# Harmonic Integrals

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# Introduction

The first part of these lectures deals with the theory of E. Cartan's exterior differential forms and of Hodge's harmonic integrals. Chapter I contains the principles of the theory of exterior differential forms; I systematically introduce two kinds of differential forms and two kinds of chains, the "even kind" and the "odd kind", thanks to which the whole theory can be applied to non orientable manifolds as well as to orientable ones. Chapter II develops in a new way, with the help of the idea of distribution of Mr. Laurent Schwartz, the concept of current, that I have introduced in order to make a synthesis of the theory of differential forms and the topological theory of chains (references in §26). Chapter III is an exposition of the theory of harmonic differential forms, in which the currents are utilized to simplify some recent results of Mr. K. Kodaira and myself. Chapter IV is a complement to Chapter II and sketches a new method for studying the homology properties of differential forms and of currents.

I am most thankful to Professor J. W. Alexander for helping me to revise the English version of the text.

Georges de Rham

The last Chapter V is concerned with the application of harmonic forms to the theory of analytic functions on compact complex analytic manifolds. First, in §27, I show the existence of "analytic currents" with given singularities and then, in §28, I prove two theorems on the existence of many valued meromorphic functions with given divisors.

Kunihiko Kodaira

# Chapter I.

## Differential Forms on a Manifold.

### §1. Manifold $C^\infty$. Partition of Unity.

An n-dimensional manifold $C^\infty$, M, is a topological n-dimensional manifold with an infinitely differentiable structure, or, as we shall say briefly, a $C^\infty$ structure. The general concept of a $C^\infty$ structure can be defined axiomatically, with the help of one primitive concept, that of a function $C^\infty$ at a point, and the two following axioms.

Axiom 1. $f(x)$ being a real valued function defined in a neighborhood U of a point $x \in M$, f is either $C^\infty$ at x or it is not.

Axiom 2. To each point $x_o \in M$, there is a neighborhood U and n functions $x_1(x), \ldots, x_n(x)$ defined in U, such that

a) the mapping $x \to (x_1(x), \ldots, x_n(x))$ is a topological mapping of U on an open set of $E^n$. Consequently, each function f defined in U can be expressed with the help of $x_1, \ldots, x_n$, $f(x) = f(x_1, \ldots, x_n)$.

b) $f(x)$ is $C^\infty$ at a point of U if and only if $f(x_1, \ldots, x_n)$ is infinitely differentiable for the corresponding values of $x_1, \ldots, x_n$.

The functions $x_1, \ldots, x_n$ are called local coordinates in U. According to 2b, they are $C^\infty$ in U (i.e. at each point of U). Axiom 2 asserts the existence of local coordinates in a neighborhood of each point of M.

The $C^{\infty}$ structure will be determined as soon as we have a rule allowing us to recognize if a given function is $C^{\infty}$ or not. Practically, this is done by giving an open covering $\{U_i\}$ of the manifold and a system of local coordinates in each $U_i$. But, in the intersection $U_i \cap U_j$ the coordinates of one system must be infinitely differentiable functions of the coordinates of the other system, according to 2b.

A function is said to be $C^r$, for any given integer $r \geq 0$, if, when expressed with the help of local coordinates, it has continuous derivatives of orders up to and including r. A function $C^0$ is simply a continuous function.

We shall say that the <u>carrier</u> of a function f is the smallest closed set of points outside of which $f = 0$. Thus, it is the closure of the set of points x such that $f(x) \neq 0$. Its complement is the largest open set in which $f = 0$.

<u>Theorem.</u> Given an open covering $\{U_i\}$ of $M^n$, it is possible to find a set of functions $\varphi_j$ such that

1) $$1 = \sum_j \varphi_j$$

2) $\varphi_j$ is $C^{\infty}$ and $0 \leq \varphi_j \leq 1$ everywhere, its carrier is compact and contained in one of the open sets $U_i$.

3) Every point of M has a neighborhood which is met by only a finite number of the carriers of the $\varphi_j$.

From the theorem of Borel-Lebesgue, it follows that the condition 3) is equivalent to the following

3') Only a finite number of carriers of the $\varphi_j$ can meet a given compact set.

Formula 1) is called a partition of unity, which is said to be locally finite if it satisfies 3). If the manifold is compact, the set of the $\varphi_i$ is finite. If not, it is enumerable.

Since this theorem deals only with well known facts, its proof is not given here.

## §2. Differential Forms.

We get the differential forms of the first degree on M as sums of products

$$\sum g\, d f ,$$

where f, g, ... are functions on M.

With the help of local coordinates $x^1, \ldots, x^n$, any such a form can be reduced to the expression

$$\sum_{i=1}^{n} a_i dx^i$$

and when we change the coordinates, the coefficients $a_i$ change like the components of a covariant-vector. The form is said to be equal to zero at a point, if all coefficients vanish at this point. It is said $C^r$, if its coefficients are $C^r$.

From the forms of degree 1, we get the exterior differential forms of higher degree, with the help of a new kind of multiplication, exterior multiplication, which is represented by the symbol $\wedge$. This last obeys to the following rules

1) Associativity and distributivity

2) Pseudo commutativity

$$dx^i \wedge dx^j = -dx^j \wedge dx^i , \qquad dx^i \wedge dx^i = 0$$

$$a \wedge dx^i = dx^i \wedge a = a\, dx^i$$

$$dx^i \wedge a\, dx^j = a\, dx^i \wedge dx^j \qquad \text{(a being a scalar).}$$

With the help of these rules, every form of degree p can be reduced to the canonical expression

$$\alpha = \sum_{i_1 < i_2 < \ldots < i_p} a_{i_1 \ldots i_p}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

The form $\alpha$ is said to be <u>equal to zero</u> at a point, if all coefficients $a_{i_1 \ldots i_p}$ vanish at this point.

If $\alpha$ and $\beta$ are two forms of degrees p and q respectively, we have

$$\alpha \wedge \beta = (-1)^{pq}\, \beta \wedge \alpha ,$$

and it is clear that Rules 1) and 2) are independent of the coordinate system.

Now, the <u>differential</u> of the form $\alpha$ is the form $d\alpha$ defined by

$$d\alpha = \sum_{i_1 < \ldots < i_p} da_{i_1 \ldots i_p} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

From this definition follow the rules

$$d(\alpha_1 + \alpha_2) = d\alpha_1 + d\alpha_2$$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \quad (p = \text{degree of } \alpha)$$

$$d^2\alpha = 0 \quad \text{for every form } \alpha.$$

From these rules it follows that the definition of $d\alpha$ is invariant with respect to changes of coordinates.

Let us now consider these forms from the point of view of the tensor calculus. The coefficients $a_{i_1 \ldots i_p}$ of $\alpha$ being defined for $i_1 < \ldots < i_p$, we can define them for all values of the indices by the condition of <u>skew-symmetry</u>. Then, taking the sum over <u>all</u> values of the indices, we can write

$$\alpha = \frac{1}{p!} \sum_{i_1 \ldots i_p} a_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

If we change the coordinates, we get $dx^i = \sum_j \frac{\partial x_i}{\partial \bar{x}_j} d\bar{x}_j$ and, according to the preceding rules,

$$\alpha = \frac{1}{p!} \sum_{j_1 \ldots j_p} \bar{a}_{j_1 \ldots j_p} d\bar{x}^{j_1} \wedge \ldots \wedge d\bar{x}^{j_p}$$

with

$$\bar{a}_{j_1 \ldots j_p} = \sum_{i_1 \ldots i_p} a_{i_1 \ldots i_p} \frac{\partial x_{i_1}}{\partial \bar{x}_{j_1}} \cdots \frac{\partial x_{i_p}}{\partial \bar{x}_{j_p}}$$

We see, the coefficients change according to the same law as the components of a <u>covariant tensor of rank p.</u>

Following E. Cartan, a skew symmetric tensor of rank p will be called a <u>p-vector.</u> An exterior differential form of degree p will be called a p-form. To give a p-form is the same as to give, at each point, a covariant p-vector. Hence, the concepts of a p-form and of a covariant p-vector field or covariant p-vector function are exactly equivalent.

In order to write the canonical expressions of $\alpha \wedge \beta$ and $d\alpha$, the Kronecker symbol $\delta^{j_1 \ldots j_p}_{i_1 \ldots i_p}$ will be useful. For p = 1, it is defined by $\delta^i_i = 1$, $\delta^j_i = 0$ $(i \neq j)$. For $p > 1$, $\delta^{j_1 \ldots j_p}_{i_1 \ldots i_p}$ is the determinant $\| \delta^{j_\ell}_{i_k} \|$ of order p ($k, \ell = 1 \ldots p$).

The following two properties are equivalent to this definition

1) $\delta^{j_1 \ldots j_p}_{i_1 \ldots i_p}$ is skew symmetric in the $i_k$ and in the $j_k$.

2) For $i_1 < \ldots < i_p$ and $j_1 < \ldots < j_p$,

$$\delta^{j_1 j_2 \ldots j_p}_{i_1 i_2 \ldots i_p} = \delta^{j_1}_{i_1} \delta^{j_2}_{i_2} \ldots \delta^{j_p}_{i_p} = \begin{cases} 1 & \text{if } i_k = j_k \ (k=1 \ldots p) \\ 0 & \text{if } i_k \neq j_k \text{ for at least one } k. \end{cases}$$

Hence, the $\delta^{j_1 \ldots j_p}_{i_1 \ldots i_p}$ can be considered as the elements of a unit matrix with $\binom{n}{p}$ rows and $\binom{n}{p}$ columns.

Condition 2) is equivalent to

(2') For every system of $\binom{n}{p}$ numbers $a_{i_1 \ldots i_p}$ $(i_1 < \ldots < i_p)$,

$$a_{i_1 \ldots i_p} = \sum_{j_1 < \ldots < j_p} \delta^{j_i \ldots j_p}_{i_1 \ldots i_p} a_{j_1 \ldots j_p}$$

On account of the skew symmetry, this is also equivalent to the following condition

2") For every covariant p-vector $a_{i_1 \dots i_p}$,

$$a_{i_1 \dots i_p} = \frac{1}{p!} \sum_{j_1 \dots j_p} \delta^{j_1 \dots j_p}_{i_1 \dots i_p} \, a_{j_1 \dots j_p}$$

This identity shows the tensorial character of the Kronecker symbol. For each p ($0 < p \leq n$), the $\delta^{j_1 \dots j_p}_{i_1 \dots i_p}$ are the components of a tensor which has the same components in any coordinate system.

