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Volume I

Seminar I - Theory of Functions of
Several Complex Variables

Seminar II - Conformal Mapping and
Schlicht Functions

Institute for Advanced Study
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PREFACE

These two volumes contain the half-hour addresses delivered in the five seminars at the Conference on Analytic Functions held September 2-14, 1957, at the Institute for Advanced Study, Princeton, New Jersey. The principal addresses have been published as a book under the title ANALYTIC FUNCTIONS by the Princeton University Press.

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Marston Morse, Chairman
Arne Beurling
Atle Selberg
Organizing Committee of the Conference

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INTRODUCTORY REMARKS

Dr. Carl Kaplan
Chief of Scientific Research
Air Force Office of Scientific Research

I would like to convey to you the regrets of General Gregory, Commander of the Air Force Office of Scientific Research, for not being able to greet you in person. He is at present in England attending the Anglo-American Conference of the Royal Aeronautical Society. He has assured me, however, that it is a great honor for the Air Force Office of Scientific Research to be a co-partner with the Institute for Advanced Study in sponsoring this distinguished symposium.

I am certain that General Gregory would have taken advantage of the opportunity to remind this audience of the importance of the accumulated store of knowledge in the Theory of Functions of a Complex Variable to the remarkably rapid progress of aeronautics during the past 50 years. It would be no exaggeration to say that, had this knowledge not been available beforehand, the aeronautical achievements of this century would have been postponed many years.

For example, the relatively recent implementation of the Riemann Mapping Theorem with practical mathematical techniques enables the aerodynamicist to calculate velocity and pressure distributions over arbitrarily prescribed bodies in both two dimensional and axi-symmetrical incompressible flows.

Also, one of the most remarkable developments in the period 1932-1942 were the various attempts to treat the dynamics of a compressible fluid in a manner similar to that of the incompressible fluid. These efforts culminated in the successful application of conformal mapping techniques and the utilization of z and $\bar{z}$ as independent variables to the solution of problems in compressible flow. It should be mentioned, however, that the Russian applied mathematician Muskilesvilli and his co-workers had already in earlier years applied similar techniques to problems in theoretical elasticity. Unfortunately, the work of the Russians remained long unknown to the aerodynamicists of the Western world.

With these few remarks, I would like to close with the thought that, although you have added several complex variables to function theory, the results of your researches will undoubtedly contribute to future progress in the aeronautical sciences to at least as high a degree as function theory of a single complex variable contributed to aeronautical progress of the past 50 years.

Seminar I. THEORY OF FUNCTIONS OF SEVERAL COMPLEX VARIABLES

ON PROPERTIES OF DOMAIN FUNCTIONALS

Stefan Bergman

In the following we consider some problems relating to the distributions of values of certain domain functionals.

§1. CRITICAL POINTS OF THE KERNEL FUNCTION K_B AND OF CERTAIN INVARIANTS UNDER PSEUDO-CONFORMAL TRANSFORMATIONS

The study of value-distribution of invariants, such as J_B or $\mathcal{R}_B$ (see [1], pp. 54 and 55), is in many instances simpler than that of K_B. If we pass from one domain to another by a pseudo-conformal transformation, the value of an invariant does not change. Therefore in the case of a domain which can be mapped onto a Reinhardt circular domain R we can restrict ourselves to the study of invariants for the latter domains. A Reinhardt circular domain admits the transformations $z_k = z_k e^{i\theta_k}$, $k = 1, 2$, into itself and it is, therefore, sufficient to consider an invariant, say $J_R(r_1 e^{i\phi_1}, r_2 e^{i\phi_2})$, for special values of ϕ_1, ϕ_2, say for $\phi_1 = \phi_2 = 0$. Let us suppose that the boundary r^3 of R is such that at every boundary point of r^3 the kernel function is infinite of third order and sufficiently regular, (i.e. satisfies the hypothesis 1^0 of [2], p. 9). Then the invariants J_R and $\mathcal{R}_R$ assume constant boundary values $2/9\pi^2$ and -1, respectively. According to [1], p. 55 and [2], p. 19, $J_B = \lambda^{01}\lambda^{001}/(\lambda^1)^3$, $\mathcal{R}_B = \lambda^{[2]}/\lambda^{[3]}$, where the quantities $\lambda \equiv \lambda_B$ (being solutions of certain minimum problems) are always positive (see [2], p. 3). By the relation $J_R((-1)^k r_1, (-1)^s r_2) = J_R(r_1, r_2)$, $k, s = 1$ or 2, we define J_R for positive and negative values of the r_ν. Then $r_3 = J_R(r_1, r_2)$ defines a segment of a surface bounded by a curve in the plane $r_3 = 2/9\pi^2$. Inside the curve, $J_R > 0$.

REMARK. If J_R is not constant on the boundary, we proceed as follows: By the relation $r_3 = -J_R(r_1, r_2)$ we define the reflected segment, and by using a segment of a sufficiently smooth surface T we connect both segments to one simply-connected closed surface. Most of our further considerations hold also for closed surfaces.

$J_R(r_1, r_2)$ is an analytic function of two real variables r_1, r_2 inside the domain. By a slight modification of the values in a sufficiently thin strip around the boundary we can replace $J_R(r_1, r_2)$ by a function $\tilde{J}_R(r_1, r_2)$ which has the same critical set and which is twice continuously differentiable in the closed domain.

In general the invariants J_R (and consequently $\tilde{J}_R$) have degenerate critical sets. According to the theorem of [8], p. 178, one can approximate the surface $r_3 = \tilde{J}_R(r_1, r_2)$ by another surface $r_3 = \tilde{\tilde{J}}_R(r_1, r_2)$ arbitrarily close to the original one which has only critical points and for which the Morse relations, e.g. $m + M - S = 2 - \nu$, hold *) ([7], p. 35). Here m, M, S denote the number of minimum, maximum, and saddle points, respectively, and $\nu = 1$.

As mentioned before, in our approach the question arises of approximating one-dimensional critical sets by critical points. If the maximum (minimum) curve has no double points it can be deformed into a curve which has a maximum (minimum) point and one minimax point.

The same methods can be applied for the study of other invariants, e.g. $\mathcal{R}$ and ds^2 which can be represented in terms of minimum values λ of minimum problems, see [2], p. 3 ff., p. 19, formula (33).

In the case of the kernel-function the considerations are more complicated and we shall consider at first the two-dimensional case. We can, however, investigate kernel-functions of a more

*) ν is the number of the boundary curves.

general class. We assume that the corresponding set of functions ϕ_ν is orthonormal in the sense that $\iint_B \phi_\nu \overline{\phi}_\mu \omega \, dxdy = \delta_{\nu\mu}$, where ω is a conveniently chosen positive weighting function, (see [3], p. 143 ff.). The kernel-function becomes infinite on the boundary, and therefore the function $k(x, y) = 1/K(z, \overline{z})$, will vanish on the boundary of the domain. The surface $\tau = k(x, y)$ in the three-dimensional x, y, τ space can have critical lines and points. If the surface has critical lines, we always can deform it in the neighborhood of the line, so that instead of the line we have critical points. For the surface deformed in this way the relation $m - S + M = 2 - \nu$ (see Morse [7]) holds. Here m, S, M denote the number of minima, minimaxes and maxima, respectively.

REMARK. In these considerations we assume that the function is twice continuously differentiable in the closed domain. To prove that this is the case in the closed domain often causes difficulties. If, however, the curvature of the boundary curve is uniformly bounded, so that we can use circles of radius r, ($r > 0$ being <u>independent</u> of the boundary point) as an exterior and an interior domain of comparison (see [2], p. 7) we can consider in place of the boundary the curve $K(z, \overline{z}) = \lambda_0$, where λ_0 is sufficiently large. The methods discussed above can be generalized to the case of the kernel-function of domains in the space of two complex variables.

§2. PROPERTIES OF CRITICAL VALUES IN ANALYTIC HYPERSURFACES

When passing from the theory of functions of one complex variable to the case of two (or several) variables we encounter the following difficulty: A function $f(z)$ of one variable vanishes at isolated points. With a set of isolated points we can associate certain functionals (say the number $N(r)$ of zero points in a circle of radius r); the growth of these functionals is connected with the growth of the function $|f(z)|$. A function $f(z_1, z_2)$ of two variables assumes a constant value in segments of two-dimensional surfaces, and the question arises of determining functionals of these segments such that the growth of these functionals is connected with the growth of $|f(z_1, z_2)|$.

Let $f(z_1, z_2)$ be an entire function of z_1 and z_2. We assume that for every z_1 the equation $f_{z_1}(z_1, z_2) = 0$, $(f_{z_1} \equiv \partial f/\partial z_1)$, has in $|z_2| < \rho_2$ finitely many solutions $A_h(z_1)$, $h = 1, 2, \ldots$. Therefore the surface $f_{z_1}(z_1, z_2) = 0$ can be written in the form $z_2 = A_h(z_1)$,

$h = 1, 2, \ldots$. The equation $f_{z_1}(z_1, z_2) = 0$ will represent a surface in the z_1, z_2-space whose intersection with the hypersurface

$z_1 = \rho_1 e^{i\phi_1}$, ρ_1 fixed, $0 \leq \phi_1 \leq 2\pi$, consists of segments of curves. In the following we shall determine a functional of these segments and derive for this functional bounds depending upon the growth of $f(z_1, z_2)$ in certain three-dimensional manifolds of the z_1, z_2-space.

Let $a_\mu(z_2)$ be the zeros of $f(z_1, z_2)$ in the plane $z_2 = \text{const.}$. Then, (see [4], [5]),

$$(2.1) \quad \frac{f_{z_1}(z_1, z_2)}{f(z_1, z_2)} = \frac{1}{\pi} \int_0^{2\pi} \log \left| \frac{f_{z_1}(\rho_1 e^{i\theta_1}, z_2)}{f(\rho_1 e^{i\theta_1}, z_2)} \right| \frac{\rho_1 e^{i\theta_1} d\theta_1}{(\rho_1 e^{i\theta_1} - z_1)^2}$$

$$- \sum_{|a_\mu(z_2)| < \rho_1} \frac{\rho_1^2 - |a_\mu(z_1)|^2}{(a_\mu(z_2) - z_1)(\rho_1^2 - \overline{a_\mu(z_2)} z_1)}, \quad |z_k| < \rho_k.$$

Since, for every fixed z_1, $f_{z_1}(z_1, z_2)/f(z_1, z_2)$ is a meromorphic function of z_2, we have:

$$(2.2)\quad \log\left|\frac{f_{z_1}(\rho_1 e^{i\theta_1}, z_2)}{f(\rho_1 e^{i\theta_1}, z_2)}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f_{z_1}(\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2})}{f(\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2})}\right| \cdot$$

$$\cdot \frac{(\rho_2^2 - r_2^2)d\theta_2}{\rho_2^2 + r_2^2 - 2\rho_2 r_2 \cos(\theta_2 - \phi_2)} - \sum_{|A_h(\rho_1 e^{i\theta_1})|<\rho_2} \log\left|\frac{\rho_2^2 - \overline{A_h(\rho_1 e^{i\theta_1})} z_2}{\rho_2(z_2 - A_h(\rho_1 e^{i\theta_1}))}\right|$$

$$+ \sum_{|B_h(\rho_1 e^{i\theta_1})|<\rho_2} \log\left|\frac{\rho_2^2 - B_h(\rho_1 e^{i\theta_1}) z_2}{\rho_2(z_2 - B_h(\rho_1 e^{i\theta_1}))}\right| \quad .$$

Here $B_h(z_1^0)$, $h = 1, 2, \ldots,$ are the zeros of $f(z_1, z_2)$ in the plane $z_1 = z_1^0$. Combining (2.1) and (2.2) and setting $z_1 = z_2 = 0$, we obtain, if $f(0,0) \neq 0$:

$$(2.3)\quad \frac{f_{z_1}(0,0)}{f(0,0)} = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} \log\left|\frac{f_{z_1}(\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2})}{f(\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2})}\right| \frac{d\theta_1 d\theta_2}{\rho_1 e^{i\theta_1}}$$

$$- \frac{1}{\pi}\int_0^{2\pi} \sum_{|A_h(\rho_1 e^{i\theta_1})|<\rho_2} \log\left|\frac{\rho_2}{A_h(\rho_1 e^{i\theta_1})}\right| \frac{d\theta_1}{\rho_1 e^{i\theta_1}}$$

$$+\frac{1}{\pi}\int_0^{2\pi} \sum_{|B_h(\rho_1 e^{i\theta_1})|<\rho_2} \log\left|\frac{\rho_2}{B_h(\rho_1 e^{i\theta_1})}\right| \frac{d\theta_1}{\rho_1 e^{i\theta_1}}$$

$$- \sum_{|a_\mu(0)|<\rho_1} \frac{\rho_1^2 - |a_\mu(0)|^2}{\rho_1^2 a_\mu(0)} .$$

Results obtained in the present section can be generalized to the case of distinguished boundary surfaces of more complicated structure.

§3. CRITICAL POINTS IN A SEGMENT OF AN ANALYTIC HYPERSURFACE ON THE BOUNDARY OF A DOMAIN WITH A DISTINGUISHED BOUNDARY SURFACE

Many results in the theory of functions of one complex variable can be generalized to the case of two (and several) variables when we consider a special class of domains, namely those which are bounded by finitely many segments h_ν^3, $\nu = 1, 2, \ldots, n$, of analytic hypersurfaces. On the boundary $b^3 = \sum_{\nu=1}^{n} h_\nu^3$ of such a domain B, lies a two-dimensional "distinguished boundary surface" $D^2 = \sum_{\nu,\mu=1}^{n} S_{\nu\mu}^2$, $S_{\nu\mu}^2 = h_\nu^3 \cap h_\mu^3$, $\nu \neq \mu$, which from a function-theoretic point of view plays a role similar to the boundary curve in the case of one complex variable. If $n > 2$ the intersections $\ell^1_{\nu jk} = h^3_\nu \cap h^3_j \cap h^3_k$ form the distinguished boundary line ([5], p. 171). In the above paper, in the case of domains with distinguished boundary surface, we introduced "generalized harmonic measures" which assume, on one closed component of the distinguished boundary surface, the value one and vanish on all other components. Generalizing this approach one can consider generalized harmonic measure $\omega(z_1, z_2; P^2; B)$ of the extended class, which is characterized by the following properties:

1) $\omega(z_1, z_2; P^2; B)$ is a function of the extended class in the domain B

2) it assumes the value 1 on the segment P^2 of the distinguished boundary surface D^2 and vanishes on the complementary part $D^2 - P^2$.

Using the "projection" of $\omega(z_1, z_2; P^2; B)$ on the space of real parts of functions of two complex variables, one obtains bounds for analytic functions $f(z_1, z_2)$ in terms of bounds on the distinguished boundary surface. Further, using $\omega(z_1, z_2; S_{\nu k}; B)$ one obtains invariants with respect to pseudo-conformal transformations. It is of interest to investigate the value distribution of $\omega(z_1, z_2; S^2_{\nu k}; B)$. Suppose that h^3_ν is simply connected and bounded by four surfaces $S^2_{\nu k}$, $k = 1, 2, 3, 4$, where closed segments $\overline{S^2_{\nu 1}}$ and $\overline{S^2_{\nu 3}}$ have no points in common. Then $\omega(z_1, z_2; S^2_{\nu 1}; B) + \omega(z_1, z_2; S^2_{\nu 3}; B)$ has in h^3_ν at least one minimax line. Our approach can also be used to show that the functions of extended class in certain cases have minimaxes inside the domain.

As indicated before, to given boundary values on the distinguished boundary surface one can determine functions of the extended class in different ways and one chooses them according to the purpose. Here we consider functions described in [6]. We shall discuss only an example from which one can see how to proceed in the general case.

We consider a domain bounded by five hypersurfaces. Four segments h^3_ν, $\nu = 1, 2, 3, 4$ of the boundary b^3 belong to hypersurfaces of the form $z_1 = H_\nu(z_2, \lambda_\nu)$, respectively, while the fifth segment, h^3_5, lies in $z_2 = e^{i\chi}$, $0 \leq \chi \leq 2\pi$.

The domain B under consideration has the distinguished boundary surface consisting of the following segments: $S^2_{\mu\,\mu+1} = h^3_\mu \cap h^3_\mu$

$\mu = 1, 2, 3$, $S^2_{41} = h^3_4 \cap h^3_1$ and $S^2_{\mu 5} = h^3_\mu \cap h^3_5$, $\mu = 1, 2, 3, 4$. We consider a function of the extended class which assumes on S^2_{15} and S^2_{35} the value 1, on S^2_{25} and S^2_{45} the value -1, while on $S^2_{\mu\,\mu+1}$, $\mu = 1, 2, 3$ and on S^2_{14} it vanishes. Using results of Morse [7], [8], one can show that the function $\omega(z_1, z_2; S^2_{15} + S^2_{35}; B) - \omega(z_1, z_2; S^2_{25} + S^2_{45}; B)$ with the above described boundary values on $P^2_1 = \sum_{\mu=1}^{3} S^2_{\mu\,\mu+1} + S^2_{14} + \sum_{\mu=1}^{4} S^2_{\mu 5}$ has in B at least one critical surface.

Results of this paper can be extended to the study of functions which represent a generalization of pseudo-harmonic functions and of solutions of systems of certain linear partial differential equations.

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ON A GENERALIZED DIRICHLET PROBLEM FOR PLURISUBHARMONIC FUNCTIONS AND PSEUDO-CONVEX DOMAINS. CHARACTERIZATION OF ŠILOV BOUNDARIES

H. J. Bremermann

To continuous real boundary values prescribed on the boundary of a domain D in the complex number space C^n, $n > 1$, there does not exist, in general, a pluriharmonic function (i. e. real part of a holomorphic function) that assumes the given boundary values.

For $n > 1$ the real parts of holomorphic functions (satisfying $\frac{\partial^2 h}{\partial z_\mu \partial \bar{z}_\nu} = 0$ for $\mu, \nu = 1, \ldots, n$) are a proper subclass of, and no longer coincide with, the harmonic functions (satisfying $\sum_{\mu=1}^{n} \frac{\partial^2 h}{\partial z_\mu \partial \bar{z}_\mu} = 0$). The harmonic functions solve the Dirichlet problem uniquely but they are not invariant under holomorphic transformations. The latter property, however, seems to be quite desirable and is indispensable if the theory is to be extended to complex manifolds.

S. Bergman [3], [4], [5] has been the first to suggest that one not give up at this point but extend the class of pluriharmonic functions such that the boundary value problem becomes uniquely solvable, the functions are pseudo-conformally invariant, and as many properties of the pluriharmonic functions are preserved as possible. His method, while having certain advantages, is limited to a special class of domains.

We suggest in the following a different method that applies to a larger class of domains and can be considered as the direct extension of the Perron-Carathéodory method (Perron [11], Carathéodory [8]) from one to several complex variables. Given a domain D in the plane of one complex variable, one takes the subharmonic functions in $\overline{D}$ that are less than or equal to the prescribed values on the boundary ∂D of D. It is easy to show that the upper envelope of this class of functions is harmonic, and that if the boundary of D is "sufficiently nice", this upper envelope function assumes exactly the given boundary values, and hence the Dirichlet problem is solved.

To obtain the analog for several complex variables, the subharmonic functions (of one complex variable) are replaced by the plurisubharmonic functions, which constitute their proper generalization to several variables. The big difference that arises is that in general the upper envelope is not pluriharmonic, but only plurisubharmonic.

Also, while for complex dimension one (and sufficiently nice boundaries) we may prescribe arbitrary boundary values on the whole boundary, this is no longer true for higher dimension. Only for strictly pseudo-convex domains are the values prescribed on the whole boundary, while, for instance, in the case of analytic polyhedra the values are prescribed on a subset of real dimension n of the ($2n-1$ dimensional) boundary.

The treatment of general pseudo-convex domains requires the notion of "Šilov boundary", a notion which so far has played a role in Banach algebras (Arens and Singer [1]). Our main result, utilizing this notion, is:

THEOREM: Let D be a bounded domain of the form $D = \{z \mid V(z) < 0\}$, $V(z)$ plurisubharmonic and continuous in a neighborhood of D. (This implies that D is pseudo-convex.) Then the Dirichlet problem can be solved for the upper envelope $\phi(z)$ of the plurisubharmonic functions in $\overline{D}$ that are less than or equal to the given continuous boundary values $b(z)$ (at those points of ∂D where $b(z)$ is defined) if and only if the boundary values $b(z)$ are prescribed on, and only on, the Šilov boundary

S(D) of D. (Bremermann [7].)

For complex dimension one the conditions on D imply that $S(D) = \partial D$ and hence we obtain as corollary that the (ordinary) Dirichlet problem has a solution for domains $D = \{z \mid V(z) < 0\}$, $V(z)$ subharmonic and continuous in a neighborhood of D.

For dimension higher than one we are faced with the problem of characterizing the Šilov boundary. This has been done so far only for a rather special class of domains, the Reinhardt circular domains (de Leeuw [9], Behnke-Stein [2]). We have shown:

If $D = \{z \mid V(z) < 0\}$ as above and $V(z)$ is (C^2), then all those points on ∂D where $\bar{\partial}\partial V = \Sigma \frac{\partial^2 V}{\partial z_\mu \partial \bar{z}_\nu} dz_\mu d\bar{z}_\nu$ is positive definite belong to $S(D)$, and all points on ∂D where in a neighborhood $\bar{\partial}\partial V = 0$ for all directions do not belong to $S(D)$.

In particular: If D is strictly pseudo-convex then $S(D) = \partial D$; and if D is an analytic polyhedron, S(D) is the distinguished boundary surface of D, in the sense of S. Bergman.

The Šilov boundary of a non-pseudo-convex domain is contained in the Šilov boundary of its (pseudo-convex) envelope of holomorphy $E(D)$: $S(D) \subset S(E(D))$.

Besides the plurisubharmonic solution $\varphi(z)$ one can introduce analogously the plurisuperharmonic solution $\Phi(z)$ which exists if and only if $\varphi(z)$ exists. Let $h(z)$ be the harmonic solution; then $\phi(z) \leq h(z) \leq \Phi(z)$ with equality in all points of D if equality holds in one point. While $h(z)$ is not pseudo-conformally invariant, $\phi(z)$ and $\Phi(z)$ are and provide an estimate:

Let D^* be pseudo-conformally equivalent to D (of the form $D = \{z \mid V(z) < 0\}$), let $h(z)$ be harmonic in D, $h^*(z^*)$ in D^* and $h(z) = h^*(z^*(z))$ on ∂D. Then $|h(z) - h^*(z^*(z))| \leq \Phi(z) - \phi(z)$.

The plurisubharmonic solutions have the property of being only sub-additive; this excludes the representation of $\phi(z)$ in the form of an integral over the boundary values. Therefore the following property is the more important because it provides us with a different method to calculate or estimate $\varphi(z)$: Let $b^*(z) = b(z)$ on $S(D)$, $b^*(z) = \max_{z \in S(D)} b(z)$ on $\overline{D} - S(D)$. Then the envelope of holomorphy of the point set

$$H = \{(z, w) \mid z \in \overline{D},\ |w| < e^{-b^*(z)}\}$$

<u>is</u>

$$E(H) = \{(z, w) \mid z \in \overline{D},\ |w| < e^{-\phi(z)}\}.$$

Both H and E(H) are schlicht. Therefore E(H) can be calculated explicitly by a method previously described by the author [6]. <u>Hence $\phi(z)$ can be calculated.</u>

Most of these results can be extended to Stein manifolds and some results to complex Banach spaces of infinite dimension. Also, the condition that the boundary values be continuous can be relaxed so that upper semi-continuous boundary values and the values of log $|f|$, f holomorphic, are admissible. The latter indicates the possibility of Nevanlinna-type applications. Such applications, and others, have been made by S. Bergman [4], [5] for his extended class, and it is to be expected that some of these can be done for the plurisubharmonic solutions also. Furthermore it is expected that results obtained by the author [7] will enable one to generalize parts of the Morse theory [10] to several variables.

Finally, examples have been given in the form of tube domains, which are pseudo-convex if and only if they are convex. Here the Šilov boundary is given exactly by the strictly convex points of the base.

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ON OKA'S THEOREM FOR STEIN MANIFOLDS

H. J. Bremermann

The present note outlines a solution of the following problem posed in 1953 by H. Cartan [4]:

> Let X be a (complex) Stein manifold. Given an open set $U \subset X$, under what conditions is U a Stein manifold? A necessary condition is that every point in $\overline{U}$ possesses in X an open neighborhood V, such that $V \cap U$ is a Stein manifold. Is this necessary condition sufficient?

It is shown that the answer to the latter question is affirmative: U <u>is a Stein manifold if it is locally a Stein manifold.</u> Also other necessary and sufficient conditions for U to be a Stein manifold are given.

For schlicht domains the above problem is equivalent to whether a "pseudo-convex domain" is a domain of holomorphy. This question has stimulated research ever since it was first posed in 1911 by E. E. Levi [8]. It was first solved for complex dimension 2 in 1942 by K. Oka [10]. B. A. Fuks [5] gave an improved version of Oka's proof, and in 1954 F. Norguet [9] and H. J. Bremermann [2] overcame a difficulty that stood in the way of the extension of

Oka's method to prove the result for arbitrary finite dimension n. In 1955 another paper of Oka [11] appeared that proved the result also for locally schlicht domains over the C^n. For the case of general Stein manifolds, however, the problem as posed by H. Cartan remained open even after the two Cousin problems had been settled for Stein manifolds (H. Cartan [4], J. P. Serre [12]).

The methods used for schlicht domains seem not to be applicable to Stein manifolds without strong modifications. Our proof reduces the problem for Stein manifolds by an imbedding to the schlicht case. In the following we assume U to be relatively compact in X; the general case follows by an approximation and the "theorem of Behnke-Stein" which for Stein manifolds has been proved (H. Grauert and R. Remmert [6]). U is imbedded as an analytic surface U^* without self-intersections into a schlicht poly-cylinder of higher dimension. In order to construct around U^* a pseudo-convex domain of the imbedding space Cartan's condition is transformed into an equivalent one: In U there exists a pluri-subharmonic function $\phi(P)$ such that the point set $\{P \mid \phi(P) < M, P \varepsilon U\}$ is relatively compact in U for every real M; in other words $\phi(P)$ tends to infinity everywhere at the boundary of U.

If U is schlicht $\phi(P)$ can easily be found by introducing the euclidean distance $d_U(P)$ of the point P from the boundary of U. It is well known that under Cartan's condition $-\log d_U(P)$ is plurisubharmonic in U (Bremermann [3]), and tends to infinity everywhere at the boundary of U. If U is a relatively compact submanifold of a Stein manifold it can be covered by a finite number of coordinate systems, each system failing only in an analytic set S of dimension $n-1$. With one of these local coordinate systems an analogue to $-\log d_U(P)$ is formed which is plurisubharmonic in U except in a neighborhood of $\partial U \cap S$.

S is prescribed as the zero surface of an analytic function (second Cousin problem). Let the solution be $g(P)$. Then for a sufficiently large positive constant c the sum $-\log d_U(P) + c \log|g(P)|$ becomes $-\infty$ on S. The upper envelope of this function with a suitable constant K is plurisubharmonic throughout U, and the upper envelope of the corresponding functions for the finitely many different coordinate systems covering U is plurisubharmonic and tends to infinity at ∂U as required. Thus the function $\phi(P)$ has been found.

$\phi(P)$ is now used to "inflate" U^* to a pseudo-convex domain

U^{**} of the imbedding space, whose coordinates we denote by $w_1, \ldots, w_k$, as follows: Let

$$U^* = \{w : w_j = X_j(P),\ P \varepsilon\ U\},\ X_j \text{ holomorphic in } U,\quad j = 1, 2, \ldots, k$$

then

$$U^{**} = \{w : ||W - X(P)|| < \inf(\varepsilon,\ e^{-\phi(P)}),\ P \varepsilon\ U\},\ \varepsilon > 0 \text{ sufficiently small.}$$

It can be shown that to any boundary point of U^{**} a neighborhood can be found such that the intersection with U^{**} is pseudo-convex, hence U^{**} is pseudo-convex.

Now Oka's solution of the problem for schlicht domains is applied and we conclude that U^{**} is holomorph-convex, which implies that U^* and U are holomorph-convex, and hence U is a Stein manifold.

A submanifold U satisfying Cartan's condition we call, by analogy with schlicht domains, pseudo-convex. Necessary and sufficient for U to be pseudo-convex is each one of the following conditions:

1) There exists a function $\phi(P)$ plurisubharmonic in U that tends to infinity everywhere at the boundary of U.

2) To every point of $\overline{U}$ there exists a neighborhood V such that $U \cap V$ is pseudo-convex.

3) U is the limit of domains U_ν such that $U_\nu \subset\subset U_{\nu+1} \subset\subset U$, $\lim U_\nu = U$, and such that for every U_ν the Levi-Krzoska condition (cf. Behnke-Thullen [1]) holds.

4) The "Kontinuitätssatz" holds for U, that is: If $\{S_\nu\}$ are analytic sets in U, and $\partial S_\nu = T_\nu$, such that $S_\nu \cup T_\nu \subset U$ for every ν, then $\lim T_\nu \subset U$ implies $\lim S \subset U$. Further equivalences can easily be derived.

Finally it should be noted that using equivalence 1) H. Cartan's problem can be sharpened to the following problem (unsolved):

Let U be a complex manifold with countable base such that

1) To every point P in U there exist n functions $h_1, \ldots, h_n$ holomorphic in all of U that are local coordinates in a neighborhood of P.

2) To any two different points P and Q in U there exists a function f holomorphic in U such that $f(P) \neq f(Q)$.

3) There exists a plurisubharmonic function $\phi(P)$ in U

such that $\{P | \phi(P) < M\}$ is relatively compact in U for every real M.

Under these conditions, is U a Stein manifold?

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FUNCTIONAL EQUATIONS AND DIRICHLET SERIES

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We shall review some recent applications of the theory of location of singularities of analytic functions, represented by Dirichlet series, to the problem of determining the maximum number of linearly independent solutions of Riemann's functional equation as generalized by Bochner [1]*.

1. SINGULARITIES OF FUNCTIONS REPRESENTED BY DIRICHLET SERIES

Let s be a complex variable, $s = \sigma + i\tau$, and $0 < \lambda_1 < \lambda_2 < \ldots \longrightarrow \infty$. Let us consider the function $f(s) = \Sigma a_n \exp(-2\pi\lambda_n s)$, where the series is assumed to have a finite abscissa of convergence. If $\lim\inf(\lambda_{n+1} - \lambda_n) > 0$, and $\lim n/\lambda_n = 0$, then by a theorem of F. Carlson, E. Landau and O. Szasz, which generalizes Fabry's gap theorem on power series, every point of the axis of convergence of the Dirichlet series is a singularity of $f(s)$. This theorem was generalized by Ostrowski [3] who proved that if $\lim\inf(\lambda_{n+1} - \lambda_n) \geq 2$,

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* Numbers in square brackets refer to the bibliography at the end of this paper. Relevant references not listed explicitly can be found in the cited papers.

and $D^\lambda \equiv \limsup n/\lambda_n \leq \theta$, then there exists a function r depending only on θ, with the property: $r(\theta) \longrightarrow 0$ as $\theta \longrightarrow 0$, such that $f(s)$ has at least one singularity in each circle of which the centre is any point on the axis of convergence, and the radius is r. This theorem can also be extended to the case: $\liminf(\lambda_{n+1} - \lambda_n) > 0$. Ostrowski's theorem was considerably sharpened by Polya [3], who proved that if $h_\lambda \equiv \liminf(\lambda_{n+1} - \lambda_n) > 0$, and the sequence $\{\lambda_n\}$ has a finite <u>maximal density</u> M, then in every interval of length greater than M of the axis of convergence of $\Sigma a_n \exp(-2\pi\lambda_n s)$ there is at least one singularity of $f(s)$. Since the maximal density of $\{\lambda_n\}$ has the properties: (a) if $\lim n/\lambda_n$ exists, then $M = \lim n/\lambda_n$; (b) $M \geq \limsup n/\lambda$ (c) $h_\lambda \cdot M \leq 1$, it follows from Polya's theorem that if $h_\lambda > 0$, and $n/\lambda_n \longrightarrow D < \infty$, then in every interval of length greater than D of the axis of convergence, there exists at least one singularity of $f(s)$; secondly, if only $h_\lambda > 0$, then $f(s)$ has an infinity of singular points on the axis of convergence.

A related type of theorem, based on the analogy of Cauchy's estimate for the coefficients of power series, is one in which we make assumptions about λ_n as above, and about the <u>absence</u> of singularities of $f(s)$ in a certain region, for example in a strip parallel to the real axis, and deduce, as a consequence, an estimate for a_n, which in special cases helps us to prove that $a_n \equiv 0$. Since we generally start with series which do not vanish identically, this would imply a contradiction of our assumption on the absence of singularities, and thereby yield an indirect method of locating the singularities. The following theorem of Mandelbrojt [5] is an example: if σ_c is the (finite) abscissa of convergence of the Dirichlet series $\Sigma a_n \exp(-2\pi\lambda_n s) = f(s)$, with $D^\lambda < \infty$, and if $f(s)$ can be continued analytically to the left of $\sigma = \sigma_c$, in a strip $a \leq \tau \leq b$, with $b - a > D^\lambda$, and R is such that $D^\lambda < 2R < b - a$, then

$$(1.1) \qquad |a_n| \leq A(R) \cdot \lambda_n \cdot M_n^* \cdot M(\sigma_o, R) \exp(2\pi\lambda_n\sigma_o)$$

where σ_o is any real number, $A(R)$ depends only on R, M_n^*, the <u>associated sequence</u> of the $\{\lambda_n\}$ (see [5]), is finite for each n, and $M(\sigma_o, R) = \max|f(s)|$ for $|s - \sigma_o - \frac{1}{2}(a+b)| \leq R$. If, for example, $f(s)$ is bounded in the strip considered, then by letting $\sigma_o \longrightarrow -\infty$, we can, in special cases, infer that $a_n = 0$, $n \geq 1$.

A somewhat different, but related, type of theorem, with an arithmetical bias, is due to Agmon; we consider the following variant [5]. Let S stand for the set of singularities of $f(s)$ on the axis of convergence of $\Sigma a_n \exp(-2\pi\lambda_n s)$, say $\sigma = \sigma_c$. If on a segment of that axis, say $\sigma = \sigma_c$, $a \leq \tau \leq b$, with $b - a > D^\lambda + h_\lambda^{-1}$, there lie k

points of the set S, which are denoted by $\sigma_c + i\alpha_q$, $q = 1, 2, \ldots, k$, and all of them are simple poles of f, and f can be continued analytically into the half-strip: $\sigma < \sigma_c$, $a \leq \tau \leq b$, this continuation being bounded in $\sigma \leq \sigma_c - 1$, $a \leq \tau \leq b$, then the only singularities of f are the points of S. f(s) is uniform in the whole s-plane punctured by the points of S; every isolated point of S is a simple pole of f. If $\sigma_c + i\alpha$ is such a point, then $\alpha = m_1\alpha_1 + \ldots + m_k\alpha_k$, where the m's are integers. For $\sigma < \sigma_c$, the function f is represented by a Dirichlet series of the form $\Sigma\, a_n' \exp(2\pi\lambda_n' s)$, with $\lambda_{n+1}' - \lambda_n' \geq h_\lambda'$, and actually $\lambda_{n+1} - \lambda_n \geq h_\lambda$.

We shall presently see how all these results can be applied to the solution of functional equations of Riemann's type.

2. GENERAL DIRICHLET SERIES AS SOLUTIONS OF FUNCTIONAL EQUATIONS

It is well known that the zeta-function of Riemann satisfies the equation

$$\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma\{\tfrac{1}{2}(1-s)\}\zeta(1-s).$$

According to a classical theorem of Hamburger [3] it is essentially uniquely defined by the functional equation. To be precise, if we start with two functions f and g which can be represented by the simple Dirichlet series $f(s) = \Sigma a_n n^{-s}$, $g(s) = \Sigma b_n n^{-s}$, the first series converging absolutely for $\sigma > 1$, and the second for $\sigma > \beta > 0$, and f is of the form: $f(s) = F(s)/G(s)$, where F is an entire function of finite order and G a polynomial, then the functional equation

$$(2.1) \qquad \pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)f(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma\{\tfrac{1}{2}(1-s)\}g(1-s)$$

implies that $f(s) = g(s) = a_1 \cdot \zeta(s)$. More generally, if g is of the form $g(s) = \Sigma b_n \mu_n^{-s}$, where $0 < \mu_1 < \mu_2 < \ldots \longrightarrow \infty$, then the inequality $\mu_1 \geq 1$ implies again that there is essentially (i.e. except for a constant factor) only one solution, namely $\zeta(s)$, whereas if $\mu_1 < 1$, then the number of solutions is not necessarily finite.

There are two simple proofs of Hamburger's theorem, one by Siegel [3] and another by Hecke [1]. Siegel's proof consists in showing that equation (2.1) implies the relation

$$(2.2) \qquad \sum_1^\infty a_n \exp(-\pi n^2 x) = x^{-\frac{1}{2}} \sum_1^\infty b_n \exp(-\pi n^2/x) + \frac{1}{2}Q(x),$$

for $\operatorname{Re} x > 0$, where $Q(x)$ is a finite sum of terms of the form $x^{-r}Q_r(\log x)$, Q_r being a polynomial; and this in return implies another relation

$$(2.3)\quad \sum_{1}^{\infty} \frac{a_n t}{\pi(t^2 + n^2)} - H(t) = \sum_{1}^{\infty} b_n \exp(-2\pi n t), \qquad \operatorname{Re} t > 0,$$

where $H(t)$ is a sum of terms of the form $t^a \log^b t$. The periodicity of th right side of (2.3) then implies that $a_n = a_{n+1}$, for $n \geq 1$.

Hecke's proof, on the other hand, depends on the fact that the functional equation implies the summation formula

$$(2.4)\qquad B(x) \equiv \sum_{n \leq x} b_n (x - n) = -2 \sum_{1}^{\infty} \frac{a_n}{(2\pi n)^2} (\cos 2\pi n x - 1) + R(x),$$

where $R(x)$ is a sum of terms of the form $x^a \log^b x$. The series on the right side of (2.4) is again a periodic function of x with period 1, so that $B(x + 1) - B(x) = R(x + 1) - R(x)$. Here the right side is, for $x > 0$, a continuously differentiable function, while the left side is piece-wise linear, with a piece-wise continuous derivative, equal to b_{m+1} in $m \leq x < m+1$; hence $b_m = b_{m+1}$, $m \geq 1$. Hecke's method is quite general, and can always lead to a summation-formula from a functional equation.

A more general equation than (2.1) was first formulated and studied by Bochner [1], who established a kind of reciprocity between such a functional equation and what he calls a modular relation (which is really a generalized theta-relation of the form (2.2)) by means of his concept of residual function (of which $Q(x)$ in (2.2) is a particular case) [2]. If we start with a real number $\delta > 0$, and two sequences of exponents $\{\lambda_n\}$, $\{\mu_n\}$ such that $0 < \lambda_1 < \lambda_2 < \ldots \longrightarrow \infty$, $0 < \mu_1 < \mu_2 < \cdots \longrightarrow \infty$, we speak of a solution (ϕ, ψ) of the functional equation

$$(2.5) \qquad \pi^{-\frac{1}{2}} \Gamma(\tfrac{1}{2}s)\phi(s) = \pi^{-\frac{1}{2}(\delta - s)} \Gamma\{\tfrac{1}{2}(\delta - s)\}\psi(\delta - s),$$

<u>pertaining to the label</u> $(\delta, \lambda_n, \mu_n)$, if there exists a sequence of complex numbers a_n, and a sequence b_n, not all the terms being zero, such that the Dirichlet series $\phi(s) = \sum_1^\infty a_n \lambda_n^{-s}$, and $\psi(s) = \sum_1^\infty b_n \mu_n^{-s}$ admit finite abscissae of absolute convergence, and there exists a function χ which is holomorphic in a region $|s| > R$, such that $\lim_{|\tau| \to \infty} \chi(\sigma + i\tau) = 0$ uniformly in each strip $\sigma_1 \le \sigma \le \sigma_2$, and such that $\chi(s) = \pi^{-\frac{1}{2}s} \Gamma(\frac{1}{2}s)\phi(s)$ for $\sigma > \alpha$, and $\chi(s) = \pi^{-\frac{1}{2}(\delta - s)} \Gamma\{\frac{1}{2}(\delta - s)\} \psi(\delta - s)$ for $\sigma < \beta$, where α, β are some constants [3].

In the case of such a generalized functional equation as Bochner's, one can, if one wishes, again follow Hecke's method of proof of Hamburger's theorem, and establish a generalized summation-formula of the type (2.4), by taking if necessary the higher averages $\sum_{n<x} b_n (x - n)^\alpha$, $\alpha > 0$, and draw certain conclusions, at least for the special cases $\delta = 1$, or 3. An important instance of this procedure arises when the coefficients b_n have some arithmetical interest, as for example, when b_n is the number of representations of a number λ_n as a quadratic form; the summation formulae which then arise will be generalizations of the Hardy-Voronoi identity, and will involve zeta-functions of indefinite quadratic forms [cf. V. V. Rao, Doctoral Thesis, Bombay, 1957]. It seems also possible to reverse Hecke's procedure and to deduce the functional equation from the summation formula, at least in some special cases.

On the other hand one can generalize Siegel's method of proof of Hamburger's theorem, by starting with Bochner's result that equation (2.5) implies the modular relation

$$\sum_{1}^{\infty} a_n \exp(-\pi\lambda_n^2 x) = x^{-\delta} \sum_{1}^{\infty} b_n \exp(-\pi\mu_n^2/x) + P(x), \tag{2.6}$$

where $P(x)$ is a residual function. (2.6) implies another relation similar to (2.3) which is fundamental for our purposes.

THEOREM 1 [3]. Functional equation (2.5) implies, for a sufficiently large integer r, the following relation:

$$\frac{\Gamma\{\frac{1}{2}(\delta+1)\}}{\pi^{\frac{1}{2}(\delta+1)}} \sum_{1}^{\infty} a_n \frac{d^{2r}}{dt^{2r}} \left[\frac{t}{(t^2+\lambda_n^2)^{\frac{1}{2}(\delta+1)}}\right] - K_r(t) \tag{2.7}$$

$$= (2\pi)^{2r} \sum_{1}^{\infty} b_n \mu_n^{2r} e^{-2\pi\mu_n t}$$

where $K_r(t)$ is holomorphic on the surface on which $\log t$ is defined, and $K_r(t) = 0(|t|^{-\varepsilon})$, $\varepsilon > 0$, as $|t| \longrightarrow \infty$ in any angle $|\arg t| \leq \theta_o$.

We shall see that relation (2. 7) implies certain restrictions on the existence of solutions for equation (2. 5), and that these restrictions have to do with the density of the sequences λ_n and μ_n. Since in (2. 7) the μ_n's are the exponents of a Dirichlet series, and the λ_n's are the singularities of the function represented by the series, on the axis of convergence, we can call into play all the results described in §1.

Before so doing, it may be pertinent to remark that the proof of Theorem 1 [3] depends on the modular relation of Bochner [1], which when multiplied by the smoothing kernel $H_o(t, x) = t \exp(-\pi t^2 x)$, and integrated with respect to x from 0 to ∞, yields

$$(2.8) \qquad \frac{\Gamma\{\frac{1}{2}(\delta + 1)\}}{\pi^{\frac{1}{2}(\delta + 1)}} \sum_{1}^{\infty} \frac{a_n t}{(t^2 + \lambda_n^2)^{\frac{1}{2}(\delta+1)}} - K_o(t) = \sum_{1}^{\infty} b_n \exp(-2\pi\mu_n t).$$

The series on the left does not, in general, converge, without some additional assumptions on a_n or λ_n, and if we wish to uphold something like (2. 8) without such restrictions, an obvious course would be formally to "differentiate" it a sufficiently large number of times. It is proved in [3] that the differentiated relation can indeed be obtained directly from the modular relation (2.6) (without differentiation), if instead of using the kernel $H_0(t, x)$ one uses the differentiated kernel $H_r(t, x) = (d^{2r}/dt^{2r})[t \exp(-\pi t^2 x)]$.

I have since given a proof of Theorem 1 which does not require the modular relation (2.6), and which does not require even the differentiated kernel $H_r(t, x)$, by directly showing, with the aid of Mellin's transformation of the Beta function, that the right side of (2.7) is equal to the left. Such a proof incidentally implies a further simplification of Siegel's proof of Hamburger's theorem; it also permits the application, as J. P. Kahane has observed, of the theory of mean-periodic distributions to the solution of functional equations.

3. APPLICATION OF DENSITY THEOREMS

Because of our assumption, in equation (2.5), that not all the numbers $\{a_n\}$ or $\{b_n\}$ vanish, and that $\Sigma\, b_n \mu_n^{-s}$ admits a finite abscissa of convergence, it follows that $(2\pi)^{2r} \sum_1^\infty b_n \mu_n^{2r} \exp(-2\pi\mu_n t)$ in (2.7) converges absolutely for $\operatorname{Re} t > 0$, and the function represented by it must have a singularity somewhere on the axis $\operatorname{Re} t = 0$.

The only possible singularities are at $t = \pm i\lambda_n$, $n \geq 1$, and at $t = 0$ ($\equiv \lambda_o$). If we now assume that $h_\mu \equiv \lim\inf(\mu_{n+1} - \mu_n) > 0$ and $\lim n/\mu_n = D < \infty$, then by Polya's theorem (cf. §1) it follows that every interval of length greater than D on the imaginary axis contains at least one λ_n; rather, the non-existence of any λ_n in such an interval would imply a contradiction. Next, if there is an interval L, of length exceeding D, which contains exactly one λ_n, say λ_{r_o}, then there can be at most one solution, for if there were two solutions $\{\phi, \psi\}$, $\{\phi', \psi'\}$ with the corresponding coefficients (a_n, b_n), (a'_n, b'_n), then by writing (2.7) for each of the two solutions, and subtracting after multiplication by a suitable constant, one obtains the series $(2\pi)^{2r} \sum_1^\infty (b_n - cb'_n)\mu_n^{2r} \exp(-2\pi\mu_n t)$ which again has the line Re $t = 0$ as its axis of convergence, but which has no singularity in the interval L, since the term involving a_{r_o} and λ_{r_o} disappears in the subtraction. Hence $b_n - cb'_n = 0$ for $n \geq 1$, and the two solutions are linearly dependent. By means of this argument one obtains

THEOREM 2 [3]. If $\lim\inf(\mu_{n+1} - \mu_n) > 0$, and $\lim n/\mu_n = D < \infty$, then the number of linearly independent solutions of functional equation (2.5) is equal to the minimum number of points among the $\{\lambda_n\}$ that lie in an interval, of the positive real axis, of length greater than D.

It is obvious that in this theorem one can replace the condition $\lim n/\mu_n = D < \infty$ by the less stringent one $D^{\mu} \equiv \limsup n/\mu_n < \infty$, by applying Ostrowski's theorem (cf. §1) or its extension due to Mandelbrojt [3]. As a matter of fact, however, it was shown recently in [5] that the first condition $\liminf (\mu_{n+1} - \mu_n) > 0$ can be dropped altogether, if we appeal to Mandelbrojt's inequality (1.1). For, if there is an interval: $\sigma = 0$, $a \leq \tau \leq b$, with $b - a > D^{\mu}$, which is free from any λ_n, then in the strip $a \leq \tau \leq b$, the function $f(s) = (2\pi)^{2r} \sum_1^{\infty} b_n \mu_n^{2r} \exp(-2\pi\mu_n s)$ is bounded, (or is at most $O(|s|^a)$), and if we let $\sigma_o \longrightarrow -\infty$ in (1.1), we obtain: $a_n = 0$, $n \geq 1$. Then by repeating the argument described above, we obtain

THEOREM 3 [5]. *If $D^{\mu} \equiv \limsup n/\mu_n < \infty$, then the maximum number of linearly independent solutions of the functional equation (2. 5) is equal to the minimum number of points among the $\{\lambda_n\}$ that lie in an interval of length greater than D^{μ}.*

It follows therefore that if (2.5) holds, and $D^{\mu} < \infty$, then $\lambda_{n+1} - \lambda_n \leq D^{\mu}$ for every n. Hence $\lambda_r \leq r \cdot D^{\mu}$ or $(r/\lambda_r) \geq (1/D^{\mu})$ or $D^{\lambda} \geq (1/D^{\mu})$ or

(3.1) $$D^{\lambda} \cdot D^{\mu} \geq 1,$$

(with the understanding that if one of the D's is zero, then the other is ∞). See [3] for the first formulation. Inequality (3.1) suggests that we investigate the extremal cases $D^{\lambda} = 1$, $D^{\mu} = 1$. If, in these cases, we make some additional assumption on the numbers a_n, then we can assert that there can be at most one solution. We have

THEOREM 4 [3]. Let

(3.2) $$D^{\lambda} = D^{\mu} = 1.$$

Define $A(n) = \sum_{n \leq \lambda_r \leq n+1} |a_r|$, $A(0) = \sum_{\lambda_r < 1} |a_r|$, and let

(3.3) $$A(n) = 0(n^{\frac{1}{2}(\delta - 1)}).$$

Then $\lambda_{n+1} - \lambda_n = 1$, for every n, so that (by Theorem 3) equation (2.5) has at most one solution.

The proof follows from the fact that if there exists a single index r_o such that $\lambda_{r_o+1} - \lambda_{r_o} < 1 - 4a$, $0 < a < \frac{1}{4}$, $a < \lambda_{r_o}$, and $a_{r_o} \neq 0$, then there exists an infinity of such indices, which will contradict the assumption $D^\lambda = 1$. One uses the fact that (i) $f(s) = (2\pi)^{2r} \sum_1^\infty b_n \mu_n^{2r} \exp(-2\pi\mu_n s)$ is almost periodic (Bohr) on any line $\text{Re } s = \sigma_o > 0$, and therefore has a relatively dense set of translation numbers τ_ε, and (ii) if u is such that $(\lambda_{r_o} + u - a, \lambda_{r_o} + u + a)$ is free from any λ_n, then in the rectangle R_{r_o} : $\lambda_{r_o} - \frac{1}{2} a \leq \tau \leq \lambda_{r_o} + \frac{1}{2} a$, $0 < \sigma < 1$, the functions $\{g_u(s) = f(s) - f(s+iu)\}$ form a normal family. If $\varepsilon > 0$ is given, and u is chosen as a translation number of $f(s)$ corresponding to it, then (i) and (ii) can be shown to be contradictory, thereby establishing the existence of a new λ, say λ_p, in $(\lambda_{r_o} + u - a, \lambda_{r_o} + u + a)$ for each translation number u. The same argument when applied to λ_{r_o+1} will lead to a λ_{p+1}, such that the pair $(\lambda_p, \lambda_{p+1})$ have the property $\lambda_{p+1} - \lambda_p < 1 - 2a$. For details, see [3].

It is desirable to prove Theorem 4 without the explicit assumption (3.3). I learn from Professor Mandelbrojt that (3.3) can be shown to be a consequence of the other hypotheses, by means of a Tauberian theorem in complex variables.

If we retain (3.2), and assume, instead of (3.3), that

(3.4) $$\mu_{n+1} - \mu_n = 1 \text{ for } n \geq n_o,$$

then we have, as an easy consequence of Theorem 1,

THEOREM 5 [3]. If (3.2) and (3.4) hold, then equation (2.5) implies that $\lambda_n = n$, or $n - \frac{1}{2}$; $\mu_n = n$, or $n - \frac{1}{2}$; $\delta = 1$, or 3, and the solutions are $\zeta(s)$, $(2_1^{1-s}-1)\zeta(s)$, and $2^{s-1}L(s-1)$. [$L(s) = \sum_1^\infty d_n n^{-s}$, $d_n = (-1)^{\frac{1}{2}(n-1)}$ for n odd, and $= 0$ for n even.]

If we retain only assumption (3.4), and have, instead, the assumption that there are only finitely many different b_n's, then by Szegö's theorem on power series, we obtain

THEOREM 6 [3]. If there are only finitely many different b_n's, then under assumptions (3.4) and (3.5), functional equation (2.5) implies that $\delta = 1$, $\mu_n = n$, $\lambda_n = n/k$ for a certain integer k, $a_n = a_{n+k}$ and $b_n = \sum_{q=0}^{n-1} a_n \cos(2\pi qn/k)$.

On the other hand, if we assume $\delta = 1$, or 3, then by integrating relation (2.7) one can obtain a similar relation in which λ_n's occur as simple poles of a Dirichlet series on its axis of convergence. Hence by appealing to Agmon's theorem [cf. §1] one obtains

THEOREM 7 [5]. If $h_\mu \equiv \liminf(\mu_{n+1} - \mu_n) > 0$, and $\delta = 1$, or 3, then the set of numbers $\{\lambda_n\}$ has a finite base (with integer coefficients); any (finite) subset contained in an interval of length greater than $D^\mu + h_\mu^{-1}$ serves as a base for generating all the λ_n.

It is a simple corollary of Theorem 7 that if $\delta = 1$ or 3, and $\mu_n = n$, then the condition $\lambda_1 \geq 1$ implies that $\phi(s) = \psi(s) = a_1 \zeta(s)$.

4. SOME PROBLEMS

As I have pointed out elsewhere, the problem of the existence of solutions is not at all touched in the above considerations. It would be interesting to construct Dirichlet series $\Sigma a_n \lambda_n^{-s}$, where not all the a_n are zero, not all the λ_n are integers, and say λ_2/λ_1 is irrational. Theorem 7 suggests a method. It is also a problem to compare Theorem 7 with Ostrowski's theorem [6] that if a Dirichlet series of the type $\Sigma a_n e^{-\lambda_n s}$ has a half-plane of absolute convergence, and satisfies an algebraic difference-differential equation, then only a finite number of the λ_n's are linearly independent. It is a problem also to see whether the number of solutions can be reduced by assuming that, along with

$\Sigma\, a_n \lambda_n^{-s}$, the series $\Sigma\, a_n \chi(n) \lambda_n^{-s}$ is also a solution, where $\chi(n)$ is a character modulo m.

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(Other relevant references can be found in these papers.)

ON CURRENTS IN AN ANALYTIC COMPLEX MANIFOLD

Georges de Rham

Let M be an analytic complex manifold with n complex dimensions. To each analytic complex submanifold W of M, with p complex dimensions, without singularities and closed in M, a closed current is associated, defined by

$$W[\phi] = \int_W \phi,$$

where ϕ is an arbitrary exterior differential form with a compact carrier in M, of degree 2p, of class C^∞. If W has singularities, it is not evident that the integral of ϕ over W can be defined and the problem arises whether a closed current can still be associated with W. This problem has been treated recently by Pierre Lelong [3]. I will sketch here, for the same problem, a method based upon the integral geometry of the Hermitean elliptic space.

Let $z_o, z_1, \ldots, z_n$ be homogeneous complex coordinates in the complex n-dimensional projective space P, and

$$H = \sum_{k=0}^{n} z_k \bar{z}_k, \qquad ds^2 = \sum_{k=0}^{n} \frac{\partial^2 \log H}{\partial z_k \partial \bar{z}_k} dz_k d\bar{z}_k .$$

With this ds^2, P is the Hermitean elliptic space. The unitary transformations on $z_o, z_1, \ldots, z_n$, leave H and ds^2 invariant; they define motions of the space P and the group G of all these motions is compact. The exterior differential form

$$\omega = \frac{i}{2} \sum_{k=0}^{n} \frac{\partial^2 \log H}{\partial z_k \, \partial \bar{z}_k} dz_k \wedge d\bar{z}_k$$

is closed and invariant by the group G. Elie Cartan has shown that, up to a constant factor, the only invariant exterior differential forms are the powers ω^p of ω $(p = 0, 1, \ldots, n)$.

With the help of this Riemannian metric, one defines in the usual way the measure of any differentiable real q-dimensional submanifold of P. For $1 < q < 2n$, I will call it the q-dimensional area or simply the area. For an analytic complex submanifold, the area can be expressed by the forms ω^p. But let us first recall the definition of an analytic set [4].

A subset V of P (or of any analytic complex manifold) is called <u>analytic</u> at the point z if there is an open neighborhood U of z such that $V \cap U$ can be defined by a system of equations

$$f_j = 0 \qquad (j = 1, 2, \ldots, \nu)$$

where the f_j are holomorphic functions in U. If $z \in V$ and if furthermore U and the equations can be chosen in such a way that the differentials df_j are linearly independent at z and $\nu = n-p$, we say that V is regular p-dimensional at z. Then there exists a well determined p-dimensional linear space tangent to V at z (p means here, of course, the complex dimension, the real dimension will be $2p$). If $z \notin V$, according to the above definition, V will be analytic at z if and only if $z \notin \overline{V}$ (the topological closure of V).

Now, a set $W \subset P$ which is analytic and regular p-dimensional at each of its points is a differentiable submanifold of P and has a 2p-dimensional area, finite or infinite, which is given by

$$\text{area of } W = \frac{1}{p!} \int_W \omega^p . \tag{1}$$

Of course, one has to take the natural orientation of W in such a way that the integral will be positive.

We shall denote by E the Grassmannian manifold of all linear subspaces L^{n-p} of P with $n-p$ complex dimensions. It is a symmetric Riemannian space for G, with an invariant positive definite Riemannian metric. It is also an analytic complex manifold, with $p(n - p + 1)$ complex dimensions.

For every subset K of P, let $\hat{K}$ be the set of all L^{n-p} which intersect K. If $O \varepsilon P$, $\hat{O}$ will be the set of all L^{n-p} passing through O; it is a submanifold of E, with $p(n-p)$ complex dimensions. The complex codimension of $\hat{O}$ (i.e. the difference $\dim E - \dim \hat{O}$) is equal to p.

