

ATYPICAL HODGE LOCI

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I. INTRODUCTION

We will be concerned with the question

What can one say about Hodge loci?

Specifically,

- B is a smooth, connected quasi-projective variety;
- $\mathbb{V} \rightarrow B$ is the local system underlying a variation of polarized Hodge structure of weight n ;
- $\mathrm{HL}(B)$ is the set of points $b \in B$ where there are more Hodge classes in the tensor algebra $\mathbb{V}_b^\otimes := \bigoplus_k (\mathbb{V}_b \otimes_k \mathbb{V}_b)$ than there are at a very general point of B .

Q: *What can we say about $\mathrm{HL}(B)$?*

- In [CDK] it is proved that $\mathrm{HL}(B)$ is a countable union of algebraic varieties;

*Talk based on the paper [BKU] and related works given in the references in that work, and on extensive discussions with Mark Green and Colleen Robles.

- very informally stated, denoting by $\mathrm{HL}(B)_{\mathrm{pos}}$ the positive dimensional components of $\mathrm{HL}(B)$ the result of [BKU] is

(I.1) *For $n \geq 3$ and aside from exceptional cases, every irreducible component of $\mathrm{HL}(B)_{\mathrm{pos}}$ has less than the expected codimension.*

Thus if there are Hodge classes that vary in a positive dimensional family, then there are strictly more of these than suggested by a dimension count. As explained below the reason for this will be the integrability conditions arising from the differential constraint of the variation of Hodge structure. For this a key ingredient is the argument in [R] relating the Hodge and root space decompositions of a semi-simple Hodge Lie algebra.

Although we shall not discuss it, in [BKU] it is proved that if (I.2) is satisfied, then $\mathrm{HL}(B)_{\mathrm{pos}}$ is a *finite* union of irreducible algebraic varieties.

An interesting point is that whereas in general integrability conditions *decrease* the expected dimension of the space of solutions to a system of differential equations, due to the special circumstances in the case at hand here this dimension is actually *increased*. The mechanism behind this will be illustrated in Example 2 below.

With the notation to be explained below a sufficient condition for (I.1) to hold is

$$(I.2) \quad \mathfrak{g}^{-k,k} \neq 0 \text{ for some } k \geq 3.$$

The case $\mathfrak{g}^{-k,k} = 0$ for $k \geq 2$ includes the *classical case* where the period domain is Hermitian symmetric. The case $\mathfrak{g}^{-2,2} \neq 0$ but $\mathfrak{g}^{-k,k} = 0$ for $k \geq 3$ includes the case of weight $n = 2$ Hodge structures; here the differential constraint is non-trivial but the corresponding integrability conditions do not enter into the dimension count.

In Remark IV.14 the criterion to have the exceptional cases where

$$\mathfrak{g}^{-k,k} = 0 \text{ for } k \geq 3$$

will be given.

In section V we will recall the definition of the coupling length $\zeta(\mathfrak{A})$ ([VZ]) where $\mathfrak{A} \subset \mathfrak{g}^{-1,1}$ is the image of the differential of the period mapping at a general point. Then we will show that

$$(I.3) \quad \zeta(\mathfrak{A}) \geq 3 \implies (I.2).$$

The *geometric case* is when the variation of Hodge structure arises from the cohomology along the fibres of a smooth family $\mathcal{X} \rightarrow B$ of projective varieties. In this case, assuming the Hodge conjecture the result gives that if there are algebraic cycles $Z_b \subset X_b$ whose cohomology classes are not found on a general X_b and that non-trivially vary with b , then there are strictly more such cycles than one naively expects to find.

Examples where the coupling length $\zeta(\mathfrak{A}) \geq 3$ include the family of smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$ with $n \geq 3$, $\deg X > 5$ ([BKU]), and the moduli family of Calabi-Yau's of dimension ≥ 3 whose Yukawa coupling is generically non-zero.

II. TWO EXAMPLES

We consider the geometric case where $\mathcal{X} \rightarrow B$ is a family of surfaces with $X_{b_0} = X$ and $\lambda \in \mathrm{Hg}^1(X)_{\mathrm{prim}}$ is a primitive Hodge class.

