\alpha, \tau) = \operatorname{average}\{\mathcal{C}^1_0(\beta, \tau), \mathcal{C}^1_0(\gamma, \tau)\},$$

we have $|\mathcal{C}^1_0(\alpha, \tau) - \mathcal{C}^1_0(\beta, \tau)| \leq 2^{n-1}$. By a similar, but easier, argument one concludes also $|\mathcal{C}^1_0(\alpha, \tau) - \mathcal{C}^1_0(\beta, \tau)| \leq 2^{n-1}$ whenever $\alpha \in \mathcal{J}^{\#}(m, 2)_0 - \mathcal{J}^{\#}(m, 1)_0$ and $\beta \in \mathcal{J}^{\#}(m, 1)_0$ are the endpoints of some 1-cell in $\mathcal{J}^{\#}(m, 2)_1$ and $(\alpha, \tau), (\beta, \tau) \in \operatorname{dmn}(\mathcal{C}^1)$.

For $\alpha \in \mathcal{J}^{\#}(m, 2)$ and $(\alpha, \tau) \in \operatorname{dmn}(\mathcal{C}^1)$ we define

$$\mathcal{C}^1(\alpha, \tau) = J(\alpha, \tau) \cdot 2^{n-1} \in \{0 \cdot 2^{n-1}, 1 \cdot 2^{n-1}, 2 \cdot 2^{n-1}\}$$

where $J(\alpha, \tau)$ is that unique integer satisfying

$$(J(\alpha, \tau) - 1) \cdot 2^{n-1} < \mathcal{C}^1_0(\alpha, \tau) \leq J(\alpha, \tau) 2^{n-1}.$$

Note that $|\mathcal{C}^1(\alpha, \tau) - \mathcal{C}^1(\beta, \tau)| \leq 2^{n-1}$ whenever $\alpha, \beta \in \mathcal{J}^{\#}(m, 2)_0$ are the endpoints of some 1-cell in $\mathcal{J}^{\#}(m, 2)_1$ and $(\alpha, \tau), (\beta, \tau) \in \operatorname{dmn}(\mathcal{C}^1)$. Note also that $\mathcal{C}^1(\alpha, \tau) = 0$ whenever $\alpha \in \mathcal{J}^{\#}(m, 2)_0$ lies in the 2-nd # subdivision of some cell σ of $\mathcal{J}^{\#}(m)$ with $\dim(\sigma) < \dim(\tau)$, $(\alpha, \tau) \in \operatorname{dmn}(\mathcal{C}^1)$, since $\zeta_2(\alpha)$ will be a cell in the 1-st # subdivision of σ. For each m-cell $\gamma \in \mathcal{J}^{\#}(m, 2)_m$ there exists [illegible] one cell $\tau \in \mathcal{J}^{\#}(m)$ such that $\mathcal{C}^1(\alpha, \tau) = 2^n$ for each vertex α of γ.

(e) Assume inductively that $2 \leq k < n+1$ and

$$\mathcal{C}^1 | \operatorname{dmn}(\mathcal{C}^1) \cap \{(a, \tau) : a \in \mathcal{J}^{\#}(m, k)_0\}$$
$$\longrightarrow \{0, 1\cdot 2^{n-k+1}, 2\cdot 2^{n-k+1}, \ldots, 2^{k-1}\cdot 2^{n-k+1}\}$$

has been defined with the following properties:

(i) $|\mathcal{C}^1(a, \tau) - \mathcal{C}^1(\beta, \tau)| \leq 2^{n-k+1}$ whenever $a, \beta \in \mathcal{J}^{\#}(m, k)_0$ are the endpoints of some 1-cell in $\mathcal{J}^{\#}(m, k)_1$, $(a, \tau), (\beta, \tau) \in \operatorname{dmn}(\mathcal{C}^1)$;

(ii) $\mathcal{C}^1(a, \tau) = 0$ whenever $(a, \tau) \in \operatorname{dmn}(\mathcal{C}^1)$ and $a \in \mathcal{J}^{\#}(m, k)_0$ lies in the k-th $\#$ subdivision of some cell σ of $\mathcal{J}^{\#}(m)$ where $\dim(\sigma) < \dim(\tau)$; and

(iii) For each m-cell $\gamma \in \mathcal{J}^{\#}(m, k)_m$ there exists at least one cell $\tau \in \mathcal{J}^{\#}(m)$ such that $\mathcal{C}^1(a, \tau) = 2^n$ for each vertex a of γ.

For $a \in \mathcal{J}^{\#}(m, k+1)_0$ and $(a, \tau) \in \operatorname{dmn}(\mathcal{C}^1)$ we set

$$\mathcal{C}^1_0(a, \tau) = \operatorname{average}\{\mathcal{C}^1(\beta, \tau) : \beta \in \mathcal{J}^{\#}(m, k)_0 \text{ is a vertex of } \zeta_k(a)\}$$
$$\mathcal{C}^1(a, \tau) = J(a, \tau)\cdot 2^{n-k} \in \{0, 1\cdot 2^{n-k}, 2\cdot 2^{n-k}, \ldots, 2^k\cdot 2^{n-k}\}$$

where $J(a, \tau)$ is that unique integer satisfying

$$(J(a, \tau) - 1)\cdot 2^{n-k} < \mathcal{C}^1_0(a, \tau) \leq J(a, \tau)\cdot 2^{n-k}.$$

One verifies as in (d):

(i) $|\mathcal{C}^1(a, \tau) - \mathcal{C}^1(\beta, \tau)| \leq 2^{n-k}$ whenever $a, \beta \in \mathcal{J}^{\#}(m, k+1)_0$ are endpoints of some 1-cell in $\mathcal{J}^{\#}(m, k+1)_1$, $(a, \tau), (\beta, \tau) \in \operatorname{dmn}(\mathcal{C}^1)$.

(ii) $\mathcal{C}^1(a, \tau) = 0$ whenever $(a, \tau) \in \operatorname{dmn}(\mathcal{C}^1)$ and $a \in \mathcal{J}^{\#}(m, k+1)_0$ lies in the (k+1)-st $\#$ subdivision of some cell σ of $\mathcal{J}^{\#}(m)$ where $\dim(\sigma) < \dim(\tau)$.

(iii) For each m-cell $\gamma \in \mathcal{J}^{\#}(m, k+1)_m$ there exists at least one cell $\tau \in \mathcal{J}^{\#}(m)$ such that $\mathcal{C}^1(a, \tau) = 2^n$ for each vertex a of γ.

The definition of $\mathcal{C}^1$ is completed by the inductive procedure.

(9) $\square$ will be called a <u>cubical complex for</u> R if and only if there exists a doubly infinite sequence of numbers

$$\ldots < a_{-3} < a_{-2} < a_{-1} < a_0 < a_1 < a_2 < a_3 < \ldots$$

with

$$\lim_{i \to \infty} a_i = \infty$$
$$\lim_{i \to -\infty} a_i = -\infty$$

and the 1 cells of $\square$, denoted $\square_1$, are given by

$$\square_1 = \{\ldots, [a_{-3}, a_{-2}], [a_{-2}, a_{-1}], [a_{-1}, a_0], [a_0, a_1], [a_1, a_2], [a_2, a_3], \ldots\}$$

and the 0-cells of $\square$, denoted $\square_0$, are given by

$$\square_0 = \{\ldots, [a_{-3}], [a_{-2}], [a_{-1}], [a_0], [a_1], [a_2], [a_3], \ldots\}$$

(10) Let m be a positive integer. $\square$ will be called a cubical complex for R^m if and only if there exists cubical complexes $\square^1, \square^2, \ldots, \square^m$ for R such that $\square = \square^1 \otimes \square^2 \otimes \ldots \otimes \square^m$. We denote by $\square_p$ the p-dimensional cells of $\square$ for each $p = 0, 1, \ldots, m$. We define the distance, denoted $\mathrm{dist}_{\square}(\alpha, \beta)$, between two vertices α and β in $\square_0$ to be the smallest number of 1-cells in $\square_1$ which comprise a path connecting α to β. Cell $\sigma = \sigma^1 \otimes \sigma^2 \otimes \ldots \otimes \sigma^m$ is called a face of cell $\tau = \tau^1 \otimes \tau^2 \otimes \ldots \otimes \tau^m$ if and only if $\sigma^i = \tau^i$ whenever $\tau^i \in \square_0^i$ and if $\tau_i \in \square_1^i$ then either $\sigma^i = \tau^i$ or σ^i is an endpoint of τ^i, $i = 1, 2, \ldots, m$. A 0-dimensional face is called a vertex.

(11) Let m be a positive integer and $\square$ a cubical complex for R^m. $\nabla \subset \square$ is called a subcomplex of $\square$ if and only if $\sigma \in \nabla$ whenever σ is a cell of which is a face of some cell in ∇. $\Diamond \subset \square$ is a subcomplex of ∇ if and only if $\Diamond \subset \square$ and $\Diamond$ is a subcomplex of $\square$. For $p = 0, 1, \ldots, m$, we set $\nabla_p = \nabla \cap \square_p$ and $\Diamond_p = \Diamond \cap \square_p$.

(12) Let m be a positive integer and $\square$ a cubical complex for R^m. Let $\gamma \in$ [illegible] and $\mathcal{J}^{\#}_{\gamma}(m)$ be the subcomplex of $\square$ generated by γ and all its faces. $\mathcal{J}^{\#}_{\gamma}(m)$ is isomorphic with $\mathcal{J}^{\#}(m)$ in the obvious natural way preserving coordinate direc[illegible] Let $\varphi_\gamma : \mathcal{J}^{\#}_{\gamma}(m) \longrightarrow \mathcal{J}^{\#}(m)$ denote this isomorphism. Then there exists an affine mapping $f_\gamma : R^m \longrightarrow R^m$ having each coordinate direction as an eigen-direction with a positive eigenvalue such that for each cell a of $\mathcal{J}^{\#}_{\gamma}(m)$, $f_\gamma(|a|) = |\varphi_\gamma(a)|$. Here $|a|, |\varphi_\gamma(a)| \subset R^m$ are the closed sets correspon[illegible]

in the obvious way, to a and $\varphi_\gamma(a)$ respectively. Again, in the obvious way, for each non-negative integer n, the n-th $\#$ subdivision $\mathcal{J}^{\#}(m, n)$ of $\mathcal{J}^{\#}(m)$ determines a subdivision $\mathcal{J}_\gamma^{\#}(m, n)$ of $\mathcal{J}_\gamma^{\#}(m)$, also induced by f_γ and called the n-th $\#$ <u>subdivision of</u> $\mathcal{J}_\gamma^{\#}(m)$. φ_γ extends to be an isomorphism $\psi_\gamma : \mathcal{J}_\gamma^{\#}(m, n) \longrightarrow \mathcal{J}^{\#}(m, n)$.

We define

$$\mathcal{C}_\gamma : \mathcal{J}_\gamma^{\#}(m, n)_0 \times \bigcup_j \mathcal{J}_\gamma^{\#}(m)_j \longrightarrow \{0, 1, 2, \ldots, 2^n\}$$

by setting $\mathcal{C}_\gamma(a, \tau) = \mathcal{C}^1(\psi_\gamma(a), \psi_\gamma(\tau))$ for each $(a, \tau) \in \operatorname{dmn}(\mathcal{C}_\gamma)$.

To each m-cell $\gamma \in \square_m$ we have associated the n-th $\#$ subdivision of the subcomplex $\mathcal{J}_\gamma^{\#}(m)$ of $\square$. We define the n-th $\#$ <u>subdivision of</u> $\square$ as $\bigcup\{\mathcal{J}_\gamma^{\#}(m, n) : \gamma \in \square_m\}$. This union is, of course, not a disjoint union.

Let ${}_1\square$ be the (n+1)-st $\#$ subdivision of $\square$. We define

$$\mathcal{C} : ({}_1\square_0 \times \bigcup_j \square_j) \cap \{(a, \tau) : a \text{ is a vertex of } \tau\} \longrightarrow \{0, 1, 2, \ldots, 2^n\}$$

$$= \bigcup\{\mathcal{C}_\gamma : \gamma \in \square_m\} \mid \operatorname{dmn}(\mathcal{C}).$$

One verifies that $\mathcal{C}$ is well defined and has the following properties:

(i) For each $a \in {}_1\square_0$ there exists some $(a, \tau) \in \operatorname{dmn}(\mathcal{C})$ with $\mathcal{C}(a, \tau) = 2^n$;

(ii) For $(a, \tau), (\beta, \tau) \in \operatorname{dmn}(\mathcal{C})$, $|\mathcal{C}(a, \tau) - \mathcal{C}(\beta, \tau)| \leq \operatorname{dist}_{{}_1\square}(a, \beta)$;

(iii) If $(a, \tau) \in \operatorname{dmn}(\mathcal{C})$ and $\beta \in {}_1\square_0$, $(\beta, \tau) \notin \operatorname{dmn}(\mathcal{C})$, then $\mathcal{C}(a, \beta) \leq \operatorname{dist}_{{}_1\square}(a, \beta)$.

(13) Let m be a positive integer and $\square$ a cubical complex for R^m. $\square$ has the usual boundary homomorphism $d : \square \longrightarrow \square$ induced for example from (7) by the various isomorphisms $\{\varphi_\gamma\}_\gamma$ of (12). Note that if ∇ is a subcomplex of $\square$ then $d(\nabla) \subset \nabla$.

12.2 DEFINITIONS. <u>Homotopy relations between mappings.</u>

(1) Let m be a positive integer and $\square$ a cubical complex for R^m. Let ∇ be a subcomplex of $\square$ and $\diamondsuit$ a subcomplex of ∇. Let K be a metric space with

distance function, dist_K, and $L \subset K$. For each $\varphi : (\nabla_0, \Diamond_0) \longrightarrow (K, L)$ we define the <u>fineness of</u> φ to be

$$\sup\{(\mathrm{dist}_{\Box}(\alpha, \beta))^{-1}\mathrm{dist}_K(\varphi(\alpha), \varphi(\beta)) : \alpha, \beta \in \nabla_0, \alpha \neq \beta\}.$$

We say that φ <u>has fineness</u> δ if and only if the fineness of φ does not exceed δ.

(2) Let m be a positive integer and ${}_1\Box$, ${}_2\Box$ be cubical complexes for R^m. Let ${}_i\nabla$ be a subcomplex of ${}_i\Box$, let ${}_i\Diamond$ be a subcomplex of ${}_i\nabla$, and let ${}_i\varphi : ({}_i\nabla_0, {}_i\Diamond_0) \longrightarrow (K, L)$ for $i = 1, 2$ where K and L are as in (1). Let $\delta \in R^+$. One says that ${}_1\Box$ <u>is simply homotopic to</u> ${}_2\Box$ <u>with fineness</u> δ if and only if:

(a) Either ${}_1\Box = {}_2\Box$ or there exists a non-negative integer n_1 such that ${}_2\Box$ is the n_1-th $\#$ subdivision of ${}_1\Box$, ${}_2\nabla$ is the n_1-th $\#$ subdivision of ${}_1\nabla$, and ${}_2\Diamond$ is the n_1-th $\#$ subdivision of ${}_1\Diamond$;

(b) There exists a non-negative integer n_2 and a mapping

$$\psi : ([\mathcal{I}^{\#}(1, n_2) \otimes {}_1\nabla]_0, [\mathcal{I}^{\#}(1, n_2) \otimes {}_1\Diamond]_0) \longrightarrow (K, L)$$

having fineness δ such that $\psi([0] \otimes \alpha) = {}_1\varphi(\alpha)$ and $\psi([3] \otimes \alpha) = {}_2\varphi(\alpha)$ for each $\alpha \in {}_1\nabla_0 \subset {}_2\nabla_0$; and

(c) ${}_2\varphi(\alpha) = {}_2\varphi(\eta(\alpha))$ for each $\alpha \in {}_2\nabla_0$. Here $\eta(\alpha)$ is the nearest vertex in ${}_1\nabla_0$ to α.

We say that ${}_1\varphi$ <u>is homotopic with</u> ${}_2\varphi$ <u>with fineness</u> δ if and only if there exists a finite sequence of mappings

$$\varphi^i : (\nabla^i_0, \Diamond^i_0) \longrightarrow (K, L), \qquad i = 1, 2, 3, \ldots, q,$$

where for each i, ∇^i is a subcomplex of some cubical subdivision of R^m, $\Diamond^i$ is a subcomplex of ∇^i, and:

(a) Either ${}_1\varphi$ is simply homotopic to φ^1 with fineness δ or φ^1 is simply homotopic to ${}_1\varphi$ with fineness δ;

(b) Either ${}_2\varphi$ is simply homotopic to φ^q with fineness δ or φ^q is simply homotopic to ${}_2\varphi$ with fineness δ.

(c) For each $i = 1, 2, 3, \ldots, q-1$, either φ^i is simply homotopic to φ^{i+1} with fineness δ or φ^{i+1} is simply homotopic to φ^i with fineness δ.

(3) Let m be a positive integer and $\square$ be a cubical complex for R^m. Let ${}_1\nabla$, ${}_2\nabla$ be subcomplexes of $\square$. We say ${}_1\nabla$ and ${}_2\nabla$ are <u>isomorphic</u> if and only if there exists a chain map $\theta : {}_1\nabla \longrightarrow {}_2\nabla$ of degree 0 which is a chain isomorphism.

12.3 DEFINITIONS. <u>Critical sequences of mappings.</u> Let m be a positive integer and $\square$ a cubical complex for R^m. Let ∇ be a subcomplex of $\square$ and $\Diamond$ a subcomplex of ∇. Let K be a complete metric space, $L \subset K$ a closed subset, and $f : K \longrightarrow R^+$ be lower semi-continuous such that for each $r \in R^+$, $K \cap \{x : f(x) \leq r\}$ is compact. <u>A homotopy sequence of mappings</u> $(\nabla, \Diamond) \longrightarrow (K,$ <u>with respect to</u> f is a sequence S of mappings

$$\varphi^i : (\nabla_0^i, \Diamond_0^i) \longrightarrow (K, L), \qquad i = 1, 2, 3, \ldots,$$

together with $\delta^1, \delta^2, \delta^3, \ldots \in R^+$ such that

(a) For each $i = 1, 2, 3, \ldots$ there exists a finite sequence $n_1(i), n_2(i), n_3(i), \ldots, n_q(i)$ of non-negative integers such that

(i) ∇^i is the $n_1(i)$-th # subdivision of the $n_2(i)$-th # subdivision of ... of the $n_q(i)$-th # subdivision of ∇, and

(ii) $\Diamond^i$ is the $n_1(i)$-th # subdivision of the $n_2(i)$-th # subdivision of ... of the $n_q(i)$-th # subdivision of $\Diamond$;

(b) φ^i is homotopic with φ^{i+1} with fineness δ^i for each $i = 1, 2, 3,$

(c) $\lim_i \delta^i = 0$; and

(d) $\sup\{f \circ \varphi^i(a) : a \in \nabla_0^i, \ i = 1, 2, 3, \ldots\} < \infty$.

If ${}_1S$ and ${}_2S$ are homotopy sequences of mappings $(\nabla, \Diamond) \longrightarrow (K,$ with respect to f given by

$${}_jS = \{{}_j\varphi^i : ({}_j\nabla^i, {}_j\Diamond^i) \longrightarrow (K, L)\}_i \qquad (j = 1, 2),$$

we say that ${}_1S$ <u>is homotopic with</u> ${}_2S$ if and only if there exist $\delta^1, \delta^2, \ldots$

with $\lim_i \delta^i = 0$ such that ${}_1\varphi^i$ is homotopic with ${}_2\varphi^i$ with fineness δ^i for each $i = 1, 2, 3, \ldots$. One verifies that "is homotopic with" is an equivalence relation in the space of homotopy sequences of mappings $(\nabla, \Diamond) \longrightarrow (K, L)$ with respect to f. We call an equivalence class of such sequences a <u>homotopy class of mappings</u> $(\nabla, \Diamond) \longrightarrow (K, L)$ <u>with respect to</u> f.

Let Π be a homotopy class of mappings $(\nabla, \Diamond) \longrightarrow (K, L)$ with respect to f. We define

$$\underline{\underline{K}} : \Pi \longrightarrow \{A : A \subset K \text{ is compact}\}$$

$$\underline{\underline{K}}(S) = \{x : \text{for some sequence } i_1 < i_2 < i_3 < \ldots \text{ and choice of } a_j \in \nabla_0^{i_j},\ x = \lim_j \varphi^{i_j}(a_j)\}$$

for each $S = \{\varphi^i : (\nabla_0^i, \Diamond_0^i) \longrightarrow (K, L)\}_i \in \Pi$. We define

$$\underline{\underline{L}} : \Pi \longrightarrow R^+$$

$$\underline{\underline{L}}(S) = \sup\{f(x) : x \in \underline{\underline{K}}(S)\}$$

for each $S \in \Pi$. For each $S \in \Pi$ we define the <u>critical set of</u> S to be

$$\underline{\underline{C}}(S) = \underline{\underline{K}}(S) \cap \{x : f(x) = \underline{\underline{L}}(S)\}.$$

Unless f is continuous, $\underline{\underline{C}}(S)$ may be empty. We define also

$$\underline{\underline{L}}(\Pi) = \inf\{\underline{\underline{L}}(S) : S \in \Pi\}$$

to be the <u>critical level of</u> Π. $S \in \Pi$ is called a <u>critical sequence for</u> Π if and only if $\underline{\underline{L}}(S) = \underline{\underline{L}}(\Pi)$.

12.4 PROPOSITION. <u>Let</u> K <u>be a complete metric space,</u> $L \subset K$ <u>a closed subset, and</u> $f : K \longrightarrow R^+$ <u>be lower semi-continuous such that for each</u> $r \in R^+$, $K \cap \{x : f(x) \le r\}$ <u>is compact. Let</u> m <u>be a positive integer. Then each homotopy class</u> Π <u>of mappings</u> $(\nabla, \Diamond) \longrightarrow (K, L)$ <u>with respect to</u> f <u>contains a critical sequence</u> S. <u>Here for some cubical complex</u> $\Box$ <u>for</u> R^m, ∇ <u>is a subcomplex of</u> $\Box$ <u>and</u> $\Diamond$ <u>is a subcomplex of</u> ∇.

