COMPLETIONS OF PERIOD MAPPINGS: PROGRESS REPORT

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Abstract. We give an informal, expository account of a project to construct completions of period maps.

1. Introduction

The purpose of this this paper is to give an expository overview, with examples to illustrate some of the main points, of recent work [GGR21b] to construct "maximal" completions of period mappings. This work is part of an ongoing project, including [GGLR20, GGR21a], to study the global properties of period mappings at infinity.

1.1. **Completions of period mappings.** We consider triples $(\overline{B}, Z; \Phi)$ consisting of a smooth projective variety $\overline{B}$ and a reduced normal crossing divisor Z whose complement

$$B \;=\; \overline{B} \backslash Z$$

has a variation of (pure) polarized Hodge structure

$$\begin{array}{c} \mathcal{F}^p \subset \mathcal{V} = \tilde{B} \times_{\pi_1(B)} V \\ \downarrow \\ B \end{array} \tag{1.1a}$$

inducing a period map

$$\Phi : B \;\to\; \Gamma \backslash D\,. \tag{1.1b}$$

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Here D is a period domain parameterizing pure, weight n, Q–polarized Hodge structures on the vector space V, and $\pi_1(B) \twoheadrightarrow \Gamma \subset \mathrm{Aut}(V,Q)$ is the monodromy representation. Without loss of generality, $\Phi : B \to \Gamma\backslash D$ is proper [Gri70a]. Let

$$\wp \ = \ \Phi(B)$$

denote the image. The goal is to construct *both* a projective completion $\overline{\wp}$ of $\wp$ *and* a surjective extension $\Phi^{\mathsf{e}} : \overline{B} \to \overline{\wp}$ of the period map. We propose two such completions

$$\begin{tikzcd} \overline{B} \arrow[r, "\Phi^{\mathsf{T}}"] \arrow[dr, "\Phi^{\mathsf{S}}"'] & \overline{\wp}^{\mathsf{T}} \arrow[d] \\ & \overline{\wp}^{\mathsf{S}} . \end{tikzcd} \tag{1.2}$$

The completion $\Phi^{\mathsf{T}} : \overline{B} \to \overline{\wp}^{\mathsf{T}}$ is maximal, in the sense that it encodes all the Hodge-theoretic information associated with the triple $(\overline{B}, Z; \Phi)$. The second completion $\Phi^{\mathsf{S}} : \overline{B} \to \overline{\wp}^{\mathsf{S}}$ is a quotient encoding the minimal amount of Hodge-theoretically meaningful data. The nilpotent orbit theorem [Sch73] indicates how one might try to do this, at least set-theoretically (§2.1): boundary points of $\overline{\wp}^{\mathsf{T}}$ should parameterize (equivalence classes of) nilpotent orbits (or limiting mixed Hodge structures) [CCK80, Cat84, Hof84, KU09, KP16], and points of $\overline{\wp}^{\mathsf{S}}$ should parameterize (equivalence classes of) Hodge structures on associated "weight-graded quotients" [Gri70b, CK77]. It is conjectured (and proven in a few special cases) that the spaces $\overline{\wp}^{\mathsf{T}}$ and $\overline{\wp}^{\mathsf{S}}$ are algebraic, and that the maps in (1.2) are morphisms [GGLR20, GGR21b].

Remark 1.3. In the classical case that D is Hermitian and Γ is arithmetic (which includes period mappings for curves and principally polarized abelian varieties, and K3 surfaces), Borel's theorem yields an extension $\Phi^{\mathsf{S}} : \overline{B} \to \overline{\Gamma\backslash D}^{\mathsf{S}}$ of the period map $\Phi : B \to \Gamma\backslash D$ to the Satake–Baily–Borel compactification, and we may take $\overline{\wp}^{\mathsf{S}} = \Phi^{\mathsf{S}}(\overline{B})$, [BB66, Bor72]. In particular, the conjectured algebraic structure holds for $\Phi^{\mathsf{S}} : \overline{B} \to \overline{\wp}^{\mathsf{S}}$. The maximal completion $\overline{\wp}^{\mathsf{T}}$ is what one would expect to get if one took the closure of $\wp$ inside a toroidal desingularization $\overline{\Gamma\backslash D}^{\mathsf{T}} \twoheadrightarrow \overline{\Gamma\backslash D}^{\mathsf{S}}$, [Mum75, AMRT75, CCK80, Cat84, Hof84].

Remark 1.4. Consider $B = \overline{\mathcal{M}}_g$ the Deligne–Mumford compactification of the moduli space $\mathcal{M}_g$ of smooth complete curves of genus g [DM69]. In this setting, our Φ^{T} is closely related to the Torelli map studied by Namikawa, Mumford and others [Nam76a, Nam76b].

Remark 1.5. In general $\overline{\wp}^{\mathsf{T}}$ has the flavor of what one would expect to obtain by taking the closure of $\wp$ in a Kato–Usui horizontal completion $\overline{\Gamma\backslash D}^{\Sigma}$ when Γ is arithmetic [KU09]. We will *not* need to work work with fans. Our construction is *relative*, in the sense that it depends on choice of triple $(\overline{B}, Z; \Phi)$ and the pair $(\overline{B}, Z)$ provides the boundary structure. We are not constructing a compactification, or horizontal completion, of $\Gamma\backslash D$.

Evidence for the conjectural algebraic structure on $\overline{\wp}^{\mathsf{T}}$ includes Theorem 1.7. Let

$$\begin{tikzcd} B \arrow[r, "\hat{\Phi}"'] \arrow[rr, bend left, "\Phi"] & \hat{\wp} \arrow[r] & \wp \end{tikzcd} \tag{1.6}$$

be the Stein factorization of the period map (1.1b); the fibres of $\hat{\Phi}$ are connected, the fibres of $\hat{\wp} \to \wp$ are finite, and $\hat{\wp}$ is a normal complex analytic space.

Theorem 1.7 ([GGR21b])**.** *Assume Γ is neat. The complex analytic variety $\hat{\wp}$ is Zariski open in a compact, normal Moishezon variety $\hat{\wp}^{\mathsf{T}}$, the map $\hat{\Phi} : B \to \hat{\wp}$ extends to a morphism $\hat{\Phi}^{\mathsf{T}} : \overline{B} \to \hat{\wp}^{\mathsf{T}}$ of algebraic spaces, and there is a map $\hat{\wp}^{\mathsf{T}} \to \overline{\wp}^{\mathsf{T}}$ with finite fibres so that the diagram*

$$\begin{tikzcd} \overline{B} \arrow[r, "\hat{\Phi}^{\mathsf{T}}"] \arrow[dr, "\Phi^{\mathsf{T}}"'] & \hat{\wp}^{\mathsf{T}} \arrow[d, "finite"] \\ & \overline{\wp}^{\mathsf{T}} \end{tikzcd} \tag{1.8}$$

commutes.

Remark 1.9. Let $\mathcal{F}^p_{\mathrm{e}} \subset \mathcal{V}_{\mathrm{e}}$ denote Deligne's extension of the Hodge vector bundles (1.1a) to $\overline{B}$. The restriction of the (*extended, augmented*) *Hodge line bundle*

$$\Lambda_{\mathrm{e}} \ = \ \det(\mathcal{F}^n_{\mathrm{e}}) \otimes \det(\mathcal{F}^{n-1}_{\mathrm{e}}) \otimes \cdots \otimes \det(\mathcal{F}^{\lceil (n+1)/2 \rceil}_{\mathrm{e}}) \tag{1.10}$$

to B is semi-ample, and realizes the image $\wp$ as a quasi-projective variety [BBT18]. This immediately gives a projective completion $\overline{\wp}$. However, this falls short of what we want as it is not known what Hodge-theoretic information is encoded in the boundary $\overline{\wp} \backslash \wp$, or whether there is an extension $\Phi^{\mathsf{e}} : \overline{B} \to \overline{\wp}$ (both of which are important for applications). However, if one could show that Λ_{e} is semi-ample over $\overline{B}$, then

$$\overline{\wp}^{\mathsf{S}} \ = \ \mathrm{Proj} \oplus_k \ H^0(\overline{B}, \Lambda_{\mathrm{e}}^{\otimes k})\,.$$

This naturally raises the question of whether or not the Base Point Free Theorem can be applied to show that Λ_{e} is semi-ample; unfortunately, this does not seem to be the case, cf. Example 1.11.

Example 1.11. If $\dim \overline{B} = 2$, then Λ_{e} is semi-ample [GGLR20]. Let's try to prove this using the Base Point Free Theorem. For convenience of exposition, assume that the cone $\mathrm{Eff}^1(\overline{B})$ of effective algebraic 2-cycles is finitely generated. We need to show that $m\Lambda_{\mathrm{e}} - K_{\overline{B}}$ is nef. Taking $m \gg 0$ this is equivalent to

$$-K_{\overline{B}} \cdot Z_i \ \geq 0\,. \tag{1.12}$$

Let Z_i denote the irreducible components of $Z = \cup Z_i$. Let g_i be the genus of the curve Z_i. Then $K_{Z_i} = (K_{\overline{B}} + [Z_i])|_{Z_i}$ implies that

$$-K_{\overline{B}} \cdot Z_i \ = \ -\deg K_{Z_i} + Z_i^2 \ = \ 2 - 2g_i + Z_i^2\,.$$

So (1.12) holds if and only if

$$2g_i \ \leq \ Z_i^2 + 2\,. \tag{1.13}$$

Suppose that $\Phi^0(Z)$ is a point, then $Z_i^2 < 0$ [GGLR20]. Then (1.13) holds if and only if $g_i = 0$ and $-2 \leq Z_i^2$. However, there are examples in which this fails. One is given by taking $\overline{B}$ to be Mok's "Mumford compactification" $\overline{X}_M$ of the ball quotient $X = B^2/\Gamma$ [Mok12]. Then Φ^0 is precisely Mok's map $\overline{X}_M \to \overline{X}_{\min}$, and the connected components of $Z = \overline{X}_M \backslash X$ are complex tori that are collapsed to points.

1.2. **Boundary points and fibres.** The completions $\overline{\wp}^{\mathsf{T}} \to \overline{\wp}^{\mathsf{S}}$ will be described in greater detail in §2.1. Here we give a brief overview of the boundary points, and the fibres

$$\begin{tikzcd} \mathcal{F} \arrow[r, hook] & \overline{\wp}^{\mathsf{T}} \arrow[d] \\ & \overline{\wp}^{\mathsf{S}} \end{tikzcd} \tag{1.14}$$

Points of $\hat{\wp}^{\mathsf{T}}$ parameterize equivalence classes of limiting mixed Hodge structures (§2.1.2). The image of one such equivalence class $[W, F, \sigma]^{\mathsf{T}}$ under the map $\overline{\wp}^{\mathsf{T}} \to \overline{\wp}^{\mathsf{S}}$ is an equivalence class $[\mathrm{Gr}^W, F, \sigma]^{\mathsf{S}}$ of Hodge structures $F^p(\mathrm{Gr}^W_\ell)$ on the graded quotients $\mathrm{Gr}^W_\ell = W_\ell/W_{\ell-1}$. Here elements of the cone σ determine subspaces of Gr^W_ℓ which admit induced polarized Hodge structures (§2.1.3). The fibres $\mathcal{F}$ parameterize limiting mixed Hodge structures with the same associated graded; equivalently they parameterize extension data (§§3.2, 3.4).

1.2.1. *The classical case: Hermitian symmetric period domains.* Suppose that Γ is neat. Let $\overline{\Gamma\backslash D}^{\mathsf{T}} \to \overline{\Gamma\backslash D}^{\mathsf{S}}$ be a toroidal desingularization of the Satake–Baily–Borel compactification. It is well-known [CCK80, Hof84] that points of $\overline{\Gamma\backslash D}^{\mathsf{T}}$ parameterize equivalence classes $[W, F, \sigma]^{\mathsf{T}}$ of limiting mixed Hodge structures, and that the map $\overline{\Gamma\backslash D}^{\mathsf{T}} \to \overline{\Gamma\backslash D}^{\mathsf{S}}$ sends $[W, F, \sigma]^{\mathsf{T}} \mapsto [\mathrm{Gr}^W, F, \sigma]^{\mathsf{S}}$. If we fix a point $[\mathrm{Gr}^W, F, \sigma]^{\mathsf{S}} \in \overline{\Gamma\backslash D}^{\mathsf{S}}$, the fibre $\mathcal{F} \subset \overline{\Gamma\backslash D}^{\mathsf{T}}$ over the point has the structure of a semi-abelian variety

$$1 \;\to\; (\mathbb{C}^*)^k \;\to\; \mathcal{F} \;\to\; \mathcal{J} \;\to\; 1\,.$$

In particular, the fibre $\mathcal{F}$ admits a fibration $\mathcal{F} \to \mathcal{J}$ over an abelian variety. A somewhat similar, but richer, structure holds in general.

1.2.2. *The general case.* Returning to the general case (1.14), we have $\mathcal{F} = \Phi^{\mathsf{T}}(A)$, where $A \subset \overline{B}$ is a Φ^{S}–fibre. The weight filtration induces an iterated fibration

$$\mathcal{F} = \mathcal{F}^{2n} \to \mathcal{F}^{2n-1} \to \cdots \to \mathcal{F}^{2} \to \mathcal{F}^{1} \tag{1.15}$$

of the fibre $\mathcal{F}$. In the classical case (§1.2.1), we have $n = 1$ and

$$\begin{array}{ccc} \mathcal{F}^2 & \longrightarrow & \mathcal{F}^1 \\ \| & & \| \\ \mathcal{F} & \longrightarrow & \mathcal{J}\,. \end{array}$$

In general, the $\mathcal{F}^1$ are not abelian varieties, or even complex tori. However, the connected components of $\mathcal{F}^1$ are subvarieties of compact complex tori $\mathbb{T}$, and the irreducible components of $\mathcal{F}^1$ are subvarieties of abelian varieties $\mathcal{J} \subset \mathbb{T}$ (§3.5). The tori (abelian varieties) parameterize level one extension data of (limiting) mixed Hodge structures (Remark 3.8). The continuous data in the fibre $\mathcal{F}^2 \to \mathcal{F}^1$ (which parameterizes level two extension data) are encoded by canonical sections s_M of a family of line bundles $\{L_M\}$. The line bundles relate the geometry of the fibre A to the normal bundles $\mathcal{N}_{Z_i/\overline{B}}$, (3.23). The sections s_M essentially capture the nilpotent orbits approximating the period map along A (§3.5.3). The map $\mathcal{F} \to \mathcal{F}^2$ is finite (Remark 3.24). Both the finiteness of this map, and the fact that the continuous data in $\mathcal{F}^2 \to \mathcal{F}^1$ is given by the sections s_M are consequences of the infinitesimal period relation, the compactness of A, and the structure of the extension data of a mixed Hodge structure.

1.3. **Contents.** Theorem 1.7 is discussed in §2. Two key ingredients here are period matrix representations of Φ and extension data of limiting mixed Hodge structures; these are discussed in §3. Finally, in the spirit of [Car87], we discuss geometric interpretations of the extension data in limiting mixed Hodge structures, and their relationship to the period matrix representations in §§4–6.

Remark 1.16. In general the monodromy $\gamma_i = \gamma_{i,\mathsf{s}} \exp(N_i)$ about Z_i^* is quasi-unipotent: here $\gamma_{i,\mathsf{s}}^{m_i} = \mathbf{1}$ for some $m_i \geq 1$, N_i is nilpotent, and $\gamma_{i,\mathsf{s}} \exp(N_i) = \exp(N_i)\gamma_{i,\mathsf{s}}$. After a base change, the monodromy $\gamma_i = \exp(N_i)$ becomes unipotent, with an action of the semisimple $\gamma_{i,\mathsf{s}}$ on the limiting mixed Hodge structures. We will not consider the action of the semisimple factor $\gamma_{i,\mathsf{s}}$ and so, for the purposes of this paper, will assume $\gamma_i = \exp(N_i)$. We do note that $\gamma_{i,\mathsf{s}}$ is of significant geometric interest; the assumption is made here primarily for the purpose of exposition.

Contents

2. Construction of maximally extended period mapping

2.1. **Set-theoretic construction.** We begin our discussion of the extensions (1.2) with an informal description highlighting the underlying geometric ideas.

2.1.1. Let Z_i denote the irreducible components of

$$Z \;=\; Z_1 \cup \cdots \cup Z_\nu\,.$$

Define

$$Z_I \;=\; \bigcup_{i\in I} Z_i\,.$$

Then $Z_J \subset Z_I$ if and only if $I \subset J$. Set

$$Z_I^* \;=\; Z_I \backslash Z_I'\,, \qquad Z_I' \;=\; \bigcup_{J \supsetneq I} Z_J\,.$$

Given point $b \in Z_I^*$ we assume that the local monodromy about Z_i is unipotent, and let N_i denote the nilpotent logarithm of monodromy, and

$$\sigma_I \;=\; \mathrm{span}_{\mathbb{R}_{>0}}\{N_i \mid i \in I\}$$

the local monodromy cone. In general, σ_I depends on a local lift of the period map, and so is defined only up to the action of Γ.

2.1.2. The nilpotent orbit theorem [Sch73] associates to any point $b \in Z_I^*$ an equivalence class of *limiting mixed Hodge structures* (LMHS) (W, F, σ_I). The weight filtration $W = W(N)$ is independent of $N \in \sigma_I$. The Hodge filtration $F \in \check{D}$ satisfies the first Hodge–Riemann bilinear relation, but not necessarily the second. It also depends on the choice of local coordinates at b. The nilpotent orbit $\exp(\mathbb{C}\sigma_I) \cdot F$ is independent of the choice. Two such LMHS (W, F, σ_I) and (W', F', σ_I') are *equivalent* if and only if there exists $\gamma \in \Gamma$ such that $\sigma_I' = \mathrm{Ad}_\gamma \sigma_I$ (which implies $W' = \gamma W$) and $F' \in \gamma \exp(\mathbb{C}\sigma_I)F$. Let

$$\Phi^{\mathsf{T}}(b) \;=\; [W, F, \sigma_I]^{\mathsf{T}} \tag{2.1}$$

be the associated equivalence class.

Definition 2.2. The completion $\overline{\wp}^{\mathsf{T}}$ is the set of equivalence classes of LMHS associated to $(\overline{B}, Z; \Phi)$ and $\Phi^{\mathsf{T}} : \overline{B} \to \overline{\wp}^{\mathsf{T}}$ is the map $b \mapsto [W, F, \sigma_I]^{\mathsf{T}}$.

Remark 2.3. At least at the set theoretic level it is clear that Φ^{T} retains the maximal amount of Hodge theoretic data in the triple $(\overline{B}, Z; \Phi)$.

Conjecture 2.4. *The set $\overline{\wp}^{\mathsf{T}}$ is a projective algebraic variety and Φ^{T} is a morphism.*

As evidence for the conjecture we have Theorem 1.7, which will be discussed in §2.2.

2.1.3. Given a limiting mixed Hodge structure, (W, F, σ_I), the Hodge filtration F determines a pure Hodge structure of weight ℓ on

$$\mathrm{Gr}_\ell^W \;=\; W_\ell / W_{\ell-1}\,.$$

Any other $F' \in \exp(\mathbb{C}\sigma_I)F$ determines the same Hodge filtration $\mathcal{F}^p(\mathrm{Gr}_\ell^W)$. Let $f_\ell^p = \dim_\mathbb{C} \mathcal{F}^p(\mathrm{Gr}_\ell^W)$ denote the Hodge numbers.

Given $N \in \sigma_I$, the subspace

$$\mathrm{Prim}_{n+k}^N \;=\; \ker\{N^{k+1} : \mathrm{Gr}_{n+k}^W \to \mathrm{Gr}_{n-k-2}^W\}$$

inherits the Hodge structure, and the later is polarized by $Q(\cdot, N^k\cdot)$.