(II.1) **Q:** *How many conditions is it for λ to vary with X as a Hodge class?*

We restrict to a neighborhood U of $b_0 \in B$ so that $\lambda \in H^2(X_b)_{\mathrm{prim}}$ is well defined. Setting

$$\mathrm{NL}_\lambda = \{b \in U : \lambda \in \mathrm{Hg}^1(X_b)\}$$

we are asking *What is the codimension of NL_λ in U ?* For this there is the classical estimate

$$(II.2) \quad \mathrm{codim}_U \mathrm{NL}_\lambda \leq h^{2,0}(X).$$

This bound is achieved; e.g., for smooth surfaces $X \subset \mathbb{P}^3$ with $d = \deg X \geq 4$ (cf. [G1] and [G2]).¹ In this case one also has

$$(II.3) \quad d - 3 \leq \text{codim}_B \text{NL}_\lambda$$

with equality holding only for surfaces containing a line.

For the second example we let $\mathcal{X} \rightarrow B$ be a family of 4-folds with $\lambda \in \text{Hg}^2(X)_{\text{prim}}$ a primitive Hodge class for $X = X_{b_0}$ and ask the same question. The analogue of (II.2) is

$$(II.4) \quad \text{codim}_B \text{NL}_\lambda \leq h^{4,0}(X) + h^{3,1}(X).$$

However due to transversality of the period mapping we have that for the first order variation of X in a direction $\theta \in T := T_{b_0}B$ the product $\theta \cdot \lambda \in H^{1,3}(X)$ and therefore for any $\omega \in H^{4,0}(X)$

$$(II.5) \quad \langle \omega, \theta \cdot \lambda \rangle = 0.$$

Thus a refinement of (II.4) is

$$(II.6) \quad \text{codim}_B \text{NL}_\lambda \leq h^{3,1}(X).$$

We refer to the right-hand side of (II.6) as the *expected codimension* of NL_λ in B .

At this juncture a new consideration enters. Setting

$$(II.7) \quad \begin{aligned} T_\lambda &:= \{\theta \in T : \theta \cdot \lambda = 0 \text{ in } H^{1,3}(X)\}, \\ \sigma(\lambda) &= \text{Image}\{T_\lambda \otimes H^{4,0}(X) \rightarrow H^{3,1}(X)\} \end{aligned}$$

we have

$$(II.8) \quad \text{codim}_B \text{NL}_\lambda \leq h^{3,1}(X) - \dim \sigma(\lambda).$$

Proof. For $\theta \in T_\lambda$ and any $\theta' \in T$, $\omega \in H^{4,0}(X)$

$$\begin{aligned} \langle \theta \cdot \omega, \theta' \lambda \rangle &= -\langle \omega, \theta \theta' \lambda \rangle \\ &= \langle \omega, \theta' \theta \lambda \rangle \\ &= 0 \end{aligned}$$

where the second step follows from the integrability condition $\theta \theta' = \theta' \theta$ arising from transversality. $\square$

¹A general treatment of Noether-Lefschetz loci given in [G3].

Assuming that the map defining $\sigma(\lambda)$ in (II.7) is non-zero we see that due to integrability the actual codimension of NL_λ is strictly less than the expected codimension.

One may show that the estimate (II.8) is sharp; e.g., by taking $X \subset \mathbb{P}^5$ a hypersurface of degree 6 containing a 2-plane Λ and for $\lambda \in \text{Hg}^2(X)_{\text{prim}}$ the primitive part of the class of Λ (cf. [GG]).

Remark II.9: Let I', I'' be distributions given in the tangent bundle of a manifold M by the vanishing of sets $\{\omega'_i\}, \{\omega''_\alpha\}$ of linearly independent 1-forms. Let N', N'' be variable integral manifolds of I', I'' . We want to estimate the codimension in M of the set of $N' \cap N''$'s. This intersection is an integral manifold of the $\{\omega''_\alpha|_{N'}\}$. However the integrability conditions given by the $d\omega'_i$ may impose linear relations on the $\omega''_\alpha|_{N'}$, thus allowing for more than the expected number of $N' \cap N''$'s. This is what happens in the second example.

III. THE MAIN RESULT

(i) *Hodge structures and Mumford-Tate groups.* A polarized Hodge structure of weight n is given by the data $(V, Q, F^\bullet)$ where

- V is a $\mathbb{Q}$ -vector space and $Q: V \otimes V \rightarrow \mathbb{Q}$ is a non-degenerate bilinear form with $Q(u, v) = (-1)^n Q(v, u)$;
- $F^n \subset F^{n-1} \subset \dots \subset F^0 \subset V_{\mathbb{C}}$ is a Hodge filtration satisfying

$$F^p \oplus \overline{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}}, \quad 0 \leq p \leq n;$$

and

- the two Hodge-Riemann bilinear relations are satisfied (we do not need their explicit form).