PROOF. Let Π be a homotopy class of mappings $(\nabla, \Diamond) \longrightarrow (K,$

with respect to f (where ∇ and $\Diamond$ are as above) and ${}_jS = \{{}_j\varphi^i : ({}_j\nabla_0^i, {}_j\Diamond_0^i) \to (K, L)\}_i \in \Pi$ for $j = 1, 2, 3, \ldots$ such that $\lim_j \underline{\underline{L}}({}_jS) = \underline{\underline{L}}(\Pi)$. Noting that for each $j = 1, 2, 3, \ldots$

$$\lim \sup_i \{f \circ {}_j\varphi^i(a) : a \in {}_j\nabla_0^i\} = \underline{\underline{L}}({}_jS),$$

$$\lim \inf_i \{\delta : {}_j\varphi^i \text{ is homotopic with } {}_j\varphi^{i+1} \text{ with fineness } \delta\} = 0,$$

and

$$\lim \inf_i \{\delta : {}_j\varphi^i \text{ is homotopic with } {}_{j+1}\varphi^i \text{ with fineness } \delta\} = 0;$$

one chooses integers $1 \leq n(1) \leq n(2) \leq n(3) \leq \ldots$ such that

(a) $\displaystyle\sup_{i \geq n(h-1)} \sup\{f \circ {}_h\varphi^i(a) : a \in {}_h\nabla^i\} \leq \underline{\underline{L}}({}_hS) + 2^{-h}$ for $h = 2, 3, 4, \ldots,$

(b) $\displaystyle\sup_{i \geq n(h-1)} \inf\{\delta : {}_h\varphi^i$ is homotopic with ${}_h\varphi^{i+1}$ with fineness $\delta\} \leq 2^{-h}$ for $h = 2, 3, 4, \ldots,$ and

(c) $\displaystyle\sup_{1 \leq \ell \leq h} \sup_{i \geq n(h)} \inf\{\delta : {}_\ell\varphi^i$ is homotopic with ${}_{\ell+1}\varphi^i$ with fineness $\delta\} \leq 2^{-h}$

for $h = 1, 2, 3, \ldots$.

We define $S = \{\varphi^i\}_i \in \Pi$ by setting $\varphi^i = {}_1\varphi^i$ for $1 \leq i \leq n(1) - 1$, and $\varphi^i = {}_j\varphi^i$ for $n(j-1) \leq i \leq n(j) - 1$ for each $j = 2, 3, 4, \ldots$. One verifies that $S \in \Pi$ and $\underline{\underline{L}}(S) = \underline{\underline{L}}(\Pi)$.

12.5 THEOREM. <u>Let</u> $1 \leq k \leq n$ <u>and</u> $0 \leq m$ <u>be integers and</u> G <u>an admissible group. Let</u> A <u>be a compact submanifold of</u> R^n <u>with boundary</u> B, <u>and</u> C <u>be a compact submanifold of</u> A <u>with boundary</u> D. <u>Let</u> $c \in R^+$ <u>and</u> Π <u>denote a homotopy class of mappings</u> $(\nabla, \Diamond) \to (\underline{\underline{VZ}}_k(A, B \cup C; G), \underline{\underline{VZ}}_k(A, B \cup C; G) \cap \{\nu : \underline{\underline{W}}(\nu) \leq c\})$ <u>with respect to</u> $\underline{\underline{W}}$. (<u>Here for some cubical complex</u> $\Box$ <u>for</u> R^m, ∇ <u>is a subcomplex of</u> $\Box$, <u>and</u> $\Diamond$ <u>is a subcomplex of</u> ∇). <u>Then there exists a critical sequence</u> S <u>for</u> Π <u>such that:</u>

(1) $\underline{\underline{C}}(S)$ <u>is not empty</u>,

(2) <u>For each</u> $(V, W; T, \partial T) \in \underline{\underline{C}}(S)$, $\underline{\underline{S}}(R^n, A, B, C)(V, W) = 0$, $\underline{\underline{P}}^-(R^n, A)(V, W) = 0$, <u>and</u> $\underline{\underline{P}}^-(R^n, B, C)(W, 0) \leq 1$.

(3) <u>In case</u> $B = D = \emptyset$ <u>then</u> $\underline{\underline{P}}(R^n, A)(V, W) = 0$ <u>and</u> $\underline{\underline{P}}(R^n, C)(W, 0) \leq 1$ <u>for each</u> $(V, W; T, \partial T) \in \underline{\underline{C}}(S)$.

PROOF. Let ${}_1S = \{{}_1\varphi^i\}_i$ be a critical sequence for Π (12.4) and $c_3 = \sup\{{}_1\varphi^i(\alpha) : \alpha \in \operatorname{dmn}({}_1\varphi^i),\ i = 1, 2, 3, \ldots\} < \infty$. The theorem is immediate for $k \geq \dim(A)$. We assume therefore $k < \dim(A)$.

For $i = 0, 1, 2$ we define

$$G^i : R^+ \times \underline{\underline{X}}(R^n) \times \underline{\underline{VZ}}_k(R^n, R^n; G) \longrightarrow \underline{\underline{VZ}}_k(R^n, R^n; G)$$

by setting for each $t \in R^+$, $g \in \underline{\underline{X}}(R^n)$, and $(V, W; T, \partial T) \in \underline{\underline{VZ}}_k(A, B; G)$,

$$G^0(t, g, (V, W; T, \partial T)) = (f(t, \cdot)_\# V,\ f(t, \cdot)_\# W;\ f(t, \cdot)_\# T,\ f(t, \cdot)_\# \partial T),$$

$$G^1(t, g, (V, W; T, \partial T)) = (f(t, \cdot)_\# V + f_\#([0, t] \times W),\ W;\ f(t, \cdot)_\# T + f_\#([0, t] \times \partial T),\ \partial T),$$

$$G^2(t, g, (V, W; T, \partial T)) = (V + f_\#([0, t] \times W),\ f(t, \cdot)_\# W;\ T - f_\#([0, t] \times \partial T),\ f(t, \cdot)_\# \partial t).$$

Here $f : R \times R^n \longrightarrow R$ is that unique deformation characterized by $f(0, x) = x$ and $(\partial f / \partial t)(t, x) = g(x)$ for each $x \in R^n$ and $t \in R$. The continuity properties of G^0, G^1, and G^2 were noted in 4.1 and 6.4 (5).

We define for $\alpha = 0, 1, 2$ and $i = 2, 3, 4, \ldots$

$$\underline{\underline{A}} = \underline{\underline{VZ}}_k(A, B \cup C; G) \cap \{\nu : \underline{\underline{W}}(\nu) \leq c_3\}$$

$$A^0_\infty = \underline{\underline{A}} \cap \{(V, W; T, \partial T) : \underline{\underline{S}}(R^n, A, B, C)(V, W) = 0\}$$

$$A^1_\infty = \underline{\underline{A}} \cap \{V, W; T, \partial T) : \underline{\underline{P}}^-(R^n, A)(V, W) = 0\}$$

$$A^2_\infty = \underline{\underline{A}} \cap \{(V, W; T, \partial T) : \underline{\underline{P}}^-(R^n, B, C)(W, 0) \leq 1$$

$$A^\alpha_1 = \underline{\underline{A}} \cap \{\nu : \underline{\underline{F}}(\nu, A^\alpha_\infty) \geq 2^{-1}\}$$

$$A^\alpha_i = \underline{\underline{A}} \cap \{\nu : 2^{-i} \leq \underline{\underline{F}}(\nu; A^\alpha_\infty) \leq 2^{-i+1}\}$$

where for each $B \subset \underline{\underline{A}}$ and $\nu \in \underline{\underline{A}}$, $\underline{\underline{F}}(\nu, B) = \inf\{\underline{\underline{F}}(\nu, \mu) : \mu \in B\}$.

For $\alpha = 0, 1, 2$ and $j = 1, 2, 3, \ldots$ one chooses a positive integer $q(\alpha, j)$ and:

$$u^a(j, 1),\ u^a(j, 2),\ \ldots,\ u^a(j, q(a, j)) \in A^a_j,$$

$$r^a(j, 1),\ r^a(j, 2),\ \ldots,\ r^a(j, q(a, j)) \in R^+_0 \cap \{r : 0 < r < 2^{-j}\},$$

$$g^a(j, 1),\ g^a(j, 2),\ \ldots,\ g^a(j, q(a, j)) \in \underline{\underline{X}}^a \cap \{g : \|g\| \leq 1\}$$

(where $\underline{\underline{X}}^0 = \underline{\underline{X}}(R^n, A, B, C)$, $\underline{\underline{X}}^1 = \underline{\underline{X}}(R^n, A)$, and $\underline{\underline{X}}^2 = X(R^n, B, C)$) such that for each $i = 1, 2, 3, \ldots$ the open balls

$$U^a(j, i) = \underline{\underline{A}} \cap \{\nu : \underline{\underline{F}}(\nu, u^a(j, i)) < r^a(j, i)\}$$

have the following properties:

(a) $$A^a_j \subset \bigcup_{i=1}^{q(a,j)} U^a(j, i)$$

(b) $$-[\underline{\underline{S}}(V^0, g^0(j, i)) + \underline{\underline{S}}(W^0, g^0(j, i))] \geq 2^{-1} \inf\{\underline{\underline{S}}(R^n, A, B, C)(V, W) : (V, W; T, \partial T) \in A^0_j \text{ for some } T\} > 0$$

for each $(V^0, W^0; T^0, \partial T^0) \in U^0(j, i)$, $i = 1, 2, \ldots, q(0, j)$ and $j = 1, 2, 3, \ldots$.

(c) $$-\underline{\underline{S}}(V^1, g^1(j, i)) - \underline{\underline{T}}(W^1, g^1(j, i)) \geq 2^{-1} \inf\{\sup\{-\underline{\underline{S}}(V, g) - \underline{\underline{T}}(W, g) : g \in \underline{\underline{X}}(R^n, A) \text{ and } \|g\| \leq 1\} : (V, W; T, \partial T) \in A^1_j \text{ for some } T\} > 0$$

for each $(V^1, W^1; T^1, \partial T^1) \in U^1(j, i)$, $i = 1, 2, \ldots, q(1, j)$ and $j = 1, 2, 3, \ldots$.

(d) $$-\underline{\underline{S}}(W^2, g^2(j, i)) - \underline{\underline{T}}(W^2, g^2(j, i)) \geq 2^{-1} \inf\{\sup\{-\underline{\underline{S}}(W, g) - \underline{\underline{T}}(W, g) : g \in \underline{\underline{X}}(R^n, B, C) \text{ and } \|g\| \leq 1\} : (V, W; T, \partial T) \in A^2_j \text{ for some } V \text{ and } T\} > 0$$

for each $(V^2, W^2; T^2, \partial T^2) \in U^2(j, i)$, $i = 1, 2, \ldots, q(2, j)$ and $j = 1, 2, 3, \ldots$.

The choices above are possible by the compactness of each A^a_j and the continuity of $\underline{\underline{S}}(\cdot, g)$ and $\underline{\underline{T}}(\cdot, g)$ for each $g \in \underline{\underline{X}}(R^n)$. Note that $A^a_j \cap U^a(j+2, i) = \emptyset$ for each $i = 1, 2, \ldots, q(a, j+2)$ and that $A^a_\infty \cap U^a(j, i) = \emptyset$ for each $i = 1, 2, \ldots, q(a, j)$, for $j = 1, 2, 3, \ldots$ and $a = 0, 1, 2$.

For $a = 0, 1, 2$, $j = 1, 2, 3, \ldots$, and $i = 1, 2, \ldots, q(a, j)$ we define continuous functions

$$\psi^a(j, i) : \underline{\underline{A}} \longrightarrow R^+$$

given for $\nu \in \underline{\underline{A}}$ by

$$\psi^a(j, i)(\nu) = \underline{\underline{F}}(\nu, \underline{\underline{A}} - U^a(j, i)).$$

For the same a, j, i we define

$$\varphi^a(j, i) : \underline{\underline{A}} \longrightarrow R^+$$

by setting for each $\nu \in \underline{\underline{A}}$

$$\varphi^a(j, i)(\nu) = (\sum\{\psi^a(\ell, m)(\nu) : \ell = 1, 2, 3, \ldots \text{ and } m = 1, 2, \ldots, q(a, \ell)\})^{-1} \psi^a(j, i).$$

Note that in each sum appearing in the denominator only finitely many summands are non-zero for each $\nu \in \underline{\underline{A}}$. Note also that for each a, $\{\varphi^a(j, i)\}_{i,j}$ is a partition of unity on $\underline{\underline{A}} - A^a_\infty$.

We define for $a = 0, 1, 2$

$$g^a : \underline{\underline{A}} \longrightarrow \underline{\underline{X}}^a \cap \{g : \|g\| \leq 1\},$$

$$g^a(\nu) = \sum\{[\varphi^a(j, i)(\nu)]g^a(j, i) : j = 1, 2, 3, \ldots \text{ and } i = 1, 2, \ldots, q(a, j)\}$$

whenever $\nu \in \underline{\underline{A}} - A^a_\infty$,

$$g^a(\nu) = 0 \text{ for } \nu \in A^a_\infty.$$

Note that $g^a | \underline{\underline{A}} - A^a_\infty$ is continuous in the $\underline{\underline{F}}$ metric topology on $\underline{\underline{A}} - A^a_\infty$ and the $\| \ \|$ topology on $\underline{\underline{X}}^a$.

Now choose for $a = 0, 1, 2$ a continuous function

$$h^a : \underline{\underline{A}} \longrightarrow \{t : 0 \leq t \leq 1\}$$

such that for each $\nu \in \underline{\underline{A}}$

(a) $h^a(\nu) = 0$ if $\nu \in A^a_\infty$

(b) $h^a(\nu) > 0$ if $\nu \in \underline{\underline{A}} - A^a_\infty$

(c) $\underline{\underline{W}}(G^a(t_2, g^a(\nu), \nu)) < \underline{\underline{W}}(G^a(t_1, g^a(\nu), \nu))$

whenever $0 \leq t_1 < t_2 \leq h^a(\nu)$.

We are using 6.9 (6) at this point.

We define for $a = 0, 1, 2$

$$H^a : \{t : 0 \le t \le 1\} \times \underline{\underline{A}} \longrightarrow \underline{\underline{A}}$$

given for $0 \le t \le 1$ and $\nu \in \underline{\underline{A}}$ by

$$H^a(t, \nu) = G^a(t, g^a(\nu), \nu) \text{ if } 0 \le t \le h^a(\nu)$$

$$H^a(t, \nu) = H^a(h^a(\nu), \nu) \text{ if } h^a(\nu) \le t \le 1.$$

One verifies for $a = 0, 1, 2$

(a) H^a is continuous in the product topology on $\operatorname{dmn}(H^a)$.

(b) $H^a(t, \nu) = \nu$ whenever $0 \le t \le 1$ and $\nu \in A^a_\infty$.

(c) $\underline{\underline{W}}(H^a(1, \nu)) < \underline{\underline{W}}(\nu)$ whenever $\nu \in \underline{\underline{A}} - A^a_\infty$.

The desired critical sequence $\{\varphi^i\}_i$ for Π is given by

$$\varphi^i(a) = H^2(1, H^1(1, H^0(1, {}_1\varphi^i(a))))$$

for each $a \in \operatorname{dmn}(\varphi^i) = \operatorname{dmn}({}_1\varphi^i)$, $i = 1, 2, 3, \ldots$.

13. THE HOMOTOPY GROUPS OF THE VZ SPACES.

13.1 DEFINITIONS.

(1) For each $n = 0, 1, 2, \ldots$ let $\mathscr{I}(1, n)$ be the cell complex of the unit interval $I^1 = R \cap \{x: 0 \le x \le 1\}$ whose 1-cells are the subintervals $[0, 1\cdot 2^{-n}]$, $[1\cdot 2^{-n}, 2\cdot 2^{-n}], \ldots, [(2^n-1)\cdot 2^{-n}, 1]$ and whose 0-cells are the endpoints $[0], [1\cdot 2^{-n}], [2\cdot 2^{-n}], \ldots, [(2^n-1)\cdot 2^{-n}], [1]$.

(2) for each $m = 1, 2, 3, \ldots$ and $n = 0, 1, 2, \ldots$ let

$$\mathscr{I}(m, n) = \mathscr{I}(1, n) \otimes \mathscr{I}(1, n) \otimes \ldots \otimes \mathscr{I}(1, n) \qquad (m \text{ times})$$

be a cell complex of the unit m-cube $I^m = R^m \cap \{x: 0 \le x^i \le 1 \text{ for each } i = 1, 2, \ldots, m\}$. For each cell $\alpha \in \mathscr{I}(m, n)$ we denote by $|\alpha|$ the closed subset of I^m corresponding to α,

$$|\alpha| = |\alpha^1 \otimes \alpha^2 \otimes \ldots \otimes \alpha^m| = I^m \cap \{x: x^i \in |\alpha^i|\}$$

where $|[a, b]| = \{x: a \le x \le b\}$ and $|[c]| = \{c\}$ whenever $[a, b]$ and $[c]$ are cells of $\mathscr{I}(1, n)$. Note also that if α is an m-cell of $\mathscr{I}(m, n)$ the subcomplex $\mathscr{I}(m, n; \alpha)$ of $\mathscr{I}(m, n)$ generated by α and all its faces is naturally isomorphic with $\mathscr{I}(m, 0)$ by an isomorphism $\theta_\alpha : \mathscr{I}(m, n; \alpha) \longrightarrow \mathscr{I}(m, 0)$ induced by an affine mapping $f_\alpha : R^m \longrightarrow R^m$ whose Jacobian has each $v \in \underline{\underline{\Lambda}}_1(R^m)$ as an eigenvector with eigenvalues 2^n, i.e. for each $\beta \in \mathscr{I}(m, n; \alpha)$, $|\theta_\alpha(\beta)| = f_\alpha(|\beta|)$.

(3) Let $m \in \{1, 2, 3, \ldots\}$ be fixed. We choose and fix a function

$$\xi_1 : \{\sigma: \sigma \text{ is an m-cell of } \mathscr{I}(m, 1)\} \longrightarrow \{0, 1, 2, \ldots, 2^m-1\}.$$

with the property that $\xi_1(\sigma) > \xi_1(\tau)$ whenever

$$\sup\{\sum_{i=1}^m x_i : x \in |\sigma|\} < \sup\{\sum_{i=1}^m x_i : x \in |\tau|\}.$$

For each $n = 2, 3, 4, \ldots$ we define

$$\xi_n : \{\sigma : \sigma \text{ is an m-cell of } \mathcal{L}(m, n)\} \longrightarrow \{0, 1, 2, \ldots, 2^m - 1\}$$

$$\xi_n(\sigma) = \xi_1(\theta_\alpha(\sigma))$$

or each m-cell σ of $\mathcal{L}(m, n)$ where α is that unique m-cell in $\mathcal{L}(m, n-1)$ or which $\sigma \in \operatorname{dmn}(\theta_\alpha)$.

) Let $m \in \{1, 2, 3, \ldots\}$ be fixed. We define

$$\xi : \bigcup_{n=1}^{\infty} \{\alpha : \alpha \text{ is an m-cell of } \mathcal{L}(m, n)\} \longrightarrow \{t : 0 \leq t \leq 1\}$$

$$\xi(\alpha_n) = 2^{-m}\, \xi_1(\alpha_1) + 2^{-2m} \xi_2(\alpha_2) + 2^{-3m}\, \xi_3(\alpha_3) + \ldots + 2^{-nm} \xi_n(\alpha_n)$$

or each m-cell $\alpha_n \in \mathcal{L}(m, n)$, $n = 1, 2, 3, \ldots$. Here α_i is that unique n-cell in $\mathcal{L}(m, i)$, $i = 1, 2, \ldots, n-1$ for which

$$|\alpha_n| \subset |\alpha_{n-1}| \subset |\alpha_{n-2}| \subset \cdots\ |\alpha_1| \subset I^m \ .$$

5) Let $m \in \{1, 2, 3, \ldots\}$ be fixed. We define for each t, $0 \leq t \leq 1$,

$$A_t = I^m \cap \operatorname{clos} \bigcup \{ |\alpha| : \alpha \text{ is an m-cell of } \mathcal{L}(m, n) \text{ for some } n = 1, 2, 3, \ldots \text{ and } \xi(\alpha) \geq t\}.$$

Each A_t is, of course, compact.