Now, if the coefficients of $\alpha$ are $a_{i_1 \dots i_p}$ and those of $\beta$ are $b_{j_1 \dots j_q}$, the coefficients $c_{k_1 \dots k_{p+q}}$ of $\alpha \wedge \beta$ are

$$c_{k_1 \dots k_{p+q}} = \sum_{\substack{i_1 < \dots < i_p \\ j_1 < \dots < j_q}} \delta^{i_1 \dots i_p j_1 \dots j_q}_{k_1 \dots k_{p+q}} \, a_{i_1 \dots i_p} b_{j_1 \dots j_q}$$

and the coefficients $a'_{k_1 \dots k_{p+1}}$ of $d\alpha$ are

$$a'_{k_1 \dots k_{p+1}} = \sum_{\substack{i_1 < \dots < i_p \\ j}} \delta^{j\, i_1 \dots i_p}_{k_1 \dots k_{p+1}} \frac{\partial a_{i_1 \dots i_p}}{\partial x^j}$$

as can be immediately verified.

## §3. Forms and Tensors of Odd Kind.

To each type of tensor in the usual sense, or, as we shall say, of <u>even kind</u>, we can associate another type of tensor, which will be said to be of <u>odd kind</u>.

Let us begin with the scalars. A <u>scalar of odd kind</u> is determined, at each point, by one component f, which depends on the coordinate system according to the following law; if $\bar{f}$ is its value in another coordinate system $\bar{x}^1,\dots,\bar{x}^n$, $\bar{f} = f$ if the Jacobian $J = \dfrac{D(x^1\dots x^n)}{D(\bar{x}^1,\dots\bar{x}^n)}$ is positive, $\bar{f} = -f$ if $J$ is negative. We can write

$$\bar{f} = \frac{J}{|J|}\, f$$

A <u>covariant vector of odd kind</u> is determined by components $a_1\dots a_n$ which change according to the rule

$$\bar{a}_i = \frac{J}{|J|} \sum_j \frac{\partial x^j}{\partial \bar{x}^i}\, a_j .$$

In the same way, by introducing the factor $\frac{J}{|J|}$ in the transformation equations, to each type of tensor of even kind is associated a type of tensor of odd kind.

We can multiply these tensors by a number and add two tensors of odd kind and of the same type. We can also take the product of two such tensors, but we notice that <u>the product of two tensors of the same kind is a tensor of even kind</u>, and <u>the product of two tensors of different kinds is a tensor of odd kind.</u>

We are particularly interested in the covariant p-vectors, or, what is the same, the differential forms. To each covariant p-vector function $a_{i_1\dots i_p}$ of odd kind, we also associate a differential form

$$\sum_{i_1<\dots<i_p} a_{i_1\dots i_p}\, dx^{i_1}\wedge\dots\wedge dx^{i_p}$$

which will be called a <u>differential form of odd kind.</u>

We can multiply and differentiate these forms in exactly the same way as the forms of <u>even kind</u>, as we shall call the forms introduced in §2.

We notice that the exterior product of two forms of the same kind is a form of even kind. The exterior product of two forms of different kinds is of odd kind. The differential $d\alpha$ of a form $\alpha$ is of the same kind as $\alpha$.

Orientability can be defined as follows. The manifold M is said to be orientable, if there exists a continuous scalar of odd kind on M, $\varepsilon$, such that $\varepsilon^2 = 1$.

The square of a scalar of odd kind is a scalar of even kind, i.e., an ordinary scalar. Suppose $\varepsilon$ and $\varepsilon_1$ are two continuous scalars of odd kind, such that $\varepsilon^2 = \varepsilon_1^2 = 1$. Then, $\varepsilon\,\varepsilon_1$ is a continuous scalar of even kind, i.e., an ordinary scalar, whose value at each point is $\pm 1$. If the manifold is connected, either $\varepsilon\,\varepsilon_1 = +1$ everywhere, or $\varepsilon\,\varepsilon_1 = -1$ everywhere, consequently $\varepsilon_1 = \varepsilon$ or $\varepsilon_1 = -\varepsilon$. Hence, on an orientable and connected manifold M, there are exactly two continuous scalars of odd kind with square equal to 1. Each of these scalars $\varepsilon$ and $-\varepsilon$ can be called an orientation of M.

This definition is equivalent to the usual one. Indeed, let us denote as positive the coordinate systems with respect to which $\varepsilon = +1$, and as negative the others. The Jacobian relative to two coordinate systems is clearly positive or negative, according as the systems are of the same sign or not. Conversely, if we have such a repartition of the coordinate systems in two classes, we can define $\varepsilon$ by the condition that $\varepsilon = 1$ with respect to any coordinate system of the positive class.

Now, if the manifold M is orientable and if we choose an orientation $\varepsilon$, we can associate, with each form of odd kind $\alpha$, a form of even kind $\varepsilon\alpha$.

Hence, in the case of orientable manifolds, it is possible to avoid the tensors of odd kind, by choosing an orientation. But for the non-orientable manifolds, the concept of "odd kind" is useful and natural.

## §4. Integral of an n-Form of Odd Kind.

Let $\alpha = a_{12\ldots n} dx^1 \wedge \ldots \wedge dx^n$ a n-form of odd kind. If we change the coordinates, it becomes

$$\alpha = \bar{a}_{12\ldots n} d\bar{x}^1 \wedge \ldots \wedge d\bar{x}^n$$

where

$$\bar{a}_{12\ldots n} = \frac{J}{|J|} \sum_{i_1\ldots i_n} a_{i_1\ldots i_n} \frac{\partial x^{i_1}}{\partial \bar{x}^1} \cdots \frac{\partial x^{i_n}}{\partial \bar{x}^n}$$

or, as

$$a_{i_1\ldots i_n} = \delta^{1\ldots n}_{i_1\ldots i_n} a_{1\ldots n}$$

and

$$\sum_{i_1\ldots i_n} \delta^{1\ldots n}_{i_1\ldots i_n} \frac{\partial x^{i_1}}{\partial \bar{x}^1} \cdots \frac{\partial x^{i_n}}{\partial \bar{x}^n} = J ,$$

$$\bar{a}_{12\ldots n} = |J| \, a_{12\ldots n}$$

We see that the coefficient $a_{12\ldots n}$ of an n-form of odd kind changes according to the same rule as a scalar density.

Now, suppose the carrier of $\alpha$ is contained in the domain U of a coordinate system $x^1 \ldots x^n$. (The carrier of any form is defined as the smallest closed set, outside of which the form is equal to zero.) Then, $a_{12\ldots n}$ is a function of $x^1 \ldots x^n$, which we can extend to the whole Euclidean space $E^n$

by putting $a_{12\ldots n} = 0$ outside the domain of $E^n$ corresponding to U. Moreover we define <u>the integral of</u> $\alpha$ <u>by</u>

$$\int \alpha = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} a_{12\ldots n} dx^1 dx^2 \ldots dx^n.$$

Clearly, this definition is independent of the choice of the coordinate system in U.

For an arbitrary form $\alpha$ of degree n and of odd kind, we use a locally finite partition of unity,

$$1 = \sum_i \varphi_i ,$$

with $0 \leq \varphi_i \leq 1$ and such that for each i the carrier of $\varphi_i$ is compact and is contained in the domain of some coordinate system. Then $\int \varphi_i \alpha$ is defined. We say $\int \alpha$ is <u>convergent</u> and we set

$$\int \alpha = \sum_i \int \varphi_i \alpha$$

if the series is convergent <u>for each partition of unity satisfying to the above conditions</u>.

If this condition of convergence is satisfied, the series is absolutely convergent, because it must remain convergent under any permutation of its terms. It is easy to see that the values corresponding to two partitions $1 = \sum_i \varphi_i$ and $1 = \sum_j \psi_j$ are equal, by comparing them with the value corresponding to the third partition $1 = \sum_{i,j} \varphi_i \psi_j$. If the carrier of $\alpha$ is compact, the number of $\varphi_i$'s such that $\varphi_i \alpha$ is not identical to zero is finite and the integral is always convergent.

<u>Theorem.</u> If $\beta$ is an (n-1)-form $C^1$ of odd kind with a <u>compact</u> carrier, $\int d\beta = 0$.

To prove this, we can suppose the carrier of $\beta$ is contained in the domain of some coordinate system $x^1 \ldots x^n$. Otherwise, using a partition of unity, we would replace $\beta$ by $\sum \varphi_i \beta$. Now, if

$$\beta = b_{23\ldots n} dx^2 \wedge dx^3 \wedge \ldots \wedge dx^n + \ldots$$

$$d\beta = \frac{\partial b_{2\ldots n}}{\partial x^1} dx^1 \wedge \ldots \wedge dx^n + \ldots$$

$$\int d\beta = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \frac{\partial b_{2\ldots n}}{\partial x^1} dx^1 \ldots dx^n + \ldots$$

and as the carrier of $b_{2\ldots n}$ is compact, $\int_{-\infty}^{+\infty} \frac{\partial b_{2\ldots n}}{\partial x^1} dx^1 = 0$, the first term vanishes and also the others in the same way.

<u>Remark about densities.</u> As we have seen, a scalar density is the same thing as a covariant n-vector of odd kind. Now, consider a tensor of odd kind $a^{i_1 \ldots i_p}_{j_1 \ldots j_n}$ which is skewsymmetric in p contravariant indices and skew symmetric in n covariant indices. On account of its skew symmetry, it is completely determined by the components $a^{i_1 \ldots i_p}_{1 \ldots n} = cr^{i_1 \ldots i_p}$, which are the components of a contravariant skew symmetric density of rank p. But, to each such a density, we can associate a covariant (n-p)-vector of odd kind $b_{j_1 \ldots j_{n-p}}$, and conversely, by the

following two equivalent formulae (in which the indices $j_1 \dots j_{n-p}$ are assumed different from $k_1 \dots k_p$)

$$b_{j_1 \dots j_{n-p}} = a_{1 \dots n}^{k_1 \dots k_p} \delta_{k_1 \dots k_p j_1 \dots j_{n-p}}^{1 \dots n} = \sum_{i_1 < \dots < i_p} a_{1 \dots n}^{i_1 \dots i_p} \delta_{i_1 \dots i_p j_1 \dots j_{n-p}}^{1 \dots n}$$

$$a_{1 \dots n}^{k_1 \dots k_p} = b_{j_1 \dots j_{n-p}} \delta_{1 \dots n}^{k_1 \dots k_p j_1 \dots j_{n-p}} = \sum_{\ell_1 < \dots < \ell_p} b_{\ell_1 \dots \ell_{n-p}} \delta_{1 \dots n}^{k_1 \dots k_p \ell_1 \dots \ell_{n-p}}$$

Thus, a covariant (n-p)-vector of odd kind is exactly equivalent to a contravariant skew symmetric density of rank p. Hence, it will not be necessary to use the densities explicitly; the forms of the two kinds will be sufficient.