Given σ_I, the weight filtration $W = W(\sigma_I)$ is uniquely determined, and we consider the triples $(\mathrm{Gr}_\bullet^W, \mathcal{F}, \sigma_I)$, with the Hodge filtration $\mathcal{F}(\mathrm{Gr}_\ell^W)$ having the same Hodge numbers f_ℓ^p, and the properties:

(i) Every $N \in \sigma_I$ maps $\mathcal{F}^p(\mathrm{Gr}_\ell^W) \to \mathcal{F}^{p-1}(\mathrm{Gr}_{\ell-2}^W)$.

(ii) For every $N \in \sigma_I$, the induced Hodge structure $\mathcal{F}(\mathrm{Prim}_{n+k}^N)$ is polarized by $Q(\cdot, N^k\cdot)$.

We say two such $(\mathrm{Gr}_\ell^W, \mathcal{F}, \sigma_I)$ and $(\mathrm{Gr}_\ell^{W'}, \mathcal{F}', \sigma_I')$ are *equivalent* if there exists $\gamma \in \Gamma$ so that $\sigma_I' = \mathrm{Ad}_\gamma \sigma_I$ (which implies $W' = \gamma W$), and the induced action $\gamma : \mathrm{Gr}_\ell^W = \mathrm{Gr}_\ell^{W'}$ satisfies $\gamma(\mathcal{F}) = \mathcal{F}'$.

Let

$$\Phi^{\mathsf{S}}(b) \;=\; [\mathrm{Gr}^W, F, \sigma_I]^{\mathsf{S}} \tag{2.5}$$

be the associated equivalence class.

Definition 2.6. The completion $\overline{\wp}^{\mathsf{S}}$ is the set of equivalence classes of graded quotients of LMHS associated to $(\overline{B}, Z; \Phi)$ and $\Phi^{\mathsf{S}} : \overline{B} \to \overline{\wp}^{\mathsf{S}}$ is the map $b \mapsto [\mathrm{Gr}^W, F, \sigma_I]^{\mathsf{S}}$.

The space $\overline{\wp}^{\mathsf{S}}$ is endowed with a natural compact Hausdorff topology with respect to which Φ^{S} is continuous and proper, and the restriction of Φ^{S} to Z_I^* is analytic, [GGR21b].

Conjecture 2.7. *The set $\overline{\wp}^{\mathsf{S}}$ is a projective algebraic variety, Φ^{S} is a morphism, and we have a commutative diagram* (1.2). This conjecture is discussed in general and proved in special cases (including $\dim \wp \leq 2$) in [GGLR20].

Remark 2.8. The extended Hodge line bundle Λ_{e} is a natural candidate for an ample line bundle on $\overline{\wp}^{\mathsf{S}}$. The identification of an ample line bundle on $\overline{\wp}^{\mathsf{T}}$ seems to be a more subtle question. Even in the classical case (§1.2.1), while $\overline{\Gamma\backslash D}^{\mathsf{T}}$ is known to be projective, to the best of our knowledge no explicit ample line bundle is known. One candidate is the line bundle of Conjecture 3.29.

2.2. **Proof of Theorem 1.7.** There are three steps.

(i) The first is to apply the Cattani–Deligne–Kaplan result on the algebraicity of Hodge loci [CDK95] to deduce that the closure of
$$\{(b,b') \in B \times B \mid \hat{\Phi}(b) = \hat{\Phi}(b')\}$$
in $\overline{B} \times \overline{B}$ is an algebraic variety $\hat{X}$.

(ii) The second step is to show that $\hat{X}$ defines a (proper, holomorphic) equivalence relation on $\overline{B}$. It then follows from [Gra83] that the quotient
$$\hat{\wp}^{\mathsf{T}} \;=\; \overline{B}/\sim$$
is a compact complex analytic variety, and the quotient map $\hat{\Phi}^{\mathsf{T}} : \overline{B} \to \hat{\wp}^{\mathsf{T}}$ is holomorphic.

(iii) Since $\overline{B}$ is projective, it follows that $\hat{\wp}^{\mathsf{T}}$ is Moishezon [AT82, §5, Corollary 11], and Serre's GAGA implies $\hat{\Phi}^{\mathsf{T}}$ is a morphism [Art70, §7].

It is in the second step where the new work comes. We informally summarize it here. The problem is to show that $\hat{X}$ is a proper, holomorphic equivalence relation. For this we must show that every point $b \in \overline{B}$ admits neighborhood $\overline{\mathcal{O}} \subset \overline{B}$ and a *proper* holomorphic map

$$f : \overline{\mathcal{O}} \;\to\; \mathbb{C}^d \tag{2.9}$$

whose Stein factorization $\overline{\mathcal{O}} \xrightarrow{\hat{f}} \hat{\mathcal{O}} \to \mathbb{C}^d$ has the property that the (connected) fibres of $\hat{f}$ coincide with those of $\hat{\Phi}$ over $\mathcal{O} = B \cap \overline{\mathcal{O}}$. Note that $\hat{\Phi}$ will then be proper on $\mathcal{O}$.

The basic idea is that any point $b \in \overline{B}$ admits a neighborhood $\overline{\mathcal{O}} \subset \overline{B}$ with the properties that:

(a) Over $\mathcal{O} = B \cap \overline{\mathcal{O}}$ the period map is represented by a period matrix (§2.3). This period matrix may be multi-valued – this multivaluedness comes from the monodromy over $\mathcal{O}$.

(b) The infinitesimal period relation implies that the full period matrix is determined (up to constants of integration) by a subset of the coefficients; we call these the horizontal coefficients. The horizontal coefficients $(\varepsilon_1, \ldots, \varepsilon_d) : \mathcal{O} \to \mathbb{C}^d$ are of two types. Either the coefficient $\varepsilon_j : \mathcal{O} \to \mathbb{C}$ is well-defined (single-valued) and extends to a holomorphic function on all of $\overline{\mathcal{O}}$; or ε_j is multivalued (and does not extend to $\overline{\mathcal{O}}$), but $\tau_j = \exp 2\pi \mathbf{i} \varepsilon_j$ is well-defined (single-valued) and extends to a holomorphic function on $\overline{\mathcal{O}}$.

(c) Suppose that we index the horizontal coefficients so that ε_j is holomorphic if and only if $j \le c$. Then $f = (\varepsilon_1, \ldots, \varepsilon_c, \tau_{c+1}, \ldots, \tau_d) : \overline{\mathcal{O}} \to \mathbb{C}^d$ is proper over $\overline{\mathcal{O}}$, and the fibres of $\hat{f}$ and $\hat{\Phi}$ coincide.

Given this structure, the existence of the finite map $\hat{\wp}^{\mathsf{T}} \to \overline{\wp}^{\mathsf{T}}$ follows from this period matrix representation, properness and the infinitesimal period relation.

The sticking point here is the requirement that f be proper. It is relatively easy to see that every point $b \in \overline{B}$ admits a local coordinate neighborhood $\overline{\mathcal{U}} \subset \overline{B}$ with the property that Φ may be represented by a period matrix over $\mathcal{U} = B \cap \overline{\mathcal{U}}$ (Remark 2.16). The issue is that the function $f : \overline{\mathcal{U}} \to \mathbb{C}^d$ constructed from this matrix representation need not be proper. We will outline in §3.1 how to obtain the proper map $f : \overline{\mathcal{O}} \to \mathbb{C}^d$ of (c).

2.3. **Period matrix representations.**

2.3.1. *Hermitian D.* The period domain D parameterizing polarized Hodge structures of weight $n = 1$ with Hodge numbers $\mathbf{h} = (g, g)$ is naturally identified with the Siegel space $\mathcal{H}_g$ of symmetric $g \times g$ complex matrices with positive definite imaginary part. In this way each Hodge structure $F \in D$ admits a period matrix representation. We think of the period map $\Phi : B \to \Gamma\backslash D$ as admitting a multivalued period matrix representation.

More generally, any bounded symmetric domain may be parameterized by matrices [Ise71]. So any period map $\Phi : B \to \Gamma\backslash D$ into a locally Hermitian symmetric space will admit a multivalued period matrix representation.

Remark 2.10. In general there are various realizations of D as an open domain in $\mathbb{C}^d$, $d = \dim D$. The realizations are used in both the Satake–Baily–Borel and toroidal compactifications of $\Gamma\backslash D$, [BB66, Mum75].

In contrast, the non-Hermitian period domains contain compact subvarieties of positive dimension (Example 2.11), and so can not be realized as subsets of any complex affine space. These domains do not admit (global) period matrix representations.

Example 2.11. Suppose that D is the non-Hermitian period domain parameterizing Q–polarized Hodge structures of weight $n = 2$ and with Hodge number $p_g = h^{2,0} \geq 2$. Given $(F^2 \subset F^1) \in D$, set $\mathbb{C}^{2p_g} = F^2 \oplus \overline{F^2}$. Then the isotropic Grassmannian

$$C \;=\; \mathrm{Gr}^Q(p_g, \mathbb{C}^{2p_g}) \;=\; \{E \in \mathrm{Gr}(p_g, \mathbb{C}^{2p_g}) \mid \; Q|_E = 0\}$$

naturally injects into D by sending $E \in \mathrm{Gr}^Q(p_g, \mathbb{C}^{2p_g})$ to the Hodge decomposition $V_\mathbb{C} = E \oplus (E \oplus \overline{E})^\perp \oplus \overline{E}$.

2.3.2. *Schubert cells and Plücker coordinates.* The compact dual $\check{D}$ is covered by Zariski open Schubert cells $\mathcal{S} \simeq \mathbb{C}^N$; this biholomorphism is nothing more than the Plücker coordinates on $\mathcal{S}$. These coordinates are the period matrix representation of $F \in \mathcal{S}$.

Example 2.12. Let D be the Hermitian period domain parameterizing weight one, Q-polarized Hodge structures on $V \simeq \mathbb{Q}^{2g}$. The compact dual is the Lagrangian grassmannian

$$\check{D} \;=\; \mathrm{LG}(g, V_\mathbb{C}) \;=\; \{E \in \mathrm{Gr}(g,V)_\mathbb{C} \mid \; Q|_E = 0\}\,.$$

Fix a basis $\{v_1, \ldots, v_{2g}\}$ of $V_\mathbb{C}$ so that

$$Q(v_i, v_j) \;=\; \begin{cases} \delta^{2g+1}_{i+j}\,, & 1 \le i \le g\,, \\ -\delta^{2g+1}_{i+j}\,, & g+1 \le i \le 2g\,. \end{cases}$$

Set $E = \mathrm{span}\{v_{g+1}, \ldots, v_{2g}\} \in \check{D}$. Any element in the Schubert cell

$$\mathcal{S} \;=\; \{F \in \check{D} \mid F \cap E = 0\}$$

admits a unique basis of the form

$$F \;=\; \mathrm{span}_\mathbb{C}\{v_a + \xi^s_a v_s \mid 1 \le a \le g\}\,,$$

where we sum over $g+1 \le s \le 2g$. The condition that F be Q–isotropic (the first Hodge–Riemann bilinear relation) is

$$0 \;=\; Q(v_a + \xi^s_a v_s\,,\, v_b + \xi^r_b v_r) \;=\; \xi^{\bar{a}}_b \,-\, \xi^{\bar{b}}_a\,,$$

where $\bar{a} = 2g+1-a$. So the ξ define a biholomorphism $\mathcal{S} \to \mathbb{C}^{g(g+1)/2}$. These are the Plücker coordinates on $\mathcal{S} \subset \check{D}$, and we say

$$\left[\begin{array}{ccc|ccc} 1 & & & \xi^{g+1}_1 & \cdots & \xi^{2g}_1 \\ & \ddots & & \vdots & & \vdots \\ & & 1 & \xi^{g+1}_g & \cdots & \xi^{2g}_g \end{array}\right]^{\mathsf{t}}$$

is the *matrix representation* of $F \in \mathcal{S}$.

Example 2.13. Suppose that D is the non-Hermitian period domain parameterizing Q–polarized Hodge structures of weight 2 and with Hodge number $p_g = h^{2,0} = 2$.

The compact dual is the Q–isotropic Grassmannian

$$\check{D} \;=\; \mathrm{Gr}^Q(2, V_{\mathbb{C}}) \;=\; \{E \in \mathrm{Gr}(2, V_{\mathbb{C}}) \mid \left.Q\right|_E = 0\}\,.$$

Given $E \in \check{D}$, the Hodge filtration $F^2 \subset F^1 \subset V_{\mathbb{C}}$ is $F^2 = E$ and $F^1 = E^\perp$.

Fix a basis $\{v_0, \ldots, v_r\}$ of $V_{\mathbb{C}}$ so that $Q(v_i, v_j) = \delta^r_{i+j}$. Set $E = \mathrm{span}\{v_{r-1}, v_r\} \in \check{D}$ so that $E^\perp = \mathrm{span}\{v_2, \ldots, v_r\}$. Any element in the Schubert cell

$$\mathcal{S} \;=\; \{F^2 \in \check{D} \mid F^2 \cap E^\perp = 0\}$$

admits a unique basis of the form

$$F^2 \;=\; \mathrm{span}_{\mathbb{C}} \left\{ v_0 + \sum_{i=2}^r \xi_0^i v_i \,,\; v_1 + \sum_{i=2}^r \xi_1^i v_i \right\} .$$

Set $\bar{i} = r - i$. The condition that F^2 be Q–isotropic (the first Hodge–Riemann bilinear relation) is

$$\begin{aligned} 0 \;&=\; 2\xi_0^{\bar{0}} + \sum_{a=2}^{\bar{2}} \xi_0^a \xi_0^{\bar{a}} \,, \\ 0 \;&=\; 2\xi_1^{\bar{1}} + \sum_{a=2}^{\bar{2}} \xi_1^a \xi_1^{\bar{a}} \,, \qquad (2.14) \\ 0 \;&=\; \xi_1^{\bar{0}} + \xi_0^{\bar{1}} + \sum_{a=2}^{\bar{2}} \xi_0^a \xi_1^{\bar{a}} \,. \end{aligned}$$

So the ξ define a biholomorphism $\mathcal{S} \to \mathbb{C}^{2r-5}$. These are the Plücker coordinates on $\mathcal{S} \subset \check{D}$, and we say

$$\left[\begin{array}{cc|ccc} 1 & 0 & \xi_0^2 & \cdots & \xi_0^r \\ 0 & 1 & \xi_1^2 & \cdots & \xi_1^r \end{array} \right]^{\mathsf{t}}$$

is the *matrix representation* of $F^2 \in \mathcal{S}$.

Definition 2.15. A period map $\Phi : B \to \Gamma\backslash D$ *admits a period matrix representation* over an open subset $\mathcal{O} \subset B$ if there is an open Schubert cell $\mathcal{S} \subset \check{D}$ such that:

(i) The monodromy $\Gamma_{\mathcal{O}}$ over $\mathcal{O}$ preserves $D \cap \mathcal{S}$.

(ii) The restricted period map $\mathcal{O} \to \Gamma_{\mathcal{O}}\backslash D$ takes value in $\Gamma_{\mathcal{O}}\backslash(D \cap \mathcal{S})$.

In this case, the period matrix representation of $\Phi|_{\mathcal{O}}$ is given by the Plücker coordinates on $\mathcal{S}$. The matrix coefficients will be multivalued when the action of $\Gamma_{\mathcal{O}}$ on $D \cap \mathcal{S}$ is nontrivial.

Remark 2.16. Every $b \in \overline{B}$ admits a local coordinate chart $\overline{\mathcal{U}}$ with the property that Φ admits a matrix representation over $\mathcal{U} = B \cap \overline{\mathcal{U}}$. When $b \in B$, this is an immediate consequence of the fact that period maps are locally liftable and that $\check{D} \supset D$ is covered by Zariski open Schubert cells. In the case that $b \in Z$, this is a consequence of the nilpotent orbit theorem [Sch73].

3. Period mappings at infinity

The restriction of the map Φ^{T} in (2.1) to Z_I^* defines a variation of limiting mixed Hodge structures that is encoded by a holomorphic "period map"

$$\Phi_I : Z_I^* \;\to\; (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I\,. \tag{3.1}$$

Here D_I is the set of all $F \in \check{D}$ with the property that (W, F, σ_I) is a polarized mixed Hodge structure. It is a homogeneous submanifold of $\check{D}$ with automorphism group $\mathrm{Aut}(D_I)$ containing both $\exp(\mathbb{C}\sigma_I)$ and Γ_I, with the latter a subgroup of Γ centralizing the cone σ_I [KP16]. Likewise, the restriction of the map Φ^{S} of (2.5) to Z_I^* defines a period mapping

$$\Phi_I^0 : Z_I^* \;\to\; \Gamma_I\backslash D_I^0\,. \tag{3.2}$$

Here D_I^0 is a Mumford–Tate domain and a quotient of D_I by a normal subgroup of $\mathrm{Aut}(D_I)$ containing $\exp(\mathbb{C}\sigma_I)$. The map Φ_I^0 factors through (3.1), and we have a commutative diagram

$$\begin{tikzcd} Z_I^* \arrow[r, "\Phi_I"] \arrow[dr, "\Phi_I^0"'] & (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I \arrow[d] \\ & \Gamma_I\backslash D_I^0\,. \end{tikzcd} \tag{3.3}$$

See [GGR21b, §2] for details.

Modulo some finite identifications (which we shall not go into here) the restriction of (1.2) to Z_I^* is (3.3). In particular (and modulo those identifications), any fibre $\mathcal{F}$ of $\overline{\wp}^{\mathsf{T}} \to \overline{\wp}^{\mathsf{S}}$ is contained in a fibre of

$$(\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I \;\to\; \Gamma_I\backslash D_I^0 . \tag{3.4}$$

The latter parameterizes limiting mixed Hodge structures (W, F, σ_I) with fixed/constant associated graded Hodge structures $F^p(\mathrm{Gr}_\ell^W)$. In particular, what varies along the fibre is the extension data of (W, F, σ_I). This extension data encodes a rich geometric structure on the fibres of Φ^{S}. It is reviewed in §§3.2–3.4.

3.1. **Period matrix representations at infinity.** The purpose of this section is to sketch where the map f of §2.2(c) comes from.

As will be discussed in §§3.2–3.4, the extension data is filtered, and filtration factors (3.4) as

$$(\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I \;\to\; \Gamma_I\backslash D_I^1 \;\to\; \Gamma_I\backslash D_I^0 . \tag{3.5}$$

The fibres of $\Gamma_I\backslash D_I^1 \;\to\; \Gamma_I\backslash D_I^0$ parameterize "level one" extension data (cf. §3.5.1). The diagram (3.3) in turn factors as

$$\begin{tikzcd} Z_I^* \arrow[r, "\Phi_I"] \arrow[dr, "\Phi_I^1"'] \arrow[ddr, bend right, "\Phi_I^0"'] & (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I \arrow[d, two heads] \\ & \Gamma_I\backslash D_I^1 \arrow[d, two heads] \\ & \Gamma_I\backslash D_I^0 . \end{tikzcd} \tag{3.6}$$

The map Φ^1_I is holomorphic. Both the Φ^0_I and the Φ^1_I "patch together" to define proper, continuous maps onto compact Hausdorff topological spaces

$$\begin{tikzcd} \overline{B} \arrow[r, "\Phi^1"] \arrow[dr, "\Phi^0"'] & \overline{\wp}^1 \arrow[d] \\ & \overline{\wp}^0 . \end{tikzcd} \tag{3.7}$$

See [GGR21b] for details.[1] Notice that $\Phi = \Phi^0|_B = \Phi^1|_B$.

Remark 3.8. If A is a Φ^0–fibre, then the fibre $\mathcal{F}$ of (1.15) is $\Phi^\mathsf{T}(A)$ and $\Phi^1(A) = \mathcal{F}^1$.