Setting

$$V^{p,q} = F^p \cap \overline{F}^q$$

the second condition above is equivalent to the *Hodge decomposition*

$$(III.1) \quad V_{\mathbb{C}} = \bigoplus V^{p,q}, \quad \overline{V^{p,q}} = V^{q,p}.$$

Using Q we have an identification

$$(III.2) \quad V \cong V^*.$$

We will generally omit reference to Q , its presence being understood.

When the weight $n = 2m$ the *Hodge classes* are

$$\mathrm{Hg}^m(V) = V^{m,m} \cap V,$$

the rational vectors of type (m, m) .

We denote by

$$V^\otimes := \bigoplus^k (\otimes^k V)$$

the tensor algebra of V ;

$$\mathrm{Hg}^\bullet(V^\otimes) := \bigoplus^k \mathrm{Hg}^{kn/2}(\otimes^k V);$$

denotes the sub-algebra of Hodge tensors.

Definition: The *Mumford-Tate group* $\mathrm{MT}(V)$ is the sub-group of $\mathrm{Aut}(V, Q)$ that fixes $\mathrm{Hg}^\bullet(V^\otimes)$.²

It is a reductive $\mathbb{Q}$ -algebraic group frequently denoted by G . Its Lie algebra

$$\mathfrak{g} = \mathrm{End}(V, Q)$$

is a *Hodge Lie algebra*; i.e., it has a Hodge structure of weight zero with Hodge decomposition

$$\mathfrak{g}_\mathbb{C} = \bigoplus \mathfrak{g}^{-k,k}$$

where

$$\mathfrak{g}^{-k} := \mathfrak{g}^{-k,k} = \{A \in \mathfrak{g}_\mathbb{C} \text{ such that } A : V^{p,q} \rightarrow V^{p-k,q+k}\} = \overline{\mathfrak{g}^k}.$$

We note that the real Lie group

$$G(\mathbb{R}) = G_1 \times \cdots \times G_k \times T$$

is a product of simple Lie groups with a compact torus. However we will not have a corresponding product decomposition of G .

Let $\mathfrak{g}_\mathbb{R} = \bigoplus \mathfrak{g}_{i,\mathbb{R}}$ be the decomposition of $\mathfrak{g}_\mathbb{R}$ into simple factors. Following [BKU] we have the

Definition III.3: The *level* $\ell(\mathfrak{g})$ is the smallest k such that all $\mathfrak{g}_i^k \neq 0$.

²The general reference for Mumford-Tate groups is [GGK], whose notations and terminology we will generally follow here.

(ii) *Variation of Hodge structure.* This is given by the data $(\mathbb{V}, \mathcal{F}^\bullet; B)$ where

- $\mathbb{V} \rightarrow B$ is a local system over a smooth, connected quasi-projective variety B ;
- $\mathcal{F}^\bullet$ is a filtration of $\mathcal{V} = \mathbb{V}_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_B$ by holomorphic sub-bundles that induce on each $\mathbb{V}_b$ a Hodge structure, and where for ∇ the Gauss-Manin connection corresponding to $\mathbb{V} \subset \mathcal{V}$ the transversality condition

$$(III.4) \quad \nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega_B^1$$

is satisfied.

It is understood that there is a horizontal section Q of $(\mathbb{V}_{\mathbb{Q}} \otimes \mathbb{V}_{\mathbb{Q}})^*$ that polarizes the Hodge structures.

At each point of B there is an algebra of Hodge tensors and Mumford-Tate group. Outside of a countable union of proper subvarieties of B these algebras are locally constant. We denote by $V = \mathbb{V}_{b_0}$ the fibre of $\mathbb{V}$ at such a very general point and by $G \subset \text{Aut}(V, Q)$ the corresponding Mumford-Tate group of the variation of Hodge structure.

The action of $\pi_1(B, b_0)$ on V induces the monodromy group $\Gamma \subset \text{Aut}(V, Q)$. It is known that $\Gamma \subset G$ and in [GGK] there is a general structure theorem describing their relation. In these notes in order to isolate the central points we will make the assumption

$$(III.5) \quad G \text{ is a simple } \mathbb{Q}\text{-algebraic group equal to the } \mathbb{Q}\text{-Zariski closure } \overline{\Gamma}^{\mathbb{Q}} \text{ of the monodromy group.}$$

(iii) *Period mappings.* Given (V, Q) the set of polarized Hodge structures with given Hodge numbers $h^{p,q} = \dim V^{p,q}$ is a homogeneous complex manifold called a period domain. The set of those polarized Hodge structures whose Mumford-Tate group is contained in G gives a *Mumford-Tate domain*

$$D = G(\mathbb{R})/G_0$$

where $G(\mathbb{R})$ is the real Lie group associated to G and G_0 is a compact subgroup.

