Lectures

by

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on

THEORY OF INVARIANTS

Fall Term 1936

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APOLOGIA

It was at first decided that no notes would be taken of these lectures. But the listeners finally rebelled, and requested that some sort of mentally refreshing outline, however sketchy, be made available. The following meager response to their cry certainly satisfies the sketchiness requirement, yet at the same time an effort at completeness has been made by giving exact references to the literature and to notes of previous years.

A fervent apology must be made for the poor stencilling. The following list may give the reader a clue in his moments of despair:

p.3 l.19. $$\frac{\partial x^i}{\partial s^\alpha} = \sum_\sigma u^i_\sigma(x)\,\lambda^\sigma_\alpha(s)$$

l.24 $$\frac{\partial t^\alpha}{\partial s^\beta} = \sum_\sigma \mu^\alpha_\sigma(t)\,\lambda^\sigma_\beta(s)$$

l.25 $$t^\alpha = \psi^\alpha(t_0\,;s)$$

p.4 l.1 $$\varphi^i(\varphi(x_0\,;t_0);s) = \varphi^i(x_0\,;\psi(t_0\,;s))$$

l.8. $$u_i(x) = \sum_k a_{ik}\,x_k = (a\,x)_i \;;\quad a = \| a_{ik} \|$$

p.6 l9. $$d_\ell f = \sum \alpha_{ij}\,x_j\,\frac{\partial f}{\partial x_i} + \sum \beta_{ij}\,y_j\,\frac{\partial f}{\partial y_i} + \dots + \sum \gamma_{ij}\,z_j\,\frac{\partial f}{\partial z_i}$$

l18 $d_\ell f = c_\ell f$ where $c_\ell = c_1 c_{\ell_1} + \dots + c_n c_{\ell_n}$

p7 l4. $$\mu(s) = |s|^g\,\overline{|s|}^{g'}$$

l15 $$j = [f] = \frac{1}{h}\sum_{s \text{ in } G} s f$$

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p8 l16 $\quad e = \sum \delta s_{ik}\, e_{ik}$

l18 $\quad \sum \delta s_{ik}\, d_{e_{ik}} J = 0$

p9 ll1-3
$$\begin{cases} \delta s_{ii} = \sqrt{-1}\, \delta\sigma_{ii} \\ \delta s_{ik} = \delta\sigma_{ki} + \sqrt{-1}\cdot\delta\sigma_{ik} \\ \delta s_{ki} = -\delta\sigma_{ki} + \sqrt{-1}\,\delta\sigma_{ik} \end{cases}$$

p12
$$\begin{vmatrix} \frac{\partial}{\partial s_{11}} & \cdots & \frac{\partial}{\partial s_{1n}} \\ \cdots & \cdots & \cdots \\ \frac{\partial}{\partial s_{n1}} & \cdots & \frac{\partial}{\partial s_{nn}} \end{vmatrix} = \sum_{(i)} \pm \frac{\partial^n}{\partial s_{1i_1}\,\partial s_{2i_2}\cdots\partial s_{ni_n}}$$

p16
$$\underline{A} = \sum \frac{r!}{r_0!\cdots r_n!}\,\frac{\rho!}{\rho_0!\cdots\rho_n!}\,A_{r_0\cdots r_n\rho_0\cdots\rho_n} \times \xi_0^{r_0}\cdots\xi_n^{r_n}\,x_0^{\rho_0}\cdots x_n^{\rho_n}$$

p19 l14 $\quad \left[\sum_k \gamma_{ik} x_k, \ldots, \sum_k \gamma_{ik} y_k, \xi, \ldots, \eta\right]$

p25 $\quad \delta y_1 = y_2,\ \delta y_2 = y_3, \ldots, \delta y_{\mu-1} = y_\mu,\ \delta y_\mu = 0$

p26 l3
$$\varphi_y(\xi,\eta) = y_\mu\,\xi^{\mu-1} + (\mu-1)\,y_{\mu-1}\,\xi^{\mu-2}\eta + (\mu-1)(\mu-2)\,y_{\mu-2}\,\xi^{\mu-3}\eta^2 + \cdots + (\mu-1)!\,y_1\,\eta^{\mu-1}$$

p26 l11. $\delta\varphi_y = (\mu-1)y_\mu\,\xi^{\mu-2}\eta + (\mu-1)(\mu-2)y_{\mu-1}\,\xi^{\mu-3}\eta^2 + \dots$

p29. $$\prod_{k=1}^{h}\left(u + u_1 x_1^{(k)} + \dots + u_n x_n^{(k)}\right) =$$

$$\sum u^\alpha u_1^{\alpha_1}\dots u_n^{\alpha_n}\, G_{\alpha\alpha_1\dots\alpha_n}(x_1,\dots,x_n)$$

$(\alpha + \alpha_1 + \dots + \alpha_n = h)$

p30 l11

$$\bar{g}\left(\frac{\partial}{\partial x}\right)f(x) = \sum_{(k)} \bar{b}_{k_1\dots k_n}\frac{\partial^q f}{\partial x_1^{k_1}\dots\partial x_n^{k_n}}$$

p36 $$F(z_1,\dots,z_e) =$$

$$\frac{|1\ z\ \dots\ z^{e-n-1}\ z^{l_n+e-n}\ z^{l_{n-1}+e-n}\ \dots\ z^{l_1+e-n}|}{|1\ z\ z^2\ \dots\ z^{e-1}|}$$

THEORY OF INVARIANTS

Introduction

Let G be an abstract group with elements $s, t, \ldots$. Let $s \to A(s)$, $s \to B(s), \ldots, s \to C(s)$ be a finite number of representations of G in a field K. Let $x, y, \ldots, z$ denote variable vectors in the respective representation spaces; the element s of G induces the linear transformations $x \to A(s)x$, $y \to B(s)y, \ldots, z \to C(s)z$. A form $f(x, y, \ldots, z)$, homogeneous in the components of each variable vector $x, y, \ldots, z$ separately, and with coefficients in K, is an invariant of G, relative to the given representations, if for all s in G

$$f(A(s)x, B(s)y, \ldots, C(s)z) = f(x, y, \ldots, z).$$

This is the situation with which the "general problem of invariants" has to deal [pp. 5-7 of Elementary Theory of Invariants, by Prof. Hermann Weyl, 1935-36. Notes by Prof. Weyl and Dr. Leonard M. Blumenthal; mimeographed. Cited as 1935-36 Notes.] The "first main problem" is to discover whether or not there exist a finite number of invariants in terms of which all others are rationally and integrally expressible ("rational integral basis"), and to construct such a set if possible. No example is known of a group for which this "first main theorem" (existence of a rational integral basis) does not hold. The "second main problem" is to discover whether or not there exist a finite number of relations ("syzygies") among the basic invariants such that all other relations are algebraic consequences of them, and to construct such a set if possible. The existential part is answered in the affirmative ("second main theorem") for all groups in which the first main theorem holds [1935-36 Notes, pp. 13-20].

In the classical theory, G is the full linear group, and the representations are those generated by indeterminate algebraic forms (the "symmetric tensor representations" of G) [1935-36 Notes, pp. 22-26, 110-130]. Strictly speaking, the invariants considered are relative and not absolute (they are absolute invariants of the unimodular group). If in the above definition $f(x, y, \ldots, z)$ appears on the right multiplied by a numerical function $\mu(s)$, it is a relative invariant and $\mu(s)$ is its weight [1935-36 Notes, pp. 26-30]; if $\mu(s) = 1$ for all s in G, it is an absolute invariant.

The "special problem" ("vector invariants") deals with the particular case of a linear group G in which the representations are either G itself or its contragredient representation, and the corresponding variable vectors $x, y, \ldots, z$ are called respectively covariant or contravariant [1935-36 Notes, pp. 20-22]. In Chapter I of the 1935-36 Notes both the first and second main problems are solved constructively for vector invariants of the more important groups: unimodular, orthogonal, complex, and their extensions.

CHAPTER I. GENERAL THEORY OF INVARIANTS

A. Differential and Integral Methods

1. Groups, their realizations and representations

By a realization of an abstract group G we mean a homomorphism of G onto a group H of transformations (one-to-one correspondences) of any abstract class ("point field") onto itself. The realization is true if the homeomorphism is one-to-one (isomorphism). Every group G has a true realization, namely that afforded by the left-translations $s \to as$ of G [pp. 2-4 of The Structure and Representation of Continuous Groups, by Prof. Hermann Weyl, 1934 (second semester). Notes by N. Jacobson; mimeographed. Cited as 1934 Notes.]

A representation of G is a realization of G by means of a group H of linear transformations; the point field is in this case a linear space $\mathcal{R}$. To each element s of G there corresponds a linear transformation or matrix $A(s)$

obeying the law $A(st) = A(s)A(t)$ for all s, t in G. We refer to the 1934 Notes for the following fundamental definitions: equivalence of two representations (p. 11); invariant subspaces of $\mathcal{H}$ reducible and fully reducible representations (p. 10); character of a representation (p. 16). The whole first chapter of the 1934 Notes is recommended as an introduction to representation theory; the same material will be found in Chapters I and III of Prof. Weyl's book, The Theory of Groups and Quantum Mechanics.

2. Lie's infinitesimal approach: first stage of differentiation

3. Second stage of differentiation: commutator product

4. Third stage of differentiation: Jacobi's law

5. Inversion problems

Those four sections are a repetition of Chapter 2 of the 1934 Notes (pp. 32-44). By "inversion problems" we mean Lie's fundamental theorems: first, the realization of an abstract infinitesimal group (Lie algebra) by a "velocity field"; second, the extension of a velocity field to a "group germ". The proof of the latter amounts to showing that we can determine functions $\lambda_\alpha^\sigma(s)$ such that the equations (1.9) on p. 34 of the 1934 Notes are completely integrable:

$$\frac{\partial x^i}{\partial s^\alpha} = \sum_\sigma u_\sigma^i(x)\,\lambda_\alpha^\sigma(s)$$

The theory of total differential equations tells us that a solution $x^i = \varphi^i(x_0; s)$, depending on arbitrary parameters x_0^j, exists for a finite domain D in the space of the s^α containing the origin $s^\alpha = 0$. The equations (1.11), p. 34

$$\frac{\partial t^\alpha}{\partial s^\beta} = \sum_\sigma \mu_\sigma^\alpha(t)\,\lambda_\beta^\sigma(s)$$

are likewise completely integrable, and have a solution $t^\alpha = \psi^\alpha(t_0; s)$ in some finite domain about the origin. We can restrict the latter to a domain D' such that if t_0 and s are in D', then t will be in D. Within D' the finite conditions for a group are satisfied:

$$\varphi^i(\varphi(x_0;t_0);s) = \varphi^i(x_0;\psi(t_0;s)).$$

That this "group germ" can be extended -- by some sort of continuation process -- to a continuous group in the large has been shown by Cartan [Theorie des Groupes Finis et Continus et l'Analysis Situs (Memorial 42), p. 18].