Let $A^1 \subset A^0$ be connected components of a Φ^1 and Φ^0–fibre, respectively. Both are compact, complex subvarieties of $\overline{B}$. Given $i = 0, 1$, the fibre A^i admits a neighborhood $\overline{\mathcal{O}}^i \subset \overline{B}$ with the properties:

(i) The restriction $\Phi^i|_{\overline{\mathcal{O}}^i}$ is proper.

(ii) The period map Φ admits a matrix representation over $\mathcal{O}^i = B \cap \overline{\mathcal{O}}^i$

(Definition 2.15). The monodromy Γ_1 about A^1 is particularly simple. Simple enough in fact that the matrix coefficients of the representation over $\mathcal{O}^1$ satisfy §2.2(b). Whence we obtain the map f of §2.2(c). See [GGR21b] for details.

For both $\overline{\mathcal{O}}^1 \subset \overline{\mathcal{O}}^0$, the Schubert cell giving the period matrix representation is given by (3.9) below. Fix any $b \in A^0$ and consider any representative (F, σ) of $\Phi^\mathsf{T}(b) = [W, F, \sigma]^\mathsf{T}$. The limit

$$F_\infty \;:=\; \lim_{y \to +\infty} \exp(\mathbf{i} y N) \cdot F \;\in\; \check{D}$$

is independent of all these choices, cf. [GGR21b], as is

$$\mathcal{S} \;=\; \left\{ E \in \check{D} \mid \dim\left(E^a \cap \overline{F^b_\infty}\right) = \dim\left(F^a \cap \overline{F^b_\infty}\right),\ \forall\ a, b \right\} . \tag{3.9}$$

The monodromy Γ_0 over $\mathcal{O}^0$ fixes F_∞, and therefore $\overline{F_\infty}$, and so preserves $\mathcal{S}$.

[1] Caveat emptor: the "patching" may require that we make some additional identifications. That is, $\Phi^0|_{Z^*_I}$ will factor through Φ^0_I, and $\Phi^1|_{Z^*_I}$ will factor through Φ^1_I. However, those identifications are *finite*, so that the maps $\Phi^0_I(Z^*_I) \to \Phi^0(Z^*_I)$ and $\Phi^1_I(Z^*_I) \to \Phi^1(Z^*_I)$ have finite fibres.

3.2. **Extension data for a mixed Hodge structure.** To begin, fix a mixed Hodge structure (W, F) on V with weight filtration

$$0 \;=\; W_{-1} \;\subset\; W_0 \;\subset\; W_1 \;\subset \cdots \subset\; W_{2n} \;=\; V\,,$$

and Hodge filtration

$$0 \;=\; F^{n+1} \;\subset\; F^n \;\subset \cdots \subset\; F^1 \;\subset\; F^0 \;=\; V_{\mathbb{C}}\,.$$

Set associated graded

$$\mathrm{Gr}^W_\bullet \;=\; \bigoplus_{\ell=0}^{2n} \mathrm{Gr}^W_\ell\,.$$

Let H^ℓ denote $\mathrm{Gr}^W_\ell = W_\ell / W_{\ell-1}$ equipped with the pure weight ℓ Hodge structure $F^p(\mathrm{Gr}^W_\ell)$.

3.2.1. *Description.* The *extension data* of the mixed Hodge structure (W, F) is the set $\mathcal{E}_{W,F}$ of Γ_W–equivalence classes of mixed Hodge structures $(W, \tilde F)$ with the same associated graded Hodge structure $\tilde H^\bullet = H^\bullet$. As will be summarized in §A.1, it is a discrete quotient

$$\mathcal{E}_{W,F} \;=\; \Gamma_W \backslash (P^1_W \cdot F) \tag{3.10}$$

of a complex homogeneous manifold, and it is an iterated fibre bundle

$$\mathcal{E}_{W,F} \;=\; \mathcal{E}^{2n}_{W,F} \;\twoheadrightarrow\; \mathcal{E}^{2n-1}_{W,F} \;\twoheadrightarrow \cdots \twoheadrightarrow\; \mathcal{E}^2_{W,F} \;\twoheadrightarrow\; \mathcal{E}^1_{W,F}\,. \tag{3.11}$$

To describe the fibres, given $\ell \geq 1$, let

$$\mathrm{Ext}(H^k, H^{k-\ell}) \;=\; \frac{\mathrm{Hom}(H^k, H^{k-\ell})}{F^0\mathrm{Hom}(H^k, H^{k-\ell}) \,+\, \mathrm{Hom}_{\mathbb{Z}}(H^k, H^{k-\ell})}\,,$$

be the set of congruence classes of short exact sequences

$$0 \;\to\; H^{k-\ell} \;\to\; W_k / W_{k-\ell-1} \;\to\; H^k \;\to\; 0$$

of mixed Hodge structures; see [Car87] for further discussion. The base space $\mathcal{E}^1_{W,F}$ parameterizes the *level one extension data*; it is a product

$$\mathcal{E}^1_{W,F} \;=\; \bigoplus_{k=1}^{2n} \mathrm{Ext}(H^k, H^{k-1})$$

of compact complex tori. The fibres

$$\begin{array}{ccc} \mathrm{Ext}_\ell(W,F) & \hookrightarrow & \mathcal{E}^\ell_{W,F} \\ & & \downarrow\!\!\downarrow \\ & & \mathcal{E}^{\ell-1}_{W,F} \end{array}$$

parameterize the *level ℓ extension data*. It also a product

$$\mathrm{Ext}_\ell(W,F) \;=\; \bigoplus_{k=\ell}^{2n} \mathrm{Ext}(H^k, H^{k-\ell})$$

of (in general, noncompact) complex tori. The space $\mathcal{E}^\ell_{W,F}$ parameterizes the *extension data of level $\leq \ell$*.

3.2.2. *Constraints imposed by the infinitesimal period relation.* Consider a complex analytic map $\psi : \mathcal{Z} \to \mathcal{E}^\ell_{W,F}$. In general, the maps that we are interested in will satisfy the infinitesimal period relation $\mathrm{d}F^p \subset F^{p-1}$, and this has some important implications for the map ψ. To explain these, define

$$F^{-1}\mathrm{Ext}(H^k, H^{k-\ell}) \;=\; \frac{F^{-1}\mathrm{Hom}(H^k, H^{k-\ell})}{F^0\mathrm{Hom}(H^k, H^{k-\ell}) \,+\, \mathrm{Hom}_{\mathbb{Z}}(H^k, H^{k-\ell})} \;\subset\; \mathrm{Ext}(H^k, H^{k-\ell})$$

and

$$F^{-1}\mathrm{Ext}_\ell(W,F) \;=\; \bigoplus_{k=1}^{2n} F^{-1}\mathrm{Ext}(H^k, H^{k-\ell}) \;\subset\; \mathrm{Ext}_\ell(W,F)\,.$$

Note that $F^{-1}\mathrm{Ext}_\ell(W,F)$ is the product of a complex torus with an affine space $\mathbb{C}^d$. Keeping in mind that $\mathrm{Ext}_\ell(W,F)$ is also a complex torus, we may consider translations of $F^{-1}\mathrm{Ext}_\ell(W,F)$ in $\mathrm{Ext}_\ell(W,F)$.

If $\ell = 1$, then the infinitesimal period relation implies that $\psi : \mathcal{Z} \to \mathcal{E}^1_{W,F}$ takes value in a translate of $F^{-1}\mathrm{Ext}_1(W,F)$. More generally, if $\ell \geq 2$, and the composition $\mathcal{Z} \xrightarrow{\psi} \mathcal{E}^\ell_{W,F} \twoheadrightarrow \mathcal{E}^{\ell-1}_{W,F}$ of ψ with the projection of (3.11) is the constant map, then ψ

takes value in a fibre $\mathrm{Ext}_\ell(W,F)$ of $\mathcal{E}^\ell_{W,F} \twoheadrightarrow \mathcal{E}^{\ell-1}_{W,F}$ and the infinitesimal period relation implies that the map takes value in a translate of $F^{-1}\mathrm{Ext}_\ell(W,F)$.

If $\mathcal{Z}$ is compact and connected, then we obtain further restrictions on the map $\psi : \mathcal{Z} \to \mathcal{E}^\ell_{W,F}$. To see this, notice that $F^{-1}\mathrm{Ext}_\ell(W,F)$ is the product of a complex torus with an affine space $\mathbb{C}^{d_\ell}$. If $\ell \geq 2$, then the complex torus has no compact factor, so that $F^{-1}\mathrm{Ext}_\ell(W,F) = \mathbb{C}^{d_\ell} \times (\mathbb{C}^*)^{d'_\ell}$. In the case that $\ell = 1$, let $\mathbb{T} \subset F^{-1}\mathrm{Ext}_1(W,F)$ denote the maximal compact complex torus. Since $\mathcal{Z}$ is compact (and connected), $\psi : \mathcal{Z} \to \mathcal{E}^1_{W,F}$ must take value in a translate of $\mathbb{T}$. If $\ell \geq 2$ and the composition $\mathcal{Z} \to \mathcal{E}^\ell_{W,F} \twoheadrightarrow \mathcal{E}^{\ell-1}_{W,F}$ is constant, then $\psi : \mathcal{Z} \to \mathcal{E}^\ell_{W,F}$ must be constant. This establishes

Lemma 3.12. *Let $\mathcal{Z}$ be a compact, connected, complex analytic variety and $\psi : \mathcal{Z} \to \mathcal{E}_{W,F}$ an analytic map satisfying the infinitesimal period relation $\mathrm{d}F^p \subset F^{p-1}$. Then*

(i) *The map $\pi^1 \circ \psi : \mathcal{Z} \to \mathcal{E}^1_{W,F}$ takes value in a translate of the maximal compact complex torus $\mathbb{T} \subset F^{-1}\mathrm{Ext}_1(W,F)$.*

(ii) *If $\ell \geq 2$ and $\pi^{\ell-1} \circ \psi$ is constant, then $\pi^\ell \circ \psi : \mathcal{Z} \to \mathcal{E}^\ell_{W,F}$ must be constant. In particular, ψ is locally constant on the fibres of $\pi^1 \circ \psi$.*

3.2.3. *Extensions of maps to the compact torus.* In practice the maps to extension data that arise when considering period maps at infinity are defined on (noncompact) quasi-projective varieties. So it is interesting to note that the maps to level one extension data will extend to smooth projective completions.

Lemma 3.13. *Suppose that $\mathcal{Z}^*$ is Zariski open in a smooth algebraic variety $\mathcal{Z}$. Then any holomorphic map $\psi^1 : \mathcal{Z}^* \to \mathbb{T}$ extends to $\psi^1 : \mathcal{Z} \to \mathbb{T}$.*

Proof. Let $H_1(\cdot)$ denote the first homology group with $\mathbb{Z}$ coefficients, modulo torsion. The induced $\psi^1_* : H_1(\mathcal{Z}^*) \to H_1(\mathbb{T})$ is a morphism of mixed Hodge structures. The mixed Hodge structure on $H_1(\mathcal{Z}^*) = W_{-1}(H_1(\mathcal{Z}^*))$ has weights ≤ -1, while $H_1(\mathbb{T})$ is a pure Hodge structure of weight -1. Thus we have an induced map

$$\alpha : H_1(\mathcal{Z}) \xrightarrow{\simeq} \mathrm{Gr}^W_{-1} H_1(\mathcal{Z}^*) \to H_1(\mathbb{T}) .$$

The morphism of Hodge structure α is, up to translation, induced by a holomorphic map $\mathrm{Alb}(\mathcal{Z}) \to \mathbb{T}$. And the desired extension of ψ^1 to $\mathcal{Z}$ is given by the composition $\mathcal{Z} \to \mathrm{Alb}(\mathcal{Z}) \to \mathbb{T}$. □

3.3. **Limiting mixed Hodge structures.** A limiting mixed Hodge structure is an equivalence class of polarized mixed Hodge structures (§3.3.1). As such it both carries a richer structure than a mixed Hodge structure (coming from a cone of polarizing nilpotent operators), and is a slightly coarser object (virtue of working with equivalence classes. The coarser nature is due to the additional quotient by $\mathrm{span}_{\mathbb{C}}\{\sigma\}$ in (3.16) compared with (3.10)). This dichotomy is seen when juxtaposing Lemma 3.12 above with §3.5 and Lemma 3.20 below. The richer structure gives us the *ample* line bundles $\mathcal{L}_M$ over the irreducible components of Φ^0–fibres, and their relationship (3.23) to the normal bundles $\mathcal{N}_{Z_i/\overline{B}}$ (§3.5). This strengthens the first statement of the lemma, and encodes the central geometric information arising when considering the variation of limiting mixed Hodge structure (3.1) along Z_I^*. The coarser nature gives us Lemma 3.20 as the analog of Lemma 3.12(ii).

3.3.1. *Definition.* A mixed Hodge structure (W, F), with $F \in \check{D}$, is *polarized* by a nilpotent operator $N \in \mathrm{End}(V, Q)$ if

(i) the action of N satisfies $N(F^p) \subset F^{p-1}$ for all p,

(ii) and $N(W_\ell) \subset W_{\ell-2}$ for all ℓ;

(iii) the induced map $N^k : \mathrm{Gr}^W_{n+k} \to \mathrm{Gr}^W_{n-k}$ is an isomorphism;

(iv) the weight $n+k$ Hodge structure on $\mathrm{Prim}^N_{n+k} = \ker\{N^{k+1} : \mathrm{Gr}^W_{n+k} \to \mathrm{Gr}^W_{n-k-2}\}$ that is induced by F is polarized by $Q^N_k(\cdot,\cdot) = Q(\cdot, N^k\cdot)$.

The triple (W, F, N) is a *polarized mixed Hodge structure.* Given commuting nilpotent operators $N_1, \ldots, N_m \in \mathrm{End}(V, Q)$, the *nilpotent cone*

$$\sigma \;=\; \mathrm{span}_{\mathbb{R}_{>0}}\{N_1, \ldots, N_m\} \;\subset\; \mathrm{End}(V_{\mathbb{R}}, Q) \tag{3.14}$$

polarizes (W, F) if (W, F, N) is a polarized mixed Hodge structure for every $N \in \sigma$. This is the case if and only if

$$(t_1, \ldots, t_m) \ \mapsto \ \exp(\textstyle\sum_i t_i N_i) \cdot F$$

is a nilpotent orbit [CKS86]. The associated *limiting mixed Hodge structure* is the equivalence class

$$[W, F, \sigma] \ = \ \{(W, F', \sigma) \mid F' \in \exp(\mathbb{C}\sigma) \cdot F\}\,.$$

Given a polarized mixed Hodge structure (W, F, σ), W is the unique filtration satisfying (ii) and (iii). So we will sometimes let (F, σ) and $[F, \sigma]$ denote the polarized and limiting mixed Hodge structures.

Remark 3.15 (Relationship to Deligne's mixed Hodge structure). Deligne has shown that the cohomology $H^k(X_0)$ of any quasi-projective variety X_0 admits a functorial mixed Hodge structure [Del74, PS08]. Suppose that X_0 is a projective variety and admits a smoothing: this means that X_0 can be realized as the central fibre of a family $\{X_t\}_{t \in \Delta}$ parameterized by the unit disc $\Delta = \{|t| < 1\} \subset \mathbb{C}$ with X_t smooth for all $t \neq 0$. Fix $t_o \neq 0$. Then Schmid's nilpotent orbit theorem [Sch73] endows $H^k(X_{t_o})$ with a limiting mixed Hodge structure that depends only on the family $\{X_t\}_{t \in \Delta^*}$ of smooth varieties, $\Delta^* = \{0 < |t| < 1\}$, not the central fibre X_0. It is with these limiting mixed Hodge structures that we are concerned in this note. However, we note that the two mixed Hodge structures are related by the Clemens–Schmid exact sequence [Cle77] and its generalizations [KL19, KL20].

3.4. **Extension data for a limiting mixed Hodge structure.** As in §3.2 we fix a limiting mixed Hodge structure $[W, F, \sigma]$. As above, we let H^ℓ denote $\mathrm{Gr}^W_\ell = W_\ell / W_{\ell-1}$ equipped with the pure weight ℓ Hodge structure $F^p(\mathrm{Gr}^W_\ell)$. The condition §3.3.1(ii) implies that this Hodge structure does not depend on the choice of $(W, F, \sigma) \in [W, F, \sigma]$; that is, if $\tilde{F} \in \exp(\mathbb{C}\sigma) \cdot F$, then $\tilde{H}^\ell = H^\ell$.

3.4.1. *Description.* The *extension data* of the limiting mixed Hodge structure $[W, F, \sigma]$ is the a set $\mathcal{E}_{\sigma,F}$ of equivalence classes of limiting mixed Hodge structures $[W, \tilde{F}, \sigma]$ with the same associated graded Hodge structure $\tilde{H}^\bullet = H^\bullet$. As will be summarized in §A.2 it is a discrete quotient

$$\mathcal{E}_{\sigma,F} \;=\; (\exp(\mathbb{C}\sigma)\Gamma_\sigma)\backslash(C^1_\sigma \cdot F) \tag{3.16}$$

of a complex homogeneous manifold, and has the structure of an iterated fibre bundle

$$\mathcal{E}_{\sigma,F} \;=\; \mathcal{E}^{2n}_{\sigma,F} \;\twoheadrightarrow\; \mathcal{E}^{2n-1}_{\sigma,F} \;\twoheadrightarrow \cdots \twoheadrightarrow\; \mathcal{E}^{2}_{\sigma,F} \;\twoheadrightarrow\; \mathcal{E}^{1}_{\sigma,F}\,. \tag{3.17}$$

To describe the fibres, set $\ell \geq 1$ and let

$$\mathrm{Ext}_\sigma(H^k, H^{k-\ell}) \;=\; \frac{\mathrm{Hom}_\sigma(H^k, H^{k-\ell})}{F^0\mathrm{Hom}_\sigma(H^k, H^{k-\ell}) \,+\, \mathrm{Hom}_{\sigma,\mathbb{Z}}(H^k, H^{k-\ell})}\,,$$

Here, Ext_σ denotes extension classes of *polarized mixed Hodge structures*; in particular, Hom_σ denotes homomorphisms in the category of polarized mixed Hodge structures, and $\mathrm{Ext}_\sigma(H^k, H^{k-\ell})$ is the space of congruence classes of short exact sequences

$$0 \;\to\; H^{k-\ell} \;\to\; W_k/W_{k-\ell-1} \;\to\; H^k \;\to\; 0$$

of polarized mixed Hodge structures. The base of the fibration (3.17) is

$$\mathcal{E}^1_{\sigma,F} \;=\; \bigoplus_{k=1}^{2n} \mathrm{Ext}_\sigma(H^k, H^{k-1})\,,$$

parameterizes the *level one extension data*, and is a product of compact complex tori.

More generally, if $\ell \geq 3$, then the fibre of

$$\begin{tikzcd} \mathrm{Ext}_\ell(\sigma, F) \arrow[r, hook] & \mathcal{E}^\ell_{\sigma,F} \arrow[d, two heads] \\ & \mathcal{E}^{\ell-1}_{\sigma,F} \end{tikzcd} \tag{3.18}$$

parameterizes the *level ℓ extension data*

$$\mathrm{Ext}_\ell(\sigma, F) \;=\; \bigoplus_{k=\ell}^{2n} \mathrm{Ext}_\sigma(H^k, H^{k-\ell})\,.$$

It also a product of (in general, noncompact) complex tori.