6) We define for each $m = 1, 2, 3, \ldots$ and each t, $0 \leq t \leq 1$,

$$f(m, m, t) : I^m \longrightarrow I^m,$$

given for $x \in I^m$ by

$$f(m, m, t)(x) = (x^1 - s,\ x^2 - s,\ \ldots,\ x^m - s)$$

where $s \in R^+$ is the smallest non-negative number for which either $(x^1 - s, x^2 - s, \ldots, x^m - s) \in A_t$ or $x^i - s = 0$ for some $i = 1, 2, \ldots, m$. Geometrically, $f(m, m, t)$ is the diagonal retraction of I^m onto the union of A_t with those $(m-1)$-dimensional faces of I^m containing the origin 0. It is a consequence of the fact that a uniform limit of a sequence of functions having a common Lipschitz constant is a function with the same Lipschitz constant that $\mathrm{Lip}(f(m, m, t)) \leq m^{\frac{1}{2}}$ for $0 \leq t \leq 1$. Note that

$$f(m, m, 1)(I^m) \subset \bigcup_{i=1}^{m} I^m \cap \{x: x^i = 0\} .$$

We define for each $m = 1, 2, 3, \ldots$ and t, $1 \leq t \leq 2$,

$$f(m, m-1, t) : \bigcup_{i=1}^{m} I^m \cap \{x: x^i = 0\} \longrightarrow \bigcup_{i=1}^{m} I^m \cap \{x: x^i = 0\}$$

given for $x = (x^1, x^2, \ldots, x^{j-2}, 0, x^{j+1}, \ldots, x^m) \in I^m$ by

$$f(m, m-1, t)(x) = \sigma_j(f(m-1, m-1, t-1)[\lambda_j(x)])$$

where $\sigma_j : I^{m-1} \longrightarrow I^m$ sends $y \in I^{m-1}$ to $(y^1, \ldots, y^{j-1}, 0, y^{j+1}, \ldots, y^{m-1})$ $\in I^m$, and $\lambda_j : I^m \longrightarrow I^{m-1}$ sends $x \in I^m$ to $(x^1, \ldots, x^{j-1}, x^{j+1}, \ldots, x^m) \in I^{m-1}$. Note that $\mathrm{Lip}(f(m, m-1, t)) \leq (m-1)^{\frac{1}{2}}$ for $0 \leq t \leq 1$ and

$$f(m, m-1, 2) \circ f(m, m, 1)(I^m) \subset \bigcup_{1 \leq i < j \leq m} I^m \cap \{x : x^i = x^j = 0\} .$$

One makes successive definition of $f(m, m-k, t)$ for each $m = 1, 2, 3, \ldots$, $k = 2, 3, \ldots, m-1$, and $k \leq t \leq k+1$ as follows:

$$f(m, m-k, t) : \bigcup_{1 \leq i_1 < i_2 < \ldots < i_k \leq m} I^m \cap \{x : x^{i_1} = x^{i_2} = \ldots = x^{i_k} = 0\}$$
$$\longrightarrow \bigcup_{1 \leq i_1 < i_2 < \ldots < i_k \leq m} I^m \cap \{x : x^{i_1} = x^{i_2} = \ldots = x^{i_k} = 0\}$$

by setting for each $1 \leq i(1) < i(2) < \ldots < i(k) \leq m$, $x \in I^m$ for which $x^{i(j)} = 0$ for $j = 1, 2, \ldots, k$, and $k \leq t \leq k+1$,

$$f(m, m-k, t)(x) = \sigma_{i(1), i(2), \ldots, i(k)}(f(m-k, m-k, t-k)[\lambda_{i(1), i(2), \ldots, i(k)}(x)])$$

where $\sigma_{i(1), i(2), \ldots, i(k)} : I^{m-k} \longrightarrow I^m$ sends $y \in I^{m-k}$ to

$(y^1, y^2, \ldots y^{i_1-1}, 0, y^{i_1}, \ldots, y^{i_k-1}, 0, y^{i_k}, \ldots, y^{m-k}) \in I^m$, and $\lambda_{i(1), i(2), \ldots, i(k)}: I^m \longrightarrow I^{m-k}$ sends $x \in I^m$ to $(x^1, x^2, \ldots, x^{i_1-1}, x^{i_1+1}, \ldots, x^{i_k-1}, x^{i_k+1}, \ldots, x^{i_m}) \in I^{m-k}$. Note that $\mathrm{Lip}(f(m, m-k, t)) \le (m-k)^{\frac{1}{2}}$ for $k \le t \le k+1$ and

$$f(m, m-k, k+1) \circ f(m, m-k+1, k) \circ f(m, m-k+2, k-1) \circ \ldots \circ f(m, m, 1)(I^m) \subset \bigcup_{1 \le i(1) < i(2) < \ldots < i(k+1) \le m} I^m \cap \{x : x^{i(1)} = x^{i(2)} = x^{i(m)} = 0\} .$$

We define for each $m = 1, 2, 3, \ldots$

$$f(t, x) = f(m, m-k, t) \circ f(m, m-k+1, k) \circ f(m, m-k+2, k-1) \circ \ldots \circ f(m, m, 1)(x)$$

whenever $x \in I^m$ and $k \le t \le k+1$ for $k = 0, 1, 2, \ldots, m-1$. Note that $\mathrm{Lip}(f(t, \cdot)) \le (m!)^{\frac{1}{2}}$ and $f(m, x) = 0 \in I^m$ for each $x \in I^m$.

13.2 THEOREM. <u>Let</u> $1 \le k \le m$ <u>be integers and</u> G <u>an admissible group. Then the mapping</u>

$$\mathcal{D}: \{t : 0 \le t \le m\} \times \underset{=}{Z}_k(I^m, I^m; G) \longrightarrow \underset{=}{Z}_k(I^m, I^m; G),$$

<u>given for</u> $0 \le t \le m$ <u>and</u> $T \in \underset{=}{Z}_k(I^m, I^m; G)$ <u>by</u> $\mathcal{D}(t, T) = f(t, \cdot)_{\#}T$, <u>is continuous with respect to the</u> $\underset{=}{M}$ <u>metric topology on</u> $\underset{=}{Z}_k(I^m, I^m; G)$.

PROOF.

<u>Part 1.</u> Let m be a positive integer and $f: \{t : 0 \le t \le m\} \times I^m \longrightarrow I^m$ be as in 12.1(5). Then corresponding to each t with $0 \le t \le 1$ there exist $p(t), q(t) \in I^m$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ satisfying

$$I^m \cap \{x : f(s, x) \ne f(t, x)\} \subset I^m \cap \{x : (x^1 - s, x^2 - s, \ldots, x^m - s) \in \underset{=}{D}^m(p(t), \varepsilon) \cup \underset{=}{D}^m(q(t), \varepsilon) \text{ for some } r \in R\}$$

whenever $|s-t| < \delta$.

Proof of part 1. One notes that if

$$a_1, \ldots a_2, \ldots, a_{n-1} \in \{0, 1, \ldots, 2^m-1\}, \; a_n \in \{0, 1, \ldots, 2^m-2\} ,$$

and

$$(*) \qquad \sum_{i=1}^{n} 2^{-im} a_i \leq s \leq t \leq \sum_{i=1}^{n} 2^{-im} + 2^{nm} ,$$

then $A_s - |\alpha| = A_t - |\alpha|$ where α is that unique m-cell of $\mathcal{J}(m, n)$ for which $\xi(\alpha) = \sum_{i=1}^{n} 2^{-im} a_i$. We consider two cases:

Case (a). There exist $a_1, a_2, \ldots, a_n \in \{0, 1, \ldots, 2^m-1\}$ such that $t = \sum_{i=1}^{n} 2^{-im} a_i$. In this case one chooses $\alpha_i, \beta_i \in \mathcal{J}(m, n)$ for $i = n+2, n+3, n+4, \ldots$ such that $\xi(\alpha_i) = t - 2^{-im}$ and $\xi(\beta_i) = t$. One notes that $|\alpha_{n+2}| \supset |\alpha_{n+3}| \supset |\alpha_{n+4}| \supset \ldots$ and $|\beta_{n+2}| \supset |\beta_{n+3}| \supset |\beta_{n+4}| \supset \ldots$ and sets $\{p(t)\} = \bigcap_i |\alpha_i|$, and $\{q(t)\} = \bigcap_i |\beta_i|$.

Case (b). t has no finite expansion as in case (a). In this case for each $n = 1, 2, 3, \ldots$ one can find numbers $a_1, a_2, \ldots, a_{n-1} \in \{0, 1, \ldots, 2^n-1\}$ and $a_n \in \{0, 1, \ldots, 2^n-2\}$ such that (*) holds (with $s = t$) and $A_s - |\alpha_n| = A_t - |\alpha_n|$ for each s with $|s-t| < \min\{t - \sum_{i=1}^{n} 2^{-im} a_i, \sum_{i=1}^{n} 2^{-im} a_i + 2^{-nm} - t\}$. Here α_n is that unique m-cell of $\mathcal{J}(m, n)$ for which $\xi(\alpha_n) = \sum_{i=1}^{n} 2^{-im} a_i$. We have furthermore that $|\alpha_1| \supset |\alpha_2| \supset |\alpha_3| \supset \ldots$. One sets $\{p(t)\} = \{q(t)\} = \bigcap_i |\alpha_i|$.

Part 2. For $k \geq 2$ $\mathfrak{D} | \{t : 0 \leq t \leq 1\} \times \underline{\underline{Z}}_k(I^m, I^m; G)$ is easily verified to be continuous since $\underline{\underline{M}}T(L) = 0$ for each $T \in \underline{\underline{Z}}_k(I^m, I^m; G)$ and each straight line $L \subset I^m$. For $k = 1$ the desired continuity follows using the fact that $\underline{\underline{M}}T \cap L$ is absolutely continuous with respect to $\underline{\underline{H}}^1 \cap L$ and the continuity of $\mathfrak{I} | \{t : 0 \leq t \leq 1\} \times L \longrightarrow L$ for each $T \in \underline{\underline{Z}}_1(I^m, I^m; G)$ and straight line $L \subset I^m$ parallel with $(1, 1, \ldots, 1)$. The theorem follows from the sequential definition of f.

13.3 DEFINITIONS.

(1) Let m be a non-negative integer and $\Box$ a cubical complex for R^m. Let γ be some m-cell, ∇ the subcomplex of $\Box$ generated by γ and all its faces, and $\Diamond$ the subcomplex of ∇ generated by all the proper faces of γ, not including γ itself. Let K, L, f be as in 12.3. We denote by $\Pi_m^{\#}(K, L)$ the homotopy classes of mappings $(\nabla, \Diamond) \longrightarrow (K, L)$ with respect to f. Assuming now that L consists of a single point , one gives $\Pi_1^{\#}(K, L)$ a group structure and gives $\Pi_m^{\#}(K, L)$, $m \geq 2$, the structure of an abelian group.

We denote by $\Pi_m(K, L)$ the usual m-dimensional homotopy group, i.e. the homotopy equivalence classes of continuous mappings $(I^m, \dot{I}^m) \longrightarrow (K, L)$, again assuming that L is a point.

(2) Let $1 \leq k \leq n$ be integers, $B \subset A \subset R^n$, and G be an admissible group. $\underset{=}{Z}_k(A, B; G)$ has been defined and given the $\underset{=}{F}$ metric topology. We define $\underset{=}{Z}_k(A, B; G; \underset{=}{M})$ to be the space $\underset{=}{Z}_k(A, B; G)$ with the metric topology given for $S, T \in \underset{=}{Z}_k(A, B; G)$ by

$$\operatorname{dist}(S, T) = \underset{=}{M}(S, T) + \underset{=}{M}(\partial S, \partial T) \text{ for } k \geq 2, \text{ and}$$
$$\operatorname{dist}(S, T) = \underset{=}{M}(S, T) + \underset{=}{F}(\partial S, \partial T) \text{ for } k = 1 .$$

$\underset{=}{V}\underset{=}{Z}_k(A, B; G)$ has been defined and given the $\underset{=}{F}$ metric topology. We define $\underset{=}{V}\underset{=}{Z}_k(A, B; G; \underset{=}{W}\underset{=}{M})$ to be the space $\underset{=}{V}\underset{=}{Z}_k(A, B; G)$ with the metric topology given for $(V, W; T, \partial T), (V', W'; T', \partial T') \in \underset{=}{V}\underset{=}{Z}_k(A, B; G)$ by

$$\operatorname{dist}[(V, W; T, \partial T), (V', W'; T', \partial T')] =$$
$$\underset{=}{W}(V, V') + \underset{=}{W}(W, W') + \underset{=}{M}(T, T') + \underset{=}{M}(\partial T, \partial T') \text{ for } k \geq 2, \text{ and}$$

$$\operatorname{dist}[(V, W; T, \partial T), (V', W'; T', \partial T')]$$
$$= \underset{=}{W}(V, V') + \underset{=}{F}(W, W') + \underset{=}{M}(T, T') + \underset{=}{F}(\partial T, \partial T') \text{ for } k = 1.$$

We define also

$$\underset{=}{V}\underset{=}{Z}_k^{\natural}(A, B; G) = \underset{=}{V}\underset{=}{Z}_k(A, B; G) \cap \{(|T|, |\partial T|; T, \partial T) : T \in \underset{=}{Z}_k(A, B; G)\}$$

$$\mathbf{VZ}^{\natural}_k(A,\ B;\ G;\ \mathbf{WM}) = \mathbf{VZ}_k(A,\ B;\ G;\ \mathbf{WM}) \cap \{|T|,\ |\partial T|;\ T,\ \partial T) : T \in \mathbf{Z}_k(A,\ B;\ G;\ \mathbf{M})\}.$$

The following diagram is a commutative diagram of continuous maps:

$$\begin{tikzcd}
\mathbf{VZ}_k(A,\ B;\ G;\ \mathbf{WM}) \arrow[rr, shift left, "z_1"] \arrow[ddd, "i_2"] & & \mathbf{Z}_k(A,\ B;\ G;\ \mathbf{M}) \arrow[ll, shift left, "v_1"] \arrow[dl, shift left, "v_2"] \arrow[ddd, "j"] \\
& \mathbf{VZ}^{\natural}_k(A,\ B;\ G;\ \mathbf{WM}) \arrow[ul, "i_3"] \arrow[ur, shift left, "z_2"] \arrow[d, "i_1"] & \\
& \mathbf{VZ}^{\natural}_k(A,\ B;\ G) \arrow[dl, "i_4"] \arrow[dr, "z_4"] & \\
\mathbf{VZ}_k(A, B;\ G) \arrow[rr, "z_3"] & & \mathbf{Z}_k(A,\ B;\ G)
\end{tikzcd}$$

$i_k(\nu) = \nu$ for $\nu \in \mathrm{dmn}(i_h)$, $h = 1, 2, 3, 4$

$j(T) = T$ for $T \in \mathrm{dmn}(j)$

$v_h(T) = (|T|,\ |\partial T|;\ T,\ \partial T)$ for $T \in \mathrm{dmn}(v_h)$, $h = 1, 2$

Note that $v_1 \circ z_1 =$ (identity), $v_2 \circ z_2 =$ (identity), $z_2 \circ v_2 =$ (identity), but in general, $z_1 \circ v_1 \neq$ (identity).

13.4 THEOREM. Let $1 \leq k \leq n$ and $0 \leq m$ be integers. Let $A \subset R^n$ be a compact submanifold of R^n of class 3 with boundary B. Let $C \subset A$ be a compact submanifold of A of class 3 with boundary D. Let G be an admissible *group. The following groups are naturally isomorphic* : $H_{m+k}(A,\ B\cup C;\ G)$,

$\Pi_m(\mathbf{Z}_k(A,\ B\cup C;\ G),\ \{0\})$, $\Pi^{\#}_m(\mathbf{Z}_k(A,\ B\cup C;\ G),\ \{0\})$,

$\Pi_m(\mathbf{Z}_k(A,\ B\cup C;\ G;\ \mathbf{WM}),\ \{0\})$, *and*

$\Pi^{\#}_m(\mathbf{Z}_k(A,\ B\cup C;\ G;\ \mathbf{WM}),\ \{0\})$.

PROOF. For each of the above homotopy groups the methods of [Al 3.2] give a homomorphism into the homology group. For the homotopy groups based on the $\mathbf{F}$ topology, the methods of [Al] generalize immediately to give the desired isomorphism. For the homotopy groups of $\mathbf{Z}_k(A,\ B\ \ C;\ G;\ \mathbf{WM})$ these arguments fail because the deformation D[Al 5.2] no longer provides continuous paths in the stronger topology. With D replaced by $\mathfrak{D}$, the arguments of [Al] give the isomorphism. One, of course, uses the interpolation formula [Al .6.5] to show the surjective property rather than [Al 3.4], and uses a differentiable triangulation of (A, B, C, D) rather than the simplicial complex of [Al 5].

13.5 THEOREM. *Let* k, n, m, A, B, C, D, G *be as in* 13.4. *There exist natural isomorphisms*

$$\Pi^{\#}_m(\mathbf{VZ}^{\natural}_k(A,\ B\cup C;\ G;\ \mathbf{WM}),\ \{0\}) \cong H_{m+k}(A,\ B\cup C;\ G)\ ,$$

$$\Pi^{\#}_m(\mathbf{VZ}^{\natural}_k(A,\ B\cup C;\ G),\ \{0\}) \cong \Pi^{\#}_m(\mathbf{VZ}_k(A,\ B\cup C;\ G),\ \{0\}).$$

Also $H_{m+k}(A,\ B\cup C;\ G)$ *is naturally a direct summand of* $\Pi^{\#}_m(\mathbf{VZ}_k(A,\ B\cup C;\ G),\ \{0\})$.

PROOF. The first conclusion is a consequence of 13.4 and the homeomorphism v_2. The second conclusion follows since $\mathbf{VZ}^{\natural}_k(A,\ B\cup C;\ G)$ is dense in $\mathbf{VZ}_k(A,\ B\cup C;\ G)$. The third conclusion follows from the first two and the obvious isomorphism $\Pi^{\#}_m(z_4 \circ i_1) = \Pi^{\#}_m(j \circ v_2)$. Here the symbol $\Pi^{\#}_m$ refers to the induced mapping on homotopy groups.

13.6 COROLLARY. Let k, n, m, A, B, C, D be as in 13.4 with $k \leq \dim(A) = m$. Let G be the group of integers modulo 2 with $|(1)| = 1$. Then there exists $\Pi \in \Pi^{\#}_{m-k}(\underline{\underline{V}}\,\underline{\underline{Z}}_k(A, B \cup C; G), \{0\})$ such that $\underline{\underline{L}}(T) > 0$.

PROOF. Note that $H_m(A, B \cup C; G) \neq 0$. The corollary follows from 13.4, 13.5, and [Al 8.2].

14. CONSTRUCTION FOR DECREASING AREA.

14.1. ASSUMPTIONS. We assume the following throughout this chapter:

(1) m and n are positive integers.

(2) A is a compact submanifold of R^n of class 3 having boundary B.

(3) C is a compact submanifold of A of class 3 having boundary D.

(4) $1 \leq k < \dim(A)$ is an integer.

(5) $L(m)$ is the number of cells of all dimensions in $\mathcal{J}(m)$.

(6) $M(n)$ is the positive integer defined in 13.10.

(7) $N(n)$ is the positive integer defined in 8.6

(8) $a(h)$ is the number defined in 8.7 for $1 \leq h \leq k$.

(9) $b(h, n)$ is the number defined in 8.7 for $1 \leq h \leq k$.

(10) $c(m, n) = (M(n)L(m))^{3^m}$,

(11) $c_2 \in R_0^+$.

(12) $d(h) = a(h)^{-1}[3b(h)]^{1-k}$ for $1 \leq h \leq k$. We set $d(0) = 0$.

(13) $\varphi(n, k) : R^+ \times R_0^+ \longrightarrow R_0^+$ is as in 9.10.

(14) G is an admissible group.

14.2. DEFINITIONS. We partition $\underline{\underline{VZ}}_k(A, B \cup C; G)$ into six classes as follows:

<u>Class</u> (0). Class (0) consists of all $(V, W; T, \partial T) \in \underline{\underline{VZ}}_k(A, B \cup C; G)$ for which $\underline{\underline{S}}(R^n, A, B, C)(V, W) > 0$ or $\underline{\underline{P}}^-(R^n, A)(V, W) > 0$ or $\underline{\underline{P}}^-(R^n, B, C)(W, 0) > 1$.

<u>Class</u> (a). Class (a) consists of all $(V, W; T, \partial T) \in \underline{\underline{VZ}}_k(A, B \cup C; G)$ which are not in class (0) and for which there exist $c(m, n)$ distinct points $p_1, p_2, \ldots, p_{c(m,n)} \in \operatorname{spt}(V) - \operatorname{spt}(W)$ such that $\Theta^k(\underline{\underline{W}}V, p_i) \leq d(k)$ for each i.

<u>Class</u> (b). Class (b) consists of all $(V, W; T, \partial T) \in \underline{\underline{VZ}}_k(A, B \cup C; G)$ which are not in classes (0) or (a) and for which there exist $c(m, n)$ distinct points $p_1, p_2, \ldots, p_{c(m,n)} \in \operatorname{spt}(V) - \operatorname{spt}(W)$ such that the tangent cone to $\operatorname{spt}(V)$ at p_i is a k-dimensional disk but $\Theta^k(\underline{\underline{W}}V, p_i)$ is not an integer for each i. Note that 10.7 implies that if $(V, W; T, \partial T)$ is in class (b) then $V \cap (R^n - \operatorname{spt}(W)) \in \underline{\underline{RV}}_k(R^n)$.

<u>Class</u> (c). Class (c) consists of all $(V, W; T, \partial T) \in \underline{\underline{VZ}}_k(A, B \cup C; G)$

which are not in classes (0), (a), or (b) and for which there exist $c(m, n)$ distinct points $p_1, p_2, \ldots, p_{c(m,n)} \in \operatorname{spt}(W)$ such that $\Theta^{k-1}(\underline{\underline{W}}W, p_i) \leq d(k-1)$ for each i. Note that 5.4(c) implies that if $(V, W; T, \partial T)$ is in class (c) then $V \cap (R^n - \operatorname{spt}(W)) \in \underline{\underline{IV}}_k(R^n)$.

Class (d). Class (d) consists of all $(V, W; T, \partial T) \in \underline{\underline{VZ}}_k(A, B \cup C; G)$ which are not in classes (0), (a), (b), or (c) and for which there exist $c(m, n)$ distinct points $p_1, p_2, \ldots, p_{c(m,n)} \in \operatorname{spt}(W)$ such that the tangent cone to $\operatorname{spt}(W)$ at p_i is a $(k-1)$-dimensional disk but $\Theta^{k-1}(\underline{\underline{W}}W, p_i)$ is not an integer for each i. Note that 10.7 and 5.4(c) imply that if $(V, W; T, \partial T)$ is in class (d) then $V \cap (R^n - \operatorname{spt}(W)) \in \underline{\underline{IV}}_k(R^n)$ and $W \in \underline{\underline{RV}}_{k-1}(R^n)$.

Class (e). Class (e) consists of all $(V, W; T, \partial T) \in \underline{\underline{VZ}}_k(A, B\ \ C; G)$ which are not in classes (0), (a), (b), (c), or (d). 10.7, 10.8, and 5.4(c) imply that if $(V, W; T, \partial T)$ is in class (e) then $V \in \underline{\underline{IV}}_k(R^n)$, $W \in \underline{\underline{IV}}_{k-1}(R^n)$, $\underline{\underline{S}}(R^n, A, B, C)(V, W) = 0$, $\underline{\underline{P}}^-(R^n, A)(V, W) = 0$, and $\underline{\underline{P}}^-(R^n, B, C)(W, 0) = 0$.