6. Representations of abstract infinitesimal groups; the derived group

If the given continuous group is a linear group Γ its velocity vectors u, v, ... are linear:

$$u_i(x) = \sum_k a_{ik} x_k = (ax)_i \quad , \quad a = \| a_{ik} \|$$

$$v_i(x) = \sum_k b_{ik} x_k = (bx)_i \quad ; \quad b = \| b_{ik} \|.$$

Conversely, a Lie algebra γ of linear velocity vectors will generate a linear continuous group Γ.

The commutator-product [uv] of u and v is

$$[uv]_i = \sum_j \left(u_j \frac{\partial v_i}{\partial x_j} - v_j \frac{\partial u_i}{\partial x_j}\right)$$

$$= \sum_j (b_{ij} a_{jk} - a_{ij} b_{jk}) x_k .$$

Hence $[uv]x = (ba-ab)x$;

the matrices a, b, ... themselves constitute a Lie algebra under the definition $[ab] = ab - ba$.

In general, by a representation of an infinitesimal group $\mathfrak{g}$ (or an abstract Lie algebra) we mean a homeomorphism of $\mathfrak{g}$ onto a Lie algebra γ of matrices: if a and b in $\mathfrak{g}$ correspond respectively to the matrices A and B in γ, and λ is a number, then

$$\lambda a \to \lambda A, \quad a + b \to A + B, \quad [ab] \to AB - BA.$$

If a linear group Γ is a representation of a continuous group G, and if γ and $\mathfrak{g}$ are their respective infinitesimal groups, then γ is a representation of $\mathfrak{g}$ in the sense just defined. Conversely, if γ is a representation of the infinitesimal group $\mathfrak{g}$ of a given G, then will the linear group Γ generated by γ be a representation of G? See §7 below.

If $\mathfrak{g}$ is an infinitesimal group, then the linear space spanned by all the commutator products [ab] of elements a, b in $\mathfrak{g}$ is an invariant subgroup $\mathfrak{g}'$ (invariant Lie subalgebra) of $\mathfrak{g}$. $\mathfrak{g}'$ is called the derived group of $\mathfrak{g}$; it is the infinitesimal group of the commutator group G' of the finite continuous group G generated by $\mathfrak{g}$ (G' is the group generated by all the elements $sts^{-1}t^{-1}$ with s and t in G). $\mathfrak{g}' = \mathfrak{g}$ for the unimodular group and for the orthogonal group in n dimensions $(n > 2)$.

7. Topological link between representations of the infinitesimal group and the whole group

To answer the question in the last section as to whether or not Γ is a representation of G, we can set up a single-valued continuous homeomorphic correspondence φ of G onto Γ within a certain domain of the identity element of G; that is, the group germ of Γ is a representation of the group germ of G. Assuming that the group manifold of G is connected, we can extend this correspondence φ along any arc in the manifold of G from the identity element to any element s of G. But if G is not simply connected, we cannot be sure that the element $\varphi(s)$ of Γ will be independent of the path chosen. If, however, we pass to the universal covering manifold $\bar{G}$ of G [1934 Notes, pp. 106-110], then $\bar{G}$ is a simply-connected continuous group locally isomorphic with G (hence having the same infinitesimal group and the same group germ), and we can be certain that the continuation process defines the homeomorphism $\bar{s} \rightarrow \varphi(\bar{s})$ of $\bar{G}$ onto Γ in unambiguous fashion.

The answer is, therefore, that Γ is a representation of the universal covering group $\bar{G}$ of G. For example, the orthogonal group in n dimensions is not simply connected; its universal covering group is two-sheeted, and this accounts for the occurrence of double-valued representations.

8. Invariants characterized by differential equations: absolute and relative invariants

Returning to the general scheme presented in §1, we replace the abstract group there by an r-parameter continuous Lie group G_r (i.e. one which can be generated by its infinitesimal elements). Under the influence of an infinitesimal element e of G_r, the vectors x, y, ..., z are transformed into x + dx, y + dy, ..., z + dz, where the increments dx, dy, ..., dz are linear (§5):

$$dx_i = \sum_j \alpha_{ij} x_j, \quad dy_i = \sum_j \beta_{ij} y_j, \; \ldots; \; dz_i = \sum_j \gamma_{ij} z_j$$

A form f(x, y, ..., z) is transformed correspondingly into $f + d_e f$, where

$$d_e f = \sum \frac{\partial f}{\partial x_i} dx_i + \sum_i \frac{\partial f}{\partial y_i} dy_i + \cdots + \sum_i \frac{\partial f}{\partial z_i} dz_i$$
$$= \sum \alpha_{ij} x_j \frac{\partial f}{\partial x_i} + \sum_i \beta_{ij} y_j \frac{\partial f}{\partial y_i} + \cdots + \sum_i \gamma_{ij} z_j \frac{\partial f}{\partial z_i}.$$

f will be an (absolute) invariant of G_r if and only if $d_e f = 0$ for every infinitesimal element e of G_r. If $e_1, e_2, \ldots, e_r$ is a basis of the infinitesimal group $\mathfrak{g}$ of G_r, then the invariants of G are characterized as the solutions f of the linear homogeneous differential equations

$$d_{e_1} f = 0, \; d_{e_2} f = 0, \; \ldots, \; d_{e_r} f = 0.$$

If f is a relative invariant, then

$$d_{e_\alpha} f = c_{e_\alpha} f \quad (\alpha = 1, 2, \ldots, r),$$

where the c_{e_α} are constants. If $e = \tau_1 e_1 + \ldots + \tau_r e_r$ is any element of $\mathfrak{g}$ then $d_e f = c_e f$, where $c_e = \tau_1 c_{e_1} + \cdots + \tau_r c_{e_r}$.

Since
$$(d_e d_{e'} - d_{e'} d_e) f = (c_e c_{e'} - c_{e'} c_e) f = 0,$$

every relative invariant f is an absolute invariant under the derived group $\mathfrak{g}'$ of $\mathfrak{g}$. This is evident also from the fact that the weight function $\mu(s)$ is a one-dimensional representation of G: $\mu(st) = \mu(s)\mu(t)$; hence for commutators $sts^{-1}t^{-1}$ we have $\mu(sts^{-1}t^{-1}) = 1$.

In the case of the real full linear group, $\mu(s)$ must always be a power of the determinant of s, while for the complex full linear group it always has the form

$$\mu(s) = |s|^{q} |\bar{s}|^{q'},$$

the bar indicating complex conjugate.

9. Taking the average over a finite group

If f is a form, and s an element of the group G (as in §1), then we define the form sf by

$$sf(A(s)x, B(s)y, \ldots, C(s)z) = f(x,y,\ldots,z).$$

We then have the rules

$$s(f+g) = sf + sg; \quad s(fg) = (sf)(sg); \quad (ts)f = t(sf).$$

G is an automorphism group of the ring generated by all forms: in what follows we might replace it by any ring containing the rational numbers.

If G is a finite group of order h, and f is any form, then

$$j = [f] = \frac{1}{h} \sum_{s \in G} sf$$

is an invariant: sj = j for all s in G. This averaging operator [ ] is linear,

$$[\alpha f + \beta g] = \alpha[f] + \beta[g] \qquad (\alpha, \beta \text{ numbers}),$$

and if j is any invariant,

$$[jf] = j[f].$$

As a first application, suppose j is a form such that sj = j for all s in G, not for all x, y, ..., z but only for those points $(\xi, \eta, \ldots, \zeta)$ lying on some manifold $\mathfrak{M}$ (in the product-space $\mathcal{K}_x \times \mathcal{K}_y \times \ldots \times \mathcal{K}_z$). Then there exists a "free" invariant J coinciding with the "bound" invariant j on $\mathfrak{M}$ namely J = [j].

As a second application, every representation of a finite group is equivalent to a unitary representation [the "hermitian argument" - 1934 Notes, p. 31].

10. Invariant volume measure on a compact Lie group

[1934 Notes, pp. 76-78.]

11. The unitarian trick

It will be shown in §13 that the first main theorem holds for any compact Lie group [1934 Notes, pp. 79-80]; the most familiar examples are the unitary group and the real orthogonal group. The real unimodular group is not compact, but we can reduce the proof of the first main theorem for it to that of the unimodular unitary group in the following way.

Consider a relative invariant J of the full linear group relative to representations $s \to A(s)$, $s \to B(s)$, ... which are of homogeneous, integral rational character, like those defined by forms. The invariance of J will hold for all complex transformations if it holds for all real ones: reality restrictions are algebraically irrelevant. In restriction to unimodular transformations, J is an absolute invariant, and satisfies the differential equation

$$d_e J = 0,$$

where

$$e = \sum \delta s_{ik} \cdot e_{ik}$$

is any infinitesimal unimodular transformation. That is,

(11.1) $$\sum \delta s_{ik} \, d_{e_{ik}} J = 0$$

for arbitrary δs_{ik} subject to the single restriction

(11.2) $$\delta s_{11} + \delta s_{22} + \dots + \delta s_{nn} = 0.$$

If (11.1) holds for all real δs_{ik} subject to (11.2), it will hold for all complex δs_{ik} subject to (11.2), and hence in particular for all δs_{ik} satisfying (11.2) and

(11.3) $$\delta s_{ik} = - \overline{\delta s_{ki}}.$$

J is thus an (absolute) invariant of the unimodular unitary group.

