

## Hodge theory and moduli

Phillip Griffiths

### Outline

- I.A Introductory comments
- I.B Classification problem; first case of relation between moduli and Hodge theory
- II.A Moduli
- II.B Hodge theory
- III.A Generalities on Hodge theory and moduli
- III.B  $I$ -surfaces and their period mappings
- Appendix; Application of Hodge theory to  $I$ -surfaces with a degree 2 elliptic singularity

**I.A. Introductory comments.** Algebraic geometry is frequently seen as a very interesting and beautiful subject, but one that is also very difficult to get into. This is partly due to its breadth, as traditionally algebra (commutative and homological), topology, analysis, differential geometry, Lie theory — and more recently combinatorics, logic, categorical and derived algebraic techniques, ... — are used to study it. I have tried to make these notes accessible to a general audience by reviewing some of the most basic concepts, illustrating the material with elementary examples and informal geometric and heuristic arguments, and with occasional side comments for experts in the subject.

Although algebra, both commutative and homological, are the central tools in algebraic geometry, their use in many current areas of research (birational geometry and the minimal model program) is frequently quite technical and will not be extensively discussed in these notes. On the other hand, partly because Hodge theory involves analysis, Lie theory and differential geometry as well as a wide variety of homological methods, it is perhaps less prevalent in the more algebraic works in the field. One objective of these talks is to illustrate how, in partnership with the more algebraic and homological methods, Hodge theory may be used to study interesting and important geometric questions. In the appendix we have sketched how this may be carried out in a particular example.

---

Notes based on the lecture presented at the conference “Geometry at the frontier” held at Pucon, Chile during November 2018.

Parts of this paper represents joint work in progress with Mark Green, Radu Laza and Colleen Robles.

In summary the theme of these notes is to discuss and illustrate how complex analysis, differential geometry and Lie theory may be combined to study a basic problem in algebraic geometry. Although the main techniques in much of contemporary algebraic geometry are algebraic, some of the most interesting questions in the subject require non-algebraic methods for their study and this is what we hope to illustrate.<sup>1</sup>

As a warm up and to establish some notations we recall that an affine algebraic variety is given by the solution space over the complex numbers to polynomial equations

$$(*) \quad f_i(x_1, \dots, x_n) = 0 \quad i = 1, \dots, m.$$

The “elementary” examples are

linear spaces  $ax + by + c = 0$



conics  $ax^2 + 2bxy + cy^2 + ex + fy + g = 0$

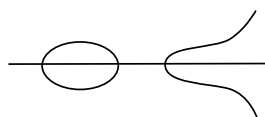


quadrics  $Q(x) = \sum_{i,j=1}^n a_{ij}x_i x_j + \sum_{i=1}^n b_i x_i + c = 0, \quad a_{ij}, a_{ji}$



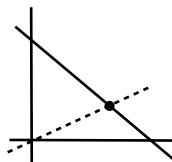
The first non-elementary examples are cubics

$$y^2 = 4x^3 + ax + b$$



The first two elementary examples and the non-elementary example are algebraic curves. One of course considers higher dimensional varieties; surfaces, threefolds, . . . We will be primarily concerned with curves and surfaces.

The above algebraic curves are all affine algebraic varieties in  $\mathbb{C}^2$ . In general one adds to an affine variety the asymptotes or “points at infinity” to obtain the projective space  $\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$ . Equivalently,  $\mathbb{P}^n$  is the quotient of  $\mathbb{C}^{n+1} \setminus \{0\}$  by the scaling action  $z_i \rightarrow \lambda z_i$  where  $\lambda \in \mathbb{C}^*$ . Geometrically  $\mathbb{P}^n$  is the set of lines through the origin in  $\mathbb{C}^{n+1}$ . The picture of the projective plane  $\mathbb{P}^2$  is



line at infinity

<sup>1</sup>This paper is written in an informal style, usually without precise definitions and statements of results. At the end there is a fairly extensive set of references where the more standard presentation of the material to algebraic geometers may be found. In particular we suggest [CMSP17] as a general reference relating Hodge theory and algebraic geometry and where several examples relevant to these notes are discussed in detail.



choosing a path  $\gamma$  of integration in the complex plane and a branch  $y(x)$  of  $f(x, y(x)) = 0$  along  $\gamma$ .<sup>2</sup> Thus

$$w = \int^{\sin w} \frac{dx}{\sqrt{1-x^2}} = \int^{\sin w} \frac{dx}{y(x)}$$

where  $x^2 + y^2 = 1$  and  $\sqrt{dx^2 + dy^2} = dx/y$  gives the parametrization  $w \rightarrow (\sin w, \cos w)$  of the circle by arclength in terms of “elementary” functions (trigonometric functions and logarithms). However parametrizing the ellipse by arclength led to integrals such as

$$w = \int^{p(w)} \frac{dx}{\sqrt{4x^3 + ax + b}}$$

which gave non-elementary functions and led Euler, Legendre, Abel, Gauss, Jacobi, Riemann, . . . to the beginnings of the rich and deep interplay between analysis and algebraic geometry. This evolved into modern *Hodge theory*, and it is this interface between analysis and algebraic geometry that is a main theme of these talks.

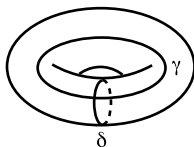
Part of the richness of the subject of algebraic geometry are the multiple perspectives that may be used in its study:

- *geometric*;
- *algebraic* — e.g., as we will briefly discuss, in birational geometry the algebraic classification of certain classes of singularities



of algebraic varieties plays a central role;<sup>3</sup>

- *analytic*; we have mentioned complex analysis and the integrals of algebraic functions;
- *topological*; one always has in mind the first picture



that is encountered.

**Example (running example for illustrative purposes):** The compact 1-dimensional complex manifold associated to the algebraic curve

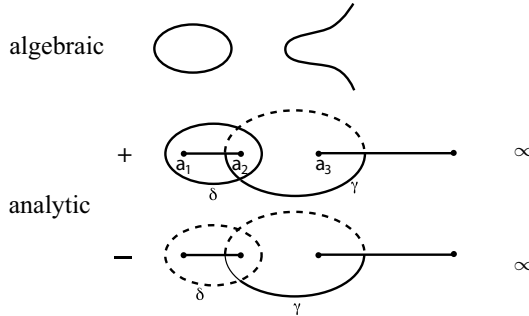
$$X = \{y^2 = (x - a_1)(x - a_2) \cdots (x - a_{2g+1})\}$$

where the  $a_i$  are distinct is a compact Riemann surface of genus  $g$  which can be studied from all of these perspectives.

<sup>2</sup>A more correct notation would be  $w = \int^{(x(w), y(w))} r(x, y(x)) dx$  where  $(x(w), y(w))$  is the point on the curve  $f(x, y) = 0$  in  $\mathbb{C}^2$  that is given by the integral along the path  $\gamma$  in the curve. To simplify the notation the second coordinate  $y(w)$  will be implicit in what follows.

<sup>3</sup>Cf. [Kol13b] in which the extent and complexity of this story are on full display.

For  $g = 1$  as in the cubic above, in addition to the topological one the pictures are



where  $a_1, a_2, a_3$  are the roots of the cubic equation in the RHS of the above equation. The  $+$  and  $-$  represent the two possible values of  $y(x)$  with the understanding that analytic construction across a slit changes the sign of  $y(x) = \sqrt{(x - a_1)(x - a_2) \cdots (x - a_{2g+1})}$ .

The integral

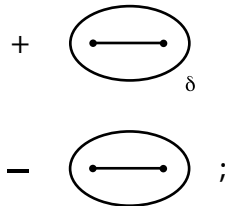
$$\int^w \frac{dx}{y}$$

is determined up to the *periods*

$$\pi_1 = \int_{\delta} \frac{dx}{y}, \quad \pi_2 = \int_{\gamma} \frac{dx}{y};$$

one may show that  $\pi_1 \neq 0$  and  $\text{Im}(\pi_2/\pi_1) > 0$ .

Incidentally for the hyperbola  $y^2 = x^2 - 1$  the picture is

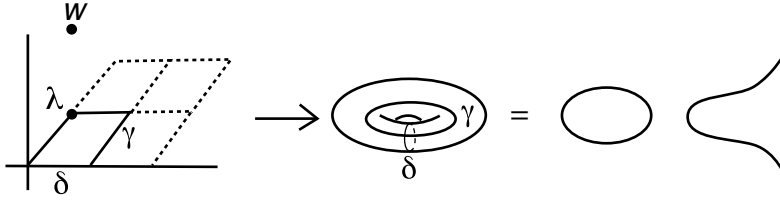


there is only one cycle  $\delta$ . In this case there is only one period  $\int_{\delta} \frac{dx}{y}$ .

For another analytic perspective, as will be further discussed below the function  $p(w)$  given by inverting the elliptic integral is a doubly periodic entire function (doubly due to the periods  $\int_{\delta} dx/y$  and  $\int_{\gamma} dx/y$ )<sup>4</sup> that leads to the parametrization of the cubic curve

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & X \\ \Psi & & \Psi \\ w & \longrightarrow & (p(w), p'(w)) \end{array}$$

<sup>4</sup>For the hyperbola above there is only one period and inverting the integral gives singly periodic trigonometric functions.



We note that  $y(w) = p'(w)$  arises from

$$dw = d \int^{p(w)} \frac{dx}{y} = \frac{p'(w)dw}{y(w)}.$$

Here the ratio  $\lambda = \pi_2/\pi_1$  of the periods is determined up to

$$\lambda \rightarrow \frac{a\lambda + b}{c\lambda + d}$$

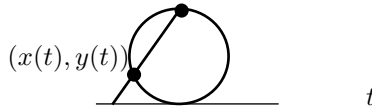
with  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  reflecting the choices of the basis  $\delta, \gamma \in H_1(X, \mathbb{Z})$  with  $(\delta, \gamma) = 1$ .

**I.B. Classification problem; first case of relation between moduli and Hodge theory.** A central problem in algebraic geometry is to classify the equivalence classes of algebraic varieties. For this there are two types of parameters:

- discrete (e.g., the genus  $g = (1/2)b_1(X)$  for smooth algebraic curves)
- continuous (*moduli*; the smooth curves of genus  $g$  form a  $(3g-3+\rho)$ -dimensional family  $\mathcal{M}_g$ , where  $\rho = \dim \mathrm{Aut}(X)$ ).

**Example (curves):**

- $g = 0$  there is only one equivalence class; for example using the birational map given by stereographic projection all non-singular conics are projectively equivalent and are birationally equivalent to a line  $\mathbb{P}^1$



where  $x(t), y(t)$  are rational functions of  $t$ .

- $g = 1$   $\dim \mathcal{M}_1 = 1$ , from the theory of elliptic curves one has that

$$j = \frac{1728a^3}{\Delta}, \quad \Delta = a^3 - 27b^2$$

gives a set-theoretically 1-1 map  $\mathcal{M}_1 \rightarrow \mathbb{C}$ ; thus the curves

$$\begin{cases} x_0^3 + x_1^3 + x_2^3 = x_0x_1x_2 & (\text{in } \mathbb{P}^2) \\ Q_1(x_0, x_1, x_2, x_3) = Q_2(x_0, x_1, x_2, x_3) = 0 & (\text{in } \mathbb{P}^3) \end{cases}$$

are both equivalent to  $y^2 = 4x^3 + ax + b$  for a unique value of  $j$ .

Here  $Q_1(x) = 0$  and  $Q_2(x) = 0$  are smooth quadrics that intersect transitively.<sup>5</sup>

In general one hopes that

<sup>5</sup>We are finessing the subtle issues that arise when the curve has non-trivial automorphisms. We refer to [Kol13a] and the references given there for a discussion of the structure of moduli spaces at points where the corresponding algebraic variety has automorphisms.

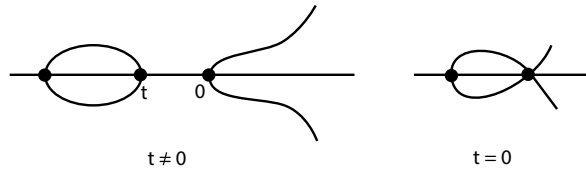
- (i) a moduli space  $\mathcal{M}$  will be an algebraic variety, generally *not* complete (compact);
- (ii) there will be a canonical completion  $\overline{\mathcal{M}}$  corresponding to adding *certain* singular varieties.

In these notes we will assume (i) and will be primarily concerned with (ii). What does (ii) mean?

We imagine a family of plane curves

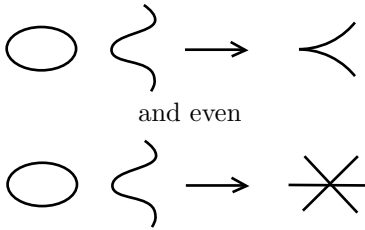
$$X_t = \{f(x, y, t) = 0\}, \quad t \in \Delta$$

that are smooth for  $t \in \Delta^*$  but may be singular for  $t = 0$ . A picture like



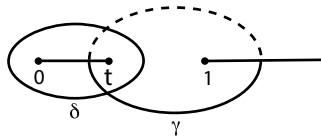
will give such a family.<sup>6</sup>

Applying coordinate changes depending on  $t$  can give a family  $\tilde{X}_t$  such that  $\tilde{X}_t$  is equivalent to  $X_t$  for  $t \neq 0$  but  $\tilde{X}_0$  is quite different from  $X_0$ . For example we can have

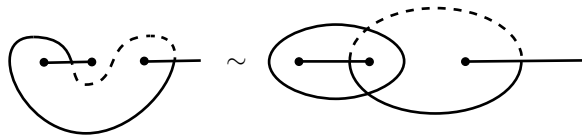


How can we say what a **canonical** choice for  $X_0$  should be?