Associated to a variation of Hodge structure with Mumford-Tate group G there is a period mapping

$$(III.6) \quad \Phi : B \rightarrow \Gamma \backslash D.$$

It may be assumed that Φ is proper with image $P \subset \Gamma \backslash D$ a quasi-projective variety. In order to simplify the notation we will make the assumption

$$(III.7) \quad P \text{ is smooth, and we identify } \Phi(B) = P \subset \Gamma \backslash D.$$

All of the results discussed below hold without this assumption.

For $b \in B$ and using a lift to D of $\Phi(b)$ the tangent space to D at the point is identified with $\mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}}$. Using (III.4) the differential of the period mapping is

$$(III.8) \quad \Phi_* : T_b B \rightarrow F^{-1}\mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}}.$$

We may identify the right-hand side of (III.8) with $\mathfrak{g}^{-1,1}$. For later use we note

$$(III.9) \quad \Phi_*(T_b B) := \mathfrak{A} \text{ is an abelian sub-algebra of } \mathfrak{g}^{-1,1}.$$

The “abelian” is a consequence of the integrability conditions imposed by the transversality property (III.8).

(iv) *Hodge loci.* We are interested in proper, irreducible subvarieties $Z \subset B$ along which the corresponding Hodge structures have extra Hodge tensors. Equivalently the Mumford-Tate group H at a general point of Z should be strictly contained in G ; i.e.,

$$\mathfrak{h} \subsetneq \mathfrak{g}.$$

A basic observation is

$$(III.10) \quad \text{If } D_H \subset D \text{ is the } H(\mathbb{R})\text{-orbit of a very general point of } Z, \\ \text{then for } \Gamma_H = \Gamma \cap H \text{ we have}$$

$$\Phi(Z) \subset \Gamma_H \backslash D_H.$$

Definition ([BKU]): If $H \subset G$ is a Mumford-Tate subgroup, then

$$(III.11) \quad \Phi^{-1}(\Phi(B) \cap (\Gamma_H \backslash D_H))^0$$

is a *special subvariety* of B .

Here the exponent 0 means to take an irreducible component of the intersection. Using the notation $P = \Phi(B)$ we set

$$P_H = (P \cap (\Gamma_H \backslash D_H))^0.$$

The subvariety Z may not be maximal with Mumford-Tate group H . In order to consider irreducible subvarieties that have extra Hodge classes rather than a particular Z one should use the intersection (III.11). We have

$$(III.12) \quad \text{codim}_{\Gamma \backslash D}(\Phi(B) \cap (\Gamma_H \backslash D_H))^0 \leq \text{codim}_{\Gamma \backslash D}(\Gamma_H \backslash D_H) + \text{codim}_{\Gamma \backslash D} \Phi(B),$$

or in the notation just introduced

$$(III.13) \quad \text{codim}_{\Gamma \backslash D} P_H \leq \text{codim}_{\Gamma \backslash D}(\Gamma_H \backslash D_H) + \text{codim}_{\Gamma \backslash D} P.$$

Definition ([BKU]): The subvariety $\Phi^{-1}(\Gamma_H \backslash D_H) \subset B$ is *atypical* if we have strict inequality in (III.12).

In other words, atypical means there are strictly more Hodge tensors than suggested by intersection theoretic dimension counts.

MAIN THEOREM III.14: *If the variation of Hodge structure has level at least three, then every positive dimensional special subvariety is atypical.*

As will be explained below, the proof will be to show that the condition to have level at least three will imply that the integrability conditions arising from transversality are non-trivial for *every* algebra of Hodge tensors defined over a positive dimensional subvariety of B .

Example III.15 ([BKU]): For B the family of smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$ with $n \geq 3$ and $\text{degree}(X) > 5$, $\mathfrak{g}^{-3,3} \neq 0$ and the theorem applies.

A proof of this is given in Section V below.

IV. PROOF OF THE MAIN RESULT

(i) *Sketch of the argument for Main Theorem III.14.* The strategy is to assume equality in (III.12), or equivalently (III.13), and from this

infer that

$$(IV.1) \quad \mathfrak{h}^- = \mathfrak{g}^-.$$

This implies that

$$(IV.2) \quad D_H = D,$$

i.e., the special subvariety is all of B .

