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THE THEORY OF THE POSITRON AND RELATED TOPICS

(Report of a Seminar)

by  
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(Notes by Dr. Banesh Hoffmann)

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### Errata

The following errata have been noticed. The list can make no pretence of completeness.

Page 11. Eq. (34) should read

$$\vec{g}_k = 2e \hbar c \vec{k} \psi_R^* \psi_R.$$

Page 12. In Eq. (36), the \* on  $b_k$  should be omitted in the formula for  $\psi_R^*$ .

Page 24. In Eq. (73), the last term should be  $\psi^* \psi$  and not  $\bar{\psi} \psi$ .

Page 27. Eq. (87) should have the  $E_m^* E_n$  in the integral replaced by  $u_m^* u_n$ .

Page 33, below Eq. (109). The suffixes on the N's in  $\mathcal{E}$  are missing; it should read

$$\mathcal{E}_n(N_1, \dots, N_n, \dots).$$

Page 94, line 11. Replace  $\beta = \frac{1}{2}b$  by  $\beta = \frac{1}{2B}$ .

# THE THEORY OF THE POSITRON AND RELATED TOPICS

by

WOLFGANG PAULI

The general program of the seminar, so far as the early part is concerned, can be stated now. Later there will be time for the discussion of topics that can be decided on during the course of the lectures.

We shall discuss first the recent theory of Pauli and Weisskopf dealing with the relativistic wave equation for particles having zero spin. It may well be described as the Anti-Dirac Theory! We shall then consider the Theory of Holes in the Heisenberg formulation, which encounters very bad singularities, and, later, an application of the theory, by Euler and Kochel, to the problem of the scattering of two light quanta with each other. This process has the same relation to the generation of pairs as dispersion to absorption, so it is probable that such a process takes place. But it is not yet decided whether the theory agrees well, quantitatively, with experiment.

The scattering of light quanta by light quanta occurs on any theory in which the Maxwell equations for the vacuum are modified by the introduction of non-linear terms. Two recent theories of this type, the one starting out from the theory of holes and the other, due to Born, from the electromagnetic field, have recently been propounded. There is an interesting connection between these two theories that we shall discuss in its turn.

-----

We begin with a discussion of the scalar theory of Pauli and Weisskopf. First of all, some general remarks.

It would seem that Dirac overestimated the rigor of the a priori argument that the density of the particles must be of the general form

$$(1) \quad \rho(x) = \sum_r \Psi_r^* \Psi_r .$$

This density requires the use of linear differential equations of the first order, of the form

$$(2) \quad \frac{\hbar}{i} \frac{\partial \Psi_r}{\partial t} = H \Psi_r ,$$

where H is a hermitean operator.

Now before Dirac's work there already existed the relativistic wave equation, due to Schrödinger, which has the form

$$(3) \quad \hbar^2 \frac{\partial^2 \psi}{\partial t^2} = \hbar^2 c^2 \Delta \psi - m^2 c^4 \psi ,$$

(m will throughout denote the rest mass)

with the corresponding density of charge

$$(4) \quad \rho = -\hbar i \left( \frac{\partial \psi^*}{\partial t} \psi - \frac{\partial \psi}{\partial t} \psi^* \right) .$$

Dirac thought that one could exclude this  $\rho$  for reasons taken from the general transformation theory of quantum mechanics. These reasons are correct only if we have to do with a one-body problem, and the whole idea of Dirac's theory was to treat the relativistic one-body problem. But the further developments of the Dirac theory have led to the discovery of the significance of the negative energy states, and to their interpretation in terms of positrons. And in this new interpretation of the Dirac theory we no longer have a one-body problem since the generation of electron-positron pairs is an intrinsic part of the whole theory. We shall see, in fact, that the number of particles is, in general, no longer an integral of the motion. Thus Dirac's argument for his  $\rho$  cannot hold.

However, we know that the electron has a spin and for this reason we must introduce spinors. The most convenient way to do this is as in the Dirac equations.

The scalar theory will not give spin. But it has some advantages over the Dirac theory from a logical point of view. For, when there are pairs, we have not to look for the density of particles but for the density of charge, which is independent of pair production. This charge can take on both positive and negative values. Thus we should no longer require that the density of energy shall not be positive definite, but that the density of charge shall be capable of taking both signs.

In the Schrödinger theory the energy density is actually positive definite, being, in fact,

$$(5) \quad W = \hbar^2 \frac{\partial \psi^*}{\partial t} \frac{\partial \psi}{\partial t} + \hbar^2 c^2 \text{grad} \psi^* \text{grad} \psi + m^2 c^4 \psi^* \psi.$$

In the Maxwell theory the energy density is also positive definite. But this is not the case in the spinor theory, for there it has the form

$$(6) \quad \psi^* H \psi, \quad (H = \sum \alpha^k \frac{\partial}{\partial x^k} + \beta m c^2).$$

From the point of view of pure invariant theory, one could try to introduce the different expression

$$(7) \quad \frac{\partial \psi^*}{\partial t} \beta \frac{\partial \psi}{\partial t}$$

for the energy density. But this expression too can take on negative values because the matrix

$$(8) \quad \beta \sim \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

has negative eigenvalues. It seems that the spinor representation can never give a positive definite energy density.

In the scalar equation, then, the charge density can take both signs while the energy density is positive definite. But if we want a quantitative theory of pair production we must, as in the theory of holes, introduce second quantization. In the theory of holes this step is necessary in order to make transmutation between energy and matter discrete instead of continuous. The introduction of second quantization in the scalar theory makes no difference to the annihilation of matter, but is necessary for a good theory of the production of pairs. This is quite analogous to the effect of the field quantization on the theory of absorption and emission of radiation.

The Dirac theory and the theory of holes are in some ways similar; but in the latter we must introduce some artificial operations to obtain the right expressions for the density of energy. The second quantization and this subtraction process save the theory from the dangers of the infinities, as we shall see later.

Artificial subtractions of infinities are not necessary in the scalar theory with second quantization, but the results of this theory are very similar to those of the theory of holes, even quantitatively. For example, the probability of pair production takes the same form in both theories, only the numerical coefficients being different. The same is true for other effects, but this unfortunately includes some undesirable results. Thus the polarization of the vacuum and the self-energy of particles become infinite, just as in the theory of holes. In both theories the polarization comes out as the logarithmically divergent integral in momentum space

$$(9) \quad \int \frac{d|\mathbf{k}|}{|\mathbf{k}|} .$$

The infinite polarization of the vacuum implies that no free charge can exist, for if we should place a charge in a vacuum it would surround itself with an

equal and opposite charge. And, again, since the photon cannot exist without virtually generating pairs, it follows that its self-energy is also infinite in the same way as the particle self-energy.

All this indicates that the electromagnetic field and the material wave field are not to be considered as independent entities. It is not natural to make a difference between the electromagnetic energy and the energy of pairs. We cannot even know how much of the energy is due to pairs since we cannot know how many pairs there are --- only how much energy and charge. In all experiments that measure the density of particles, we always produce, if the wave-length is small enough, an undetermined number of pairs; but the charge density (as also the current density) has a well defined localization in space and time so that we can have a well defined function  $\bar{\rho}(x, t)$  to describe it. When second quantization is introduced this still holds, since in this case, if the charge densities at two different points of space-time are  $\rho(x, t)$ ,  $\rho(x', t')$ , there exists the commutation rule (at least for  $t' = t$ )

$$(10) \quad [\rho(x, t), \rho(x', t)] = 0.$$

We do not know about the possibility of measurements of field quantities in regions whose spatial dimensions are of the order of the radius of the electron, but such distances are outside the domain of the present scalar theory. It is indeed characteristic of wave mechanics in general that one always assumes that the apparatus of measurement is itself not atomistic.

The relationship between spin and statistics has an interesting mathematical interpretation in the quantized scalar theory. Experimentally we know that if we have Bose statistics the spin will be zero or an integer, while if the spin is an odd multiple of a half the exclusion principle holds. At first this was merely an initial postulate in theoretical work, but it seems that in the relativistic theory the spin and the type of statistics are mathe-

matically connected since the exclusion principle is necessary for the development of the theory of holes. We ask, therefore, whether it would be possible to exclude mathematically the possibility of particles having zero spin but obeying the exclusion principle. Such particles, of course, do not exist physically. It turns out that they can be excluded mathematically by the following reasonable postulates:

- 1) the theory must be relativistically invariant;
- 2) the operators of physical quantities, such as energy-momentum and density-current, must be hermitean;
- 3) 
$$[\rho(x,t), \rho(x',t)] = 0.$$

This result is an advance over the non-relativistic case. We shall prove it later.

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We shall now consider the mathematical details of the Anti-Dirac Theory.

It is convenient to begin with the case where no external forces are present since the whole of the mathematical structure is exhibited in this case, and we can then discuss the general case rather briefly later. We shall find that the use of second quantization in the manner of Heisenberg and Pauli leads automatically to the result that in this case we have to do with particles of opposite electric charges which will be able to transmute into each other.

As usual, we must start with the wave equation. We can derive a relativistically invariant wave equation from the variational principle

$$(11) \quad \delta \int L dx^1 dx^2 dx^3 dx^4 = 0,$$

where\*

---

\* We shall always use Greek indices to take on the values 1, 2, 3, 4, and Latin indices for the values 1, 2, 3. The coordinate  $x^4$  is ict.

---

$$\begin{aligned}
 (12) \quad L &= -\hbar^2 c^2 \sum_{\nu=1}^4 \frac{\partial \psi^*}{\partial x^\nu} \frac{\partial \psi}{\partial x^\nu} - m^2 c^4 \psi^* \psi = \\
 &= \hbar^2 \frac{\partial \psi^*}{\partial t} \frac{\partial \psi}{\partial t} - \hbar^2 c^2 \sum_{k=1}^3 \frac{\partial \psi^*}{\partial x^k} \frac{\partial \psi}{\partial x^k} - m^2 c^4 \psi^* \psi.
 \end{aligned}$$

(Note that, as always in the relativity theory, the signs relating to space and to time come out opposite.)

The wave equation obtained in the usual manner from this variational principle is

$$(13) \quad \Delta \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \frac{m^2 c^2}{\hbar^2} \psi = 0,$$

and the energy-density tensor takes the form

$$(14) \quad T_{\mu\nu} = -\hbar^2 c^2 \left( \frac{\partial \psi^*}{\partial x^\mu} \frac{\partial \psi}{\partial x^\nu} + \frac{\partial \psi^*}{\partial x^\nu} \frac{\partial \psi}{\partial x^\mu} \right) - L \delta_{\mu\nu}.$$

Thus the total energy is

$$\begin{aligned}
 (15) \quad \bar{H} &= \int T_{44} dV = \\
 &= \int \left\{ \hbar^2 \frac{\partial \psi^*}{\partial t} \frac{\partial \psi}{\partial t} + \hbar^2 c^2 \sum_{k=1}^3 \frac{\partial \psi^*}{\partial x^k} \frac{\partial \psi}{\partial x^k} + m^2 c^4 \psi^* \psi \right\} dV,
 \end{aligned}$$

and the total momentum

$$(16) \quad \vec{G}_k = \frac{i}{c} \int T_{4k} dV = - \int \hbar^2 \left( \frac{\partial \psi^*}{\partial t} \frac{\partial \psi}{\partial x^k} + \frac{\partial \psi^*}{\partial x^k} \frac{\partial \psi}{\partial t} \right) dV.$$

(We shall always place a bar over integral quantities to distinguish them from local quantities. The arrow denotes that the quantity beneath it is a vector.)

The component  $T_{44}$  of (14) is positive definite so that the total energy  $\bar{H}$  is also positive definite. However the energy scalar  $\sum_{\nu=1}^4 T_{\nu\nu}$  can take negative values.

Before we can introduce second quantization we must bring the theory to canonical form. Denoting hermitean conjugates by means of a \*, we introduce

the canonical conjugates  $\pi, \pi^*$  of  $\psi, \psi^*$  by the usual equations

$$(17) \quad \pi = \frac{1}{\hbar} \frac{\partial L}{\partial \left( \frac{\partial \psi}{\partial t} \right)} = \hbar \frac{\partial \psi^*}{\partial t}, \quad \pi^* = \frac{1}{\hbar} \frac{\partial L}{\partial \left( \frac{\partial \psi^*}{\partial t} \right)} = \hbar \frac{\partial \psi}{\partial t}.$$

The equation of continuity is

$$(18) \quad \sum_{\nu=1}^4 \frac{\partial S_{\nu}}{\partial x^{\nu}} = 0,$$

where

$$(19) \quad S^{\nu} = e \hbar c i \left( \frac{\partial \psi^*}{\partial x^{\nu}} \psi - \frac{\partial \psi}{\partial x^{\nu}} \psi^* \right);$$

thus the charge density is

$$(20) \quad \rho = -e \hbar c i \left( \frac{\partial \psi^*}{\partial t} \psi - \frac{\partial \psi}{\partial t} \psi^* \right),$$

which we can now write in the canonical form

$$(21) \quad \rho = -e i (\pi \psi - \pi^* \psi^*).$$

Dirac himself long ago pointed out that this charge density is not positive definite.

To carry out the second quantization we introduce the formalism developed by Heisenberg and Pauli. It is a generalization of the case of a finite number of degrees of freedom, for we now have a non-denumerably infinite number. We shall later see how we can replace this continuum by a denumerable set of values.

In the second quantization  $(\pi, \psi)$  will be analogous to the  $(p, q)$  of the ordinary theory. But this analogy is not exact since here  $\pi, \psi$  are not hermitean operators. Were they hermitean the density-current vector would vanish identically!

The Heisenberg-Pauli formalism of second quantization is applicable to any classical relativistic field theory and will always lead to a relativistic quantum theory. But this theory will always deal with particles obeying the Einstein-Bose statistics. This is because in the case of large quantum num-

bers the fields become classical only if we have Einstein-Bose statistics. It is not always possible to find a scheme of quantization that will lead to the exclusion principle.

The commutation rules of the present theory are

$$(22) \quad i [\pi(x,t), \psi(x',t)] = \delta(x-x'),$$

$$(23) \quad i [\pi^*(x,t), \psi^*(x',t)] = \delta(x-x'),$$

there being commutation between  $\pi$  and  $\pi^*$ ,  $\pi$  and  $\psi^*$ ,  $\pi^*$  and  $\psi$ , and  $\psi$  and  $\psi^*$ .

In the above commutation rules the time is always the same for both quantities. It will appear later that if no forces are present we can write down explicit formulas for the commutation rules for different times. The advantage in writing down the rules for the same times is that if we use different times, the quantities  $\pi, \psi$  must satisfy the wave equation identically and so the commutation laws must agree with the wave equation. But if we take the times to be the same, the values of  $\pi, \psi$  are arbitrary for the common time and so in writing down the commutation rules we no longer need to take account of the wave equation. This procedure, however, causes the relativistic invariance to be far less obvious than in the classical theory; but this is perhaps merely an aesthetic inconvenience. Dirac has occupied himself in bringing the relativistic invariance back to the surface.

Since  $(\bar{H}, \bar{G}_R)$  are the canonical conjugates of  $(t, x^R)$ , we obtain the well-known rules that, if  $f$  is a function that can be expanded as a power series in  $\pi, \pi^*, \psi, \psi^*$ , then

$$(24) \quad \frac{\partial f}{\partial t} = \frac{i}{\hbar} [\bar{H}, f],$$

$$(25) \quad \frac{\partial f}{\partial x^R} = -\frac{i}{\hbar} [\bar{G}_R, f].$$

A very important consequence of the formalism, and one, moreover, that is trivial mathematically, is that

$$(26) \quad [\rho(x), \rho(x')] = 0, \quad (t = t').$$

For, if  $x = x'$ , this is obvious, while, if  $(x-x')$  is finite, it follows from the infinitesimal character of the commutation rules. (We shall talk of commutation rules as having an infinitesimal character when all quantities commute for  $t = t'$  and  $(x-x')$  finite. The commutation rules of the present theory are obviously of this type.)

In order to see the physical significance of the theory it is convenient to introduce the momentum space. To go over to momentum space, we expand all quantities in terms of plane waves. We can avoid Fourier integrals and accomplish this expansion by means of Fourier series by employing some convenient artifice. For example, we could assume reflecting walls. However, we shall here accomplish it by introducing, merely formally, the condition of periodicity, which means that the functions are not changed when the variables are increased by an amount  $L$ . If we write  $v$  for the volume  $L^3$ , we may then expand any  $\psi$  in the form

$$\psi(x) = \frac{1}{\sqrt{v}} \sum_{\mathbf{k}} \psi_{\mathbf{k}} \cdot e^{i(\vec{\mathbf{k}} \cdot \vec{\mathbf{x}})}$$

From (17) it follows that any  $\pi^*(x)$  may be expanded in the same form, while the  $\psi^*(x)$  and  $\pi(x)$  will require minus signs in the exponents. Thus we may write

$$(27) \quad \left\{ \begin{array}{l} \psi(x) = \frac{1}{\sqrt{v}} \sum_{\mathbf{k}} \psi_{\mathbf{k}} \cdot e^{i(\vec{\mathbf{k}} \cdot \vec{\mathbf{x}})}, \quad \pi^*(x) = \frac{1}{\sqrt{v}} \sum_{\mathbf{k}} \pi_{\mathbf{k}}^* \cdot e^{i(\vec{\mathbf{k}} \cdot \vec{\mathbf{x}})} \\ \psi^*(x) = \frac{1}{\sqrt{v}} \sum_{\mathbf{k}} \psi_{\mathbf{k}}^* \cdot e^{-i(\vec{\mathbf{k}} \cdot \vec{\mathbf{x}})}, \quad \pi(x) = \frac{1}{\sqrt{v}} \sum_{\mathbf{k}} \pi_{\mathbf{k}} \cdot e^{-i(\vec{\mathbf{k}} \cdot \vec{\mathbf{x}})} \end{array} \right.$$

and inversely,

$$(28) \begin{cases} \psi_{\vec{k}} = \frac{1}{\sqrt{v}} \int \psi(x) \cdot e^{i(\vec{k} \cdot \vec{x})} dv, & \pi_{\vec{k}}^* = \frac{1}{\sqrt{v}} \int \pi^*(x) \cdot e^{i(\vec{k} \cdot \vec{x})} dv, \\ \psi_{\vec{k}}^* = \frac{1}{\sqrt{v}} \int \psi^*(x) \cdot e^{-i(\vec{k} \cdot \vec{x})} dv, & \pi_{\vec{k}} = \frac{1}{\sqrt{v}} \int \pi(x) \cdot e^{-i(\vec{k} \cdot \vec{x})} dv. \end{cases}$$

The vector  $\vec{k}$  (the propagation vector) will now always have components that are integral multiples of  $2\pi/L$ . By this artifice, applicable to every quantity, we obtain a denumerable number of variables.

The  $x$ 's are c-numbers, but the Fourier coefficients  $f_{\vec{k}}$  are q-numbers, and it is easy to verify that the commutation relations conserve their canonical form:

$$(29) \quad i[\pi_{\vec{k}}, \psi_{\vec{l}}] = \delta_{\vec{k}\vec{l}}, \quad i[\pi_{\vec{k}}^*, \psi_{\vec{l}}^*] = \delta_{\vec{k}\vec{l}}.$$

This conservation of the form of the commutation relations is due not so much to the fact that we have used plane waves, as that these plane waves form a complete orthogonal set of functions. The form will be conserved if we expand in terms of any complete orthogonal set of functions.

Let us, for the sake of brevity, write

$$(30) \quad E_{\vec{k}} = +c \sqrt{\hbar^2 k^2 + m^2 c^2}.$$

Then we find that the Fourier coefficients of the energy and momentum are

$$(31) \quad \bar{H} = \sum_{\vec{k}} \bar{H}_{\vec{k}}; \quad \bar{H}_{\vec{k}} = \pi_{\vec{k}}^* \pi_{\vec{k}} + E_{\vec{k}}^2 \psi_{\vec{k}}^* \psi_{\vec{k}},$$

$$(32) \quad \vec{\bar{G}} = -i\hbar \sum_{\vec{k}} \vec{\bar{G}}_{\vec{k}}; \quad \vec{\bar{G}}_{\vec{k}} \equiv \vec{k} (\pi_{\vec{k}} \psi_{\vec{k}} - \pi_{\vec{k}}^* \psi_{\vec{k}}^*).$$

In the same way we find that, if  $\bar{e}$ ,  $\vec{\bar{g}}$  are respectively the total charge  $\int \rho dv$  and the total current  $\int \vec{s} dv$ , then the corresponding Fourier coefficients are

$$(33) \quad \bar{e}_{\vec{k}} = -ei(\pi_{\vec{k}} \psi_{\vec{k}} - \pi_{\vec{k}}^* \psi_{\vec{k}}^*),$$

and

$$(34) \quad \vec{\bar{g}}_{\vec{k}} = 2e\hbar c \vec{k} \psi_{\vec{k}}^* \psi_{\vec{k}}.$$

It will turn out that, unlike the other quantities, the  $\vec{G}_k$  will still depend on the time.

We have, for  $\bar{H}_k$ ,  $\vec{G}_k$ , and  $\bar{e}_k$ , quadratic forms in the variables  $\pi_k$ ,  $\psi_k$ ,  $\pi_k^*$ ,  $\psi_k^*$ , and it is easy to see that these forms commute with each other. We may therefore bring them simultaneously to the normal form.

We do this by means of the interesting change of variables

$$(35) \quad (\pi_k, \psi_k; \pi_k^*, \psi_k^*) \rightarrow (a_k, b_k; a_k^*, b_k^*),$$

where

$$(36) \quad \begin{cases} \pi_k = \frac{\sqrt{E_k}}{\sqrt{2}} (a_k^* + b_k), & \psi_k = \frac{-i}{\sqrt{2}\sqrt{E_k}} (-a_k + b_k^*), \\ \pi_k^* = \frac{\sqrt{E_k}}{\sqrt{2}} (a_k + b_k^*), & \psi_k^* = \frac{-i}{\sqrt{2}\sqrt{E_k}} (a_k^* - b_k^*), \end{cases}$$

with the inverse

$$(37) \quad \begin{cases} a_k = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{E_k}} \pi_k^* - i\sqrt{E_k} \psi_k \right), & a_k^* = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{E_k}} \pi_k + i\sqrt{E_k} \psi_k^* \right), \\ b_k = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{E_k}} \pi_k - i\sqrt{E_k} \psi_k^* \right), & b_k^* = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{E_k}} \pi_k^* + i\sqrt{E_k} \psi_k \right). \end{cases}$$

We find that the commutation rules still retain the canonical form:

$$(38) \quad [a_k, a_l^*] = \delta_{kl}, \quad [b_k, b_l^*] = \delta_{kl},$$

while the various Fourier coefficients take the forms

$$(39) \quad \bar{H}_k = E_k (a_k^* a_k + b_k^* b_k + 1),$$

$$(40) \quad \vec{G}_k = \hbar \vec{k} (a_k^* a_k - b_k^* b_k),$$

$$(41) \quad \bar{e}_k = e (a_k^* a_k - b_k^* b_k),$$

$$(42) \quad \frac{i}{c} \vec{G}_k = \frac{e \hbar c \vec{k}}{E_k} (a_k^* a_k + b_k^* b_k - a_k^* b_k^* - a_k b_k + 1).$$

The (+1) in the formula for  $\bar{H}_k$  is the zero point energy.

The total current does not commute with the energy:

$$(43) \quad [\vec{J}, H] \neq 0.$$

Thus we cannot simultaneously bring the total current into a similar form.

From (43) and (24) it follows that the total current is not independent of the time.

It is the quantities

$$(44) \quad \overrightarrow{g}_k \quad a_k^* b_k^* , \quad a_k b_k$$

in  $\overrightarrow{g}_k$  that prevent it from commuting with the other expressions. Such quantities are well known from Dirac's theory of radiation. Indeed, we know that if we write

$$(45) \quad N_k^+ = a^* a , \quad N_k^- = b^* b ,$$

the quantities  $N_k^+$ ,  $N_k^-$  will, in view of (38), have the integral eigen-values 0, 1, 2, 3, ... . And from the relation between the energy (39), the momentum (40), and the charge (41), it follows that  $N_k^+$  is the number of particles having momentum  $\hbar \vec{k}$  and charge  $+e$  while  $N_k^-$  is the number of particles having momentum  $(-\hbar \vec{k})$  and charge  $(-e)$ . We can therefore interpret  $a_k$ ,  $b_k$ ,  $a_k^*$ ,  $b_k^*$  as operators of a very simple character. We must introduce the Schrödinger functional  $\Phi(\dots, N_k^+, \dots, N_k^-, \dots)$ , and then we have

$$(46) \quad \left\{ \begin{array}{l} a_k \Phi(\dots, N_k^+, \dots) = \sqrt{N_k^+} \Phi(\dots, N_k^+ - 1, \dots), \\ a_k^* \Phi(\dots, N_k^+, \dots) = \sqrt{N_k^+ + 1} \Phi(\dots, N_k^+ + 1, \dots), \\ b_k \Phi(\dots, N_k^-, \dots) = \sqrt{N_k^-} \Phi(\dots, N_k^- - 1, \dots), \\ b_k^* \Phi(\dots, N_k^-, \dots) = \sqrt{N_k^- + 1} \Phi(\dots, N_k^- + 1, \dots). \end{array} \right.$$

The operators  $N_k^+$  and  $N_k^-$  correspond to the production and annihilation of pairs.

Let us investigate how the expressions (44) that enter  $\overrightarrow{g}$  vary with the time. Applying (24) we find that

$$(47) \quad \begin{cases} \dot{a}_k = -i \frac{E_k}{\hbar} a_k, & \dot{a}_k^* = +i \frac{E_k}{\hbar} a_k^*, \\ \dot{b}_k = -i \frac{E_k}{\hbar} b_k, & \dot{b}_k^* = +i \frac{E_k}{\hbar} b_k^*, \end{cases}$$

which, on integration, give

$$(48) \quad \begin{cases} a_k = a_k(0) \cdot \exp(-i \frac{E_k}{\hbar} t), & a_k^* = a_k^*(0) \cdot \exp(+i \frac{E_k}{\hbar} t), \\ b_k = b_k(0) \cdot \exp(-i \frac{E_k}{\hbar} t), & b_k^* = b_k^*(0) \cdot \exp(+i \frac{E_k}{\hbar} t). \end{cases}$$

Hence the terms (44) oscillate with the frequency  $2E_k/\hbar$ . This corresponds to the trembling movement of Schrödinger's interpretation of Dirac's equations.

It is characteristic of the present theory that though we have both positive and negative frequencies, we have only positive energy states. Physically, the formulas (36) mean that we have made a separation of the quantities  $\pi_k, \psi_k, \pi_k^*, \psi_k^*$  into terms having positive frequencies and terms having negative frequencies. This will turn out to be of great importance when we come to discuss the exclusion principle.

It is not possible to find a relativistically invariant expression for the density of the particles since we cannot define, in an unambiguous way, quantities of the type  $N^+(\mathbf{x}), N^-(\mathbf{x})$  that would have the significance of the density of particles of charge (+e) and (-e) respectively. This is also the case in the theory of holes.

The charge density  $\rho(\mathbf{x})$  does not commute with total number of positively charged particles  $N^+ = \sum_k N_k^+$ , nor with the total number of negatively charged particles  $N^- = \sum_k N_k^-$ , but only with their difference  $(N^+ - N^-)$ . Hence we must conclude that, in this theory, every measurement of localization of the charge density will make impossible a simultaneous knowledge of the total number of particles, and this means that an undetermined number of pairs will be produced whenever such a measurement is made.

The whole theory is actually symmetric between positive and negative charges. We can see this directly if we apply the transformation  $\psi \rightarrow \psi^*$ ,  $\psi^* \rightarrow \psi$  to the original formulas since this leaves the energy and momentum and the commutation rules unaltered and merely changes the sign of the density-current vector. Thus this transformation amounts to merely changing the sign of  $e$  throughout. We shall find that this symmetry between positive and negative charges is characteristic also of the theory of holes.

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We proceed to the question of the possibility of obtaining the Fermi statistics while retaining the relativistic invariance of the theory. This work is as yet unpublished and will appear in the Annales Poincaré.

The question is not a trivial one. For, though we can always obtain a theory leading to the Einstein-Bose statistics, it is only a fortunate accident when we can make a theory that will give the exclusion principle.

We must first consider in more detail the significance of the variables  $a$  and  $b$ , and the possibility of generalizing the Einstein-Bose case. Only then can we discuss the possibility of a theory leading to the Fermi statistics.

Now equations (48) show that there is a separation of the  $\psi_R$ 's and  $\pi_R$ 's into two parts, one part having positive frequency, the other negative frequency. We can go back from the  $\psi_R$ 's to the original  $\psi$ 's by means of (27) and the parts of positive and negative frequency will still be separated. Since a Lorentz transformation does not mix up positive and negative frequencies, the two parts will each be relativistically invariant -- at least for the case in which no external forces are present.

We find that

$$(49) \quad \psi = \psi^+ + \bar{\psi}^- ,$$

where

$$(50) \quad \begin{cases} \bar{\Psi}(x,t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{R}} \frac{i}{\sqrt{2}} \frac{a_{\mathbf{R}}}{\sqrt{E_{\mathbf{R}}}} e^{i(\mathbf{R} \cdot \vec{x})}, & \bar{\Psi}^*(x,t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{R}} \frac{-i}{\sqrt{2}} \frac{a_{\mathbf{R}}^*}{\sqrt{E_{\mathbf{R}}}} e^{-i(\mathbf{R} \cdot \vec{x})}, \\ \Psi(x,t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{R}} \frac{-i}{\sqrt{2}} \frac{b_{\mathbf{R}}^*}{\sqrt{E_{\mathbf{R}}}} e^{i(\mathbf{R} \cdot \vec{x})}, & \Psi^*(x,t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{R}} \frac{i}{\sqrt{2}} \frac{b_{\mathbf{R}}}{\sqrt{E_{\mathbf{R}}}} e^{-i(\mathbf{R} \cdot \vec{x})}. \end{cases}$$

These two parts must now satisfy differential equations of the first order in the time derivatives, namely

$$(51) \quad \begin{cases} \hbar \frac{\partial \Psi}{\partial t} = +i \sqrt{-\hbar^2 \Delta + m^2 c^2} \Psi, \\ \hbar \frac{\partial \bar{\Psi}}{\partial t} = -i \sqrt{-\hbar^2 \Delta + m^2 c^2} \bar{\Psi}. \end{cases}$$

These equations contain the very disagreeable operator  $\sqrt{-\hbar^2 \Delta + m^2 c^2}$ .

It is really an abbreviation for the introduction of the Fourier series since we can only interpret it in terms of a Fourier expansion.

The equation (51) is equivalent to

$$\hbar \frac{\partial \Psi_{\mathbf{R}}}{\partial t} = +i \sqrt{\hbar^2 \mathbf{R}^2 + m^2 c^2} \Psi_{\mathbf{R}}, \text{ etc.,}$$

so that the operator  $\sqrt{-\hbar^2 \Delta + m^2 c^2}$  is equivalent, in the Fourier case, to multiplication of each component by a constant. It is not a point operator, but owing to the infinitesimal nature of the commutation rules, it actually depends only on a finite domain of the small dimensions  $\hbar/mc$ . The introduction of this operator was not necessary for the results previously obtained, but for the discussion of possible generalizations of the theory it is essential.

We shall need to introduce a function that is a generalization of the Dirac  $\delta$ -function. The Dirac  $\delta$ -function can be put in the form

$$(52) \quad \delta(\mathbf{x}) = \frac{1}{(2\pi)^3} \int e^{i(\mathbf{R} \cdot \vec{x})} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$$

which converges after integration of the  $\mathbf{x}$ 's over a finite domain. We introduce the  $g$ -function by

$$(53) \quad g(x) = \frac{1}{(2\pi)^3} \int \frac{e^{i(\vec{k} \cdot \vec{x})}}{\sqrt{\hbar^2 k^2 + m^2 c^2}} d\hbar_1 d\hbar_2 d\hbar_3.$$

Thus

$$(54) \quad g(x) = \frac{1}{\sqrt{-\hbar^2 \Delta + m^2 c^2}} \delta(x),$$

and we have

$$(55) \quad \frac{1}{\sqrt{-\hbar^2 \Delta + m^2 c^2}} f(x) = \int g(x-x') f(x') dV'.$$

It is now possible to formulate the commutation rules for the  $\bar{\Psi}^+$  and  $\bar{\Psi}$ . Using (50) and (38), we find

$$(56) \quad \begin{cases} [\bar{\Psi}^+(x, t), \bar{\Psi}^*(x', t)] = -\frac{1}{2} g(x-x'), \\ [\bar{\Psi}(x, t), \bar{\Psi}^*(x', t)] = +\frac{1}{2} g(x-x'), \end{cases}$$

while there is commutation whenever a  $+$  and  $-$  are concerned together.

If we want commutation rules for  $t \neq t'$  we must first generalize the definition of the  $g$ -function by writing

$$(57) \quad \begin{cases} g_+(x, t) = \frac{1}{(2\pi)^3} \int \frac{\exp. i\{\hbar \cdot \vec{x} + \sqrt{\hbar^2 k^2 + m^2 c^2} \cdot ct\}}{\sqrt{\hbar^2 k^2 + m^2 c^2}} d\hbar_1 d\hbar_2 d\hbar_3, \\ g_-(x, t) = g_+(x, -t). \end{cases}$$

We can also give an implicit definition of these  $g$ 's owing to the fact that they must satisfy the wave equation. We have, in fact, for some arbitrary time  $t = 0$ ,

$$(58) \quad \begin{cases} g_+(x, 0) = g_-(x, 0) = \frac{1}{\sqrt{-\hbar^2 \Delta + m^2 c^2}} \delta(x), \\ \frac{1}{c} \left( \frac{\partial g_+}{\partial t} \right)_{x,0} = -\frac{1}{c} \left( \frac{\partial g_-}{\partial t} \right)_{x,0} = i \delta(x), \\ \left( \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{m^2 c^2}{\hbar^2} \right) g_{\pm} = 0. \end{cases}$$

The commutation rules are now expressible in the form

$$(59) \quad \begin{cases} [\bar{\Psi}^+(x, t), \bar{\Psi}^*(x', t')] = -\frac{1}{2} g_+(x-x', t-t'), \\ [\bar{\Psi}(x, t), \bar{\Psi}^*(x', t')] = +\frac{1}{2} g_-(x-x', t-t'), \end{cases}$$

all this being for the case of no external forces.

If we put into the expression for the density-current vector the separation of  $\psi$  into two parts, we find, after some rearrangement whose purpose will appear later, that

$$(60) \quad S_\nu = e\hbar c i \left[ \left( \frac{\partial \bar{\psi}^*}{\partial x^\nu} \bar{\psi} - \bar{\psi}^* \frac{\partial \bar{\psi}}{\partial x^\nu} \right) + \left( \frac{\partial \psi^{*+}}{\partial x^\nu} \psi^+ - \psi^{*+} \frac{\partial \psi^+}{\partial x^\nu} \right) + \left( \frac{\partial \bar{\psi}^{*+}}{\partial x^\nu} \psi^+ - \psi^{*+} \frac{\partial \bar{\psi}}{\partial x^\nu} \right) \right].$$

We may now consider the important question of the possible ways in which the theory can be generalized.

If we require that all physical observables shall be represented by hermitean operators, that the theory shall be relativistically invariant, and that the density-current vector shall satisfy the equation of continuity, but no longer insist on the Hamiltonian canonical formalism, nor on the infinitesimal character of the commutation rules, we may generalize the theory by introducing three arbitrary constants in  $S_\nu$ , so that it now reads

$$(61) \quad S_\nu = e\hbar c i \left[ c_1 ( \quad ) + c_2 ( \quad ) + c_3 ( \quad ) \right].$$

It is easily verified that the three parts separately satisfy the equation of continuity, and the Lorentz invariance of each parenthesis is obvious from its form when it is remembered that each  $\psi$  is a scalar under the Lorentz group.

Let us investigate the physical significance of these constants. The value of the total charge is now given by

$$(62) \quad \bar{e} = \int \rho dV = e \sum_R (c_1 a_R^* a_R - c_2 b_R^* b_R),$$

the  $c_3$  not entering this expression. It follows from the form of  $\bar{e}$  that  $(c_1, e)$  and  $(-c_2, e)$  are respectively the charges of the positive and negative particles,  $c_3$  determines the magnitude of the part of the current density  $\rho(x)$  that gives

rise to the process of annihilation and generation of pairs, and if it is zero there will be no such annihilation and generation.

It must be admitted that we have not shown that the extension to this more general theory is possible if external fields are present, and it is almost certain that difficulties will arise in this case.

The canonical or anti-Dirac case is given by  $c_1 = c_2 = c_3$  and in this case we can eliminate the non-point operators. We have proved that it is an immediate consequence of the canonical formalism and of the infinitesimal character of the commutation rules that in this case

$$(26) \quad [\rho(x,t), \rho(x',t)] = 0.$$

This result is very desirable on physical grounds and we therefore ask what values we can give to the c's without destroying it. The c's must all be real if  $\rho$  is to be hermitean, but if we work out the conditions that (26) shall hold we find we must have

$$c_1^2 = c_2^2 = c_3^2 ; \quad c_1 c_3 = c_2 c_3$$

so that

$$(63) \quad c_1 = c_2, \quad c_3 = \pm c_1.$$

This seems to hold the new possibility  $c_1 = c_2 = -c_3$  but it is actually the canonical case over again. This is easily seen if we change the sign of  $\Psi^\dagger$  relative to that of  $\Psi$  throughout the canonical theory, since the only

We are now prepared to investigate the possibility of obtaining the exclusion principle. We make the postulate that the equations for the Fourier components of the energy and momentum, (31) and (32), shall still hold but that the N's, instead of having the eigen values 0, 1, 2, 3, ..., shall have only the eigen values 0, 1. We can satisfy this condition, in the well-known manner due to Wigner and Jordan, by means of the equations

effect of this change is to alter the sign of  $c_3$ . Thus the canonical case is the only case in which we may still have (26) satisfied.

$$(64) \quad [a_k, a_l^*]_+ = \delta_{kl}, \quad [b_k, b_l^*]_+ = \delta_{kl},$$

where

$$(65) \quad [A, B]_+ \equiv AB + BA.$$

With (64) we of course also require, not that  $a_k, a_l$ , for example, shall commute, but that  $[a_k, a_l]_+$  shall be zero.

Now these rules mean that the N's of different proper oscillations will commute and every N will have only the eigen values 0, 1. Therefore this mathematical scheme is completely equivalent to the exclusion principle without any arbitrary postulates, and it does not seem as if we can introduce the principle in an essentially different manner. We cannot postulate an equation of the type  $[a_k, a_l^*]_+ = -\delta_{kl}$  if we want  $a_k$  to be hermitean.

We can work out the commutation rules for the  $\psi$ 's for the present case, and it is necessary to be very careful of the signs. It turns out that we now have

$$(66) \quad \begin{cases} [\psi^\dagger(x,t), \psi^\dagger(x',t)]_+ = +\frac{1}{2} g(x-x'), & \rightarrow [\psi^\dagger, \psi^\dagger]_+ = g(x-x') \\ [\bar{\psi}(x,t), \bar{\psi}^*(x',t)]_+ = +\frac{1}{2} g(x-x'), & \text{Further } [\psi, \psi']_+ = 0 \\ \text{with } [\psi^\dagger, \bar{\psi}^*]_+ = 0 \text{ etc.} & [\psi^\dagger, \psi']_+ = 0 \end{cases}$$

It is easy to see that the plus signs on the left of (66) must prevent the setting up of quantities obeying commutation rules having an infinitesimal character. For let us evaluate  $[\pi(x,t), \psi(x',t)]_+$  for this theory.

We have

$$\begin{aligned} [\pi(x,t), \psi(x',t)]_+ &= \left[ \hbar \frac{\partial \psi^*(x,t)}{\partial t}, \psi(x',t) \right]_+ = \\ &= \left[ \hbar \frac{\partial \psi^\dagger(x,t)}{\partial t}, \psi^\dagger(x',t) \right]_+ + \left[ \hbar \frac{\partial \bar{\psi}^*(x,t)}{\partial t}, \bar{\psi}(x',t) \right]_+ = \\ &= +i \sqrt{-\hbar^2 \Delta + m^2 c^2} [\psi^\dagger(x,t), \psi^\dagger(x',t)]_+ - i \sqrt{-\hbar^2 \Delta + m^2 c^2} [\bar{\psi}^*(x,t), \bar{\psi}(x',t)]_+ \quad (\text{by (51)}) \\ &= +i \sqrt{-\hbar^2 \Delta + m^2 c^2} \left\{ -\frac{1}{2} g(x-x') \right\} - i \sqrt{-\hbar^2 \Delta + m^2 c^2} \left\{ -\frac{1}{2} g(x-x') \right\} \quad (\text{by (66)}) \\ &= 0. \end{aligned}$$

Thus  $\pi$  and  $\psi$  will not commute at all, but will anticommute everywhere. In the Einstein-Bose case the single minus sign on the right of (56) changes the sign of the second term in the analogous argument for  $[\pi, \psi]_-$  so that instead of zero we get  $-i \delta(x-x')$  by (54). On the other hand, we have

$$[\psi^*(x,t), \psi(x',t)]_+ = [\psi^*(x,t), \psi(x,t)]_+ + [\bar{\psi}(x,t), \bar{\psi}(x',t)]_+ = g(x-x')$$

whereas in the Einstein-Bose case, there was

$$[\psi^*(x,t), \psi(x',t)] = 0.$$

We have shown, however, that it is possible to construct a relativistically invariant theory that leads to the exclusion principle.

We can introduce the same  $S_\nu$  as before, and the  $c$ 's must still be real. If we now require that (26) shall be satisfied in this theory, we find that

$$(67) \quad c_1^2 = c_2^2 = -c_3^2 ; , c_1 c_3 = c_2 c_3$$

which have no real solutions!

Thus if we want (26) to be satisfied, the theory to be relativistically invariant, and physical observables to be represented by hermitean operators, the exclusion principle is impossible. But if we renounce (26) we can obtain a theory leading to the exclusion principle, though even in this case difficulties will arise when external forces are present.

In the non-relativistic theory both types of statistics are possible, but in the theory of holes the Einstein-Bose statistics is not possible, while in the scalar theory, which has zero spin, the Fermi statistics cannot occur. We may say therefore that there is, in the relativistic theories, a close connection between the spin and the statistics.

The mathematical significance of the requirement of relativistic invariance can be seen from the following considerations. In the commutation rules (56) and (66) the operator  $g(x)$  is relativistically invariant. It is, moreover, the only relativistically invariant operator of this type, for in so far as the relation

$$(68) \quad k_0^2 - (k_1^2 + k_2^2 + k_3^2) = m^2 c^2 / \hbar^2$$

is fulfilled, we have the invariant expression

$$(69) \quad \frac{dk_1 dk_2 dk_3}{\sqrt{k^2 + m^2 c^2 / \hbar^2}} = \frac{dk_1 dk_2 dk_3}{k_0}$$

which is a surface element of a three-dimensional surface in a four-dimensional space.

The form of the denominator in  $g(x)$  is therefore determined by the requirement of relativistic invariance. If we were dealing with a non-relativistic theory we could substitute any other function for this denominator -- for example, a constant. Thus the requirement of relativistic invariance imposes a very stringent limitation on the choice of the function  $g(x)$ .

We can save the infinitesimal character of the commutation rules if we are prepared to sacrifice the requirement that the operators be hermitean. For, instead of  $\psi, \pi, \psi^*, \pi^*$ , where the \* denotes the hermitean conjugate, we can introduce  $\psi, \pi, \psi^\dagger, \pi^\dagger$  by means of the formulas

$$(70) \quad \left\{ \begin{array}{l} \psi_k = \frac{-i}{\sqrt{2} \sqrt{E_k}} (-a_k + b_k^*), \quad \pi_k = \frac{\sqrt{E_k}}{\sqrt{2}} (a_k^* \overset{N.B.}{\leftarrow} b_k), \\ \psi_k^\dagger = \frac{-i}{\sqrt{2} \sqrt{E_k}} (a_k^* \overset{N.B.}{\leftarrow} b_k), \quad \pi_k^\dagger = \frac{\sqrt{E_k}}{\sqrt{2}} (a_k + b_k^*), \end{array} \right.$$

which differ from the corresponding formulas (36) only in the signs specially indicated above. If we use these  $\psi$ 's and  $\pi$ 's we now find, from (64), the

infinitesimal commutation rules

$$(71) \left\{ \begin{array}{l} i[\pi(x,t), \psi(x',t)]_+ = \delta(x-x'), \text{ etc.} \\ [\psi(x,t), \psi^\dagger(x',t)]_+ = 0, [\psi(x,t), \psi(x',t)]_+ = 0, \text{ etc.} \end{array} \right.$$

To sum up the results of our discussion of the possibility of generalizing the original theory, we may say that, if we are prepared to ignore one of the three postulates given on page 6, we can obtain a theory that will lead to the exclusion principle, but if we insist on all three postulates we must have the Einstein-Bose statistics. It should be noted, though, that if

$[\rho(x,t), \rho(x',t)]$  is different from zero this does not mean that the expectation value of  $\rho$  cannot be concentrated, and in many physical problems it is only the expectation value of  $\rho$  that is important.

We now go back to the original canonical theory leading to the Bose-Einstein statistics and discuss the effects of introducing external forces. We shall introduce these forces by means of the electromagnetic potentials which we shall treat as c-numbers. They will thus be treated as being unrelated to the field of the particles themselves, and in this way we shall avoid the difficulty of the infinite self-energy of the particles. Since we do not include in the Lagrangian or Hamiltonian the energy-momentum of the external field, we shall, as in the classical case, no longer have a conservation of momentum, though if the potentials do not contain the time explicitly we shall still have a conservation of energy.