Historically one suggested answer to this question was provided by Hodge theory; i.e., considering the *period matrix* associated to the curve. For the example  $y^2 = x(x-t)(x-1)$



as  $t$  turns around 0 by following what  $\gamma$  does we get a picture like the figure on the left



and denoting homology by  $\sim$  the picture on the right leads to the result that as  $t$  turns around the origin in homology we have the *Picard-Lefschetz transformation*

$$\begin{cases} \delta \rightarrow \delta \\ \gamma \rightarrow \gamma + \delta. \end{cases}$$

<sup>6</sup>Below we will give equations for such a picture.

For the periods by  $\pi_1 = \int_{\delta} dx/y$ ,  $\pi_2 = \int_{\gamma} dx/y$ , using elementary complex analysis one may show that

- $\pi_1(t)$  is non-zero and holomorphic for  $t \in \Delta$ ;
- $\pi_2(t) = \pi_1(t) \frac{\log t}{2\pi i} + (\text{holomorphic function of } t \in \Delta)$ .

In general for *any* family  $X_t$ ,  $t \in \Delta^*$ , of smooth genus 1 curves we will have periods  $\pi_1, \pi_2$  as above where

$$\lambda = \pi_2/\pi_1, \quad \text{Im } \lambda > 0;$$

here we are thinking of  $\lambda$  as a point in  $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$  where  $\mathbb{H} = \{w \in \mathbb{C} : \text{Im } w > 0\}$  is the upper half plane. The periods are locally holomorphic functions of  $t \in \Delta^*$ , and as  $t$  turns around the origin the cycles  $\delta, \gamma$  will undergo a *monodromy transformation*

$$\begin{cases} \begin{pmatrix} \pi_2 \\ \pi_1 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \pi_2 \\ \pi_1 \end{pmatrix}, \text{ where} \\ T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \text{ is the monodromy matrix.} \end{cases}$$

For  $w$  a lifting of  $\lambda$  to  $\mathbb{H}$  this gives a diagram

$$\begin{array}{ccccc} z \in & \mathbb{H} & \xrightarrow{w} & \mathbb{H} \\ \downarrow & \downarrow & & \downarrow \\ e^{2\pi iz} = & t \in \Delta^* & \xrightarrow{\pi} & \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \end{array}$$

where  $\pi(t) = \lambda$  and

$$w(z+1) = Tw(z).$$

LEMMA 1. *The eigenvalues  $\mu$  of  $T$  satisfy  $|\mu| = 1$ .*

Since the characteristic polynomial of  $T$  has integral coefficients, by the Gelfand-Schneider theorem from analytic number theory

$$\mu = e^{2\pi ip/q}$$

is root of unity. Replacing  $t$  by  $t^q$  gives that  $T$  is unipotent, and for a suitable choice of generators of  $H_1(X_t, \mathbb{Z})$  we may assume that

$$T = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \quad m \in \mathbb{Z}^+.$$

Thus *any* monodromy matrix is equivalent to a power of a Picard-Lefschetz transformation.

LEMMA 2. *Given a holomorphic mapping  $w : \mathbb{H} \rightarrow \mathbb{H}$  satisfying*

$$w(z+1) = w(z) + m,$$

*it follows that*

$$w(z) = mz + u(e^{2\pi iz})$$

*where  $u$  is bounded as  $\text{Im } z \rightarrow \infty$ .*

Taking  $m = 1$  for simplicity this gives

$$\pi(t) = \frac{\log t}{2\pi i} + u(t),$$

where  $u(t)$  is holomorphic in all of  $\Delta$ .

Both of these lemmas are proved by complex analysis arguments using the Schwarz lemma in the form



$w$  is distance decreasing in the  $\mathrm{SL}_2(\mathbb{R})$ -invariant hyperbolic (Poincaré) metric  $\frac{dw \, d\bar{w}}{|w|^2}$

and the observation that

the length of the circles  $|t| = \epsilon$  tends to zero as  $\epsilon \rightarrow 0$ .

**Sketch of the proof of Lemma 1:** Let  $z_n \in \mathbb{H}$  be a sequence with  $\mathrm{Re} \, z_n = 0$ ,  $\mathrm{Im} \, z_n \rightarrow \infty$  and set

$$w_n = w(z_n).$$

Then

$$w(z_n + 1) = Tw(z_n) = Tw_n.$$

The hyperbolic distance  $d(z_n, z_n + 1) \rightarrow 0$ , and by the distance decreasing property of  $w$  we have

$$d(w_n, Tw_n) \rightarrow 0.^7$$

Now  $w_n = A_n \cdot w$  for some  $A_n \in \mathrm{SL}_2(\mathbb{R})$  and fixed  $w \in \mathbb{H}$ , and using the invariance of the metric from

$$d(w, A_n^{-1}TA_n w) \rightarrow 0$$

a little argument shows that by passing to a subsequence we will have

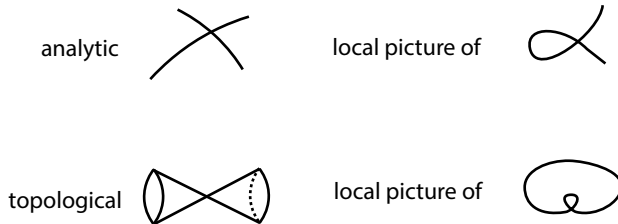
$$A_n^{-1}TA_n \rightarrow H = \{\text{isotropy group of } w\}.$$

Since  $H$  is compact, all its eigenvalues have absolute value 1 and this implies the same for  $T$ .<sup>8</sup> Since all that we have used is that the ratio of the periods  $\lambda(t)$  of *any* holomorphic family of genus 1 curve parametrized by  $\Delta^*$  has the same analytic behavior as for the above family has a pair of roots of the cubic coming together at  $t = 0$  we may draw the

**Conclusion:** *The periods of an arbitrary family of genus 1 algebraic curves over  $\Delta^*$  have the same asymptotic behavior as a family acquiring nodal singularities given locally analytically by*

$$x^2 = y^2 + tf(x, y).$$

The local pictures are

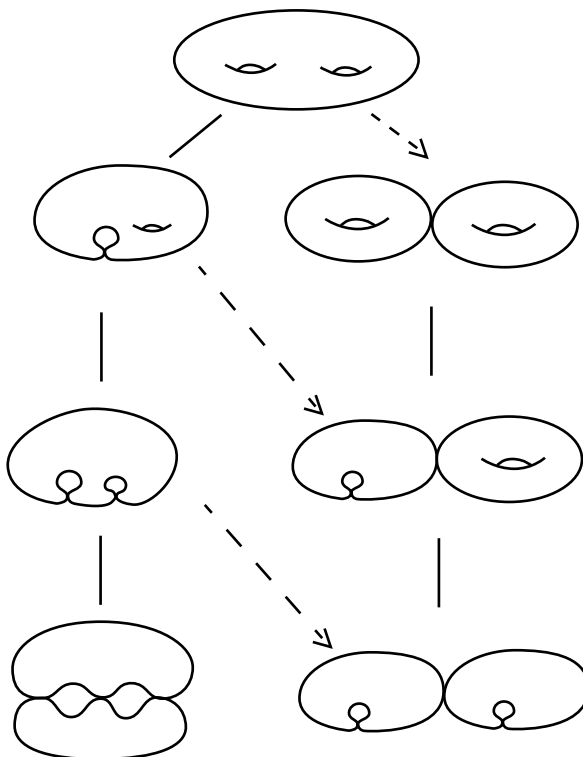


This analysis of the asymptotics of the period matrix (Hodge structure) extends to that of algebraic curves of any genus  $g \geq 2$  and provided an early suggestion as to what  $\overline{\mathcal{M}}_g$  should be.

<sup>7</sup>The horizontal segments from  $z_n$  to  $z_n + 1$  map down to circles in the punctured disc.

<sup>8</sup>Since  $\mathrm{SL}(\mathbb{R})$  acts transitively on  $\mathbb{H}$ , the isotropy group of any point is conjugate in  $\mathrm{SL}_2(\mathbb{R})$  to the isotropy group  $\left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right\}$  of  $i \in \mathbb{H}$ .

The schematic of  $\overline{\mathcal{M}}_2$  is



This gives the stratification of  $\overline{\mathcal{M}}_2$  together with the incidence (degeneration) relations among the strata. (The solid and dotted arrows will be explained later.) We will see that this stratification is captured by the Hodge structures and their limits.

The objective of these notes is to discuss how this picture might be extended to some completed moduli spaces of varieties of general type (analogues of curves of genus  $g \geq 2$ ) and to illustrate how this works for the first non-classical algebraic surfaces (called  $I$ -surfaces and which have the invariants  $p_g(X) = 2$ ,  $q(X) = 0$ ,  $K_X^2 = 1$ ).

To jump ahead and anticipate some of the main points to be made; with notations and terminology to be explained, there are first the general results (some of which are work in progress).

- For a given class of surfaces of general type the moduli space  $\mathcal{M}$  exists and has a canonical completion  $\overline{\mathcal{M}}$ .
- There is a period mapping

$$\mathcal{M} \xrightarrow{\Phi} \mathcal{P} \subset \Gamma \backslash D$$

that associates to each surface  $X$  the Hodge structure on  $H^2(X, \mathbb{Z})$ .

- There is a canonical minimal completion  $\overline{\mathcal{P}}$  of  $\mathcal{P}$  to projective variety, and the period mapping extends to

$$\overline{\mathcal{M}} \xrightarrow{\Phi_e} \overline{\mathcal{P}}.$$

There are then the specific results for the  $I$ -surface  $X$ . We will foreshadow these here with the explanation of the notations and terminology to also be given later. The objective is to give something of the flavor of what is to come. We begin with the

**Picture/equations:**

- $\left\{ \begin{array}{l} \text{Diagram: A cone with vertex } P \text{ and base } V. \text{ Inside the cone, there are two dashed circles representing the intersection of the cone with a surface } X. \end{array} \right. \quad \mathbb{P}(1, 1, 2) \hookrightarrow \mathbb{P}^3 \text{ given by } (t_0, t_1) \hookrightarrow [t_0^2, t_0 t_1, t_1^2, y]$
- $X \rightarrow \mathbb{P}(1, 1, 2)$  is a 2:1 map branched over the vertex  $P$  and the intersection of  $\mathbb{P}(1, 1, 2)$  with a quintic surface  $V \in |\mathcal{O}_{\mathbb{P}^3}(5)|$ ;
- $X$  is realized as the weighted hypersurface in  $\mathbb{P}(1, 1, 2, 5)$  given by an equation  $z^2 = F_{10}(t_0, t_1, y)$ .<sup>9</sup>

For the moduli space,  $\mathcal{M}_I$  is smooth and

- $\dim \mathcal{M}_I = h^1(T_X) = 28$ ;<sup>10</sup>
- the period domain  $D_I$  is a homogeneous contact manifold with  $\dim D_I = 57 = 2 \dim \mathcal{M}_I + 1$ ;
- $\Phi = \mathcal{M}_I \rightarrow \Gamma_I \backslash D_I$  and  $\Phi_*$  is injective (local Torelli)

$\Downarrow$

$$\Phi(\mathcal{M}_I) = \text{contact submanifold } \mathcal{P} \hookrightarrow \Gamma_I \backslash D_I.$$

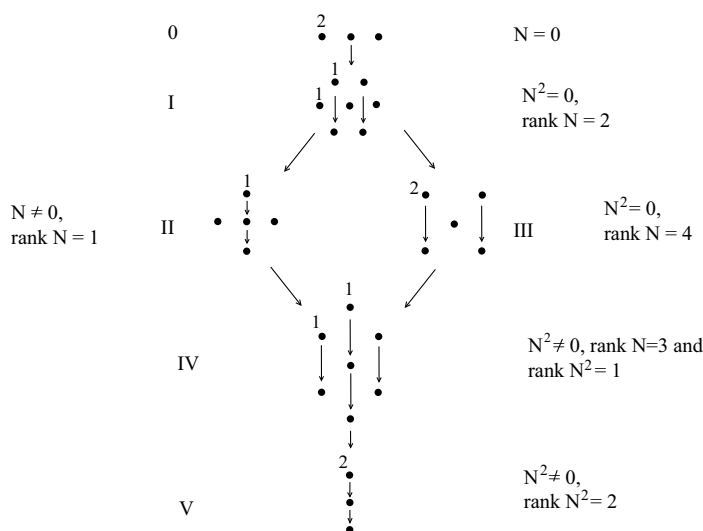
The contact structure on  $\mathcal{P}$  will be explained below. In the appendix by analyzing the Hodge structure at the boundary of moduli we will sketch a proof of generic local Torelli.

On the Hodge theoretic side we have the

**Picture of the stratification of  $\overline{\mathcal{P}}$  ( $N = \text{logarithm of monodromy}$ ):**

<sup>9</sup>This is the bicanonical model of a smooth  $I$ -surface.

<sup>10</sup>In fact, it can be shown that for a smooth  $X$ ,  $h^0(T_X) = h^2(T_X) = 0$ , and from the Hirzebruch Riemann-Roch theorem the Euler characteristic  $\chi(T_X) = -28$ . One may also “count parameters” in the above equation for  $X$  and arrive at the same conclusion.



The main result, here stated informally, is

*The Hodge theoretic stratification of  $\overline{\mathcal{P}}$  uniquely determines the stratification of  $\overline{\mathcal{M}}_I^{\text{Gor}}$ .*<sup>11</sup>

Rather than display the whole picture, the following table is just the part for simple elliptic singularities (types I and III). They have  $N^2 = 0$  since for the semi-stable-reduction (SSR) of such a degeneration only double curves (and no triple points) occur; all of the other types occur if we include cusp singularities.

stratum	dimension	minimal resolution $\tilde{X}$	$\sum_{i=1}^k (9 - d_i)$	$k$	$\text{codim in } \overline{\mathcal{M}}_I$
$I_0$	28	canonical singularities	0	0	0
$I_2$	20	blow up of a K3-surface	7	1	8
$I_1$	19	minimal elliptic surface with $\chi(\tilde{X}) = 2$	8	1	9
$III_{2,2}$	12	rational surface	14	2	16
$III_{1,2}$	11	rational surface	15	2	17
$III_{1,1,R}$	10	rational surface	16	2	18
$III_{1,1,E}$	10	blow up of an Enriques surface	16	2	18
$III_{1,1,2}$	2	ruled surface with $\chi(\tilde{X}) = 0$	23	3	26
$III_{1,1,1}$	1	ruled surface with $\chi(\tilde{X}) = 0$	24	3	27

The particular K3, elliptic surfaces, rational surfaces and ruled surfaces can be specified. Note that the last column is the sum of the two columns preceding it.