Conversely, if J is a real invariant of the unitary unimodular group, then (11.1) holds for all complex δs_{ik} satisfying (11.2) and (11.3). We can

write
$$\begin{aligned} \delta s_{ii} &= \sqrt{-1}\,\delta\sigma_{ii} \\ \delta s_{ik} &= \delta\sigma_{ki} + \sqrt{-1}\,\delta\sigma_{ik} \\ \delta s_{ki} &= -\delta\sigma_{ki} + \sqrt{-1}\,\delta\sigma_{ik} \end{aligned} \Bigg\} \quad (i < k),$$

where the $\delta\sigma_{ik}$ are n^2 real parameters subject only to

$$\delta\sigma_{11} + \delta\sigma_{22} + \cdots + \delta\sigma_{nn} = 0;$$

When we substitute these in (11.1) and observe that the $d_{e_{ik}} J$ are all real, we get, on equating to zero the imaginary part,

$$\sum \delta\sigma_{ik}\, d_{e_{ik}} J = 0.$$

Hence J is an invariant of the real unimodular group. The two groups thus have the same invariants, and once we have proved the first main theorem for the unimodular unitary group (§13) the same follows immediately for the real unimodular group.

In general the procedure would be like this. Let $e_1, e_2, \ldots, e_r$ be a basis of the infinitesimal group of a real continuous group G. Every infinitesimal element e of G is uniquely expressible in the form

$$e = \sigma_1 e_1 + \cdots + \sigma_r e_r,$$

with real numbers $\sigma_1, \ldots, \sigma_r$. We now endeavor to pass to a new basis

$$e_i' = \sum_k \lambda_{ik} e_k$$

with <u>complex</u> λ_{ik} such that

(i) the commutators $[e_i' e_k']$ are <u>real</u> linear combinations of the e_i'; and

(ii) the group G' generated by $e_1', e_2', \ldots, e_r'$ is compact.

If we can do this, our unitarian trick is successful: G and G' have the same invariants, and we know that the first main theorem holds for G'.

As a second example, let G be the group of all real linear transformations leaving invariant a non-degenerate quadratic form $\sum g^{ij} x_i x_j$ (Lorentz group). The infinitesimal elements of G,

$$dx_i = \sum_i s_i^j x_j,$$

are characterized by the assertion that the matrix

$$s^{ij} = \sum_r g^{ir} s_r^j$$

is skew-symmetric.

By a linear transformation A we can change $\sum g^{ij} x_i x_j$ into $x_1^2 + x_2^2 + \dots + x_n^2$. $A^{-1}SA$ leaves the latter invariant, and hence $A^{-1}SA = \| \sigma_{ij} \|$ is skew-symmetric. On restricting σ_{ij} to real values we are led to the real orthogonal group, which is compact.

12. Simple connectivity of the unimodular unitary group

[1934 Notes, pp. 111-113.]

B. Algebraic Methods - Hilbert's Theory

13. First Main Theorem

Section B deals largely with Hilbert's two fundamental papers on the classical theory of invariants:

Ueber die Theorie der algebraischen Formen [Math. Ann. 36, 1890, 473-534; number 16 in his Ges. Abh., vol. 2, 199-257. Cited as H 16.]

Ueber die vollen Invariantensysteme [Math. Ann. 42, 1893, 313-373; number 19 in his Ges. Abh., vol. 2, 287-344. Cited as H 19. All page numbers cited will refer to the Ges. Abh.]

The two pillars upon which Hilbert lays the massive lintel of the classical theory of invariants are his finite-basis theorem for polynomial ideals and the Cayley Ω-process. An easy proof of the Hilbert basis-theorem will be found in van der Waerden, Moderne Algebra, vol. 2, pp. 23-25. On p. 25 we find the corollary: if $\mathfrak{M}$ is any set of polynomials whatever (not necessarily an ideal) then there exist in $\mathfrak{M}$ a finite number of polynomials $f_1, f_2, \dots, f_r$ such that every polynomial f in $\mathfrak{M}$ is expressible in the form

$$f = a_1 f_1 + a_2 f_2 + \dots + a_r f_r,$$

where $a_1, a_2, \ldots, a_r$ are polynomials. In H 16 [Ges. Abh., pp. 199-203] Hilbert proves this directly without using the concept of an ideal. If the polynomials in $\mathfrak{M}$ are forms, then (by omitting superfluous terms) we can assume that $a_1, a_2, \ldots, a_r$ are forms.

If we take for $\mathfrak{M}$ the set of all non-constant invariants $J(x, y, \ldots, z)$ of a group G relative to certain representations (as in §1), Hilbert's basis-theorem asserts the existence of a finite number of invariants $J_1, \ldots, J_r$ such that every other invariant J can be expressed in the form

$$J = A_1J_1 + \ldots + A_rJ_r, \tag{13.1}$$

where the A's are forms (not necessarily invariants). If J_i is of degree λ_i in x, μ_i in y, ..., ν_i in z, and J is of degree λ in x, μ in y, ..., ν in z, then A_i will be of degree $\lambda - \lambda_i$ in x, $\mu - \mu_i$ in y, ..., $\nu - \nu_i$ in z, or will be identically zero if any of these numbers are negative.

If we can show that every invariant J of G can be expressed in the form (13.1) in such a way that the A's are <u>invariants</u>, then $J_1, \ldots, J_r$ constitute a rational integral basis, and the first main theorem holds for G. We merely use induction on the total degree of J: noting that the total degree of each non-vanishing A_i is less than that of J, the A_i are polynomials in $J_1, \ldots, J_r$ and hence so also is J. This is the essence of Hilbert's proof; the Ω-operator applied to (13.1) will convert the A's into invariants (§16 below).

We can effect this second part in the case of a finite group G by means of the averaging process (§9). Applying it to (13.1) we obtain

$$J = [A_1]J_1 + \ldots + [A_r]J_r,$$

and each coefficient $[A_i]$ is an invariant. This proves the first main theorem for all finite groups. We can do the same for any compact Lie group by integration over the group manifold [§10; i.e. 1934 Notes, pp. 79-80]. The first main theorem is thus established for all compact Lie groups, and for all groups

-- such as the unimodular group -- which are reducible to compact groups by the unitarian trick (§11).

14. Second main theorem

Hilbert applied his basis-theorem also to show that there exist a finite number of relations ("syzygies") among any set of basic invariants $J_1, \ldots, J_r$ such that all others are algebraic consequences of them. By a "relation" we mean a polynomial $\varphi(u_1, \ldots, u_r)$ in r variables $u_1, \ldots, u_r$ such that

$$\varphi(J_1, \ldots, J_r) = 0$$

identically in the variables x, y, ..., z upon which the J's depend. The set of all such polynomials is evidently an ideal; a basis $\varphi_1, \ldots, \varphi_k$ thereof evidently has the desired property [1936 Notes, pp. 19-20].

15. Proof of Hilbert's basis-theorem

[Van der Waerden, Moderne Algebra, vol. 2, pp. 23-25.]

16. The Cayley Ω-process

Let G in §1 be the full linear group over any field of characteristic zero, and let the representations be rational, integral, and homogeneous: each component of the matrices A(s), B(s), ..., C(s) shall be homogeneous polynomials in the variable components s_{ij} of the matrix s. Ω_s is defined to be the differential operator [1936 Notes, p. 41]

$$\begin{vmatrix} \frac{\partial}{\partial s_{11}} & \cdots & \frac{\partial}{\partial s_{1n}} \\ \cdots & \cdots & \cdots \\ \frac{\partial}{\partial s_{n1}} & \cdots & \frac{\partial}{\partial s_{nn}} \end{vmatrix} = \sum_{(i)} \pm \frac{\partial^n}{\partial s_{1i_1} \partial s_{2i_2} \cdots \partial s_{ni_n}}$$

If f(x, y, ..., z) is any form, then $|s|^q f(A(s)x, B(s)y, \ldots, C(s)z)$, where $|s|$ is the determinant of s, and q is an integer $\geqq 0$, is a homogeneous polynomial in the variables s_{ij}, say of degree p.

$$I(x, y, \ldots, z) = \Omega_s^p\{|s|^q f(A(s)x, B(s)y, \ldots, C(s)z)\}$$

is a form no longer involving the s_{ij}; Hilbert shows that it is a relative invariant of weight p-q [H 16 Ges. Abh., pp. 249-250]. While ostensibly dealing with ground-forms, Hilbert's proof holds without modification for this more general scheme.

A second essential property of Ω is that

$$\Omega_s^p\{|s|^p\} = N_p,$$

where N_p is a positive integer ($\neq 0$) [H 16 Ges. Abh., p. 250].

If now we transform (13.1) by a variable s in G we obtain

$$|s|^p J = A_1(A(s)x, B(s)y, \ldots)|s|^{p_1} J_1 + \ldots + A_r(A(s)x, B(s)y, \ldots)|s|^{p_r} J_r,$$

where $p, p_1, \ldots, p_r$ are the weights of $J, J_1, \ldots, J_r$. Applying the operator Ω_s^p, we get

$$N_p J = I_1 J_1 + \ldots + I_r J_r$$

where $N_p \neq 0$ and

$$I_\alpha = \Omega_s^p\{|s|^{p_\alpha} A_\alpha (A(s)x, B(s)y, \ldots)\}$$

are invariants ($\alpha = 1, \ldots, r$). It then follows immediately (§13) that $J_1, \ldots, J_r$ constitute a rational integral basis for the invariants of G.

It should be observed that Hilbert's proof, which is of a purely rational character, holds for any underlying number field of characteristic zero, whereas the representations considered are restricted to be homogeneous, rational, and integral. On the other hand, when we restrict the number field to the real or complex numbers, we are allowed a compensating freedom in the representations; the method of integration over the group manifold is at least valid for all continuous representations.

All this may be extended to allow among the given representations those of a contragredient nature, such as that afforded by tensors with some contravariant and some covariant indices. We simply modify the law of transformation of a contravariant vector in that we replace the matrix contragredient to s by the matrix of the minors of s. The latter differs from the former only by the factor $|s|$, and the only effect is to change an old invariant into a new invariant of different weight. In particular the first main theorem is thus proved for <u>covariants</u> $I(a, b, \ldots; x)$ as well as for <u>invariants</u> $I(a, b, \ldots)$ of any number of ground forms $\underline{a}(x)$, $\underline{b}(x)$, ...; the variable x is of course transformed contragrediently. The same is proved for ground-forms involving both covariant and contravariant vectors.