Note that if $k = 1$ then classes (c) and (d) are empty.

14.3. DEFINITIONS. For each sufficiently small positive number η we classify fourteen types of subsets of A.

Type (1). $A \cap \underline{\underline{D}}^n(a, r)$ for some $a \in A$ and $0 < r < \operatorname{dist}(a, B \cup C)$.

Type (2). $A \cap \underline{\underline{D}}^n(b, r)$ for some $b \in B$ and $0 < r < \operatorname{dist}(b, C)$.

Type (3). $A \cap \underline{\underline{D}}^n(c, r)$ for some $c \in C$ and $0 < r < \operatorname{dist}(c, B \cup D)$.

Type (4). $A \cap \underline{\underline{D}}^n(d, r)$ for some $d \in D$ and $0 < r < \operatorname{dist}(d, B)$.

Type (5). $A \cap \underline{\underline{D}}^n(a, r)\ \ \{x : \operatorname{dist}(x, \Pi^k) \leq \eta r\}$ for some $a \in A$ and $0 < r < \operatorname{dist}(a, B \cup C)$ where Π^k is a k-plane in R^n containing a and parallel with the tangent plane to A at a.

Type (6). $A \cap \underline{\underline{D}}^n(b, r) \cap \{x : \operatorname{dist}(x, \Pi^k) \leq \eta r\}$ for some $b \in B$ and $0 < r < \operatorname{dist}(b, C)$ where Π^k is a k-plane in R^n containing b and parallel with the tangent plane to B at b.

Type (7). $A \cap \underline{\underline{D}}^n(c, r) \cap \{x : \operatorname{dist}(x, \Pi^k) \leq \eta r\}$ for some $c \in C$ and $0 < r < \operatorname{dist}(c, B \cup D)$ where Π^k is a k-plane in R^n containing c and parallel with the tangent plane to A at c.

Type (8). $A \cap \underline{\underline{D}}^n(d, r) \cap \{x : \operatorname{dist}(x, \Pi^k) \leq \eta r\}$ for some $d \in D$ and

$0 < r < \text{dist}(d, B)$ where Π^k is a k-plane in R^n containing d and parallel with the tangent plane to A at d.

Type (9). $B \cap E$ where $E \subset A$ is of type (2).

Type (10). $C \cap E$ where $E \subset A$ is of type (3).

Type (11). $C \cap E$ where $E \subset A$ is of type (4).

Type (12). $B \cap \underline{\underline{D}}^n(b, r) \cap \{x : \text{dist}(x, \Pi^{k-1}) \leq \eta r\}$ for some $b \in B$ and $0 < r < \text{dist}(b, C)$ where Π^{k-1} is a (k-1)-plane in R^n containing b and parallel with the tangent plane to B at b.

Type (13). $C \cap \underline{\underline{D}}^n(c, r) \cap \{x : \text{dist}(x, \Pi^{k-1}) < \eta r\}$ for some $c \in C$ and $0 < r < \text{dist}(c, B \cup D)$ where Π^{k-1} is a (k-1)-plane in R^n containing c and parallel with the tangent plane to C at c.

Type (14). $C \cap \underline{\underline{D}}^n(d, r) \cap \{x : \text{dist}(x, \Pi^{k-1}) < \eta r\}$ for some $d \in D$ and $0 < r < \text{dist}(d, B)$ where Π^{k-1} is a (k-1)-plane in R^n containing d and parallel with the tangent (half)-plane to C at d.

14.4. LEMMA. Let $\eta > 0$ be as in 14.3. Then there exist $\delta, \Delta \in R_0^+$ (depending on η, A, B, C, D) with the following property: Let $E^0 \subset A$ be of type (i) for some $i = 1, 2, \ldots, 14$ with $\text{diam}(E^0) < \delta$. Then for each $\varepsilon > 0$ there exists $E^\varepsilon \subset R^n$ satisfying:

(a) If $i = 1, 3, 10$ then $E^\varepsilon = E^0$ is a compact submanifold of R^n of class 3 with boundary and is diffeomorphic with a closed disk of appropriate dimension.

(b) If $i = 2, 4, 5, 6, 7, 8$ then $E^0 \subset E^\varepsilon \subset A \cap \{x : \text{dist}(x, E^0) < \varepsilon\}$ is a compact submanifold of R^n of class 3 with boundary and is diffeomorphic with a closed disk having the same dimension as A.

(c) If $i = 9, 12$ then $E^0 \subset E^\varepsilon \subset B \cap \{x : \text{dist}(x, E^0) < \varepsilon\}$ is a compact submanifold of R^n of class 3 with boundary and is diffeomorphic with a closed disk having the same dimension as B.

(d) If $i = 11, 13, 14$ then $E^0 \subset E^\varepsilon \subset C \cap \{x : \text{dist}(x, E^0) < \varepsilon\}$ is a compact submanifold of R^n of class 3 with boundary and is diffeomorphic with a closed disk having the same dimension as C.

(e) If $i = 1, 2, 3, 4, 5, 6, 7, 8$ then $\Delta_k^-(E^\varepsilon, R^n) \leq \Delta$.

(f) If $i = 9, 10, 11, 12, 13, 14$ then $\Delta_{k-1}^-(E^\varepsilon, R^n) \leq \Delta$ whenever $\text{dim}(E^\varepsilon) \geq$

PROOF. One can realize the ε-neighborhoods of E^0 as star-shaped regions with respect to a suitably chosen center and the exponential mapping. Smoothing the radius function defining the boundary yields the desired smooth manifolds with boundary. The estimates one obtains by this process are sufficient, together with small δ, to prove the lemma.

14.5. DEFINITION. Let $r, \varepsilon, \delta \in R_0^+$ and $p \in A$. We denote by $\underline{\underline{R}}(A, B \cup C; \varepsilon, \delta; p, r)$ the family of all sets $\{\nu_1^1, \nu_2^1, \ldots, \nu_{L(m)}^1\}$ of $L(m)$, not necessarily distinct, elements of $\underline{\underline{VZ}}_k(A, B \cup C; G)$ for which there exists a positive integer s and a family of sequences

$$\begin{array}{l} \nu_1^1, \nu_1^2, \nu_1^3, \ldots, \nu_1^s \\ \nu_2^1, \nu_2^2, \nu_2^3, \ldots, \nu_2^s \\ \nu_3^1, \nu_3^2, \nu_3^3, \ldots, \nu_3^s \\ \cdot \\ \cdot \\ \cdot \\ \nu_{L(m)}^1, \nu_{L(m)}^2, \nu_{L(m)}^3, \ldots, \nu_{L(m)}^s \end{array}$$

of elements of $\underline{\underline{VZ}}_k(A, B \cup C; G)$ satisfying the following conditions:

(1) $\underline{\underline{F}}(\nu_i^j, \nu_i^{j+1}) \leq \delta$ for each $i = 1, 2, \ldots, L(m)$ and $j = 1, 2, \ldots, s$,

(2) $\underline{\underline{F}}(\nu_i^j, \nu_h^j) \leq \delta$ for each $i, h = 1, 2, \ldots, L(m)$ and $j = 1, 2, \ldots, s$,

(3) $\sup_{i,j} \underline{\underline{W}}(\nu_i^j) \leq \sup_i \underline{\underline{W}}(\nu_i^1) + \delta$,

(4) $\sup_i \underline{\underline{W}}(\nu_i^s) \leq \sup_i \underline{\underline{W}}(\nu_i^1) - \varepsilon$, and

(5) $\nu_i^j \cap [R^n - \underline{\underline{D}}^n(p, r)] = \nu_i^1 \cap [R^n - \underline{\underline{D}}^n(p, r)]$ for each $i = 1, 2, \ldots, L(m)$ and $j = 1, 2, \ldots, s$, where the intersections in this context have the obvious meaning.

14.6. LEMMA. Let E be a compact submanifold of R^n of class 3 with boundary F. For each $b \in R_0^+$, $N \in R^+$, and positive integer m there exists a continuous function $\xi_b : R^+ \longrightarrow R^+$ with $\xi_b^{-1}\{0\} = \{0\}$ and $\xi_b(r) \geq \xi_c(r)$ whenever $0 < b < c$ and $r \in R^+$ having the following property. Let $f : R^n \longrightarrow R$ be of class 1 with $\mathrm{Lip}(f) \leq 1$, and let

$T_1, T_2, \ldots, T_m \in \underline{\underline{Z}}_k(E, F; G)$ with $\underline{\underline{M}}(T_i) + \underline{\underline{M}}(\partial T_i) \leq N$ for each $i = 1, 2, \ldots, m$. Then there exist $0 < r < b$ and $T_1^*, T_2^*, \ldots, T_m^* \in \underline{\underline{Z}}_k(E, F; G)$ such that for each $i = 2, 3, \ldots, m$,

(1) $T_i^* \cap \{x : f(x) < r\} = T_1 \cap \{x : f(x) < r\}$,

(2) $T_i^* \cap \{x : f(x) > r\} = T_i \cap \{x : f(x) > r\}$, and

(3) $\underline{\underline{F}}[(|T_i^*|, |\partial T_i^*|; T_i^*, \partial T_i^*), (|T_1|, |\partial T_1|; T_1, \partial T_1)] \leq \xi_b(\sup_j \underline{\underline{F}}[(|T_j|, |\partial T_j|; T_j, \partial T_j), (|T_1|, |\partial T_1|; T_1, \partial T_1)])$.

PROOF. Let E, T, b, N, m, and f be as above. We will have proved the lemma if we can show that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $T_1, T_2, \ldots, T_m$ are as above with

$$\sup_j \underline{\underline{F}}[(|T_j|, |\partial T_j|; T_j, \partial T_j), (|T_1|, |\partial T_1|; T_1, \partial T_1)] < \delta$$

then there exist $T_1^*, T_2^*, \ldots, T_m^*$ as above satisfying (1) and (2) above and

$$\sup_j \underline{\underline{F}}[(|T_j^*|, |\partial T_j^*|; T_j^*, \partial T_j^*), (|T_1|, |\partial T_1|; T_1, \partial T_1)] < \varepsilon \quad .$$

To show this we now choose several small numbers whose choice depends on E, F, b, N, and m but does not depend on $T_1, T_2, \ldots, T_m$.

(a) $\varepsilon_1 = \varepsilon_1(\varepsilon)$. Choose $\varepsilon_1 = 11^{-1}\varepsilon$.

(b) $\delta_1 = \delta_1(E, F, b, N, m, \varepsilon_1)$. Choose $0 < \delta < 4^{-1}b$ with the following property: For each $g : R^n \longrightarrow R$ of class 1 with $\operatorname{Lip}(g) \leq 1$ there exists some $0 < r_1 < b - 3\delta_1$ for which

$$\sup_j \underline{\underline{M}} T_j(\{x : r_1 < f(x) < r_1 + 3\delta_1\}) < c^{-1}\varepsilon_1$$

$$\sup_j \underline{\underline{M}} \partial T_j(\{x : r_1 < f(x) < r_1 + 3\delta_1\}) < c^{-1}\varepsilon_1 \quad .$$

Here $c \in R_0^+$ is the constant of 2.4(2) and 2.5.

(c) $\delta_2 = \delta_2(\varepsilon_1, \delta_1)$. Choose $\delta_2 = (1 + \delta_1^{-1})^{-1}\varepsilon_1$. Note that 2.5 implies that if $\underline{\underline{F}}(|T_i|, |T_1|) \leq \delta_2$ and $\underline{\underline{F}}(|\partial T_i|, |\partial T_1|) \leq \delta_2$ for each i, then for each r, $r_1 + \delta_1 < r < r_1 + 2\delta_1$, with r_1 as in (b), and for each $i = 1, 2, \ldots, m$

$$\underline{\underline{F}}(|T_i| \cap \{x : f(x) > r\},\ |T_1| \cap \{x : f(x) > r\})$$
$$\leq \underline{\underline{F}}(|T_i|,\ |T_1|)(1+\delta^{-1}) + c\underline{\underline{W}}T_1(\{x : r_1 < f(x) < r_1 + 3\delta_1\})$$
$$+ c\underline{\underline{W}}T_i(\{x : r_1 < f(x) < r_1 + 3\delta_1\})$$
$$\leq 3\varepsilon_1 \text{ , and}$$
$$\underline{\underline{F}}(|\partial T_i| \cap \{x : f(x) > r\},\ |\partial T_1| \cap \{x : f(x) > r\}) \leq 3\varepsilon_1 \text{ .}$$

(d) $\delta_3 = \delta_3(m, \varepsilon_1, \delta_1, E, F)$. Choose $0 < \delta_3 < \varepsilon_1$ to have the following property. Let $r_1 \in R$ as in (b). Then whenever $\underline{\underline{F}}(T_i, T_1) \leq \min\{c^{-1}\delta_3, \delta_3\}$, there exists some r_2, $r_1 + \delta_1 < r_2 < r_1 + 2\delta_1$, such that $T_i \cap \{x : f(x) = r_2\} = 0$ and $\partial T_i \cap \{x : f(x) = r_2\} = 0$ for each $i = 1, 2, \ldots, m$, while for each $i = 2, 3, \ldots, m$ there exists $K_i \in \underline{\underline{Z}}_{k+1}(E, E; G)$ such that

$$\partial K_i - (T_1 - T_i) \in \underline{\underline{Z}}_k(F, F; G) \text{ ,}$$
$$\underline{\underline{M}}(\partial K_i - (T_1 - T_i)) + \underline{\underline{M}}(K_i) < \varepsilon_1 \text{ ,}$$
$$\underline{\underline{M}}(\partial P_i(r_2) - \partial P_i \cap \{x : f(x) > r_2\}) \leq \varepsilon_1, \text{ and}$$
$$\underline{\underline{M}}(Q_i(r_2)) \leq \min\{c^{-1}\varepsilon_1, \varepsilon_1\} \text{ .}$$

Here for $i = 2, 3, \ldots, m$,

$$P_i = T_1 - T_i - \partial K_i \in \underline{\underline{Z}}_k(F, F; G) \text{ ,}$$
$$P_i(r_2) = P_i \cap \{x : f(x) > r\} \in \underline{\underline{Z}}_k(F, F; G) \text{ ,}$$
$$Q_i(r_2) = \partial(K_i \cap \{x : f(x) > r\}) - \partial K_i \cap \{x : f(x) > r\}$$
$$\in \underline{\underline{Z}}_k(E, E; G) \text{ .}$$

The choice of r_2 is possible by the methods of [A1, 1.18].

For each $i = 2, 3, \ldots, m$ we define

$$T_i^* = T_1 \cap \{x : f(x) < r_2\} + T_i \cap \{x : f(x) > r_2\} + Q_i(r_2) \text{ .}$$

Under the hypotheses that $\sup_i \underline{\underline{F}}(|T_i|, |T_1|) \leq \delta_2$, $\sup_i \underline{\underline{F}}(|\partial T_i|, |\partial T_1|) \leq \delta_2$, and $\sup_i \underline{\underline{F}}(T_i, T_1) \leq \delta_3$ we make the following assertions:

Assertion 1. $\sup_i \underline{\underline{F}}(T_i^*, T_1) \leq 2\varepsilon_1$.

Proof of assertion 1. Observe that

$$\begin{aligned}
&\partial(K_i \cap \{x : f(x) > r_2\}) - (T_1 - T_i) \\
&\quad = \partial(K_i \cap \{x : f(x) > r_2\}) - T_1 + T_1 \cap \{x : f(x) < r_2\} \\
&\qquad + T_i^* \cap \{x : f(x) > r_2\} + Q(r_2) \\
&\quad = \partial(K_i \cap \{x : f(x) > r_2\}) - (T_1 - T_i^*) \cap \{x : f(x) > r_2) \\
&\qquad + Q_i(r_2) \\
&\quad = P_i(r_2)
\end{aligned}$$

has mass no larger than ε_1 for each $i = 2, 3, \ldots, m$. Hence

$$\begin{aligned}
\underline{\underline{F}}(T_i^*, T_1) &\leq \underline{\underline{M}}(\partial(K_i \cap \{x : f(x) > r_2\}) - (T_1 - T_i^*)) \\
&\qquad + \underline{\underline{M}}(K_i \cap \{x : f(x) > r_2\}) \\
&\leq 2\varepsilon_1 .
\end{aligned}$$

Assertion 2. $\sup_i \underline{\underline{F}}(|T_i^*|, |T_1|) \leq 4\varepsilon_1$.

Proof of assertion 2. For each $i = 2, 3, \ldots, m$

$$\begin{aligned}
\underline{\underline{F}}(T_i^*, T_1) &= \underline{\underline{F}}(|T_i^*| \cap \{x : f(x) < r_2\} + |T_i^*| \cap \{x : f(x) > r_2\} \\
&\qquad + Q_i(r_2), |T_1| \cap \{x : f(x) < r_2\} \\
&\qquad + |T_1| \cap \{x : f(x) > r_2\}) \\
&= \underline{\underline{F}}(|T_i| \cap \{x : f(x) > r_2\} + \underline{\underline{Q}}(r_2) \;, \\
&\qquad |T_1| \cap \{x : f(x) > r_2\}) \\
&\leq \underline{\underline{F}}(|T_i| \cap \{x : f(x) > r_2\}, \ |T_1| \cap \{x : f(x) > r_2\}) \\
&\qquad + c\underline{\underline{W}}(Q_i(r_2)) \\
&\leq 4\varepsilon_1 .
\end{aligned}$$

Assertion 3. $\sup_i \underline{\underline{F}}(|\partial T_i^*|, |\partial T_1|) \leq 4\varepsilon_1$.

Proof of assertion 3. For each $i = 2, 3, \ldots, m$

$$\begin{aligned}
\partial T_i^* &= \partial(T_1 \cap \{x : f(x) < r_2\}) + \partial(T_i \cap \{x : f(x) > r_2\}) + \partial Q(r_2) \\
&= \partial(T_1 \cap \{x : f(x) < r_2\}) + \partial(T_i \cap \{x : f(x) > r_2\}) \\
&\quad + \partial[(-P_i + T_1 - T_i) \cap \{x : f(x) > r_2\}] \\
&= \partial T_1 - \partial P_i \cap \{x : f(x) > r_2\} + \partial P_i \cap \{x : f(x) > r_2\} \\
&\quad - \partial P_i(r_2) \\
&= \partial T_1 - (\partial T_1 - \partial T_i) \cap \{x : f(x) > r_2\} \\
&\quad + \partial P_i \cap \{x : f(x) > r_2\} - \partial P_i(r_2) \\
&= \partial T_1 \cap \{x : f(x) < r_2\} + \partial T_i \cap \{x : f(x) > r_2\} \\
&\quad + \partial P_i \cap \{x : f(x) > r_2\} - \partial P_i(r_2) \ .
\end{aligned}$$

Thus

$$\begin{aligned}
\underline{\underline{F}}(|\partial T_i^*|, |\partial T_1|) &= \underline{\underline{F}}(|\partial T_i^*| \cap \{x : f(x) < r_2\} + |\partial T_i^*| \cap \{x : f(x) > r_2\} \\
&\qquad + |\partial P_i \cap \{x : f(x) > r_2\} - \partial P_i(r_2)| \ , \\
&\qquad |\partial T_1| \cap \{x : f(x) < r_2\} + |\partial T_1| \cap \{x : f(x) > r_2\}) \\
&\leq \underline{\underline{F}}(|\partial T_i^*| \cap \{x : f(x) > r_2\}, |\partial T_1| \cap \{x : f(x) > r_2\}) \\
&\qquad + c\underline{\underline{M}}(\partial P_i \cap \{x : f(x) > r_2\} - \partial P_i(r_2)) \\
&\leq 4\varepsilon_1 \ .
\end{aligned}$$

Assertion 4. $\underline{\underline{F}}[(|T_i^*|, |\partial T_i^*|; T_i^*, \partial T_i^*),$

$$(|T_1|, |\partial T_1|; T_1, \partial T_1)] \leq 10\varepsilon_1 < \varepsilon$$

whenever $\delta = \min\{\delta_2, \delta_3\}$ and

$$\sup_i \underline{\underline{F}}[(|T_i|, |\partial T_i|; T_i, \partial T_i), (|T_1|, |\partial T_1|; T_1, \partial T_1)] < \delta \ .$$

Proof of assertion 4. Obvious as a consequence of the definitions and assertions (1), (2), (3).

This completes the proof of the lemma.

14.7. LEMMA. Let E be a compact submanifold of R^n of class 3 with boundary F, and let $(V_1, W_1; T_1, \partial T_1) \in \underline{\underline{VZ}}_k(E, F; G)$. Then for each $\varepsilon > 0$ there exists a positive integer q and a sequence

$$(V_1, W_1; T_1, \partial T_1), (V_2, W_2; T_1, \partial T_1), \ldots, (V_q, W_q; T_1, \partial T_1)$$

of elements of $\underline{\underline{VZ}}_k(E, F; G)$ such that

(1) $(V_q, W_q; T_1, \partial T_1) = (|T_1|, |\partial T_1|; T_1, \partial T_1)$;

(2) $\underline{\underline{W}}(V_i, W_i; T_1, \partial T_1) > \underline{\underline{W}}(V_{i+1}, W_{i+1}; T_1, \partial T_1)$ for each $i = 1, 2, \ldots, q-1$;

(3) If $k \geq 2$,

$$\underline{\underline{F}}[(V_i, W_i; T_1, \partial T_1), (V_{i+1}, W_{i+1}; T_1, \partial T_1)] \leq \varepsilon$$

for each $i = 1, 2, \ldots, q-1$;

(4) If $k = 1$ and G is the additive group of integers then $\underline{\underline{F}}(V_i, V_{i+1}) < \varepsilon$ and $\underline{\underline{W}}(W_i, W_{i+1}) \leq 2$ for each $i = 1, 2, \ldots, q-1$;

(5) If $k = 1$ and G is a finite group, then $\underline{\underline{F}}(V_i, V_{i+1}) \leq \varepsilon$ and $\underline{\underline{W}}(W_i, W_{i+1}) \leq 1 + 2\max\{|g| : g \in G\}$.

PROOF.