The set of all pairs (z, L^{n-p}) such that $z \varepsilon L^{n-p}$ is a fibre space with base space P, fibres isomorphic to $\hat{O}$ and with projection $pr_1(z, L^{n-p}) = z$. It is also a fibre space with base space E, fibres isomorphic to L^{n-p} and with projection $pr_2(z, L^{n-p}) = L^{n-p}$. Then we have

$$\hat{K} = pr_2 \circ pr_1^{-1}(K)$$

and from this one can easily prove the following propositions:

I. If K is a differentiable submanifold of P or a differentiable chain of real dimension $q < 2p$, $\hat{K}$ is a set of measure zero in E.

II. If the subset W_k of P is analytic regular $(p-k)$-dimensional at each of its points, $k \geq 0$, $\hat{W}$ is analytic regular $(pn - p^2 + p - k)$-dimensional at each of its points: the complex codimension of $\hat{W}_k$ is equal to k.

If U is a subset of a Riemannian space and $\varepsilon > 0$, the set of all points whose distance to U is $< \varepsilon$ will be called the tube of center U and radius ε and denoted by $T(U, \varepsilon)$. If U is a subset of finite area of a differentiable submanifold of real codimension d, the volume of $T(U, \varepsilon)$, for $\varepsilon \longrightarrow 0$, tends to zero as the volume of a ball of radius ε in the Euclidean space R^d, i.e.

(2) $$\text{vol } T(U, \varepsilon) = 0(\varepsilon^d).$$

From the definition of the Riemannian symmetric metrics in P and in E, it follows that the distance from a point 0 to a linear space L^{n-p}, in P, is equal to the distance from $\hat{L}^{n-p}$ to $\hat{O}$ in E (up to a constant factor, which we may assume to be equal to 1). Therefore, for any subset K of P,

(3) $$\hat{T}(K, \varepsilon) = T(\hat{K}, \varepsilon).$$

Now, if K is a compact subset of W_k, $\hat{K}$ will be a compact subset of $\hat{W}_k$, with finite area, and as the real codimension of $\hat{W}_k$ is equal to 2k (according to II), we obtain from (2) and (3):

(4) $$\text{vol } \hat{T}(K, \varepsilon) = 0(\varepsilon^{2k}).$$

Let now S be a differentiable 2p-dimensional chain in P. The Kronecker index $I(S.L^{n-p})$ or algebraic number of intersections of S with L^{n-p} is well determined when L^{n-p} does not meet the boundary of S. By (I), it follows that $I(S.L^{n-p})$ is a function of L^{n-p} defined almost everywhere in E; it can be integrated over E, with respect to the invariant volume in E, and we have the integral-geometric formula

(5) $$\int_E I(S.L^{n-p}) = C_1 \int_S \omega^p,$$

with $C_1 = \pi^{-p} \text{ vol } E.$

This can be proved [1], [2], by showing that the left hand side of (5) is the integral over S of a differential form. Then, as it is clearly invariant under G, that differential form must be a multiple $C_1\omega^p$ of ω^p, and the constant C_1 can be calculated by taking a linear space L^p for S.

From (1) and (5) it follows that

(6) $$\text{area of } W = C \int_E I(W.L^{n-p}),$$

with $$C = \frac{\pi^p}{p!\ \mathrm{vol}\ E} .$$

Let us say that W and L^{n-p} intersect <u>properly</u>, if every point $z \in W \cap L^{n-p}$ is the unique common point of L^{n-p} and the tangent space of W at z; then $I(W.L^{n-p})$ is nothing else than the number of points of $W \cap L^{n-p}$. But, the L^{n-p} which do not intersect W properly form a submanifold of E of complex codimension 1, and therefore a set of measure zero. Hence, for almost all L^{n-p}, $I(W.L^{n-p})$ is the number of points of $W \cap L^{n-p}$.

From (6) we deduce immediately: if D is a domain in P, $W \subset D$ and $I(W.L^{n-p}) \leq m$ (for almost all L^{n-p}), then

(7) $$\text{area of } W \leq mC \text{ vol } \hat{D} .$$

Let now V be an <u>algebraic</u> manifold in P, of complex dimension p and of degree m, let V_1 be the set of singular points of V, and set $W = V - V_1$. Since $I(W.L^{n-p}) = m$ for almost all L^{n-p}, it follows from (6):

The area of an algebraic manifold of complex dimension p and of degree m in P is always equal to $m\frac{\pi^p}{p!}$.

This theorem has been proved by Wirtinger [5] with the help of formula (1), using the fact that the algebraic manifold can be considered as a closed chain and is homologous to mL^{n-p}. This is very clear if the manifold V has no singularities, but Wirtinger did not mention the difficulty arising from the singularities in the definition of the integral in (1).

Since the area of W is finite, the integral

$$\int_W \phi$$

is always convergent, for every continuous differential form ϕ of degree 2p, and defines the current associated with V. Thus, the first part of the problem is solved for algebraic submanifolds.

For the second part, we have to show that this current is closed, or that

$$\int_W d\psi = 0 \tag{8}$$

for every form ψ of class C^1 and of degree 2p-1.

The set V_1 of the singular points of V is an algebraic manifold of at most p-1 dimensions. Let W_1 be the set of points in V_1 at which V_1 is regular (p-1)-dimensional; then $V_2 = V_1 - W_1$

is an algebraic manifold of at most p-2 dimensions. In this manner, we define V_k and W_k for $k = 1, 2, \ldots, p$: V_k is an algebraic manifold of at most p-k dimensions, W_k is the set of points in V_k at which V_k is regular (p-k)-dimensional and $V_{k+1} = V_k - W_k$. We have $V \supset V_1 \supset V_2 \supset \ldots \supset V_p$ and V_{p+1} is the empty set.

I shall now show that if (8) holds for all forms ψ whose carriers do not meet V_k, it also holds for all forms ψ whose carriers do not meet V_{k+1}. Since (8) holds when the carrier of ψ does not meet V_1 and since $V_{p+1} = \emptyset$, (8) will be completely proved by induction.

Assuming that the carrier of ψ does not meet V_{k+1}, let us take $\delta > 0$ smaller than the distance between V_{k+1} and the carrier of ψ. Let $r = r(z)$ be the distance of $z \in P$ to V_k. From the properties of the geodesic distance we know that, for ε sufficiently small, r is differentiable in the domain

$$\Delta = T(V_k, \varepsilon) \cap CT(V_{k+1}, \delta)$$

except at the points of V_k. Moreover, the coefficients of dr are continuous and bounded in $\Delta \cap CV_k$. Take a real function $\alpha(t)$ of a real variable t, of class C^1, such that $\alpha(t) = 1$ for $t < \frac{1}{2}$ and

$\alpha(t) = 0$ for $t > 1$. The form $\alpha(\frac{r}{\varepsilon})\psi$ is then of class C^1, its carrier is contained in Δ, and it is equal to ψ in $T(V_k, \frac{\varepsilon}{2})$. Hence, the carrier of $\psi - \alpha(\frac{r}{\varepsilon})\psi$ does not meet V_{k+1} and it follows from our hypothesis

$$(9) \qquad \int_W d\psi = \int_W \alpha(\tfrac{r}{\varepsilon})d\psi + \frac{1}{\varepsilon}\int_W \alpha'(\tfrac{r}{\varepsilon})dr \wedge \psi.$$

Let us set $K = W_k \cap CT(V_{k+1}, \frac{\delta}{2})$. It follows from the triangle inequality that, for $\varepsilon < \frac{\delta}{2}$, $\Delta \subset T(K, \varepsilon)$. In the integrals on the right hand side of (9), we can replace W by $W \cap T(K, \varepsilon)$, because the forms under the $\int$ signs vanish outside of Δ. Since their coefficients are bounded, it follows from (7) that

$$\left|\int_W d\psi\right| \leq (A + \frac{B}{\varepsilon})\, mC \operatorname{vol} \hat{T}(K, \varepsilon)$$

with A and B constant, and since (4) holds with $k \geq 1$ and ε is arbitrarily small, we get (8). Q. E. D.

The same method can be applied to prove that, if D is a domain in P and if V is a p-dimensional complex analytic set in D, a current is associated to V, which is closed in D. The only supplementary difficulty consists in proving that, for almost all L^{n-p}, the number of points of $W \cap L^{n-p}$ lying in a compact subset of D is bounded.

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SOME APPLICATIONS OF THE THEORY OF DISTRIBUTIONS

TO SEVERAL COMPLEX VARIABLES

Leon Ehrenpreis

1. INTRODUCTION

Let Ω be a bounded (schlicht) domain in C^n with a C^∞ boundary. By D'_q (E_q) we denote the space of distribution (C^∞) forms of type $(0, q)$ on Ω. D_q (E'_q) is the dual of D'_q (E_q); D_q (E'_q) is the space of C^∞ (distribution) $(0, q)$ forms on Ω of compact carrier. We give all these spaces the Schwartz topology (see [9]).

If V is a space of functions or distributions we shall denote by $S(V)$ the sheaf of germs of sections of V.

The differential d defines (see [3], [6]) maps $\bar\partial: SE_q \longrightarrow SE_{q+1}$ (or $\bar\partial: SD'_q \longrightarrow SD'_{q+1}$); the adjoint define maps $\delta: SE_q \longrightarrow SE_{q-1}$ (or $\delta: SD'_q \longrightarrow SD'_{q-1}$). We define the Laplacian $\Delta = \delta\bar\partial + \bar\partial\delta$ so Δ defines maps of $SE_q \longrightarrow SE_q$ or $SD'_q \longrightarrow SD'_q$. Δ is a constant multiple of the ordinary Laplacian on $R^{2n} = C^n$.

By A_q (A'_q) we denote the subspace of E_q (D'_q) consisting of $\bar\partial$-closed $(0, q)$ forms; B_q (B'_q) is the subspace of E_q (D'_q) consisting of δ-closed $(0, q)$ forms. (Note that A'_q (B'_q) is <u>not</u> the dual of A_q (B_q); we denote by A^*_q (B^*_q) the dual of A_q (B_q).)

We are interested in $H^q(\Omega, SA_o)$. By Dolbeault's theorem (see [3], [10]) we have $H^q(\Omega, SA_o) \approx A_q/\bar{\partial}E_{q-1}$.

We say that Ω is q-pseudoconvex if each point x on the boundary of Ω possesses a neighborhood N in C^n for which $N \cap \Omega$ is q-convex in the sense of Rothstein [8], that is, if U is any relatively compact subset of $N \cap \Omega$ then there exists a relatively compact subset U' of $N \cap \Omega$ with $\overline{U} \subset U'$, having the following properties: for any boundary point z' of U' we can find $n - q$ functions $f_1, \ldots, f_{n-q}$ analytic in $N \cap \Omega$, with $|f_j(z')| > 1$, and for each $z \in U$, $\min |f_j(z)| < 1$. Our main result is the following generalization of Oka's theorem (see [1]):

THEOREM 1. Ω is q-pseudoconvex if and only if Ω is q-convex.

We can extend the result to the case where Ω does not have a C^∞ boundary, or is unbounded, by means of an analog of the Behnke-Stein theorem.

In order to prove Theorem 1 we need the following generalization of results of Serre, Cartan, Oka (see [2]):

THEOREM 2. Ω is q-convex if and only if $H^j(\Omega, SA_o) = 0$ for $j \geq n-q$.

Theorem 2 is much easier to prove than Theorem 1, and the implication "q-convex implies q-pseudoconvex" is easy. In section 2 we shall assume Theorem 2 and use it to prove Theorem 1 in case $n = 2$, $q = 1$. This proof can be extended to the general case, but the extension requires some considerations which we shall not give. Our proof is entirely different from the previous proofs of Oka's theorem, and much simpler than the usual proofs.

We can also formulate other necessary and sufficient conditions for q-convexity:

(1) Let Ω be defined in the neighborhood of a boundary point p by $u > 0$. Then the Levi quadratic form $\Sigma \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz^j d\bar{z}^k$ has (in normal form) on the plane $\Sigma \frac{\partial u}{\partial z_j} dz^i = 0$ not more than $n-q-1$ negative terms in the neighborhood of p in Ω.

(2) Let $\rho(z)$ denote the distance (measured in any convenient metric) from $z \in \Omega$ to the boundary of Ω. Then the quadratic

form $\frac{\partial^2(-\log \rho(z))}{\partial z_j \partial \bar{z}_k} dz^j d\bar{z}^k$ has not more than n-q-1 negative terms (in normal form).

(3) The analog of the "theorem of continuity" (see [1]) holds for analytic planes of dimension n-q (in place of analytic lines for usual convexity).

Let us define the sheaves S_j by: $S_o = A_o$; if $S_o, \ldots, S_{j-1}$ have been defined, then S_j is defined so as to make the following sequence exact: $0 \longrightarrow S_j \longrightarrow S_{j-1} \xrightarrow{\partial_j} S_{j-1} \longrightarrow 0$ where $\partial_j = \frac{\partial}{\partial z_j}$. Then by taking the cohomology of these exact sequences and using Theorem 2 we can prove

THEOREM 3. Let Ω be q-pseudoconvex; then $H^j(\Omega, C) = 0$ for $j \geq 2n-q$.

REMARK. A result stronger than Theorem 3, namely that $H^j(\Omega, Z) = 0$ for $j \geq 2n-q$, was proven at about the same time by R. Thom by means of the theory of critical points of M. Morse.

In Section 3 we discuss extensions of closed forms and Runge domains. In particular, we give a characterization of Runge domains in terms of extensions of closed forms. In Section 4 we give a generalization of the classical potential theory of Riesz to plurisubharmonic functions by means of the Schwartz kernel theorem (see [5]).

It is possible to give other applications of the theory of distributions. In particular, Theorem 1 can be extended to classes of functions which are defined as the solutions of equations $\partial_j f = 0$ for suitable systems $\{\partial_j\}$.

The results of Sections 3 and 4 can be extended to manifolds. It seems unlikely that Theorem 1 is true for manifolds in general. The difficulty is that there exist manifolds Ω e.g. complex projective spaces, for which $H^j(\Omega, SA_o) = 0$ for $j \geq 1$, but such that Ω is not a Stein manifold. However, we can prove that, with a general definition of q-pseudoconvexity, if Ω is a q-pseudoconvex manifold then $H^j(\Omega, SA_o) = 0$ for $j \geq n-q$.

2. PROOF OF THEOREM 1 IN CASE n = 2.

We have to show that $\bar{\partial}: E_o \longrightarrow A_1$ is onto. Now, A_1 is a closed subspace of E_1, so the dual A_1^* of A_1 can be identified with E_1' modulo the closure of $\delta E_2'$ in E_1'. To show $\bar{\partial}$ is onto, it is sufficient, by the Hahn-Banach theorem and the fact that E_o is metrizable, to show that if B is a subset of A_1^* such that δB is bounded in E'_o then B is bounded in A_1^*. By the definition of the topology of E'_o, there exists a compact set $K \subset \Omega$ such that all δT for $T \in B$ have their carriers (supports) contained in K.

LEMMA 1. There is a compact set $K' \subset \Omega$ so that each $T \in B$ can be represented by a distribution whose carrier lies in K'.

Proof. (Pie-nibbling method.) For $k = 1, 2, 3, 4$ we choose a finite covering $\{U_j^k\}$ of the boundary of Ω by open sets U_j^k such that

1. $U_j^k \cap \Omega$ are (connected) domains of holomorphy and U_j^k meets the boundary of Ω in connected sets.

2. For each j we have the following property: Let f be analytic on an arbitrary neighborhood in Ω of the intersection of U_j^4 with the boundary of Ω. Then f can be extended to a function which is holomorphic on U_j^3.

3. For no j does the closure of U_j^4 meet K.

4. The closure of U_j^k is contained in the union of U_j^{k+1} and the boundary of Ω for $k = 1, 2, 3$.

We shall show that each $T \in B$ can be represented by a distribution S whose carrier does not meet $\cup U_j^1$. For this purpose, let S^1 be any element of E'_1 which represents T, say $S^1 = (S_1^1, S_2^1)$ where $S_j^1 \in E'_o$. We know that $\delta T \cdot g = 0$ for all $g \in C^\infty$ of compact carrier with carrier $g \subset U_1^4$. It follows that $0 = T \cdot \bar{\partial} g$ $= (S_1^1, S_2^1) \cdot \bar{\partial} g = S_1^1 \cdot \frac{\partial g}{\partial \bar{z}_1} + S_2^1 \cdot \frac{\partial g}{\partial \bar{z}_2} = \left(- \frac{\partial S_1^1}{\partial \bar{z}_1} - \frac{\partial S_2^1}{\partial \bar{z}_2} \right) \cdot g$. Thus, $\frac{\partial S_1^1}{\partial \bar{z}_1} + \frac{\partial S_2^1}{\partial \bar{z}_2} = 0$ in U_1^4. Since U_1^4 is a domain of holomorphy, there exists an S^2 $D'_2(U_1^4)$ with $\frac{\partial S^2}{\partial \bar{z}_1} = S_2^1$, $\frac{\partial S^2}{\partial \bar{z}_2} = -S_1^1$, i.e. $\delta S^2 = S^1$ in U_1^3.

Now, $\delta S^2 = S^1 = 0$ on a neighborhood N in Ω of the intersection of U_1^4 with the boundary of Ω. Thus, S^2 coincides with a function S^3 which is analytic on N; by our construction, S^3 can be extended to an analytic function (which we again denote by S^3) on all of U_1^3.

For any $f \in A_1$, we can write

(1) $$S^1 \cdot f = S^1 \cdot \phi f + S^1 \cdot (1-\phi) f$$

where $\phi \in E_o$, $0 \leq \phi \leq 1$, $\phi = 1$ on a neighborhood of U_1^1 and $\phi = 0$ outside U_1^2. Then we deduce

(2) $S^1 \cdot \phi f = \delta(S^2 - S^3) \cdot \phi f$

because $\partial S^3 = 0$ on $U_1^3 \supset$ carrier ϕf, and if y is any boundary point of U_1^3 then at least one of ϕf, $S^2 - S^3$ vanishes in the neighborhood of y. Thus we also have

(3) $S^1 \cdot \phi f = (S^2 - S^3) \cdot \bar{\partial} \phi f.$

But $\phi = 1$ on a neighborhood of U_1^1, so that $\bar{\partial} \phi f = \bar{\partial} f = 0$ on a neighborhood of U_1^1. This means that $S^1 \cdot \phi f$ can be made small if we make f and its derivatives small on a relatively compact subset of $U_1^4 - U_1^1$. That is, we can write

(4) $S^1 \cdot \phi f = S^4 \cdot f$

where $S^4 \in E_1'$, and the carrier of S^4 does not meet a neighborhood of U_1.

Combining (4) with (1) we have

(5) $T \cdot f = S^5 \cdot f$

where $S^5 \in E_1^1$ and the carrier of S^5 does not meet a neighborhood of U_1^1.

Equation (5) means that U_1^1 can be "nibbled" out of the carrier of S^1.

If we already know that the carrier of S^1 does not meet a neighborhood of $U_2^1 \cup \ldots \cup U_r^1$ (say), then we can modify the above construction to show that the carrier of S^5 does not meet a neighborhood of $U_1^1 \cup U_2^1 \cup \ldots \cup U_r^1$. Thus Lemma 1 is proved.

Proof of Theorem, continued. Let B be a set in A_1^* with δB bounded in E'_o. Let K be a compact set containing the carriers of all $T \varepsilon B$ (by Lemma 1, K exists). We may assume K has a connected C^∞ boundary.

Now, there exist many $T' \varepsilon E'_k$ with $\delta T' = \delta T$; in fact, any $T' = T + \delta T''$ with $T'' \varepsilon E'_2$ has this property. We want to determine that T'' which makes T' "smallest". A heuristic calculus of variations argument shows that we should try to take T'' satisfying

(6) $\Delta T'' = \overline{\partial} T.$

Now, it is easily seen by Fourier transform arguments that (6) does not always have a solution $T'' \varepsilon E'_2$ for given $T \varepsilon E'_1$. However we have

LEMMA 2. There exists a bounded set $B' \subset E'_2$ such that all $T' \varepsilon B'$ have their carriers in K, and

If we already know that the carrier of S^1 does not meet a neighborhood of $U_2^1 \ldots U_r^1$ (say), then we can modify the above construction to show that the carrier of S^5 does not meet a neighborhood of $U_1^1 U_2^1 \ldots U_r^1$. Thus Lemma 1 is proven.

Proof of Theorem, continued. Let B be a set in A_1^* with δB bounded in E_o'. Let K be a compact set containing the carriers of all $T \varepsilon B$ (by Lemma 1, K exists). We may assume K has a connected C^∞ boundary.

Now, there exist many $T' \varepsilon E_k'$ with $\delta T' = \delta T$; in fact, any $T' = T + \delta T''$ with $T'' \varepsilon E_2'$ has this property. We want to determine that T'' which makes T' "smallest". A heuristic calculus of variations argument shows that we should try to take T'' satisfying

(6) $\Delta T'' = \bar{\partial} T$.

Now, it is easily seen by Fourier transform arguments that (6) does not always have a solution $T'' \varepsilon E_2'$ for given $T \varepsilon E_1'$. However we have

LEMMA 2. There exists a bounded set $B' \subset E_2'$ such that all $T' \varepsilon B'$ have their carriers in K, and for each $T \varepsilon B$ we can find a $T' \varepsilon B'$ and a $T'' \varepsilon E_2'$ with carrier $T'' \subset K$ and

(7) $\Delta T'' = \overline{\partial} T - T'$.

To complete the proof of Theorem 1 we shall show that $\{T - \delta T''\}$, $T \varepsilon B$, T'' satisfying (7) is bounded in E_1'. By a result of Malgrange, it is sufficient to show $\{\Delta(T - T'')\}$ is bounded (see [7]). We have

$$\begin{aligned} \Delta(T - \delta T'') &= \overline{\partial}\delta T - \overline{\partial}\delta\delta T'' + \delta\overline{\partial} T - \delta\overline{\partial}\delta T'' \\ &= \overline{\partial}\delta T + \delta(\overline{\partial} T - \overline{\partial}\delta T'') \\ &= \overline{\partial}\delta T + \delta(\overline{\partial} T - \Delta T'') \\ &= \overline{\partial}(\delta B) + \delta B' \end{aligned}$$

by (7). Since $\overline{\partial}$ and δ are continuous, $\{\Delta(T - \delta T'')\}$ is bounded, which completes the proof of Theorem 1.

3. EXTENSION OF CLOSED FORMS AND RUNGE DOMAINS

Let U be a relatively compact subset of Ω, and set $F = \Omega - U$. We obtain the exact sequence of cohomology groups

(8) $H^j(\Omega, SA_o) \longrightarrow H^j(F, SA_o) \longrightarrow H_*^{j+1}(U, SA_o)$

for any $j \geq 0$, where H_* denotes cohomology with compact carriers. Using Theorem 2, Serre's duality theorem [10], and Dolbeault's theorem we deduce

THEOREM 4. Let U be q-convex; then for $j \leq q - 1$ every $\bar{\partial}$-closed $(0, j)$ form on F is cohomologous on F to a form which is $\bar{\partial}$-closed on all of Ω.

Theorem 4 can be considered as an extension theorem for $\bar{\partial}$-closed forms which is in some sense dual to the extension theory of analytic varieties (see [8]).

We could ask what happens for $j = q$. In case $q = n - 1$ (domain of holomorphy) this is answered by

THEOREM 5. Let $U \subset \Omega$ be domains of holomorphy. Then U is a relative Runge domain if and only if for any C^∞ $(0, n-1)$ form f which is defined and $\bar{\partial}$-closed on a neighborhood in Ω of F, and which vanishes outside of a compact set of Ω, there exists a $\bar{\partial}$-closed $(0, n-1)$ form g which is C^∞ and of compact carrier on Ω such that $g > f$ on a neighborhood of F.

In case $n = 1$ we see readily that Theorem 5 is just Runge's theorem that U is a relative Runge domain in Ω if and only if U is relatively simply connected in Ω.

4. POTENTIAL THEORY

We can give the following extension of the classical Riesz theory:

THEOREM 6. Let Ω be a domain of holomorphy whose second Betti number is zero. Then we can find kernels $k_{i\bar{j}}$ (see [5]) which are continuous linear maps of $E_o(\Omega)$ into $E_o(\Omega)$ so that if $\Sigma g_{i\bar{j}} dz^i d\bar{z}^j$ is any C^∞-closed (1, 1) form then the C^∞ function

$$(9) \qquad g = \Sigma k_{i\bar{j}} \cdot g_{i\bar{j}}$$

satisfies $\dfrac{\partial^2 g}{\partial z_i \partial \bar{z}_j} = g_{i\bar{j}}$. In particular, we have the Riesz decomposition formula: Any $f \in E_o(\Omega)$ can be written in the form

$$(10) \qquad f = h + \Sigma k_{i\bar{j}} \cdot \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}$$

where h is pluriharmonic.

For $n > 1$ we do not know any general method of computing the $k_{i\bar{j}}$ (which depend on Ω for $n > 1$).

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ANALYTIC FIBRE BUNDLES

OVER HOLOMORPHICALLY COMPLETE SPACES

Hans Grauert

INTRODUCTION*)

About twenty years ago fibre bundles appeared in mathematical literature for the first time. Their first adequate definition was given by Whitney in the year 1935. In the following decade this newly introduced concept gave rise to many new investigations. The methods developed through these investigations have been essential to modern topology ever since.

Because of the close relations between topology and complex analysis it seems plausible to use fibre bundles for the function theory of several complex variables as well. The first step in this direction was made by H. Cartan [2] when he showed in 1950 that the well-known Cousin problems can be formulated in the terminology of this new theory. Following this, F. Hirzebruch used special complex line bundles in order to carry over the theorem of Riemann-Roch to algebraic manifolds of higher dimensions. Further

*) The square brackets refer to the bibliography at the end of this note. The note contains results of [11], [12], [13]. See also [5] and [10].

extensive investigations about problems pertaining to the subject were made by Kodaira, Spencer, Serre, Atiyah, Frenkel and others.

The essential purpose of these papers was to express analytic properties as consequences of topological data. Thus, for instance, the II. Cousin problem (see [16], [18]) is solvable in domains of holomorphy if and only if a certain cohomology condition is satisfied. In the theorem of Riemann-Roch the number of linearly independent meromorphic functions belonging to a given divisor is related to an algebraic combination of Chern classes. It will be shown here that the so-called complex analytic fibre bundles are determined by their topological properties if their base space is a non-compact complex space which is meaningful for function theory (see I and II in §2.4)[1)].

This has as a consequence that the theory of topological fibre bundles becomes applicable to our analytic fibre bundles; hence

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1) We assume that the base space is a holomorphically complete space. These spaces which form a proper subclass K of the complex spaces have to be regarded as the correct generalizations of the non-compact Riemann surfaces. The domains of holomorphy of C^n and the frequently investigated Stein manifolds (variétés de Stein, for def. see [3], p. 49) belong to K. Already in 1953 F. Frenkel proved theorems I and II for the special case that the structure groups of the considered fibre bundles are solvable. See [6] and [7].

a number of assertions, of which examples are given in § 3. Among these examples it is mentioned, for instance, that an analytic fibre bundle R over a holomorphically complete space B is always analytically trivial if B can be continuously retracted to a point (theorem 6). If B is a non-compact Riemann surface, then each analytic fibre bundle over B with a connected structure group is analytically trivial (theorem 7). After that, statements are proved about the possibility of reducing structure groups of fibre bundles R to subgroups (theorems 4 and 5). All these statements can be successfully applied in function theory of several complex variables (see the investigations by H. Röhrl [18] on the Riemann-Hilbert problem and by H. Holmann [15] on mapping theory, see investigations [13] on vector fields on complex manifolds, etc.).

A short summary of the contents of this paper will be given here: In §1 the concepts of the complex Lie group, of the analytic fibre bundle and of the equivalence of fibre bundles etc. are defined. §2 deals with the relations of these concepts to sheaf theory. Furthermore it gives the main results and shows how they are obtained. The last section deals, as mentioned before, with the consequences which follow from the application of theorems of the theory of topological fibre bundles.

1. FIBRE BUNDLES

1. In this section a short summary of known definitions and theorems will be given. A complex Lie group L^m is an m-dimensional complex manifold (not necessarily connected) with the following properties:

1) A group operation $\cdot$ is defined between the points $\ell \varepsilon L^m$.

2) Let ℓ^{-1} designate the inverse of a point $\ell \varepsilon L^m$; then $(\ell_1, \ell_2) \longrightarrow \ell_1 \cdot \ell_2^{-1}$ is a holomorphic mapping of $L^m \times L^m$ onto L^m.

The points of a complex Lie group can simultaneously be holomorphic automorphisms of a complex space.

We say that a complex Lie group $L = L^m$ operates holomorphically on a complex space[2)] F if:

2) In this note complex spaces are always C-spaces (see: [8], [9] and [11].) H. Cartan called the C-spaces "normal complex spaces" (see: [4], p. 96).

1) a group homomorphism j of the group L into the group G of the holomorphic automorphisms of F is defined,

2) the mapping of the pairs $(\ell, x) \longrightarrow j(\ell) \otimes x$: $L \times F \longrightarrow F$ is holomorphic.

In this $\otimes$ designates the application of the automorphism $j(\ell)$ to x. From now on we shall write ℓ for $j(\ell)$. If j is a homomorphism of L into G we say that L operates effectively holomorphically on F.

2. In order to define the concept of complex fibre bundle we shall introduce some necessary terminology. Let there be given complex spaces B and F, a complex Lie group L operating effectively on F, and a Hausdorff space R.

DEFINITION 1. A (B, F)-map in R is a triple $(U, h, W \times F)$ where U is an open set in R, W an open set in B and h a topological mapping of U onto $W \times F$. Two (B, F)-maps $K_1 = (U_1, h_1, W_1)$ and $K_2 = (U_2, h_2, W_2)$ are continuously (analytically) compatible, if the following holds:

a) $h_1(U_1 \cap U_2) = h_2(U_1 \cap U_2) = (W_1 \cap W_2) \times F$.

b) The mapping $h_1 \cdot h_2^{-1}$ can be given in the form:

$(x, y) \longrightarrow (x, \phi(x)) \otimes y$.

c) The mapping $\phi(x)\colon W_1 \cap W_2 \longrightarrow L$ is continuous (is holomorphic).

Obviously b) contains the statement that $h_2 \cdot h_1^{-1}$ is a "fibre preserving" mapping. If $\phi(x)$ is holomorphic, $h_2 \cdot h_1^{-1}$ and $h_1 \cdot h_2^{-1}$ are also holomorphic mappings. Then the complex structures induced in $U_1 \cap U_2$ by K_v, $v = 1, 2$ are identical. The (B, F)-maps will now be combined to fibre atlasses.

DEFINITION 2. A topological (an analytic) (B, F)-atlas in R is a system $\{U_\nu, h_\nu, W_\nu \times F, \nu \in I\}$ of pairwise continuously (analytically) compatible (B, F)-maps in R, with $\cup U_\nu = R$ and $\cup W_\nu = B$.

A continuous (an analytic) (B, F)-atlas $A = \{K_\nu, \nu \in I\}$ is called <u>complete</u>, if each (B, F)-map in R continuously (analytically) compatible with all K_ν, $\nu \in I$, belongs to A. An incomplete (B, F)-atlas can always be completed. A complete topological (analytical) (B, F)-atlas is called a topological (an analytic) fibre structure. Since all the mappings $h_{\nu_2} \cdot h_{\nu_1}^{-1}$ are fibre preserving, R is "fibred" through each fibre structure in a natural way.

DEFINITION 3. A complex topological (analytic) fibre bundle is a Hausdorff space R with a topological (analytic) fibre structure.

Each complex analytic fibre bundle is naturally also a complex space. Its complex structure is given by the maps of its fibre structure. We shall refer to the following terminology:

F is called the fibre of R, B the base space of R, L the structure group of R. The natural decomposition mapping $\pi: R \longrightarrow B = R/F$ is called the fibre projection.

3. Now let us consider two fibre bundles to be equivalent when they possess the same structure.

DEFINITION 4. Let R_1, R_2 be topological (analytic) fibre bundles with the same base space B, the same fibre F and the same structure group L. Let $\{U_\nu, h_\nu, W_\nu \times F)$, $\nu \in I\}$ and $\{(U'_\nu, h'_\nu, W'_\nu \times F)$, $\nu \in I'\}$ be the topological (analytic) fibre structures of R_1 and R_2 respectively,

π_1 and π_2 the corresponding fibre projections. Then R_1 and R_2 are called topologically (analytically) equivalent, if there exists a topological mapping α of R_1 onto R_2 such that:

1) α is fibre preserving: $\pi_1 = \pi_2 \cdot \alpha$.

2) $\{(\alpha^{-1}(U'_\nu, h'_\nu \cdot \alpha, W'_\nu \times F), \nu \in I'\} = \{(U_\nu, h_\nu, W_\nu \times F), \nu \in I\}$

and consequently:

$$\{(U'_\nu, h'_\nu, W'_\nu \times F), \nu \in I'\} = \{(\alpha(U_\nu), h_\nu \cdot \alpha^{-1}, W_\nu \times F), \nu \in I\}.$$

It may be noted here that if R_1 and R_2 are analytic fibre bundles analytically equivalent to each other by means of α, then α is an invertible holomorphic mapping of the complex space R_1 onto the complex space R_2. Each analytic fibre bundle becomes a topological fibre bundle, if one completes the analytic fibre structure of R co a topological fibre structure. Two arbitrary complex fibre bundles are called topologically equivalent if, conceived as topological fibre bundles, they are topologically equivalent.

The Cartesian product $B \times F$ carries the atlas $A = \{(B\times F, i, B\times F)\}$, which consists only of one map (i designates the identity mapping $B \times F \longrightarrow B \times F$). When A is completed, $B \times F$ becomes an analytic fibre bundle. Complex fibre bundles which are topologically (analytically) equivalent to $B \times F$ are called topologically (analytically) trivial.

2. THE ADJOINT STRUCTURE SHEAF. MAIN RESULTS.

1. Let L be an arbitrary complex Lie group. Let B be a complex space and $\underline{L}^c$ (or $\underline{L}^a$) over B be the sheaf of germs of continuous (or holomorphic) mappings into L. $\underline{L}^c$ and $\underline{L}^a$ are sheaves of groups that are Abelian groups if and only if L is an Abelian Lie group. It is clear that $\underline{L}^a$ is a subsheaf of $\underline{L}^c$.

Although in general a theory of cohomology groups with coefficients in $\underline{L}^c$ or in $\underline{L}^a$ is impossible, the first sets of cohomology of B with coefficients in $\underline{L}^c$ and $\underline{L}^a$ are defined (see [14]). Let these be denoted by $H^1(B, \underline{L}^c)$ and $H^1(B, \underline{L}^a)$. $H^1(B, \underline{L}^c)$ and $H^1(B, \underline{L}^a)$ do not carry group structure. But there exists a well determined neutral element $e \in H^1(B, \underline{L}^c)$ and $e \in H^1(B, \underline{L}^a)$.

$H^1(B, \underline{L}^c)$ and $H^1(B, \underline{L}^a)$ are defined as inductive limits of the cohomology sets $H^1(W, \underline{L}^c)$ and $H^1(W, \underline{L}^a)$ which are corresponded to each open covering W of B. There exists a canonical monomorphism[3)] t_c: $H^1(W, \underline{L}^c) \longrightarrow H^1(B, \underline{L}^c)$ and t_a: $H^1(W, \underline{L}^a) \longrightarrow H^1(B, \underline{L}^a)$.

We will go through the construction of the sets $H^1(W, \underline{L})$ (with $\underline{L} = \underline{L}^c$ or $= \underline{L}^a$). Let $W = \{W_\nu, \nu \in I\}$, $W_{\nu_1\nu_2} = W_{\nu_1} \cap W_{\nu_2}$, and let $s_{\nu_1\nu_2}$ be a cross-section over $W_{\nu_1\nu_2}$ in $\underline{L}$. We denote by s a collection $\{s_{\nu_1\nu_2}, \nu_1 \in I, \nu_2 \in I\}$ of cross-sections and by $C^1(W, \underline{L})$ the set of such collections s. Furthermore we put $e_o = \{s_{\nu_1\nu_2} \equiv 1 \in \underline{L}, \nu_1, \nu_2 \in I\}$. s is called a cocycle if the equation $s_{\nu_1\nu_2} \cdot s_{\nu_2\nu_3} = s_{\nu_1\nu_3}$ is valid for all ν_1, ν_2, ν_3 in $W_{\nu_1\nu_2\nu_3} = W_{\nu_1} \cap W_{\nu_2} \cap W_{\nu_3}$ ($\cdot$ designates the group operation of $\underline{L}$).

We shall now introduce an equivalence relation into the set $Z(W, \underline{L})$ of the cocycles: $s_1 = \{s^{(1)}_{\nu_1\nu_2}\} \in Z(W, \underline{L})$ and $s_2 = \{s^{(2)}_{\nu_1\nu_2}\} \in Z(W, \underline{L})$ are called equivalent if there is a collection of cross-sections s_ν over W_ν such that in $W_{\nu_1\nu_2}$ we have $s_{\nu_1} \cdot s^{(2)}_{\nu_1\nu_2} \cdot s^{-1}_{\nu_2} = s^{(1)}_{\nu_1\nu_2}$ always. $H^1(W, \underline{L})$ is the set of equivalence classes of

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3) A monomorphism (homomorphism, isomorphism, etc.) is always a one-to-one (a univalent, etc.) mapping which is compatible with the algebraic structures considered. In the case considered t and t_a map the neutral element onto the neutral element.

$Z(W, \underline{L})$, $e \in H^1(W, \underline{L})$ the equivalence class generated by $e_o \in Z(W, \underline{L})$.

2. We now assume that L operates effectively holomorphically on a complex space F. Let R be a topological (an analytic) fibre bundle with L for structure group, F for fibre and B for base space. Since two maps of the fibre structure $\{(U_\nu, h_\nu, W_\nu), \nu \in I\}$ of R are always continuously (analytically) compatible, the mappings $h_{\nu_1} \cdot h_{\nu_2}^{-1}: W_{\nu_1\nu_2} \times F \longrightarrow W_{\nu_1\nu_2} \times F$ are given by expressions $(x, y) \longrightarrow (x, \phi_{\nu_1\nu_2}(x) \otimes y)$ where the $\phi_{\nu_1\nu_2}(x)$ are continuous (holomorphic) mappings $W_{\nu_1\nu_2} \longrightarrow L$. Now there exists the relation $\phi_{\nu_1\nu_2}(x) \cdot \phi_{\nu_2\nu_3}(x) = \phi_{\nu_1\nu_3}(x)$ in all of $W_{\nu_1\nu_2\nu_3}$. If one regards $\phi_{\nu_1\nu_2}(x)$ as a cross-section in $\underline{L}^c$ (in $\underline{L}^a$) it follows that $\{\phi_{\nu_1\nu_2}(x), \nu_1, \nu_2 \in I\}$ is an element of $Z(W, \underline{L}^c)$ (of $Z(W, \underline{L}^a)$) and it generates an element of $H^1(W, \underline{L}^c)$ (of $H^1(W, \underline{L}^a)$). So each topological (analytic) fibre bundle R is put into correspondence with an element $\xi_c(R) \in H^1(W, \underline{L}^c)$ or $\xi_a(R) \in H^1(W, \underline{L}^a)$. If $R = B \times F$ it follows that $\xi_a(R) = e$. According to [14] the following elementary theorem holds:

THEOREM 1. Let K^c (or K^a) be the set of topological (or analytic) fibre bundles with B for base space, F for fibre and L for structure group. Then the system of topological (or analytic) equivalence classes of K^c (or K^a) is mapped monomorphically on $H^1(B, \underline{L}^c)$ [or $H^1(B, \underline{L}^a)$] by the mapping $j_c : R \longrightarrow t(\xi_c(R))$ [or $j_a : R \longrightarrow t_a(\xi_a(R))$]. j_c and j_a are independent of W.

3. The injection $i: \underline{L}^a \longrightarrow \underline{L}^c$ generates a homomorphism $H^1(W, \underline{L}^a) \longrightarrow H^1(W, \underline{L}^c)$ and thereby a homomorphism i^* of the inductive limits: $H^1(B, L^a) \longrightarrow H^1(B, L^c)$. It is plausible that ξ and $i^*(\xi)$ in the sense of theorem 1 always correspond to topologically equivalent fibre bundles. In [12] the following theorem was proved:

THEOREM 2. If B is a holomorphically complete space, then i^* is an isomorphism of $H^1(B, \underline{L}^a)$ onto $H^1(B, \underline{L}^c)$.

In this the concept of holomorphically complete space has to be defined in the following way:

DEFINITION 5. A complex space is called holomorphically complete, if it possesses the following two additional properties:

1) It is holomorphically convex: If M is a compact subset of B, then the set $\hat{M} = \{x \in B: |f(x)| \leq \sup|f(M)|$ for all $f \in H\}$ is compact as well (H = set of the holomorphic functions in B).

2) It is K-complete: To each point $x \in B$ there exists a neighborhood $U(x)$ and a holomorphic mapping q of B into a complex number space C^k such that q is nowhere degenerate in $U(x)$.

Nowhere degenerate means that the set $q^{-1}(z) \cap U(x)$ for each $z \in C^k$ is discrete in U. Therefore K-completeness of B means nothing other than that there exist sufficiently many holomorphic functions in B to ensure a meaningful function theory.

4. Following §1.3 we can regard an analytic fibre bundle of K^a to be a topological fibre bundle of K^c as well. The hereby produced mapping $k: K^a \longrightarrow K^c$ generates a homomorphism k^* of the system S^a of analytic equivalence classes of K^a into the system S^c of topological equivalence classes of K^c. From theorem 1

and theorem 2 it follows by commutativity of the diagram:

$$\begin{array}{ccc} S^a & \xrightarrow{k^*} & S^c \\ {\scriptstyle j_a}\downarrow & & \downarrow{\scriptstyle j_c} \\ H^1(B, L^a) & \xrightarrow{i^*} & H^1(B, L^c) \end{array}$$

that:

THEOREM 3. If B is a holomorphically complete space, then k^* is an isomorphism of S^a onto S^c.

This means in other words:

THEOREM I. Let B be a holomorphically complete space. Then two analytic topologically equivalent fibre bundles with the same fibre and structure group over B are always analytically equivalent.

THEOREM II. If B is a holomorphically complete space then there exists a topologically equivalent analytic fibre bundle with the same fibre and structure group for each topological complex fibre bundle over B.

5. The author will show in a future paper that Cartan's and Serre's concept of the analytic sheaf of Abelian groups can be generalized such that the sheaves $\underline{L}^a$ belong to it as well. This new concept may be called "general analytic sheaf". It can be shown that each general analytic sheaf S^a is a subsheaf of a sheaf S^c which possesses properties similar to $\underline{L}^c$. In order to obtain interesting theorems, a coherence condition like that for the Cartan sheaves has to be satisfied. This being the case, an analogue of theorem 2 holds:

THEOREM 2a. If B is a holomorphically complete space, the injection $i: S^a \longrightarrow S^c$ generates an isomorphism of $H^1(B, S^a)$ onto $H^1(B, S^c)$.

3. CONSEQUENCES

By theorems I and II in §2.4 the theory of topological fibre bundles as developed in Steenrod's book [19] becomes applicable to analytic fibre bundles. There follow a number of consequences, some of which are listed below:

1. Let L be a complex Lie group, L_1 a closed subgroup of L

which is a complex Lie group. Let L/L_1 designate the set of right cosets of L_1 in L. Since each connected component of L/L_1 is a manifold with a countable topology it follows: L/L_1 is solid if and only if L/L_1 is retractable into itself (see [19]). Herein a topological space S is called retractable into itself if there exists a continuous mapping $\phi(x,t)$: $S \times \{t, 0 \leq t \leq 1\} \longrightarrow S$ for which $\phi(x,1) = x$, $\phi(x,0) = x_o \varepsilon S$.

Let R be an (analytic) topological fibre bundle with L for structure group. One then says that it is possible to reduce the structure group L of R continuously (analytically) to L_1 if there exists for R a topological (analytic) fibre bundle R_1 with structure group L_1 which - regarded as fibre bundle with structure group L - is topologically (analytically) equivalent to R. Since according to [19], p. 56, this reduction is always possible for complex fibre bundles when L/L_1 is solid, there follows from theorems I and II:

THEOREM 4. Let B be a holomorphically complete space, R a complex analytic fibre bundle over B which has L for structure group. If L/L_1 is retractable into itself, the structure group L of R can be analytically reduced to L_1.

This theorem can also be successfully applied in mapping theory (see: [15], chapter III, theorem 2).

2. Let R be an analytic principal bundle over B with structure group L. The fibre of R is equal to L, the structure group L operates on the fibre L as left translation: $\ell \longrightarrow \ell_o \cdot \ell$. Because of this a product $y \cdot \ell_o$ of the points $y \varepsilon R$ by $\ell_o \varepsilon L$ is defined: If in a map (U, h) the point y has the coordinates $h(y) = (x, \ell)$ then $y \cdot \ell_o$ has the coordinates $(x, \ell \cdot \ell_o)$ in the same map. If two points $y_1, y_2 \varepsilon R$ are called equivalent when $y_1 = y_2 \cdot \ell$ for one ℓ of the subgroup L_1, then the equivalence relation thereby defined decomposes the space R analytically[4]. The quotient space $Q = R/L_1$ is an analytic fibre bundle over B with the complex manifold of the right cosets L/L_1 for fibres. If one puts $L_1^* = \{\ell \varepsilon L_1 : \ell_o^{-1} \cdot \ell \cdot \ell_o \varepsilon L_1 \text{ for all } \ell_o \varepsilon L\}$, L_1^* is a normal subgroup of L and the Lie group $L^* = L/L_1^*$ is the effectively operating structure group in L/L_1 of $Q = R/L_1$. Hence:

THEOREM 5. If L_1 is retractable into itself, every analytic principal bundle R with fibre L over a holo-

4) K. Stein has generally investigated such equivalence relations. See [20].

morphically complete space B is analytically trivial if and only if $Q = R/L_1$ has a continuous cross-section.

Proof. If R is trivial Q is also trivial and possesses a continuous cross-section. It only remains to be proved that our criterion suffices for the triviality of R.

As is commonly known L is analytic principal bundle over L/L_1. Therefore R can be regarded as an analytic principal bundle over $Q = R/L_1$ (with fibre L_1). Since L_1 is retractable, it follows from [19] that R is topologically trivial over Q. We can put $R = Q \times L_1$. Let s be a continuous cross-section of Q over B. Then $\hat{s} = s \times 1$, $1 \varepsilon L_1$, is a continuous cross-section of $R = Q \times L_1$. A principal bundle with a cross-section is, however, topologically trivial ([19], p. 36). Therefore it follows from theorem I that R is analytically trivial as well.

As is well known each analytic fibre bundle R over B possesses an adjoint principal bundle $\check{R}$ over B with the same structure group as R [5]. R is trivial if and only if $\check{R}$ is trivial.

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5) [19], 35 ff.

Theorem 5 therefore gives a criterion for the triviality of analytic fibre bundles. We consider a simple example:

Let the complex Lie group L operate effectively on a complex manifold F, let each automorphism $\ell \varepsilon L$ possess a fixed point $x_o \varepsilon F$ which is independent of ℓ. If then $z_1, \ldots, z_n$ are complex coordinates of the complex structure of F in a neighborhood $U(x_o)$ and if $x_o = (0, \ldots, 0)$, each automorphism $\ell \varepsilon L$ has a development in x_o:

$$z_\nu^* = \sum_{\mu=1}^{n} a_{\nu\mu}(\ell) \cdot z_\mu + \text{higher links}, \quad \nu = 1, \ldots, n.$$

If we let $A(\ell)$ denote the matrix $((a_{\nu\mu}(\ell)))$, then the determinant $|A(\ell)|$ is different from 0. Now let L_1 be the subgroup of those $\ell \varepsilon L$ for which $A(\ell) = E$, the unit matrix. In this case L_1 is even a normal subgroup of L: L/L_1 is canonically isomorphic to the linear group of the $A(\ell)$. If we assume further that L_1 is connected it can easily be proved that L_1 is retractable into itself, therefore it follows from theorem 5 that for the triviality of each analytic fibre bundle with F for fibre, L for structure group and a holomorphically complete space B for base space only the group $A(\ell)$ - giving the behavior of L in x_o - is of decisive importance.

3. We now consider the case where B is a holomorphically complete space which is continuously retractable into itself. Following a known theorem ([19], p. 53, §11.6) every fibre bundle with such a base space is topologically trivial. It follows from theorem I that:

THEOREM 6. Every analytic fibre bundle R with a retractable, holomorphically complete base space is analytically trivial.

A similar result can be obtained for the case that B is a non-compact (connected) Riemann surface. Here the following topological theorem, which can easily be proved by application of obstruction theory, holds[6]:

Every fibre bundle over B whose structure group is a connected Lie group is topologically trivial.

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6) See: [19], p. 148 ff. The asserted theorem follows, because the second group of cohomology with coefficients in an arbitrary Abelian group vanishes for every non-compact connected Riemann surface.

Since B is known to be a holomorphically complete space [1], one obtains from theorem I again:

THEOREM 7. Let R be an analytic fibre bundle over a non-compact connected Riemann surface B. Let the structure group of R be a connected complex Lie group. Then R is analytically trivial. [7)]

7) H. Röhrl was the first to prove this theorem. See [17].

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MULTIPLIERS ON COMPLEX HOMOGENEOUS SPACES

R. C. Gunning

1. Several problems which arise in the study of automorphic functions of one and several complex variables involve the so-called factors of automorphy associated to a group G of analytic automorphisms of a complex manifold M; these are C^{∞} complex-valued functions $f(x;z)$ on $G \times M$, holomorphic on M, and satisfying a functional equation of the form $f(x_1x_2; z) = f(x_1; z)f(x_2; x_1^{-1}z)$, where $x_1^{-1}z$ denotes the action of the transformation x_1^{-1} on the point z. A classification of such factors of automorphy for discrete groups G, the usual case in the theory of automorphic functions, has been given in [2]; but there is also some interest in classifying factors of automorphy for continuous groups. As an example, let M be the punctured complex affine space, consisting of n-tuples of complex numbers $(z_1, \ldots, z_n)$ not all of which are 0, and let G be the multiplicative group of non-zero complex numbers, where the action of an element x in G is defined by $x(z_1, \ldots, z_n) = (xz_1, \ldots, xz_n)$; the orbit space is then just the (n-1)-dimensional complex projective space. In this case it is easy to see that the most general factor of automorphy must be of the form $f(x; z) = x^m h(x^{-1}z)/h(z)$, where m is a positive or negative integer and $h(z)$ is a holomorphic and nowhere-vanishing function on M.

Now any meromorphic function $g(z)$ on M whose zeros and poles are invariant under the action of the group G (that is to say, whose zeros and poles can be defined on the orbit space) must satisfy an equation of the form $g(x^{-1}z) = f(x; z)g(z)$ for some factor of automorphy $f(x; z)$; but then $G(z) = g(z)/h(z)$ has the same zeros and poles, and since it satisfies the functional equation $G(x^{-1}z) = x^{-m}G(z)$ it is a homogeneous polynomial of degree m, which of course was to be expected as a consequence of Chow's theorem.

The problem to be considered here is that of formulating a classification theorem for factors of automorphy which embraces both the case of a discrete group and the case of a continuous group of transformations. Let M be a connected complex analytic manifold, and G be an arbitrary Lie group (connected or not, hence possibly discrete) of analytic automorphisms acting on M; we shall merely assume that the action of G, as well as the natural projection $M \longrightarrow M/G$, admit local cross-sections, although even this restriction can be weakened to include, for example, the case of discrete groups of transformations with fixed points. Let $\psi^{r,s}(M)$ denote the space of C^∞ complex-valued exterior differential forms of type (r,s) on the manifold M, and $\phi^{r,s}(M)$ denote the subspace consisting of $\overline{\partial}$-closed differential forms; in particular, $\psi^{0,0}(M)$ is the space of C^∞ complex-valued functions on M, and $\phi^{0,0}(M)$ is the subspace of holomorphic functions on M. The bundle of forms of type (r, s) over M induces a bundle over $G^p \times M$ under the natural map from the Cartesian product onto its second factor; the space of C^∞ cross-sections of this bundle will be denoted by $C^p(G, \psi^{r,s}(M))$, and will be called the group of p-cochains of G with coefficients in $\psi^{r,s}(M)$. Roughly speaking, this is just the group of C^∞ maps from G^p into the group $\psi^{r,s}(M)$. Introduce the homomorphisms $\delta: C^p(G, \psi^{r,s}(M)) \longrightarrow C^{p+1}(G, \psi^{r,s}(M))$ defined by: $\delta f(x_1, \ldots, x_{p+1}; z)$ $= f(x_2, \ldots, x_{p+1}; x_1^{-1} z) + \Sigma_{j=1}^{p} (-1)^j f(x_1, \ldots, x_j x_{j+1}, \ldots, x_{p+1}; z)$ $+(-1^{p+1} f(x_1, \ldots, x_p; z)$. These homomorphisms satisfy the condition $\delta\delta = 0$, so that the groups introduced above form a cochain complex under the maps δ; the cohomology groups of this cochain complex will be denoted by $H^p(G, \psi^{r,s}(M))$. For a discrete group G, this reduces to the customary non-homogeneous formulation of the cohomology groups of G with coefficients in $\psi^{r,s}(M)$, [1]. The corresponding group $H^p(G, \phi^{r,s}(M))$, in the particular case $p = 1$, $r = s = 0$, is just the additive analogue of the group of factors of automorphy modulo the subgroup consisting of the trivial factors of the form $h(x^{-1}z)/h(z)$ where $h(z)$ is holomorphic and nowhere vanishing on M; so that the classification problem, in a more general setting, can be expressed as the problem of calculating these cohomology groups.

2. First consider the case in which M is a Stein manifold and the isotropy subgroup of $\mathcal{G}$ at any point of M is a compact group; this includes, among others, the entire complex affine group acting on itself and the bounded, homogeneous, symmetric domains in several complex variables. As usual it is easier to treat the continuous case first, and then to use the results so obtained in handling the holomorphic or $\bar{\partial}$-closed case. It follows from the compactness of the isotropy subgroups that $H^p(\mathcal{G}, \psi^{r,s}(M)) = 0$ whenever $p > 0$; and it is clear from the definition that $H^0(\mathcal{G}, \psi^{r,s}(M))$ is simply the space of $\mathcal{G}$-invariant differential forms of type (r, s) on M. Then from the exact sequence of cochain groups $0 \longrightarrow C^p(\mathcal{G}, \phi^{r,s}(M)) \longrightarrow C^p(\mathcal{G}, \psi^{r,s}(M)) \xrightarrow{\bar{\partial}} C^p(\mathcal{G}, \phi^{r,s+1}(M)) \longrightarrow 0$ there follows in the usual manner the exact sequence of cohomology groups $0 \longrightarrow H^0(\mathcal{G}, \phi^{r,s}(M))$

$\longrightarrow H^0(G, \psi^{r,s}(M)) \longrightarrow H^0(G, \phi^{r,s+1}(M)) \longrightarrow H^1(G, \phi^{r,s}(M))$
$\longrightarrow H^1(G, \psi^{r,s}(M)) \longrightarrow \dots \longrightarrow H^p(G, \psi^{r,s}(M)) \longrightarrow H^p(G, \phi^{r,s+1}(M)$
$\longrightarrow H^{p+1}(G, \phi^{r,s}(M)) \longrightarrow H^{p+1}(G, \psi^{r,s}(M)) \longrightarrow \dots$; and therefore $H^p(G, \phi^{r,s+1}(M)) = H^{p+1}(G, \phi^{r,s}(M))$ whenever $p > 0$ while $H^1(G, \phi^{r,s}(M)) = H^0(G, \phi^{r,s+1}(M) / \bar{\partial} H^0(G, \psi^{r,s}(M))$ is just the Dolbeault group of G-invariant forms, or in other words the Dolbeault group of M/G. Therefore:

THEOREM 1. $H^p(G, \phi^{r,s}(M)) = H^{r,p+s}(M/G)$, the Dolbeault group of M/G, when M is Stein and G has compact isotropy subgroups.

For another case in which these cohomology groups are easy to calculate, suppose that G is a compact Lie group and that M is a homogeneous space of G. Then the Dolbeault groups $H^{r,s}(M)$ are isomorphic to the groups of harmonic differential forms of type (r, s) on M, and are therefore finite-dimensional complex vector spaces. Since G is compact, it follows readily that $H^p(G, H^{r,s}(M)) = 0$ for $p > 0$, and as in the first example considered, that $H^p(G, \psi^{r,s}(M)) = 0$ for $p > 0$ as well. There is then an exact sequence of cochain groups: $0 \longrightarrow C^p(G, \phi^{r,s}(M))$

$\longrightarrow C^p(G, \psi^{r,s}(M)) \xrightarrow{\bar\partial} C^p(G, \phi^{r,s+1}(M)) \longrightarrow C^p(G, H^{r,s+1}(M))$ $\longrightarrow 0$; then upon rewriting this exact sequence as a pair of exact sequences of length five, and passing to the associated exact cohomology sequences, it follows immediately that:

THEOREM 2. $H^p(G, \phi^{r,s}(M)) = 0$ for $p > 0$, when G is a compact Lie group which is transitive on M.