17. The adjunction argument

Every invariant $j(f, f^*, \ldots)$ of a set of forms $f, f^*, \ldots$ under the orthogonal group is obtainable from a projective invariant $J(f, f^*, \ldots; \gamma)$ of the set of forms $f, f^*, \ldots; \gamma$, where γ is an indeterminate quadratic form transforming contragrediently to the forms $f, f^*, \ldots$, by specializing γ to be the unit form $\gamma_0(x) = x_1^2 + x_2^2 + \ldots + x_n^2$:

$$j(f, f^*, \ldots) = J(f, f^*, \ldots; \gamma_0).$$

[1936 Notes, pp. 128-129.] The first main theorem for the orthogonal invariants j can then be inferred from its holding for projective invariants J. We proceed to apply the same argument to the extensions [1936 Notes, pp. 56-63] of the unimodular and orthogonal groups.

We take first the group $\mathfrak{A}_1$ of affine geometry:

$$\begin{aligned}
x_0 &= x_0' + s_1x_1' + \ldots + s_nx_n' \\
x_1 &= \phantom{x_0' + {}} s_{11}x_1' + \ldots + s_{1n}x_n' \qquad \text{(homogeneous plane-coordinates).} \\
&\ldots\ldots \\
x_n &= \phantom{x_0' + {}} s_{n1}x_1' + \ldots + s_{nn}x_n'
\end{aligned}$$

$$\begin{aligned}
\xi_0' &= \xi_0 \\
\xi_1' &= s_1\xi_0 + s_{11}\xi_1 + \ldots + s_{n1}\xi_n \qquad \text{(homogeneous point-coordinates),} \\
&\ldots\ldots \\
\xi_n' &= s_n\xi_0 + s_{1n}\xi_1 + \ldots + s_{nn}\xi_n
\end{aligned}$$

$$\begin{vmatrix} s_{11} & \cdots & s_{1n} \\ \cdots & \cdots & \cdots \\ s_{n1} & \cdots & s_{nn} \end{vmatrix} = 1.$$

We consider the invariants of one or more ground-forms $\underline{a}$, $\underline{b}$, ...:

$$\underline{a} = \sum \frac{r!}{r_o! \ldots r_n!} a_{r_o \ldots r_n} \xi_o^{r_o} \ldots \xi_n^{r_n}, \quad r_o + r_1 + \ldots + r_n = r, \text{ etc.}$$

Adjoin to this set an indeterminate linear form

$$\ell = \ell_o \xi_o + \ell_1 \xi_1 + \ldots + \ell_n \xi_n .$$

Our group $\mathfrak{A}_1$ is characterized by the property that it leaves invariant the particular linear form $e = (1, 0, \ldots, 0) = \xi_o$. We assert that every invariant $j(\underline{a}, \underline{b}, \ldots)$ under $\mathfrak{A}_1$ is obtainable from a projective invariant $J(\underline{a}, \underline{b}, \ldots; \ell)$ by specializing ℓ to e:

$$j(\underline{a}, \underline{b}, \ldots) = J(\underline{a}, \underline{b}, \ldots; e).$$

This means that every affine invariant of a set of algebraic surfaces is a projective invariant of these surfaces together with the plane at infinity. The converse is evident.

Proof:

Write the ground-forms symbolically [1936 Notes, pp. 113-115]:

$$\underline{a} = (a\xi)^r, \quad a_{r_o \ldots r_n} = a_o^{r_o} \ldots a_n^{r_n};$$

$$\underline{b} = (b\xi)^s, \quad b_{s_o \ldots s_n} = b_o^{s_o} \ldots b_n^{s_n}; \quad \text{etc.}$$

Expressing j symbolically,

$$j = j(a, a', \ldots; b, b', \ldots; \ldots),$$

we see that j becomes a vector invariant of $\mathfrak{A}_1$. A full table of typical basic vector invariants of $\mathfrak{A}_1$ is [1936 Notes, p. 58]:

$$[xy \ldots z]_{n+1} = \begin{vmatrix} x_o x_1 & \cdots & x_n \\ y_o y_1 & \cdots & y_n \\ \cdots & \cdots & \cdots \\ z_o z_1 & \cdots & z_n \end{vmatrix}; \qquad [y \ldots z] = \begin{vmatrix} y_1 & \cdots & y_n \\ \cdots & \cdots & \cdots \\ z_1 & \cdots & z_n \end{vmatrix}$$

Hence $j(a, a', \ldots; b, b', \ldots; \ldots)$ is a polynomial in bracket factors of the type $[ab \ldots c]_{n+1}$ and $[b \ldots c]_n$. But

$$[b \ldots c]_n = \begin{vmatrix} 1 & 0 & \ldots & 0 \\ 0 & b_1 & \ldots & b_n \\ \ldots & \ldots & \ldots & \ldots \\ 0 & c_1 & \ldots & c_n \end{vmatrix} = [eb \ldots c]_{n+1}.$$

If in j we replace terms of type $[b \ldots c]_n$ by $[\ell b \ldots c]_{n+1}$ we obtain a projective invariant J of the forms $\underline{a}$, $\underline{b}$, ...; ℓ which reduces to j under the specialization $\ell = e$.

We can apply the same argument to mixed forms

$$\underline{A} = \sum \frac{r!}{r_0! \cdots r_n!} \frac{\rho!}{\rho_0! \cdots \rho_n!} A_{r_0 \cdots r_n \rho_0 \cdots \rho_n} \xi_0^{r_0} \cdots \xi_n^{r_n} x_0^{\rho_0} \cdots x_n^{\rho_n},$$
$$(r_0 + \cdots + r_n = r, \ \rho_0 + \cdots + \rho_n = \rho).$$

Writing $\underline{A}$ symbolically as $(a\xi)^r(\alpha x)^\rho$ an invariant $j(\underline{A}, \underline{B}, \ldots)$ under $\mathfrak{A}$, becomes in symbolic notation a vector invariant $j(a, b, \ldots; \alpha, \beta, \ldots)$ of $\mathfrak{A}$, depending on both covariant and contravariant vectors. Hence [1936 Notes, p. 78, appropriately specialized] j is expressible as a polynomial in invariants of the following type:

(a) $[ab \ldots c]_{n+1}$ (b) $[b \ldots c]_n$ (covariant)

(c) $(a\alpha) = a_0\alpha_0 + a_1\alpha_1 + \ldots + a_n\alpha_n$ (mixed)

(d) $[\alpha\beta \ldots \gamma]_{n+1}$ (e) α_0 (contravariant).

Again a single linear form ℓ suffices. For types (a), (c), and (d) are already projective invariants, while for (b) and (e) we have respectively

$$[b \ldots c]_n = [eb \ldots c]_{n+1},$$
$$\alpha_0 = (e\alpha).$$

We take next the affine group $\mathfrak{A}_2$ of rank two. Its elements have the form

$$\begin{array}{c|cc|ccc} & x_1' & x_2' & x_3' & \cdots & x_n' \\ \hline x_1 & \multicolumn{2}{c|}{} & * & \cdots & * \\ & \multicolumn{2}{c|}{S_2} & & & \\ x_2 & & & * & \cdots & * \\ \hline x_3 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & & S_{n-2} & \\ x_n & 0 & 0 & & & \end{array}$$

(homogeneous plane coordinates),

$$\begin{array}{c|cc|ccc} & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_n \\ \hline \xi_1' & & & 0 & \cdots & 0 \\ & \multicolumn{2}{c|}{S_2'} & & & \\ \xi_2' & & & 0 & \cdots & 0 \\ \hline \xi_3' & * & * & & & \\ \vdots & \vdots & \vdots & & S_{n-2}' & \\ \xi_n' & * & * & & & \end{array}$$

(homogeneous point coordinates),

where S_2 and S_{n-2} are arbitrary unimodular matrices of degrees 2 and n-2 respectively.

Applying the same method as before, our symbolic j is a polynomial in the typical basic vector invariants of $\mathfrak{A}_2$ [1936 Notes, p. 78]:

(a) $[ab \dots c]_n$ (b) $[a \dots b]_{n-2} = \begin{vmatrix} a_3 & \cdots & a_n \\ \cdots & \cdots & \cdots \\ b_3 & \cdots & b_n \end{vmatrix}$

(c) $[\alpha\beta \cdots \gamma]_n$ (d) $[\alpha\beta]_2 = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}$

In this case we introduce as absolute an arbitrary skew-symmetric bilinear form $\ell = \sum_{i,k} \ell_{ik}\xi_i\eta_k$, which we specialize to $e = \xi_1\eta_2 - \xi_2\eta_1 = \begin{vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{vmatrix}$.

(a) and (c) are already projective invariants, while for (b) and (d) we have respectively

$[a \ldots b]_{n-2} = \sum_{(i)} \pm e_{i_1 i_2} a_{i_3} \ldots b_{i_n}$ (alternating sum), $[\alpha\beta]_2 = \sum e_{ik}\alpha_i\beta_k$.
But $\sum \pm \ell_{i_1 i_2} a_{i_3} \ldots b_{i_n}$ and $\sum \ell_{ik}\alpha_i\beta_k$ are projective invariants, involving covariant vectors a, b, ..., contravariant vectors α, β, ..., and the covariant skew-symmetric tensor ℓ_{ij}. Putting these in j in place of $[a \ldots b]_{n-2}$ and $[\alpha\beta]_2$, we obtain a projective invariant J of the forms $\underline{a}$, $\underline{b}$, ... together with the indeterminate skew-symmetric bilinear form $\underline{\ell}$, which reduces to j under the specialization $\underline{\ell} \to e$.

Finally we consider the group O_1 of Euclidean motions [1936 Notes, p. 56; enumeration of basic vector invariants, pp. 79-87, listed on extra sheet preceding p. 75]:

	x'_0	x'_1	...	x'_n
x_0	1	*	...	*
x_1	0	O_{11}	...	O_{1n}
⋮	⋮	⋮		⋮
x_n	0	O_{n1}	...	O_{nn}

	ξ_0	ξ_1	...	ξ_n
ξ'_0	1	0	...	0
ξ'_1	*	O_{11}	...	O_{n1}
⋮	⋮	⋮		⋮
ξ'_n	*	O_{1n}	...	O_{nn}

$\| O_{ij} \|$ being a proper orthogonal matrix. The typical basic vector invariants for covariant vectors only are

$$[xy \ldots z]_{n+1}, \qquad [y \ldots z]_n, \qquad (x|y) = x_1y_1 + \ldots + x_ny_n.$$

Their transcription into projective invariants is respectively

$$[xy \ldots z]_{n+1}, \qquad [\ell y \ldots z]_{n+1}, \qquad \sum \gamma_{ik}x_iy_k.$$

Hence we need only adjoin a covariant linear form $\underline{\ell} = \sum \ell_i \xi_i$ and a contravariant symmetric bilinear form $\underline{\gamma} = \sum \gamma_{ik}x_iy_k$ to the given ground-forms under consideration; we specialize $\underline{\ell}$ to ξ_0 and $\underline{\gamma}$ to $x_1y_1 + \ldots + x_ny_n$. The projectively invariant relation

$$\sum_k \gamma_{ik}\ell_k = 0,$$

holding for our special $\underline{\ell}$ and $\underline{\gamma}$ is disregarded.