## §5. Mappings and Chains.

Let M and $M_1$ be two manifolds $C^\infty$, of dimensions n and m respectively, and $\mu$ a mapping $C^2$ of $M_1$ in M. Each point $y \in M_1$ is mapped on a point $x = \mu y \in M$.

Then, to each p-form $\alpha$ in M, of even kind, corresponds a p-form $\mu^* \alpha$ in $M_1$, of even kind. The operation $\mu^*$ has the following properties, which are characteristic.

1) For p = 0, i.e. for a function f, $\mu^* f(y) = f(\mu y)$

2) $\mu^* d\alpha = d\mu^*\alpha$ , $\mu^*(\alpha \wedge \beta) = \mu^*\alpha \wedge \mu^*\beta$.

So, in order to get the expression of $\mu^*\alpha$ in a neighborhood of $y_o \in M_1$ with the help of local coordinates $y^1 \ldots y^m$, we have only to take the expression of $\alpha$ with the help of local coordinates $x^1 \ldots x^n$ in a neighborhood of $x_o = \mu y_o$ and to replace $x^1 \ldots x^n$ by the functions of $y^1 \ldots y^n$ which determine the mapping $x = \mu\, y$.

In order to do the same with forms of odd kind, we need the concept of the orientation of a mapping.

An orientation of the mapping $\mu$ is a law associating in a continuous way to an orientation $\varepsilon_1$ of the neighborhood of $y \in M_1$ an orientation $\varepsilon$ of the neighborhood of $x = \mu\, y \in M$.

If this is impossible, we say $\mu$ is non orientable. If this is possible and if $M_1$ is connected, this is possible in exactly two manners, one associating $\varepsilon$ and $\varepsilon_1$ (or, what is the same, $-\varepsilon$ and $-\varepsilon_1$), the other associating $\varepsilon$ and $-\varepsilon_1$ (or $-\varepsilon$ and $\varepsilon_1$). This is always possible if $M_1$ is simply connected (monodromie theorem).

Let us consider the case where $M_1$ is regularly imbedded in M and $\mu$ is the identical mapping. If the dimensions m of $M_1$ and n of M are equal, $M_1$ is a domain in M, and there is an orientation in which $\varepsilon = \varepsilon_1$; this last will be called the absolutely positive or natural orientation.

Suppose now $m = n-1$, let $e_1$ a vector at y in M normal to $M_1$, and $e_2, \ldots, e_n$ m linearly independent vectors tangent to $M_1$ at y. The vectors determine an orientation $\varepsilon_1$ of $M_1$ in the neighborhood of y, while $e_1, e_2, \ldots, e_n$ determine an orientation of M in the neighborhood of y. We see that the orientation of the identical mapping in which $\varepsilon$ corresponds to $\varepsilon_1$ is completely determined by the choice of the positive direction of $e_1$ along the normal to $M_1$.

Now, let us go on to the definition of $\mu^*\alpha$ in case $\alpha$ is of odd kind and $\mu$ is oriented. We take an orientation $\varepsilon_1$ of a neighborhood of a point y and the corresponding orientation $\varepsilon$ of the neighborhood of $x = \mu y$, and we set

$$\mu^*\alpha = \varepsilon_1\mu^*(\varepsilon\alpha).$$

Thus, $\mu^*\alpha$ is defined in the neighborhood of each point y, i.e. on the whole $M_1$. It is therefore a form of odd kind, and the properties 2) are still valid.

Differentiable chains.

A p-dimensional simplex of odd kind in M, $s^p$, is defined by a mapping $\pi$ of a rectilinear p-dimensional simplex $S^p$, contained in a Euclidean space $E^p$, and an orientation of $S^p$,

$$s^p = (S^p, \pi, \text{ orientation of } S^p).$$

If instead of an orientation of $S^p$, we consider an orientation of $\pi$, we get the definition of a simplex of even kind in M

$$s^p = (S^p, \pi, \text{ orientation of } \pi).$$

The simplex is said to be $C^r$, if the mapping $\pi$ is $C^r$, more precisely if it can be extended to a mapping in M of a domain D of $E^p$ containing $S^p$ so as to be $C^r$ in D. We shall always suppose $r \geq 1$, the simplex is then differentiable.

A <u>chain in M</u> is a linear combination of simplexes

$$c^p = \sum_i k_i \; s_i^p$$

with real coefficients $k_i$. We shall always suppose all the simplexes of the same dimension and the same kind, which will be the dimension and the kind of the chain. Further we agree that if $s_1^p$ and $s_2^p$ are defined by the same mapping $\pi$ but with opposite orientations (of $\pi$ or of $S^p$), $s_1^p = -s_2^p$. The expression for $c^p$ is <u>reduced</u>, if all coefficients $k_i$ are $\neq 0$ and if all mappings $\pi$ of the $s_i^p$ are different. Two chains are <u>identical</u>, if they have the same reduced expression.

If the number of terms occurring in the reduced expression for $c^p$ is finite, $c^p$ is <u>finite</u>. It will be useful to consider also <u>infinite chains</u>, but we shall always suppose they are <u>locally finite</u>, i.e. only a finite number of $s_i^p$ can meet a same compact set of M. If M is compact, there are only finite chains.

It must be noticed that the usual kind of chains are the chains of odd kind, while the usual kind of forms are the forms of even kind.

<u>Transformation of a chain by a mapping</u>. Let $\mu$ be a mapping of $M_1$ in M, $s^p = (S^p, \pi, \varepsilon)$ a simplex of odd kind in $M_1$, where $\varepsilon$ is an orientation of $S^p$. Then $\mu s^p = (S^p, \mu\pi, \varepsilon)$ is a simplex in M: the image of $s^p$ by $\mu$. If the mapping $\mu$ is oriented, we can define in the same way the image of a simplex of even kind, because if $\pi$ is oriented, the product $\mu\pi$ is then oriented in a natural way.

Now, for any chain $c^p = \sum_i k_i s_i^p$, we define the transformed chain $\mu c^p = \sum_i k_i \mu \; s_i^p$. If $c^p$ is finite, $\mu c^p$ is also finite. But if $c^p$ is infinite and locally finite, $\mu c^p$ is not necessarily locally finite. $\mu c^p$ will be locally finite if $\mu$ satisfies the following condition: for each compact set $K \subset M$, $\mu^{-1}(K)$ is compact.

Integral of a p-form on a p-dimensional chain. Let $\alpha$ be a p-form of even kind and $s^p = (S^p, \pi, \varepsilon)$ a simplex of odd kind. Then we set

$$\int_{s^p} \alpha = \int_{S^p} \varepsilon \pi^* \alpha = \int f \varepsilon \pi^* \alpha$$

In the last integral, f is the characteristic function of $S^p$ in $E^p$ ($f = 1$ in $S^p$, $f = 0$ outside $S^p$) and $\varepsilon$ an orientation of $E^p$ (scalar of odd kind $= \pm 1$), thus $f \varepsilon \pi^* \alpha$ is a p-form of odd kind with a compact carrier in $E^p$ and the integral is determined by the definition of §4.

If $\alpha$ is a p-form of odd kind and $s^p$ a simplex of even kind, $\pi$ being oriented, $\pi^* \alpha$ is a p-form of odd kind in $E^p$ and we set

$$\int_{s^p} \alpha = \int f \, \pi^* \alpha \, .$$

Now, for every finite chain $c^p = \sum_i k_i s_i^p$ and every p-form $\alpha$ of different kind than $c^p$, we define the integral of $\alpha$ extended to $c^p$ by

$$\int_{c^p} \alpha = \sum_i k_i \int_{s_i^p} \alpha$$

If the chain is infinite (but locally finite) and the carrier of $\alpha$ is compact, we define the integral by the same formula; only a finite number of terms of the sum can be different from zero.

If $c^p$ is infinite and the carrier of $\alpha$ is not compact, we say $\int_{c^p} \alpha$ is convergent and we set

$$\int_{c^p} \alpha = \sum_i \int_{c^p} \varphi_i \alpha$$

if the series is convergent for each partition of unity, $1 = \sum_i \varphi_i$, satisfying to the same conditions as in §4.

In the singular homology theory, one used to say that a chain is equal to zero if its reduced expression vanishes. Here we shall use another concept. A chain c will be said to be negligible, if $\int_c \alpha = 0$ for each form $\alpha$. It is sufficient that this hold for each $\alpha$ $C^\infty$ with a compact carrier. The definition of the sum and the difference of two chains are obvious. We shall say further that two chains are identical, if they have the same reduced expression, and that they are equal, if their difference is negligible.

§6. Boundary. Stokes' Formula.

From the definition of $\mu^*\alpha$ and $\mu c$, we obtain

(1) $$\int_c \mu^*\alpha = \int_{\mu c} \alpha$$

Boundary of a chain. Let $S^p$ a rectilinear simplex, $S_i^{p-1}$ $(i=0,1,\ldots,p)$ its (p-1)-dimensional sides. If we agree that the positive direction along the normal to $S_i^{p-1}$ is the direction towards the outside of $S^p$, we determine, as noticed, a relation between the orientations of $S^p$ and $S_i^{p-1}$. Then, if $s^p$ is a simplex in M determined by the mapping $\pi$ of $S^p$, the restriction of $\pi$ on $S_i^{p-1}$, together with that relation between the orientations of $S^p$ and $S_i^{p-1}$, defines a well determined simplex $s_i^{p-1}$ in M, of the same class as $s^p$. The chain $B\, s^p = \sum_i s_i^{p-1}$ is the boundary of $s^p$. The boundary of any chain $c^p = \sum_j k_j s_j^p$ is then defined by $B\, c^p = \sum_j k_j B\, s_j^p$. This is a chain of dimension p-1 and of the same kind as $c^p$. As is well known and can be immediately verified, the boundary of a boundary is identical to zero. Further, for any mapping $\mu$, $\mu\, B\, c^p = B\, \mu\, c^p$.