3. Now let $C^*(M)$ be the multiplicative group of non-vanishing C^∞ complex-valued functions on M, and let $A^*(M)$ be the subgroup consisting of the functions of $C^*(M)$ which are holomorphic. The problem originally posed, that of classifying the factors of automorphy for G, is then just that of calculating the group $H^1(G, A^*(M))$. To show how such a calculation can be carried out, let us consider again the first case treated in Section 2, in which M is Stein and the isotropy subgroups of G are compact. There is then an exact cochain sequence $0 \longrightarrow C^p(G, A^*(M))$ $\longrightarrow C^p(G, C^*(M)) \xrightarrow{\bar\partial^*} C^p(G, \phi^{0,1}(M)) \longrightarrow 0$, where $\bar\partial^* f(x_1, \ldots, x_p; z) = \bar\partial_z f(x_1, \ldots, x_p; z) / f(x_1, \ldots, x_p; z)$; and from this there follows the exact cohomology sequence $0 \longrightarrow H^0(G, A^*(M))$

$\longrightarrow H^0(G, C^*(M)) \longrightarrow H^0(G, \phi^{0,1}(M)) \longrightarrow H^1(G, A^*(M))$ $\longrightarrow H^1(G, C^*(M)) \longrightarrow H^1(G, \phi^{0,1}(M)) \longrightarrow \ldots$. This therefore determines the desired group in terms of the groups $H^p(G, C^*(M))$ and $H^p(G, \phi^{0,1}(M))$, the latter of which have already been calculated.

Suppose in particular that the maximal connected subgroup of G and the space M are both simply-connected. There is then an exact cochain sequence $0 \longrightarrow C^p(G, Z) \longrightarrow C^p(G, C(M))$ $\longrightarrow C^p(G, C^*(M)) \longrightarrow 0$, where Z denotes the additive group of the integers. Since $H^p(G, C(M)) = 0$ as noted already, it follows that $H^p(G, C^*(M)) = H^{p+1}(G, Z)$ for all $p > 0$. And substituting this into the exact sequence of the last paragraph, there follows the exact sequence: $0 \longrightarrow A^*(M/G) \longrightarrow C^*(M/G) \longrightarrow \phi^{0,1}(M/G)$ $\longrightarrow H^1(G, A^*(M)) \longrightarrow H^2(G, Z) \longrightarrow H^{0,2}(M/G) \longrightarrow \ldots$, which is precisely the form of the classification theorem determined in [2] in the case in which G is a discrete group. Other calculations can be carried out in a similar fashion.

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AN ELEMENTARY METHOD FOR THE LOCAL STUDY OF AN ANALYTIC SET

Michel Hervé

INTRODUCTION

We consider <u>complex</u> analytic sets in the space C^n: a subset M of an open set Ω contained in C^n is said to be an <u>analytic set</u> in Ω if each point of Ω possesses a neighborhood U such that $M \cap U$ coincides with the set of common zeros of a finite number of functions holomorphic in U; from this definition it follows at once that M is closed in Ω. The set M is said to be <u>reducible</u> at a point $x \in M$ if x possesses a neighborhood U such that $M \cap U$ is the union of two analytic sets in U, each of which contains x and does not coincide with $M \cap U$ in any neighborhood of x.

As to the <u>local study</u> of an analytic set M, two main purposes may be assigned to it:

<u>First purpose</u>. To define the set M, in the neighborhood of a given point of M, by formulae as explicit as possible in terms of independent parameters.

<u>Second purpose</u>. To define the <u>dimension</u> of M at a point $x \in M$ and see how that dimension is altered:

1) for a given set M, by moving the point x along M;

2) for a given point x, by considering, instead of M, another analytic set included in M or including M;

3) by an analytic transformation with a Jacobian $\neq 0$.

The first of these purposes is fulfilled by two classical theorems:

THEOREM 1. A point x of the set M possesses a neighborhood U such that $M \cap U$ is the union of a finite number of analytic sets in U, each of which goes through x and is irreducible at the point x; these sets M_i are uniquely determined under the additional assumption that, however two different indices i, j are chosen, the inclusion $M_i \subset M_j$ does not hold in any neighborhood of x.

These uniquely determined sets are called the irreducible components of M at the point x. Thanks to theorem 1, theorem 2 may consider only the case when M is irreducible at the point x, which may be assumed to be the origin.

A few notations. The n variables (or coordinates) will be denoted by $x_1, \ldots, x_n$; the functions of these variables which are holomorphic in some neighborhood of the origin form a ring denoted by H_n; those functions which depend only on the first variables form a subring denoted by H_p. Given a ring A and a variable z, the polynomials with z as a variable and coefficients in A form the ring A[z]. A polynomial $\varepsilon H_p[z]$ is said to be <u>distinguished</u> if the coefficient of the term of highest degree is 1 and all other coefficients are non-units of the ring H_p, i.e. are functions of H_p which vanish at the origin.

THEOREM 2. Given an analytic set M which contains the origin, is irreducible there, and neither has the origin as an isolated point nor fills up a neighborhood of the origin, then, after a suitable linear change of coordinates, there exist:

an integer m ($1 \leq m \leq n-1$), which turns out to be the dimension of M at the origin, when that is defined;

a distinguished polynomial $R \varepsilon H_m[x_n]$, which is irreducible in the ring $H_m[x_n]$;

for each integer i from m+1 to n-1, another polynomial $S_i \varepsilon H_m[x_n]$, which has a degree smaller than that of R and coefficients all vanishing at the origin;

thus M coincides in a suitable neighborhood of the origin, with the closure of the set defined by the following conditions:

$$\begin{cases} R(x_1, \ldots, x_m, x_n) = 0 \qquad \dfrac{\partial R}{\partial x_n}(x_1, \ldots, x_m, x_n) \neq 0 \\ x_i \dfrac{\partial R}{\partial x_n}(x_1, \ldots, x_m, x_n) - S_i(x_1, \ldots, x_m, x_n) = 0 \quad \text{for} \quad i = m+1, \ldots, n-1. \end{cases}$$

The proof of theorems 1 and 2 which is now to be found in classical books (see for instance Bochner and Martin, Several Complex Variables, 1948) is due to W. Rückert (Mathematische Annalen, vol. 107, 1933), who described it as formal, that is to say, involving no function theory, but only algebraic theories: those of ideals and fields. Rückert's method, because of its formal character, is clear and relatively simple, but also ill-adapted to the second purpose stated above and to many problems requiring a close local scrutiny of an analytic set. As a consequence of this, Remmert and Stein (Mathematische Annalen, vol. 126, 1953) had to set up a new method of investigation when they generalized Thullen's theorem on the continuation of an analytic set. The method sketched in this paper may prove useful for other problems; a more detailed account will appear elsewhere.

A SKETCH OF THE METHOD

The starting stage. Let M be an analytic set which contains the origin, is irreducible there, and neither has the origin as an isolated point, nor fills up a neighborhood of the origin: the functions of the ring H_p which vanish identically on M (in some neighborhood of the origin) form an ideal I_p which is prime, since M is irreducible at the origin. After a suitable linear change of coordinates, the ideal I_n is regular (see for instance Rückert): that means, there exists an integer m $(1 \leq m \leq n-1)$ such that I_m is reduced to 0 and, for each integer p from $m+1$ to n, I_p contains a distinguished polynomial $Q_p \in H_{p-1}[x_p]$.

For a given p $(m+1 \leq p \leq n)$, the polynomials of $H_{p-1}[x_p]$ which belong to I_p form an ideal i_p in the ring $H_{p-1}[x_p]$; i_p is prime and has a finite basis, which in general cannot be reduced to one element. So we consider the rings of integrity $A_p = H_p/I_p$ and their quotient fields K_p: the ideal i_p in the ring $H_{p-1}[x_p]$ generates, first an ideal i_p^* in the ring $A_{p-1}[x_p]$, which also is prime and has a finite basis, then an ideal in the ring $K_{p-1}[x_p]$, which admits as a basis a single element $P_p^* \in i_p^*$; since the constant 1 does not belong to any I_p, the distinguished polynomial Q_p generates a polynomial $\neq 0$ in $K_{p-1}[x_p]$, and therefore $P_p^* \neq 0$. Since a polynomial $\varepsilon\ i_p^*$ which is of degree 0 must be $\equiv 0$, the degree of P_p^* is at least 1. Now A_{p-1} contains an element $\phi_{p-1}^* \neq 0$ such that P_p^* divides, in the ring $A_{p-1}[x_p]$, the product of ϕ_{p-1}^* by any polynomial $\varepsilon\ i_p^*$; ϕ_{p-1}^* is generated by a function ϕ_{p-1} belonging to H_{p-1}, but not to I_{p-1}, and P_p^* by a polynomial $P_p \varepsilon\ i_p$, which may be chosen <u>of the same degree</u> as P_p^*, so that the coefficient of the term of highest degree in P_p does not belong to I_{p-1}. The product of ϕ_{p-1} by any polynomial $\varepsilon\ i_p$ can be written as $P_p U+V$, where $U \varepsilon\ H_{p-1}[x_p]$ and $V \varepsilon\ I_{p-1}[x_p]$. In the particular case $p = m+1$, that result means only that P_{m+1} divides any polynomial $\varepsilon\ i_{m+1}$ in the ring $K_m[x_{m+1}]$; but P_{m+1} may be assumed to be primitive and then divides any polynomial $\varepsilon\ i_{m+1}$ in the ring $H_m[x_{m+1}]$. So we can state:

LEMMA 1. The ideal I_{m+1} admits as a basis the single element $P_{m+1} \varepsilon\ H_m[x_{m+1}]$, of degree ≥ 1. For each integer p from m+2 to n, there exist a polynomial $P_p \varepsilon\ i_p$, $i_p = I_p \cap H_{p-1}[x_p]$, and a function $\phi_{p-1} \varepsilon\ H_{p-1}$, so that, if P_p and every function of I_{p-1}, but not ϕ_{p-1}, vanish at a common point, then every function of I_p vanishes at that point; P_p has degree ≥ 1; the coefficient

a_{p-1} of the term of highest degree in P_p does not belong to I_{p-1}, nor does ϕ_{p-1}.

Now consider a polynomial $T \in H_{p-1}[x_p]$, $T \notin i_p$: T generates $T^* \not\equiv 0$ in the ring $A_{p-1}[x_p]$; in that ring can be found 2 polynomials U^*, V^*, which are not both $\equiv 0$, V^* being of degree less than that of P_p^*, such that $P_p^* U^* + T^* V^*$ is of degree 0. Suppose that $P_p^* U^* + T^* V^* \equiv 0$: then, as i_p^* is prime and does not contain T^*, $V^* \in i_p^*$, P_p^* divides V^* in the ring $K_{p-1}[x_p]$, which is impossible. So:

LEMMA 2. If a polynomial $T \in H_{p-1}[x_p]$ does not belong to i_p (for instance, after the above underlined property of P_p, if T has a degree smaller than that of P_p and $T \notin I_{p-1}[x_p]$), then there exist two polynomials $U, V \in H_{p-1}[x_p]$ and a function δ belonging to H_{p-1}, but not to I_{p-1}, such that $(P_p U + TV - \delta) \in I_{p-1}[x_p]$.

<u>The construction of a dense subset of M.</u> Every point of M is a common zero of the $n-m$ polynomials $P_{m+1}, \ldots, P_n$; conversely, after lemma 1, any common zero of $P_{m+1}, \ldots, P_n$ where <u>none</u> of the $n-m-1$ functions $\phi_{m+1}, \ldots, \phi_{n-1}$ vanishes, is a point of M.

In order to gain more information on M, we apply lemma 2 to the polynomial $T = \frac{\partial P_p}{\partial x_p}$: we get 2 polynomials U_p, $V_p \in H_{p-1}[x_p]$ and a function δ_{p-1} belonging to H_{p-1}, but not to I_{p-1}, such that $(P_p U_p + \frac{\partial P_p}{\partial x_p} V_p - \delta_{p-1}) \in I_{p-1}[x_p]$; since I_{p-1} is prime, the product $a_{p-1}\delta_{p-1}\phi_{p-1} \notin I_{p-1}$. Using the distinguished polynomial $Q_{p-1} \in I_{p-1}$ and a classical lemma due to Späth (Crelle's Journal, vol. 161, 1929), we get a polynomial T_{p-1} of $H_{p-2}[x_{p-1}]$ such that $(a_{p-1}\delta_{p-1}\phi_{p-1} - T_{p-1}) \in I_{p-1}$; thus $T_{p-1} \notin i_{p-1}$ and lemma 2 may be applied with p-1 instead of p and T_{p-1} instead of T: we get 2 polynomials U, V $\in H_{p-2}[x_{p-1}]$ and a function ρ_{p-2} belonging to H_{p-2}, but not to I_{p-2}, such that $(P_{p-1}U + T_{p-1}V - \rho_{p-2}) \in I_{p-2}[x_{p-1}]$; therefore, if every function of I_{p-1}, but not ρ_{p-2}, vanishes at a fixed point, $a_{p-1}\delta_{p-1}\phi_{p-1}$ does not vanish at that point.

Since $\rho_{p-2} \notin I_{p-2}$ and I_{p-2} is prime, $a_{p-2}\delta_{p-2}\phi_{p-2}\rho_{p-2} \notin I_{p-2}$, and the same argument gives a function ρ_{p-3} belonging to H_{p-3}, but not to I_{p-3}, such that, if every function of I_{p-2}, but not ρ_{p-3}, vanishes at a fixed point, $a_{p-2}\delta_{p-2}\phi_{p-2}\rho_{p-2}$ does not vanish at that point. The induction finally leads to a function $\rho_m \in H_m$, $\rho_m \not\equiv 0$, with the following property: for each integer p from m+1 to n, if every function of I_{p-1}, but not ρ_m, vanishes at a fixed point, $\alpha_{p-1}\delta_{p-1}\phi_{p-1}$ does not vanish at that point; for $p = m+1$, since I_m is reduced to 0 and ϕ_m does not exist, that property means only this: $\rho_m \neq 0$ implies $\alpha_m \delta_m \neq 0$.

As a first consequence of that, a common zero of P_{m+1}, ..., P_n where $\rho_m \neq 0$ belongs to M; as a second consequence, a dense subset $\underline{M}$ of M (which of course is not an analytic set) can be constructed as follows: if $x_1, \ldots, x_m$ are given numerical values such that $\rho_m \neq 0$, then P_{m+1}, $\frac{\partial P_{m+1}}{\partial x_{m+1}}$, U_{m+1}, V_{m+1} become polynomials in x_{m+1} with numerical coefficients; the coefficient of the term of highest degree in P_{m+1} is $\alpha_m \neq 0$ and $P_{m+1}U_{m+1} \frac{\partial P_{m+1}}{\partial x_{m+1}} V_{m+1} \equiv \delta_m \neq 0$, so that P_{m+1} and $\frac{\partial P_{m+1}}{\partial x_{m+1}}$ have no common root, finite or infinite; in other words, P_{m+1} has a number of distinct roots equal to its degree in $H_m[x_{m+1}]$; if we choose one of those roots as a numerical value for x_{m+1}, we get a point $(x_1, \ldots, x_m, x_{m+1})$ where every function of I_{m+1} vanishes (see lemma 1), but not ρ_m; so $\alpha_{m+1}\delta_{m+1}\phi_{m+1}$ does not vanish at that point; since $\alpha_{m+1} \neq 0$ and $\delta_{m+1} \neq 0$, P_{m+2} has a number of distinct roots equal to its degree in $H_{m+1}[x_{m+2}]$; if we choose one of those roots as a numerical value for x_{m+2}, we get a point

$(x_1, \ldots, x_m, x_{m+1}, x_{m+2})$ where P_{m+2} vanishes, every function of I_{m+1} too, but not ϕ_{m+1}, hence every function of I_{m+2} vanishes too (see lemma 1), and so on. The induction finally gives us a fixed number k (the product of the degrees of P_{m+1} in $H_m[x_{m+1}], \ldots, P_n$ in $H_{n-1}[x_n]$) of distinct points $X_1(x_1, \ldots, x_m), \ldots, X_k(x_1, \ldots, x_m)$, with the preassigned values $x_1, \ldots, x_m$ as their first m coordinates, where all functions of I_n vanish; thus those points belong to M; moreover, their coordinates x_{m+1} are also roots of the distinguished polynomial Q_{m+1}, their coordinates x_{m+2} are also roots of the distinguished polynomial Q_{m+2}, and so on. As the roots of a distinguished polynomial depend continuously on its coefficients, we can state:

LEMMA 3. As the point $(x_1, \ldots, x_m)$ varies in a suitable neighborhood of the origin and outside the set of zeros of the function ρ_m, the points of M which have $x_1, \ldots, x_m$ as their first m coordinates remain in a fixed number k; their last $n-m$ coordinates are locally holomorphic functions of $x_1, \ldots, x_m$ which remain bounded and tend to 0 as $x_1, \ldots, x_m$ tend to 0. Let $\underline{M}$ be the subset of M generated by those points $X_1, \ldots, X_k$; then $\underline{M}$ is dense in M.

Now we want to prove that $\underline{M}$ is dense in M. First we suppose that a function f_p belonging to H_p, but not to I_p, vanishes identically (in some neighborhood of the origin) on $\underline{M}$; the argument already used for the proof of lemma 3 gives us a function f_{p-1} belonging to H_{p-1}, but not to I_{p-1}, such that if every function of I_p, but not f_{p-1}, vanishes at a fixed point, f_p does not vanish at that point; consequently f_{p-1} vanishes identically (in some neighborhood of the origin) on $\underline{M}$. The induction finally leads to a function $f_m \varepsilon H_m$ such that $f_m \not\equiv 0$, but $f_m \equiv 0$ on $\underline{M}$, i.e. $f_m \equiv 0$ outside an analytic set in C^m, which is absurd. So:

LEMMA 4. Any function of H_n which vanishes everywhere (in some neighborhood of the origin) on $\underline{M}$ belongs to the ideal I_n and, therefore, vanishes everywhere (in some neighborhood of the origin) on M.

Consider any function $f \varepsilon H_n$ with $f(0) = 0$: any symmetric entire function of $f[X_1(x_1, \ldots, x_m)], \ldots, f[X_k(x_1, \ldots, x_m)]$ (see lemma 3) may be expressed, in terms of $x_1, \ldots, x_m$, as a function holomorphic and bounded outside the set of zeros of ρ_m, which moreover tends to 0 as $x_1, \ldots, x_m$ tend to 0, hence as a function of H_m, which moreover vanishes at the origin. By applying lemma 4 we thus get the decisive:

LEMMA 5. To any function $f \varepsilon H_n$ with $f(0) = 0$, there exists a distinguished polynomial R, with coefficients εH_m and a degree equal to the integer k in lemma 3, such that $R(x_1, \ldots, x_m, f) \varepsilon I_n$.

Now let $x^* = (x_1^*, \ldots, x_n^*)$ be a point belonging to M, but not to $\underline{M}$; $\rho_m(x_1^*, \ldots, x_m^*) = 0$, but $x_1', \ldots, x_m'$ may be chosen arbitrarily near to $x_1^*, \ldots, x_m^*$ so that $\rho_m(x_1', \ldots, x_m') \neq 0$; for $i = m+1, \ldots, n$, the i^{th} coordinates of the points $X_1(x_1', \ldots, x_m')$, $\ldots, X_k(x_1', \ldots, x_m')$ (see lemma 3) are the roots (see lemma 5) of a distinguished polynomial with degree k and coefficients depending continuously on $x_1', \ldots, x_m'$; so the points $X_1(x_1', \ldots, x_m'), \ldots,$ $X_k(x_1', \ldots, x_m')$ have at most k^{n-m} distinct limit points X_j^* as $x_1', \ldots, x_m'$ tend to $x_1^*, \ldots, x_m^*$. Suppose that $X_j^* \neq x^*$ for each j: then a linear form u, depending only on the last $n-m$ variables, may be found so that $u(X_j^*) \neq u(x^*)$ for each j; lemma 5 yields a distinguished polynomial R such that $R(x_1, \ldots, x_m, u) \varepsilon I_n$; $u(x^*)$ is a root of $R(x_1^*, \ldots, x_m^*, u)$, hence the limit, as $x_1', \ldots, x_m'$ tend to $x_1^*, \ldots, x_m^*$, of a root of $R(x_1', \ldots, x_m', u)$, hence a $u(X_j^*)$; that contradiction proves that x^* is an X_j^*, and therefore that $\underline{M}$ is a dense subset of M.

Theorem 2 (see Introduction) is now easily proved by a similar argument, together with the following addition: to any function $f \varepsilon H_n$ with $f(0) = 0$, there exists a polynomial $S \varepsilon H_m[x_n]$, which has a degree smaller than that of R and coefficients all vanishing at the origin, such that $(f \frac{\partial R}{\partial x_n} - S) \varepsilon I_n$.

SOME APPLICATIONS TO THE LOCAL DIMENSION THEORY OF AN ANALYTIC SET

A definition of the dimension. A point x of an analytic set M is said to be an ordinary point of M if there exists an analytic transformation F, with Jacobian $\neq 0$, which maps a neighborhood of the point x on M, onto a neighborhood of the point F(x) on some complex plane, the dimension of which defines the dimension of M at the point x, and also at any point sufficiently near to x. Then every point of the subset $\underline{M}$ constructed above is an ordinary point of M, with the dimension m; the fact that $\underline{M}$ is dense on M implies, first that the ordinary points of M are dense on M, and secondly, that m is the dimension of M at any ordinary point sufficiently near to the origin, be it in <u>M</u> or not. Since the dimension of M has a constant value at all ordinary points sufficiently near to a given non-ordinary point x, the dimension of M at the point x may be defined as that constant value, and that definition is, in particular, valid for the origin. Hence:

PROPOSITION 1. If an analytic set M is irreducible and has the dimension m at the origin, it also has the dimension m at every point of M in a suitable neighborhood U of the origin; in particular, any irreducible component of M at any point in $M \cap U$ has the dimension m at that point.

If, on the contrary, M is reducible at the origin, proposition 1 does not hold, and the dimension of M at the origin is not defined yet; it may be defined either as the maximum value of the dimension of M at neighboring ordinary points, or as the maximum dimension, at the origin, of the irreducible components of M at the origin. Now the dimension of M is defined at every point of M, and in such a way that:

PROPOSITION 2. The dimension of an analytic set is unaltered by an analytic transformation with non-vanishing Jacobian.

Let us now compare the dimensions, at the origin, of two analytic sets M, M' which are irreducible at the origin, with $M' \subset M$, but $M' \neq M$ in any neighborhood of the origin. The corresponding ideals I_p, I'_p (see the starting stage of the method) satisfy $I'_p \supset I_p$ for each p, but $I'_n \neq I_n$; so there is a function f_n belonging to I'_n, but not to I_n, and the argument already used for the proof of lemmas 3 and 4 gives a function f_{n-1} belonging to I'_{n-1}, but not to I_{n-1}. If m is the dimension of M at the origin, the induction finally leads to a function $f_m \in I'_m$, $f_m \not\equiv 0$; so I'_m is not reduced to 0 and the dimension of M' at the origin is smaller than m:

PROPOSITION 3. If M is irreducible and has the dimension m at the origin, any analytic subset of M either coincides with M in some neighborhood of the origin or has a dimension smaller than m at the origin; in particular, the non-ordinary points of M are included in an analytic subset of M which has a dimension smaller than m at any of its points.

The latter subset is the set of all points of M where $\rho_m = 0$. By putting together Propositions 1 and 3, we get the following important one:

PROPOSITION 4. Given an analytic set M which is irreducible at the origin, and a function $f \in H_n$ which does not vanish identically on M in any neighborhood of the origin, then the origin possesses a neighborhood U such that, for any point $x \in M \cap U$, f does not vanish identically on any irreducible component of M at the point x, in any neighborhood of x.

Now we consider again an analytic set M which is irreducible and has the dimension m at the origin, and a function f belonging to H_n, but not to I_n, with $f(0) = 0$; let M_f be the set of all points of M where $f = 0$, M', M'', ... the irreducible components of M_f at the origin (which, after proposition 3, all have dimensions $< m$), and I'_p, I''_p,... the corresponding ideals. The polynomial R of lemma 5, which satisfies the condition

$R(x_1, \ldots, x_m, f) \varepsilon I_n$, was formed in such a way that, provided the coefficient of the term of degree 0 in R vanishes at the point $(x_1, \ldots, x_m)$, M_f contains at least one point with preassigned first m coordinates $x_1, \ldots, x_m$; therefore, after a suitable linear change of coordinates, one at least of the ideals $I'_{m-1}, I''_{m-1}, \ldots$ is reduced to 0; that is to say, one at least of the components $M', M'', \ldots$ has the dimension m-1 at the origin. But, as the same result holds for the irreducible components of M_f at any point of M_f, M' must have the dimension m-1 at a point of M' which belongs to none of the other components $M'', M''', \ldots$. So:

PROPOSITION 5. Given an analytic set M which is irreducible and has the dimension m at the origin, and a function $f \varepsilon H_n$ which vanishes at the origin, but does not vanish identically on M in any neighborhood of the origin, then the points of M where $f = 0$ form an analytic set which has the dimension m-1 at every point sufficiently near the origin.

COROLLARY 1. Given an analytic set M which is irreducible and has the dimension m at the origin, and q functions $f_1, \ldots, f_q \varepsilon H_n$ with $f_1(0) = \ldots = f_q(0) = 0$, then the points of M where $f_1 = \ldots = f_q = 0$ form an analytic set which has a dimension $\leq m$ and $\geq m-q$ at every point sufficiently near the origin.

Corollary 1 is obtained through iteration of proposition 5, and proves the equivalence between the definition of the dimension given above, and that stated by Remmert and Stein at the beginning of their paper:

COROLLARY 2. The analytic set M has the dimension m at the origin if and only if n-m is the maximum dimension of a complex plane P going through the origin and such that the origin is an isolated point of $M \cap P$.

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IDEALS OF MEROMORPHIC FUNCTIONS OF SEVERAL VARIABLES

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1. INTRODUCTION

During the past several years, there has been much progress in the theory of sheaves (faisceaux), which introduced a unified aspect into the theory of functions of several variables (cf. Cartan [1], [2], Serre [6]). In the present note, I shall consider, in place of the holomorphic functions in the above theory, the sheaf of fractional ideals of meromorphic functions, and I would like to give a generalization of two fundamental theorems on coherent sheaves on a Stein manifold to the meromorphic case. The essential part has been already discussed in our former paper (Hitotumatu-Kôta [4]), but here I shall give some generalizations and simplifications of the proofs.

2. DEFINITIONS AND NOTATIONS

Let X be a complex analytic manifold. Two functions f and g defined in a neighborhood of a point x on X are called equivalent at x if f = g in a suitable neighborhood of x. An equivalence class f_x of a function f classified by this equivalence relation is called the jet or germ of f at x. If f is holomorphic at x, the jet f_x is called holomorphic at x.

We denote by O_x the collection of all holomorphic jets at a point x. This is nothing but the ring consisting of all power series around x with non-empty convergence region. As is well known, O_x is a noetherian ring of integrity, in which the theorem of unique factorization holds.

We denote by K_x the quotient-field of the ring O_x. Let B be a subset of K_x. If there exists a jet $\phi \varepsilon K_x$ not identically zero, such that

$$\phi \cdot B = \{\phi\psi \mid \psi \varepsilon B\} \subset O_x,$$

then ϕ is called an integralizator of B. A subset $A \neq \{o\}$ of K_x is called an ideal or more precisely an ideal with respect to O_x, if A is an O_x-module with integralizator. The set of all integralizators of a given ideal A forms with o an ideal which is called the inverse-ideal of the ideal A and denoted by A^{-1}.

The collection

$$O = \{(f,x) \mid f \varepsilon O_x, x \varepsilon X\}$$

has the structure of a <u>sheaf of rings</u> on X, and similarly

$$K = \{(\phi, x) \mid \phi \in K_x, x \in X\}$$

has the structure of a sheaf of fields on X. K is also a sheaf of O-modules on X.

A cross-section

$$\phi \in \Gamma(K; U) = K_U$$

of the sheaf K in an open set U in X is called a <u>meromorphic function</u> in U. The field $K_U = \Gamma(K; U)$ evidently contains the quotient field of the ring $O_U = \Gamma(O; U)$, and these two fields are not equal in general. As we have remarked (Hitotumatu and Kôta [4]), they coincide with each other when U is a Stein manifold.

Let S be a subsheaf of O-modules in K, and $U \subset X$. A cross-section $\phi \in K_U$ is called an integralizator of S in U, if the jet ϕ_x integralizes the fiber S_x in K_x at every point x in U. We say that S has a local integralizator at a point x, if there exists a neighborhood U of x in which an integralizator of S exists. When S has local integralizators at every point x on X, S is called a <u>sheaf of ideals</u> on X. Next, suppose that, in a neighborhood U of a point x, there exist a finite number of cross-sections $\phi_1, \ldots, \phi_k \in \Gamma(S;U)$ of the subsheaf S, such that they generate the fiber S_y as O_y-module at every point y in U. Then these functions $\phi_1, \ldots, \phi_k$ are called a local generator of S at the point x. When k = 1, i.e., when S has a local generator consisting of only one element, we say that S is locally principal at x. A sheaf of ideals S having a local generator at every point x on X is called a coherent sheaf on X.

For example, let A be a fixed ideal in K_X. We denote by $[A]_x$ the set of all jets of functions in A at the point x. Then we can define a subsheaf of K by

$$(1) \qquad F(A) = \{(f, x) \mid f \in [A]_x, \; x \in X\},$$

which is called a sheaf associated to the ideal A. It is easy to see that the sheaf of ideals F(A) is coherent.

3. TRANSPORTER-SHEAF AND INVERSE-SHEAF

Let S and T be two coherent subsheaves of ideals in K. Then we define the transporter-sheaf S : T as follows:

$$S : T = \{(\psi, x) \mid \psi \in (S : T)_x, \; x \in X\},$$

where

$$(2) \qquad (S : T)_x = \{\psi \in K_x \mid \psi \cdot T_x \subset S_x\}.$$

If S is the sheaf O of all holomorphic functions, we write

$$T^{-1} = O : T$$

and call it the inverse-sheaf of T. $(T^{-1})_x$ is nothing but the set of all integralizators of T_x.

THEOREM 1. The transporter-sheaf S : T is again coherent. When S is locally principal, so is S : T, and in particular the inverse-sheaf T^{-1} of a coherent sheaf T is always locally principal.

PROOF. First we remark that S : T is a sheaf. We have a neighborhood U of a point x analytically isomorphic to a polycylinder and local generators $\phi_1, \ldots, \phi_\ell$ of S and $\psi_1, \ldots, \psi_m$ of T in U respectively. If $f_x \in (S : T)_x$, the jets of the functions $f \cdot \psi_1, \ldots, f \cdot \psi_m$ at a point y belong to S_y for every point y sufficiently near to x. Since $\psi_1, \ldots, \psi_m$ generates T_y, we have $f_y \in (S : T)_y$ which implies that S : T is a sheaf.

Next we prove the coherence. We may assume that none of the generators ϕ_j and ψ_i are identically zero. We proceed by induction on m. If m = 1, this is evident, because the functions $\phi_1/\psi_1, \ldots, \phi_\ell/\psi_1$ are the generator of S : T in U. Next we assume that the assertion has been proved for m-1. If we denote by T* the sheaf on U generated by $\psi_1, \ldots, \psi_{m-1}$, we have local generators $\chi_1, \ldots, \chi_p$ of S : T* in a suitable neighborhood U_1. By Oka's theorem, there exists a finite number of generators

$$(a_1^{(\lambda)}, \ldots, a_p^{(\lambda)}, b_1^{(\lambda)}, \ldots, b_\ell^{(\lambda)}) \qquad (\lambda = 1, \ldots, q)$$

of the sheaf of relations R among the functions $\chi_1\psi_m, \ldots, \chi_p\psi_m, -\phi_1, \ldots, -\phi_\ell$ in a neighborhood U_2. Though these functions are meromorphic in U_2, we can apply Oka's theorem in the holomorphic case by multiplying by a common multiple of the denominators. We shall show that the functions $\omega_\lambda = \sum_{k=1}^{p} a_k^{(\lambda)} \chi_k$ are the local generator of S : T in U_2. It is evident that ω_λ belongs to S : T, because

$$\chi_k \cdot T^* \subset S \quad \text{and} \quad \sum_{k=1}^{p} a_k^{(\lambda)} \chi_k \psi_m = \sum_{j=1}^{\ell} b_j^{(\lambda)} \phi_j \in S.$$

On the other hand, if we have a transporter $f \in S : T$ in U_2, we have the expressions

$$f = \sum_{k=1}^{p} a_k \chi_k \quad \text{and} \quad f\psi_m = \sum_{k=1}^{p} a_k \chi_k \psi_m = \sum_{j=1}^{\ell} b_j \phi_j$$

where a_k and b_j are holomorphic in U_2. Since $(a_1, \ldots, a_p, b_1, \ldots, b_\ell)$ belongs to the sheaf of relations R, we have the expression $a_k = \sum_{\lambda=1}^{q} \alpha_\lambda a_k^{(\lambda)}$, which implies

$$f = \sum_\lambda \sum_k \alpha_\lambda a_k^{(\lambda)} \chi_k = \sum_\lambda \alpha_\lambda \omega_\lambda.$$

Hence the first part of our assertion is proved.

When S is locally principal (i. e., $\ell = 1$), we may assume that $p = 1$, and then we have $q = 1$, which implies that $S : T$ is locally principal. Of course the latter can be proved directly.

REMARK 1. When T is locally principal, this theorem was given by Oka [5], and the fact that $O \cap (S : T)$ is coherent is remarked in Serre [6]. However, I think our Theorem 1 is not trivial, since K is not coherent.

REMARK 2. If T is not coherent, the collection $S : T$ defined by (2) is not a sheaf in general. For example, take $X = C^2(z_1, z_2)$, $S = O$, and put $T_x = O_x$ for the point $x = (z_1, z_2)$ if $z_1 \neq o$ and T_x = [the ideal generated by $(z_1)^n$] if $z_1 = o$

and $n \leq |z_2| < n+1$. Then the inverse T^{-1} is not a sheaf of ideals in K.

4. COHERENT SHEAVES ON A STEIN MANIFOLD

Now we shall state and prove two fundamental theorems on coherent sheaves S on a Stein manifold X in the case of meromorphic functions. (For the definition of Stein manifolds, see Cartan [1], IX.)

THEOREM A. The set of cross-sections $\Gamma(S\,;\,X)$ generates the fiber S_x as O_x-module at every point x of X.

THEOREM B. When $q \geq 1$, the cohomology group $H^q(S\,;\,X) = o$.

To prove these, we shall show

THEOREM 2. Every coherent sheaf S of ideals on a Stein manifold X has an integralizator in the whole space X.

If this is proved, multiplying by the integralizator of the sheaf S, it is easy to reduce Theorems A and B to the known case of holomorphic functions (Cartan [1], XIX).

PROOF OF THEOREM 2. At every point x, we have a local generator $\psi^{(x)}$ of the inverse-sheaf S^{-1} in a neighborhood U (Theorem 1). By the theorem of unique factorization, we can write $\psi^{(x)} = p^{(x)}/q^{(x)}$, where $p^{(x)}$ and $q^{(x)}$ are holomorphic and coprime at every point in U (Siegel [7], p. 9). The systems $\{\psi^{(x)}\}$ and $\{p^{(x)}\}$ give multiplicative Cousin distributions in X (Siegel [7], p. 16). Denoting by P_x the ideal in O_x generated by $p^{(x)}$, the collection

$$P = \{(f, x) \mid f \in P_x, \ x \in X\}$$

is a locally principal subsheaf of O. Since Theorem A is already proved for coherent subsheaf <u>in O,</u> we have a non-zero cross-section $f \in \Gamma(P;X)$. At every point $x \in X$, the jet f_x is a multiple of $(p^{(x)})_x = (q^{(x)})_x \cdot (\psi^{(x)})_x$, where the jet $(\psi^{(x)})_x$ is a generator of the ideal $(S_x)^{-1}$. This means that the function f is an integralizator of the sheaf S in X.

Now, it is well-known that the fundamental Theorems A and B (in the holomorphic case) imply various interesting properties on Stein manifolds (see for example, Cartan [1], [2], Hitotumatu [3], Serre [6]). We are ready to generalize them to the case of meromorphic functions in a similar manner. However, we shall note only a few results.

Hereafter, we always assume that X is a Stein manifold.

THEOREM 3. Let A be an ideal in K_X. The sheaf associated to the inverse-sheaf A^{-1} is equal to the inverse-sheaf of the sheaf associated to A, i.e., if we define $F(A)$ by (1), we have

$$F(A^{-1}) = F(A)^{-1}.$$

PROOF. We first prove the equality

$$\Gamma(F(A)^{-1}; X) = A^{-1} \text{ in } K. \tag{3}$$

For simplicity, we denote by B the left hand side of (3). From relation (2), we have

$$(F(A)^{-1})_x = (F(A))_x^{-1} = ([A]_x)^{-1},$$

which obviously contains the ideal $[A^{-1}]_x = (F(A^{-1}))_x$. Hence we have $A^{-1} \subset B$. On the other hand, every element $\beta \in B$ integralizes A, because $\beta_x \in ([A]_x)^{-1}$ at every point $x \in X$. Hence

$B \subset A^{-1}$, and so the equality (3) is proved. Since the sheaf $F(A)^{-1}$ is coherent by Theorem 1, the ideal B generates the fiber $(F(A)^{-1})_x = ([A]_x)^{-1}$ at every point x, due to Theorem A. Therefore we have

$$(4) \qquad [A^{-1}]_x = [B]_x = ([A]_x)^{-1}, \text{ i.e., } (F(A^{-1}))_x = (F(A)^{-1})_x,$$

where the operation of the identification of the fibers in (4) is compatible with the structures of sheaves. Thus Theorem 3 is proved.

THEOREM 4. (Additive Cousin problem for ideals.) Let A be a given ideal in K_X and suppose that there exist an open covering $X = \bigcup_\nu U_\nu$ and functions ψ_ν satisfying:

1° ψ_ν is meromorphic in U_ν,

2° unless $U_\mu \cap U_\nu$ is empty, the jet $(\psi_\mu - \psi_\nu)_x$ belongs to the ideal $[A]_x$ at every point $x \in U_\mu \cap U_\nu$.

Then there exists a function ψ meromorphic in X such that the jet $(\psi - \psi_\nu)_x$ belongs to $[A]_x$ at every point x in U_ν.

Multiplying by an integralizator of the ideal A, this is reduced to the case when the ideal A is in O_X. However, this is not trivial, because the functions ψ_ν are not necessarily holomorphic even if A is in O_X. Theorem 4 was given by Hitotumatu and Kôta [4], but here I give a simpler proof based on the notion of sheaves.

PROOF. We take the coherent sheaf F(A) associated to the ideal A. If we use the terminology of sheaves, the assertion of Theorem 4 is equivalent to the fact that the canonical mapping

$$K_X = \Gamma(K\ ;\ X) \longrightarrow \Gamma(K/F(A)\ ;\ X) \tag{5}$$

is onto. Now, from an exact sequence

$$o \longrightarrow F(A) \longrightarrow K \longrightarrow K/F(A) \longrightarrow o$$

(o means the zero-sheaf),

we have the exact sequence

$$o \longrightarrow \Gamma(F(A)\ ;\ X) \longrightarrow K_X \longrightarrow \Gamma(K/F(A)\ ;\ X) \longrightarrow H^1(F(A)\ ;\ X) \longrightarrow \cdots, \tag{6}$$

where the fifth term of (6) vanishes by Theorem B. This means that (5) is onto.

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POSITIVE CURRENTS AND ANALYTIC SETS

P. Lelong

INTRODUCTION

In this paper we give the definition and the properties of a class of currents (in sense of G. de Rham) - the positive currents - and applications to the integration of forms on analytic sets. Conditions of sign arise in a natural way in the theory of analytic functions and plurisubharmonic functions and are often significant (cf. for example pseudo-convexity). The class of positive currents will be useful for such properties. Further, the class of positive and closed currents seems suitable for studying the metric properties of analytic sets. We give here an application to our existence theorem for the operator $t(\phi)$ of integration of a form ϕ on an analytic set, the main result being: this operator is a sum of closed and positive currents (see [9] and [10]).

1. DEFINITION. A current t is called a positive current of degree p (and complex dimension $n-p$) on a complex manifold D of complex dimension n if:

1) t is a homogeneous current of degree (p, p); $t(\phi) = 0$ for the homogeneous forms ϕ which are not of degree $(n-p, n-p)$ in dz_i, $d\bar{z}_j$.

2) For every system of linear forms $g_1, \dots, g_{n-p}$ with C^∞ coefficients $g_s = \Sigma_k a_s^k(z_i, \bar{z}_j) dz_k$, and with conjugates $\bar{g}_s = \Sigma \bar{a}_s^k(z_i, \bar{z}_j) dz_k$, the current of maximal degree

$$t \wedge (\tfrac{i}{2} g_1 \wedge \bar{g}_1) \wedge (\tfrac{i}{2} g_2 \wedge \bar{g}_2) \wedge \dots \wedge (\tfrac{i}{2} g_{n-p} \wedge \bar{g}_{n-p})$$

is a distribution (in the sense of L. Schwartz) which is positive; this distribution is a positive measure. (In this paper distributions and measures are operators on functions, or currents of maximal degree.)

We denote by (T_p^+) the class of the positive currents of degree p. A form ϕ is called <u>a positive form of degree</u> p, if $\phi \in (T_p^+)$ and ϕ has coefficients of class C^o; (F_p^+) is the class of positive forms of degree p.

<u>Examples</u>. 1) For $p = n$, (T_n^+) is the class of the positive measures. For $p = 0$, (F_o^+) is the class of positive functions.

2) An important class is obtained for $p = 1$. If we write the current

$$t = \frac{i}{2} \sum_{p,q} t_{p\bar{q}} \, dz_p \wedge d\bar{z}_q,$$

t is positive if and only if the distribution

$$(t, \vec{\lambda}) = T(\vec{\lambda}) = (\sum_{p,q} t_{p\overline{q}} \lambda_p \overline{\lambda}_q) \, d\tau_{2n}$$

is a positive measure for every complex vector $\vec{\lambda}$. A form

$$\phi = \frac{i}{2} \sum_{p,q} \phi_{p\overline{q}} \, dz_p \wedge d\overline{z}_q$$

is positive in D if and only if the function

$$F(\vec{\lambda}) = \sum_{p,q} \phi_{p\overline{q}}(z_i, \overline{z}_j) \lambda_p \overline{\lambda}_q$$

is positive (or zero) in each point of D and for every complex vector $\vec{\lambda}$.

It is convenient to consider exterior forms and currents instead of hermitian forms. We denote by d_z the differential with respect to the z_i.

This case has connections with the following results:

THEOREM 1. A function $V(z_i, \overline{z}_j)$, defined in a complex manifold D, is a plurisubharmonic function if and only if

a) V is real valued, $-\infty \leq V < +\infty$; V is locally summable in D.

b) The current $2i d_z d_{\bar{z}} V = 2i \sum \frac{\partial^2 V}{\partial z_p \partial \bar{z}_q} dz_p \wedge d\bar{z}_q$ is a positive current.

c) $V(P) = V_m(P)$ in each point $P \in D$; $V_m(P)$ is the maximum of V at the point P, when sets of measure zero are neglected.

These properties are characteristic.

THEOREM 2. If t is a given current in D, with properties:

a) t is positive of degree 1

b) t is closed

then, to each point $M \in D$ corresponds a neighborhood ω_M in D, and a function V_M plurisubharmonic in ω_M such that $t = 2i d_z d_{\bar{z}} V_M$. The positive and closed currents are therefore the currents which are locally associated with the plurisubharmonic functions.

To an analytic set W^1 defined by data of Cousin of zeros in D, there corresponds a positive and closed current of degree 1 which is defined in each neighborhood ω_M of $M \in D$ by $t = 2i d_z d_{\bar{z}} \log |f_M|$, $f_M(z_1, \ldots, z_n) = 0$ being the data of zeros in

ω_M. The current of integration on W^1 is then $\frac{1}{2\pi}$ t; this result was first given by Poincaré, in an elementary form and with an insignificant error in the coefficient; it was given in 1945 in [4], in connection with the study of plurisubharmonic functions, and in 1950 in [5], where the metric properties of the analytic sets $f = 0$ are investigated. It is given in the language of currents in the lectures of G. de Rham and Kodaira [2], (1950). It can be considered as a particular case (relative to the data of Cousin) of the general theorem that we have stated in the Introduction.

Further examples of positive currents and forms of degree p are obtained by the multiplication of the positive forms and currents.

2. We give now properties of the classes (T_p^+), (F_p^+). We denote by A^p a complex p-vector defined by means of the equations

(1) $$z_i = z_i^o + \Sigma_k a_i^k u_k, \qquad 1 \leq k \leq p$$

giving an analytic plane of dimension p; we suppose (1) to be a unitary representation and consider the fundamental form

$$d\tau_{2p} = g(A^p) = (\tfrac{i}{2})^p (-1)^{\frac{p(p-1)}{2}} du_1 \wedge d\bar{u}_1 \wedge \ldots du_p \wedge d\bar{u}_p$$

The adjoint form $^*g(A^p)$ is the fundamental form of the complex n-p vector B^{n-p} orthogonal to A^p. To a homogeneous current t of degree (p, p) we define a corresponding distribution

$$(2) \qquad (t, A^p) \longrightarrow T(A^p) = t \wedge {}^*g(A^p)$$

which is explicitly given by

$$(3) \qquad T(A^p) = k_p \sum_{(i),(j)} t_{(i)(\overline{j})} \alpha_{(i)} \overline{\alpha}_{(j)}, \quad k_p = \left(\frac{i}{2}\right)^p (-1)^{\frac{p(p-1)}{2}}$$

$$(i) = (i_1 < i_2 \dots < i_p); \quad (j) = (j_1 < j_2 \dots < j_p)$$

$\alpha_{(i)} = |a_i^k|^{k=1,\dots,p}_{i=i_1,\dots,i_p}$ are the parameters of A^p (of coordinates a_i^k).

We obtain the following property of positive currents:

THEOREM 3. A homogeneous (p, p) current t is positive if and only if the distribution $T(A^p)$ is a positive measure for every complex p-vector A^p.

Originally we introduced (cf. [7]) the positive currents by using this property as a definition.

The definition of the classes (T_p^+) does not make use of the connection between the variables z_i, z_j of D and the differentials

dz_i, $d\overline{z}_j$. It is therefore possible to consider positive currents as relative to an exterior algebra with basis $(g_i, \overline{g}_j)$. We obtain the same classes T_p^+ if the g_i are linear forms in the dz_i, with continuous coefficients in D. This fact yields the following result:

THEOREM 4. The image $t' = Ft$ of a positive current by a locally one-to-one analytic transformation $z' = Fz$, is a positive current.

We recall that the image $t'(\phi)$ is, by definition, $t'(\phi) = t[F^*\phi]$ where $F^*\phi$ is obtained from ϕ by substituting the z_i and dz_i for the z'_i and dz'_i.

From the equations (3), we obtain also:

THEOREM 5. A positive current is continuous of order zero: the distributions $T_{(i)(\overline{j})} = k_p^{-1} t_{(i)(\overline{j})} d\tau_{2n}$ are complex measures; $T_{(i)(\overline{j})}$ is the conjugate of $T_{(j)(\overline{i})}$; $T_{(i)(\overline{i})}$ are positive measures.

We consider now a system $|\Sigma| = \{A_1^p \ldots A_N^p\}$ of N complex p-vectors of coordinates $(a_{i,s}^k)$, $N = (C_n^p)^2$. Such a system (Σ) is called regular if the system of linear equations

(4) $$T(A_s^p) = k_p \sum_{(i),(j)} t_{(i)(\bar{j})} \alpha_{(i),s} \bar{\alpha}_{(j),s}$$

is a regular one. Using such a system (Σ), from the equations (4), the distributions $T_{(i)(\bar{j})}$ appear as sums of positive measures $T(A_s^p)$ with complex numerical coefficients. The determinant

$$\Delta = |\alpha_{(i),s} \bar{\alpha}_{(j),s}|^{(i)\times(j)}_{s=1,\ldots,N}$$

vanishes in the space $C^\nu = R^{2\nu}$ $(\nu = N\cdot p\cdot n)$ of the ν complex coordinates $(a_{j,s}^k = a'^k_{j,s} + ia''^k_{j,s})$ on an algebraic set given by an equation $P(a'^k_{j,s}, a''^k_{j,s}) = 0$, P being a polynomial with real coefficients. As a consequence, we obtain:

PROPOSITION 1. Let $(a_i^k)_o$ be the coordinates of a complex p-vector A_o^p, with a unitary representation; then if $\varepsilon > 0$ is given, there exists a regular system (Σ) of p-vectors A_s^p $(1 \leq s \leq N)$, of coordinates $a_{i,s}^k$ (in unitary representation), satisfying

$$|a_{i,s}^k - (a_i^k)_o| < \varepsilon$$

The norm of a positive current t in a domain D is defined by $|t|_D = \sup |t(\phi)|$ for the forms ϕ of the space $\mathcal{D}^o(D)$ of the forms (C^o) with compact support $K(\phi) \subset D$, and $|\phi| \leq 1$, where $|\phi|$ is the maximum of the modulus of the coefficients. We have

PROPOSITION 2. To every regular system (Σ), there corresponds a coefficient $C(\Sigma) > 0$, such that

$$|t|_D \leq C(\Sigma) \sup_s |T(A_s^p)|_D, \; A_s^p \in (\Sigma).$$

THEOREM 6 (Multiplication). If $t \in (T_p^+)$, and $\phi \in (F_1^+)$, then $t \wedge \phi \in (T_{p+1}^+)$. Conversely, if $t \in (T_1^+)$, and $\phi \in (F_p^+)$, $t \wedge \phi \in (T_{p+1}^+)$.

We have also the following theorem of division:

THEOREM 7 (Division). If $t \in (T_p^+)$ and $\phi \in (F_1^+)$, if $t \wedge \phi = 0$, if $\phi^q \neq 0$ and $\phi^{q+1} = 0$ in D, then we have $t = t_1 \wedge \phi^q$, with $t_1 \in (T_{p-q}^+)$ ($t_1 = 0$, $t = 0$, if $p < q$).

3. INTEGRATION ON AN ANALYTIC SET

We consider an analytic set A defined in D and of complex dimension p in each point $M \in A$. We denote by A^* the set of the ordinary points of A, by $A' = A - A^*$ the closed set of the non-ordinary points of A. The current $t_o(\phi) = \int_{A^*} \phi$ is defined for the forms ϕ of $\mathcal{D}^o(D - A')$. The integration $t(\phi) = \int_A \phi$ will be a continuation of the current $t_o(\phi)$, on the forms of $\mathcal{D}^o(D)$. The current $t_o(\phi)$ has the following properties:

THEOREM 8. a) t_o is positive: $t_o \in (T^+_{n-p})$,

b) t_o is closed,

c) to every domain G which is compactly contained in D $(G \subset\subset D)$, there corresponds a finite number $k(G)$ such that for every sphere $B \subset G$ of radius r, the following majorization holds:

(5) $$|t_o|_B \leq k(G) r^{2p}.$$

The norm in (5) is calculated on the ϕ of $\mathcal{D}^o(D - A')$. The property (5) is a significant step in the proof of the existence theorem and is based on the relations between the norm of a positive current and the norm of the measures $T(A^{n-p})$. A second step of the proof is a theorem on the continuation of a closed current, continuous of order zero, defined in D - E, [more precisely: on the forms of $\mathcal{D}^o(D - E)$] by a current of the same kind. For our purpose it is sufficient to consider the case: D is a domain in the space $R^m(x_1, \ldots, x_m)$

and E is a subspace R^s: $x_{s+1} = x_{s+2} = \dots = x_m = 0$. We denote by $\alpha_1(x_{s+1}, \dots, x_m)$ a kernel of class C^∞, $0 \leq \alpha_1 \leq 1$, depending only on the distance $\delta(x)$ of (x) from the set $E = R^s$ with $\alpha_1(x) = 1$ for $0 \leq \delta(x) \leq \frac{1}{2}$, $\alpha_1 = 0$ for $\delta \geq 1$. We consider the kernels

$$\alpha_r(x_j) = \alpha_1\left(\frac{x_j}{r}\right), \qquad s+1 \leq j \leq m.$$

Then we have:

$$|d\alpha_r| < Ar^{-1}$$

for the form $d\alpha_r$. A continuation of a current t_o continuous of order zero in $D - E$ by such a current in D exists if and only if $|t_o|_G$ is bounded for every domain $G \subset\subset D$. Then the continuation

$$t = \lim_{r=0} t_o(1 - \alpha_r)$$

exists; we call such a continuation t the simple extension of t; t is continuous of order zero. A characteristic property of the simple extension is: the norm is not increased by such a continuation.

THEOREM 9. The simple extension t of the closed current t_o is a closed current if and only if the current $\theta_r = t_o \wedge d\alpha_r$ tends to zero on every form ϕ of $\mathcal{D}^o(D)$.

A sufficient condition is therefore: $r^{-1}|t_o|_G^r \longrightarrow 0$, where $|t_o|_G^r$ is the norm of t_o on the open set $G \cap \{x | 0 < \delta(x) < r\}$.

The third step in the proof is to use theorem 9 by the following process. The set $M' \subset M$ of the non-ordinary points of an analytic set of homogeneous complex dimension p, is an analytic set of dimension $\leq p-1$. If $P \in M'$ is an ordinary point of M', there exists a one-to-one analytic mapping $z' = F(z)$ and a neighbourhood ω_P such that $F(\omega_P \cap M') = F(\omega_P) \cap C^s$, where C^s is a subspace of C^n $(s \leq p-1)$. Then in the space of the z_i', Theorem 9 gives: the simple extension of the image Ft_o of t_o is a closed and positive current. Using the inverse transformation, we get: the current t is defined by the simple extension of t_o in ω_P. The norm is unchanged by this continuation and the majorization given by Theorem 8 remains valid for this continuation. The process can be continued and applied to non-ordinary points of M, and so on, see [10]. We obtain:

THEOREM 10. 1) If M is an analytic set of homogeneous dimension p in D, the current $t_o(\phi) = \int_{M^*} \phi$ of integration on the ordinary points of M, possesses a simple extension $t(\phi)$; this continuation is by definition the operator:

$$t(\phi) = \int_M \phi.$$

2) $t(\phi)$ is a positive current of degree $n-p$; $t(\phi)$ is closed.

THEOREM 11. By a decomposition $M = \Sigma M_s$ of M in analytic sets irreducible in D, we have $t(\phi) = \Sigma_s t_s(\phi)$.

In the general case, for an analytic set M of maximal complex dimension p, we have:

THEOREM 12. The current $t(\phi) = \int_M \phi$ is the sum of p positive and closed currents $t^{(q)}$, $(1 \leq q \leq p)$, of degrees $n-1, \ldots, n-p$. Each $t^{(q)}$ is the simple extension of the integration $t_o^{(q)}$ on the set of ordinary points of complex dimension q in M.

4. POSITIVE AND CLOSED CURRENTS

We consider now the forms:

$$\beta_1 = \frac{i}{2} \Sigma\, dz_j \wedge d\bar{z}_j, \qquad \beta_q = \frac{1}{q!} \beta_1^q$$

$$\alpha = \frac{i}{2} d_z d_{\bar{z}} \log. \Sigma z_j \bar{z}_j$$

and the forms α^q, which are positive forms defined for $|z| \neq 0$. For $t \in (T^+_{n-p})$ in D, the distributions

(5) $$\sigma = t \wedge \beta_p$$

(6) $$\nu = \pi^{-p} t \wedge \alpha^p$$

are positive measures by theorem 6. Let t be closed. In a sphere $|z| < R$, contained in D, we have $t = d\theta$, because t is closed and homologous to zero. We denote by $\nu(r, R)$ the ν-measure of the domain $r < |z| < R$ and by $\sigma(R)$ the σ-measure of the domain $|z| < R$. Then, by applying Stokes' theorem, we find:

$$\nu(r,\ R) = p!\ \pi^{-p}\left[\frac{\sigma(R)}{R^{2p}} - \frac{\sigma(r)}{r^{2p}}\right] \geq 0.$$

THEOREM 13. If t is a positive and closed current of degree $n-p$, the measure $\sigma(R)$ has the following property:

$\sigma(R)R^{-2p}$ is an increasing function of R, and has a finite limit when R tends to zero; moreover, the positive measure ν is bounded on every compact set contained in D.

If t is the current of integration on an analytic set (M^p) of homogeneous dimension p, $d\sigma$ given by (5) is the area of (M^p); this yields:

THEOREM 14. The 2p-dimensional area of an analytic set (M^p) is the positive measure defined by (5); it is bounded on every domain compactly contained in D.

The fact that the current is closed gives the following more precise result:

THEOREM 15. Let M^p be an analytic set in D, of homogeneous dimension p; let $B(O, R)$ be the sphere of center $O \varepsilon M^p$ and radius R, contained in D. The area $\sigma(R)$ of $M^p \cap B(O, R)$ has the property: $\sigma(R)R^{-2p}$ is an increasing function of R; it has a limit when R tends to zero.

The geometrical interpretation of the property of ν is the following: in the projective complex space P^{n-1} of the vector OQ, the projective domain described by OQ has a finite projective area ν when Q $(Q \neq O)$, describes a compact subset of M^p.

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CAUCHY'S PROBLEM IN THE LARGE FOR LINEAR ANALYTIC PARTIAL DIFFERENTIAL EQUATIONS

Jean Leray

1. THE SINGULARITIES OF THE SOLUTION

The fundamental property of linear analytic ordinary differential equations is the well-known Cauchy theorem: the singularities of their solutions are singularities of the equation. That theorem can be extended as follows to linear analytic partial differential equations:

THEOREM 1. The singularities of the solutions of the linear analytic Cauchy problem belong to the characteristics issued from the singularities of the data or tangent to the variety S carrying Cauchy's data.