The typical basic vector invariants of O_1 for both covariant and contravariant vectors are:

(a) $[xy \dots z]_{n+1}$; (b) $[y \dots z]_n$; (c) $(x|y) = x_1y_1 + \dots + x_ny_n$;

(d) $[\xi\eta \dots \zeta]_{n+1}$; (e) ξ_0

(f) $(\xi\eta \| \xi'\eta') = \sum_i (\xi_0\eta_i - \eta_0\xi_i)(\xi'_0\eta'_i - \eta'_0\xi'_i)$, etc.

(g) $(x\xi) = x_0\xi_0 + x_1\xi_1 + \dots + x_n\xi_n$

(h) $[\underbrace{x|, \dots, y|}_{a}, \underbrace{\xi, \dots, \eta}_{n+1-a}]$ $(a = 1, 2, \dots, n)$; $x| = (0, x_1, \dots, x_n)$.

Their projective transcription is as follows:

(a) $[xy \dots z]_{n+1}$

(b) $[\ell y \dots z]_{n+1}$

(c) $\sum \gamma_{ik}x_iy_k$

(d) $[\xi\eta \dots \zeta]_{n+1}$

(e) $(\ell\xi)$

(f)
$$\begin{vmatrix} \gamma_{00} & \gamma_{01} & \dots & \gamma_{0n} & \xi_0 & \eta_0 \\ \gamma_{10} & \gamma_{11} & \dots & \gamma_{1n} & \xi_1 & \eta_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_{n0} & \gamma_{n1} & \dots & \gamma_{nn} & \xi_n & \eta_n \\ \xi'_0 & \xi'_1 & \dots & \xi'_n & 0 & 0 \\ \eta'_0 & \eta'_1 & \dots & \eta'_n & 0 & 0 \end{vmatrix}$$

(g) $(x\xi)$

(h) $[\sum_k \gamma_{ik}x_k, \dots, \sum_k \gamma_{ik}y_k, \xi, \dots, \eta]$

The adjunction of ℓ and γ is sufficient in this case also.

18. Hilbert's zero-theorem

Hilbert's zero-theorem (Nullstellensatz) states that if a polynomial $f(x_1, \dots, x_n)$ vanishes at every common zero $(\xi_1, \dots, \xi_n)$ of the polynomials

$$f_i(x_1, \dots, x_n), \quad (i = 1, 2, \dots, r),$$

then some power f^σ of f is in the ideal generated by $f_1, \dots, f_r$:

$$f^\sigma = a_1f_1 + \dots + a_rf_r,$$

where $a_1, \dots, a_r$ are polynomials in $x_1, \dots, x_n$ [H 19, Ges. Abh., p. 294].

Van der Waerden, Modern Algebra, vol. 2, gives two proofs: the first (p. 11) by means of elimination theory, and the second by means of polynomial ideals and algebraic manifolds (top of p. 54 in combination with the result on p. 61 that every prime ideal "belongs" to the algebraic manifold it defines).

19. Using the zero-manifold of all invariants

Hilbert's proof of the first main theorem for invariants of forms under the full linear group [§§13, 16 above; H 16, I and V] consisted in showing first the existence of an ideal-basis $J_1, \ldots, J_\ell$ for all invariants, and second that this ideal-basis is also a rational integral basis. The first part -- which is true for any class of polynomials whatever -- is of a highly set-theoretical character. In his second paper [H 19], Hilbert attempts to put the whole matter on a more constructive basis. The essence of this paper is that he departs completely from that old hobby-horse of classical invariant theory -- the symbolic calculus -- and subjects the whole theory of invariants to the arithmetic theory of function-fields, already largely developed by Kronecker.

By a complete null-set of invariants $J_1, J_2, \ldots, J_\ell$ we mean a set such that their vanishing for particular values of the argument vectors (i.e., for particular values of the coefficients of the ground-forms) entails the vanishing of all invariants. In other words, the equations

$$J_1 = 0, \quad J_2 = 0, \ldots, \quad J_h = 0$$

define the zero-manifold (Nullstellenmanigfaltigkeit) of all invariants.

The first result is that if $J_1, J_2, \ldots, J_h$ is a complete null-set, then every invariant i is integral with respect to $J_1, \ldots, J_h$, i.e. it satisfies an equation

$$i^n + C_1 i^{n-1} + \ldots + C_n = 0$$

with leading coefficient unity, $C_1, \ldots, C_n$ being polynomials in $J_1, \ldots, J_h$ [H 19, Ges. Abh., pp. 299-300]. The proof involves both Hilbert's zero-theorem (§18) and the Ω-process (§16).

On p. 342 [H 19, Ges. Abh.] Hilbert summarizes the three steps by which he constructs a finite rational integral basis. This theorem reduces the first step to the construction of a complete null-set. Assume that this has been done. The second step is to construct from $J_1, \ldots, J_h$ a rational basis for all invariants. In the last section of H 19 [Ges. Abh., pp. 324-344] Hilbert manages to avoid the second step entirely. None the less, if we wish to retain it, we can use a general theorem of E. Noether [Gött. Nach., 1926, 28-35] to the effect that any algebraic extension within the field of all rational functions of a field of polynomials is finite. In our case, if K denote the underlying number field, we have shown that every invariant i is <u>integral</u> with respect to the <u>ring</u> $K[J_1, \ldots, J_h]$, hence in particular <u>algebraic</u> with respect to the <u>field</u> $K(J_1, \ldots, J_h)$,-- brackets denoting ring-adjunction, parentheses denoting field-adjunction. The field generated by all invariants ("Invariantenkörper") is thus algebraic over $K(J_1, \ldots, J_h)$, and by Noether's theorem it is a finite extension thereof. Since K is of characteristic zero, the field of invariants can be obtained from $K(J_1, \ldots, J_h)$ by the adjunction of a single polynomial J ["primitive element" -- van der Waerden, vol. 1, p. 120]; this abstract theorem may replace Hilbert's proof [H 19, Ges. Abh., pp. 291-292]. $J, J_1, \ldots, J_h$ then constitute a rational basis for all invariants.

But we know more than this: every invariant i is <u>integral</u> with respect to $K[J_1, \ldots, J_h]$. The converse is obvious: every rational function belonging to the field $K(J, J_1, \ldots, J_h)$ and integral with respect to $K[J_1, \ldots, J_h]$ is necessarily a polynomial, and hence an integral invariant [H 19, Ges. Abh., p. 292]. The ring generated by all (integral) invariants thus coincides with the ring of all "algebraic integers" of the field $K(J, J_1, \ldots, J_h)$ relative to $K[J_1, \ldots, J_h]$. Hilbert's third and final step is the determination of a "modul-basis" $j_1, \ldots, j_m$: every "algebraic integer" i is expressible in the form

$$i = A_1 j_1 + \cdots + A_m j_m$$

with $A_1, \ldots, A_m$ in $K[J_1, \ldots, J_h]$. The h+m invariants $J_1, \ldots, J_h, j_1, \ldots, j_m$ are then obviously a rational integral basis for all invariants. This third step is of a purely arithmetic character [H 19, Ges. Abh., p. 293; van der Waerden, vol. 2, pp. 93-94].

The problem is thus reduced to that of finding a complete null-set $J_1, \ldots, J_h$.

20. Construction of a complete set of null-invariants

This is the most difficult of the three steps. Hilbert shows how it can be done for the case of invariants of a single ternary form a of degree n [H 19 IV, Ges. Abh., pp. 319-325], by showing (p. 325) that if, for particular values of the coefficients of a, there exists a non-vanishing invariant J(a) of a, then there must exist a non-vanishing invariant of a of weight $\leq 9n(3n+1)^8$. Hence we may take as a complete null-set a vector basis $J_1, \ldots, J_h$ of the (finite) linear space of all invariants of weight $\leq 9n(3n+1)^8$.

The procedure is as follows. If a possesses a non-vanishing invariant J(a) of weight g, then the equation

$$\delta^g J(a) - J(a(s)) = 0, \qquad \delta = \begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix},$$

shows that the function δ of the nine independent variables s_{ij} is an integral algebraic function of the transformed coefficients a(s) of a. The converse is then shown (pp. 320-321). Hence a possesses a non-vanishing invariant J(a) of weight g if and only if δ is an integral algebraic function of degree g of the transformed coefficients a(s) of a.

On p. 324 Hilbert proves the lemma that among any h+1 forms of degree m in h variables there must exist an algebraic relation of degree $\leq h(m+1)^{h-1}$. Of the transformed coefficients a(s) of a there are at most nine which are algebraically independent. Suppose for concreteness there are seven. Then by a lemma on p. 290 we can find linear combinations $b_1, \ldots, b_7$ of these such that all the a(s) are integral algebraic functions of them. We can supplement $b_1, \ldots, b_7$ with two of the s_{ij}, which we call s_8 and s_9, such that $b_1, \ldots, b_7$,

s_8, s_9 are algebraically independent. Applying the lemma to the 10 forms δ^n, b_1^3, ..., b_7^3, s_8^{3n}, s_9^{3n} of degree 3n (h = 9, m = 3n), we find that δ^n satisfies an equation of degree $\leq 9(3n+1)^8$, and hence δ itself an equation of degree $\leq 9n(3n+1)^8$, in b_1, ..., b_7, s_8, s_9.

If now a has a non-vanishing invariant, then we know that δ is an integral algebraic function of b_1, ..., b_7 alone, since the a(s) are integral with respect to b_1, ..., b_7. The irreducible equation satisfied by δ in $K[b_1, \ldots, b_7]$ has leading coefficient unity, and cannot become reducible under the adjunction of s_8 and s_9, which are transcendental over $K(b_1, \ldots, b_7)$. Hence its degree g must be $\leq 9n(3n+1)^8$; we then infer the existence of a non-vanishing invariant of a of weight $g \leq 9n(3n+1)^8$.

21. Chain of syzygies

[H 16 III, Ges, Abh., pp. 215-233.]