Stoke's Formula. If $\alpha$ is a (p-1)-form $C^1$ and c a finite p-dimensional chain $C^2$, we have

$$\int_c d\alpha = \int_{B\,c} \alpha$$

It is sufficient to prove this in the case c is a simplex $s^p = (S^p, \pi, \varepsilon)$. Then, as $s^p = \pi\, S^p$, $B\, s^p = \pi\, B\, S^p$, according to (1), we have only to prove

$$\int_{S^p} d\omega = \int_{B\, S^p} \omega$$

where $\omega = \pi^* \alpha$ is a form $C^1$ in the Euclidean space $E^p$ containing $S^p$. Let us suppose, to make matters definite, that the form is of even kind. Then $S^p$ is of odd kind, i.e. $S^p$ is taken with a given orientation.

The summits of $S^p$ being $P_0, P_1, \ldots, P_p$, we consider the coordinate system $t_1 \ldots t_p$ defined in $E^p$ by

$$P = P_0 + \sum_{i=1}^{p} t_i (P_i - P_0)$$

or

$$P = \sum_{i=0}^{p} t_i P_i \quad \text{with} \quad t_0 = 1 - \sum_{i=1}^{p} t_i .$$

We can suppose this coordinate system is positive with respect to the given orientation of $S^p$.

Then, the form $\omega$ is a sum of p terms, and it is sufficient to prove the formula for one of them, let us say the first. We can thus assume

$$\omega = a(t_1, t_2, \dots, t_p) dt_2 \wedge \dots \wedge dt_p \ .$$

Let $S_i^{p-1}$ be the side of $S^p$ opposite to $P_i$. We have $t_i = 0$ on $S_i^{p-1}$, hence the integral of $\omega$ on $S_i^{p-1}$ is zero except for i = 0 or 1. According to the definition of the boundary, $t_2 \dots t_p$ is a negative coordinate system in $S_1^p$ and a positive one in $S_o^p$. Consequently

$$\int_{BS^p} \omega = \int_{S_o^{p-1}+S_1^{p-1}} \omega = \int \dots \int f \cdot [a(\tau, t_2, \dots, t_p) - a(0, t_2, \dots, t_p)] dt_2 dt_3 \dots dt_p$$

where $\tau = 1 - \sum_{i=2}^{p} t_i$ is the value of $t_1$ on $S_o^p$ and f is the characteristic function of $S_1^{p-1}$ (or, what is the same, of $S_o^p$) expressed with the help of $t_2 \dots t_p$.

On the other hand, as $d\omega = \dfrac{\partial a(t_1, \dots, t_p)}{\partial t_1} dt_1 \wedge dt_2 \wedge \dots \wedge dt_p$, we have

$$\int_{S^p} d\omega = \int \dots \int g \cdot \frac{\partial a(t_1 \dots t_p)}{\partial t_1} dt_1 dt_2 \dots dt_p$$

where g is the characteristic function of $S^p$. But $g = f\ h(t_1)$ where $h(t_1)$ is the characteristic function of the interval $0 \leq t_1 \leq \tau$. Integrating with respect to $t_1$, we get for $\int_{S^p} d\omega$ precisely the above expression of $\int_{BS^p} \omega$.

In the above general statement, the chain c is assumed to be $C^2$ in order that the mappings $\pi$ defining its simplexes are $C^2$ and $\omega = \pi^* \alpha$ is $C^1$. But as every simplex $C^1$ can be represented by the limit of a series of simplexes $C^2$, the formula still holds for chains $C^1$.

It is essential that the chain c be finite. If the chain c is infinite and the carrier of $\alpha$ is compact, the chain c can be replaced by a finite one and the formula holds. But if c is infinite and the carrier of $\alpha$ is not compact, the formula is not valid.

It is convenient to consider also the case p = 1. A form of degree 0 and of even kind is a function $f(x)$, a chain of dimension 0 and of odd kind is a linear combination of points, $c^o = \sum_i k_i s_i^o$, and the "integral" is then defined by

$$\int_{c^o} f = \sum_i k_i f(s_i^o)$$

Now, for the case p = 1, the formula means only that, if $B\, s^1 = s_1^o - s_o^o$,

$$\int_{s^1} df = f(s_1^o) - f(s_o^o) .$$

# Chapter II.

## Currents and Distributions.

### §7. Definition and Examples.

There is a deep analogy, in an n-dimensional manifold, between the p-dimensional chains and the forms of degree n-p. It suggests that they be considered as particular cases of a more general concept, which will be called current. From this point of view, the operation d (differential) and B (boundary) will be particular cases of the same operation, and the exterior product of differential forms will correspond to the topological intersection of chains. The exact definition given here was first found by Laurent Schwartz, with his theory of distributions.

Definition.

In an n-dimensional $C^\infty$ manifold M, a current T of odd kind and dimension p is a functional $T[\varphi]$, defined on the linear space of all $C^\infty$ forms $\varphi$ of degree p and even kind with a compact carrier, which is linear, i.e. such that

$$T[k_1\varphi_1 + k_2\varphi_2] = k_1T[\varphi_1] + k_2T[\varphi_2]$$

for any constants $k_1, k_2$ and any forms $\varphi_1$, $\varphi_2$ of our linear space, and which is continuous in the following way:

if $\varphi_1, \varphi_2, \dots, \varphi_h, \dots$ is a series of forms of our linear space, such that their carriers are all contained in the same compact set K, itself contained in the domain of some coordinate system $x^1, \dots, x^n$, and such that each derivative of each coefficient of $\varphi_h$ (expressed with the help of $x^1 \dots x^n$

tends uniformly to zero for $h \longrightarrow \infty$, then $T[\varphi_h] \longrightarrow 0$.

By replacing $\varphi$ by a form of odd kind, we get the definition of a current of even kind.

In the above definition, among the derivatives of the coefficients of $\varphi_h$ are included the derivatives of order zero, i.e. the coefficients themselves.

The number n-p will be called the degree of the current T. Thus the sum of the degree and the dimension of a current is always equal to the dimension of the whole manifold M.

1st example. Let $\alpha$ be a differential form of degree n-p, of odd kind, whose coefficients are locally integrable functions. Then, $\varphi$ being $C^\infty$ of degree p and of even kind with a compact carrier, $\alpha \wedge \varphi$ is of degree n and of odd kind, and its integral, as defined in §4, gives a current

$$\alpha[\varphi] = \int \alpha \wedge \varphi$$

We shall say that this current is equal to the form $\alpha$.

In the same way, a form of even kind defines a current of even kind.

2nd example. A chain c defines a current, of the same dimension and of the same kind as c,

$$c[\varphi] = \int_c \varphi$$

We shall say that this current is equal to the chain c.

The chain can either be finite or infinite, but locally finite.

3rd example. Let y be a point of M, and v a contravariant p-vector at y, whose components $v^{i_1 \ldots i_p}$ with respect to a coordinate system $x^1, \ldots, x^n$ are numerically determined. Then, if $\varphi = \sum_{i_1 < \ldots < i_p} \varphi_{i_1 \ldots i_p}(x) dx^{i_1} \wedge \ldots \wedge dx^{i_p}$

$$v[\varphi] = \sum_{i_1 < \dots < i_p} \varphi_{i_1 \dots i_p}(y)\, v^{i_1 \dots i_p}$$

is independent of the coordinate system and defines a current.

Let D be an open set in M. We shall say that the current T is equal to zero in D, if $T[\varphi] = 0$ for each form $\varphi$ with a compact carrier contained in D.

Theorem. If T = 0 in a neighborhood of each point of D, T = 0 in D.

Indeed, the neighborhoods in which T = 0 form a covering $\{U_j\}$ of D and, according to §1, there is a partition of unity in D, $1 = \sum_i \varphi_i$ in D, such that the carrier of each $\varphi_i$ is contained in some $U_j$ and consequently $T[\varphi_i \varphi] = 0$. As the partition is locally finite in D and as the carrier of $\varphi$ is compact and contained in D, $\varphi = \sum_i \varphi_i \varphi$, with only a finite number of terms different from zero and consequently

$$T[\varphi] = \sum_i T[\varphi_i \varphi] = 0 .$$

On account of this theorem, there is a largest open set in which T = 0: the set of all points which have a neighborhood in which T = 0. The complement of this set will be called the carrier of T. In the case where T is equal to a form $C^0$, this definition agrees with the previous one.

Generalized Chains. Two chains define the same current if, and only if, their difference is a negligible chain, in the sense of §5. In this sense, they can be equal without being identical; they then give two different representations of the same current by linear combinations of simplexes; for the calculation of $c[\varphi] = \int_c \varphi$ we can use either the one or the other. But we can also make the calculation without any such a representation, by using a partition of unity. We shall say that the current T is a <u>generalized chain</u>, if any point is contained in a neighborhood U in which T is equal to a chain $c_U$, i.e. if, for any $\varphi$ $C^\infty$ with a compact carrier contained in U, we have $T[\varphi] = \int_{c_U} \varphi$ ; then, using a partition of unity, we can calculate $T[\varphi]$ for any $\varphi$ with a compact carrier; it is not necessary to know whether T can be represented in the large by a linear combination of simplexes.

The whole n-dimensional manifold M can be considered as a generalized n-chain of even kind. Each sufficiently small neighborhood U is indeed contained in a regularly imbedded n-simplex $s^n$; with the natural (absolutely positive) orientation of the identical mapping, $s^n$ is an n-chain of even kind and the integral over M of any n-form of odd kind whose carrier is contained in U is equal to its integral on $s^n$. As a current, this n-dimensional chain of even kind represented by M is equal to the function 1, because

$$\int_M \varphi = 1[\varphi] = \int \varphi$$

according to the above definitions.

More generally, it is easy to see that <u>each n-chain of even kind is equal to a function</u>, i.e., that to every such chain $c^n$ there is associated a function f such that

$$\int_{c^n} \varphi = \int f \varphi$$

for each n-form $\varphi$ of odd kind, with a compact carrier.