To introduce the external field we make the usual substitution

$$(72) \quad p_\mu \rightarrow p_\mu - \frac{e}{c} \mathcal{P}_\mu$$

which is in conformity with the principle of gauge invariance. The minus sign implies that  $e$  is the actual charge, so that the charge of an electron will be  $(-e)$ .

The resulting Lagrangian, Hamiltonian, wave equation, energy-momentum tensor, and density-current vector are very well known, having been discussed in print by many theoretical physicists.

The Lagrangian is

$$(73) \quad L = -\frac{\hbar^2}{2c^2} \sum_{\nu=1}^4 \left( \frac{\partial \Psi^*}{\partial x^\nu} + \frac{ie}{\hbar c} \phi_\nu \Psi^* \right) \left( \frac{\partial \Psi}{\partial x^\nu} - \frac{ie}{\hbar c} \phi_\nu \Psi \right) - m^2 c^4 \Psi^* \Psi,$$

and the wave equation is

$$(74) \quad \sum_{\kappa=1}^3 \left( \frac{\partial}{\partial x^\kappa} - \frac{ie}{\hbar c} \phi_\kappa \right) \left( \frac{\partial}{\partial x^\kappa} - \frac{ie}{\hbar c} \phi_\kappa \right) \Psi - \left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{ie}{\hbar c} \phi_0 \right) \left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{ie}{\hbar c} \phi_0 \right) \Psi - \frac{m^2 c^2}{\hbar^2} \Psi = 0,$$

The  $\pi$ 's are defined formally as before, and we now have

$$(75) \quad \begin{cases} \pi = \frac{1}{\hbar} \frac{\partial L}{\partial \left( \frac{\partial \Psi}{\partial t} \right)} = \frac{1}{\hbar} \frac{\partial \Psi^*}{\partial t} - ie \phi_0 \Psi^*, \\ \pi^* = \frac{1}{\hbar} \frac{\partial L}{\partial \left( \frac{\partial \Psi^*}{\partial t} \right)} = \frac{1}{\hbar} \frac{\partial \Psi}{\partial t} + ie \phi_0 \Psi. \end{cases}$$

The above relationship between the  $\pi$ 's and the time derivatives of the  $\Psi$ 's is no longer that given in (17), but the commutation rules between the present  $\pi$ 's and  $\Psi$ 's introduced by the second quantization will be of the same form as before.

The Hamiltonian  $\bar{H}$  is obtained in the usual way, and can be broken up into an unperturbed part  $\bar{H}_0$  and a perturbation energy  $\bar{H}_1$ . We have, in fact,

$$(76) \quad \bar{H} = \int \left( \frac{\hbar}{2} \pi \dot{\Psi} + \frac{\hbar}{2} \pi^* \dot{\Psi}^* - L \right) d\tau = \bar{H}_0 + \bar{H}_1,$$

where

$$(77) \quad \bar{H}_0 = \int \left\{ \pi \pi^* + \frac{\hbar^2}{2c^2} \sum_{\kappa=1}^3 \frac{\partial \Psi^*}{\partial x^\kappa} \frac{\partial \Psi}{\partial x^\kappa} + m^2 c^4 \Psi^* \Psi \right\} d\tau$$

and

$$(78) \quad \bar{H}_1 = ie \int \phi_0 (\pi^* \psi^* - \pi \psi) dv + \\ + \left\{ ie \hbar c \sum_{k=1}^3 \phi_k \left( \psi^* \frac{\partial \psi}{\partial x^k} - \frac{\partial \psi^*}{\partial x^k} \psi \right) + e^2 \sum_{k=1}^3 \phi_k^2 \psi^* \psi \right\} dv.$$

Note that  $\bar{H}_0$  is not independent of the  $\phi$ 's since they enter in the  $\pi \pi^*$  term.

The density current vector is

$$(79) \quad s_\nu = e \hbar c i \left[ \left( \frac{\partial \psi^*}{\partial x^\nu} + \frac{ie}{\hbar c} \phi_\nu \psi^* \right) \psi - \left( \frac{\partial \psi}{\partial x^\nu} - \frac{ie}{\hbar c} \phi_\nu \psi \right) \psi^* \right]$$

so that, in particular,

$$(80) \quad \rho = -i s_4 = -ei \left[ \left( \hbar \frac{\partial \psi^*}{\partial t} + ie \phi_0 \psi^* \right) \psi - \left( \hbar \frac{\partial \psi}{\partial t} - ie \phi_0 \psi \right) \psi^* \right] = \\ = -ei (\pi \psi - \pi^* \psi^*).$$

This  $\rho$  has the same form in terms of the  $\pi$ 's and  $\psi$ 's as before, and since the commutation rules for the new  $\pi$ 's and  $\psi$ 's are also the same as before, we shall retain the result that  $[\rho(x, t), \rho(x', t)] = 0$ .

The perturbation energy  $\bar{H}_1$  can be expressed in terms of plane waves, and we find that

$$(81) \quad \bar{H}_1 = \frac{1}{2} e \sum_k \sum_l \phi_{kl}^0 \left[ \frac{E_k + E_l}{\sqrt{E_k E_l}} (a_k^* a_l - b_l^* b_k) + \frac{E_k - E_l}{\sqrt{E_k E_l}} (a_l b_k - a_k^* b_l^*) \right] + \\ + \frac{1}{2} \sum_k \sum_l \frac{1}{\sqrt{E_k E_l}} \left[ \hbar c e (\vec{\phi}_{kl}, \vec{k} + \vec{l}) - e^2 (\phi^2)_{kl} \right] (a_k^* a_l + b_l^* b_k - a_k^* b_l^* - b_k a_l).$$

It will be seen that the operators  $a_k b_k$  and  $a_k^* b_k^*$  that correspond to the generation and annihilation of pairs are present in this perturbation energy. It turns out that this property of the theory of accounting for the production and annihilation of pairs has the very disagreeable consequence that it leads to an

infinite polarizability of the vacuum of the same form as in the theory of holes. Owing to the presence of the non-diagonal terms  $a_{\mathbf{k}} b_{\mathbf{k}}$ ,  $a_{\mathbf{k}}^* b_{\mathbf{k}}^*$  in  $\bar{H}_1$  no stationary solution exists that will correspond to the vacuum.

As a final point in the present theory we discuss the treatment of the hydrogen atom.

It is clear that we shall no longer have an exact solution because of the infinite polarizability of the vacuum, and the question we wish to discuss is to what extent an approximate solution is possible.

We shall show that we get the right values for the energy levels if we ignore the commutation rules. We take  $\phi_{\mathbf{k}}$  to be zero and  $\phi_0$  to be independent of the time. Then we can always find solutions of the wave equation that are stationary in the sense that their absolute values are independent of  $t$ . Thus we take

$$(82) \quad \psi = \sum_n c_n u_n(x) e^{-\frac{i}{\hbar} E_n t},$$

where, as yet, we do not know that  $E_n$  is an energy.

The wave equation now gives

$$(83) \quad (E_n - e\phi_0)^2 u_n = \mathcal{H} u_n,$$

where  $\mathcal{H}$  is hermitean. And, instead of the usual orthogonality relations, we find

$$(84) \quad \int u_n^* u_m (E_n + E_m - 2e\phi_0) dv = 0, \quad E_n \neq E_m.$$

If we assume that the  $c$ 's are  $c$ -numbers, the total charge takes the form

$$(85) \quad \bar{e} = \int \rho dv = e \sum_n c_n^* c_n \cdot E_n$$

where  $E_n$  is a normalization factor given by

$$(86) \quad E_n = \int |u_n|^2 \cdot 2(E_n - e\phi_0) dv.$$

We cannot be sure that  $E_n$  will be normalized to +1 since  $(E_n - e\phi_0)$  can be negative and no normalization of  $u_n$  will change the sign of  $|u_n|^2$ . So we can normalize  $E_n$  only to within the choice  $\pm 1$ .

If we calculate the energy we find, after making some simple transformations that use the wave equation, that

$$(37) \quad \bar{E} = \sum_{n,m} c_m^* c_n \int \frac{u_m^*}{K_m} \frac{u_n}{K_n} (E_n + E_m - 2e\phi_0) E_m dv$$

which, by (84), reduces to

$$(88) \quad \bar{E} = \sum_n \epsilon_n E_n c_n^* c_n.$$

Thus  $E_n$  is seen to be the energy of the  $n^{\text{th}}$  state.

If the nuclear charge  $e$  is small, the sign of  $E_n$  will be, to a first approximation, the same as that of  $E_n$  so that  $\bar{E}$  will be positive; but  $\bar{e}$  will have both positive and negative values. It is, however, not necessary that the energy  $E_n$  shall be positive since it contains the potential energy.

All this was on the assumption that the  $c$ 's are  $c$ -numbers, but actually, owing to the second quantization, they must be  $q$ -numbers. And when we try to take account of this fact we meet great difficulties. If the commutation rules were of the type  $[c_m^*, c_n] = \delta_{mn}$  it would not be very bad, but owing to the peculiar form of the "orthogonality" relations (84), such simple commutation rules do not hold.

### The Theory of Holes

We shall now consider Heisenberg's extension of Dirac's theory of the positron. This theory is an attempt to find a new interpretation for the negative energy states, and makes an essential use of the exclusion principle. Indeed, the exclusion principle finds its most natural expression in the  $q$ -number

formalism of the theory of holes, for physically the second quantization of the  $\Psi$ -function does not introduce a new assumption but merely brings into the open the fact that the number of particles in each state can only be zero or one.

We must start from the  $\Psi$ -functions that satisfy the Dirac equations. These equations, in the field-free case, may be written in the matrix form

$$(89) \quad \frac{1}{c} \frac{\partial \Psi}{\partial t} + \sum_{\mathbf{R}} \alpha^{\mathbf{R}} \frac{\partial \Psi}{\partial x^{\mathbf{R}}} + i \frac{mc}{\hbar} \beta \Psi = 0,$$

with the conjugate complex set

$$(90) \quad -\frac{1}{c} \frac{\partial \Psi^*}{\partial t} + \sum_{\mathbf{R}} \frac{\partial \Psi^*}{\partial x^{\mathbf{R}}} \alpha^{\mathbf{R}} - i \frac{mc}{\hbar} \Psi^* \beta = 0.$$

The  $\alpha$ 's and the  $\beta$  obey the commutation rules

$$(91) \quad \begin{cases} \frac{1}{2} (\alpha^i \alpha^{\mathbf{R}} + \alpha^{\mathbf{R}} \alpha^i) = \delta_{i\mathbf{R}}, \\ \beta^2 = 1, \quad \alpha^{\mathbf{R}} \beta + \beta \alpha^{\mathbf{R}} = 0. \end{cases}$$

The form of the Dirac equations given in (89) is such that the time coordinate is distinguished from the other coordinates in that the coefficient of the time derivative is a multiple of the unit matrix. This is useful if we wish to discuss questions of reality, but if we are interested in the relativistic invariance we must put the equations (89) into a more symmetric form. This can be done by multiplying them by  $(i\beta)$  on the left and writing  $\gamma^{\mathbf{R}}$  for  $-i\beta \alpha^{\mathbf{R}}$  and  $\gamma^4$  for  $\beta$ . We then find

$$(92) \quad \sum_{\mu=1}^4 \gamma^{\mu} \frac{\partial \Psi}{\partial x^{\mu}} + \frac{mc}{\hbar} \Psi = 0,$$

where

$$(93) \quad \frac{1}{2} (\gamma^{\mu} \gamma^{\nu} + \gamma^{\nu} \gamma^{\mu}) = \delta_{\mu\nu},$$

and  $x^4 = ict$ . This form of the Dirac equations is very convenient for proving the relativistic invariance but, owing to the imaginary character of  $x^4$ , it is now unsuitable for the discussion of reality properties.

We shall first introduce the second quantization for the case in which there is no external field, and we shall see that the existence of both positive and negative energy states gives rise to the possibility of modifying the interpretation of the theory.

We must begin with the normal, orthogonal set of  $\psi$ -functions for the force-free case. These are the plane waves, but unlike the corresponding plane waves of the scalar theory they cannot be characterized by the values of the momentum  $p$  alone. We must add an index  $n$  that can take four values. This corresponds to the notion of the polarization of waves and is quite analogous to the electromagnetic case where we have two orthogonal wave functions corresponding to a given amplitude and direction. In the present case two of the four states corresponding to a given  $p$  will belong to a positive frequency, and the other two to a negative frequency.

If we let  $u^{(n,\mu)}(\rho, \vec{x})$  be the part of a monochromatic  $\psi$  that is independent of the time we may express it in the form

$$(94) \quad u^{(n,\mu)}(\rho, \vec{x}) = c_{\rho}^{(n)}(h) \cdot \frac{1}{(2\pi h)^{\frac{3}{2}}} e^{\frac{i}{h}(\vec{p} \cdot \vec{x})}$$

where the momentum  $p$  is here a constant. Some remarks are needed in order to clarify the significance of the various suffixes. The index  $n$  is the "polarization" index and takes the values 1, 2, 3, 4. The  $\vec{x}$  is the set of spatial coordinates, each of which has a continuous range of values. The  $\vec{p}$ , as already explained, defines the momentum, and  $\rho$  is the spin variable, taking on the four values 1, 2, 3, 4. It is important to distinguish between the polarization index  $n$  and the spin index  $\rho$ .

Owing to the multiplicity of indices the normalization and orthogonality relations will be more complicated than in the scalar case. We have, in fact, the relations

$$(95) \quad \left\{ \begin{array}{l} \sum_{\rho=1}^4 C_{\rho}^{*n}(k) \cdot C_{\rho}^m(k) = \delta_{nm} \\ \sum_{n=1}^4 C_{\rho}^{*n}(k) \cdot C_{\rho'}^n(k) = \delta_{\rho\rho'} \end{array} \right.$$

which are respectively consequences of the corresponding relations on the  $u$ 's, namely

$$(96) \quad \left\{ \begin{array}{l} \int d\omega. \sum_{\rho} u_{(\rho,x)}^{*(n,k)} \cdot u_{(\rho,x)}^{(m,k')} = \delta_{nm} \delta(t-t'), \\ \int d\rho \sum_n u_{(\rho,x)}^{*(n,k)} \cdot u_{(\rho',x')}^{(n,k)} = \delta_{\rho\rho'} \delta(x-x'). \end{array} \right.$$

(Note that in these and several other formulas the arrows denoting that a quantity is a vector have been omitted. There should be no confusion if the possibility of such an omission is always borne in mind.)

A general  $\Psi$  will be expanded in the form

$$(97) \quad \Psi_{\rho}(x,t) = \sum_n \int d\rho. \underline{a}^n(k,t) \cdot u_{(\rho,x)}^{(n,k)},$$

where the  $a$ 's will contain the dependence on the time. Thus

$$(98) \quad a(t) = a(0) \cdot e^{\frac{-i}{\hbar} Et}.$$

The minus sign in the exponent is inconvenient in that it means that a positive energy must be associated with a negative frequency, and vice versa; but it seems to be an unavoidable consequence of the Lorentz invariance since it is  $(\vec{p} \cdot \vec{x} - Et)$  that is invariant.

Now in the relations (95) and (96) the summation is over all states. But for many physical applications it is necessary to have a knowledge of the values of the sums when taken over only the positive energy states, and when

taken over only the negative energy states. We shall therefore calculate these partial sums. We want to compute the two matrices

$$(99) \quad \begin{cases} S_{\rho\sigma}^+(k) \equiv \sum_{n=1,2} c_{\rho}^{+n}(k) \cdot c_{\sigma}^{+*n}(k), \\ S_{\rho\sigma}^-(k) \equiv \sum_{n=1,2} \bar{c}_{\rho}^{-n}(k) \cdot \bar{c}_{\sigma}^{-*n}(k), \end{cases}$$

where we know, of course, from (95), that

$$(100) \quad S_{\rho\sigma}^+(k) + S_{\rho\sigma}^-(k) = \delta_{\rho\sigma}.$$

Note that the \* is placed over the second c in each sum in (99). It was Dirac who showed the importance of this order, and its significance will appear shortly.

The calculation of  $S_{\rho\sigma}^+$  and  $S_{\rho\sigma}^-$  is made very simple by the introduction of the following artifice. We merely note that, in virtue of the field equations (89), the operator

$$(101) \quad \frac{1}{2} \left[ 1 + \frac{(\vec{\alpha} \cdot \vec{k}) + mc\beta}{\sqrt{k^2 + m^2c^2}} \right]$$

will take the value (+1) when operating on a state of positive energy, and the value 0 for a negative energy state, provided we always interpret the  $\sqrt{\quad}$  as having a positive sign. Thus we have the matrix equations

$$(102) \quad \begin{cases} \frac{1}{2} \left[ 1 + \frac{(\vec{\alpha} \cdot \vec{k}) + mc\beta}{\sqrt{k^2 + m^2c^2}} \right] S^+ = S^+ \\ \frac{1}{2} \left[ 1 + \frac{(\vec{\alpha} \cdot \vec{k}) + mc\beta}{\sqrt{k^2 + m^2c^2}} \right] S^- = 0. \end{cases}$$

From these relations and (100) we at once obtain the matrix equations

$$(103) \quad \begin{cases} S^+ = \frac{1}{2} \left[ 1 + \frac{(\vec{\alpha} \cdot \vec{p}) + mc\beta}{\sqrt{p^2 + m^2 c^2}} \right], \\ S^- = \frac{1}{2} \left[ 1 - \frac{(\vec{\alpha} \cdot \vec{p}) + mc\beta}{\sqrt{p^2 + m^2 c^2}} \right]. \end{cases}$$

From (99) and (95) we at once obtain the further matrix relations

$$(104) \quad \begin{cases} S^+(k) \cdot S^+(k) = S^+(k), \\ S^-(k) \cdot S^-(k) = S^-(k), \\ S^+(k) \cdot S^-(k) = 0. \end{cases}$$

These would not be true if the \*'s in (97) had been placed over the left-hand c's.

We can rewrite the values of the matrices  $S^\pm$  in terms of the representation in which  $(\vec{x}, \rho)$  are diagonal. Introducing Dirac's notation, we write

$$(105) \quad \int d^3p \cdot \sum_{n=1,2} u_{\rho'}^{\pm(n,p)}(x'), u_{\rho''}^{\pm(n,p)}(x'') = (\rho' x' | S^\pm | \rho'' x''),$$

and then we have directly

$$(106) \quad (\rho' x' | S^\pm | \rho'' x'') = \frac{1}{2} \delta_{\rho' \rho''} \delta(x' - x'') \pm \frac{1}{2} \left\{ (\vec{\alpha} \cdot \frac{\vec{p}}{c} \frac{\partial}{\partial x'}) + \beta mc \right\}_{\rho' \rho''} g(x' - x''),$$

where  $g(x-x')$  is the same operator as that defined in (53). In our present slightly different notation it is given by

$$(107) \quad g(\vec{x}) = \frac{1}{(2\pi\hbar)^3} \int d^3p \cdot \frac{e^{\frac{i}{\hbar}(\vec{p} \cdot \vec{x})}}{\sqrt{p^2 + m^2 c^2}}.$$

We now introduce the second quantization. This seems a trivial step since it leaves everything as before except that the  $\psi$ 's are now q-numbers; the u's are still c-numbers. We assume that

$$(108) \quad \begin{cases} a_n a_m^* + a_m^* a_n = \delta_{nm}, \\ a_n a_m + a_m a_n = 0, \\ a_n^* a_m^* + a_m^* a_n^* = 0, \end{cases}$$

or, more accurately,

$$(108a) \quad a_n(k) a_m^*(k') + a_m^*(k') a_n(k) = \delta_{nm} \delta(k-k'), \text{ etc.}$$

This merely means that the quantities  $N_n = a_n^* a_n$  can have only the eigen-values 0 and 1. If we introduce the Schrödinger functional  $\Phi(N_1, N_2, \dots, N_m, \dots)$ ,

we have

$$(109) \quad \begin{cases} a_n^* \Phi(N_1, \dots, N_m, \dots) = (1 - N_m) \cdot \epsilon_n(N_1, \dots, N_m, \dots) \cdot \Phi(N_1, \dots, 1 - N_m, \dots), \\ a_n \Phi(N_1, \dots, N_m, \dots) = N_m \cdot \epsilon_n(N_1, \dots, 1 - N_m, \dots) \cdot \Phi(N_1, \dots, 1 - N_m, \dots), \end{cases}$$

where  $\epsilon_n(N_1, \dots, N_m, \dots)$  has the values  $\pm 1$ , the sign depending on an arbitrary ordering of the states and fulfilling the relation

$$(110) \quad \epsilon_n(N_1, \dots, 1 - N_m, \dots) = -\epsilon_n(N_1, \dots, N_m, \dots).$$

Since (108) is not changed if we interchange  $a$  and  $a^*$ , we may replace  $N_n$  by  $(1 - N_n)$  for some states. In particular, it is convenient to make use of this possibility for the negative energy states and to introduce <sup>the</sup> following definitions which correspond physically to the introduction of holes instead of the negative energy states. For positive energy states we write

$$(111) \quad N_n^+ = a_n^+ a_n^{+*},$$

and for negative energy states

$$(112) \quad N_n^- = a_n^- a_n^{-*}.$$

The whole theory can now be transformed in the following way. The Hamiltonian can be written in the form

$$(113) \quad \begin{aligned} \bar{H} &= \frac{1}{2} \int d\omega \sum_{\rho} \left\{ \psi_{\rho}^* \left( -\frac{\hbar}{i} \frac{\partial \psi_{\rho}}{\partial t} \right) - \left( -\frac{\hbar}{i} \right) \frac{\partial \psi_{\rho}}{\partial t} \psi_{\rho}^* \right\} \equiv \\ &\equiv \frac{1}{2} \sum_n E_n (a_n^* a_n - a_n a_n^*), \end{aligned}$$

where the order of the quantities has been altered, and  $E_n$  can take on both signs:

$$(114) \quad E_n = \pm c \sqrt{p^2 + m^2 c^2}.$$

The total charge, in a similar way, can be written as

$$(115) \quad \bar{e} = \frac{1}{2} e \int d\omega, \sum_{\rho} (\Psi_{\rho}^* \Psi_{\rho} - \Psi_{\rho} \Psi_{\rho}^*) \equiv \\ \equiv \frac{e}{2} \sum_n (a_n^* a_n - a_n a_n^*).$$

By means of (111) and (112) we now have

$$(116) \quad \bar{H} = \sum_n |E_n| \cdot \left\{ (N_n^+ - \frac{1}{2}) + (N_n^- - \frac{1}{2}) \right\}$$

and

$$(117) \quad \bar{e} = e \sum_n \left\{ (N_n^+ - \frac{1}{2}) - (N_n^- - \frac{1}{2}) \right\},$$

the plus sign coming in (116) because of the change of the sign of  $E_n$  when we go from states of positive to states of negative energy. Thus  $\bar{H}$  is always positive, but  $\bar{e}$  can take both signs. There is, unfortunately, a kind of zero-point energy in  $\bar{H}$  which is infinite, owing to the summation, and when we come to the case where external forces are present we shall have to alter  $\bar{H}$  to remove this infinity. There is no corresponding infinity in  $\bar{e}$  — but, in general, we must be careful in making cancelations like that of the  $\frac{1}{2}$ 's in  $\bar{e}$ .

We separate  $\Psi$  into the two parts corresponding to the positive and negative energy states by means of the definitions

$$(118) \quad \Psi = \Psi^+ + \Psi^-,$$

$$(119) \quad \Psi_{\rho'}^{\pm}(x') = \int d\omega'' \sum_{\rho''} (\rho' x' | S^{\pm} | \rho'' x'') \Psi_{\rho''}(x''),$$

and we obtain from (97) and (105) the commutation rules

$$(120) \quad \left\{ \begin{array}{l} [\Psi_{\rho'}^+(x'), \Psi_{\rho''}^{+*}(x'')]_{+} = (x' \rho' | S^+ | x'' \rho''), \\ [\Psi_{\rho'}^-(x'), \Psi_{\rho''}^{-*}(x'')]_{+} = (x' \rho' | S^- | x'' \rho''), \\ [\Psi^+, \Psi^-]_{+} = 0, \text{ etc.} \end{array} \right.$$

When no external field is present it is not difficult to generalize these commutation rules to the case when the times of the two  $\psi$ 's are different. The method is similar to that used in the anti-Dirac theory, the general commutation rules being completely determined by the wave equation once their form for  $t' = t''$  is known. In order to write the results we shall require some new quantities. We shall first recall the generalized g-function already considered in the scalar theory (cf. (57)). In our present notation it is given by

$$(121) \quad \begin{cases} g_+(x,t) = \frac{1}{(2\pi\hbar)^3} \int \frac{\exp i \left[ (\vec{p}, \vec{x}) + \sqrt{p^2 + m^2 c^2} \cdot ct \right]}{\sqrt{p^2 + m^2 c^2}} d^3 p, \\ g_-(x,t) = g_+(x,-t), \end{cases}$$

and is such that

$$(122) \quad \frac{\hbar}{i} \left( \frac{1}{c} \frac{\partial g_{\pm}}{\partial t} \right)_{t=0} = \pm \delta(x).$$

We shall find it convenient to introduce the operators

$$(123) \quad D = \frac{1}{2}(g_+ - g_-), \quad F = \frac{1}{2}(g_+ + g_-)$$

which satisfy the simpler initial conditions

$$(124) \quad \begin{cases} D(x,0) = 0, & \frac{\hbar}{i} \left( \frac{1}{c} \frac{\partial D}{\partial t} \right)_{t=0} = \delta(x), \\ F(x,0) = g(x), & \frac{\hbar}{i} \left( \frac{1}{c} \frac{\partial F}{\partial t} \right)_{t=0} = 0. \end{cases}$$

The operators  $g_+(x, t)$ ,  $g_-(x, t)$ ,  $D(x, t)$ ,  $F(x, t)$  all satisfy the second order relativistic wave equation for no external field, i.e. the DeBroglie wave equation.

We next introduce generalizations of the operators  $S^+$  and  $S^-$  given in (103) and which enter the restricted commutation rules (120). The generalized forms are given symbolically by the equations

$$(125) \quad \begin{cases} S^+ = -\frac{1}{2} \left( \frac{\hbar}{i} \frac{1}{c} \frac{\partial}{\partial t} - \alpha \frac{\hbar}{i} \frac{\partial}{\partial x} - mc\beta \right) \cdot g_-(x, t), \\ S^- = +\frac{1}{2} \left( \frac{\hbar}{i} \frac{1}{c} \frac{\partial}{\partial t} - \alpha \frac{\hbar}{i} \frac{\partial}{\partial x} - mc\beta \right) \cdot g_+(x, t). \end{cases}$$

Finally we shall find it useful to introduce the notations

$$(126) \quad \begin{cases} S_F \equiv S^+ S^- = - \left( \frac{\hbar}{i} \frac{1}{c} \frac{\partial}{\partial t} - \alpha \frac{\hbar}{i} \frac{\partial}{\partial x} - mc\beta \right) \cdot D(x, t), \\ S \equiv \frac{1}{2} (S^- S^+) = \frac{1}{2} \left( \frac{\hbar}{i} \frac{1}{c} \frac{\partial}{\partial t} - \alpha \frac{\hbar}{i} \frac{\partial}{\partial x} - mc\beta \right) \cdot F(x, t). \end{cases}$$

The commutation rules for the full  $\psi$ 's when  $t' \neq t''$  are

$$(127) \quad [\psi_{\rho'}(x', t'), \psi_{\rho''}^*(x'', t'')]_+ = (\rho', x', t' | S_F | \rho'', x'', t''),$$

while those for the partial  $\psi$ 's are

$$(128) \quad \begin{cases} [\psi_{\rho'}^+(x', t'), \psi_{\rho''}^{+*}(x'', t'')]_+ = (\rho', x', t' | S^+ | \rho'', x'', t''), \\ [\psi_{\rho'}^-(x', t'), \psi_{\rho''}^{-*}(x'', t'')]_+ = (\rho', x', t' | S^- | \rho'', x'', t''). \end{cases}$$

That these are the correct relations is easily verified. For they reduce to the correct form for  $t' = t''$ , and they each satisfy the first order wave equation since the application of the first order operator of the Dirac wave equation to  $S(x, t)$  gives, by (125), the same result as operating on  $g_{\pm}(x, t)$  with the second order DeBroglie operator, and we know that this result is zero.

That  $S_F$  also satisfies the first order wave equation is now obvious.

The matrix  $S_F$  corresponds physically to the case where all the states are occupied while  $S$  corresponds to the difference in the number of positive and negative states that are filled. The matter was discussed by Dirac in Camb. Phil. Soc. XXX. (1934), 150.

For most physical applications we shall not need the generalized commutation relations (127), (128), and we have written them here only for completeness and to show the relativistic invariance.

The matrix  $S$  has singularities everywhere on the light cone, the singularities being all of the type

$$(129) \quad S = C \frac{ct + (\vec{\alpha} \cdot \vec{x})}{(c^2 t^2 - r^2)^2} + \frac{v}{c^2 t^2 - r^2} + w \log(c^2 t^2 - r^2),$$

where  $C$  is a constant and  $u, v$  are regular functions of the coordinates and the spin indices.  $F$  has singularities of the  $\delta$ -type but fortunately we shall not require  $F$  in future work.

We must now consider the density matrix belonging to the present theory. We can define what we may call the formal density matrix by the equation

$$(130) \quad (\rho', x' | R | \rho'', x'') = \frac{1}{2} \{ \Psi_{\rho''}^*(x'') \Psi_{\rho'}(x') - \Psi_{\rho'}(x') \Psi_{\rho''}^*(x'') \}.$$

This is the mathematical expression for the particle density, but it contains elements on the diagonal which, on integration over all space, give rise to terms corresponding to a kind of negative zero-point energy which is infinite.

We therefore introduce what we may call the physical density matrix given by

$$(131) \quad (\rho', x' | r | \rho'', x'') = \Psi_{\rho''}^+(x'') \Psi_{\rho'}^+(x') - \bar{\Psi}_{\rho'}(x') \bar{\Psi}_{\rho''}^*(x'') + \\ + \Psi_{\rho''}^+(x'') \bar{\Psi}_{\rho'}(x') + \bar{\Psi}_{\rho''}^*(x'') \Psi_{\rho'}^+(x'),$$

which no longer has these zero-point terms.

The total charge and total energy are now given by

$$(132) \quad \bar{e} = e \int dx \sum_{\rho} (\rho, x | r | \rho, x) = e \sum_n (N_n^+ - N_n^-),$$

$$(133) \quad \bar{E} = \int dx \sum_{\rho} (\rho, x | H r | \rho, x) = \sum_n (N_n^+ + N_n^-) |E_n|,$$

where

$$(134) \quad H \equiv \alpha \frac{\hbar c}{i} \frac{\partial}{\partial x} + \beta m c^2$$

and acts, in  $(\rho, x) | \text{Hr} | \rho, x$ , only on  $(x', t')$  and not on  $(x'', t'')$ .

The extra terms in (131) vanish when integrated over all space, but they prevent the charge density from commuting with the total number of particles. These extra terms have interesting consequences for the fluctuation of the density in the vacuum. Similar terms, of the type  $\psi^+ \psi^-$ ,  $\psi^{+*} \psi^{-*}$ , arose in the scalar theory in the formula for the current vector. (Compare (60) and (131).)

The difference between the matrices  $R$  and  $r$  is of special interest. We have in fact, by simple algebra,

$$\begin{aligned} (\rho', x' | R - r | \rho'', x'') &= -\frac{1}{2} \left[ \psi_{\rho'}^+(x'), \psi_{\rho''}^{+*}(x'') \right]_+ + \frac{1}{2} \left[ \bar{\psi}_{\rho'}(x'), \bar{\psi}_{\rho''}^{*}(x'') \right]_+ = \\ &= S_{(t'=t'')} \quad (\text{by (128) and (126)}). \end{aligned}$$

Thus

$$(135) \quad (\rho', x' | r | \rho'', x'') = (\rho', x' | R | \rho'', x'') - (\rho', x' | S | \rho'', x''),$$

where, of course,  $t' = t''$  throughout.

The matrix  $S$ , for  $t' = t''$ , is given by

$$(136) \quad S = -\left( \alpha \frac{\hbar}{i} \frac{\partial}{\partial x} + \beta mc \right) g(x)$$

so that it is a universal function, independent of the  $N$ 's and thus independent of which states are filled. It is remarkable that the separation of the  $\psi$ 's into positive and negative parts does not enter the formula for  $S$ .

The matrix  $R$  is infinite for  $x \equiv x' - x'' = 0$ , but so also is  $S$  in this case, and their difference  $r$  is finite. Physically we need only the expressions for the energy-momentum and charge-current, and these involve the values of the elements of  $r$  for  $x = 0$ . The values of the non-diagonal elements, where  $x \neq 0$ , are thus a luxury. We have here a method of giving the difference of two indefinite quantities a definite meaning; we introduce the difference of two functions of an auxiliary variable  $x$  and proceed to the limit as  $x$  approaches

zero. This process is not very satisfying physically, but it is unambiguous and it eliminates the infinity that is inherent in the Dirac theory. The results of this terrible formalism are comparatively simple, and are no more complicated than are the results of the scalar theory, in which, since no such infinities exist in the first place, no artificial subtractions are necessary.

If external forces are present we can no longer give a definite meaning to the separation of  $\Psi$  into  $\Psi^+$  and  $\bar{\Psi}$ . Dirac has shown that the singularities of R will be of the same type as before. The simplest assumption about the value of S turns out to be

$$(137) \quad S = S_0 \exp. \left\{ -\frac{ei}{\hbar c} \int_{x',t'}^{x'',t''} \mathcal{G}_\nu dx^\nu \right\},$$

where  $S_0$  is the value when no external field is present (i.e.  $S_0$  is the S of (136)) and the integral is to be taken over the straight line joining  $(x', t')$  to  $(x'', t'')$ . This S is relativistically invariant and gauge invariant and is a universal function depending only on the electromagnetic field. It leads to an infinite polarizability of the vacuum.

We can try introducing a more complicated S to avoid the infinite polarizability. For example we could try taking

$$(138) \quad S = S_0 \exp. \left\{ -\frac{ei}{\hbar c} \int_{x',t'}^{x'',t''} \mathcal{G}_\nu dx^\nu \right\} + S_1.$$

For many applications we do not need the  $S_1$  at all, but it is useful for getting finite results for the polarizability of the vacuum. It is doubtful if it can really be justified since we still get a self-energy that is infinite, and if the self-energy is infinite there is no reason why we should complain of an infinite polarizability of the vacuum.

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Remarks and Addenda

Though equation (109) is correct if we define both  $N_m^+$  and  $N_m^-$  with the  $a^*$  to the left of the  $a$  it must be changed for the definition (112) of  $N^-$  here used. For in (112) we have made use of the possibility of changing  $N_m^-$  into  $1-N_m^-$  without altering the commutation rules (108). Thus we are now talking directly about positrons and not of holes, and the equation (109), which is still correct for  $a_m^+$  and  $a_m^{+*}$  must be modified for  $a_m^-$  and  $a_m^{-*}$ . We require, in fact, to replace the two equations (109) by the four equations

$$(109') \left\{ \begin{array}{l} a_m^{+*} \Phi(N_1^+, \dots, N_m^+, \dots) = (1-N_m^+) \cdot \epsilon_m(N_1^+, \dots, N_m^+, \dots) \cdot \Phi(N_1^+, \dots, 1-N_m^+, \dots), \\ a_m^+ \Phi(N_1^+, \dots, N_m^+, \dots) = N_m^+ \cdot \epsilon_m(N_1^+, \dots, 1-N_m^+, \dots) \cdot \Phi(N_1^+, \dots, 1-N_m^+, \dots), \\ a_m^{-*} \Phi(N_1^+, \dots, N_m^-, \dots) = N_m^- \cdot \epsilon_m(N_1^+, \dots, 1-N_m^-, \dots) \cdot \Phi(N_1^+, \dots, 1-N_m^-, \dots), \\ a_m^- \Phi(N_1^+, \dots, N_m^-, \dots) = (1-N_m^-) \cdot \epsilon_m(N_1^+, \dots, N_m^-, \dots) \cdot \Phi(N_1^+, \dots, 1-N_m^-, \dots). \end{array} \right.$$

It was not explained how the formula (131) for the physical density matrix  $\tau$  was obtained.

The density matrix is identically the same as

$$\Psi_{\rho''}^*(x'') \Psi_{\rho'}(x') = (\rho'; x' | S^- | \rho''; x'').$$

This expression was derived by Dirac in a previous formulation of the theory in which all the quantities were c-numbers. The matrix  $S$  corresponded, in that formalism, to the contribution to the density when all negative states were occupied and was subtracted from the formal density matrix in order to remove the infinite electromagnetic field due to such a distribution. The matrix  $\tau$  of the present theory is just the q-number analogue of the physical density matrix of

the c-number theory.

Finally, it is to be remarked that in the theory of holes the transformation

$$\psi \rightleftharpoons \psi^*, \quad \rho', x' \rightleftharpoons \rho'', x''$$

changes the sign of the charge  $e$  but does not affect the theory in any other way. Thus the theory of holes is symmetric between positive and negative charge.

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### The Scattering of Light by Light

(Reported by M.H.L.Pryce)

We shall discuss briefly the work of Euler and Kockel on the scattering of light by light according to the theory of holes. A short account of this work has appeared in Die Naturwissenschaften, 23 (1935), 246, but the lengthy computations have not yet been published.

Breit and Wheeler have discussed the scattering of high energy photons by high energy photons. This scattering takes place indirectly; the incident photons create an electron-positron pair which, on recombination, gives another pair of photons.

Such a process cannot occur with ordinary light since the energy is far too small for actual pair formation. But there is still the possibility of virtual pair formation and this gives rise to a definite probability of photon-photon interaction. This process involves four stages and the final energy and

momentum must be the same as the initial energy and momentum, but only the momentum can hope to be conserved at each separate stage. It turns out that momentum is actually conserved at each stage.

Euler and Kockel, using the perturbation theory, were able to obtain a value for the cross-section of a photon for such photon-photon scattering.

If the energy is expanded in terms of the parameter  $e$ , the zeroth order term corresponds to the case of free particles and free radiation, while the perturbation energy consists of the four terms

$$(139) \quad \left\{ \begin{array}{l} V^{(1)} = e \int \psi^* (\alpha, A) \psi d\tau, \\ V^{(2)} = e^2 \int [\text{Quantity of the second degree in } A] d\tau, \\ V^{(3)} = e^3 \int [\text{Quantity of the third degree in } A] d\tau, \\ V^{(4)} = \frac{-e^4}{12\pi^2 \hbar^3 c^3} \lim_{r \rightarrow 0} \int \frac{(A(\xi) \cdot r)^4}{|r|^4} d\tau, \end{array} \right.$$

where  $A(\xi)$  is the four-potential of the radiation field and  $r$  is the quantity  $(\xi - \xi')$ . The last three terms arise from the subtraction process that eliminates singularities in the theory of holes, and if  $S$  were replaced by Heisenberg's  $S + S$ , the final result of the present work would be unaffected.

The matrix elements of  $V^{(1)}$  are

$$(140) \quad V_{ik}^{(1)} \left\{ \begin{array}{l} = e \hbar \sqrt{\frac{2\pi c}{|\mathbf{g}|v}} (\mathbf{g}^{\lambda\sigma} | (\alpha, \hbar \mathbf{g}) | \mathbf{g}'^{\lambda'\sigma'}), \\ = 0, \end{array} \right.$$

the upper value applying if momentum is conserved while the zero is valid if conservation of momentum does not take place. The suffix  $i$  denotes a state with no radiation, the electronic states being described by  $\lambda, \sigma$ , while  $k$  refers to a state with one photon  $\mathbf{g}$  of polarization  $\hbar \mathbf{g}$  and with the electron states given by  $\lambda', \sigma'$ . All transitions due to the first order interaction energy  $V^{(1)}$  will thus be such that the momentum is conserved.

The scattering of a photon by a photon involves four stages since it requires the absorption of two photons and the emission of two others.  $V^{(1)}$  is linear in  $A$  and gives rise to only one absorption or emission of a photon. It must therefore enter the transition probability we seek in the fourth order. It turns out that the transition probability can contain only terms of the fourth order. For  $V^{(2)}$  corresponds to cases in which no electron jump takes place and thus, from considerations of energy conservation, cannot give photon-photon scattering in the first order. Also  $V^{(3)}$  combines three photons with conservation of momentum and thus cannot apply to the present process in the first order. Similarly  $V^{(2)} V^{(1)}$  gives no contribution. Thus the transition probability becomes

$$(141) \quad H_{i,n}^{(4)} = \sum_{k,l,m} \frac{V_{i,k}^{(1)} V_{k,l}^{(1)} V_{l,m}^{(1)} V_{m,n}^{(1)}}{(E_i - E_k)(E_k - E_l)(E_l - E_m)} + \dots + V_{i,n}^{(4)}$$

where the dots represent the interaction terms involving  $V^{(2)} V^{(2)}$ ,  $V^{(2)} V^{(1)} V^{(1)}$ , and  $V^{(3)} V^{(1)}$ . From general considerations Euler and Kockel were able to show that these terms cannot contribute to the transition probabilities. We are thus left with only the terms explicitly indicated in (141)

The following table shows the various ways in which the photon-photon scattering can take place through the first term of (141). Since momentum is conserved at each stage, we may permute the order of the columns\* in any way so that there are twenty-four times as many possibilities as are actually shown in the table.

\* Relative to their labels on the top line

Absorption of $\sigma^{(1)}$	Absorption of $\sigma^{(2)}$	Emission of $-\sigma^{(3)}$	Emission of $-\sigma^{(4)}$
Pair production	Electron jump	Electron jump	Pair annihilation
"	"	Positron jump	"
"	Positron jump	Electron jump	"
"	"	Positron jump	"
"	Pair production	Electron 1 and positron 2 combine	Electron 2 and positron 1 combine
"	"	Electron 2 and positron 1 combine	Electron 1 and positron 2 combine

It is shown by Euler and Kockel that  $H_{i\hbar}^{(4)}$  can be regarded as the matrix elements of the space integral of a function that is of the fourth degree in the potentials, and on expanding the first term in  $H_{i\hbar}^{(4)}$  according to frequency this integral can be expressed as a sum of integrals of the type

$\int (AAAA) dv$ ,  $\int \left( \frac{\partial A}{\partial x} AAA \right) dv$ ,  $\int \left( \frac{\partial A}{\partial x} \frac{\partial A}{\partial x} AA \right) dv$ ,  $\int \left( \frac{\partial A}{\partial x} \frac{\partial A}{\partial x} \frac{\partial A}{\partial x} A \right) dv$ ,  
 and  $\int \left( \frac{\partial A}{\partial x} \frac{\partial A}{\partial x} \frac{\partial A}{\partial x} \frac{\partial A}{\partial x} \right) dv$ , where these expressions are meant to give merely an indication of the type of fourth order homogeneity that is involved. Now

since we must have gauge invariance, it is evident that only the last integral can ultimately survive. It can be shown directly that the zeroth order term

$(\sigma^{(1)} \sigma^{(2)} | \int (AAAA) dv | \sigma^{(3)} \sigma^{(4)})$  cancels the Heisenberg term  $V^{(4)}$ . That the other terms vanish is not so easily proved explicitly. It has been proved for the first order term, but only for special cases for the second and third order terms. If the zero order term did not vanish we would find a finite cross-section.

tion of scattering for light of any frequency. The introduction of Heisenberg's more complicated subtraction process involving  $S_1$ , makes no difference to the terms we are considering here.

Instead of working out the fourth order term Euler and Kockel restricted the possibilities by considerations of relativistic invariance. They proved that since the first order perturbation is relativistically invariant, the  $H_{if}^{(4)}$  must also be. Thus the general form of the integral of which  $H_{if}^{(4)}$  are the matrix elements must be

$$(142) \quad -\frac{\hbar c}{e^2} \frac{1}{E_0} \int [\alpha (B^2 - D^2)^2 + \beta (B \cdot D)^2] dv, \quad \left( E_0 = \left( \frac{mc^2}{e^3} \right)^3 \right).$$

A term involving  $(B \cdot D) (B^2 - D^2)$  would change sign under an improper Lorentz transformation and is therefore inadmissible.

The values of the pure numbers  $\alpha$  and  $\beta$  were found by considering two special cases. In the first case two equal photons with the same polarizations were converted into each other (i.e. they went right through each other!) while in the second case the polarizations were changed by a right angle. The comparison of the results given by the two forms for the transition probability gave the values

$$(143) \quad \alpha = \frac{1}{360\pi^2}, \quad \beta = \frac{7}{360\pi^2}.$$

The differential cross-section  $dQ$  for the solid angle  $d\Omega$  in the case when equal photons traveling towards each other are scattered through an angle

$\varphi$  comes out as

$$(144) \quad dQ = \frac{2 d\Omega}{3^4 \cdot 5^2} (3 + \cos^2 \varphi)^2 \left( \frac{e^2}{mc^2} \right)^4 \left( \frac{\hbar}{mc} \right)^4 \lambda^{-6}$$

where  $\lambda$  is the wave-length. It thus varies inversely as the sixth power of the wave-length.

For visible light the cross-section is of the order  $10^{-36}$  cm., for X-rays it is  $10^{-24}$  cm., and for  $\gamma$ -rays,  $10^{-15}$  cm. However, the approximation will break down for high energy photons.

Let us imagine two 1000 Watt beams of visible light concentrated into the thinnest pencil allowed by the diffraction. If these beams meet head on one photon will be scattered, on the average, every  $10^{16}$  years!

Euler and Kockel point out that the theory is beyond the range of experimental confirmation!