We let $\tilde{P} \subset D$ be the inverse image in D of $P \subset \Gamma \backslash D$ and $\tilde{P}_H = \tilde{P} \cap D_H$. From (III.13) we have

$$(IV.3) \quad \text{codim}_D(\tilde{P}_H) = \text{codim}_D \tilde{P} + \text{codim}_D D_H.$$

Working infinitesimally in the tangent space $T_0 D \cong \mathfrak{g}^-$ with $T_0 D_H \cong \mathfrak{h}^-$ this gives

$$\begin{aligned} \text{codim}_D \tilde{P}_H &= \dim \mathfrak{g}^- - \dim T_0 \tilde{P}_H, \\ \text{codim}_D \tilde{P} &= \dim \mathfrak{g}^- - \dim T_0 \tilde{P}, \\ \text{codim}_D D_H &= \dim \mathfrak{g}^- - \dim \mathfrak{h}^-. \end{aligned}$$

Then (IV.3) yields

$$\dim \mathfrak{h}^- + \dim T_0 \tilde{P} - \dim T_0 \tilde{P} \cap \mathfrak{h}^- = \dim \mathfrak{g}^-.$$

Rewrite this as

$$\sum_{p \geq 2} \dim \mathfrak{h}^{-p} + \text{codim}_{\mathfrak{h}^{-1}} T_0 \tilde{P}_H = \sum_{p \geq 2} \dim \mathfrak{g}^{-p} + \text{codim}_{\mathfrak{g}^{-1}} T_0 \tilde{P}.$$

Since $\dim \mathfrak{h}^{-p} \leq \dim \mathfrak{g}^{-p}$ and $\text{codim}_{\mathfrak{h}^{-1}} T_0 \tilde{P}_H \leq \text{codim}_{\mathfrak{g}^{-1}} T_0 \tilde{P}$ this forces

$$(IV.4) \quad \begin{aligned} \text{codim}_{\mathfrak{h}^{-1}} T_0 \tilde{P}_H &= \text{codim}_{\mathfrak{g}^{-1}} T_0 \tilde{P} \quad \text{and} \\ \mathfrak{h}^{-p} &= \mathfrak{g}^{-p}, \quad p \geq 2. \end{aligned}$$

From this we want to conclude (IV.1), i.e.,

$$T_0 D_H = T_0 D$$

which implies (IV.2).

At this point the idea is that from the second condition in (IV.4) the Lie algebra $\mathcal{L}$ generated by the $\mathfrak{h}^k$, $|k| \geq 2$ is equal to the Lie algebra

generated by the $\mathfrak{g}^{-k}$, $|k| \geq 2$. If we can show that

$$\ell(\mathfrak{g}) \geq 3 \text{ implies that } \mathcal{L} = \mathfrak{g}^-,$$

then we are done. The intuition is that the bracket generation property will capture the integrability conditions. This is the basic idea; the actual argument is a bit more involved.

(ii) *Proof of Theorem III.14 under the assumption (IV.5) below.* We will first give the argument under the assumption

$$(IV.5) \quad \mathfrak{h}^\pm \text{ bracket generates all of } \mathfrak{h}_\mathbb{C}.$$

This assumption may not be satisfied, but we shall show that in the particular circumstances at hand an adaptation of the argument assuming it gives the result.

Recall that the semi-simple Lie algebra $\mathfrak{g}$ has a weight zero polarized Hodge structure with Hodge decomposition

$$\mathfrak{g}_\mathbb{C} = \bigoplus \mathfrak{g}^k, \quad \mathfrak{g}^{-k} = \mathfrak{g}^{-k,k} = \overline{\mathfrak{g}^k}.$$

Let $\mathfrak{g}'_\mathbb{C}$ be the complex Lie sub-algebra generated by $\mathfrak{g}^\pm$. Then $\mathfrak{g}'_\mathbb{C}$ is the complexification of a real sub-algebra $\mathfrak{g}'_\mathbb{R}$. Writing

$$\mathfrak{g}_\mathbb{R} = \mathfrak{g}'_\mathbb{R} \oplus \mathfrak{l}_\mathbb{R}$$

then

$$\mathfrak{l}_\mathbb{R} = (\mathfrak{g}'_\mathbb{R})^\perp \subset \mathfrak{g}_\mathbb{R}^{0,0}$$

implies that $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is semi-simple and $[\mathfrak{g}^k, \mathfrak{g}^\ell] \subset \mathfrak{g}^{k+\ell}$ it follows that $\mathfrak{g}'$ is a direct sum of factors $\tilde{\mathfrak{g}}_i$ of $\mathfrak{g}$ where all $\tilde{\mathfrak{g}}_i^\pm = (0)$. For the conclusions to be drawn below we may assume $\mathfrak{l}_\mathbb{R} = 0$ and thus that $\mathfrak{g}$ is generated by $\mathfrak{g}^\pm$.