<sup>11</sup>Here  $\overline{\mathcal{M}}_I^{\text{Gor}}$  is the completed moduli space for Gorenstein  $I$ -surfaces. Work in progress suggests that the Hodge theoretic structure of  $\overline{\mathcal{P}}$  may go a long way towards determining the full stratification of  $\overline{\mathcal{M}}_I$  by singularity type.

(This may be explained using Hodge theory; in the appendix we will sketch how this may be done for  $I_2$ .)

**II.C. Moduli and Hodge theory.** We have seen that Hodge theory, in the classical form of periods of integrals of algebraic functions together with some complex analysis and differential geometry, suggests what singular curves should be included to compactify the moduli space  $\mathcal{M}_g$ , leading to an essentially smooth  $\overline{\mathcal{M}}_g$ .

For surfaces (and higher dimensional) varieties of general type the story thus far is both similar and different,<sup>12</sup> especially in the non-classical (term to be explained) case.

Birational geometry tells us that it is possible to define a moduli space  $\mathcal{M}$  with a canonical completion  $\overline{\mathcal{M}}$ . It does not

- (i) tell us what the singular surfaces  $X$  corresponding to the boundary  $\partial\mathcal{M} = \overline{\mathcal{M}} \setminus \mathcal{M}$  are;<sup>13</sup>
- (ii) tell us the stratification of  $\overline{\mathcal{M}}$ ; and
- (iii) in contrast to the curve case,  $\overline{\mathcal{M}}$  may be highly singular along  $\overline{\mathcal{M}} \setminus \mathcal{M}$ , and it does not suggest how to desingularize it.

We will explain and illustrate how Hodge theory, in partnership with birational geometry, helps us understand points (i)–(iii).

**A. Moduli** Invariants of a smooth projective variety  $X$  are basically

- discrete 
 $\swarrow$  Kodaira dimension  $\kappa(X)$   
 $\searrow$  topological (Chern numbers); and
- continuous (moduli).

$X$  is a compact, complex manifold and a basic invariant is the space  $H^0(K_X^m)$  of global holomorphic forms expressed locally in holomorphic coordinates  $x_1, \dots, x_n$  as

$$\varphi = f(x)(dx_1 \wedge \cdots \wedge dx_n)^m$$

where  $f(x)$  is holomorphic and transforms by the  $m^{\text{th}}$  power of the Jacobian determinant when we change coordinates.

The *Kodaira dimension*  $\kappa(X)$  is defined by

$$\dim H^0(mK_X) = h^0(mK_X) = Cm^{\kappa(X)} + \cdots, C > 0.$$

By convention we set  $\kappa(X) = -\infty$  if all  $h^0(mK_X) = 0$ .

The purpose of this part of these notes is to give an informal introduction to moduli, to describe two simple classes of algebraic curves and surfaces, and to illustrate the semi-log-canonical (slc) singularities that arise for surfaces and to begin the discussion of how Hodge theory relates to them.

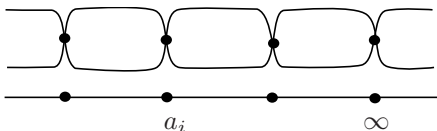
**Examples:** When  $n = 1$  and  $X$  is the smooth algebraic curve (compact Riemann surface) with affine equation

$$y^2 = \prod_{i=1}^{2g+1} (x - a_i), \quad a_i \text{ distinct},$$

<sup>12</sup>Cf. the papers [Kol13a], [Kol13b] and the recent Séminaire Bourbaki by Benoist in the references.

<sup>13</sup>In fact, even though the non-Gorenstein isolated singularities are all rational, there does not yet seem to be a practical bound on their index.

we may picture  $X$  as a 2-sheeted branched covering of  $\mathbb{P}^1$



and the holomorphic 1-forms on  $X$  are

$$\varphi = \frac{p(x)dx}{y}, \quad \deg p(x) \leq g-1.$$

For  $m \geq 2$  there are similar expressions  $\frac{q(x)dx^m}{y}$  where  $\deg q(x) \leq (2g-2)m - g$  for the holomorphic  $m$ -forms.

The genus  $g(X) = g$  and the number of parameters of  $X$ 's given as above is  $2g+1-2 = 2g-1$ ; a general curve of genus  $g$  is represented this way for  $g = 1, 2$ . The  $\varphi$ 's above give the space  $H^0(K_X) = H^0(\Omega_X^1)$  of holomorphic differentials on  $X$ , and the first de Rham cohomology group is a direct sum

$$H_{\text{DR}}^1(X) \cong H^0(\Omega_X^1) \oplus \overline{H^0(\Omega_X^1)}$$

which using de Rham's theorem gives the *Hodge structure* on  $H^1(X, \mathbb{C}) \cong H_{\text{DR}}^1(X, \mathbb{C})$ . Thus for  $h^0(K_X) = \dim H^0(K_X)$  we have

$$h^0(K_X) = \left(\frac{1}{2}\right) b_1(X) = g,$$

the first result in Hodge theory relating the algebro-geometric invariant  $h^0(K_X)$  to the topological invariant  $b_1(X)$ .

Now take  $X$  to be the smooth algebraic surface with affine equation

$$(*) \quad z^2 = f(x, y), \quad \deg f(x, y) = 2k$$

where  $f(x, y) = 0$  defines a smooth algebraic curve  $C \subset \mathbb{P}^2$ .<sup>14</sup> The holomorphic 2-forms on  $X$  are

$$\varphi = \frac{p(x, y)dx \wedge dy}{z}, \quad \deg p(x, y) \leq k-3.<sup>15</sup>$$

There are formulas similar to the above in the curve case for the  $H^0(mK_X)$ 's.

For  $H^0(\Omega_X^2) = H^0(K_X)$  the direct sum decomposition

$$(**) \quad H_{\text{DR}}^2(X) \cong H^0(\Omega_X^2) \oplus H^1(\Omega_X^1) \oplus \overline{H^0(\Omega_X^2)}, \quad H^1(\Omega_X) = \overline{H^1(\Omega_X^1)}$$

gives the Hodge structure on  $H^2(X, \mathbb{C}) \cong H_{\text{DR}}^2(X)$ .<sup>16</sup>

For an initial explanation of the  $H^1(\Omega_X^1)$  term, if  $Q$  is the bilinear form on  $H^2(X, \mathbb{C})$  given by the cup-product, then

$$\begin{aligned} F^2 H^2(X, \mathbb{C}) &:= H^0(\Omega_X^2) \\ &\cap \\ F^1 H^2(X, \mathbb{C}) &:= H^0(\Omega_X^2) \oplus H^1(\Omega_X^1) \end{aligned}$$

where under the cup product  $Q$  in cohomology

$$F^1 = F^{2\perp}$$

<sup>14</sup>This surface is similar to but both simpler and more complicated than the  $I$ -surface.

<sup>15</sup>Here  $p(x, y)$  is arbitrary. Later in these notes we will encounter the case  $k = 3$ , i.e.,  $X$  is a 2-sheeted covering branched over a sextic curve in  $\mathbb{P}^2$ : this is a K3 surface with  $h^0(\Omega_X^2) = 1$ .

<sup>16</sup>In contrast to the curve case, for any  $k \geq 4$  a general surface in the moduli space  $\mathcal{M}$  of surfaces of the above numerical type is equivalent to one given by  $(*)$ .

and

$$F^1 \cap \overline{F}^1 = H^1(\Omega_X^1)$$

so that the Hodge decomposition  $(**)$  is determined by  $H^0(\Omega_X^2)$  and  $Q$ . Thus in both these cases,  $H^0(\Omega_X^n)$  together with  $Q$  determines the Hodge decomposition and resulting Hodge structure on the cohomology  $H^n(X, \mathbb{C})$ ,  $n = \dim X$ .

The *Kodaira number*  $\kappa(X)$  for curves is

$$\kappa(X) = \begin{cases} -\infty & \text{for } g(X) = 0 \\ 0 & \text{for } g(X) = 1 \\ 1 & \text{for } g(X) \geq 2. \end{cases}$$

For the above algebraic surfaces

$$\kappa(X) = \begin{cases} -\infty & k \leq 2 & \text{(rational)} \\ 0 & k = 3 & \text{(K3)} \\ 2 & k \geq 4 & \text{(general type)} \end{cases} \quad (\text{K3})$$

to get  $\kappa(X) = 1$  you have to allow  $C$  to be quite singular.

*General type* surfaces are those with  $\kappa(X) = 2$ ; for these the important numerical invariants are

- $p_g = h^0(K_X) = h^0(\Omega_X^2)$  = geometric genus;
- $q = h^0(\Omega_X^1)$  = irregularity;
- $K_X^2 = c_1(X)^2$ .

They are related by

- $p_g - q + 1 = \frac{1}{12}(K_X^2 + \chi_{\text{top}}(X))$  (Noether's formula);
- $p_g \leq \frac{K_X^2}{2} + 2$  (Noether's inequality).<sup>17</sup>

**THEOREM ([KSB88], [Ale94]).** *For general type surfaces with given numerical invariants there exists a moduli space  $\mathcal{M}$  with a canonical completion  $\overline{\mathcal{M}}$ .*

As noted above the proof is via birational geometry. It describes in principle what the singularities of a surface  $X$  corresponding to a boundary point in  $\overline{\mathcal{M}} \setminus \mathcal{M}$  can be.<sup>18</sup> For surfaces there is no description, nor examples that I know other than the work of [FPR15a, FPR15b, FPR17], of the global structure of what the  $X$ 's can be and how they fit together.

Some guiding questions are

- *How does Hodge theory limit what the singular  $X$ 's can be?*
- *Which Hodge-theoretically possible degenerations are realized algebro-geometrically?*
- *Does the Hodge theoretic stratification capture the algebro-geometric one?*

Given a family  $\mathcal{X}^* \xrightarrow{\pi} \Delta^*$  of smooth surfaces  $X_t = \pi^{-1}(t)$  for  $t \neq 0$ , by the theorem there is defined a *unique* limit surface  $X_0 = X$  that fills in the family  $\mathcal{X} \rightarrow \Delta$  where the conditions

<sup>17</sup>For the above surface we have  $K_X^2 = 2$  and  $p_g = 3$ , so that it is extremal for Noether's inequality. For the  $I$ -surface we have  $K_X^2 = 1$  and  $p_g = 2$  so that it is also extremal.

<sup>18</sup>In the paper [Kol13a] by Kollár in the references there is a fairly short list of the singularity types that can occur. However within each non-Gorenstein type there is an invariant, the *index*. The recent paper [RU19] seems to be a promising approach to bounding the index in terms of  $K_X^2$ .

- $\mathcal{X}$  has *canonical singularities* over  $X_{\text{sing}}$ ;<sup>19</sup> and
- $\mathcal{X}$  is of relative general type and minimal (more precisely, the relative dualizing sheaf  $\omega_{\mathcal{X}/\Delta}$  is  $\mathbb{Q}$ -Cartier and relatively ample).

The first condition is local along  $X$ ; the second is global.

As is the case for any analytic variety,  $\overline{\mathcal{M}}$  has a *stratification*

- $\overline{\mathcal{M}}$  is a union of irreducible subvarieties  $\mathcal{Z}_i$ ;
- the incidence relation  $\mathcal{Z}_j \subset \overline{\mathcal{Z}}_i$  means that singular varieties parametrized by  $\mathcal{Z}_i$  can degenerate further into those parametrized by  $\mathcal{Z}_j$ .

The proof of the general existence theorem does not suggest what the stratification should be; again aside from [FPR15a, FPR15b, FPR17] I know of no other examples where it has been analyzed.

To give some flavor of how Hodge theory helps to organize the singularities, we note that  $\mathcal{X}^* \rightarrow \Delta^*$  is topologically a fibre bundle over the circle, and thus there is a monodromy operator (here  $t_0 \in \Delta^*$  is a base point)

$$T : H^2(X_{t_0}, \mathbb{Z}) \rightarrow H^2(X_{t_0}, \mathbb{Z}).^{20}$$

Denoting by

$$T = T_{\text{ss}} T_{\text{u}}$$

the Jordan decomposition of  $T$  where  $T_{\text{ss}}$  is semi-simple and  $T_{\text{u}}$  is unipotent with logarithm  $N$ , using analytic arguments arising from Hodge theory that extend the one given above in the case of elliptic curves leads to a proof the *monodromy theorem* [PS08], [Sch73]

$$T_{\text{ss}}^m = 1 \text{ (i.e., the eigenvalues of } T \text{ are roots of unity)}$$

$$N^3 = 0 \text{ (i.e., the Jordan blocks of } T \text{ have length } \leq 2).$$

A crude Hodge theoretic classification of the singularities of  $X$  is given by

- normal (a):  $N = 0$
- normal (b):  $N \neq 0, N^2 = 0$
- normal (c):  $N^2 \neq 0$
- non-normal (a):  $N \neq 0$  but  $N^2 = 0$ <sup>21</sup>
- non-normal (b):  $N^2 \neq 0$ .

This may be refined by putting in the ranks of  $N$  and of  $N^2$ .

A much finer invariant is given by also including the conjugacy class of  $T_{\text{ss}}$ , usually expressed in terms of the spectrum. And if we include the extension data in the limiting mixed Hodge structure (LMHS), we obtain even more Hodge-theoretic information.<sup>22</sup>

The following is an informal discussion of typical singularities of each of the above types.