Let us state the assumptions to be made and the precise definition of those characteristics; for the sake of simplicity let us restrict the problem to a sufficiently small <u>neighborhood of</u> S.

Let X be an analytic ℓ-dimensional variety; we denote by x a point of X, by $(x_1, \ldots, x_\ell)$ local coordinates, by $D_1, \ldots, D_\ell$ the partial derivations relative to $x_1, \ldots, x_\ell$. Let $a(x, D)$ be a linear analytic differential operator of order m, defined on X; locally

$$a(x, D)\, u(x) = \sum_{i+j+\ldots+k \leq m} a_{ij\ldots k}(x)\, D_1^i D_2^j \ldots D_\ell^k\, u(x),$$

the $a_{ij\ldots k}(x)$ being analytic regular functions. Let S be in X an analytic regular $(\ell-1)$-dimensional variety, whose local equation is

$$s(x) = 0; \qquad Ds = (D_1 s, \ldots, D_\ell s) \neq 0.$$

Let $b(x, D)$ be a first order differential operator such that

$$b(x, 0) = 0, \qquad b(x, D)s \neq 0.$$

Let $v(x)$ and $w_j(x)$ $(j = 0, \ldots, m-1)$ be analytic regular functions defined respectively on X and on S. Cauchy's problem asks for a function $u(x)$ such that:

$$a(x, D)\, u(x) = v(x);$$

$$u(x) = w_0(x),\ b(x, D)\, u(x) = w_1(x), \ldots, [b(x, D)]^{m-1} u(x) = w_{m-1}(x) \quad \text{on } S.$$

Either X and S are complex analytic of complex dimensions ℓ and ℓ-1, or they are real analytic of real dimensions ℓ

and ℓ-1; but then we have to assume a(x, D) to be hyperbolic and Cauchy's problem to be well-settled: see for instance [1].

Denote by h(x, p) the homogeneous polynomial in $p = (p_1, \ldots, p_\ell)$

$$h(x,p) = \sum_{i+j+\ldots+k=m} a_{ij\ldots k}(x)\, p_1^i p_2^j \ldots p_\ell^k.$$

A point x of S is said to be characteristic when the first order contact element (x, Ds) of S at x satisfies the characteristic equation h(x, Ds) = 0. An (ℓ-1)-dimensional variety of X is said to be a characteristic when all its contact elements are characteristic. S can have characteristic points, but we assume that S is not a characteristic.

The theory of non-linear first-order differential equations leads to the notion of bicharacteristics: a bicharacteristic of a(x, D) is a first-order contact element (x(t), p(t)), function of the numerical parameter t, satisfying the ordinary differential system:

$$x_t = h_p(x,p), \quad p_t = -h_x(x,p), \quad h(x,p) = 0.$$

The main result of that theory is the following theorem and its converse: the bicharacteristic issued from a contact element of a characteristic belongs to that characteristic.

We are now able to give the precise definition of the characteristic K tangent to the variety S: it is the set of all the bicharacteristics issued from the characteristic contact elements of S. Where K is an $(\ell-1)$-dimensional variety, $k(x) = 0$, there it satisfies the characteristic equation

$$h(x, Dk) = 0 \qquad \text{for} \quad k(x) = 0.$$

As for the characteristic issued from the singularities of the data, it is, in the real case, the set of all the bicharacteristics issued from the characteristic contact elements tangent to the set of the singularities of the $v(x)$ and $w_j(x)$. In the complex case, a more elaborate definition has to be used: consider all the characteristic contact elements (x, p) of $(\ell-1)$-dimensional complex analytic varieties, such that the set of the singularities of the $v(x)$ and $w_j(x)$ is tangent to the $(2\ell-1)$-dimensional real-linear variety

$$\mathrm{Re}\,(p \cdot dx) = 0;$$

(Re: real part of ...; $p \cdot dx$: scalar product; dx: vector from x to a point of that real-linear variety); the set of all the bicharacteristics issued from those contact elements (x, p) is the characteristic issued from the singularities of the data. Where it is a $(2\ell-1)$-real-dimensional variety, there its tangent real-linear variety $\mathrm{Re}(p\cdot dx) = 0$ satisfies the characteristic equation $h(x,p) = 0$; where it is an $(\ell-1)$- complex-dimensional variety, there its tangent variety $p\cdot dx = 0$ satisfies the same equation.

Now, in the neighborhood of S, Theorem 1 has a precise meaning.

Its proof uses a linear transformation, defined by integrals, which yields the solution of all the Cauchy problems for $a(x,D)$, by means of the solution of the special one:

$$a(x,D)\, u(x,p') = 1;$$

$$u(x) \text{ vanishes } m \text{ times for } p'\cdot x = 0.$$

We denote

$$p'\cdot x = p_o + p_1 x_1 + \dots + p_\ell x_\ell.$$

In the real case, that transformation is an extension of the inverse Laplace transformation. In the complex case, it is an extension of the convolution by an inverse Laplace transformation: see [2]. In both cases, for using that transformation, we have to know the behavior of $u(x, p')$ near the variety $p' \cdot x = 0$, that is the behavior of the solution of the linear analytic Cauchy problem on the characteristic K tangent to S.

From now on we shall talk about that behavior; its study has no analogue in the theory of ordinary differential equations; its simplicity is a striking feature of linear partial differential equations.

2. THE REGULARITY OF THE SOLUTION OF CAUCHY'S PROBLEM ON A CHARACTERISTIC NEIGHBORHOOD OF THE VARIETY S CARRYING CAUCHY'S DATA

What we call a neighborhood of S over X is composed of

1) a complex analytic space Y; $\dim Y = \dim X = \ell$;
2) a complex analytic variety V in Y; $\dim V = \ell - 1$;
3) an analytic mapping $x(y)$ of Y into X; it has to be a homeomorphism of V onto S; V shall not be enclosed in the variety W of Y where the Jacobian $\frac{D(x)}{D(y)}$ vanishes.

We identify V with S; we say that Y is branched over $x(W)$. Let $u(y)$ be a function defined on Y; its projection onto X is the function $u(x)$ resulting from the elimination of y from $u(y)$ and $x(y)$; generally $u(x)$ is a multivalued function.

Let Y' be another neighborhood of S over X; there is at most one analytic homeomorphism between Y and Y' such that $x(y) = x(y')$; it is indeed obvious near $S - W - W'$; if such a homeomorphism exists, then let it identify Y with Y' and, of course, $x(y)$ with $x(y')$, W with W'.

In order to define now the characteristic neighborhoods of S, denote by $g(x,p)$ any function with the following properties:

1) $g(x,p)$ is homogeneous of degree 1 in p;
2) $\dfrac{g(x,p)}{h(x,p)}$ is regular and non-vanishing, for all contact elements (x,p) of S.

For instance one can choose $g(x,p) = h(x,p)[b(x,p)]^{1-m}$.

Let us call "the differential equation of the characteristic projection" the following ordinary differential system

$$x_t = g_p(x,p), \qquad p_t = -g_x(x,p),$$

where t is the independent variable, x and p the unknown functions of t; it has the following properties:

1) it has the first integral $g(x, p)$;

2) it has the absolute integral invariant $\int p \cdot dx - g\, dt$;

3) its solutions satisfying $g = 0$ become, by a change of parameter t, solutions of the equation defining the bicharacteristics.

Denote by $x(t, z)$, $p(t, z)$ the solution of the preceding equation issued from the contact element $(z, Ds(z))$ of S; a change of the local equation $s(x) = 0$ of S does not change $x(t, z)$, which is called the characteristic projection and which is regular analytic for:

$$z \in S, \qquad |t| \text{ small enough.}$$

Let us denote by y any couple (t, z) satisfying that condition, by Y the set of all such couples. Y is a complex analytic space, which $x(y)$ maps into X, mapping S onto itself identically; $\frac{D(x)}{D(y)} = 0$ on S only if y is a characteristic point of S. Thus Y is a neighborhood of S over X. Such a neighborhood of S over X is said to be <u>a characteristic neighborhood of</u> S.

Now we can state <u>the main theorem</u>:

THEOREM 2. The solution $u(x)$ of Cauchy's problem is the projection on X of a function $u(y)$ analytic and regular on a characteristic neighborhood of S. That function $u(y)$ depends linearly on the data v and w_j. That neighborhood depends only on X, S and h.

The following theorem justifies our terminology:

THEOREM 3. The variety W of Y where $\frac{D(x)}{D(y)} = 0$ is the set of the points $y = (t, z)$ of Y such that z is a characteristic point of S. Let us draw on W the fibers:

t arbitrary, z fixed;

the projections of those fibers are the bicharacteristics tangent to S. Thus the projection $x(W)$ of W is the characteristic K tangent to S.

The following theorem shows that the main theorem is not ambiguous:

THEOREM 4. Let Y and Y' be two characteristic neighborhoods of S; there is another one Y'' such that

$$Y'' \subset Y, \qquad Y'' \subset Y'.$$

Each fiber of W'' belongs to a fiber of W and to a fiber of W'.

It can happen that the projection of a characteristic neighborhood Y of S is not a neighborhood of S, that the projection $u(x)$ of a regular analytic function $u(y)$ has an infinite number of branches; but that happens only under very special conditions which we shall now state.

A point x of S is said to be exceptional when the characteristic conoid of vertex x touches S along a curve containing x. More precisely: the point x of S is exceptional if and only if it has a contact element $(x, p(t))$, an analytic function of t, such that S contains the contact element with parameter t of the bicharacteristic issued from $(x, p(t))$.

A point is ordinary if it is not exceptional. For such a point:

THEOREM 5. Let us replace X by a sufficiently small neighborhood of an ordinary point of S. Then:

1) K is an analytic set, which can be defined by a unique equation: $k(x) = 0$.

2) Each point of $X - K$ is the projection of the same finite positive number of points of Y.

3) If $u(y)$ is regular analytic on Y, then its projection $u(x)$ is an algebraic function of regular analytic functions of x.

The simplest ordinary points are of the following kinds:

1) x is a non-characteristic point of X, then K is empty; the projection of Y into X is a homeomorphism; $u(x)$ is regular.

2) x is a characteristic point of S, where the bicharacteristic direction $h_p(x, Ds)$ does not belong to the variety whose points are the characteristic points of S, then: K is regular and has a contact of order 1 with S; the projection of Y into X has two inverses, which become equal on K and only there; $u(x)$ is a regular function of x and $\sqrt{k(x)}$.

The proof of those theorems begins by the study of the ordinary points of those two kinds; the first kind has been studied by Cauchy, Kowalewski, Schauder, Petrowsky; they used majorant functions;

the study of the second kind requires some other processes; general properties of analytic functions and integral invariants are used.

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A COMPLEX FROBENIUS THEOREM

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1. It was recently shown by Newlander and Nirenberg [3] that a sufficiently differentiable almost complex manifold satisfying the so-called "complete integrability conditions" is in fact a complex analytic manifold. This result can be formulated as a special "complex Frobenius theorem"; in fact, if the almost complex structure is real analytic the result follows easily from the theorem of Frobenius.

We recall briefly this formulation: an almost complex manifold is a 2n-dimensional real manifold on which there is given a real tensor field h_λ^μ satisfying $h_\lambda^\mu h_\nu^\lambda = -\delta_\nu^\mu$; here summation convention is employed. The problem of reducing this to a complex analytic structure is that of finding new local coordinates $x = (x^1, \ldots, x^{2n})$ so that operating with the tensor h_λ^μ is equivalent to transforming the form $dx^a + idx^{a+n}$ into $i(dx^a + idx^{a+n})$, $a = 1, \ldots, n$, i.e. so that in the new coordinate system we have $h_{a+n}^a = 1$, $h_a^{a+n} = -1$, $a = 1, \ldots, n$, with $h_\lambda^\mu = 0$ otherwise. The complete integrability conditions are easily seen to be necessary: we may suppose that the h_μ^λ have the special values above at some particular point, in a neighborhood of which we have coordinates

$y = (y^1, \ldots, y^{2n})$. Then $dx^a + idx^{a+n} = (x^a_\mu + ix^{a+n}_\mu)dy^\mu$ and $i(dx^a + idx^{a+n}) = (x^a_\mu + ix^{a+n}_\mu)h^\mu_\lambda dy^\lambda$, $a = 1, \ldots, n$; here $x^\lambda_\mu = \partial x^\lambda / \partial y^\mu$. One then verifies easily that the linear space of forms given by linear combinations of the $dx^a + idx^{a+n}$, $a = 1, \ldots, n$, with complex coefficients is equivalent to the space of forms Ω spanned by the forms $(h^\mu_\lambda + i\delta^\mu_\lambda)dy^\lambda$, $\mu = 1, \ldots, 2n$, of which the first n are independent. As a consequence we have the necessary condition: the exterior differential of any form in Ω may be expressed as a sum of exterior products of forms of Ω with first order forms. This may be expressed as

(1) $\qquad d\Omega \subset$ ideal generated by Ω.

We observe also that $\Omega \cap \overline{\Omega} = 0$, where $\overline{\Omega}$ consists of the complex conjugates of the forms in Ω. In [3] these conditions were shown to be sufficient for the equivalence of the almost complex structure to a complex analytic structure, under sufficient differentiability assumptions on the h^μ_λ.

The Frobenius theorem, we recall (see for instance [2]), asserts that if $\Omega = \Omega(y)$ is a K-dimensional subspace of the linear space of real first order differential forms at any point $y = (y^1, \ldots, y^N$ in a neighborhood in an N-dimensional real manifold (varying sufficiently smoothly with y) then a necessary and sufficient condition that we may find new local coordinates x so that $\Omega(y)$ is spanned by $dx^1, \ldots, dx^K$ is that

$$(2) \qquad d\Omega \subset \text{ideal generated by } \Omega.$$

Thus the result above may be considered as a "complex Frobenius theorem".

Here we prove a general complex Frobenius theorem containing the above as a special case. The general theorem is, however, easily derived from this special case with the aid of the real Frobenius theorem. Before formulating the theorem we state a result equivalent to the real Frobenius theorem above:

THEOREM A: Suppose, in the above, that for $j = 1, \ldots, K$, dy^j = linear combination of $dy^{K+1}, \ldots, dy^N$ (mod Ω), with coefficients that are functions of all the y^k. Then (2) is necessary and sufficient that the system of differential equations $\Omega = 0$ have a unique solution $y^j(y^{K+1}, \ldots, y^N)$, $j = 1, \ldots, K$, having arbitrary prescribed initial values $y^j(0, \ldots, 0)$ (see for example [2]).

Since our complex Frobenius theorem is local we shall assume that we are operating in a neighborhood of the origin of N-space $y = (y^1, \ldots, y^N)$. For convenience we shall also suppose that the coefficients of the forms occurring are of class C^∞. $\overline{\Omega}$ will denote the space of forms which are complex conjugates of the forms of a space Ω.

THEOREM 1: Let $\Omega = \Omega(y)$ be a K-dimensional subspace of the space of complex-valued forms defined at every point y (and varying in a C^∞ way) in a neighborhood of the origin. Set $\Lambda = \Omega \cap \overline{\Omega}$ and assume that K' = dimension of Λ is constant and that Λ has a basis with C^∞ coefficients (we note that necessarily $K' \geq 2K - N$); set $L = K - K'$. Necessary and sufficient for the existence of new local coordinates x so that Ω is equivalent to the space spanned by

$$dx^a + idx^{a+L}, \quad dx^\sigma, \quad a = 1, \ldots, L; \quad \sigma = N - K' + 1, \ldots, N,$$

are the conditions

(3) $\qquad d\Omega \subset$ ideal generated by Ω

(4) $\qquad d\Lambda \subset$ ideal generated by Λ.

We observe that if $\tilde{x}$ is another coordinate system with Ω equivalent to the space spanned by $d\tilde{x}^a + i d\tilde{x}^{a+L}$, $d\tilde{x}^\sigma$, $a \leq L$, $\sigma > N - K'$, then the coordinates $\tilde{x}^\sigma$ are functions of the x^τ alone. $\sigma, \tau > N - K'$, and for $a \leq L$ the coordinates $\tilde{x}^a + i\tilde{x}^{a+L}$ are holomorphic functions of the $x^b + ix^{b+L}$, $b = 1, \ldots, L$, and depend in addition only on the variables x^σ, $\sigma > N - K'$.

The necessity of (3), (4) is immediately verified. We shall describe only the proof of sufficiency.

If $\Omega = \Lambda$, so that $L = 0$ and any form in Ω equals a real form in Ω plus i times another, we see that the theorem is simply the real Frobenius theorem. If N is even, $K = N/2$ and $K' = 0$ the theorem is simply the result described earlier and proved in [3].

It is easily seen that Theorem 1 may be restated as a result concerning first order differential operators with complex coefficients: $\Sigma a^j \dfrac{\partial}{\partial y^j}$.

THEOREM 1': Let S be an (N-K)-dimensional linear space of first order differential operators (spanned by $N - K$ such operators with C^∞ coefficients) in a neighborhood of the origin in N-space. Let $\overline{S}$ denote the space of operators obtained by replacing the coefficients in the operators of S by their complex conjugates, and let $\widetilde{S}$ denote the space of operators spanned by those in S and $\overline{S}$. Assume that the dimension of $\widetilde{S} = N - K + L$ is constant, and that $\widetilde{S}$ can be spanned by operators with C^{∞} coefficients. Then necessary and sufficient for the existence of new local coordinates x so that S is equivalent to the space of operators spanned by

$$\frac{\partial}{\partial x^a} + i\frac{\partial}{\partial x^{a+L}}, \quad \frac{\partial}{\partial x^\sigma}, \quad a = 1, \ldots, L; \quad \sigma = L+K+1, \ldots, N$$

are the conditions:

(3') the commutator of any two operators in S belongs to S,

(4') the commutator of any two operators in $\widetilde{S}$ belongs to $\widetilde{S}$.

A special case of this for $N - K = L = 1$, was announced in [3], p. 393.

Before proving Theorem 1 a few remarks about the case N even, $K = N/2$, $K' = 0$, treated in [3], where the new coordinates x

were obtained by solving a system of integral equations in a neighborhood U of the origin: (a) If a given system of basis elements of Ω depends differentiably (say C^∞) on some parameters then the new coordinates x so constructed are also continuously differentiable in these parameters (also C^∞) provided U is chosen sufficiently small (this was stated at the end of [3]). Indeed it is not difficult to show that the first derivatives of x with respect to the parameters will be as small as desired provided U is sufficiently small.

(b) The new coordinates x constructed in [3] have the following property: if, on a neighborhood, a basis for Ω can be chosen so that a finite number of forms $dy^j + idy^{j+K}$ are basis elements, then for these values of j we have $x^j = y^j$, $x^{j+K} = y^{j+K}$.

In §2 we prove Theorem 1, and in §3 we prove an analogue of Theorem A.

2. PROOF OF THEOREM 1.

2.1. We consider first the case $K' = 0$, i.e. $L = K$, and construct new coordinates x in a neighborhood of the origin so that Ω is the space spanned by

$$dx^a + idx^{a+L}, \qquad a = 1, \ldots, L.$$

We use summation convention and assume that the indices a, b run from 1 to L, and λ, μ, ν from $2L+1$ to N. By suitably modifying the coordinates y we may suppose that $\Omega(y)$ is spanned by forms $\omega^a(y)$ with $\omega^a(0) = dy^a + idy^{a+L}$ (mod dy^λ). In a neighborhood of the origin we may therefore express the dy^a, dy^{a+L} as linear combinations of the forms dy^λ, ω^b, $\overline{\omega^b}$.

To construct the new coordinates we shall (following [3]) consider the y coordinates as functions of the x coordinates, with

$$y^\lambda \equiv x^\lambda.$$

We construct the functions $y^a(x)$, $y^{a+L}(x)$ in two steps: (i) Define them on the $2L$-dimensional submanifold M: $x^\lambda = 0$ for all $\lambda > 2L$. (ii) Extend the functions so defined to a full neighborhood of $x = 0$.

To carry out step (i) consider the forms ω^a on M: $x^\lambda = y^\lambda = 0$; there, near the origin, the forms ω^a, $\overline{\omega^b}$, $a, b = 1, \ldots, L$ are linearly independent. By (3) we have the situation of [3], i.e. an almost complex structure, and we may therefore choose the y^a, y^{a+L} as functions of x^b, x^{b+L} so that Ω is spanned by the forms $dx^a + idx^{a+L}$ on $x^\lambda = 0$. We remark that the functions as constructured in [3] are such that $\omega^a = dx^a + idx^{a+L}$ at the origin.

To extend the functions y^a, y^{a+L} to a full neighborhood of $x = 0$ we now make the requirement that for every fixed $x^1, \ldots, x^{2L}$, the functions y^a, y^{a+L} satisfy the differential equations $\Omega = 0$ identically in the x^λ. By a remark above this system of equations may be expressed thus: dy^a, dy^{a+L} are linear combinations of the $dx^\lambda = dy^\lambda$ with coefficients that are functions of all the y^j. Because of (3) and Theorem A on page 3 there exists for every fixed $(x^1, \ldots, x^{2L})$ a unique solution $y^1, \ldots, y^{2L}$ of this system with given initial values for $x^\lambda = 0$, which we take to be the values determined in (i). We have thus defined a change of coordinates $y(x)$ near $x = 0$, which in a sufficiently small neighborhood may be seen to have any given number of derivatives and to be non-singular.

Since the forms of Ω vanish whenever $dx^1 = \ldots = dx^{2L} = 0$, it follows that they are linear combinations of $dx^1, \ldots, dx^{2L}$. It remains still to show that they are linear combinations of the $dx^a + idx^{a+L}$ alone; by (i) this holds on $x^\lambda = 0$. To verify this in general we choose a basis for Ω of the form

$$dx^a + idx^{a+L} + \beta^a_b(dx^b - idx^{b+L}), \quad a = 1, \ldots, L$$

with the $\beta^a_b = 0$ on $x^\lambda = 0$. From (3) it follows directly, however, that $\partial\beta^a_b/\partial x^\lambda = 0$, and we conclude that the β^a_b vanish identically. This completes the proof of the theorem for $K' = 0$.

Remarks: (a) The remark (a) at the end of §1 still holds. (b) The remark (b) at the end of that section also holds. Furthermore, we see from our proof that if the coordinates y are such that we have a basis $\omega^a(y)$ of $\Omega(y)$ with

$$\omega^a(0) = dy^a + i dy^{a+L} \text{ (mod the } dy^\lambda)$$

then we may choose $x^\mu = y^\mu$, $\mu = 2L+1, \ldots, N$.

2.2. We prove the general case of the theorem, $K' \neq 0$, by a reduction to the special case 2.1. It is to be shown that new coordinates x may be found so that Ω is spanned by

$$(5) \qquad dx^a + i dx^{a+L}, \quad dx^\sigma, \quad a = 1, , , . L; \sigma = N-K'+1, \ldots, N,$$

We shall assume here that σ runs from $N-K'+1$ to N. Since $\Lambda = \overline{\Lambda}$ we may, by virtue of (4), apply the real Frobenius theorem, and conclude that we have new coordinates u so that Λ is equivalent to the space spanned by the du^σ. We may write Ω as a direct sum $\Omega = \Lambda \oplus \Gamma$ where Γ is spanned by forms which do not involve the du^σ. Clearly

$$(6) \qquad \Gamma \cap \overline{\Gamma} = 0,$$

and, from (3), it follows that if the coordinates u^σ are kept fixed then

$$(7) \qquad d\Gamma \subset \text{ideal generated by } \Gamma.$$

To prove the general case it suffices to find new coordinates x so that the last K' are unchanged, $x^\sigma \equiv u^\sigma$, and so that when these coordinates are held fixed the space Γ is spanned by the forms $dx^a + idx^{a+L}$, $a = 1, \ldots, L$; for then, in general, Γ, and hence Ω, is spanned by the forms (5). But if we now regard the x^σ as parameters, the problem of making Γ equivalent to the space spanned by $dx^a + idx^{a+L}$ is, because of (6), simply the special case of the theorem treated in 2.1, with N there replaced by $N - K'$, and (3) there being expressed by (7). Thus such a change of coordinates $(u^1, \ldots, u^{N-K'}) \longrightarrow (x^1, \ldots, x^{N-K'})$ is possible for every fixed $(u^{N-K'+1}, \ldots, u^N)$. That this change of coordinates is sufficiently differentiable in the x and that the full change of coordinates

$(u^1, \ldots, u^N) \longrightarrow (x^1, \ldots, x^N)$ is non-singular follows from the remarks (a) at the end of sections 1 and 2.1.

This completes the proof of Theorem 1.

As above, we note that if the given Ω depends differentiably (say C^∞) on some parameters then our new coordinates have continuous derivatives (also C^∞) in these parameters provided we operate in a sufficiently small neighborhood.

In case a given basis for Ω depends analytically on some parameters we may show, for $K' = 0$, the following.

THEOREM 2: Let $\Omega = \Omega(y)$ be a space of complex forms as in Theorem 1, with $K' = 0$; assume that a basis for Ω depends analytically on a finite number of real parameters t and holomorphically on a finite number of complex parameters τ, and that for each fixed t, τ (in some t, τ neighborhood) condition (3) is satisfied. Then we may construct new coordinates x satisfying the conditions in Theorem 1 so that $x^a + ix^{a+L}$ are analytic in t and holomorphic in τ, for $a = 1, \ldots, L$ and the coordinates x^s, $s > 2L$ are independent of t and τ.

Proof: Because of the analyticity in t we may extend the forms to be holomorphic for complex values of t (still keeping $K' = 0$); thus it suffices to consider the case that a basis for Ω is holomorphic in some complex parameters $\tau = (\tau^1, \dots, \tau^k)$.

We now extend the system of variables by considering the total system $\tilde{y} = (y^1, \dots, y^N, \operatorname{Re} \tau^1, \operatorname{Re} \tau^2, \dots, \operatorname{Im} \tau^k)$, and consider an enlarged system $\tilde{\Omega}$ of forms spanned by Ω and $d\tau^1, \dots, d\tau^k$. Because of (3), and the holomorphic character of Ω, the enlarged system in $\tilde{y}$ satisfies

$$d\tilde{\Omega} \subset \text{ideal generated by } \tilde{\Omega}.$$

Furthermore $\tilde{\Omega} \cap \overline{\tilde{\Omega}} = 0$. We may therefore apply Theorem 1 and construct new coordinates $\tilde{x} = (x^1, \dots, x^N, \tilde{x}^1, \dots, \tilde{x}^{2k})$ with $\tilde{\Omega}$ spanned by

$$dx^a + i dx^{a+L}, \quad d\tilde{x}^j + i d\tilde{x}^{j+k}, \quad a = 1, \dots, L;\ j = 1, \dots, k.$$

By the remark (b) at the end of sections 1 and 2.1 we have

$$\tilde{x}^j + i\tilde{x}^{j+k} = \tau^j .$$

Thus we have $\tilde{\Omega}$ spanned by $dx^a + i dx^{a+L}$, $d\tau^j$, $a = 1, \dots, L$, $j = 1, \dots, k$. For every fixed τ the new coordinates $x = (x^1, \dots, x^N)$

are now easily seen to have the properties asserted in the theorem.

An analogous statement should hold for $K' \neq 0$, but this simple argument does not seem to be directly applicable - on going from real to complex values of t the value of K' may change.

3. AN ANALOGUE OF THEOREM A.

In response to a question of K. Kodaira and D. C. Spencer[1] and as another illustration of the technique used above of extending the number of variables we prove an analogue of Theorem A.

We consider a system of first order partial differential equations for K complex-valued functions $(u^1, \dots, u^K) = u$ of the real variables $(x^1, \dots, x^n) = x$. For convenience we write $n = 2m+s$ and express the last $2m$ variables $x^{s+1}, x^{s+2}, x^{s+3}, \dots, x^n$ by m complex variables $z^1, \dots, z^m$, setting $z^\lambda = x^{s+\lambda} + i x^{s+m+\lambda}$, $\lambda = 1, \dots, m$. In the following the indices j, k will run from 1 to K, the indices p, q from 1 to s, and the indices λ, μ from 1 to m; summation convention will be used.

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[1] They recently proved a special case of Theorem A' using methods similar to those used in [3], and asked the author whether it could be deduced from the result in [3]; this we do here.

The system to be studied is

(8)

$$(8)_\lambda \qquad \frac{\partial u^j}{\partial \bar{z}^\lambda} + \alpha^j_k \frac{\partial \bar{u}^k}{\partial \bar{z}^\lambda} = a^j_\lambda$$

$$(8)'_p \qquad \frac{\partial u^j}{\partial x^p} + \alpha^j_k \frac{\partial \bar{u}^k}{\partial x^p} = b^j_p$$

for $j = 1, \ldots, K$; $\lambda = 1, \ldots, m$; $p = 1, \ldots, s$. Here the coefficients α, a, b are given functions (of class C^∞, for convenience) of the variables u, x, defined in some neighborhood of the origin in the product space. Thus the system is non-linear. We shall assume that $\alpha(0,0) = 0$ (though it would suffice to assume the $\alpha^j_k(0,0)$ small).

THEOREM A': The following is necessary and sufficient for the system (8) to have solutions $u(x, u_o)$, in a neighborhood of $x = 0$, with given initial values $u(0, u_o) = u_o$ in a neighborhood of the origin in the u space, and such that $u(x, u_o)$ is of class C^∞ in x and u_o: The space Ω of complex Pfaffian forms, in the $2K+n$ variables $(\operatorname{Re} u, \operatorname{Im} u, x)$, which is spanned by the forms

$$du^j + a^j_k d\bar{u}^k - a^j_\lambda d\bar{z}^\lambda - b^j_p dx^p, \qquad j = 1, \ldots, K$$

(9)

$$dz^\lambda \qquad\qquad , \qquad \lambda = 1, \ldots, m$$

satisfies the conditions

(10) $$d\Omega \subset \text{ideal generated by } \Omega.$$

Proof: The necessity of (10) is verified by direct calculation: Differentiate $(8)_\lambda$ with respect to $\bar{z}^\mu$, $(8)_\mu$ with respect to $\bar{z}^\lambda$ and subtract. Form similarly (speaking loosely) $\frac{\partial}{\partial x^p}(8)_\lambda - \frac{\partial}{\partial \bar{z}^\lambda}(8)'_p$ and $\frac{\partial}{\partial x^p}(8)'_q - \frac{\partial}{\partial x^q}(8)'_p$. One sees easily that the resulting relations on the coefficients α, a, b are equivalent to conditions (10).

To prove the sufficiency of (10) we apply Theorem 1, noting that $\Omega \cap \bar{\Omega} = 0$. Applying the theorem, for the case $K' = 0$, to the forms Ω in the $2K+n$ variables (Re u, Im u, x), we infer that we may introduce $2K+n$ new variables, which we may write as (Re v, Im v, ξ), so that Ω is spanned by the forms

$$dv^j, \quad d\xi^{s+\lambda} + i d\xi^{s+m+\lambda}, \quad j = 1, \ldots, K; \quad \lambda = 1, \ldots, m.$$

Here $v = (v^1, \ldots, v^K)$ is complex and $\xi = (\xi^1, \ldots, \xi^n)$ is real.

By Remark (b) at the end of §2.1 we may choose $\xi \equiv x$. It follows that the transformation of variables (Re v, Im v) to (Re u, Im u), for fixed x, is non-singular, and hence that this transformation for $x = 0$ is one-to-one and maps a neighborhood of $v = 0$ onto a neighborhood of $u = 0$, i. e. onto a full neighborhood of initial values u_o for u.

Thus to complete the proof of Theorem A' we need only show that u satisfies (8) as a function of $\xi = x$ and v. To see this write the form (9) in terms of the new variables, v, $\xi = x$; this form equals

$$\frac{\partial u^j}{\partial \bar{z}^\lambda} d\bar{z}^\lambda + \frac{\partial u^j}{\partial \bar{v}^\ell} d\bar{v} + \frac{\partial u^j}{\partial x^p} dx^p + a^j_k\left(\frac{\partial \bar{u}^k}{\partial \bar{z}^\lambda} d\bar{z}^\lambda + \frac{\partial \bar{u}^k}{\partial \bar{v}^\ell} d\bar{v} + \frac{\partial \bar{u}^k}{\partial x^p} dx^p\right) - a^j_\lambda d\bar{z}^\lambda - b^j_p dx^p$$

modulo the forms of (11). Since this form belongs to Ω the coefficients of $d\bar{z}^\lambda$, $d\bar{v}^\ell$ and dx^p vanish. Setting the coefficients of $d\bar{z}^\lambda$ and dx^p equal to zero we obtain equations (8). Q. E. D.

In conclusion we mention that the complex Frobenius theorem can be applied to give conditions for a non-analytic hypersurface in a domain of several complex variables to contain families of (complex) analytic subsurfaces. In studying such questions Behnke and Sommer [1] implicitly assumed a form of the theorem.

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AN ELEMENTARY METHOD FOR THE LOCAL STUDY OF AN ANALYTIC SET

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INTRODUCTION

We consider *complex* analytic sets in the space C^n: a subset M of an open set Ω contained in C^n is said to be an *analytic set* in Ω if each point of Ω possesses a neighborhood U such that $M \cap U$ coincides with the set of common zeros of a finite number of functions holomorphic in U; from this definition it follows at once that M is closed in Ω. The set M is said to be *reducible* at a point $x \in M$ if x possesses a neighborhood U such that $M \cap U$ is the union of two analytic sets in U, each of which contains x and does not coincide with $M \cap U$ in any neighborhood of x.

As to the *local study* of an analytic set M, two main purposes may be assigned to it:

First purpose. To define the set M, in the neighborhood of a given point of M, by formulae as explicit as possible in terms of independent parameters.

Second purpose. To define the *dimension* of M at a point $x \in M$ and see how that dimension is altered:

1) for a given set M, by moving the point x along M;

2) for a given point x, by considering, instead of M, another analytic set included in M or including M;

3) by an analytic transformation with a Jacobian $\neq 0$.

The first of these purposes is fulfilled by two classical theorems:

THEOREM 1. A point x of the set M possesses a neighborhood U such that $M \cap U$ is the union of a finite number of analytic sets in U, each of which goes through x and is irreducible at the point x; these sets M_i are uniquely determined under the additional assumption that, however two different indices i, j are chosen, the inclusion $M_i \subset M_j$ does not hold in any neighborhood of x.

These uniquely determined sets are called the <u>irreducible components</u> of M at the point x. Thanks to theorem 1, theorem 2 may consider only the case when M is irreducible at the point x, which may be assumed to be the origin.

<u>A few notations</u>. The n variables (or coordinates) will be denoted by $x_1, \ldots, x_n$; the functions of these variables which are holomorphic in some neighborhood of the origin form a ring denoted by H_n; those functions which depend only on the first variables

form a subring denoted by H_p. Given a ring A and a variable z, the polynomials with z as a variable and coefficients in A form the ring A[z]. A polynomial $\varepsilon H_p[z]$ is said to be <u>distinguished</u> if the coefficient of the term of highest degree is 1 and all other coefficients are non-units of the ring H_p, i.e. are functions of H_p which vanish at the origin.

THEOREM 2. Given an analytic set M which contains the origin, is irreducible there, and neither has the origin as an isolated point nor fills up a neighborhood of the origin, then, after a suitable linear change of coordinates, there exist:

an integer m ($1 \leq m \leq n-1$), which turns out to be the dimension of M at the origin, when that is defined;

a distinguished polynomial $R \varepsilon H_m[x_n]$, which is irreducible in the ring $H_m[x_n]$;

for each integer i from m+1 to n-1, another polynomial $S_i \varepsilon H_m[x_n]$, which has a degree smaller than that of R and coefficients all vanishing at the origin;

thus M coincides in a suitable neighborhood of the origin, with the closure of the set defined by the following conditions:

$$\begin{cases} R(x_1, \ldots, x_m, x_n) = 0 \qquad \dfrac{\partial R}{\partial x_n}(x_1, \ldots, x_m, x_n) \neq 0 \\ x_i \dfrac{\partial R}{\partial x_n}(x_1, \ldots, x_m, x_n) - S_i(x_1, \ldots, x_m, x_n) = 0 \quad \text{for} \quad i = m+1, \ldots, n-1. \end{cases}$$

The proof of theorems 1 and 2 which is now to be found in classical books (see for instance Bochner and Martin, Several Complex Variables, 1948) is due to W. Rückert (Mathematische Annalen, vol. 107, 1933), who described it as formal, that is to say, involving no function theory, but only algebraic theories: those of ideals and fields. Rückert's method, because of its formal character, is clear and relatively simple, but also ill-adapted to the second purpose stated above and to many problems requiring a close local scrutiny of an analytic set. As a consequence of this, Remmert and Stein (Mathematische Annalen, vol. 126, 1953) had to set up a new method of investigation when they generalized Thullen's theorem on the continuation of an analytic set. The method sketched in this paper may prove useful for other problems; a more detailed account will appear elsewhere.

A SKETCH OF THE METHOD

The starting stage. Let M be an analytic set which contains the origin, is irreducible there, and neither has the origin as an isolated point, nor fills up a neighborhood of the origin: the functions of the ring H_p which vanish identically on M (in some neighborhood of the origin) form an ideal I_p which is prime, since M is irreducible at the origin. After a suitable linear change of coordinates, the ideal I_n is regular (see for instance Rückert): that means, there exists an integer m $(1 \leq m \leq n-1)$ such that I_m is reduced to 0 and, for each integer p from $m+1$ to n, I_p contains a distinguished polynomial $Q_p \in H_{p-1}[x_p]$.

For a given p $(m+1 \leq p \leq n)$, the polynomials of $H_{p-1}[x_p]$ which belong to I_p form an ideal i_p in the ring $H_{p-1}[x_p]$; i_p is prime and has a finite basis, which in general cannot be reduced to one element. So we consider the rings of integrity $A_p = H_p/I_p$ and their quotient fields K_p: the ideal i_p in the ring $H_{p-1}[x_p]$ generates, first an ideal i_p^* in the ring $A_{p-1}[x_p]$, which also is prime and has a finite basis, then an ideal in the ring $K_{p-1}[x_p]$, which admits as a basis a single element $P_p^* \in i_p^*$; since the constant 1 does not belong to any I_p, the distinguished polynomial Q_p generates a polynomial $\not\equiv 0$ in $K_{p-1}[x_p]$, and therefore $P_p^* \not\equiv 0$. Since a polynomial $\varepsilon\ i_p^*$ which is of degree 0 must be $\equiv 0$, the degree of P_p^* is at least 1. Now A_{p-1} contains an element $\phi_{p-1}^* \neq 0$ such that P_p^* divides, in the ring $A_{p-1}[x_p]$, the product of ϕ_{p-1}^* by any polynomial $\varepsilon\ i_p^*$; ϕ_{p-1}^* is generated by a function ϕ_{p-1} belonging to H_{p-1}, but not to I_{p-1}, and P_p^* by a polynomial $P_p \ \varepsilon\ i_p$, which may be chosen <u>of the same degree</u> as P_p^*, so that the coefficient of the term of highest degree in P_p does not belong to I_{p-1}. The product of ϕ_{p-1} by any polynomial $\varepsilon\ i_p$ can be written as $P_p U+V$, where $U\ \varepsilon\ H_{p-1}[x_p]$ and $V\ \varepsilon\ I_{p-1}[x_p]$. In the particular case $p = m+1$, that result means only that P_{m+1} divides any polynomial $\varepsilon\ i_{m+1}$ in the ring $K_m[x_{m+1}]$; but P_{m+1} may be assumed to be primitive and then divides any polynomial $\varepsilon\ i_{m+1}$ in the ring $H_m[x_{m+1}]$. So we can state:

LEMMA 1. The ideal I_{m+1} admits as a basis the single element $P_{m+1}\ \varepsilon\ H_m[x_{m+1}]$, of degree ≥ 1. For each integer p from m+2 to n, there exist a polynomial $P_p\ \varepsilon\ i_p$, $i_p = I_p \cap H_{p-1}[x_p]$, and a function $\phi_{p-1}\ \varepsilon\ H_{p-1}$, so that, if P_p and every function of I_{p-1}, but not ϕ_{p-1}, vanish at a common point, then every function of I_p vanishes at that point; P_p has degree ≥ 1; the coefficient

a_{p-1} of the term of highest degree in P_p does not belong to I_{p-1}, nor does ϕ_{p-1}.

Now consider a polynomial $T \in H_{p-1}[x_p]$, $T \notin i_p$: T generates $T^* \not\equiv 0$ in the ring $A_{p-1}[x_p]$; in that ring can be found 2 polynomials U^*, V^*, which are not both $\equiv 0$, V^* being of degree less than that of P_p^*, such that $P_p^* U^* + T^* V^*$ is of degree 0. Suppose that $P_p^* U^* + T^* V^* \equiv 0$: then, as i_p^* is prime and does not contain T^*, $V^* \in i_p^*$, P_p^* divides V^* in the ring $K_{p-1}[x_p]$, which is impossible. So:

LEMMA 2. If a polynomial $T \in H_{p-1}[x_p]$ does not belong to i_p (for instance, after the above underlined property of P_p, if T has a degree smaller than that of P_p and $T \notin I_{p-1}[x_p]$), then there exist two polynomials $U, V \in H_{p-1}[x_p]$ and a function δ belonging to H_{p-1}, but not to I_{p-1}, such that $(P_p U + TV - \delta) \in I_{p-1}[x_p]$.

<u>The construction of a dense subset of M.</u> Every point of M is a common zero of the n-m polynomials $P_{m+1}, \ldots, P_n$; conversely, after lemma 1, any common zero of $P_{m+1}, \ldots, P_n$ where <u>none</u> of the n-m-1 functions $\phi_{m+1}, \ldots, \phi_{n-1}$ vanishes, is a point of M.

In order to gain more information on M, we apply lemma 2 to the polynomial $T = \frac{\partial P_p}{\partial x_p}$: we get 2 polynomials U_p, $V_p \varepsilon H_{p-1}[x_p]$ and a function δ_{p-1} belonging to H_{p-1}, but not to I_{p-1}, such that

$(P_p U_p + \frac{\partial P_p}{\partial x_p} V_p - \delta_{p-1}) \varepsilon I_{p-1}[x_p]$; since I_{p-1} is prime, the product $a_{p-1} \delta_{p-1} \phi_{p-1} \not\varepsilon I_{p-1}$. Using the distinguished polynomial $Q_{p-1} \varepsilon I_{p-1}$ and a classical lemma due to Späth (Crelle's Journal, vol. 161, 1929), we get a polynomial T_{p-1} of $H_{p-2}[x_{p-1}]$ such that $(a_{p-1} \delta_{p-1} \phi_{p-1} - T_{p-1}) \varepsilon I_{p-1}$; thus $T_{p-1} \not\varepsilon i_{p-1}$ and lemma 2 may be applied with p-1 instead of p and T_{p-1} instead of T: we get 2 polynomials U, $V \varepsilon H_{p-2}[x_{p-1}]$ and a function ρ_{p-2} belonging to H_{p-2}, but not to I_{p-2}, such that $(P_{p-1} U + T_{p-1} V - \rho_{p-2}) \varepsilon I_{p-2}[x_{p-1}]$; therefore, if every function of I_{p-1}, but not ρ_{p-2}, vanishes at a fixed point, $a_{p-1} \delta_{p-1} \phi_{p-1}$ does not vanish at that point.

Since $\rho_{p-2} \not\varepsilon I_{p-2}$ and I_{p-2} is prime, $a_{p-2} \delta_{p-2} \phi_{p-2} \rho_{p-2} \not\varepsilon I_{p-2}$, and the same argument gives a function ρ_{p-3} belonging to H_{p-3}, but not to I_{p-3}, such that, if every function of I_{p-2}, but not ρ_{p-3}, vanishes at a fixed point, $a_{p-2} \delta_{p-2} \phi_{p-2} \rho_{p-2}$ does not vanish at that point. The induction finally leads to a function $\rho_m \varepsilon H_m$, $\rho_m \not\equiv 0$, with the following property: for each integer p from m+1 to n, if every function of I_{p-1}, but not ρ_m, vanishes at a fixed point, $\alpha_{p-1}\delta_{p-1}\phi_{p-1}$ does not vanish at that point; for $p = m+1$, since I_m is reduced to 0 and ϕ_m does not exist, that property means only this: $\rho_m \neq 0$ implies $\alpha_m \delta_m \neq 0$.

As a first consequence of that, a common zero of P_{m+1}, ..., P_n where $\rho_m \neq 0$ belongs to M; as a second consequence, a dense subset $\underline{M}$ of M (which of course is not an analytic set) can be constructed as follows: if $x_1, \ldots, x_m$ are given numerical values such that $\rho_m \neq 0$, then P_{m+1}, $\frac{\partial P_{m+1}}{\partial x_{m+1}}$, U_{m+1}, V_{m+1} become polynomials in x_{m+1} with numerical coefficients; the coefficient of the term of highest degree in P_{m+1} is $\alpha_m \neq 0$ and $P_{m+1} U_{m+1} \frac{\partial P_{m+1}}{\partial x_{m+1}} V_{m+1} \equiv \delta_m \neq 0$, so that P_{m+1} and $\frac{\partial P_{m+1}}{\partial x_{m+1}}$ have no common root, finite or infinite; in other words, P_{m+1} has a number of distinct roots equal to its degree in $H_m[x_{m+1}]$; if we choose one of those roots as a numerical value for x_{m+1}, we get a point $(x_1, \ldots, x_m, x_{m+1})$ where every function of I_{m+1} vanishes (see lemma 1), but not ρ_m; so $\alpha_{m+1}\delta_{m+1}\phi_{m+1}$ does not vanish at that point; since $\alpha_{m+1} \neq 0$ and $\delta_{m+1} \neq 0$, P_{m+2} has a number of distinct roots equal to its degree in $H_{m+1}[x_{m+2}]$; if we choose one of those roots as a numerical value for x_{m+2}, we get a point

$(x_1, \ldots, x_m, x_{m+1}, x_{m+2})$ where P_{m+2} vanishes, every function of I_{m+1} too, but not ϕ_{m+1}, hence every function of I_{m+2} vanishes too (see lemma 1), and so on. The induction finally gives us a fixed number k (the product of the degrees of P_{m+1} in $H_m[x_{m+1}], \ldots, P_n$ in $H_{n-1}[x_n]$) of distinct points $X_1(x_1, \ldots, x_m), \ldots, X_k(x_1, \ldots, x_m)$, with the preassigned values $x_1, \ldots, x_m$ as their first m coordinates, where all functions of I_n vanish; thus those points belong to M; moreover, their coordinates x_{m+1} are also roots of the distinguished polynomial Q_{m+1}, their coordinates x_{m+2} are also roots of the distinguished polynomial Q_{m+2}, and so on. As the roots of a distinguished polynomial depend continuously on its coefficients, we can state:

LEMMA 3. As the point $(x_1, \ldots, x_m)$ varies in a suitable neighborhood of the origin and outside the set of zeros of the function ρ_m, the points of M which have $x_1, \ldots, x_m$ as their first m coordinates remain in a fixed number k; their last $n-m$ coordinates are locally holomorphic functions of $x_1, \ldots, x_m$ which remain bounded and tend to 0 as $x_1, \ldots, x_m$ tend to 0. Let $\underline{M}$ be the subset of M generated by those points $X_1, \ldots, X_k$; then $\underline{M}$ is dense in M.

Now we want to prove that $\underline{M}$ is dense in M. First we suppose that a function f_p belonging to H_p, but not to I_p, vanishes identically (in some neighborhood of the origin) on $\underline{M}$; the argument already used for the proof of lemma 3 gives us a function f_{p-1} belonging to H_{p-1}, but not to I_{p-1}, such that if every function of I_p, but not f_{p-1}, vanishes at a fixed point, f_p does not vanish at that point; consequently f_{p-1} vanishes identically (in some neighborhood of the origin) on $\underline{M}$. The induction finally leads to a function $f_m \in H_m$ such that $f_m \not\equiv 0$, but $f_m \equiv 0$ on $\underline{M}$, i.e. $f_m \equiv 0$ outside an analytic set in C^m, which is absurd. So:

LEMMA 4. Any function of H_n which vanishes everywhere (in some neighborhood of the origin) on $\underline{M}$ belongs to the ideal I_n and, therefore, vanishes everywhere (in some neighborhood of the origin) on M.

Consider any function $f \in H_n$ with $f(0) = 0$: any symmetric entire function of $f[X_1(x_1, \ldots, x_m)], \ldots, f[X_k(x_1, \ldots, x_m)]$ (see lemma 3) may be expressed, in terms of $x_1, \ldots, x_m$, as a function holomorphic and bounded outside the set of zeros of ρ_m, which moreover tends to 0 as $x_1, \ldots, x_m$ tend to 0, hence as a function of H_m, which moreover vanishes at the origin. By applying lemma 4 we thus get the decisive:

LEMMA 5. To any function $f \in H_n$ with $f(0) = 0$, there exists a distinguished polynomial R, with coefficients $\in H_m$ and a degree equal to the integer k in lemma 3, such that $R(x_1, \ldots, x_m, f) \in I_n$.

Now let $x^* = (x_1^*, \ldots, x_n^*)$ be a point belonging to M, but not to $\underline{M}$; $\rho_m(x_1^*, \ldots, x_m^*) = 0$, but $x_1', \ldots, x_m'$ may be chosen arbitrarily near to $x_1^*, \ldots, x_m^*$ so that $\rho_m(x_1', \ldots, x_m') \neq 0$; for $i = m+1, \ldots, n$, the i^{th} coordinates of the points $X_1(x_1', \ldots, x_m'), \ldots, X_k(x_1', \ldots, x_m')$ (see lemma 3) are the roots (see lemma 5) of a distinguished polynomial with degree k and coefficients depending continuously on $x_1', \ldots, x_m'$; so the points $X_1(x_1', \ldots, x_m'), \ldots, X_k(x_1', \ldots, x_m')$ have at most k^{n-m} distinct limit points X_j^* as $x_1', \ldots, x_m'$ tend to $x_1^*, \ldots, x_m^*$. Suppose that $X_j^* \neq x^*$ for each j: then a linear form u, depending only on the last $n-m$ variables, may be found so that $u(X_j^*) \neq u(x^*)$ for each j; lemma 5 yields a distinguished polynomial R such that $R(x_1, \ldots, x_m, u) \in I_n$; $u(x^*)$ is a root of $R(x_1^*, \ldots, x_m^*, u)$, hence the limit, as $x_1', \ldots, x_m'$ tend to $x_1^*, \ldots, x_m^*$, of a root of $R(x_1', \ldots, x_m', u)$, hence a $u(X_j^*)$; that contradiction proves that x^* is an X_j^*, and therefore that $\underline{M}$ is a dense subset of M.

Theorem 2 (see Introduction) is now easily proved by a similar argument, together with the following addition: to any function $f \varepsilon H_n$ with $f(0) = 0$, there exists a polynomial $S \varepsilon H_m[x_n]$, which has a degree smaller than that of R and coefficients all vanishing at the origin, such that $(f \frac{\partial R}{\partial x_n} - S) \varepsilon I_n$.

SOME APPLICATIONS TO THE LOCAL DIMENSION THEORY OF AN ANALYTIC SET

A definition of the dimension. A point x of an analytic set M is said to be an ordinary point of M if there exists an analytic transformation F, with Jacobian $\neq 0$, which maps a neighborhood of the point x on M, onto a neighborhood of the point $F(x)$ on some complex plane, the dimension of which defines the dimension of M at the point x, and also at any point sufficiently near to x. Then every point of the subset $\underline{M}$ constructed above is an ordinary point of M, with the dimension m; the fact that $\underline{M}$ is dense on M implies, first that the ordinary points of M are dense on M, and secondly, that m is the dimension of M at any ordinary point sufficiently near to the origin, be it in $\underline{M}$ or not. Since the dimension of M has a constant value at all ordinary points sufficiently near to a given non-ordinary point x, the dimension of M at the point x may be defined as that constant value, and that definition is, in particular, valid for the origin. Hence:

PROPOSITION 1. If an analytic set M is irreducible and has the dimension m at the origin, it also has the dimension m at every point of M in a suitable neighborhood U of the origin; in particular, any irreducible component of M at any point in $M \cap U$ has the dimension m at that point.

If, on the contrary, M is reducible at the origin, proposition 1 does not hold, and the dimension of M at the origin is not defined yet; it may be defined either as the maximum value of the dimension of M at neighboring ordinary points, or as the maximum dimension, at the origin, of the irreducible components of M at the origin. Now the dimension of M is defined at every point of M, and in such a way that:

PROPOSITION 2. The dimension of an analytic set is unaltered by an analytic transformation with non-vanishing Jacobian.

Let us now compare the dimensions, at the origin, of two analytic sets M, M' which are irreducible at the origin, with $M' \subset M$, but $M' \neq M$ in any neighborhood of the origin. The corresponding ideals I_p, I'_p (see the starting stage of the method) satisfy $I'_p \supset I_p$ for each p, but $I'_n \neq I_n$; so there is a function f_n belonging to I'_n, but not to I_n, and the argument already used for the proof of lemmas 3 and 4 gives a function f_{n-1} belonging to I'_{n-1}, but not to I_{n-1}. If m is the dimension of M at the origin, the induction finally leads to a function $f_m \in I'_m$, $f_m \not\equiv 0$; so I'_m is not reduced to 0 and the dimension of M' at the origin is smaller than m:

PROPOSITION 3. If M is irreducible and has the dimension m at the origin, any analytic subset of M either coincides with M in some neighborhood of the origin or has a dimension smaller than m at the origin; in particular, the non-ordinary points of M are included in an analytic subset of M which has a dimension smaller than m at any of its points.

The latter subset is the set of all points of M where $\rho_m = 0$. By putting together Propositions 1 and 3, we get the following important one:

PROPOSITION 4. Given an analytic set M which is irreducible at the origin, and a function $f \in H_n$ which does not vanish identically on M in any neighborhood of the origin, then the origin possesses a neighborhood U such that, for any point $x \in M \cap U$, f does not vanish identically on any irreducible component of M at the point x, in any neighborhood of x.

Now we consider again an analytic set M which is irreducible and has the dimension m at the origin, and a function f belonging to H_n, but not to I_n, with $f(0) = 0$; let M_f be the set of all points of M where $f = 0$, M', M'', ... the irreducible components of M_f at the origin (which, after proposition 3, all have dimensions < m), and I'_p, I''_p,... the corresponding ideals. The polynomial R of lemma 5, which satisfies the condition

$R(x_1, \ldots, x_m, f) \varepsilon I_n$, was formed in such a way that, provided the coefficient of the term of degree 0 in R vanishes at the point $(x_1, \ldots, x_m)$, M_f contains at least one point with preassigned first m coordinates $x_1, \ldots, x_m$; therefore, after a suitable linear change of coordinates, one at least of the ideals I'_{m-1}, $I''_{m-1}, \ldots$ is reduced to 0; that is to say, one at least of the components M', $M'', \ldots$ has the dimension m-1 at the origin. But, as the same result holds for the irreducible components of M_f at any point of M_f, M' must have the dimension m-1 at a point of M' which belongs to none of the other components M'', $M''', \ldots$. So:

PROPOSITION 5. Given an analytic set M which is irreducible and has the dimension m at the origin, and a function $f \varepsilon H_n$ which vanishes at the origin, but does not vanish identically on M in any neighborhood of the origin, then the points of M where $f = 0$ form an analytic set which has the dimension m-1 at every point sufficiently near the origin.

COROLLARY 1. Given an analytic set M which is irreducible and has the dimension m at the origin, and q functions $f_1, \ldots, f_q \in H_n$ with $f_1(0) = \ldots = f_q(0) = 0$, then the points of M where $f_1 = \ldots = f_q = 0$ form an analytic set which has a dimension $\leq m$ and $\geq m-q$ at every point sufficiently near the origin.

Corollary 1 is obtained through iteration of proposition 5, and proves the equivalence between the definition of the dimension given above, and that stated by Remmert and Stein at the beginning of their paper:

COROLLARY 2. The analytic set M has the dimension m at the origin if and only if $n-m$ is the maximum dimension of a complex plane P going through the origin and such that the origin is an isolated point of $M \cap P$.

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IDEALS OF MEROMORPHIC FUNCTIONS OF SEVERAL VARIABLES

Sin Hitotumatu

1. INTRODUCTION

During the past several years, there has been much progress in the theory of sheaves (faisceaux), which introduced a unified aspect into the theory of functions of several variables (cf. Cartan [1], [2], Serre [6]). In the present note, I shall consider, in place of the holomorphic functions in the above theory, the sheaf of fractional ideals of meromorphic functions, and I would like to give a generalization of two fundamental theorems on coherent sheaves on a Stein manifold to the meromorphic case. The essential part has been already discussed in our former paper (Hitotumatu-Kôta [4]), but here I shall give some generalizations and simplifications of the proofs.

2. DEFINITIONS AND NOTATIONS

Let X be a complex analytic manifold. Two functions f and g defined in a neighborhood of a point x on X are called equivalent at x if f = g in a suitable neighborhood of x. An equivalence class f_x of a function f classified by this equivalence relation is called the jet or germ of f at x. If f is holomorphic at x, the jet f_x is called holomorphic at x.

We denote by O_x the collection of all holomorphic jets at a point x. This is nothing but the ring consisting of all power series around x with non-empty convergence region. As is well known, O_x is a noetherian ring of integrity, in which the theorem of unique factorization holds.

We denote by K_x the quotient-field of the ring O_x. Let B be a subset of K_x. If there exists a jet $\phi \in K_x$ not identically zero, such that

$$\phi \cdot B = \{\phi\psi \mid \psi \in B\} \subset O_x,$$

then ϕ is called an integralizator of B. A subset $A \neq \{o\}$ of K_x is called an ideal or more precisely an ideal with respect to O_x, if A is an O_x-module with integralizator. The set of all integralizators of a given ideal A forms with o an ideal which is called the inverse-ideal of the ideal A and denoted by A^{-1}.