If $\ell = 2$, then there is a natural map

$$\mathrm{span}_{\mathbb{C}}\{\sigma\} \;\to\; \mathrm{Hom}_\sigma(H^k, H^{k-2})$$

(we say the nilpotent cone is "level two extension data"), and we may consider the quotient

$$\frac{\mathrm{Ext}_\sigma(H^k, H^{k-2})}{\mathrm{span}_{\mathbb{C}}\{\sigma\}} \;=\; \frac{\mathrm{Hom}_\sigma(H^k, H^{k-2})}{F^0\mathrm{Hom}_\sigma(H^k, H^{k-2}) \,+\, \mathrm{Hom}_{\sigma,\mathbb{Z}}(H^k, H^{k-2}) \,+\, \mathrm{span}_{\mathbb{C}}\{\sigma\}}\,.$$

The fibre of

$$\begin{tikzcd} \dfrac{\mathrm{Ext}_2(\sigma, F)}{\mathrm{span}_{\mathbb{C}}\{\sigma\}} \arrow[r, hook] & \mathcal{E}^2_{\sigma,F} \arrow[d, two heads] \\ & \mathcal{E}^1_{\sigma,F} \end{tikzcd} \tag{3.19}$$

is the *quotient*

$$\frac{\mathrm{Ext}_2(\sigma, F)}{\mathrm{span}_{\mathbb{C}}\{\sigma\}} \;=\; \bigoplus_{k=2}^{2n} \frac{\mathrm{Ext}_\sigma(H^k, H^{k-2})}{\mathrm{span}_{\mathbb{C}}\{\sigma\}}$$

of the level two extension data by the nilpotent cone. The additional quotient by $\mathrm{span}_{\mathbb{C}}\{\sigma\}$ here is due to the coarser nature of limiting mixed Hodge structures (versus polarized mixed Hodge Hodge structures). The fibre is again a product of (in general, noncompact) complex tori.

3.4.2. *Constraints imposed by the infinitesimal period relation.* In general the maps $\psi : \mathcal{Z} \to \mathcal{E}_{\sigma,F}$ that we are interested in satisfy the infinitesimal period relation $\mathrm{d}F^p \subset F^{p-1}$, and this imposes constraints on ψ analogous to those in §3.2.2.

Briefly, we let $F^{-1}\mathrm{Ext}_\ell(\sigma, F)$ be the image of $\oplus_{k\ge\ell}\, F^{-1}\mathrm{Hom}_\sigma(H^k, H^{k-\ell})$ under the projection $\oplus_{k\ge\ell}\mathrm{Hom}_\sigma(H^k, H^{k-\ell}) \twoheadrightarrow \mathrm{Ext}_\ell(\sigma, F)$. Again $F^{-1}\mathrm{Ext}_\ell(\sigma, F)$ is the product of a complex torus with an affine space $\mathbb{C}^d$. The torus has no compact factor if $\ell \ge 2$. Let $\mathcal{J}$ be the compact factor of the torus in $F^{-1}\mathrm{Ext}_1(\sigma, F)$. The argument of §3.2.2 yields

Lemma 3.20. *Let $\mathcal{Z}$ be a compact, connected, complex analytic variety.*

(i) *Any holomorphic map* $\psi^1 : \mathcal{Z} \to \mathcal{E}^1_{\sigma,F}$ *satisfying the infinitesimal period relation takes value in a translate of the maximal compact complex torus* $\mathcal{J} \subset F^{-1}\mathrm{Ext}_1(\sigma, F)$.

(ii) *Let* $\ell \geq 2$. *If a map* $\psi^\ell : \mathcal{Z} \to \mathcal{E}^\ell_{\sigma,F}$ *satisfies the infinitesimal period relation and the composition* $\mathcal{Z} \to \mathcal{E}^\ell_{\sigma,F} \twoheadrightarrow \mathcal{E}^{\ell-1}_{\sigma,F}$ *is constant, then* ψ^ℓ *must be constant. In particular, any map* $\psi : \mathcal{Z} \to \mathcal{E}_{\sigma,F}$ *is locally constant on the fibres of* $\rho^1 \circ \psi$.

Remark 3.21. We are primarily interested in the case that $\mathcal{Z}$ is a fibre of one of the two maps in (3.7). (These fibres are compact.) If A^0 is a connected component of a Φ^0–fibre, then the restriction $\Phi^1|_{A^0}$ takes value in $\mathcal{E}^1_{\sigma,F}$. Part (i) of the lemma implies that this map takes value in a translate of the compact torus $\mathcal{J} \subset \mathcal{E}^1_{\sigma,F}$. The implication of part (ii) is more subtle to state. Informally it implies that the variation of limiting mixed Hodge structure $\Phi^{\mathsf{T}}\big|_{A^1}$ is determined up to constants of integration by a nilpotent cone. (See [GGR21b, Proposition 5.1] for a precise statement.) As will be discussed next, the level two extension data coming from the cone is encoded by sections of certain line bundles, and some of these line bundles polarize $\mathcal{J}$.

3.5. **Geometry of period maps at infinity.** We now turn to the geometry of a connected component A^0 of a Φ^0–fibre as in §3.1. The fibre parameterizes limiting mixed Hodge structures with fixed associated graded $H^\bullet$. So what varies along A^0 is extension data. This extension data encodes rich geometric information via line bundles over compact tori that are related to the normal bundles $\mathcal{N}_{Z_i/\overline{B}}$.

3.5.1. *Extension data as fibres.* The fibres of the projections in (3.5) are

$$\begin{tikzcd} \mathcal{E}_{\sigma_I,F} \arrow[r, hook] & (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I \arrow[d, twoheadrightarrow] \\ & \Gamma_I\backslash D_I^0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{E}^1_{\sigma_I,F} \arrow[r, hook] & \Gamma_I\backslash D_I^1 \arrow[d, twoheadrightarrow] \\ & \Gamma_I\backslash D_I^0 \end{tikzcd}$$

By definition $\Phi^0_I(A^0 \cap Z_I^*)$ is a point of $\Gamma_I\backslash D_I^0$. So $\Phi^1_I(A^0 \cap Z_I^*) \subset \mathcal{E}^1_{\sigma_I,F}$. It is a nontrivial fact that Φ^0_I extends to $A^0 \cap Z_I$, and $\Phi^1_I(A^0 \cap Z_I) \subset \mathcal{E}^1_{\sigma_I,F}$ as well. (In the case that A^0 is smooth, this is Lemma 3.13; see [GGR21b] for the general case.) Likewise $\Phi^1(A^0) \subset \mathcal{E}^1_{W,F}$. Lemmas 3.12 and 3.20 imply that

$$\Phi^1(A^0) \;\subset\; \mathbb{T} \;\subset\; \mathcal{E}^1_{W,F} \quad\text{and}\quad \Phi^1(A^0 \cap Z_I) \;\subset\; \mathfrak{J}_I \;\subset\; \mathcal{E}^1_{\sigma_I,F}\,. \tag{3.22}$$

See [GGR21b, §4] for details.

3.5.2. *Ample line bundles over level one extension data of polarized mixed Hodge structures.* Given $\sigma = \mathrm{span}_{\mathbb{R}_{>0}}\{N_1,\ldots,N_m\}$, as (3.14), let $\mathbf{N} = \mathrm{span}_{\mathbb{Z}}\{N_1,\ldots,N_m\}$ be the $\mathbb{Z}$–module generated by the N_i, and let $\check{\mathbf{N}} = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{N},\mathbb{Z})$ be the dual. Every $M \in \check{\mathbf{N}}$ determines a line bundle $\mathcal{L}_M$ over the level one extension data $\mathcal{E}^1_{\sigma,F}$. The fact that the cone σ polarizes the mixed Hodge structure (W,F) implies that there is a nonempty subset $\mathbf{N}^{\mathfrak{sl}_2} \subset \check{\mathbf{N}}$ of line bundles $\mathcal{L}^*_M$ that polarize the compact complex torus $\mathfrak{J} \subset \mathcal{E}^1_{\sigma,F}$; that is $\mathfrak{J}$ is an abelian variety, see [GGR21b, §4] for details. This positivity is a general property of extension data of a *polarized* mixed Hodge structure.[2] Given a *limiting* mixed Hodge structure arising in the context of a triple $(\overline{B},Z;\Phi)$, as in §1.1, these line bundles are part of a rich structure relating the geometry of the Φ^0–fibres to the normal bundles $\mathcal{N}_{Z_i/\overline{B}}$.

3.5.3. *The central geometric information at infinity.* Let $\mathbf{N}_{A^0} = \mathrm{span}_{\mathbb{Z}}\{N_i \mid Z^*_i \cap A^0 \neq \emptyset\}$.[3] Given any $M \in \check{\mathbf{N}}_{A^0}$, the neighborhood $\overline{\mathcal{O}}^0$ of A^0 (cf. §3.1) admits a line bundle L_M with canonical section s_M having divisor

$$(s_M) \;=\; \sum \langle M, N_i\rangle (Z_i \cap \overline{\mathcal{O}}^0)\,.$$

In particular,

$$L_M \;=\; \sum \langle M, N_i\rangle \; [Z_i]|_{\overline{\mathcal{O}}^0}$$

[2]See also [BBT20] where this positivity is used to prove that the image of a mixed period map is quasi-projective.

[3]As defined, $\mathbf{N}_{A^0} \subset W_{-2}\mathrm{End}(V,Q)$. A subtle point (which we elide here) is that the $\mathbb{Z}$–module $\mathbf{N}_{A^0}$ is well-defined modulo $W_{-3}\mathrm{End}(V,Q)$, and so should be regarded as a subset of $\mathrm{Gr}^W_{-2}\mathrm{End}(V,Q)$.

The sections s_M encode the information in the nilpotent orbit that is lost in the map Φ_I of (3.1) when we quotient by $\exp(\mathbb{C}\sigma_I)$; see [GGR21b, §3] for details.

As in §3.5.2, each $M \in \check{\mathbf{N}}_{A^0}$ determines a line bundle $\mathcal{L}_M$ over the torus $\mathbb{T}$ in (3.22). These line bundles are related to the normal bundles

$$L_M\big|_{A^0} \;=\; (\Phi^1\big|_{A^0})^*(\mathcal{L}_M) \;=\; \sum \langle M, N_i\rangle \, \mathcal{N}_{Z_i/\overline{B}}\Big|_{A^0} \,. \tag{3.23}$$

An illustrative example is given in §4.3.5.

Remark 3.24. It follows from Lemma 3.20 that the extension data along A^0 that is *not* encoded by the map Φ^1 to level one extension data, and the sections s_M (which is level two extension data) is discrete. This is the sense in which (3.23) is the *central geometric information at infinity.*

3.6. **Applications.** We now turn to some applications of §3.5.2 and (3.23). The irreducible components of the Φ^0–fibre A^0 are all of the form $A^0_I = A^0 \cap Z_I$ with $A^0 \cap Z_I^*$ (nonempty and) Zariski open in A^0. Given any such I there is a nonempty subset $\mathbf{N}_I^{\mathfrak{sl}_2} \subset \check{\mathbf{N}}_{A^0}$ with the property that the $\mathcal{L}^*_M$, with $M \in \mathbf{N}_I^{\mathfrak{sl}_2}$, polarize $\mathcal{J}_I \subset \mathbb{T}$, cf. §3.5.2. Furthermore, the set

$$\mathbf{N}_I^{\mathfrak{sl}_2,+} \;=\; \{M \in \mathbf{N}_I^{\mathfrak{sl}_2} \mid \langle M, N_i\rangle > 0\,,\ \forall\, i \in I\}$$

is nonempty [GGR21b, Theorem 4.3].

Example 3.25. Suppose that $I = \{i\}$ and that $A^0_i = A^0$ is irreducible. We may choose $M \in \mathbf{N}_i^{\mathfrak{sl}_2,+}$, so that $\mathcal{L}^*_M \to J_i$ is ample and $\kappa(M, N_i) > 0$. Then $\mathcal{N}^*_{Z_i/\overline{B}}\big|_{A^0}$ is ample if the differential of $\Phi^1\big|_{A^0}$ is injective.

More generally, we have

Corollary 3.26 ([GGR21b]). *Suppose the differential $\Phi^1|_{A^0_I}$ is injective. Then the line bundle*

$$\sum \kappa(M, N_j)\, \mathcal{N}^*_{Z_j/\overline{B}}\Big|_{A^0_I} \tag{3.27}$$

is ample.

Remark 3.28. The sum (3.27) is over those j with $Z_j \cap A^0_I$ nonempty; this includes those $j \in I$, but will be a larger set when $A^0_I \not\subset Z^*_I$. A subtle point is that we may choose M so that the integers $\langle M, N_j\rangle$ are positive when $j \in I$; we are not able to say the same when $j \notin I$. This gives (3.27) somewhat the character of a negative definite matrix whose diagonal entries are negative but whose off-diagonal entries are non-negative. Suppose that we may choose $M \in \mathbf{N}_I^{\mathfrak{sl}_2,+}$ so that $\langle M, N_j\rangle \geq 0$ for all j (in the sum), and that $\dim A^0_I = 1$. Then $\deg\left(\Phi^1|_{A^0_I}\right)^*(\mathcal{L}_M) > 0$. For $j \notin I$ we have $Z_j \cdot A^0_I \geq 0$, so that $\langle M, N_j\rangle \deg\ [Z_j]|_{A^0_I} \geq 0$. This suggests that $\langle M, N_i\rangle \deg\ [Z_i]|_{A^0_I} < 0$ for $i \in I$.

The next two applications of (3.23) are special cases of

Conjecture 3.29 ([GGR21a])**.** *Under suitable local Torelli-type assumptions, there exist integers $0 \leq a_i \in \mathbb{Z}$ and m_0 so that $m\Lambda_{\mathrm{e}} - \sum a_i[Z_i]$ is ample for $m \geq m_0$.*

Proposition 3.30 ([GGR21a])**.** *Suppose that $Z = Z_1$ consists of a single irreducible component, and $\mathrm{d}\Phi^1$ is injective on Φ^0–fibres. Assume also that the effective cone $\mathrm{Eff}^1(\overline{B})$ of 1–cycles is finitely generated. Then the line bundle $\Pi = m\Lambda_{\mathrm{e}} - [Z]$ is ample for $m \geq m_0$.*

Outline of proof. It suffices to show that there exists m_0 so that $(m\Lambda_{\mathrm{e}} - [Z]) \cdot C > 0$ for all curves $C \subset \overline{B}$ and $m \geq m_0$. Without loss of generality, we may assume that C is an irreducible curve.

If the image $\Phi^0(C)$ is also a curve, then $\Lambda_{\mathrm{e}} \cdot C > 0$. So we will have $(m\Lambda_{\mathrm{e}} - [Z]) \cdot C > 0$ when $m \gg 0$. Now suppose that $C \subset A^0$ is contained in a Φ^0–fibre. Then $\Lambda_{\mathrm{e}} \cdot C = 0$. However, the hypothesis that $\mathrm{d}\Phi^1$ is injective and §3.5.2 imply that $\mathcal{N}^*_{Z/\overline{B}}\big|_C$ is ample. In particular, $-[Z] \cdot C > 0$. $\square$

Theorem 3.31 ([GGR21a])**.** *Suppose that* $\dim B = 2$ *and that the cone of* $\mathrm{Eff}^1(\overline{B})$ *of effective algebraic 2-cycles is finitely generated. Assume that the period map* Φ :

$B \to \Gamma\backslash D$ is locally injective, Z is connected, $\Phi^0(Z)$ is a point, and that there exists m_0 and integers $a_i \geq 0$ so that

$$L_m \;=\; m\Lambda_{\mathrm{e}} - \sum a_i[Z_i]$$

is ample for all $m \geq m_0$.

Remark 3.32. Several of the hypotheses in the theorem may be dropped. See [GGR21a] for a more general result.

Outline of proof. Briefly, the argument is as follows.

The local Torelli hypothesis implies $\Lambda_{\mathrm{e}}^2 > 0$. The hypothesis that $\Phi^0(Z_i)$ is a point implies $\Lambda_{\mathrm{e}} \cdot Z_i = 0$. The Hodge index theorem then implies that the intersection matrix $\|Z_i \cdot Z_j\|$ is negative definite. There exist $a_i > 0$ so that $Z_j \cdot \sum_i a_i Z_i < 0$, for all j [GGR21a, Lemma 2.3].

If $C \subset \overline{B}$ is an irreducible curve, then either $C \cap B$ is Zariski open in C in which case $\Lambda_{\mathrm{e}} \cdot C > 0$, or $C = Z_j$ for some j. It follows that there exists $m_0(C)$ so that $L_m \cdot C > 0$ for $m \geq m_0(C)$. The finite generation of $\mathrm{Eff}^1(\overline{B})$ implies that we may choose m_0 independent of C. The desired result now follows from the Nakai–Moishezon criterion for ampleness. □

Question 3.33. The proof above motives the following question. Suppose that $\overline{\mathcal{O}}$ is a smooth complex surface containing a reduced normal crossing divisor $Z = \cup_i Z_i$. (What we have in mind here is that we replace the projective $\overline{B}$ with an analytic neighborhood $\overline{\mathcal{O}}$ of Z.) Given a period map Φ on $\mathcal{O} = \overline{\mathcal{O}}\backslash Z$, suppose that the map Φ^0 collapses the Z_i to points. Can we conclude that the intersection matrix $\|Z_i \cdot Z_j\|$ is negative definite?

The final application is a constraint on the variations of limiting mixed Hodge structure that may arise along the divisor Z when $\dim B = 2$.

Proposition 3.34 ([GGR21a]). *Assume that* $\dim B = 2$ *and that* $\Phi : B \to \Gamma\backslash D$ *satisfies generic local Torelli (equivalently,* Φ_* *is injective at some point* $b \in B$*, so that* $\dim \wp = 2$*). Then* Φ^1 *is necessarily non-constant on some irreducible component* Z_i *of* Z*.*

Definition 3.35. The variation $(\mathcal{W}^I, \mathcal{F}_{\mathrm{e}}|_{Z_I^*})$ of limiting mixed Hodge structure along Z_I^* is of *Hodge–Tate type*, if the associated graded variation $\mathcal{F}_{\mathrm{e}}^p(\mathrm{Gr}_a^{\mathcal{W}^I})$ of Hodge structure is Hodge–Tate.

Remark 3.36. When the variation along Z_I^* is of Hodge–Tate type, both the period map $\Phi^0|_{Z_I^*}$ and map $\Phi^1|_{Z_I^*}$ of (3.7) are locally constant along Z_I^*. (The level one extension data $\mathrm{Ext}_1(\sigma, F)$ is zero along fibre A^0 of Φ^0.) In this case all the information in $\Phi^{\mathsf{T}}\big|_{Z_I^*}$, up to constants of integration, is encoded in the sections s_M of the line bundles $L_M \to \overline{\mathcal{O}}^0$ (Remark 3.24).

Corollary 3.37. *Suppose that* B *is a surface and that the limiting mixed Hodge structures along all of* Z *is of Hodge–Tate type. Then* $\dim \wp \leq 1$*. Equivalently, if* $\dim \wp = 2$*, then there is at least one* Z_i *such that the variation of limiting mixed Hodge structure along* Z_i^* *is not of Hodge–Tate type.*

Proof of Proposition 3.34. We argue by contradiction. Suppose that Φ^1 is constant along all of Z. Then Φ^0 is necessarily constant along all of Z; that is, $Z = A^0$. Since $\Phi^1(Z) = \Phi^1(A^0)$ is a point in the compact torus $\mathbb{T}$, it follows from (3.23) that

$$(\Phi^1\big|_Z)^*(\mathcal{L}_M) \;=\; \sum_{i=1}^{\nu} \langle M, N_i\rangle [Z_i]\big|_Z \quad \text{is trivial.}$$

So

$$0 \;=\; \left(\sum_{i=1}^{\nu} \langle M, N_i\rangle [Z_i]\right)^2 \;=\; \sum_{i,j=1}^{\nu} \langle M, N_i\rangle \langle M, N_j\rangle\, Z_i \cdot Z_j$$

The Hodge index theorem implies that the intersection matrix $\|Z_i \cdot Z_j\|$ is negative definite [GGLR20, Lemma 3.1.1], and this forces $\langle M, N_i\rangle = 0$ for all i. As M is arbitrary, this is a contradiction. □

4. Discussion of weight $n = 1$

The goal of this section is to illustrate the constructions above in the classical weight $n = 1$ case. The geometric interpretations of level one extension data for a nodal curve C (as discussed in §4.1.1 and §4.2) go back to [Car87]. In that work Carlson considers Deligne's mixed Hodge structure on $H^1(C)$, for which all extension data is of level one. The limiting mixed Hodge structure of a smoothing deformation of C also has level two extension data, and this is discussed in §4.1.2.