---

<sup>19</sup>For normal  $\mathcal{X}$  this means that for  $U$  any open set in  $\mathcal{X}$  any holomorphic  $\omega \in H^0(\mathcal{U} \cap \mathcal{X}_{\text{reg}}, K_{\mathcal{X}})$  satisfies

$$\int_{\mathcal{X}_{\text{reg}}} \omega \wedge \overline{\omega} < \infty$$

<sup>20</sup>We will consider integral cohomology modulo torsion.

<sup>21</sup>In the examples I know of, non-normal  $\implies N \neq 0$ .

<sup>22</sup>The above crude Hodge theoretic classification is extracted from its associated graded. The LMHS will be defined below.



**Normal (a):** Then the monodromy is finite and the Hodge structures on the  $H^2(X_t, \mathbb{C})$  extend across  $t = 0$ .<sup>23</sup> These include a number of finite quotient singularities; typically among them are those denoted

$$\frac{1}{d}(1, a), \quad \gcd(d, a) = 1$$

and which is given by the quotient of  $\mathbb{C}^2$  acted on by the cyclic group generated by

$$(x, y) \rightarrow (\zeta x, \zeta^a y)$$

where  $\zeta = e^{2\pi i/d}$ . Among these are the Wahl singularities  $\frac{1}{n^2}(1, na - 1)$ . For  $n = 2$  this is a cone over a rational normal curve  $C$  in  $\mathbb{P}^4$ . It is noteworthy in that for this singularity  $T = \text{Id}$ .

**Normal (b): simple elliptic singularities.** Here  $(X, p)$  is a normal surface  $X$  having an isolated singular point  $p$ , the minimal resolution  $(\tilde{X}, C) \rightarrow (X, p)$  is then given by contracting an elliptic curve  $C \subset \tilde{X}$  with  $C^2 = -d$  where  $d > 0$  is the degree of the elliptic singularity. To say that  $(\tilde{X}, C)$  is minimal means that there are no  $(-1)$ -curves not meeting  $\tilde{C}$ .<sup>24</sup> The assumption that  $(X, p)$  is smoothable implies that  $1 \leq d \leq 9$ .

For  $d \geq 3$ , one may think of the cone over an elliptic normal curve in  $\mathbb{P}^{d-1}$ . One typically pictures such a singularity as



There are two types of restriction here:

- (i) the cone is over a smooth elliptic curve as opposed to a cone over a curve of genus  $g \geq 2$ ;
- (ii) the restriction  $d \leq 9$  for the elliptic curve.

An analytic explanation may be given for (i); as noted above, (ii) is the condition that the isolated singularity be smoothable.

**Note:** We are finessing the subtlety that in order to fit the desingularization  $\tilde{X}$  of  $X$  into a family  $\tilde{X} \rightarrow \tilde{\Delta}$  we have to do semi-stable-reduction (SSR), which involves a base change  $t = \tilde{t}^m$  where  $T_{ss}^m = \text{Id}$ . The fibre over the origin in  $\tilde{X} \rightarrow \tilde{\Delta}$  has  $\tilde{X}$  as one component. The other component is a rational surface  $Y$  meeting  $\tilde{X}$  along  $C$ , and  $p_g(Y) = 0$  so that  $\lim_{t \rightarrow 0} H^0(\Omega_{X_t}^2)$  lives on  $\tilde{X}$ .<sup>25</sup>

For the normal (b) degeneration a similar argument applies except now for  $\omega_t \in H^0(\Omega_{X_t}^2)$  and  $\lim_{t \rightarrow 0} \omega_t := \omega \in H^0(\Omega_{\tilde{X}}^2)$

$$\text{Res}_C(\omega) \in H^0(\Omega_C^1) \cong \mathbb{C}.$$

<sup>23</sup>There is a Riemann extension theorem for a family of Hodge structures over  $\Delta^*$  having finite monodromy. Thus even though  $X_0$  may be singular, there is associated to it a polarized Hodge structure.

<sup>24</sup>These are curves  $E \cong \mathbb{P}^1$  with  $E^2 = -1$ .

<sup>25</sup>An additional subtlety is that in order to have some degree of uniqueness one may want to allow  $\tilde{X}$  to have cyclic quotient singularities of a “simpler” type than the ones that were started with. The meaning of  $\lim_{t \rightarrow 0} H^0(\Omega_{X_t}^2)$  will be discussed below.

If there are  $e$  elliptic singularities, this type of argument leads to the bound

$$e \geq \text{rank } N.$$

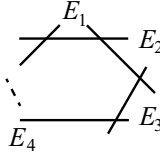
Another type of Hodge theoretic argument gives

$$e \leq \text{rank } N + 1.$$

We will see that both bounds are sharp for  $I$ -surfaces.

We will also see that for  $I$ -surfaces the degrees of the elliptic singularities are determined by the eigenvalues of  $T_{\text{ss}}$ . An explicit such singular surface will be discussed in the appendix.

**Normal (c): cusp singularity.**  $(X, p)$  where the minimal resolution  $(X, D) \rightarrow (X, p)$  has for  $D$  a cycle of  $\mathbb{P}^1$ 's  $E_i$  with all  $E_i^2 \leq -2$



and the least one  $E_i^2 \leq -3$ . It seems plausible, and may in fact be known, that the  $-E_i^2$  are determined by the spectrum of  $T_{\text{ss}}$ .

For the cusp,  $\text{Res}_{E_i}(\omega)$  is a 1-form on  $\mathbb{P}^1$  with log poles at the two intersection points. At a point of  $E_i \cap E_{i+1}$  the residues are opposite. Hence for each cusp the  $p_g$  can drop by at most 1 in the limit.

**Non-normal (a):**  $X$  has a smooth double curve  $C$  with pinch points (Whitney swallowtail given locally by  $x^2y = z^2$ ).

**Non-normal (b):** Informally stated,  $X$  has a nodal double curve with pinch points whose minimal resolution has a cycle of  $\mathbb{P}^1$ 's. These surfaces are frequently constructed by a gluing construction that will be illustrated below.

**Example of non-normal (a):** Let  $C \subset \mathbb{P}^2$  be a smooth plane quartic having an involution  $\tau : C \rightarrow C$  with quotient  $D = C/\tau$  an elliptic curve. Then  $X = \mathbb{P}^2/\tau$  is a surface having a smooth double curve  $D$  with pinch points at the 4 branch points of  $C \rightarrow D$ . The desingularization  $\tilde{X}$  of  $X$  is  $\mathbb{P}^2$  and by pulling back 2-forms one has that

$$H^0(K_X) \cong H^0(\Omega^2(\log C))^- \cong H^0(\mathcal{O}_{\mathbb{P}^2}(1))^-$$

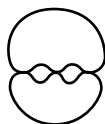
are the  $\tau$  anti-invariant 2-forms on  $\mathbb{P}^2$  having a log pole on  $C$ . Thus

$$\begin{cases} h^0(K_X) = 2 \\ K_X^2 = 1 \end{cases}$$

so that  $X$  “looks like” an  $I$ -surface. In fact  $X$  can be smoothed to such ([FPR15a, FPR15b, FPR17]).

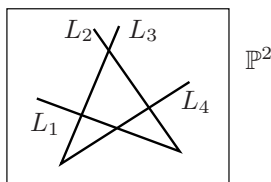
**non-normal (b) ([LR16]):** This is a degeneration of the preceeding example. Before explaining it we will give a general contextual comment.

Conjecturally an  $I$ -surface analogue of the “most degenerate,” meaning no equisingular deformations, genus 2 curve



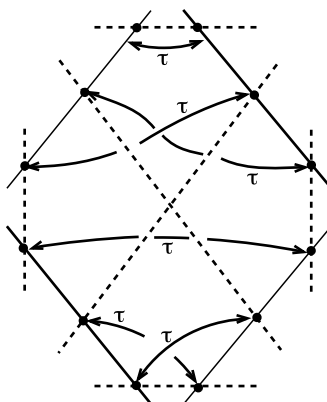
or \$ (dollar bill curve)

obtained by identifying three distinct points on each of two  $\mathbb{P}^1$ 's is the surface obtained by identifying pairs of lines in a quadrilateral



Here to obtain a well-defined involution  $\tau$  of the quartic curve given by the quadrilateral we identify  $L_1$  and  $L_2$  by  $\begin{cases} 12 \longleftrightarrow 21 \\ 13 \longleftrightarrow 24 \\ 14 \longleftrightarrow 23 \end{cases}$  and similarly for  $L_3$  and  $L_4$ . In more

detail, to identify two  $\mathbb{P}^1$ 's we need to say how three points on each are identified. Setting  $ij = L_i \cap L_j$  the identification of  $L_1$  with  $L_2$  is described by the procedure in the brackets. The construction is illustrated by the picture



Here the dotted lines are the blowups of the intersection points of the original four lines. On this blowup  $\mathbb{P}^2$  the involution  $\tau$  is well defined, and the cycles of  $\mathbb{P}^1$ 's are obtained from those in the picture. We will return to this example later.

**B. Hodge theory:** Traditionally there have been two principal ways in which Hodge theory interacts with algebraic geometry:

- topology; as previously noted many of the deeper aspects of the topology of an algebraic variety  $X$  are proved via Hodge theory;<sup>26</sup>

<sup>26</sup>This was true initially when  $X$  is smooth. Using mixed Hodge theory it is now the case when  $X$  is arbitrary (singular, non-complete or both), and as will be discussed below it is also the case when we have a degeneration  $X_t \rightarrow X$  leading for example to a proof of the above monodromy theorem and definition of the definition of  $\lim_{t \rightarrow 0} H^n(X_t)$ . There is also a very rich and beautiful Hodge theory associated to isolated hypersurface singularities.

- geometry; the Hodge structure on cohomology and its 1<sup>st</sup> order variations have been used to study the geometry of an algebraic variety  $X$ , especially the algebraic cycles that lie in  $X$  and in varieties constructed from  $X$ .

A central point of these notes is to possibly add a third point to this list:

- it is now well understood how Hodge structures can degenerate to a *limiting mixed Hodge structure*; this can then be used to guide and complement the study of algebraic varieties acquiring singularities as occur in moduli.

*What is meant by a Hodge structure (HS), a mixed Hodge structure (MHS) and a limiting mixed Hodge structure (LMHS)?*

Traditionally a HS or a MHS was given by a period matrix

$$\left\| \int_{\Gamma_\alpha} \omega_i \right\|$$

where the  $\omega_i$  are rational (meromorphic) differential forms on an algebraic variety  $X$  and the  $\Gamma_\alpha$  are cycles (including relative ones).<sup>27</sup> When  $X$  is smooth and the  $\omega_i$  are regular (holomorphic)  $n$ -forms this gives a holomorphic part

$$H^0(\Omega_X^n) = H^{n,0}(X) \subset H_{\text{DR}}^n(X) \cong H^n(X, \mathbb{C})$$

of the cohomology of  $X$ . As noted above, when  $n = \dim X = 1, 2$  using conjugation and the cup product in cohomology the holomorphic part  $H^0(\Omega_X^n)$  determines the *Hodge decomposition*

$$\begin{cases} H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X) \\ \overline{H^{p,q}(X)} = H^{q,p}(X) \end{cases}$$

on cohomology, where using the isomorphism given by de Rham's theorem

$$H^{p,q}(X) = \left\{ \begin{array}{l} \text{cohomology classes represented by } C^\infty \\ \text{differential forms of type } (p, q) \end{array} \right\}$$

One defines a *Hodge structure of weight  $n$*   $(V, F^\bullet)$  to be given by a  $\mathbb{Q}$ -vector space  $V$  and a decreasing *Hodge filtration*

$$F^n \subset F^{n-1} \subset \dots \subset F^0 = V_{\mathbb{C}}$$

that satisfies

$$F^p \oplus \overline{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}}$$

---

We note that much of the use of Hodge theory in birational geometry centers around the implications (primarily to vanishing theorems) of the surjectivity of the natural map  $H^1(X, \mathbb{C}) \rightarrow H^1(\mathcal{O}_X)$ . There is a natural splitting of this map. Also, of course the mere existence of functorial mixed Hodge structures and the strictness property of morphisms between them is of frequent use (cf. [KK10]). The actual geometry arising from Hodge structures together with their extensions and 1<sup>st</sup> order variations has thus far not played a particularly significant role.

<sup>27</sup>Classically (Riemann, Picard, Lefschetz, ...) there were differentials of 1<sup>st</sup> 2<sup>nd</sup> and 3<sup>rd</sup> kinds. It is now understood that the first kind deals with the holomorphic part of the Hodge theory of smooth varieties, the second kind with the full cohomology of smooth varieties and the third with the mixed Hodge theory of singular varieties.

for  $0 \leq p \leq n$ .<sup>28</sup> The relations

$$\begin{cases} F^p = \bigoplus_{p' \geq p} V^{p',q} \\ V^{p,q} = F^p \cap \overline{F}^q \end{cases}$$

give a 1-1 correspondence between Hodge filtrations and Hodge decompositions

$$\begin{cases} V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q} \\ \overline{V}^{p,q} = V^{q,p}. \end{cases}$$

The reason for using Hodge filtrations is that the  $F^p(X)$  vary holomorphically with  $X$ . In practice there will also be a lattice  $V_{\mathbb{Z}} \subset V$  that represents integral cohomology.

When  $X$  is of dimension  $n$  the cup products on cohomology and relations extending

$$\begin{cases} \int_X \omega \wedge \omega' = 0 & (\text{because } \omega \wedge \omega' = 0) \\ c_n \int_X \omega \wedge \overline{\omega} > 0 & (\text{because } c_n \omega \wedge \overline{\omega} > 0)^{29} \end{cases}$$

for holomorphic  $n$ -forms lead to the definition of a *polarized Hodge structure*  $(V, Q, F^{\bullet})$  where

$$Q : V \otimes V \rightarrow \mathbb{Q}, \quad Q(u, v) = (-1)^n Q(v, u)$$

and the following two *Hodge-Riemann bilinear relations* are satisfied:

- (I)  $Q(F^p, F^{n-p+1}) = 0$ ;
- (II)  $i^{p-q} Q(V^{p,q}, \overline{V}^{p,q}) > 0$ .