Convergence. Suppose $\varphi$ is $C^\infty$ with a non-compact carrier. Then we shall say that $T[\varphi]$ is convergent and that

$$T[\varphi] = \sum_i T[\varphi_i \varphi]$$

if the series is convergent for each partition of unity $1 = \sum_i \varphi_i$ (locally finite, $0 \leq \varphi_i \leq 1$, $\varphi_i$ $C^\infty$ with a compact carrier). Then the series is absolutely convergent and the sum is the same for all partitions (the proof is the same as in §4 for $1[\varphi]$).

In the case where the carrier of T is compact, there is always convergence. Thus, $T[\varphi]$ is defined for all $\varphi$ $C^\infty$.

§8. Products, Differentiation, Transformation.

If $\alpha$ and $\beta$ are forms of degree p and q respectively, we have

$$(\alpha \wedge \beta)[\varphi] = (-1)^{pq}(\beta \wedge \alpha)[\varphi] = \int \alpha \wedge \beta \wedge \varphi = \alpha[\beta \wedge \varphi]$$

Now, for any current T of degree p and any form $\beta$ $C^\infty$ of degree q, we define the products $T \wedge \beta$ and $\beta \wedge T$ by

$$(T \wedge \beta)[\varphi] = (-1)^{pq}(\beta \wedge T)[\varphi] = T[\beta \wedge \varphi]$$

In particular, for any function f $C^\infty$, we have

$$f\,T[\varphi] = T\,f[\varphi] = T[f\,\varphi]$$

We shall say that a current is $C^\infty$ in a domain D, if it is equal in D to a form $C^\infty$. The above formula gives the definition of the product of two currents in the case where one of them is $C^\infty$.

The following notations will sometimes be useful. If T is a current of degree n and of odd kind, provided T[1] is convergent, which is always the case if the carrier of T is compact, we shall write

$$\int T \quad \text{for} \quad T[1]$$

as in the case where T is equal to a form. If T is of degree n-p and $\varphi$ a form $C^\infty$ of degree p with a compact carrier, T and $\varphi$ being of different kinds, since $T[\varphi] = (T \wedge \varphi)[1]$, we shall also write

$$T[\varphi] = \int T \wedge \varphi$$

In particular, for a chain c,

$$\int c \wedge \varphi = \int_c \varphi$$

<u>Differential of a current.</u> Suppose $\alpha$ is a p-form and $\varphi$ an (n-p-1)-form with a compact carrier, not of the same kind as $\alpha$. Then $d(\alpha \wedge \varphi)$ is an n-form of odd kind with a compact carrier and according to the theorem of §4, $\int d(\alpha \wedge \varphi) = 0$. But, as $d(\alpha \wedge \varphi) = d\alpha \wedge \varphi + (-1)^p \alpha \wedge d\varphi$, this means that

$$d\alpha[\varphi] = (-1)^{p+1} \alpha[d\varphi] \; .$$

Now, <u>for any current</u> T <u>of degree</u> p, <u>we define</u> d T <u>by</u>

$$d\, T[\varphi] = (-1)^{p+1}\, T[d\varphi]$$

If T is equal to a form $\alpha$, $d\,T = d\alpha$. If T is equal to a (n-p)chain $c^{n-p}$,

$$dc^{n-p}[\varphi] = (-1)^{p+1} c^{n-p}[d\varphi] = (-1)^{p+1} \int_{c^{n-p}} d\varphi = (-1)^{p+1} \int_{Bc^{n-p}} \varphi = (-1)^{p+1} Bc^{n-p}[\varphi]$$

according to the Stokes' formula. This means that, <u>except for the sign, the differential of a chain is its boundary</u>: $dc^{n-p} = (-1)^{p+1} Bc^{n-p}$. For a chain of dimension p, we would have $dc^p = (-1)^{n-p-1} Bc^p$.

From the definition of d T, it follows immediately that

$$d^2 T = 0 \quad \text{and} \quad d(T \wedge \beta) = dT \wedge \beta + (-1)^p T \wedge d\beta$$

where p is the degree of T.

In the defining formula $dT[\varphi] = (-1)^{p+1} T[d\varphi]$, the carrier of $\varphi$ is supposed to be compact. But, <u>if the carrier of</u> T <u>is compact, this formula holds for any</u> $\varphi$ $C^\infty$. In particular, <u>if</u> T <u>is of degree</u> n-1, <u>of odd kind and with a compact carrier</u>, $dT[1] = (-1)^n T[d1] = 0$, i.e.,

$$\int dT = 0 .$$

<u>Transformation of a current by a mapping</u>. Let $\mu$ be a mapping $C^\infty$ of the manifold $M_1$ in the manifold M. Then, to every form $\varphi$ $C^\infty$ in M corresponds a form $\mu^*\varphi$ in $M_1$, which is $C^\infty$ and has the same degree as $\varphi$, and we have $d\mu^*\varphi = \mu^* d\varphi$, $\mu^*(\varphi \wedge \psi) = \mu^*\varphi \wedge \mu^*\psi$, $\mu^* 1 = 1$.

The carrier of $\mu^*\varphi$ need not be compact even if the carrier of $\varphi$ is compact. For instance, if M is compact and $M_1$ not compact, the carrier of 1 in M is M (compact) while the carrier of $\mu^* 1 = 1$ in $M_1$ is $M_1$ (not compact).

<u>Definition</u>. If $T_1$ is a current in $M_1$, with a compact carrier, its transformed by $\mu$ is the current $\mu T_1$ in M defined by

$$\mu T_1[\varphi] = T_1[\mu^*\varphi].$$

Since the carrier of $T_1$ is compact and since $\mu^*\varphi$ is $C^\infty$, $T_1[\mu^*\varphi]$ is always convergent and the definition is correct.

Since $\varphi$ and $\mu^*\varphi$ have the same degree and are of the same kind, $T_1$ and $\mu T_1$ have the same dimension and are of the same kind, but not of the same degree except in the case where $M$ and $M_1$ are of the same dimension. In case $T_1$ is of even kind, $\varphi$ being of odd kind, we must suppose that $\mu$ is oriented.

For chains, this definition agrees with the usual one given in §5. The image of an 0-simplex consisting of the point $y \in M_1$ is the point $x = \mu y \in M$.

The carrier of $\mu^*\varphi$ is contained in the converse image $\mu^{-1}C$ of the carrier $C$ of $\varphi$ (set of all points $y \in M_1$ such that $\mu y \in C$). Consequently, the carrier of $\mu T_1$ is contained in the image $\mu K_1$ of the carrier $K_1$ of $T_1$, and, as $K_1$ is compact, it is also compact.

If the mapping $\mu$ is such that, for each compact set $C \subset M$, $\mu^{-1}C$ is also compact, then, for each $\varphi$ with a compact carrier, $\mu^*\varphi$ has also a compact carrier, and the above formula gives the definition of $\mu T_1$ for all currents $T_1$, even with a non compact carrier. In the contrary case, let $M_\mu$ be the set of all points of $M$ which have a neighborhood $U$ such that $\mu^{-1}U$ is compact; $M_\mu$ is a domain in $M$ and for any compact set $K \subset M_\mu$, $\mu^{-1}K$ is compact. Consequently, if the carrier of $\varphi$ is compact and contained in $M_\mu$, the carrier of $\mu^*\varphi$ is compact and $\mu T_1[\varphi] = T_1[\mu^*\varphi]$ is well determined. Hence, the image $\mu T_1$ of any current in $M_1$ is a current well determined in $M_\mu$, but not, in general, in the whole manifold $M$.

In the case where $M$ and $M_1$ have the same dimension, $\mu 1$ is a current of degree zero in $M_\mu$, whose differential vanishes, because $d\mu 1 = \mu d1 = 0$.

It can be proved that such a current is a function (cf.§7) which must be constant in each open connected component of $M_\mu$. This is nothing else than the topological degree of the mapping $\mu$.

## §9. The Symbolic Form Associated with a Current.

If $T_{i_1 \ldots i_p}$ $(i_1 < i_2 < \ldots < i_p)$ are $\binom{n}{p}$ currents of degree zero defined in the domain D of a coordinate system $x^1, \ldots, x^n$, then, according to the above definitions,

$$(1) \qquad T = \sum_{i_1 < \ldots < i_p} T_{i_1 \ldots i_p} \, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

is a current of degree p, defined in D.

Now, any current T of degree p defined in D can be represented by such an expression. Indeed, T being given in D, if we define $T_{i_1 \ldots i_p}$ by

$$(2) \qquad T_{i_1 \ldots i_p}[a dx^1 \wedge \ldots \wedge dx^n] = \delta^{1 \; \ldots \; n}_{i_1 \ldots i_p j_1 \ldots j_{n-p}} \, T[a dx^{j_1} \wedge \ldots \wedge dx^{j_{n-p}}],$$

where $i_1 \ldots i_p j_1 \ldots j_{n-p}$ is a permutation of $1 \ldots n$, the formula (1) holds.

The currents of degree zero are the <u>distributions</u> introduced by L. Schwartz. We see <u>that every current can be represented by a differential form whose coefficients are distributions</u>, or, as we shall say, by a <u>symbolic differential form.</u> The coefficients are well determined by (2).

<u>Derivatives.</u> For a form $\alpha = \sum_{i_1 < \ldots < i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$, the derivative $\frac{\partial \alpha}{\partial x^i}$ is defined by

$$(3) \qquad \frac{\partial \alpha}{\partial x^i} = \sum_{i_1 < \ldots < i_p} \frac{\partial a_{i_1 \ldots i_p}}{\partial x^i} \, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

We suppose, of course, that $\alpha$ is $C^1$. The derivative is thus dedined in the domain D of the coordinate system. But if the carrier of $\alpha$ is contained in D, we agree that $\frac{\partial \alpha}{\partial x^i} = 0$ outside of D, and it is then defined everywhere. Under this condition, if $\alpha$ is of degree n and of odd kind, we have

$$\int \frac{\partial \alpha}{\partial x^i} = 0$$

Indeed, $\alpha = a\, dx^1 \wedge \dots \wedge dx^n$, $\frac{\partial \alpha}{\partial x^1} = d(a\, dx^2 \wedge \dots \wedge dx^n)$ and our assertion follows (for i = 1 and in the same way for each i) from the theorem of §4.