A close analogy exists between the present work and the theory of Van der Waals forces between two atoms since these forces arise from virtual transitions of electrons in the atoms from low energy states to higher and then back. The ordinary Coulomb interaction is not sufficient to let the electrons stay in the excited states, just as the photons of visible light will not permit actual pair formation, but the virtual electron transitions caused by the Coulomb interaction lead to second order interaction terms which are just the Van der Waals forces.

Note:

We add some details of the reasons for rejecting the terms of (141) involving  $V^{(2)}V^{(2)}$ ,  $V^{(2)}V^{(1)}V^{(1)}$ , and  $V^{(3)}V^{(1)}$ .

$V^{(2)}$  is independent of the material field and its matrix elements will therefore vanish except for transitions in which two photons interact and conserve their total momentum. It is easy to see that, in the process we are interested in, this momentum conservation places restrictions on the directions of the quanta. For if we want to make two  $V^{(2)}$ 's convert two light quanta into two others we must consider them as absorbed in one process and emitted in

the other, or else we must consider each process as changing one photon into another. The latter is trivial since the conservation of momentum at once shows that nothing can happen to the photon. The other possibility is restricted, by the conservation of momentum, to equal and opposite photons in each case, and we may ignore such processes since they are not relativistically invariant.

Similar considerations will dispose of the possibility of having a term involving  $V^{(2)}V^{(1)}V^{(1)}$ .

Finally, since  $V^{(3)}$  combines three light quanta with total momentum equal to zero and  $V^{(1)}$  involves an electron jump, a term containing  $V^{(3)}V^{(1)}$  will always imply a resultant electron jump so that such a term cannot be concerned with the present scattering process.

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On Fluctuations of Charge Density in the Vacuum

(Reported by G. Breit)

We discuss a paper by Heisenberg appearing in the Sächsischen Akademie der Wissenschaften, Leipzig, 86 (1934), 317.

Heisenberg considers the problem of taking some portion of space of volume  $v$ , finding out how much charge is in this volume, and seeing how this charge fluctuates. We consider here only the part of Heisenberg's work referring to fluctuations in a vacuum. This work is done without the use of the subtraction process and it can be understood from the original Dirac point of view.

It is sufficient to assume that we have a wave function  $\Psi$  of the type

$$(143) \quad \Psi = \sum_n a_n u_n(x, t, \sigma),$$

where the  $u_n$  represent plane waves and the  $a_n$  are quantized, satisfying the usual commutation relations

$$(144) \quad a_n a_m^* + a_m^* a_n = \delta_{nm}, \text{ etc.}$$

The index  $n$  is the  $(n, p)$  of previous lectures, and  $\sigma$  is the spin variable.

This  $\psi$  is really an operator and it is only a convention to continue to talk of it as a wave function.

The total charge in the volume is given by the operator

$$(145) \quad e = \epsilon \int_V \psi^* \psi d\tau,$$

where  $\epsilon$  is the electronic charge (a c-number!). We may write this in the form

$$(146) \quad e = \epsilon \int_V \sum_{n,m} a_n^* a_m u_n^* u_m d\tau.$$

Let us now consider the expectation value  $\bar{e}$  of  $e$ . It is given by

$$\bar{e} = \epsilon \int_V \overline{\sum_{n,m} a_n^* a_m} \cdot u_n^* u_m d\tau,$$

but since the "vacuum" is the configuration in which every negative state is filled while every positive state is empty on the average, we must have

$$(147) \quad \overline{a_n^* a_m} = \delta_{nm} \{0\},$$

the zero going with the positive energy states.

Thus

$$(148) \quad \begin{aligned} \bar{e} &= \epsilon \sum_{E_n < 0} N_n \int_V u_n^* u_n d\tau = \\ &= \epsilon \sum_{E_n < 0} N_n \frac{V}{V}, \end{aligned}$$

where  $V$  is the volume of the "box" in terms of which the second quantization is defined and  $N_n$  (the number of electrons in state  $n$ ) is 1 for all terms in the sum.  $\bar{e}$  is actually infinite since we are ignoring the subtraction process.

This infinity does not constitute a fundamental difficulty in any case, and particularly in the calculation of the fluctuation. The operator which represents

the fluctuation of charge from its mean value is

$$e - \bar{e} = \varepsilon \left[ \sum_{E_n < 0} a_n^* a_m \int_V u_n^*(x, t, \sigma) u_m(x, t, \sigma) d\sigma - \sum_{E_n < 0} N_n \right].$$

One may replace here the number  $\sum_{E_n < 0} N_n$  by the operator  $\sum a_n^* a_n$  because

we are only interested in the expectation value which is expressible in terms of

the functional  $\Phi$  as  $\overline{(e - \bar{e})^2} = ((\Phi^*(e - \bar{e}))((e - \bar{e}) \Phi))$  and because

$$\sum a_n^* a_n \Phi = \sum_{E_n < 0} N_n \Phi. \quad \text{We may use therefore}$$

$$(149) \quad e - \bar{e} = \varepsilon \sum_{n \neq m} a_n^* a_m \int_V \sum_{\sigma} u_n^*(x, t, \sigma) u_m(x, t, \sigma) d\sigma$$

and obtain

$$(150) \quad \overline{(e - \bar{e})^2} = \varepsilon^2 \sum_{\substack{n \neq m \\ n' \neq m'}} a_n^* a_m a_{n'}^* a_{m'} \iint_{V, V'} \sum_{\sigma, \sigma'} u_n^* u_m u_{n'}^* u_{m'} d\sigma d\sigma',$$

where  $u'$  denotes  $u(x', t', \sigma')$  and  $\sigma$  denotes the spin coordinate.

In evaluating the expectation values occurring in (150) we have to take into account the terms with  $n = m'$  and  $m = n'$ . We therefore bring the  $a_m$  two places to the right. The first interchange merely alters the sign since  $m' \neq n'$ , while the second interchange brings the sign back to its original value since two unstarred  $a$ 's always anti-commute. Thus we have

$$\sum_{\substack{n \neq m \\ n' \neq m'}} a_n^* a_m a_{n'}^* a_{m'} = \sum_{\substack{n \neq m \\ n' \neq m'}} (a_n^* a_{m'}) (a_m a_{n'}^*)$$

and if we now take the expectation value the terms on the right for  $n \neq m'$  and for  $m \neq n'$  will vanish, and we are thus left with

$$\sum_{n \neq m} N_n (1 - N_m)$$

the summation being essentially over all occupied states  $n$  and all unoccupied states  $m$ .

The suffixes of the u's in (150) must be changed in accordance with the above manipulation and it is easily seen that we have the final result that

$$(151) \quad \overline{(e - \bar{e})^2} = \sum_{n \neq m} \varepsilon^2 N_n (1 - N_m) J_{nm},$$

where  $J_{nm}$  are the exchange integrals

$$(152) \quad J_{nm} = \iint_{v, v'} \sum_{\sigma, \sigma'} u_n^* u_m u_m^* u_n' dv dv'$$

[ It is interesting to see how this result comes out of the more elementary treatment using the configuration space.

In this method the charge density operator is given by

$$(153) \quad \rho(x) = \varepsilon \{ \delta(x-x_1) + \delta(x-x_2) + \dots + \delta(x-x_n) \}$$

so that the total charge is

$$(154) \quad e = \varepsilon \int \{ \delta(x-x_1) + \delta(x-x_2) + \dots + \delta(x-x_n) \} dx$$

To find  $\bar{e}$  we must take the sum of the diagonal elements of the matrix  $e$  for a particular state and for this purpose we must introduce the antisymmetric wave function

$$(155) \quad \Psi = \frac{1}{\sqrt{n!}} \begin{vmatrix} u_1(1), & u_2(1), & \dots, & u_n(1) \\ u_1(2), & u_2(2), & \dots, & u_n(2) \\ \vdots & \vdots & \vdots & \vdots \\ u_1(n), & u_2(n), & \dots, & u_n(n) \end{vmatrix}$$

As is well known

$$(156) \quad \bar{e} = (\Psi | e | \Psi) = \varepsilon \int \sum_i u_i^*(x) u_i(x) dx,$$

where for simplicity we omit writing explicitly the spin coordinate. The operator for  $e^2$  is

$$(157) \quad e^2 = \varepsilon^2 \left( \sum_{i=j} + \sum_{i \neq j} \right) \iint \delta(x-x_i) \delta(x-x_j) dx dx' = \\ = \varepsilon^2 \sum_i A_i + \varepsilon^2 \sum_{i \neq j} B_{ij}.$$

We let  $e^2 = (\Psi | e^2 | \Psi)$  and we have

$$(158) \quad \overline{e^2} = \varepsilon^2 \sum_i \int u_i^*(x_i) A_i(x_i) dx_i + \\ + \varepsilon^2 \sum_{i \neq j} \iint u_i^*(x_i) u_j^*(x_j) B_{ij} \{u_i(x_i) u_j(x_j) - u_i(x_j) u_j(x_i)\} dx_i dx_j.$$

Here  $\int u_i^*(x_i) A_i u_i(x_i) dx_i = \int u_i^*(x) u_i(x) dx$ ,

while the term under  $\sum_{i \neq j}$  is

$$(159) \quad B_{ij,ij} - B_{ij,ji} = \iint dx dx' u_i^*(x) u_j^*(x') [u_i(x) u_j(x') - u_i(x') u_j(x)].$$

In order to transform the above result into the form of (151) we introduce an index  $k$  which runs over all unoccupied levels and an index  $s$  which runs over all levels. We have

$$\sum_{i \neq j} B_{ij,ij} = \sum_{i,s} B_{is,si} - \sum_i B_{ii,ii} - \sum_{i,k} B_{ik,ki} = \\ = \sum_i \int u_i^*(x) u_i(x) dx - \sum_i \left\{ \int u_i^*(x) u_i(x) dx \right\}^2 - \sum_{i,k} B_{ik,ki},$$

and using this with (158), (159)

$$(160) \quad \overline{(e-\bar{e})^2} = \overline{e^2} - \bar{e}^2 = \varepsilon^2 \sum_i \int u_i^*(x) u_i(x) dx + \varepsilon^2 \sum_{i \neq j} B_{ij,ij} - \\ - \varepsilon^2 \left[ \sum_i \int u_i^*(x) u_i(x) dx - \sum_i \left\{ \int u_i^*(x) u_i(x) dx \right\}^2 + \sum_{i,k} B_{ik,ki} \right] - \\ - \varepsilon^2 \left( \sum_i \int u_i^*(x) u_i(x) dx \right)^2 = \varepsilon^2 \sum_{i,k} B_{ik,ki},$$

which is the same as obtained in (151). It will be noted that an essential part of (157) is an operator which represents the simultaneous probability of two electrons being found in the volume  $v$ . ]

To return to the main argument, we must now calculate the values of the integrals  $J_{nm}$ . The averaging can be extended over a four-dimensional volume and the  $J_{nm}$  are then generalized into

$$(161) \quad J_{nm} = \iint \sum_{\sigma, \sigma'} u_n^*(x, t, \sigma) u_m(x, t, \sigma) u_m^*(x', t', \sigma') u_n(x', t', \sigma') d\vec{x} d\vec{x}' dt dt'$$

And they give infinite results!

In order to obtain finite results it is necessary to generalize these integrals by assigning different importance to charges in different parts of the volume and also for different times. We therefore introduce weight functions  $g(\vec{x})$ ,  $f(t)$ , normalized by

$$(162) \quad \int g(\vec{x}) d\vec{x} = 1, \quad \int f(t) dt = 1.$$

The integrals  $J_{nm}$  now become

$$(163) \quad \begin{cases} J_{nm} = \iint \sum_{\sigma, \sigma'} u_n^* u_m u_m^* u_n \cdot K d\vec{x} d\vec{x}' dt dt', \\ K \equiv g(\vec{x}) g(\vec{x}') f(t) f(t'). \end{cases}$$

Now from (151) we are actually interested only in the summation of  $J_{nm}$  over all occupied states  $n$  and all unoccupied states  $m$ . That is, we want to find the value of

$$\sum_{\substack{E_n > 0 \\ E_m < 0}} J_{nm}.$$

When we sum  $u_n^* u_n$  over all negative energy states, we obtain, by (103), the matrix

$$\frac{1}{2} \left( 1 - \frac{\alpha_i p_i + \beta mc}{p_0} \right)_{\sigma\sigma'}$$

Similarly, on summing  $u_m u_m^*$  over all positive energy states we get

$$\frac{1}{2} \left( 1 + \frac{\alpha_i p_i + \beta mc}{p_0} \right)_{\sigma\sigma'}$$

Further, it is the matrix product of these two matrices that enters the integrand; in fact, only the spur of this resulting matrix. And since the spur of the product of two matrices is independent of the order of the matrices we find that

$$(164) \sum_{\substack{E_m > 0 \\ E_n < 0}} J_{nm} = \int \frac{dt dt'}{d\vec{r} d\vec{r}'} K \frac{p_0 p'_0 - \vec{r} \cdot \vec{r}' - m^2 c^2}{p_0 p'_0 h^6} e^{\frac{i}{h} [(p_i - p'_i)(x'_i - x_i) - c(p_0 + p'_0)(t' - t)]}$$

where the primes refer to negative energy terms.

If we now introduce the quantities  $f(p_0)$ ,  $g(\vec{p})$  defined by

$$(165) \begin{cases} f(p_0) = \int dt. f(t) e^{\frac{i}{h} p_0 c t} \\ g(\vec{p}) = \int d\vec{x}. g(\vec{x}) e^{\frac{i}{h} \vec{r} \cdot \vec{x}} \end{cases}$$

which are the functions of which the weight functions  $f(t)$ ,  $g(x)$  of (162) are the Fourier components, the integral in (164) becomes

$$\int \frac{d\vec{r} d\vec{r}'}{h^6} \frac{p_0 p'_0 - \vec{r} \cdot \vec{r}' - m^2 c^2}{p_0 p'_0} |g(\vec{r} - \vec{r}')|^2 |f(p_0 + p'_0)|^2,$$

and if we now write

$$(166) \quad \vec{r} - \vec{r}' = \vec{R}, \quad \frac{1}{2}(\vec{r} + \vec{r}') = \vec{P},$$

we have

$$(167) \sum_{\substack{E_m > 0 \\ E_n < 0}} J_{nm} = \int \frac{d\vec{R} d\vec{P}}{h^6} |f(p_0 + p'_0)|^2 |g(\vec{R})|^2 \frac{\sqrt{A - p^2 + \frac{R^2}{4} - m^2 c^2}}{\sqrt{A}},$$

where

$$(168) \quad A \sim (p_0 p'_0)^2 = \left(p^2 + \frac{R^2}{4} + m^2 c^2\right)^2 - (\vec{R} \cdot \vec{P})^2.$$

If we take the weight factors to have infinitely sharp boundaries we get an infinite result. For in this case we have

$$(169) \quad \begin{cases} |g(\vec{k})|^2 = \frac{\hbar^4}{k_x^2 k_y^2 k_z^2} \cdot 64 \sin^2 \frac{k_x l_1}{2\hbar} \cdot \sin^2 \frac{k_x l_2}{2\hbar} \cdot \sin^2 \frac{k_x l_3}{2\hbar}, \\ |f(p_0 + p_0')|^2 = \frac{4 \sin^2 \left\{ (p_0 + p_0') c T / 2\hbar \right\}}{\left\{ (p_0 + p_0') c T / 2\hbar \right\}^2}; \end{cases}$$

for large  $\vec{P}$ ,  $p_0 + p_0' \rightarrow 2P$  so that the integration over  $\vec{P}$  involves approximately

$$\int d\vec{P} \frac{\sin^2(PcT/\hbar)}{(PcT/\hbar)^2} \cdot \frac{\frac{\hbar^2}{2} - \frac{(\vec{k} \cdot \vec{P}/P)^2}{2}}{P^2} \rightarrow \hbar^2 \frac{\hbar}{cT},$$

and the integration over  $k$  now becomes of the type

$$\int \frac{\hbar^2}{k_x^2 k_y^2 k_z^2} \sin^2 \frac{k_x l_1}{2\hbar} \cdot \sin^2 \frac{k_x l_2}{2\hbar} \cdot \sin^2 \frac{k_x l_3}{2\hbar} dk_x dk_y dk_z$$

which is infinite. [The first two integrations are convergent but the third diverges.]

We shall obtain finite results if we let the spatial weight function have gentle ends, even if we take the boundaries of the temporal weight function to be abrupt. We take, in fact,

$$(170) \quad \begin{cases} f(t) \sim \text{[rectangular pulse of width } T \text{]}, \\ g(\vec{x}) \sim \text{[smooth profile with Gaussian tails]}, \end{cases}$$

where the diagram for  $g(\vec{x})$  is a profile of the actual three-dimensional function and the ends are the Gauss error curves  $g(\vec{k}) \sim e^{-b^2 k^2 / \hbar^2}$ . It is possible to evaluate the resulting integrals for certain limiting cases:

If  $T \ll \frac{b}{c}$  and  $T \ll \hbar/mc^2$ , it is found that

$$(171) \quad \sum J_{nm} \sim \frac{1}{\hbar^2 c T} \int \frac{d\vec{k}}{\hbar^3} \hbar^2 |g(\vec{k})|^2 \sim \frac{v^{2/3}}{bcT},$$

while if  $T \ll \frac{b}{c}$  and  $T \gg \hbar/mc^2$ , it turns out that

$$\begin{aligned}
 (172) \quad \sum J_{nm} &\sim \int \frac{d\vec{k}}{h^6} |g|^2 \frac{\sin^2\left(\frac{mc^2 T}{\hbar}\right)}{\left(\frac{mc^2 T}{\hbar}\right)^2} \cdot \frac{\hbar^2 P^2 dP}{P^2 + m^2 c^2} \sim \\
 &\sim \frac{1}{\hbar^4 (mc)(cT)^2} \int \hbar^2 g(\vec{k}) d\vec{k} \sim \hbar v^{2/3} / mc \mathcal{O}(cT)^2.
 \end{aligned}$$

The occurrence of the factor  $v^{2/3}$  in both (171) and (172) is significant since it shows that the value of the charge depends not so much on a volume as on a surface, and this is related to the fact that the best way to measure the charge in a given volume is to measure the total normal flux over the closed surface bounding this volume.

It can be shown directly from the form of (171) that the charge fluctuation will depend on the gradient of  $g(\vec{x})$  through a factor of the form

$$\int |\text{grad } g(\vec{x})|^2 d\vec{x}.$$

It is thus apparent that an infinitely sharp boundary must lead to infinite fluctuations.

There is a physical reason for the inclusion of the spatial weight function. For, the nearer a given charge is to the boundary, the more difficult it becomes to decide which side it is; and the greater the accuracy with which this is to be determined, the greater will be the chance of producing pairs which give rise to fluctuations. For sharp boundaries the fluctuation is infinite.

Bohr and Rosenfeld have shown that the mean value of the electromagnetic field components over a finite time interval is always finite. To find  $e$  we have to calculate  $\int E_n dS$  and the introduction of the weight factor  $f(t)$  will perhaps correspond to taking the integral over a three-dimensional sub-space of space-time:

$$\bar{e} = \frac{1}{T} \int_t^{t+T} dt \int_{r_1}^{r_2} r^2 dr d\theta d\phi \cdot E_r.$$

The question whether we can really make such measurements in small regions is not a simple one since in a formalism containing no limitations concerning the use of heavy particles in small volumes we can make measurements in volumes whose diameter is small compared with the Compton wave-length of the electron. We do not know anything about the possible limitations of measurements with heavy particles, but we cannot draw any conclusions about this from the present theory.

The result of Heisenberg's calculation can be understood qualitatively from the following point of view. If all  $\overline{e}^n$  were  $= \overline{e}^n$  we would have a condition in which the charge in  $v$  would be  $\overline{e}$  with certainty. Such a condition could be obtained if it were possible to transform the wave functions so as to have a division into two classes of functions: (a) Those wholly confined to  $v$  and (b) those wholly outside  $v$ . The states available in Dirac's negative energy distribution do not form a complete set of functions and it is impossible to form out of them linear combinations which will fall into the above classes. It is particularly difficult to do so if the boundary is sharp because of the well known difficulty in forming localized wave packets with Dirac's functions of one sign of energy. Heisenberg's calculation can be interpreted to mean that such a localization is quite impossible if the boundary is sharp and that it can be approximately satisfied if the boundary is diffuse. The occurrence of  $(\hbar/mc\ell)$  in the result is also readily understandable from this point of view as well as dimensionally.

According to (149), (151) the value of  $\overline{(e - \overline{e})^2}$  is obtained because when  $(e - \overline{e})$  operates on the functional  $\Phi$ , new states are created in which a particle has been transferred from the state  $n$  to the state  $m$ . In this sense the

value of  $\overline{(e-\bar{e})^2}$  owes its origin to pair production. This fact is also in agreement with the qualitative explanation of localization of wavepackets because the occurrence of pairs in  $(e-\bar{e})\bar{\Phi}$  owes its origin to the fact that an eigenfunction of  $e-\bar{e}$  is a linear combination of functionals referring to populations of negative as well as positive energy states.

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The Polarizability of the Vacuum

(Reported by W. Pauli)

We discuss the polarizability of the vacuum by an external electromagnetic field. This is an application of the theory of holes which is very characteristic of the theory, and though we do not know if it is physically true it leads to consequences that can be compared with experiment. It is also interesting historically as the starting point of the whole subtraction formalism.

If we introduce an external field  $(\varphi_0, \vec{\varphi})$ , electrons in the negative energy states will change their wave functions, and this can give rise to the generation of an electric current and charge density.

Let  $u_n(x, \rho)$  be the eigen-functions for the force-free case, and let  $v_n(x, \rho)$  be the eigen-functions when an external field is present. (To be accurate we should not refer to the  $v$ 's as eigen-functions since if the external potentials depend explicitly on the time, the  $v$ 's, though still a complete set of orthogonal functions, are no longer eigen-functions.)

To avoid infinities we use the old trick of starting with expressions for the elements of the constituents of the density matrix for which  $x' \neq x''$ , and later proceed to the limit as  $x'$  and  $x''$  approach each other. But, through-

out, we shall take  $t' = t''$ .

As Dirac discovered, it is more convenient to deal with the difference between all positive and all negative energy states than to deal with only the negative energy states. The matrix corresponding to the case when all states are occupied is always the  $\delta$ -function. Actually, when an external field is present the states are in the most general case no longer separated into positive and negative states, but for many fields, including the important case of the H-atom, we can still speak of the difference between the positive and negative energy states.

We consider, then, the matrices  $T$ ,  $S_0$ , defined by

$$(173) \quad (x', \rho' | T | x'', \rho'') = \sum_{E_n < 0} v_n^*(x'', \rho'') v_n(x', \rho') - \sum_{E_n > 0} v_n^*(x'', \rho'') v_n(x', \rho'),$$

$$(174) \quad (x', \rho' | S_0 | x'', \rho'') = \sum_{E_n < 0} u_n^*(x'', \rho'') u_n(x', \rho') - \sum_{E_n > 0} u_n^*(x'', \rho'') u_n(x', \rho').$$

The matrix  $T$  will later be seen to be closely connected with the matrix  $R$  of the  $q$ -number theory. We are here concerned, of course, with the  $c$ -number theory.

For the physical density matrix  $r$  we must have

$$(x', \rho' | r | x'', \rho'') = (x', \rho' | T - S | x'', \rho''),$$

but the question is, what are we to take for  $S$ ?

We might try taking  $S = S_0$  but, as Dirac showed in his Solvay report, this leads to an infinite electric density even for weak fields. We might also try the other extreme  $S = T$ , but this would give zero electric density by definition and would thus spoil the possibility of pair production,

It is reasonable to postulate that  $S$  shall be independent of the previous history of the electromagnetic field, and that if this field should vanish on the straight line between  $x'$ ,  $t'$  and  $x''$ ,  $t''$ , then we shall have  $S = S_0$ . The

simplest assumption for  $S$  would be that considered some time back, namely

$$(137) \quad S = S_0 \exp \left\{ \frac{-ei}{\hbar c} \int_{x', t'}^{x'', t''} \varphi_V dx^V \right\},$$

but we shall see that to get finite results for the electric density and current we shall have to subtract more terms than this.

Let us consider the quantity  $(T-S_0)$ . We can compute its matrix elements approximately if we use the perturbation theory. We shall retain only first order terms, neglecting terms involving  $e^2$ . In his paper on the theory of holes Heisenberg calculated  $(T-S_0)$  for the special case in which the external field has only a scalar potential. If we assume this at first, for simplicity, to be of the form

$$(175) \quad \varphi_0(x) = a e^{i\vec{k} \cdot \vec{x}} + a^* e^{-i\vec{k} \cdot \vec{x}},$$

and introduce the notation

$$(176) \quad \left\{ \begin{array}{l} \vec{\xi} = \frac{\vec{x}' + \vec{x}''}{2}, \quad \vec{x} = \vec{x}'' - \vec{x}'; \\ \varepsilon = \sqrt{\frac{m^2 c^2}{\hbar^2} + \left(\varrho + \frac{\hbar}{2}\right)^2}, \quad \varepsilon' = \sqrt{\frac{m^2 c^2}{\hbar^2} + \left(\varrho - \frac{\hbar}{2}\right)^2}, \end{array} \right.$$

his result may be written

$$(177) \quad (x', \rho' | T - S_0 | x'', \rho'') = -\frac{1}{8} e a e^{i\vec{k} \cdot \vec{x}} \int \frac{d^3 \varrho}{(2\pi)^3} \Omega,$$

where the integrand  $\Omega$  is given by

$$(178) \quad \Omega = \frac{e^{i\vec{v} \cdot \vec{x}}}{\varepsilon + \varepsilon'} \left[ \left\{ 1 + \frac{\vec{x} \cdot \left( \vec{v} + \frac{\vec{h}}{2} \right) + \frac{\beta m c}{2}}{\varepsilon} \right\} \left\{ 1 - \frac{\vec{x} \cdot \left( \vec{v} - \frac{\vec{h}}{2} \right) + \frac{\beta m c}{\hbar}}{\varepsilon'} \right\} + \right. \\ \left. + \left\{ 1 - \frac{\vec{x} \cdot \left( \vec{v} + \frac{\vec{h}}{2} \right) + \frac{\beta m c}{\hbar}}{\varepsilon} \right\} \left\{ 1 + \frac{\vec{x} \cdot \left( \vec{v} - \frac{\vec{h}}{2} \right) + \frac{\beta m c}{\hbar}}{\varepsilon'} \right\} + \right. \\ \left. + \text{(the conjugates of these quantities)} \right]$$

We shall not work out here the terms that contain the products of two different

Dirac matrices since we shall ultimately have to take the spur either of the matrix (177) itself, or of this matrix multiplied by one  $\alpha$ , and these terms will thus not enter the result. Omitting these terms, then, and also the higher order terms, we find that  $(T-S_0)$  becomes, in units such that the unit of length is  $\hbar/mc$ ,

$$(179) \quad (x', \rho' | T-S_0 | x'', \rho'') = -\frac{1}{2} \phi_0(\xi) \frac{1}{(2\pi)^3} \int d^3 q \cdot e^{i\vec{q} \cdot \vec{x}} \left[ \frac{\epsilon \epsilon' - q^2 - 1 + R^2/4}{(\epsilon + \epsilon') \epsilon \epsilon'} \right].$$

This integral diverges logarithmically when  $x \rightarrow 0$ . It is this fact, which was contained in Dirac's Solvay paper, that led to the introduction of the subtraction process.

We shall develop the subtraction formalism in terms of the above formula. If we develop in powers of  $\vec{k}$  we find that

$$(180) \quad \epsilon \epsilon' \sim 1 + q^2 + \frac{1}{4} R^2 - \frac{1}{2} \frac{(\vec{q} \cdot \vec{R})^2}{1 + q^2}$$

and that the part of (179) in square brackets thus becomes

$$(181) \quad \frac{1}{4} \frac{R^2 - (\vec{q} \cdot \vec{R})^2 / (1 + q^2)}{(1 + q^2)} \frac{1}{\sqrt{1 + q^2}} + \dots \equiv f_0(q) + \dots \text{ (say).}$$

We shall now make use of this quantity  $f_0(q)$  in order to obtain an additional subtraction term. We write, in fact,

$$(182) \quad (x', \rho' | S_1 | x'', \rho'') = \frac{-e \phi_0}{2(2\pi)^3} \int d^3 q \cdot f_0(q) \cdot e^{i\vec{q} \cdot \vec{x}},$$

and subtract this also from T. If we write

$$(183) \quad \chi(\vec{k}) = \int \left\{ \frac{\epsilon \epsilon' - (q^2 + 1 - \frac{1}{4} R^2)}{\epsilon \epsilon' (\epsilon + \epsilon')} - f_0(q) \right\} d^3 q,$$

we now have

$$(184) \quad \lim_{x \rightarrow 0} e \sum_{\sigma} \left( \xi - \frac{x}{2}, \sigma | T-S_0-S_1 | \xi + \frac{x}{2}, \sigma \right) = \delta p = \\ = \frac{e^2}{\hbar c} \frac{1}{\pi} \mathcal{F}(R^2) \cdot \rho_{\pm},$$

where

$$(185) \quad \mathcal{F} = -\frac{1}{\pi R^2} \chi.$$

By the above subterfuge we avoid a divergent integral and are thus led to finite results. The subtraction terms we have used are really the same as those obtained quite generally by Dirac and Heisenberg to within the approximation in  $\epsilon$  considered; we have obtained them here in terms of the perturbation method. The subtraction has been made in such a way that the total charge is not changed by the polarization.

It will be noted that the induced charge  $\delta\rho$  is proportional to the original charge  $\rho_0$  in the unperturbed state. This is not trivial physically since it contains the result that a light wave does not give rise to a polarization density proportional to  $e^2$ . Furthermore, in the case of a pure scalar potential field, the polarization current vanishes.

When the magnetostatic case is treated it turns out that we now have an induced current given by

$$(186) \quad \delta\vec{j} = \frac{e^2}{\hbar c} \frac{1}{\pi} \mathcal{F}(R^2) \vec{j}_0.$$

This induced current is proportional to the current in the unperturbed state, and the factor of proportionality is the same as in the electrostatic result (184).

The integral  $F$  was computed by Serber in The Physical Review, 48 (1935), 49, but it is possible to improve his result and express  $F$  in terms of elementary transcendental functions. We find that

$$(187) \quad \mathcal{F}(R^2) = \frac{-5}{9} + \frac{4}{3R^2} + \frac{2}{3} \frac{\sqrt{R^2+4}}{R^3} (R^2-2) \log \frac{\sqrt{R^2+4} + R}{2}.$$

At first sight the right-hand side does not look like an even function of  $k$ , as is implied by the notation  $F(k^2)$ . Actually the log term is even, for  $\frac{1}{2}(\sqrt{k^2+4} + k)$  and  $\frac{1}{2}(\sqrt{k^2+4} - k)$  are reciprocals so that

$$\frac{1}{\hbar} \log \frac{\sqrt{\hbar^2 + 4} + \hbar}{2} \equiv \frac{1}{(-\hbar)} \log \frac{\sqrt{\hbar^2 + 4} - \hbar}{2}.$$

The extension to the non-static case

$$\phi_0 = a e^{i(\vec{k} \cdot \vec{x} - \nu t)} + \text{conj.}, \quad \vec{\phi} = \vec{a} e^{i(\vec{k} \cdot \vec{x} - \nu t)} + \text{conj.},$$

was begun by Heisenberg and finished by Serber. Heisenberg found that we get the same integral as before except that we must substitute for  $1/(\epsilon + \epsilon')$  the expression  $(\epsilon + \epsilon') / \{(\epsilon + \epsilon')^2 - \hbar_0^2\}$ , where  $k_0 = \nu/c$ ,  $\nu$  being the frequency of the time dependence. Serber treated the non-static case by arguing that the equations must be Lorentz invariant and so if we know the result for the case of no time dependence we can introduce a moving coordinate system and obtain the solution for the time-dependent case. It is known that the expression

$$(188) \quad K^2 = k^2 - k_0^2$$

is Lorentz invariant. Serber states that the general result may be obtained by substituting  $K^2$  for the previous  $k^2$ . Thus for the time-dependent case we shall have

$$(189) \quad \begin{cases} \delta \rho = \frac{e^2}{\hbar c} \frac{1}{\pi} \mathcal{F}(K^2), \rho_0, \\ \delta \vec{j} = \frac{e^2}{\hbar c} \frac{1}{\pi} \mathcal{F}(K^2), \vec{j}_0. \end{cases}$$

There is one difficulty with the above argument since it is applicable only if  $K$  is space-like, i.e. if  $K^2 > 0$ . But Serber, from an argument of continuity, was able to show that the result is valid also for the time-like case. Actually it is possible to compute the integral for the case where  $k$  is small and  $k_0$  arbitrary, and also for the general case, and to check Serber's result directly.

$\mathcal{F}$  can be expanded in terms of  $k^2$  and the series will converge if  $-4 < k^2 < 4$ . There is an essential singularity at  $k^2 = -4$ , but none at  $k = 0$  or  $k = +4$ . This corresponds to the energy  $2mc^2$  and is thus at the point where

pair production by the external field begins. The singularity is therefore a real one physically.

The singularity in  $F(K^2)$  is connected with the integral

$$(190) \quad \mathcal{G}(K^2) = \int_0^1 \frac{dx}{4 + K^2 - K^2 x^2}$$

since, when  $K^2 < -4$ , the integrand passes through an infinite value. Thus  $F(K^2)$  must be defined by analytic continuation. We may write it as follows:

For  $0 < K^2$ :

$$(191) \quad \mathcal{G}(K^2) = \frac{1}{\sqrt{K^2(4+K^2)}} \log \frac{\sqrt{4+K^2} + \sqrt{4K^2}}{2}.$$

For  $-4 < K^2 < 0$ :

$$(192) \quad \mathcal{G}(K^2) = \frac{1}{\sqrt{|K^2|(4-|K^2|)}} \text{arc. sin } \frac{K}{2}.$$

(There is no singularity at  $K = 0$ , but the above form is preferable to the logarithmic form in this range since it is explicitly real, whereas the logarithmic form, though implicitly real, contains imaginaries.)

Finally, for  $K^2 < -4$ , owing to the infinity in the integrand, we must take the principal value

$$(193) \quad \mathcal{G}(K^2) = \lim_{\epsilon \rightarrow 0} \left( \int_0^{x_0 - \epsilon} + \int_{x_0 + \epsilon}^1 \right) \frac{dx}{4 + K^2 - K^2 x^2},$$

where  $x_0$  is the point of singularity of the integrand. The  $\epsilon$ 's on each side of  $x_0$  must be equal. If we write  $a^2$  for  $(-K^2)$  this principal value becomes

$$\frac{-1}{\sqrt{a^2(a^2-4)}} \log \frac{\sqrt{a^2-4} + a}{2}$$

and thus, since  $(K^2 - 2) = -(a^2 + 2)$ ,  $F(K^2)$  is given in the present range by

$$(194) \quad \mathcal{F}(K^2) = -\frac{5}{9} - \frac{4}{3a^2} + \frac{2}{3} \frac{\sqrt{a^2-4}}{a^3} (a^2+2) \log \frac{\sqrt{a^2-4} + a}{2}.$$

For very long wave-lengths compared with the Compton wave-length, i.e. for  $k^2 \ll 1$ , we have

$$(195) \quad \mathcal{F}(k^2) \sim \frac{k^3}{15}.$$

This corresponds exactly to the result obtained by Heisenberg in the q-number formulation since  $k^2$  corresponds to the Laplacian operator and we get therefore

$$(196) \quad \delta\rho = -\frac{1}{15\pi} \frac{e^2}{\hbar c} \left(\frac{\hbar}{mc}\right)^2 (\Delta\rho) + \text{higher powers of the operator } \Delta.$$

For the general case  $k^2$  is replaced by  $K^2$  and thus  $\Delta$  must be replaced by  $\square$ .

If  $k^2 \gg 1$  we have

$$(197) \quad \mathcal{F}(k^2) \sim \frac{2}{3} \log(k) - \frac{5}{9}$$

which approaches infinity.

We are now in a position to discuss the effect that the polarization of the vacuum will have on the Klein-Nishina formula. Heisenberg and Dirac have remarked that such an effect could occur since a light wave will give rise to a scattering momentum for the electron and this in turn will give rise to polarization currents which will thus alter the scattered radiation. However, we have seen that since, to the order of approximation considered here, and also by Heisenberg and Dirac, only the terms in the current induced by the external radiation with a vanishing  $K^2 \equiv k^2 - k_0^2$  give rise to a scattered radiation field in the wave zone, there will be no polarization effect. Thus we shall have no correction terms in the Klein-Nishina formula up to terms of order  $e^2$  in  $\delta I/I$ . This is contrary to a statement in Heisenberg's paper. In the next higher approximation correction terms will occur already owing to the damping radiation.

It is interesting to consider the relationship between this c-number theory and the q-number formulation.

In the q-number formulation the matrix R was defined as

$$(x', \rho' | R | x'', \rho'') = \frac{1}{2} \{ \Psi_{\rho''}^*(x'') \Psi_{\rho'}(x') - \Psi_{\rho'}(x') \Psi_{\rho''}^*(x'') \}.$$

When an external field  $(\phi_0, \vec{\phi})$  is present, the  $\Psi$ 's can be expanded in terms of the eigen-values of the Hamiltonian

$$H = -e\phi_0 + \vec{\alpha} \left( \frac{\hbar}{c} \frac{\partial}{\partial \vec{x}} + \frac{e}{c} \vec{\phi} \right) + \beta m c$$

that corresponds to this field. We shall thus have

$$H v_n(x, \rho) = E_n v_n(x, \rho)$$

and the  $v$ 's will be orthogonal so that the difficulties encountered in the scalar theory do not arise and when we write

$$\Psi = \sum_m b_m v_m(x, \rho)$$

the  $b$ 's will obey the simple commutation rules

$$[b_m^*, b_n]_+ = \delta_{mn}.$$

It follows now that

$$(198) \quad (x', \rho' | R | x'', \rho'') = \sum_m \sum_n v_n^*(x'', \rho'') v_m(x', \rho'), \frac{1}{2} (b_m^* b_n - b_n b_m^* + \epsilon_n \delta_{mn}) - (x', \rho' | T | x'', \rho''),$$

where

$$(199) \quad \epsilon_n = \begin{cases} +1 & \text{for } E_n > 0 \\ -1 & \text{for } E_n < 0 \end{cases}$$

and T does not involve q-numbers.

The physical density matrix r is given by

$$(200) \quad (x, \rho' | r | x, \rho'') = \lim_{x' \rightarrow x'' \rightarrow x} (x', \rho' | R - S | x'', \rho'') = \\ = \sum_{E_n > 0} v_n^*(x, \rho'') v_n(x, \rho') N_{n^+} - \sum_{E_n < 0} v_n^*(x, \rho'') v_n(x, \rho') N_{n^-} + \\ + \sum_{n \neq m} (v_n v_m) + \lim_{x' \rightarrow x'' \rightarrow x} (x', \rho' | T - S | x'', \rho'').$$

The terms indicated by  $\sum_{n \neq m} (v_n v_m)$  will vanish on integration owing to the orthogonality of the  $v$ 's. The first two summations refer only to electric particles that are really present and are therefore both zero for the case of a vacuum. Thus in this case the physical density matrix reduces essentially to  $(T - S)$ .

The energy is given by

$$(201) \quad \bar{E} = \sum_n |E_{n+}| N_{n+} + \sum_n |E_{n-}| N_{n-} + \int d\nu \sum_{\rho} (x, \rho | HT - HS | x, \rho).$$

Here again, in the case of a vacuum the first two terms will be zero and only the integral will survive. In the H-atom it corresponds to the nuclear field. This is, however, only a first approximation since it neglects the change in the wave functions caused by the induced density; and to this approximation it turns out that all the energy levels are shifted by the same amount so that the spectral lines remain unaltered.

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Polarisation Effects in the Positron Theory

(Reported by M. E. Rose)

We discuss a paper by Uehling in The Physical Review, 48 (1935), 55.

Uehling considers, from the point of view of the c-number formulation of the theory of holes, some special effects resulting from the polarizability of the vacuum. He first investigates the charge density induced by an arbitrary static external field. Then, in the special case where the external field is due to two point charges, he considers the effect of the induced distribution on the energy of the system and thus obtains a modification of the

Coulomb field. The effect of this modified interaction on the Rutherford scattering formula is investigated, and also its effect on atomic energy levels.

Let there be an external electrostatic field given by the electrostatic potential  $\phi_0(\vec{\xi})$ . The physical density matrix  $r$  can be computed in the manner explained on pp. 57-61. If we write  $(\vec{\xi} | r | \vec{\xi})$  for  $\sum_{\sigma} (\vec{\xi}, \sigma | r | \vec{\xi}, \sigma)$  we find that, if  $\rho$  is the charge density when no account is taken of the polarization effects, i.e.

$$(202) \quad \Delta \phi_0(\vec{\xi}) = -4\pi\rho(\vec{\xi}),$$

then

$$(203) \quad (\vec{\xi} | r | \vec{\xi}) = \frac{\alpha}{4\pi^2 e} \int U(|\vec{r}' - \vec{\xi}|) \cdot \Delta \rho(\vec{r}') d\vec{r}',$$

where

$$(204) \quad U(\vec{\xi}) = \frac{1}{\lambda_0 \xi} \int_0^1 (1-u^2) \text{li}(e^{-2\xi/\sqrt{1-u^2}}) d\mu, \quad (\lambda_0 = \hbar/mc),$$

the logarithmic integral  $\text{li}(e^{-y})$  being defined by

$$\text{li}(e^{-y}) = \int_0^{e^{-y}} \frac{dt}{\log t} = \int_{\infty}^1 e^{-y^2} \frac{dy}{y} = \text{Ei}(y).$$

The function  $U(\vec{\xi})$  can also be written as

$$U(\vec{\xi}) = \frac{-1}{\pi \xi \lambda_0} \int_0^{\pi/2} \cos^3 \psi d\psi \int_0^{\infty} \sin \frac{\hbar}{\xi} \xi \cdot \log \left( 1 + \frac{\hbar^2}{4} \cos^2 \psi \right) \frac{d\hbar}{\hbar}.$$

It turns out that this can be expressed in terms of Bessel functions. We have, in fact,

$$(205) \quad U(\xi) = \frac{-1}{3\xi\lambda_0} \left\{ 2\left(\frac{\xi^2}{3} + 1\right) K_0(2\xi) - \frac{2\xi^2}{3} (2\xi^2 + 5) K_1(2\xi) + \frac{\xi^2}{3} \left(\frac{4\xi^2}{3} + 3\right) B(2\xi) \right\},$$

where  $K_0, K_1$  are Bessel functions of the second kind, and  $B$  is given by

$$B(\xi) = - \int_{\infty}^{\xi} K_0(\xi) d\xi.$$

The asymptotic behavior of  $U$  is important. For small  $\xi$  we have from (205)

$$(206) \quad U(\xi) = \frac{2}{3\xi\lambda_0} \left\{ \gamma + \frac{5}{6} + \log(\xi) + O(\xi) \right\},$$

where  $\gamma$  is Euler's constant, while for large  $\xi$

$$(207) \quad U(\xi) = \frac{-\pi^{1/2}}{4\xi^{3/2}\lambda_0} e^{-2\xi} \left\{ 1 + O\left(\frac{1}{\xi}\right) \right\}.$$

The induced charge density is given by

$$(208) \quad \delta\rho = e \left( \vec{\xi} | r | \vec{\xi} \right).$$

If we take a slowly varying external field we can expand the  $\Delta\rho$  in (203), retaining only the first term, and thus obtain

$$(209) \quad \delta\rho \left( \vec{\xi} \right) = -\frac{\alpha}{15\pi} \lambda_0^2 \Delta\rho \left( \vec{\xi} \right),$$

which agrees with Heisenberg's result for slowly varying fields.

Let us consider two point charges  $Z'e$ ,  $Z''e$ , distant  $R$  apart. If we calculate the energy of this system taking account of the polarization of the vacuum we find that in addition to the Coulomb energy there are extra terms also representing an interaction energy between the particles which must therefore be regarded as correction terms to the Coulomb law.

The additional energy due to the polarization is given by

$$(210) \quad E = \text{spur} (x', \sigma' | H r | x'', \sigma''),$$

where  $r$  is the physical density matrix. This energy can be broken up into two parts,  $(E_e + E_{\text{kin}})$ , where  $E_e$  is electromagnetic energy while  $E_{\text{kin}}$  is kinetic energy. It turns out that  $E_{\text{kin}} = -\frac{1}{2}E_e$  so that  $E = \frac{1}{2}E_e$ . Since

$$E_e = -4e\lambda_0^3 \int \phi_0(\vec{\xi}) \left( \vec{\xi} | r | \vec{\xi} \right) d\vec{\xi}$$

there will arise two kinds of terms; terms involving  $Z'^2$ ,  $Z''^2$ , and terms involving  $Z'Z''$ . The former consist of divergent integrals but may be ignored for our present purposes since they refer to that part of the self-energies of the particles due to the polarization. The term in  $Z'Z''$  is evidently an interaction energy between the two particles. This energy is finite (for finite  $R$ ) and is found to have the value

$$(211) \quad E_{int} = -\frac{\alpha}{\pi} Z' Z'' e^2 U(R),$$

where  $U(R)$  is the function defined in (205). This energy constitutes the correction to the Coulomb energy according to the present theory.

It can be seen from (204) that  $U(R)$  is always negative. Therefore the correction to the Coulomb energy is of the same sign as the Coulomb energy itself. For large distances the correction is negligible, but when the distance between the charges is small the correction term can dominate the Coulomb energy. However the present theory is not valid for very small distances. (It is asserted that the limiting distance for which the result is valid is  $\sqrt{\alpha} \lambda_0$  where  $\alpha$  is the fine structure constant. At this distance the correction term is less than one percent of the Coulomb potential.)

The deviation from the Coulomb potential implies an alteration in the Rutherford scattering formula.