We have

- $\mathfrak{g}$ is a Hodge Lie algebra;
- $\mathfrak{g}^+$ is a nilpotent sub-algebra of $\mathfrak{g}$;
- $\mathfrak{g}^+ \oplus \mathfrak{g}^0$ is a parabolic sub-algebra with Levi factor $\mathfrak{g}^0$;
- the center of $\mathfrak{g}^0$ is contained in a Cartan sub-algebra $\mathfrak{t}$, and we denote by β_i , $i \in I$, the positive simple $\mathfrak{t}$ -roots of $\mathfrak{g}^+$ with corresponding root spaces $\mathfrak{g}_{\beta_i}$.

All of the above are sub-Hodge structures of $\mathfrak{g}$.

LEMMA IV.6: $\mathfrak{g}^+$ is generated by $\mathfrak{g}^1$ if and only if $\mathfrak{g}_{\beta_i} \subset \mathfrak{g}^1$ for all $i \in I$.

Proof. We first note that since every positive root is a sum of simple positive roots the simple root spaces $\bigoplus_{i \in I} \mathfrak{g}_{\beta_i}$ generate $\mathfrak{g}^+$. Since every positive root is $\sum n_i \beta_i$ with $0 \leq n_i \in \mathbb{Z}$, if some $\mathfrak{g}_{\beta_i} \not\subset \mathfrak{g}^1$, then the algebra generated by $\mathfrak{g}^1$ will not contain $\mathfrak{g}_{\beta_i}$. $\square$

Now let $\mathcal{L} \subset \mathfrak{g}$ be the reductive sub-algebra of $\mathfrak{g}$ generated by the $\mathfrak{h}^{\pm i}$, $i \geq 2$. It is a sub-Hodge structure and by assumption (IV.5) is satisfied.

PROPOSITION IV.7: If the level $\ell(\mathfrak{g}) \geq 3$ and $\mathfrak{h}^k = \mathfrak{g}^k$ for all $k \geq 2$, then $\mathfrak{h}^+ = \mathfrak{g}^+$.

Proof. From the above we see that $\mathfrak{h}^+$ is a direct sum of positive root spaces. The proof of Lemma IV.6 then gives that there is a subset, which may be empty, $J \subset I$ such that $\mathfrak{h}^+ \cap \mathfrak{g}^1$ is the direct sum of the root spaces generated by the $\mathfrak{g}_{\beta_j}$, $j \in J$. We claim that we then have

$$(IV.8) \quad [\mathfrak{g}_{\beta_i}, \mathfrak{g}_{\beta_j}] = 0, \quad i \in I \setminus J, j \in J.$$

Indeed if this bracket is non-zero, then since $[\mathfrak{g}_{\beta_i}, \mathfrak{g}_{\beta_j}] \subset \mathfrak{g}^{\geq 2} = \mathfrak{h}^{\geq 2}$

$$\mathfrak{g}_{\beta_i} = \left[[\mathfrak{g}_{\beta_i}, \mathfrak{g}_{\beta_j}], \mathfrak{g}_{-\beta_j} \right] \subset \mathfrak{h}_{\mathbb{C}}$$

which is a contradiction. $\square$

The final step using $\ell(\mathfrak{g}) \geq 3$ is the

LEMMA IV.9: $\ell(\mathfrak{g}) \geq 3 \implies [\mathfrak{g}^{-2}, \mathfrak{g}^3] \subset \mathfrak{h} \cap \mathfrak{g}^1$ is non-zero.