Now, if $\alpha$ is of degree p and $\varphi$ of degree n-p with a compact carrier contained in D, $\alpha$ and $\varphi$ being of different kinds, since

$$\frac{\partial}{\partial x^i}(\alpha \wedge \varphi) = \frac{\partial \alpha}{\partial x^i} \wedge \varphi + \alpha \wedge \frac{\partial \varphi}{\partial x^i} \quad ,$$

we get

$$\frac{\partial \alpha}{\partial x^i}[\varphi] = -\alpha\left[\frac{\partial \varphi}{\partial x^i}\right]$$

Now, following Laurent Schwartz, <u>for any current</u> T, <u>we define</u> $\frac{\partial T}{\partial x^i}$ <u>in</u> D <u>by</u>

$$\frac{\partial T}{\partial x^i}[\varphi] = -T\left[\frac{\partial \varphi}{\partial x^i}\right]$$

It can be immediately verified that the formula (3) is still valid for a current T instead of for a form $\alpha$. Furthermore, the differential operator d can be expressed, for currents as well as for forms, by

$$d = \sum_i dx^i \wedge \frac{\partial}{\partial x^i}$$

Hence, in the same way as we got the differential forms from the functions, we can get the currents from the distributions.

Under a change of coordinates, the distribution coefficients $T_{i_1 \ldots i_p}$ change according to the same rule as the coefficients of a form, i.e., the components of a covariant p-vector. The currents are particular cases of distribution tensors.

## §10. Differentiation with Respect to Parameters. Tensor Product of Distributions and Currents.

Theorem I. If the form $\varphi = \varphi(x;t)$ depends on a parameter t and if its coefficients are $C^\infty$ with respect to the coordinates of x and t together while its carrier remains in a fixed compact set for $t_0 < t < t_1$, then, for any current T, $T[\varphi]$ is a function $C^\infty$ of t for $t_0 < t < t_1$ and

$$\frac{\partial T[\varphi]}{\partial t} = T\left[\frac{\partial \varphi}{\partial t}\right].$$

This theorem is a generalization of the theorem of the Calculus according to which an integral can be differentiated with respect to a parameter under the integral sign.

To prove the theorem, we can suppose the carrier of $\varphi$ remains in a fixed compact set contained in the domain of some coordinate system $x^1 \ldots x^n$. For any function of t let us set

$$\Delta_h F(t) = \frac{F(t+h) - F(t)}{h}$$

On account of the linearity of $T[\varphi]$, we have $\Delta_h T[\varphi] = T[\Delta_h \varphi]$, and on account of the continuity condition which has to be satisfied by T,

$$\lim_{h=0} T[\Delta_h \varphi] = T\left[\frac{\partial\varphi}{\partial t}\right]$$

because each derivative of any order with respect to $x^1 \ldots x^n$ of each coefficient of $\Delta_h\varphi - \frac{\partial\varphi}{\partial t}$ tends uniformly to zero for $h \longrightarrow 0$, while the carrier of this form remains in a fixed compact set.

This proves the existence of $\frac{\partial T[\varphi]}{\partial t}$ together with the equality $\frac{\partial T[\varphi]}{\partial t} = T[\frac{\partial\varphi}{\partial t}]$. The existence of the derivatives of higher orders follows all at once from this.

Let us now consider two manifolds $M$ and $M_1$, of dimensions $n$ and $m$ respectively. If $x$ is a variable point on $M$, $y$ a variable point on $M_1$, $(x,y)$ represents a variable point on the product space $M \times M_1$. Let $T = T(x)$ be a current of degree $n$ and of odd kind on $M$, $S = S(y)$ a current of degree $m$ and of odd kind on $M_1$. Then, if $\varphi(x,y)$ is a function $C^\infty$ with a compact carrier on $M \times M_1$, for each $y$ it is a function $C^\infty$ of $x$ with a compact carrier on $M$; according to Theorem I, $T(x)[\varphi(x,y)]$ is a function $C^\infty$ of $y$ on $M_1$, whose carrier is obviously compact; consequently $S(y)[T(x)[\varphi(x,y)]]$ is a well determined linear functional of $\varphi(x,y)$. Following Laurent Schwartz, we shall call it the tensor product of $S$ and $T$ and we shall write

$$S(y)T(x)[\varphi(x,y)] \quad \text{for} \quad S(y)[T(x)[\varphi(x,y)]]$$

Theorem II. The tensor product of currents of dimension zero and of odd kind is a current and is commutative

$$(1) \qquad S(y)T(x)[\varphi(x,y)] = T(x)S(y)[\varphi(x,y)]$$

In the first place, we have to prove that the linear functional $S(y)T(x)[\varphi(x,y)]$ satisfies the continuity condition, i.e. tends to zero if each derivative of $\varphi(x,y)$ tends uniformly to zero while its carrier remains in a compact set. We can suppose this compact set K is contained in the product $U \times V$ of a neighborhood U on M and another one V on $M_1$. Then it is clear that the carrier of $T(x)[\varphi(x,y)]$ remains in the projection of K on $M_1$ (set of all y corresponding to which there exists an x such that $(x,y) \in K$), which projection is a compact set contained in V. Moreover, as each derivative of $\varphi(x,y)$ with respect to x and y tends uniformly to zero, $T(x)[\varphi(x,y)]$ tends to zero uniformly with respect to y, and, according to Theorem I, the same holds for each of its derivatives. Consequently, $S(y)T(x)[\varphi(x,y)]$ tends to zero and $S(y)T(x)$ is a current.

To prove the commutativity, we remark in the first place that the equality (1) is evident if $\varphi(x,y)$ is a product $\varphi_1(x)\ \varphi_2(y)$ of a function of x and a function of y. Consequently, (1) still holds if $\varphi(x,y)$ is a finite sum $\sigma(x,y)$ of such products. Accordingly, on account of the continuity condition which is satisfied by $S(y)T(x)$ and by $T(x)S(y)$, the extension of (1) to the case of any $\varphi(x,y)$ follows immediately from the following lemma (which can be considered as well known).

<u>Lemma</u>. Given a function $\varphi(x,y)$ $C^\infty$ with a compact carrier contained in $U \times V$, it is possible to find a series of functions $\sigma_k(x,y)$ $(k = 1,2,\ldots)$, where each $\sigma_k(x,y)$ is a finite sum of products $\varphi_1(x)\varphi_2(y)$ of a function $\varphi_1(x)$ $C^\infty$ with a carrier contained in a fixed compact set in U and a function $\varphi_2(y)$ $C^\infty$ with a carrier contained in a fixed compact set

in V, such that each derivative of $\sigma_k(x,y)$ tends uniformly, for $k \longrightarrow \infty$, to the corresponding derivative of $\varphi(x,y)$.

As an example, if T and S are simplexes of dimension zero, i.e. points $t \in M$ and $s \in M_1$ respectively, their tensor product is the simplex of dimension zero which consists of the point $(t,s)$ of $M \times M_1$.

We shall now define the tensor product of two currents of degree zero and of even kind.

Given a coordinate system $x^1 \ldots x^n$ in U, let us consider the form of degree n and of odd kind equal to $dx^1 \wedge \ldots \wedge dx^n$, and similarly the form of degree m and of odd kind equal to $dy^1 \wedge \ldots \wedge dy^m$ in V, where $y^1 \ldots y^m$ is a coordinate system in V. We shall represent the first by dx, the second by dy. They are currents of dimension zero. According to our definition, the tensor product dxdy is represented in $U \times V$, with the coordinate system $x^1 \ldots x^n y^1 \ldots y^m$, by

$$dxdy = dx^1 \wedge \ldots \wedge dx^n \wedge dy^1 \wedge \ldots \wedge dy^m$$

With the coordinate system $y^1 \ldots y^m x^1 \ldots x^n$ we get in the same way

$$dydx = dy^1 \wedge \ldots \wedge dy^m \wedge dx^1 \wedge \ldots \wedge dx^n .$$

Now, let us consider two currents of degree zero and of even kind, $T_o(x)$ on M and $S_o(y)$ on $M_1$. Then $T_o(x)dx$ and $S_o(y)dy$ are currents of dimension zero and of odd kind well determined in U and V respectively. According to §9, their tensor product $T_o(x)dxS_o(y)dy$ can be represented in $U \times V$ by an expression like $R_o(x,y)dxdy$ where $R_o(x,y)$ is a current of degree zero and of even kind, which, as can be easily verified, is independent of the choice of the coordinate systems and depends only on $T_o(x)$ and $S_o(y)$. It is consequently well determined on the whole manifold $M \times M_1$. We call it the <u>tensor product</u> of $T_o(x)$ and $S_o(y)$ and we write $R_o(x,y) = T_o(x)S_o(y)$. Clearly, the commutativity law holds.

In the case where $T_o(x)$ and $S_o(y)$ are two functions, this is nothing else than their product in the usual sense. The name tensor product comes from the analogy with the tensor product $a_i b_j$ of two vectors, the indices i and j being replaced by x and y.

It is easy to extend the definition of the tensor product to the case of two currents of arbitrary degrees and of <u>the same kind</u>. But we shall not use it here.

## §11. <u>Harmonic Distributions.</u>

According to the Laurent Schwartzs' definition of the partial derivatives given in §9, the Laplacian of a distribution, in the Euclidean space $E^3$, is a well determined distribution. If this last is equal to zero in a domain D, the former is said to be <u>harmonic in D</u>. We shall now prove the following

<u>Theorem:</u> <u>If a distribution is harmonic in D, it is equal in D to a function $C^{\infty}$.</u>

The coordinates of a point x being $x_1, x_2, x_3$, we note by lap or $\text{lap}_x$ the Laplacian

$$\text{lap}_x = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^3}{\partial x_3^2}$$

and by dx the volume element $dx = dx_1 dx_2 dx_3$ or $dx_1 \wedge dx_2 \wedge dx_3$, considered as a 3-form of odd kind. For another point $y = (y_1, y_2, y_3)$ of the same space, $\mathrm{lap}_y$ and dy are defined in the same way.