Uehling argues that if the incident particles are to feel the deviation from the Coulomb potential they must come very close to the nucleus. They must therefore be particles of high energy, and in this case the Born approximation may be used. For the scattering of protons we may consider energies up to the order of a million electron volts before the theory becomes invalid. For such energies, and for a considerable range of smaller energies, the parameter  $Z'Z''e^2/hv$ , the smallness of which determines the validity of the Born approximation, is indeed smaller than unity.

The application of the Born method leads to a modified scattering law:

$$(212) \quad f(\theta) = -\frac{2M}{h^2} \int_0^{\infty} \frac{\sin Kr}{Kr} V(r) r^2 dr,$$

where  $\chi = \frac{2mv}{\hbar} \sin \frac{\theta}{2}$ , and  $V(r)$  is the modified potential. Here  $M$  is the mass of the incident particle and  $v$  is its velocity at infinity. On integration this gives

$$(213) \quad \begin{cases} f(\theta) = \frac{-Z'Z''e^2}{2Mv^2} \operatorname{cosec}^2 \left\{ \frac{\theta}{2} \left[ 1 + \frac{\alpha}{\pi} \mathcal{F}(R^2) \right] \right\}, \\ k_2 = \frac{2Mv}{mc} \sin \frac{\theta}{2}, \end{cases}$$

where  $\mathcal{F}$  is defined in (187). Unfortunately the effect of the extra energy terms is too small and varies too slowly to permit of verification by scattering experiments.

The extra terms in the interaction energy lead to a displacement of the spectral lines of atoms.

The change in the energy of an extra-nuclear electron due to the polarization caused by the field of the nucleus is given by

$$(214) \quad \delta E = \int E_{int} |\Psi_{nlm}|^2 d\tau.$$

For 1s and 2s levels the displacement is downwards:

$$(215) \quad \delta E_{1s} = \frac{-8\alpha^3 Z^2 R}{15\pi} (1 + O(\alpha Z)),$$

$$(216) \quad \delta E_{2s} = \frac{-\alpha^3 Z^2 R}{15\pi} (1 + O(\alpha Z)),$$

where  $R$  is the Rydberg constant.

For p-terms the displacement is of the next order in  $(\alpha Z)$ . Thus, at least for light atoms, it is only in the s-terms that the effect will be noticeable. But the polarization effect leads to an increase in the doublet separation and experiment shows a decrease to be necessary.

On the Production of Pairs by Photons in the Field of a Nucleus

(Reported by G. Breit)

The theory of the production and annihilation of pairs was initiated by Dirac with his theory of the negative energy states of the relativistic electron. At the time, the positron had not been detected and Dirac tried to regard his theory as referring to electrons and protons, but his computations of the mutual annihilation of positive and negative particles are applicable when his theory is considered as applying to electrons and positrons.

If we begin with no positive energy electrons and with all the negative energy levels filled, two photons, of sufficient energy, can give up their energy and momentum to an electron in a negative energy state and raise it to a state of positive energy. This "creates" an electron and a positron. The inverse process, when a positive energy electron jumps into a hole in the negative energy levels, corresponds to the annihilation of an electron-positron pair and the creation of a pair of photons. These simple processes can be applied to complicated situations such as arise in the interaction of cosmic rays with matter, by means of a method due to Weizsäcker.

We shall begin with a brief discussion of the theory of the collision of two light quanta, following a paper by G. Breit and J. A. Wheeler in The Physical Review, 46 (1934), p. 1087.

The two photons may be represented by two plane waves whose potentials are

$$\left\{ \begin{array}{l} a_1 e^{i(\nu_1 t - \vec{k}_1 \cdot \vec{r})} + \text{conjugate,} \\ a_2 e^{i(\nu_2 t - \vec{k}_2 \cdot \vec{r})} + \text{conjugate.} \end{array} \right.$$

If these photons act simultaneously on an electron in a negative energy state they will cause a perturbation so that the final state will lie partly in the positive

energy region, thus giving a definite probability of an electron and positron being created. The density of electrons represented by the perturbed wave function can be put in the form

$$(217) \quad \left(\frac{e^2}{mc^2}\right)^2 |a_1|^2 |a_2|^2 B \left( \frac{1 - e^{-\epsilon \delta W t / \hbar}}{\delta W} \right),$$

where B is a function depending on the initial polarizations of the photons and on the momenta and spins of the positrons. The parenthesis contains a resonance term,  $\delta W$  being the change in the total energy of the system.

Let us suppose that there are  $N_1$  photons in the first beam and  $N_2$  in the second. Then we must have

$$(218) \quad |a_1|^2 = \frac{\hbar c}{2\pi v_1} N_1, \quad |a_2|^2 = \frac{\hbar c}{2\pi v_2} N_2.$$

If we now denote the energy-momentum of the positron by  $(W_1, p_1)$  and of the electron by  $(W_2, p_2)$  we may express the probability of pair formation as

$$(219) \quad \left(\frac{e^2}{mc^2}\right)^2 \left(\frac{\hbar^2 c^2 N_1 N_2}{4\pi^2 v_1 v_2}\right) \left(\frac{2p_1^2 d\omega_1}{\hbar^3}\right) \frac{\bar{B}^\sigma \left(\frac{4\pi^2 t}{\hbar}\right)}{c^2 \left[ \vec{p}_1/W_1 + \vec{p}_2/W_2 \right]},$$

where the term involving the N's reduces this to a probability per unit volume of the space in which the photons collide while the factor  $(2p_1^2 d\omega_1/\hbar^3)$  reduces it to a probability per unit solid angle for the positron for both directions of polarization. The quantity  $\bar{B}^\sigma$  is the average of B over the spin of the positron, the spin of the electron being completely determined once we know the spin of the positron and the polarizations of the initial photons.

To define a collision cross-section  $\sigma$  for the photons we equate this probability to  $N_1 N_2 \frac{\sigma t}{c} d\omega_1$ . Thus  $\sigma$  is given by

$$(220) \quad \sigma = \left(\frac{e^2}{mc^2}\right)^2 \frac{2p_1^2 c \bar{B}^\sigma}{\hbar v_1 \hbar v_2 \left[ \vec{p}_1/W_1 + \vec{p}_2/W_2 \right]}.$$

Actually this  $\sigma$  is twice the ordinary cross-section in the special case in which the photons travel against each other in the same line; for  $\sigma$  refers to the situ-

ation where one volume of photons is regarded as fixed and the other as moving against it. Though, according to the theory of relativity, if both groups of photons were moving, their relative velocity would still be that of light, the number of photons passing each other per second would be twice what it was before.

The probability of emission can be computed in a similar way. It was first calculated by Dirac. The emission probability comes out as

$$(221) \left(\frac{e^2}{mc^2}\right)^2 \frac{c^4 \hbar^2}{4\pi^2 v_1 v_2} \cdot \frac{1}{V^2} \left(\frac{P_1^2}{\hbar^3} V d\Omega\right) \left(2\beta(s_1, \bar{s}_2)\right) \frac{\frac{4\pi^2 t}{\hbar}}{c [1 - \vec{P}_1 \vec{P}_2 / P_1 P_2]} V^2 \frac{1}{V},$$

where the final  $V^2$  reduces it to a probability per unit density while the  $1/V$  makes it per cubic centimeter. The  $P$ 's refer to the momenta of the photons and  $d\Omega$  to the element of solid angle in which they are found. The quantity  $\beta(s_1, \bar{s}_2)$  is a function similar to the  $B$  of (217),  $s_1, s_2$  referring to the polarizations of the photons. When the photons move in opposite directions

$$\beta \sim \frac{1}{2} B.$$

In a frame of reference in which the photons travel in opposite directions the emission probability becomes

$$(222) c \left(\frac{e^2}{mc^2}\right)^2 \int \frac{(v_1/v_2) \bar{B}^\sigma}{[1 - \vec{P}_1 \vec{P}_2 / P_1 P_2]} d\Omega_1 \rightarrow \frac{c}{2} \left(\frac{e^2}{mc^2}\right)^2 \int \bar{B}^\sigma d\Omega_1,$$

where  $\bar{B}^\sigma$  is the average of  $B^\sigma$  over all polarizations of the photons. We define a cross-section of capture,  $\sigma_c$ , by setting the above equal to  $\sigma_c \cdot \frac{2c\hbar}{W}$ ,  $\frac{2c\hbar}{W}$  being the relative velocity. Thus

$$(223) \sigma_c \cdot \frac{2c\hbar}{W} = \frac{c}{2} \left(\frac{e^2}{mc^2}\right)^2 \int \bar{B}^\sigma d\Omega_1.$$

If we take the average of  $\sigma$  in (220) we find

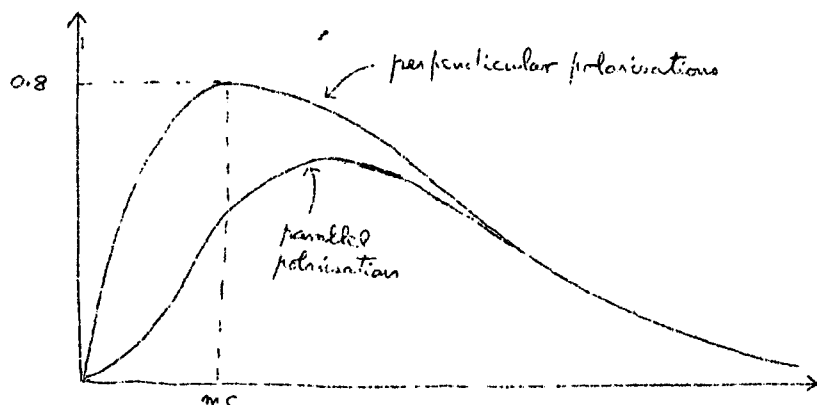
$$(224) \bar{\sigma} = 2 \left(\frac{e^2}{mc^2}\right)^2 \int \frac{c\hbar^2}{\hbar v_1 \hbar v_2} \frac{\bar{B}^\sigma d\omega_1}{[(\hbar_1/W_1) + (\hbar_1 \vec{P}_1 / \hbar_1 W_2)]} \rightarrow \left(\frac{e^2}{mc^2}\right)^2 \frac{c\hbar}{W} \int \bar{B}^\sigma d\omega_1.$$

Thus we actually find

$$(225) \quad \frac{\bar{\sigma}_c}{(\hbar/\mu)^2} = \frac{1}{4} \frac{\bar{\sigma}}{(c/v)^2}.$$

The relation is even closer since the factor  $1/4$  can be explained away. It arises from two sources. Firstly a factor  $1/2$  comes from the fact that  $\bar{\sigma}$  is not the ordinary cross-section and secondly a further factor  $1/2$  enters because when the final product is a pair of photons we cannot make a distinction between  $\overset{1}{\leftarrow} \overset{2}{\rightarrow}$  and  $\overset{2}{\leftarrow} \overset{1}{\rightarrow}$ .

It is possible to calculate the collision cross-sections for photons of parallel spin and for photons with perpendicular spins. The results are similar but their differences are significant for small energies. Roughly we find



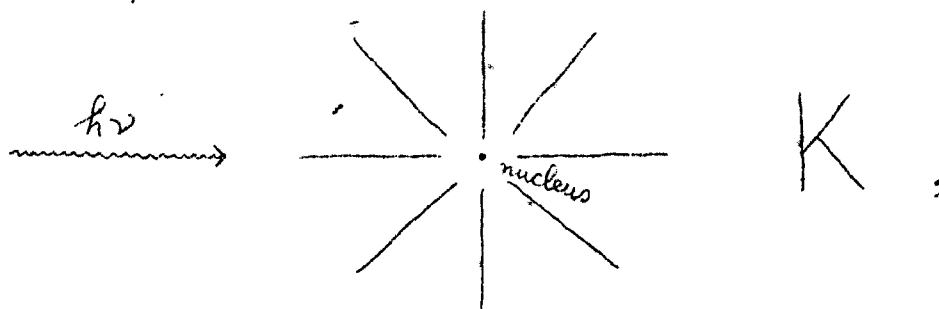
where the horizontal axis refers to the momentum of the positrons while the vertical axis gives the collision cross-section of the photons in units  $2\pi \left(\frac{e^2}{mc^2}\right)^2$ . For large energies the two curves coalesce and behave like  $[\log(E/mc^2)] / (E/mc^2)^2$ .

It is possible to integrate (224). We find that

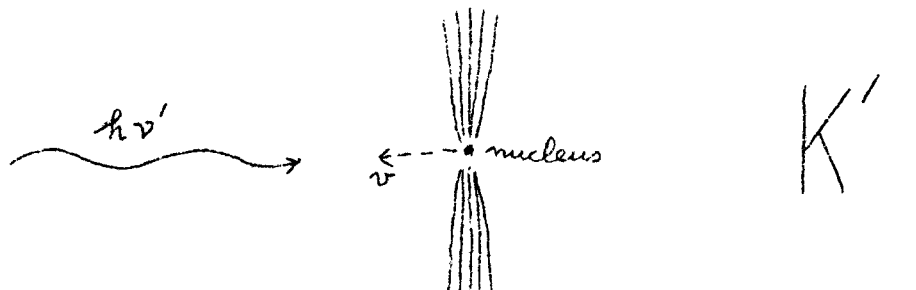
$$(226) \quad \left\{ \begin{array}{l} \bar{\sigma} = 2\pi \left(\frac{e^2}{mc^2}\right)^2 \left[ \frac{2\theta}{c^4} + \frac{2\theta}{c^2} - \frac{S}{c^3} - \frac{S}{c^5} - \frac{\theta}{c^6} \right], \\ \text{where } \frac{E}{mc^2} = c = \cosh \theta, \\ \frac{\mu}{mc} = S = \sinh \theta. \end{array} \right.$$

The above results for the collision of photons with photons can be used in calculating the effect of the collision of a photon with the field of a nucleus. The method was first applied to allied phenomena in detail by v. Weizsäcker, Zeitschrift für Physik, 88 (1934), p. 612, although it had been previously used by Fermi and the qualitative ideas for high energy particles were also used previously by Williams.

The trick is to discuss the problem in a frame of reference in which the nucleus is moving with a velocity approaching the velocity of light. Let  $K$  be the frame in which the nucleus is at rest and let  $K'$  be the frame in which it is moving with approximately the velocity  $c$ . In  $K$  the field of the nucleus is symmetrical



but in  $K'$  the lines of force will bunch together perpendicularly to the direction of the relative motion



To an observer fixed in  $K'$  the field of the nucleus will now appear as a sudden pulse moving with approximately the velocity of light, and the motion will also introduce a magnetic component in such a way that the whole field approaches a transverse field of photons. This field can now be analyzed into a set of pho-

tons, and since the original photon has merely undergone a Doppler change the problem has been reduced to that of the collision of photons with photons. This treatment, however, ignores the phase relations that must exist between the photons into which the nuclear field is analyzed.

In  $K$  the field of the nucleus is inverse square. If we use cylindrical coordinates with  $x$  as the direction of relative motion and  $\rho$  as the radius vector perpendicular to  $x$ , then the component of the electric intensity in  $K$  perpendicular to  $x$  is

$$(227) \quad \mathcal{E}_\rho = \frac{e\rho}{[x^2 + \rho^2]^{3/2}}.$$

In  $K'$  it turns out that only this component of the electric intensity is important since it acquires a factor  $\xi \equiv 1/\sqrt{1-v^2/c^2}$ . The magnetic intensity induced by the motion is approximately perpendicular to this  $\mathcal{E}_\rho$  and to  $x$ . In  $K'$  we have, in fact,

$$(228) \quad \mathcal{E}_\rho = \frac{e\rho\xi}{[\xi^2(x+vt)^2 + \rho^2]^{3/2}},$$

and we analyze this into photons by means of the Fourier integral

$$(229) \quad \mathcal{E}_\rho = \int_{-\infty}^{\infty} \mathcal{E}_{\rho\nu} e^{-2\pi i\nu t} d\nu.$$

At this point Weizsäcker rather brutally says that we can obtain the number of photons by picking out a particular  $\rho$  and then taking the average of  $\mathcal{E}_\rho^2$  over a long time interval  $T$ . This gives

$$(230) \quad T \overline{\mathcal{E}_\rho^2} = \int_{-\infty}^{\infty} 2 |\mathcal{E}_{\rho\nu}|^2 d\nu$$

and we can now treat the Fourier components separately. For if  $N_\nu$  photons are in the range  $d\nu$  their energy is  $h\nu N_\nu d\nu$  and the contribution to  $h\nu N_\nu d\nu$  due to a unit area at a distance  $\rho$  is

$$(231) \quad \frac{c}{4\pi} 2 |\mathcal{E}_{\rho\nu}|^2 d\nu.$$

If we sum over a small ring from  $\rho$  to  $\rho + d\rho$  we get

$$(232) \quad h\nu N_\nu = \frac{c}{4\pi} \int 2 |E_{\rho\nu}|^2 2\pi\rho d\rho = c \int |E_{\rho\nu}|^2 \rho d\rho.$$

If we write

$$(233) \quad \frac{\xi(x+vt)}{\rho} = S$$

then

$$E_\rho = \frac{e\xi}{\rho^2(1+S^2)^{3/2}}$$

and

$$t = \frac{\rho S}{\xi v} - \frac{x}{v}.$$

so that

$$(234) \quad E_{\rho\nu} = \frac{e\xi}{\rho^2} \cdot e^{-\frac{2\pi i x}{\lambda}} \int_{-\infty}^{\infty} \frac{e^{-2\pi i \nu \rho S / \xi v}}{(1+S^2)^{3/2}} \cdot \frac{\rho}{\xi v} ds.$$

Weizsäcker has pointed out that if  $\rho > \xi\lambda$ , ( $\lambda = v/v$ ) then  $E_{\rho\nu}$  will be approximately zero, while if  $\rho < \xi\lambda$

$$(235) \quad E_{\rho\nu} \sim \frac{ze}{\rho v} e^{-\frac{2\pi i x}{\lambda}} \sim \frac{ze}{\rho c} e^{-\frac{2\pi i x}{\lambda}}.$$

Thus the number of photons is now given by the equation (232) with the above value of  $E_{\rho\nu}$  and with the upper limit of the integral taken as  $\xi\lambda$ . It is not possible to take the lower limit as zero since this introduces a logarithmic infinity. One must therefore integrate from some lower limit  $\rho_0$ . At first it was thought that  $\rho_0 \sim \hbar/\mu c$  since in a smaller region it is impossible to build wave-packets.

The number of photons, when we leave the lower limit of  $\rho$  as  $\rho_0$ , comes out as

$$(236) \quad N_\nu = \frac{z}{\pi} \frac{Z^2 \alpha}{v} \log\left(\frac{\lambda \xi}{\rho_0}\right).$$

After we have found from this the number of pairs produced, we must go back to the

original frame  $K$  to compute the effective cross-section. We find

$$(237) \quad \sigma_{\text{eff}} = \frac{4}{\pi} Z^2 \int_{\bar{\nu} = \frac{mc^2}{h}}^{\bar{\nu} = \frac{\sqrt{E_1^2}}{2\rho}} \sigma(\bar{\nu}) \log \frac{\sqrt{c\nu_1/2\rho_0}}{\bar{\nu}} \cdot \frac{d\bar{\nu}}{\bar{\nu}},$$

where  $\bar{\nu}$  is the frequency of the two photons when they are transformed so that they are equal and opposite. The quantity  $\xi$  no longer enters the formulas. This was to be expected since  $\xi$  contains the relative velocity  $v$  which has a merely auxiliary significance. Actually we have

$$\bar{\nu}^2 = \nu_1' \nu_2' = \sqrt{\frac{c-v}{c+v}} \nu_1 \nu_2',$$

the change to  $\nu_1$  from  $\nu_1'$  being necessary because  $\nu_1'$  was measured in the moving system. Thus

$$\frac{c\xi}{\rho_0 \nu_2'} = \frac{c\xi \nu_1}{\rho_0 \bar{\nu}^2} \sqrt{\frac{1-v/c}{1+v/c}},$$

and the  $\sqrt{1-v/c}$  cancels the  $\xi$  while the  $(1/\sqrt{1+v/c})$  contributes, in the limit  $v \rightarrow c$ , a factor  $(1/\sqrt{2})$ .

In the above calculations it is assumed that the nucleus does not absorb energy.

The effective cross-section for a Coulomb nuclear field can be computed.

If we write

$$(238) \quad C_1 = \cosh \theta_1 = \sqrt{\left(\frac{h\nu_1}{2mc^2}\right) \left(\frac{h/mc}{\rho_0}\right)},$$

it is found that for this case

$$(239) \quad \sigma_{\text{eff}} = \frac{Z^2 (e^2)^2}{137 (mc^2)^2} 8 \left\{ \left[ \frac{7}{9} + \frac{1}{2C_1^2} + \frac{1}{8C_1^4} - \frac{1}{36C_1^6} \right] \theta_1 - \frac{43}{36} \frac{S_1}{C_1} - \frac{11}{72} \frac{S_1}{C_1^3} - \frac{1}{36} \frac{S_1}{C_1^5} \right\} \sim$$

$$\sim \frac{Z^2 (e^2)^2}{137 (mc^2)^2} \left\{ \frac{28}{9} \log \frac{2E_1 (h/mc)}{\rho_0} - \frac{86}{9} \right\},$$

where  $E_1 = h\nu_1/mc^2$ , the energy of the photon in units  $mc^2$ ,

It is interesting to compare this result with the result obtained for the same problem by H. Bethe and W. Heitler, Proceedings of the Royal Society of London, 146A (1934), p. 83, using a direct method that avoids the introduction of a moving reference system. Their result is (ibid. p. 109)

$$(240) \quad \sigma \sim \frac{Z^2}{137} \left( \frac{28}{9} \log 2\varepsilon_1 - \frac{218}{27} \right).$$

The fact that the dependence on  $\varepsilon_1$  is the same in the two results shows that Weizsäcker's approximations are not unsatisfactory. The two formulas can be brought into agreement by a suitable choice of  $\rho_0$  in (240). It turns out that we must take  $\rho_0 = (\hbar/mc) e^{-40/27} \sim (\hbar/mc) (1/4.4)$ . But we should not take this value of  $\rho_0$  too seriously in view of the approximations that have been made in applying Weizsäcker's method and the uncertain validity of the Bethe-Heitler formula as applied to experiment.

The problem first worked out by Weizsäcker when he introduced the idea of using a moving frame of reference was not the one we have been considering, but had to do with the emission of photons by the collision of fast electrons with a nucleus. There are some slight differences in the two cases, the chief of which is that the passage to the limit  $v \rightarrow c$  is no longer necessary when we are dealing with electrons since the velocity  $v$  is part of the physical problem and is not a mathematical auxiliary; it is in fact most convenient to transform to a system in which the electron is at rest.

As in the case of the impact of photons with a nucleus, the nuclear field in the moving frame of reference is analyzed into a set of photons.

The value of the impact parameter  $p$ , the classical distance of closest approach, is essentially bounded both above and below. The lower limit is of the

order of the Compton wave-length  $h/mc$  and arises from the fact that it is impossible to construct a wave-packet for the electron of smaller dimensions than this, using states with one sign of energy. Thus if the nucleus approaches closer than  $h/mc$ , its effect on the wave-packet will be partly in one direction and partly in the opposite direction, so that the net acceleration of the packet will be small and therefore the radiation will also be small.

The upper limit of the impact parameter is given by

$$(241) \quad b < \xi (h/mc)$$

and arises from the fact that the width of the moving field depends on  $\xi$ , being of the order  $(p/\xi)$ , and that this, for reasons similar to those used in establishing the lower limit, may not be greater than the order of magnitude of the Compton wave-length.

Now the Klein-Nishina formula shows that the scattering of the photons will be strongest in the original direction, with the wave-length not much changed, while the long wave-length photons are scattered backwards with a smaller intensity. But in the frame of reference in which the nucleus is at rest and the electron moves, the forward photons appear softer and with lessened intensity while the backward ones become harder and their intensity increases. It thus turns out that quite large energy losses in the frame of the nucleus are due to the photons that are scattered backwards and sideways, and these photons have the by no means small wave-length  $\sim (h/mc)$ .

The total energy loss increases with the velocity of the electron since as  $v$  increases the quantity  $\xi$  increases and thus the range of values of the impact parameter  $p$  increases. However, experimentally, we have to do with screened nuclei for which the field vanishes if  $p$  is too great. As a result, the increase of energy losses with velocity turns out to be much smaller for screened nuclei.

Weizsäcker found that the differential cross-section for a collision in the range of energy  $(mc^2 \xi) (\epsilon, \epsilon + \delta\epsilon)$  is given by

$$(242) \quad d\sigma = \frac{4Z^2}{137} \left(\frac{e^2}{mc^2}\right)^2 \left[\frac{4}{3}(1-\epsilon) + \epsilon^2\right] \log \left\{ \frac{2\xi(1-\epsilon)}{\epsilon} k \right\} \frac{d\epsilon}{\epsilon},$$

where we cannot be sure of the value of the constant  $k$ .

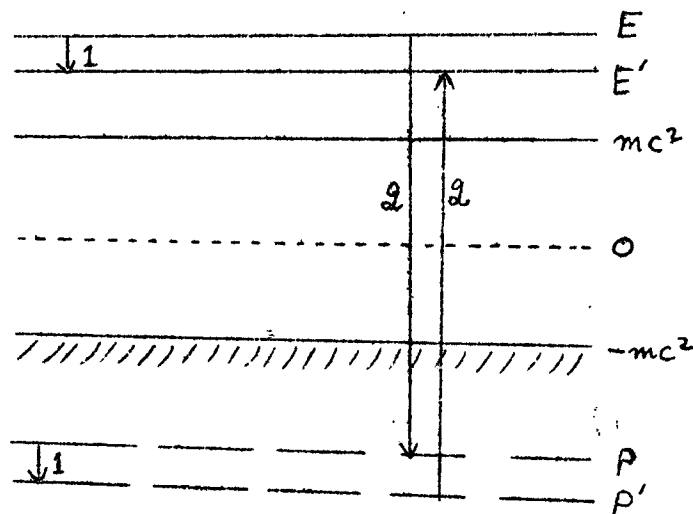
When screening is taken into account the total collision cross-section becomes

$$(243) \quad \sigma = \frac{Z^2 (e^2/mc^2)^2}{137} \log (137/Z^{1/3}),$$

where the  $Z^{1/3}$  comes from the fact that, according to the Fermi-Thomas distribution, the effectiveness of the charge of the nucleus reaches only to a distance  $a_H Z^{-1/3}$  from the center of the nucleus.

A recent paper by Bhabha is of interest in that he considers an important effect that leads to the possibility of an experiment that will tell about the nature of the positron. For, if positrons are scattered by electrons, the hypothesis that the positrons are holes in the negative energy states of the electrons should show itself as an exchange phenomenon.

For example, suppose that initially we have an electron in the state  $E$  and an electron hole in the negative energy state  $P$ , and that after the collision we find an electron in the state  $E'$  and a hole in the negative energy state  $P'$ .



If the electron and positron had no relationship there would be no exchange and the only way in which the collision could occur would be

$$E \rightarrow E', \quad P \rightarrow P'$$

as shown by arrows 1 in the figure. But if we have exchange there is, in addition to the above, the possibility that the electron in state E jumps to fill the hole P while the electron originally in the negative energy state P' jumps to the positive energy state E', thus leaving a hole at P'. This process is indicated by arrows 2 in the figure.

Bhabha says that the transitions of the second type can be considered as occurring in two steps. The first jump will create a virtual photon which will be reabsorbed to produce the two final particles arising from the second jump. Thus the second type of process can happen on any theory <sup>of</sup> which the possibility of creation and annihilation of light and matter is a part. In particular it will occur in the theory of Pauli and Weisskopf, even though this theory does not contain the exclusion principle. The results, of course, will be different for different theories, in general.

This point of Bhabha's is, however, somewhat misleading without the following consideration. The photon emission which is considered is a virtual one, without conservation of energy. Such emissions also occur in the theory of the interaction of two charged particles of any type and it is, therefore, misleading to attribute the exchange effect to the possibility of electron-positron recombination; in fact, exchange terms were already known to be of importance from Møller's calculations on scattering of electrons by electrons. It is also wrong to attribute them to virtual photon emission because there are no exchange terms for scattering of electrons by protons while virtual photon emission is present.

From the point of view of virtual photon emission, the condition is as follows. There are two kinds of virtual photon emission possible: (a) Electron

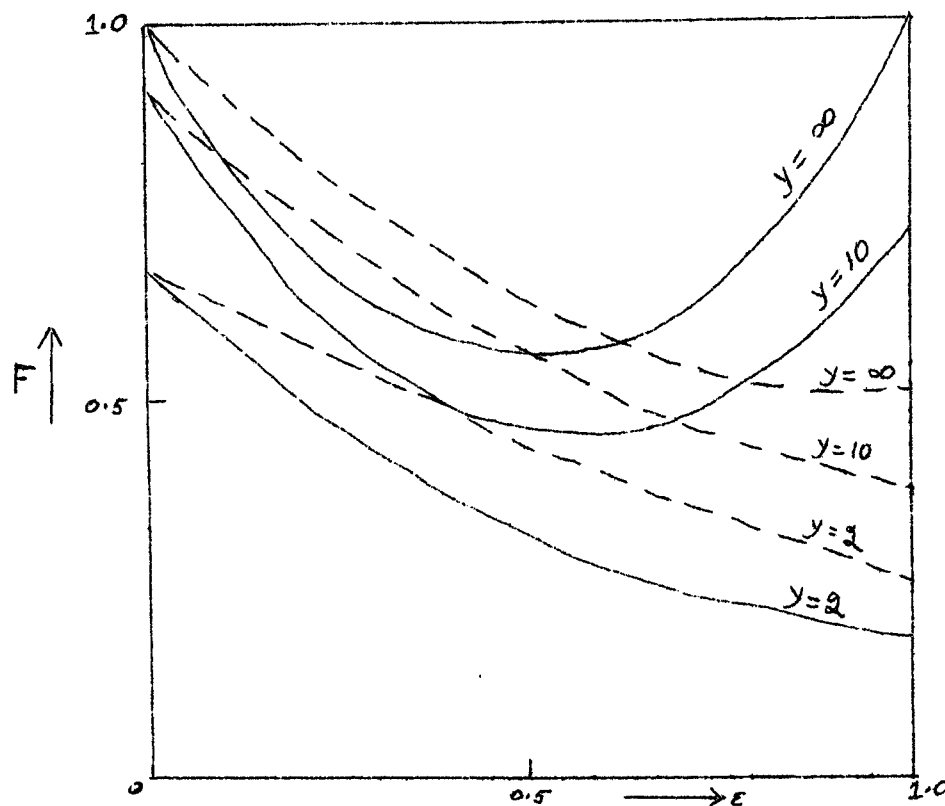
E emits a photon and goes into E'; the photon is absorbed by positron P which then changes into P'; an essentially similar process occurs in reverse order  $P \rightarrow P' + h\nu$ ;  $E + h\nu \rightarrow E'$ ; (b) E and P disappear and form  $h\nu$ ;  $h\nu$  disappears and forms E' and P'. Or else E', P' and  $h\nu$  are first created and then  $h\nu$ , E, P disappear. Only type (a) is possible for unrelated particles like electrons scattered by protons, while on the Dirac and the Pauli-Weisskopf theories both (a) and (b) can occur.

The actual calculations are quite straight-forward and the interesting point is that they show that there is an appreciable difference between the scattering cross-sections with and without exchange. The formula obtained is

$$(244) \quad dQ = 2\pi \left(\frac{e^2}{mc^2}\right)^2 \frac{y F(y, \epsilon)}{(y-1)^2 \epsilon^2} d\epsilon,$$

where  $y$  is the original energy of one particle in units  $mc^2$  when the other particle is taken to be at rest and  $\epsilon$  is the fraction of the original kinetic energy that has been transferred to the second particle.

The figure shows the behavior of the function F under various conditions.



The solid lines refer to cases when exchange is taken into account, the broken lines to cases when exchange is ignored. There is seen to be a large difference between the two formulas for large energy transfers. The first term in the difference is of the order  $(v^2/c^2)$  and the second of order  $(v^4/c^4)$ .

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On the Failure of Quantum Mechanics for High Energies

(Reported by W. Pauli)

We have seen that the theories of radiative collision cross-sections lead to a logarithmic increase of these cross-sections with high energies. It seems that this result is a fundamental one and is not due to the approximations that are used since all investigations show that these approximations are good for high energies; furthermore a similar result comes <sup>in</sup> other effects, notably in the theory of pair production and the theory of energy losses by ionization impacts.

On the other hand, experiments by Neddermeyer seem to show that the energy losses in ionization impacts are, for energies of the order  $137mc^2$ , much smaller than those given by theory.

The method of Weizsäcker is particularly suited to our present discussion of a possible change in the theory owing to its mathematical simplicity. We shall rewrite the formula for the total cross-section for the radiative collision of an electron with a nucleus in a new notation. We write

$$(245) \quad \left\{ \begin{array}{l} \alpha = \frac{e^2}{\hbar c} \sim \frac{1}{137}, \quad d = \frac{e^2}{mc^2} = \text{electron radius,} \\ E = \frac{1}{2}mc^2 = \text{relative total energy,} \\ \Lambda = \hbar/mc = d/\alpha. \end{array} \right.$$

Then the cross-section is

$$(246) \quad \sigma = 4 \alpha Z^2 d^2 \log (k_2/k_1),$$

where, theoretically,

$$(247) \quad \left\{ \begin{array}{l} k_1 = \Lambda, \\ k_2 = \begin{cases} \Lambda \} & \text{for no screening,} \\ \frac{\hbar^2}{m e^2} Z^{-1/3} & \text{for screening, } (\hbar^2/m e^2) \text{ being the} \\ & \text{Bohr radius.} \end{cases} \end{array} \right.$$

Different proposals have been made for changing the theory. Oppenheimer in The Physical Review, 47 (1935), p. 44 has discussed the question in some detail. An interesting point is that we cannot resolve the difficulty by introducing a finite shape for the electron since we have seen that in the coordinate system of Weizsäcker it is not the short wave-lengths that are important but only those larger than the Compton wave-length.

Also, though we have impacts of a very short duration, we make a Fourier analysis of these impacts and then say that it is only the longer wave-lengths that are important. These wave-lengths form only a small fraction of the whole field.

Thus the theory must be so changed that we can justify the fact that the Fourier expansion is no longer valid. If we no longer have a linear relation between the acceleration of the electron and the magnitude of the external field we can hope to explain why the theory no longer holds. We ought then to show that the amended theory will give a smaller cross-section but that is a much more complicated matter and we shall be content here with discussing merely the first point.

Oppenheimer begins by making the quasi-empirical assumption that all impacts whose duration is less than  $(d/c)$  have no effect in producing radiation from

the electron. This restriction leaves the upper limit  $p_2$  unchanged but the lower limit becomes smaller:

$$(248) \quad \mu_1 = \frac{2}{3} d = \frac{1}{137} \wedge \frac{2}{3}.$$

The cross-section is now given by

$$(249) \quad \sigma \sim 4\pi Z^2 d^2 \log(1/x), \text{ for no screening.}$$

For screening we have

$$(250) \quad \sigma \sim 4\pi Z^2 d^2 \log(1/x^2 Z^{1/3} \frac{2}{3})$$

if  $\alpha^2 Z^{1/3} \frac{2}{3} < 1$ , i.e. if  $p_1$  is smaller than the atomic radius. If  $p_1$  is greater than the atomic radius, the cross-section is zero since by the original assumption all impacts are then excluded.

The above assumption is, of course, rather arbitrary. We can introduce a different point of view by requiring that the Fourier analysis breaks down when the field strength becomes greater than some critical value. It seems reasonable to take this critical field strength to be the field at the "surface" of the electron; thus

$$(251) \quad E_{cr} = e/d^2 = m^2 c^4 / e^3.$$

This restriction leads to

$$(252) \quad \mu_1 = d \cdot (Z \frac{2}{3})^{1/2},$$

which is smaller than the previous value (248), at least for  $\frac{2}{3} > Z^{1/2}$ . In the present case it turns out that when there is no screening the cross-section increases to infinity for increasing  $\frac{2}{3}$ , but since our experiments are all with screened nuclei we cannot say that such a result is in contradiction with experiment.

Oppenheimer discusses Lagrangian functions that depend not only on the field strengths but also on their derivatives, and finds that the break-down now takes place earlier. If he introduces the  $n^{\text{th}}$  derivative in a non-linear manner

in the field equations -- that is, in a non-quadratic manner in the Lagrangian function -- he finds that

$$(253) \quad \mu_1 \sim d \cdot Z^{1/(n+1)} \int \xi^{(n+1)/(n+2)}.$$

In the limit  $n \rightarrow \infty$  which corresponds to an integral operator, this gives

$$\mu_1 \sim \int \xi d$$

which is just (248). Oppenheimer therefore regards this limit as coming from the entry, in the field equations, of non-linear terms depending on the derivatives of the field strengths.

Recently Nordheim in The Physical Review, 49 (1936), p. 189, has proposed the critical field strength

$$(254) \quad E_{cr} = \alpha \frac{e}{d^2} = \frac{1}{137} \frac{e}{d^2}.$$

This gives

$$(255) \quad \mu_1 = \Lambda (Z \int \xi \alpha)^{1/2}$$

and leads, for screened nuclei, <sup>to</sup> the cross-section

$$(256) \quad \sigma = 4\alpha Z^2 d^2 \log \left( \frac{1}{\xi} \alpha^{1/2} Z^{5/6} \right),$$

if  $\frac{e^{1/2}}{\xi} \alpha^{3/2} Z^{5/6} < 1$ , and to zero for the other case. However, it does not seem that the experiments are good enough to distinguish between the proposals of Oppenheimer and Nordheim, and there are no essentially new ideas in Nordheim's paper.

Casimir, in Helvetica Physica Acta 6 (1933), p. 287, has made calculations on the scattering of photons by electrons bound in atoms. It would seem that, in the case of photon energies large compared with the binding energy of the electrons, we could treat the electrons as free and apply the Klein-Nishina formula; but the calculations show that there enters an extra term that, relative to the Klein-Nishina part, is of the order  $(\alpha Z)^5 \log(\hbar\nu/mc^2)$ . For very large energies this term will be bigger than the Klein-Nishina term. There is, how-

ever, no contradiction with experiment since we do not find such large energy photons in practice.

Another application of the theory would be to cosmic rays. Here we would obtain theoretically so large a cross-section for fast electrons that they could not be responsible for the penetrating components of the cosmic rays. It is not easy to decide experimentally whether these components are electrons or heavy particles, but the energy distribution of the secondary particles seems to be in favor of the first possibility. If this were the case, the present theory of energy loss would be completely wrong for such fast electrons.

It seems on the whole that the characteristic logarithmic terms of the present theory are not correct for energies of the order  $(m c^2 / \alpha)$ , although for wave-lengths of the order of the Compton wave-length they give no bad results. The above considerations are not sufficient to show that the cross-sections should be smaller, but they do show that the present theory is incorrect and that it is the principle of superposition that is probably at fault.

### On Polarization Effects in Homogeneous Electric and Magnetic Fields

(Reported by W. Pauli)

We shall discuss a new paper by Heisenberg and Euler that is to appear in Zeitschrift für Physik.

The paper deals with a particular application of the general theory of the polarizability of the vacuum. The case considered is that in which the external electromagnetic field is homogeneous and in which there are initially no electrons or positrons. The discussion is restricted to the case of unquantized wave func-

tions and for this case the correction to the Maxwellian Lagrangian function due to the pair generation is obtained.

The general procedure for subtracting the singularities has already been explained in this seminar. The mathematical density matrix (see p. 58, Eq. (133), (134)) is given by

$$\begin{aligned} (x', \rho' | T | x'', \rho'') &= \sum_{E_n < 0} v_n^*(x'', \rho'') v_n(x', \rho') = \\ &= \frac{1}{2} \left( \sum_{E_n < 0} - \sum_{E_n > 0} \right) v_n^*(x'', \rho'') v_n(x', \rho') + \frac{1}{2} \delta_{\rho', \rho''} \delta(x' - x''), \end{aligned}$$

where the  $v$ 's form a complete orthogonal set of eigen-functions of the Dirac equations for the given external field. The first subtraction term  $S_0$  involves the matrix corresponding to  $T$  in the case when no forces are present, while the second subtraction term  $S_1$  involves the external field quadratically in the field strengths and linearly in their derivatives. After the subtraction we have to proceed to the limit  $x \equiv x' - x'' \rightarrow 0$ .

In the present case we are interested in an external field that is static and homogeneous. This means that we neglect all terms containing the derivatives of the field strengths as factors, but we place no restriction necessarily on the magnitudes of the field strengths. We compute the quantum mechanical energy density corresponding to the transformation of matter into radiation and of radiation into matter and express this as a function of the initial field strengths. The total energy density of the electro-magnetic field in the vacuum is now obtained by adding the energy density so obtained to the ordinary energy-density of the Maxwell theory

$$(257) \quad U_M = \frac{1}{8\pi} (E^2 + B^2).$$

Now the energy density is not a relativistic invariant since it is really the 4-4-component of a second order tensor. However, the Lagrangian function  $L$  which is a true relativistic scalar can be defined in terms of  $E$ ,  $B$  and  $U$ . We have

$$(258) \quad U = \left( \vec{E} \cdot \frac{\partial L}{\partial \vec{E}} \right) - L,$$

where  $L$  is a function of  $\vec{E}$  and  $\vec{B}$ . The vectors  $\vec{D}$  and  $\vec{H}$  can now be defined in the usual manner by

$$(259) \quad \vec{D} = \frac{\partial L}{\partial \vec{E}}, \quad \vec{H} = - \frac{\partial L}{\partial \vec{B}}.$$

Since the whole subtraction is made in a relativistic manner, the  $L$  that we shall ultimately obtain will be a relativistic invariant. It will thus be a function of the two invariants

$$(E^2 - B^2), \quad (E \cdot B)^2,$$

because these are the only independent scalars that do not involve the derivatives of the field strengths. The reason for taking the square in the second is that without it we would not have invariance under reflections. Since  $L$  will depend only on two functions of the field strengths, it will suffice if we carry out the calculations for the case in which the electric and magnetic fields are parallel.

Before we give the result for this case we shall consider in more detail the case in which only a magnetic field is present. This case is much simpler than the more general case, and yet shows up the mechanism of the calculations. There is an essential physical difference between electric and magnetic fields that we shall discuss later.

The Dirac wave functions for a homogeneous magnetic field were first discussed by Rabi in Zeitschrift für Physik, 49 (1928), p. 507.

In this case we can choose the gauge so that

$$(260) \quad \mathcal{A}_y = 0, \quad \mathcal{A}_z = \frac{e}{c} |B| y,$$

and then the Dirac equations will not contain  $x$  or  $z$  explicitly. We may therefore take plane wave solutions for the  $x$ - and  $z$ -directions and write

$$(261) \quad \Psi_{np} = e^{\frac{i}{\hbar} (p_x x + p_z z - E t)} u_{np}(y),$$

where  $p_x$ ,  $p_z$  and  $E$  are numbers and not operators.

The Dirac equations are now

$$(262) \quad (E + \alpha_1 p_x + K) \Psi = 0,$$

where

$$(263) \quad K = -\alpha_2 i \hbar \frac{\partial}{\partial y} - \alpha_3 \left( -p_z - \frac{e}{c} |B| y \right) + \beta m c.$$

The operator  $K$  has two interesting properties. It anticommutes with  $\alpha_1$ ,

$$(264) \quad \alpha_1 K + K \alpha_1 = 0,$$

and its square has the simple form

$$(265) \quad K^2 = -\hbar^2 \frac{\partial^2}{\partial y^2} + \hbar^2 \sigma_1 \frac{e|B|}{c} + \left( p_z + \frac{e}{c} |B| y \right)^2 + m^2 c^2,$$

where  $\sigma_1$  is  $(-i \alpha_2 \alpha_3)$ .

Now we use the following representation of the matrices  $\sigma_1$ ,  $\alpha_1$ , and  $K$ :

$$\alpha_1 = \begin{pmatrix} +1 & & 0 \\ & -1 & \\ 0 & & +1 \\ & & & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} +1 & & 0 \\ & +1 & \\ 0 & & -1 \\ & & & -1 \end{pmatrix},$$

$$K = \sqrt{K^2} \cdot \begin{pmatrix} & & -i \\ 0 & +i & \\ & -i & \\ +i & & 0 \end{pmatrix}.$$

Then we have

$$(266) \quad K = \sqrt{m^2 c^2 + \frac{e}{c} |B| \hbar} (2n+1 + \sigma_1),$$

where we can treat  $\sigma_1$  as the pair of possible values  $\pm 1$ .

This operator is essentially the operator corresponding to the harmonic

oscillator in the Schrödinger theory and so, if we introduce the new variable  $\eta$

by

$$(267) \quad \eta = \gamma \sqrt{\frac{\hbar c}{e|B|}} - \frac{c p_z}{e|B|},$$

we can write the dependence of  $\Psi$  on  $y$ , namely the part  $u_{np}$  in (261), as

$$(268) \quad u_{np}(y) = H_n(\eta) e^{-\eta^2/\gamma} \cdot c_{np}.$$

The energy is related to  $n$  and  $p_x$  by

$$(269) \quad \frac{1}{c} E_n = \sqrt{p_x^2 + K^2} = \sqrt{m^2 c^2 + p_x^2 + \frac{e|B| \hbar}{c} (2n+1 + \sigma_1)}.$$

The  $\psi$ 's are normalized by the condition

$$(270) \quad \sum_{\rho} |c_{n\rho}|^2 \int_{-\infty}^{\infty} |u_n(y)|^2 dy = 1.$$

In order to give small weight to the higher energy components of the wave-packet Heisenberg introduced a factor

$$(271) \quad e^{-\alpha (E^2/c^2 - m^2c^2)}$$

where  $\alpha$  is a constant. For the case we are investigating we shall introduce the factor

$$(272) \quad e^{-\alpha \left[ p_x^2 + \frac{e|B|\hbar}{c} (2n+1+\sigma_1) \right]}.$$

Merely for the sake of simplicity we take  $\alpha$  instead of  $x \equiv x' - x''$  as our extra parameter. In the end we have to go to the limit  $\alpha \rightarrow 0$ .