Part 1. Let $\varepsilon = 2^{-2}c\delta$ where $c \in R_0^+$ is the constant of 2.4(2) and notice that if $V, W \in \underline{\underline{V}}_k(R^n)$, $p \in R^n$, $r \in R_0^+$, $V \cap [R^n - \underline{\underline{D}}^n(p, r)] = W \cap [R^n - \underline{\underline{D}}^n(p, r)]$, $\underline{\underline{W}}(V \cap \underline{\underline{D}}^n(p, r)) \leq \delta$, and $\underline{\underline{W}}(W \cap \underline{\underline{D}}^n(p, r)) \leq \delta$, then $\underline{\underline{F}}(V, W) \leq 2^{-1}\varepsilon$.

Part 2. Let $V, V_1, V_2, V_3, \ldots \in \underline{\underline{V}}_k(R^n)$ with $\lim_i V_i = V$, $r, r_1, r_2, r_3, \ldots \in R^+$ with $\lim_i r_i = r$, and $p \in R^n$. If $V \cap \partial \underline{\underline{D}}^n(p, r) = 0$ then Fatou's lemma implies $\lim_i V_i \cap \underline{\underline{D}}^n(p, r_i) = V \cap \underline{\underline{D}}^n(p, r_i)$ and $\lim_i V_i \cap [R^n - \underline{\underline{D}}^n(p, r_i)] = V \cap [R^n - \underline{\underline{D}}^n(p, r_i)]$.

Part 3. Suppose for each $p \in R^n$, $\underline{\underline{W}}(V_1 \cap \{p\}) < \delta$ and $\underline{\underline{W}}(W_1 \cap \{p\}) < \delta$. One then chooses $p_1, p_2, \ldots, p_{q-1} \in E$ and $r_1, r_2, \ldots, r_{q-1} \in R_0^+$ such that

$E \subset \bigcup_i \underline{\underline{D}}_0^n(p_i, r_i)$, $V_1 \cap \bigcup_i \partial \underline{\underline{D}}^n(p_i, r_i) = 0$, $W_1 \cap \bigcup_i \partial \underline{\underline{D}}^n(p_i, r_i) = 0$, $\sup_i \underline{\underline{W}}(V_1 \cap \underline{\underline{D}}^n(p_i, r_i)) < \delta$, and $\underline{\underline{W}}(W_1 \cap \underline{\underline{D}}^n(p_i, r_i)) < \delta$. We define

$$V_1 = V_1 \cap R^n \ ,$$

$$V_2 = V_1 \cap [R^n - \underline{\underline{D}}^n(p_1, r_1)] + |T_1| \cap \underline{\underline{D}}^n(p_1, r_1)$$

$$V_3 = V_1 \cap [R^n - \bigcup_{i=1}^{2} \underline{\underline{D}}^n(p_i, r_i)] + |T_1| \cap \bigcup_{i=1}^{2} \underline{\underline{D}}^n(p_i, r_i)$$

$$\vdots$$

$$V_q = V_1 \cap [R^n - \bigcup_{i=1}^{q-1} \underline{\underline{D}}^n(p_i, r_i)] + |T_1| \cap \bigcup_{i=1}^{q-1} \underline{\underline{D}}^n(p_i, r_i)$$

$$= |T_1| .$$

$$W_1 = W_1 \cap R^n$$

$$W_2 = W_1 \cap [R^n - \underline{\underline{D}}^n(p_1, r_1)] + |\partial T_1| \cap \underline{\underline{D}}^n(p_1, r_1)$$

$$\vdots$$

$$W_q = W_1 \cap R^n - \bigcup_{i=1}^{q-1} \underline{\underline{D}}^n(p_i, r_i) + |\partial T_1| \cap \bigcup_{i=1}^{q-1} \underline{\underline{D}}^n(p_i, r_i)$$

$$= |\partial T_1| .$$

One observes that $\underline{\underline{F}}(V_i, V_{i+1}) \leq 2^{-1}\varepsilon$ and $\underline{\underline{F}}(W_i, W_{i+1}) \leq 2^{-1}\varepsilon$ for each $i = 1, 2, \ldots, q-1$. We will have established the lemma in this case if we can show $(V_i, W_i; T_1, \partial T_1) \in \underline{\underline{VZ}}_k(E, F; G)$ for each i. It is clearly sufficient to show $(V_2, W_2; T_1, \partial T_1) \in \underline{\underline{VZ}}_k(E, F; G)$. Let $T^1, T^2, T^3, \ldots \in \underline{\underline{Z}}_k(E, F; G)$ with $\lim_i T^i = T_1$, $\lim \partial T^i = \partial T_1$, $\lim_i |T^i| = V_1$, and $\lim_i |\partial T^i| = W_1$. Let $P^1, P^2, P^3, \ldots \in \underline{\underline{Z}}_k(F, F; G)$ and $Q^1, Q^2, Q^3, \ldots \in \underline{\underline{Z}}_{k+1}(E, E; G)$ such that $\partial Q^i - (T^i - T_1) = P^i$ and $\lim_i \underline{\underline{M}}(P^i) = \lim_i \underline{\underline{M}}(Q^i) = 0$. One chooses $r^1, r^2, r^3, \ldots$ such that $\lim_i r^i = r_1$, $\lim_i \partial[Q^i \cap \underline{\underline{D}}^n(p_1, r_i)] - \partial Q^i \cap \underline{\underline{D}}^n(p_1, r_i) = 0$, and $\lim_i \partial[P^i \quad \underline{\underline{D}}^n(p_1, r_i)] - \partial(T_1 - T^i) \quad \underline{\underline{D}}^n(p_1, r_i) = 0$ [FF, 3.9, 3.10], [FL3, 5.7]. We set for each $i = 1, 2, 3, \ldots$,

$$S^i = T_1 + \partial(Q^i \cap [R^n - \underline{\underline{D}}^n(p_1, r_i)]) - P^i \cap [R^n - \underline{\underline{D}}^n(p_1, r_i)] \ ,$$

and observes using part 2 that $\lim_i S^i = T_1$, $\lim_i \partial S^i = \partial T_1$, $\lim_i |S^i| = V_2$, ...

$\lim_i |\partial S^i| = W_2$.

Part 4. Suppose $k \geq 2$, $p \in R^n$, and $W = \max\{\underline{\underline{W}}(V_1 \cap \{p\}), \underline{\underline{W}}(W_1 \cap \{p\})\} \geq \delta$. For $0 = t_q < t_{q-1} < \ldots < t_2 < t_1 = 1$ chosen so that $\max_i \{(t_{i+1})^k - (t_i)^k, (t_{i+1})^{k-1} - (t_i)^{k-1}\} < 2^{-1} W^{-1} \varepsilon$ we define

$$V_2 = V_1 \cap (R^n - \{p\}) + (t_2)^k V_1 \cap \{p\}$$
$$V_3 = V_1 \cap (R^n - \{p\}) + (t_3)^k V_1 \cap \{p\}$$
$$\vdots$$
$$V_q = V_1 \cap (R^n - \{p\})$$
$$W_2 = W_1 \cap (R^n - \{p\}) + (t_2)^{k-1} W_1 \cap \{p\}$$
$$W_3 = W_1 \cap (R^n - \{p\}) + (t_3)^{k-1} W_1 \cap \{p\}$$
$$\vdots$$
$$W_m = W_1 \cap (R^n - \{p\}) .$$

We now show $(V_i, W_i; T_1, \partial T_1) \in \underline{\underline{VZ}}_k(E, F; G)$ for each i. Clearly it is sufficient to consider only $i = 2$. Let $T^1, T^2, T^3, \ldots$ be as in part 3. One verifies the existence of a sequence of diffeomorphisms $f^i : (R^n, E, F, \{p\}) \longrightarrow (R^n, E, F, \{p\})$ $i = 1, 2, 3, \ldots$, with uniform convergence to the identity on R^n together with uniform convergence on compact subsets of $R^n - \{p\}$ of first derivatives, such that $Df_i(p, v) = t_2 v$ for each $v \in \underline{\underline{\Lambda}}_1(R^n)$ and $\lim_i f^i_{\#} T^i = T_1$, $\lim_i f^i \partial T^i = \partial T_1$ $\lim_i |f^i_{\#} T^i| = V_2$, and $\lim_i |f^i_{\#} \partial T^i| = W_2$. The above construction together with parts 1 and 3 imply the lemma for $k \geq 2$ since there are at most finitely many points $p \in R^n$ with

$$\max\{\underline{\underline{W}}(V_1 \cap \{p\}), \underline{\underline{W}}(W_1 \cap \{p\})\} \geq \delta .$$

For $k = 1$ the constructions necessary to prove the lemma are easily obtained and are left to the reader.

14.8. LEMMA. Let Δ, $c \in R^+$ and $1 \leq h \leq k$. Let $\underline{\underline{m}} : R^+ \dashrightarrow R^+$ satisfy

(1) $r^{-h}\underline{\underline{m}}(r)\exp(\Delta r)$ is non-decreasing in r for $r \in R^+$,

(2) $\lim_{r \to 0^+} r^{-h}\underline{\underline{m}}(r) < c^{1-h}$.

Then there exist arbitrarily small values of $r \in R_0^+$ for which

$$\underline{\underline{m}}(r)^{h-1} > c^{h-1}\underline{\underline{m}}'(r)^h .$$

PROOF. Suppose $\varepsilon > 0$ and for $0 < r < \varepsilon$, $\underline{\underline{m}}(r)^{h-1} \le c^{h-1}\underline{\underline{m}}'(r)^h$. Integration of this inequality yields $r^{-h}\underline{\underline{m}}(r) \ge c^{1-h}$ for $0 < r < \varepsilon$.

14.9. THEOREM. Let $\nu \in \underline{\underline{VZ}}_k(A, B \cup C; G)$ be in some class (a), (b), (c), or (d) with $\underline{\underline{W}}(\nu) \le c_2$. Then there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in R_0^+$ and $p_1, p_2, \ldots, p_{c(m,n)} \in A$ with $|p_i - p_j| > 10\varepsilon_1$ whenever $i \ne j$ such that

$$\{\nu_1^1, \nu_2^1, \ldots, \nu_{L(m)}^1\} \in \underline{\underline{R}}(A, B \cup C; \varepsilon_3, 2\gamma(n, k)\alpha(n:k)\xi_{\varepsilon_1}(\sup_{i,j}\underline{\underline{F}}[\nu_i^1, \nu_j^1]); p_h, 5\varepsilon_1)$$

for each $h = 1, 2, \ldots, c(m, n)$ whenever $\nu_1^1, \nu_2^1, \ldots, \nu_{L(m)}^1 \in \underline{\underline{VZ}}_k(A, B \cup C; G)$ with $\sup_i \underline{\underline{F}}(\nu, \nu_i^1) \le \varepsilon_2$.

PROOF.

Part 1. Let $\nu = (V, W; T, \partial T)$ be as above and $\delta, \Delta \in R_0^+$ as in 14.4. Since ν is in some class (a), (b), (c), or (d) we can find $0 < \varepsilon_1 < 5^{-1}\delta$; $c(m, n)$ distinct points $p_1, p_2, \ldots, p_{c(m,n)} \in A$ with $|p_i - p_j| > 10\varepsilon_1$ whenever $i \ne j$; and $s_1, s_2, \ldots, s_{c(m,n)}, t_1, t_2, \ldots, t_{c(m,n)} \in R$ with $0 < s_i < t_i < \varepsilon_1$ for each i; satisfying the following conditions (a), (b), (c), and (d).

Condition (a). If ν is in class (a) then

(1) $p_1, p_2, \ldots, p_{c(m,n)} \in \operatorname{spt}(V) - \{x : \operatorname{dist}(x, \operatorname{spt}(W)) > 5\varepsilon_1\}$;

(2) For each $i = 1, 2, \ldots, c(m, n)$, $p_i \in D$, or $p_i \in B$, or $p_i \in C$ and $\operatorname{dist}(p_i, B \cup D) > 5\varepsilon_1$, or $p_i \in A$ and $\operatorname{dist}(p_i, B \cup C) > 5\varepsilon_1$;

(3) $\underline{\underline{W}}V(\{x : |x - p_i| \in \{s_i, t_i\}\}) = 0$ for each $i = 1, 2, \ldots, c(m, n)$;

(4) $\underline{\underline{W}}V(\underline{\underline{D}}^n(p_i, t_i)) \le 2^{-1}a(k)(2+\Delta)^{-k}$ for each $i = 1, 2, \ldots, c(m, n)$;

(5) If $k \ge 2$, then

$$\underline{\underline{W}}(V \cap \underline{\underline{D}}^n(p_i, s_i)) \ge 3b(k, n)[(t_i - s_i)^{-1}\underline{\underline{W}}(V \cap [\underline{\underline{D}}^n(p_i, t_i) - \underline{\underline{D}}^n(p_i, s_i)])]^{k/(k-1)}$$

for each $i = 1, 2, \ldots, c(m, n)$; and

(6) If $k = 1$, then

$$(t_i - s_i)^{-1} \underline{\underline{W}}(W \cap [\underline{\underline{D}}^n(p_i, t_i) - \underline{\underline{D}}^n(p_i, s_i)]) < 1$$

for each $i = 1, 2, \ldots, c(m, n)$.

14.8 guarantees that condition (a) can be realized.

Condition (b). If ν is in class (b) then there exist $\eta, N \in R_0^+$ with $\eta\alpha(k) \leq N$; and

(1) $p_1, p_2, \ldots, p_{c(m,n)} \in \operatorname{spt}(V) - \{x : \operatorname{dist}(x, \operatorname{spt}(W)) > 5\varepsilon_1\}$;

(2) For each $i = 1, 2, \ldots, c(m, n)$, $p_i \in D$, or $p_i \in B$, or $p_i \in C$ and $\operatorname{dist}(p_i, B \cup D) > 5\varepsilon_1$, or $p_i \in A$ and $\operatorname{dist}(p_i, B \cup C) > 5\varepsilon_1$; and

(3) There exist k-planes $\Pi_1, \Pi_2, \ldots, \Pi_{c(m,n)}$ in R^n such that for each $i = 1, 2, \ldots, c(m, n)$, $p_i \in \Pi_i$, Π_i is parallel with the tangent plane to A at p_i, in case $p_i \in B$ then Π_i is parallel with the tangent plane to B at p_i, and

$$\operatorname{spt}(V) \cap \underline{\underline{D}}^n(p_i, 5\varepsilon_1) \subset \{x : \operatorname{dist}(x, \Pi_i) < 4^{-1}\varepsilon_1 \varphi(n, k)(2N, 2^{-1}\eta)\} \ ;$$

(4) $|\alpha(k)^{-1} r^{-k} \underline{\underline{W}}(V \cap \underline{\underline{D}}^n(p_i, r)) - Z| \geq 2\eta$ for each integer Z, each $\varepsilon_1 < r < 2\varepsilon_1$, and each $i = 1, 2, \ldots, c(m, n)$;

(5) $\underline{\underline{W}}(V \cap \underline{\underline{D}}^n(p_i, 3\varepsilon_1)) \leq \varepsilon_1^k$; for each $i = 1, 2, \ldots, c(m, n)$; and

(6) $\varepsilon_1(2+\Delta) < N$.

Condition (c). If ν is in class (c) then

(1) $p_1, p_2, \ldots, p_{c(m,n)} \in \operatorname{spt}(W)$;

(2) For each $i = 1, 2, \ldots, c(m, n)$, $p_i \in D$, or $p_i \in B$, or $p_i \in C$ and $\operatorname{dist}(p_i, B \cup D) > 5\varepsilon_1$;

(3) $\underline{\underline{W}}W(\{x : |x - p_i| \in \{s_i, t_i\}\}) = 0$ for each $i = 1, 2, \ldots, c(m, n)$;

(4) $\underline{\underline{W}}(W \cap \underline{\underline{D}}^n(p_i, t_i)) \leq 2^{-1}\alpha(k-1)(2+\Delta)^{-k+1}$ for each $i = 1, 2, \ldots, c(m, n)$;

(5) If $k \geq 3$, then

$$\underline{\underline{W}}(W \cap \underline{\underline{D}}^n(p_i, s_i)) \geq 3b(k-1, n)[(t_i - s_i)^{-1} \underline{\underline{W}}(W \cap [\underline{\underline{D}}^n(p_i, t_i) - \underline{\underline{D}}^n(p_i, s_i)])]^{(k-1)/(k\cdots}$$

for each $i = 1, 2, \ldots, c(m, n)$; and

(6) If $k = 2$, then

$$(t_i - s_i)^{-1}\underline{\underline{W}}(W \cap [\underline{\underline{D}}^n(p_i, t_i) - \underline{\underline{D}}^n(p_i, s_i)]) < 1$$

for each $i = 1, 2, \ldots, c(m, n)$.

14.8 guarantees that condition (c) can be realized.

Condition (d). If ν is in class (d) then there exist η, $N \in R_0^+$ with $\eta\alpha(k-1) \leq N$; and

(1) $p_1, p_2, \ldots, p_{c(m,n)} \in \operatorname{spt}(W)$;

(2) For each $i = 1, 2, \ldots, c(m, n)$, $p_i \in D$, or $p_i \in B$, or $p_i \in C$ and $\operatorname{dist}(p_i, B \cup D) > 5\varepsilon_1$;

(3) There exist $(k-1)$-planes $\Pi_1, \Pi_2, \ldots, \Pi_{c(m,n)}$ in R^n such that for each $i = 1, 2, \ldots, c(m, n)$, $p_i \in \Pi_i$, Π_i is parallel with the tangent plane to $B \cup C$ at p_i, in case $p_i \in D$ then Π_i is parallel with the tangent plane to D at p_i, and

$$\operatorname{spt}(W) \cap \underline{\underline{D}}^n(p_i, 5\varepsilon_1) \subset \{x : \operatorname{dist}(x, \Pi_i) < 4^{-1}\varepsilon_1 \varphi(n, k-1)(2N, 2^{-1}\eta)\} ;$$

(4) $|\alpha(k-1)^{-1} r^{-k+1} \underline{\underline{W}}(W \cap \underline{\underline{D}}^n(p_i, r)) - Z| \geq 2\eta$ for each integer Z, each $\varepsilon_1 < r < 2\varepsilon_1$, and each $i = 1, 2, \ldots, c(m, n)$;

(5) $\underline{\underline{W}}(V \cap D^n(p_i, 3\varepsilon_1)) \leq N\varepsilon_1^{k-1}$ for each $i = 1, 2, \ldots, c(m, n)$; and

(6) $2\varepsilon_1(2+\Delta) < \varphi(n, k-1)(2N, 2^{-1}\eta)$.

<u>Part</u> 2. Since $\underline{\underline{VZ}}_k^\natural(A, B \cup C; G)$ is dense in $\underline{\underline{VZ}}_k(A, B \cup C; G)$ it is sufficient to prove the theorem when there exist $S_1, S_2, \ldots, S_{L(m)} \in \underline{\underline{Z}}_k(A, B \cup C; G)$ with $\nu_i^1 = (|S_i|, |\partial S_i|; S_i, \partial S_i)$ for each $i = 1, 2, \ldots, L(m)$. We therefore, without loss of generality, make this additional assumption.

<u>Part</u> 3. Let $\nu, \nu_1^1, \nu_2^1, \ldots, \nu_{L(m)}^1$ be given as above and suppose $\varepsilon_1, p_1, p_2, \ldots, p_{c(m,n)}$ have been chosen as in part 1 to satisfy conditions (a), (b), (c), and (d). In view of 2.4(2d) and 14.6 we can assume, without loss of generality, that

$$\nu_i^1 \cap \underline{\underline{D}}^n(p_1, 2\varepsilon_1) = \nu_1^1 \cap \underline{\underline{D}}^n(p_1, 2\varepsilon_1)$$

for each $i = 2, 3, \ldots, L(m)$ provided we then show that for some $\varepsilon_2 > 0$, some $\varepsilon_3 > 0$, and each $\delta > 0$,

$$\{\nu_1^1, \nu_1^1, \ldots, \nu_1^1\} \in \underline{\underline{R}}(A, B \cup C; \varepsilon_3, \delta; p_1, \varepsilon_1)$$

whenever $\underline{\underline{F}}(\nu, \nu_i^1) < \varepsilon_2$. We therefore make this additional assumption and will verify the stronger conclusion.

Part 4. Let $E \subset R^n$ be compact, $h \in \{k, k-1\} - \{0\}$, $S^1 \in \underline{\underline{Z}}_h(E, E; G)$, and $\delta \in R_0^+$. We denote by Γ the set of all finite sequences $\gamma^1, \gamma^2, \ldots, \gamma^u$ of triples

$$\gamma^i = [T^i, W^i, S^i] \in \underline{\underline{Z}}_{h+1}(E, E; G) \times \underline{\underline{GV}}_h(E) \times \underline{\underline{Z}}_h(E, E; G) \quad ,$$

$i = 1, 2, \ldots, u$ for which

(1) $\partial S^i = \partial S^1$ for each $i = 2, 3, \ldots, u$;

(2) $\partial T^i = S^i - S^{i-1}$ for each $i = 2, 3, \ldots, u$;

(3) For each $i = 2, 3, \ldots, u$ there exist $S_1^i, S_2^i, S_3^i, \ldots \in \underline{\underline{Z}}_h(E, E; G)$ such that $\lim_j S_j^i = S^i$, $\lim_j |S_j^i| = W^i$, and $\partial S_j^i = \partial S^1$ for each $j = 1, 2, 3, \ldots$;

(4) $\underline{\underline{F}}(S^i, S^{i+1}) \leq \delta$ for each $i = 1, 2, \ldots, u-1$;

(5) $\underline{\underline{M}}(T^i) \leq \min\{\delta, 2^{-1}(\underline{\underline{W}}(W^{i-1}) - \underline{\underline{W}}(W^i))\}$ for each $i = 2, 3, \ldots, u$;

(6) $\underline{\underline{W}}(W^{i+1}) \leq \underline{\underline{W}}(W^i)$ for each $i = 1, 2, \ldots, u-1$;

(7) $\underline{\underline{F}}(W^i, W^{i+1}) \leq \delta$ for each $i = 1, 2, \ldots, u-1$;

(8) $W^1 = |S^1|$.