The period domain parameterizing pure polarized Hodge structures of weight $n = 1$ and Hodge numbers $\mathbf{h} = (g, g)$ is the generalized Siegel upper half space $\mathcal{H}_g$. For our illustrative example (§§4.2–4.3) we will take the case $g = 2$, as the significant (classical) phenomena are all present in this case.

4.1. **Geometric interpretation of extension data.** Let C be an irreducible curve with μ nodes $\{r_i\}_{i=1}^{\mu}$. Let $\pi : \widetilde{C} \to C$ denote the normalization. A smoothing deformation of C produces a limiting mixed Hodge structure (W, F, σ), with

$$H^0 \;\simeq\; H^0(\{r_i\}_{i=1}^{\mu})\,, \quad H^1 \;\simeq\; H^1(\widetilde{C})\,, \quad H^2 \;\simeq\; H^0(\{r_i\}_{i=1}^{\mu})(-1)$$

and

$$\sigma \;=\; \mathrm{span}_{\mathbb{R}_{>0}}\{N_1, \ldots, N_\mu\}\,,$$

where N_i corresponds to smoothing the i-th node $r_i \in C$. We may fix a basis of $\oplus_\ell \mathrm{Gr}^W_\ell = H^2 \oplus H^1 \oplus H^0$ that respects this direct sum, and with respect to which

$$Q \;=\; \left[\begin{array}{ccc} 0 & 0 & I_\mu \\ 0 & \hat{Q} & 0 \\ -I_\mu & 0 & 0 \end{array}\right],$$

with $\hat{Q}$ the intersection form on H^1, and

$$N_i \;=\; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \nu_i & 0 & 0 \end{bmatrix},$$

where ν_i is the $\mu \times \mu$ matrix whose only nonzero entry is the i-th diagonal entry.

4.1.1. *Level one extension data.* Let $\pi^{-1}(r_i) = \{p_i, q_i\}$ denote the preimages of the nodes. A neighborhood of C in the corresponding stratum of moduli is swept out by varying $\widetilde{C}$ and the $\{p_i, q_i\}$. Restricting to a Φ^0–fibre A corresponds to fixing $\widetilde{C}$. On that fibre, the level one extension data is $\mathrm{Ext}_1(W, F) = \mathrm{Ext}(H^1, H^0) \oplus \mathrm{Ext}(H^0(-1), H^1)$. Setting $D = \cup_i \{p_i, q_i\} \subset \widetilde{C}$, the group $\mathrm{Ext}(H^1, H^0)$ parameterizes the extension data in the sequence

$$\begin{tikzcd} 0 \arrow[r] & W_0 \arrow[r] \arrow[d, hook] & W_1 \arrow[r] \arrow[d, equal] & \mathrm{Gr}^W_1 \arrow[r] \arrow[d, equal] & 0 \\ & H^0(D) \arrow[r] & H^1(\widetilde{C}, D) \arrow[r] & H^1(\widetilde{C}) \arrow[r] & 0\,. \end{tikzcd} \tag{4.1}$$

It is

$$\frac{\mathrm{Hom}_{\mathbb{C}}(H^1, H^0)}{F^0\mathrm{Hom}_{\mathbb{C}}(H^1, H^0) \,+\, \mathrm{Hom}_{\mathbb{Z}}(H^1, H^0)} \;\simeq\; (H^{0,1}/H^1_{\mathbb{Z}}) \otimes H^0_{\mathbb{Z}} \;\simeq\; \bigoplus_1^{h^0} J(\widetilde{C})\,,$$

where $J(\widetilde{C})$ is the Jacobian variety of $\widetilde{C}$ and h^0 is the rank of $H^0_{\mathbb{Z}}$. Then

$$\Phi^1(C) \;=\; \sum_i \mathrm{AJ}_{\widetilde{C}}(p_i - q_i) \;\in\; \bigoplus_{i=1}^{h^0} J(\widetilde{C})\,.$$

(Here, we fix an ordering of $\{p_i, q_i\} \subset \widetilde{C}$.[4]) In the classical formulation using differential forms, we have

$$J(\widetilde{C}) \;=\; H^{0,1}/H^1_{\mathbb{Z}} \;\simeq\; H^0(\Omega^1_{\widetilde{C}})^*/H_1(\widetilde{C}, \mathbb{Z})\,.$$

Given $\omega \in H^0(\Omega^1_{\widetilde{C}})$ we choose a path γ with $\partial\gamma = \sum p_i - q_i$. Then Φ^1 is given by the map $\omega \mapsto \int_\gamma \omega$ modulo periods.

[4]This may require that we take a branched cover for the family of curves.

The group $\mathrm{Ext}(H^0(-1), H^1)$ parameterizes the extension data in the sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_1/W_0 & \longrightarrow & W_2/W_0 & \longrightarrow & \mathrm{Gr}_2^W & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow & & \\ 0 & \longrightarrow & H^1(\widetilde{C}) & \longrightarrow & H^1(\widetilde{C}\backslash D) & \longrightarrow & H^0(D)(-1)\,. & & \end{array}$$

We have

$$\mathrm{Ext}(H^0(-1), H^1) \;=\; \frac{\mathrm{Hom}_{\mathbb{C}}(H^0(-1), H^1)}{F^0\mathrm{Hom}_{\mathbb{C}}(H^0(-1), H^1) \,+\, \mathrm{Hom}_{\mathbb{Z}}(H^0(-1), H^1)}\,.$$

For each $\{p_i, q_i\}$ we choose $\eta_i \in H^0(\Omega^1_{\widetilde{C}}(\log D))$ with $\mathrm{Res}_{p_i}\eta_i = 1$ and $\mathrm{Res}_{q_i}\eta_i = -1$, and $\mathrm{Res}_{p_j}\eta_i = 0 = \mathrm{Res}_{q_j}\eta_i$ for all $i \neq j$. Then

$$\eta \;=\; \sum_i \eta_i \;\in\; H^0(\Omega^1_{\widetilde{C}}(\log D)) \;\subset\; \mathbb{H}^1(\Omega^\bullet_{\widetilde{C}}(\log D)) \;\simeq\; H^1(\widetilde{C}\backslash D)$$

lifts $\sum p_i - q_i \in H^0(D)(-1)$ and is well-defined modulo $H^0(\Omega^1_{\widetilde{C}})$.

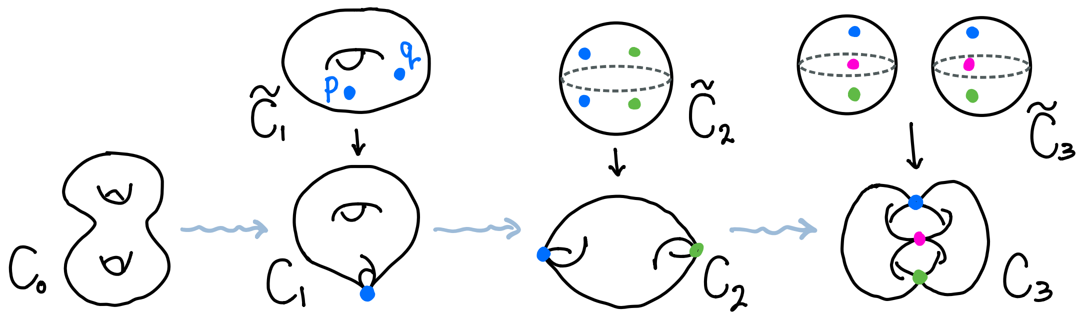
4.1.2. *Level two extension data.* The above is standard [Car87]. Perhaps less familiar is the geometric expression of the level two extension data in terms of differential forms and integrals. (As noted above, Deligne's mixed Hodge structure on $H^1(C)$ does *not* have level two extension data; but the limiting mixed Hodge structure given by a smoothing deformation *does.*)

We are considering equivalence classes of limiting mixed Hodge structures with monodromy weight filtrations $\{0\} \subset W_0 \subset W_1 \subset W_2 = V$, and where both the Hodge structures H^ℓ and the level one extensions of the mixed Hodge structure are fixed. The fibre (3.19) is given the symmetric part of

$$\frac{\mathrm{Ext}(H^0(-1), H^0)}{\exp(\mathbb{C}\sigma)} \;\simeq\; \frac{\mathrm{Hom}^{\mathrm{sym}}_{\mathbb{Z}}(H^0(-1), H^0)}{\mathrm{span}_{\mathbb{Z}}\{N_1, \ldots, N_{h^0}\}} \otimes \mathbb{C}^*\,.$$

Using the identification above, this data is represented by the off-diagonal terms in $h^0 \times h^0$ symmetric matrices whose entries are in $\mathbb{C}/2\pi\mathbf{i}\mathbb{Z}$. Those off-diagonal entries are obtained as follows. For each i, we choose a path γ_i with $\partial\gamma_i = p_i - q_i$. Then for

Figure 1. Degenerations of C_0 and their normalizations



$i \neq j$, the bilinear relations for differentials of the third kind give the classical

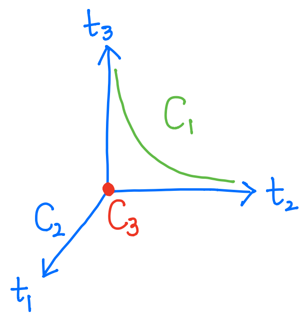
$$\int_{\gamma_i} \eta_j \equiv \int_{\gamma_j} \eta_i \quad \text{modulo periods.}$$

Example 4.2. The simplest and most classical example is when $\widetilde{C} = \mathbb{P}^1$ and $h^0 = 2$: in this case, the above construction produces the cross ratio of 2 pairs of ordered distinct points in $\mathbb{P}^1$.[5]

4.2. **Example: a 3-dimensional family of branched covers of $\mathbb{P}^1$.** Consider the family $\overline{\mathcal{C}} = \{C_{a,b,c}\} \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of double covers $C_{a,b,c} \xrightarrow{2:1} \mathbb{P}^1$ branched over six points $\{0, 1, \infty, a, b, c\} \subset \mathbb{P}^1$. Over the locus $B \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ where the six points are pairwise distinct, the branched cover $C_{a,b,c}$ is a smooth curve of genus $g = 2$. The curve $C_{0,1,\infty}$ is singular, consisting of two copies of $\mathbb{P}^1$ identified at three points (the curve C_3 in Figure 1). We will consider the family $\{C_{a,b,c}\}$ in a neighborhood of this curve. To that end, fix local coordinates $t = (t_1, t_2, t_3)$ at $(0, 1, \infty) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ so that $t_1 = 0 - a$, $t_2 = 1 - b$ and $t_3 = 1/\infty - 1/t_3$. At a general point of $\{t_1 t_2 t_3 \neq 0\}$ the $\{a, b, c\}$ are pairwise distinct and the curve is smooth. Our goal in this section is to describe the maps Φ^0 and Φ^1 along the strata Z_I^*.

[5]When C is not irreducible it is necessary to introduce combinatorial data arising from its dual graph.

FIGURE 2. Parameter space for curves



4.2.1. *Behavior along codimension one strata.* A general point in $Z_1^* = \{t_1 = 0\,,\ t_2 t_3 \neq 0\}$ corresponds to a nodal curve C_1 with $p_a = 2$ and normalization $\widetilde{C}_1 \to C_1$ (Figure 1). The restriction $\Phi^0|_{Z_1^*}$ is the period map for the family of elliptic curves $\{\widetilde{C}_1\}$. In particular, the fibres of $\Phi^0|_{Z_1^*}$ are the curves where the cross ratio $(1,\infty;t_2,t_3) = (1-t_2)/(1-t_3)$ is constant. (This is represented by the green curves in Figure 2.)

The nilpotent logarithm of monodromy Z_1^* is

$$N_1 \ = \ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} .$$

(This matrix representation is with respect to the basis $\{v_1, v_2, v_3, v_4\}$ below.) Here the level one extension data $\mathcal{J}_1$ is equivalent to the Jacobian $\mathcal{J}(\widetilde{C}_1)$ and the mapping Φ^1 along Z_1^* is essentially the Abel–Jacobi map $\mathrm{AJ}_{\widetilde{C}_1}(p-q)$. The level two extension data is $\mathrm{Ext}_2(\sigma, F) = \mathrm{Ext}_\sigma(H^0(-1), H^0) \simeq \mathbb{C}^*$. This extension data corresponds to the nilpotent cone $\sigma = \mathrm{span}_{\mathbb{R}_{>0}}\{N_1\}$ (or rather its complexification, as in (3.19)). The nilpotent orbit is encoded by the canonical section s_M of a line bundle $L_M \to \overline{\mathcal{O}}^0$ with divisor $(s_M) = Z_1 \cap \overline{\mathcal{O}}^0$ (as in §3.5.3) that vanishes along $Z_1 \cap \overline{\mathcal{O}}^0$.

The descriptions over the strata $Z_2^* = \{t_2 = 0\,,\ t_1 t_3 \neq 0\}$ and $Z_3^* = \{t_3 = 0\,,\ t_1 t_2 \neq 0\}$ are similar, and the nilpotent logarithms of monodromy about Z_2^* and

Z_3^* are

$$N_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad N_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

respectively.

4.2.2. *Behavior along the coordinate axes.* A general point of $Z_{12}^* = \{t_1, t_2 = 0\,,\ t_3 \neq 0\}$ corresponds to a curve C_2 with two nodes, and a normalization $\mathbb{P}^1 = \widetilde{C}_2 \simeq C_2$ (Figure 1). Over Z_{12} the limiting mixed Hodge structures are of Hodge–Tate type. In particular, the period map $\Phi^0|_{Z_{12}^*}$ is constant (Remark 3.36), as is $\Phi^1|_{Z_{12}^*}$. In particular, both maps collapse the coordinate axes in Figure 2 to a point. The level two extension data is $\mathrm{Ext}_2(\sigma, F) = \mathrm{Ext}_\sigma(H^0(-1), H^0) \simeq \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$. These three copies of $\mathbb{C}^*$ corresponds to the nilpotent logarithms $\{N_1, N_2, N_3\}$ of monodromy at $t = (0,0,0)$. As noted in Remark 3.36, all the information in $\Phi^\mathsf{T}\big|_{Z_I^*}$, up to constants of integration, is encoded in the sections s_M of the line bundles $L_M \to \overline{\mathcal{O}}^0$, where $M \in \check{\mathbf{N}}$ with $\mathbf{N} = \mathrm{span}_\mathbb{Z}\{N_1, N_2, N_3\}$.

4.3. **Example: period matrix representations and local analytic structure about Z_1^*.**

4.3.1. *Limiting mixed Hodge structure along Z_1^*.* Fix a basis $\{v_1, v_2, v_3, v_4\}$ of $V_\mathbb{C}$ so that $\bar{v}_1 = v_1, \bar{v}_4 = v_4$ and $\bar{v}_2 = -\mathbf{i}v_3$. Let Q be the skew-symmetric bilinear form that is represented by the matrix

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

with respect to this basis. Then

$$F^1 \;=\; \mathrm{span}\{v_1, v_2\}$$

and

$$W_0 \;=\; \mathrm{span}\{v_4\} \quad\text{and}\quad W_1 \;=\; \mathrm{span}\{v_2, v_3, v_4\}$$

defines a limiting mixed Hodge structure (W, F, N_1).

4.3.2. *Schubert cell.* The reduced period limit is $F^1_\infty = \mathrm{span}\{v_2, v_4\}$, and has complex conjugate is

$$\overline{F^1_\infty} \;=\; \mathrm{span}\{v_3, v_4\}\,.$$

The Schubert cell (3.9) is the set

$$\mathcal{S} \;=\; \{E \in \check{D} \mid E \cap \overline{F^1_\infty} = 0\}$$

of all 2-planes $E \subset V_{\mathbb{C}}$ that satisfy the first Hodge–Riemann bilinear relation $Q|_E = 0$ and have trivial intersection with $\overline{F^1_\infty}$. This is precisely the set of two planes admitting a basis of the form

$$E \;=\; \mathrm{span}\{v_1 + \alpha v_3 + \nu v_4\,,\ v_2 + \lambda v_3 + \alpha v_4\}\,.$$

The coefficients define a biholomorphism $(\alpha, \lambda, \nu) : \mathcal{S} \to \mathbb{C}^3$.

4.3.3. *Period matrix representation.* If the limiting mixed Hodge structures along the fibre A^0 are polarized by N_1, then A^0 admits a neighborhood $\overline{\mathcal{O}}^0 \subset \overline{B}$ so that the matrix representation of Φ over $\mathcal{O}^0 = B \cap \overline{\mathcal{O}}^0$ is given by

$$\Phi|_{\mathcal{O}^0} \;=\; \left[\begin{array}{cccc} 1 & 0 & \alpha & \nu \\ 0 & 1 & \lambda & \alpha \end{array}\right]^{\mathsf{t}}; \tag{4.3}$$

that is, given $b \in \mathcal{O}^0$, the Hodge filtration $F^1(b) \subset V_{\mathbb{C}}$ parameterized by $\Phi(b)$ is given by

$$F^1(b) \;=\; \mathrm{span}\{v_1 + \alpha v_3 + \nu v_4\,,\ v_2 = \lambda v_3 + \alpha v_4\}\,.$$

In general $(\lambda, \alpha, \nu) : \mathcal{O}^0 \to \mathbb{C}^3$ is multivalued (§4.3.4). However, if we restrict to a local coordinate chart $(t, w) : \overline{\mathcal{U}} \to \mathbb{C}^3$ centered at $b \in \overline{\mathcal{O}}^0$ and so that N is the logarithm of local monodromy about $\{t = 0\}$, then the nilpotent orbit theorem implies that each of $\lambda, \alpha, \nu - (\log t_1)/2\pi\mathbf{i}$ have well-defined holomorphic determinations.

4.3.4. *Action of monodromy.* Here the α, λ, ν are multi-valued holomorphic functions. The multivalued-ness is due to the monodromy about A^0, which is given by

$$\gamma \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathsf{a} & 1 & 0 & 0 \\ \mathsf{b} & 0 & 1 & 0 \\ \mathsf{c} & \mathsf{b} & -\mathsf{a} & 1 \end{bmatrix},$$

with $\mathsf{c} \in \mathbb{Z}$, $\mathsf{a} \in \mathbb{Z} + \mathbf{i}\mathbb{Z}$ and $\mathsf{b} = -\mathbf{i}\bar{\mathsf{a}}$. The period matrix representation transforms as

$$\gamma \cdot \Phi|_{\mathcal{O}^0} \;=\; \begin{bmatrix} 1 & 0 & \alpha + \mathsf{b} - \mathsf{a}\lambda & \nu + \mathsf{c} - \mathsf{ab} + 2\mathsf{a}\alpha + \mathsf{a}^2\lambda \\ 0 & 1 & \lambda & \alpha + \mathsf{b} - \mathsf{a}\lambda \end{bmatrix}^{\mathsf{t}}. \tag{4.4}$$

Note that λ is a well-defined function $\overline{\mathcal{O}}^0 \to \mathbb{C}$.