Suppose J_1, ..., J_h is a rational integral basis, and $F_1(t_1, \ldots, t_h)$, ..., $F_r(t_1, \ldots, t_h)$ a basic set of syzygies:

$$F_i(J_1, \ldots, J_h) = 0, \qquad (i = 1, 2, \ldots, r),$$

and every other syzygy $F(t_1, \ldots, t_h)$ is expressible in the form

$$F(t) = A_1(t)F_1(t) + \ldots + A_r(t)F_r(t).$$

The question arises as to the unicity of the expression for F(t); we are led to examine solutions $(A_1, A_2, \ldots, A_r)$ of

$$A_1F_1 + A_2F_2 + \ldots + A_rF_r = 0. \tag{21.1}$$

As an easy consequence of Hilbert's basis-theorem, there exist a finite number of such solution-vectors

$$A' = (A_1', \ldots, A_r'), \ldots, A^{(s)} = (A_1^{(s)}, \ldots, A_r^{(s)})$$

such that every solution-vector $A = (A_1, \ldots, A_r)$ is a linear combination of these with polynomial coefficients:

$$A = U'A' + U''A'' + \ldots + U^{(s)}A^{(s)},$$

that is,

$$A_i = U'A'_i + U''A''_i + \dots + U^{(s)}A_i^{(s)}, \qquad (i = 1, 2, \dots, r).$$

[H 16, Ges. Abh., p. 208. An easier proof is to associate with $(A_1, A_2, \dots, A_r)$ the form $u_1A_1 + u_2A_2 + \dots + u_rA_r$ with new indeterminates $u_1, \dots, u_r$; an ideal-basis of the latter leads back immediately to a basis of the solution-vectors.]

We then consider all solution-vectors $(U', U'', \dots, U^{(s)})$ of the r equations

$$(21.2) \qquad U'A'_i + U''A''_i + \dots + U^{(s)}A_i^{(s)} = 0, \qquad (i = 1, 2, \dots, r).$$

Again there exist a finite number of solution-vectors of these equations such that all others are linear combinations of them with polynomial coefficients, and again we consider the vanishing linear combinations of these solution-vectors.

In this way we get a "chain of syzygies" (21.1), (21.2), ..., and Hilbert shows [loc. cit.] that if n is the number of variables on which $J_1, \dots, J_h$ depend, then the chain must end after at most n links, the last basic set of solution-vectors being linearly independent with respect to polynomial multipliers. The proof is too long even to outline in these brief notes. Suffice it to say that induction is used on the number n of variables. It is true for n=1, for every modul of vectors $(A_1, \dots, A_r)$ of polynomials in a single variable has a linearly independent modul-basis ("lattice argument", using division algorithm). In the general case, the chain of equations can be so fashioned that we can associate with it a parallel chain involving n-1 variables; by hypothesis for induction the latter ends after n-1 steps at most, and from the manner of its construction it is then clear that the given chain must end after one step more.

22. Invariants of a single infinitesimal transformation

This is an account of the first two sections, pp. 236-242, of a paper by R. Weitzenböck, "Ueber die Invarianten von Linearen Gruppen" [Acta Math. 58, 1932, 231-293], which follows early attempts by L. Maurer [Bayer Ak. Wiss. 29, 1899; Math. Ann. 57, 1903] to prove the first main theorem for all solvable linear groups. In view of the fact that the first main theorem has been shown for all semi-simple groups, reducing them to compact groups by the unitarian trick [Weyl, Math. Zeit. 23, 24], the next step would logically be to attack the other extreme -- the solvable groups. Unfortunately there is a gap in this paper by Weitzenböck; none the less, his proof for a one-parameter group is correct.

Starting with a single infinitesimal linear transformation $dx_i = \sum_k a_{ik} x_k$, Weitzenböck throws the matrix $\| a_{ik} \|$ into the Jordan normal form: diagonal except for occasional 1's immediately to the right of the main diagonal. We may correspondingly write the differential operator d as a sum $d = d' + \delta$, where d' is the diagonal part. A relative invariant J satisfies $dJ = \alpha J$, where α is a number (§8). Evidently $d'J = \alpha' J$, and it is shown that $\alpha' = \alpha$. Hence $\delta J = 0$.

If we break the vector space up according to the consecutive 1's in the matrix of δ:

$$(x_1, \ldots, x_n) = (y_1, \ldots, y_\mu ; \; z_1, \ldots, z_\nu ; \; \ldots),$$

then

$$\begin{array}{lll} & \delta y_1 = y_2 & \delta z_1 = z_2 \\ & \delta y_2 = y_3 & \delta z_2 = z_3 \\ (22.1) & \cdots\cdots & \cdots\cdots \qquad \text{etc.,} \\ & \delta y_{\mu-1} = y_\mu & \delta z_{\nu-1} = z_\nu \\ & \delta y_\mu = 0 & \delta z_\nu = 0 \end{array}$$

and the equation $\delta J = 0$ takes the form (16), p. 241:

$$\left(y_2 \frac{\partial J}{\partial y_1} + y_3 \frac{\partial J}{\partial y_2} + \cdots + y_\mu \frac{\partial J}{\partial y_{\mu-1}} \right) + \left(z_2 \frac{\partial J}{\partial z_1} + z_3 \frac{\partial J}{\partial z_2} + \cdots + z_\nu \frac{\partial J}{\partial z_{\nu-1}} \right) + \cdots =$$

But this is the differential equation which characterizes the semi-invariants of the binary forms (18), which we modify slightly:

$$\varphi_y(\xi,\eta) = y_\mu \xi^{\mu-1} + (\mu-1)y_{\mu-1}\xi^{\mu-2}\eta + (\mu-1)(\mu-2)y_{\mu-2}\xi^{\mu-3}\eta^2 + \cdots + (\mu-1)!\,y_1\eta^{\mu-1}$$
$$\varphi_z(\xi,\eta) = z_\nu \xi^{\nu-1} + (\nu-1)z_{\nu-1}\xi^{\nu-2}\eta + \cdots + (\nu-1)!\,z_1\eta^{\nu-1}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

For a semi-invariant is by definition an invariant of these forms under the one-parameter group of transformations

(22.2) $$\xi' = \xi + \lambda\eta\,, \quad \eta' = \eta\,,$$

which is generated by the infinitesimal linear transformation

$$\delta\xi = \eta\,, \quad \delta\eta = 0\,.$$

Under this infinitesimal transformation $\varphi_y \to \varphi_y + \delta\varphi_y$ where

$$\delta\varphi_y = (\mu-1)y_\mu\xi^{\mu-2}\eta + (\mu-1)(\mu-2)y_{\mu-1}\xi^{\mu-3}\eta^2 + \cdots\,,$$

and similarly for $\varphi_z, \ldots$ But this means that the coefficients $y_\mu, y_{\mu-1}, \ldots, y_1$ of φ_y have suffered precisely the increments (22.1).

The group (22.2) is a "step-group" [1936 Notes, p. 61, heading 5)], and we know a full table of typical basic invariants: $[\xi\eta]$ and η. By the adjunction argument (§17), we can infer the first main theorem for semi-invariants (in general, as well as in the binary case), and hence for the invariants of a single infinitesimal linear transformation.

This is extended to absolute invariants in §5 (pp. 246-247).

Passing now to the case of a solvable group, we might attempt a proof by induction on the number of parameters. Assuming it true for every r-parameter solvable linear group, let Γ' be the infinitesimal group of an (r+1)-parameter solvable group G_{r+1}. By definition of solvability, G_{r+1} has a composition-series $G_{r+1} \supset G_r \supset G_{r-1} \supset \ldots \supset G_1$, and hence in particular it has an r-parameter invariant solvable subgroup G_r. The infinitesimal group γ of G_r is an invariant Lie subalgebra of Γ of one less dimension. Γ is therefore obtained from γ by adjoining a single matrix c, and if x is any matrix in γ then $x' = [xc] = xc - cx$ is in γ.

If φ is a (relative) invariant of γ, then so also is $c\varphi$. To show this, let x be any matrix in γ. Then

$$x' = [xc],\ x'' = [x'c],\ x'' = [x''c],\ \ldots$$

are all in γ and hence

$$x\varphi = \xi\varphi,\ x'\varphi = \xi'\varphi,\ x''\varphi = \xi''\varphi, \ldots$$

where $\xi, \xi', \xi'', \ldots$ are numbers. Set

$$\varphi^{(i)} = c^i\varphi, \quad (i = 1, 2, 3, \ldots).$$

Let $\varphi^{(h)}$ be the first which depends linearly on the preceding; such an h exists since the matrix c satisfies an algebraic equation with numerical coefficients (e.g. its characteristic equation). We have then

$$\begin{aligned}
c\varphi &= \varphi' \\
c\varphi' &= \varphi'' \\
&\cdots\cdots\cdots \\
c\varphi^{(h-2)} &= \varphi^{(h-1)} \\
c\varphi^{(h-1)} &= \alpha_1\varphi^{(h-1)} + \alpha_2\varphi^{(h-2)} + \cdots + \alpha_h\varphi.
\end{aligned}$$

Now

$$\begin{aligned}
x\varphi' &= xc\varphi = [xc]\varphi + cx\varphi = \xi'\varphi + \xi\varphi' \\
x\varphi'' &= xc\varphi' \\
&= [xc]\varphi' + cx\varphi' \\
&= x'c\varphi + c(\xi'\varphi + \xi\varphi') \\
&= [x'c]\varphi + cx'\varphi + \xi'\varphi' + \xi\varphi'' \\
&= \xi''\varphi + c\xi'\varphi + \xi'\varphi' + \xi\varphi'' \\
&= \xi''\varphi + 2\xi'\varphi' + \xi\varphi''.
\end{aligned}$$

Continuing in this way we get a sequence of equations

$$x\varphi = \xi\varphi$$
$$x\varphi' = \xi'\varphi + \xi\varphi'$$
$$x\varphi'' = \xi''\varphi + 2\xi'\varphi' + \xi\varphi''$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$x\varphi^{(h-1)} = \xi^{(h-1)}\varphi + \binom{h-1}{1}\xi^{(h-2)}\varphi' + \dots + \xi\varphi^{(h-1)}.$$

The h-dimensional linear space spanned by $\varphi, \varphi', \dots, \varphi^{(h-1)}$ is in- is invariant under c and under every x in γ. It thus affords a representation $x \to X$, $c \to C$ of Γ. The trace of X is $h\xi$ where ξ is given by $x\varphi = \xi\varphi$. But $x'\varphi = \xi'\varphi$, and hence the trace of the matrix X' corresponding to x' is $h\xi'$. But since $x' = xc - cx$, it follows that $X' = XC - CX$, and hence the trace of X' is zero. We conclude that $\xi' = 0$, $x\varphi' = \xi\varphi'$, and $\varphi' = c\varphi$ is invariant under γ.