The collection

$$O = \{(f, x) \mid f \in O_x, x \in X\}$$

has the structure of a sheaf of rings on X, and similarly

$$K = \{(\phi, x) \mid \phi \in K_x, x \in X\}$$

has the structure of a sheaf of fields on X. K is also a sheaf of O-modules on X.

A cross-section

$$\phi \in \Gamma(K; U) = K_U$$

of the sheaf K in an open set U in X is called a meromorphic function in U. The field $K_U = \Gamma(K; U)$ evidently contains the quotient field of the ring $O_U = \Gamma(O; U)$, and these two fields are not equal in general. As we have remarked (Hitotumatu and Kôta [4]), they coincide with each other when U is a Stein manifold.

Let S be a subsheaf of O-modules in K, and $U \subset X$. A cross-section $\phi \in K_U$ is called an integralizator of S in U, if the jet ϕ_x integralizes the fiber S_x in K_x at every point x in U. We say that S has a local integralizator at a point x, if there exists a neighborhood U of x in which an integralizator of S exists. When S has local integralizators at every point x on X, S is called a sheaf of ideals on X. Next, suppose that, in a neighborhood U of a point x, there exist a finite number of cross-sections $\phi_1, \ldots, \phi_k \varepsilon \Gamma(S;U)$ of the subsheaf S, such that they generate the fiber S_y as O_y-module at every point y in U. Then these functions $\phi_1, \ldots, \phi_k$ are called a local generator of S at the point x. When $k = 1$, i.e., when S has a local generator consisting of only one element, we say that S is locally principal at x. A sheaf of ideals S having a local generator at every point x on X is called a coherent sheaf on X.

For example, let A be a fixed ideal in K_X. We denote by $[A]_x$ the set of all jets of functions in A at the point x. Then we can define a subsheaf of K by

(1) $$F(A) = \{(f, x) \mid f \varepsilon [A]_x, \; x \varepsilon X\},$$

which is called a sheaf associated to the ideal A. It is easy to see that the sheaf of ideals F(A) is coherent.

3. TRANSPORTER-SHEAF AND INVERSE-SHEAF

Let S and T be two coherent subsheaves of ideals in K. Then we define the transporter-sheaf S : T as follows:

$$S : T = \{(\psi, x) \mid \psi \varepsilon (S : T)_x, \; x \varepsilon X\},$$

where

(2) $$(S : T)_x = \{\psi \varepsilon K_x \mid \psi \cdot T_x \subset S_x\}.$$

If S is the sheaf O of all holomorphic functions, we write

$$T^{-1} = O : T$$

and call it the <u>inverse-sheaf</u> of T. $(T^{-1})_x$ is nothing but the set of all integralizators of T_x.

THEOREM 1. The transporter-sheaf S : T is again coherent. When S is locally principal, so is S : T, and in particular the inverse-sheaf T^{-1} of a coherent sheaf T is always locally principal.

PROOF. First we remark that S : T is a sheaf. We have a neighborhood U of a point x analytically isomorphic to a poly-cylinder and local generators $\phi_1, \ldots, \phi_\ell$ of S and $\psi_1, \ldots, \psi_m$ of T in U respectively. If $f_x \varepsilon (S : T)_x$, the jets of the functions $f \cdot \psi_1, \ldots, f \cdot \psi_m$ at a point y belong to S_y for every point y sufficiently near to x. Since $\psi_1, \ldots, \psi_m$ generates T_y, we have $f_y \varepsilon (S : T)_y$ which implies that S : T is a sheaf.

Next we prove the coherence. We may assume that none of the generators ϕ_j and ψ_i are identically zero. We proceed by induction on m. If $m = 1$, this is evident, because the functions $\phi_1/\psi_1, \ldots, \phi_\ell/\psi_1$ are the generator of $S : T$ in U. Next we assume that the assertion has been proved for $m-1$. If we denote by T^* the sheaf on U generated by $\psi_1, \ldots, \psi_{m-1}$, we have local generators $\chi_1, \ldots, \chi_p$ of $S : T^*$ in a suitable neighborhood U_1. By Oka's theorem, there exists a finite number of generators

$$(a_1^{(\lambda)}, \ldots, a_p^{(\lambda)}, b_1^{(\lambda)}, \ldots, b_\ell^{(\lambda)}) \qquad (\lambda = 1, \ldots, q)$$

of the sheaf of relations R among the functions $\chi_1\psi_m, \ldots, \chi_p\psi_m$, $-\phi_1, \ldots, -\phi_\ell$ in a neighborhood U_2. Though these functions are meromorphic in U_2, we can apply Oka's theorem in the holomorphic case by multiplying by a common multiple of the denominators. We shall show that the functions $\omega_\lambda = \sum_{k=1}^{p} a_k^{(\lambda)}\chi_k$ are the local generator of $S : T$ in U_2. It is evident that ω_λ belongs to $S : T$, because

$$\chi_k \cdot T^* \subset S \quad \text{and} \quad \sum_{k=1}^{p} a_k^{(\lambda)}\chi_k\psi_m = \sum_{j=1}^{\ell} b_j^{(\lambda)}\phi_j \in S.$$

On the other hand, if we have a transporter $f \in S : T$ in U_2, we have the expressions

$$f = \sum_{k=1}^{p} a_k \chi_k \quad \text{and} \quad f\psi_m = \sum_{k=1}^{p} a_k \chi_k \psi_m = \sum_{j=1}^{\ell} b_j \phi_j$$

where a_k and b_j are holomorphic in U_2. Since $(a_1, \ldots, a_p, b_1, \ldots, b_\ell)$ belongs to the sheaf of relations R, we have the expression $a_k = \sum_{\lambda=1}^{q} \alpha_\lambda a_k^{(\lambda)}$, which implies

$$f = \sum_{\lambda} \sum_{k} \alpha_\lambda a_k^{(\lambda)} \chi_k = \sum_{\lambda} \alpha_\lambda \omega_\lambda .$$

Hence the first part of our assertion is proved.

When S is locally principal (i.e., $\ell = 1$), we may assume that $p = 1$, and then we have $q = 1$, which implies that $S : T$ is locally principal. Of course the latter can be proved directly.

REMARK 1. When T is locally principal, this theorem was given by Oka [5], and the fact that $O \cap (S : T)$ is coherent is remarked in Serre [6]. However, I think our Theorem 1 is not trivial, since K is not coherent.

REMARK 2. If T is not coherent, the collection $S : T$ defined by (2) is not a sheaf in general. For example, take $X = C^2(z_1, z_2)$, $S = O$, and put

$T_x = O_x$ for the point $x = (z_1, z_2)$ if $z_1 \neq o$

and T_x = [the ideal generated by $(z_1)^n$] if $z_1 = o$

and $n \leq |z_2| < n+1$. Then the inverse T^{-1} is not a sheaf of ideals in K.

4. COHERENT SHEAVES ON A STEIN MANIFOLD

Now we shall state and prove two fundamental theorems on coherent sheaves S on a Stein manifold X in the case of meromorphic functions. (For the definition of Stein manifolds, see Cartan [1], IX.)

THEOREM A. The set of cross-sections $\Gamma(S ; X)$ generates the fiber S_x as O_x-module at every point x of X.

THEOREM B. When $q \geq 1$, the cohomology group $H^q(S ; X) = o$.

To prove these, we shall show

THEOREM 2. Every coherent sheaf S of ideals on a Stein manifold X has an integralizator in the whole space X.

If this is proved, multiplying by the integralizator of the sheaf S, it is easy to reduce Theorems A and B to the known case of holomorphic functions (Cartan [1], XIX).

PROOF OF THEOREM 2. At every point x, we have a local generator $\psi^{(x)}$ of the inverse-sheaf S^{-1} in a neighborhood U (Theorem 1). By the theorem of unique factorization, we can write $\psi^{(x)} = p^{(x)}/q^{(x)}$, where $p^{(x)}$ and $q^{(x)}$ are holomorphic and coprime at every point in U (Siegel [7], p. 9). The systems $\{\psi^{(x)}\}$ and $\{p^{(x)}\}$ give multiplicative Cousin distributions in X (Siegel [7], p. 16). Denoting by P_x the ideal in O_x generated by $p^{(x)}$, the collection

$$P = \{(f,x) \mid f \varepsilon P_x, \ x \varepsilon X\}$$

is a locally principal subsheaf of O. Since Theorem A is already proved for coherent subsheaf in O, we have a non-zero cross-section $f \varepsilon \Gamma(P;X)$. At every point $x \varepsilon X$, the jet f_x is a multiple of $(p^{(x)})_x = (q^{(x)})_x \cdot (\psi^{(x)})_x$, where the jet $(\psi^{(x)})_x$ is a generator of the ideal $(S_x)^{-1}$. This means that the function f is an integralizator of the sheaf S in X.

Now, it is well-known that the fundamental Theorems A and B (in the holomorphic case) imply various interesting properties on Stein manifolds (see for example, Cartan [1], [2], Hitotumatu [3], Serre [6]). We are ready to generalize them to the case of meromorphic functions in a similar manner. However, we shall note only a few results.

Hereafter, we always assume that X is a Stein manifold.

THEOREM 3. Let A be an ideal in K_X. The sheaf associated to the inverse-sheaf A^{-1} is equal to the inverse-sheaf of the sheaf associated to A, i.e., if we define F(A) by (1), we have

$$F(A^{-1}) = F(A)^{-1}.$$

PROOF. We first prove the equality

$$\Gamma(F(A)^{-1}; X) = A^{-1} \text{ in } K. \tag{3}$$

For simplicity, we denote by B the left hand side of (3). From relation (2), we have

$$(F(A)^{-1})_x = (F(A))_x^{-1} = ([A]_x)^{-1},$$

which obviously contains the ideal $[A^{-1}]_x = (F(A^{-1}))_x$. Hence we have $A^{-1} \subset B$. On the other hand, every element $\beta \in B$ integralizes A, because $\beta_x \in ([A]_x)^{-1}$ at every point $x \in X$. Hence

$B \subset A^{-1}$, and so the equality (3) is proved. Since the sheaf $F(A)^{-1}$ is coherent by Theorem 1, the ideal B generates the fiber $(F(A)^{-1})_x$ $= ([A]_x)^{-1}$ at every point x, due to Theorem A. Therefore we have

$$(4) \qquad [A^{-1}]_x = [B]_x = ([A]_x)^{-1}, \quad \text{i.e.,} \quad (F(A^{-1}))_x = (F(A)^{-1})_x,$$

where the operation of the identification of the fibers in (4) is compatible with the structures of sheaves. Thus Theorem 3 is proved.

THEOREM 4. (Additive Cousin problem for ideals.) Let A be a given ideal in K_X and suppose that there exist an open covering $X = \bigcup_\nu U_\nu$ and functions ψ_ν satisfying:

1° ψ_ν is meromorphic in U_ν,

2° unless $U_\mu \cap U_\nu$ is empty, the jet $(\psi_\mu - \psi_\nu)_x$ belongs to the ideal $[A]_x$ at every point $x \in U_\mu \cap U_\nu$.

Then there exists a function ψ meromorphic in X such that the jet $(\psi - \psi_\nu)_x$ belongs to $[A]_x$ at every point x in U_ν.

Multiplying by an integralizator of the ideal A, this is reduced to the case when the ideal A is in O_X. However, this is not trivial, because the functions ψ_ν are not necessarily holomorphic even if A is in O_X. Theorem 4 was given by Hitotumatu and Kôta [4], but here I give a simpler proof based on the notion of sheaves.

PROOF. We take the coherent sheaf F(A) associated to the ideal A. If we use the terminology of sheaves, the assertion of Theorem 4 is equivalent to the fact that the canonical mapping

$$(5) \qquad K_X = \Gamma(K\,;\,X) \longrightarrow \Gamma(K/F(A)\,;\,X)$$

is onto. Now, from an exact sequence

$$o \longrightarrow F(A) \longrightarrow K \longrightarrow K/F(A) \longrightarrow o$$

(o means the zero-sheaf),

we have the exact sequence

$$(6) \qquad o \longrightarrow \Gamma(F(A)\,;\,X) \longrightarrow K_X \longrightarrow \Gamma(K/F(A)\,;\,X) \longrightarrow H^1(F(A)\,;\,X) \longrightarrow \dots ,$$

where the fifth term of (6) vanishes by Theorem B. This means that (5) is onto.

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POSITIVE CURRENTS AND ANALYTIC SETS

P. Lelong

INTRODUCTION

In this paper we give the definition and the properties of a class of currents (in sense of G. de Rham) - the positive currents - and applications to the integration of forms on analytic sets. Conditions of sign arise in a natural way in the theory of analytic functions and plurisubharmonic functions and are often significant (cf. for example pseudo-convexity). The class of positive currents will be useful for such properties. Further, the class of positive and closed currents seems suitable for studying the metric properties of analytic sets. We give here an application to our existence theorem for the operator $t(\phi)$ of integration of a form ϕ on an analytic set, the main result being: this operator is a sum of closed and positive currents (see [9] and [10]).

1. DEFINITION. A current t is called a positive current of degree p (and complex dimension $n-p$) on a complex manifold D of complex dimension n if:

1) t is a homogeneous current of degree (p, p); $t(\phi) = 0$ for the homogeneous forms ϕ which are not of degree $(n-p, n-p)$ in dz_i, $d\bar{z}_j$.

2) For every system of linear forms $g_1, \ldots, g_{n-p}$ with C^∞ coefficients $g_s = \Sigma_k a_s^k(z_i, \bar{z}_j) dz_k$, and with conjugates $\bar{g}_s = \Sigma \bar{a}_s^k(z_i, \bar{z}_j) d\bar{z}_k$, the current of maximal degree

$$t \wedge (\frac{i}{2} g_1 \wedge \bar{g}_1) \wedge (\frac{i}{2} g_2 \wedge \bar{g}_2) \wedge \ldots \wedge (\frac{i}{2} g_{n-p} \wedge \bar{g}_{n-p})$$

is a distribution (in the sense of L. Schwartz) which is positive; this distribution is a positive measure. (In this paper distributions and measures are operators on functions, or currents of maximal degree.)

We denote by (T_p^+) the class of the positive currents of degree p. A form ϕ is called <u>a positive form of degree</u> p, if $\phi \in (T_p^+)$ and ϕ has coefficients of class C^0; (F_p^+) is the class of positive forms of degree p.

<u>Examples</u>. 1) For $p = n$, (T_n^+) is the class of the positive measures. For $p = 0$, (F_0^+) is the class of positive functions.

2) An important class is obtained for $p = 1$. If we write the current

$$t = \frac{i}{2} \sum_{p,q} t_{p\bar{q}} \, dz_p \wedge d\bar{z}_q,$$

t is positive if and only if the distribution

$$(t, \vec{\lambda}) = T(\vec{\lambda}) = (\sum_{p,q} t_{p\bar{q}} \lambda_p \overline{\lambda}_q) \, d\tau_{2n}$$

is a positive measure for every complex vector $\vec{\lambda}$. A form

$$\phi = \frac{i}{2} \sum_{p,q} \phi_{p\bar{q}} \, dz_p \wedge d\overline{z}_q$$

is positive in D if and only if the function

$$F(\vec{\lambda}) = \sum_{p,q} \phi_{p\bar{q}}(z_i, \overline{z}_j) \lambda_p \overline{\lambda}_q$$

is positive (or zero) in each point of D and for every complex vector $\vec{\lambda}$.

It is convenient to consider exterior forms and currents instead of hermitian forms. We denote by d_z the differential with respect to the z_i.

This case has connections with the following results:

THEOREM 1. A function $V(z_i, \overline{z}_j)$, defined in a complex manifold D, is a plurisubharmonic function if and only if

a) V is real valued, $-\infty \leq V < +\infty$; V is locally summable in D.

b) The current $2i d_z d_{\bar{z}} V = 2i \sum \frac{\partial^2 V}{\partial z_p \partial \bar{z}_q} dz_p \wedge d\bar{z}_q$ is a positive current.

c) $V(P) = V_m(P)$ in each point $P \varepsilon D$; $V_m(P)$ is the maximum of V at the point P, when sets of measure zero are neglected.

These properties are characteristic.

THEOREM 2. If t is a given current in D, with properties:

a) t is positive of degree 1

b) t is closed

then, to each point $M \varepsilon D$ corresponds a neighborhood ω_M in D, and a function V_M plurisubharmonic in ω_M such that $t = 2i d_z d_{\bar{z}} V_M$. The positive and closed currents are therefore the currents which are locally associated with the plurisubharmonic functions.

To an analytic set W^1 defined by data of Cousin of zeros in D, there corresponds a positive and closed current of degree 1 which is defined in each neighborhood ω_M of $M \varepsilon D$ by $t = 2i d_z d_{\bar{z}} \log |f_M|$, $f_M(z_1, \ldots, z_n) = 0$ being the data of zeros in

ω_M. The current of integration on W^1 is then $\frac{1}{2\pi}t$; this result was first given by Poincaré, in an elementary form and with an insignificant error in the coefficient; it was given in 1945 in [4], in connection with the study of plurisubharmonic functions, and in 1950 in [5], where the metric properties of the analytic sets $f = 0$ are investigated. It is given in the language of currents in the lectures of G. de Rham and Kodaira [2], (1950). It can be considered as a particular case (relative to the data of Cousin) of the general theorem that we have stated in the Introduction.

Further examples of positive currents and forms of degree p are obtained by the multiplication of the positive forms and currents.

2. We give now properties of the classes (T_p^+), (F_p^+). We denote by A^p a complex p-vector defined by means of the equations

$$z_i = z_i^o + \Sigma_k a_i^k u_k, \qquad 1 \leq k \leq p \tag{1}$$

giving an analytic plane of dimension p; we suppose (1) to be a unitary representation and consider the fundamental form

$$d\tau_{2p} = g(A^p) = (\tfrac{i}{2})^p (-1)^{\frac{p(p-1)}{2}} du_1 \wedge d\bar{u}_1 \wedge \dots du_p \wedge d\bar{u}_p$$

The adjoint form ${}^*g(A^p)$ is the fundamental form of the complex n-p vector B^{n-p} orthogonal to A^p. To a homogeneous current t of degree (p,p) we define a corresponding distribution

$$(2) \qquad (t, A^p) \longrightarrow T(A^p) = t \wedge {}^*g(A^p)$$

which is explicitly given by

$$(3) \qquad T(A^p) = k_p \sum_{(i),(j)} t_{(i)(\overline{j})}\, a_{(i)} \overline{a}_{(j)}, \quad k_p = \left(\frac{i}{2}\right)^p (-1)^{\frac{p(p-1)}{2}}$$

$$(i) = (i_1 < i_2 \dots < i_p); \quad (j) = (j_1 < j_2 \dots < j_p)$$

$$a_{(i)} = |a_i^k|^{k=1,\dots,p}_{i=i_1,\dots,i_p}$$ are the parameters of A^p

(of coordinates a_i^k).

We obtain the following property of positive currents:

THEOREM 3. A homogeneous (p,p) current t is positive if and only if the distribution $T(A^p)$ is a positive measure for every complex p-vector A^p.

Originally we introduced (cf. [7]) the positive currents by using this property as a definition.

The definition of the classes (T_p^+) does not make use of the connection between the variables z_i, z_j of D and the differentials

dz_i, $d\bar{z}_j$. It is therefore possible to consider positive currents as relative to an exterior algebra with basis $(g_i, \bar{g}_j)$. We obtain the same classes T_p^+ if the g_i are linear forms in the dz_i, with continuous coefficients in D. This fact yields the following result:

THEOREM 4. The image $t' = Ft$ of a positive current by a locally one-to-one analytic transformation $z' = Fz$, is a positive current.

We recall that the image $t'(\phi)$ is, by definition, $t'(\phi) = t[F^*\phi]$ where $F^*\phi$ is obtained from ϕ by substituting the z_i and dz_i for the z'_i and dz'_i.

From the equations (3), we obtain also:

THEOREM 5. A positive current is continuous of order zero: the distributions $T_{(i)(\bar{j})} = k_p^{-1} t_{(i)(\bar{j})} d\tau_{2n}$ are complex measures; $T_{(i)(\bar{j})}$ is the conjugate of $T_{(j)(\bar{i})}$; $T_{(i)(\bar{i})}$ are positive measures.

We consider now a system $|\Sigma| = \{A_1^p \ldots A_N^p\}$ of N complex p-vectors of coordinates $(a_{i,s}^k)$, $N = (C_n^p)^2$. Such a system (Σ) is called regular if the system of linear equations

(4) $$T(A_s^p) = k_p \sum_{(i),(j)} t_{(i)(\overline{j})} \, a_{(i),s} \, \overline{a}_{(j),s}$$

is a regular one. Using such a system (Σ), from the equations (4), the distributions $T_{(i)(\overline{j})}$ appear as sums of positive measures $T(A_s^p)$ with complex numerical coefficients. The determinant

$$\Delta = \left| a_{(i),s} \, \overline{a}_{(j),s} \right|^{(i)\times(j)}_{s=1,\ldots,N}$$

vanishes in the space $C^\nu = R^{2\nu}$ $(\nu = N\cdot p\cdot n)$ of the ν complex coordinates $(a^k_{j,s} = a'^k_{j,s} + ia''^k_{j,s})$ on an algebraic set given by an equation $P(a'^k_{j,s}, a''^k_{j,s}) = 0$, P being a polynomial with real coefficients. As a consequence, we obtain:

PROPOSITION 1. Let $(a^k_i)_o$ be the coordinates of a complex p-vector A^p_o, with a unitary representation; then if $\varepsilon > 0$ is given, there exists a regular system (Σ) of p-vectors A^p_s $(1 \leq s \leq N)$, of coordinates $a^k_{i,s}$ (in unitary representation), satisfying

$$|a^k_{i,s} - (a^k_i)_o| < \varepsilon$$

The norm of a positive current t in a domain D is defined by $|t|_D = \sup |t(\phi)|$ for the forms ϕ of the space $\mathcal{B}^o(D)$ of the forms (C^o) with compact support $K(\phi) \subset D$, and $|\phi| \leq 1$, where $|\phi|$ is the maximum of the modulus of the coefficients. We have

PROPOSITION 2. To every regular system (Σ), there corresponds a coefficient $C(\Sigma) > 0$, such that

$$|t|_D \leq C(\Sigma) \sup_s |T(A_s^p)|_D, \; A_s^p \in (\Sigma).$$

THEOREM 6 (Multiplication). If $t \in (T_p^+)$, and $\phi \in (F_1^+)$, then $t \wedge \phi \in (T_{p+1}^+)$. Conversely, if $t \in (T_1^+)$, and $\phi \in (F_p^+)$, $t \wedge \phi \in (T_{p+1}^+)$.

We have also the following theorem of division:

THEOREM 7 (Division). If $t \in (T_p^+)$ and $\phi \in (F_1^+)$, if $t \wedge \phi = 0$, if $\phi^q \neq 0$ and $\phi^{q+1} = 0$ in D, then we have $t = t_1 \wedge \phi^q$, with $t_1 \in (T_{p-q}^+)$ $(t_1 = 0,\ t = 0$, if $p < q)$.

3. INTEGRATION ON AN ANALYTIC SET

We consider an analytic set A defined in D and of complex dimension p in each point $M \in A$. We denote by A^* the set of the ordinary points of A, by $A' = A - A^*$ the closed set of the non-ordinary points of A. The current $t_o(\phi) = \int_{A^*} \phi$ is defined for the forms ϕ of $\mathcal{D}^o(D - A')$. The integration $t(\phi) = \int_A \phi$ will be a continuation of the current $t_o(\phi)$, on the forms of $\mathcal{D}^o(D)$. The current $t_o(\phi)$ has the following properties:

THEOREM 8. a) t_o is positive: $t_o \in (T^+_{n-p})$,

b) t_o is closed,

c) to every domain G which is compactly contained in D ($G \subset\subset D$), there corresponds a finite number k(G) such that for every sphere $B \subset G$ of radius r, the following majorization holds:

(5) $$|t_o|_B \leq k(G) r^{2p}.$$

The norm in (5) is calculated on the ϕ of $\mathcal{D}^o(D - A')$. The property (5) is a significant step in the proof of the existence theorem and is based on the relations between the norm of a positive current and the norm of the measures $T(A^{n-p})$. A second step of the proof is a theorem on the continuation of a closed current, continuous of order zero, defined in D - E, [more precisely: on the forms of $\mathcal{D}^o(D - E)$] by a current of the same kind. For our purpose it is sufficient to consider the case: D is a domain in the space $R^m(x_1, \ldots, x_m$

and E is a subspace $R^s : x_{s+1} = x_{s+2} = \dots = x_m = 0$. We denote by $\alpha_1(x_{s+1}, \dots, x_m)$ a kernel of class C^∞, $0 \leq \alpha_1 \leq 1$, depending only on the distance $\delta(x)$ of (x) from the set $E = R^s$ with $\alpha_1(x) = 1$ for $0 \leq \delta(x) \leq \frac{1}{2}$, $\alpha_1 = 0$ for $\delta \geq 1$. We consider the kernels

$$\alpha_r(x_j) = \alpha_1\left(\frac{x_j}{r}\right), \qquad s+1 \leq j \leq m.$$

Then we have:

$$|d\alpha_r| < Ar^{-1}$$

for the form $d\alpha_r$. A continuation of a current t_o continuous of order zero in D - E by such a current in D exists if and only if $|t_o|_G$ is bounded for every domain $G \subset\subset D$. Then the continuation

$$t = \lim_{r=0} t_o(1 - \alpha_r)$$

exists; we call such a continuation t the <u>simple extension</u> of t; t is continuous of order zero. A characteristic property of the simple extension is: the norm is not increased by such a continuation.

THEOREM 9. The simple extension t of the closed current t_o is a closed current if and only if the current $\theta_r = t_o \wedge d\alpha_r$ tends to zero on every form ϕ of $\mathcal{D}^o(D)$.

A sufficient condition is therefore: $r^{-1}|t_o|_G^r \longrightarrow 0$, where $|t_o|_G^r$ is the norm of t_o on the open set $G \cap \{x | 0 < \delta(x) < r\}$.

The third step in the proof is to use theorem 9 by the following process. The set $M' \subset M$ of the non-ordinary points of an analytic set of homogeneous complex dimension p, is an analytic set of dimension $\leq$ p-1. If $P \in M'$ is an ordinary point of M', there exists a one-to-one analytic mapping $z' = F(z)$ and a neighbourhood ω_P such that $F(\omega_P \cap M') = F(\omega_P) \cap C^s$, where C^s is a subspace of C^n ($s \leq p-1$). Then in the space of the z'_i, Theorem 9 gives: the simple extension of the image Ft_o of t_o is a closed and positive current. Using the inverse transformation, we get: the current t is defined by the simple extension of t_o in ω_P. The norm is unchanged by this continuation and the majorization given by Theorem 8 remains valid for this continuation. The process can be continued and applied to non-ordinary points of M, and so on, see [10]. We obtain:

THEOREM 10. 1) If M is an analytic set of homogeneous dimension p in D, the current $t_o(\phi) = \int_{M^*} \phi$ of integration on the ordinary points of M, possesses a simple extension $t(\phi)$; this continuation is by definition the operator:

$$t(\phi) = \int_M \phi.$$

2) $t(\phi)$ is a positive current of degree $n-p$; $t(\phi)$ is closed.

THEOREM 11. By a decomposition $M = \Sigma M_s$ of M in analytic sets irreducible in D, we have $t(\phi) = \Sigma_s t_s(\phi)$.

In the general case, for an analytic set M of maximal complex dimension p, we have:

THEOREM 12. The current $t(\phi) = \int_M \phi$ is the sum of p positive and closed currents $t^{(q)}$, $(1 \leq q \leq p)$, of degrees $n-1, \ldots, n-p$. Each $t^{(q)}$ is the simple extension of the integration $t_o^{(q)}$ on the set of ordinary points of complex dimension q in M.

4. POSITIVE AND CLOSED CURRENTS

We consider now the forms:

$$\beta_1 = \frac{i}{2} \Sigma\, dz_j \wedge d\bar{z}_j, \qquad \beta_q = \frac{1}{q!} \beta_1^q$$

$$\alpha = \frac{i}{2} d_z d_{\bar{z}} \log \cdot \Sigma z_j \bar{z}_j$$

and the forms α^q, which are positive forms defined for $|z| \neq 0$.

For $t \in (T^+_{n-p})$ in D, the distributions

(5) $$\sigma = t \wedge \beta_p$$

(6) $$\nu = \pi^{-p} t \wedge \alpha^p$$

are positive measures by theorem 6. Let t be closed. In a sphere $|z| < R$, contained in D, we have $t = d\theta$, because t is closed and homologous to zero. We denote by $\nu(r, R)$ the ν-measure of the domain $r < |z| < R$ and by $\sigma(R)$ the σ-measure of the domain $|z| < R$. Then, by applying Stokes' theorem, we find:

$$\nu(r,\ R) = p!\ \pi^{-p}\left[\frac{\sigma(R)}{R^{2p}} - \frac{\sigma(r)}{r^{2p}}\right] \geq 0.$$

THEOREM 13. If t is a positive and closed current of degree $n-p$, the measure $\sigma(R)$ has the following property:

$\sigma(R)R^{-2p}$ is an increasing function of R, and has a finite limit when R tends to zero; moreover, the positive measure ν is bounded on every compact set contained in D.

If t is the current of integration on an analytic set (M^p) of homogeneous dimension p, $d\sigma$ given by (5) is the area of (M^p); this yields:

THEOREM 14. The 2p-dimensional area of an analytic set (M^p) is the positive measure defined by (5); it is bounded on every domain compactly contained in D.

The fact that the current is closed gives the following more precise result:

THEOREM 15. Let M^p be an analytic set in D, of homogeneous dimension p; let $B(O,R)$ be the sphere of center $O \in M^p$ and radius R, contained in D. The area $\sigma(R)$ of $M^p \cap B(O,R)$ has the property: $\sigma(R)R^{-2p}$ is an increasing function of R; it has a limit when R tends to zero.

The geometrical interpretation of the property of ν is the following: in the projective complex space P^{n-1} of the vector OQ, the projective domain described by OQ has a finite projective area ν when Q $(Q \neq O)$, describes a compact subset of M^p.

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CAUCHY'S PROBLEM IN THE LARGE FOR LINEAR ANALYTIC PARTIAL DIFFERENTIAL EQUATIONS

Jean Leray

1. THE SINGULARITIES OF THE SOLUTION

The fundamental property of linear analytic ordinary differential equations is the well-known Cauchy theorem: the singularities of their solutions are singularities of the equation. That theorem can be extended as follows to linear analytic partial differential equations:

THEOREM 1. The singularities of the solutions of the linear analytic Cauchy problem belong to the characteristics issued from the singularities of the data or tangent to the variety S carrying Cauchy's data.

Let us state the assumptions to be made and the precise definition of those characteristics; for the sake of simplicity let us restrict the problem to a sufficiently small neighborhood of S.

Let X be an analytic ℓ-dimensional variety; we denote by x a point of X, by $(x_1, \ldots, x_\ell)$ local coordinates, by $D_1, \ldots, D_\ell$ the partial derivations relative to $x_1, \ldots, x_\ell$. Let $a(x, D)$ be a linear analytic differential operator of order m, defined on X; locally

$$a(x,D)\,u(x) = \sum_{i+j+\dots+k\le m} a_{ij\dots k}(x)\, D_1^i D_2^j \dots D_\ell^k\, u(x),$$

the $a_{ij\dots k}(x)$ being analytic regular functions. Let S be in X an analytic regular $(\ell-1)$-dimensional variety, whose local equation is

$$s(x) = 0; \qquad Ds = (D_1 s, \dots, D_\ell s) \neq 0.$$

Let $b(x,D)$ be a first order differential operator such that

$$b(x,0) = 0, \qquad b(x,D)s \neq 0.$$

Let $v(x)$ and $w_j(x)$ $(j = 0,\dots,m-1)$ be analytic regular functions defined respectively on X and on S. Cauchy's problem asks for a function $u(x)$ such that:

$$a(x,D)\,u(x) = v(x);$$

$$u(x) = w_o(x),\ b(x,D)\,u(x) = w_1(x), \dots, [b(x,D)]^{m-1}u(x) = w_{m-1}(x) \quad \text{on } S.$$

Either X and S are complex analytic of complex dimensions ℓ and $\ell-1$, or they are real analytic of real dimensions ℓ

and ℓ-1; but then we have to assume $a(x, D)$ to be hyperbolic and Cauchy's problem to be well-settled: see for instance [1].

Denote by $h(x, p)$ the homogeneous polynomial in

$p = (p_1, \ldots, p_\ell)$

$$h(x,p) = \sum_{i+j+\ldots+k=m} a_{ij\ldots k}(x)\, p_1^i p_2^j \ldots p_\ell^k.$$

A point x of S is said to be characteristic when the first order contact element (x, Ds) of S at x satisfies the characteristic equation $h(x, Ds) = 0$. An $(\ell-1)$-dimensional variety of X is said to be a characteristic when all its contact elements are characteristic. S can have characteristic points, but we assume that S is not a characteristic.

The theory of non-linear first-order differential equations leads to the notion of bicharacteristics: a bicharacteristic of $a(x, D)$ is a first-order contact element $(x(t), p(t))$, function of the numerical parameter t, satisfying the ordinary differential system:

$$x_t = h_p(x,p), \quad p_t = -h_x(x,p), \quad h(x,p) = 0.$$

The main result of that theory is the following theorem and its converse: the bicharacteristic issued from a contact element of a characteristic belongs to that characteristic.

We are now able to give the precise definition of the characteristic K tangent to the variety S: it is the set of all the bicharacteristics issued from the characteristic contact elements of S. Where K is an $(\ell-1)$-dimensional variety, $k(x) = 0$, there it satisfies the characteristic equation

$$h(x, Dk) = 0 \qquad \text{for} \quad k(x) = 0.$$

As for the characteristic issued from the singularities of the data, it is, in the real case, the set of all the bicharacteristics issued from the characteristic contact elements tangent to the set of the singularities of the $v(x)$ and $w_j(x)$. In the complex case, a more elaborate definition has to be used: consider all the characteristic contact elements (x, p) of $(\ell-1)$-dimensional complex analytic varieties, such that the set of the singularities of the $v(x)$ and $w_j(x)$ is tangent to the $(2\ell-1)$-dimensional real-linear variety

$$\text{Re}\,(p \cdot dx) = 0;$$

(Re: real part of ...; $p \cdot dx$: scalar product; dx: vector from x to a point of that real-linear variety); the set of all the bicharacteristics issued from those contact elements (x, p) is the characteristic issued from the singularities of the data. Where it is a $(2\ell-1)$-real-dimensional variety, there its tangent real-linear variety $\mathrm{Re}(p\cdot dx) = 0$ satisfies the characteristic equation $h(x,p) = 0$; where it is an $(\ell-1)$- complex-dimensional variety, there its tangent variety $p\cdot dx = 0$ satisfies the same equation.

Now, in the neighborhood of S, Theorem 1 has a precise meaning.

Its proof uses a linear transformation, defined by integrals, which yields the solution of all the Cauchy problems for $a(x,D)$, by means of the solution of the special one:

$$a(x,D)\, u(x,p') = 1;$$

$$u(x) \text{ vanishes } m \text{ times for } p'\cdot x = 0.$$

We denote

$$p'\cdot x = p_0 + p_1x_1 + \dots + p_\ell x_\ell.$$

In the real case, that transformation is an extension of the inverse Laplace transformation. In the complex case, it is an extension of the convolution by an inverse Laplace transformation: see [2]. In both cases, for using that transformation, we have to know the behavior of $u(x, p')$ near the variety $p' \cdot x = 0$, that is the behavior of the solution of the linear analytic Cauchy problem on the characteristic K tangent to S.

From now on we shall talk about that behavior; its study has no analogue in the theory of ordinary differential equations; its simplicity is a striking feature of linear partial differential equations.

2. THE REGULARITY OF THE SOLUTION OF CAUCHY'S PROBLEM ON A CHARACTERISTIC NEIGHBORHOOD OF THE VARIETY S CARRYING CAUCHY'S DATA

What we call a neighborhood of S over X is composed of

1) a complex analytic space Y; $\dim Y = \dim X = \ell$;

2) a complex analytic variety V in Y; $\dim V = \ell - 1$;

3) an analytic mapping $x(y)$ of Y into X; it has to be a homeomorphism of V onto S; V shall not be enclosed in the variety W of Y where the Jacobian $\frac{D(x)}{D(y)}$ vanishes.

We identify V with S; we say that Y is branched over $x(W)$. Let $u(y)$ be a function defined on Y; its projection onto X is the function $u(x)$ resulting from the elimination of y from $u(y)$ and $x(y)$; generally $u(x)$ is a multivalued function.

Let Y' be another neighborhood of S over X; there is at most one analytic homeomorphism between Y and Y' such that $x(y) = x(y')$; it is indeed obvious near $S - W - W'$; if such a homeomorphism exists, then let it identify Y with Y' and, of course, $x(y)$ with $x(y')$, W with W'.

In order to define now the characteristic neighborhoods of S, denote by $g(x,p)$ any function with the following properties:

1) $g(x,p)$ is homogeneous of degree 1 in p;
2) $\dfrac{g(x,p)}{h(x,p)}$ is regular and non-vanishing, for all contact elements (x,p) of S.

For instance one can choose $g(x,p) = h(x,p)[b(x,p)]^{1-m}$.

Let us call "the differential equation of the characteristic projection" the following ordinary differential system

$$x_t = g_p(x,p), \qquad p_t = -g_x(x,p),$$

where t is the independent variable, x and p the unknown functions of t; it has the following properties:

1) it has the first integral $g(x, p)$;

2) it has the absolute integral invariant $\int p \cdot dx - g\, dt$;

3) its solutions satisfying $g = 0$ become, by a change of parameter t, solutions of the equation defining the bicharacteristics.

Denote by $x(t, z)$, $p(t, z)$ the solution of the preceding equation issued from the contact element $(z, Ds(z))$ of S; a change of the local equation $s(x) = 0$ of S does not change $x(t, z)$, which is called the characteristic projection and which is regular analytic for:

$$z \in S, \qquad |t| \text{ small enough.}$$

Let us denote by y any couple (t, z) satisfying that condition, by Y the set of all such couples. Y is a complex analytic space, which $x(y)$ maps into X, mapping S onto itself identically; $\frac{D(x)}{D(y)} = 0$ on S only if y is a characteristic point of S. Thus Y is a neighborhood of S over X. Such a neighborhood of S over X is said to be <u>a characteristic neighborhood of</u> S.

Now we can state <u>the main theorem</u>:

THEOREM 2. The solution $u(x)$ of Cauchy's problem is the projection on X of a function $u(y)$ analytic and regular on a characteristic neighborhood of S. That function $u(y)$ depends linearly on the data v and w_j. That neighborhood depends only on X, S and h.

The following theorem justifies our terminology:

THEOREM 3. The variety W of Y where $\frac{D(x)}{D(y)} = 0$ is the set of the points $y = (t, z)$ of Y such that z is a characteristic point of S. Let us draw on W the fibers:

$$t \text{ arbitrary}, \quad z \text{ fixed};$$

the projections of those fibers are the bicharacteristics tangent to S. Thus the projection $x(W)$ of W is the characteristic K tangent to S.

The following theorem shows that the main theorem is not ambiguous:

THEOREM 4. Let Y and Y' be two characteristic neighborhoods of S; there is another one Y'' such that

$$Y'' \subset Y, \qquad Y'' \subset Y'.$$

Each fiber of W'' belongs to a fiber of W and to a fiber of W'.

It can happen that the projection of a characteristic neighborhood Y of S is not a neighborhood of S, that the projection $u(x)$ of a regular analytic function $u(y)$ has an infinite number of branches; but that happens only under very special conditions which we shall now state.

A point x of S is said to be <u>exceptional</u> when <u>the characteristic conoid of vertex</u> x <u>touches</u> S <u>along a curve containing</u> x. More precisely: the point x of S is exceptional if and only if it has a contact element $(x, p(t))$, an analytic function of t, such that S contains the contact element with parameter t of the bicharacteristic issued from $(x, p(t))$.

A point is <u>ordinary</u> if it is not exceptional. For such a point:

THEOREM 5. Let us replace X by a sufficiently small neighborhood of an ordinary point of S. Then:

1) K is an analytic set, which can be defined by a unique equation: $k(x) = 0$.

2) Each point of $X - K$ is the projection of the same finite positive number of points of Y.

3) If $u(y)$ is regular analytic on Y, then its projection $u(x)$ is an algebraic function of regular analytic functions of x.

The simplest ordinary points are of the following kinds:

1) x is a non-characteristic point of X, then K is empty; the projection of Y into X is a homeomorphism; $u(x)$ is regular.

2) x is a characteristic point of S, where the bicharacteristic direction $h_p(x, Ds)$ does not belong to the variety whose points are the characteristic points of S, then: K is regular and has a contact of order 1 with S; the projection of Y into X has two inverses, which become equal on K and only there; $u(x)$ is a regular function of x and $\sqrt{k(x)}$.

The proof of those theorems begins by the study of the ordinary points of those two kinds; the first kind has been studied by Cauchy, Kowalewski, Schauder, Petrowsky; they used majorant functions;

the study of the second kind requires some other processes; general properties of analytic functions and integral invariants are used.

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A COMPLEX FROBENIUS THEOREM

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1. It was recently shown by Newlander and Nirenberg [3] that a sufficiently differentiable almost complex manifold satisfying the so-called "complete integrability conditions" is in fact a complex analytic manifold. This result can be formulated as a special "complex Frobenius theorem"; in fact, if the almost complex structure is real analytic the result follows easily from the theorem of Frobenius.

We recall briefly this formulation: an almost complex manifold is a 2n-dimensional real manifold on which there is given a real tensor field h^{μ}_{λ} satisfying $h^{\mu}_{\lambda} h^{\lambda}_{\nu} = -\delta^{\mu}_{\nu}$; here summation convention is employed. The problem of reducing this to a complex analytic structure is that of finding new local coordinates $x = (x^1, \ldots, x^{2n})$ so that operating with the tensor h^{μ}_{λ} is equivalent to transforming the form $dx^a + idx^{a+n}$ into $i(dx^a + idx^{a+n})$, $a = 1, \ldots, n$, i.e. so that in the new coordinate system we have $h^{a}_{a+n} = 1$, $h^{a+n}_{a} = -1$, $a = 1, \ldots, n$, with $h^{\mu}_{\lambda} = 0$ otherwise. The complete integrability conditions are easily seen to be necessary: we may suppose that the h^{λ}_{μ} have the special values above at some particular point, in a neighborhood of which we have coordinates

$y = (y^1, \ldots, y^{2n})$. Then $dx^a + idx^{a+n} = (x^a_\mu + ix^{a+n}_\mu)dy^\mu$ and $i(dx^a + idx^{a+n}) = (x^a_\mu + ix^{a+n}_\mu)h^\mu_\lambda dy^\lambda$, $a = 1, \ldots, n$; here $x^\lambda_\mu = \partial x^\lambda / \partial y^\mu$. One then verifies easily that the linear space of forms given by linear combinations of the $dx^a + idx^{a+n}$, $a = 1, \ldots, n$, with complex coefficients is equivalent to the space of forms Ω spanned by the forms $(h^\mu_\lambda + i\delta^\mu_\lambda)dy^\lambda$, $\mu = 1, \ldots, 2n$, of which the first n are independent. As a consequence we have the necessary condition: the exterior differential of any form in Ω may be expressed as a sum of exterior products of forms of Ω with first order forms. This may be expressed as

(1) $\quad d\Omega \subset$ ideal generated by Ω.

We observe also that $\Omega \cap \overline{\Omega} = 0$, where $\overline{\Omega}$ consists of the complex conjugates of the forms in Ω. In [3] these conditions were shown to be sufficient for the equivalence of the almost complex structure to a complex analytic structure, under sufficient differentiability assumptions on the h^μ_λ.

The Frobenius theorem, we recall (see for instance [2]), asserts that if $\Omega = \Omega(y)$ is a K-dimensional subspace of the linear space of real first order differential forms at any point $y = (y^1, \ldots, y^N)$ in a neighborhood in an N-dimensional real manifold (varying sufficiently smoothly with y) then a necessary and sufficient condition that we may find new local coordinates x so that $\Omega(y)$ is spanned by $dx^1, \ldots, dx^K$ is that

(2) $d\Omega \subset$ ideal generated by Ω.

Thus the result above may be considered as a "complex Frobenius theorem".

Here we prove a general complex Frobenius theorem containing the above as a special case. The general theorem is, however, easily derived from this special case with the aid of the real Frobenius theorem. Before formulating the theorem we state a result equivalent to the real Frobenius theorem above:

THEOREM A: Suppose, in the above, that for $j = 1, \ldots, K$, dy^j = linear combination of $dy^{K+1}, \ldots, dy^N$ (mod Ω), with coefficients that are functions of all the y^k. Then (2) is necessary and sufficient that the system of differential equations $\Omega = 0$ have a unique solution $y^j(y^{K+1}, \ldots, y^N)$, $j = 1, \ldots, K$, having arbitrary prescribed initial values $y^j(0, \ldots, 0)$ (see for example [2]).

Since our complex Frobenius theorem is local we shall assume that we are operating in a neighborhood of the origin of N-space $y = (y^1, \ldots, y^N)$. For convenience we shall also suppose that the coefficients of the forms occurring are of class C^∞. $\overline{\Omega}$ will denote the space of forms which are complex conjugates of the forms of a space Ω.

THEOREM 1: Let $\Omega = \Omega(y)$ be a K-dimensional subspace of the space of complex-valued forms defined at every point y (and varying in a C^∞ way) in a neighborhood of the origin. Set $\Lambda = \Omega \cap \overline{\Omega}$ and assume that K' = dimension of Λ is constant and that Λ has a basis with C^∞ coefficients (we note that necessarily $K' \geq 2K - N$); set $L = K - K'$. Necessary and sufficient for the existence of new local coordinates x so that Ω is equivalent to the space spanned by

$$dx^a + i dx^{a+L}, \quad dx^\sigma, \quad a = 1, \ldots, L; \ \sigma = N - K' + 1, \ldots, N,$$

are the conditions

(3) $\quad d\Omega \subset$ ideal generated by Ω

(4) $\quad d\Lambda \subset$ ideal generated by Λ.

We observe that if $\tilde{x}$ is another coordinate system with Ω equivalent to the space spanned by $d\tilde{x}^a + id\tilde{x}^{a+L}$, $d\tilde{x}^\sigma$, $a \leq L$, $\sigma > N - K'$, then the coordinates $\tilde{x}^\sigma$ are functions of the x^τ alone. $\sigma, \tau > N - K'$, and for $a \leq L$ the coordinates $\tilde{x}^a + i\tilde{x}^{a+L}$ are holomorphic functions of the $x^b + ix^{b+L}$, $b = 1, \ldots, L$, and depend in addition only on the variables x^σ, $\sigma > N - K'$.

The necessity of (3), (4) is immediately verified. We shall describe only the proof of sufficiency.

If $\Omega = \Lambda$, so that $L = 0$ and any form in Ω equals a real form in Ω plus i times another, we see that the theorem is simply the real Frobenius theorem. If N is even, $K = N/2$ and $K' = 0$ the theorem is simply the result described earlier and proved in [3].

It is easily seen that Theorem 1 may be restated as a result concerning first order differential operators with complex coefficients: $\sum a^j \frac{\partial}{\partial y^j}$.

THEOREM 1': Let S be an $(N-K)$-dimensional linear space of first order differential operators (spanned by $N - K$ such operators with C^∞ coefficients) in a neighborhood of the origin in N-space. Let $\bar{S}$ denote the space of operators obtained by replacing the coefficients in the operators of S by their complex conjugates, and let $\widetilde{S}$ denote the space of operators spanned by those in S and $\overline{S}$. Assume that the dimension of $\widetilde{S} = N - K + L$ is constant, and that $\widetilde{S}$ can be spanned by operators with C^∞ coefficients. Then necessary and sufficient for the existence of new local coordinates x so that S is equivalent to the space of operators spanned by

$$\frac{\partial}{\partial x^a} + i \frac{\partial}{\partial x^{a+L}}, \quad \frac{\partial}{\partial x^\sigma}, \quad a = 1, \dots, L; \quad \sigma = L+K+1, \dots, N$$

are the conditions:

(3') the commutator of any two operators in S belongs to S,

(4') the commutator of any two operators in $\widetilde{S}$ belongs to $\widetilde{S}$.

A special case of this for $N - K = L = 1$, was announced in [3], p. 393.

Before proving Theorem 1 a few remarks about the case N even, $K = N/2$, $K' = 0$, treated in [3], where the new coordinates x

were obtained by solving a system of integral equations in a neighborhood U of the origin: (a) If a given system of basis elements of Ω depends differentiably (say C^∞) on some parameters then the new coordinates x so constructed are also continuously differentiable in these parameters (also C^∞) provided U is chosen sufficiently small (this was stated at the end of [3]). Indeed it is not difficult to show that the first derivatives of x with respect to the parameters will be as small as desired provided U is sufficiently small.

(b) The new coordinates x constructed in [3] have the following property: if, on a neighborhood, a basis for Ω can be chosen so that a finite number of forms $dy^j + idy^{j+K}$ are basis elements, then for these values of j we have $x^j = y^j$, $x^{j+K} = y^{j+K}$.

In §2 we prove Theorem 1, and in §3 we prove an analogue of Theorem A.

2. PROOF OF THEOREM 1.

2.1. We consider first the case $K' = 0$, i.e. $L = K$, and construct new coordinates x in a neighborhood of the origin so that Ω is the space spanned by

$$dx^a + i dx^{a+L}, \qquad a = 1, \ldots, L.$$

We use summation convention and assume that the indices a, b run from 1 to L, and λ, μ, ν from $2L+1$ to N. By suitably modifying the coordinates y we may suppose that $\Omega(y)$ is spanned by forms $\omega^a(y)$ with $\omega^a(0) = dy^a + i dy^{a+L}$ (mod dy^λ). In a neighborhood of the origin we may therefore express the dy^a, dy^{a+L} as linear combinations of the forms dy^λ, ω^b, $\overline{\omega^b}$.

To construct the new coordinates we shall (following [3]) consider the y coordinates as functions of the x coordinates, with

$$y^\lambda \equiv x^\lambda.$$

We construct the functions $y^a(x)$, $y^{a+L}(x)$ in two steps: (i) Define them on the 2L-dimensional submanifold M: $x^\lambda = 0$ for all $\lambda > 2L$. (ii) Extend the functions so defined to a full neighborhood of $x = 0$.

To carry out step (i) consider the forms ω^a on M: $x^\lambda = y^\lambda = 0$; there, near the origin, the forms ω^a, $\overline{\omega^b}$, $a, b = 1, \ldots, L$ are linearly independent. By (3) we have the situation of [3], i.e. an almost complex structure, and we may therefore choose the y^a, y^{a+L} as functions of x^b, x^{b+L} so that Ω is spanned by the forms $dx^a + i dx^{a+L}$ on $x^\lambda = 0$. We remark that the functions as constructured in [3] are such that $\omega^a = dx^a + idx^{a+L}$ at the origin.

To extend the functions y^a, y^{a+L} to a full neighborhood of $x = 0$ we now make the requirement that for every fixed $x^1, \ldots, x^{2L}$, the functions y^a, y^{a+L} satisfy the differential equations $\Omega = 0$ identically in the x^λ. By a remark above this system of equations may be expressed thus: dy^a, dy^{a+L} are linear combinations of the $dx^\lambda = dy^\lambda$ with coefficients that are functions of all the y^j. Because of (3) and Theorem A on page 3 there exists for every fixed $(x^1, \ldots, x^{2L})$ a unique solution $y^1, \ldots, y^{2L}$ of this system with given initial values for $x^\lambda = 0$, which we take to be the values determined in (i). We have thus defined a change of coordinates $y(x)$ near $x = 0$, which in a sufficiently small neighborhood may be seen to have any given number of derivatives and to be non-singular.

Since the forms of Ω vanish whenever $dx^1 = \ldots = dx^{2L} = 0$, it follows that they are linear combinations of $dx^1, \ldots, dx^{2L}$. It remains still to show that they are linear combinations of the $dx^a + idx^{a+L}$ alone; by (i) this holds on $x^\lambda = 0$. To verify this in general we choose a basis for Ω of the form

$$dx^a + idx^{a+L} + \beta^a_b(dx^b - idx^{b+L}), \quad a = 1, \ldots, L$$

with the $\beta_b^a = 0$ on $x^\lambda = 0$. From (3) it follows directly, however, that $\partial\beta_b^a/\partial x^\lambda = 0$, and we conclude that the β_b^a vanish identically. This completes the proof of the theorem for $K' = 0$.

Remarks: (a) The remark (a) at the end of §1 still holds. (b) The remark (b) at the end of that section also holds. Furthermore, we see from our proof that if the coordinates y are such that we have a basis $\omega^a(y)$ of $\Omega(y)$ with

$$\omega^a(0) = dy^a + idy^{a+L} \text{ (mod the } dy^\lambda)$$

then we may choose $x^\mu = y^\mu$, $\mu = 2L+1, \ldots, N$.

2.2. We prove the general case of the theorem, $K' \neq 0$, by a reduction to the special case 2.1. It is to be shown that new coordinates x may be found so that Ω is spanned by

$$(5) \qquad dx^a + idx^{a+L}, \quad dx^\sigma, \quad a = 1, , , .L; \sigma = N-K'+1, \ldots, N,$$

We shall assume here that σ runs from $N-K'+1$ to N. Since $\Lambda = \overline{\Lambda}$ we may, by virtue of (4), apply the real Frobenius theorem, and conclude that we have new coordinates u so that Λ is equivalent to the space spanned by the du^σ. We may write Ω as a direct sum $\Omega = \Lambda \oplus \Gamma$ where Γ is spanned by forms which do not involve the du^{σ}. Clearly

(6) $$\Gamma \cap \overline{\Gamma} = 0,$$

and, from (3), it follows that if the coordinates u^{σ} are kept fixed then

(7) $$d\Gamma \subset \text{ideal generated by } \Gamma.$$

To prove the general case it suffices to find new coordinates x so that the last K' are unchanged, $x^{\sigma} \equiv u^{\sigma}$, and so that when these coordinates are held fixed the space Γ is spanned by the forms $dx^{a} + idx^{a+L}$, $a = 1, \ldots, L$; for then, in general, Γ, and hence Ω, is spanned by the forms (5). But if we now regard the x^{σ} as parameters, the problem of making Γ equivalent to the space spanned by $dx^{a} + idx^{a+L}$ is, because of (6), simply the special case of the theorem treated in 2.1, with N there replaced by $N - K'$, and (3) there being expressed by (7). Thus such a change of coordinates $(u^{1}, \ldots, u^{N-K'}) \longrightarrow (x^{1}, \ldots, x^{N-K'})$ is possible for every fixed $(u^{N-K'+1}, \ldots, u^{N})$. That this change of coordinates is sufficiently differentiable in the x and that the full change of coordinates

$(u^1, \ldots, u^N) \longrightarrow (x^1, \ldots, x^N)$ is non-singular follows from the remarks (a) at the end of sections 1 and 2.1.

This completes the proof of Theorem 1.

As above, we note that if the given Ω depends differentiably (say C^∞) on some parameters then our new coordinates have continuous derivatives (also C^∞) in these parameters provided we operate in a sufficiently small neighborhood.

In case a given basis for Ω depends analytically on some parameters we may show, for $K' = 0$, the following.

THEOREM 2: Let $\Omega = \Omega(y)$ be a space of complex forms as in Theorem 1, with $K' = 0$; assume that a basis for Ω depends analytically on a finite number of real parameters t and holomorphically on a finite number of complex parameters τ, and that for each fixed t, τ (in some t, τ neighborhood) condition (3) is satisfied. Then we may construct new coordinates x satisfying the conditions in Theorem 1 so that $x^a + ix^{a+L}$ are analytic in t and holomorphic in τ, for $a = 1, \ldots, L$ and the coordinates x^s, $s > 2L$ are independent of t and τ.

Proof: Because of the analyticity in t we may extend the forms to be holomorphic for complex values of t (still keeping $K' = 0$); thus it suffices to consider the case that a basis for Ω is holomorphic in some complex parameters $\tau = (\tau^1, \ldots, \tau^k)$.

We now extend the system of variables by considering the total system $\tilde{y} = (y^1, \ldots, y^N, \operatorname{Re} \tau^1, \operatorname{Re} \tau^2, \ldots, \operatorname{Im} \tau^k)$, and consider an enlarged system $\tilde{\Omega}$ of forms spanned by Ω and $d\tau^1, \ldots, d^\tau k$. Because of (3), and the holomorphic character of Ω, the enlarged system in $\tilde{y}$ satisfies

$$d\tilde{\Omega} \subset \text{ ideal generated by } \tilde{\Omega}.$$

Furthermore $\tilde{\Omega} \cap \overline{\Omega} = 0$. We may therefore apply Theorem 1 and construct new coordinates $\tilde{x} = (x^1, \ldots, x^N, \tilde{x}^1, \ldots, \tilde{x}^{2k})$ with $\tilde{\Omega}$ spanned by

$$dx^a + idx^{a+L}, \quad d\tilde{x}^j + id\tilde{x}^{j+k}, \quad a = 1, \ldots, L; \; j = 1, \ldots, k.$$

By the remark (b) at the end of sections 1 and 2.1 we have

$$\tilde{x}^j + i\tilde{x}^{j+k} = \tau^j .$$

Thus we have $\tilde{\Omega}$ spanned by $dx^a + idx^{a+L}$, $d\tau^j$, $a = 1, \ldots, L$, $j = 1, \ldots, k$. For every fixed τ the new coordinates $x = (x^1, \ldots, x^N)$

are now easily seen to have the properties asserted in the theorem.