4.3.5. *Theta line bundle.* If we write $m = \kappa(M, N) \in \mathbb{Z}$, then locally the line bundle L_M admits a trivialization with respect to which the canonical section s_M is given by the function

$$\tau_M(t) \;=\; t_1^m \exp(2\pi\mathbf{i}\, m\, \nu(t)) .$$

While $\tau_M(t)$ is invariant under the local monodromy $\exp(N_1)$, we must also account for the monodromy about the fibre A^0. From (4.4) we see that $\tau_M(t)$ transforms as

$$\begin{aligned} \tau_M(t) \;&\mapsto\; t_1^m \exp 2\pi\mathbf{i}\, m\, (\nu(t) + \mathsf{a}^2\lambda(t) - 2\mathsf{a}\alpha(t)) \\ &=\; \tau_M(t) \exp 2\pi\mathbf{i}\, m\, (\mathsf{a}^2\lambda(t) - 2\mathsf{a}\alpha(t)) . \end{aligned}$$

This is the functional equation for the classical theta function. We may normalize our choice of coordinates t so that $\nu(t) = (\log t_1)/2\pi\mathbf{i}$. Then, taking $m = 1$, this computation implies that $t_1 \cdot \vartheta$, with ϑ a section of the dual to the theta line bundle, is globally well-defined along A^0; that is, the pull-back of the theta line bundle is the conormal bundle.

4.3.6. *Local analytic structure.* Upon restricting to a neighborhood $\overline{\mathcal{O}}^1 \subset \overline{\mathcal{O}}^0$ of a fibre $A^1 \subset A^0$, the monodromy about A^1 simplifies to $\mathsf{a}, \mathsf{b} = 0$. In particular, the functions $\alpha, \lambda, \exp(2\pi\mathbf{i}\nu)$ are well-defined over $\mathcal{O}^1 = B \cap \overline{\mathcal{O}}^1$, and extend (again, by the nilpotent orbit theorem) to holomorphic functions on $\overline{\mathcal{O}}^1$. This gives us the map $f = (\alpha, \lambda, \exp(2\pi\mathbf{i}\nu)) : \overline{\mathcal{O}}^1 \to \mathbb{C}^3$ in step (c) of the proof of Theorem 1.7.

5. Discussion of weight $n = 2$

The goal of this section is to illustrate the constructions above in a non-classical weight $n = 2$ case. We begin with a review of level one extension data in §5.1, and then specialize to a class of surfaces with moduli space admitting a canonical projective compactification $\overline{\mathcal{M}}_I$ that has been extensively studied by Franciosi–Pardini–Rollenske [FPR15a, FPR15b, FPR17]. The compactification $\overline{\mathcal{M}}_I$ is highly singular along the boundary, and it seems that it is exactly the extension data in the limiting mixed Hodge structure that may guide a desingularization at a general point of the boundary (Remark 5.4). The period matrix representation for family of surfaces with $p_g = 2$ is discussed in §5.3, and some geometric interpretation of this is given in §5.4.

5.1. **Geometric interpretation of level one extension data.**

5.1.1. *Extension data for $C \subset X$.* We begin with the review the geometric interpretation of the extension data for a smooth, but possibly reducible, curve C on a smooth surface X. Then the relevant dual exact sequences are

$$0 \longrightarrow \frac{H^1(C)}{H^1(X)} \longrightarrow H^2(X,C) \longrightarrow \ker\left\{H^2(X) \to H^2(C)\right\} \longrightarrow 0$$

$$0 \longrightarrow \frac{H^2(X)}{\mathrm{Gys} H^0(C)} \longrightarrow H^2(X \backslash C) \longrightarrow \ker\left\{\mathrm{Gys} : H^1(C)(-1) \to H^3(X)\right\} \longrightarrow 0$$

with Gys the Gysin map. Assuming for simplicity that X is regular, the numerator of $\mathrm{Ext}_{\mathrm{MHS}}$ for the first sequence is

$$\mathrm{Hom}_{\mathbb{C}}\left(\ker\{H^2(X) \to H^2(C)\}\,,\, H^1(C)\right)\,.$$

It is convenient to write $H^2(X) = \mathrm{Hg}^1(X) \oplus H^2(X)_{\mathrm{tr}}$, with $H^2(X)_{\mathrm{tr}}$ the transcendental part of $H^2(X)$, and where the summands are orthogonal with respect to the intersection pairing. Assuming for the moment that C is irreducible, elements of $\ker\{\mathrm{Hg}^1(X) \to H^2(C)\}$ are given by divisors D on X such that $D \cdot C = 0$. Unwinding the definitions, we see that the extension class corresponding to $\mathrm{Hg}^1(X) \subset H^2(X)$ is given by the

$$D \;\mapsto\; \mathrm{AJ}_C(D \cdot C)\,.$$

For the transcendental part $H^2(X)_{\mathrm{tr}}$ of $H^2(X)$, after factoring by the F^0–part of the denominator, a typical element is $\xi = \alpha + \beta$ with $\alpha \in H^{2,0}(X) = H^0(\Omega^2_X)$ and $\beta \in H^{1,1}(X)$ with $\beta|_C = \mathrm{d}\gamma$, with γ a $(1,0)$–form on C that is orthogonal to the harmonic forms $H^{1,0}(C)$. Given $\delta \in H_1(C,\mathbb{Z})$ we have $\delta = \partial\Delta$ for a 2-chain Δ in X. The transcendental part of the extension class is then given by

$$\langle \xi, \delta\rangle \;=\; \int_\Delta \alpha \;-\; \int_\delta \gamma\,,$$

modulo periods. The term $\int_\Delta \alpha$ is a *membrane integral.* (For more on membrane integrals, see [KLMS06] and the expository [Lew06].)

5.1.2. *Extension data for a pair of surfaces glued together along a curve.* Consider two smooth surfaces X_1 and X_2 with smooth curves $C_1 \subset X_1$ and $C_2 \subset X_2$ together with an isomorphism $C_1 \simeq C_2$. Let X be the surface obtained by gluing X_1 and X_2 together along the curves (via the isomorphism). For simplicity of notation, we identify the curves and denote them by C. Then

$$X \;=\; X_1 \cup_C X_2\,.$$

(The extension data for Deligne's mixed Hodge structure on $H^2(X)$ was first studied in [Car87].) A necessary condition [Fri83] for X to be smoothable is

$$N_{C/X_1} \simeq N^*_{C/X_2}\,. \tag{5.1}$$

In this case, there is a well-defined equivalence class of limiting mixed Hodge structures [PS08, Ste76]. Again for simplicity we assume that X_1 and X_2 are regular. Then the limiting mixed Hodge structure has

$$H^1 \simeq H^1(C)\,.$$

To describe H^2, consider the complex

$$H^0(C)(-1) \xrightarrow{\alpha} H^2(X_1) \oplus H^2(X_2) \xrightarrow{\beta} H^2(C)\,;$$

here α is the direct sum of the Gysin maps and β is the difference of the restriction mappings. The smoothing condition (5.1) implies $C_1^2 + C_2^2 = 0$, as line bundles, so that $\beta \circ \alpha = 0$. Then

$$H^2 = \frac{\ker \beta}{\operatorname{im} \alpha}$$

is the cohomology of this complex.

5.2. **I-surfaces.** For a specific illustration of §5.1 we consider the Kollár–Shepherd-Barron–Alexeev (KSBA) moduli space $\mathcal{M}_I$ of smooth, minimal, regular ($q(X) = 0$), general type surfaces X with $K_X^2 = 1$ and $p_g(X) = 2$.[6] These surfaces are in many ways the analog of genus two curves. The moduli space $\mathcal{M}_I$ is essentially smooth and of dimension 28.[7] The period domain D is of dimension 57 and the IPR is a contact structure on D. The period mapping

$$\Phi : \mathcal{M}_I \to \Gamma\backslash D$$

[6]The discussion that follows is cursory. Some of this is discussed in more detail in [Gri19, Gri18, Gri20]. A reader with a working knowledge of surface theory and mixed Hodge theory, and the papers [FPR15a, FPR15b, FPR17] will be able to to fill in the details.

[7]This means that $H^1(TX)$ is unobstructed. In particular, the Kuranishi space is smooth and $\mathcal{M}_I$ is locally the quotient of an open set in $\mathbb{C}^{28}$ by a finite group.

is locally injective, and the image $\Phi(\mathcal{M}_I)$ is a contact submanifold.[8]

5.2.1. *The KSBA compactification.* Unlike $\overline{\mathcal{M}}_2$, the space $\overline{\mathcal{M}}_I$ is highly singular along the boundary. It is exactly the extension data in the limiting mixed Hodge structure that may guide a desingularization of the boundary (Remark 5.4).

The surfaces X_0 parameterized by the boundary $\partial\mathcal{M}_I = \overline{\mathcal{M}}_I \backslash \mathcal{M}_I$ have $\mathbb{Q}$-Gorenstein canonical divisor class K_{X_0} and semi-log-canonical (slc) singularities. These slc singularities have been classified [Kol13]. In the case that X_0 is normal, and $p \in X_0$ is an isolated singular point:

(i) If X_0 is Gorenstein, then p is either simple elliptic, a cusp or a du Val singularity.

(ii) If X_0 is non-Gorenstein, then p is a rational singularity. If X_0 is smoothable, then we may assume that X_0 has $\mathbb{Q}$–Gorenstein smoothable singularities [Hac16, Kov13].

The period map $\Phi : \mathcal{M}_I \to \Gamma\backslash D$ admits an extension $\Phi^0 : \overline{\mathcal{M}}_I \to \overline{\wp}^0$, ibid. The monodromy about points of type (ii) is finite. The monodromy about points of type (i) is infinite and there is a nontrivial limiting mixed Hodge structure (W, F, σ) associated with a degeneration $X \to X_0$.

5.2.2. *The stratum $\mathcal{N}_2$.* There is a 20-dimensional boundary component $\mathcal{N}_2 \subset \overline{\mathcal{M}}_I$ whose general point corresponds to a singular I-surface X_0 that is normal, Gorenstein and with a simple elliptic singularity of degree 2.[9] The resolution $(\widetilde{X}, \widetilde{C}) \to (X_0, p)$ of this singularity is a smooth surface $\widetilde{X}$, whose minimal model is a K3 surface X, with an elliptic curve $\widetilde{C} \subset \widetilde{X}$ of self-intersection $\widetilde{C}^2 = -2$. The map $\widetilde{X} \to X$ contracts a

[8]The monodromy group Γ is of finite index in $\mathrm{Aut}(V_{\mathbb{Z}}, Q)$. Since $K_X^2 = 1$, the intersection form is unimodular on the primitive cohomology. The ideal situation would be that $\Gamma = \mathrm{Aut}(V_{\mathbb{Z}}, Q)$ and that generic Torelli holds; but this is an important open issue.

[9]For us this example arose in the September 2017 meeting at Duke with Radu Laza, Marco Franciosi, Rita Pardini and Sönke Rollenske and was instrumental in suggesting the use, in general, of extension data of the period mapping at infinity as a method of (at least partially) desingularizing the boundary of KSBA moduli spaces of surfaces.

-1 curve E with $E \cdot \widetilde{C} = 2$. In particular, the image $C \subset X$ of $\widetilde{C}$ is a curve with one node and self-intersection $C^2 = 2$. From this it follows that X is a 2:1 cover of $\mathbb{P}^2$ branched over a sextic curve C', and that C is a double cover of a tangent line ℓ to C'.

Given $X_0 \in \mathcal{N}_2$ consider a one-parameter degeneration $X_t \to X_0$ and do a semi-stable reduction to have a smooth total space with normal crossing divisor $\widetilde{X}_0$ over the origin. From the Clemens–Schmid exact sequence [Cle77, Mor84] the simplest possibility is that $\widetilde{X}_0$ has a double curve isomorphic to $\widetilde{C}$; that is,

$$\widetilde{X}_0 \;=\; \widetilde{X} \,\cup_{\widetilde{C}}\, Y\,,$$

with $Y \supset \widetilde{C}$ a smooth surface. Friedman's smoothability condition (5.1) implies

$$N^*_{\widetilde{C}/\widetilde{X}} \;\simeq\; N_{\widetilde{C}/Y}\,.$$

The line bundle $N^*_{\widetilde{C}/\widetilde{X}}$ has degree 2. And if we think of Y as obtained from a smooth cubic $\widetilde{C}$ in $\mathbb{P}^2$ by blowing up points $\{q_i\}$ on the cubic, then there must be seven points q_i in order to have $\deg N_{\widetilde{C}/Y} = 2$.

5.2.3. *The limiting mixed Hodge structure.* The nilpotent logarithm N of monodromy for this degeneration has rank 2 and satisfies $N^2 = 0$. From the general procedure for computing the limiting mixed Hodge structure over $\mathbb{Q}$ [PS08] one has the following

$$H^1 \;=\; \mathrm{Gr}^W_1 \;=\; H^1(\widetilde{C})\,, \qquad H^3 \;=\; \mathrm{Gr}^W_3 \;=\; H^1(\widetilde{C})(-1)$$

and H^2 is the cohomology at the middle of the complex

$$\begin{array}{c}\\ (5.2)\\ \\ \end{array} \qquad \begin{tikzcd} & H^2(\widetilde{X}) \arrow[dr, "\rho_{\widetilde{X}}"] & \\ H^0(\widetilde{C})(-1) \arrow[ur, "\mathrm{Gys}_{\widetilde{X}}"] \arrow[dr, "\mathrm{Gys}_Y"'] & \bigoplus & H^2(\widetilde{C})\,; \\ & H^2(Y) \arrow[ur, "\rho_Y"'] & \end{tikzcd}$$

here $\mathrm{Gys}_{\widetilde{X}}$ and Gys_Y are Gysin maps, and $\rho_{\widetilde{X}}$ and ρ_Y are signed restriction maps.

In more detail we will think of $\widetilde{C}$ as a cubic curve in $\mathbb{P}^2$ and will denote by C_1 the curve $\widetilde{C}$ in $X_1 := \widetilde{X}$, by C_2 the curve $\widetilde{C}$ in $X_2 := Y$, and by $f : C_2 \xrightarrow{\simeq} C_1$ the identification that glues $\widetilde{X}$ and Y. Denoting by E_i the blow-up of $q_i \in C_2$ in $\mathbb{P}^2$, we have ($\mathbb{Q}$–coefficients)

$$H^2(Y) \;\simeq\; \mathbb{Q}[C_2] \,\oplus\, \left(\oplus_{i=1}^7 \mathbb{Q}[E_i]\right) .$$

The sum $\mathrm{Gys}_{\widetilde{X}} + \mathrm{Gys}_Y$ maps $1 \mapsto [C_1] + [C_2]$. The fact that (5.2) is a complex is the topological consequence of $C_1^2 + C_2^2 = 0$ of (5.1).

5.2.4. *Level one extension data.* We will be concerned with the *algebraic part* of the level one extension data, which is defined to be the part of $\mathrm{Ext}^1_{\mathrm{MHS}}(H^1(\widetilde{C})(-1), H^2)$ corresponding to the subgroup $\mathrm{Hg}^1(\widetilde{X}) = H^{1,1}(\widetilde{X}) \cap H^2(\widetilde{X}, \mathbb{Q})$ of Hodge classes. It will follow from the discussion below that, for a generic point of $\mathcal{N}_2$, the group $\mathrm{Hg}^1(\widetilde{X})$ is freely generated by the classes $[C_1]$ and $[E]$.

Proposition 5.3. *The algebraic part of the extension data is isomorphic to a direct sum of copies of the Jacobian variety $\mathfrak{J}(C_2)$. Letting h be the hyperplane class of $\widetilde{C} \subset \mathbb{P}^2$, the algebraic part of the level one extension data is given by the points $\mathrm{AJ}_{\widetilde{C}}(h - 3q_i)$ and $\mathrm{AJ}_{\widetilde{C}}(h - \sum b_i\, q_i)$, where $\sum b_i = 3$. These points determine the $q_i \in \widetilde{C}$ up to adding a common element of order there to each of them.*

Proof. If $\xi \in \mathrm{Hg}^1(X_1) = \mathrm{Hg}^1(\widetilde{X})$ and $a, b_i \in \mathbb{Z}$, then map $\rho = \rho_{\widetilde{X}} + \rho_Y$ sends

$$\xi \,\oplus\, (a[C_2] + \textstyle\sum b_i[E_i]) \;\mapsto\; \xi \cdot C_1 \,+\, 2a \,+\, \textstyle\sum b_i\,.$$

Then $q(\widetilde{X}) = 0$ implies that ξ determines an element of $\mathrm{Pic}(\widetilde{X})$, and

$$D \;=\; f^*(\xi) \,+\, a\,(3h - \textstyle\sum q_i) \,+\, \textstyle\sum b_i q_i$$

is a divisor of degree zero on $C_1 \simeq \widetilde{C}$. So $\mathrm{AJ}_{\widetilde{C}}(D)$ is defined, and gives the level one part of the extension data. This is equivalent to giving $\mathrm{AJ}_{\widetilde{C}}(h - 3q_i)$ and $\mathrm{AJ}_{\widetilde{C}}(h - \sum b_i\, q_i)$, where $\sum b_i = 3$, and these determine the q_i up to adding to each a common multiple of order 2 in $\mathfrak{J}(\widetilde{C})$. $\square$

Remark 5.4. For the dimension count of $27 = 19 + 1 + 7$, note that we have 19 parameters for X, one for $\widetilde{C}$ and seven for the $\{q_i\}$. The mapping Φ^{T} is locally 1-1 on the data $(X, \widetilde{C}, \{q_i\})$. One may check that its image is locally isomorphic to the blowup of $\mathcal{N}_2 \subset \overline{\mathcal{M}}_I$.

5.2.5. *Level two extension data.* What follows is a very brief discussion to suggest the general nature of level two extension data in this case; details will be discussed elsewhere.

For a general I-surface X with one simple elliptic singularity the level two extension data is relatively uninteresting.[10] However if X has two elliptic singularities,[11] then there is a interaction between them, somewhat in analogy to an irreducible algebraic curve having two nodes that interact through a cross ratio type construction as in Example 4.2 and (6.2). The desingularization $\widetilde{X}$ will have two disjoint elliptic curves $\widetilde{C}_1$ and $\widetilde{C}_2$ that contract to singular points $p_1, p_2 \in X$. We choose $\omega_i \in H^0(\Omega^2_{\widetilde{X}}(\widetilde{C}_i))$, $i = 1, 2$, and with $\varphi_i = \mathrm{Res}_{\widetilde{C}_i}(\omega_i)$ a nonzero generator of $H^0(\Omega^1_{\widetilde{C}_i})$. The vector space $\mathrm{Ext}^1_{\mathrm{MHS}}(H^1(\widetilde{C}_1)(-1), H^1(\widetilde{C}_2))$ maps to level two extension data, and similarly with 1 and 2 swapped. We thus have a map

$$\mathrm{Hom}(H^{1,0}(\widetilde{C}_1), H_1(\widetilde{C}_2)^*) \;\to\; \mathrm{Ext}_{\mathrm{MHS}}(H^1(-1), H^1)\,.$$

Without giving details, an element of $\mathrm{Hom}(H^{1,0}(\widetilde{C}), H_1(\widetilde{C}_2)^*)$ is given by

$$(\varphi_1, \delta_2) \;\mapsto \int_{\Delta_2} \omega_1\,,$$

where $\delta_2 \in H^1(\widetilde{C}_2)$ and Δ_2 is a 2-chain in $\widetilde{X}$ with $\partial\Delta_2 = \delta_2$. An interaction between p_1 and p_2 is provided by using that ω_i is the limit of $\omega_i(t) \in H^0(\Omega^2_{X_t})$ with $\omega_1(t) \wedge \omega_2(t) = 0$ and the alternating bilinear form on $H^1(-1)$ given by $(\alpha, \beta) = Q(N\alpha, \beta)$.

[10] To obtain more interesting classes in $\mathrm{Ext}^1_{\mathrm{MHS}}(H^1(\widetilde{C})(-1), H^1(\widetilde{C}))$ one needs the elliptic curve to have complex multiplication.

[11] The cases when this happens are classified by [FPR15a, FPR15b, FPR17].