If we limit the x- and z-directions by the usual box, we can sum over the states for these directions and obtain the energy density in the form

$$(273) \quad U = -\sum_{n=0}^{\infty} \sum_{\sigma_1=\pm 1} \int_{-\infty}^{\infty} \frac{dp_x}{\hbar} \frac{dp_z}{\hbar} u_n^2(y) \sqrt{m^2c^2 + p_x^2 + \frac{e|B|\hbar}{c} (2n+1+\sigma_1)} e^{-\alpha \left[ p_x^2 + \frac{e|B|\hbar}{c} (2n+1+\sigma_1) \right]}.$$

We introduce  $E_{cr} = \frac{m^2c^3}{e\hbar} = \frac{e^2}{\hbar c} \cdot \frac{e}{d^2}$  with  $d = \frac{e^2}{mc^2}$ , and write

$$(274) \quad b = |B|/E_{cr},$$

so that b is the value of  $|B|$  in terms of the limiting electric field strength.

Then we have, after integrating with respect to  $p_x$ ,

$$(275) \quad U = \left( \frac{mc}{\hbar} \right)^3, mc^2, \pi, \sum,$$

where

$$(276) \quad -\sum = 4b \left[ \int_{-\infty}^{\infty} dx \sum_{n=0}^{\infty} \left( \sqrt{x^2 + 1 + 2bn} \cdot e^{-\varepsilon(x^2 + 2bn)} \right) - \frac{1}{2} \int_{-\infty}^{\infty} dx \cdot \sqrt{x^2 + 1} \cdot e^{-\varepsilon x^2} \right],$$

$\varepsilon$  being written for  $(\alpha/mc)$ . The subtraction of the second integral in this formula is not part of the general subtraction formalism for the removal of singularities. It arises from the fact that when we took the  $(2n+1)$  and the  $\sigma_1 = \pm 1$  together as one coefficient, all the values of  $2n$  occurred twice except the value

$n = 0$ , and it is precisely a half of the integral for the case  $n = 0$  that is subtracted off in (276).

If  $b$  is small we can replace the summation over  $n$  by an integral and it is easily found that this corresponds exactly to the case of no external forces.

We have in fact

$$(277) \quad -\Sigma_0 = 2 \int_{-\infty}^{\infty} \int_0^{\infty} dx d\rho \sqrt{x^2 + 1 + \rho} \cdot e^{-\varepsilon(x^2 + \rho)}$$

For small fields we may use Euler's summation formula which gives the difference between a sum and the related integral. The first term is

$$(278) \quad -\Sigma_1 = -\frac{1}{12}(2b)^2 \cdot 4 \int_0^{\infty} \left( \frac{1}{2\sqrt{x^2+1}} - \varepsilon \sqrt{x^2+1} \right) e^{-\varepsilon x^2} dx.$$

This is infinite for  $\varepsilon = 0$  and corresponds to the subtraction term that refers to the infinite constant polarizability that is independent of the field strength.

The next term is finite for  $\varepsilon \neq 0$ :

$$(279) \quad \lim_{\varepsilon \rightarrow 0} \Sigma_2 = \frac{-b^4}{45}.$$

We seek an exact expression for the regular function

$$(280) \quad \mathcal{Y} = \lim_{\varepsilon \rightarrow 0} [\Sigma - \Sigma_0 - \Sigma_1].$$

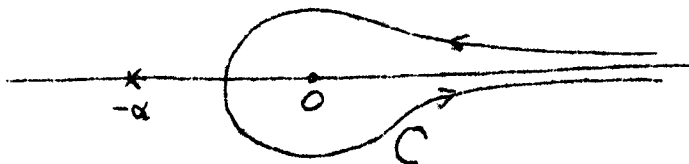
This may be obtained in terms of the generalized zeta function defined by

$$(281) \quad \zeta(s, a, \alpha) = \sum_{n=0}^{\infty} \frac{e^{-\alpha n}}{(a+n)^s}.$$

It is known that we have

$$(282) \quad \zeta(s) = \frac{-\Gamma(1-s)}{2\pi i} \int_C (-t)^{s-1} \frac{e^{-at}}{1-e^{-(t \cdot a)}} dt,$$

where the path of integration is as shown:



For our purposes we need the case  $s = -\frac{1}{2}$  for the part  $\Sigma$ .

It turns out that the subtraction terms correspond to the residue of this function at the pole  $t = -\alpha$ . If we expand in the neighborhood of the pole  $t = -\alpha$  we find, on letting  $\epsilon$  tend to zero, that

$$(283) \quad \mathcal{F} = \int_0^{\infty} \frac{dt}{t^3} e^{-t} \left\{ bt \coth bt - 1 - \frac{1}{3} b^2 t^2 \right\},$$

where the  $(-1)$  in the brackets is the contribution of the field-free term and the  $(-\frac{1}{3} b^2 t^2)$  corresponds to the subtraction term  $\Sigma$ , that depends on the field. The whole bracket is of the order  $t^4$  for small  $t$  so that the  $1/t^3$  is overcompensated.

To obtain the asymptotic behavior of  $F$  for  $b \gg 1$  it is convenient to introduce  $\beta = \frac{1}{b}$ . Then (283) can be put in the form

$$\mathcal{F} = \frac{1}{\beta^2} \int_0^{\infty} \frac{dv}{v^2} e^{-\beta v} \left\{ \frac{1}{e^v - 1} + \frac{1}{2} - \frac{1}{v} - \frac{v}{12} \right\}.$$

If we now differentiate the integrand with respect to  $\beta$  we can express  $\mathcal{F}$  as

$$(284) \quad \mathcal{F} = \frac{1}{\beta^2} \left\{ \frac{1}{12} \log \beta + C - \int_0^{\beta} dx \left\{ \log \Gamma(x) - (x - \frac{1}{2}) \log x + x - \frac{1}{2} \log 2\pi \right\} \right\};$$

the constant  $C$  (Compare Whittaker and Watson, Modern Analysis, p. 249) is given by

$$(285) \quad C = \lim_{\beta \rightarrow 0} \left\{ \beta^2 \mathcal{F} - \frac{1}{12} \log \beta \right\} = \int_0^{\infty} \frac{dv}{v^2} \left\{ \frac{1}{e^v - 1} - \frac{1}{v} + \frac{1}{2} - \frac{v}{12} e^{-v} \right\} = \\ = \frac{1}{2} \left\{ \frac{\gamma}{6} + \sum_{n=2}^{\infty} \frac{S_n}{(n+1)(n+2)} \right\} = 0.401\dots,$$

where

$$(286) \quad \left\{ \begin{array}{l} S_n = \sum_{\gamma=1}^{\infty} \frac{1}{\gamma^n} \\ y = 0.577\dots \end{array} \right. \quad \text{and} \quad \text{is Euler's constant.}$$

For  $b \gg 1$  we now have

$$(287) \quad \mathcal{F} \sim - \left\{ b^2 \left[ \frac{1}{12} \log 2b - C \right] + \frac{b}{4} \left[ \log 2b + 1 - \log 2\pi \right] + \right. \\ \left. + \frac{1}{8} \log 2b - \frac{1}{4} \left( \frac{\gamma}{2} - \frac{3}{4} \right) \right\},$$

or

$$(288) \quad G_{\nu} = \left\{ \ell^2 \left[ \frac{1}{17} \log \ell - 0.191 \right] + \frac{\ell}{4} \left[ \log \ell - 0.145 \right] + \frac{1}{8} \log \ell + 0.202 \right\}.$$

Let us now consider the case discussed by Heisenberg and Euler in which the external field consists of parallel homogeneous electric and magnetic intensities. Exact eigen-functions for this case have been obtained by Sauter in Zeitschrift für Physik, 69 (1931), p. 742. But in the present case we meet a physical complication that was absent from the case of a purely magnetic field since a homogeneous electric field, even if very weak, will induce pairs, while a homogeneous magnetic field has no such power. This is connected with the fact that a homogeneous magnetic field exerts a force perpendicular to the motion of a classical electron and thus cannot alter its kinetic energy. If we have a homogeneous electric field in the x-direction given by the potential

$$\phi = -e \mathcal{E} |x|,$$

then classically the domain where the electron is present is limited only by the condition

$$E - e \mathcal{E} |x| > mc^2$$

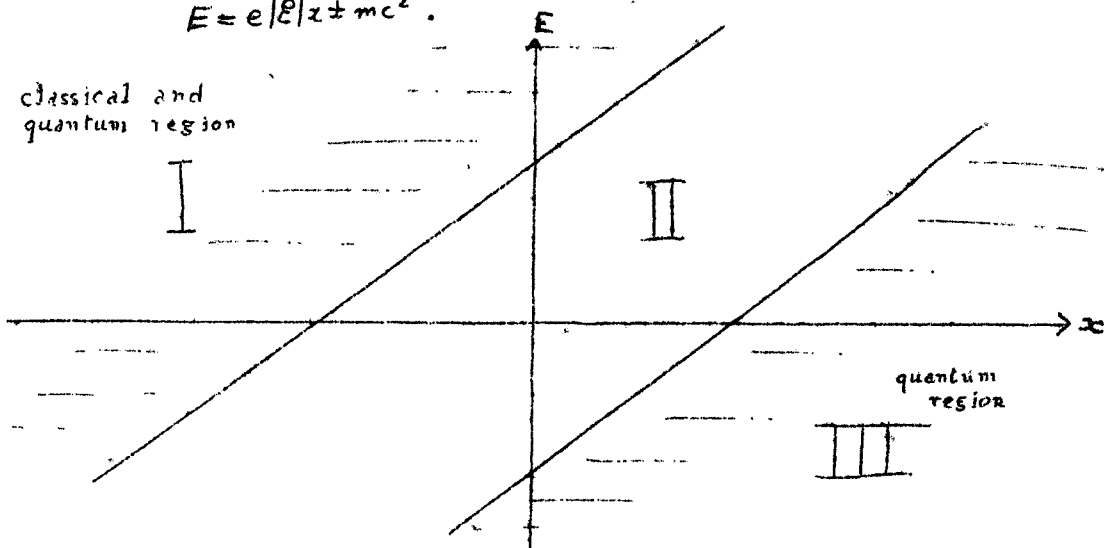
while in quantum mechanics we also have the different domain

$$E - e \mathcal{E} |x| < -mc^2.$$

These regions are shown in the figure. The boundaries of the regions are the

lines

$$E = e \mathcal{E} |x| \pm mc^2.$$



Thus a mere translation along the x-axis can carry us from any region to any other, and it turns out that even if we start with a wave function that is large only in region I, say, then after a time we shall find that it has seeped through to region III as if region II were a Gamow barrier. This process corresponds to the Klein paradox and the transparency coefficient for these transitions is given by

$$D = e^{- (E/E_{cr}) \pi}$$

It is thus not possible to make a sharp distinction between positive and negative energy states. But if the electric intensity is small compared to the critical electric intensity at the surface of the classical electron, the pair production will not occur with great frequency and so an approximate separation of positive and negative energy states will be possible. Thus if we write

$$(289) \quad |\mathcal{E}|/E_{cr} = a$$

we must take  $a \ll 1$  in order to be sure that the calculations will have meaning.

Heisenberg and Euler begin with such an approximate separation of the positive and negative energy states and make use of the relativistic invariance in the manner already explained. The calculations are very complicated and lead to the result

$$(290) \quad \mathcal{J} = \int_0^{\infty} \frac{dt}{t^3} e^{-t} \left[ at \cot at, bt \coth bt - 1 - \frac{t^2}{3} (b^2 - a^2) \right].$$

This can be written as

$$(291) \quad \mathcal{J} = \int_0^{\infty} e^{-t} \frac{dt}{t^3} \left[ iabt^2 \frac{\cosh(bt+ia)t + \cosh(b-ia)t}{\cosh(bt+ia)t - \cosh(b-ia)t} - 1 - \frac{t^2}{3} (b^2 - a^2) \right]$$

which is more convenient for our purposes since we can expand in even powers of the arguments and

$$\begin{aligned} (b+ia)^2 &= (b^2 - a^2) + 2i(ab) \\ (b-ia)^2 &= (b^2 - a^2) - 2i(ab) \end{aligned}$$

and since the result is real we shall have  $a$  and  $b$  entering only in the combination  $(b^2 - a^2)^2$  and  $(ab)^2$  which are just the combinations needed for the relativistically invariant expression of the result,

For  $a \ll 1$ ,  $b \ll 1$ , we have

$$(292) \quad \mathcal{F} = - \left[ \frac{(a^2 - b^2)^2 + 4(ab)^2}{45} \right]$$

which leads to the Lagrangian function (compare (279), obtained for the case  $a = 0$ )

$$(293) \quad \mathcal{L} = \frac{1}{8\pi} (E^2 - B^2) + mc^2 \left( \frac{mc}{\hbar} \right)^3 \pi \frac{(a^2 - b^2)^2 + 4(ab)^2}{45} + \dots$$

This corresponds exactly to the result obtained by Euler and Kockel given in eqs. (142), (143) on p. 45, as can be verified at once when account is taken of the fact that Euler and Kockel use  $\hbar$  and not  $h$  and also use a different value for the critical field. (The verification is much facilitated if the misprint in (142) is corrected! The definition of  $E_0$  should be  $E_0 = (m^2 c^4 / e^3)$ .) The difference in sign arises from the fact that Euler and Kockel give the change in the Hamiltonian while Heisenberg and Euler discuss the change in the Lagrangian; for an infinitesimal alteration in the field strengths we have actually

$$\delta U(D, B) = -\delta L(E, B).$$

Euler and Kockel expanded in powers of the frequency for large wave lengths and this corresponds to the assumption of homogeneous fields. The present calculations are not so complicated as those of Euler and Kockel, so that we have a real step forward as well as a confirmation of the result they obtained.

In the case of the electric field the integrand in (290) has poles at  $t = N\pi/a$ , the residues being of the order  $a^2 e^{-\pi/a}$ . The existence of these poles corresponds exactly to the real pair production in the electric field.

If we had started with quantized wave functions there would have been a big difference in the whole computation since we should have matrix elements corresponding to radiation and pair generation from the initial state, and this would again lead to infinities. Thus we do not know what our result means for the quantized case. We are not even sure whether it has meaning unless a  $\ll 1$ .

The connection with the Born theory is not very close. The Lagrangian function is quite different, the main difference being that there is, apparently, no direct connection with the question of the self-energy in the present theory. And, again, in the Born theory there is no pair production.

#### On the Neutrino Theory of Light

(Reported by M.H.L.Pryce)

The neutrino theory of light was initiated by L. de Broglie in a series of papers in Comptes Rendus, 195 (1932), p. 536, p. 577, p. 862; 197 (1933), p. 1377; 198(1934), p. 135; 199 (1934), p. 445, p. 1165; and 199 (1934), p. 813, the last with J. Winter. The theory has been further developed by P. Jordan in Zeitschrift für Physik, 93 (1935), p. 464; 98 (1936), p. 759, by R. de L. Kronig in Physica, 2 (1935), p. 491, p. 854, p. 965, and by O. Scherzer in Zeitschrift für Physik, 97 (1935), p. 725.

In the neutrino theory of light photons are regarded as physical consequences of the interaction of matter with neutrinos. Neutrinos are particles of spin one-half and rest mass zero. There are two types of neutrino transitions which correspond to the absorption of light; (1) Absorption of a neutrino and emission of another, or re-emission of the original neutrino, the two cases being

of course, indistinguishable; (2) Simultaneous absorption of two neutrinos. In each case the two neutrinos entering the reaction, or the two states of the same neutrino, must have exactly the same direction of motion since if the whole process is to be equivalent to the absorption of a photon the change in energy must be equal to the change in momentum multiplied by  $c$ . In the first case the energy of the photon is the difference of the two neutrino energies while in the second case it is the sum. Since the photon has zero spin the two neutrinos in the second case must have opposite spins.

De Broglie considered the Dirac equation for the case in which the rest mass is taken to be zero. Let  $\varphi, \psi$  be two solutions of this equation. Then we can form a set of six quantities from them that will transform like a six-vector and which we can identify with the electromagnetic six-vector. If we write

$$(294) \quad \begin{cases} E \sim i\varphi^\dagger \alpha_1 \alpha_4 \psi, \varphi^\dagger \alpha_2 \alpha_4 \psi, \varphi^\dagger \alpha_3 \alpha_4 \psi, \\ H \sim i\varphi^\dagger \alpha_2 \alpha_3 \alpha_4 \psi, \varphi^\dagger \alpha_3 \alpha_1 \alpha_4 \psi, \varphi^\dagger \alpha_1 \alpha_2 \alpha_4 \psi, \end{cases}$$

this six-vector  $(E, H)$  will satisfy the Maxwell equations provided  $\varphi$  and  $\psi$  refer to neutrinos that have the same direction of motion.

Let the common direction of motion be the  $z$ -axis. Then we may take a representation of the  $\alpha$ 's with  $\alpha_3$  diagonal:

$$(295) \quad \alpha_1 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} i & -i \\ -i & i \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 & -1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} & & & \\ & & & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}.$$

We consider the neutrino wave functions to be analyzed into Fourier components and take only those components that represent motion in the  $z$ -direction. We consider four solutions of the Dirac equations for zero rest mass of the forms

$$(296) \quad \Psi_I = \begin{pmatrix} \sum_{-\infty}^{\infty} A_k e^{2\pi i k y (\frac{z}{c} - t)} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_{II} = \begin{pmatrix} \sum_{-\infty}^{\infty} B_k e^{-2\pi i k y (\frac{z}{c} - t)} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\Psi_{III} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sum_{-\infty}^{\infty} C_k e^{2\pi i k y (\frac{z}{c} - t)} \end{pmatrix}, \quad \Psi_{IV} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sum_{-\infty}^{\infty} D_k e^{-2\pi i k y (\frac{z}{c} - t)} \end{pmatrix}.$$

These four  $\Psi$ 's contain all possible types of neutrino moving along the z-axis;

$\Psi_I$  refers to neutrinos having positive momentum and positive spin,  $\Psi_{II}$  to those having negative momentum and positive spin,  $\Psi_{III}$  to positive momentum and negative spin, and  $\Psi_{IV}$  to negative momentum and negative spin.

We now have

$$(297) \quad E_i = \frac{i}{2} \left[ e^{i\varphi} \Psi_I \alpha_i \alpha_4 \Psi_{III} + e^{-i\varphi} \Psi_{III} \alpha_i \alpha_4 \Psi_I + \right. \\ \left. + i e^{i\varphi} \Psi_{II} \alpha_i \alpha_4 \Psi_{IV} + i e^{-i\varphi} \Psi_{IV} \alpha_i \alpha_4 \Psi_{II} \right],$$

etc., where  $\varphi$  is a phase.

The corresponding Fourier series for  $\vec{E}$ ,  $\vec{H}$  are now given by

$$(298) \quad \begin{cases} \vec{E} = \sum_{\vec{k}} \left[ \vec{F}(\vec{k}) e^{2\pi i \vec{k} \cdot \vec{r} (\frac{z}{c} - t)} + \vec{F}^*(\vec{k}) e^{-2\pi i \vec{k} \cdot \vec{r} (\frac{z}{c} - t)} \right], \\ \vec{H} = \vec{k} \times \vec{E}, \end{cases}$$

where

$$(299) \quad \begin{cases} F_1(\vec{k}) = i \left[ \cos \varphi \sum_{-\infty}^{\infty} A_k C_{k-e} + \sin \varphi \sum_{-\infty}^{\infty} B_k D_{k-e} \right] e^{-2\pi i k y t} \\ F_2(\vec{k}) = i \left[ \sin \varphi \sum_{-\infty}^{\infty} A_k C_{k-e} - \cos \varphi \sum_{-\infty}^{\infty} B_k D_{k-e} \right] e^{-2\pi i k y t} \\ F_3(\vec{k}) = 0. \end{cases}$$

From (298) we see at once that  $\vec{H}$  is perpendicular to  $\vec{E}$  and to the direction of

motion  $\vec{k}$ .

The theory of De Broglie was concerned mainly with the possibility of linearizing the wave-equation for photons and of obtaining in terms of the solutions of the linearized equations a six-vector that would represent the electromagnetic intensity six-vector.

Jordan and Kronig are concerned with a different aspect of the theory. They start with the quantized wave functions of the neutrinos and build up from these the quantized wave functions of the photons. The artificiality caused by the need for exact equality of direction of the two neutrinos representing a photon remains, of course, in the Jordan-Kronig formulation. Jordan has considered the simplified problem of the absorption and emission of light in one dimension and here this artificiality is not so prominent. We may take the direction of the neutrinos to be from left to right along the x-axis, and can separate them into two classes having opposite spins. Then one class can be represented by a wave function  $\phi$  and the other by a wave function  $\chi$ . If we assume that the wave functions are periodic with periodicity  $d$  we may express  $\phi$  and  $\chi$  as Fourier series

$$(300) \quad \begin{cases} \phi = \sum_{k=0}^{\infty} a_k e^{i k v_1 (t - \frac{x}{c})} \\ \chi = \sum_{k=0}^{\infty} c_k e^{i k v_1 (t - \frac{x}{c})} \end{cases}$$

where  $v_1 = c/d$ .

In the same way we may represent the corresponding photons by a wave function  $\psi$  expressible as the Fourier series

$$(301) \quad \psi = \sum_{k=1}^{\infty} b_k e^{i k v_1 (t - \frac{x}{c})}$$

The photons (301) are not to be regarded as existing in addition to the neutrinos (300). They are merely the expression of a part of the neutrino field. We

must therefore express the Fourier coefficients  $b_k$  of the photons in terms of the Fourier coefficients  $a_k, c_k$  of the neutrinos. Before we discuss this we shall consider the commutation rules for the neutrino amplitudes. The neutrinos obey the Fermi-Dirac statistics. We must therefore introduce commutation rules appropriate to these statistics. We make the convention

$$(302) \quad a_k^+ = a_{-k}, \quad c_k^+ = c_{-k} \quad (k \neq 0)$$

and take the commutation rules to be

$$(303) \quad \begin{cases} a_k a_l + a_l a_k = \delta_{k, -l}, \\ c_k c_l + c_l c_k = \delta_{k, -l}, \\ a_k c_l + c_l a_k = 0. \end{cases}$$

These hold for  $k, \overset{-l \neq 0}{\wedge} \overset{\wedge}{l}$ . To extend them we write

$$(304) \quad \begin{cases} \sqrt{2} a_0 = \alpha_0 + \alpha_0^+ \\ \sqrt{2} c_0 = \gamma_0 + \gamma_0^+ \end{cases}$$

and then the  $\alpha_0, \gamma_0$  fit into the scheme (303).

Since only pairs of neutrinos of opposite spin can represent photons, Jordan relates the b's to the a's and c's by the equation

$$(305) \quad b_k = \frac{i}{\sqrt{|k|}} \sum_{-\infty}^{\infty} a_l c_{k-l}.$$

In view of the requirement of relativistic invariance it does not seem possible to take a simple relationship essentially different from this. It takes account of the conservation laws and includes all the possible types of interaction that we would expect to represent photon emission or absorption; for  $a_k, c_k$  represent the creation of neutrinos in the  $k$  state if  $k$  is positive while, by (302), if  $k$  is negative they represent the annihilation of neutrinos. Thus if, for example, both  $\overset{\wedge}{k}$  and  $(k - \overset{\wedge}{l})$  happen to be positive, the operator  $a_{\overset{\wedge}{k}} c_{k - \overset{\wedge}{l}}$  corresponds to the creation of a pair of neutrinos of opposite spin, which in turn corresponds to the absorption of a photon.

It is essential to show that the photons defined by (305), (301) obey the Bose-Einstein statistics and it was shown by Jordan that the b's defined in (305) satisfy the commutation rules

$$(306) \quad \left\{ \begin{array}{l} b_h b_l^\dagger - b_l^\dagger b_h = \delta_{h,l}, \\ b_h b_l - b_l b_h = 0, \\ b_h^\dagger b_l^\dagger - b_l^\dagger b_h^\dagger = 0, \end{array} \right.$$

which are precisely the commutation rules that lead to the Bose-Einstein statistics.

There is an apparent difficulty connected with the fact that the total energy of the neutrinos is not the same as the total energy of the photons. We should, of course, expect this to be the case since it is essential for the theory that there be a reservoir of unattached neutrinos that are not paired off as photons, but a difference in the Hamiltonians for the photons and the neutrinos seems to imply a difference in the equations of motion and a falsification of the energy relations of photons. However, it is merely necessary that the difference between the two energies shall commute with all the b's since then the photonic processes predicted by the neutrino Hamiltonian will be identical with those predicted by the photon Hamiltonian so that, so far as the behavior of photons is concerned we shall not notice the difference between the Hamiltonians in experiments in which the unpaired neutrinos are not detected.

Now the occupation operators for the neutrinos are

$$L_h = a_h^\dagger a_h = a_{-h} a_h; \quad N_h = c_h^\dagger c_h = c_{-h} c_h,$$

these operators having the eigen-values 0, 1. The occupation operators for the photons are

$$M_h = b_h^\dagger b_h = b_{-h} b_h,$$

which have the eigen-values 0, 1, 2, 3, ... . Thus the energy of the neutrinos is given by

$$(307) \quad E = h\nu_1 \sum_1^{\infty} \frac{1}{2} (a_{-k} a_k + c_{-k} c_k),$$

and that for the photons is

$$(308) \quad W = h\nu_2 \sum_1^{\infty} \frac{1}{2} b_{-k} b_k;$$

E and W are not equal. Kronig shows that (E-W) actually commutes with the b's,

and we shall indicate the line of proof. We introduce the quantity

$$(309) \quad B = i \sum_{-\infty}^{\infty} a_l c_{-l}.$$

Then, by (305) and the commutation rules (303), we have, ignoring the numerical factors,

$$\begin{aligned} B b_k &\sim \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} a_l c_{-l} a_m c_{k-m} = \\ &= \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} (a_m a_l - \delta_{m,-l}) c_{-l} c_{k-m} = \\ &= \sum \sum a_m a_l c_{-l} c_{k-m} - \sum c_{-l} c_{k+l} = \\ &= \sum \sum a_m c_{k-m} a_l c_{-l} - \sum (a_l a_{k-l} + c_l c_{k+l}) = \\ &= b_k B - \sum (a_l a_{k-l} + c_l c_{k+l}). \end{aligned}$$

But

$$\sum_{l=-\infty}^{\infty} a_l a_{k-l} = \sum_{l'=-\infty}^{\infty} a_{k-l'} a_{l'} \quad (l' = k-l)$$

and therefore

$$\sum_{l=-\infty}^{\infty} a_l a_{k-l} = \frac{1}{2} \sum_{l=-\infty}^{\infty} (a_l a_{k-l} + a_{k-l} a_l) = 0, \text{ by (303).}$$

Similarly

$$\sum_{l=-\infty}^{\infty} c_l c_{k-l} = 0.$$

Hence

$$(310) \quad B b_k = b_k B.$$

Kronig now proves that

$$(311) \quad \frac{2}{h\nu_1} (E-W) = B^2 - \frac{1}{4},$$

from which it now follows that (E-W) commutes with the b's so that there is actually no difficulty with the fact that E and W are not equal.

Kronig has shown that B can take the values

$$(312) \quad B = \pm n + \frac{1}{2},$$

where n is an integer. Thus

$$(313) \quad E = W + \frac{1}{2}n(n+1).$$

Kronig discusses the derivation of the Planck radiation formula.

Since we have shown that the photons defined by double neutrino interactions obey the Bose-Einstein statistics it is rather redundant to prove in detail that we also obtain the Planck radiation formula. However, the derivation is not without interest and it introduces the concept of the thermal equilibrium of neutrinos with matter.

We can take a representation in which the operators  $L_k, N_k$ , that denote the numbers of neutrinos in each state, are diagonal. The corresponding operators  $M_k$  for the photons will not be diagonal in this representation owing to the non-commutation. But this does not constitute a real difficulty since in deriving a statistical formula such as the Planck formula we are interested only in the expectation values of the M's, and thus only in their diagonal elements. It turns out that the diagonal element corresponding to the  $k^{\text{th}}$  state is given by

$$(314) \quad \bar{M}_k = \frac{1}{2\hbar} \left[ L_0 N_k + L_k N_0 + L_k (1 - N_0) + N_k (1 - L_0) \right] + \\ + \frac{1}{\hbar} \sum_{m=1}^k L_m N_{k-m} + \frac{1}{\hbar} \sum_{m=1}^k \left[ L_{m+k} (1 - N_m) + N_{m+k} (1 - L_m) \right].$$

We now assume that the neutrinos are in thermal equilibrium with the walls so that, since the neutrinos obey the Fermi-Dirac statistics,

$$(315) \quad \bar{N}_k = \bar{L}_k = \frac{e^{-\hbar\nu_k \hbar / \hbar' T}}{1 + e^{-\hbar\nu_k \hbar / \hbar' T}},$$

where we have written  $k'$  for the Boltzmann constant to avoid confusion with the momentum k. If we take the volume of the enclosure to be large so that we may

replace the sums in (314) by integrals, we have, on writing

$$(316) \quad e^{h\nu/kT} = y, \quad \frac{m}{k} = \omega,$$

$$(317) \quad \begin{aligned} \bar{M}_k &= \int_0^1 \frac{y \, dy}{(1+y^\omega)(1+y^{(1-\omega)})} + 2 \int_0^{\infty} \frac{y^{\omega+1}}{(1+y^{(\omega+1)})(1+y^\omega)} = \\ &= \frac{y}{1-y} \left[ 1 + \frac{2 \log \left( \frac{2}{1+y} \right)}{\log y} \right] - \frac{y}{1-y} \left[ \frac{\log \left( \frac{2}{1+y} \right)}{\log y} \right] = \\ &= \frac{y}{1-y} = \frac{e^{h\nu/kT}}{1 - e^{h\nu/kT}}, \end{aligned}$$

which is Planck's distribution law.

The two integrals in (317) have a physical interpretation. If we have two neutrinos, of energy  $\omega h\nu$  and  $(1-\omega)h\nu$  the probability of finding one of them in thermal equilibrium is proportional to

$$\frac{e^{\omega h\nu/kT}}{1 + e^{\omega h\nu/kT}}$$

and of finding the other is

$$\frac{e^{(1-\omega)h\nu/kT}}{1 + e^{(1-\omega)h\nu/kT}}.$$

Thus the first integral is the total probability of finding two neutrinos simultaneously absorbed as a photon of frequency  $h\nu$ . The second integral, on the other hand, corresponds to the probability of a neutrino being absorbed and a different neutrino being simultaneously emitted in the same direction, the whole effect being once more that of the absorption of a photon of frequency  $h\nu$ . From the fact that a factor 2 multiplies the second integral, we must infer that the second process is twice as probable as the first.

In his second paper Jordan gives a better interpretation to some of Kronig's results. Instead of the quantized wave functions used by Kronig, he

introduces new wave functions whose quantized amplitudes  $\alpha_r, \gamma_r$  are given in terms of the quantized amplitudes  $a_r, c_r$  of Kronig's work by the relations

$$(318) \quad \begin{cases} \alpha_r = \frac{a_r + ic_r}{\sqrt{2}}, & \alpha_r^+ = \frac{a_r^+ - ic_r^+}{\sqrt{2}} = \frac{a_{-r} - ic_{-r}}{\sqrt{2}} = \gamma_{-r}, \\ \gamma_r = \frac{a_r - ic_r}{\sqrt{2}}, & \gamma_r^+ = \alpha_{-r}. \end{cases}$$

The relation (305) between neutrinos and photons is now given by

$$(319) \quad \sqrt{\hbar} b_r = \sum_1^{\infty} (\alpha_r^+ \alpha_{\hbar+r} - \gamma_r^+ \gamma_{\hbar+r}) - \sum_1^{\hbar-1} \alpha_r \gamma_{\hbar-r}, \quad \hbar > 0,$$

and the quantity  $B$  defined in (303) now takes the form

$$(320) \quad B = \sum_1^{\infty} (\alpha_r^+ \alpha_r - \gamma_r^+ \gamma_r) + a_0 c_0.$$

The commutation relations satisfied by the  $\alpha$ 's and  $\gamma$ 's are

$$(321) \quad \begin{cases} \alpha_r \alpha_s + \alpha_s \alpha_r = \delta_{-r,s}, \\ \gamma_r \gamma_s + \gamma_s \gamma_r = \delta_{-r,s}, \\ \alpha_r \gamma_s + \gamma_s \alpha_r = 0, \end{cases}$$

which are of the same form as before and show that the new quantized waves also obey the Fermi-Dirac statistics. The transition from the  $a$ 's and  $c$ 's to the  $\alpha$ 's and  $\gamma$ 's corresponds to the rotation of the directions of the spins of the fundamental spin states through a right angle.

If we call the neutrinos in states given by the  $\alpha$ 's positive neutrinos and those given by  $\gamma$ 's negative neutrinos, the number of positive neutrinos in the  $r^{\text{th}}$  state is given by the eigenvalues of the operator

$$N_r^{(+)} = \alpha_r^+ \alpha_r,$$

and the number of negative neutrinos in the  $r^{\text{th}}$  state by the eigenvalues of the operator

$$N_r^{(-)} = \gamma_r^+ \gamma_r.$$

So (320) can be written as

$$(322) \quad B - a_0 c_0 = \sum_1^{\infty} (N_r^{(+)} - N_r^{(-)}),$$

where  $a_0 c_0$  has the eigenvalues  $\pm \frac{1}{2}$ . This gives a physical meaning to B, for it shows that B measures the excess of positive over negative neutrinos.  $\mathcal{P}$ We have already seen that the photonic energy is

$$W = \sum_1^{\infty} \hbar \omega_R b_R^+ b_R$$

in natural units. The energy of the neutrinos can be expressed in terms of the  $\alpha$ 's and  $\gamma$ 's as

$$(323) \quad \begin{aligned} E &= \sum_1^{\infty} \hbar \omega_R (a_R^+ a_R + c_R^+ c_R) = \\ &= \sum_1^{\infty} \hbar \omega_R (\alpha_R^+ \alpha_R + \gamma_R^+ \gamma_R). \end{aligned}$$

With the present B we have

$$W = E - \frac{1}{2} B^2.$$

Thus the neutrino energy and the photon energy will be the same only if  $B = 0$  which requires that  $N^{(+)} = N^{(-)}$  so that the neutrinos and antineutrinos will be paired off. The other extreme case where there is no radiation but we still have neutrinos is given by  $W = 0$  so that  $E = \frac{1}{2} B^2$ .

It is possible to improve on Jordan's formalism. We retain the relations (318) but work only with the  $\alpha$ 's and  $\alpha^+$ 's, the range of  $r$  being  $(-\infty, +\infty)$ . For symmetry we introduce

$$\alpha_0^+ \alpha_0 = \frac{1}{2} (1 + \sigma), \quad \sigma^2 = 1.$$

Then we have, without exception for  $r = 0$ ,

$$(324) \quad \begin{cases} \alpha_r \alpha_s^+ + \alpha_s^+ \alpha_r = \delta_{r,s}, \\ \alpha_r \alpha_s + \alpha_s \alpha_r = 0. \end{cases}$$

The expression for  $b_k$  becomes simply

$$(325) \quad \sqrt{\hbar \omega_R} b_R = \sum_{-\infty}^{\infty} \alpha_r^+ \alpha_{R+r}.$$

We define the occupation operators by

$$(326) \quad n_r^{(+)} = \alpha_r^+ \alpha_r \quad ; \quad n_r^{(-)} = \alpha_{-r} \alpha_{-r}^+ \quad , \quad r > 0,$$

and assume that only a finite number of energy states are occupied. This means that the eigenvalues of the  $n$ 's are to be taken as zero for large  $r$ , and this ensures the convergence of the series for  $b_k$  for such states.

For  $k \neq 0$  it is easy to prove that  $[b_k^+, b_l] = 0$ . The case  $k = 0$ , however, involves an interesting point. We have

$$b_k (b_k b_k^+ - b_k^+ b_k) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} (\alpha_r^+ \alpha_{k+r} \alpha_{k+s}^+ \alpha_s - \alpha_{k+s}^+ \alpha_s \alpha_r^+ \alpha_{k+r}),$$

where  $k \neq 0$ . By means of the commutation rules (324) we may reduce this to

$$(327) \quad b_k (b_k b_k^+ - b_k^+ b_k) = \sum_{r=-\infty}^{\infty} (\alpha_r^+ \alpha_r - \alpha_{k+r}^+ \alpha_{k+r})$$

and apparently the right-hand side is zero! But actually, though the summation on the right converges it is the difference of two divergent series since  $\alpha_r^+ \alpha_r$  is unity and not zero for  $r \rightarrow \infty$ . If we were to write (327) with the  $\alpha$ 's in the order  $\alpha \alpha^+$  on the right, we would meet a similar condition for

$r \rightarrow +\infty$ . It is a formal matter to show that the sum on the right is actually  $k$ . The situation can be illustrated in terms of the limiting case of continuous  $r$ . The dotted line represents  $\alpha_{k+r}^+ \alpha_{k+r}$  while the solid line represents



$\alpha_r^+ \alpha_r$  The partial sums are the areas under the two curves, but though these are each infinite their difference is the shaded part of the diagram which has finite area since the two curves are identical in shape and merely separated by a translation along the  $r$ -axis.

If we write

$$(328) \quad \sum_1^{\infty} n_r^{(+)} = N^{(+)}, \quad \sum_1^{\infty} n_r^{(-)} = N^{(-)},$$

we may define B by

$$(329) \quad B = (N^{(+)} \pm N^{(-)} + \frac{1}{2}\sigma)$$

from which it at once follows that its eigenvalues are  $(n + \frac{1}{2})$  where n is any integer. The proof that B commutes with every  $b_k$  is now quite short, and the derivation of the relation between E and W is simpler than before. We omit the details.

Scherzer attacks the problem from a different angle. He considers that part of the Hamiltonian that refers to the interaction of the neutrinos with matter. His neutrinos are not quite the same as those previously considered since they have transverse polarizations just as if they were photons. This is not a very important matter, however. The interesting thing is that he obtains results that agree with those obtained from the quantum theory of light, with a few exceptions that can give no experimentally detectable effect.

The energy of interaction between matter and radiation can be written

$$(330) \quad W = \vec{\alpha} \cdot \vec{A},$$

where  $\vec{A}$  is the vector potential of the radiation field and  $\vec{\alpha}$  is the vector whose components are the Dirac matrices.

For "classical" light we have, assuming periodicity, a Fourier expansion of  $\vec{A}$  in the form

$$(331) \quad \vec{A} = 2e \sqrt{\frac{c}{V}} \sum_k \vec{a}_k \sqrt{\frac{N_k}{V_k}} \cos(\omega_k t + \frac{1}{2} n_k \cdot \vec{x}),$$

where  $\vec{a}_k$  is a unit vector in the direction of  $\vec{A}$ . There are two mutually orthogonal basic directions for  $\vec{a}_k$  corresponding to the two possible states of polarization of the light wave. The  $N_k$  are, as yet, still c-numbers.

We introduce the abbreviation

$$(332) \quad \gamma_R \equiv \vec{p}_R \cdot \vec{n}_R \vec{x}.$$

It is convenient for the later work to rewrite (331) in the form

$$(333) \quad \vec{A} = 2 \sqrt{\frac{2\pi}{c}} \sum_R \vec{a}_R \sqrt{N_R} \nu_R \cos(\Theta_R + \gamma_R) \sqrt{d\omega_R d\nu_R},$$

where  $c^3/V = \nu_R^2 d\omega_R d\nu_R$  and  $d\omega_R$  is the element of solid angle.

The advantage in using this form comes from the fact that we shall be interested in double processes for the neutrinos and shall thus have this expression entering quadratically.

The interaction obtained from (333) and (330) will contain one cosine term, but in the neutrino theory we deal always with pairs of neutrinos and thus we shall want two cosine terms. Scherzer takes for neutrinos

$$(334) \quad \vec{A} = 2 \sqrt{\frac{2\pi}{c}} \sum_{r,s} \vec{a}_r \sqrt{N_r} \cos(\Theta_r + \gamma_r) \cos(\Theta_s + \gamma_s) (\vec{a}_r \cdot \vec{a}_s) \sqrt{d\omega_r d\nu_r} \sqrt{d\omega_s d\nu_s} \delta(\vec{n}_r - \vec{n}_s),$$

where the factor  $(\vec{a}_r \cdot \vec{a}_s)$  ensures that only neutrinos with parallel polarizations can interact to give photons since if  $\vec{a}_r, \vec{a}_s$  are perpendicular this term vanishes. The  $\vec{n}_r, \vec{n}_s$  are unit vectors in the directions of the wave normals and the entry of the  $\delta$ -function prevents the combination of neutrinos having different directions of motion.

The corresponding interaction energy is given by (330) with the expression (334) used for  $\vec{A}$ .

The neutrinos obey the Fermi-Dirac statistics and we therefore impose the scheme of second quantization first discussed by Jordan and Wigner. It was shown by Jordan and Wigner that we can represent the quantized N's diagonally as matrices of the type

$$(335) \quad N_r = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and that the corresponding phase  $\Theta_r$  is given by

$$(336) \quad e^{i\Theta_r} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \epsilon, \text{ say,}$$

the following commutation rules then being satisfied:

$$(337) \quad \begin{cases} N_r \varepsilon_r = \varepsilon_r (1 - N_r) = a_r^+, \text{ say,} \\ \varepsilon_r N_r = (1 - N_r) \varepsilon_r = a_r, \text{ say.} \end{cases}$$

The  $a_r^+$  and  $a_r$  thus defined differ from the usual quantized amplitudes by not possessing the Jordan-Wigner sign function. This means they commute for  $r \neq s$ .

We have now

$$(338) \quad 2 N_r \cos(\Theta_r + \gamma_r) = a e^{-i\gamma_r} + a^+ e^{i\gamma_r},$$

the  $\gamma$ 's still being c-numbers. Thus we find

$$(339) \quad W = \frac{e}{2} \sqrt{\frac{\hbar}{c}} \sum_{r,s} (W_1 + W_2 + W_3) (\vec{a}_r \cdot \vec{a}_r) (\vec{a}_r \cdot \vec{a}_s) \sqrt{\delta(\vec{n}_r - \vec{n}_s)} \sqrt{d\omega_r d\omega_s d\nu_r d\nu_s} \cdot \varepsilon_r \varepsilon_s,$$

where

$$(340) \quad \begin{cases} W_1 = (1 - N_r)(1 - N_s) e^{-i(\gamma_r + \gamma_s)} \\ W_2 = N_r N_s e^{+i(\gamma_r + \gamma_s)} \\ W_3 = 2 N_r (1 - N_s) e^{i(\gamma_r - \gamma_s)} \end{cases}$$

$W_1$  corresponding to the spontaneous emission,  $W_2$  to the absorption, and  $W_3$  to the stimulated emission.

We can write this in analogy with the theory of light and find what the interaction of the light field with matter will be in terms of the neutrinos.

If  $b_k^+$ ,  $b_k$  are the quantized amplitudes of the light, we have

$$(341) \quad 2 \sqrt{N_R} \cos(\Theta_R + \gamma_R) = b_R e^{-i\gamma_R} + b_R e^{+i\gamma_R}.$$

If we now compare the two values (331), (334) of  $\vec{A}$ , we find, writing  $b_k^+ = b_{-k}$ ,

$$(342) \quad \frac{b_R}{\sqrt{|R|}} = \sum'_s \frac{a_s a_{R-s}}{|s(R-s)|},$$

the summation avoiding  $s = k$ .

Scherzer's paper implies that a neutrino has two states of polarization and that only similar polarizations can combine, their law of combination being (342). However, there seems to be a flaw in Scherzer's scheme since it can be shown that

$$(343) \quad [b_{\mathbf{k}}, b_{\mathbf{k}'}] \neq 0, \quad \mathbf{k} \neq \mathbf{k}',$$

so that the  $b$ 's do not satisfy the Einstein-Bose statistics. Thus the theory cannot be equivalent to the usual theory of light, and indeed, Scherzer finds that the stimulated emission given by his theory for the case in which one neutrino is absorbed and re-emitted in the same direction does not agree with the result obtained from the theory of light. He assumes that such a case should correspond to absorption of a photon, and even in the Jordan theory a field with one neutrino present does not correspond to a field with a definite number of photons present.

Remarks on Theories of Light

(By W. Pauli)

The quantity

$$B = \sum (N^{(+)} - N^{(-)})$$

introduced by Kronig and Jordan is an important operator. For if we start from the Dirac equations for the neutrino we cannot easily avoid the procedure of the theory of holes that makes the energy positive; i.e. we must separate the states into two parts corresponding to positive and negative energy and we must then replace  $N^{(-)}$  by  $(1 - N^{(-)})$  as already explained for the case of electrons and positrons. In the neutrino case we shall then find, in complete analogy with the electron-positron case, that there exists a "current" density vector satisfying the equation of continuity, and the corresponding 4-component will have both positive and negative eigenvalues.  $B$  is the volume-integral of this density and will be the direct analogue of the total electric charge, but we do not know what its physical meaning is when we are dealing with neutrinos. Naturally it

cannot be regarded as an electric charge, but it will imply that there is a real physical difference between a neutrino and an anti-neutrino. Whether the Fermi theory of  $\beta$ -emission requires that we treat neutrinos and anti-neutrinos as different we do not know.

The whole theory of Jordan and Kronig may be right or wrong, but even if we avoid entirely the idea that photons are manifestations of pairs of neutrinos we cannot, in the theory of neutrinos, avoid the operator  $B$ !