T being a distribution in $E^3$, according to the definition of the partial derivatives, we have, for each function $\varphi$ $C^\infty$ with a compact carrier,

$$\mathrm{lap}\, T[\varphi\, dx] = T[\mathrm{lap}\, \varphi\, dx] .$$

Let $\varepsilon$ be a positive number and $\rho(x,y)$ a function $C^\infty$ of x and y, which depends only on the distance $r(x,y)$ of the two points x and y, such that $0 \leq \rho \leq 1$, $\rho = 0$ if $r > \varepsilon$, $\rho = 1$ if $r < \frac{\varepsilon}{2}$. We consider the function

$$\gamma(x,y) = \frac{\rho(x,y)}{4\pi r(x,y)}$$

and, following K. Kodaira, we call it the <u>modified elementary solution.</u> It has the following properties

1) The function $\bar{\gamma}(x,y) = \mathrm{lap}_x \gamma(x,y)$ for $x \neq y$, $= 0$ for $x = y$ is $C^\infty$ with respect to x and y, even for $x = y$.

2) If $\varphi$ is a function $C^\infty$ with a compact carrier,

$$\psi(x) = \int \gamma(x,y)\varphi(y)dy \text{ is } C^\infty \text{ and}$$
$$\mathrm{lap}\, \psi(x) = -\varphi(x) + \int \bar{\gamma}(x,y)\varphi(y)dy$$

This last equation is equivalent to Poissons' equation.

Now, let $D_\varepsilon$ be the set of all points of D whose distance from the complement of D is greater than $\varepsilon$. Suppose T is harmonic in D and the carrier of $\varphi$ is contained in $D_\varepsilon$. Then, the carrier of $\psi$ is in D and consequently

$$T[\text{lap}\, \psi\, dx] = \text{lap}\, T[\psi\, dx] = 0$$

On account of 2) and of Theorem II of §11, we have

$$T[\varphi\, dx] = T(x)dx[\int \varphi(y)\bar{\gamma}(x,y)dy] = T(x)dx\,\varphi(y)dy[\bar{\gamma}(x,y)] =$$

$$= \varphi(y)dy\; T(x)dx[\bar{\gamma}(x,y)] = \int f_\varepsilon(y)\varphi(y)dy$$

where $f_\varepsilon(y) = T(x)dx[\bar{\gamma}(x,y)]$ is $C^\infty$, according to Theorem I of §11. This means that $T = f_\varepsilon$ in $D_\varepsilon$. $f_\varepsilon$ depends on $\varepsilon$ because $\gamma$ and $\bar{\gamma}$ depend on $\varepsilon$. But if $\varepsilon' > \varepsilon$, as $D_{\varepsilon'} \subset D_\varepsilon$, $f_{\varepsilon'} = f_\varepsilon$ in $D_{\varepsilon'}$. Hence $f_0 = \lim_{\varepsilon = 0} f_\varepsilon$ exists and is $C^\infty$ in D and $T = f_0$ in D.

Of course, since T is harmonic in D, $f_0$ has also to be harmonic in D.

With the same argument, we can prove:

<u>Theorem.</u> If lap T is $C^\infty$ in D, T is $C^\infty$ in D.

Indeed, if lap $T = h$ in D, h being $C^\infty$ in D. we have

$$T[\text{lap}\,\psi\, dx] = \text{lap}\, T[\psi\, dx] = h[\psi\, dx] =$$

$$= \iint h(x)\gamma(x,y)\varphi(y)dxdy = h_\varepsilon(y)[\varphi(y)dy]$$

where $h_\varepsilon(y) = \int h(x)\gamma(x,y)dx$ is $C^\infty$ in D.

The above argument now gives $T = f_\varepsilon - h_\varepsilon$ in D, instead of $T = f_\varepsilon$, and $T = \lim_{\varepsilon = 0}(f_\varepsilon - h_\varepsilon)$ in D, the limit existing and being $C^\infty$ in D.

The same proof is valid for any linear differential operator $\Lambda$ for which we have a "modified elementary solution" with the properties corresponding to 1) and 2). This is the case for **any** total elliptic differential operator. It is not essential that $\Lambda$ be self adjoint. The proof and the theorem are also valid for systems of equations. We shall use it in the theory of harmonic differentials.

## §12. Homology.

Here, we shall only state, without proof, the main theorems concerning homologies between currents, forms and chains in a manifold.

Definition. The current T is said to be closed, if dT = 0. It is said to be homologous to zero, if there is a current S such that T = dS. It is said to be compact homologous to zero, if there is a current S with a compact carrier such that T = dS.

If T = dS, we shall also say that T bounds S.

Clearly, if T is homologous to zero, T is closed. If T is compact homologous to zero, the carrier of T is compact. All closed currents, of a given degree and a given kind, which are homologous to one of them, constitute an homology class.

Theorem A. 1) Each closed current is homologous to a form $C^{\infty}$. Each closed current with a compact carrier is compact homologous to a form $C^{\infty}$.

2) If a form $C^r (0 \leq r \leq \infty)$ bounds a current [ a current with a compact carrier], it bounds a form $C^r$[a form with a compact carrier].

<u>Theorem B.</u> 1) Each closed current is homologous to a chain. Each closed current with a compact carrier is compact homologous to a finite closed chain.

2) If a closed chain bounds a current, it bounds a chain. If a finite closed chain bounds a current with a compact carrier, it bounds a finite chain.

<u>Theorem C.</u> (Duality theorem)

The current $T$ is homologous to zero, if and only if $T[\varphi] = 0$ for each closed form $\varphi$ with a compact carrier. The current $T$, with a compact carrier, is compact homologous to zero, if and only if $T[\varphi] = 0$ for each closed form $\varphi$.

These theorems generalize those that I proved in my Thesis. They can be proved by the same method, using a triangulation of the manifold. For Theorem C, in the non compact case, the duality theorem of Pontrjagin has to be used instead of that of Poincaré. Another method for proving A and B, the "smoothing method", will be given elsewhere.

For compact manifolds, Theorems A and C and part 2 of B will be deduced from the central theorem concerning harmonic differentials.

Let us remark that the "only if" part of Theorem C is immediate. Indeed, if $T = dS$, $T[\varphi] = dS[\varphi] = \pm S[d\varphi] = 0$, provided $d\varphi = 0$ and the carrier of $\varphi$ or that of $S$ is compact. Consequently, <u>if</u> $T_1$ <u>is homologous to</u> $T_2$, $T_1[\varphi] = T_2[\varphi]$ <u>for each closed form</u> $\varphi$ <u>with a compact carrier.</u> And <u>if</u> $T_1$ <u>is compact homologous to</u> $T_2$, $T_1[\varphi] = T_2[\varphi]$ <u>for each closed</u> $\varphi$.

Here are a few consequences of these theorems.

Suppose T is a closed form $\omega$, which is not homologous to zero. Then, according to C, there exists a closed form $\varphi$, with a compact carrier, such that $\omega[\varphi] = \int \omega \wedge \varphi \neq 0$. Now, according to B, $\varphi$ is compact homologous to a finite closed chain c, and according to the above remark we have

$$c[\omega] = \int_c \omega = \varphi[\omega] = \pm\, \omega[\varphi] \neq 0 .$$

c being a closed chain, $\int_c \omega$ is called a <u>period</u> of $\omega$. Our result shows that:

<u>A closed form all of whose periods with respect to finite closed chains are zero is homologous to zero.</u>

In the same way, we get:

<u>A closed form with a compact carrier all of whose periods, with respect to finite and infinite closed chains, are zero, is compact homologous to zero.</u>

Now suppose that T is a closed chain c which is not homologous to zero. Then, according to C, there is a closed form $\varphi$ with a compact carrier whose period relative to c, $c[\varphi]$, is not zero. More generally:

<u>If</u> $c_1, \dots, c_r$ <u>are</u> r <u>closed chains such that</u> $c = \sum_i k_i c_i$ <u>is not homologous to zero unless all coefficients</u> $k_i$ <u>are zero, and if</u> $a_1, \dots, a_r$ <u>are</u> r <u>arbitrary given numbers, there is a closed form</u> $\varphi$ <u>with a compact carrier whose period relative to</u> $c_i$ <u>is equal to</u> $a_i$, $c_i[\varphi] = a_i \; (i=1,\dots,r)$.

Indeed, if this were impossible for some set of numbers $a_i$, the periods $c_i[\varphi]$ of any closed form $\varphi$ with a compact carrier would all satisfy the same relation of the form $\sum_i k_i c_i[\varphi] = 0$ and we would have $c[\varphi] = 0$, contrary to the above result.

In the same way we get:

There exists a closed form whose r periods relative to r given finite closed chains no combination of which is compact homologous to zero have r arbitrary given values.

# Chapter III.

## Harmonic Differentials on a Riemann Space.

### §13. Scalar Product. Adjoint Form.

On an n-dimensional manifold $C^{\infty}$, M, let $g_{ij}$ be a covariant tensor function of rank 2, $C^{\infty}$, symmetric and positive definite;

$$g_{ij} = g_{ji}, \qquad ds^2 = \sum_{i,j} g_{ij}\, dx^i\, dx^j > 0 .$$

The manifold M with this tensor function is a Riemann Space.

We shall use the associated contravariant tensor $g^{ij}$ determined by the equations

$$\sum_k g^{ik} g_{kj} = \delta^i_j$$

and we shall use the operation of raising or lowering indices according to the formulae

$$a^{i_1 \dots i_p} = \sum_{k_1 \dots k_p} g^{i_1 k_1} \dots g^{i_p k_p} a_{k_1 \dots k_p} ,$$

$$a_{i_1 \dots i_p} = \sum_{k_1 \dots k_p} g_{i_1 k_1} \dots g_{i_p k_p} a^{k_1 \dots k_p} .$$

The coordinate system is said to be orthonormal at a given point, if at this point $g_{ij} = \delta^j_i$. Then, at this point, $g^{ij} = \delta^j_i$ and $a^{i_1 \dots i_p} = a_{i_1 \dots i_p}$.