A *mixed Hodge structure* is given by  $(V, W_{\bullet}, F^{\bullet})$  where the increasing weight filtration

$$W_0 \subset W_1 \subset \cdots \subset W_m$$

is defined over  $\mathbb{Q}$ , and where the Hodge filtration  $F^{\bullet}$  induces on the graded quotients

$$\mathrm{Gr}_n^W V = W_n(V)/W_{n-1}(V)$$

a Hodge structure of weight  $n$ . Here  $F^p \mathrm{Gr}_n^W V = F^p \cap W_n(V)/W_{n-1}(V)$ .

Mixed Hodge structures have wonderful linear algebra properties. They form an abelian category,<sup>30</sup> and any morphism  $\varphi : (V, W, F) \rightarrow (V', W', F')$  is *strict* in the sense that  $\varphi(V) \cap W'_k = \varphi(W_k)$  and  $\varphi(V) \cap F'^p = \varphi(F^p)$ .

The basic results connecting Hodge theory to the cohomology of algebraic varieties are

- for  $X$  smooth and complete,  $H^n(X, \mathbb{Q})$  has a Hodge structure of weight  $n$  (Hodge).

As noted above, for  $m = n = 1$  the HS is determined by the period matrix

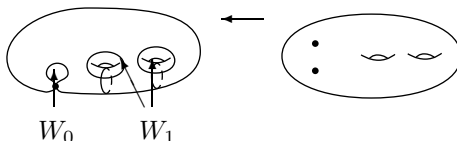
$$\Omega = \left\| \int_{\gamma_i} \omega_{\alpha} \right\| \begin{cases} \omega_{\alpha} \in H^0(\Omega_X^1) & (\dim = g) \\ \gamma_i \in H_1(X, \mathbb{Z}) & (\cong \mathbb{Z}^{2g}) \end{cases}$$

<sup>28</sup>One may think of  $F^p$  as represented by differential forms of degree  $n$  having in holomorphic local coordinates  $z_1, \dots, z_n$  at least  $p$   $dz_i$ 's.

<sup>30</sup>Polarized Hodge structures constitute a semi-simple category. Although many Hodge structures occurring in geometry have no natural polarization, the proofs of the deeper results in Hodge theory and its applications to topology require the Hodge structures to be polarizable.

For  $m = n = 2$  the HS on  $H^2(X)$  is determined by  $H^0(\Omega_X^2) = F^1$  by  $F^2 = F^{1\perp}$ ; as in the curve case  $H^0(\Omega_X^2)$  is given by the period matrix for the holomorphic 2-forms. Thus for both curves and surfaces the PHS is determined by the classical period matrix.

For a general complete algebraic variety  $X$ ,  $H^m(X, \mathbb{Q})$  has a mixed Hodge structure where the weight filtration is  $W_0 \subset \cdots \subset W_m$  (Deligne).



The picture is meant to suggest that MHS on  $H^m(X, \mathbb{Q})$  is constructed from the pure HS's on the strata of a desingularization of  $X$ ; this is (very non-trivially using homological algebra constructions — cf. [CMSP17] and [PS08]) indeed the case.

The use of Hodge theory to study a degenerating family  $X_t \rightarrow X_0 = X$  of algebraic varieties leads to the notion of a very special type of mixed Hodge structures, namely that of a *limiting mixed Hodge structure*  $(V, W(N), F^\bullet)$ . Here  $W(N)$  is the *monodromy weight filtration* constructed from the logarithm  $N$  of the unipotent part of monodromy. Assuming  $N^{n+1} = 0$ ,  $N^n \neq 0$ , it is the unique filtration

$$W_0(N) \subset W_1(N) \subset \cdots \subset W_{2n}(N)$$

satisfying

$$\begin{cases} N : W_k(N) \rightarrow W_{k-2}(N) \\ N^k : W_{n+k}(N) \xrightarrow{\sim} W_{n-k}(N). \end{cases}^{31}$$

Then a LMHS is given by a MHS  $(V, W(N), F_{\lim}^\bullet)$  where

$$N : F_{\lim}^p \rightarrow F_{\lim}^{p-1}.$$

The associated graded  $\text{Gr}(\text{LMHS}) \cong \bigoplus_{\ell=0}^{2m} H^\ell$  where  $H^\ell$  is a HS of weight  $\ell$ ; the picture is a Hodge diamond. Here  $m = 2$  and  $N$  is the vertical arrows; the dots are the  $H^{p,q}$ 's:

$$\begin{array}{ccccc} & & (2,2) & & \\ & & \bullet & & \\ (2,1) & \bullet & \downarrow & \bullet & (1,2) \\ (2,0) & \bullet & \downarrow & \bullet & \downarrow & \bullet & (0,2) \\ & (1,0) & \bullet & \downarrow & \bullet & (0,1) \\ & & (0,0) & & \bullet & \end{array}$$

We will set  $h^{p,q}$  = dimension of the  $(p, q)$  dot.

THEOREM (Schmid).<sup>32</sup> Given  $\mathcal{X} \rightarrow \Delta$  as above

$$\lim_{t \rightarrow 0} H^m(X_t) = \text{LMHS}.$$

The proof is a combination of

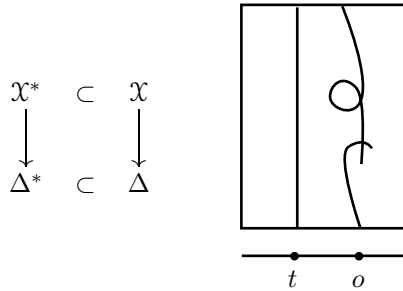
- Lie theory

<sup>32</sup>Cf. [Sch73] and [CKS86] in the references. An algebraic approach may be found in in [PS08].

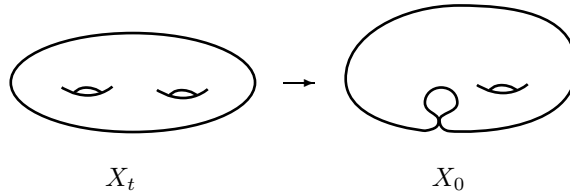
- complex analysis
- differential geometry

In simplest terms, the period matrices of a degenerating family of varieties have entries that are polynomials in  $\log t$  and with holomorphic functions as coefficients. Monodromy is given by analytic continuation of  $\log t$  around  $t = 0$ . The above ingredients, especially Lie theory, are then used to give a precise description of the asymptotic behavior of the period matrix.

The following is a typical picture one has in mind of a family of varieties  $X_t$  parametrized by the disc  $\Delta$  and which are smooth for  $t \in \Delta^* = \{t \in \Delta : t \neq 0\}$  while  $X_0$  has acquired singularities.



The following picture represents a family of genus 2 curves acquiring a node:



Associated to a general family of degenerating smooth varieties is a monodromy operator  $T : H^n(X_t) \rightarrow H^n(X_t)$

$$\begin{cases} T = T_{ss}T_u & (\text{Jordan decomposition}) \\ T_{ss}^k = I, T_u = e^N & \text{with } N^{n+1} = 0. \end{cases}$$

It is the basic topological invariant associated to the family  $X^* \rightarrow \Delta^*$ .



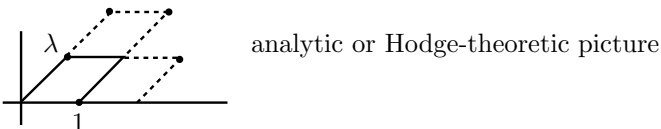
For the above degeneration of a genus 2 curve the LMHS is

$$\begin{array}{ccccc} & & (1, 1) & & \\ & & \bullet & & \\ (1, 0) & \bullet & \downarrow & \bullet & (0, 1) \\ & & \bullet & & \\ & & (0, 0) & & \end{array}$$

The solid vertical line represents the action of  $N$  on the associated graded to the LMHS (cf. Chapter 4 in [CMSP17]).

Referring to the picture of  $\overline{\mathcal{M}}_2$  in the introduction, the solid lines in the diagram represent degenerations with  $N \neq 0$ .

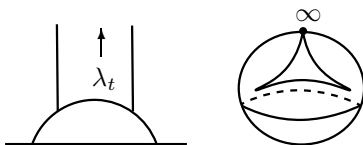
The following illustrates the type of pictures one has in mind for a LMHS:

-  topological picture
  -   $y^2 = x(x-1)(x-t)$   
algebraic picture
  - $X = \mathbb{C}/\Lambda$ ,  $\Lambda = \{1, \lambda\}$
-  analytic or Hodge-theoretic picture

Here  $\lambda$  is determined up to  $\lambda \rightarrow \frac{a\lambda+b}{c\lambda+d}$  where  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  and  $\mathcal{M}_1 \cong \mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$ ,  $\mathbb{H} = \{\lambda : \mathrm{Im} \lambda > 0\}$

In this case  $V = \begin{pmatrix} * \\ * \end{pmatrix}$ ,  $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ,  $F^1 = \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \in \mathbb{P}^1$ , HR II corresponds to  $\mathrm{Im} \lambda > 0$  and the space of PHS's is  $\mathbb{H} \subset \mathbb{P}^1$ . The monodromy  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  is translation in the upper half plane, and as  $\lambda \rightarrow i\infty$  we have  $F^1 \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = F_{\mathrm{lim}}^1$ .<sup>33</sup>

$$\lambda_t = \frac{\log t}{2\pi i}$$



**How does Lie theory enter?** The actors in the story are

- *Period domain*  $D = \{F^\bullet = \text{flag } \{F^n \subset \cdots \subset F^0 = V_{\mathbb{C}}\} \text{ in } V_{\mathbb{C}} : (V, Q, F^\bullet) = \text{PHS}\}$ ;
- *compact dual*  $\check{D} = \{F^\bullet \text{ is a flag with } Q(F^p, F^{n-p+1}) = 0\}$ ;
- $G = \mathrm{Aut}(V, Q)$  is a  $\mathbb{Q}$ -algebraic group;

<sup>33</sup>This picture is not indicative of what happens in the non-classical case. The  $F'_{\mathrm{lim}}$  will not in general lie in the boundary of the period domain in its compact dual (see below).



- $G_{\mathbb{R}}$  acts transitively on  $D$  and  $G_{\mathbb{C}}$  acts transitively on  $\check{D}$  so that we have

$$\begin{aligned} D &= G_{\mathbb{R}}/H \text{ with } H \text{ compact} \\ &\cap \\ \check{D} &= G_{\mathbb{C}}/P \text{ with } P \text{ parabolic} \quad \begin{pmatrix} * & * & * & * \\ \circ & * & * & * \\ \circ & \circ & * & * \\ \circ & \circ & \circ & * \end{pmatrix} \end{aligned}$$

where  $D$  is an open  $G_{\mathbb{R}}$ -orbit in  $\check{D}$ .

**Examples:**

- $m = 1$ :  $D = \mathrm{Sp}(2g, \mathbb{R})/\mathcal{U}(g) = \mathcal{H}_g$  where  $g = h^{1,0}$ ;
- $m = 2$ :  $D = \mathrm{SO}(2k, \ell)/\mathcal{U}(k) \times \mathrm{SO}(\ell)$  where  $k = h^{2,0}$ ,  $\ell = h^{1,1}$ .

The classical case is when we have

$$\begin{aligned} D &= \text{Hermitian symmetric domain (HSD)} \\ &\parallel \\ G_R/K, \quad K &= \text{maximal compact.} \end{aligned}$$

In algebraic geometry these arise as

$$\begin{aligned} m = 1 &\text{ (curves, abelian varieties);} \\ m = 2 &\text{ is HSD} \iff k = 1 \text{ (K3's)}.^{34} \end{aligned}$$

Thus  $h^{2,0} \geq 2$  is non-classical. For  $n \geq 3$  and  $X$  Calabi-Yau, the  $D$  corresponding to  $H^n(X)$  is also non-classical.

Period domains have sub-domains corresponding to PHS's with additional structure; e.g.,

$$\begin{aligned} D' &\subset D \\ &\parallel \\ &\{ \text{reducible PHS's} \\ &\quad \text{that are } \oplus\text{'s} \} \end{aligned}$$

This is what the dotted lines represent in the diagram in the Section I.A for  $\overline{\mathcal{M}}_2$ . In general one has Mumford-Tate sub-domains of  $D$ , defined to be those PHS's with a given algebra of Hodge tensors.

*Period mappings* arise from holomorphic mappings

$$\Phi : B \rightarrow \left\{ \begin{array}{l} \text{equivalence} \\ \text{classes of} \\ \text{PHS's} \end{array} \right\} = \Gamma \backslash D$$

where  $B$  is a complex manifold and  $\Gamma \subset G_{\mathbb{Z}}$  contains the monodromy group. One may think of  $B$  as the parameter space for a family of smooth algebraic varieties  $X_b$ ,  $b \in B$ , whose cohomology groups can be identified with  $H^n(X_{b_0})$  for a base point  $b_0 \in B$  up to the action by monodromy of  $\pi_1(B, b_0)$  on  $H^n(X_{b_0})$ .

**Example:** As noted above the first non-classical case is weight  $n = 2$  when  $h^{2,0} = 2$ . In this case  $D$  has an invariant contact structure and the image of any period mapping  $\Phi$  is an integral variety of that structure. This means that if the contact structure is given by a 1-form  $\theta$ , which up to scaling is invariant by  $G_{\mathbb{R}}$ , then

$$\Phi^*(\theta) = \Phi^*(d\theta) = 0.$$

---

<sup>34</sup>In this case  $D$  is a type IV HSD; it may be equivariantly embedded on  $\mathcal{H}_g$ . Also the unit ball may be equivariantly in  $\mathcal{H}_g$ . Thus the classical case should probably be defined as referring to algebraic varieties whose PHS's lie in a Mumford-Tate sub-domain of  $\mathcal{H}_g$ .