Proof. Since $\mathfrak{g}^1$ generates $\mathfrak{g}^+$ and $\mathfrak{g}^1$ is spanned by simple positive root vectors, there exist simple positive roots $\beta_1, \beta_2, \beta_3$ such that $\beta_1 + \beta_2 + \beta_3$ is a root. Then $\mathfrak{g}_{-\beta_1 - \beta_2} \in \mathfrak{g}^{-2}$ and

$$\left[\mathfrak{g}_{\beta_1 + \beta_2 + \beta_3}, \mathfrak{g}_{-\beta_1 - \beta_2} \right] = \mathfrak{g}_{\beta_3} \neq 0. \quad \square$$

(iii) *Discussion of assumption of (IV.5).* The fact that we may assume (IV.5) uses the following result from [R],

THEOREM IV.10: *Let H be a Mumford-Tate group and $D_H = H(\mathbb{R})/H_0$ a Mumford-Tate domain. Any point $o \in D_H$ defines a weight zero Hodge structure on the Lie algebra $\mathfrak{h}$ of H . Let $\tilde{\mathfrak{h}} \subset \mathfrak{h}$ be the real semi-simple Lie subalgebra generated by $\mathfrak{h}^{-1,1} \oplus \mathfrak{h}^{1,-1}$. Then $\tilde{\mathfrak{h}} \subset \mathfrak{h}$ is a real sub-Hodge structure. Any connected integral manifold of the horizontal sub-bundle corresponding to $\mathfrak{h}^{-1,1} \subset T_0 D_H$ is contained in $D = \tilde{H}(\mathbb{R})o$ where $\tilde{H}(\mathbb{R})$ is the connected real Lie group with Lie algebra $\tilde{\mathfrak{h}}$. The horizontal sub-bundle $T^h D \subset TD$ is bracket generating; equivalently, $\tilde{\mathfrak{h}}^{-1,1}$ generates $\tilde{\mathfrak{h}}^-$ under Lie bracket.*

The general picture is that $\tilde{\mathfrak{h}}$ generates an integrable sub-bundle of TD_H and the $D = \tilde{H}(\mathbb{R})o$ above is a leaf of the corresponding foliation (cf. [R] for details and further discussion).

Using this theorem the assumption (IV.5) may be dropped thus completing the proof of Theorem III.14. $\square$

The result IV.10, specifically the last sentence, is the key to where the integrability conditions imposed by transversality kick in to give atypicality of Hodge loci.

We also note that using the result in [R] it can be shown that for the grading element $E \in \mathfrak{h}$ defined by condition $[E, X] = pX$ for all $X \in \mathfrak{h}^{p,-p}$ (thus $E \in$ center of $\mathfrak{h}^{0,0}$)

$$(IV.11) \quad \ell(\mathfrak{h}) = \tilde{\alpha}(E)$$

where $\tilde{\alpha}$ is the highest root.

Remark IV.12: It is *not* always the case that $\mathfrak{h}^{-1,1}$ generates $\mathfrak{h}^{-,+}$. To construct examples we let

$$\varphi : S^1 \rightarrow H(\mathbb{R})$$

be the circle defining the complex structure on D_H . Then for the E defined above, we can choose a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{h}_{\mathbb{C}}$ and a set $\Delta \subset \mathfrak{t}$ of simple roots such that $E \in \mathfrak{t}$ and $0 \leq \alpha(E) \in \mathbb{Z}$ for all $\alpha \in \Delta$. Then it can be shown that

$$(IV.13) \quad \mathfrak{h}^{1,-1} \text{ generates } \mathfrak{h}^{+,-} \iff \alpha(E) \in \{0, 1\}, \quad \alpha \in \Delta.$$

Using this one can construct many examples where $\mathfrak{h}^{+, -}$ is not generated by $\mathfrak{h}^{1, -1}$.

Remark IV.14: Finally from (IV.11)

$$\ell(\mathfrak{g}) \leq 2 \iff \tilde{\alpha}(E) \leq 2$$

where $\tilde{\alpha}$ is the highest root.

Thus, if we know the Mumford-Tate Lie algebra $\mathfrak{g}$, then this provides a test for when the main theorem applies.

V. REPRISÉ

(i) In the second example above if we assume that the algebra $\mathrm{Hg}^\bullet(X)_{\mathrm{prim}}$ corresponding to $H^4(X)_{\mathrm{prim}}$ is generated by Q and λ , then

$$G_\lambda := \{g \in \mathrm{Aut}(V, Q) : g\lambda = \lambda\}$$

is the Mumford-Tate group. In (III.12) we take $H = G_\lambda$, $\Gamma_\lambda = \Gamma \cap G_\lambda$ and set $D_H = D_\lambda \subset D$. Then

(V.1)

$$\mathrm{codim}_{\Gamma \backslash D} (\Phi(B) \cap (\Gamma_\lambda \backslash D_\lambda)) = \mathrm{codim}_{\Gamma \backslash D} (\Gamma_\lambda \backslash D_\lambda) + \mathrm{codim}_{\Gamma \backslash D} \Phi(B) - \dim \sigma(\lambda).$$

Thus $\dim \sigma(\lambda)$ is the correction term needed to convert the inequality (III.12) into an equality.