5.3. **Example: period matrix representations and local analytic structure.** This is a continuation of Example 2.13. Consider the case that D is the period domain parameterizing pure, weight $n = 2$, Q–polarized Hodge structures on V with Hodge numbers $\mathbf{h} = (2, r-3, 2)$.

5.3.1. *Limiting mixed Hodge structure.* Fix a basis $\{v_0, \ldots, v_r\}$ of $V_{\mathbb{C}}$ so that $Q(v_i, v_j) = \delta^r_{i+j}$ and

$$\bar v_0 \;=\; -v_{\bar 0}\,, \quad \bar v_1 \;=\; v_2\,, \quad \bar v_{\bar 1} \;=\; v_{\bar 2}\,, \quad \text{and} \quad \bar v_\alpha \;=\; v_{\bar\alpha} \quad \forall\; 3 \le \alpha \le \bar 3\,.$$

Let $\{v^0, \ldots, v^r\}$ be the dual basis of $V^*_{\mathbb{C}}$, so that $\{v_a \otimes v^b\}$ is a basis of $\mathrm{End}(V_{\mathbb{C}})$, and define a nilpotent operator by

$$N \;=\; \mathbf{i} v_1 \otimes v^{\bar 2} \;-\; \mathbf{i} v_2 \otimes v^{\bar 1}\,.$$

Then

$$\begin{aligned} F^2 &\;=\; \mathrm{span}\{v_0, v_1\}\,, \\ F^1 &\;=\; (F^2)^\perp \;=\; \mathrm{span}\{v_0, \ldots, v_{\bar 2}\} \end{aligned}$$

defines $F \in \check D$, and and $W_0 = 0$,

$$\begin{aligned} W_1 &\;=\; \mathrm{span}\{v_{\bar 1}, v_{\bar 2}\}\,, \\ W_2 &\;=\; (W_1)^\perp \;=\; \mathrm{span}\{v_0, v_3, \ldots v_r\}\,, \end{aligned}$$

$W_3 = V$, defines a limiting mixed Hodge structure (W, F, N) on D.

5.3.2. *Schubert cell.* The reduced period limit is the flag F_∞ given by $F^2_\infty = \mathrm{span}\{v_0, v_{\bar 2}\}$ and $F^1_\infty = (F^2_\infty)^\perp = \mathrm{span}\{v_0, v_1, v_3, \ldots, v_{\bar 1}\}$. The complex conjugate $\overline{F_\infty}$ is

$$\overline{F^2_\infty} \;=\; \mathrm{span}\{v_{\bar 0}, v_{\bar 1}\}\,, \quad \overline{F^1_\infty} \;=\; (\overline{F^2_\infty})^\perp \;=\; \mathrm{span}\{v_2, \ldots, v_r\}\,.$$

The Schubert cell

$$\mathcal{S} \;=\; \{E \in \check D \mid E \cap \overline{F^1_\infty} = 0\}$$

is the set of all 2-planes $E \subset V_{\mathbb{C}}$ that satisfy the first Hodge–Riemann bilinear relation $Q|_E = 0$ and have trivial intersection with $\overline{F^1_\infty}$. This is precisely the set of two planes admitting a basis of the form

$$E \;=\; \mathrm{span}\left\{ v_0 + \sum_{a\geq 2} \xi_0^a\, v_a\,,\; v_1 + \sum_{a \geq 2} \xi_1^a\, v_a \right\}\,,$$

as in Example 2.13. The infinitesimal period relation is given by either of the following two equivalent expressions

$$-\mathrm{d}\xi_0^{\bar 1} \;=\; \sum_{a=2}^{\bar 2} \xi_1^a\, \mathrm{d}\xi_0^{\bar a} \quad \text{and} \quad -\,\mathrm{d}\xi_1^{\bar 0} \;=\; \sum_{a=2}^{\bar 2} \xi_0^a\, \mathrm{d}\xi_1^{\bar a}\,. \tag{5.5}$$

5.3.3. *Period matrix representation.* If (W, F, N) is a limiting mixed Hodge structures along the fibre A^0, then A^0 admits a neighborhood $\overline{\mathcal{O}}^0 \subset \overline{B}$ so that the matrix representation of Φ over $\mathcal{O}^0 = B \cap \overline{\mathcal{O}}^0$ is given by

$$\Phi\big|_{\mathcal{O}^0} \;=\; \left[\begin{array}{cc|ccc|cc} 1 & 0 & \xi_0^2 & \xi_0^\alpha & \xi_0^{\bar 2} & \xi_0^{\bar 1} & \xi_0^{\bar 0} \\ 0 & 1 & \xi_1^2 & \xi_1^\alpha & \xi_1^{\bar 2} & \xi_1^{\bar 1} & \xi_1^{\bar 0} \end{array}\right]^{\mathsf{t}}\,, \qquad 3 \leq \alpha \leq \bar 3\,. \tag{5.6}$$

The horizontal entries of the period matrix representation are the $\{\xi_0^a, \xi_1^a\}_{2\leq a \leq \bar 2}$. The matrix coefficients $\xi_i^a \not\in \{\xi_0^{\bar 1}, \xi_1^{\bar 1}, \xi_1^{\bar 2}\}$ are all holomorphic on $\overline{\mathcal{O}}$ (but possibly multivalued). Additionally ξ_0^2 vanishes along $Z \cap \overline{\mathcal{O}}$.

5.3.4. *Local coordinate expressions.* As noted above, the matrix coefficients $\xi_i^a \not\in \{\xi_0^{\bar 1}, \xi_1^{\bar 1}, \xi_1^{\bar 2}\}$ are all holomorphic. Working in local coordinates $(t, w) \in \Delta^* \times \Delta^{d-1} = \mathcal{U}$, the three non-holomorphic matrix coefficients are given by

$$\xi_1^{\bar 2}(t,w) \;\equiv\; \mathbf{i}\ell(t)\,, \quad \xi_0^{\bar 1}(t,w) \;\equiv\; -\mathbf{i}\ell(t)\xi_0^2\,, \quad \xi_1^{\bar 1}(t,w) \;\equiv\; -\mathbf{i}\ell(t)\xi_1^2 \qquad \mathrm{mod}\;\; \Omega^0(\overline{\mathcal{U}})\,,$$

where

$$\ell(t) \;=\; \frac{\log t}{2\pi\mathbf{i}}\,.$$

5.3.5. *Action of monodromy.* The ξ_i^a are multi-valued holomorphic functions on $\mathcal{O}^0 = B \cap \overline{\mathcal{O}}^0$. The multivalued-ness is due to the monodromy about A^0. This monodromy has matrix representation

$$\gamma \;=\; \begin{bmatrix} 1 & \gamma_1^0 & 0 & & & & \\ 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & & & & \\ 0 & \gamma_1^\alpha & \gamma_2^\alpha & \delta_\beta^\alpha & & & \\ \gamma_0^{\bar 2} & \gamma_1^{\bar 2} & \gamma_2^{\bar 2} & \gamma_\beta^{\bar 2} & 1 & 0 & 0 \\ 0 & \gamma_1^{\bar 1} & \gamma_2^{\bar 1} & \gamma_\beta^{\bar 1} & 0 & 1 & \gamma_{\bar 0}^{\bar 1} \\ 0 & 0 & \gamma_2^{\bar 0} & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad 3 \le \alpha, \beta \le \bar 3\,. \tag{5.7}$$

The condition that $\gamma \in \mathrm{Aut}(V,Q)$ preserve the polarization Q is $\delta_{a\bar b} = \sum \gamma_a^c\, \gamma_b^{\bar c}$. The condition that $\gamma \in \mathrm{Aut}(V_{\mathbb{R}},Q)$ is real is $\overline{\gamma_b^a\, v_a \otimes v^b} = \overline{\gamma_b^a}\, \bar v_a \otimes \bar v^b$.

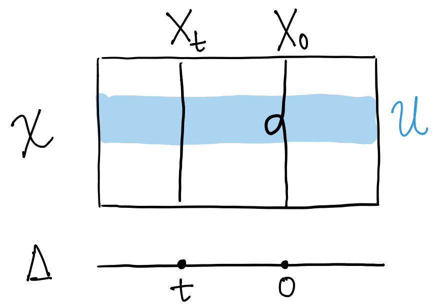
The functions ξ_0^2 and $\xi_0^2\,\xi_1^{\bar 0} - \xi_1^2\,\xi_0^{\bar 0}$ are well-defined (Γ_A–invariant) functions. The other functions transformation in a somewhat complicated manner, cf. Table 5.1, where $3 \le \alpha, \beta \le \bar 3$.

TABLE 5.1. Action of monodromy on period matrix representation

$$\begin{aligned}
&\xi_0^{\bar 0} \;\mapsto\; \xi_0^{\bar 0} + \gamma_2^{\bar 0}\,\xi_0^2\,, \quad \xi_0^\alpha \;\mapsto\; \xi_0^\alpha + \gamma_2^\alpha\,\xi_0^2\,, \quad \xi_1^2 \;\mapsto\; \xi_1^2 - \gamma_1^0\,\xi_0^2\,;\\
&\xi_0^{\bar 2} \;\mapsto\; \xi_0^{\bar 2} + \gamma_0^{\bar 2} + \gamma_2^{\bar 2}\,\xi_0^2 + \textstyle\sum \gamma_\alpha^{\bar 2}\,\xi_0^\alpha\,, \quad \xi_0^{\bar 1} \;\mapsto\; \xi_0^{\bar 1} + \gamma_{\bar 0}^{\bar 1}\,\xi_0^{\bar 0} + \gamma_2^{\bar 1}\,\xi_0^2 + \textstyle\sum \gamma_\alpha^{\bar 1}\,\xi_0^\alpha\,,\\
&\xi_1^{\bar 0} \;\mapsto\; \xi_1^{\bar 0} + \gamma_2^{\bar 0}\,\xi_1^2 + \gamma_{\bar 0}^{\bar 1}(\xi_0^{\bar 0} + \gamma_2^{\bar 0}\,\xi_0^2)\,, \quad \xi_1^\alpha \;\mapsto\; \xi_1^\alpha + \gamma_1^\alpha + \gamma_2^\alpha\,\xi_1^2 - \gamma_1^0\,(\xi_0^\alpha + \gamma_2^\alpha\,\xi_0^2)\,;\\
&\xi_1^{\bar 2} \;\mapsto\; \xi_1^{\bar 2} + \gamma_1^{\bar 2} + \gamma_2^{\bar 2}\,\xi_1^2 + \textstyle\sum \gamma_\alpha^{\bar 2}\,\xi_1^\alpha \;-\; \gamma_1^0 \left(\xi_0^{\bar 2} + \gamma_0^{\bar 2} + \gamma_2^{\bar 2}\,\xi_0^2 + \sum \gamma_\alpha^{\bar 2}\,\xi_0^\alpha\right)\,,\\
&\xi_1^{\bar 1} \;\mapsto\; \xi_1^{\bar 1} + \gamma_1^{\bar 1} + \gamma_2^{\bar 1}\,\xi_1^2 + \gamma_\alpha^{\bar 2}\,\xi_1^\alpha + \gamma_{\bar 0}^{\bar 1}\,\xi_1^{\bar 0} \;-\; \gamma_1^0 \left(\xi_0^{\bar 1} + \gamma_{\bar 0}^{\bar 1}\,\xi_0^{\bar 0} + \gamma_2^{\bar 1}\,\xi_0^2 + \sum \gamma_\alpha^{\bar 1}\,\xi_0^\alpha\right)\,.
\end{aligned}$$

5.3.6. *Local analytic structure.* Upon restricting to a neighborhood $\overline{\mathcal{O}}^1 \subset \overline{\mathcal{O}}^0$ of a fibre $A^1 \subset A^0$, the monodromy about A^1 simplifies to $\gamma_{\bar 2}^1 + \gamma_{\bar 1}^2 = 0$, and $\gamma_j^i = 0$ for all other

FIGURE 3. Semi-stable reduction family



$i < j$. All the horizontal entries $\{\xi_0^a, \xi_1^a\}_{2 \le a \le \bar{2}}$ of the period matrix representation are well-defined and holomorphic on $\overline{\mathcal{O}}^1$, except $\xi_1^{\bar{2}}$. And while $\xi_1^{\bar{2}}$ is not well-defined, $\exp(2m\pi\xi_1^{\bar{2}})$ is for some $1 \le m \in \mathbb{Z}$. This gives us the map f in step §2.2(c) of the proof of Theorem 1.7.

5.4. **Classical period matrix interpretation for a smoothing deformation of two surfaces glued along a curve.** The objective of this section is to relate the period matrix representation in §5.3.3 to the classical geometric presentation of the period matrix. This discussion will be informal, and follows from [Cle69, Cle77].

The basic set-up is a semi-stable reduction family

$$\begin{array}{c} \mathcal{X} \\ \downarrow{\scriptstyle \pi} \\ \Delta\,. \end{array} \tag{5.8}$$

Here $\Delta = \{|t| < 1\} \subset \mathbb{C}$ is the unit disc, the fibres $X_t = \pi^{-1}(t)$ are compact, X_0 is a reduced normal crossing divisor, and the restriction $\mathcal{X}^* = \pi^{-1}(\Delta^*)$ to the punctured disc $\Delta^* = \{0 < |t| < 1\}$ is a holomorphic fibration. Fix a fibered neighborhood $\mathcal{U} \subset \mathcal{X}$ of the singular locus of X_0, and set $U_t = \mathcal{U} \cap X_t$ (Figure 3).

5.4.1. *A toy model: nodal curve.* It will be helpful to warm-up with the $n = 1$ case that X_t is a curve. Consider a neighborhood of the singular point p in a nodal curve (Figure 4). Because the nodal curve is only a local normal crossing divisor, it must

FIGURE 4. Nodal curve and neighborhood of singular point

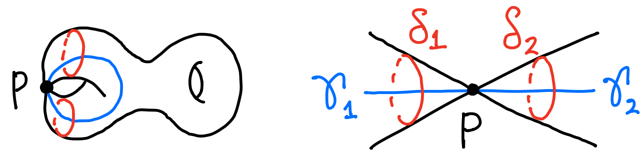


FIGURE 5. Fibres X_0 and X_t

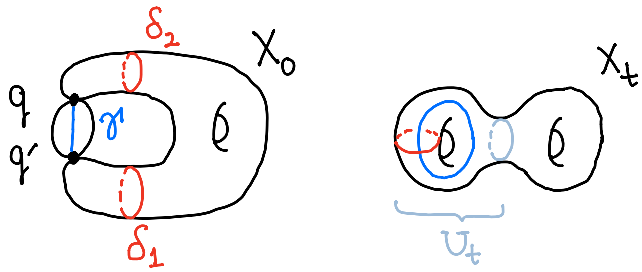
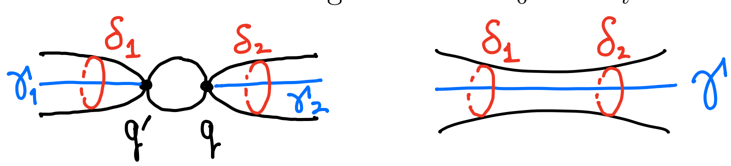


FIGURE 6. Neighborhoods U_0 and U_t



be replaced by $X_0 = \mathbb{P}^1 \cup E$ with $\mathbb{P}^1 \cap E = \{q, q'\}$ (Figure 5). The normalization

$$\begin{array}{l} \widetilde{X}_0 = \mathbb{P}^1 \sqcup \widehat{X}_0 \\ \downarrow \\ X_0 \end{array}$$

of X_0 is the disjoint union $\widetilde{X}_0 = \mathbb{P}^1 \sqcup \widehat{X}_0$ of the projective line with an elliptic curve $\widehat{X}_0$. The degeneration $X_t \to X_0$ is obtained by shrinking the vanishing cycle $\delta \in H_1(X_t)$ to a point p, and then replacing p by a $\mathbb{P}^1$ with two marked points $\{q, q'\}$.

We consider a neighborhood U as pictured in Figure 6 with homology classes $\delta_1 \sim \delta_2 \in H_1(U_t)$ and $\gamma \in H_1(U_t, \partial U_t)$ satisfying $\delta_i \cdot \gamma = 1$. Here curves δ_1 and δ_2 are homologous in both the local and global pictures. When we move to the $n = 2$ case (§5.4.2) this will not be automatic; rather it is a consequence of the necessary condition (5.1) for the smoothability of X_0.

We now turn to monodromy, the limiting Hodge filtration and the period matrix representation. The nilpotent logarithm of monodromy is given by

$$N\gamma \;=\; \delta \quad \text{and} \quad N\delta \;=\; 0\,.$$

For $F^1_t = H^0(\omega_{X_t})$ we may choose a framing $\{\omega_1(t), \omega_2(t)\}$ with the following properties:

- We have $\omega_1(t) \in H^0(\Omega^1_{X_t})$ when $t \neq 0$, and $\omega_1(0) \in H^0(\omega_{X_0})$ has residues ± 1 at q, q'. More specifically
$$\omega_1(0) \;\in\; H^0(\Omega^1_{\widehat{X}_0}(q+q'))$$
determines a unique differential in $H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(q+q'))$ by prescribing opposite residues at the points $\{q, q'\}$ of $\widehat{X}_0$ and $\mathbb{P}^1$.
- We have $\omega_2(t) \in H^0(\Omega^1_{X_t})$ and the pull-back of $\omega_1(0)$ to the normalization $\widetilde{X}_0 = \mathbb{P}^1 \sqcup \widehat{X}_0$ vanishes on $\mathbb{P}^1$ and is a holomorphic $(1,0)$-form on $\widehat{X}_0$.

The period matrix for $\{\omega_1(t), \omega_2(t)\}$ may be normalized to take the form (4.3). That is, $\omega_1(t)$ is represented by the first column $[\; 1 \quad 0 \quad \alpha(t) \quad \nu(t) \;]^{\mathsf{t}}$ and $\omega_2(t)$ is represented by the second column $[\; 0 \quad 1 \quad \lambda(t) \quad \alpha(t) \;]^{\mathsf{t}}$. Here $\alpha(t)$, $\lambda(t)$ are holomorphic, with $\operatorname{Im}\lambda > 0$; and $\beta(t) = \nu(t) - \ell(t)$ also holomorphic. Specifically,

$$\begin{bmatrix} \int_\delta \omega_1 & \int_\delta \omega_2 \\ \int_\gamma \omega_1 & \int_\gamma \omega_2 \end{bmatrix} \;=\; \begin{bmatrix} 1 & 0 \\ \nu(t) & \alpha(t) \end{bmatrix} \;=\; \begin{bmatrix} 1 & 0 \\ \beta(t) + \ell(t) & \alpha(t) \end{bmatrix}.$$

The entry $\alpha(0)$ encodes the level one extension data, which is

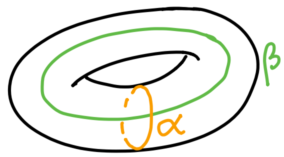
$$\mathrm{AJ}_{\widehat{X}_0}(q - q') \;\in\; \mathcal{J}(\widehat{X}_0)\,,$$

where $\widehat{X}_0$ is the elliptic curve $\mathbb{C}/(\mathbb{Z} + \lambda\mathbb{Z})$, given by $\lambda = \lambda(0)$.

5.4.2. *The example: smoothing of two surfaces glued along a curve.* We now turn to the $n = 2$ case that X_t is a surface. We want an analog of §5.4.1 when the central singular fibre

$$X_0 \;=\; Y_1 \cup_C Y_2$$

FIGURE 7. The curve C



consists of two smooth surfaces Y_i glued along a smooth double curve C. With the I-surface example (§5.2) in mind, we assume

$$p_g(X_t) \;=\; h^{2,0}(X_t) \;=\; 2 \quad \text{and} \quad g(C) \;=\; 1$$

(Figure 7), and denote by C_i the curve C in Y_i. We will have

$$C_1^2 \;=\; -d\,,$$

with $d > 0$, and C_1 may be contracted to a simple elliptic singularity in a normal surface. We assume that X_0 admits a smoothing deformation as in (5.8). By (5.1) we must have

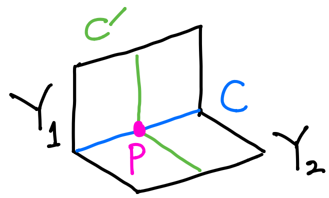
$$C_2^2 \;=\; d\,.$$

We will take Y_2 to be a del Pezzo surface.