(9) $T^1 = 0$.

Since $\operatorname{clos}\{\gamma : \gamma$ is a term in a sequence of $\Gamma\}$ is compact in the $\underline{\underline{F}} \times \underline{\underline{F}} \times \underline{\underline{F}}$ topology there exists a positive integer μ such that whenever $\gamma = \{\gamma^1, \gamma^2, \ldots, \gamma^u\} \in \Gamma$, $\gamma^i = [T^i, W^i, S^i]$, one can find $1 = i(1) < i(2) < \ldots < i(\mu) = u$ such that $\underline{\underline{F}}(W^{i(j)}, W^{i(j+1)}) \leq \delta$ and $\underline{\underline{F}}(S^{i(j)}, S^{i(j+1)}) \leq \delta$ for each $j = 1, 2, \ldots, \mu$. In view of (5) above and the preceding observation one concludes by elementary arguments that if $\gamma = [T, W, S]$ is the last element of some sequence in Γ, then γ is also the last element of some sequence in Γ having no more than $\mu(1 + 2\delta^{-1}[\underline{\underline{M}}(S_1) - \underline{\underline{W}}(W)])$ terms. The compactness noted above together with the preceding bound on the length of sequences having a given last term implies the existence of a sequence $\gamma^1, \gamma^2, \ldots, \gamma^u$ in Γ whose last term $\gamma^u = [T^u, W^u, S^u]$ satisfies

$W(W^u) = \inf\{\underline{\underline{W}}(W)$: for some T and S, $[T, W, S]$ is an element of some sequence in $\Gamma\} = \underline{\underline{L}}(\Gamma)$.

14.7 implies $W^u = |S^u|$.

Now suppose E is compact submanifold of R^n of class 3 with $\Delta_h(E, R^n) \leq \Delta$ and $\gamma^u = [T, |S|, S]$ is the last term of a sequence $\gamma^u, \gamma^u, \ldots, \gamma^u$ in Γ with $\underline{\underline{W}}(|S|) = \underline{\underline{L}}(\Gamma)$. We assert that $\underline{\underline{P}}(R^n)(|S|, |\partial S|) \leq 2 + \Delta$. If not then $\underline{\underline{P}}(R^n)(|S|, |\partial S|) > 2 + \Delta$ and, in particular, $\underline{\underline{P}}^-(R^n, E)(|S|, |\partial S|) > 2$. Hence there exists a differentiable deformation $f : R \times R^n \longrightarrow R^n$ with $f(t, \cdot)(E) \subset E$ for each $t \in R^+$ and $f(0, x) = x$ for each $x \in R^n$ such that

$$-\underline{\underline{S}}(|S|, (\partial f/\partial t)(0, \cdot)) - \underline{\underline{T}}(|\partial S|, (\partial f/\partial t)(0, \cdot)) > 2\underline{\underline{T}}(|S|, (\partial f/\partial t)(0, \cdot));$$

and, in particular, for some $t > 0$ if one sets

$$\gamma^{u+1} = [-f_{\#}([0, t] \times S), |f(t, \cdot)_{\#}(S) - f_{\#}([0, t] \times \partial S)|, f(t, \cdot)_{\#}(S) - f_{\#}([0, t] \times \partial S)]$$

then the sequence $\gamma^1, \gamma^2, \ldots, \gamma^u, \gamma^{u+1}$ is in Γ and $\underline{\underline{W}}(|f(t, \cdot)_{\#}(S) - f_{\#}([0, t] \times \partial S)|) < \underline{\underline{L}}(\Gamma)$ which is a contradiction.

Part 5. We give three applications of the results of part 4. Let $E, h, S_1, \delta, \Gamma, \underline{\underline{L}}(\Gamma), \Delta$ be as in part 4.

Application (1). Let $E^0 \subset E$ be closed and

$$\underline{\underline{M}}(S^1) < r^h \exp(-r(2+\Delta))$$

where $r = \operatorname{dist}(E^0, \operatorname{spt}(\partial S^1))$. 8.2, 8.3, and part 4 imply that if $\gamma_u = [T, |S|, S]$ is the last term in a sequence in Γ with $\underline{\underline{W}}(|S|) = \underline{\underline{L}}(\Gamma)$, then $\operatorname{spt}(S) \cap E^0 = \emptyset$.

Application (2). Suppose

(1) $(2+\Delta)^h \underline{\underline{M}}(S^1) \leq a(h)$;

(2) $\underline{\underline{M}}(S^1) \geq 2b(h, n)\underline{\underline{M}}(\partial S^1)^{h/(h-1)}$ if $h \geq 2$; and

(3) $\underline{\underline{M}}(\partial S^1) < 1$ if $h = 1$.

8.7, part 4, and the fact that if $h = 1$, $\underline{\underline{M}}(\partial S^1)$ must be a non-negative integer imply that if $\gamma_u = [T, |S|, S]$ is the last term of a sequence in Γ with $\underline{\underline{W}}(|S|) = \underline{\underline{L}}(\Gamma)$, then

(1) $\underline{\underline{W}}(|S|) \leq b(h, n)\underline{\underline{W}}(\partial S^1)^{h/(h-1)}$ if $h \geq 2$, and

(2) $S = 0$ if $h = 1$, and hence

$$\underline{\underline{W}}(|S^1|) - \underline{\underline{W}}(|S|) \geq 2^{-1}\underline{\underline{W}}(|S^1|) .$$

Application (3). Let r, N, $\eta \in R_0^+$ with $\eta\alpha(k) \leq N$ and suppose

(1) $E \subset \underline{\underline{D}}^h(0, r[1 + \varphi(n, h)(2N, 2^{-1}\eta)]) \times \underline{\underline{D}}^{n-h}(0, r\,\varphi(n, h)(2N, 2^{-1}\eta))$;

(2) $\operatorname{spt}(\partial S^1) \subset [\underline{\underline{D}}^h(0, r[1 + \varphi(n, h)(2N, 2^{-1}\eta)] - \underline{\underline{D}}^h(0, r)] \times \underline{\underline{D}}^{n-h}(0, r\,\varphi(n, h)(2N, 2^{-1}\eta$

(3) $|\alpha(h)^{-1}r^{-h}\underline{\underline{M}}(S^1) - Z| > \eta$ for each integer Z;

(4) $r(2+\Delta) \leq 2N$;

(5) $r^{-h}\underline{\underline{M}}(S^1) \leq 2N$; and

(6) $r^{-h-1}\cdot\underline{\underline{M}}(\partial S^1) \leq 2N$.

9.10 and part 4 imply that if $\gamma_u = [T, |S|, S]$ is the last term in a sequence in Γ with $\underline{\underline{W}}(|S|) = \underline{\underline{L}}(\Gamma)$, then

$$|\alpha(h)r^h\underline{\underline{W}}(|S|) - Z| \leq 2^{-1}\eta$$

for some integer Z, and hence

$$\underline{\underline{W}}(|S^1|) - \underline{\underline{W}}(|S|) \geq 2^{-1}\eta r^h .$$

Part 6. Our arguments now separate into fourteen cases. The following table gives information characterizing the separate cases.

Case no.	Class of ν = Condition ν satisfies	location of p_1	relevant Types of subsets	Applications used from part 4
(1)	(a)	$A - (B\cup C)$	1	(2)
(2)	(a)	B	2, 9	(1), (2)
(3)	(a)	C - D	3, 10	(1), (2)
(4)	(a)	D	4, 11	(1), (2)
(5)	(b)	$A - (B\cup C)$	5	(3)
(6)	(b)	B	6, 9	(1), (3)
(7)	(b)	C - D	7, 10	(1), (3)
(8)	(b)	D	8, 11	(1), (3)
(9)	(c)	B	9	(2)
(10)	(c)	C - D	10	(2)
(11)	(c)	D	11	(2)
(12)	(d)	B	12	(3)
(13)	(d)	C - D	13	(3)
(14)	(d)	D	14	(3)

We will provide the details of the proof only for cases (1), (6), and (11). Case (1) is probably the easiest case and case (6) the most intricate. The arguments in these three cases are representative of the arguments necessary for the proof of the other cases. One obtains from each case (i) numbers ε_2^i, ε_3^i from which ε_2 and ε_3 are computed in part 7.

Case (1). Let $\nu = (V, W; T, \partial T)$ be in class (a) and satisfy condition (a). Suppose $p_1 \in A - (B\cup C)$ and $A \cap \underline{\underline{D}}^n(p_1, 5\varepsilon_1)$ is of type (1) with $\Delta_k^-(A\cap \underline{\underline{D}}^n(p_1, 5\varepsilon_1)) \leq \Delta$. Choose $\varepsilon_2^1 \in R_0^+$ sufficiently small so that if

$$\nu_1^1 = (|S_1|, |\partial S_1|; S_1, \partial S_1) \in \underline{\underline{VZ}}_k(A, B\cup C; G)$$

with $\underline{\underline{F}}(\nu, \nu_1^1) < \varepsilon_2^1$, then

(1) $\underline{\underline{W}}(|S_1| \cap \underline{\underline{D}}^n(p_1, t_1)) \leq (2+\Delta)^{-k} a(k)$;

(2) $\underline{\underline{W}}(|S_1| \cap \underline{\underline{D}}^n(p_1, s_1)) \geq 2^{-1} \underline{\underline{W}}(V\cap \underline{\underline{D}}^n(p_1, s_1))$;

(3) $\underset{=}{W}(|S_1| \cap \underset{=}{D}^n(p_1, s_1)) \geq 2b(k, n)[(t_1 - s_1)^{-1}\underset{=}{W}(V \cap [\underset{=}{D}^n(p_1, t_1) - \underset{=}{D}^n(p_1, s_1)])]^{k/(k-1)}$ if $k \geq 2$; and

(4) $(t_1 - s_1)^{-1}\underset{=}{W}(V \cap [\underset{=}{D}^n(p_1, t_1) - \underset{=}{D}^n(p_1, s_1)]) < 1$ if $k = 1$. [FF, 3.9, 3.10] and [FL3, 5.7] imply the existence of $s_1 < r < t_1$ such that $\partial[S_1 \cap \underset{=}{D}^n(p_1, r)]$ exists and

(1) $\underset{=}{W}(|S_1| \cap \underset{=}{D}^n(p_1, r)) \geq 2b(k, n)\underset{=}{M}(\partial[S_1 \cap \underset{=}{D}^n(p_1, r)])^{k/k-1}$ if $k \geq 2$, and

(2) $\underset{=}{M}(\partial[S_1 \cap \underset{=}{D}^n(p_1, r)]) < 1$ if $k = 1$.

We set $S^1 = S_1 \cap \underset{=}{D}^n(p_1, r)$. Part 5, application (2) implies, with $h = k$, the existence of a sequence

$$[0, |S^1|, S^1], [T^2, W^2, S^2], [T^3, W^3, S^3], \ldots, [T^n, |S^n|; S^n]$$

as in part 4 with $\underset{=}{W}(|S^u|) = \underset{=}{L}(\Gamma)$ for $\underset{=}{L}(\Gamma)$ as in part 4. The desired sequence $\nu_1^1, \nu_1^2, \nu_1^3, \ldots, \nu_1^u$ by virtue of which

$$\{\nu_1^1, \nu_1^1, \ldots, \nu_1^1\} \quad \underset{=}{R}(A, B \cup C; \varepsilon_3^1, \delta; p_1, 5\varepsilon_1)$$

is defined as follows:

$$\nu_1^1 = (|S_1|, |\partial S_1|; S_1, \partial S_1)$$
$$\nu_1^2 = (|S_1 - S^1| + W^2, |\partial S_1|; S_1 - S^1 + S^2, \partial S_1)$$
$$\nu_1^3 = (|S_1 - S^1| + W^3, |\partial S_1|; S_1 - S^1 + S^3, \partial S_1)$$
$$\vdots$$
$$\nu_1^u = (|S_1 - S^1| + W^u, |\partial S_1|; S_1 - S^1 + S^u, \partial S_1) .$$

Application (2) implies, in particular,

$$\underset{=}{W}(\nu_1^1) - \underset{=}{W}(\nu_1^u) \geq 4^{-1}\underset{=}{W}(V \cap \underset{=}{D}^n(p_1, s_1)) = \varepsilon_3^1 .$$

Case (6). Let $\nu = (V, W; T, \partial T)$ be in class (b) and satisfy condition (b). Suppose $p_1 \in B$ and Π_1 is a k-dimensional plane in R^n containing p_1 and tangent to B at p_1 such that for N, η as in condition (b),

$$A \cap \underline{\underline{D}}^n(p_1, r) \cap \{x : \operatorname{dist}(x, \Pi_1) < r\,\varphi(n, k)(2N, 2^{-1}\eta)\}$$

is a subset of A of type (6). Choose $\varepsilon_2^6 \in R_0^+$ sufficiently small so that if

$$\nu_1^1 = (|S_1|, |\partial S_1|; S_1, \partial S_1) \in \underline{\underline{VZ}}_k(A, B \cup C; G)$$

with $\underline{\underline{F}}(\nu, \nu_1^1) < \varepsilon_2^6$, then
(1) $\underline{\underline{W}}(|\partial S_1| \cap \underline{\underline{D}}^n(p_1, 5\varepsilon_1)) < \varepsilon_1^{k-1}\exp(-\varepsilon_1(2+\Delta))$;
(2) $|\underline{\underline{W}}(|S_1| \cap \underline{\underline{D}}^n(p_1, r) \cap \{x : \operatorname{dist}(x, \Pi_1) < 2^{-1}\varepsilon_1\,\varphi(n, k)(2N, 2^{-1}\eta)\})$
$- \underline{\underline{W}}(V \cap \underline{\underline{D}}^n(p_1, r))| < 2^{-1}\eta\alpha(k)\varepsilon_1^k$ for each $\varepsilon_1 < r < 2\varepsilon_1$;
(3) $\underline{\underline{W}}(|S_1| \cap \underline{\underline{D}}^n(p_1, 3\varepsilon_1) \cap \{x : \operatorname{dist}(x, \Pi_1) \geq 2^{-1}\varepsilon_1\,\varphi(n, k)(2N, 2^{-1}\eta)\})$
$+ 2^{-1}\underline{\underline{W}}(|\partial S_1| \cap \underline{\underline{D}}^n(p_1, 4\varepsilon_1)) < \min\{2^{-1}\eta\alpha(k)\varepsilon_1^k, (\min\{\varepsilon_1, 4^{-1}\varepsilon_1\,\varphi(n, k)(2N, 2^{-1}\eta)\})^k$
$\exp(-(2+\Delta)\min\{\varepsilon_1, 4^{-1}\varepsilon_1\,\varphi(n, k)(2N, 2^{-1}\eta)\})\}$.

The construction of the desired sequence

$$\nu_1^1, \nu_1^2, \nu_1^3, \ldots, \nu_1^u, \nu_1^{u+1}, \nu_1^{u+2}, \ldots, \nu_1^v, \nu_1^{v+1}, \nu_1^{v+2}, \ldots, \nu_1^w$$

by virtue of which

$$\{\nu_1^1, \nu_1^1, \ldots, \nu_1^1\} \in \underline{\underline{R}}(A, B \cup C; \varepsilon_3^6, \delta; p_1, 5\varepsilon_1)$$

takes place in three steps:

<u>Step</u> (1). We construct terms $\nu_1^1, \nu_1^2, \ldots, \nu_1^u$ of the sequence to remove $\partial S_1 \cap \underline{\underline{D}}^n(p_1, 3\varepsilon_1)$. Use [FF, 3.9, 3.10] and [FL3, 5.7] to choose $4\varepsilon_1 < r < 5\varepsilon_1$ such that $\partial[\partial S_1 \cap \underline{\underline{D}}^n(p_1, r)]$ exists. We set $S^1 = \partial S_1 \cap \underline{\underline{D}}^n(p_1, r)$ and note that
(1) $\operatorname{spt}(\partial S^1) \subset B \cap \underline{\underline{D}}^n(p_1, 5\varepsilon_1)$
(2) $\operatorname{dist}(B \cap \underline{\underline{D}}^n(p_1, 3\varepsilon_1), \operatorname{spt}(\partial S^1)) > \varepsilon_1$
(3) $\underline{\underline{M}}(S^1) < \varepsilon_1^k\exp(-\varepsilon_1(2+\Delta))$
(4) $B \cap \underline{\underline{D}}^n(p_1, 5\varepsilon_1)$ is a subset of A of type (9) with $\Delta_k^-(B \cap \underline{\underline{D}}^n(p_1, 5\varepsilon_1), R^n) \leq \Delta$.
If one takes $h = k-1$ in part 5, application (1), there exists a sequence

$$[0, |S^1|, S^1], [T^2, W^2, S^2], [T^3, W^3, S^3], \ldots, [T^u, |S^u|, S^u]$$

as in part 4 with $\underline{\underline{W}}(|S^u|) = \underline{\underline{L}}(\Gamma)$ as in part 4, such that $\operatorname{spt}(|S^u|) \cap \underline{\underline{D}}^n(p_1, 3\varepsilon_1)$ -

and

$$\sum_{i=2}^{u} \underline{\underline{M}}(T^i) \le 2^{-1}\underline{\underline{M}}(S^1) < 2^{-1}\eta\varepsilon_1^k .$$

We define

$$\nu_1^1 = (|S_1|, |\partial S_1|; S_1, \partial S_1)$$
$$\nu_1^2 = (|S_1| + |T^2|, |\partial S_1 - S^1| + W^2; S_1 + T^2, \partial S_1 - S^1 + S^2)$$
$$\nu_1^3 = (|S_1| + |T^2| + |T^3|, |\partial S_1 - S^1| + W^3; S_1 + T^2 + T^3, \partial S_1 - S^1 + S^3)$$
$$\vdots$$
$$\nu_1^u = (|S_1| + \sum_{i=2}^{u}|T^i|, |\partial S_1 - S^1| + |S^u|; S_1 + \sum_{i=2}^{u} T^i, \partial S_1 - S^1 + S^u) .$$

We have, in particular,

(1) $\underline{\underline{W}}(\nu_1^{i+1}) \le \underline{\underline{W}}(\nu^i)$ for each $i = 1, 2, \ldots, u-1$;

(2) $\underline{\underline{W}}(|S_1| + \sum_{i=2}^{u} T^i) - \underline{\underline{W}}(|S_1|) < 2^{-1}\eta\varepsilon_1^k$; and

(3) $\operatorname{spt}(|\partial S_1 - S^1| + |S^u|) \cap \underline{\underline{D}}^n(p_1, 3\varepsilon_1) = \emptyset$.

Step (2). We now construct terms $\nu_1^{u+1}, \nu_1^{u+2}, \ldots, \nu_1^v$ of the sequence to remove $|S_1| + \sum_{i=2}^{u}|T^i|$ from the region

$$A \cap \underline{\underline{D}}^n(p_1, 2\varepsilon_1) - \{x : \operatorname{dist}(x, \Pi_1) \le \varepsilon_1 \varphi(n, k)(2N, 2^{-1}\eta)\} .$$

14.7 implies the existence of a sequence $\nu_1^{u+1}, \nu_1^{u+2}, \ldots, \nu_1^q$ for which $\nu_1^q = (|S_1^q|, |\partial S_1^q|; S_1^q, \partial S_1^q)$ where $S_1^q = S_1 + \sum_{i=2}^{u} T^i$ and $\partial S_1^q = \partial S_1 - S^1 + S^u$. Use [FF, 3.9, 3.10] and [FL3, 5.7] to choose $3\varepsilon_1 < r < 4\varepsilon_1$ such that $\partial[S_1^q \cap \underline{\underline{D}}^n(p_1, r)]$ exists and to choose

$$(1/2)\varepsilon_1 \varphi(n, k)(2N, 2^{-1}\eta) < s < (3/4)\varepsilon_1 \varphi(n, k)(2N, 2^{-1}\eta)$$

such that if

$$S^q = S_1^q \cap \underline{\underline{D}}^n(p_1, r) \cap \{x : \operatorname{dist}(x, \Pi_1) > s\} ,$$

then ∂S^q exists. One notes that

(1) $\underline{\underline{M}}(S^q) < (\min\{\varepsilon_1, 4^{-1}\varepsilon_1 \varphi(n, k)(2N, 2^{-1}\eta)\})^k \exp(-(2+\Delta)\min\{\varepsilon_1, 4^{-1}\varepsilon_1\varphi(n, k)(2N, 2^{-1}\eta)\})$

and

(2) $\operatorname{dist}(\operatorname{spt}(\partial S^q), \underline{\underline{D}}^n(p_1, 2\varepsilon_1)\cap\{x : \operatorname{dist}(x, \Pi) \geq \varepsilon_1 \varphi(n, k)(2N, 2^{-1}\eta)\})$
$> \min\{\varepsilon_1, 4^{-1}\varepsilon_1 \varphi(n, k)(2N, 2^{-1}\eta)\}$.

If one takes $h = k$ in part 5, application (1), there exists a sequence

$$[0, |S^q|, S^q], [T^{q+1}, W^{q+1}, S^{q+1}], [T^{q+2}, W^{q+2}, S^{q+2}], \ldots, [T^v, |S^v|, S^v]$$

as in part 4, with $\underline{\underline{W}}(|S^v|) = \underline{\underline{L}}(\Gamma)$ as in part 4, such that

$$\operatorname{spt}(|S^v|)\cap\underline{\underline{D}}^n(p_1, 2\varepsilon_1)\cap\{x : \operatorname{dist}(x, \Pi_1) \geq \varepsilon_1 \varphi(n, k)(2N, 2^{-1}\eta)\} = \emptyset$$

and $\underline{\underline{W}}(|S^v|) \leq \underline{\underline{W}}(S^q)$.