An analogous statement should hold for $K' \neq 0$, but this simple argument does not seem to be directly applicable - on going from real to complex values of t the value of K' may change.

3. AN ANALOGUE OF THEOREM A.

In response to a question of K. Kodaira and D. C. Spencer[1] and as another illustration of the technique used above of extending the number of variables we prove an analogue of Theorem A.

We consider a system of first order partial differential equations for K complex-valued functions $(u^1, \ldots, u^K) = u$ of the real variables $(x^1, \ldots, x^n) = x$. For convenience we write $n = 2m+s$ and express the last $2m$ variables $x^{s+1}, x^{s+2}, x^{s+3}, \ldots, x^n$ by m complex variables $z^1, \ldots, z^m$, setting $z^\lambda = x^{s+\lambda} + ix^{s+m+\lambda}$, $\lambda = 1, \ldots, m$. In the following the indices j, k will run from 1 to K, the indices p, q from 1 to s, and the indices λ, μ from 1 to m; summation convention will be used.

[1] They recently proved a special case of Theorem A' using methods similar to those used in [3], and asked the author whether it could be deduced from the result in [3]; this we do here.

The system to be studied is

$$(8) \quad \begin{cases} (8)_\lambda & \dfrac{\partial u^j}{\partial \bar{z}^\lambda} + \alpha^j_k \dfrac{\partial \bar{u}^k}{\partial \bar{z}^\lambda} = a^j_\lambda \\[2ex] (8)'_p & \dfrac{\partial u^j}{\partial x^p} + \alpha^j_k \dfrac{\partial \bar{u}^k}{\partial x^p} = b^j_p \end{cases}$$

for $j = 1, \ldots, K$; $\lambda = 1, \ldots, m$; $p = 1, \ldots, s$. Here the coefficients α, a, b are given functions (of class C^∞, for convenience) of the variables u, x, defined in some neighborhood of the origin in the product space. Thus the system is non-linear. We shall assume that $\alpha(0, 0) = 0$ (though it would suffice to assume the $\alpha^j_k(0, 0)$ small).

THEOREM A': The following is necessary and sufficient for the system (8) to have solutions $u(x, u_o)$, in a neighborhood of $x = 0$, with given initial values $u(0, u_o) = u_o$ in a neighborhood of the origin in the u space, and such that $u(x, u_o)$ is of class C^∞ in x and u_o: The space Ω of complex Pfaffian forms, in the $2K+n$ variables (Re u, Im u, x), which is spanned by the forms

$$
(9)\qquad \begin{array}{ll} du^j + a^j_k d\bar{u}^k - a^j_\lambda d\bar{z}^\lambda - b^j_p dx^p, & j = 1, \ldots, K \\ dz^\lambda \qquad\qquad\qquad\qquad\qquad\qquad, & \lambda = 1, \ldots, m \end{array}
$$

satisfies the conditions

$$
(10)\qquad d\Omega \subset \text{ideal generated by } \Omega.
$$

Proof: The necessity of (10) is verified by direct calculation: Differentiate $(8)_\lambda$ with respect to $\bar{z}^\mu$, $(8)_\mu$ with respect to $\bar{z}^\lambda$ and subtract. Form similarly (speaking loosely) $\frac{\partial}{\partial x^p}(8)_\lambda - \frac{\partial}{\partial \bar{z}^\lambda}(8)'_p$ and $\frac{\partial}{\partial x^p}(8)'_q - \frac{\partial}{\partial x^q}(8)'_p$. One sees easily that the resulting relations on the coefficients α, a, b are equivalent to conditions (10).

To prove the sufficiency of (10) we apply Theorem 1, noting that $\Omega \cap \bar{\Omega} = 0$. Applying the theorem, for the case $K' = 0$, to the forms Ω in the $2K+n$ variables (Re u, Im u, x), we infer that we may introduce $2K+n$ new variables, which we may write as (Re v, Im v, ξ), so that Ω is spanned by the forms

$$
dv^j,\ \ d\xi^{s+\lambda} + i d\xi^{s+m+\lambda},\ \ j = 1, \ldots, K;\ \ \lambda = 1, \ldots, m.
$$

Here $v = (v^1, \ldots, v^K)$ is complex and $\xi = (\xi^1, \ldots, \xi^n)$ is real.

By Remark (b) at the end of §2.1 we may choose $\xi \equiv x$. It follows that the transformation of variables (Re v, Im v) to (Re u, Im u), for fixed x, is non-singular, and hence that this transformation for $x = 0$ is one-to-one and maps a neighborhood of $v = 0$ onto a neighborhood of $u = 0$, i. e. onto a full neighborhood of initial values u_o for u.

Thus to complete the proof of Theorem A' we need only show that u satisfies (8) as a function of $\xi = x$ and v. To see this write the form (9) in terms of the new variables, v, $\xi = x$; this form equals

$$\frac{\partial u^j}{\partial \bar{z}^\lambda} d\bar{z}^\lambda + \frac{\partial u^j}{\partial \bar{v}^\ell} d\bar{v} + \frac{\partial u^j}{\partial x^p} dx^p + a_k^j\left(\frac{\partial \bar{u}^k}{\partial \bar{z}^\lambda} d\bar{z}^\lambda + \frac{\partial \bar{u}^k}{\partial \bar{v}^\ell} d\bar{v} + \frac{\partial \bar{u}^k}{\partial x^p} dx^p\right) - a_\lambda^j\, d\bar{z}^\lambda - b_p^j\, dx^p$$

modulo the forms of (11). Since this form belongs to Ω the coefficients of $d\bar{z}^\lambda$, $d\bar{v}^\ell$ and dx^p vanish. Setting the coefficients of $d\bar{z}^\lambda$ and dx^p equal to zero we obtain equations (8). Q. E. D.

In conclusion we mention that the complex Frobenius theorem can be applied to give conditions for a non-analytic hypersurface in a domain of several complex variables to contain families of (complex) analytic subsurfaces. In studying such questions Behnke and Sommer [1] implicitly assumed a form of the theorem.

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REDUCTION OF COMPLEX SPACES*

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1. In this paper we consider only connected normal complex spaces in the sense of H. Cartan [3]. By a recent result of H. Grauert and the author complex spaces as defined by H. Behnke and K. Stein ([1], also [5]) are always of this type.

Let X be a complex space. We denote by $I(X)$ the integral domain of holomorphic functions in X. There may exist a complex space X^* and a holomorphic, non-invertible mapping $\zeta: X \longrightarrow X^*$ of X onto X^*, which induces an isomorphism $\zeta^*: I(X^*) \longrightarrow I(X)$ of $I(X^*)$ onto $I(X)$. For example, this is the case if X is a compact complex space and ζ the projection of X onto a single point. Non-trivial examples are given in the general theory of modifications [5]. The space X^* is called a <u>holomorphic reduction</u> of X. In this paper we will give an introduction to the theory of reductions and state some theorems on the existence of holomorphic reductions of a given complex space. We are especially interested in holomorphic reductions X^* of X, which are holomorphically irreducible.

For formulating our results we need some concepts of K. Stein [12] on analytic decompositions. We will recall the principal results of Stein in §4.

2. RELATED COMPLEX SPACES, REDUCTIONS AND KERNEL SPACES.

In this section we introduce some basic concepts.

DEFINITION 1: Two complex spaces X, Y are called holomorphically related, if there exists a continuous isomorphism $\zeta^*: I(Y) \longrightarrow I(X)$ of $I(Y)$ onto $I(X)$.[1)]

X and Y are called meromorphically related, if ζ^* can be continued to an isomorphism of $K(Y)$ onto $K(X)$.[2)]

These notions define two equivalence relations in the "set" of all complex spaces. Evidently, meromorphically related complex spaces are always holomorphically related. The converse is not true, however. If we consider, for instance, the product $C^k \times P^\ell$ of an affine space C^k with a projective space P^ℓ, $\ell > 0$, we see that the spaces $C^k \times P^\ell$ and C^k are holomorphically related, for there are no holomorphic functions on P^ℓ except constants. But $C^k \times P^\ell$ and C^k are not meromorphically related, since there are non-constant meromorphic functions on P^ℓ.

We notice at once: Two holomorphically related complex spaces X and Y are meromorphically related, if every meromorphic function on X (on Y) is the quotient of two holomorphic functions.

In particular this condition is fulfilled if the spaces X and Y are holomorphically complete.[3)] But in this case one can even prove

THEOREM 1: Two holomorphically related, holomorphically complete complex spaces are always analytically isomorphic.

This theorem is a generalization of a well known theorem of Igusa [6]. The proof of theorem 1 is based on some general results on coherent analytic sheaves due to H. Cartan [2], and on the fact that there is always a finite number of holomorphic functions separating the spaces X, Y (cf. [11]).

DEFINITION 2: Let X be a complex space. A couple (X^*, ζ) is called a holomorphic reduction of X, if the following conditions are satisfied:

a) X^* is a complex space, ζ is a holomorphic mapping of X onto X^*.

b) ζ induces an isomorphism $\zeta^*: I(X^*) \longrightarrow I(X)$ of $I(X^*)$ onto $I(X)$.

A holomorphic reduction (X^*, ζ) of X is called a meromorphic reduction, if ζ induces an isomorphism of $K(X^*)$ onto $K(X)$.

If (X^*, ζ) and $('X^*, '\zeta)$ are two holomorphic (meromorphic) reductions of a complex space X, then the spaces X^* and $'X^*$ are holomorphically (meromorphically) related.

Two (holomorphic or meromorphic) reductions (X^*, ζ) and $('X^*, '\zeta)$ of a complex space X are called <u>equivalent</u> if there exists a bi-holomorphic mapping η of X^* onto $'X^*$, such that $'\zeta = \eta \circ \zeta$.

Every complex space X admits a trivial reduction (x, i), where i is the identical map of X onto X. A reduction is called <u>substantial</u>, if it is not equivalent to the trivial reduction.

DEFINITION 3: A reduction (X^*, ζ) of a complex space X is called proper, if $\zeta: X \longrightarrow X^*$ is a proper mapping.

Example: The couple (C^k, ζ), where $\zeta: C^k \times P^\ell \longrightarrow C^k$ denotes the natural projection, is a proper holomorphic reduction of $C^k \times P^\ell$. This reduction is substantial, if $\ell > 0$.

Non-trivial examples for proper meromorphic reductions are given by modifications. H. Grauert and the author proved

([5], p. 288):

PROPOSITION 1: If the triple (X, ζ, X^*) is a proper (and substantial) modification of the complex space X^*, then (X^*, ζ) is a proper (and substantial) meromorphic reduction of X.

Complex spaces which do not have substantial reductions are of special interest. We define:

DEFINITION 4: A complex space X is called a (proper) holomorphic or meromorphic kernel space, if there exists no (proper) substantial holomorphic or meromorphic reduction of X.

A holomorphic (meromorphic) reduction (X^*, ζ) of a complex space X is called a holomorphic (meromorphic) kernel of X, if X^* is a holomorphic (meromorphic) kernel space.

Evidently, (C^k, ζ) is a proper holomorphic (but not meromorphic kernel of $C^k \times P^\ell$, $\ell > 0$.

Immediately, we get:

PROPOSITION 2: Every holomorphically separable complex space X is a holomorphic kernel space [4].

3. MEROMORPHICALLY SEPARABLE COMPLEX SPACES AND MEROMORPHIC KERNEL SPACES

We generalize the notion of a holomorphically separable complex space as follows:

DEFINITION 5: A complex space X is called meromorphically separable, if for two given points $x_1, x_2 \in X$, $x_1 \neq x_2$, there always exists a meromorphic function $h \in K(X)$, which is holomorphic at x_1 and x_2 and separates these points: $h(x_1) \neq h(x_2)$.

In the following we consider only compact meromorphically separable complex spaces. All algebraic spaces are of this type. We assert:

THEOREM 2: If (X^*, ζ) is a meromorphic reduction of a compact meromorphically separable complex space X, then (X, ζ, X^*) is a proper modification of X^*.

PROOF: First we will show that X^* and X have the same complex dimension. Let us denote the complex dimensions of X, X^* by d, d^* respectively. It is evident that $d^* \leq d$. There are exactly d algebraically independent meromorphic functions on X, for X is meromorphically separable. Since the fields $K(X^*)$ and $K(X)$ are isomorphic, they have the same degree of transcendence. Thus there are also d algebraically independent meromorphic functions on X^*. By a general theorem on algebraic dependence, $d^* + 1$ meromorphic functions on X^* are always algebraically dependent (cf. [8], Satz 6''). Therefore we have: $d < d^* + 1$. Together with $d^* \leq d$ this yields $d^* = d$.

Let us now consider the holomorphic mapping $\zeta: X \longrightarrow X^*$. By a general theorem on holomorphic mappings it follows (cf. [10], Satz 18), that there exists an analytic set $N \neq X$ in X, such that ζ is almost one-to-one in $X - N$. Therefore, ζ must be a one-to-one mapping of $X - N$ into X^*, since X is meromorphically separable. This proves that (X, ζ, X^*) is a proper modification of X.

We are now able to give some general examples of meromorphic kernel spaces. By Theorem 2, a compact meromorphically separable complex space X is a complex kernel space if and only if X cannot be derived by a substantial proper modification from a complex space X^*. Complex spaces with this property have been called primitive (Urräume, cf. [7]). It has been proved (cf. [5], [7]) that all product spaces $\times_{\rho=1}^{r} P^{n_\rho}$, where P^n denotes the complex projective space of dimension n, and all homogeneous compact complex manifolds[5)] are primitive. Thus we have:

THEOREM 2': Every product space $\times_{\rho=1}^{r} P^{n_\rho}$ is a meromorphic kernel space. Every homogeneous meromorphically separable compact complex manifold is a meromorphic kernel space.

Note that not all algebraic spaces are meromorphic kernel spaces. For example, the algebraic manifold $'P^2$, derived from P^2 by the σ-process, is not a meromorphic kernel space.

A well known theorem on periodic functions may be expressed as a theorem on reductions in the following way:

THEOREM 3: Every n-dimensional complex torus T^n with n algebraically independent meromorphic functions is a meromorphic kernel space.

For every n-dimensional complex torus T^n with k algebraically independent meromorphic functions, $0 \leq k < n$, there exists a meromorphic kernel (T^k, ζ), where T^k is a complex torus of dimension k.

4. HOLOMORPHIC REDUCTIONS AND HOLOMORPHIC SEPARATIONS

If (X^*, ζ) is a holomorphic reduction of a complex space X, the mapping $\zeta\colon X \longrightarrow X^*$ defines an analytic decomposition Z of X in the sense of K. Stein [12]: the elements of Z are the fibres $\zeta^{-1}(\zeta(x))$, $x \in X$, of the mapping ζ, the complex structure on the quotient set X/Z $(= X^*)$ is the complex structure of the complex space X^*.

For every complex space X we define a natural decomposition S by analytic sets.

DEFINITION 6 (Holomorphic separation): Two points x_1, x_2 of a complex space X are called holomorphically non-separable, if for every holomorphic function $f \in I(X)$ we have $f(x_1) = f(x_2)$. The decomposition of X induced by this equivalence relation is called the holomorphic separation S of X.

The following statements are obvious (we denote by $S(x)$ that element of S which contains the point $x \in X$).

a) Every element $S(x_o)$, $x_o \in X$, of S is an analytic set (defined by all holomorphic functions $f \in I(X)$, which vanish in x_o).

b) Every element $S(x_o)$, $x_o \in X$ of S is the holomorphically convex envelope of the set $\{x_o\}$.

A decomposition Z_1 of X is called a <u>refinement</u> of a decomposition Z_2 of X, $Z_1 \subset Z_2$, if each element of Z_1 is contained in an element of Z_2. A refinement Z_1 of Z_2 is called a <u>substantial refinement</u> of Z_2, $Z_1 \Subset Z_2$, if $Z_1 \neq Z_2$.

We denote by Λ the finest decomposition of X (the elements of Λ are the points of X). Then it can be proved:

THEOREM 4: A complex space X admits a substantial holomorphic reduction, if and only if there exists an analytic decomposition Z of X such that $\Lambda \Subset Z \subset S$.

If the holomorphic separation S of X is an analytic decomposition of X, then the couple $(X/S, \zeta)$, where ζ denotes the natural projection of X onto X/S, is a holomorphic kernel of X.

5. EXISTENCE OF HOLOMORPHIC KERNELS

In this section we study the holomorphic separation S of a complex space X. By using a general theorem on decompositions, we will prove that in special cases S is an analytic decomposition.

A decomposition Z of a complex space X is called proper, if the saturated envelope of each compact set of X is a compact set. The decomposition Z is called simple, if all elements of Z are connected sets. Every decomposition Z of X defines a simple decomposition of X in a natural way; this decomposition is denoted by Z'.

The following theorem is a slight generalization of a theorem due to K. Stein [12].

THEOREM: Let Z be a proper and simple decomposition of a complex space X by analytic sets. For every element N of Z let there exist a saturated relatively compact neighbourhood V of N and a holomorphic mapping λ of V into a complex space X_V, such that the simple decomposition of V defined by λ is identical with the restriction of Z to V. Then Z is an analytic decomposition, if

1) X is a complex manifold,

2) almost all elements of Z are of dimension lower than 2.

We wish to apply this theorem to the simple decomposition S' of X defined by the holomorphic separation S. It can easily be proved that S' is proper, if the space X is holomorphically convex, that is, if the holomorphically convex envelope of each compact set is compact. We assert:

THEOREM 5: a) If X is a holomorphically convex complex manifold, then the holomorphic separation S of X is a simple and proper analytic decomposition of X.

b) If X is an n-dimensional holomorphically convex complex space, and if there are at least $(n-1)$ analytically independent holomorphic functions on X, then the holomorphic separation S of X is a simple and proper analytic decomposition of X.

Let us indicate the proof of a). (The proof of b) is quite analogous.) We denote by S' the simple decomposition of X defined by S; let N be any element of S'. Then there exist a finite number of holomorphic functions $f_1, \ldots, f_s \in I(X)$, such that the analytic set

$$\{x' \in X \mid f_1(x') = 0, \ldots, f_s(x') = 0\}$$

is identical with N in a neighborhood of N. Let $\lambda_o : X \longrightarrow C^s$ denote the holomorphic mapping defined by these functions; evidently N is a connected fibre of λ_o. It follows (from [12], Satz 9,1) that there exists a connected relatively compact neighborhood V of N, such that

a) V is saturated by the connected fibres of λ_o, and

b) all fibres $\lambda_o^{-1}(\lambda_o(x))$, $x \in V$, are compact.

It is obvious that S' is a refinement of the simple decomposition $Z'(\lambda_o)$ defined by the mapping λ_o. In particular V is saturated with respect to S'. Now it can easily be seen that by enlarging the number s of holomorphic functions we can define a mapping λ_o, such that $S' = Z'(\lambda_o)$ on $V - E$, where $E \neq V$ is an analytic set. It follows (by induction) that by a further enlargement of the number s we can define a holomorphic mapping $\lambda: V \longrightarrow C^s$, such that $S' = Z'(\lambda)$. Thus the conditions of the theorem of Stein are fulfilled, therefore S' is a proper analytic decomposition of X.

It remains to show that $S = S'$. We consider the quotient space X/S'. It can easily be proved that X/S' is holomorphically convex. By definition of S' it follows that for each point $y_o \varepsilon X/S'$ there exists a holomorphic mapping τ of X/S' into an m-dimensional space C_m of complex numbers, such that y_o is an isolated point of the fibre $\tau^{-1}(\tau(y_o))$. Therefore the space X/S' is K-complete (cf. [4]), and it follows by a fundamental theorem of H. Grauert, [4], that X/S' is holomorphically complete. This shows $S = S'$.

Now one can easily prove

THEOREM 6: If X is a holomorphically convex complex manifold, then there exists a proper kernel (X^*, ζ) of X. The kernel space X^* is uniquely determined and analytically isomorphic to the complex quotient space X/S; in particular, the kernel space is holomorphically complete.

The preceding statement remains true if X is an n-dimensional holomorphically convex complex space with at least $(n-1)$ analytically independent holomorphic functions.

It must only be proved, that (X^{*}, ζ) is unique up to analytic isomorphisms. But this follows immediately from the results of Stein on proper analytic decompositions (cf. [12]).

From theorem 6 and the results of [4] we get, in particular:

THEOREM 7: Each holomorphically convex complex manifold has a countable basis of open sets.

FOOTNOTES

*) The results of this paper have been announced to some extent in [9].

1) The rings I(X), I(Y) become topological rings, if we introduce the topology of compact convergence. We suppose that ζ^{*} carries constant functions into themselves.

2) We denote by K(Y), K(X) the field of all meromorphic functions on Y, X respectively.

3) For this notion cf. [4].

4) Cf. [9].

5) Every homogeneous complex space is a complex manifold!

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is introduced for every $K \subseteq M$ and $R \subseteq G$. The study of these "scatter" sets can be regarded as a contribution to the above problem as was done by Thimm [16] in the case where H is the projective space P^m, and the map τ is given by meromorphic functions.

An analytic map τ of an open subset A of a complex manifold G into the complex manifold H is said to be meromorphic [2)] in G if $M = G - A$ is thin, and if the scatter set $\Sigma_\tau(\{P_o\}, L)$ consists of at most one point for every point $P_o \in M$ and for every 1-dimensional complex submanifold L of G with

$$L \cap M = \overline{L} \cap M = \{P_o\}.$$

In the case where H is the projective space P^m, the map τ is meromorphic if, and only if, the map τ is given by meromorphic functions.

III. Restriction: The map τ is required to be "convertible in the large" by a map of the same type; that is τ defines a

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2) Stoll [12]. Another definition of a meromorphic map is given by Remmert [9].

"modification".

$$\tau: \ (G, M) \longrightarrow (H, N)$$

is said to be a <u>modification</u>[3] T if G and H are complex manifolds of the same dimension n, and if M is a thin subset of G and N is a thin subset of H, and if τ maps $A = G - M$ pseudoconformally onto $B = H - N$.

$$\tau^{-1}: \ (G, M) \longrightarrow (H, N)$$

is said to be the <u>inverse modification</u> T^{-1}.

The modification T is said to be <u>open</u> if $\tau(A \cap U) \cup N$ is open for every open neighborhood U of M. This modification T is said to be <u>open in both directions</u> if both T and T^{-1} are open. For instance, a modification between compact manifolds G and H is always open in both directions.

The modification T is said to be <u>meromorphic</u> if τ is meromorphic, and T is said to be <u>meromorphic in both directions</u> if T and T^{-1} are meromorphic. If $n = 2$, a meromorphic modification is meromorphic in both directions. A meromorphic modification

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[3] Stoll [11]. For other different definitions of modifications see Behnke and Stein [2], Hopf [5], Grauert and Remmert [3], Kreyszig [7] and Aeppli [1].

between algebraic manifolds is a birational transformation and vice versa.

The modification T is said to be analytic if τ^{-1} can be analytically continued into H. A modification is said to be trivial if T and T^{-1} are analytic. That is the case if and only if τ can be analytically continued to a one-to-one map of G onto H. If n = 1, every modification open in both directions is trivial.

IV. Restriction: n = 2.

A simple example of an analytic, non-trivial modification was given by Hopf ([5], [6]):

Let $\zeta_1:\zeta_2$ be homogeneous coordinates of the Riemannian sphere P^1 and

$$D = \{(z_1, z_2) \mid |z_1| < 1, \ |z_2| < 1\},$$

$$H = \{(z_1, z_2, \zeta_1:\zeta_2) \mid z_1\zeta_2 - z_2\zeta_1 = 0, \ (z_1, z_2) \in D, \ (\zeta_1:\zeta_2) \in P^1\},$$

$$S = \{(0, 0, \zeta_1:\zeta_2) \in P^1\},$$

$$P_o = (0, 0) \in D$$

$$\sigma(z_1, z_2) = (z_1, z_2, z_1:z_2) \in H \text{ for } P_o \neq (z_1, z_2) \in D.$$

Then

$$\sigma: (D, \{P_o\}) \longrightarrow (H, S),$$

is an analytic modification where the origin is replaced by a Riemannian sphere. This process has only local character and therefore it can be done simultaneously in each point of a set M of isolated points on every complex manifold G. An analytic modification S_M

$$\sigma: (G, M) \longrightarrow (H, N)$$

is defined replacing each point of M by a sphere called a σ-sphere. Consequently, N is a union of disjoint spheres. S_M is called a σ-process.

Hopf ([5], [6]) proved the theorem:

If G and H are compact and

$$\tau: (G, \{P_o\}) \longrightarrow (H, N)$$

is an analytic modification T in the single point P_o, then T can be reconstructed by a finite sequence of σ-processes

$$\sigma_\nu: (G_{\nu-1}, M_{\nu-1}) \longrightarrow (G_\nu, N_\nu) \quad (\nu = 1, \ldots, r)$$

with

$$M_o = \{P_o\}, \quad M_\nu \subseteq N_\nu \quad \text{for} \quad \nu = 1, \ldots, r,$$

$$G_o = G, \quad G_r = H,$$

$$\tau(P) = \sigma_{r-1}\sigma_{r-2} \ldots \sigma_1(P) \quad \text{for} \quad P \in G - M,$$

$$N = \bigcup_{\nu=1}^{r-1} \sigma_{r-1}\sigma_{r-2} \ldots \sigma_\nu(N_\nu - M_\nu) \cup N_r.$$

Under certain conditions this theorem can be generalized to every analytic modification (Stoll [14], [15]).

Hopf's Theorem leads to the question:

Is it possible to reconstruct a given meromorphic modification by a finite or infinite sequence of σ-processes or inverse σ-processes?

The affirmative answer is based on the theorem of H. Hopf and the following theorem:

THEOREM 1. (Stoll [13]).

If a modification T is open in both directions and meromorphic, if $n = 2$ and if $P_o \in M$ is a singular point of τ, then there is a pure one-dimensional analytic set $C = C(P_o) \subseteq \Sigma_\tau(P_o) \subseteq N$ such that τ^{-1} is regular on $C(P_o)$

with the exception of a set D of isolated points, and if β is the analytic continuation of τ^{-1} on C - D, then

$$\beta(Q) = P_o \text{ for } Q \varepsilon C - D$$

is constant and equal to P_o.

The meaning of this theorem is that the set M_o of singular points of M is mapped one-to-one by $P_o \longrightarrow C(P_o)$ into the set S of pure one-dimensional analytic sets of H contained in N. If H is compact, S is finite. Consequently τ has only a finite number of singularities. If H has a countable base of open sets, S is countable. Therefore τ has only a countable set of singularities, which are isolated as can be easily shown.

The proof of Theorem 1 is very complicated. But when G and H have a countable base of open sets, a much simpler proof can now be given which is based on the following

LEMMA[4]:

If H is an open subset of the complex vector space C^n of dimension n, if D is a region of C^p, if

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4) A similar lemma was proved by Rothstein [10].

$D \times (0,\dots,0) = S \subseteq H$ is closed in H, if N is a pure p-dimensional analytic subset of $H - S$, if N is singular on S, and if

$$N_x = \{z \mid z = (x,w) \in N \text{ and } w \in C^{n-p}\}$$

then

$$F = \{x \mid (x,0) \notin \overline{N}_x \text{ and } x \in D\}$$

has measure zero with respect to Lebesgue measure of 2p real dimensions in C^p.

A consequence of this Lemma is that the graph

$$\Gamma = \{(P, \tau(P)) \mid P \in A\} \subseteq G \times H$$

of the modification T has a closure $\overline{\Gamma}$, which is an analytic subset of $G \times H$.

The projections $\omega_o : G \times H \longrightarrow G$ and $\tilde{\omega}_o : G \times H \longrightarrow H$ induce proper analytic maps onto: $\omega_o : \overline{\Gamma} \longrightarrow G$ and $\tilde{\omega}_o : \overline{\Gamma} \longrightarrow H$ and analytic modifications:

$$\omega_o^{-1} : (G, M_o) \longrightarrow (\overline{\Gamma}, C) \quad \text{with } \omega_o^{-1}(M_o) = C,$$

$$\tilde{\omega}_o^{-1}: (H, N_o) \longrightarrow (\overline{\Gamma}, \tilde{C}) \text{ with } \tilde{\omega}_o^{-1}(N_o) = \tilde{C},$$

where M_o is the set of singularities of τ, and N_o is the set of singularities of τ^{-1}. The zero set of the Jacobian of ω_o is C, and the zero set of the Jacobian of $\tilde{\omega}_o$ is $\tilde{C}$. Therefore C and $\tilde{C}$ are analytic sets of pure dimension one, or empty. Because the analytic maps ω_o and $\tilde{\omega}_o$ are proper, $M_o = \omega_o(C)$ and $N_o = \tilde{\omega}_o(\tilde{C})$ are 0-dimensional analytic sets, therefore sets of isolated points. If P_o is a singular point of τ, then it can be proven that

$$\Sigma_\tau(P_o) = \tilde{\omega}_o \omega^{-1}(P_o) \subseteq N$$

is an analytic set of pure dimension one (Remmert [8], Grauert and Remmert [3]). If β is the analytic continuation of τ^{-1} onto $\Sigma_\tau(P_o) - N_o$, then $\beta(Q) = P_o$ for $Q \varepsilon \Sigma_\tau(P_o) - N_o$. A proof of Theorem 1 has been sketched.

In general, $\overline{\Gamma}$ is not a manifold but a complex space. If G and H are compact, $\overline{\Gamma}$ is compact. Then $\overline{\Gamma}$ has only a finite set V of non-uniform points. According to a theorem of Hirzebruch [4] there is a compact complex manifold F, an analytic map χ from F onto $\overline{\Gamma}$ and an analytic subset E of F of pure dimension one, such that

$$\chi^{-1}: (\overline{\Gamma}, V) \longrightarrow (F, E)$$

is an analytic modification. The analytic maps

$$\omega = \omega_o \chi: F \longrightarrow G \quad \text{and} \quad \tilde{\omega} = \tilde{\omega}_o \chi: F \longrightarrow H$$

induce analytic modifications

$$\omega^{-1}: (G, M) \longrightarrow (F, W) \quad \text{and} \quad \tilde{\omega}^{-1}: (H, N) \longrightarrow (F, W)$$

with

$$W = \omega^{-1}(M) = \tilde{\omega}^{-1}(N)$$

and

$$\tilde{\omega}\, \omega^{-1}(P) = \tau(P) \quad \text{for } P \varepsilon A.$$

Hopf's theorem about the reconstruction of an analytic modification by σ-processes gives

THEOREM 2 (Stoll [14]):

If G and H are compact complex manifolds of dimension 2, and if

$$\tau: (G, M) \longrightarrow (H, N)$$

is a meromorphic modification T, then there are two finite sequences of σ-processes:

$$\sigma_\nu : (G_{\nu-1}, M_{\nu-1}) \longrightarrow (G_\nu, N_\nu) \qquad (\nu = 1, \ldots, r)$$
$$\tilde{\sigma}_\nu : (H_{\nu-1}, \tilde{M}_{\nu-1}) \longrightarrow (H_\nu, \tilde{N}_\nu) \qquad (\nu = 1, \ldots, s)$$

with

$$G_o = G, \quad N_o = M, \quad H_o = H, \quad \tilde{N}_o = N$$
$$G_r = H_s = F$$

and

$$\tilde{\sigma}_1^{-1} \tilde{\sigma}_2^{-1} \ldots \tilde{\sigma}_{s-1}^{-1} \sigma_{r-1} \sigma_{r-2} \ldots \sigma_2 \sigma_1 (P) = \tau(P)$$
for $P \varepsilon A$.

$M_\nu \subseteq N_\nu$ is the set of singularities of $\tau \sigma_1^{-1} \ldots \sigma_\nu^{-1}$ for $\nu = 0, 1, \ldots, r$ and $\tilde{M}_\nu \subseteq \tilde{N}_\nu$ is the set of singularities of $\tau^{-1} \tilde{\sigma}_1^{-1} \ldots \tilde{\sigma}_\nu^{-1}$ for $\nu = 0, 1, \ldots, s$, where

$$M_\nu \neq \emptyset \text{ for } \nu = 0, 1, \ldots, r-1 \text{ and } M_r = \emptyset,$$
$$\tilde{M}_\nu \neq \emptyset \text{ for } \nu = 0, 1, \ldots, s-1 \text{ and } \tilde{M}_s = \emptyset.$$

Therefore the modification T is reconstructed by σ-processes and inverse σ-processes.[5)]

Hirzebruch's theorem about the reduction of the non-uniform points of a complex space was used to prove theorem 2. The use of Hirzebruch's theorem could have been avoided because it is now possible to prove the special case of Hirzebruch's theorem which was used above by means of Hopf's theorem for a non-compact manifold.

Theorem 2 can also be proven if G or H is not compact, but the modification T is open in both directions and meromorphic (Stoll [15]). Both sequences of σ-processes are constructed in the same way, but may be infinite. The manifolds G_ν and H_ν "converge" to the same "limit" manifold F. The maps $\sigma_1^{-1} \ldots \sigma_r^{-1}$ and $\tilde{\sigma}_1^{-1} \ldots \tilde{\sigma}_s^{-1}$ "converge" to analytic maps which can be analytically continued to

$$\omega: F \longrightarrow G \qquad \text{and} \qquad \tilde{\omega}: F \longrightarrow H$$

such that

$$\tau(P) = \tilde{\omega}\, \omega^{-1}(P) \qquad \text{for } P \varepsilon A.$$

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5) Here $\emptyset$ denotes the empty set.

Both ω and $\tilde{\omega}$ are maps onto.

If G or H does not have a countable base of open sets and if τ: (G, M) $\longrightarrow$ (H, N) is a meromorphic modification T open in both directions, then T can be reconstructed by σ-processes and inverse σ-processes; but trees of σ-processes have to be used instead of sequences.

In the case $n > 2$, the much more difficult reconstruction problem is still unsolved. Aeppli [1] proved: If G is a complex manifold of dimension n, and M a regularly imbedded, compact, connected complex submanifold of dimension $q < n$, then there is (up to analytic equivalence) one and only one analytic modification

$$\tau: (G, M) \longrightarrow (H, N)$$

such that N is a regularly imbedded, connected, compact, complex submanifold of dimension n-1. It is called the $\sigma^{n,q}$ process. According to E. Kreyszig a modification

$$\tau: (P^{m_1} \times \ldots \times P^{m_r}, M) \longrightarrow (P^{n_1} \times \ldots \times P^{n_s}, N)$$

can be constructed by σ-processes and inverse σ-processes if P^{μ} is the μ-dimensional complex projective space, if

$$m_1 + \ldots + m_r = n_1 + \ldots + n_s = n$$

and if $P^{m_1} \times \ldots \times P^{m_r} - M$ and $P^{n_1} \times \ldots \times P^{n_s} - N$ are the imbedded affine spaces C^n.

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POLYNOMIAL APPROXIMATION ON CURVES

John Wermer

Let Γ be a simple closed Jordan curve in the space C^n of the complex variables $z_1, \ldots, z_n$. By the <u>hull</u> of Γ we mean the set

$$h(\Gamma) = \left\{(z_1^o, \ldots, z_n^o) \mid \text{for every polynomial } P \quad |P(z_1^o, \ldots, z_n^o)| \leq \max_{\Gamma} |P(z_1, \ldots, z_n)|\right\}.$$

It is easy to see that $h(\Gamma)$ is a compact set containing Γ. It is also true but non-trivial that $h(\Gamma)$ is connected. We are interested in getting more detailed information about the nature of $h(\Gamma)$.

EXAMPLE 1: Γ is a plane curve $(n = 1)$. Then $h(\Gamma)$ is the union of Γ and its interior.

EXAMPLE 2: $n = 2$ and Γ is given by:

$$z_1 = e^{i\theta}, \quad z_2 = e^{-i\theta}, \quad 0 \leq \theta \leq 2\pi.$$

Here $h(\Gamma) = \Gamma$. No new points arise in $h(\Gamma)$. For $z_1 z_2 - 1 = 0$ on Γ. Hence if $(z_1^o, z_2^o) \in h(\Gamma)$, $z_1^o z_2^o - 1 = 0$. Both $|z_1^o|$ and $|z_2^o| = 1$. Else one of these, e.g. $|z_1^o|$, > 1. But then the polynomial z_1 has a larger modulus at (z_1^o, z_2^o) than on Γ. Hence $(z_1^o, z_2^o) \in \Gamma$.

EXAMPLE 3: Let Γ lie on an analytic surface Σ, where Σ has one complex dimension. Assume Γ is the boundary of a region D on Σ such that $D \cup \Gamma$ is compact. Then by the maximum principle applied to D, each point in D belongs to $h(\Gamma)$.

In order to obtain a complete description of $h(\Gamma)$, we assume that Γ admits the parametrization:

$$z_i = \phi_i(u), \quad i = 1, \ldots, n, \quad |u| = 1,$$

where the ϕ_i satisfy

(1) Each ϕ_i is analytic in an annulus $a < |u| < b$ which contains the circle $|u| = 1$.

(2) The ϕ_i together map $|u| = 1$ one-one onto Γ.

(3) $\phi_1' \neq 0$ on $|u| = 1$.

The geometric significance of these assumptions is that Γ lies on a little annular strip of an analytic surface, namely the strip Σ_o onto which the annulus $a < |u| < b$ is sent by the map $u \longrightarrow (\phi_1(u), \ldots, \phi_n(u))$.

THEOREM: Under hypotheses (1), (2), (3), we have either

(a) $h(\Gamma) = \Gamma$, or

(b) $h(\Gamma) - \Gamma$ is an analytic surface of one complex dimension having at most finitely many singularities.

The proof of this result for arbitrary n can be reduced to the case $n = 2$, and in giving an outline of the proof I shall restrict myself to that case. In geometric terms, what we shall show is this: if $h(\Gamma) \neq \Gamma$, then the little strip Σ_o can be extended to a larger analytic surface Σ in such a way that on Σ our curve Γ bounds a compact piece, i.e. there exists a region Σ^* on Σ with $\Sigma^* \cup \Gamma$ compact. As we saw above, then, each point in Σ^* lies in $h(\Gamma)$, and so $\Sigma^* \subseteq h(\Gamma) - \Gamma$. To prove (b) we must then still show that <u>every</u> point on $h(\Gamma) - \Gamma$ lies on Σ^*. In this paper I shall not go into that part of the argument.

Let then the equations $z_1 = \phi_1(u)$, $z_2 = \phi_2(u)$, $|u| = 1$, define a curve Γ in C^2 and assume (1), (2), (3). Also assume $h(\Gamma) \neq \Gamma$. Denote by γ the image curve of $|u| = 1$ under ϕ_1. Without loss of generality, we may assume γ has only finitely many multiple points.

Set $f = \phi_2(\phi_1^{-1})$. Then on Γ we have $z_2 = f(z_1)$, locally.

Because of (1) f is analytic on γ, multiple-valued but without branch-points, the last because of (3). By a <u>place</u> we mean a pair (z,h) consisting of a point z in the plane and an algebraic function element h at z.

Let S denote the Riemann surface of f, i.e. the set of places obtained by continuing f. On S we have two distinguished functions Z and F where $Z(z,h) = z$, $F(z,h) = h(z)$, for (z,h) a place on S. Let $\mathring{\gamma}$ be the set of places obtained by continuing a function-element of f along γ. Because of (2) $\mathring{\gamma}$ is a simple closed curve on S. Also $\mathring{\gamma}$ projects on γ.

LEMMA 1:[1] The complement of $\mathring{\gamma}$ on S has a component $\mathring{D}$ such that $\mathring{D} \cup \mathring{\gamma}$ is compact and such that Z and F are analytic on $\mathring{D} \cup \mathring{\gamma}$.

Once this Lemma has been proved, we can map S into C^2 by the map: $p \longrightarrow (Z(p), F(p))$. This map carries S on an analytic surface Σ, carries $\mathring{\gamma}$ onto the given curve Γ and carries the

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[1] This result is proved in detail in the author's paper <u>Function rings and Riemann surfaces</u>, to appear in the Annals of Mathematics.

region $\mathring{D}$ on a region Σ^* on Σ. Since $\mathring{D} \cup \mathring{\gamma}$ is compact, the same is true of $\Sigma^* \cup \Gamma$. Thus our theorem reduces to the Lemma, (except for the part whose proof we said we should omit).

To prove the Lemma we first note that there exists a complex-valued measure $d\sigma$ on $\mathring{\gamma}$, $d\sigma \not\equiv 0$, such that

$$\int_{\mathring{\gamma}} Z^n F^m d\sigma = 0, \quad n, m \geq 0. \tag{4}$$

This follows easily by the Riesz representation theorem from the hypothesis that $h(\Gamma) - \Gamma$ is non-empty. We also may assume that $d\sigma$ has no point-masses at any point on $\mathring{\gamma}$ which Z maps on a multiple point of γ.

We write $[Z, F]$ for the algebra of functions on S expressible as polynomials in Z and F and $C[Z, F]$ for the uniform closure of that algebra on $\mathring{\gamma}$. Let Ω denote the complement of γ in the plane. Fix a component W of Ω. Choose g in $C[Z, F]$. Then set for z_o in W:

$$\Phi(W, g, z_o) = \frac{1}{2\pi i} \int_{\mathring{\gamma}} \frac{g \, d\sigma}{Z - z_o} . \tag{5}$$

Then $\Phi(W, g)$ is an analytic function in W. Let α be a boundary arc of W (which of course is part of γ) and for each β in α let

$\Phi(W, g, \beta)$ be the non-tangential limit of $\Phi(W, g, z)$ as $z \longrightarrow \beta$, if this limit exists. One can then show

(6) $\Phi(W, g, \beta)$ exists for a. a. β on α.

Let next W' be a second component of Ω having α as boundary arc. Then for a. a. β on α, with $p(\beta)$ denoting the point $\mathring{\gamma}$ lying over β, we have

(7) $\Phi(W', g, \beta) - \Phi(W, g, \beta) = g(p(\beta))\sigma'(p(\beta))$

where σ' denotes a certain "derivative" of the measure $d\sigma$.

Let Ω_o be the unbounded component of Ω and let W_1 be another component which shares a boundary arc α_1 with Ω_o.

LEMMA 2: Under the hypothesis

(8) $\sigma'(p(\beta)) \neq 0$ a. e. on α_1

there exists a one-sheeted region $\mathring{W}_1$ on S lying over W_1 such that for every g in [Z, F]:

(9) $g(p(z))k(z) = \Phi(W_1, g, z), \quad z \in W_1$

where $p(z)$ is the place on $\mathring{W}_1$ lying over z and $k(z)$ is independent of g.

Proof: The function $(Z - z_o)^{-1}$ is uniformly approximable by polynomials in Z on $\mathring{\gamma}$ as long as $z_o \varepsilon \Omega_o$. Hence (4) gives that

(10) $\Phi(\Omega_o, g, z) = 0$, all z in Ω_o, $g \varepsilon C[Z, F]$.

Applying (7) to W_1 and Ω_o and using (10) we get for all g in $C[Z, F]$

(11) $\Phi(W_1, g, \beta) = g(p(\beta))\sigma'(p(\beta))$, a. a. β on α_1

and so in particular

(12) $\Phi(W_1, 1, \beta) = \sigma'(p(\beta))$.

For z in W, we now put

(13) $$\mathring{g}(z) = \Phi(W_1, g, z)/\Phi(W_1, 1, z).$$

The denominator does not vanish identically, because of (8). Thus for each g in $C[Z, F]$ there exists a meromorphic function $\mathring{g}$ on W_1 whose boundary value, in virtue of (11), coincides with $g(p(\beta))$ a. e. on α_1. Fix z_1 in W_1 with $\Phi(W_1, 1, z_1) \neq 0$. Then the map $g \longrightarrow g(z_1)$ is a multiplicative linear functional on the Banach algebra

$C(Z, F)$. By a basic property of such functionals, we get $|g(z_1)| \leq ||g||$, where $||g|| = \max |g|$. Hence $\mathring{g}$ is a bounded analytic function on W_1. It follows that $\mathring{g}$ assumes the boundary values $g(p(\beta))$ continuously on α_1.

In particular, $F(p(\beta)) = f(\beta)$ admits a single-valued analytic extension $\mathring{F}$ from α_1 to W_1. Denote by $\mathring{W}_1$ the one-sheeted Riemann surface over W_1 determined by $\mathring{F}$. Clearly $\mathring{W}_1$ lies on S. From (13) and the fact that $\mathring{g}(z) = g(p(z))$ we get assertion (9) with $k(z) = \Phi(W_1, 1, z)$. So Lemma 2 is established.

NOTE: If (8) is false, a modification has to be made in the statement of Lemma 2.

We now define the component $\mathring{D}$ of the complement of $\mathring{\gamma}$ on S which appears in the statement of Lemma 1 as that component which contains the region $\mathring{W}_1$ which we have just constructed. We have to prove that $\mathring{D} \cup \mathring{\gamma}$ is compact and shall do this in the following way: For each component W of Ω we construct an m-sheeted Riemann surface $\mathring{W}$ over W by an equation

$$(14) \qquad \sum_{i=0}^{m} c_i(z) w^i = 0, \quad z \text{ in } W$$

with the c_i analytic on W and $c_m \equiv 1$. Here m depends on W and may equal 0, i.e. $\overset{\circ}{W}$ may be empty.

The functions c_i will be rational functions of the $\Phi(W, g)$ defined in (5). The multiple-valued function on W defined by (14) gives rise to a single-valued analytic function on $\overset{\circ}{W}$. This we denote by w. The characteristic properties of $\overset{\circ}{W}$ will then be the following: If $z \varepsilon W$ and is not a branch-point for (14) and if $p_1, \ldots, p_m$ are the places on W over z, then we can find constants $k_1(z), \ldots, k_m(z)$ such that

(15) $$\sum_{i=1}^{m} w(p_i)^{\nu}\, k_i(z) = \Phi(W, F^{\nu}, z), \quad \nu = 0, 1, 2, \ldots$$

and

(16) $$\Delta(W, z) \not\equiv 0 \text{ for } z \text{ in } W$$

where $\Delta(W, z)$ is the $m \times m$ determinant whose entry in the i'th column and j'th row is $\Phi(W, F^{i+j-2}, z)$.

We can then prove the following Lemmas.

LEMMA 3: Over a given W there exists at most one finite sheeted Riemann surface satisfying (15) and (16).

The proof is a matter of elementary algebra.

LEMMA 4: Fix a component W^* and assume some place on $\mathring{D}$ projects into W^*. Then a surface $\mathring{W}^*$ over W^* satisfying (15) and (16) exists and contains all the places of $\mathring{D}$ that project into $\mathring{W}^*$. **F**urthermore, the coefficients c_i in the equation (14) of $\mathring{W}^*$ have analytic extensions to an open set which contains the closure of W^*. Also W^* is bounded.

The complete proof of this Lemma is long and complicated. We shall content ourselves here with an outline of the construction of $\mathring{W}^*$.

We choose a certain sequence of components of Ω, W_1, $W_2, \dots,$ W_s, where W_1 is the component of Lemma 2 and $W_s = W^*$, such that W_i and W_{i+1} share a boundary arc α_i. Now by Lemma 2 a surface $\mathring{W}_1$ obeying (15) and (16) with $m = 1$ exists over W_1.

We shall prove the existence of W^* by induction, using the fact that W_1 exists and showing that if $\mathring{W}_i$ exists over W_i satisfying (15) and (16), then $\mathring{W}_{i+1}$ exists over W_{i+1} satisfying (15) and (16). Write W for W_i, W' for W_{i+1} and α for α_i. By induction hypothesis we have

$$(17) \qquad \sum_{i=1}^{m} w^{\nu}(p_i)k_i(z) = \Phi(W, F^{\nu}, z), \quad \nu = 0,1,2,\dots$$

for z in W and $p_1, \ldots, p_m$ on W.

For a. a. β in α we can find a triangle T_β lying in W with vertex at β such that over T_β lie m distinct sheets T^i_β, $i = 1, \ldots, m$, of $\overset{o}{W}$. Further, as $z \longrightarrow \beta$ from with T_β $w(p_i(z))$ has a unique limit $w_i(\beta)$, where $p_i(z)$ is the place in T^i_β lying over z. Also $k_i(\beta) = \lim_{z \longrightarrow \beta} k_i(z)$ exists. From (17) we then get

$$(18) \qquad \sum_{i=1}^{m} w_i^\nu(\beta) k_i(\beta) = \Phi(W, F^\nu, \beta), \quad \nu = 0, 1, \ldots$$

a.e. on α. Combining (18) with (7) we have

$$(19) \qquad \sum_{i=1}^{m+1} w_i^\nu(\beta) k_i(\beta) = \Phi(W', F^\nu, \beta), \quad \nu = 0, 1, \ldots$$

where we have put $w_{m+1}(\beta) = F(p(\beta))$ and $k_{m+1}(\beta) = \sigma'(p(\beta))$, with $p(\beta)$ in $\overset{\bullet}{\gamma}$ and projecting on β.

Let now $\sigma_1'(\beta), \ldots, \sigma_{m+1}'(\beta)$ be the elementary symmetric functions of $w_1(\beta), \ldots, w_{m+1}(\beta)$. Computing suitably with (19) we get:

$$(20) \qquad \sigma_j'(\beta) = \Delta_j(\beta)/\Delta(\beta), \quad j = 1, \ldots, m+1$$

where Δ_j and Δ are certain polynomials in the functions $\Phi(W', 1)$, $\Phi(W', F), \ldots, \Phi(W', F^{2m})$, <u>provided</u> $\Delta \not\equiv 0$. We shall only discuss this case. When $\Delta \equiv 0$, a similar argument goes through.

Since Δ_j and Δ are polynomials in the $\Phi(W', F^{\nu})$ and these functions are analytic in W', we can set

(21) $$\sigma_j'(z) = \Delta_j(z)/\Delta(z), \quad z \in W', \quad j = 1, \ldots, m+1$$

and thereby get functions σ_j' meromorphic on W' which agree a. e. on α with the symmetric functions of $w_1(\beta), \ldots, w_{m+1}(\beta)$. One can show that the σ_j' have no poles in W', but we shall not go into this.

Let now $\mathring{W}'$ be the surface over W' whose equation is:

(22) $$w'^{m+1} - \sigma_1'(z)w'^{m} + \ldots + (-1)^{m+1}\sigma_{m+1}'(z) = 0.$$

We assert that (15) and (16) hold for $\mathring{W}'$, where (15) has to be interpreted as saying:

(23) $$\sum_{i=1}^{m+1} w'(p_i)^{\nu} k_i'(z) = \Phi(W', F^{\nu}, z), \quad \nu = 0, 1, 2, \ldots$$

if $p_1, \ldots, p_{m+1}$ are the places on $\mathring{W}'$ over z, w' is the function on W' defined by (22) and $k_1'(z), \ldots, k_{m+1}'(z)$ are certain constants depending only on z.

To show this we first show by direct computation that there exists a function k' on W' of the form:

(24) $$k'(p) = \sum_{j=0}^{m} A_j(z) w'(p)^j, \quad p \in W'$$

where z is the projection of p and the A_j are certain meromorphic functions on W' such that (23) is satisfied for $\nu = 0,1,2,\ldots,m$ if we set $k_i'(z) = k'(p_i)$, $i = 1,\ldots,m+1$. Next let z approach a point β on α from within W'. Then $k_i'(z)$ has a limit $k_i'(\beta)$ for a. a. β on α. Also limits $w_i'(\beta)$, defined in analogy with our earlier definition of $w_i(\beta)$, exist a. e. on α for each i. Since (23) holds on W' for $\nu = 0,1,\ldots,m$, we get a. e. on α

$$(25) \qquad \sum_{i=1}^{m+1} w_i'(\beta)^{\nu} k_i'(\beta) = \Phi(W', F^{\nu}, \beta), \quad \nu = 0,1,\ldots,m.$$

Now (22) remains true a. e. on α. Also $\sigma_j'(\beta)$ is the j'th elementary symmetric function of $w_1(\beta),\ldots,w_{m+1}(\beta)$. It follows that the sets of numbers $\{w_i(\beta)\}_1^{m+1}$ and $\{w_i'(\beta)\}_1^{m+1}$ coincide for a. a. β on α. For a suitable reordering of the $w_i'(\beta)$ we then get

$$(26) \qquad w_j'(\beta) = w_j(\beta), \quad j = 1,\ldots,m+1.$$

Compare (25) and (19) and use (26). Because of the uniqueness of the solution of a system of $m+1$ linear equations in $m+1$ unknowns with non-vanishing determinant, we get

$$(27) \qquad k_i'(\beta) = k_i(\beta), \quad i = 1,\ldots,m+1, \quad \text{a. e. on } \alpha.$$

Set now $B_{\nu}(z) = \sum_{i=1}^{m+1} w'(p_i)^{\nu} k'(p_i)$, $\nu = 0,1,2,\ldots,$ where the p_i

project on z, z in W'. Applying (26) and (27) to (19) we find that $B_\nu(\beta) = \Phi(W', F^\nu, \beta)$ a. e. on α for all $\nu \geq 0$. Since a meromorphic function is determined by its non-tangential a. e. boundary values on any arc, we get that $B_\nu \equiv \Phi(W', F^\nu)$ on W', for all ν. But this is just (23).

Hence (15) holds for $\mathring{W}'$. One can next show that (16) also holds for W'. The induction is thus complete from W_i to W_{i+1}. Hence $\mathring{W}^*$ exists satisfying (15) and (16).

Proof of Lemma 1: For each component W of Ω, let $Z^{-1}(W)$ be the set on $\mathring{D}$ which projects into W. Then $\mathring{D} \cup \mathring{\gamma}$ is the closure on S of the union of the $Z^{-1}(W)$ as W ranges over the various components. Since the number of components is finite, to prove $\mathring{D} \cup \mathring{\gamma}$ compact it suffices to prove that each $Z^{-1}(W)$ has compact closure.

Fix a component W with $Z^{-1}(W)$ non-empty. By Lemma 4, W is bounded and there exists an equation

$$(28) \qquad \sum_{i=1}^{m} c_i(z)w^i = 0, \quad z \in W$$

with coefficients c_i analytic on the closure $\overline{W}$ of W such that if $\mathring{W}$ is the surface defined by (28) over W, then $Z^{-1}(W)$ is contained in W. It follows that the closure of $Z^{-1}(W)$ is contained in the set of places defined by (28) over $\overline{W}$. Since the c_i remain analytic on $\overline{W}$, that set of places is compact. Hence $Z^{-1}(W)$ has compact closure and so $\mathring{D} \cup \mathring{\gamma}$ is compact.

Further, since W is bounded and since the c_i are analytic on $\overline{W}$, the functions Z and F are analytic on the closure of $Z^{-1}(W)$. Hence Z and F are analytic on all of $\mathring{D} \cup \mathring{\gamma}$. Thus Lemma 1 holds.

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ON THE BOUNDARY VALUES OF FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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I wish to give a brief account of results about the existence of boundary values of functions of several complex variables. Most of the results I am going to discuss are well known and have been published (see Bibliography). But there are a number of problems still awaiting solution, and one of the purposes of the lecture is to call attention to these problems.

We shall consider functions $f(z_1, z_2, \ldots, z_m)$ of m complex variables defined and analytic in <u>product domains</u>, that is to say domains which are topological products of m one-dimensional domains. Without losing much generality we may suppose that these one-dimensional domains are either half-planes or unit discs. In this case we have at our disposal the apparatus of Fourier integrals or Fourier series which, though insufficient to give solution to our problems, is helpful at some stages. Of course the domains which we consider are of very special nature, but the character of the results obtained may give useful hints about the situation in more general cases.

We consider functions $f(z_1, \ldots, z_m)$ regular in the m-cylinder

$$\Delta_m: |z_1| < 1, \ \ldots, \ |z_m| < 1,$$

and we study this behavior as $(z_1, \ldots, z_m)$ approaches the boundary B of Δ_m. The most interesting part of B is the distinguished (or extremal) boundary of Δ_m:

$$\Gamma_m: |z_1| = 1, \ldots, |z_m| = 1.$$

The remainder of B consists of various parts, typical of which is the topological product of Δ_k $(|z_1| < 1, \ldots, |z_k| < 1)$ and Γ_{m-k} $(|z_{k+1}| = 1, \ldots, |z_m| = 1)$.

Consider the integral

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \log^+ |f(r_1 e^{i\theta_1}, \ldots, r_m e^{i\theta_m})| \, d\theta_1 \ldots d\theta_m .$$

The class of functions f for which this integral remains bounded (for all $r_1, \ldots, r_m$ less than 1) will be denoted by N. Class N has been thoroughly investigated in the case $m = 1$, and in particular it is very well known that in this case the function f has a finite non-tangential limit at almost all points of the unit circumference. The case $m > 1$ is much more difficult, the main reason being that we do not have here a simple analogue of Jensen's formula; nor do we have an analogue of the Blaschke product.

Let $\psi(u)$ be a function non-negative, non-decreasing and convex for $u > 0$. We assume that $\psi \not\equiv 0$, so that $\psi(u) > Cu$ for u large enough, C being a positive constant. We may consider the class N_ψ of functions f such that the integral

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \psi\left\{\log^+|f(r_1e^{i\theta_1}, \ldots, r_me^{i\theta_m})|\right\} d\theta_1 \ldots d\theta_m$$

is bounded for all r_j less than 1. In view of our hypotheses about ψ, we have $N_\psi \subset N$. Of special interest are two cases:

(i) $\psi(u) = e^{\alpha u}$, where $\alpha > 0$. The class N_ψ will then be denoted by H^α, and it consists of functions f such that the integral

$$\int_0^{2\pi} \cdots \int_0^{2\pi} |f(r_1e^{i\theta_1}, \ldots, r_me^{i\theta_m})|^\alpha d\theta_1 \ldots d\theta_m$$

remains bounded;

(ii) $\psi(u) = u\,(\log^+ u)^\alpha$, $\alpha > 0$. The class N_ψ will then be denoted by N^α; it consists of functions for which the integral

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \log^+|f|(\log^+ \log^+|f|)^\alpha d\theta_1 \ldots d\theta_m$$

where $f = f(r_1e^{i\theta_1}, \ldots, r_me^{i\theta_m})$, remains bounded. Clearly $H^\alpha \subset N^\beta$ for all positive α, β.