(ii) Referring to the first example, for $d = \deg X \geq 5$

$$(V.2) \quad d - 3 = \mathrm{codim} \mathrm{NL}_\lambda < h^{2,0}(X)$$

holds only for X 's containing a line Λ . In this case the strict inequality holds for geometric, not Hodge theoretic, reasons. If $\omega \in H^0(\Omega_X^2)$ has divisor $(\omega) \supset \Lambda$ containing the line, then for any $\theta \in T$ we have

$$\langle \omega, \theta \cdot \lambda \rangle = 0$$

due to $\theta \cdot \omega|_\Lambda = 0$ at the form level.

This phenomenon is general; we may say that the correction term needed to have equality in (III.13) is always positive if $\mathfrak{g}^3 \neq (0)$, and in particular cases it may be greater than it is for a general Hodge locus in B due to geometric reasons peculiar to the particular Hodge locus.

These considerations raise the following

Q: Given $(V, Q, F^\bullet)$ with Mumford-Tate group G where $\mathfrak{g}^3 \neq 0$, is there a *uniform bound* depending only $\mathfrak{g}$ for the correction term needed to convert (III.12) into an equality?

(iii) *Proof of Example III.15.* Let $X \subset \mathbb{P}^{n+1}$ be a smooth degree d hypersurface given by an equation

$$(V.3) \quad F(x) = 0$$

where $x = [x_0, \dots, x_{n+1}]$ and $F(x)$ is homogeneous of degree d . For

$$\begin{cases} S^\bullet = \mathbb{C}[x_0, \dots, x_{n+1}], \\ J_F^\bullet = \text{Jacobian ideal } \{F_{x_0}, \dots, F_{x_{n+1}}\}, \\ R^\bullet = S^\bullet / J_F^\bullet \end{cases}$$

it is well known ([G3]) that there is an isomorphism

$$H^{p, n-p}(X)_{\text{prim}} \cong R^{(n-p)d+n-2}.$$

Moreover the tangent space to the family of X 's modulo projective equivalence is

$$T \cong R^d$$

and the maps

$$(V.4)_p \quad T \otimes H^{p, n-p}(X)_{\text{prim}} \rightarrow H^{p-1, n-p+1}(X)_{\text{prim}}$$

are given by multiplication of polynomials

$$(V.5)_p \quad R^d \otimes R^{(n-p)d+n-2} \rightarrow R^{(n-p+1)d+n-2}.$$

Finally since X is non-singular, it follows from Macaulay's theorem that

$$(V.6) \quad \text{the mappings } (V.5)_p \text{ are non-zero whenever both sides are non-zero}$$

(cf. [G3]).

If G is the Mumford-Tate group for the period mapping of X 's as above, then we have

$$R^d \rightarrow \mathfrak{g}^{-1,1} \subset F^{-1} \text{End}(V, Q).$$

The image of this map is the *abelian* sub-algebra $\mathfrak{A} \subset \mathfrak{g}^{-1,1} \subset \mathfrak{g}$. There is an induced map

$$(V.7) \quad \mathrm{Sym}^k \mathfrak{A} \rightarrow \mathfrak{g}^{-k,k}.$$

This then gives

$$\mathrm{Sym}^k R^d \rightarrow \mathfrak{g}^{-k,k} \subset \oplus \mathrm{Hom} \left(H^{p,n-p}(X)_{\mathrm{prim}}, H^{n-k,n-p+k}(X)_{\mathrm{prim}} \right)$$

which is just the map

$$\mathrm{Sym}^k R^d \otimes R^{(n-p)d+n-2} \rightarrow R^{(n-p+k)d+n-2}$$

given by multiplication of polynomials. From (V.6) we may conclude that the map is non-zero whenever both sides are non-zero, which then gives that $\mathfrak{g}^{-3,3} \neq 0$ for $n \geq 3, d > 3$. $\square$

(iv) *The coupling length:* This is defined by $\zeta(\mathfrak{A}) = \max\{m : \mathrm{Sym}^m \mathfrak{A} \rightarrow \mathrm{Hom}(\mathbb{V}_b^{n,0}, \mathbb{V}_b^{n-m,m}) \text{ is } \neq 0\}$ at a general point of b . The same argument as in the hypersurfaces example then gives (I.3) above. There are many examples where this holds.

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