We define

$$\begin{aligned}
\nu_1^q &= (|S_1^q|, |\partial S_1^q|; S_1^q, \partial S_1^q)\\
\nu_1^{q+1} &= (|S_1^q - S^q| + W^{q+1}, |\partial S_1^q|; S_1^q - S^q + S^{q+1}, \partial S_1^q)\\
\nu_1^{q+2} &= (|S_1^q - S^q| + W^{q+2}, |\partial S_1^q|; S_1^q - S^q + S^{q+2}, \partial S_1^q)\\
&\ \vdots\\
\nu_1^v &= (|S_1^q - S^q| + |S^v|, |\partial S_1^q|; S_1^q - S^q + S^v, \partial S_1^q) .
\end{aligned}$$

We have, in particular,

(1) $\underline{\underline{W}}(\nu_1^{i+1}) \leq \underline{\underline{W}}(\nu_1^i)$ for each $i = u, u+1, \ldots, v-1$;

(2) $\underline{\underline{W}}(S^v) \leq 2^{-1}\eta\alpha(k)\varepsilon_1^k$.

(3) $\operatorname{spt}(S_1^q - S^q + S^v)\cap\underline{\underline{D}}^n(p_1, r) \subset \{x : \operatorname{dist}(x, \Pi_1) < \varepsilon_1 \varphi(n, k)(2N, 2^{-1}\eta)\}$.

<u>Step</u> 3. In this step we make use of the fact that $\operatorname{spt}(V)$ near p_1 lies close to Π_1 but does not have an integer density there. 14.7 implies the existence of a sequence $\nu_1^{v+1}, \nu_1^{v+2}, \ldots, \nu_1^t$ for which $\nu_1^t = (|S_1^t|, |\partial S_1^t|; S_1^t, \partial S_1^t)$ where $S_1^t = S_1^q - S^q + S^v$ and $\partial S_1^t = \partial S_1^q$. Use [FF, 3.9, 3.10] and [FL3, 5.7] to choose $\varepsilon_1 < r < \varepsilon_2$ such that if $S^t = S_1^t\cap\underline{\underline{D}}^n(p_1, r)$, then ∂S^t exists and

$$\underline{\underline{M}}(\partial S^t) \leq [N + \eta\alpha(k)]\varepsilon_1^{k-1} \leq 2N\varepsilon_1^{k-1} .$$

One notes that

(1) $r^{-k}\underline{\underline{M}}(S^t) \leq 2N$;

(2) $r^{-k+1}\underline{\underline{M}}(\partial S^t) \leq 2N$;

(3) $r(2+\Delta) \leq 2N$;

(4) $|\alpha(k)^{-1} r^{-k}\underline{\underline{M}}(S^t) - Z| \geq \eta$ for each integer Z;

(5) $\operatorname{spt}(S^t) \subset A \cap \underline{\underline{D}}^n(p_1, r) \cap \{x : \operatorname{dist}(x, \Pi_1) \leq r\,\varphi(2N, 2^{-1}\eta)\}$

(6) $\operatorname{spt}(\partial S^t) \subset A \cap \partial\underline{\underline{D}}^n(p_1, r) \cap \{x : \operatorname{dist}(x, \Pi_1) \leq r\,\varphi(2N, 2^{-1}\eta)\}$.

One uses 14.4 together with part 5, application (3) to obtain a sequence

$$[0, |S^t|, S^t], [T^{t+1}, W^{t+1}, S^{t+1}], [T^{t+2}, W^{t+2}, S^{t+2}], \ldots, [T^w, |S^w|, S^w]$$

as in part 4, with $\underline{\underline{W}}(|S^w|) = \underline{\underline{L}}(\Gamma)$ as in part 4, such that $|\alpha(k)^{-1} r^{-k}\underline{\underline{W}}(|S^w|) - Z| \leq 2^{-1}\eta$ for some integer Z and hence $\underline{\underline{W}}(|S^t|) - \underline{\underline{W}}(|S^w|) \geq 2^{-1}\eta$.

We define

$$\nu_1^t = (|S_1^t|, |\partial S_1^t|; S_1^t, \partial S_1^t)$$
$$\nu_1^{t+1} = (|S_1^t - S^t| + W^{t+1}, |\partial S_1^t|; S_1^t - S^t + S^{t+1}, \partial S_1^t)$$
$$\nu_1^{t+2} = (|S_1^t - S^t| + W^{t+2}, |\partial S_1^t|; S_1^t - S^t + S^{t+2}, \partial S_1^t)$$
$$\vdots$$
$$\nu_1^w = (|S_1^t - S^t| + |S^w|, |\partial S_1^t|; S_1^t - S^t + S^w, \partial S_1^t) .$$

We have, in particular, $\underline{\underline{W}}(\nu_1^{i+1}) \leq \underline{\underline{W}}(\nu_1^i)$ for each $i = t, t+1, \ldots, w-1$ and

$$\underline{\underline{W}}(\nu_1^1) - \underline{\underline{W}}(\nu_1^w) \geq \underline{\underline{W}}(\nu_1^t) - \underline{\underline{W}}(\nu_1^w) \geq 2^{-1}\eta\varepsilon_1^k = \varepsilon_3^6 .$$

Case (11). Let $\nu = (V, W; T, \partial T)$ be in class (c) and satisfy condition (c). Suppose $p_1 \in D$ and $D \cap \underline{\underline{D}}^n(p_1, 5\varepsilon_1)$ is a subset of A of type (11). Choose $\varepsilon_2^{11} \in R_0^+$ sufficiently small so that if

$$\nu_1^1 = (|S_1|, |\partial S_1|; S_1, \partial S_1) \in \underline{\underline{VZ}}_k(A, B \cup C; G)$$

with $\underline{\underline{F}}(\nu, \nu_1^1) < \varepsilon_2^{11}$, then

(1) $\underline{\underline{W}}(|\partial S_1| \cap \underline{\underline{D}}^n(p_1, t_1)) \leq (2+\Delta)^{-k+1} a(k-1)$;

(2) $\underline{\underline{W}}(|\partial S_1| \cap \underline{\underline{D}}^n(p_1, s_1)) \geq 2^{-1} \underline{\underline{W}}(W \cap \underline{\underline{D}}^n(p_1, s_1))$;

(3) $\underline{\underline{W}}(|\partial S_1| \cap \underline{\underline{D}}^n(p_1, s_1)) \geq$

$$2b(k-1, n)[(t_1-s_1)^{-1} \underline{\underline{W}}(|\partial S_1| \cap [\underline{\underline{D}}^n(p_1, t_1) - \underline{\underline{D}}^n(p_1, s_1)])]^{(k-1)/(k-2)}$$

if $k \geq 3$;

(4) $(t_i - s_i)^{-1} \underline{\underline{W}}(|\partial S_1| \cap [\underline{\underline{D}}^n(p_1, t_1) - \underline{\underline{D}}^n(p_1, s_1)]) < 1$ if $k = 2$.

Recall that class (c) is empty if $k = 1$.

[FF, 3.9, 3.10] and [FL3, 5.7] imply the existence of $s_1 < r < t_1$ such that if $S^1 = \partial S_1 \cap \underline{\underline{D}}^n(p_1, r)$, then ∂S^1 exists and

(1) $\underline{\underline{W}}(|S^1|) \geq 2b(k-1, n) \underline{\underline{W}}(|\partial S^1|)^{(k-1)/(k-2)}$ if $k \geq 3$, and

(2) $\underline{\underline{M}}(\partial S^1) < 1$ if $k = 2$.

Part 5, application (2) implies, with $h = k-1$, the existence of a sequence

$$[0, |S^1|, S^1], [T^2, W^2, S^2], [T^3, W^3, S^3], \ldots, [T^u, |S^u|, S^u]$$

as in part 4 with $\underline{\underline{W}}(|S^u|) = \underline{\underline{L}}(\Gamma)$ for $\underline{\underline{L}}(\Gamma)$ as in part 4. The desired sequence $\nu_1^1, \nu_1^2, \ldots, \nu_1^u$ by virtue of which

$$\{\nu_1^1, \nu_1^1, \ldots, \nu_1^1\} \in \underline{\underline{R}}(A, B \cup C; \varepsilon_3^{11}, \delta; p_1, 5\varepsilon_1)$$

is defined as follows:

$$\nu_1^1 = (|S_1|, |\partial S_1|; S_1, \partial S_1)$$
$$\nu_1^2 = (|S_1| + |T^2|, |\partial S_1 - S^1| + W^2; S_1 + T^2, \partial S_1 - S^1 + S^2)$$
$$\nu_1^3 = (|S_1| + \sum_{i=2}^{3} |T^i|, |\partial S_1 - S^1| + W^3; S_1 + \sum_{i=2}^{3} T^i, \partial S_1 - S^1 + S^3)$$
$$\vdots$$
$$\nu_1^u = (|S_1| + \sum_{i=2}^{u} |T^i|, |\partial S_1 - S^1| + W^u; S_1 + \sum_{i=2}^{u} T^i, \partial S_1 - S^1 + S^u) .$$

Application (2) implies, in particular,

$$\underline{\underline{W}}(\nu_1^1) - \underline{\underline{W}}(\nu_1^u) \geq 4^{-1}\underline{\underline{W}}(W \cap \underline{\underline{D}}^n(p_1, s_1)) = \varepsilon_3^{11} .$$

<u>Part</u> 7. We now choose ε_2 and ε_3 as required in part 3. We set

$$\varepsilon_3 = 3^{-1}\min\{1, \varepsilon_3^1, \varepsilon_3^2, \ldots, \varepsilon_3^6, \ldots, \varepsilon_3^{11}, \ldots, \varepsilon_3^{14}\} .$$

Now choose $\varepsilon_2^0 > 0$ sufficiently small so that if $\nu_1^1 \in \underline{\underline{VZ}}_k(A, B \cup C; G)$ with $\underline{\underline{F}}(\nu, \nu_1^1) \leq 2\varepsilon_2^0$, then $\underline{\underline{W}}(\nu_1^1) - \underline{\underline{W}}(\nu) \leq \varepsilon_3$ and

$$2\gamma(n, k)\alpha((n : k))\xi_{\varepsilon_1}(2\varepsilon_2) \leq \varepsilon_3 .$$

We set

$$\varepsilon_2 = \min\{\varepsilon_2^0, \varepsilon_2^1, \varepsilon_2^2, \ldots, \varepsilon_2^6, \ldots, \varepsilon_2^{11}, \ldots, \varepsilon_2^{14}\} .$$

One verifies easily with ε_1, ε_2, ε_3 as defined the conclusion desired in part 3 is true. The theorem follows.

14.10. DEFINITION. For each positive integer n we define $M(n)$ to be the smallest positive integer such that whenever $p_1, p_2, \ldots, p_{M(n)+1} \in \underline{\underline{D}}^n(0, 1)$ there exist some $i, j \in \{1, 2, \ldots, M(n)+1\}$ such that $i \neq j$ and $|p_i - p_j| < 1$.

Observe that if m is a positive integer and $b_1, b_2, \ldots, b_m$ are pairwise disjoint closed balls in R^n with $\operatorname{diam}(b_i) \geq 1$ and $b_i \cap \underline{\underline{D}}^n(0, 2^{-1}) \neq \emptyset$ for each $i = 1, 2, \ldots, m$, then $m \leq M(n)$. We see this as follows. Define $p_1, p_2, \ldots, p_m \in \underline{\underline{D}}^n(0, 1)$ by setting $p_i = \operatorname{center}(b_i)$ if $|\operatorname{center}(b_i)| \leq 1$ and $p_i = |\operatorname{center}(b_i)|^{-1}\operatorname{center}(b_i)$ if $|\operatorname{center}(b_i)| > 1$ for each $i = 1, 2, \ldots, m$. Since $\underline{\underline{D}}^n(p_i, 2^{-1}) \subset b_i$ for each i, $|p_i - p_j| > 1$ for $i \neq j$, and hence $m \leq M(n)$.

14.11. LEMMA. <u>Let</u> ℓ, m, n <u>be positive integers and</u>

$$\underline{\underline{B}} = \{b(i, j) : i = 1, 2, \ldots, \ell \text{ and } j = 1, 2, \ldots, \ell m M(n)\}$$

<u>be a set of closed balls in</u> R^n <u>such that for each</u> $i = 1, 2, \ldots, \ell$, $b(i, j) \cap b(i, k) = \emptyset$ <u>whenever</u> $j \neq k$ <u>and</u>

$$b(i, j), b(i, k) \in B_i = \{b(i, j) : j = 1, 2, \ldots, \ell m M(n)\} .$$

Then there exists $\mathbf{C} \subset \mathbf{B}$ such that $b(i, j) \cap b(h, k) = \emptyset$ whenever $b(i, j), b(h, k) \in \mathbf{C}$ and such that the number of elements in $C_i = B_i \cap \mathbf{C}$ is exactly m for each $i = 1, 2, \ldots, \ell$.

PROOF. Let $\mathbf{B}$ be as above. We choose $\mathbf{C}$ as follows. Let $b(i_1, j_1)$ be some ball in $\mathbf{B}$ of smallest diameter. Then $b(i_1, j_1)$ intersects at most $M(n)$ balls in each B_i, $i \neq i_1$, which balls we discard. Let B_i^1 denote the (at least) $(\ell m-1)M(n)$ balls remaining in each B_i, $i \neq j$, and $B_{i_1}^1 = B_{i_1} - \{b(i_1, i_2)\}$. We place $b(i_1, j_1)$ in C_{i_1}.

Now let $b(i_2, j_2)$ be some ball of smallest diameter in $\bigcup_i B_i^1$. $b(i_2, j_2)$ intersects at most $M(n)$ balls in B_i^1, $i \neq i_2$, which balls we discard. Let B_i^2 denote the (at least) $(\ell m-2)M(n)$ balls remaining in each B_i, $i \neq i_2$, and $B_{i_2}^2 = B_{i_2}^1 - \{b(i_2, j_2)\}$. We place $b(i_2, j_2)$ in C_{i_2}.

We continue choosing elements of $\mathbf{C}$ in this manner until for some i, C_i contains m elements, at which time we discard the remaining elements of B_i and continue choosing the elements of $\mathbf{C}$ as above. Since for each choice of an element of $\mathbf{C}$ each B_i loses at most $M(n)$ elements, and the selection of $\mathbf{C}$ is complete after ℓm choices, it is clear that $\mathbf{C} = \bigcup_i C_i$ can be chosen as desired.

14.12. PROPOSITION. *Let* m *and* n *be positive integers and* ∇ *a subcomplex of the unit cubical complex* $\square$ *for* R^m. *For each cell* σ *of of any dimension let* $\mathbf{B}\sigma$ *be a pairwise disjoint family of* $c(m, n)$ *closed balls in* R^n. *Then there exists a function* b *which assigns to each cell* σ *of* ∇ *a ball* $b\sigma \in \mathbf{B}\sigma$ *such that* $b\sigma \cap b\tau = \emptyset$ *whenever* σ, τ, γ *are cells of* ∇, *not necessarily of the same dimension, and* σ *and* τ *are faces of* γ.

PROOF. Two m-cells σ and τ of $\square$ will be called equivalent if and only if there exists a vector $v \in R^m$, each of whose coordinates is an integer, such that $|\sigma| = |\tau| + 3v$. The 3^m equivalence classes $E_1, E_2, \ldots, E_q$ of m-cells of $\square$ have the property that any two distinct m-cells in the same equivalence class have no faces of any dimension in common. We assume without loss of generality that $\nabla = \square$. 14.11 implies that for each m-cell σ in E_1 we can choose a subfamily $B^1\sigma \subset \mathbf{B}\sigma$ of $(M(n) \cdot L(m))^{3^m-1}$ elements for each face σ of γ such that $\bigcup\{B^1\sigma : \sigma$ is a face of $\gamma\}$ is a pairwise disjoint family of n-balls in R^n. Similarly, 14.11 implies that for each m-cell γ in E_2 we can choose a subfamily $B^2\sigma \subset \underline{\underline{B}}\sigma$ of $(M(n)L(m))^{3^m-2}$ elements for each face σ of γ such that $B^2\sigma \subset B^1\sigma$ in case σ is also a face of some m-cell of E_1 and $\bigcup\{B^2\sigma : \sigma$ is a face of $\gamma\}$ is a pairwise disjoint family of n-balls in R^n. Proceeding in this manner, one chooses for each m-cell $\gamma \in E_i$, $i = 3, 4, \ldots, 3^m$ a subfamily $B^i\sigma \subset \underline{\underline{B}}\sigma$ of $(M(n)L(m))^{3^m-i}$ elements for each face σ of γ such that $B^i\sigma \subset B^j\sigma$ whenever $j < i$ and σ is also a face of some m-cell of E_j and $\bigcup\{B^i\sigma : \sigma$ is a face of $\gamma\}$ is a pairwise disjoint family of n-balls in R^n. At the end of the 3^m-th step above we will have assigned to each cell σ of $\square$ a set $B^{i(\sigma)}\sigma$ (where $i(\sigma)$ is the largest integer i for which $B^i\sigma$ as been defined) consisting of at least one closed n-ball in R^n. Let $b\sigma \in B^{i(\sigma)}\sigma$ be any element. b so defined has the desired property.

15. STATIONARY INTEGRAL VARIFOLDS ON MANIFOLDS

15.1 THEOREM. Let m and n be positive integers and $0 \le c_1 < c_2 < \infty$. Let $A \subset R^n$ be a compact Riemannian manifold of class 3 with boundary B which is isometrically imbedded as a submanifold of R^n. Let C be a compact submanifold of $A - B$ of class 3 with boundary D. Let $1 \le k \le \dim(A)$ and G be an admissible group. Let ∇ be a finite subcomplex of some cubical complex $\square$ for R^n and $\Diamond$ be a subcomplex of ∇. Let Π be a homotopy class of mappings

$$(\nabla, \Diamond) \longrightarrow (\underline{\underline{VZ}}_k(A, B \cup C; G), \underline{\underline{VZ}}_k(A, B \cup C; G) \cap \{\nu : \underline{\underline{W}}(\nu) \le c_1\})$$

with respect to $\underline{\underline{W}}$ such that $\underline{\underline{L}}(\Pi) = c_2$. Then there exists

$$\nu = (V, W; T, \partial T) \in \underline{\underline{VZ}}_k(A, B \cup C; G)$$

with $\underline{\underline{W}}(\nu) = c_2$ such that

(1) $V \in \underline{\underline{IV}}_k(A)$,

(2) $W \in \underline{\underline{IV}}_{k-1}(B \cup C)$,

(3) $\underline{\underline{S}}(R^n, A, B, C)(V, W) = 0$,

(4) $\underline{\underline{P}}^-(R^n, A)(V, W) = 0$, and

(5) $\underline{\underline{P}}^-(R^n, B, C)(W, 0) \le 1$.

If $B = D = \emptyset$, we have also

(4') $\underline{\underline{P}}(R^n, A)(V, W) = 0$, and

(5') $\underline{\underline{P}}(R^n, C)(W, 0) \le 1$.

PROOF.

Part 1. Let A, B, C, D, Π, c_1, c_2 be as above. If $k = \dim(A)$ the theorem is trivial. We assume therefore $k < \dim(A)$. Let

$$S = \{\varphi^i : (\nabla_0^i, \Diamond_0^i) \longrightarrow (\underline{\underline{VZ}}_k(A, B \cup C; G), \underline{\underline{VZ}}_k(A, B \cup C; G) \cap \{\nu : \underline{\underline{W}}(\nu) < c_1\})\}$$

be a critical sequence for Π satisfying the conclusions of 12.5. We wish to sh there exists at least one $\nu \in \underline{\underline{C}}(S)$ in class (e). By virtue of our choice of S using 12.5 there is no $\nu \in \underline{\underline{C}}(S)$ in class (o). Assume then that there is no $\nu \in \underline{\underline{C}}(S)$ in class (e). Then each $\nu \in \underline{\underline{C}}(S)$ is in some class (a), (b), (c), or (d). Under this assumption we will now construct a new sequence $*S$ for Π with $\underline{\underline{L}}(*S) < \underline{\underline{L}}(S)$. This contradiction proves the theorem.

Part 2. Under our hypothesis that each $\nu \in \underline{\underline{C}}(S)$ is in some class (a), (b), (c) or (d), 14.9 implies the existence of functions $\varepsilon_i : \underline{\underline{C}}(S) \longrightarrow R_0^+$, $i = 1, 2, 3$, and $p_j : \underline{\underline{C}}(S) \longrightarrow A$, $j = 1, 2, \ldots, c(m, n) = (L(m)M(n))^{3^m}$ such that for each $\nu \in \underline{\underline{C}}(S)$, $|p_h(\nu) - p_j(\nu)| > 10\varepsilon_1(\nu)$ for $i \neq j$, and

$$\{\nu_1^1, \nu_2^1, \ldots, \nu_{L(m)}^1\} \in$$

$$\underline{\underline{R}}(A, B \cup C; \varepsilon_3(\nu), 2\,\gamma(n,k)\alpha(n:k)\xi_{\varepsilon_1(\nu)}(\sup_{i,j} \underline{\underline{F}}[\nu_i^1, \nu_j^1]); p_h(\nu), 5\varepsilon_1(\nu))$$

for each $h = 1, 2, \ldots, c(m, n)$ whenever $\nu_1^1, \nu_2^1, \ldots, \nu_{L(m)}^1 \in \underline{\underline{VZ}}_k(A, B \cup C; G)$ with $\sup_i \underline{\underline{F}}(\nu, \nu_i^1) < \varepsilon_2(\nu)$. The compactness of $\underline{\underline{C}}(S)$ implies the existence of a positive integer z and $\nu(1), \nu(2), \ldots, \nu(z) \in \underline{\underline{C}}(S)$ such that for each $\nu \in \underline{\underline{C}}(S)$ there is some $i = 1, 2, \ldots, z$ for which $\underline{\underline{F}}(\nu, \nu(i)) < 2^{-1}\varepsilon_2(\nu(i))$. We set $\varepsilon_j = \min_i \varepsilon_j(\nu(i))$ for $j = 1, 2, 3$, and

$$\begin{aligned}\varepsilon_4 = 3^{-1}\sup\{\varepsilon :\ & \text{for each } \nu \in \underline{\underline{K}}(S) \cap \{\nu : \underline{\underline{W}}(\nu) \geq \underline{\underline{L}}(S) - 2\varepsilon\},\\ & \underline{\underline{F}}(\nu, \nu(i)) < 2^{-1}\varepsilon_2(\nu(i)) \text{ for some}\\ & i = 1, 2, \ldots, z\} > 0.\end{aligned}$$

Part 3.