The two theorems which follow describe the behavior of f near component $\Gamma_1 \times \Delta_{m-1}$ of the boundary B.

THEOREM 1. Suppose that $f \varepsilon N$. Then almost all numbers θ_1^o in $(0, 2\pi)$ have the following property: if z_1 tends non-tangentially to $e^{i\theta_1^o}$, and $(z_2, \ldots, z_m)$ is confined to any $(m-1)$-cylinder

$$\Delta_{m-1}^{(\varepsilon)}: |z_2| \leq 1-\varepsilon, \ \ldots, \ |z_m| \leq 1-\varepsilon \quad (0 < \varepsilon < 1),$$

the function f tends uniformly to a finite limit $f^{\theta_1^o}(z_2, \ldots, z_m)$; this limit is regular in $\Delta_{m-1}: |z_2| < 1, \ldots, |z_m| < 1$ and belongs to N.

THEOREM 2. Theorem 1 holds if we replace N by N_ψ throughout. In particular, if $f(z_1, \ldots, z_m)$ is in H^a, then almost all $f^{\theta_1^o}(z_2, \ldots, z_m)$ are also in H^a.

Applying Theorem 1 to $f^{\theta_1^o}(z_2, \ldots, z_m)$, we see that for almost all pairs (θ_1^o, θ_2^o) the function

$$f^{\theta_1^o \theta_2^o}(z_3, \ldots, z_m) = \lim_{z_2 \to e^{i\theta_2^o}} f^{\theta_1^o}(z_2, z_3, \ldots, z_m)$$

exists and is in N, and so on. Finally, the iterated limit $f^{\theta_1^o \theta_2^o \ldots \theta_m^o}$ exists for almost all points $(e^{i\theta_1^o}, \ldots, e^{i\theta_m^o})$ of the distinguished boundary Γ_m.

The existence of the iterated limit for functions of the class N raises two problems. First, is this iterated limit independent of the order of the passage to the limit; for example, is

$$f^{\theta_1\theta_2}(z_3,\ldots,z_m) = f^{\theta_2\theta_1}(z_3,\ldots,z_m)$$

for almost all pairs (θ_1,θ_2)? The second question is, does there exist a limit when the variables $z_1, z_2, \ldots, z_m$ tend simultaneously to a point $(e^{i\theta_1^o},\ldots,e^{i\theta_m^o})$ of the distinguished boundary Γ_m? We discuss the second problem first.

We say that f has a restricted non-tangential limit s at $(e^{i\theta_1^o},\ldots,e^{i\theta_m^o})$ if $f \longrightarrow s$ as each z_j tends non-tangentially to $e^{i\theta_j^o}$ in such a way, however, that all the ratios

$$\frac{1-r_k}{1-r_\ell}$$

remain bounded.

THEOREM 3. If $f \varepsilon N$, then f has a finite restricted non-tangential limit at almost all points of Γ_m.

More generally:

THEOREM 4. If $f \varepsilon N$, $k < m$, then almost all points $(e^{i\theta_1^o},\ldots,e^{i\theta_m^o})$ of Γ_k have the following property:

if $(z_1, \ldots, z_k)$ tends restrictedly non-tangentially to $(e^{i\theta_1^o}, \ldots, e^{i\theta_k^o})$, f tends, uniformly in each (m-k)-cylinder

$$\Delta_{m-k}^{(\varepsilon)} : |z_{k+1}| \leq 1-\varepsilon, \ \ldots, \ |z_m| \leq 1-\varepsilon \qquad (0 < \varepsilon < 1),$$

to a function $f_{\theta_1^o \ldots \theta_k^o}(z_{k+1}, \ldots, z_m) \in N$. If $f(z_1, \ldots, z_m)$ is in N_ψ, then almost all functions $f_{\theta_1^o \ldots \theta_k^o}(z_{k+1}, \ldots, z_m)$ are also in N_ψ.

We may ask if the condition of restrictedness of the non-tangential approach in Theorem 3 can be dropped. The answer is probably no, though no examples are known. The answer is, however, affirmative if we limit ourselves to the class N^{m-1} of functions f such that the integral

$$\int_0^{2\pi} \ldots \int_0^{2\pi} \log^+|f|(\log^+ \log^+ |f|)^{m-1} d\theta_1 \ldots d\theta_m,$$

where $f = f(r_1 e^{i\theta_1}, \ldots, r_m e^{i\theta_m})$, remains bounded.

THEOREM 5. If $f(z_1, \ldots, z_m) \in N^{m-1}$ then f has an unrestricted non-tangential limit at almost all points of the distinguished boundary Γ_m.

More generally we have the following

THEOREM 6. Suppose that $1 \leq k < m$ and that $f(z_1, z_2, \ldots, z_m) \varepsilon N^{k-1}$. Then (i) f has a limit at almost all points $(e^{i\theta_1^o}, \ldots, e^{i\theta_m^o})$ of Γ_m if $(z_1, \ldots, z_k)$ tends non-tangentially to $(e^{i\theta_1^o}, \ldots, e^{i\theta_k^o})$, and $(z_{k+1}, \ldots, z_m)$ tends restrictedly non-tangentially to $(e^{i\theta_{k+1}^o}, \ldots, e^{i\theta_m^o})$ (ii) almost all points $(e^{i\theta_1^o}, \ldots, e^{i\theta_k^o})$ of Γ_k have the following property: if $(z_1, \ldots, z_k)$ tends unrestrictedly non-tangentially to $(e^{i\theta_1^o}, \ldots, e^{i\theta_k^o})$ the function f tends to a limit $f_{\theta_1^o, \ldots, \theta_k^o}(z_k, \ldots, z_m)$, uniformly in every (m-k)-cylinder

$$\Gamma_{m-k}: |z_{k+1}| < 1, \ \ldots, \ |z_m| < 1.$$

(By the second part of Theorem 4, $f_{\theta_1^o \ldots \theta_k^o}(z_{k+1}, \ldots, z_m)$ is in N^{k-1} for almost all $(\theta_1^o, \ldots, \theta_k^o)$.

The theorem which follows partly answers the problem about the independence of the iterated limit from the order of the passage to the limit.

THEOREM 7. Suppose that $f(z_1, \ldots, z_m) \varepsilon N^{m-1}$. Then for almost all points $(e^{i\theta_1}, \ldots, e^{i\theta_m})$ of Γ_m the non-tangential limit $f_{\theta_1 \ldots \theta_m}$ coincides with the iterated limit $f^{\theta_1 \ldots \theta_m}$. Hence, if $f \varepsilon N^{m-1}$, then at almost all points of Γ_m the value of the iterated limit is independent of the order of the passage to the limit.

The last conclusion applies, in particular, to functions from any H. Theorem 7 can be completed as follows.

THEOREM 8. Suppose that $1 \leq k < m$ and that $f(z_1, \ldots, z_m)$ is in N^{k-1}. Then (compare Theorems 4 and 6)

$$f_{\theta_1^o \ldots \theta_k^o}(z_{k+1}, \ldots, z_m) = f^{\theta_1^o \ldots \theta_k^o}(z_{k+1}, \ldots, z_m)$$

for almost all points $(e^{i\theta_1^o}, \ldots, e^{i\theta_k^o})$ of Γ_k.

Theorem 9 which follows is of a slightly different nature. It is not difficult, but is useful.

THEOREM 9. Suppose that $f(z_1, \ldots, z_m)$ is in some N_ψ, where ψ satisfies the condition $\lim_{u \to +\infty} \psi(u)/u = +\infty$. Let $\phi(u)$, $-\infty < u < +\infty$, be non-negative, non-decreasing and convex, and suppose that

$$\phi[\log |f_{\theta_1 \ldots \theta_m}|]$$

is integrable over Γ_m, where $f_{\theta_1 \ldots \theta_m}$ is the restricted non-tangential limit of $f(z_1, \ldots, z_m)$ (compare Theorem 3). Then

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \phi[\log |f(r_1 e^{i\theta_1}, \ldots, r_m e^{i\theta_m})|] d\theta_1 \ldots d\theta_m$$

$$\leq \int_0^{2\pi} \cdots \int_0^{2\pi} \phi[\log |f_{\theta_1 \ldots \theta_m}|] d\theta_1 \ldots d\theta_m.$$

In particular, if $f(z_1, \ldots, z_m)$ is in H^α, and $f_{\theta_1 \ldots \theta_m}$ is in L^β, then $f(z_1, \ldots, z_m)$ is in H^β.

The rest of this paper we devote to a slightly different problem whose solution seems to be rather difficult and may require subtle arguments from the theory of real variables. It is the problem of the existence of boundary values of analytic functions represented by Cauchy's integral. Consider first the case of functions of a single complex variable.

Let $F(\zeta)$ be an integrable, in general complex-valued, function defined on a simple closed curve C, sufficiently smooth, and consider the function

$$f(z) = \frac{1}{2\pi i} \int_C \frac{F(\zeta)}{\zeta - z} d\zeta$$

defined and regular in the interior of C. It is very well known that at almost all points of C a finite non-tangential limit exists. The result, though classical, is still non-trivial and the main difficulties arise already in the simplest case when C is the circumference of the unit circle. The general case when C is sufficiently smooth is reducible to the case of the circle.

A similar problem arises when we consider m-dimensional generalizations of Cauchy's integral. There are many generalizations possible but the main difficulties appear already in the simplest possible case of

$$f(z_1, \ldots, z_m) = \frac{1}{(2\pi i)^m} \int_{C_1} \cdots \int_{C_m} \frac{F(\zeta_1, \ldots, \zeta_m)}{(\zeta_1 - z_1) \ldots (\zeta_m - z_m)} d\zeta_1 \ldots d\zeta_m,$$

where $C_1, \ldots, C_m$ are circumferences of unit circles and $(z_1, \ldots, z_m)$ is in the m-cylinder Δ_m. The situation is relatively simple when F is in L^p on Γ_m, where p is greater than 1. A finite non-tangential limit then exists at almost all points of Γ_m.

The result still holds, though the proof is definitely more difficult if f is in $L(\log^+ L)^{m-1}$, that is if $|F|(\log^+ |F|)^{m-1}$ is integrable on Γ_m. This is the limit of our knowledge, and possibly also of the validity of the theorem. It is quite likely that the result is false if we impose less stringent conditions on the integrability of F, and in particular if we only assume that F is integrable. The main conjecture here is that the result holds if we consider restricted non-tangential passage to the limit. The problem is essentially of the same degree of difficulty as that of the existence almost everywhere of the singular integral

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{F(t_1, \ldots, t_m)}{(t_1 - x_1) \ldots (t_m - x_m)} dt_1 \ldots dt_m$$

in the sense of $\lim \int_{|x_1 - t_1| \geq \varepsilon_1} \cdots \int_{|x_m - t_m| \geq \varepsilon_m}$,

where F is integrable and the ε_j on the right are all equal, or, slightly more generally, tend to 0 in such a way that the ratio of any two of them remains bounded. In the absence of much knowledge about the boundary behavior of analytic functions of several variables the problem must be tackled by means of real variables only, and approached this way it seems to be quite difficult.

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Seminar II. CONFORMAL MAPPING AND SCHLICHT FUNCTIONS

ON FREE-BOUNDARY PROBLEMS
FOR THE LAPLACE EQUATION

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INTRODUCTION

The following typical free-boundary problem will be considered. Given is a doubly-connected region R limited by the infinite point and a compact boundary component Γ which does not reduce to a point. A positive continuous function Q is given in R. The problem is to find a finite annulus $\omega \subset R$ having Γ as one of its boundary components while the unknown component γ (the "free boundary") has to be determined so that there exists in ω an harmonic function V satisfying these conditions:

(1) $V = 0$ on Γ ,

(2) $V = 1$ on γ ,

(3) $|\text{grad } V| = Q$ on γ .

Since already the first two conditions determine V uniquely, the third condition is in general not satisfied for a given ω.

In spite of its hydrodynamic applications the theory of this kind of problem is still very incomplete, a state which might be explained by the following circumstances. Unless specific conditions are imposed on Q the problem may have none or an infinity of solutions, and even a two parametric family of solutions may exist. Another complicating factor is the occurrence of various types of solutions having quite different characteristics and requiring different approaches. Among the various types of solutions we shall distinguish three, defined as follows. Replace the right hand side of (3) by λQ where λ is a positive parameter and denote by ω_λ a solution of the new problem. Assume that for some $\varepsilon > 0$ and for $1 - \varepsilon < \lambda < 1 + \varepsilon$ a solution ω_λ exists with free boundary γ_λ contained in some given neighborhood of the free boundary of ω. If ω_λ is shrinking as λ increases, the solution $\omega = \omega_1$ of the original problem shall be called elliptic. If ω_λ is monotonic increasing with λ, ω shall be called hyperbolic. Finally, if ω_λ does not exist for $\lambda > 1$ (or for $\lambda < 1$) but two solutions ω_λ and ω'_λ do exist for $1 - \varepsilon \leq 1$ (or for $1 \leq \lambda < 1 + \varepsilon$) such that $\omega_\lambda \subset \omega \subset \omega'_\lambda$, where both regions tend to ω as $\lambda \longrightarrow 1$, then ω shall be called a parabolic solution.

The present paper is devoted to a method applicable, in particular, to elliptic and parabolic solutions. The hyperbolic solutions, however, are not obtainable by the procedure to be developed in this paper, nor do they seem to fall within the scope of any other general method.

THE EXISTENCE PROBLEM

We shall admit Q to tend to infinity at Γ within this limitation:

(4) $$Q = o\left(\frac{|\mathrm{grad}\ u|}{u}\right)$$

where u stands for any harmonic function which vanishes on Γ and is positive and regular in some annulus ω contained in R and having Γ as one boundary component. It should be pointed out that this condition is independent of the particular choice of u. We should also observe that our problem remains invariant under a conformal mapping $z \longrightarrow z_1$ of R. Conditions (1) and (2) will obviously remain the same while (3) will be changed into

(3') $$|\mathrm{grad}\ V|_{z_1} = Q(z(z_1)) \left|\frac{dz}{dz_1}\right| = Q_1(z_1) .$$

Q is thus transformed in such a way that the Riemann metric $ds = Q|dz|$ stays invariant, i.e.

$$Q(z)\ |dz| = Q_1(z_1)\ |dz_1| .$$

We recall at this instance that Gaussian curvature K of this metric is a true conformal invariant:

$$K = -\frac{\Delta z \log Q}{Q^2} = -\frac{\Delta z_1 \log Q_1}{Q_1^{\,2}} .$$

A convenient choice for R is the half-cylinder realized by the half-plane $z = x + iy$, $y > 0$, where points z, z' are identified if $z = z' \pmod{2\pi}$. From now on we will assume that this region is our R, and consequently, that $Q(x, y)$ is periodic in x and such that

(4')
$$\lim_{y \downarrow o} y\,Q(x, y) = 0 ,$$

which relation is the equivalent of (4).

The following definitions and notations will be used. By C we denote the set of all finite subannuli ω of R having $y = 0$ as one boundary component. For a given $\omega \in C$, the harmonic function with boundary values (1), (2) will be denoted V_ω and referred to as the stream function of ω. If γ is the free boundary of ω, $\bar{\gamma}$ will denote the reflexion of γ in $y = 0$, and $\bar{\omega}$ will denote the annulus limited by γ and $\bar{\gamma}$. Each stream function V_ω can be continued analytically across $y = 0$ into $\bar{\omega}$ and assumes the value -1 on $\bar{\gamma}$. The notation p_ω will stand for the function $\log |\text{grad } V_\omega|$; p_ω is harmonic in $\bar{\omega}$ and symmetric with respect to $y = 0$. For ω given,

q_ω shall denote the solution of the Dirichlet problem for $\bar{\omega}$ with boundary values $\log Q(x, |y|)$. Outside $\bar{\omega}$, $q_\omega(z)$ shall be defined as $\log Q(x, |y|)$. Frequent use will be made of the facts that q_ω is everywhere continuous and $\frac{\partial q_\omega}{\partial y} = 0$ on $y = 0$.

To each $\omega \in C$ we assign these quantities:

$$a(\omega, Q) = a(\omega) = \liminf Q^{-1} |\text{grad } V_\omega| ,$$

$$b(\omega, Q) = b(\omega) = \limsup Q^{-1} |\text{grad } V_\omega| ,$$

where the limits are taken for $z \in \omega$ tending to the free boundary. By means of these limits we define the following three sets of annuli:

$$A(Q) = A = \{\omega; \omega \in C, a(\omega) \geq 1\} ,$$

$$B(Q) = B = \{\omega; \omega \in C, b(\omega) \leq 1\} ,$$

$$B_o(Q) = B_o = \{\omega; \omega \in C, b(\omega) < 1\} .$$

The proper definition of the somewhat ambiguous condition (3) is $a(\omega) = b(\omega) = 1$. The intersection $A \cap B$ is thus identic with the set of solutions.

On comparing the two harmonic functions p_ω and q_ω in $\bar{\omega}$ we find by the minimum principle that the property $\omega \in A$ is equivalent with the inequality

$$p_\omega = q_\omega \geq 0, \qquad z \in \bar{\omega} .$$

Similarly, $\omega \in B$ if and only if the same difference is ≤ 0 in $\bar{\omega}$. From this we derive

LEMMA I. Let $\{\omega_n\}_1^\infty$ be a sequence of increasing annuli belonging to A (B, or $A \cap B$) and having a limit $\omega \in C$, then ω belongs to A (B, or $A \cap B$).

If $\omega_n \in A$, then $p_{\omega_n} \geq q_{\omega_n}$ in $\bar{\omega}_n$. At each fixed point in $\bar{\omega}$ we will have as $n \longrightarrow \infty$

$$q_{\omega_n} \longrightarrow q_\omega \, , \; V_{\omega_n} \longrightarrow V_\omega \, , \; p_{\omega_n} \longrightarrow p_\omega \; .$$

Therefore $p_\omega \geq q_\omega$ in $\bar{\omega}$, and hence $\omega \in A$. The alternate cases have similar proofs.

LEMMA II. Assume $\omega_1 \subset \omega_2$, $\omega_1 \in A$, $\omega_2 \in B_o$. Then ω_2 contains the free boundary of ω_1.

Let V_1, V_2 be the stream functions of ω_1, ω_2 respectively. By the maximum principle we find that

$$(5) \qquad V_2(z) \leq V_1(z) \, , \qquad z \in \omega_1 \, .$$

Assume that the free boundaries of ω_1, ω_2 have a point z_o in common. Let α be the curve ending at z_o which is orthogonal to the level lines of V_1. The existence of α is assured by the fact that each $\omega \in A$ has a rectifiable boundary. This curve α is geodesic in the metric $|\text{grad } V_1|\,|dz|$. If z is a point on α we will have

$$1 - V_1(z) = \int_{z_o}^{z} |\text{grad } V_1|\,|dz| \quad ,$$

$$1 - V_2(z) \leq \int_{z_o}^{z} |\text{grad } V_2|\,|dz| \quad ,$$

where the integration is taken along α. According to our assumptions there exists an $\varepsilon > 0$ such that on the path of integration

$$|\text{grad } V_2| < (1-\varepsilon)Q < |\text{grad } V_1| \ ,$$

provided z is sufficiently close to z_o. Thus, $1 - V_2(z) < 1 - V_1(z)$, contradictory to (5).

Our next lemma requires these definitions:

The extended union of two annuli ω_1, $\omega_2 \in C$ consists of those points $z \in R$ which can be separated from the infinite point of R by a Jordan curve homotopic with Γ and contained in the union of the sets ω_1, ω_2.

The reduced intersection of ω_1, ω_2 consists of those points

$z \in R$ which can be joined with Γ by a Jordan arc, which, except for its endpoint on Γ, is contained in the intersection of ω_1, ω_2.

Both the extended union and the reduced intersection of ω_1, $\omega_2 \in C$ are annuli $\in C$.

LEMMA III. If ω_1, ω_2 belong to A, so does their extended union. Similarly, if ω_1, ω_2 belong to B, so does their reduced intersection.

The lemma is an elementary consequence of the maximum-minimum principle. The reader is referred to a previous paper (Acta Math. V. 90, 1953, pp. 117-130) where a detailed proof is given for a similar case. After these preliminaries we are now able to establish the following result concerning existence and localization of solutions.

THEOREM I. Assume $\Omega \subset \Omega_o$, $\Omega \in A$, $\Omega_o \in B_o$. Then there exists a solution ω^* such that $\Omega \subset \omega^* \subset \Omega_o$.

The set

$$S = \{\omega; \omega \in A, \Omega \subset \omega \subset \Omega_o\}$$

is not void because it contains Ω. Let us first show that S includes an element ω^* which is maximal in the sense that ω^* contains each $\omega \in S$. To this purpose we form the ordinary union U of all $\omega \in S$ and choose a sequence $\{D_n\}_1^\infty \in S$ such that each point $z \in U$ is contained in some D_n. We next define a sequence $\{\omega_n\}_1^\infty$ by the following procedure: $\omega_1 = D_1$, ω_{n+1} = the extended union of ω_n and D_{n+1}, $n = 2, 3, \ldots$. It follows by the three previous lemmas that $\{\omega_n\}_1^\infty$ as well as its limit ω^* belong to S and, moreover, that the free boundary of ω^* is contained in Ω_o. This annulus ω^* is not only maximal in S, it is also locally maximal in A in the sense that there exists a neighborhood N of its free boundary such that ω^* is not contained in any $\omega \in A$, $\omega \neq \omega^*$, with free boundary located in N. We shall show that each locally maximal annulus $\omega \in A$ is a solution. If this were not true the non-negative harmonic function $p_\omega - q_\omega$ would be positive throughout $\bar{\omega}$, and consequently

$$|\text{grad } V_\omega| > e^{q_\omega}, \quad z \in \omega .$$

From this we conclude that the distance $[z, \omega]$ from a point z in ω to $y = 0$, measured within ω in the metric $e^{q_\omega}|dz|$ would be $< V_\omega(z)$, and in particular that

$$k = \min_{z \in \gamma} [z, \omega] < 1 .$$

The annulus $D = \{z; 0 < [z, \omega] < k\}$ would therefore be contained in ω and have at least one boundary point, say z_o, in common with the free boundary of ω. Let r be a conjugate harmonic function of $q = q_\omega$. Since $\frac{\partial q}{\partial y} = 0$ on $y = 0$ we may normalize r by the condition $r(x, 0) = 0$. Consider the analytic function

$$f = \varphi + i\psi = \int_o^z e^{q+ir}\, dz, \qquad z \in \omega .$$

Its imaginary part ψ will coincide with $[z, \omega]$ in the region D and the stream function V of D is therefore $= \psi/k$. Consequently

$$|\text{grad } V| = \frac{e^{q_\omega}}{k} , \quad z \in D .$$

We will show that this relation implies that ω is not locally maximal. To this purpose consider a small translation, $z \longrightarrow z + \delta$, of the free boundary β of D taking it into β_δ. Let D_δ be the annulus having β_δ as free boundary and let V_δ be its stream function. It is elementary to show that if $m(\delta)$ is the maximum of $V_\delta(x + iy)$ for $0 < y \le |\delta|$, then

$$\frac{|\text{grad } V_\delta(z+\delta)|}{|\text{grad } V(z)|} \geq 1 - m(\delta) , \qquad z \in D .$$

Since $m(\delta)$ tends to 0 with $|\delta|$, we can choose δ arbitrarily small but so that D_δ contains the point z_o while D_δ itself is contained in the set $A(e^{q_\omega})$. By Lemma III we conclude that the extended union ω_δ of ω and D_δ belongs to $A(e^{q_\omega})$. Since however $e^{q_\omega} = Q$ on the free boundary of ω_δ it follows that $\omega_\delta \in A(Q)$, which contradicts our assumption that ω is locally maximal. This proves our statement and also finishes the proof of Theorem I.

THEOREM II. A necessary and sufficient condition for the existence of a solution is that the set $B(Q)$ is not void.

The necessity is obvious since each solution belongs to B. Assume first that there exists a region $\Omega \in B_o$, and let V be its stream function. The stream function of the annulus Ω_t, having the level line $V = t$ as free boundary, is then V/t. If t is small, then

$$|\text{grad } V/t| > \frac{\text{const.}}{y} > Q(x, y)$$

on the free boundary of Ω_t. Hence $\Omega_t \in$ A and Theorem I asserts the existence of a solution $\omega \subset \Omega$. If Ω belongs to B but not to B_o we replace the right hand side of (3) by λQ, where λ is a parameter > 1. Theorem I applies to this case and the problem has a solution ω_λ defined as the maximal region of the set $\{\omega; \omega \in A(\lambda Q), \omega \subset \Omega\}$. For λ decreasing to 1, ω_λ will increase and tend to a limit $\omega \subset \Omega$, which by Lemma I has to be a solution of the original problem.

THEOREM III. Assume that ω_1 and ω_2 are solutions, none contained in the other. Then there exists a third solution ω_3 contained in both ω_1 and ω_2.

The reduced intersection Ω of ω_1, ω_2 belongs to B and it follows by the previous discussion that Ω contains a solution.

THE UNIQUENESS PROBLEM

We have already pointed out that uniqueness requires specific conditions imposed on Q. In this section we shall derive some conditions implying either global or local uniqueness.

We set

$$\ell = \inf \int_\gamma Q(z)\,|dz|$$

where γ is a rectifiable Jordan curve homotopic with Γ and contained in R. The length of Γ shall be defined as

$$L = \inf \int_{\gamma} Q(z)\,|dz|$$

for γ tending to Γ. In case $L = \ell$ we shall say that Γ is convex in the metric $Q|dz|$.

THEOREM IV. Let the given metric have Gaussian curvature ≤ 0 (in the sense that log Q is subharmonic in R) and let Γ be convex. Then there cannot exist more than one solution.

It is well known that the quantity

$$\ell(\omega) = \int_{\Gamma} |\text{grad } V_{\omega}|\ |dz|$$

is strictly decreasing for ω increasing: if $\omega_1 \subset \omega_2$, $\omega_1 \neq \omega_2$, then $\ell(\omega_1) > \ell(\omega_2)$. From our conditions on Q and Γ we can conclude that $\log Q(x, |y|)$ is subharmonic not only in $y > 0$ but in each finite region of the z-plane as well. This implies that q_{ω} is non-decreasing for ω increasing. By Theorem III we know that if the problem has more than one solution, then there are solutions ω_1, ω_2 such that $\omega_1 \subset \omega_2$, $\omega_1 \neq \omega_2$. On Γ we would therefore have

$$|\operatorname{grad} V_{\omega_1}| = e^{q_{\omega_1}} \leq e^{q_{\omega_2}} = |\operatorname{grad} V_{\omega_2}| .$$

Thus, $\ell(\omega_1) \leq \ell(\omega_2)$, and this contradiction proves Theorem IV.

THEOREM V. Let ω be a solution such that the Gaussian curvature of the metric $Q|dz|$ is < -1 throughout ω. Then ω cannot contain any other solution, nor can any solution have a free boundary point in ω.

In this theorem we assume that Q has continuous derivatives up to the second order. Let V be the stream function of ω and set

$$\varphi(z) = \frac{|\operatorname{grad} V(z)|}{V(z)Q(z)} .$$

We shall show that the assumptions on Q imply $\varphi(z) > 1$, $z \in \omega$. For z tending to the free boundary we have $\varphi(z) \longrightarrow 1$. As z tends to Γ, $|\operatorname{grad} V|$ is bounded away from 0 while VQ tends to 0 as $y\, Q(x, y)$ and thus $\varphi(z) \longrightarrow +\infty$. If the inequality $\varphi > 1$ were not true φ would have a minimum $m \leq 1$ at some point $z_o \in \omega$ and we would have $\Delta \log \varphi \geq 0$ at z_o. We find that

$$\Delta \log \frac{|\text{grad } V|}{V} = \frac{|\text{grad } V|^2}{V^2} .$$

Hence, at z_o

$$0 \leq \Delta \log \varphi = \frac{|\text{grad } V|^2}{V^2} - \Delta \log Q = m^2 Q^2 - \Delta \log Q .$$

For the Gaussian curvature at z_o we obtain

$$K(z_o) = - \frac{\Delta \log Q}{Q^2} \geq - m^2 \geq -1 ,$$

which is contradictory to our assumption.

Assume now that there exists a solution, say D, with a free boundary intersecting ω. Let t be the largest number such that the annulus $\omega_t = \{z;\ 0 < V(z) < t\}$ is contained in D. The stream function of ω_t is V/t. We have just shown that on the level line V = t,

$$|\text{grad } V/t| > Q .$$

Since both functions involved in this inequality are continuous there exists a number $\lambda > 1$ such that ω_t belongs to the set $A(\lambda Q)$. Since D is a solution it will belong to $B_o(\lambda Q)$. Thus,

$\omega_t \subset D$ while the free boundaries of the two regions have a point in common. This is contradictory to Lemma II and the existence of D is thus disproved.

As a concluding remark we wish to recall that according to the definitions given in the introduction an elliptic solution is locally maximal in the set A and hyperbolic solutions locally minimal in the same set. The important difference between these cases is that each locally maximal region is a solution while minimal regions in general are not solutions. In this connection it should be pointed out that a great variety of problems in hydrodynamics are of elliptic type and their solutions obtainable by the procedure developed in this paper. This is for example the case with the jet and wake problems solved by Weinstein and Leray respectively. The classical wave problems, on the other hand, are examples of hyperbolic nature and our method does not apply.

SOME SOLVED AND UNSOLVED COEFFICIENT PROBLEMS FOR SCHLICHT FUNCTIONS

W. K. Hayman

Let S be the class of functions

$$w = f(z) = z + \sum_{1}^{\infty} a_n z^n$$

univalent in $|z| < 1$. I propose to discuss certain problems relating to the coefficients a_n of these functions and also to deal briefly with some other classes of univalent functions.

A basic problem concerns the upper bound of the $|a_n|$. It was conjectured by Bieberbach [2] in 1916 that

$$|a_n| \leq n$$

holds for all $f(z) \in S$, with equality only for the extremals $f(z) = z(1 - ze^{i\theta})^{-2}$. This was proved by him for $n = 2$, by Löwner [15] for $n = 3$ and Garabedian and Schiffer [6] for $n = 4$. It was also proved for functions mapping the unit circle onto a star domain by Nevanlinna [18], and the more general close-to-convex functions by Reade [19] and for functions with real coefficients by Rogosinski [20].

In the general case the best results have been obtained by a method initiated by Littlewood [12] and refined by Bazilevic [1], who proved

$$|a_n| < \frac{1}{2} en + 1.51, \qquad n \geq 2.$$

Let

$$M(r, f) = \sup_{|z|=r} |f(z)|$$

and

$$\psi(r) = \frac{(1-r)^2}{r} M(r, f).$$

Recently I proved [9] that, except for the above extremals, $\psi(r)$ decreases steadily with increasing r, $0 < r < 1$, and so approaches a limit $\alpha < 1$ as $r \longrightarrow 1$. Further

$$\frac{|a_n|}{n} \longrightarrow \alpha, \text{ as } n \longrightarrow \infty.$$

The method is highly non-uniform and although it shows that $|a_n| \leq n$ from a certain n_o onwards, n_o depends essentially on the function $f(z)$. The best we can prove is

THEOREM I. Suppose that $f(z) \in S$. Then, given $\varepsilon > 0$, the inequality

$$\left| \frac{|a_n|}{n} - \psi(1 - \frac{1}{n}) \right| < \varepsilon$$

holds for all n except possibly those in ranges

$n_\nu \leq n \leq 2n_\nu$, $\nu = 1$ to p, where $p \leq A(\varepsilon)$.

Essentially the exceptional ranges are those in which $\psi(1 - \frac{1}{n})$ is still greater than a positive constant but decreasing relatively rapidly. To show that no upper bound can be placed on the exceptional suffixes n_ν in the above theorem, consider

$$f_\theta(z) = \frac{z}{1 - 2z \cos \theta + z^2} = \sum_1^\infty z^n \frac{\sin n\theta}{\sin \theta} .$$

It is easily seen that $f_\theta(z) \varepsilon S$ for $0 < \theta < \frac{\pi}{2}$. Also

$$\psi(r) = \frac{(1 - r)^2}{(1-r)^2 + 2r(1-\cos \theta)}, \text{ if } r + \frac{1}{r} > 2 \sec \theta;$$

$$\psi(r) = \frac{(1 - r)}{(1+r)\sin \theta}, \text{ if } r + \frac{1}{r} \leq 2 \sec \theta.$$

If we make $\theta \longrightarrow 0$ and choose $n = t/\theta$, where t remains between positive constants then

$$\frac{a_n}{n} = [1+ O(1)] \frac{\sin t}{t}$$

$$\psi(1 - \frac{1}{n}) = \frac{1 + O(1)}{1 + t^2}, \quad \text{if } t \leq 1,$$

$$\psi(1 - \frac{1}{n}) = \frac{1 + O(1)}{2t}, \quad \text{if } t \geq 1.$$

Thus a_n/n and $\psi(1 - 1/n)$ are both nearly 1 if $n = O(\theta^{-1})$ and both small if $\theta^{-1} = O(n)$. If n is comparable with θ^{-1}, $\psi(1 - \frac{1}{n})$ decreases relatively rapidly with n and a_n/n oscillates between $\mp 2\psi(1 - \frac{1}{n})$. In particular $a_n = 0$ and $\psi(1 - \frac{1}{n}) > (1-\varepsilon)/(2\pi)$ are compatible for sufficiently large n and f depending on n and so are the inequalities

$$(1 + \varepsilon)\frac{a_n}{n} > 2\psi(1 - \frac{1}{n}) > \frac{2 - \varepsilon}{\pi}.$$

Thus the question of the behavior of

$$A_n = \sup_{f \varepsilon S} |a_n|$$

does not seem to be accessible by the methods of Theorem I alone. By using other methods as well one can show, however, that

$$\frac{A_n}{n} \longrightarrow K_o, \quad \text{as } n \longrightarrow \infty$$

where $1 \leq K_o < \frac{1}{2} e$, [10]. The constant K_o is the upper bound for the inverse Laplace transform of a certain class of functions univalent in a half plane. The inequality $K_o < \frac{1}{2} e$, is based on Bazilevic's result and gives a slight improvement of his bounds for large n. Further Nehari [16] has recently proved the inequality

$$|a_n| \leq 4K_o \, dn, \qquad n = 1, 2, 3, \ldots$$

where d is the modulus of any value w such that $f(z) \neq w$ in $|z| < 1$. If $K_o = 1$, then the resulting inequality would be sharp for every n, though not as strong as Bieberbach's conjecture. Thus the conjecture $K_o = 1$ represents in several ways an intermediate result which may be easier to prove than the full Bieberbach conjecture.

Let us now briefly consider what we can say about the coefficients of certain subclasses of S. Here the key result is the following due implicitly to Paley and Littlewood [14] and explicitly to Spencer [23].

THEOREM 2. Suppose that $f(z) \in S$ and that there exist constants $C > 0$ and $\alpha > \frac{1}{2}$, such that

(1) $M(r, f) < C(1-r)^{-\alpha}, \qquad 0 < r < 1;$

then we have

(2) $$|a_n| < A(\alpha)Cn^{\alpha-1}, \qquad n \geq 1.$$

If (1) holds with $\alpha = \frac{1}{2}$, we can deduce only

$$|a_n| < An^{-\frac{1}{2}} \log(1 + n),$$

and if $\alpha < \frac{1}{2}$

$$|a_n| < A(\alpha)n^{-\frac{1}{2}} [\log (1 + n)]^{\frac{1}{2}},$$

instead of (2).

If $f(z) \varepsilon S$ and is of the form

(3) $$f(z) = z + a_{k+1}z^{k+1} + a_{2k+1}z^{2k+1} + \ldots$$

then we easily see that $[f(z^{\frac{1}{k}})]^k \varepsilon S$, so that (1) holds with $\alpha = 2/k$. Thus in this case (2) gives for $k = 1, 2, 3$

(4) $$|a_n| = O(n^{\frac{2}{k}-1}).$$

One can also prove by a somewhat different method (4) that (1) holds with $\alpha = 2/k$, if for each r, $0 < r < 1$, we have

$$|f(z)| > C_1 M(r, f)$$

at k points $z = re^{i\theta_\nu}$, where $\theta_{\nu+1} - \theta_\nu > C_2$, $\nu = 1$ to k, $\theta_{k+1} = \theta_o + 2\pi$, and C_1, C_2 are constants. This hypothesis is satisfied with $k = 2$, for instance, if $a_n = 0$ for some sequence of suffixes n in arithmetic progression with common difference a say. Hence in this case it follows from (2) that all the other coefficients are uniformly bounded, [12]. On the other hand our above example of the functions $f_\theta(z)$ shows that the other coefficients may be as large as cosec (π/a) (if a is even, and slightly less if a is odd). We do not know what effect on the other coefficients an irregular or a more rapidly growing sequence a_{n_ν} of zero coefficients may have. However Biernacki [3] has recently proved that for any $f(z) \varepsilon S$ we have

$$||a_{n+1}| - |a_n|| < A\{\log(n+1)\}^{\frac{3}{2}} .$$

This result, although probably not best possible, represents a very strong advance on what was known previously and is our only tool available so far for tackling the general problem.

Another open problem concerns the order of magnitude of the coefficients of bounded schlicht functions, or more generally those satisfying (1) with $\alpha \leq \frac{1}{2}$. If $f(z)$ is bounded, the area of the image domain is finite, so that $\sum n|a_n|^2$ converges and so

$$(5) \qquad |a_n| = o(n^{-\frac{1}{2}}), \text{ as } n \longrightarrow \infty.$$

Nothing stronger than this is known. In the opposite direction Littlewood [13] has constructed examples of bounded functions $f(z) \in S$ of the form (3) which satisfy

$$(6) \qquad |a_n| > n^{-1 + \frac{A_o}{\log k}}$$

for some arbitrarily large n, where A_o is a small absolute constant. Thus (4) breaks down for large k and the implication from (1) to (2) breaks down for $\alpha < \frac{A_o}{\log 2}$. My own conjecture would be that (5) is the strongest result that is true in any of these cases, when $f(z)$ is bounded. This is known to be the case for mean 1-valent functions [22], which otherwise have much the same growth properties as univalent functions. However it is much more difficult to construct schlicht counterexamples.

Let us now consider the class Σ of functions

$$f(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

schlicht in $|z| > 1$, and with a simple pole of residue 1 at infinity. For these functions we have again

$$\Sigma\, n|a_n|^2 \leq 1 \tag{7}$$

and this suggests an analogy between them and bounded functions in S. In particular the functions in Σ also satisfy (5). It follows trivially from (7) that $|a_1| \leq 1$ for $f(z) \in \Sigma$. Next Schiffer [21] and Golusin [7] showed that $|a_2| \leq \frac{2}{3}$, with equality only for $(z^3 + 2 + z^{-3})^{\frac{2}{3}} = z + \frac{2}{3}z^{-2} + \ldots$. This led to the conjecture that $|a_n| \leq 2/(n+1)$, for $f(z) \in \Sigma$, with equality for

$$f(z) = (z^{n+1} + 2 + z^{-n-1})^{\frac{2}{n+1}} \in \Sigma.$$

This result has recently been proved (by Nehari and Netanyahu [17]) for $n = 3, 4, 5$ and 6 in the special case when the set of values not taken by $f(z)$ is star-shaped with respect to the origin. Nevertheless the conjecture is false. Garabedian and Schiffer [5], showed that the correct bound for $|a_3|$ is $\frac{1}{2} + e^{-6}$. (This had previously been proved for odd functions by Golusin [8].) Further Clunie [4] has recently shown that (6) is possible for $f(z) \in \Sigma$ and of the form

$$f(z) = z + \frac{a_{k-1}}{z^{k-1}} + \frac{a_{2k-1}}{z^{2k-1}} + \ldots .$$

It may be worth while to say a little more about Clunie's examples. Let k be a positive integer and

$$\phi(\zeta) = \exp \{\lambda \sum_{t=1}^{\infty} \zeta^{k^t} \} = \sum_0^{\infty} c_p \zeta^p .$$

Then Clunie has shown that if $1 < \lambda < \frac{\pi}{2}$ and $k > k_o(\lambda)$, then

$$\phi_1(z) = \int_0^z \phi(\zeta) d\zeta = \sum_0^{\infty} \frac{c_p \zeta^{p+1}}{p+1} \in S,$$

and

$$\phi_2(z) = \int^z \phi(\frac{1}{\zeta}) d\zeta = \sum_0^{\infty} \frac{c_p \zeta^{-p+1}}{1-p} \in \Sigma .$$

In each case the function remains continuous on the unit circle $|z| = 1$, which corresponds to a Jordan curve.

It is evident that $\phi(\zeta) = (1 + \lambda\zeta^k + \ldots)(1 + \lambda\zeta^{k^2} + \ldots)\ldots$ has non-negative coefficients and if p is of the form $p = k + k^2 + \ldots + k^{\nu}$ then

$$c_p > \lambda^{\nu} = \exp(\nu \log \lambda) \geq \exp \frac{\log(\frac{1}{2}p) \log \lambda}{\log k} = (\frac{p}{2})^{\frac{\log \lambda}{\log k}},$$

since $p < 2k^{\nu}$. Now the results follow. We may take for A_o any constant less than $\log(\frac{\pi}{2})$ in (6) provided k is large enough.

Finally by giving λ and k suitable numerical values, Clunie has constructed a function $f(z) \varepsilon \Sigma$, for which

$$a_n > n^{-\frac{13}{14}}$$

for some arbitrarily large n.

In conclusion let me mention briefly the results for inverse functions of S and Σ. If $w = f(z) \varepsilon S$ let

$$z = f^{-1}(w) = w + b_2 w^2 + b_3 w^3 + \dots .$$

Then Löwner's method enabled him to show that for every n the sharp inequality

$$|b_n| \leq \frac{2^n . 1 . 3 \dots (2n-1)}{(n+1)!}$$

holds with equality when

$$w = f(z) = \frac{z}{(1+z)^2}, \quad z = f^{-1}(w) = \frac{1 - 2w - (1-4w)^{\frac{1}{2}}}{2w} .$$

If

$$w = f(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots \varepsilon \Sigma$$

let

$$z = w + \frac{b_1}{w} + \frac{b_2}{w^2} + \ldots .$$

Then it is easily seen that $b_1 = -a_1$, $b_2 = -a_2$, so that the sharp inequalities $|b_1| \leq 1$, $|b_2| \leq \frac{2}{3}$ hold [7, 21]. However, for $|b_3|$ we have the sharp bound $|b_3| \leq 1$ due to Springer [24], with equality when

$$w = f(z) = z + \frac{1}{2}, \quad z = w - \frac{1}{w} - \frac{1}{w^3} + \ldots .$$

Springer also proved that for larger n

$$|b_n| \leq \frac{2^n}{n}$$

and constructed examples to show that for large odd n and any $\varepsilon > 0$

$$|b_n| > (1 - \varepsilon) \frac{2^{2n-2} e}{(\pi n^3)^{1/2}}$$

is possible, and he conjectured that his examples give the largest coefficients that can occur for odd n.

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SEMIGROUPS OF CONFORMAL MAPPINGS

C. Loewner

This paper is concerned with a study of analytic functions $f(z)$, $z = x + iy$, which map the upper half-plane $y > 0$ schlicht into itself, and have real boundary values on an open interval (a, b) of the x-axis (which may depend on f). We are in particular interested in the behavior of f in (a, b). The totality of real transformations $x' = f(x)$ thus obtained forms a semigroup and it is our objective to characterize it in a purely group theoretic way. The result was announced by the author, without proof, in an address [1] before the American Mathematical Society.

The concept of a transformation semigroup or transformation will be used in a way slightly deviating from the usual one.[1] It is therefore necessary to give the relevant definitions. A transformation f on the x-axis is described by a function $x' = f(x)$ which we shall always assume to be defined on a finite or infinite open interval (a, b). We further make the restrictive assumption that $f(x)$ is continuous and strictly increasing in (a, b).

A system S of transformations on the x-axis will be called a transformation semigroup if the following conditions are satisfied:

[1] The expression "pseudogroup" instead of "group" would perhaps be more appropriate.

(a) If $f \varepsilon S$ then the same f considered only in a subinterval (a', b') of the original interval (a, b) should also belong to S.

(b) If $f \varepsilon S$ and $g \varepsilon S$ and the range of g belongs to the domain of f then the composite function $f(g(x))$ should also represent a transformation of S.

(c) Let f_n $(n = 1, 2, 3, \ldots)$ be a sequence of transformations of S all defined on the same interval (a, b) which converge uniformly in any closed subinterval of (a, b) to a strictly monotonic function f. Then f also belongs to S.

A system G of transformations will be called a group if the further condition

(d) $f \varepsilon G$ implies $f^{-1} \varepsilon G$, f^{-1} being the function inverse to f

is satisfied.

One verifies easily that the above introduced real functions derived from schlicht mappings of the upper half-plane form a semigroup T of transformations on the x-axis. As emphasized before, our objective is to characterize T in a purely group-theoretic way.

We remark first that T contains the group P of all proper projective mappings $x' = \frac{\alpha x + \beta}{\gamma x + \delta}$ $(\alpha\delta - \beta\gamma > 0)$ considered in intervals (a, b) not containing the pole $(-\frac{\delta}{\gamma})$. The inverses of all transformations of T again form a semigroup T' containing the group P. We now will prove the

THEOREM: Each proper extension of P to a semigroup S which contains three times continuously differentiable non-projective transformations must contain either T or T' completely. If it contains non-projective transformations from both T and T' it coincides with the group of all continuous transformations.

REMARK: It would be desirable to eliminate the assumption of the existence of a three times continuously differentiable non-projective mapping. I was not able to do so.

Let now $f_o(x)$ be a special non-projective three times continuously differentiable mapping of S. Since the projective mappings p(x) are characterized by the differential equation $p'p''' - \frac{3}{2}(p'')^2 = 0$ one may assume that $f_o(x)$, in the interval I in which it is defined, satisfies the inequality

(1) $$f_o'(x)f_o'''(x) - \frac{3}{2}f_o''^2(x) \neq 0 .$$

One may further assume that in I

(2) $$f_o'(x) > 0.$$

This is the case because $f_o(x)$ is monotonically increasing and I can be replaced by a smaller interval if necessary.

New functions of S satisfying (1) and (2) can be constructed by composing f_o with projective mappings. One can thus arrange that $I = (-c, c)$ contains the origin $x = 0$ and that

(3) $$f_o(0) = 0, \quad f_o'(0) = 1, \quad f_o''(0) = 0, \quad f_o'''(0) = \pm 6.$$

The Taylor expansion of $f_o(x)$ at $x = 0$ then has the form

(4) $$f_o(x) = x \pm x^3 + o(x^3) \qquad (|x| < c).$$

We will assume that the positive sign holds in (4). The negative sign would be obtained if S were replaced by the semigroup S' consisting of the inverses of the mappings of S.

From (4) we will construct infinitesimal transformations of S. We form the one-parameter family of transformations of S depending on a parameter $t \geq 0$

$$f_t(x) = \frac{1}{t} f_o(tx) = x + t^2x^3 + t^2o(x^3) \quad (|x| < \frac{c}{t}) \tag{5}$$

which for $t = 0$ yields the identity mapping of the whole real axis. Changing to the new parameter $\tau = t^2$ we obtain in

$$\xi_o(x) = \left(\frac{\partial f_t(x)}{\partial \tau}\right)_{\tau=0} = x^3 \tag{6}$$

<u>an infinitesimal transformation of S</u> defined on the whole real axis. From (6) new infinitesimal transformations of S can be obtained by subjecting $\xi_o(x)$ to proper projective transformations. Let $x_1 = \frac{\alpha x + \beta}{\gamma x + \delta}$ $(\alpha\delta - \beta\gamma > 0)$. Then

$$\xi_1(x) = x_1^3 \frac{dx}{dx_1} = \frac{(\alpha x + \beta)^3}{(\alpha\delta - \beta\gamma)(\gamma x + \delta)} \tag{7}$$

again represents an infinitesimal transformation of S since we have assumed that S contains the projective group P. Further infinitesimal transformations of S can be constructed by multiplying $\xi_1(x)$ by an arbitrary constant positive factor and adding an arbitrary infinitesimal transformation of P. It is well known that the latter is represented by an arbitrary quadratic expression in x. This leads, as one easily sees, to an expression of the form

$$\xi(x) = \frac{P_3(x)}{\gamma x + \delta} \tag{8}$$

where $P_3(x)$ represents a polynomial of not higher than the third degree subject only to the condition that the residue of $\xi(x)$ at the pole $x = -\frac{\delta}{\gamma}$ is negative.

The knowledge of the infinitesimal transformations (8) of S permits us to construct finite transformations of S by the process of composition of infinitesimal transformations. The fact that the generated mappings belong to S follows from the assumption (c) in the definition of semigroup. Our essential objective is to show that every transformation generated in this way belongs to T and that every mapping of T can be arbitrarily closely approximated by mappings obtained in this way. Since projective mappings are available there is no loss of generality if we restrict ourselves to mappings $x_1 = f(x)$ defined in the interval $(-1, 1)$ and we may assume that this interval is transformed onto itself so that $f(0) = 0$. We have therefore

(9) $$a = -1 \qquad b = 1$$

(10) $$f(-1 + 0) = -1, \quad f(0) = 0, \quad f(1 - 0) = 1.$$

Correspondingly we make use only of those infinitesimal transformations of the form (8) which keep the points $x = \pm 1$ and $x = 0$ fixed;

i.e., for which

(11) $$\xi(-1) = \xi(0) = \xi(1) = 0.$$

Then $\xi(x)$ is, except for a positive constant factor, of the form

(12) $$\xi(x) = -\frac{x(1 - x^2)}{1 - kx} \qquad (|k| < 1).$$

Now we are interested in the transformations composed of infinitesimal transformations of type (12) and their behavior after continuation into the complex domain. This leads to the study of differential equations in the complex domain of the form

(13) $$\frac{dw}{dt} = -\frac{w(1 - w^2)}{1 - k(t)w} \qquad (|k(t)| < 1)$$

with an arbitrary real continuous function $k(t)$ defined in some interval $0 \leq t \leq t_o$ subject only to the condition $|k(t)| < 1$. Let

(14) $$w = g(z, t)$$

be the solution of (13) satisfying the initial condition

(15) $$g(z, 0) = z.$$

The function $g(z, t_o)$ represents, in the complex domain, the mapping generated by the infinitesimal transformations described by the right hand side of (13). We shall first prove that the solution exists in the whole interval $[0, t_o]$ for all real values z between -1 and +1 and all complex values z.

The first part of this assertion follows immediately from the fact that -1 and 1 remain fixed for the whole interval $[0, t_o]$ and therefore any initial value between -1 and 1 leads to a solution which cannot leave the interval (-1, 1).

The second part of the assertion can be proved in the following way: We introduce in the upper half of the w-plane the Poincaré model of the Bolyai-Lobatschewski geometry whose line element is given by

$$(16) \qquad ds^2 = \frac{dw\, \overline{dw}}{v^2} \qquad (w = u + iv) .$$

We will now show that any infinitesimal transformation (8) continued into the complex domain decreases non-Euclidean distances. Since linear mappings of the upper half-plane onto itself leave distances invariant it is sufficient to verify this only for the infinitesimal transformation $\omega = w^3$. The change of ds^2 effected by it is given by

$$-4\,\frac{d\omega\,\overline{dw} + \overline{d\omega}\,dw}{(w - \overline{w})^2} + 8\,\frac{dw\,\overline{dw}}{(w - \overline{w})^3}\,(\omega - \overline{\omega})$$

$$= 4\left|\frac{-3(w^2 + \overline{w}^2)}{(w - \overline{w})^2} + 2\,\frac{w^2 + w\overline{w} + \overline{w}^2}{(w - \overline{w})^2}\right| = -dw\,\overline{dw} < 0$$

which proves that the line element is decreased.

Because of this decrease in distances, given any initial value z in the upper half-plane, the corresponding solution of (13) exists in the whole interval $[0, t_o]$. For the proof we may assume, since arbitrary linear mappings of the upper half-plane are available, that for some $z = z_o$ in this half-plane $g(z_o, t) \equiv z_o$. But then the distances of $g(z, t)$ from z_o decrease with t, which ensures the existence of $g(z, t)$ for the whole interval $[0, t_o]$.

We know now that $g(z, t)$ represents for each t of $[0, t_o]$ a schlicht mapping of the upper half-plane into itself which becomes real on the interval $[-1, 1]$. This interval is transformed onto itself with $z = 0$ remaining fixed. The mapping can be continued into the lower half-plane.

Let now $g(z)$ be an arbitrary schlicht mapping of the upper half-plane into itself which becomes real on the interval $(-1, 1)$, transforming this interval onto itself with $g(0) = 0$. We want to show that $g(x)$ on $(-1, 1)$ belongs to the semigroup T. On each closed sub-interval $g(x)$ can be uniformly approximated by functions $g(x, t_o)$ originating from solutions of equations of the form (11) in which $k(t)$ is suitably chosen. We construct mappings $g(z)$ on domains bounded by the real axis and a slit domain emanating from a point outside of the interval $[-1, 1]$ but lying otherwise completely in the upper half-plane. The mapping can be arranged so that the points $-1, 0, 1$ remain fixed. In the same way as in the author's paper [2] it can be shown that the mapping function can be obtained as a $g(z, t_o)$ by suitable choice of $k(t)$ in (12). On the other hand, it can be proved in a well-known manner that to every domain which is part of the upper half-plane and is partly bounded by the interval $(-1, 1)$, a slit domain of the above shape can be constructed such that the following holds: The function mapping the upper half-plane onto this slit domain which keeps $-1, 0, 1$ fixed, differs from $g(z, t_o)$ by as little as we wish in any given compact subset of the z-plane from which the half-lines $(-\infty, -1)$ and $[1, \infty]$ have been deleted.

We have finally to show that a semigroup S which contains non-projective transformations from T and T' must coincide with the semigroup of all continuous mappings.

PROOF: All the infinitesimal transformations (8) with arbitrary $P_3(x)$ are now infinitesimal transformations of S. We obtain more general infinitesimal transformations of S by the process of addition. But evidently every continuous function on (-1, 1) can, in a closed subinterval, be uniformly approximated by functions obtained by this superposition process.

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CONFORMAL INVARIANTS AND LINEAR DIFFERENTIAL EQUATIONS

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The method for generating conformal invariants to be briefly described in this paper is based on the complex eigenvalue problem

$$(1) \qquad \left(\frac{y'(z)}{f'(z)}\right)' + \lambda p(z)y(z) = 0, \qquad y(a) = y(b) = 0,$$

where $f(z)$ and $p(z)$ are regular and single-valued analytic functions in a plane domain D of finite connectivity, $z = a$ and $z = b$ are points of D, and λ is to be determined in such a way that the (non-trivial) solution $y(z)$ of (1) vanishes at the two specified points. To avoid certain complications, we shall further assume that $f(z)$ is univalent in D.

If the point $z = a$ is kept fixed, it is not difficult to show that for every point $b \in D$ $(b \neq a)$ -- with the possible exception of a number of isolated points -- the problem (1) has an infinite sequence of eigenvalues λ which converges to infinity. If $y(z, \lambda)$ denotes the solution of (1) determined by the initial conditions $y(a, \lambda) = 0$, $y'(a, \lambda) = f'(a)$, an elementary estimate of the coefficients $A_n(z)$ in the expansion

$$y(z, \lambda) = f(z) + \sum_{n=1}^{\infty} A_n(z)\lambda^n \qquad (A_n(a) = A_n'(a) = 0)$$

shows that $y(b, \lambda)$ is an entire function of λ of order not exceeding $\frac{1}{2}$.

The equation $y(b, \lambda) = 0$, which determines the eigenvalues of (1), will therefore have an infinity of solutions which converge to infinity, unless $y(b, \lambda)$ reduces to a polynomial. The latter case, however, can only occur at isolated points $z = b$, since an elementary argument shows that the existence of a limit point of such points within D would lead to an absurdity. It may also be noted that the eigenvalue $\lambda = 0$ is excluded. Indeed, (1) shows that $y(b, 0) = f(b) - f(a)$, and this does not vanish since $f(z)$ is univalent in D.

Unless D is simply-connected, the eigensolutions of (1) will in general not be single-valued in D. In this case the set of eigenvalues associated with the pair of points (a, b) will also depend on the homotopic type of the curve connecting $z = a$ and $z = b$ along which the equation (1) is solved. To avoid these complications, we shall restrict ourselves to pairs of points (a, b) for which there exist eigenfunctions of (1) which are single-valued in D. We note that, for any given $f(z)$ and for any pair (a, b), there always exist functions $p(z)$ for which this is the case. For instance, if $p(z) = c^2 f'(z)$, where c is a non-zero constant, equation (1) reduces to

$$(2) \qquad \left(\frac{y'(z)}{f'(z)}\right)' + \lambda c^2 f'(z) y(z) = 0,$$

and this has the solution

$$y(z) = \sin\{c\sqrt{\lambda}\,[f(z) - f(a)]\}$$

which vanishes for $z = a$ and is single-valued for all values of λ. Accordingly, the eigenvalues determined by the condition $y(b) = 0$ are

$$\lambda_n = \pi^2 n^2 c^{-2}[f(b) - f(a)]^{-2}, \quad n = 1, 2, \ldots . \tag{3}$$

If the analytic function $w = w(z)$ maps D conformally onto a domain D_1, the system (1) transforms into

$$\left(\frac{y_1'(w)}{f_1'(w)}\right)' + \lambda p_1(w) y_1(w) = 0, \quad y_1(a_1) = y_1(b_1) = 0, \tag{4}$$

where $y_1(w) = y[z(w)]$, $f_1(w) = f[z(w)]$, $p_1(w) = \frac{dz}{dw}\, p[z(w)]$, $a_1 = w(a)$, $b_1 = w(b)$. Accordingly, the systems (1) and (4) have the same set of eigenvalues. If the pairs (a, b), and the classes of functions from which $f(z)$ and $p(z)$ are to be selected, are defined in a conformally invariant manner, it is therefore possible to obtain conformal invariants by imposing suitable extremal conditions on the eigenvalues λ.