If we consider only processes of light emission and absorption we can simply add as an extra condition on the neutrino theory of light that  $B$  shall be zero. We shall then get a theory that is absolutely identical with the present theory of light; but to explain  $\beta$ -ray disintegration we require that  $B$  shall not be zero.

In the three-dimensional case, the whole neutrino theory of light becomes very artificial because only neutrinos having exactly the same direction can combine to give a photonic interaction with matter. This makes the whole formalism very inelegant. For instance, we shall have a conservation of  $B$  for every direction.

It is not easy to see how we can express the electromagnetic potentials in a gauge invariant way in terms of the  $\alpha$ 's and  $\gamma$ 's, though there is no necessary contradiction if we cannot do this. If we consider the interaction with matter, the potentials occur explicitly in the Hamiltonian. Therefore it would not be very satisfactory in this case to take some fixed gauge since this particular gauge would not be Lorentz invariant.

There are theories analogous to the neutrino theory of light in which, however, the particles of which the photons are composed are identified with the

electron and positron and not with the neutrino and anti-neutrino. We encounter here a serious difficulty since we must assume that in the first approximation the rest mass is zero and must try to understand the actual rest mass as a result of the interaction between particles. Bloch tried to do this, but without success.

Heisenberg has made the attempt to describe the external field in terms of the singularities of the mathematical density matrix of the theory of holes. The work was done some years ago, but has not been published.

Heisenberg argued that since the singularities of the mathematical density matrix depend on the external electro-magnetic field, if we know the type of singularity we can calculate the external field.

If we write  $p_0$  for  $(m^2 c^2 + p^2)^{\frac{1}{2}}$ , we may write the mathematical density matrix for the quantized wave functions in the momentum space as

$$\begin{aligned}
 (\vec{k}'', \rho'' | R | \vec{k}', \rho') = & -\frac{1}{2} \frac{\vec{\alpha} \cdot \vec{k}' + \beta m c}{k_0'} \delta(\vec{k}' - \vec{k}'') - \\
 (344) \quad & -\frac{1}{8} \left\{ \left[ 1 + \frac{\vec{\alpha} \cdot \vec{k}'' + \beta m c}{k_0''} - \frac{\vec{\alpha} \cdot \vec{\epsilon} \vec{\Phi}(\vec{k}' - \vec{k}'')}{k_0' + k_0''} \left( 1 - \frac{\vec{\alpha} \cdot \vec{k}' + \beta m c}{k_0} \right) \right] + \right. \\
 & \left. + \left[ 1 - \frac{\vec{\alpha} \cdot \vec{k}'' + \beta m c}{k_0''} + \frac{\vec{\alpha} \cdot \vec{\epsilon} \vec{\Phi}(\vec{k}' - \vec{k}'')}{k_0' + k_0''} \left( 1 + \frac{\vec{\alpha} \cdot \vec{k}' + \beta m c}{k_0} \right) \right] + \right. \\
 & \left. + \text{hermitean conjugate} \right\} + \\
 & + \text{higher order terms.}
 \end{aligned}$$

We see that for given  $(\vec{k}' - \vec{k}'')$  and large  $\vec{\Phi}$  we can calculate  $\vec{\Phi}$  from this and thus also the field strengths. To do this, however, we must introduce a limiting process that involves a function  $f(p, \epsilon)$  such that

$$(345) \quad \begin{cases} \int p \, d^3 p \, f(p, \epsilon) = 1, \\ \lim_{\epsilon \rightarrow 0} f(p, \epsilon) = 0. \end{cases}$$

As an example, we could take

$$(346) \quad f = c \varepsilon^2 e^{-\varepsilon k}$$

In terms of such a function the Fourier components of the electromagnetic potential are given by

$$(347) \quad \frac{e}{c} \varphi_{\rho}(P) = \lim_{\varepsilon \rightarrow 0} \frac{3}{16\pi} \int (k' / \alpha^{\rho} / k'') (k'', \rho'' / R / k'' + \rho, \rho') f(k'', \varepsilon) d^3 k'',$$

and the Fourier components of the corresponding field strengths by

$$(348) \quad \begin{cases} \frac{e}{c} E_{\rho}(P) = \lim_{\varepsilon \rightarrow 0} \frac{3}{8\pi i \hbar} \int \sum_{k'} (k' / \alpha^{\rho} \alpha^{\rho} / k'') k''_{\rho} (k'', \rho'' / R / k'' + \rho, \rho') f(k'', \varepsilon) d^3 k'', \\ H_{\rho}(P) = [P \times \varphi(P)] \end{cases}$$

The corresponding formulas in ordinary space are

$$(349) \quad \varphi_{\rho}(x') = \frac{3}{2} \frac{\hbar c}{e} \pi^2 \lim \int \sum_{\rho', \rho''} \alpha^{\rho} \alpha^{\rho'} (\rho', x' / R / \rho'', x'') D(x' - x'') |\vec{x}' - \vec{x}''|^2 dV'',$$

$$(350) \quad E_{\rho}(x') = 3 \frac{\hbar c}{e} \pi^2 \lim \int \left\{ \sum_{\rho', \rho''} \sum_{\rho, \rho'} (\alpha^{\rho} \alpha^{\rho'}) (\rho', x' / R / \rho'', x'') \times \right. \\ \left. \times \frac{\partial}{\partial x^{\rho}} [D(x' - x'') |\vec{x}' - \vec{x}''|^2] \right\} dV'',$$

where the limiting process is the tending of  $D(x)$  to the Dirac  $\delta$ -function in such a way that  $\lim D(x) = 0$  for  $x \neq 0$  while

$$\int_{-a}^{+a} D(x) dx = 1 \quad \text{for all } a > 0.$$

These expressions contain only the density matrix for a given time  $t$  so that we do not have to make use of a wave equation. It is essential, however, that the density matrix be singular since if it were regular the field obtained by the above process would be identically zero.

There is a similarity between this and the neutrino theory of light since  $R$  is expressed as a bilinear expression in the  $\psi$ 's and it is a bilinear composition of the neutrino spinor field  $\psi$ , satisfying the Jordan-Wigner commutation relations, that leads, in the neutrino theory, to the Bose-Einstein statistics.

Heisenberg tried, by introducing the commutation rules for E and H, to obtain a relation for  $e^2/\hbar c$  but this was not possible because of the ambiguity of the limiting process.

### The New Field Theory

(Reported by M. H. L. Pryce)

The theory we shall discuss was proposed by Born in The Proceedings of the Royal Society of London 143A (1934), p. 410, and has been developed in further papers with Infeld, and by Pryce and others.

We begin by discussing the derivation of the Maxwell electromagnetic field equations from a variational principle, since this will form a convenient introduction to the generalization that will follow.

In the Maxwell theory there is a magnetic vector potential  $\vec{A}$  and a scalar potential  $\phi$ , in terms of which the field strengths  $\vec{E}$ ,  $\vec{B}$  are given by

$$(351) \quad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \text{grad } \phi, \quad \vec{B} = \text{curl } \vec{A}.$$

We use  $\vec{B}$  instead of the more usual  $\vec{H}$  for reasons that will appear later.

These equations are equivalent to

$$(352) \quad \text{div } \vec{B} = 0, \quad \frac{\partial \vec{B}}{\partial t} + \text{curl } \vec{E} = 0$$

which form one part of the Maxwell equations. The other part is, for empty space

$$(353) \quad \text{div } \vec{E} = 0, \quad \frac{\partial \vec{E}}{\partial t} - \text{curl } \vec{B} = 0.$$

It is well-known that if we begin with the first set of equations (352) we can obtain the second set (353) from the variational principle

$$(354) \quad \delta \int \frac{1}{2} (E^2 - B^2) dv dt = 0,$$

where the variation of  $\vec{E}$  and  $\vec{B}$  is such that the equations (352) are satisfied.

This means that we may consider the variations of  $\vec{E}$  and  $\vec{B}$  as caused by variations of  $\vec{A}$  and  $\phi$ .

The formalism becomes clearer in the notation of the theory of relativity. We introduce the four-vector  $\mathcal{G}_i$  by

$$(355) \quad \mathcal{G}_i \sim (\vec{A}, -\phi)$$

and the antisymmetric second-order tensor  $f_{ij}$  by

$$(356) \quad \left\{ \begin{array}{l} f_{ij} = -f_{ji}, \\ f_{12}, f_{23}, f_{31} \sim E_1, E_2, E_3, \\ f_{13}, f_{31}, f_{12} \sim H_1, H_2, H_3. \end{array} \right.$$

We may now form the relativistic scalar

$$(357) \quad \dots \dots \dots F = \frac{1}{2} f^{ik} f_{ik},$$

and it can be easily shown that, in the Galilean case,

$$(358) \quad F = E^2 - B^2.$$

The variational principle (354) is thus

$$(359) \quad \delta \int \frac{1}{2} F \sqrt{-g} dx^1 dx^2 dx^3 dx^4,$$

where the  $\sqrt{-g}$  is necessary in order to give, in combination with the

$dx^1 dx^2 dx^3 dx^4$  an invariant volume element in accordance with Jacobi's theorem.

If we try to obtain a theory of electromagnetic mass in terms of the Maxwell equations, we encounter the serious difficulty that the total electrostatic energy of a point charge is infinite. Born decided that this difficulty could be avoided by a suitable change in the field equations, and that this change could best be made by keeping the relation between the potentials and the field strengths unchanged and altering only the Lagrangian function. We shall discuss first some of the general statements that can be made about this procedure before we consider

Born's particular choice of Lagrangian. Born has advanced various arguments to show that his Lagrangian is specially significant, but these arguments are of doubtful value and his Lagrangian is not elegant.

In addition to the scalar  $F$  defined in (358) we can form just one other scalar that does not involve the derivatives of the field strengths, namely  $(\vec{E} \cdot \vec{B})^2$ . In this form it is not obviously invariant since  $\vec{E}$  and  $\vec{B}$  are not vectors but merely parts of an antisymmetric tensor. To bring the invariance into prominence we introduce the completely antisymmetric tensor of the fourth rank  $j^{klmn}$  given by

$$(360) \quad j^{klmn} = \begin{cases} 0 & \text{if two or more indices are equal,} \\ +\frac{1}{2} \frac{1}{\sqrt{-g}} & \text{if } k, l, m, n \text{ form an even permutation of } 1, 2, 3, 4, \\ -\frac{1}{2} \frac{1}{\sqrt{-g}} & \text{if they form an odd permutation.} \end{cases}$$

That a quantity so defined is actually a tensor is a standard theorem of the tensor calculus. In terms of this tensor we can form the scalar

$$(361) \quad G = \frac{1}{4} \text{tr} f_{mn} j^{klmn}$$

which, in the Galilean case, reduces to

$$(362) \quad G = (\vec{E} \cdot \vec{B}).$$

Since  $j^{klmn}$  changes sign under an improper Lorentz transformation it is  $G^2$  that is a true scalar under reflections.

We may introduce the notion of the dual  $f^{*ik}$  of an antisymmetric tensor  $f_{ik}$  by the definition

$$(363) \quad f^{*ik} = j^{iklm} f_{lm},$$

which is such that

$$(364) \quad f^{**ik} = -f^{ik}.$$

In the vector notation, if  $f_{ik} \sim (\vec{E}, \vec{B})$  then  $f^{*ik} \sim (\vec{B}, \vec{E})$ . In terms of the

dual we now have

$$(365) \quad G = f^{*ik} f_{ik}.$$

If we wish to obtain a relativistically invariant theory we must take a Lagrangian that is invariant. And if we require gauge invariance we cannot have  $\phi_i$  entering except in the combination of its curl. Mia's theory used a Lagrangian that involved the  $\phi_i$  not in this combination and was therefore not gauge invariant. Thus for relativistic invariance and gauge invariance, if we assume that the derivatives of the  $f_{ik}$  do not enter the Lagrangian, we must take

$$(366) \quad \mathcal{L} = \mathcal{L}(F, G^2).$$

In the Maxwell theory we have simply

$$(367) \quad \mathcal{L} = \frac{1}{2} F.$$

We introduce the antisymmetric tensor  $p^{ik}$  by

$$(368) \quad 2 \frac{\partial \mathcal{L}}{\partial f_{ik}} = p^{ik},$$

the  $p^{ik}$  in the Maxwell theory being just  $f^{ik}$ , as is readily verified.

The equations (351) can be written as

$$(369) \quad f_{ik} = \frac{\partial \phi_k}{\partial x^i} - \frac{\partial \phi_i}{\partial x^k}$$

and we must take as our variational principle

$$(370) \quad \delta \int \mathcal{L}(F, G^2) \sqrt{-g} d\tau_4 = 0,$$

where  $d\tau_4$  is written for  $dx^1 dx^2 dx^3 dx^4$ .

On performing the variation we find

$$\begin{aligned} 0 &= \int \frac{\partial \mathcal{L}}{\partial f_{ik}} \delta f_{ik} \sqrt{-g} d\tau_4 = \\ &= \frac{1}{2} \int p^{ik} \left( \frac{\partial}{\partial x^i} \delta \phi_k - \frac{\partial}{\partial x^k} \delta \phi_i \right) \sqrt{-g} d\tau_4 = \\ &= \int p^{ik} \left( \frac{\partial}{\partial x^i} \delta \phi_k \right) \sqrt{-g} d\tau_4 \quad (\text{since } p^{ik} = -p^{ki}), \end{aligned}$$

and on integrating by parts and taking the variations to be zero at the boundary

of the region of integration we find that this gives

$$0 = - \int \frac{\partial (h^{ik} \sqrt{-g})}{\partial x^i} \delta q_k dT_4$$

so that we have the field equations

$$\frac{\partial}{\partial x^i} (\sqrt{-g} h^{ik}) = 0$$

or

$$(371) \quad h^{ik}_{,k} = 0.$$

If we introduce the vector notation by writing

$$(372) \quad \begin{cases} h_{14}, h_{24}, h_{34} \sim \vec{D}, \\ h_{23}, h_{31}, h_{12} \sim \vec{H}, \end{cases}$$

we find that these equations are

$$(373) \quad \text{div } \vec{D} = 0, \quad \frac{\partial \vec{D}}{\partial t} = \text{curl } \vec{H}.$$

In the Maxwell theory  $\vec{D} = \vec{E}$  and  $\vec{H} = \vec{B}$ , but in the general case the relationship is not so simple, being given by the definition (368) of  $p^{ik}$ . The equations (373) are analogous to the Maxwell equations for a medium whose dielectric constant and magnetic susceptibility are functions of the field strengths.

The field equations are now (369), (371), and (368). We can make them more symmetrical if we note that (369) can be replaced by

$$(374) \quad \frac{\partial f_{hl}}{\partial x^i} + \frac{\partial f_{li}}{\partial x^h} + \frac{\partial f_{ih}}{\partial x^l} = 0$$

which is the same as

$$(375) \quad \frac{\partial (f^{*ik} \sqrt{-g})}{\partial x^k} = 0.$$

Thus, starting with (375) and the Lagrangian (366), we have deduced the second set of field equations (371). We shall show that it is possible to begin with (371) and, by a suitable choice of Lagrangian, to deduce the first set (375).

We introduce

$$(376) \quad \mathcal{H} = \mathcal{L} - \frac{1}{2} f_{ik} p^{ik}$$

Then

$$(377) \quad \begin{aligned} \delta \mathcal{H} &= \delta \mathcal{L} - \frac{1}{2} \delta f_{ik} \cdot p^{ik} - \frac{1}{2} f_{ik} \cdot \delta p^{ik} = \\ &= \frac{1}{2} p^{ik} \delta f_{ik} - \frac{1}{2} \delta f_{ik} \cdot p^{ik} - \frac{1}{2} f_{ik} \cdot \delta p^{ik} = \\ &= -\frac{1}{2} f_{ik} \delta p^{ik}, \end{aligned}$$

so that if we regard  $\mathcal{H}$  as a function of the  $p^{ik}$  we have

$$\frac{\partial \mathcal{H}}{\partial p^{ik}} = -\frac{1}{2} f_{ik},$$

or

$$(378) \quad 2 \frac{\partial \mathcal{H}}{\partial p^{ik}} = -f^{ik},$$

which is analogous to (368). We have here merely performed the usual Legendre transformation.

Thus if we assume (371) we can infer the existence of an "anti-potential"

$\psi_i$  such that

$$(379) \quad p^{*ik} = \frac{\partial \psi_k}{\partial x^i} - \frac{\partial \psi_i}{\partial x^k}$$

and, from the variation principle

$$(380) \quad \delta \int \mathcal{H} \sqrt{-g} \, d\tau_4 = 0,$$

we shall now obtain (375).

We thus have two different methods for obtaining the field equations from a variational principle. We shall return to this matter when we consider the case of point charges.

We must now introduce the stress-energy-momentum tensor of the electromagnetic field. This is defined by

$$(381) \quad T_i^k = \mathcal{L} \delta_i^k - p^{lk} f_{li},$$

where we are now considering only the Galilean case. In the Maxwell theory this is

$$(382) \quad T_i^k = F \delta_i^k - f_{li}^k$$

and in this case the symmetry of  $T^{ik}$  is quite obvious. In the general case (381) the symmetry is not so easily seen, but it has been proved by Mie that if we begin with  $\mathcal{L} = \mathcal{L}(F, G^2)$  the tensor  $T^{ik}$  defined in (381) will always be symmetric.

We shall show that this tensor satisfies the conservation laws. These laws are contained in the statement

$$(383) \quad T_{,k}^{ik} = 0,$$

and their significance is best understood in terms of the physical meaning of the various components of  $T^{ik}$ . We may write

$$(384) \quad \frac{1}{4\pi} T^{ik} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} X_x & X_y & X_z & S_x \\ Y_x & Y_y & Y_z & S_y \\ Z_x & Z_y & Z_z & S_z \\ S_x & S_y & S_z & U \end{pmatrix} \end{matrix},$$

and then the X's, Y's, and Z's are the components of the ordinary stress tensor,  $\vec{S}$  is the Poynting vector measuring the flow of momentum, and U is the energy.

The conservation laws in the Galilean case, when expressed in this non-relativistic notation are just

$$(385) \quad \left\{ \begin{array}{l} \frac{\partial S_x}{\partial t} + \operatorname{div} \vec{X} = 0, \\ \frac{\partial S_y}{\partial t} + \operatorname{div} \vec{Y} = 0, \\ \frac{\partial S_z}{\partial t} + \operatorname{div} \vec{Z} = 0, \\ \frac{\partial U}{\partial t} + \operatorname{div} \vec{S} = 0. \end{array} \right.$$

The first three state that the rate at which momentum is leaving a given volume is equal to the total normal component of stress across its surface, while the fourth

shows that the rate at which energy is leaving is equal to the total normal flux of the momentum  $\vec{S}$  across the surface.

We wish, therefore, to show that (383) holds for the tensor defined in (381). We have

$$\frac{\partial}{\partial x^k} T_i^k = \frac{\partial \mathcal{L}}{\partial f_{ij}} \frac{\partial f_{ij}}{\partial x^k} \delta_i^k - f_{li} \left( \frac{\partial}{\partial x^k} p^{lk} \right) - p^{lk} \left( \frac{\partial}{\partial x^k} f_{li} \right),$$

but by the field equations (371) we have, in the Galilean case,

$$\frac{\partial}{\partial x^k} p^{lk} = 0,$$

and, from the definition (368) of  $p^{ik}$ , we now find

$$\begin{aligned} \frac{\partial}{\partial x^k} T_i^k &= \frac{1}{2} p^{lk} \left( \frac{\partial}{\partial x^i} f_{lk} \right) - p^{lk} \left( \frac{\partial}{\partial x^k} f_{li} \right) = \\ &= \frac{1}{2} p^{lk} \left[ \frac{\partial}{\partial x^i} f_{lk} + \frac{\partial}{\partial x^k} f_{il} + \frac{\partial}{\partial x^l} f_{ki} \right] \text{ (from the antisymmetry of } p \text{ and } f \text{),} \end{aligned}$$

which vanishes because of the field equations (374).

Thus we have shown that

$$(386) \quad \frac{\partial}{\partial x^k} T_i^k = 0$$

which is the Galilean form of (383).

Let us consider the spherically symmetric electrostatic case. Here we shall have  $\vec{B}$  and  $\vec{H}$  zero, and the field equations will reduce to

$$(387) \quad \text{div } \vec{D} = 0, \quad \text{curl } \vec{E} = 0,$$

while, since  $G^2$  vanishes in this case, we shall have

$$(388) \quad \mathcal{L} = \mathcal{L}(E^2)$$

so that

$$(389) \quad \vec{D} = - \frac{\partial \mathcal{L}}{\partial \vec{E}} = -2E \frac{\partial \mathcal{L}}{\partial F}.$$

Using spherical polar coordinates, we obtain, from the first equation of (387),

$$(390) \quad D_r = \frac{e}{r^2},$$

which shows that  $\vec{D}$  is the usual Coulomb field. The quantity  $e$  is here an integration constant. The second equation of (387) is identically satisfied. The

potential is not prominent in these equations but it can be obtained from the relation

$$(391) \quad \vec{E} = \text{grad } \phi,$$

and the value of  $\vec{E}$  is found from (390) and the relation (389) between  $\vec{D}$  and  $\vec{E}$ .

The energy density  $U$ , which is here just  $\mathcal{L} + \vec{E} \cdot \vec{D}$ , can now be obtained as a function of  $r$ .

The condition that the energy of the point charge shall be finite is the same as the condition that  $\vec{E}$  shall be regular when  $\vec{D}$  is infinite since  $\mathcal{L}$  contains only  $\vec{E}$  in the present case.

Born chose as his Lagrangian

$$(392) \quad \mathcal{L} = b^2 \left\{ \sqrt{1 - F/b^2 - G^2/b^4} - 1 \right\},$$

where  $b$  has the dimensions of an electric or magnetic field,  $c$  being taken as unity.

If we take units such that  $b = 1$ , we have

$$\mathcal{L} = \sqrt{1 - F - G^2} - 1$$

and for small fields this tends to the Maxwell value

$$\mathcal{L} = -\frac{1}{2} F.$$

In the spherically symmetric case we find in the Born case that

$$\mathcal{L} = \sqrt{1 - E^2}$$

so that

$$D = \frac{E}{\sqrt{1 - E^2}}$$

or

$$(393) \quad E = \frac{D}{\sqrt{1 + D^2}},$$

and since

$$D = \frac{e}{r^2}$$

we have

$$(394) \quad E_r = \frac{e}{\sqrt{e^2 + r^4}}.$$

As  $r$  tends to zero this tends to the finite value unity instead of the infinite value of the Maxwell case. But, unfortunately,  $E_r$  contains a singularity since the direction of  $E$  at  $r = 0$  is not determined. If we wish to avoid a singularity in  $E_r$  at  $r = 0$  we must have  $E_r \rightarrow 0$  as  $r \rightarrow 0$  and the fact that in the Born case  $E_r \rightarrow 1$  is merely due to the inconvenient type of Lagrangian discussed by Born. We could have any functional relation between  $\vec{E}$  and  $\vec{D}$  instead of (393) and the relativistic invariance would then give the dependence of  $\mathcal{L}$  on  $\vec{E}$  and  $\vec{H}$  correctly.

The energy density in the Born case is given by

$$(395) \quad 4\pi U = \frac{1}{\sqrt{1-E^2}} - 1 = \sqrt{\frac{e^2+r^4}{r^4}} - 1$$

so that the total energy of the field is,

$$(396) \quad W = \int_0^\infty U r^2 dr = \int_0^\infty (\sqrt{e^2+r^4} - r^2) dr,$$

which is finite. If we go back to conventional units so that  $b$  is no longer ignored, we find that

$$(397) \quad W = (1.2361\dots) e^{3/2} \sqrt{b}.$$

If we equate this energy to the  $mc^2$  of an electron we find

$$(398) \quad b = 9 \times 10^{15} \text{ e.s.u.}$$

which is a very large field. The largeness of  $b$  ensures that for ordinary fields the Maxwell equations will hold to a high order of accuracy.

The field equations are just sufficient to give a solution in every case when the world lines of the singularities are given. But they do not determine these world lines and we therefore need further conditions that will make the theory complete. These conditions will be, essentially, the equations of

motion of the singularities, and we may obtain them from a variational principle leading to the field equations. It is necessary to use the second form of the variational derivation of the field equations involving (380) since it is only when we use this method that we are able to introduce the variations of the coordinates of the singularities. This is done by replacing (371) by

$$(399) \quad \frac{\partial \rho^{ik}}{\partial x^k} = j^i,$$

$j^i$  being the charge-current density caused by the singularities, and then building the corresponding  $\vec{H}$ , which will now involve the coordinates of the singularities, and performing the variation in accordance with the condition (399).

We shall begin by assuming that the particles are represented by world lines whose spatial singularity is in the form of a  $\delta$ -function:

$$(400) \quad \vec{j}_n = \rho_n \vec{v}_n, \quad j_{i4} = \rho_n \delta(\vec{x} - \vec{x}_n),$$

where  $\vec{x}_n$  is the position of the  $n^{\text{th}}$  singularity. This is actually a relativistically invariant relation.

We assume that the effects of the singularities are additive so far as their charge-current densities are concerned. Thus we take

$$(401) \quad j_i = \sum_n j_{ni}.$$

We now write

$$(402) \quad \rho^{ik} = \rho_0^{ik} + \sum_n \rho_n^{ik},$$

where the  $\rho_n^{ik}$  satisfy

$$(403) \quad \frac{\partial \rho_n^{ik}}{\partial x^k} = j_n^i.$$

The manner in which we have broken up  $\rho^{ik}$  is, of course, by no means unique, but we shall see that the final result will be independent of the particular manner in which (402) is satisfied. We want the  $\rho_n^{ik}$  to be functions of  $\vec{x}_n$  and  $\dot{\vec{x}}_n$ ; that this is always possible is evident from the example

$$(404) \quad \begin{cases} \vec{D} = \frac{e}{r^2} \vec{r}, & (\vec{r} = |\vec{x} - \vec{x}_n|), \\ \vec{H} = \dot{\vec{x}}_n \times \vec{D}. \end{cases}$$

From (399), (401), (402), and (403), we have

$$(405) \quad \frac{\partial \rho_0^{ik}}{\partial x^k} = 0$$

which implies the existence of an anti-potential  $\psi_R$  such that

$$(406) \quad \rho_0^{ik} = \frac{\partial \psi_R}{\partial x^i} - \frac{\partial \psi_i}{\partial x^k}.$$

If we proceed at this stage to the variation of (407) subject to the conditions to be satisfied by the  $p^{ik}$ , we shall encounter a difficulty in connection with the entry of the  $\delta$ -functions. For since  $\vec{D}$  is infinite in the neighborhood of a point charge, we shall cause arbitrarily large changes in  $\vec{D}$  with small changes in the  $\xi$ 's. We must therefore smooth out the  $\delta$ -functions and suppose that the charge is spread out in a small volume that is constant along the world line. This finite size is not related to the structure already obtained for a charged particle; it is to be regarded, for example, as having a radius much smaller than the radius of the classical electron.

There are certain fundamental properties of  $\vec{D}$  and  $\vec{H}$  that we shall need.

We have, from the equations (403),

$$(408) \quad \text{div } \vec{D}_n = \rho_n(\vec{x} - \vec{\xi}_n),$$

$$(409) \quad \dot{\vec{D}}_n = \text{curl } \vec{H}_n - \rho_n(\vec{x} - \vec{\xi}_n) \cdot \dot{\vec{\xi}}_n,$$

where  $\rho_n(\vec{x} - \vec{\xi}_n)$  is the distribution function of the charge. From

(409) we can show that  $\vec{D}_n$  will be a function of  $(\vec{x} - \vec{\xi}_n)$  alone and not of its time derivative. For we have

$$\delta \dot{\vec{D}}_n = \frac{\partial \vec{D}_n}{\partial \dot{\xi}_{na}} \delta \dot{\xi}_{na} + \frac{\partial \vec{D}_n}{\partial \xi_{na}} \delta \xi_{na},$$

and a comparison with (409) shows at once that

$$(410) \quad \frac{\partial \vec{D}_n}{\partial \dot{\xi}_{na}} = 0.$$

Again, since  $\frac{\partial}{\partial t} = -\frac{\partial}{\partial r}$  when operating on  $\vec{D}$ , we have

$$(411) \quad \dot{\vec{D}}_n = -\left(\dot{\sum}_n \cdot \nabla\right) \vec{D}_n.$$

From (409) and (411) we now have

$$\text{curl } \vec{H}_n = \dot{\vec{D}}_n + \rho_n \dot{\sum}_n = \rho_n \dot{\sum}_n - \left(\dot{\sum}_n \cdot \nabla\right) \vec{D}_n,$$

and, by elementary vector analysis, since  $\dot{\sum}_n$  is independent of  $x, y, z$ , being a function of  $t$  alone, we have

$$\begin{aligned} \text{curl} \left( \dot{\sum}_n \times \vec{D} \right) &= \text{div } \vec{D}_n \dot{\sum}_n - \left( \dot{\sum}_n \cdot \nabla \right) \vec{D}_n = \\ &= \rho \dot{\sum}_n - \left( \dot{\sum}_n \cdot \nabla \right) \vec{D}_n, \end{aligned}$$

and a comparison with the previous result gives at once

$$\text{curl} \left( \vec{H}_n - \dot{\sum}_n \times \vec{D}_n \right) = 0$$

from which we obtain the important relation

$$(412) \quad \vec{H}_n = \dot{\sum}_n \times \vec{D}_n + \text{grad } \chi_n,$$

$\chi_n$  being an arbitrary function.

We must now consider what modification of the conservation laws is caused by the introduction of the charges. The computation is straightforward and leads to the result that in the Galilean case with which we have been working

$$\begin{aligned} (413) \quad \frac{\partial}{\partial x^h} T_i^h &= \frac{\partial}{\partial x^i} \mathcal{L} - j^{lh} \left( \frac{\partial}{\partial x^h} \beta_{li} \right) - \beta_{li} \left( \frac{\partial}{\partial x^h} v^{lh} \right) = \\ &= \frac{1}{2} j^{lh} \left( \frac{\partial}{\partial x^i} \beta_{lh} - 2 \frac{\partial}{\partial x^h} \beta_{li} \right) + j^l \beta_{il} = \\ &= j^l \beta_{il}, \end{aligned}$$

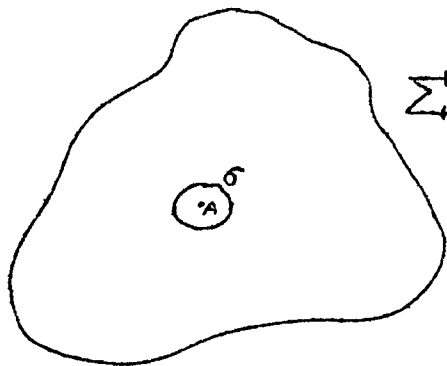
a result that is of exactly the same type as that in the Maxwell theory.

We must introduce the concept of the total force acting on a charge; first for the case of real point charges given in terms of  $\delta$ -functions. In this case  $T$  behaves like  $(1/r^2)$  and not like the  $(1/r^4)$  of the Maxwell theory, but in the Maxwell theory the field is always symmetrical and so the term in  $(1/r^4)$  vanishes on integration over a small closed region surrounding a charge. In the Born-Infeld theory we have no symmetry for  $\vec{D}$  in the neighborhood of a charge; for instance, in an extreme case we could find even



It is precisely this asymmetry that can lead to a non-vanishing limit to the value of the surface integrals of stress and momentum in (414), etc.

If we have a charge at  $A$  we may surround it by a small region  $\Sigma$  and may then omit the charge by cutting out



a still smaller region  $\sigma$  that contains it.

If we consider the conservation theorems as applied to the region bounded by  $\Sigma$  and  $\sigma$  we find

$$(414) \quad \frac{\partial}{\partial t} \int_{(\Sigma-\sigma)} S_r \, dv = - \int_{\Sigma} \vec{X} \cdot d\vec{\sigma} + \int_{\sigma} \vec{X} \cdot d\vec{\sigma}, \text{ etc.}$$

As  $\sigma \rightarrow 0$  the left-hand side tends to a finite value while the first integral on the right is not altered. Thus the second integral on the right has a definite value and we call this value the value of the x-component of the total force on the charge, which we denote by  $F_x$ :

$$(415) \quad \frac{\partial}{\partial t} \int_{(\Sigma)} S_x d\sigma = - \int_{\Sigma} \vec{X} \cdot d\vec{\sigma} + F_x, \text{ etc.};$$

similarly

$$(416) \quad \frac{\partial}{\partial t} \int_{(\Sigma)} U dv = - \int_{\Sigma} \vec{S} \cdot d\vec{\sigma} + G,$$

where  $G$  is the total rate of doing work on the charge.

For the case of the smoothed out charge we have from (413)

$$(417) \quad \begin{aligned} \operatorname{div} \vec{X} + \frac{\partial S_x}{\partial t} &= \sum_n \rho_n (\vec{E} + \dot{\vec{z}}_n \times \vec{B}), \text{ etc.}, \\ \operatorname{div} \vec{S} + \frac{\partial U}{\partial t} &= \sum_n \rho_n \vec{E} \cdot \dot{\vec{z}}_n. \end{aligned}$$

If we integrate over a volume containing the charges we have

$$(418) \quad \frac{\partial}{\partial t} \int S_x d\sigma = - \int_{\Sigma} \vec{X} \cdot d\vec{\sigma} + \sum_n \int_{(\Sigma)} \rho_n (\vec{E} + \dot{\vec{z}}_n \times \vec{B}) dv, \text{ etc.},$$

and by comparison with the previous discussion we have

$$(419) \quad \begin{cases} \vec{F}_n = \int \rho_n (\vec{E} + \dot{\vec{z}}_n \times \vec{B}) d\sigma, \\ G_n = \int \rho_n \vec{E} \cdot \dot{\vec{z}}_n dv. \end{cases}$$

Thus

$$(420) \quad G_n = \dot{\vec{z}}_n \cdot \vec{F}_n.$$

We can now perform the variation

$$\delta \int \mathcal{H} d\tau_4 = 0.$$

We have

$$\begin{aligned}
 \delta \mathcal{H} &= \frac{\partial \mathcal{H}}{\partial p_{i\alpha}^*} \delta p_{i\alpha}^* = \frac{1}{2} f^{*i\hbar} \left\{ \delta p_{0i\hbar}^* + \sum_n \delta p_{ni\hbar}^* \right\} = \\
 &= -f^{*i\hbar} \frac{\partial}{\partial x^\alpha} \delta \psi_i + \sum_n \left( \vec{E} \cdot \frac{\partial \vec{D}_n}{\partial \xi_{na}} - \vec{B} \cdot \frac{\partial \vec{H}_n}{\partial \xi_{na}} \right) \delta \xi_{na} + \\
 &\quad + \sum_n \left( \vec{E}_n \cdot \frac{\partial \vec{D}_n}{\partial \xi_{na}} - \vec{B}_n \cdot \frac{\partial \vec{H}_n}{\partial \xi_{na}} \right) \delta \xi_{na} = \\
 &= -f^{*i\hbar} \frac{\partial}{\partial x^\alpha} \delta \psi_i + \sum_n \left( \vec{B}_n \cdot \frac{\partial \vec{H}_n}{\partial x^\alpha} - \vec{E}_n \cdot \frac{\partial \vec{D}_n}{\partial x^\alpha} \right) \delta \xi_{na} + \\
 &\quad + \sum_n \left\{ (\vec{D}_n \times \vec{B})_\alpha - \frac{\partial \nabla \chi_n}{\partial \xi_{na}} \cdot \vec{B} \right\} \delta \xi_{na},
 \end{aligned}$$

where in the last step we have made use of the fact that  $\vec{D}_n$  is a function of  $(\vec{x} - \vec{x}_n)$  while  $\vec{H}_n$  is given by (412). The variational principle now gives, since the  $\xi_{na}$ 's are functions only of  $t$ ,

$$\begin{aligned}
 0 &= \int \delta \mathcal{H} d\tau_4 = \int -f^{*i\hbar} \frac{\partial}{\partial x^\alpha} \delta \psi_i d\tau_4 + \\
 &\quad + \int dt \left[ \sum_n \delta \xi_{na} \int ( \quad ) dv \right] + \\
 &\quad + \int dt \left[ \sum_n \delta \xi_{na} \int ( \quad ) dv \right] = \\
 &= \int \frac{\partial}{\partial x^\alpha} f^{*i\hbar} \delta \psi_i d\tau + \\
 &\quad + \int dt \left[ \sum_n \delta \xi_{na} \int \left\{ \vec{B}_n \cdot \frac{\partial \vec{H}_n}{\partial x^\alpha} + \vec{D}_n \cdot \frac{\partial \vec{E}}{\partial x^\alpha} - \frac{\partial}{\partial t} (\vec{D}_n \times \vec{B}) - \frac{\partial \chi_n}{\partial \xi_{na}} \text{div} \vec{B} \right\} dv \right],
 \end{aligned}$$

where the term  $(\vec{D}_n \cdot \frac{\partial \vec{E}}{\partial x^\alpha})$  comes from an integration by parts of the original term  $(-\vec{E} \cdot \frac{\partial \vec{D}_n}{\partial x^\alpha})$ .

From this we obtain, in addition to the usual field equations

$$\frac{\partial}{\partial x^\alpha} f^{*i\hbar} = 0,$$

the extra conditions

$$(422) \quad 0 = \int \left[ \vec{B}_n \cdot \frac{\partial \vec{H}_n}{\partial x^\alpha} + \vec{D}_n \cdot \frac{\partial \vec{E}}{\partial x^\alpha} - \frac{\partial}{\partial t} (\vec{D}_n \times \vec{B}) \right] dv.$$

In the latter we have used the fact that  $\text{div} \vec{B} = 0$  which comes out of the above

field equations. We are interested in these extra conditions. We can transform them by means of the field equations (403) which take the vector form

$$(423) \quad \text{curl } \vec{H}_n - \dot{\vec{D}}_n = \rho_n \vec{\xi}_n, \quad \text{div } \vec{D}_n = \rho_n,$$

and the field equations

$$(351) \quad \text{curl } \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \quad \text{div } \vec{B} = 0.$$

We obtain, in fact

$$0 = \int \left[ \vec{B} \cdot \frac{\partial \vec{H}_n}{\partial x_a} - (\vec{B} \times \text{curl } \vec{H}_n)_a + \vec{D}_n \cdot \frac{\partial \vec{E}}{\partial x_a} - (\vec{D}_n \times \text{curl } \vec{E})_a - \rho_n (\vec{\xi}_n \times \vec{B})_a \right] dv,$$

and further manipulation involving various identities of the vector calculus finally leads to

$$(424) \quad 0 = - \int \rho_n (\vec{E} + \vec{\xi}_n \times \vec{B})_a dv = -F_n a.$$

Thus, just as in the classical theory of Maxwell, a charged particle moves in such a way that the total force  $\vec{F}_n$  acting on it will vanish. The total force here used is a well defined quantity and does not depend on the way in which the field was broken up. This condition that the total force on a charge shall be zero is equivalent to the conservation laws for momentum and energy.

It is of interest to compare the result just obtained with the corresponding result in the Maxwell theory. In the Maxwell theory we may not go to the limit of a point charge, but if we introduce some rigid distribution of electric density we can write

$$(425) \quad \text{div } \vec{E} = \rho(\vec{x} - \vec{\xi}), \quad \text{curl } \vec{H} - \frac{1}{c} \dot{\vec{E}} = \frac{1}{c} \rho(\vec{x} - \vec{\xi}).$$

If we now break up the field by

$$(426) \quad \left\{ \begin{array}{l} \vec{E} = \vec{E}' + \vec{E}_0, \quad \vec{H} = \vec{H}' + \vec{H}_0, \\ \text{with } \vec{E}' = \int \frac{\rho}{r} d\omega, \quad \vec{H}' = \left[ \frac{\dot{\vec{r}}}{r} \times \vec{E}' \right], \\ \vec{E}_0 = \text{curl } \vec{A}, \quad \vec{H}_0 = \frac{1}{c} \dot{\vec{A}} + \text{grad } A_0, \end{array} \right.$$

we obtain from

$$(427) \quad \delta \int (E^2 - H^2) d\omega = 0$$

on varying  $\vec{A}$ ,  $A_0$ ,  $\vec{r}(t)$ , firstly the field equations

$$\text{div } \vec{H} = 0, \quad \text{curl } \vec{E} + \frac{1}{c} \dot{\vec{H}} = 0,$$

and secondly the equations of motion

$$0 = \int \rho \left( \vec{E} + \frac{1}{c} \dot{\vec{r}} \times \vec{H} \right) d\omega$$

which give the condition that the total force on the charge is zero. However, if we have a rigid distribution of charge density we cannot have Lorentz invariance, while if we try to use a point charge we encounter infinities that spoil the theory. The advantage of the Born-Infeld theory is that it allows us to go to the limit of a point charge and thus to retain the Lorentz invariance. If  $\vec{E}$  itself were regular at the point charge we could write the condition governing the motion as

$$(428) \quad \left( \vec{E} + \left[ \frac{\dot{\vec{r}}}{r} \times \vec{B} \right] \right)_{r=\xi} = 0.$$

It is of great interest to discuss the motion of a charge with the approximations allowed when the external field is weak -- that is, weak compared with the large absolute field given by  $b$ . We may divide space into three regions, the interior of a sphere whose center is the center of the charge and whose radius is of the order of the electronic radius, the "intermediate region" lying between this sphere and a concentric sphere  $\Sigma$  of such a radius that the field at this distance is approximately classical, and, finally, the region exterior to this second sphere, where the field may be treated as Maxwellian. We may assume that in this macroscopic region the superposition principle will hold, but a rigorous

discussion of this point has not been given. Again, since we are dealing with weak external fields we may assume that the acceleration of the charge will be small compared with  $(c^2/r_0)$ ,  $r_0$  being the electronic radius. This means that the reaction of the radiation will be small. The field in the classical region will now be just the sum of an external field and the field of a uniformly moving electron according to the usual Maxwell theory. From the relations

$$\frac{\partial}{\partial t} \int_{(\Sigma)} S_x dv = - \int_{\Sigma} \vec{X} \cdot d\vec{\sigma}, \text{ etc.},$$

since the right-hand sides are just the classical force on the charge, we have

$$(429) \quad \frac{\partial}{\partial t} \int_{(\Sigma)} \vec{S} dv = e(\vec{E} + \dot{\vec{x}} \times \vec{B}).$$

But the volume  $\Sigma$  is such that the Poynting vector of the external field gives only a small contribution so that we may regard the integral on the left as a measure of the electromagnetic momentum  $\vec{p}$  of the moving charge. Thus we may

write (429) as

$$(430) \quad \frac{\partial \vec{p}}{\partial t} = (e\vec{E} + \dot{\vec{x}} \times e\vec{B}).$$

By a suitable Lorentz transformation we can reduce the charge to rest. The momentum will now be zero and the total energy  $W$  can be regarded as a measure of the rest mass  $m_0$  of the particle. [Actually we want  $W = m_0 c^2$ , but the units are such that  $c = 1$ .] From the relativistic invariance we now have the well-known relation

$$(431) \quad h = \frac{m_0 \dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}}^2}},$$

with the important advantage over previous theories that the  $m_0$  entering this formula has been related directly to the total energy of the electromagnetic field of the particle.

We can put the condition that the total force on a charge must vanish into a relativistic form. We have

$$\frac{\partial}{\partial t} \int_{(\Sigma)} S_x dv + \int_{\Sigma} \vec{X} \cdot d\vec{\sigma} = 0, \text{ etc.},$$

whether the surface  $\Sigma$  encloses charges or not. If we integrate this with respect to  $t$ , and let the three-dimensional hypersurface bounding the new region of integration be given by

$$(432) \quad x^i = x^i(u, v, w), \quad (i = 1, 2, 3, 4),$$

we get just

$$(433) \quad \int T^{ik} \sigma_i du dv dw = 0,$$

where  $\sigma_i$  are components of the hypersurface element and are given by the four functional determinants that form the three-rowed minors of the matrix

$$\left\| \begin{array}{cccc} \frac{\partial x^1}{\partial u} & \frac{\partial x^2}{\partial u} & \frac{\partial x^3}{\partial u} & \frac{\partial x^4}{\partial u} \\ \frac{\partial x^1}{\partial v} & - & - & - \\ \frac{\partial x^1}{\partial w} & - & - & \frac{\partial x^4}{\partial w} \end{array} \right\|$$

By Gauss's theorem the left-hand side of (433) can be transformed into the four-dimensional volume integrals

$$\int \frac{\partial T^{ik}}{\partial x^k} d\tau,$$

the domain of integration being the four-dimensional volume contained in the hypersurface (432). We thus find that

$$\frac{\partial T^{ik}}{\partial x^k} = 0,$$

even where there are singularities, and this equation is, in fact, just the relativistic formulation of the law that the total force on a charge is always zero, when it is applied to singular regions. The connection between the laws of motion and the laws of conservation is now apparent.

Remarks on the New Field Theory

(by W. Pauli)

The derivation of the value of the absolute field strength  $b$  is not very remarkable, for from (397) it is evident that  $b$  is essentially the critical field strength already discussed in this seminar; the numerical value  $1 \cdot 2361$  of the integral

$$\int_0^{\infty} (\sqrt{1+x^4} - x^2) dx$$

is rather accidental, the important point being merely that it should be of the order of unity. And all integrals in physics are of the order of unity!

The Lagrangian of Heisenberg and Euler previously reported on in this seminar does not possess the property that the total energy of a point charge is finite but leads to a logarithmically infinite value.

In a sense the Born theory is not a unitary theory since in addition to the field quantities the positions of the charges must be treated as independent variables. This is not in accordance with Mie's intention; for Mie wanted a completely regular field and hoped to derive from the field equations alone the complete behavior of what we call charges. However, in the more restricted sense that energy and momentum are purely electromagnetic, the theory is unitary.