(1) Choose a positive integer n_1 sufficiently large so that whenever $i \geq n$ and $\alpha \in \nabla_0^i$ either (a) $\underline{\underline{W}}(\varphi^i(\alpha)) < \underline{\underline{L}}(S) - 2\varepsilon_4$, or (b) $\underline{\underline{W}}(\varphi^i(\alpha)) \geq \underline{\underline{L}}(S) - 2\varepsilon_4$ and $\underline{\underline{F}}(\varphi^i(\alpha), \nu(j)) < \varepsilon_2(\nu(j))$ for some $j = 1, 2, \ldots, z$.

(2) Choose for each $i \geq n_1$

$$f_1^i : \nabla_0^i \cap \{\alpha : \underline{\underline{W}}(\varphi^i(\alpha)) \geq \underline{\underline{L}}(S) - 2\varepsilon_4\} \longrightarrow \{1, 2, \ldots, z\}$$

so that $\underline{\underline{F}}(\varphi^i(\alpha), \nu(i)) < \varepsilon_2(\nu(f_1^i(\alpha)))$ for each $\alpha \in \mathrm{dmn}(f_1^i)$.

(3) Choose for each $i \geq n_1$ a positive integer $n_2(i)$ such that whenever $a_1, a_2, \ldots, a_{L(m)} \in \nabla_0^i$ are (not necessarily distinct) vertices of a co m-cell with $\min_j \underline{\underline{W}}(\varphi^i(a_j)) \geq \underline{\underline{L}}(S) - 2\varepsilon_4$, then there exists a family of

$$\begin{aligned}
\varphi^i(a_1) &= \nu_1^1,\ \nu_1^2,\ \nu_1^3,\ \ldots,\ \nu_1^u \\
\varphi^i(a_2) &= \nu_2^1,\ \nu_2^2,\ \nu_2^3,\ \ldots,\ \nu_2^u \\
\varphi^i(a_3) &= \nu_3^1,\ \nu_3^2,\ \nu_3^3,\ \ldots,\ \nu_3^u \\
&\vdots \\
\varphi^i(a_{L(m)}) &= \nu_{L(m)}^1,\ \nu_{L(m)}^2,\ \ldots,\ \nu_{L(m)}^u
\end{aligned}$$

of elements of $\underline{\underline{VZ}}_k(A, B \cup C; G)$ as in 14.5, with $u \leq 2^{n_2(i)}$, by virtu

$$\{\varphi^i(a_1), \varphi^i(a_2), \ldots, \varphi^i(a_{L(m)})\} \in$$

$$\underline{\underline{R}}(A, B \cup C; \varepsilon_3(\nu(f_1^i(a_1))), 2\gamma(n, k)a(n:k)\cdot \xi_{\varepsilon_1(\nu(f_1^i(a_1)))}(\sup_{h,j} \underline{\underline{F}}(\varphi^i(a_h), p_\ell(\nu(f_1^i(a_1))), 5\varepsilon_i(\nu(f_1^i(a_1))))$$

for each $\ell = 1, 2, \ldots, c(m, n)$.

(4) Choose a positive integer $n_3 \geq n_1$ such that if $i \geq n_3$ then

$$\underline{\underline{F}}(\varphi^i(\alpha), \varphi^i(\beta)) \leq \inf\{\underline{\underline{F}}(\nu, \nu') : \nu, \nu' \in \underline{\underline{VZ}}_k(A, B \cup C; G) \text{ and } \underline{\underline{W}}(\nu) - \underline{\underline{W}}(\nu') \geq 3^{-1}\varepsilon_4\} > 0$$

whenever $\alpha, \beta \in \nabla^i$ are vertices of some common m-cell in $\square$.

(5) Choose for each $i \geq n_3$ a positive integer $n_4(i)$ sufficiently large s choice of f_3^i below is possible. First for each $i \geq n_3$ denote by ${}_1\nabla^i$ tl $\#$ subdivision of ∇^i considered as a subcomplex of the $n_4(i)$-th $\#$ ${}_1\square^i$ of $\square^i$. (Here ∇^i is a subcomplex of a cubical complex $\square^i$ fo: For each $i \geq n_3$ define

$$f_2^i : {}_1\nabla_0^i \longrightarrow \nabla_0^i$$

by setting $f_2^i(a)$ equal to the nearest vertex in ∇_0^i to a for each $a \in$. Now define for each $i \geq n_3$

$$_1\varphi^i : {}_1\nabla_0^i \longrightarrow \underline{\underline{VZ}}_k(A, B \cup C; G),$$
$$_1\varphi^i = \varphi^i \circ f_2^i.$$

ally choose

$$f_3^i : {}_1\nabla_0^i \longrightarrow \{0, 1, 2, \ldots, 2^{2^{n_2(i)}}\}$$

r each $i \geq n_3$ to have the following properties:

(a) $f_3^i(\alpha) = 2^{2^{n_2(i)}}$ whenever $\alpha \in {}_1\nabla_0^i$ and $\underline{\underline{W}}({}_1\varphi^i(\alpha)) \geq \underline{\underline{L}}(S) - \varepsilon_4$;

(b) $f_3^i(\alpha) = 0$ whenever $\alpha \in {}_1\nabla_0^i$ and $\underline{\underline{W}}({}_1\varphi^i(\alpha)) < \underline{\underline{L}}(S) - 2\varepsilon_4$; and

(c) $f_3^i(\alpha) - f_3^i(\beta) \leq 1$ whenever $\alpha, \beta \in {}_1\nabla_0^i$ are vertices of some common n-cell in ${}_1\square$.

(6) For each $i \geq n_3$ let ${}_2\nabla^i$ be the $n_2(i)$-th # subdivision of ${}_1\nabla^i$, considered as a subcomplex of the $n_2(i)$-th # subdivision ${}_2\square^i$ of ${}_1\square^i$ and define

$$f_4^i : {}_2\nabla_0^i \longrightarrow {}_1\nabla_0^i$$

by setting $f_4^i(\alpha)$ equal to the nearest vertex in ${}_1\nabla_0^i$ to α for each $\alpha \in {}_2\nabla_0^i$.

(7) For each $i \geq n_3$ choose

$$f_5^i : \bigcup_j {}_1\nabla_j^i \longrightarrow {}_1\nabla_0^i$$

such that for each cell $\tau \in {}_1\nabla^i$, $f_5^i(\tau)$ is a vertex of τ.

(8) For each $i \geq n_3$ choose

$$f_6^i : {}_1\nabla^i \cap \{\tau : \tau \text{ is a cell and } f_3^i(\alpha) > 0 \text{ for each vertex } \alpha \text{ of } \tau\} \longrightarrow A \times \mathbb{R}_{..}$$

to have the following properties:

(a) $f_6^i(\tau) \in \{(p_j(\nu(f_1^i \circ f_2^i \circ f_5^i(\tau))], 5\varepsilon_i[\nu(f_1^i \circ f_2^i \circ f_5^i(\tau))]) : j = 1, 2, \ldots, c(m,$

for each cell τ of ${}_1\nabla^i$, and

(b) $\underline{\underline{D}}^n(f_6^i(\sigma)) \cap \underline{\underline{D}}^n(f_6^i(\tau)) = \emptyset$ whenever $\sigma \neq \tau$ are faces of a common c... in ${}_1\square^i$.

14.12 implies the possibility of choosing f_6^i. We observe

$$0 < \inf\{\operatorname{dist}(\underline{\underline{D}}^n(f_6^i(\sigma)),\ \underline{\underline{D}}^n(f_6^i(\tau))) : i = n_3,\ n_3+1,\ n_3+2,\ \ldots \text{ and } \sigma, \tau \in \operatorname{dmn}(f_6^i) \text{ are distinct faces of some common } m\text{-cell in } {}_1\square^i\}$$

because there are only finitely many choices involved in selecting $\{\nu(j)\}_j$, $\{p_h(\nu(j))\}_{h,j}$, and $\{\varepsilon_h(\nu(j))\}_{h,j}$; these choices, in particular, do not depend on i.

(9) For each $i \geq n_3$ choose

$$f_7^i : [(\bigcup_j {}_1\nabla_j^i \times {}_1\nabla_0^i) \cap \{(\tau, a) : a \text{ is a vertex of } \tau\}] \times \{0, 1, 2, \ldots, 2^{n_2(i)}\} \longrightarrow \underline{\underline{VZ}}_k(A, B\cup C; G)$$

as follows: Let τ be a cell in ${}_1\nabla^i$ with $f_3^i \circ f_5^i(\tau) > 0$. For some $u \leq 2^{n_2(i)}$, (3) guarantees the existence of a sequence

$${}_1\varphi^i(a) = \nu_a^1,\ \nu_a^2,\ \nu_a^3,\ \ldots,\ \nu_a^u \in \underline{\underline{VZ}}_k(A, B\cup C; G)$$

for each $a \in {}_1\nabla_0^i$ which is a face of τ, constructed by 14.9 utilizing the fact that

$$\underline{\underline{F}}({}_1\varphi^i \circ f_5^i(\tau),\ \nu(f_1^i \circ f_2^i \circ f_5^i(\tau))) < \varepsilon_2(\nu(f_1^i \circ f_2^i \circ f_5^i(\tau))),$$

and by virtue of which

$$\{{}_1\varphi^i(a) : a \in {}_1\nabla_0^i \text{ is a vertex of } \tau\} \in$$

$$\underline{\underline{R}}(A, B\cup C; \varepsilon_3, 2\gamma(n, k)a(n:k)\xi_{\varepsilon_1}(\delta^i); f_6^i(\tau)).$$

Here δ^i is the fineness of φ^i. If then $a \in {}_1\nabla_0^i$ is a vertex of τ we set

$$f_7^i(\tau, a, 0) = {}_1\varphi^i(a);$$

$$f_7^i(\tau, a, j) = \nu_a^j \text{ for } 1 \leq j \leq \min\{u, f_3^i(a)\}; \text{ and}$$

$$f_7^i(\tau, a, j) = \nu_a^{\min\{u, f_3^i(a)\}} \text{ for}$$

$\min\{u, f_3^i(a)\} \leq j \leq 2^{n_2(i)}$; provided $f_3^i(a) > 0$. If $f_3^i(a) = 0$ we set $f_7^i(\tau, a, j)$ ·: for each $0 \leq j \leq 2^{n_2(i)}$.

(10) For each $i \geq n_3$ define

$$f_8^i : {}_2\nabla_0^i \longrightarrow {}_1\nabla^i$$

by setting for each $a \in {}_2\nabla^i_0$, $f^i_8(a)$ to be that unique cell $\tau \in {}_1\nabla^i$ of highest dimension such that a is a vertex of some $\#$ subdivision of τ but a is not a vertex of any $\#$ subdivision of any proper face of τ.

(11) For each $i \geq n_3$ define

$$f^i_9 : ({}_2\nabla^i_0 \times {}_1\nabla^i) \cap \{(a, \tau) : \tau \text{ is a (not necessarily proper) face of } f^i_8(a)\} \longrightarrow \{0, 1, 2, \ldots, 2^{n_2(i)}\},$$

$$f^i_9 = \mathcal{C} \,|\, \mathrm{dmn}(f^i_9) \qquad (12.1\ (12)).$$

(12) For each $i \geq n_3$ let ${}_2\mathcal{I}^i$ be the cell complex of the interval $\{t : 0 \leq t \leq 2^{n_2(i)}\}$ given by

$${}_1\mathcal{I}^i_1 = \{[0, 1], [1, 2], [2, 3], \ldots, [2^{n_2(i)} - 1, 2^{n_2(i)}]\},$$

$${}_1\mathcal{I}^i_0 = \{[0], [1], [2], \ldots, [2^{n_2(i)}]\}.$$

(13) For each $i \geq n_3$ define

$${}_2\psi^i : [{}_2\mathcal{I}^i \otimes {}_2\nabla^i]_0 \longrightarrow \underline{\underline{VZ}}_k(A, B \cup C; G)$$

as follows: If $a \in {}_2\nabla^i_0$ and $f^i_3 \circ f^i_4(a) = 0$ we set ${}_2\psi^i([j] \otimes a) = {}_1\varphi^i \circ f^i_4(a)$ for each $j = 0, 1, 2, \ldots, 2^{n_2(i)}$.

If $a \in {}_2\nabla^i_0$ and $f^i_3 \circ f^i_4(a) > 0$ we set

$$\begin{aligned}
{}_2\psi^i([j] \otimes a) \cap \underline{\underline{D}}^n(f^i_6(\tau)) &= {}_1\varphi^i \circ f^i_4(a) \cap \underline{\underline{D}}^n(f^i_6(\tau)) \quad \text{for } j = 0, 1 \\
&= f^i_7(\tau, f^i_4(a), j) \cap \underline{\underline{D}}^n(f^i_6(\tau)) \quad \text{for } 1 \leq j \leq \min\{f^i_3 \circ f^i_4(\tau), f^i_9(a, \tau)\}, \\
&= f^i_7(\tau, f^i_4(a), \min\{f^i_3 \circ f^i_4(\tau), f^i_9(a, \tau)\}) \cap \underline{\underline{D}}^n(f^i_6(\tau)) \\
&\qquad \text{for } \min\{f^i_3 \circ f^i_4(a), f^i_9(a, \tau)\} \leq j \leq 2^{n_2(i)}
\end{aligned}$$

whenever τ is a (not necessarily proper) face of $f^i_8(a)$; and

$$\begin{aligned}
&{}_2\psi^i([j] \otimes a) \cap (R^n - \bigcup\{\underline{\underline{D}}^n(f^i_6(\tau)) : \tau \text{ is a (not necessarily proper) face of } f^i_8(a)\}) \\
&= {}_1\varphi^i \circ f^i_4(a) \cap (R^n - \bigcup\{\underline{\underline{D}}^n(f^i_6(\tau)) : \tau \text{ is a (not necessarily proper) face of } f^i_8(a)\})
\end{aligned}$$

for $0 \le j \le 2^{n_2(i)}$. The intersections above have the obvious meaning.

(14) We define for $i = 1, 2, 3, \ldots, n_3-1$, $*\varphi^i = \varphi^i$, $*\nabla^i = \nabla^i$, and $*\Diamond^i = \Diamond^i$. For $i = n_3, n_3+1, n_3+2, \ldots$ we define $*\nabla^i = {}_2\nabla^i$, $*\Diamond^i = {}_2\Diamond^i$, and

$$*\varphi^i : (*\nabla_0^i, *\Diamond_0^i) \longrightarrow (\underline{\underline{VZ}}_k(A, B\cup C; G), \underline{\underline{VZ}}_k(A, B\cup C; G)\cap\{\nu : \underline{\underline{W}}(\nu) \le c_1\})$$

by setting for each $\alpha \in *\nabla_0^i$,

$$*\varphi^i(\alpha) = {}_2\psi^i([2^{n_2(i)}] \otimes \alpha).$$

Part 4. It is straightforward to verify that $*S = \{*\varphi^i\}_i \in \Pi$ using the homotopies $\{{}_2\psi^i\}_i$. Furthermore

$$\underline{\underline{L}}(*S) \le \max\{c_1, \underline{\underline{L}}(S) - \varepsilon_3\} < \underline{\underline{L}}(\Pi).$$

This is the contradiction desired in part 1, and the theorem is proved.

15.2 COROLLARY. *Let* A, B, C, D, k *be as in* 15.1. *Then there exists at least one*

$$\nu = (V, W, T, \partial T) \in \underline{\underline{VZ}}_k(A, B\cup C; G) - \{0\}$$

satisfying the conclusions of 15.1. *In particular, whenever* $1 \le k \le m$ *are integers, each compact* m-*dimensional Riemannian manifold* M *of class* 3 *supports at least one non-zero* k-*dimensional stationary integral varifold pair* $(V, W) \in \underline{\underline{IV}}_k(M, \partial M)$. *Furthermore* W *will be* (k-1)-*dimensional regular integral varifold.*

15.3 REMARK. Note that any compact Riemannian manifold M of class 3 can occur as a stationary integral varifold in some manifold. For example $(|M \times \{p\}|, |\partial M \times \{p\}|)$ is a stationary integral varifold pair in $M \times S^1$ whenever $p \in S^1$.

15.4 REMARK. One might conjecture that the proof of 15.1 could be simplified by requiring

$$\bigcup_i \operatorname{range}(\varphi^i) \subset \underline{\underline{VZ}}_k(A, B\cup C; G)\cap\{(V, W; T, \partial T) : \underline{\underline{P}}(R^n, A, B, C)(V, W) \le c\}$$

for some $c \in R_0^+$ and using the compactness theorem 10.8. The difficulty in such an approach is illustrated in the following example by the fact that varifolds (curves in this case) lying near the desired stationary varifold have large $\underline{\underline{P}}$ valve (curvature).

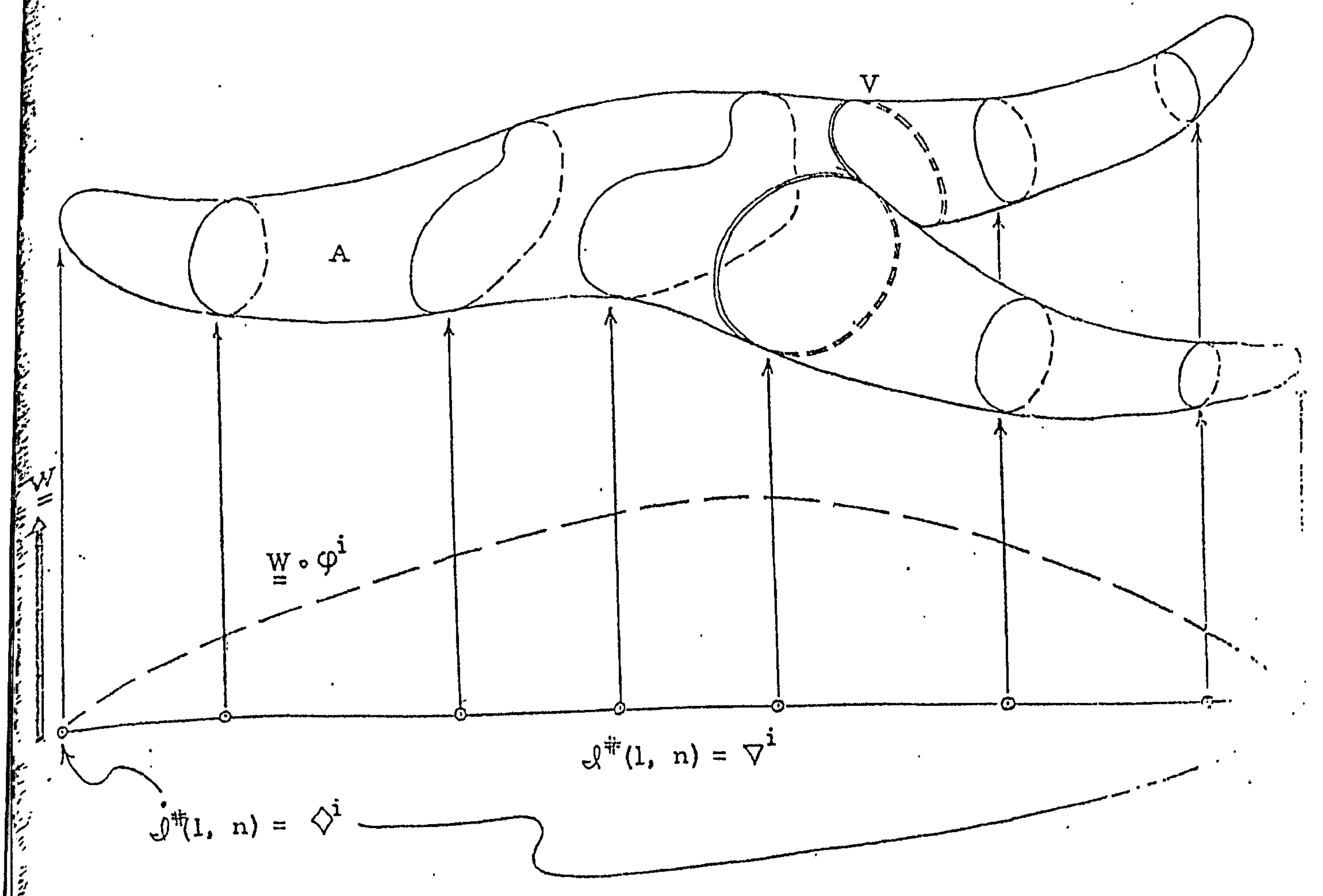
Example. Let A be the Riemannian manifold illustrated (A is diffeomorphic with S^2 and isometrically like the skin of a three-legged starfish). Let G be the integers. We have illustrated a mapping $\varphi^i \in S$ where S is a homotopy sequence of mappings

$$(\mathcal{I}^{\#}(1),\ \dot{\mathcal{I}}^{\#}(1)) \longrightarrow (\underline{\underline{V}}\underline{\underline{Z}}_1(A, \emptyset; G), \{0\})$$

with respect to $\underline{\underline{W}}$ representing the homotopy class Π corresponding to a generator of

$$H_2(A; G) \cong \Pi_1(\underline{\underline{Z}}_1(A, \emptyset; G), \{0\}) \subseteq \Pi_1^{\#}(\underline{\underline{V}}\underline{\underline{Z}}_1(A, \emptyset; G), \{0\}).$$

Also $\underline{\underline{L}}(S) = \underline{\underline{L}}(\Pi)$.



The vertical arrows represent the mapping φ^i. Observe that the 1-dimensional stationary integral varifold V occurs at the maximum of the function $\underline{\underline{W}} \circ \varphi^i$.

15.5 REMARK. The topological invariants used in 15.1 are (appropriately defined) homotopy groups. A similar theorem is true based on (appropriately defined) homology groups, and [A1] and [M] guarantee the, at least partial, computability of these groups. The chief utility of the homology approach would lie in the attempt to assign a topological index to stationary integral varifolds in some analytically useful way. The example of 15.4 illustrates the difficulty in this attemp The stationary curve V should have a topological index of 1. This curve, however, is stable (6.21) and would imply an analytic index of 0 unless there is some way of computing part of the analytic index from the singularity of V.

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