The operator c thus maps the ring $\mathcal{K}$ generated by the invariants of γ upon itself. By hypothesis for induction, $\mathcal{K}$ possesses a finite rational integral basis $\varphi_1, \dots, \varphi_k$. The question then reduces to this: given such a finite ring $\mathcal{K}$ of polynomials, and a linear differential operator c mapping $\mathcal{K}$ upon itself, does the ring generated by all polynomials φ in $\mathcal{K}$ for which $c\varphi = \xi\varphi$ possess a finite rational integral basis?

As we have shown in the case of a single differential operator, the answer is in the affirmative when $\mathcal{K}$ is the ring of all polynomials, but the general case seems to be much more difficult.

23. E.Noether's elementary proof of the first main theorem for finite groups

[Math. Ann. 77, 1916, 89-91]

Given a finite group $\mathcal{G}$ of linear transformations

$$x^{(k)} = A^{(k)}x, \qquad (k = 1, 2, \dots, h),$$

if F(x) is an invariant of $\mathcal{G}$,

$$F(x^{(k)}) = F(x), \qquad (k = 1, 2, \dots, h),$$

then
$$F(x) = \frac{1}{h}\sum_{k=1}^{h} F(x^{(k)}).$$

Hence if we write out the components:

$$x^{(1)} = (x_1^{(1)}, \dots, x_n^{(1)})$$
$$x^{(2)} = (x_1^{(2)}, \dots, x_n^{(2)})$$
$$\dots\dots\dots\dots$$
$$x^{(h)} = (x_1^{(h)}, \dots, x_n^{(h)}),$$

we see that $F(x)$ is a symmetric function of the columns:

$$x_1 = (x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(h)})$$
$$\dots\dots\dots\dots$$
$$x_n = (x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(h)}),$$

which we look upon as n vectors in an h-dimensional space.

If we regard all the $x_i^{(k)}$ as nh indeterminates, the function $\frac{1}{h}\sum F(x^{(k)})$ is a vector invariant $F(x_1, \dots, x_n)$ of the symmetric group, depending on n covariant vectors: the components $x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(h)}$ of each vector x_i undergo the <u>same</u> permutation. But we know a set of typical basic invariants for this group [1935-36 Notes, p. 34], namely the fully polarized elementary symmetric functions:

$$\sigma_1(x) = x^{(1)} + \dots + x^{(h)}$$
$$\sigma_2(x,y) = \sum_{i \neq j} x^{(i)}y^{(j)}$$
$$\sigma_3(x,y,z) = \sum_{i,j,k \text{ all } \neq} x^{(i)}y^{(j)}z^{(k)}$$
$$\dots\dots\dots\dots$$

These can be obtained as the coefficients $G_{\alpha\alpha_1\cdots\alpha_n}$ of the power products of the indeterminates $u, u_1, \dots, u_n$ in the product

$$\prod_{k=1}^{h}(u + u_1 x_1^{(k)} + \dots + u_n x_n^{(k)}) = \sum_{\alpha+\alpha_1+\dots+\alpha_n = h} u^{\alpha} u_1^{\alpha_1} \cdots u_n^{\alpha_n} G_{\alpha\alpha_1\cdots\alpha_n}(x_1, \dots, x_n).$$

Replace the indeterminates $x^{(k)}$ in the polynomials $G_{\alpha\alpha_1\cdots\alpha_n}$

by $A^{(k)}x$. The resulting polynomials $H_{\alpha\alpha_1\cdots\alpha_n}(x)$ are evidently invariants of $\mathcal{G}$. They constitute a rational integral basis for the invariants of $\mathcal{G}$. For $\frac{1}{h}\sum F(x^{(k)})$ is a polynomial in the G's; and if we replace $x^{(k)}$ by $A^{(k)}x$ in this identity, we obtain F(x) expressed as a polynomial in the H's.

24. E.Fischer's "Formenstaaten"

[Crelle 140, 1911, 48-81; specifically 48-52, 66-68 (Satz 1).]

If

$$f(x) = \sum_{(i)} a_{i_1\ldots i_n} x_1^{i_1}\ldots x_n^{i_n}, \quad i_1 + \ldots + i_n = p.$$

$$g(x) = \sum_{(k)} b_{k_1\ldots k_n} x_1^{k_1}\ldots x_n^{k_n}, \quad k_1 + \ldots k_n = q,$$

are any two forms of degree p and q respectively, we define

$$\bar{g}\left(\frac{\partial}{\partial x}\right)f(x) = \sum_{(k)} \bar{b}_{k_1\ldots k_n} \frac{\partial^q f}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}},$$

where the bar indicates complex conjugate. This is zero if $q > p$, otherwise a form of degree $p - q$. If $q = p$,

$$\bar{g}\left(\frac{\partial}{\partial x}\right)f(x) = \sum_{(i)} i_1!\ldots, i_n!\, \bar{b}_{i_1\ldots i_n} a_{i_1\ldots i_n},$$

and this defines an Hermitian scalar product in the (finite) linear space of all forms of degree p. If this vanishes, f and g are orthogonal; if f is orthogonal to itself:

$$\bar{f}\left(\frac{\partial}{\partial x}\right)f(x) = 0,$$

then f vanishes identically.

A Formenstaat is a set $\mathcal{R}$ of forms f, g, ... such that if f and g are in $\mathcal{R}$ then

(i) αf is in $\mathcal{R}$, where α is a number;

(ii) f + g is in $\mathcal{R}$, provided f and g have the same degree;

(iii) $\bar{g}(\frac{\partial}{\partial x})f(x)$ is in $\mathcal{R}$.

The main theorem [Satz 1, p. 68] is that if f, f_1, ..., f_h are in a Formenstaat $\mathcal{R}$, and

(24.1) $$f = \ell_1 f_1 + \dots + \ell_h f_h,$$

where $\ell_1, \dots, \ell_h$ are forms (not necessarily in $\mathcal{R}$), then f is expressible in the form

(24.2) $$f = k_1 f_1 + \dots + k_h f_h$$

with $k_1, \dots, k_h$ in $\mathcal{R}$.

Let $\mathcal{R}_p$ be the set of all forms in $\mathcal{R}$ of degree p, $\mathcal{R}_p^*$ the set of all forms in $\mathcal{R}_p$ expressible in the form (24.1), and $\mathcal{R}_p^{**}$ the set of all forms in $\mathcal{R}_p$ expressible in the form (24.2). $\mathcal{R}_p^*$ and $\mathcal{R}_p^{**}$ are linear subspaces of the linear space $\mathcal{R}_p$, and evidently $\mathcal{R}_p^* \supseteq \mathcal{R}_p^{**}$. The theorem asserts that $\mathcal{R}_p^* \subseteq \mathcal{R}_p^{**}$.

Let P_p^* and P_p^{**} be the subspaces of $\mathcal{R}_p$ orthogonal respectively to $\mathcal{R}_p^*$ and $\mathcal{R}_p^{**}$. We have only to show that $P_p^{**} \subseteq P_p^*$.

Let $\varphi(x)$ be in P_p^{**}. This means that

$$\bar{k}_\alpha\left(\frac{\partial}{\partial x}\right)\bar{f}_\alpha\left(\frac{\partial}{\partial x}\right)\varphi(x) = 0 , \qquad (\alpha = 1, 2, \dots, h)$$

for arbitrary $k_\alpha(x)$ of degree $p-p_\alpha$ in $\mathcal{R}$.

By postulate (iii), the form of degree $p-p_\alpha$,

$$\psi_\alpha(x) = \bar{f}_\alpha\left(\frac{\partial}{\partial x}\right)\varphi(x),$$

is in $\mathcal{R}$. We may therefore in particular choose $k_\alpha(x) = \psi_\alpha(x)$, and this entails the identical vanishing of $\psi_\alpha(x)$, i.e.

$$\bar{f}_\alpha\left(\frac{\partial}{\partial x}\right)\varphi(x) = 0 , \qquad (\alpha = 1, 2, \dots, h).$$

But it is then evident that $\bar{f}\left(\frac{\partial}{\partial x}\right)\varphi(x) \equiv 0$ for any f(x) of the form (24.1). Hence $\varphi(x)$ is orthogonal to all f in $\mathcal{R}_p^*$, i.e. $\varphi(x)$ is in P_p^*, and $P_p^{**} \subseteq P_p^*$.

The application to group theory is this remarkable result: The first main theorem holds for any linear group $\mathcal{G}$ with the property that if s is in $\mathcal{G}$ then the Hermitian conjugate $\bar{s}'$ of s is in $\mathcal{G}$.

We need only convince ourselves that the set of all invariants of $\mathcal{G}$ is a Formenstaat; that is, if F(x) and G(x) are invariants, then so also is $\bar{G}\left(\frac{\partial}{\partial x}\right)F(x)$. This is evididently so, for the differential operators

$$\frac{\partial^g}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}$$

transform contragrediently to the monomials $x_1^{k_1} \dots x_n^{k_n}$.

On account of the bar, the operator $\bar{G}(\frac{\partial}{\partial x}) \rightarrow \bar{H}(\frac{\partial}{\partial x})$ under the influence of $x \rightarrow sx$, where $H = \bar{s}'^{-1} G$. If $\bar{s}'$ is in $\mathcal{G}$ then $H = G$, and $\bar{G}(\frac{\partial}{\partial x})F(x)$ is invariant.

The rest is as in §13: indeed, Fischer's result is that <u>any</u> Formenstaat possesses a rational integral basis.

CHAPTER II. APPLICATIONS OF THE INTEGRATION METHOD

1. Generalization by Haar and von Neumann

This account of the Haar measure for locally Euclidean, separable groups is a repetition of pp. 105-106 of the 1934 Notes. [A. Haar, Annals of Math. 34, 1933, p. 147.]