In general the differential constraint satisfied by period mappings in the non-classical case is the basic new phenomenon that occurs. Thus  $\Phi(\mathcal{M})$  cannot contain an open set in  $D$ ,  $\Gamma$  need not be arithmetic, etc.

**Note:** For families of algebraic surfaces with  $p_g = 2$  we think of the period mapping  $\Phi$  as given by a holomorphically varying  $2 \times k$  matrix  $\Omega$  satisfying HRI expressed by  $\Omega Q^t \Omega = 0$ . Differentiating this relation gives that the  $2 \times 2$  matrix  $d\Omega Q^t \Omega = -{}^t(d\Omega Q^t \Omega)$  is skew symmetric; writing

$$d\Omega Q^t \Omega = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

the 1-form  $\theta$  gives the contact structure.

Using Lie theory the set of equivalence classes of LMHS's has been classified [Ker15]; they form a stratified object. As noted above one may informally say that we know how Hodge structures degenerate; the strategy is then to use this information to help understand how algebraic varieties degenerate.

**Examples:** For  $n = 1$  the stratification may be pictured as

$$I_0 - I_1 - \cdots - I_g .$$

reflecting (for  $g = 2$ )



For  $n = 2$  with  $h^{2,0} = 1$  the picture is

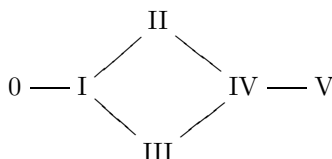
$$\text{I} \text{ --- } \text{II} \text{ --- } \text{III}$$

with corresponding Hodge diamonds

I	$\begin{array}{ccc} 1 & b & 1 \\ \bullet & \bullet & \bullet \end{array}$	
II	$\begin{array}{ccc} 1 & & 1 \\ \bullet & & \bullet \\ \downarrow & b-2 & \downarrow \\ \bullet & \bullet & \bullet \end{array}$	$N^2 = 0, \text{ rank } N = 2$
III	$\begin{array}{c} 1 \\ \bullet \\ \downarrow \\ \bullet \\ \downarrow \\ \bullet \end{array} \quad b$	$N^2 \neq 0, \text{ rank } N = \text{rank } N^2 = 1$

In the case of moduli of K3's, case I corresponds to smooth polarized K3 surfaces.<sup>35</sup> In case II, Hodge theory suggests is that the surface acquires an elliptic double curve. Case III is the case when the LMHS is of Hodge-Tate type. Dating from the work of Kulikov in the 70's, the moduli of K3's is a much studied and very beautiful story (cf. [Fri84] and [Laz16]).

For  $n = 2$  the picture is



The detailed picture with the ranks of  $N$  and  $N^2$  filled in when  $h^{2,0} = 2$  will be given below. In all of these examples the Roman numerals reflect the associated graded to the equivalence classes of LMHS's. The stratification is linear and transitive in the classical case, transitive but not linear in the non-classical  $n = 2$  case, and neither transitive nor linear in the general  $n \geq 3$  case.

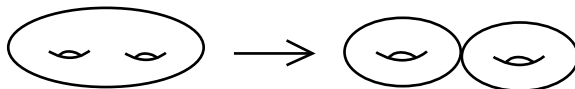
Within each of the above strata there is a refined stratification given by PHS's with "additional" Hodge tensors (Mumford-Tate sub-domains).

We emphasize that the Hodge-theoretic stratification has the following ingredients:

- (a) the equivalence classes of LMHS's over  $\mathbb{Q}$ ;
- (b) the spectra (basically the eigenvalues) of the semi-simple parts of monodromy;
- (c) for each equivalence class of LMHS's over  $\mathbb{Q}$ , within the PHS's given by the associated graded there is a stratification by Mumford-Tate sub-domains.

For the  $I$ -surface example we will see that the part of the stratification of  $\overline{\mathcal{M}}_I$  that has been determined is faithfully captured by using all of the Hodge-theoretic ingredients (i), (ii), (iii) above.

**Example: Curves with  $N = 0$ .** Using the proper Mumford-Tate sub-domain given by PHS's that are non-trivial direct sums over  $\mathbb{Z}$  one may Hodge-theoretically detect the degeneration



which has trivial monodromy. In general, in the stratification of  $\overline{\mathcal{M}}_2$  pictured in the first section of these notes the solid lines refer to degenerations where  $N \neq 0$  and the dotted lines to degenerations where the Jacobian of the normalization splits further into a direct sum of principally polarized abelian varieties.

**Example:  $n = 2$ .** At least in some examples one may Hodge theoretically detect a degeneration to a  $\frac{1}{d}(1, a)$  singularity where  $N = 0$ . The first case is the Wahl singularity  $\frac{1}{4}(1, 1)$  where  $T = \text{Id}$ ; then for  $I$ -surfaces  $X$  with one such singularity there is an outline of an argument that the image in the period domain picks up an extra Hodge class. It is known that having this singularity defines a non-empty boundary divisor in moduli, and then the tentative result is that where the image

<sup>35</sup>Suitably interpreted it also includes the case where  $X$  is nodal.

of the period mapping meets the Mumford-Tate domain corresponding to the extra Hodge class will define that boundary divisor. This will be further discussed below.

**III.A. Generalities on Hodge theory and moduli.** The first point is that there is a moduli space

$$\mathcal{H} = \Gamma \backslash D$$

for  $\Gamma$ -equivalence classes of PHS's (think of  $D = \mathbb{H}$  and  $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ ).

The second point is that there is a *period mapping*

$$(*) \quad \Phi : \mathcal{M} \rightarrow \mathcal{P} \subset \Gamma \backslash D$$

where  $\Gamma$  contains the *global monodromy group* given by the image of the monodromy representation

$$\rho : \pi_1(\mathcal{M}) \rightarrow G_{\mathbb{Z}}.$$

There are several important technicalities here, some dealing with the singularities of  $\mathcal{M}$  and some with the presence of  $X$ 's with extra automorphisms. And of course the general  $X$  may not be smooth but rather will have canonical singularities. In the example of the  $I$ -surface to be discussed next these issues can be addressed directly.

The following are statements that have been proved at the set-theoretic level and full results established under various assumptions; complete proofs are a work in progress (cf. [[Gri69]] for a discussion of this).

**THEOREM A.** *The image  $\mathcal{P} = \Phi(\mathcal{M}) \subset \Gamma \backslash D$  is a quasi-projective variety that has a canonical projective completion  $\overline{\mathcal{P}}$ .<sup>36</sup> Set-theoretically,  $\overline{\mathcal{P}}$  is obtained from  $\mathcal{P}$  by attaching the associated graded to the limiting mixed Hodge structures arising from  $\Phi$  in (\*).*

We shall call  $\overline{\mathcal{P}}$  the *Satake-Baily-Borel (SBB) completion* of  $\mathcal{P}$ . In the classical case using the Borel extension theorem the  $\overline{\mathcal{P}}$  is induced from the classical Satake-Baily-Borel compactification of arithmetic quotients of HSD's.

In general  $\overline{\mathcal{P}}$  is the minimal natural Hodge theoretic completion of  $\mathcal{P}$ . One may think of it as throwing out the extension data in the LMHS. For  $\Gamma$  arithmetic and with the assumption of the existence of a fan, Kato-Usui in [KU09] have constructed a universal maximal completion of  $\Gamma \backslash D$ , one in which the extension data is included. It may be thought of as a Hodge-theoretic toroidal compactification of  $\Gamma \backslash D$ .

**Note:** The proof of the above theorem (if completed) will have the following algebro-geometric implication: Let

$$Y \xrightarrow{f} Z$$

be a morphism of smooth, projective varieties and assume that the relative dualizing sheaf  $\omega_{Y/Z}$  is a line bundle. Then

$$(**) \quad \Lambda =: \det f_*(\omega_{Z/Y}) \text{ is semi-ample.}$$

It is known that  $\Lambda$  is nef, and if local Torelli holds for a general point of  $Z$ , then  $\Lambda$  is big [[Gri69]]. The freeness seems more subtle (witness the abundance conjecture).

---

<sup>36</sup>Cf. [BBT] and [BK] for an interesting “model-theoretic” proof of the result that when  $\Gamma$  is arithmetic  $\mathcal{P}$  is quasi-projective.

One may ask: Once you know that  $\Lambda$  is big and nef, why don't the standard methods of birational geometry (the minimal-model-program, including the base-point-free theorem) apply to give a proof? The interesting answer is that the signs needed in the base-point-free theorem are opposite to those that occur in the above situation.

The connection of this statement with the above theorem is that if

$$\overline{\mathcal{X}} \xrightarrow{f} \overline{\mathcal{M}}$$

is a versal family of general type varieties, then

$$\overline{\mathcal{P}} = \text{Proj}(\Lambda).$$

Thus assuming (\*\*) one may define the SBB completion of the image of the period mapping without using any Hodge theory.<sup>37</sup>

The second work-in-progress result is

**THEOREM B.** *The period mapping  $\Phi$  extends to*

$$(**) \quad \Phi_e : \overline{\mathcal{M}} \rightarrow \overline{\mathcal{P}}.$$

The above two structural statements provide a conceptual framework for the use of Hodge theory to partner with and help guide the standard algebro-geometric methods used to study the boundary structure for the KSBA moduli spaces for surfaces of general type.<sup>38</sup> How this works will now be illustrated.

**III.B.  $I$ -surfaces and their period mappings.** Murphy's law (Vakil): Whatever nasty property a scheme can have already occurs for the moduli spaces of general type surfaces. Thus unlike curves one should select "special" surfaces to study. In geometry extremal cases are frequently interesting; Noether's inequality

$$p_g(X) \leq \frac{K_X^2}{2} + 2$$

suggests studying surfaces close to extremal. The 1<sup>st</sup> non-classical case is given by the

**Definition:** An  $I$ -surface  $X$  is a regular ( $q(X) = 0$ ) general type surface that satisfies

$$p_g(X) = 2, K_X^2 = 1.$$

Here we are assuming that  $X$  is either smooth or has canonical (du Val) singularities. The KSBA moduli space for these surfaces will be denoted by  $\mathcal{M}_I$ .

One studies general type surfaces via their pluri-canonical maps

$$(\natural) \quad \varphi_{mK_X} : X \dashrightarrow \mathbb{P}H^0(mK_X)^* \cong \mathbb{P}^{P_m-1}$$

and pluricanonical rings  $R(X) = \bigoplus H^0(mK_X)$ .

Instead of  $(\natural)$  it is frequently better to use weighted projective spaces corresponding to when we add new generators to  $R(X)$ . From

$$P_m(X) = m(m-1)/2 + 3, \quad m \geq 2$$

---

<sup>37</sup>Of course even if (\*\*) is proved algebro-geometrically, just taking the Proj of  $\det(f_*\omega_{X/Y})$  does not seem to even roughly suggest what singularities are added on the boundary.

<sup>38</sup>It is not necessary to have proofs in order to use the statements of these results to help to guide how one may understand moduli.

and Kodaira-Kawamata-Viehweg vanishing (cf. [Dem12]) one has for the  $I$ -surface

$$\begin{aligned}\varphi_{K_X} : X &\dashrightarrow \mathbb{P}^1, \quad |K_X| = \text{pencil of hyperelliptic curves} \\ \varphi_{2K_X} : X &\rightarrow \mathbb{P}(1, 1, 2) \hookrightarrow \mathbb{P}^3 \text{ of degree 2;} \\ \varphi_{5K_X} : X &\hookrightarrow \mathbb{P}(1, 1, 2, 5) \hookrightarrow \mathbb{P}^{12} \text{ an embedding.}\end{aligned}$$

Here the homogenous coordinates of the mappings  $\varphi_{mK_X}$  are given by the generators of  $R(X)$  in the indicated degrees. If  $C \in |K_X|$  is a general smooth fibre, then by adjunction

$$2K_X|_C = K_C.$$

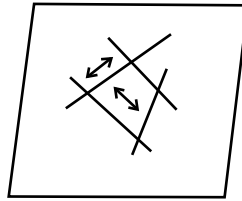
Thus the images  $\varphi_{2K_X}(C) = \varphi_{K_C}(C)$  are canonical curves. The  $I$ -surface was important classically since  $\varphi_{4K_X}$  is not birational, while for any general type surface  $\varphi_{5K_X}$  always is birational.

For the  $I$ -surface we have the following equations and picture:

- $\left\{ \begin{array}{l} \text{Diagram of a quadric cone with vertex } P \text{ and a section } V. \\ \mathbb{P}(1, 1, 2) \hookrightarrow \mathbb{P}^3 \text{ given by} \\ (t_0, t_1) \hookrightarrow [t_0^2, t_0 t_1, t_1^2, y] \\ X = 2:1 \text{ map branched over } P \text{ and } V \in |\mathcal{O}_{\mathbb{P}^3}(5)| \end{array} \right.$
- The canonical pencil  $|K_X|$  has base point  $P$  and is given by the two sheeted coverings of the lines  $L$  through the vertex in the quadric and branched over  $P$  and  $L \cap V$ .
- The equation of  $X$  is  $z^2 = F_5(t_0, t_1, y)z + F_{10}(t_0, t_1, y)$  (weighted hyper-surface in  $\mathbb{P}(1, 1, 2, 5)$ ).

One may extend the definition to include surfaces  $X$  whose canonical Weil divisor class  $K_X$  is  $\mathbb{Q}$ -Cartier and which have semi-log-canonical singularities and which have the numerical properties of smooth  $I$ -surfaces (cf. [Kol13a]).