Quantization of the New Field Theory

(by M.H.L.Pryce)

We shall discuss the problem of introducing a quantization into the classical theory previously reported.

In the Lagrangian formulation of classical dynamics we start by forming a Lagrangian function  $\mathcal{L}$  that involves the  $q$ 's and  $\dot{q}$ 's and define the conjugate

momenta by the equation

$$h_r = \frac{\partial \mathcal{L}}{\partial \dot{q}_r}.$$

In the case of the electromagnetic field, however, we meet a somewhat different situation. For the quantities  $\mathcal{L}$  and  $\mathcal{H}$  that we have been referring to as Lagrangian functions are really just the densities and it is their integrals over all space -- not space-time -- that are the true Lagrangian functions in the sense of classical dynamics. This is at once evident from the fact that classical dynamics is derivable from the principle of least action which states that

$$\delta \int \mathcal{L} dt = 0,$$

where  $\mathcal{L}$  is the Lagrangian function, while the field equations in the electromagnetic case come from

$$\delta \int \mathcal{L} dx dy dz dt = 0 \text{ or } \delta \int \mathcal{H} dx dy dz dt = 0,$$

which, when written as

$$\delta \int dt \left[ \int \mathcal{L} dx dy dz \right] = 0, \text{ etc.},$$

show that

$$\int \mathcal{L} dx dy dz, \quad \int \mathcal{H} dx dy dz$$

are the analogues of the Lagrangian function of classical dynamics. This point is of importance since the introduction of commutation rules depends upon the definition of canonically conjugate variables in accordance with the methods of classical dynamics, and if we have the Lagrangian

$$(434) \quad \Lambda = \int \mathcal{H}(\Psi_\alpha, \vec{\xi}_n, \dot{\vec{\xi}}_n) dV,$$

$\Psi_\alpha$  being the anti-potential four-vector and entering only in the form of its curl, this Lagrangian is a function of the values of the  $\Psi$ 's at all points of space and is thus a function of an infinite number of variables. It is, in fact, a functional and if we wish to define the conjugates of the quantities  $\Psi$  we must make use of the notion of the functional derivative, as was done by Pauli and Heisenberg in their pioneer work on the quantization of the Maxwell field. The

functional derivative is defined by

$$(435) \quad \frac{\delta \Lambda}{\delta \psi} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int [\mathcal{H}(\psi + \epsilon \delta(\vec{x} - \vec{x}_0), \vec{\xi}, \dot{\vec{\xi}}) - \mathcal{H}(\psi, \vec{\xi}, \dot{\vec{\xi}})] dV.$$

If we define the conjugate of  $\vec{\psi}$  to be

$$\frac{\delta \Lambda}{\delta \dot{\vec{\psi}}}$$

we find, since

$$\dot{\vec{\psi}} - \text{grad } \psi_0 = \vec{H}_0$$

and therefore for  $\wedge$

$$\frac{\partial}{\partial \dot{\vec{\psi}}} = \frac{\partial}{\partial \vec{H}_0},$$

that the conjugate of  $\vec{\psi}$  is

$$(436) \quad \frac{\delta \Lambda}{\delta \dot{\vec{\psi}}} = \frac{\partial \mathcal{H}}{\partial \vec{H}_0} = \vec{B}.$$

It turns out that the conjugate of the scalar anti-potential vanishes:

$$(437) \quad \frac{\delta \Lambda}{\delta \dot{\psi}_0} = 0.$$

Let us introduce the usual Poisson bracket notation

$$(438) \quad [\xi, \eta] \equiv \frac{1}{i\hbar} (\xi \eta - \eta \xi).$$

Then, by analogy with the case of a finite number of variables where all quantities commute unless they are canonical conjugates, in which case their Poisson bracket is equal to unity, we assume that

$$(439) \quad [\psi_\alpha, B'_\beta] = \delta(\vec{x} - \vec{x}') \delta_{\alpha\beta},$$

where the notation means that  $\psi_\alpha$  is taken at the point  $(\vec{x}, t)$  and  $B'_\beta$  at the point  $(\vec{x}', t)$ , the times being taken the same throughout all this work. The

symbol  $\delta(\vec{x} - \vec{x}_0)$  is written for the product of three  $\delta$ -functions,  
 $\delta(x - x_0) \delta(y - y_0) \delta(z - z_0).$

Since  $\vec{D}_0$ , the residual part of  $\vec{D}$  in the arbitrary separation of the field discussed in the classical part, is just the curl of  $\vec{\psi}$  we find that

$$(440) \quad [D_x, B'_x] = 0$$

while

$$(441) \quad [D_x, B_y'] = -\frac{\partial}{\partial z} \delta(\vec{x}-\vec{x}'), \text{ etc.},$$

just as in the quantized form of the Maxwell field given by Heisenberg and Pauli.

The equations (440), (441) have the advantage of being gauge invariant, and, moreover, it turns out that if we retain the  $\psi$ 's as fundamental quantities entering the commutation rules we will find the equation  $\text{div } \vec{B} = 0$  inconsistent with the commutation rule (439). In the Pauli-Heisenberg formalism the theory is formulated in terms of the ordinary electromagnetic potentials and in this case it is the equation  $\text{div } \vec{D} = \rho$  that is inconsistent with the commutation rules involving these potentials. We shall therefore use only the gauge invariant commutation rules.

When particles are present we have the extra coordinates  $\vec{r}_{\alpha n}$  and it is therefore necessary that we find their conjugate variables. We have

$$\frac{\delta \Lambda}{\delta \vec{r}_{\alpha n}} = \int \frac{\partial \mathcal{H}}{\partial \vec{r}_{\alpha n}} d\omega;$$

but  $\mathcal{H}$  is a function of  $\vec{D}$  and  $\vec{H}$ , and we have already shown that  $\vec{D}$  is independent of  $\vec{r}_{\alpha n}$ . Thus

$$\frac{\partial \mathcal{H}}{\partial \vec{r}_{\alpha n}} = \frac{\partial \mathcal{H}}{\partial \vec{H}} \frac{\partial \vec{H}}{\partial \vec{r}_{\alpha n}},$$

and we find now that the momentum  $\vec{p}_{\alpha n}$  conjugate to  $\vec{r}_{\alpha n}$  is given by

$$(442) \quad \vec{p}_{\alpha n} = \int (\vec{D}_n \times \vec{B}) d\omega.$$

Since  $\vec{r}_{\alpha n}$  and  $\vec{p}_{\alpha n}$  are conjugate we want

$$(443) \quad [\vec{r}_{\alpha n}, \vec{p}_{\alpha n}] = \delta_{\alpha\beta},$$

but with the definition (442) of  $\vec{p}_{\alpha n}$  we find that

$$[\vec{r}_{\alpha n}, \vec{p}_{\alpha n}] = 0.$$

We must have  $\vec{r}_{\alpha n}$  and  $\psi$  commute since they are not conjugate. Thus the only

simple way in which we can obtain (443) is to alter the definition of  $\vec{p}_{\alpha n}$  by

adding some quantity  $\vec{\pi}_n$  as:

$$(444) \quad \vec{p}_{\xi_n} = \int (\vec{D}_n \times \vec{B}) dv + \vec{\pi}_n.$$

We shall now obtain the correct commutation rule (443) if we take

$$(445) \quad [\xi_{m\alpha}, \pi_{n\beta}] = \delta_{mn} \delta_{\alpha\beta}.$$

This quantity  $\vec{\pi}_n$  corresponds to a spatial translation operator and can always be found. The total momentum is now

$$\int (\vec{D} \times \vec{B}) dv + \sum_n \vec{\pi}_n$$

and it is easily verified that this obeys the correct commutation rules with  $\vec{p}_{\xi_n}$ .

Since we wish to use only gauge-invariant quantities, we take as coordinates the quantities  $\vec{D}_0$ ,  $\vec{\xi}_n$  and as conjugate momenta  $\vec{B}$ ,  $\vec{p}_{\xi_n}$ . This means

that we must have

$$(446) \quad [p_{\xi_n}, \vec{D}_0] = 0$$

which is the same as

$$[\vec{\pi}_n + \int (\vec{D}_n \times \vec{B}) dv, \vec{D}_0] = 0.$$

This leads to the condition

$$[D_{0\beta}, \pi_{n\alpha}] = -\delta_{\alpha\beta} \delta(\vec{\xi}_n - \vec{x}) + \frac{\partial D_{0\beta}}{\partial x_\alpha}.$$

But

$$[D_{n\beta}, \pi_{\xi_n\alpha}] = \frac{\partial D_{n\beta}}{\partial x_\alpha}$$

and we therefore have, for the total field  $(\vec{D}_0 + \sum_n \vec{D}_n)$ ,

$$[D_{0\alpha} + \sum_n D_{n\alpha}, \pi_{n\beta}] = -\delta(\vec{\xi}_n - \vec{x}) \delta_{\alpha\beta},$$

i.e.

$$(447) \quad [D_\alpha, \pi_{n\beta}] = -\delta(\vec{\xi}_n - \vec{x}) \delta_{\alpha\beta}.$$

The commutation rule

$$(448) \quad [\vec{B}, \vec{\pi}] = 0$$

is automatically satisfied, and the rule

$$(449) \quad [p_{\xi_{m\alpha}}, p_{\xi_{n\beta}}] = 0$$

leads to the condition

$$(450) \quad [\pi_{m\alpha}, \pi_{n\beta}] = -\delta_{mn} \int \delta(\vec{\xi}_n - \vec{x}) \epsilon_{\alpha\beta\gamma} B_\gamma dv.$$

To sum up, we have found the commutation rules

$$(451) \left\{ \begin{array}{l} [D, D'] = 0, \quad [B, B'] = 0, \\ [D_x, B_x'] = 0, \text{ etc.}, \quad [D_x, B_y'] = -\frac{\partial}{\partial z} \delta(\vec{x} - \vec{x}'), \text{ etc.}; \\ [\xi, D] = 0, \quad [\xi, B] = 0; \\ [\xi_{m\alpha}, \pi_{m\beta}] = \delta_{m\alpha\beta} \delta_{\alpha\beta}; \\ [\pi, B] = 0, \quad [\pi_{m\alpha}, D_\beta] = \delta_{\alpha\beta} \delta(\vec{x} - \vec{\xi}_m); \\ [\pi_{mx}, \pi_{ny}] = -\delta_{mnm} \int \delta(\vec{\xi}_m - \vec{x}) \cdot B_z \, dv, \text{ etc.} \end{array} \right.$$

In the case of a vacuum we have, as in the Maxwell theory,  $\text{div } D = 0$ ,

and then the operator  $p$  defined by

$$\vec{p} = \int (\vec{D} \times \vec{B}) \, dv$$

acts as a space differentiation operator; that is, we find

$$[p_x, D_x] = -\frac{\partial D_y}{\partial y} - \frac{\partial D_z}{\partial z} = \frac{\partial D_x}{\partial x}, \text{ etc.}$$

In the case of the  $p$ 's defined by (444) we have

$$(452) \quad [p_x, D_x] = -\frac{\partial D_y}{\partial y} - \frac{\partial D_z}{\partial z} + \sum_n \rho_n = \frac{\partial D_x}{\partial x} + \sum_n \rho_n - \text{div } \vec{D}.$$

If, therefore, we assume that  $p$  acts as a space differentiation operator so that

$$(453) \quad [p_x, D_x] = \frac{\partial D_x}{\partial x},$$

we obtain, as a consequence of the commutation rules, the field equation

$$\text{div } \vec{D} = \sum_n \rho_n = \rho.$$

In a similar manner we obtain the field equation

$$\text{div } B = 0.$$

This procedure is the same as in the Pauli-Heisenberg theory.

It is now necessary to define time differentiation as commutation with respect to a Hamiltonian function. It happens that we cannot take as the Hamiltonian the quantity

$$W = \int U \, dv$$

since  $U$  is a function of  $\vec{D}$  and  $\vec{B}$ , and  $W$  would therefore commute with  $\vec{P}_m$ , leading to the result

$$\dot{\vec{P}}_m = 0.$$

In order to avoid such a result we add to the above  $W$  a term linear in  $\vec{\pi}$  that involves the first three Dirac  $\alpha$ 's:

$$(454) \quad W = \int U dv + \sum_n \vec{\alpha}_n \cdot \vec{\pi}_n.$$

We now find that

$$(455) \quad \dot{\vec{S}}_n = [\vec{S}_n, W] = \vec{\alpha}_n,$$

$$(456) \quad \dot{\vec{P}}_n = 0, \quad \text{and}$$

$$(457) \quad \dot{\vec{\pi}}_n = \vec{F}_n, \quad \text{the total force.}$$

Further, we find from the commutation rules, the remaining field equations

$$\dot{\vec{D}} = \text{curl } \vec{H} - \sum_n \vec{\alpha}_n \rho_n = \text{curl } \vec{H} - \sum_n \rho_n \vec{S}_n,$$

$$\dot{\vec{B}} = -\text{curl } \vec{E}.$$

The angular momentum is defined by

$$(458) \quad \vec{M} = \int \vec{x} \times (\vec{D} \times \vec{B}) dv + \sum_n (\vec{S}_n \times \vec{\pi}_n + \frac{1}{2} \vec{\sigma}_n)$$

and by means of the commutation rules we obtain the conservation law of angular momentum

$$(459) \quad \dot{\vec{M}} = [\vec{M}, W] = 0.$$

In the classical theory, there is no analogue to  $\vec{\pi}_n$ , or rather, it is zero.

The quantum dynamical condition of motion is (457) while classically we have  $\vec{F}_n = 0$ .

### Remarks on the Quantization of the New Field Theory

(by W. Pauli)

The physical significance of his quantized theory and its relation to the previous theory which was, at least in a restricted sense, a unitary classical theory, was not made sufficiently clear by Mr. Pryce. The distinction between the

classical and quantum theoretical parts of his report is that in the latter the equations  $\vec{\pi}_n = 0$  and therefore also  $\vec{F}_n = 0$ , are not valid. This has not only the consequence that Pryce's quantum theoretical assumptions are essentially dualistic but also that the classical unitary theory cannot be considered as a limiting case of the quantum theoretical part. Indeed, the non-commutability of the  $\pi$ 's expressed in equation (451) prevents one from adding the equations  $\vec{\pi}_n = 0$  as extra conditions to the theory. If we can speak at all of a classical limiting case of the quantum theoretical assumptions of Pryce, it can only be valid if  $|\pi_n| \gg \sqrt{|B(\xi_\alpha)|}$  and the limiting process then leads to a classical formulation which is distinguished from the previous classical theory by non-electromagnetic extra terms  $\sum_m \vec{\pi}_m$  in the momentum,  $\pm \sum_m \sqrt{\pi_m^2}$  in the energy, and  $(-\vec{\pi}_m)$  in the equations of motion. (These extra terms alone — without the electromagnetic terms — would describe particles having zero rest mass, moving always with the velocity of light, and having positive and negative values of the energy.)

Apart from the fact that a solution for a particular case has not yet been obtained, it is to be remarked that even the finiteness of the energy in the case of a pure radiation field with no charges present is not proved for the quantum theoretical formulation of Pryce.

Summarizing, we can say that the quantum theoretical formalism, though mathematically self-consistent, has no advantage at all over the Maxwell theory and can hardly be considered as of any physical significance. One has the impression that Born's attempt to construct a unitary electromagnetic field theory for charged particles is essentially classical and has no natural connection with the quantum theory at all.

We can characterize Pryce's assumptions formally as follows: We can obtain his Hamiltonian from that of the ordinary Maxwell-Dirac theory by first of all striking out the rest mass term and secondly replacing the Maxwell energy density  $\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$  by the more general  $U(\vec{D}, \vec{B})$  of a classical unitary theory corresponding to a finite rest mass.

As in the Maxwell-Dirac case, two different but completely equivalent formulations are possible. The first (see W. Pauli, "Quantentheorie", Handbuch der Physik XXIV/1, §6) uses only gauge-invariant quantities such as the field-strengths and the operators  $\vec{\pi}_m$ , and  $\vec{\Sigma}_m$  for the matter. The second (see W. Pauli and W. Heisenberg, Zeitschrift für Physik, 56 (1929, p. 1; 59 (1929), p. 168), which may be the more familiar formulation, uses the electromagnetic potentials and the wave functions  $\Psi(x, y, z)$  for matter. The latter fulfil the commutation rules

$$(460) \quad \Psi_{\rho'}(\vec{x}) \Psi_{\rho''}^*(\vec{x}') \pm \Psi_{\rho''}^*(\vec{x}') \Psi_{\rho'}(\vec{x}) = \delta(\vec{x} - \vec{x}') \delta_{\rho'\rho''},$$

where the upper sign corresponds to the exclusion principle and the lower to the Einstein-Bose statistics, the latter being also compatible with the first formulation. The functions  $\Psi, \Psi^*$  commute with the electromagnetic quantities.

For the sake of completeness we shall give here the equivalent formulation of the second type to Pryce's assumptions, which are of the first type. At a given time  $t$  we have, in addition to the electromagnetic potentials  $(\vec{\Phi}, \phi)$  — not the anti-potential  $\Psi$  — which determine  $\vec{B}$  and  $\vec{E}$  by the equations

$$(461) \quad \vec{B} = \text{curl } \vec{\Phi}, \quad \vec{E} = -\frac{\partial \vec{\Phi}}{\partial t} - \text{grad } \phi,$$

the quantities  $\vec{D}, \Psi_{\rho}, \Psi_{\rho}^*$ , where  $\vec{D}$  and  $\vec{\Phi}$  satisfy the commutation rule

$$(462) \quad [D_x, \Phi_y'] = \delta(\vec{x} - \vec{x}') \delta_{xy}.$$

We introduce further an auxiliary operator  $p$  which satisfies

$$(463) \quad [p, \phi_0'] = \delta(\vec{x} - \vec{x}')$$

and commutes with all quantities except  $\mathcal{G}_0$ , while  $\mathcal{G}_0$  commutes with all quantities except  $p$ .

We then have the Hamiltonian

$$(464) \quad \begin{aligned} \bar{H} = & \int \psi^* \left( \frac{\hbar}{i} \frac{\partial \psi}{\partial x^2} - e \vec{\mathcal{G}} \psi \right) d\omega + \int U(\vec{D}, \text{curl } \vec{\mathcal{G}}) d\omega + \\ & + \int \mathcal{G}_0 (e \psi^* \psi - \text{div } \vec{D}) d\omega - \int \mu \frac{\partial \mathcal{G}_0}{\partial t} d\omega, \end{aligned}$$

and the momentum

$$(465) \quad \begin{aligned} \vec{\mu} = & \int \psi^* \left( \frac{\hbar}{i} \frac{\partial \psi}{\partial x^2} - e \vec{\mathcal{G}} \psi \right) d\omega + \int [\vec{D} \times \text{curl } \vec{\mathcal{G}}] d\omega + \\ & + \int \vec{\mathcal{G}} (e \psi^* \psi - \text{div } \vec{D}) d\omega + \int \mu \frac{\partial \mathcal{G}_0}{\partial x^2} d\omega. \end{aligned}$$

Further, we have the extra conditions

$$(466) \quad \mu \mathcal{F} = 0,$$

$$(467) \quad (e \psi^* \psi - \text{div } \vec{D}) \mathcal{F} = 0,$$

where  $\mathcal{F}$  is the "Schrödinger functional". The only conditions of compatibility that such extra conditions have to fulfil is that their commutators with each other and with the energy and momentum shall vanish as a consequence of their own validity and the commutation rules of the quantities involved. It is easily seen that these conditions are fulfilled in the present case.

The last two integrals in (464) and (465) vanish as a consequence of the extra conditions (466) and (467), but it is necessary to add them in order that for every quantity  $f$  the equations

$$(468) \quad \frac{\partial f}{\partial t} = -[H, f], \quad \frac{\partial f}{\partial x^2} = [\vec{\mu}, f]$$

shall hold. According to (467) the commutation rules for  $\partial \mathcal{G}_0 / \partial t$  are completely arbitrary.

Applying the first of the rules (468) we can verify the second equation

(461) directly, and on putting

$$(469) \quad \vec{E} = \frac{\partial U}{\partial \vec{D}}, \quad \vec{H} = \frac{\partial U}{\partial \vec{B}},$$

we are now able to verify the field equation

$$(470) \quad \frac{\partial \vec{D}}{\partial t} = \text{curl } \vec{H} - e(\psi^* \vec{\alpha} \psi).$$

Furthermore we find for  $\frac{\partial \psi}{\partial t}$  the Dirac equation.

The equivalence of this formulation with that of Pryce can be proved in the same way as in the Maxwell-Dirac case.

### Quantum Mechanics of Infinite Systems

(by J. von Neumann)

I wish to discuss some rather incomplete ideas concerning difficulties that arise in some parts of quantum mechanics. In general there have been no serious difficulties when we are dealing with a finite number of particles, but very essential difficulties arise as soon as we treat a system having an infinite number of degrees of freedom; for example, the theory of holes, which, because of the pair generation, requires an indefinite number of particles; also the Dirac non-relativistic theory of light and the Pauli-Heisenberg relativistic quantum electro-dynamics, these being equivalent to systems consisting of an infinite number of particles.

In dealing with a continuum we find two types of infinity. One arises from the fact that we have an infinite space, but this does not lead to serious difficulties and can be avoided by considering a finite box, or, better, by assuming periodicity in space. The second type of infinity is much more serious. It comes from the fact that in a continuum a field quantity has an infinite number of proper values. The assumption of periodicity does not remove this difficulty at all. Neither does the assumption that space is discrete solve the essential dif-

difficulty since if we pass to the limit "lattice  $\rightarrow$  continuum" we get just the continuum result, which diverges.

It is the fact that we have an infinite number of degrees of freedom that causes the difficulties, and we shall therefore discuss how we can change the formal part of the theory in some way so that we can treat a system having an infinite number of degrees of freedom in a less divergent way.

The thing that seems to be wrong with the usual theory is the use of wave functions, or, in the language of Dirac, of maximum observations. For, suppose we wish to find out experimentally what state a given system is in. The normal method, according to the quantum theory, is to make the maximum number of compatible measurements; from these we can find not only the state of the system but also, by means of the Schrödinger time-dependent equation, what happens to this state afterwards. However, if our system has an infinite number of degrees of freedom, we require an infinite number of observations before we can determine its state, and this seems unreasonable. The difficulty enters in field theories as well as in particle theories since in the former the wave functions depend on all those field components that are simultaneously measurable and these are, in the Pauli-Heisenberg theory, for example, infinite in number.

We take the attitude that this is the real cause of the trouble and we shall show that we can perhaps avoid the divergences by using a model which avoids the wave functions. We shall first discuss not the electromagnetic field but the case of a system consisting of an infinite number of Fermi-Dirac particles, i.e. a system in which it is essential that we do not know the number of particles, as is always the case when we have pair generation. If we assume that the universe is large but finite, the energy levels will form a discrete set and we can describe the situation by telling which are the states in which a particle is present. Let us consider the question of how many particles are in a certain box. We can find

a complete set of standing waves in the box and a complete set of states outside. Then any particle definitely in the box will be represented by a wave-function consisting of a linear combination of the standing waves alone and there is thus a definite meaning to the number of particles in the box. We can actually make one simple observation of this number by merely weighing the box. But in fact the weighing will not give the number of particles in the box unless the walls are impermeable, and, moreover, impermeable not only to the particles in the box but to all types of particle or energy. Experience shows that such a box cannot be realized in practice. There is an upper limit to the molecular weight of the substance of the box and thus an upper limit to its impermeability. Even if there were a sharply defined energy at which the box changes from complete impermeability to relative impermeability we could not say that energy would be conserved within the box; for particles of very high energy could enter the box and give up part of their energy by collision with particles inside the box, thus increasing the total energy. They could also cause a decrease in the total energy by giving to particles in the box enough energy to allow them to pass through the walls.

Another question of fundamental importance is this: Can we assume that particles of extremely high energy essentially do not interfere with particles of low energy? We must certainly assume that particles of very high energy exist since with every relevant advance in experimental technique particles of higher energy are discovered and we cannot limit the number of such particles a priori since the number observed by us is dependent on how short wave lengths we are able to measure. The fact that we exist and do not feel any ill effects from the high energy particles seems to show that the extremely large interaction demanded by electrodynamic theory is incorrect and that high energy particles do not interact with those of low energy. The known deviations from the Klein-Nishina law of absorption also point to such a conclusion. We shall therefore assume that

this conclusion is correct. The situation is similar to that which arises in the Newtonian theory of gravitation. In that theory, although matter distributed uniformly over a spherical shell ( $r, r + dr$ ) will have no resultant attraction at the center, if fluctuations occur there will be a resultant attraction and in an infinite universe approximately uniformly filled with matter the effect of the distant masses will lead to divergent results. In the present electrodynamics the particles of high energy lead to an analogous divergence. One might say: The distant regions of the momentum space have the same divergence-generating effect in (Maxwellian) electrodynamics as the distant regions of common space have in Newtonian gravitation theory. Thus in both theories the fundamental postulate which requires the existence of "closed systems" is violated.

We shall look for a theory in which closed systems exist in momentum space. To discuss mathematically what sort of change we must expect, we must consider how we can describe systems of which we do not know the maximum information. We must also consider how we put two systems together since we get infinite systems by applying this process repeatedly.

A single particle of the infinite set of particles constituting the system will, if left alone, have certain stationary states, say  $\Sigma_1, \Sigma_2, \Sigma_3, \dots$ . In the Bose-Einstein case we have a complete description of the system if we know the numbers  $r_1, r_2, r_3, \dots$  of particles in the respective states  $\Sigma_1, \Sigma_2, \Sigma_3, \dots$ . These  $r$ 's are treated as coordinates and there is an infinite number of them, the system behaving as if it were the sum of the separate states  $\Sigma_1, \Sigma_2, \Sigma_3, \dots$ , corresponding to the equivalent oscillators of the classical theory. In the Fermi-Dirac case the  $r$ 's can have only the values 0, 1, and the states  $\Sigma$  behave like spins rather than oscillators. Thus a Bose-Einstein assembly of indistinguishable particles can be considered as a classical assembly of distinguishable oscillators, while a Fermi-Dirac assembly of indis-

distinguishable particles can be regarded as a classical assembly of distinguishable "spins". It is the putting together of such "spin" systems that we wish to consider. The subdivision into  $\sum$ 's is not unique, but this causes no difficulties. In order to avoid unessential infinities we make the following idealization; in quantum theory any state  $\phi$  can be expanded in terms of an infinite number of eigen-states  $\phi_r$  as

$$(471) \quad \phi = \sum_{r=1}^{\infty} c_r \phi_r,$$

but we shall assume here that only a finite number of eigen-states are necessary to form a basis for this type of expansion. Thus the state of a system is now to be regarded as given by a finite number of  $c_r$ 's. This is the same as if we had taken a lattice instead of the continuum (in a finite volume). This simplification is further justified because for the "spin" systems which we shall ultimately consider these numbers are really finite. We shall see later that the infinity we get is quite independent of what we take to characterize eigen-states -- energy or some other quantity. Let the number of eigen-states of a given system be  $N$  and let the indices  $i, j$  go from 1 to  $N$ . Then any state  $\sigma$  of the system will be described by a vector  $x_i$  such that

$$(472) \quad \sum_{i=1}^N |x_i|^2 = 1,$$

and an observable  $\mathcal{O}$  will be represented by a Hermitian matrix  $a_{ij}$ . The expectation value of  $\mathcal{O}$  for the state  $\sigma$  is given by

$$(473) \quad \text{Exp.}(\mathcal{O}, \sigma) = \sum_{i,j} a_{ij} x_i x_j^*.$$

[The usual notation is  $\int A \phi(x) \phi^*(x) dx$  and here  $\int dx \rightarrow \sum_i, A\phi \rightarrow \sum_j a_{ij}$  and  $\phi^* \rightarrow x_j^*$ .]

If  $\mathcal{A}$  and  $\mathcal{B}$  are two simultaneously observable observables, their matrices must commute.

The product of two observables is given by

$$(474) \quad \mathcal{L} = \mathcal{A}\mathcal{B}; \quad c_{ij} = \sum_k a_{ik} b_{kj},$$

and their sum by

$$(475) \quad \mathcal{L} = \mathcal{A} + \mathcal{B}; \quad c_{ij} = a_{ij} + b_{ij}.$$

The product has a meaning only when  $\mathcal{A}, \mathcal{B}$  are simultaneously observable -- that is, when the matrices commute. The sum however is always defined. The observable  $\mathcal{A}^n$  has the matrix  $(a_{ij})^n$  and we can thus compute  $\text{Exp}(\mathcal{A}, \sigma)$  for all  $n$ . It follows that the possible values of  $\mathcal{A}$  are the eigenvalues of  $a_{ij}$  and that their probabilities are the squares of the corresponding expansion coefficients of the vector  $x_1, \dots, x_N$ .

All of this describes a system given by a maximum observation. But we must consider how we can describe a system whose state might be any one of the several states  $\sigma_\nu \sim x_{i,\nu}$ ,  $\nu = 1, \dots, n$ . Let the respective probabilities be  $p_\nu$  ( $p_\nu \geq 0$ ). Then, if the totality of the states  $\sigma_\nu$  is denoted by  $\sigma'$ , we certainly want

$$\text{Exp.}(\mathcal{A}, \sigma') = \sum_\nu p_\nu \text{Exp.}(\mathcal{A}, \sigma_\nu).$$

We can write this as

$$\begin{aligned} \text{Exp.}(\mathcal{A}, \sigma') &= \sum_\nu p_\nu \sum_{i,j} a_{ij} x_{i,\nu} x_{j,\nu}^* = \\ &= \sum_{i,j} a_{ij} \left( \sum_\nu p_\nu x_{i,\nu} x_{j,\nu}^* \right). \end{aligned}$$

The parenthesis on the right depends only on the states and their probabilities and is independent of the observable  $\mathcal{A}$ . We have thus broken up  $\text{Exp.}(\mathcal{A}, \sigma')$  into two parts, the one depending only on  $\mathcal{A}$  and the other only on the states and their probabilities. We therefore regard the latter as the correct descrip-

tion of these states for our present purposes and denote it by  $u_{ji}$ :

$$(476) \quad u_{ji} = \sum_{\nu} p_{\nu} x_{i\nu} x_{j\nu}^*$$

$$(477) \quad \text{Exp.}(\alpha, \sigma') = \sum_{i,j} a_{ij} u_{ji}.$$

We shall call a state in which a maximal observation is not known a mixture, and the corresponding  $u_{ji}$  the statistical matrix of the mixture. If we denote the matrix of  $a_{ij}$  by  $a$  and that of  $u_{ji}$  by  $u$  we can write (477) as

$$(478) \quad \text{Exp.}(\alpha, \sigma') = \text{Trace}(au).$$

These matrices  $a, u$  are not arbitrary. The matrix  $a$  must be Hermitean while, since  $\sum_{\nu} p_{\nu} = 1$  and the  $x$ 's satisfy (472), it follows that  $u$  must be a positive definite Hermitean matrix having the trace unity.

There is a disadvantage in the  $u_{ji}$  in that it has two indices instead of the single index of a state  $x_i$ . However, since for a single state we have

$$u_{ij} = x_i^* x_j,$$

we see that the statistical matrix is independent of the phase of the wave function. Thus the  $u$ 's are in one-to-one correspondence with the states, whereas the  $x$ 's are not. Again for states we have the "linear superposition"

$$\psi = c_1 \psi_1 + c_2 \psi_2,$$

which must not be mixed up with the somewhat analogous operation

$$u = |c_1|^2 u_1 + |c_2|^2 u_2$$

on statistical matrices. This operation, "Uniting", as contrasted to the previously mentioned "linear superposition", deals directly with the probabilities, which are the physical quantities.

Having discussed how we shall represent a mixture, we must now consider how we can put two systems together. Let us assume there are two systems  $\sigma_1$  and see what can be said concerning the system consisting of these two together

or. Let  $\mathcal{D}$  have  $N$  different states ( $i, j = 1, \dots, N$ ) and  $\mathcal{F}$  have  $M$  ( $k, \ell = 1, \dots, M$ ). The sum  $(\mathcal{D} + \mathcal{F})$  will have  $NM$  states and we shall use the pairs of indices  $(ik), (j\ell)$  to denote them. Let the observable  $\mathcal{A}$  in  $\mathcal{D}$  have the matrix  $a_{ij}$  in  $\mathcal{D}$ . We can look for  $\mathcal{A}$  in  $(\mathcal{D} + \mathcal{F})$  and so we must have a matrix for it in  $(\mathcal{D} + \mathcal{F})$ . We write this matrix as  $a_{ik/j\ell}$  and make the customary assumption that

$$(479) \quad \bar{a}_{ik/j\ell} = a_{ij} \delta_{k\ell}.$$

This means that we do to the variable that characterizes  $\mathcal{D}$  the same as we do to it in  $\mathcal{D}$  alone, and that we do nothing to  $\mathcal{F}$ . Again, for  $\mathcal{B}$  in  $\mathcal{F}$  we take for  $(\mathcal{D} + \mathcal{F})$  the matrix

$$(480) \quad \bar{b}_{ik/j\ell} = \delta_{ij} b_{k\ell}.$$

The  $\bar{a}$  and  $\bar{b}$  so defined commute with each other, and this is necessary on physical grounds since the corresponding observations are evidently compatible.

There will be a statistical matrix  $u_{ik/j\ell}$  for the mixture  $(\mathcal{D} + \mathcal{F})$ . What can we say about  $\mathcal{D}$  alone? If there is a statistical matrix  $u_{ij}$  that refers to  $\mathcal{D}$  alone and we make an observation of  $\mathcal{A}$  on  $\mathcal{D}$  we must find the same result as if we had made the observation on  $(\mathcal{D} + \mathcal{F})$ . Thus we want

$$\begin{aligned} \sum_{i,j} a_{ij} u_{ji} &= \sum_{ik/j\ell} a_{ij} \delta_{k\ell} u_{j\ell/ik} = \\ &= \sum_{ij} a_{ij} u_{j\ell/ik}, \end{aligned}$$

and therefore

$$(481) \quad u_{ij} = \sum_k u_{ik/jk}.$$

We have here an operation that carries us from the statistical matrix of  $(\mathcal{D} + \mathcal{F})$  to the statistical matrix of  $\mathcal{D}$ . It is analogous to the operation of forming the trace but only partially carried through, and corresponds to the operation of contraction in the tensor calculus. If we forget  $\mathcal{D}$  we have for  $\mathcal{F}$  the statistical matrix

$$(482) \quad v_{kl} = \sum_j u_{ik/jl}.$$

We now ask, if we know  $u_{ij}$  about  $\mathcal{D}$  and  $v_{kl}$  about  $\mathcal{F}$ , how can we find  $u_{ik/jl}$  for  $(\mathcal{D} + \mathcal{F})$ ? Actually we have not enough information about  $(\mathcal{D} + \mathcal{F})$  since there are  $(N^2 + M^2)$  components of  $u_{ij}$  and  $v_{kl}$  while there are  $N^2 M^2$  components of  $u_{ik/jl}$ . However we can obtain the statistical matrix for  $(\mathcal{D} + \mathcal{F})$  in which there are no correlations between  $\mathcal{D}$  and  $\mathcal{F}$ . If there are no correlations we will have, if  $\mathcal{O}$  is in  $\mathcal{D}$  alone and  $\mathcal{L}$  in  $\mathcal{F}$  alone,

$$\text{Exp.}(\mathcal{O}) \text{Exp.}(\mathcal{L}) = \text{Exp.}(\mathcal{O}\mathcal{L}).$$

This leads by simple computations to the result that

$$(483) \quad u_{ik/jl} = u_{ij} v_{kl}.$$

If we start with a general  $u_{ik/jl}$ , form the corresponding  $u_{ij}$  and  $v_{kl}$ , and then form a  $u_{ik/jl}$  in accordance with (483), we will not obtain the  $u_{ik/jl}$  with which we started, but this  $u_{ik/jl}$  without the correlations.

We are particularly interested in putting together a certain amount of information with no information at all. The case in which we have no information is

$$(484) \quad u_{ij} = \frac{1}{N} \delta_{ij}.$$

(It is easy to see that this is the mixture of all states of a complete orthogonal set of wave functions, each one having the probability  $\frac{1}{N}$ . This is true for every choice of the complete orthogonal set.) Thus if we know  $u_{ij}$  about the system  $\mathcal{D}$  and nothing at all about the system  $\mathcal{F}$ , we can form the corresponding statistical matrix for  $(\mathcal{D} + \mathcal{F})$  by the formula

$$(485) \quad u_{ij} \rightarrow u_{ik/jl} = \frac{1}{N} u_{ij} \delta_{kl}.$$

This process we shall refer to as expanding  $u$ .

Again, if we know  $u_{ik/jl}$  concerning  $(\mathcal{D} + \mathcal{F})$  we have already seen (eq. (481)) that we know  $u_{ij}$  about  $\mathcal{D}$  where

$$(486) \quad u_{ik/jl} \rightarrow u_{ij} = \sum_k u_{ik/jk}.$$

This process will be called contracting  $u$ .

The operation of expanding followed by contracting gives the identity operation, but if we contract first and expand afterwards we no longer get what we started with.

At this point we shall introduce a change in normalization. We alter the normalization so that instead of the trace of  $u_{ij}$  being unity, it is equal to  $N$ :

$$(487) \quad \frac{1}{N} \text{Trace}(u) = 1.$$

With this normalization, (476) becomes

$$(476') \quad u_{ji} = N \sum_{\nu=1}^N p_{\nu} a_{i\nu} x_{j\nu},$$

while (477) becomes

$$(477') \quad \text{Exp.}(\mathcal{O}, \sigma) = \frac{1}{N} \text{Trace}(\mathcal{O}u).$$

The operations of expanding and contracting take the form

$$(485') \quad u_{ij} \rightarrow u_{ik/jl} = u_{ij} \delta_{kl},$$

$$(486') \quad u_{ik/jl} \rightarrow u_{ij} = \frac{1}{M} \sum_{k=1}^M u_{ik/jk}.$$

The operations of expanding and contracting can be applied also to observables.

If we have an observable  $\mathcal{O}$  with the matrix  $a_{ij}$  in  $\sigma$  we can call it an observable in  $(\sigma + \mathcal{F})$ :

$$(488) \quad a_{ij} \text{ in } \sigma \rightarrow c_{ik/jl} \text{ in } (\sigma + \mathcal{F}); \quad c_{ik/jl} = a_{ij} \delta_{kl}.$$

Here we have expanded the observable  $\mathcal{O}$ . The formula for contracting an observable is

$$(489) \quad c_{ik/jl} \text{ in } (\sigma + \mathcal{F}) \rightarrow a_{ij} \text{ in } \sigma; \quad a_{ij} = \frac{1}{M} \sum_k c_{ik/jk},$$

and arises in the following manner. We assume that the observable  $c_{ik/jl}$  in  $(\sigma + \mathcal{F})$  is being observed by an observer who can see only  $\sigma$ . This observer

will describe the state of things by a statistical matrix of  $\sigma$ :  $u_{ij}$ . If we

use this  $u_{ij}$  for  $(\sigma + \mathcal{F})$  we must expand it to  $u_{ik/jl} = u_{ij} \delta_{kl}$  (see (477')).

Thus the expectation value of this observable in  $(\sigma + \tau)$  for our observer is

$$\frac{1}{MN} \sum_{i,j,k,l} (u_{ji} \delta_{kl}) C_{ik} / j_l,$$

which we may write as

$$\frac{1}{N} \sum_{i,j} u_{ji} \left( \frac{1}{M} \sum_k C_{ik} / j_k \right),$$

showing that  $\left( \frac{1}{M} \sum_k C_{ik} / j_k \right)$  is, from the point of view of our observer, indistinguishable from what he would call  $a_{ij}$ . We therefore call it  $a_{ij}$  and thus obtain the formula (489).

Let us now consider the putting together of a sequence of systems

$\Sigma_1, \Sigma_2, \Sigma_3, \dots, \Sigma_\nu, \Sigma_{\nu+1}, \dots$ . We shall take these systems as referring to Fermi-Dirac particles so that  $N = 2$  for each system. (Bose-Einstein assemblies will be discussed later.) Let us assume that the first  $\nu$  states have been put together and that the  $(\nu+1)$ st has not yet been added. The total number of states at this stage will be  $2^\nu$  and we shall use as indices running from 1 to  $2^\nu$  the sets of numbers  $(i_1, i_2, \dots, i_\nu)$  and  $(j_1, j_2, \dots, j_\nu)$ . Then when we wish to describe the mixture of the first  $(\nu+1)$  states we merely add an extra number to each set, as  $(i_1, \dots, i_\nu, i_{\nu+1})$ , etc. With this notation we must write a statistical matrix for the first  $\nu$  states as

$$u(i_1, \dots, i_\nu)(j_1, \dots, j_\nu)$$

and an observable as

$$a(i_1, \dots, i_\nu)(j_1, \dots, j_\nu)$$

while for the first  $(\nu+1)$  states we must write correspondingly

$$w(i_1, \dots, i_\nu, i_{\nu+1})(j_1, \dots, j_\nu, j_{\nu+1})$$

and

$$c(i_1, \dots, i_\nu, i_{\nu+1})(j_1, \dots, j_\nu, j_{\nu+1}).$$

We expand and contract between the first  $\nu$  and the first  $(\nu+1)$  states by the following formulas:

$$(490) \left\{ \begin{array}{l} \text{expanding:} \\ W_{(i_1, \dots, i_{\nu+1})}(j_1, \dots, j_{\nu+1}) = U_{(i_1, \dots, i_{\nu})}(j_1, \dots, j_{\nu}) \delta_{i_{\nu+1}, j_{\nu+1}} \\ C_{(i_1, \dots, i_{\nu+1})}(j_1, \dots, j_{\nu+1}) = A_{(i_1, \dots, i_{\nu})}(j_1, \dots, j_{\nu}) \delta_{i_{\nu+1}, j_{\nu+1}} \\ \text{contracting:} \\ U_{(i_1, \dots, i_{\nu})}(j_1, \dots, j_{\nu}) = \frac{1}{2} \sum_{k=0,1} W_{(i_1, \dots, i_{\nu}, k)}(j_1, \dots, j_{\nu}, k) \\ A_{(i_1, \dots, i_{\nu})}(j_1, \dots, j_{\nu}) = \frac{1}{2} \sum_{k=0,1} C_{(i_1, \dots, i_{\nu}, k)}(j_1, \dots, j_{\nu}, k) \end{array} \right.$$

Also, in the present notation, we have for the first  $\nu$  states,

$$(491) \text{Exp. } (0,0) = \frac{1}{2^{\nu}} \sum_{\substack{i_1, \dots, i_{\nu} \\ j_1, \dots, j_{\nu} \in \{0,1\}}} A_{(i_1, \dots, i_{\nu})}(j_1, \dots, j_{\nu}) U_{(i_1, \dots, i_{\nu})}(j_1, \dots, j_{\nu})$$

As we have already pointed out, it is suspicious that when dealing with the properties of systems having an infinite number of degrees of freedom, i.e. systems in which each particle has an infinite number of states, we use descriptions in which it is assumed that we are able to make the infinite number of measurements necessary to obtain the maximum information. We therefore want to restrict ourselves to descriptions in which an infinite number of observations is not implied. An observer who cannot measure more states than the first  $\nu$  is to be considered as <sup>in</sup> the system corresponding to the states  $(\Sigma_1 + \Sigma_2 + \dots + \Sigma_{\nu})$ . We do not want an absolute upper limit for  $\nu$ ; we want to deal with observers for whom  $\nu$  can become arbitrarily high, but we want their measurements to be such that they approach a limit as  $\nu$  increases indefinitely.

There are measurements that are not of this type. For example, if we have a box and try to measure the energy inside, the measurement of the energy will not be approximated by any measurement restricting itself to a finite number

of states. The same is true for the total number of particles in the box. The measurements we wish to consider are those that can be approximated by taking  $\nu$  arbitrarily high.

Let  $\mathcal{O}$  be an observable referring to the sum of any number of systems.

Then we can always take an observer who sees only, say, the first  $\nu$  systems, and his observations of  $\mathcal{O}$  will give us an observable  $\mathcal{O}^{(\nu)}$  in  $(\Sigma_1 + \dots + \Sigma_\nu)$ .

Thus we have a series of observables corresponding to observers who see only the first, the first two, the first three, ..., systems:

$$\mathcal{O}^{(1)}, \mathcal{O}^{(2)}, \mathcal{O}^{(3)}, \dots$$

To these we may add the  $\mathcal{O}^{(0)}$  belonging to an observer who can make no observations within the systems  $\Sigma$ . The matrix of  $\mathcal{O}^{(0)}$  will be an ordinary number

expressing the a priori expectation value of  $\mathcal{O}$  (corresponding to "complete ignorance" of the state). Since  $\mathcal{O}^{(\nu)}$  is what an observer who can only see the

first  $\nu$  systems would see when he tried to measure  $\mathcal{O}^{(\nu+1)}$ , it follows that

$\mathcal{O}^{(\nu)}$  must be a contraction of  $\mathcal{O}^{(\nu+1)}$ . Thus we have for all  $\nu$

$$(492) \quad a_{(i_1, \dots, i_\nu)(j_1, \dots, j_\nu)}^{(\nu)} = \frac{1}{2} \sum_{k=0,1} a_{(i_1, \dots, i_\nu, k)(j_1, \dots, j_\nu, k)}^{(\nu+1)}$$

For  $\nu = 0$  we have merely a number  $a^{(0)}$ .