The idea is to use the $n = 1$ picture (§5.4.1). Fix a point $p \in C$ and let C' be a hypersurface section of X_0 that contains p and is transverse to C (Figure 8). Then p is a singular point of C' and the neighborhood $\mathcal{U} \cap C'$ of p in C' is as depicted in Figure 4. As we allow the point p to vary over the curves α and β in C (Figure 7), the 1-cycles δ_1, δ_2 in C' (§5.4.1) will trace out 2-cycles $\Delta_{1,\alpha}, \Delta_{2,\alpha}$ and $\Delta_{1,\beta}, \Delta_{2,\beta}$ in $U_t \subset X_t$. It follows from $C_1^2 + C_2^2 = 0$ that $\Delta_{1,\alpha} = \Delta_{2,\alpha}$ and $\Delta_{1,\beta} = \Delta_{2,\beta}$ in $H_2(U_t)$. So we may write Δ_α for $\Delta_{i,\alpha}$ and Δ_β for $\Delta_{i,\beta}$.

Again letting p vary over the curves α and β in C, the relative 1-cycle γ in C' (§5.4.1) traces out two 2-cycles $\Gamma_\alpha, \Gamma_\beta \in H_2(U_t, \partial U_t)$. We assume that the cycles $\{\Gamma_\alpha, \Gamma_\beta, \Delta_\alpha, \Delta_\beta\}$ are the image of cycles, denoted by the same symbols, under the

FIGURE 8. The curve $C' \subset X_0$



map ϱ of

$$\begin{array}{ccc} H^2(X_t) & \longrightarrow & H^2(U_t) \\ \downarrow\simeq & & \downarrow\simeq \\ H_2(X_t) & \xrightarrow{\varrho} & H_2(U_t, \partial U_t)\,. \end{array}$$

The vertical isomorphisms above are Poincaré and Poincaré–Lefschetz duality, respectively.

This gives us a $\mathbb{Z}^4 = \mathrm{span}_{\mathbb{Z}}\{\Delta_\alpha, \Gamma_\alpha, \Delta_\beta, \Gamma_\beta\}$ in $H_2(X_t)$ whose intersection matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The action of the nilpotent logarithm of monodromy on this $\mathbb{Z}^4$ is given by

$$N\Gamma_\alpha \;=\; \Delta_\alpha\,, \quad N\Gamma_\beta \;=\; \Delta_\beta \quad \text{and} \quad N\Delta_\alpha\,,\ N\Delta_\beta \;=\; 0\,.$$

Turning to the Hodge filtration F^2_t, the procedure of §5.4.1 applies here to give $\omega_i(t) \in H^0(\omega_{X_t})$, where $\omega_1(0) \in H^0(\omega_{X_0})$ has Poincaré residue $\mathrm{Res}_C\omega_1(0) \in H^0(\Omega^1_C)$ with

$$\int_\alpha \mathrm{Res}_C\omega_1(0) \;=\; 1\,, \quad \int_\beta \mathrm{Res}_C\omega_1(0) \;=\; \lambda\,, \quad \mathrm{Im}\,\lambda \;>\; 0\,.$$

The basis $\{v_0, \ldots, v_r\}$ of $V_{\mathbb{C}} = H^2_{\mathrm{prim}}(X_t, \mathbb{C})$ in §5.3.1 may be chosen so that $\mathrm{span}_{\mathbb{C}}\{v_1, v_2\} = \mathrm{span}_{\mathbb{C}}\{\Gamma^*_\alpha, \Gamma^*_\beta\}$ and $\mathrm{span}_{\mathbb{C}}\{v_{\bar 1}, v_{\bar 2}\} = \mathrm{span}_{\mathbb{C}}\{\Delta^*_\alpha, \Delta^*_\beta\}$. (By this notation we mean that $\langle v_i, \Gamma_\alpha\rangle\,,\ \langle v_i, \Gamma_\beta\rangle = 0$ for all $i \neq 1, 2$; and $\langle v_i, \Delta_\alpha\rangle\,,\ \langle v_i, \Delta_\beta\rangle = 0$

for all $i \neq \bar{1}, \bar{2}$.) The period matrix (5.6) is

$$F_t^2 = \left[\begin{array}{cc|ccc|cc} 1 & 0 & h(t) & P(t) & f(t) & b(t) - \mathbf{i}\ell(t)h(t) & y(t) \\ 0 & 1 & \lambda(t) & A(t) & a(t) + \mathbf{i}\ell(t) & c(t) - \mathbf{i}\ell(t)\lambda(t) & z(t) \end{array}\right]^{\mathsf{t}}$$

With the exception of $\ell(t) = (\log t)/2\pi\mathbf{i}$, all the functions above are holomorphic. Additionally $h(0) = \xi_0^2(0) = 0$. The entries

$$\begin{bmatrix} \xi_0^{\bar{1}} & \xi_0^{\bar{0}} \\ \xi_1^{\bar{1}} & \xi_1^{\bar{0}} \end{bmatrix}^{\mathsf{t}} = \begin{bmatrix} b(t) - \mathbf{i}\ell(t)h(t) & y(t) \\ c(t) - \mathbf{i}\ell(t)\lambda(t) & z(t) \end{bmatrix}^{\mathsf{t}}$$

are determined up to constants of integration by the horizontal component

$$\begin{bmatrix} \xi_0^2 & \xi_0^\alpha & \xi_0^{\bar{2}} \\ \xi_1^2 & \xi_1^\alpha & \xi_1^{\bar{2}} \end{bmatrix}^{\mathsf{t}} = \begin{bmatrix} h(t) & P(t) & f(t) \\ \lambda(t) & A(t) & a(t) + \mathbf{i}\ell(t) \end{bmatrix}^{\mathsf{t}}$$

of the period matrix. In fact, three of the four parameters $\{\xi_0^{\bar{0}}, \xi_1^{\bar{1}}, \xi_1^{\bar{0}} + \xi_0^{\bar{1}}\}$ are determined by the first Hodge–Riemann bilinear relation (2.14). The remaining degree of freedom $\xi_1^{\bar{0}} - \xi_0^{\bar{1}}$ is determined up to a constant of integration by the infinitesimal period relation (5.5).

We take Y_2 to be the del Pezzo surface obtained by blowing up $9 - d$ points on $C \subset \mathbb{P}^2$ (in order to satisfy $C_2^2 = d$). The entries $A(0) = \xi_0^\alpha(0)$ encode level one extension data that geometrically arises from the points $p_i \in \mathrm{Pic}^1(C)$. The level two extension data is encoded by $a(0) + \mathbf{i}\ell(0) = \xi_1^{\bar{2}}(0)$ and $c(0) - \mathbf{i}\ell(0)\lambda(0) = \xi_1^{\bar{1}}(0)$. Of course, $\ell(0)$ is not defined; we address this by taking exponentials: the function $\exp 2\pi\xi_1^{\bar{2}}(t) = t\exp(2\pi a(t))$ is well-defined and holomorphic on Δ.

6. Discussion of Hodge–Tate degenerations

There is an extensive and long standing body of literature on variations of *graded polarized* mixed Hodge structures of Hodge–Tate type; for recent references see [Bro14, Gon01, Hai94] and the citations therein. One may anticipate that from (3.23) those arising in this paper, as variations of *limiting* mixed Hodge structures, will have additional special properties. We close the paper with a brief discussion in this direction.

6.1. **Hodge–Tate example: weight $n = 1$.** Here we are considering limiting mixed Hodge structures (W, F, σ) on $V = \mathbb{Q}^{2g}$ with $W_0 = W_1$ and $W_2 = V$ so that $H^1 = 0$ and $\dim H^0 = g$. The extension data is all of level two, and upon a choice of basis, the set of extension data will be given by $g \times g$ symmetric matrices all of whose entries are nonzero.

By Lemma 3.20(ii) the extension data of level ≤ 2 determines, up to constants of integration, the full extension data. We wish to discuss this in the special case that the limiting mixed Hodge structure (W, F, σ) is Hodge–Tate type, cf. Definition 3.35 and Remark 3.36.

Example 6.1. In $\mathbb{P}^1$ we choose g distinct pairs of points (p_i, q_i). For each i we choose $t_i \neq 0$ which gives an identification $T_{p_i}\mathbb{P}^1 \otimes T^*_{q_i}\mathbb{P}^1 \simeq \mathbb{C}$. It is standard, and will be explained in more detail in the current context in ([Gri18] or [FGG$^+$20]), that is data gives a first-order smoothing of the g–nodal curve obtained by identifying p_i and q_i. In particular, there is a well-defined LMHS. If N_i is the logarithm of monodromy corresponding to smoothing the i–th node, then $N = N_1 + \cdots + N_g$. The diagonal entries of the symmetric matrix are the t_i. The off diagonal entries are the exponentials of

$$\int_{q_j}^{p_j} \eta_i \equiv \int_{q_i}^{p_i} \eta_j \,, \tag{6.2}$$

modulo periods, where η_i is the unique differential on $\mathbb{P}^1$ with poles at p_i, q_i and normalized to be $\mathrm{d}\log t_i$ (modulo a holomorphic 1-form) near p_i. Then (6.2) is the logarithm of the cross ratio $(p_i, q_i; p_j, q_j)$, cf. §4.1.

The number of parameters of the p_i, q_i is $2g - 3$. There are g of the t_i's, giving the total number of parameters $3g - 3$. On the other hand, as noted above there are $g(g + 1)/2$ parameters in the extension data for a general LMHS. For $g = 2, 3$, the numbers are equal, but for $g \geq 4$ we have $g(g+1)/2 > 3g-3$, so that there are algebraic Schottky relations among the cross ratios $(p_i, q_i; p_j, q_j)$.[12]

Remark 6.3. In the weight $n = 1$ case the infinitesimal period relation is trivial. For non-classical Hodge–Tate variations of limiting mixed Hodge structures there will be universal infinitesimal Schottky relations imposed by the infinitesimal period relation; see §6.2 for some discussion.

6.2. **Hodge–Tate example: weight** $n = 2$**.** Consider a local coordinate chart $t = (t_1, \ldots, t_k) : \overline{\mathcal{U}} \to \Delta^k$, with $\mathcal{U} = B \cap \overline{\mathcal{U}} \simeq (\Delta^*)^k$ at a point $b \in Z$ with a Hodge–Tate limiting mixed Hodge structure (W, F, σ). The existence of a Hodge–Tate degeneration implies that the Hodge numbers $\mathbf{h} = (h^{2,0}, h^{1,1}, h^{0,2})$ satisfy $h^{2,0} \geq h^{1,1}$. For notational convenience we will assume that $h^{2,0} = h^{1,1}$, and let h denote this common integer. Let $N_1, \ldots, N_k$ denote the nilpotent logarithms of monodromy, and set

$$N \; = \; N_1 + \cdots N_k \, .$$

We may fix a basis $\{v_1, \ldots, v_h \, ; \, Nv_1, \ldots, Nv_h \, ; \, N^2v_1, \ldots, N^2v_h\}$ of $V_{\mathbb{R}}$ that is adapted to both the weight filtration

$$\begin{aligned} W_0 \; = \; W_1 \;\; &= \;\; \mathrm{span}\{N^2v_1, \ldots, N^2v_h\} \\ W_2 \; = \; W_2 \;\; &= \;\; \mathrm{span}\{N^2v_1, \ldots, N^2v_h \, ; \, Nv_1, \ldots, Nv_h\} \, , \end{aligned}$$

and the Hodge filtration

$$\begin{aligned} F^2 \;\; &= \;\; \mathrm{span}\{v_1, \ldots, v_h\} \\ F^1 \;\; &= \;\; \mathrm{span}\{v_1, \ldots, v_h \, ; \, Nv_1, \ldots, Nv_h\} \, ; \end{aligned}$$

[12] There is extensive literature, both classical and current, concerning Schottky relations. The papers [SB19, SB20] are particularly relevant here as they involve interesting Hodge theoretic considerations. The smoothing of nodes process is also discussed. For the variations of graded-polarized mixed Hodge structures of Hodge–Tate type and that arise from arithmetic considerations the finite Schottky relations correspond to identities among polylogarithms [Bro14, Gon01, Hai94].

and so that

$$Q(v_i, N^2 v_j) \;=\; \delta_{ij} \quad \text{and} \quad Q(Nv_i, Nv_j) = -\delta_{ij}\,,$$

and all other pairings are zero. Then we have matrix representations

$$Q \;=\; \begin{bmatrix} 0 & 0 & I \\ 0 & -I & 0 \\ I & 0 & 0 \end{bmatrix} \quad \text{and} \quad N_i \;=\; \begin{bmatrix} 0 & 0 & 0 \\ \nu_i & 0 & 0 \\ 0 & \nu_i^{\mathsf{t}} & 0 \end{bmatrix}.$$

The commutativity of the N_i is equivalent to

$$\nu_i^{\mathsf{t}}\,\nu_j \;=\; \nu_j^{\mathsf{t}}\,\nu_i\,. \tag{6.4}$$

The period matrix is

$$\begin{bmatrix} F^2(t) & F^1(t) \end{bmatrix} \;=\; \begin{bmatrix} I & 0 \\ X(t) & I \\ Y(t) & X(t)^{\mathsf{t}} \end{bmatrix}.$$

Here $X(t)$ is the horizontal part of the period matrix, and is linear in the $\ell(t_i)$; the component $Y(t)$ is quadratic in the $\ell(t_i)$. (In both cases the coefficients are holomorphic functions on $\overline{\mathfrak{U}}$.) The block $X(t)$ encodes the level two extension data along the fibre A^0; the matrix $Y(t)$ encodes level four data. The first Hodge–Riemann bilinear relation yields

$$Y + Y^{\mathsf{t}} \;=\; X^{\mathsf{t}}\,X\,. \tag{6.5}$$

(The skew-symmetric component $Y - Y^{\mathsf{t}}$ involves no $\ell(t_i)$ terms; it is holomorphic on $\overline{\mathfrak{U}}$.) The infinitesimal period relation $\mathrm{d}F^2 \subset F^1$ is equivalent to $Q(\mathrm{d}F^2, F^1) = 0$; that is,

$$\mathrm{d}Y \;=\; X^{\mathsf{t}}\,\mathrm{d}X\,, \tag{6.6}$$

so that the level four extension data $Y(t)$ is determined (up to constants of integration) by the level two extension data $X(t)$. The presence of the $\ell(t_i)\mathrm{d}\ell(t_j)$ terms in the right-hand side of (6.6) give $Y(t)$ the qualitative character of a dilogarithm. For more on the connections, both established and conjectural, between polylogarithms and Hodge–Tate structures see, for example, [Gon01] and the references therein.

6.3. **The Hodge–Tate case: period matrix representations.** If the variation of limiting mixed Hodge structure along the fibre A^0 of Φ^0 is of Hodge–Tate type, then we may choose our basis of $V_{\mathbb{C}}$ so that the corresponding period matrix representation of Φ over $\mathcal{O}^0$ has the property that the horizontal matrix entries $\varepsilon_j : \mathcal{O}^0 \to \mathbb{C}$ are either well-defined (single-valued) and extend to holomorphic functions on $\overline{\mathcal{O}}^0$, or function $\tau_j = \exp(2\pi\mathbf{i}\varepsilon_j)$ is a well-defined (single-valued), holomorphic function on $\overline{\mathcal{O}}^0$. These define the functions $f = (\varepsilon_1, \ldots, \varepsilon_c\,;\, \tau_{c+1}, \ldots, \tau_d) : \overline{\mathcal{O}}^0 \to \mathbb{C}^d$ of §2.2(c). An implicit point here is that for Hodge–Tate degenerations we may take $\overline{\mathcal{O}}^0 = \overline{\mathcal{O}}^1$.

Appendix A. Lie theoretic structure of extension data

Here we summarize the structure of the extension data as the discrete quotient of a complex homogeneous manifold; for details see [Car87, KP16].

A.1. **Mixed Hodge structures.** The set of all mixed Hodge structures $(W, \tilde{F})$ with the same associated graded $\tilde{H}^\bullet = H^\bullet$ is a homogeneous complex manifold

$$P^1_{W,\mathbb{C}} \cdot F \;=\; \{\tilde{F} \mid (W, \tilde{F}) \text{ is a MHS with } \tilde{H}^\bullet = H^\bullet\}\,.$$

The automorphism group $P^1_{W,\mathbb{C}}$ is the complex unipotent radical of the parabolic subgroup

$$P_W \;=\; \{g \in \mathrm{Aut}(V) \mid g(W_\ell) = W_\ell\,,\ \forall\ \ell\}$$

of automorphisms preserving the weight filtration. Notice that every $g \in P_W$ induces a well-defined element of $\mathrm{Aut}(W_\ell/W_{\ell-k})$, which we will also denote g. Let

$$P^k_W \;=\; \{g \in P_W \mid g = \mathbf{1} \in \mathrm{Aut}(W_\ell/W_{\ell-k})\,,\ \forall\ \ell\}$$

be the normal subgroup acting trivially on the quotients. Set

$$\Gamma_W \;=\; P^1_{W,\mathbb{Z}}\,.$$

Then the extension data is given by (3.10), and the iterated fibration is given by

$$\mathcal{E}^\ell_{W,F} \;=\; (\Gamma_W \cdot P^{\ell+1}_{W,\mathbb{C}})\backslash(P^1_{W,\mathbb{C}} \cdot F) \;=\; P^{\ell+1}_{W,\mathbb{C}}\backslash\mathcal{E}_{W,F}\,.$$

A.2. **Limiting mixed Hodge structures.** The set of all polarized mixed Hodge structures $(\sigma, \tilde{F})$ with the same associated graded $\tilde{H}^\bullet = H^\bullet$ is a homogeneous complex manifold

$$C^1_{\sigma,\mathbb{C}} \cdot F \;=\; \{\tilde{F} \mid (\sigma, \tilde{F}) \text{ is a PMHS with } \tilde{H}^\bullet = H^\bullet\}\,.$$

The automorphism group $C^1_{\sigma,\mathbb{C}}$ is the complex unipotent radical of the centralizer

$$C_\sigma \;=\; \{g \in \mathrm{Aut}(V) \mid \mathrm{Ad}_g N = N\,,\ \forall\ N \in \sigma\}$$

of the cone. We have $C_\sigma \subset P_W$. Notice that every $g \in C_\sigma$ induces a well-defined element of $\mathrm{Aut}(W_\ell/W_{\ell-k})$, which we will also denote g. Let

$$C^k_\sigma \;=\; \{g \in C_\sigma \mid g = \mathbf{1} \in \mathrm{Aut}(W_\ell/W_{\ell-k})\,,\ \forall\ \ell\} \;=\; C_\sigma \cap P^k_W$$

be the normal subgroup acting trivially on the quotients. Set

$$\Gamma_\sigma \;=\; C^1_{\sigma,\mathbb{Z}}\,.$$

Then the extension data of the limiting mixed Hodge structure (σ, F) is given by (3.16), and the iterated fibration (3.17) is given by

$$\mathcal{E}^\ell_{\sigma,F} \;=\; (\exp(\mathbb{C}\sigma)\Gamma_\sigma \cdot C^{\ell+1}_{\sigma,\mathbb{C}})\backslash(C^1_{\sigma,\mathbb{C}} \cdot F) \;=\; C^{\ell+1}_{\sigma,\mathbb{C}}\backslash\mathcal{E}_{\sigma,F}\,.$$

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