**Example:** At the other extreme to the smooth  $I$ -surfaces is the surface



discussed above and which is obtained from  $(\mathbb{P}^2, 4 \text{ lines})$  by identifying opposite pairs of lines. For this surface the equation is (cf. [LR16])

$$z^2 = y(t_0^2 - y)^2(t_1^2 - y)^2.$$

Geometrically it is a double cover of a quadric cone in  $\mathbb{P}^3$  branched over the vertex, a plane section, and two double plane sections. A general curve in  $|K_X|$  is



Concerning the moduli space  $\mathcal{M}_I$  for smooth  $I$ -surfaces we have

- $\mathcal{M}_I$  is smooth and
  - $\dim \mathcal{M}_I = h^1(T_X) = 28$ ,
  - $\dim D_I = 57 = 2 \dim \mathcal{M}_X + 1$ ,
- $\Phi : \mathcal{M}_I \rightarrow \Gamma_I \backslash D_I$  has injective differential  $\Phi_*$  (local Torelli)

and this implies that

- $\Phi(\mathcal{M}_I)$  is a contact submanifold  $\mathcal{P} \hookrightarrow \Gamma_I \backslash D_I$ ,
- $\Gamma_I$  is arithmetic; it is not known whether  $\Gamma = G_{\mathbb{Z}}$  or not.

Elaborating a bit on a previous statement, in general, in the non-classical case there is a non-trivial homogeneous sub-bundle  $E \subset TD$  such that any period mapping satisfies

$$\Phi_* : T\mathcal{M} \rightarrow E \subset T(\Gamma \backslash D);^{39}$$

thus the image can never be an open set in  $\Gamma \backslash D$ . Moreover, although it is always the case that

$$\text{vol } \Phi(\mathcal{M}) < \infty,$$

It can happen that  $\Gamma \subset G_{\mathbb{Z}}$  is a thin subgroup, i.e., a subgroup with  $[\Gamma : G_{\mathbb{Z}}] = \infty$ .

**Stratification of the space of Gr(LMHS)'s:** For curves with  $\Gamma = \text{Sp}(2g, \mathbb{Z})$  we have for LMHS's

$$\begin{array}{ccccccc} I_0 & \text{---} & I_1 & \text{---} & I_2 & \text{---} & \cdots & \text{---} & I_g \\ \updownarrow & & \updownarrow & & \updownarrow & & & & \updownarrow \\ \mathcal{H}_g & & \mathcal{H}_{g-1} & & \mathcal{H}_{g-2} & & & & \mathcal{H}_0. \end{array}$$

Note that  $I_{g-m}$  corresponds to  $N : \text{Gr}_2 \rightarrow \text{Gr}_0$  with  $N^2 = 0$ ,  $\text{rank } N = m$ .

$$\begin{array}{ccc} & m & \text{Gr}_2 \\ & \bullet & \\ g-m & \downarrow & \bullet \quad \text{Gr}_1 \quad (\cong H^1(\tilde{C})) \\ & \bullet & \\ & & \text{Gr}_0 \end{array}$$

For each boundary component we have the stratification

$$H^1 = \oplus H_i^1.$$

The composite of these induces a stratification of  $\overline{\mathcal{M}}_g$  by

$$\{\# \text{ nodes}, \# \text{ components}\}.$$

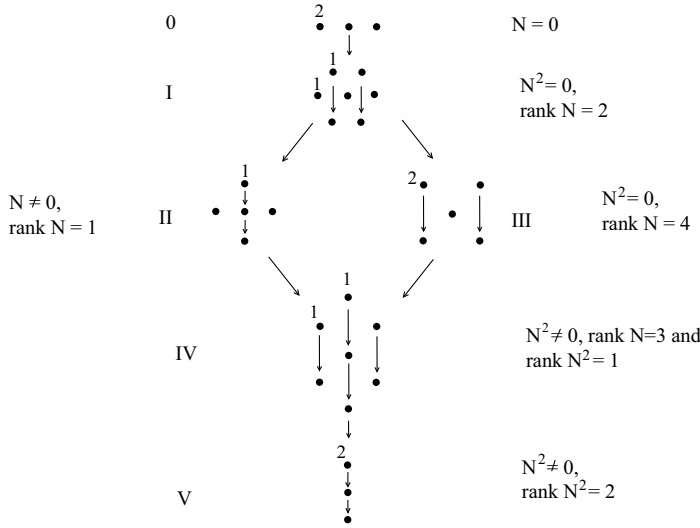
Of course this is just the beginning of the story of  $\overline{\mathcal{M}}_g$ .

For K3 surfaces there is an extensive literature dealing with the stratification of the image of the period mapping extended to include singular K3's for which  $N = 0$  (Heegard divisor, etc.); cf. [Laz16] for an analysis of degree 2 K3's and further references. The stratification diagram for the possible LMHS's was given above.

For surfaces with  $p_g = 2$  a refinement of the earlier picture by  $N \neq 0$ ,  $N^2 \neq 0$  of the classification of Gr(LMHS)'s/ $\mathbb{Q}$  is

<sup>39</sup> $E$  is defined by the differential constraint

$$\dot{F}^p \subseteq F^{p-1}.$$



For the refined Hodge-theoretic stratification of  $\text{Gr}(\text{LHHS}/\mathbb{Z})$ 's we use  $T_{\text{ss}} \rightarrow \{\text{conjugacy class } [T_{\text{ss}}] \text{ of } T_s \text{ in } \Gamma\}$ . Within each of these strata we use Mumford-Tate sub-domains appearing in  $\text{Gr}(\text{LMHS})$ 's in  $\overline{\mathcal{M}}_I$ .

We begin by considering the Gorenstein part  $\overline{\mathcal{M}}_I^{\text{Gor}} \subset \overline{\mathcal{M}}_I$ . One reason for this is the general result

*if  $X_t \rightarrow X$  is a KSBA degeneration of a surface where all the singularities of  $X$  are isolated and non-Gorenstein, then  $N = 0$ .*

Hence only Gorenstein singularities can non-trivially contribute to the LMHS/ $\mathbb{Q}$ .<sup>40</sup>

Heuristically the reason for this is the following.

- For the resolution of the singularity of a non-Gorenstein slc singularity one has a divisor  $D = \sum E_i$  where the  $E_i$  are  $\mathbb{P}^1$ 's and the dual graph is a chain or perhaps a Dynkin-like diagram with forks; there are no cycles.
- For a KSBA degeneration  $X_t \rightarrow X$  with  $\tilde{X} \rightarrow X$  a desingularization, and  $\omega_t \in H^0(\Omega_{X_t}^2)$ , the limit  $\lim_{t \rightarrow 0} \omega_t = \omega \in H^0(\Omega_{\tilde{X}}^2(\log D))$  and then  $\text{Res}_D \omega$  gives a meromorphic 1-form on the  $E_i$ 's with log poles on  $E_i \cap E_j$  and thus  $\text{Res}_D \omega = 0$ . It follows that  $p_g(\tilde{X}) = p_g(X)$ , which then implies that  $N = 0$ .<sup>41</sup>

The following results from coupling the classification in [FPR15a, FPR15b, FPR17] with the analysis of the LMHS's in the various cases.

**THEOREM B.** *The Hodge theoretic stratification of  $\overline{\mathcal{P}}$  given by the above diagram via the extended period mapping uniquely determines the stratification of  $\overline{\mathcal{M}}_I^{\text{Gor}}$ . Moreover, any Hodge-theoretic degenerations that are possible as pictured in the table below are actually realized by Gorenstein degenerations of  $I$ -surfaces.*

<sup>40</sup>We recall that for a normal surface  $X$ , Gorenstein means that the canonical Weil divisor class  $K_X$  is a line bundle. In general the *index* is the least integer  $m$  such that  $mK_X$  is a line bundle. For the  $\frac{1}{4}(1, 1)$  singularity the index is 2. A central general question in moduli is to determine a *useful* bound on the index.

<sup>41</sup>This is a consequence of the Clemens-Schmid exact sequence; cf. [CMSP17] and [PS08].



Rather than display the whole table the following is just the part for simple elliptic singularities (types  $I_k$  and  $III_k$ ). They have  $N^2 = 0$  since for the semi-stable-reduction (SSR) of a degeneration only double curves (and no triple points) occur; all of the other types occur if we include cusp singularities.

In the following,

- $X$  is irreducible (since  $K_X$  is a line bundle with  $K_X^2 = 1$  and any component of  $X$  will have positive  $K_X^2$ )
- $d_i = \text{degree of elliptic singularity}$
- $k = \# \text{ elliptic singularities}$  — in general, as previously noted using Hodge theory one may show that  $k \leq p_g + 1$
- $\tilde{X} = \text{minimal desingularization of } X$  — in a SSR given by  $\tilde{\mathcal{X}} \rightarrow \tilde{\Delta}$  the surface  $\tilde{X}$  will appear as one component of the fibre over the origin.

In the following table, in the 1<sup>st</sup> column subscripts denote the degrees of the elliptic singularities, which one can show are uniquely determined by the  $[T_{ss}]$ 's.<sup>42</sup> We will explain the  $\sum(9 - d_i)$  column below in the appendix.

stratum	dimension	minimal resolution $\tilde{X}$	$\sum_{i=1}^k (9 - d_i)$	$k$	$\text{codim}$ in $\overline{\mathcal{M}}_I$
$I_0$	28	canonical singularities	0	0	0
$I_2$	20	blow up of a K3-surface	7	1	8
$I_1$	19	minimal elliptic surface with $\chi(\tilde{X})=2$	8	1	9
$III_{2,2}$	12	rational surface	14	2	16
$III_{1,2}$	11	rational surface	15	2	17
$III_{1,1,R}$	10	rational surface	16	2	18
$III_{1,1,E}$	10	blow up of an Enriques surface	16	2	18
$III_{1,1,2}$	2	ruled surface with $\chi(\tilde{X})=0$	23	3	26
$III_{1,1,1}$	1	ruled surface with $\chi(\tilde{X})=0$	24	3	27

Note that the last column is the sum of the two columns preceding it.

**Appendix:** We will use Hodge theory as a guide to study the boundary component

$$\overline{\mathcal{M}}_2^{\text{Gor}} \subset \overline{\mathcal{M}}_I$$

whose general point is a normal  $I$ -surface  $X$  that arises from a KSBA degeneration

$$(*) \quad \mathcal{X} \rightarrow \Delta$$

whose monodromy logarithm  $N$  satisfies

$$\begin{cases} N^2 = 0 \\ \text{rank } N = 2 \end{cases}$$

<sup>42</sup>The  $T_{ss}$  contains the Coxeter element of the Dynkin diagram. By inspecting the table of Coxeter elements for simple elliptic singularities with indices 1 and 2 one may check this statement.

and where the degree of the specialization  $(*)$  is 2 (we will explain below how to define the degree of the specialization). As above,  $\overline{\mathcal{M}}^{\text{Gor}}$  denotes the part of  $\overline{\mathcal{M}}$  parametrizing surfaces with only Gorenstein singularities.

With these assumptions we will see that  $X$  is irreducible with a single normal singular point  $p$ . Hodge theory then gives that  $p$  is a simple elliptic singularity whose degree is in this case defined to be degree of the specialization. In this situation one has the following diagram:

$$(**) \quad \begin{array}{ccc} & (\tilde{X}, \tilde{C}) & \\ g \swarrow & & \searrow f \\ (X_{\min}, C) & & (X, p) \end{array}$$

where

- $(\tilde{X}, \tilde{C})$  is the minimal resolution of  $(X, p)$  where  $\tilde{C}$  is a smooth curve that contracts to  $p$  under the mapping  $f$  and that has self-intersection  $-2$ . To say that  $\tilde{X}$  is minimal means that there are no  $(-1)$ -curves in  $\tilde{X}$  that do not meet  $\tilde{C}$ ;
- $X_{\min}$  is the minimal model of  $\tilde{X}$  (all  $(-1)$ -curves in  $\tilde{X}$  have been contracted) and  $C = g(\tilde{C})$ .

We will use Hodge theory as a guide to show that

- (a)  $X_{\min}$  is a K3 surface with a degree 2 polarization. Thus  $X_{\min} \rightarrow \mathbb{P}^2$  is a 2:1 covering branched over a smooth sextic curve  $B \in |\mathcal{O}_{\mathbb{P}^2}(6)|$ , and  $C \subset X_{\min}$  is the inverse image of a tangent line to  $B$ . Thus  $C$  is irreducible with a single node; the arithmetic genus  $p_a(C) = 2$  and the normalization is  $\tilde{C} \rightarrow C$  where  $\tilde{C}$  is a smooth elliptic curve.
- (b) Any such configuration  $(X_{\min}, C)$  gives rise to a diagram  $(**)$ . From this we will infer that
  - $\dim \overline{\mathcal{M}}_2 = 20$ ;
  - a suitably interpreted version of local Torelli holds for the extended period mapping  $\Phi_e : \overline{\mathcal{M}}_2 \rightarrow \overline{\mathcal{P}}$ .
- (c) A semi-stable resolution of the KSBA degeneration may be obtained by smoothing a normal crossing surface  $\tilde{X} \cup_{\tilde{C}} Y$  where  $Y$  is a degree 2 del Pezzo surface. For a fixed  $X$  the number of parameters in such surfaces  $\tilde{X} \cup_{\tilde{C}} Y$  is 7.<sup>43</sup> These parameters correspond to the extension data in the limiting mixed Hodge structure associated to the KSBA degeneration. Inserting these amounts to blowing up  $\overline{\mathcal{M}}_2$  in  $\overline{\mathcal{M}}_I$  and leads to a desingularization of  $\overline{\mathcal{M}}_I$  along  $\overline{\mathcal{M}}_2$ .