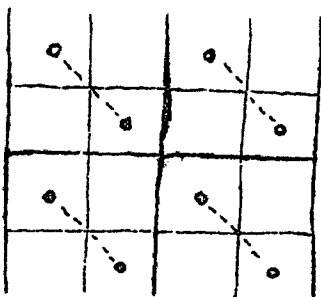
For  $\nu = 1$  we have the matrix

$$\mathcal{O}^{(1)} \sim \begin{array}{|c|c|} \hline a_{00}^{(1)} & a_{01}^{(1)} \\ \hline a_{10}^{(1)} & a_{11}^{(1)} \\ \hline \end{array},$$

and (492) says that  $a^{(0)}$  is the diagonal mean of this matrix:

$$a^{(0)} = \frac{1}{2} (a_{00}^{(1)} + a_{11}^{(1)}).$$

For  $\nu = 2$  we have a four-rowed square matrix indicated below:

$\sigma^{(2)} \sim$ 


and the means of the elements joined by dotted lines are respectively equal to the corresponding elements of the matrix of  $\sigma^{(1)}$ .

For  $\nu = 3$  we have an eight-rowed square matrix related to the matrix for  $\sigma^{(2)}$  in the same general manner, and so on for all  $\nu$ .

Thus, if we look at an observable very roughly it looks like a number, if we look more closely it turns out to be a two-rowed square matrix, if more closely still, a four-rowed square matrix, and so on.

For the states we use a similar description. Each mixture will be described by giving its statistical matrix, as seen by an observer who is restricted to the states  $\Sigma_1, \dots, \Sigma_\nu$ , this being done for every  $\nu = 1, 2, \dots$ . So the description will consist of a sequence of statistical matrices

$$U(1), U(2), \dots, U(\nu), \dots,$$

where  $U^{(\nu)}$  refers to the first  $\nu$  states. We require that  $U^{(\nu)}$  shall be a contraction of  $U^{(\nu+1)}$ :

$$(493) \quad U_{(i_1, \dots, i_\nu)(j_1, \dots, j_\nu)}^{(\nu)} = \frac{1}{2} \sum_{k=0,1} U_{(i_1, \dots, i_\nu, k)(j_1, \dots, j_\nu, k)}^{(\nu+1)}$$

This formula is of the same type as (492) for the observables.

Although the theory presented here seems to be the same as the usual theory, there is nevertheless a real difference. Consider, for example, what happens when we deal with an observable whose nature is such that we do not wish to include it in a valid theory. Let us know that there is exactly one particle in the state  $\Sigma_1$ , and none in any other of the states  $\Sigma_2, \Sigma_3, \dots, \Sigma_\nu, \dots$ . This is a statement about all the systems and if we contract to the first  $\nu$

systems this means that we know that there is no particle in the state  $\sum_{\nu} (\nu \neq 1)$ , and this is true for any large  $\nu$ . But if we know this for arbitrarily large  $\nu$  we can say further that the total number of particles is unity, which we could not say for finite  $\nu$ . Thus as  $\nu \rightarrow \infty$  we obtain more information than we could for large but finite  $\nu$  and we wish to avoid such a situation. For the statement that we have only one particle in a given box has no meaning, as we have already pointed out, because it implies a knowledge of an infinite number of systems.

We can look at this question mathematically. Let the state  $\Sigma$ , be given by the indices  $i_1 = i_2 = \dots = i_{\nu} = 0$ . Then the state vector is given by

$$(494) \quad x_{(i_1, \dots, i_{\nu})} = \begin{cases} 1 & \text{if } i_1 = i_2 = \dots = i_{\nu} = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the statistical matrix is given by

$$(495) \quad u_{(i_1, \dots, i_{\nu})(j_1, \dots, j_{\nu})} = \begin{cases} 2^{\nu} & \text{if } \{i_1, \dots, i_{\nu}\} = \{j_1, \dots, j_{\nu}\} = 0, \\ 0 & \text{otherwise,} \end{cases}$$

Thus we have the following sequence of statistical matrices:

$$u^{(1)} \sim 1,$$

$$u^{(2)} \sim \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix}$$

$$u^{(3)} \sim \begin{vmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$u^{(4)} \sim \begin{vmatrix} 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & & & & & \\ 0 & & & & & & & \\ 0 & & & & & & & \\ 0 & & & & & & & \\ 0 & & & & & & & \\ 0 & & & & & & & \\ 0 & & & & & & & \end{vmatrix}$$

and so on. And as  $\nu \rightarrow \infty$ , we see that the top diagonal element becomes arbitrarily large. We shall later make a restriction that will have the effect of preventing the occurrence of such divergences in the statistical matrices.

We shall now describe a dual procedure. We have been concerned with a state; we now discuss an observable. We want to consider a meaningless observable, such as the total energy in a box, and for simplicity we take the observable that is defined to be unity if there is one particle in the state  $\Sigma_1$  and no particle in any other state, and to be zero if this is not the case. Thus our observable represents the decision whether or not the total number of particles is one or not. Let us consider the corresponding observable referring to the first  $\nu$  states, and let us order the states as in (494). Then the observable will have the matrix

$$(496) \quad a_{(i_1, \dots, i_\nu)(j_1, \dots, j_\nu)} = \begin{cases} 1 & \text{if } \{i_1, \dots, i_\nu\} = \{j_1, \dots, j_\nu\} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Consider how this would be described by an observer who can see only the first  $\mu$  states ( $\mu < \nu$ ). Since

$$(497) \quad a^{(\nu)} = \begin{array}{c} \left| \begin{array}{cccc} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ 0 & & & & \\ \vdots & & & & \\ \dots & & & & \\ \vdots & & & & \\ 0 & \dots & \dots & \dots & 0 \end{array} \right| \begin{array}{c} \uparrow \\ \downarrow \\ 2^\nu \end{array} \end{array},$$

we have, by (492),

$$a^{(\nu-1)} = \begin{array}{c} \left| \begin{array}{cccc} \frac{1}{2} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & & & & \\ \dots & & & & \\ 0 & \dots & \dots & \dots & 0 \end{array} \right| \begin{array}{c} \uparrow \\ \downarrow \\ 2^{\nu-1} \end{array} \end{array},$$

and so on, and ultimately we find that

$$(498) \quad a^{(\mu)} = \begin{array}{c} \left| \begin{array}{cccc} \frac{1}{2^\mu} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & & & & \\ \dots & & & & \\ 0 & \dots & \dots & \dots & 0 \end{array} \right| \begin{array}{c} \uparrow \\ \downarrow \\ 2^\mu \end{array} \end{array}.$$

If we now consider a fixed  $\mu$ , and if  $\nu \rightarrow \infty$ , then this matrix tends to the zero matrix, which means that if we know we have a particle in  $\Sigma_1$  and no particles in  $\Sigma_2, \dots, \Sigma_\mu$ , then the a priori probability that this is still true

for  $\sum_2, \dots, \sum_\nu$  as  $\nu \rightarrow \infty$  is zero.

We need some regularity assumption. It turns out to be best to make this assumption refer to the observables. An instrument cannot measure over an infinite range and we therefore do not actually measure unbounded observables; only observables that are like them over a certain range but are in fact bounded. We therefore impose the condition that only bounded observables are permitted in the theory, where by a "bounded" observable we mean one whose range of values in  $\sum_1, \sum_2, \dots, \sum_\nu$  is uniformly bounded for all  $\nu$ . In symbols, our regularity condition can be written as

$$(499) \quad \overline{\lim}_{\nu \rightarrow \infty} \| a_{ij}^{(\nu)} \| \text{ shall be finite.}$$

(By  $\| b_{ij} \|$  we mean the "absolute value" of the matrix  $(b_{ij})$ : The maximum length of the vector  $(y_i)$ ,  $y_i = \sum_j b_{ij} x_j$ , if the vector  $(x_i)$  is of length 1. This is, for Hermitean  $(b_{ij})$ , the greatest absolute proper value.) Let us consider the statistical matrices. We desire the normalization

$$\frac{1}{2^\nu} \text{Trace } U^{(\nu)} = 1.$$

Since  $U^{(\nu)}$  is a contraction of  $U^{(\nu+1)}$  it is evident that the left hand side is at any rate independent of  $\nu$ . We must therefore require only that  $u^{(0)} = 1$  and then be careful that  $U^{(\nu)}$  remain definite for  $\nu > 0$ . We define  $\text{Exp.}(\mathcal{O}, \sigma)$  as

$$(500) \quad \begin{aligned} \text{Exp.}(\mathcal{O}, \sigma) &= \lim_{\nu \rightarrow \infty} \text{Exp.}_\nu(\mathcal{O}, \sigma) = \\ &= \lim_{\nu \rightarrow \infty} \frac{1}{2^\nu} \text{Trace} (\mathcal{O}^{(\nu)} U^{(\nu)}), \end{aligned}$$

and therefore we require that for all allowed  $\mathcal{O}$ 's the limit

$$(501) \quad \lim_{\nu \rightarrow \infty} \frac{1}{2^\nu} \text{Trace} (\mathcal{O}^{(\nu)} U^{(\nu)})$$

shall exist. This limitation is of the following nature; if we write  $U^{(\mu)}$  for the expansion of  $U^{(\nu)}$  to the same size matrix as  $U^{(\mu)}$ , then we want that

$$(502) \quad \lim_{\substack{\mu, \nu \rightarrow \infty \\ \mu \leq \nu}} (U^{(\nu)} - U^{(\mu)}) = 0.$$

(We will not give here an exhaustive analysis of this "limit"-relation.)

We have seen that the matrices entering the theory obey two types of restriction. Those representing observables must satisfy (499) while those representing mixtures satisfy (501). We shall refer to those of the first type as being of class  $L_{\infty}$  and those of the second type as of class  $L_1$ . We can introduce a restriction intermediate between (499) and (501) by requiring that the mean squared proper value shall be bounded. This condition can be stated in the form that for all  $\nu$ ,

$$(503) \quad \frac{1}{2^{\nu}} \sum_{i,j} |a_{ij}^{(\nu)}|^2 \quad \text{shall be bounded,}$$

and matrices satisfying it may be referred to as of class  $L_2$ . We have

$$(504) \quad \text{Class } L_{\infty} \subset \text{Class } L_2 \subset \text{Class } L_1,$$

but the advantage of (503) is that it is self-dual. Under it the range of the statistical matrices would be the same as that of the matrices of the observables.

We must now discuss the formal properties of our matrices. The following terminology will be useful; we say that two consecutive matrices in a sequence  $A(1), A(2), A(3), \dots$ , are identical if they are connected by expansion, for expansion is merely the adding of a system concerning which we know nothing new. If we take some matrix of order  $2^{\nu}$  and then form those of higher order by expansion, those of lower order being, of course, related to it by contraction, then we shall remain in class  $L_{\infty}$  and will have a system which refers to knowledge of the first  $\nu$  systems. We shall call it a finite system of order  $\nu$ . The totality of such systems will form a basis and any element can be approximated by such matrices. If we take two matrices  $A, B$  of class  $L_{\infty}$ , then

$$\text{Lim } \frac{1}{2^{\nu}} \sum_{i,j} a_{ij} b_{ij} \quad \text{exists}$$

and since this quantity has the algebraic properties of an inner product, the matrices we are considering form something analogous to a Hilbert space,

We ask, What elements of order  $(\nu+1)$  are orthogonal to all elements of order  $\nu$ ? We may consider the element of order  $\nu$  as

$$\begin{array}{|c|c|c|c|} \hline \xleftarrow{2^\nu} & a & b & c & \text{---} \\ \hline & | & | & | & | \\ & | & | & | & | \\ & | & | & | & | \\ \hline \end{array}$$

which can be expanded into the identical element

$$\begin{array}{|c|c|c|c|} \hline \xleftarrow{2^\nu \text{ matrices}} & a0 & b0 & c0 & \text{---} \\ \hline & 0a & 0b & 0c & \\ \hline & | & | & | & | \\ & | & | & | & | \\ & | & | & | & | \\ \hline \end{array}$$

Thus the general matrix of order  $(\nu+1)$

$$\begin{array}{|c|c|c|c|} \hline \xleftarrow{2^\nu \text{ matrices}} & a_{11}, a_{12} & b_{11}, b_{12} & c_{11}, c_{12} & \text{---} \\ \hline & a_{21}, a_{22} & b_{21}, b_{22} & c_{21}, c_{22} & \\ \hline & | & | & | & | \\ & | & | & | & | \\ & | & | & | & | \\ \hline \end{array}$$

will be orthogonal to this if the traces of the individual matrices  $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ , etc. are zero. This means that if we take a matrix of order  $(\nu+1)$

that is orthogonal to the matrices of order  $\nu$  and make a contraction, the result will be zero. Thus those matrices that are identical with the matrices of order  $\nu$  are essentially unaffected by contraction while those that are orthogonal to them are destroyed by this process.

We may make use of the Pauli spin matrices in forming a basis. For  $\nu = 0$  we need only the unit matrix 1. For  $\nu = 1$  we take, in addition to the expansion of the unit matrix, the three spin matrices  $\sigma_1^{(1)}, \sigma_2^{(1)}, \sigma_3^{(1)}$ . For  $\nu = 2$  we may divide all the possible matrices into two classes, those that

are identical with the previous ones and those that are orthogonal to them. The matrices of the first type are just the expansions of the previous matrices. To obtain the matrices of the second type we note that since they are orthogonal to the original matrices they must be the direct products of these matrices with two-rowed square matrices whose traces are zero, and a basis for the latter is given by the Pauli spin matrices. Thus for  $\nu = 2$  we have, in addition to the expansions of the previous matrices,  $1, \sigma_1^{(1)}, \sigma_2^{(1)}, \sigma_3^{(1)}$ , also their direct products with the Pauli matrices which we write at this stage as  $\sigma_1^{(2)}, \sigma_2^{(2)}, \sigma_3^{(2)}$ .

Thus at each stage we add a set of spin matrices  $\sigma_1^{(\nu)}, \sigma_2^{(\nu)}, \sigma_3^{(\nu)}$ , form direct products with all the matrices of the previous stage, and take these new matrices together with the expansions of the new  $\sigma$ 's and of all the previous matrices to form the new basis.

Consider some  $\sigma^{(\nu)}$ . It will be a 2-rowed square matrix having zero trace and being orthogonal to all previous  $\sigma$ 's. It will therefore be orthogonal to any linear combination of them and since any matrix will be a linear combination of all the  $\sigma$ 's, it will tend to zero in the weak sense that

$$(505) \quad \lim_{\nu \rightarrow \infty} \frac{1}{2\nu} \text{Trace} (\mathcal{O}, \sigma^{(\nu)}) = 0$$

for all  $\mathcal{O}$ . This means roughly that most of the elements in the matrix will be zero in the limit, and such a result is also evident from the fact that the matrices form a Hilbert space and (505) expresses the orthogonality of  $\sigma^{(\nu)}$  to all previous elements.

We may here make a comparison with the second quantized theory of Fermi-Dirac assemblies, as described by Jordan and Wigner, with the theory of holes and with Jordan's neutrino theory of light. We need consider only two new  $\sigma$ 's at each stage since the third is always the product of the other two. The  $\sigma$ 's previously considered are such that they anticommute if the upper indices are the same, but commute if these indices are different. However, when we are concerned

with only two new  $\sigma$ 's at each stage we can take all the  $\sigma$ 's so formed as anti-commuting by taking the new  $\sigma$ 's at each stage as, say,  $\sigma_2^{(\nu)}, \sigma_3^{(\nu)}$  and multiplying on the left by  $\sigma_1^{(\nu)}, \sigma_1^{(2)}, \sigma_1^{(3)}, \dots, \sigma_1^{(\nu-1)}$ . We may call the  $\sigma$ 's so obtained again  $\sigma_2^{(\nu)}, \sigma_3^{(\nu)}$  and now all these  $\sigma$ 's will anticommute. We may now define new matrices  $\alpha, \alpha^\dagger$  in the Jordan-Wigner manner by the equations

$$(506) \quad \sigma_2^{(\nu)} + i\sigma_3^{(\nu)} = \alpha_\nu, \quad \sigma_2^{(\nu)} - i\sigma_3^{(\nu)} = \alpha_\nu^\dagger.$$

Since  $\sigma_2^{(\nu)} \rightarrow 0$  and  $\sigma_3^{(\nu)} \rightarrow 0$  it follows that  $\alpha_\nu \rightarrow 0$  and  $\alpha_\nu^\dagger \rightarrow 0$ .

Again we have

$$\begin{aligned} \sigma_1^{(\nu)} &= i\sigma_2^{(\nu)}\sigma_3^{(\nu)} = \frac{1}{2}i[\sigma_2^{(\nu)}, \sigma_3^{(\nu)}] = \\ &= [\sigma_2^{(\nu)} - i\sigma_3^{(\nu)}, \sigma_2^{(\nu)} + i\sigma_3^{(\nu)}] = [\alpha_\nu^\dagger, \alpha_\nu]. \end{aligned}$$

Hence as  $\nu \rightarrow \infty$  we find

$$\alpha_\nu^\dagger \alpha_\nu - \alpha_\nu \alpha_\nu^\dagger \rightarrow 0.$$

But since the  $\alpha$ 's obey the commutation rules

$$[\alpha_\mu^\dagger, \alpha_\nu]_+ = \delta_{\mu\nu},$$

it follows that

$$\alpha_\nu^\dagger \alpha_\nu + \alpha_\nu \alpha_\nu^\dagger = 1$$

and therefore that

$$\alpha_\nu^\dagger \alpha_\nu \rightarrow \frac{1}{2}, \quad \alpha_\nu \alpha_\nu^\dagger \rightarrow \frac{1}{2}.$$

Thus our  $\alpha$ 's will satisfy the following limiting conditions:

$$(507) \quad \begin{cases} \alpha_\nu \rightarrow 0, & \alpha_\nu^\dagger \rightarrow 0; \\ \alpha_\nu^\dagger \alpha_\nu \rightarrow \frac{1}{2}, & \alpha_\nu \alpha_\nu^\dagger \rightarrow \frac{1}{2}. \end{cases}$$

In the original Jordan-Wigner theory we have, on the contrary, the different limiting conditions that

$$(508) \quad \begin{cases} \alpha_\nu \rightarrow 0, & \alpha_\nu^\dagger \rightarrow 0; \\ \alpha_\nu^\dagger \alpha_\nu \rightarrow 0, & \alpha_\nu \alpha_\nu^\dagger \rightarrow 1, \end{cases}$$

while in the theory of holes we have (508) if  $\nu$  runs over "positive" states, and

$$(509) \quad \begin{cases} \alpha_\nu \rightarrow 0, & \alpha_\nu^\dagger \rightarrow 0; \\ \alpha_\nu^\dagger \alpha_\nu \rightarrow 1, & \alpha_\nu \alpha_\nu^\dagger \rightarrow 0 \end{cases}$$

if  $\nu$  runs over "negative" states. The situation is the same in Jordan's neutrino theory. Thus in the theory of holes most of the positive high-energy levels are assumed to be filled, while the negative ones are assumed to be empty; but in the present theory there is an even chance that they are occupied or not.

When a statement is made within the framework of the usual quantum theory, we can transform it into a statement belonging to the present theory by simply replacing the  $\alpha$ 's of the old theory by the  $\alpha$ 's of the new.

We must now consider the role of the Schrödinger equation. In the ordinary quantum theory it is

$$(510) \quad \frac{\partial}{\partial t} \varphi = i A \varphi \quad (A = \frac{2\pi}{h} H),$$

where  $A$  is a Hermitean operator, being a multiple of the energy operator  $H$ . If

$H$  is independent of the time we can solve this equation explicitly:

$$(511) \quad \varphi_t = e^{itA} \varphi_0.$$

For the statistical matrix corresponding to this state we have

$$(512) \quad U_t = e^{itA} U_0 e^{-itA}.$$

For an observable we obtain the result

$$(513) \quad O_t = e^{itA} O_0 e^{-itA}.$$

These give an isomorphism of the system on itself of the type called inner. (An inner automorphism is of the type

$$K \rightarrow K' = V^{-1}KV,$$

where  $V$  is a fixed unitary operator.)

In the present theory the Schrödinger equation cannot have this form

since no total energy operator will exist. We must see therefore to what extent the form of the Schrödinger equation can be modified.

With the lapse of time a system, if left undisturbed, must undergo an automorphism. The observable  $\mathcal{O} = \mathcal{O}_0$  will thereby become  $\mathcal{O}_t$ . The meaning of  $\mathcal{O}_t$  is merely that instead of making the observation  $\mathcal{O}_0$  now, we wait a time  $t$  seconds and then perform it. Let us denote this isomorphism by

$$(514) \quad \mathcal{O}_t = \mathcal{J}_t \mathcal{O}_0.$$

Then since there is no difference between the moment 0 and the moment  $s$ , we have

$$\mathcal{O}_{s+t} = \mathcal{J}_t \mathcal{O}_s$$

and hence

$$(515) \quad \mathcal{J}_{s+t} = \mathcal{J}_s \mathcal{J}_t,$$

i.e. the  $\mathcal{J}_t$  form a one-parameter group.

Now if the  $\mathcal{J}_t$  are inner isomorphisms, that is if

$$(516) \quad \mathcal{O}_t = \mathcal{J}_t \mathcal{O}_0 = V_t \mathcal{O}_0 V_t^{-1} \quad (V_t \text{ unitary}),$$

then (515) implies that

$$V_{s+t} = c_{st} V_t V_s \quad (c_{st} \text{ a constant}).$$

From this one can derive, purely mathematically, the result

$$V_t = d_s e^{itA},$$

where  $d_s$  is a constant and  $A$  a fixed operator. Thus

$$(517) \quad \mathcal{O}_t = \mathcal{J}_t \mathcal{O}_0 = e^{itA} \mathcal{O}_0 e^{-itA}.$$

This gives for the observables

$$(518) \quad \frac{\partial}{\partial t} \mathcal{O}_t = i(A \mathcal{O}_t - \mathcal{O}_t A);$$

and, from the invariance of  $\text{Tr}(\mathcal{O}_t U_t)$  we obtain for the statistical matrices

$$(519) \quad \frac{\partial}{\partial t} U_t = i(A U_t - U_t A).$$

Although the deduction does not necessitate the identification of  $A$  with

$\frac{2\pi}{h} H$ , where  $H$  is the total energy operator, the well-known physical (correspondence) arguments make it very hard to have  $A$  be anything else.

So we see that the customary form of the Schrödinger equation is essentially dependent on all isomorphisms  $\mathcal{J}_t$  being inner isomorphisms.

In the usual theory, where all bounded operators correspond to observables, this is inescapable since it can be shown mathematically that the system of all bounded operators possesses inner isomorphisms only.

However, the system of matrix sequences such as we have described can be shown to possess also non-inner isomorphisms. We can thus avoid the present form of the Schrödinger equation and therefore we no longer need the existence of a total energy operator. This is necessary for formal reasons also, for it can be shown that in our present system no four-operators can form a four-vector for any unitary representation of the Lorentz group -- not even if we introduce unbounded operators. Thus, if the resulting theory is to be special-relativistically invariant, the total energy and the total momenta cannot be observables.

Consider the systems  $\Sigma_1, \Sigma_2, \Sigma_3, \dots$  with the corresponding proper energies  $\epsilon_1, \epsilon_2, \epsilon_3, \dots$  divergent. If we try to handle this as a Fermi-Dirac assembly, we must form an energy

$$\sum_{\nu} \epsilon_{\nu} \alpha_{\nu}^{\dagger} \alpha_{\nu} .$$

The only reason that this does not diverge is that in the usual theory we have

$\alpha_{\nu}^{\dagger} \alpha_{\nu} \rightarrow 0$  sufficiently strongly to overcome the divergence due to the  $\epsilon$ 's.

In the present theory, since  $\alpha_{\nu}^{\dagger} \alpha_{\nu} \rightarrow \frac{1}{2}$ , we would get a divergent result. But

though the Hamiltonian is divergent and a Schrödinger equation would thus have no meaning, we can still obtain an automorphism which will leave both the commutation

rules and the limiting values invariant. We merely replace  $\alpha_{\nu}$  by  $e^{i\epsilon_{\nu}t} \alpha_{\nu}$ .

The point is that this is not an inner automorphism of the system and yet it is

actually the common type of automorphism in the present theory; the inner isomorphisms are not the usual ones, as our simple example has shown, and thus we

cannot look for a differential equation in the present theory analogous to the

Schrödinger equation.

We cannot treat Bose-Einstein assemblies directly by the method we have just described since the corresponding distinguishable classical oscillators each have an infinite number of dimensions. It is necessary to use the De Broglie-Jordan trick of replacing each Bose-Einstein particle by several Fermi-Dirac particles. If we write  $\beta_\nu, \beta_\nu^+$  for the matrices describing the Bose-Einstein particles, we have

$$(520) \quad \begin{cases} \beta_\mu \beta_\nu^+ - \beta_\nu^+ \beta_\mu = \delta_{\mu\nu}, \\ \beta_\mu \beta_\nu - \beta_\nu \beta_\mu = 0, \end{cases}$$

while for Fermi-Dirac particles we have

$$(521) \quad \begin{cases} \alpha_\mu \alpha_\nu^+ + \alpha_\nu^+ \alpha_\mu = \delta_{\mu\nu}, \\ \alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu = 0. \end{cases}$$

Jordan succeeded in obtaining (520) from (521) by means of an algebraic trick.

He wrote a relation of the type

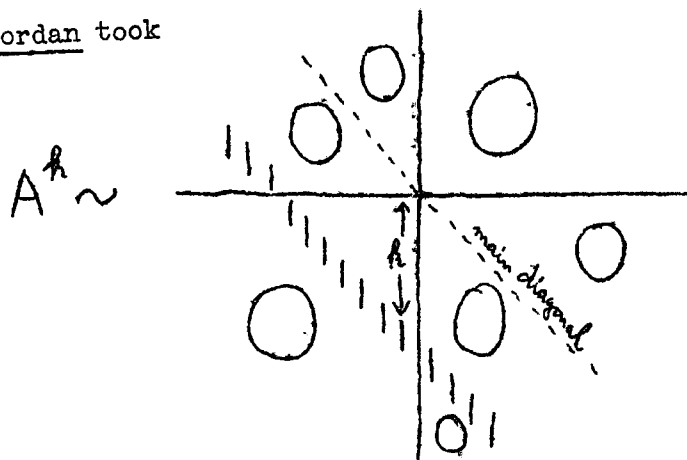
$$(522) \quad \beta_\mu = \sum_{\lambda, \nu} A_{\lambda\nu}^\mu \alpha_\lambda^+ \alpha_\nu$$

and from this it is seen that the sums of  $\beta$ 's are given by the sums of their corresponding A's, while though the relation of the products of  $\beta$ 's to those of the A's is not simple, the commutators of the  $\beta$ 's correspond to the commutators of their A's. Thus the equations (520) become

$$\begin{cases} A^\mu A^{\nu+} - A^{\nu+} A^\mu = \delta_{\mu\nu}, \\ A^\mu A^\nu - A^\nu A^\mu = 0, \end{cases}$$

where  $A^\mu$  is the matrix  $(A_{\lambda\nu}^\mu)$ . These matrix equations possess solutions.

Actually Jordan took



these matrices commute so that it seems impossible that we should get anything but zero on the right of (520). However, the actual result is

$$\alpha_\nu + \alpha_\nu \Big]_{\nu=-\infty}^{\nu=+\infty} \text{ and since } \alpha_\nu + \alpha_\nu \rightarrow 1 \text{ for } \nu \rightarrow +\infty \text{ and } \rightarrow 0 \text{ for } \nu \rightarrow -\infty,$$

we get the  $\delta_{\mu\nu}$  we need. (This is explained also on p. 109 of these notes.)

It really amounts to summing the infinite series

$$\dots + 1 - 1 + 1 - 1 + 1 - \dots$$

and in accordance with the principles of the subtraction physics we may say that since the two ends of the series are each (+1) the sum is (+1)!

Finally, a remark concerning Lorentz invariance. The systems we have described can be considered as generated by abstract elements  $\alpha_\nu$  which satisfy (a) the Jordan-Wigner commutation rules, and (b) the limiting relations (507).

Under the Lorentz group the  $\alpha$ 's are transformed by

$$(523) \quad \alpha_\nu \rightarrow \alpha'_\nu = \sum_{\mu} \sigma_{\nu\mu} \alpha_\mu,$$

where  $\sigma$  is unitary. We can show that this converges and that it gives an isomorphism of the system on itself, and this is enough to ensure the Lorentz invariance since the matrices  $(\sigma_{\nu\mu})$  may be chosen so as to form a representation of the Lorentz group. The isomorphism (523) is not an inner isomorphism so that we shall not have infinitesimal generators for these isomorphisms.

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## APPENDIX

THE FOUNDATIONS OF QUANTUM THEORY

by

Professor Wolfgang Pauli

(Lecture given before Professor Ladenburg's Seminar. Notes by M.H.L.Pryce)

It is impossible in one hour to do anything but pick out a few arbitrary things in the quantum theory that come into mind. In particular, I should like to talk about the applications of the concepts of the quantum theory to light and the electromagnetic field. But first, a few generalities about the quantum theory in its application to matter.

It is well known that it starts from the correspondence between energy and frequency given by Planck's rule

$$E = h\nu$$

and from the generalization of this by de Broglie to the relation between momentum and wave-number:

$$\vec{p} = h\vec{k}$$

in a wave given by

$$\psi = a e^{i[(\vec{k}\cdot\vec{x}) - \nu t]}$$

This is a particular case, namely a plane wave, and corresponds to a knowledge of the momentum and energy. In wider generality we apply a superposition principle, and build up a wave function:

$$\psi(x,t) = \int d\vec{k} \cdot a(k) e^{i[(\vec{k}\cdot\vec{x}) - \nu t]}$$

We take the relation between  $E$  and  $p$  from the classical theory. This gives us a dependence of  $\nu$  upon  $k$ :

Now what does  $\psi(x, t)$  signify? We interpret it as meaning that  $|a(k)|^2 d\vec{k}$  is the probability that  $\vec{k}$  will be between  $\vec{k}$  and  $\vec{k} + d\vec{k}$ , and that  $|\psi(x)|$  is the probability of the particle lying between  $\vec{x}$  and  $\vec{x} + d\vec{x}$ .

$\psi$  has therefore no direct connection with physical measurements. It is a mathematical concept that enables us to compute all probabilities that we want. This limits the applicability of the wave picture and on the other hand of the particle picture, and makes the theory self-consistent. This leads us to the fundamental uncertainty principle of quantum mechanics, which is alien to the concepts of classical theory:

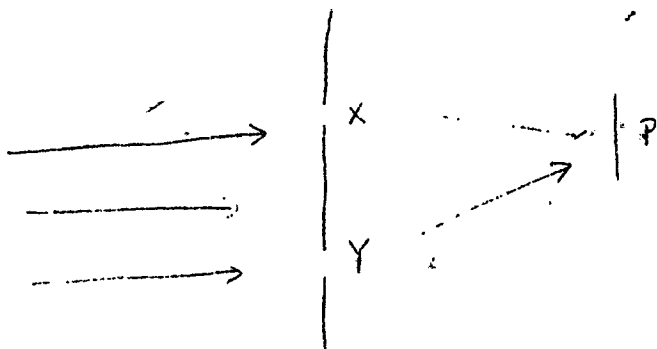
$$\Delta P \cdot \Delta X \approx h$$

The meaning of this has been widely discussed, and it will be of interest to go into it a little here. It tells us that we cannot simultaneously give exact values to two such variables as  $\underline{x}$  and  $\underline{P}$ . There is a difference between the state of affairs in quantum theory, and that described by saying that we do not know the values of  $\underline{x}$  and  $\underline{P}$  simultaneously; it insists that we cannot know them. It is very closely connected with the quantum-mechanical concept of the state of a system, and the reconciliation of the wave and particle picture.

In classical mechanics the state of a system is given by the values of all the variables simultaneously. In quantum mechanics this is impossible. The most we can give is the values of all those variables which are simultaneously measurable; this is half the number of all independent variables. The knowledge of the quantum mechanical state therefore gives much less definite information about the system than we should have in classical mechanics. Furthermore, in classical mechanics, it is possible to find out what the state of the system is by performing a measurement of all the independent variables. In quantum mechanics every measurement in general changes the state of the system. Only if we repeat a certain measurement twice in rapid succession are we sure of getting the

same answer, and so we can say that the second measurement left the state unchanged; but this is a very particular case. We can therefore know the state only after performing a maximum number of compatible observations. If we do not perform a maximum number, we cannot know which state the system is in afterwards, but only that it is distributed among certain states. Such a condition is called a mixture, and a condition where a maximum observation has been made is called a pure state.

If we have two states, we can form new states from them (mathematically) by superposition; these are pure states, and different from the mixtures that can also be formed. An illustration of this is afforded by the following system. We have a screen with two holes in it, and we allow an electron beam to strike the screen. If one hole X is closed there will be a certain probability of observing an electron



hitting a plate P (or some other means of observation). If the other hole Y is closed, and X is opened, there will be a different probability of observing an electron at P. If it is known that either one of X or Y is open and the other shut, but not which, then the condition obtaining is a mixture of the two states described before. If, however, both holes are open, the state of the system will be a superposition of the two states. In the first case the probability of finding an electron at P is a linear combination (the coefficients being the probabilities of X and Y being open respectively) of the probabilities when X or Y are open separately. In the second case it is not, but interference takes place, in a manner similar to the interference of light.

It may happen that the wave-function vanishes except at one point; there is then a certainty about the position of the particle. This is expressed statistically by a unit probability.

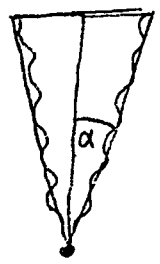
In the quantum theory knowledge can get lost by observation. In the classical theory this is not the case; one can only increase one's knowledge by observation. For instance, in astronomy one can compute the orbit of a planet with greater and greater precision by making a larger number of observations. But such a procedure is no longer possible with an electron in the region of the uncertainty principle. Every observation causes an uncertain amount of interaction with other systems, which we can only observe by making them interact with yet more systems, and so on without progress. We can never know all about the disturbance imposed by measurement.

A deep problem is connected with the composition of two systems to form another system. About this there has been a good deal of controversy. How can one conclude from the state of the whole system information about the state of one of the parts? If we have a system consisting of two particles moving on a line, with coordinates  $x_1$  and  $x_2$ , the state of it <sup>is</sup> described by  $\Psi(x_1, x_2)$ , but this is by no means equivalent to two wave-functions, one for each particle, even when there is no interaction between the systems. This is closely connected with the essentially statistical nature of the theory. The interpretation contains conditional clauses of the type: "If I measure this, I will get that", and gives correlations between the systems as well as knowledge about them independently. The viewpoint of Einstein, that the quantum mechanical state really describes an ensemble of systems, rather than one system, is probably correct. The essential difference between classical and quantum theory then lies in the fact that there are irreducible ensembles, corresponding to pure states, which cannot be split up into simpler ones, whereas classically an ensemble can always be reduced to a

single system in a known classical state. The simplest quantum mechanical ensembles must still be regarded as ensembles, and not single objects.

Let us pass to other important issues. The one which I wish to discuss now is the theory of light and electromagnetic field. The quantum theory of light started from the idea that the classical theory exaggerated the difference between light and matter. We know now that light can communicate momentum and energy in discrete amounts, and that matter shows interference phenomena. Einstein put forward the hypothesis of the light quantum to explain the phenomena in which light gives up its energy discretely. The equation  $E = h\nu$ ,  $\vec{p} = h\vec{k}$  is fundamental in such processes. We ask at once the question: "Is light really propagated in spherical waves, or is it shot out in a directional manner, like a particle?" How could we answer this question? We might take an atom, and set about verifying the coherence of the radiation waves issuing from it, or the conservation of momentum in the recoil, depending on which aspect we were investigating.

In the first case, to prove that the light from an atom is coherent and can be made to interfere in a certain region, we must know the position of the source, i.e. the atom, to a certain amount of accuracy. In fact, if the test



region subtends an angle  $\alpha$  at the atom, we must know the position to within a distance  $\Delta x$  given by

$$\Delta x < \lambda / 2 \sin \alpha$$

On the other hand, if one wanted to prove the directed nature of light by recoil with the atom,

one would measure the momentum of the atom before and after emission. To do this

with sufficient accuracy to give the experiment any sense, one must know the x-momentum to within  $\Delta p_x$  given by

$$\Delta p_x < 2h \sin \alpha / \lambda$$

The uncertainty principle, applied to the atom, therefore excludes the possibility of observing both aspects at the same time.

It is interesting to notice that directed emission is not in any way in contradiction to the Maxwell theory, because we can have currents extended in space when  $\Delta x > \lambda / 2 \sin \alpha$  which give rise to a directed emission. In fact, if the current varies like a plane sine-wave, the directed nature of the emitted radiation is complete. This of course corresponds to a knowledge of the momentum of the atom and total ignorance of its position.

All this, however, is merely an introduction to the uncertainty principle for the field. We can quite well ask what the application of quantum theory to the field in the vacuum will be. In some senses this is an empty theory, because any attempt to test it must be through the interaction of the field with matter, which is not considered in the theory of the vacuum. Nevertheless it is possible to the quantum theory of the field in a manner abstracted from any consideration of the sources of the field. The measurements can be considered phenomenologically to be performed with non-atomistic test bodies, such as mirrors and bounding walls, etc.

Classically the field is equivalent to a set of oscillators, infinite in number, and of every frequency. In the quantum theory of the field these oscillators are treated as quantized oscillators. Pauli and Jordan first investigated this, and found that it led to certain commutation rules for the field strengths:

$$[E_x^{(1)}, E_y^{(2)}] = [H_x^{(1)}, H_y^{(2)}] = \text{ctk} (A_{xy}^{(12)} - A_{xy}^{(21)})$$

$$[E_x^{(1)}, H_x^{(2)}] = 0$$

$$[E_x^{(1)}, H_y^{(2)}] = -[H_x^{(1)}, E_y^{(2)}] = \text{ctk} (B_{xy}^{(12)} - B_{xy}^{(21)})$$

where

$$A_{xx}^{(12)} = - \left( \frac{\partial^2}{\partial x_1 \partial x_2} - \frac{1}{c^2} \frac{\partial^2}{\partial t_1 \partial t_2} \right) \left\{ \frac{1}{r} \delta(t_2 - t_1 - r/c) \right\}$$

$$A_{xy}^{(12)} = - \left( \frac{\partial^2}{\partial x_1 \partial y_2} \right) \left\{ \frac{1}{r} \delta(t_2 - t_1 - r/c) \right\}$$

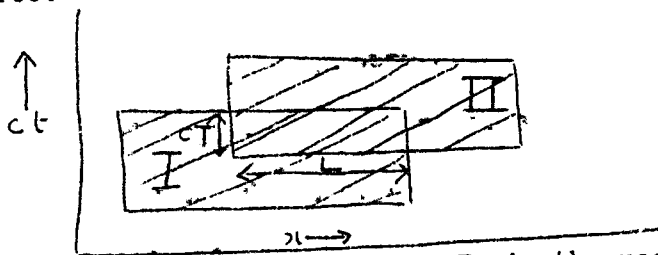
$$B_{xy}^{(12)} = - \frac{1}{c} \left( \frac{\partial^2}{\partial t_1 \partial z_2} \right) \left\{ \frac{1}{r} \delta(t_2 - t_1 - r/c) \right\}$$

and the upper index (1) or (2) indicates that the field strengths are taken at the point  $(x_1, y_1, z_1, t_1)$  or  $(x_2, y_2, z_2, t_2)$ , and the suffix  $x, y$  labels the component of a vector. The square bracket means

$$[E_x, E_y] \equiv E_x E_y - E_y E_x$$

and  $r$  is the distance between  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ .

We can interpret such formulae by finding the commutation rules for quantities which are the average values of the field in certain space-time regions. In the figure such regions are shown, with only one space dimension drawn in, instead of three.



If  $E_x^I$  etc. denote the average value of  $E_x$  in the region I, and the space extension is bigger than the time extension, i.e.  $L \gg ct$ , the uncertainty principle corresponding to the above commutation rules becomes

$$\Delta E_x^I \cdot \Delta E_x^{II} \sim \frac{h}{L^3 \cdot T}$$

So far the theory makes no appeal to the use of any test bodies, but is purely formal in character. Bohr and Rosenfeld have given a very thorough discussion of the problem of measuring field strengths from a phenomenological point of view. They find that if the dimensions of the test body used to do the measurements with are large compared to  $\hbar/Mc$ , where  $M$  is its mass, and the charge on

it is large compared to  $\sqrt{h c}$  (i.e.  $\sqrt{137}$  times the electronic charge), then it is possible to carry through the measurements consistently in regions comparable to the size of the test body, provided the atomic nature is disregarded. They reach the same uncertainty relations.

In the Pauli-Jordan theory there is no characteristic length or mass, and the theory is independent of any limitations or abstractions of matter. The uncertainty relations for the field therefore seem to me to be independent of those for matter, and to arise from the quantum treatment of the field alone. I find it difficult to believe that a return to any classical form of field theory will help.

The quantum theory is very satisfactory as it stands in two fields:

1) in the non-relativistic theory of matter, where it gives excellent results in the field of spectra, atomic collisions etc.; and 2) in the theory of the electromagnetic field so long as one does not try to treat the sources of the field as well, where it gives a correct theory of the transmission, interference and particle-like nature of light. In other words, it works very well so long as we can treat one or other of the field or matter as a given "external" influence, but when we try to unite the two to give a theory describing the interaction, we find that it is not at all so satisfactory.

This is closely connected with Dirac's attempt to give a real relativistic theory of the electron. In this attempt the success seems to have been on the side of Dirac rather than of logic. His theory consisted in a number of logical jumps. First he made a theory for one particle; this is allowed so long as the number of particles in a system is conserved. From certain postulates he derived an equation which describes the relativistic motion of an electron, and in a natural manner, the spin of the electron, and its magnetic moment. But the equation allowed the electron to be in states of negative energy, and if one allowed it to

interact with the electromagnetic field there was nothing to stop it from making transitions to these states. To avoid this conflict with observation, Dirac made a logical jump; he assumed that all the negative states were filled up with electrons, and then transitions to them were forbidden by the exclusion principle. This leads to the theory of holes for the positron, which I do not like at all. There is no longer a conservation of the total number of particles, when one considers positrons, and so Dirac's argument for the form of the wave-equation is no longer cogent because then there no longer exists any a priori reason that the wave equation shall be of the first order and the charge density shall be a sum of squares. From what has been said here, it appears that the creation and annihilation of particles in pairs is closely connected with relativity.

In spite of these logical objections, the theory gives results of the right dimensions. It has important consequences for the description of the vacuum. For instance it is no longer possible to measure the density of charge in some region, because any measurement will cause an unknown number of electron-positron pairs to be created, and so there arises an unknown fluctuation. Only the total charge present can be measured, since this is unchanged by the creation of a pair. The theory of holes in this way modifies the theory of the electromagnetic <sup>(Field)</sup>; according to the work of Bohr and Rosenfeld, it should have been possible to <sup>^</sup>measure the charge included in any region, however small, since it is given by the flux of  $E$ , if one had sufficiently small apparatus, and one could consider it as non-atomistic.

The analogy between light and matter is extended further by the similar phenomenon with light. When measuring field strengths one creates an unknown number of photons. There is, however, a fundamental difference because of the conservation of charge for matter, which has no analogue in the field. The concept of charge is very fundamental, probably more so than the concept of matter. An-

other difference is that electrons obey the exclusion principle and photons the Einstein-Bose statistics.

It seems that quantum theory is always successful when describing systems with a finite number of degrees of freedom, but that when dealing with systems possessing infinitely many degrees of freedom it causes divergent results to appear. The field in a vacuum possesses an infinite number of degrees of freedom, since it is equivalent to an infinity of oscillators, but in this case the infinities take on a fairly harmless form. It turns out that the energy of the field is infinite, but one may subtract this infinity in a well-defined way, and get finite results for the difference of the energies of two different states of the system. One already sees here the beginning of the "subtraction physics" which is such an unsatisfactory feature of the present theory. A worse infinity turns up when we evaluate the gravitational energy of a photon. The essential fact to be drawn from this is that even when the underlying classical theory is perfectly regular, the quantum theory leads to infinities; and this is quite general when we have an infinite number of degrees of freedom, and non-linear field equations.

The theory of holes postulates an infinite number of electrons, and therefore comes into the same category. Here the infinities are far worse. Not only is the energy infinite, but also the polarizability of the vacuum. An external electron field will create pairs of electrons and positrons; which polarize the vacuum, and a charged particle will surround itself with the particles of opposite charge created by its field; the theory leads to infinite results for these phenomena.

At the present moment J. von Neumann is trying to develop a theory which will avoid these difficulties for systems with infinitely many degrees of freedom, with a different concept of state for such systems.

It seems to me that our present methods are not fundamental enough, and that there are two possibilities for overcoming the difficulties. The first is to change our concept of space and time in small regions. The second, to change the concept of state for systems with an infinite number of degrees of freedom, as von Neumann does. I believe that the development of the theory along the correct lines will then lead to a numerical value of the fine-structure constant  $\alpha = e^2/\hbar c = 1/137$ , and to an explanation of the fact that arbitrarily high masses do not appear concentrated in a given space region, in nature. It seems likely that the future theory will be unitary in the sense that the duality of light and matter will disappear. By this I do not claim that we shall necessarily explain one in terms of the other, but perhaps both in terms of some more fundamental concept. Certain unitary theories have been tried, to describe the interaction of radiation and the light particles, but I do not believe that it will be possible to found a satisfactory theory without bringing in the heavy particles as well from the start. I believe that the numerical value of  $\alpha$  is closely bound up with the heavy masses and the ratio of the mass of the proton to the mass of the electron. This seems so to me, because the logical possibility of having arbitrarily high masses concentrated in small space-regions would preclude the possibility of altering the space-time concept, which seems necessary in view of the infinities inherent in unitary field theories. There is no indication in either present day theory or experimental datum that seems to me to justify the hope that a return to a classical form of field theory will help. On the contrary, the classical concept of a field seems no more fundamental than the classical concept of a particle.