For the von Neumann theory [Transactions 36, 1934, 445-492] we again refer to the 1934 Notes: periodic functions as functions on the (closed) group of rotations of a circle (p. 100); Bohr's almost-periodic functions as functions on the (open) group of translations of a straight line (pp. 102-103); von Neumann's theory of almost-periodic functions on any group whatever, using Bochner's definition (pp. 103-105). Regarding the question of the distinguishability of elements of the group by almost-periodic functions, H. Freudenthal has shown [Annals of Math. 37, 1936, p. 57] that the only groups G having the property that if s_1 and s_2 are distinct elements of G then there exists an almost periodic function f(s) on G such that $f(s_1) \neq f(s_2)$, are the compact groups and the translation groups (and direct products of these), from which the whole investigation started!

2. Volumes on the unitary group

[1934 Notes, pp. 117-121; Groups and Quantum Mechanics, pp. 386-389 in the English translation.]

Every unitary matrix is conjugate to a diagonal matrix $\begin{Vmatrix} \epsilon_1 & & \\ & \epsilon_2 & \\ & & \ddots \\ & & & \epsilon_n \end{Vmatrix}$. The ϵ's are of absolute value 1, and hence the "angles" $\varphi_1, \ldots, \varphi_n$ defined by $\epsilon_k = e^{2\pi i \varphi_k}$ are real. The volume of that portion of the unitary group consisting of all unitary matrices whose angles lie between φ_k and $\varphi_k + d\varphi_k$ $(k = 1, 2, \ldots, n)$ is found to be $\Delta\bar{\Delta}\, d\varphi_1 d\varphi_2 \cdots d\varphi_n$, where $\Delta = \prod_{i<k}(\epsilon_i - \epsilon_k)$.

3. Determination of the primitive characters

[1934 Notes, pp. 113-116; Groups and Quantum Mechanics, pp. 377-381.]

Since the character $\chi(s)$ of a representation is a "class function", $\chi(tst^{-1}) = \chi(s)$, it depends only on the angles of s and must be a symmetric function of them. Every primitive character of the unitary group (character of an irreducible representation) is shown to have the form

$$\chi(\epsilon_1, \ldots, \epsilon_n) = \frac{|\epsilon^{l_1}\ \epsilon^{l_2} \ldots \epsilon^{l_n}|}{|\epsilon^{n-1}\ \epsilon^{n-2} \ldots \epsilon\ 1|}, \quad (l_1 > l_2 > \cdots > l_n),$$

where $$|\epsilon^{l_1}\ \epsilon^{l_2} \ldots \epsilon^{l_n}| = \begin{vmatrix} \epsilon_1^{l_1} & \epsilon_1^{l_2} & \ldots & \epsilon_1^{l_n} \\ \epsilon_2^{l_1} & \epsilon_2^{l_2} & \ldots & \epsilon_2^{l_n} \\ \ldots & \ldots & \ldots & \ldots \\ \epsilon_n^{l_1} & \epsilon_n^{l_2} & \ldots & \epsilon_n^{l_n} \end{vmatrix}.$$

Conversely, every such function is a primitive character. The denominator is simply the Vandermonde determinant of $\epsilon_1, \ldots, \epsilon_n$, and is thus $\Delta(\epsilon) = \prod_{i<k}(\epsilon_i - \epsilon_k)$. It divides the numerator, so that $\chi(\epsilon_1, \ldots, \epsilon_n)$ is actually a polynomial in $\epsilon_1, \ldots, \epsilon_n$.

4. Enumeration of vector invariants and covariants

[Weyl, Acta Math. 48, 1926, 255-278.]

Every form $f(\overset{1}{x}, \overset{2}{x}, \ldots, \overset{e}{x})$ in e variable vectors $\overset{i}{x} = (\overset{i}{x}_1, \overset{i}{x}_2, \ldots, \overset{i}{x}_n)$ of degree r_i in $\overset{i}{x}$ $(i = 1, 2, \ldots, e)$ is a linear combination of monomials

$$(\overset{1}{x}{}_1^{i_{11}} \ldots \overset{1}{x}{}_n^{i_{1n}})(\overset{2}{x}{}_1^{i_{21}} \ldots \overset{2}{x}{}_n^{i_{2n}}) \ldots (\overset{e}{x}{}_1^{i_{e1}} \ldots \overset{e}{x}{}_n^{i_{en}}),$$

where
$$i_{11} + \dots + i_{1n} = r_1$$
$$i_{21} + \dots + i_{2n} = r_2$$
$$\dots\dots\dots\dots$$
$$i_{e1} + \dots + i_{en} = r_e.$$

Under transformation by a diagonal unitary matrix

$$\begin{Vmatrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \ddots & \\ & & & \epsilon_n \end{Vmatrix}$$

such a monomial is simply multiplied by

$$(\epsilon_1^{i_{11}} \dots \epsilon_n^{i_{1n}})(\epsilon_1^{i_{21}} \dots \epsilon_n^{i_{2n}}) \dots (\epsilon_1^{i_{e1}} \dots \epsilon_n^{i_{en}}).$$

Hence the character $X_{r_1 r_2 \dots r_e}(\epsilon_1, \dots, \epsilon_n)$ of the representation $\mathfrak{A}_{r_1 r_2 \dots r_e}$ of the unitary group induced in the space of all forms $f(\overset{1}{x}, \overset{2}{x}, \dots, \overset{e}{x})$ of fixed degrees $r_1, r_2, \dots, r_e$ is

$$X_{r_1 r_2 \dots r_e}(\epsilon_1, \dots, \epsilon_n) = \sum_{\substack{i_{11} + \dots + i_{1n} = r_1 \\ \dots\dots \\ i_{e1} + \dots + i_{en} = r_e}} (\epsilon_1^{i_{11}} \dots \epsilon_n^{i_{1n}})(\epsilon_1^{i_{21}} \dots \epsilon_n^{i_{2n}}) \dots (\epsilon_1^{i_{e1}} \dots \epsilon_n^{i_{en}}).$$

A set of forms $f_1, f_2, \dots, f_g$ of degrees $r_1, r_2, \dots, r_e$ such that

$$sf_i = \sum_{j=1}^{g} c_{ij}(s) f_j$$

is called a <u>covariant quantity of kind Γ</u>, where Γ is the representation $s \to \| c_{ij}(s) \|$. It is <u>primitive</u> if Γ is irreducible. The problem of finding the number of linearly independent primitive quantities of kind Γ of degrees $r_1, r_2, \dots, r_e$ respectively in $\overset{1}{x}, \overset{2}{x}, \dots, \overset{e}{x}$, is precisely that of finding how often the irreducible representation Γ occurs in the decomposition of $\mathfrak{A}_{r_1 r_2 \dots r_e}$ into its irreducible components.

If $\mathfrak{A}_{r_1 \dots r_e} \doteq \mu \Gamma + \mu' \Gamma' + \dots$, and $\chi(s)$, $\chi(s')$, are the characters of Γ, Γ', ..., then

$$X_{r_1 \dots r_e}(s) = \mu \cdot \chi(s) + \mu' \cdot \chi'(s) + \dots .$$

μ can be computed by means of the orthogonality properties of primitive characters:

$$\mu = \langle X(s) \bar{\chi}(s) \rangle$$

where $\langle \; \rangle$ indicates the average over the group manifold.

If we set

$$f(z_i \mid \epsilon) = \prod_{j=1}^{n} (1 - \epsilon_j z_i) , \qquad (i = 1, 2, \dots, e),$$

then we have formally

$$\frac{1}{f(z_1 \mid \epsilon) \cdots f(z_e \mid \epsilon)} = \sum_{(n)}^{\infty} X_{n_1 \dots n_e} z_1^{n_1} \cdots z_e^{n_e} .$$

From §3, χ must have the form

$$\chi = \frac{|\epsilon^{l_1} \cdot \epsilon^{l_n}|}{\Delta(\epsilon)}$$

If we set

$$A_{r_1 \dots r_e} = \langle X_{r_1 \dots r_e}(s) \bar{\chi}(s) \rangle$$

and $F(z_1, \dots, z_e) = \sum_{(r)}^{\infty} A_{r_1 \dots r_e} z_1^{r_1} \dots z_e^{r_e}$, then

$$F(z_1, \dots, z_e) = \frac{1}{n!} \int_0^1 \cdots \int_0^1 \frac{|\epsilon^{-l_1} \dots \epsilon^{-l_n}|}{f(z_1 \mid \epsilon) \cdots f(z_e \mid \epsilon)} \frac{\Delta \bar{\Delta}}{\Delta} d\varphi_1 \cdots d\varphi_n .$$

In the case e = n (which evidently contains the case e < n) this integral can be evaluated very simply by means of Cauchy's identity:

$$\frac{\Delta(z) \Delta(\epsilon)}{\prod_{i,k=1}^{n} (1 - \epsilon_i z_k)} = \det \left| \frac{1}{1 - \epsilon_i z_k} \right| .$$

The result is

$$F(z_1, \dots, z_n) = \frac{|z^{l_1} \dots z^{l_n}|}{|z^{n-1} \dots z \; 1|} .$$

Strangely enough, F turns out to be the same polynomial in the z's that χ is in the ϵ's. The desired integer $A_{r_1 \dots r_n}$ is the coefficient of

$z_1^{r_1} \dots z_n^{r_n}$ in the polynomial $F(z_1, \dots, z_n)$. In particular, to determine the number of linearly independent invariants of weight g we must find the number of times $\mathfrak{A}_{r_1 \dots r_n}$ contains the one-dimensional representation $s \to |s|^g$. The character of the latter is $(\epsilon_1 \epsilon_2 \dots \epsilon_n)^g$, and hence

$$F(z_1, \dots, z_n) = (z_1 z_2 \dots z_n)^g.$$

Thus every $A_{r_1 \dots r_n}$ vanishes except for $r_1 = \dots = r_n = g$, and $A_{gg \dots g} = 1$. Hence there is only one linearly independent vector invariant of weight g depending on n covariant vectors, and we know what it is: $[x^1 x^2 \dots x^n]^g$. Thus we have a new proof of the first main theorem for this case.

For $e > n$ it is shown by induction that

$$F(z_1, \dots, z_e) = \frac{|1 \;\; z \;\; \dots \;\; z^{e-n-1} \;\; z^{l_n+e-n} \;\; z^{l_{n-1}+e-n} \;\; \dots \;\; z^{l_1+e-n}|}{|1 \;\; z \;\; z^2 \;\; \dots \;\; z^{e-1}|}.$$