We will also see that the arguments used in the above lead to a proof of generic local Torelli for the period mapping

$$\Phi : \mathcal{M}_I \rightarrow \mathcal{P} \subset \Gamma \backslash D.$$

As mentioned above, using other methods one may also prove that local Torelli holds everywhere for this period mapping; a result that was independently obtained by Carlson-Toledo and by Pearlstein-Zhang. It seems quite possible that the methods

<sup>43</sup>This is the “7” that appears in the  $\sum_{i=1}^k (9 - d_i)$  column in the above table.

of Friedman (cf. [Fri84]) may be used to prove generic global Torelli, meaning that the period mapping  $\Phi : \mathcal{M}_I \rightarrow \mathcal{P}$  has degree one.

**Conclusion:** *Using Hodge theory as a guide one may determine the structure of the boundary component  $\overline{\mathcal{M}}_2$  of  $\overline{\mathcal{M}}_I$  as well as a desingularization of  $\overline{\mathcal{M}}_I$  along  $\overline{\mathcal{M}}_2$ .*

Using the results of [FPR15a, FPR15b, FPR17], together with them we are working to extend the above conclusion to the other boundary components of  $\overline{\mathcal{M}}_I^{\text{Gor}}$  as listed in the above table. We will also discuss the non-Gorenstein components below.

*Step one:* The Hodge diamond for the LMHS associated to a semi-stable reduction  $\tilde{X} \rightarrow \Delta$  arising from a KSBA family with  $X$  as central fibre in

$$\begin{array}{ccccc} & 1 & & 1 & \\ & \bullet & & \bullet & \\ 1 & \downarrow & 27 & \downarrow & 1 \\ \bullet & & \bullet & & \bullet \\ & \bullet & & \bullet & \end{array}$$

where the dimensions are written in above the dots. This suggests (does not prove) that the central fibre  $\tilde{X}_0$  of  $\tilde{X} \rightarrow \Delta$  is a normal crossing surface of the form

$$\tilde{X}_0 = \tilde{X} \cup_{\tilde{C}} Y$$

where

- $\tilde{X}$  is a desingularization of  $X$ ;
- $\tilde{C}$  is a smooth double curve.

In this situation the LMHS is computed from the groups  $H^p(\tilde{C})$ ,  $H^q(\tilde{X})$  using Gysin and restriction mappings (cf. [PS08]). Assuming that  $H^1(\tilde{X}) = 0$  and  $H^1(Y) = 0$  this gives

$$\begin{aligned} & \bullet H^1 := \text{Gr}^1(\text{LMHS}) = H^1(\tilde{C}); \\ & \bullet H^2 := \text{Gr}^2(\text{LMHS}) \text{ is the middle cohomology of the complex} \\ (\natural) \quad & H^0(\tilde{C})(-1) \xrightarrow{\text{Gy}} H^2(\tilde{X}) \oplus H^2(Y) \xrightarrow{\text{Re}} H^2(\tilde{C}) \end{aligned}$$

where Gy is the direct sum of Gysin maps and Re is the signed restriction map.

**Note:** The condition [Fri83] that  $\tilde{X} \cup_{\tilde{C}} Y$  be smoothable is

$$(\natural\!\natural) \quad N_{\tilde{C}/\tilde{X}} \cong N_{\tilde{C}/Y}^*.$$

Denoting by  $\tilde{C}_{\tilde{X}}$  the curve  $\tilde{C}$  considered in  $\tilde{X}$  and similarly for  $\tilde{C}_Y$ ,  $(\natural\!\natural)$  gives

$$\tilde{C}_{\tilde{X}}^2 + \tilde{C}_Y^2 = 0,$$

which is exactly the condition that

$$\text{Re} \circ \text{Gy} = 0$$

in  $(\natural)$  above.

*Step two:*  $\tilde{C}$  is a smooth elliptic curve and therefore can be realized as a smooth cubic in  $\mathbb{P}^2$ . In order to have  $-\deg N_{\tilde{C}/\tilde{X}} = \deg N_{\tilde{C}/Y} = 2$  in  $(\natural\!\natural)$  to obtain  $Y$  we must blow up  $\mathbb{P}^2$  in 7 points  $p_i$  and take the proper transform of the cubic curve to obtain

$$\tilde{C} \subset Y = \text{Bl}_{\{p_i\}} \mathbb{P}^2. {}^{44}$$

There is still one parameter to adjust to have (‡‡) as an equality of line bundles.

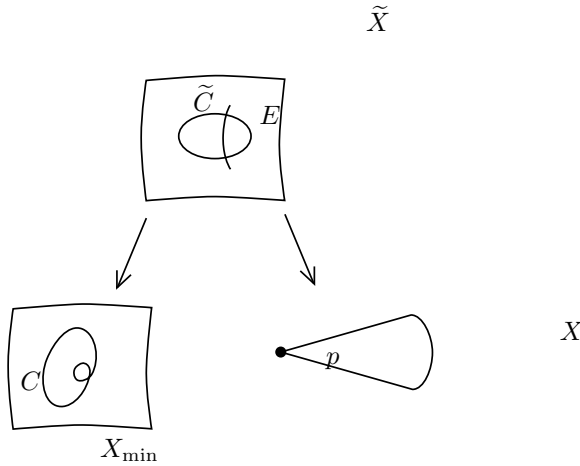
From dimensions in the Hodge diamond and  $\dim H^2(Y) = 8$  we obtain

$$\begin{cases} \dim H^{2,0}(\tilde{X}) = 1 \\ \dim H^{1,1}(\tilde{X}) = 21. \end{cases}$$

Now we use intersection numbers. From the assumption that  $(\tilde{X}, \tilde{C}) \rightarrow (X, p)$  is a minimal resolution of an  $I$ -surface and adjunction we obtain

$$\begin{cases} (K_{\tilde{X}} + \tilde{C})^2 = 1 \\ K_{\tilde{X}} \cdot \tilde{C} + \tilde{C}^2 = 0 \end{cases} \implies K_{\tilde{X}} \cdot \tilde{C} = 2.$$

From this we infer that  $K_{\tilde{X}}^2 = -1$ , and since  $h^{2,0}(\tilde{X}) = 1$  the line bundle  $K_{\tilde{X}}$  is  $[E]$  for  $E$  a  $(-1)$ -curve in  $\tilde{X}$ . This gives the diagram (\*\*) which may be pictured as



We then have

- $K_{X_{\min}} \cong \mathcal{O}_{X_{\min}}$  (since  $K_{\tilde{X}} = [E]$ );
- $h^{1,1}(X_{\min}) = 20$ ;
- $C^2 = 2 \implies p_a(C) = 2, g(\tilde{C}) = 1$ .

*Step three:* We now observe that this construction is reversible. Given a K3 surface  $X_{\min}$  with a degree 2 polarization and a curve  $C$  with  $p_a(C) = 2, g(\tilde{C}) = 1$  we have the situation described in (iii) above. We then construct  $\tilde{X}$  by blowing up the node on  $C$  to obtain  $\tilde{C}$  with  $\tilde{C}^2 = -2$ . Contracting it then gives  $(X, p)$ .

We next count parameters. First we have

$$\begin{cases} \dim X_{\min} \text{'s} = 19 \\ \dim C \text{'s} = 1 \end{cases}$$

<sup>44</sup>In [Fri84] type II degenerations are constructed by first gluing two  $\mathbb{P}^2$ 's together along a common cubic curve  $C$ . To obtain the condition (‡‡) one has to blow up 18 points on  $C$ . As shown in [Fri84] there are exactly four ways to group these points into two groups that specify in which  $\mathbb{P}^2$  the blowups occur. It may be that in the situation being discussed here there is a unique such choice.

so that the boundary component  $\overline{\mathcal{M}}_2$  has dimension 20. This gives the dimension count in (ii).

For the dimension count for the semi-stable surfaces  $\tilde{X} \cup_{\tilde{C}} Y$  with a fixed  $(X, p)$  we have

$$+ \begin{cases} \text{dimension of the spaces of } p_i \text{'s in } \mathbb{P}^2 = 14 \\ \text{dimension of cubics through the } p_i = 2 \end{cases} = \frac{2}{16}$$

and

$$- \begin{cases} \text{parameter to have } (\natural) & 1 \\ \text{dimension of } \text{Aut}(\mathbb{P}^2) & 8 \end{cases} = \frac{9}{9}$$

This gives

$$\begin{cases} \text{dimension of the surfaces } \tilde{X} \cup_{\tilde{C}} Y \\ \text{with a fixed } (X_{\min}, C) \end{cases} = 16 - 9 = 7 = 28 - 1.$$

Letting  $(X_{\min}, C)$  now vary one may show that the space of  $\tilde{X} \cup_{\tilde{C}} Y$ 's forms a smooth variety of dimension 27 biregularly equivalent to the blowup of  $\overline{\mathcal{M}}_I$  along the 20 dimensional  $\overline{\mathcal{M}}_2$ .

*Step four:* Finally how does Hodge theory enter via Torelli to relate to the above parameter counts? With the details to appear in a later work the rough idea is

- $H^1$  determines the elliptic curve  $\tilde{C}$ ;
- $H^2 = H^{2'} \oplus H^{2''}$  where  $H^{2'}$  is the Hodge structure of a K3 with a degree 2 polarization, and  $H^{2''}$  is a lattice of rank in  $\text{Hg}(H^2)$  of rank 7;
- the extension data in  $\text{Ext}_{\text{MHS}}^1(H^{2''}, H^1)$  gives the set of 7 points  $p_i$  on  $\tilde{C}$ .

There are a number of subtleties involving the quadratic forms on weight 2 PHS's which are a work in progress. We refer especially to [Fri84] where similar constructions are carried out in the case of K3's, and where a first instance of the use of extension data to provide parameters for boundary components in moduli of surfaces appeared. In summary, Hodge theory suggests where to look — the seven parameters arise from the possible extension data for  $\text{GR}(\text{LMHS})$  — and following [FPR15a, FPR15b, FPR17] in the references), and on discussions that the four of us have had with them related to a joint project that is in progress.

## References

- [Ale94] Valery Alexeev, *Boundedness and  $K^2$  for log surfaces*, Internat. J. Math. **5** (1994), no. 6, 779–810, DOI 10.1142/S0129167X94000395. MR1298994
- [CKS86] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid, *Degeneration of Hodge structures*, Ann. of Math. (2) **123** (1986), no. 3, 457–535, DOI 10.2307/1971333. MR840721
- [CMSP17] James Carlson, Stefan Müller-Stach, and Chris Peters, *Period mappings and period domains*, Cambridge Studies in Advanced Mathematics, vol. 168, Cambridge University Press, Cambridge, 2017. Second edition of [MR2012297]. MR3727160
- [Dem12] Jean-Pierre Demailly, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, vol. 1, International Press, Somerville, MA; Higher Education Press, Beijing, 2012. MR2978333
- [Fri83] Robert Friedman, *Global smoothings of varieties with normal crossings*, Ann. of Math. (2) **118** (1983), no. 1, 75–114, DOI 10.2307/2006955. MR707162
- [Fri84] Robert Friedman, *A new proof of the global Torelli theorem for K3 surfaces*, Ann. of Math. (2) **120** (1984), no. 2, 237–269, DOI 10.2307/2006942. MR763907

- [Gri69] Phillip A. Griffiths, *Hermitian differential geometry, Chern classes, and positive vector bundles*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 185–251. MR0258070
- [Ker15] C. Kerr, M. and Robles, *Partial orders and polarized relations on limited mixed Hodge structures*, 2015, in preparation.
- [KK10] János Kollár and Sándor J. Kovács, *Log canonical singularities are Du Bois*, J. Amer. Math. Soc. **23** (2010), no. 3, 791–813, DOI 10.1090/S0894-0347-10-00663-6. MR2629988
- [Kol13a] János Kollár, *Moduli of varieties of general type*, Handbook of moduli. Vol. II, Adv. Lect. Math. (ALM), vol. 25, Int. Press, Somerville, MA, 2013, pp. 131–157. MR3184176
- [Kol13b] János Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács. MR3057950
- [KSB88] J. Kollár and N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), no. 2, 299–338, DOI 10.1007/BF01389370. MR922803
- [KU09] Kazuya Kato and Sampei Usui, *Classifying spaces of degenerating polarized Hodge structures*, Annals of Mathematics Studies, vol. 169, Princeton University Press, Princeton, NJ, 2009. MR2465224
- [Laz16] Radu Laza, *The KSBA compactification for the moduli space of degree two  $K3$  pairs*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 2, 225–279, DOI 10.4171/JEMS/589. MR3459951
- [LR16] Wenfei Liu and Sönke Rollenske, *Geography of Gorenstein stable log surfaces*, Trans. Amer. Math. Soc. **368** (2016), no. 4, 2563–2588, DOI 10.1090/tran/6404. MR3449249
- [PS08] Chris A. M. Peters and Joseph H. M. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 52, Springer-Verlag, Berlin, 2008. MR2393625
- [RU19] Julie Rana and Giancarlo Urzúa, *Optimal bounds for  $T$ -singularities in stable surfaces*, Adv. Math. **345** (2019), 814–844, DOI 10.1016/j.aim.2019.01.029. MR3901675
- [Sch73] Wilfried Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319, DOI 10.1007/BF01389674. MR382272
- [FPR15a] M. Franciosi, R. Pardini, and S. Rollenske, *Computing invariants of semi-log-canonical surfaces*, Math. Z. **280** no. 3-4 (2015), 1107–1123.
- [FPR15b] M. Franciosi, R. Pardini and S. Rollenske, *Log-canonical pairs and Gorenstein stable surfaces with  $K_X^2 = 1$* , Compos. Math. **151** (2015), no. 8, 1529–1542.
- [FPR17] M. Franciosi, R. Pardini and S. Rollenske, *Gorenstein stable surfaces with  $K_X^2 = 1$  and  $p_g > 0$* , Math. Nachr. **290** (2017), no. 5-6, 794–814.

IMSA (INSTITUTE OF THE MATHEMATICAL SCIENCES OF THE AMERICAS) AT THE UNIVERSITY OF MIAMI, AND THE INSTITUTE FOR ADVANCED STUDY

Email address: pg@ias.edu