There are various ways in which this may be done. Since

$f'(z)dz = f_1'(w)dw$ and $p(z)dz = p_1(w)dw$, the classes of functions $f(z)$ and $p(z)$ for which

$$(5) \qquad \iint_D |f'(z)|^2 d\omega = 1, \qquad \iint_D |p(z)|^2 d\omega = 1$$

($d\omega$ being the area element in the z-plane), are transformed into the corresponding classes of functions $f_1(w)$ and $p_1(w)$, and $f_1(w) = f[z(w)]$ is univalent in D_1 if $f(z)$ is univalent in D. Another condition of this type is

$$(6) \qquad |f'(z)p(z)| \leq K(z, z),$$

where $K(z, z)$ is the Bergman kernel function [1] of D. Since $|f'(z)p(z)|\,|dz|^2 = |f_1'(w)p_1(w)|\,|dw|^2$ and $K(z, z)|dz|^2 = K(w, w)\,|dw|^2$, the conformal invariance of the classes of functions subject to the inequality (6) is evident.

The points $z = a$ and $z = b$ will be restricted to certain subsets A and B of D or its boundary which may be characterized in a conformally invariant manner. For instance, $z = a$ and $z = b$ may be taken to be two arbitrary distinct points on a given boundary component of D.

To define the conformal invariants associated with the normalization (5), we now proceed as follows. We choose a univalent function $f(z)$ satisfying (5), and we consider the eigenvalue problem (1), where $a \varepsilon A$, $b \varepsilon B$, and $p(z)$ is a single-valued function in D which is normalized by (5) and is, moreover, such that the solution $y(z)$ of (1) is single-valued in D. We set

$$(7) \qquad \lambda(f) = \inf|\lambda|,$$

where λ is any of the eigenvalues of (1), $p(z)$ is any single-valued function subject to (5), and a and b range over A and B, respectively. Finally, we set

$$(8) \qquad \Lambda = \sup \lambda(f),$$

where $f(z)$ ranges over all univalent functions in D which are normalized by (5). It is clear from what was said above that Λ remains unchanged if D is made subject to a conformal transformation.

If conditions (5) are replaced by (6), the same procedure leads to another conformal invariant, which will be denoted by Λ_o.

It may be pointed out that Λ and Λ_o will, in general, exist (that is, they will not reduce to zero or infinity) even if the sets A and B coincide, or if one or both of these sets reduce to a point.

It is easy to see that Λ_o cannot be larger than Λ. Indeed, if $f'(z)$ and $p(z)$ are single-valued and square-integrable in D, we have [1]

$$|f'(z)|^2 \leq K(z, z) \iint_D |f'(z)|^2 d\omega, \quad |p(z)|^2 \leq K(z, z) \iint_D |p(z)|^2 d\omega.$$

If $f(z)$ and $p(z)$ satisfy (5), it follows therefore that $|f'(z)p(z)| \leq K(z, z)$. Hence, condition (5) is more restrictive than condition (6), and we may conclude that $\lambda_o(f) \leq \lambda(f) \leq \Lambda$. If this inequality is applied to a sequence of functions $f(z)$ for which $\lambda_o(f)$ tends to its lowest upper bound, we obtain $\Lambda_o \leq \Lambda$.

The principal applications of these concepts to function-theoretic problems rest on the following lemma, which we formulate first for the case of condition (6).

LEMMA: Let

(9) $$|f'(z)p(z)| < \lambda_o(f)K(z, z)$$

and let

(10) $$F(z) = \frac{y_1(z)}{y_2(z)},$$

where $y_1(z)$ and $y_2(z)$ are two linearly independent and single-valued solutions of the equation

$$\left(\frac{y'(z)}{f'(z)}\right)' + p(z)y(z) = 0. \tag{11}$$

If a and b are points of A and B, respectively, then

$$F(a) \neq F(b).$$

Indeed, suppose that $F(a) = F(b) = \alpha$. In view of (10), it would follow that the solution $y(z) = y_1(z) - \alpha y_2(z)$ of (11) vanishes at both $z = a$ and $z = b$, that is, the system (1) would have the eigenvalue $\lambda = 1$. We now choose a positive constant ε which is small enough so that (9) may be replaced by $|p(z)f'(z)| < [\lambda_o(f) - \varepsilon]K(z, z)$, and we define the function $p_1(z)$ by $[\lambda_o(f) - \varepsilon]p_1(z) = p(z)$. We then have $|p_1(z)f'(z)| < K(z, z)$ -- i.e., condition (6) -- and the system

$$\left(\frac{y'(z)}{f'(z)}\right)' + \lambda p_1(z)y(z) = 0, \qquad y(a) = y(b) = 0,$$

has the eigenvalue $\lambda = \lambda(f) - \varepsilon$. Since this contradicts (7), the lemma is proved.

The formulation and proof of the lemma for the case corresponding to condition (5) is entirely analogous, the only difference being that (9) has to be replaced by

$$\iint_D |f'(z)|^2 d\omega \iint_D |p(z)|^2 d\omega < \lambda^2(f).$$

The conditions of the lemma insure that the values of $F(z)$ taken on the set A are different from those taken on the set B. In the particular case in which both A and B coincide with the domain D, the function $F(z)$ will therefore be univalent in D. The resulting criteria of univalence may be formulated without reference to the differential equation (11). If the symbol $\{g, z\}$ denotes the Schwarzian derivative

$$\{g, z\} = \left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2$$

of the function $g(z)$, and $F(z)$ is the ratio (10) of two independent solutions of (11), the functions $p(z)$, $f(z)$ and $F(z)$ are easily shown to satisfy the identity

$$2p(z)f'(z) = \{F, z\} - \{f, z\}.$$

If $\lambda_0(f)$ denotes the quantity (7) (corresponding to condition (6)) in the case in which both A and B coincide with the entire domain D, we are thus led to the following result.

If f(z) is univalent in D and

$$(12) \qquad |\{F, z\} - \{f, z\}| < 2\lambda_o(f)K(z, z),$$

then F(z) is likewise univalent in D.

The corresponding result associated with condition (5) is entirely analogous.

We can make this result independent of the function f(z) by passing to the conformal invariant Λ_o defined in (8). The family of univalent functions f(z) is normal, and it may be made compact by the additional condition $f'(\zeta) = 1$ ($\zeta \varepsilon D$) which does not violate (6). There will therefore exist a univalent function $f_o(z)$ for which $\lambda_o(f_o) = \Lambda_o$, and we have the following result.

If D is a domain of finite connectivity, there exists a univalent function $f_o(z)$ in D and a positive number Λ_o -- which depends only on the conformal type of D -- such that any function F(z), for which

$$|\{F, z\} - \{f_o, z\}| < 2\Lambda_o K(z, z),$$

is univalent in D.

If D_o denotes the image domain of $f_o(z)$, this result may also be formulated as follows.

If C is a conformal class of plane domains, there exists a positive constant Λ_o and a domain $D_o \varepsilon C$, such that any function $F(z)$ which satisfies

$$|\{F, z\}| < 2\Lambda_o K(z, z), \qquad z \varepsilon D_o, \tag{13}$$

is univalent in D_o.

We remark that it follows from the definition of the conformal invariant Λ_o that Λ_o furnishes the best possible constant of this type. The exact determination of Λ_o in any given case, however, does not seem to be an easy task. The same applies to the determination of the canonical domain D_o for which the condition to be imposed on $F(z)$ takes the simple form (13). In this connection it is interesting to point to the fact, proved by M. Schiffer [4], that the condition

$$|\{F, z\}| < 6\pi K(z, z)$$

is <u>necessary</u> for $F(z)$ to be univalent in D, provided D is a domain bounded by a finite number of circles. This may lead to the conjecture that the same type of domain may also play the role of the canonical domains D_o.

It is clear from the above results that Λ_o cannot be infinite. In order to show that it cannot reduce to zero, it is sufficient to show that $\lambda(f)$ is not zero for some particular $f(z)$. If D is simply-connected, this follows from a result [3] according to which $F(z)$ is univalent in the unit circle if

$$(14) \qquad |\{F, z\}| < \frac{2}{(1-|z|^2)^2}.$$

The Bergman kernel function of the unit circle is $K(z, z) = \pi^{-1}(1-|z|^2)^{-2}$, and $\{z, z\} = 0$. Since both (12) and (14) are sharp estimates [3], this shows that $\lambda_o(z) = \pi$ if D is the unit circle. For a simply-connected domain we have therefore $\Lambda \geq \Lambda_o \geq \pi$.

Lower bounds for Λ_o in the case of a doubly-connected domain can be obtained by direct computation. In the case of domains of higher connectivity, such bounds may be obtained by means of the following two lemmas.

If D has the boundary components C_ν, and $\lambda_o(f)$ and $\lambda_o^{(\nu)}(f)$ refer, respectively, to the cases in which

$A = B = D$ and $A = B = C_\nu$, then

$$\lambda_o(f) = \min_\nu \lambda_o^{(\nu)}(f).$$

We omit the proof of this result. The second lemma is an immediate consequence of the definition of $\lambda_o(f)$.

If $D' < D$ and $A \varepsilon D'$, $B \varepsilon D'$, and if $f(z)$ is univalent in D, then $\lambda_{D'}(f) \leq \lambda_D(f)$.

To obtain a lower bound for Λ_o, we take D to be a domain bounded by circular arcs and we set $f(z) = z$. By the first lemma it is sufficient to obtain a lower bound in the case in which A and B coincide with some particular boundary component of D, and by the second lemma this bound is larger than the corresponding bound for the largest annulus contained in D which is bounded by the same circle. Hence, $\Lambda_o \geq \lambda_\Delta(z)$, where Δ is a non-degenerate annulus. A positive lower bound for $\lambda_\Delta(z)$ -- depending on the Riemann modul of Δ -- can be obtained by an elementary computation.

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ON THIN SETS IN POTENTIAL THEORY

Makoto Ohtsuka

1. To find geometrical conditions for a boundary point to be regular is an important part of the Dirichlet problem. One necessary and sufficient condition is known as Wiener's theorem. Brelot [1, 2] generalized the notion of regularity of a boundary point of a set in Euclidean n-space E_n and proved a theorem corresponding to the Wiener theorem. He introduced thin sets with respect to the potential

$$\int \log \frac{1}{\overline{PQ}} \, d\mu(Q) \qquad \text{if } n = 2$$

or

$$\int \frac{1}{\overline{PQ}^{n-2}} \, d\mu(Q) \qquad \text{if } n \geq 3.$$

In the first part of the present paper, we shall define thin sets in a locally compact space with respect to general potentials and generalize the Wiener-Brelot theorem. The latter part will be devoted to applying the results obtained by the author [13] on the capacity of product sets. We shall not discuss the use of thin sets in the theory of functions; papers by Deny [5] and Lelong-Ferrand [8, 9, 10] may be referred to in this connexion.

2. Let Ω be a locally compact space. We take a real-valued function $\Phi(P, Q) > -\infty$ defined on $\Omega \times \Omega$ which is symmetric: $\Phi(P, Q) = \Phi(Q, P)$, continuous (in the extended sense that $+\infty$ is admitted), and finite if $P \neq Q$. We call this function a kernel. Let μ be a non-negative finite-valued Radon measure with compact carrier S_μ in Ω. By a mass-distribution we shall mean such a measure. The integral

$$\int \Phi(P, Q) d\mu(Q)$$

is called a potential and denoted by $U^\mu(P)$. We define the energy of μ by $\int\int \Phi(P, Q) d\mu(Q) d\mu(P)$ and denote it by (μ, μ).

It is easily seen that $U^\mu(P)$ is lower semi-continuous in Ω, and (finite) continuous outside the carrier S_μ. If we say simply that a function is continuous, we mean that the function is finite and continuous. It is also easy to see that $U^\mu(P)$ is continuous in a neighborhood of a point P_o for which $\Phi(P_o, P_o) < \infty$. The following principle which may or may not hold for a given kernel Φ is very important in the recent study of general potentials.

Continuity principle: Whenever the restriction of $U^\mu(P)$ to the carrier S_μ is continuous, $U^\mu(P)$ is continuous in the whole space Ω.

For this principle, we refer to Choquet [3] and Ohtsuka [11, 12].

Next we define some functions of sets related to the notion of capacity. Let X be a non-empty set in Ω. For a mass distribution μ we set

$$V_\mu^* = \sup_{P \varepsilon S_\mu} U^\mu(P)$$

and let $V_i^*(X)$ denote $\inf V_\mu^*$ taken with respect to the unit mass-distributions (that is, those whose total mass is one), with carriers contained in X. We remark that we put * here because $V_\mu = \sup_{P \varepsilon \Omega} U^\mu(P)$ is often considered as well. We set $V_i^*(X) = \infty$ if X is empty. If $\Phi > 0$, $1/V_i^*(X)$ may be taken as a definition of inner capacity of X. A quantity corresponding to outer capacity is defined by

$$V_e^*(X) = \sup_G V_i^*(G)$$

where G is an open set containing X.

We state without proof

PROPOSITION 1. Suppose that the kernel satisfies the continuity principle, and let G be an open set relatively compact in Ω. Given $\varepsilon > 0$, we can find a unit mass-distribution μ whose carrier is contained in the closure of G such that

$$U^{\mu}(P) \geq V_i^*(G) - \varepsilon$$

in G.

We define one more function of sets. Let K be a non-empty compact set in Ω and set

$$\inf_{P_1, \ldots, P_n \varepsilon K} \sum_{i \neq j} \Phi(P_i, P_j) = n(n-1)D_n(K).$$

As in the classical case, $D_n(K)$ decreases to a limit as $n \longrightarrow \infty$, which we denote by $D(K)$. This function of compact sets is usually used to define the transfinite diameter of K. If K is empty we set $D(K) = \infty$.

We can prove in the usual way

PROPOSITION 2. For every compact set K

$$V_i^*(K) = D(K).$$ [1)]

3. Let us now define thin sets. A set $X \subset \Omega$ is called <u>Φ-thin</u> at a point P_o if P_o is an exterior point of $X - P_o$ or there exists a potential $U^{\mu}(P)$ with kernel Φ such that

$$\underline{\lim}\, U^{\mu}(P) > U^{\mu}(P_o)$$

as P tends to P_o along $X - \{P_o\}$. If every compact subset of X is Φ-thin at P_o, X is called <u>innerly Φ-thin</u> at P_o. We can show that a boundary point P_o of a domain Δ in E_n is irregular for the Dirichlet problem if and only if the complementary set of Δ is Φ-thin at P_o with $\Phi = \log 1/\overline{PQ}$ $(n = 2)$ or $\overline{PQ}^{2-n}$ $(n \geq 3)$.

With the aid of Proposition 1 we can prove

THEOREM 1. Let Ω be a locally compact space and Φ a kernel. We set

$$X_k = \{p \in X;\ s^k \leq \Phi(P_o, P) \leq s^{k+1}\}$$

for $X \subset \Omega$ and $s > 1$. If

$$\sum_{k=1}^{\infty} \frac{s^k}{V_i^*(X_k)} < \infty \tag{1}$$

for some $s > 1$, then X is innerly Φ-thin at P_o.

If the kernel Φ satisfies the continuity principle, and if

$$\sum_{k=1}^{\infty} \frac{s^k}{V_e^*(X_k)} < \infty \tag{2}$$

for some $s > 1$, then X is Φ-thin at P_o.

To have an extension of the other part of the Wiener-Brelot theorem we need an additional condition. Let $\phi(\rho)$ be a positive decreasing continuous function of ρ. We assume the following condition on $\phi(\rho)$:

(A) There exist positive numbers ρ_o, δ_o and A such that

$$\phi(\rho) \leq A\phi(1 + \delta_o)\rho)$$

whenever $0 < \rho < \rho_o$.

THEOREM 2. Let Ω be a locally compact metric space with distance $\rho(P, Q)$ and let $\phi(\rho)$ be a positive decreasing continuous function of $\rho > 0$ which satisfies condition (A). We take $\phi(\rho(P, Q))$ as kernel Φ. If X is innerly Φ-thin at P_o then (1) holds, and if X is Φ-thin at P_o then (2) holds.

Condition (A) was used by Kunugui [7] in a discussion of equilibrium potential in a Euclidean space. We remark also that Theorems 1 and 2 for Riesz kernels $\overline{PQ}^{-\alpha}$ $(\alpha > 0)$ are found in [15]. It is easy to verify that $\log 1/\rho$ and $\rho^{-\alpha}$ $(\alpha > 0)$ satisfy (A).

4. In this last section we take a Euclidean space E_n for Ω and $\log 1/\overline{PQ}$ or $\overline{PQ}^{-\alpha}$ $(\alpha > 0)$ for Φ.

Royden [14] proved the following theorem: Let Δ be a domain in E_3 and P_o a boundary point of Δ. If there exist a spherical surface with center P_o and a bundle X of rays, issuing from P_o and lying in the complement of Δ, such that their intersection is a closed set of positive logarithmic capacity, then P_o is a regular point for the Dirichlet problem.

Let us prove this theorem by applying Wiener's theorem [2)]. We may assume that the above sphere with center P_o is a unit sphere. It is sufficient to show that X is not Φ-thin at P_o with $\Phi = \overline{PQ}^{-1}$. Take an arbitrary $s > 1$, and use the notation X_k in Theorem 1. By the definition of $D(K)$, $D(X_k) = s^k D(X_1)$. Since $D(X_k) = V_i^*(X_k)$ by Proposition 2, the series in (1) diverges if $D(X_1) = V_i^*(X_1) < \infty$, namely if the Newtonian capacity of X_1 is positive.

To show that this capacity is positive we take a small triangle T on the unit spherical surface with center P_o whose intersection with X is of positive logarithmic capacity, and consider the part S of the spherical shell $s^{-1} \leq \overline{PQ} \leq 1$, which has T as one boundary surface. We deform S into a cylinder Γ so that the base of Γ is the orthogonal projection of T into a plane touching T and so that every radial segment in S corresponds to a segment in Γ perpendicular to the base. By this deformation of S the distance between any couple of points does not change a great deal. Therefore, by the definition of D(K) which is equal to $V_i^*(K)$, the positivity of the logarithmic and Newtonian capacities of a set in S does not change under the deformation. Hence the set in the base of Γ which corresponds to $T \cap X$ has positive logarithmic capacity. We want to show that the set in Γ which corresponds to $S \cap X$ has positive Newtonian capacity. However, this follows from the following theorem of Deny-Lelong [6]:

Let $Y \subset E_m$. The (outer) capacity of order $m - 1$ [3)] of the cylinder set in E_{m+1} with Y as its base is zero if and only if the (outer) capacity of order $m - 2$ of Y is zero.

Thus the proof of the Royden theorem is completed.[4)]

What we have used in the proof are Proposition 2, Wiener's theorem and the theorem of Deny-Lelong. If we use some results by the author [13] on the capacity of product sets, various generalizations of Royden's theorem will be obtained. Let us quote the results in [13] which are useful for this purpose.

We denote $1/V_i^*(X)$ ($1/V_e^*(X)$ resp.) by $C_i^{(\alpha)*}(X)$ ($C_e^{(\alpha)*}(X)$ resp.) for the kernel $\overline{PQ}^{-\alpha}$ ($\alpha > 0$), and $\exp(-V_i^*(X))$ ($\exp(-V_e^*(X))$ resp.) by $C_i^{(o)*}(X)$ ($C_e^{(o)*}(X)$ resp.). We use $m_\alpha(X)$ ($\underline{m}_\alpha(X)$ resp.) to represent the Hausdorff (inner Hausdorff resp.) measure of dimension α of $X \subset E_n$.

(i) If Z is a set relatively compact in E_{l+m} and if $C_e^{(P)*}(Z_{P_1}) \geq b > 0$ ($\beta > 0$) for every point P_1 of a set $X \subset E_l$, where Z_{P_1} is the cross-section of Z at P_1, then

$$C_e^{(\alpha+\beta)*}(Z) \geq \gamma(\alpha, \beta)\, b\, C_e^{(\alpha)*}(X)$$

with a positive constant $\gamma(\alpha, \beta)$ depending on α, $0 < \alpha < 1$, and β.

(ii) If $X \subset E_l$ and $Y \subset E_m$, then

$$C_i^{(\alpha+\beta)*}(X \times Y) \geq \gamma(\alpha, \beta) C_i^{(\alpha)*}(X)\, C_i^{(\beta)*}(Y)$$

with the same $\gamma(\alpha, \beta)$ as above.

(iii) If $X \subset E_1$ and $Y \subset E_m$, then

$$C_i^{(1+\beta)*}(X \times Y) \geq \gamma(\beta) \underline{m}_1(X)\, C_i^{(\beta)*}(Y) \qquad (\beta > 0),$$

where $\gamma(\beta)$ is a positive constant depending on β.

(iv) If $\underline{m}_\alpha(X) > 0$ $(0 < \alpha \leq 1)$ for $X \subset E_1$ and $C_i^{(\beta)*}(Y) > 0$ $(\beta > 0)$ for $Y \subset E_m$, then

$$C_i^{(\alpha+\beta)*}(X \times Y) \geq \gamma(X, \alpha, \beta) C_i^{(\beta)*}(Y)$$

where $\gamma(X, \alpha, \beta)$ is a positive constant depending on X, α and β.

(v) If X is a segment of length ℓ and $Y \subset E_m$, then

$$C_i^{(1+\beta)*}(X \times Y) \leq \frac{(1+\ell^2)^{\frac{1+\beta}{2}}}{\gamma \cdot \left\{\frac{1}{C_i^{(\beta)*}(Y)} - 1\right\}^+} \qquad (\beta > 0),$$

where γ is a constant and $\{u\}^+$ denotes the maximum of u and 0.

We have some inequalities in (iii), (iv) and (v) in the case $\beta = 0$ too. We leave it to the reader to formulate generalizations of Royden's theorem by making use of these results.

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FOOTNOTES

1) It can be shown also that this is equal to $\inf(\mu, \mu)$ if K is not empty, where μ is a unit mass-distribution such that $S_\mu \subset K$.

2) Using this theorem Deny [4] gave another geometrical characterization of thin sets.

3) This is equal to $C_e^{(m-1)*}$ defined below.

4) Ullman [16] derived a part of the Deny-Lelong theorem from Royden's theorem.

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ON THE PHRAGMÉN-LINDELÖF THEOREM AND SOME APPLICATIONS

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The Phragmén-Lindelöf theorem plays an important role in the theory of the Riemann zeta function. It is used to obtain information about the growth of $\zeta(s)$ in a vertical parallel strip from the given growth on the boundaries of the strip. Mostly only a half strip $t > 1$ is considered. However, this is not precise enough in some other instances, where whole families of functions are considered and a uniform estimate for all these functions is needed. The purpose of this paper is to obtain a theorem with all its constants explicitly known.

The goal is reached in two steps; the first is a boundary value problem for harmonic functions, the second the application of the Phragmén-Lindelöf argument. We prove:

THEOREM 1. Let a, b, γ, δ, Q be real numbers,

$$-Q < a < b, \qquad \gamma \leq \delta.$$

Then there exists an analytic function $\phi(s) = \phi(s; Q)$ regular in the strip

$$S(a, b): a \leq \mathrm{Re}(s) \leq b,$$

and such that

$$(1)\qquad \begin{aligned} |\phi(a+it)| &= |Q+a+it|^{\gamma} \\ |\phi(b+it)| &= |Q+b+it|^{\delta} \end{aligned}$$

and that in $S(a,b)$

$$(2)\qquad |\phi(s)| \geq |Q+s|^{\gamma \frac{b-\sigma}{b-a} + \delta \frac{\sigma-a}{b-a}}$$

Moreover

$$(3)\qquad \phi(s) = O(e^{|t|^c}), \qquad |t| \longrightarrow \infty$$

for a certain $c > 0$, $s = \sigma+it$.

It is essential that the equality sign is valid in (1). Moreover (2) is sharp in the sense that it does not contain any unknown constant and requires the equality sign on the boundaries.

The second step leads to the

THEOREM 2. Let $f(s)$ be regular analytic in the strip $S(a,b)$ including its boundaries and fulfill for certain positive c, C

(4) $$|f(s)| < Ce^{|t|^c}.$$

Let moreover

(5) $$|f(a+it)| \leq A|Q+a+it|^{\alpha},$$
$$|f(b+it)| \leq B|Q+b+it|^{\beta},$$

with

(6) $$Q+a > 0, \quad \alpha \geq \beta.$$

Then in the strip $S(a,b)$

(7) $$|f(s)| \leq (A|Q+s|)^{\alpha\frac{b-\sigma}{b-a}}(B|Q+s|)^{\beta\frac{\sigma-a}{b-a}}.$$

A special case of Theorem 2 is Hadamard's three circle theorem, which is obtained for $\alpha = \beta = 0$ and for a function $f(s)$ periodic in t.

As far as the proof of Theorem 1 is concerned, we solve for the harmonic function $u(\sigma,t)$ the boundary value problem

(8) $$u(a,t) = \gamma \log|Q+a+it| = \frac{\gamma}{2}\log[(Q+a)^2+t^2].$$
$$u(b,t) = \delta \log|Q+b+it| = \frac{\delta}{2}\log[(Q+b)^2+t^2].$$

That this boundary value problem has a solution can be shown in the following way. Put

$$(9.1) \qquad \omega(\sigma, t) = \frac{1}{2} \frac{\sin \pi\sigma}{\operatorname{Cosh} \pi t - \cos \pi\sigma}.$$

Then

$$(9.2) \qquad \begin{aligned} u(\sigma, t) &= \frac{1}{b-a} \int_{-\infty}^{\infty} \omega\left(\frac{\sigma-a}{b-a}, \frac{t-y}{b-a}\right) A(y)\, dy \\ &+ \frac{1}{b-a} \int_{-\infty}^{\infty} \omega\left(\frac{b-\sigma}{b-a}, \frac{t-y}{b-a}\right) B(y)\, dy \end{aligned}$$

is a harmonic function in the strip $S(a, b)$ with the boundary values

$$u(a, t) = A(t), \quad u(b, t) = B(t).$$

The formula (9.1), (9.2) is obtained from the Poisson formula for the circle by the conformal mapping

$$t = \operatorname{tang} \frac{\pi}{2} \frac{s-m}{b-a}, \quad m = \frac{a+b}{2}.$$

The boundary values given in (8) ensure also

$$|u(\sigma, t)| < C|t|.$$

Let now $v(\sigma, t)$ be the harmonic conjugate to $u(\sigma, t)$. We put

(10) $$\phi(s) = e^{u(\sigma,t)+iv(\sigma,t)}$$

and have a regular analytic function $\phi(s)$ which satisfies (1) and (3).

Now the function

(11) $$U(\sigma,t) = (\lambda\sigma + \mu)\log|Q + \sigma + it|$$

is <u>subharmonic</u> in $S(a,b)$ for $\lambda \geq 0$, as

$$\Delta U = 4\lambda \frac{Q+\sigma}{(Q+\sigma)^2 + t^2} \geq 0$$

shows. This is seen immediately from

$$U = f \cdot g,$$

$$\Delta U = \Delta f \cdot g + 2(f_\sigma g_\sigma + f_t g_t) + f \cdot \Delta g.$$

If we determine λ, μ so that

$$\lambda a+\mu = \gamma$$

$$\lambda b+\mu = \delta,$$

i. e.

$$\lambda = \frac{\delta-\gamma}{b-a} \leq 0,$$

we see that $U(\sigma,t)$ has the same boundary values as $u(\sigma,t)$. Hence

$$U(\sigma, t) \leq u(\sigma, t)$$

and consequently

$$|\phi(s)| \geq e^{U(\sigma, t)},$$

which is (2).

For the proof of Theorem 2 we take the $\phi(s)$ of Theorem 1 with $\gamma = -\alpha$, $\delta = -\beta$ and put

$$F(s) = f(s) \cdot \phi(s) E^{-1} e^{-\nu s},$$

where E and ν are so determined that

$$\begin{aligned} A &= Ee^{\nu a} \\ B &= Ee^{\nu b} \end{aligned} \tag{12}$$

Then this together with (1) and (5) shows

$$|F(a+it)| \leq 1$$
$$|F(b+it)| \leq 1$$

and, since

$$F(s) = O(e^{|t|^c}),$$

according to the Phragmén-Lindelöf argument, also

$$|F(s)| \leq 1$$

inside $S(a, b)$. Then

$$|f(s)| \leq Ee^{\nu\sigma} |\phi(s)|^{-1},$$

which with (2) and (12) gives the desired result (7).

Through the use of other suitable subharmonic functions more theorems like Theorem 2 can be obtained, e.g.

THEOREM 3. Let $f(s)$ be regular in $S(a, b)$ and on its boundaries and

$$f(s) = O(e^{|t|^c}).$$

Let moreover $f(s)$ satisfy the conditions

$$|f(a+it)| \leq e^{\alpha |Q+a+it|^p}$$
$$|f(b+it)| \leq e^{\beta |Q+b+it|^p}$$

with

$$p > 0,\ \alpha > \beta,\ Q+a > \max(0, \frac{p(b-a)\alpha}{2(\alpha-\beta)}),$$

then for all s in $S(a, b)$

$$|f(s)| \leq e^{|Q+s|^{p}\left(\frac{b-\sigma}{b-a}\alpha + \frac{\sigma-a}{b-a}\beta\right)}.$$

More important, however, are the applications of Theorem 2 to special functions. Already for the Γ-function the following estimates seem to be new:

THEOREM 4. For $Q \geq 0$, $-\frac{1}{2} \leq \sigma \leq \frac{1}{2}$ the inequalities

$$\left| \frac{\Gamma\left(\frac{Q}{2} + \frac{1-s}{2}\right)}{\Gamma\left(\frac{Q}{2} + \frac{s}{2}\right)} \right| \leq \left(\frac{1}{2}|Q+1+s|\right)^{\frac{1}{2}-\sigma}$$

$$\left| \frac{\Gamma(Q+1-s)}{\Gamma(Q+s)} \right| \leq |Q+1+s|^{1-2\sigma}$$

hold.

It may be remarked that here equality is true in both cases for $s = \frac{1}{2} + it$, $Q \geq 0$, and $s = -\frac{1}{2} + it$, $Q = 0$.

Further applications of Theorems 2 and 4 can be made in the theory of $L(s, \chi)$ and ζ_K-functions. We obtain the following theorems:

THEOREM 5. For $0 < \eta \leq \frac{1}{2}$, for all moduli $k > 1$ and all primitive characters χ modulo k the inequality

$$|L(s,\chi)| \leq \left(\frac{k|1+s|}{2\pi}\right)^{\frac{1+\eta-\sigma}{2}} \zeta(1+\eta)$$

holds in the strip $-\eta \leq \sigma \leq 1+\eta$.

THEOREM 6. Let $\zeta_K(s)$ be the Dedekind function of the algebraic number field K of degree n and discriminant d. Then

$$|\zeta_K(s)| < 3\left|\frac{1+s}{1-s}\right|\left[|d|\left(\frac{|1+s|}{2\pi}\right)^n\right]^\tau [\zeta(1+\eta)]^n$$

where $\tau = \frac{1+\eta-\sigma}{2}$ and

$$-\frac{1}{2} \leq \eta \leq \sigma \leq 1+\eta \leq \frac{3}{2}.$$

If K is Abelian over the rational field then

$$|\zeta_K(s)| \leq \left(|d|\left(\frac{|1+s|}{2\pi}\right)^{n-1}\right)^\tau \cdot |\zeta(s)|[\zeta(1+\eta)]^{n-1}$$

in the same strip.

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A DIVISION PROBLEM FOR ANALYTIC FUNCTIONS

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1. INTRODUCTION

The Euclidean algorithm for dividing f by g consists in finding an element h of a certain class so that the remainder $f - gh$ is as small as possible. For example, if these elements are polynomials, then we may measure the size of the remainder by its degree. We shall consider here an analogous process where f and g are complex-valued functions defined on the boundary C of a domain Ω in the plane, and the quotient h is the boundary function of a function analytic in Ω. If we assume that C consists of a finite number of closed Jordan curves, then we may measure the size of the remainder by means of the L_p norm with respect to arc length.

If $f = 1$ and $g = z - z_o$, where z_o is a given interior point, and $p = 2$, then the remainder is essentially the Szegö kernel function of Ω and from this we may determine a function which gives a 1-to-n conformal mapping of Ω onto the unit circle (Szegö [29], Ahlfors [1], Garabedian [7]). Other classical cases are: f a polynomial, $g = z^n$, and Ω the unit circle, considered by F. Riesz [23]; g an arbitrary polynomial, considered by Kakeya [13]; $g = 1$ and Ω the unit circle considered for $p = 1$ by Doob [5] and for general p by Macintyre [16], Rogosinski [16], [24], and Shapiro [24], [27]; $g = 1$, general p, and general Ω, studied by Penez [21], Lax [14], and Havinson [11]. Other special cases were investigated by Garabedian and Schiffer [8], Nehari [19], and Lehto [15]. In a note to appear soon, the problem was considered with a weight factor in the L_p norm by Havinson and Tumarkin [12]. This is reducible to the problem with the ordinary L_p norm by a suitable choice of g. We wish to thank Professor Markusievič for an opportunity to examine the manuscript of this note at the recent conference in Helsinki.

The existence and uniqueness of the solution for $1 < p < \infty$ is an easy consequence of the properties of uniformly convex Banach spaces (Clarkson [4]). An abstract characterization of the solution is almost equally simple to obtain, once one takes the right point of view, by the Hahn-Banach theorem (Shapiro [27], Lax [14]), but the concrete characterization is more difficult unless the boundary is assumed to be sufficiently smooth, in which case a classical theorem of Privalov [22] (see also Golusin [9]) yields the desired result. In the general case of a rectifiable boundary, Penez [21] found a hitherto unsuspected duality relation which reduces to the simpler relations given by previous authors only when Ω satisfies Smirnov's condition [28]. In the unpublished note by Havinson and Tumarkin an alternative formulation of this duality relation is given.

Virtually the only known general method for the effective determination of the extremal function is the Rayleigh-Ritz method of exhausting the function space by finite-dimensional subspaces. For the case $p = 2$, $g = 1$, Ω simply-connected with analytic boundary, and exhaustion by the spaces of polynomials, spanned by $1, z, \ldots, z^n$, Wang [30] studied the rate of convergence, and found that the error is $O(c^n)$, where $0 < c < 1$. Conversely, we proved (see [25]) that if $f = 1$, $g = z - z_o$, $p = 2$, and Ω is simply-connected, and the space is exhausted by polynomials, then if the error $h(z) - h_n(z) = O(c^n)$, $0 < c < 1$, at a single interior point, then C must be analytic. Thus the rate of convergence of the Rayleigh-Ritz method is extremely sensitive to the regularity properties of the data, and we must expect only slow convergence if the data are only sufficiently smooth.

For the investigation of the rate of convergence, there are two essential steps. First we must know how the regularity properties of a function f on C and the smoothness of C affect the degree of best approximation to f on C by polynomials of given degree. This problem was studied by Mergelyan [17], Džrbašyan [6], and the present authors [25]. We showed that for a wide class of domains, known results on trigonometric approximation can be transferred bodily to the present problem. The second step is to obtain a priori estimates of the smoothness properties of the extremal function in terms of the properties of the data. The only previous general results known to us concern the trivial case where C is analytic and f and g are analytic on C. Of course, the extensive literature on the behavior of the Riemann mapping function on the boundary (see Ostrowski and Gattegno [20], Warschawski [31]) can be interpreted as the intensive study of a special case. We shall show that the general case can be reduced to this special case, but the process of reduction may introduce complications of its own. Hence it is desirable to seek an alternative, more direct method.

We have found in the case $p = 2$ integral equations for the quotient h and the remainder f - gh. These integral equations should be useful for the effective determination of these functions and for the direct investigation of their properties.

This paper is a preliminary report and represents work in progress. We hope that in the final version many of the natural questions, which are left unanswered here, will be more fully treated.

2. FORMULATION OF THE PROBLEM. THE DUALITY RELATIONS

It emerges from the work of Penez [21] that the problem must be formulated with a care which has not always been observed in the literature. We take Ω to be a domain bounded by a finite number of rectifiable Jordan curves $C_1, \ldots, C_n$, and set $C = \bigcup_1^n C_k$. We are given a function f of class L_p with respect to arc-length on the boundary. The function g is a given measurable function such that $\log|g|$ is of class L_1 with respect to harmonic measure. Then there exists a function $G(z)$ analytic and $\neq 0$ in Ω such that $\log G(z)$ is in the Hardy class $H_1(\Omega)$ (see below) and such that $|g/G|$ is constant on each component of C. For $1 \leq p \leq +\infty$, we say that $F \in H_p(\Omega)$ if F is analytic in Ω and if there exists a sequence Ω_k of domains such that $\overline{\Omega_k} \subset \Omega_{k+1} \subset \Omega$, $\bigcup \Omega_k = \Omega$, $\partial\Omega_k$ is rectifiable, and $||F||_{p,\partial\Omega_k} = O(1)$. Such a function has non-tangential limits almost everywhere on C and the boundary function is of class L_p.

Let $S_p(g)$ be the set of H such that $GH \varepsilon H_p(\Omega)$. If $H \varepsilon S_p(g)$, then it has non-tangential limits almost everywhere on C such that $gH \varepsilon L_p$. We shall denote the set of functions on C of the form gH, $H \varepsilon S_p(g)$, by $S_p(g)$. This is a closed subspace of the Banach space L_p.

For the sake of simplicity we shall discuss here only the case $1 < p < \infty$. The extreme cases can be treated by similar methods, but lead to certain complications because the corresponding Banach spaces are not uniformly convex. We shall also assume that Ω satisfies the Smirnov condition. For simply connected domains, if ψ is the function which maps the unit circle onto Ω, then this condition states that $\log|\psi'|$ has a Poisson-Lebesgue representation in terms of its boundary values. For a multiply connected domain Ω with boundary components $C_1, \ldots, C_n$, the condition is that for each k the component of the complement of C_k which contains Ω should be a Smirnov domain. If Ω does not satisfy this condition, then the results stated below must be modified, but the necessary changes are easily obtained from the work of Penez [21].

We can now state our problem as follows. Given $f \varepsilon L_p$

and g, such that $\log|g| \varepsilon L_1$, with respect to harmonic measure, we wish to find $f_1 \varepsilon S_p(g)$ such that $||f - f_1||_p$ = minimum. This problem always has a unique solution, which is characterized by the property that

$$f = f_1 + |f_2|^q / f_2, \qquad f_2 \varepsilon S_q(z'/g),$$

where $q = p/(p - 1)$ and z' denotes the derivative of z with respect to arc-length. The function f_2 is characterized by the dual extremal problem: to find $H \varepsilon S_q(z'/g)$ such that $||H||_q = 1$ and

$$\mathrm{Re}\{ \int_C f H \, ds \} = \text{maximum},$$

the solution of which is $H = f_2/||f_2||_q$, and the maximum is $||f - f_1||_p$.

The special case $p = 2 = q$ has another duality relation which follows from the representation $f = f_1 + \overline{f_2}$, which shows that f_2 is also the solution of the best approximation problem

$$||\overline{f} - f_2||_2 = \text{minimum}, \qquad f_2 \varepsilon S_2(z'/g),$$

and $f_1/||f_1||_p$ is the solution of the problem

$$\operatorname{Re}\{\int_C \bar{f} H \, ds\} = \text{maximum}, \quad H \in S_2(g), \quad ||H||_2 = 1.$$

If $g^2 = z'$, then the classes $S_2(g)$ and $S_2(z'/g)$ coincide, so that if f is real, then $f_1 = f_2$. Consequently the boundary value problem

$$\operatorname{Re}\{(z')^{\frac{1}{2}} h\} = f$$

always has a unique solution $h \in H_2(\Omega)$ if f is a given real function of class L_2 on C. For multiply connected domains this is in sharp contrast to the Dirichlet problem, since in general the equation $\operatorname{Re}\{h\} = f$ has no solution in the class of single valued functions. As we shall see, this remark gives an interesting new property of the Szegö kernel function.

3. DEPENDENCE OF THE SOLUTION ON f.

If $p = 2$, then the operators on f which yield f_1 and f_2 are orthogonal projections and are therefore linear. For $p \neq 2$ the situation is more complicated. It is easy to show that f_1 is a continuous function of f and that if Δf_1 is the change in f_1 corresponding to the change Δf in f, then $||\Delta f_1||_p$ is $O(||\Delta f||_p^{1/2})$ for $1 < p \leq 2$ and is $O(||\Delta f||_p^{1/p})$ for $2 \leq p < +\infty$. (See Hanner [10].)

This is certainly far from best possible.

Suppose that f is minimal with respect to $S_p(g)$, i.e. $f_1 = 0$, and assume that $f \neq 0$. Then $f \neq 0$ almost everywhere on C. If $H \in L_p$, let $f_1(\lambda)$ be the minimizing function for $f + \lambda H$, λ real. A formal calculation suggests that in general $f_1'(0)$ is determined as follows. Let $\widetilde{H}(f)$ be the real Hilbert space consisting of the measurable functions h such that $|f|^{p-2}\,|h|^2 \in L_1$, with the scalar product

$$(h_1, h_2)_f = \int_C |f|^{p-4}\{|f|^2 R(\bar{h}_1 h_2) + (p-2)R(\bar{f} h_1)R(\bar{f} h_2)\}\, ds.$$

(In the above formula $R(\ )$ denotes "real part of".) Then $f_1'(0)$ should be the projection of H onto S in $\widetilde{H}(f)$. This formal result can be proved if $1 < p \leq 2$ and $H \in \widetilde{H}(f)$ in the stronger sense that $f_1(\lambda)/\lambda$ converges in the Hilbert space $\widetilde{H}(f)$. It is, however, easy to give examples where $f_1'(0)$ exists even though $H \notin \widetilde{H}(f)$. We hope to clarify this situation in the near future.

4. INTEGRAL EQUATIONS FOR THE EXTREMAL FUNCTIONS

In the rest of this paper we consider only the case $p = 2$, and we assume, for the sake of simplicity that $\log|g|$ is bounded.

The general case can be reduced, in fact, to the special case in which $|g|$ = constant on each C_k. By using the Poincaré-Bertrand relation (see Muskhelishvili [18]) we can prove that if z' is Hölder continuous with respect to arc length on each component of C, the h satisfies the integral equation

$$h(z) = \frac{1}{2} F(z) + \frac{1}{2\pi i} \int_C^* \frac{F(\zeta)d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_C^* (F(\zeta) - h(\zeta)) K_1(z, \zeta) |d\zeta|,$$

where $F(z) = f(z)/g(z)$ and

$$K_1(z, \zeta) = \frac{1}{\pi i} \int_C^* \left|\frac{g(\zeta)}{g(t)}\right|^2 \frac{|dt|}{(t-z)(\overline{\zeta}-\overline{t})} ,$$

and the asterisks denote Cauchy principal values. The kernel K_1 has an iterate satisfying Fredholm's conditions, so that this equation can be treated by known methods. In a similar fashion, if $f_2 = z'R/g$, then R satisfies an integral equation of the same form with $F(z)$ replaced by $F_1(z) = \overline{z' f(z)}\, g(z)$ and with the kernel

$$K_2(z, \zeta) = \frac{1}{\pi i} \int_C^* \left|\frac{g(t)}{g(\zeta)}\right|^2 \frac{|dt|}{(t-z)(\overline{\zeta}-\overline{t})} .$$

We may expect to obtain by a direct study of these integral equations the most important properties of the solution of our problem. They are also well adapted to the effective computation of the unknown functions. For theoretical purposes, we may remark that if $G(z)$ is an analytic function in Ω such that $|g/G|$ is constant on each curve C_k, then $S_2(g) = S_2(g/G)$. Such a function G can be determined from the solution of the Dirichlet problem with boundary values $\log|g|$ and a knowledge of the periods of the conjugate functions of the harmonic measures of the curves C_k. Thus we can always reduce the general case to the special case where $|g|$ is constant on each component of C. Since a factor of modulus 1 can always be incorporated into f, it is sufficient, for the general theory, to consider the case where g is a positive constant e^{α_k} on each curve C_k.

5. DEPENDENCE OF THE SOLUTION ON g

Let us assume, then, that $g = e^{\alpha}$, where $\alpha = \alpha_k$ = constant on C_k. There is a kernel $K_{z_o,\alpha}(z) = K(z, z_o, \alpha)$, reducing to the Szegö kernel for $\alpha = 0$, such that

$$h(z_o) = (e^{\alpha}K_{z_o,\alpha}, f) = \int_C \overline{e^{\alpha}K(z, z_o, \alpha)}f(z)|dz|$$

for $z_o \in \Omega$. The function $K_{z_o,\alpha}$ is the quotient of

$f(z) = \overline{z'/[2\pi i(z-z_o)g]}$ by $g = e^{\alpha}$. We have the simple variational formula for the dependence of K on α:

$$dK(z_1, z_o, \alpha) = -2(e^{2\alpha}K_{z_1,\alpha}, K_{z_o,\alpha}d\alpha),$$

or

$$K_k(z_1, z_o, \alpha) = \frac{\partial K}{\partial \alpha_k} = -2e^{2\alpha_k} \int_{C_k} \overline{K(t, z_1, \alpha)}\, K(t, z_o, \alpha)\,|dt|.$$

It follows that the kernel K_k is negative definite, that is, for any complex-valued completely additive set function ϕ on Ω, the "energy"

$$\iint K(z, \zeta, \alpha)\overline{d\phi(z)}d\phi(\zeta)$$

is a decreasing function of α_k, $1 \leq k \leq n$. The kernel satisfies the differential equation

$$\sum_1^n \frac{\partial K}{\partial \alpha_k} = -2K.$$

6. TRANSFORMATION AND REGULARITY PROPERTIES OF THE SOLUTION

While the classes L_p and $S_p(g)$ are not invariant under conformal mapping, they transform in a simple way, so that for general theoretical purposes we may still use convenient normalizations. Since the solution can be computed from geometrical quantities in the sense of Bergman-Schiffer [3], while the solutions themselves are physical quantities in the above sense, these transformation properties lead to information about conformally invariant domain functions in terms of geometrical properties of Ω. This is in contrast to the situation where the measure on C is taken to be the harmonic measure (see Rudin [26]). We find that $K(z, z, \alpha)|dz|$ is an invariant metric and that $K(z, \zeta, \alpha)^2 dz\, \overline{d\zeta}$ is an invariant double form.

Suppose that we transform Ω into a domain Ω_1 whose outer boundary C_1' is the unit circle. It is then quite easy to show that the kernel $K_1(z, \zeta, \alpha)$ of Ω_1 can be continued, as a function of z, analytically across C_1' into the inverse of $\overline{\Omega}$, except for a pole at $z = 1/\overline{\zeta}$. Hence if $Z = \phi(z)$ maps the component of the complement of C_1 which contains Ω onto the interior of the unit circle, then we can deduce the regularity properties of K from the formula

$$K(z, \zeta, \alpha) = \phi'(z)^{1/2}\, \overline{\phi'(\zeta)}^{1/2}\, K_1(\phi(z), \phi(\zeta), \alpha),$$

and the known behavior of $\phi(z)$ near the boundary curve C_1. Thus if z' satisfies a Dini condition with respect to arc length on an arc γ of C_1, then $K(z, \zeta, \alpha)$ is continuous for $z \varepsilon \gamma$.

The functions h and R in the general division problem can be treated similarly. For example, if on an arc γ of C_1 the functions z', f, and g are Dini-continuous and $g \neq 0$, then h and R are continuous on γ. If, moreover, z', f, and g are Hölder-continuous on γ, then so are h and R.

The variation of the solutions with the domain can also be worked out and, in particular, generalizations of the Hadamard variational formula can be obtained.

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AN EXTREMAL PROBLEM

George Springer

For a region G of the plane, the following interpolation problem has a well known solution: Let F represent the family of holomorphic functions f in G such that $\iint_G |f(z)|^2 dx\, dy < \infty$ and $f(z_o) = 1$, where z_o is a fixed point of G. Find the function in F which minimizes the value of $\iint_G |f(z)|^2 dx\, dy$. The problem is solved by the Bergman kernel function $f_o(z) = K(z, \bar{z}_o)/K(z_o, \overline{z_o})$ (see [1]). In extending this type of problem to a region G on a Riemann surface, one does not have a natural globally defined volume element which would enable one to define the integral of $|f|^2$ over G. One procedure has been to normalize a class of differentials and pose a similar minimal problem. We now propose to formulate such an extremal problem for the class of normalized holomorphic functions on a region G of a Riemann surface.

Assume that the region G has compact closure on the Riemann surface S. Let H denote the class of holomorphic differentials in G which are continuous in $\overline{G}$ (the closure of G), do not have any zeros in $\overline{G}$, and for which $||\eta||^2 = \iint_G \eta \overset{*}{\overline{\eta}} = 1$ for all $\eta \in H$. Corresponding to any $\eta \in H$, we may define a volume element $d\tau_\eta = \eta \overset{*}{\overline{\eta}}$; that is, if $\eta = g(z)dz$ locally, then $d\tau_\eta = 2|g(z)|^2 dx\, dy$ locally.

Let F denote the class of functions holomorphic in G with the normalization $f(P_o) = 1$ for some fixed $P_o \varepsilon G$, and such that for some $\eta \varepsilon H$, $\iint_G |f|^2 d\tau_\eta < \infty$. Then we note that f is also square integrable with respect to any volume element corresponding to a differential in H.

For each $\eta \varepsilon H$, we seek

(1) $$m(\eta, P_o) = \underset{f \varepsilon F}{\text{g.l.b.}} \iint_G |f|^2 d\tau_\eta .$$

To eliminate the dependence on η, we set

(2) $$M(P_o) = \underset{\eta \varepsilon H}{\text{l.u.b.}}\ m(\eta, P_o).$$

Any two differentials $\eta_1, \eta_o \varepsilon H$ have as their ratio a holomorphic function $\rho_{1o} = \eta_1/\eta_o$ which is continuous in $\overline{G}$ and has no zeros in $\overline{G}$. Thus $\eta_1 = \rho_{1o}\eta_o$ and $d\tau_1 = \eta_1 * \overline{\eta_1} = |\rho_{1o}|^2 d\tau_o$ where $d\tau_o = \eta_o * \overline{\eta_o}$. Furthermore, if ρ_{1o} is any function holomorphic in G which is continuous and has no zeros in $\overline{G}$, and for which $\iint_G |\rho_{1o}|^2 d\tau_o = 1$, then $|\rho_{1o}|^2 d\tau_o$ is also an admissible volume element on G. Let R denote the class of functions which are holomorphic on G, continuous and having no zeros in $\overline{G}$, and satisfying $\iint_G |\rho|^2 d\tau_o = 1$. Then we may reformulate the above extremal problem as

(3) $$\underline{m}(\rho, P_o) = \underset{f \varepsilon F}{\text{g.l.b.}} \iint_G |f\rho|^2 d\tau_o$$

and

(4) $$M(P_o) = \underset{\rho \varepsilon R}{\text{l.u.b.}}\ \underline{m}(\rho, P_o).$$

The problem of finding $\underline{m}(\rho, P_o)$ is solved using the weighted kernel function (see Nehari [2]) which we denote by $K_\rho(P, P_o)$. We have $\underline{m}(\rho, P_o) = [K_\rho(P, P_o)]^{-1/2}$ and the extremal function in F for which the value $\underline{m}(\rho, P_o)$ is attained is $K_\rho(P, P_o)/ K_\rho(P_o, P_o)$. For $\rho \equiv 1$, we get the kernel $K_o(P, P_o)$. It is possible to get a simple relation between $K_\rho(P, P_o)$ and $K_o(P, P_o)$; in fact, we obtain

(5) $$K(P, P_o) = \frac{1}{\rho(P)\overline{\rho(P_o)}} K_o(P, P_o).$$

directly from the characteristic reproducing and symmetry properties of the kernel:

$$\iint_G f(P)\overline{\rho(P)^{-1}\overline{\rho(P_o)}^{-1} K_o(P, P_o)}\rho(P)\overline{\rho(P)}\, d\tau_o$$

(6) $$= \frac{1}{\rho(P_o)} \iint_G f(P)\rho(P)\overline{K_o(P, P_o)}\, d\tau_o$$

$$= \frac{f(P_o)\rho(P_o)}{\rho(P_o)} = f(P_o).$$

Thus we have

$$(7) \qquad \underline{m}(\rho, P_o) = \frac{|\rho(P_o)|}{\sqrt{K_o(P_o, P_o)}} .$$

We have therefore reduced the problem to the following. Let R be the class of all functions ρ, holomorphic in G, continuous and having no zeros in $\overline{G}$, and such that $\iint_G |\rho|^2 d\tau_o = 1$. Find the least upper bound of $|\rho(P_o)|$ for all $\rho \in R$. We have $\rho(P_o) = \iint_G \rho(P) K_o(P, P_o) d\tau_o$, so that

$$(8) \qquad |\rho(P_o)|^2 \leq \iint_G |\rho|^2 d\tau_o \iint_G K_o(P, P_o)\overline{K_o(P, P_o)} d\tau_o$$

$$\leq K_o(P_o, P_o).$$

Therefore $M(P_o = \underset{\rho \in R}{\text{l. u. b.}}\ \underline{m}(\rho, P_o) = \underset{\rho \in R}{\text{l. u. b.}}\ \frac{|\rho(P_o)|}{K_o(P_o, P_o)} \leq 1$

and equality is possible only when

$$(9) \qquad \rho(P) = \frac{K_o(P, P_o)}{K_o(P_o, P_o)}$$

is the limit in norm of functions in R.

The family of functions R is normal in G, so that from the sequence of functions ρ_n such that $\lim_{n \to \infty} |\rho_n(P_o)| = \underset{\rho \in R}{\text{l. u. b.}}\ |\rho(P_o)|$

we may extract a subsequence which converges uniformly on every compact subset of G to a limit function ρ_o. Since each function in R does not have any zeros in G, the limit function ρ_o does not have any zeros in G. Even though the function ρ_o may not belong to R because of its boundary behavior, we do have $M(P_o) = \underline{m}(\rho_o, P_o)$.

The limit function ρ_o is unique, up to a multiplicative constant of absolute value 1. For if there were another extremal function σ_o such that $\sigma_o(P_o) = \rho_o(P_o)$ and if there were a sequence $\sigma_n \in R$ converging in norm to σ_o, then we consider the sequence ρ_n' where $\rho_n' = \sqrt{\sigma_n \rho_n}/v_n$ and

$$(10) \qquad v_n^2 = \iint_G |\sigma_n \rho_n| d\tau_o \leq [\iint_G |\sigma_n|^2 d\tau_o \cdot \iint_G |\rho_n|^2 d\tau_o]^{\frac{1}{2}} = 1.$$

Each ρ_n' belongs to R and the sequence $\{\rho_n'\}$ converges in norm to $\sqrt{\sigma_o \rho_o}/v_o$, where $\lim_{n \to \infty} v_n = v_o = [\iint_G |\sigma_o \rho_o| d\tau_o]^{1/2}$. We note first that $v_o = 1$, for if $v_o < 1$ held, then $|\sigma_o(P_o)\rho_o(P_o)/v_o| > |\rho_o(P_o)|$, which contradicts the extremal nature of ρ_o. Thus

$$(11) \qquad 1 = [\iint_G |\sigma_o \rho_o| d\tau_o]^2 = \iint_G |\sigma_o|^2 d\tau_o \iint_G |\rho_o|^2 d\tau_o$$

and equality holds in the Schwarz inequality, which implies that $\sigma_o = c\rho_o$, $|c| = 1$.

In order for $M(P_o) = 1$ to hold, we now know that the kernel $K_o(P, P_o)$ must not have any zeros in G. If G is a simply connected region, we may choose as uniformizing parameter $z = z(P)$ the function mapping G onto the unit circle with $z(P_o) = 0$. We then select as the volume element $d\tau_o = (1/\pi)dx\,dy$, $z = x + iy$. The kernel expressed in terms of the parameter z is $K_o(z, 0) \equiv 1$, so that $M(P_o) = 1$.

That $M(P_o)$ is not always equal to one is made evident by considering the doubly connected region. We may take as a conformally equivalent region an annulus $r < |z| < 1$, $r > 0$, with $z(P_o) = t$. Then choosing $d\tau_o = c\,dx\,dy$, $c = [\pi(1-r^2)]^{-1}$ the kernel (see Zarankiewicz [3]) becomes

$$K_o(z, \bar{t}) = \frac{1 - r^2}{z\bar{t}} \left[\wp(\log z\bar{t}) + \frac{\eta_1}{\pi i} - \frac{1}{2 \log r}\right] \tag{12}$$

where $\wp$ is the Weierstrass elliptic function with periods $2\omega_1 = 2\pi i$ and $2\omega_2 = 2 \log r$, and $2\eta_1$ is the increment of the Weierstrass ζ-function related to the period $2\omega_1$. Here the kernel has one zero interior to G except when t lies on a certain circle with center at the origin, in which case K_o has two zeros on the boundary of the annulus. Thus, for the annulus, $M(P_o) < 1$.

It would now be of interest to get more information about the nature of the function $M(P_o)$ for an arbitrary domain G and also a characterization of the extremal function ρ_o, perhaps finding its relation to the other well known domain functions.

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