Volume Element. Taking the product of n tensors equal to $g_{ij}$ and making it skew symmetric, we get a new tensor

$$g_{i_1\ldots i_n,k_1\ldots k_n} = \sum_{j_1\ldots j_n} g_{i_1 j_1}\cdots g_{i_n j_n}\,\delta^{j_1\ldots j_n}_{k_1\ldots k_n} = \begin{vmatrix} g_{i_1k_1} & \cdots & g_{i_1k_n} \\ \vdots & & \vdots \\ g_{i_nk_1} & \cdots & g_{i_nk_n} \end{vmatrix}$$

Clearly we have $g_{i_1\ldots i_n,k_1\ldots k_n} = g_{k_1\ldots k_n,i_1\ldots i_n}$ and

$$g_{i_1\ldots i_n,k_1\ldots k_n} = \delta^{1\ \ldots\ n}_{i_1\ldots i_n}\,\delta^{1\ \ldots\ n}_{k_1\ldots k_n}\ g_{1\ldots n,1\ldots n}\,.$$

Under a change of the coordinate system, the first component of this tensor (which tensor is a double covariant n-vector) changes according to the formula

$$\bar{g}_{1\ldots n,1\ldots n} = J^2 g_{1\ldots n,1\ldots n}$$

where J is the Jacobian. If the coordinate system is orthonormal, $g_{1\ldots n,1\ldots n} = 1$. Consequently, in any coordinate system $g_{1\ldots n,1\ldots n}$ is always positive. Let us set

$$e_{1\ldots n} = +\sqrt{g_{1\ldots n,1\ldots n}}\ ,$$

we get the transformation formula

$$\bar{e}_{1\ldots n} = |J|\, e_{1\ldots n}$$

Hence, $e_{1\ldots n}$ is the first component of <u>a covariant n-vector of odd kind.</u> The corresponding differential form

$$e_{1\ldots n}\, dx^1 \wedge \ldots \wedge dx^n \ldots$$

of degree n and of odd kind is the volume element. Of course, the other components of this n-vector are $e_{i_1 \dots i_n} = \delta^{1 \dots n}_{i_1 \dots i_n} e_{1 \dots n}$.

Definition. The adjoint form to

$$\alpha = \sum_{i_1 < \dots < i_p} a_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

is the form

$$*\alpha = \sum_{j_1 < \dots < j_{n-p}} a^*_{j_1 \dots j_{n-p}} dx^{j_1} \wedge \dots \wedge dx^{j_{n-p}}$$

where

$$a^*_{j_1 \dots j_{n-p}} = \sum_{i_1 < \dots < i_p} e_{i_1 \dots i_p j_1 \dots j_{n-p}} a^{i_1 \dots i_p} .$$

Let us remark that, in this last sum, only one term can be different from zero: the term corresponding to the values of $i_1 \dots i_p$ which are different from $j_1 \dots j_{n-p}$.

The adjoint to a p-form $\alpha$ is a (n-p)-form $*\alpha$, which is of different kind, because the volume element is a form of odd kind. The adjoint to 1 (considered as a form of degree zero) is the volume element.

$$*1 = e_{1 \dots n} dx^1 \wedge \dots \wedge dx^n$$

The adjoint to any function, considered as a 0-form, is its product with the volume element.

Clearly, the operation * is a linear operation. Furthermore, if p is the degree of $\alpha$, we have

$$** \alpha = (-1)^{pn+p} \alpha$$

as can be immediately verified with the help of an orthonormal coordinate system. If pn+p is even, * is its own inverse. If pn+p is odd, i.e., if n is even and p odd, the inverse of * is -*. In all cases, *** is the inverse of *.

Now, let us consider two forms of the same degree p and of the same kind, $\alpha$ with the coefficients $a_{i_1 \dots i_p}$ and $\beta$ with the coefficients $b_{i_1 \dots i_p}$. We have

$$\alpha \wedge * \beta = \beta \wedge * \alpha = \sum_{i_1 < \dots < i_p} a^{i_1 \dots i_p} b_{i_1 \dots i_p} \cdot *1$$

as can be immediately verified with the help of an orthonormal coordinate system. Moreover, the coefficient of *1 in

$$\alpha \wedge * \alpha = \sum_{i_1 < \dots < i_p} a^{i_1 \dots i_p} a_{i_1 \dots i_p} \cdot *1$$

is always $\geq 0$ and it is $= 0$ only if $\alpha = 0$ at the corresponding point.

<u>Definition</u>. The <u>scalar product</u> of the two forms $\alpha$ and $\beta$, of the same degree and of the same kind, is the number

$$(\alpha, \beta) = \int \alpha \wedge * \beta$$

Of course, $(\alpha, \beta)$ is determined only if the above integral is convergent. This is always the case if the forms are $C^0$ and the carrier of one of them is compact.

Clearly, this scalar product is bilinear and symmetric. The scalar square is positive definite: $(\alpha,\alpha) \geq 0$. If $\alpha$ is $C^0$ and $(\alpha,\alpha) = 0$, then $\alpha = 0$ identically. We also have

$$(*\alpha, *\beta) = (\alpha, \beta)$$

because $*\alpha \wedge ** \beta = \beta \wedge * \alpha = \alpha \wedge * \beta$.

Adjoint to a Current. The above defining formula for $*\alpha$, applied to a symbolic form, gives the definition of the adjoint to a current. We shall also consider the scalar product of a current $T$ and a form $C^\infty$ $\varphi$, defined by

$$(T, \varphi) = (\varphi, T) = T[*\varphi] = \int T \wedge * \varphi.$$

It is always determined if the carrier of $\varphi$ or that of $T$ is compact. Instead of the notation $T[\varphi]$ of Chapter II, in which $T$ and $\varphi$ are not of the same kind and the dimension of $T$ has to be equal to the degree of $\varphi$, we shall now use chiefly the notation $(T, \varphi)$, in which $T$ and $\varphi$ have the same degree and are of the same kind.

## §14. The Operators $\delta$ and $\Delta$.

Suppose now that $T$ is of degree $p$ and $\varphi$ of degree $p-1$, and that the carrier of either $T$ or $\varphi$ is compact. Then, according to §8,

$$\int d(\varphi \wedge *T) = 0.$$

and consequently

$$\int d\varphi \wedge * T = (-1)^p \int \varphi \wedge d*T$$

which can be written

$$(T, d\varphi) = (-1)^{np+n+1}(*d*T, \varphi)$$

because, since the degree of $d*T$ is $n-p+1$,

$$**d*T = (-1)^{n(n-p+1)+n-p+1} d*T = (-1)^{np+n-p+1} d*T \ .$$

<u>Definition</u>. We set $\delta T = (-1)^{np+n+1} *d*T$ and we call $\delta T$ the <u>codifferential</u> of T (p is the degree of T).

With this symbol, the above formula becomes

$$(1) \qquad (T, d\varphi) = (\delta T, \varphi)$$

It is valid if the carrier of either T or $\varphi$ is compact.

In the same way, we see that, if $\psi$ is $C^{\infty}$ of degree p+1 and if the carrier of either T or $\psi$ is compact, we have

$$(2) \qquad (T, \delta\psi) = (dT, \psi)$$

It is clear that $\delta$ is a linear operator. Applied to a current or a form of degree p, it gives a current or a form of degree p-1. Applied to a function or a current of degree 0, it gives zero. Moreover, if p is the degree of T, we always have

$$(3) \qquad \delta^2 T = 0, \qquad *\delta T = (-1)^p d*T, \qquad *dT = (-1)^{p+1}\delta*T \ .$$

We shall say that a current or a form is <u>coclosed</u> if its codifferential is equal to zero. It is equivalent to say that its adjoint is closed. If $T = \delta S$, we shall also say that T <u>cobounds</u> S. A current will be said to be <u>cohomologous</u> to zero, if it cobounds another current.

It is clear that the concepts of codifferential and of adjoint form, in contrast to that of differential, depends on the Riemann metric $g_{ij}$, and not only on the differentiable structure.

The Operator $\triangle$. We set $\triangle = d\delta + \delta d$ .

This is a linear differential operator of second order. $\triangle\alpha$ is of the same degree and of the same kind as $\alpha$ . For a form of degree zero, i.e. for a function f, we have $\delta f = 0$ and $\triangle f = \delta df$. As we shall see, this is nothing else than - div grad f and $\triangle$ can be considered as a generalization of the Laplacian. This suggests the following definition:

The current T is said to be harmonic in D, if $\triangle T = 0$ in D .

In the following paragraphs, we shall prove the existence of an elementary solution corresponding to the operator $\triangle$. From this it will follow, as in §11, that a harmonic current is equal to a form $C^{\infty}$, which of course has to be harmonic.

Here, we shall prove the following

Theorem. A form which is harmonic in a compact Riemann space is closed and coclosed.

Proof: According to (1) and (2), we have

$$(\alpha, \triangle\alpha) = (\alpha, d\delta\alpha) + (\alpha, \delta d\alpha) = (\delta\alpha, \delta\alpha) + (d\alpha, d\alpha).$$

Hence, if $\triangle\alpha = 0$, $(\delta\alpha, \delta\alpha) = 0$ and $(d\alpha, d\alpha) = 0$ and consequently $\delta\alpha = 0$ and $d\alpha = 0$.

According to the definition given by Hodge (see: W. V. D. Hodge, The Theory and Applications of Harmonic Integrals, Cambridge University Press 1941), a form is said harmonic only if it is closed and coclosed.

We see that, in the case of a compact space, the two definitions are equivalent. If the space is not compact, they are not equivalent and a harmonic form in our sense need not be closed and coclosed. The operator $\triangle$ has been first introduced in this theory by K. Kodaira (see: K. Kodaira, Harmonic Fields in Riemannian Manifolds, Annals of Mathematics, Vol. 50, 1949) and, independently by P. Bidal et G. de Rham (see the paper: Les formes différentielles harmoniques - reference below in §16).

An important property of the operator $\triangle$ is that it is permutable with d, $\delta$ and $*$. As a matter of fact, we have

$$d\triangle = \triangle d = d\delta d\,, \quad \delta\triangle = \triangle\delta = \delta d\delta\,,$$

and, on account of (3),

$$*d\delta = \delta d*\,, \quad *\delta d = d\delta* \quad \text{and} \quad *\triangle = \triangle*\,.$$

## §15. Explicit Formulae for $\delta$ and $\triangle$.

It is convenient to use covariant differentiation. In the following formulae, all summations are explicitly indicated.

The covariant derivative of the covariant p-vector $a_{i_1\ldots i_p}$ is the tensor

$$a_{i_1\ldots i_p,i} = \frac{\partial a_{i_1\ldots i_p}}{\partial x^i} - \sum_{v=1}^{p}\sum_a a_{i_1\ldots i_{v-1}a i_{v+1}\ldots i_p}\Gamma^a_{i_v i}$$

in which the $\Gamma^i_{jk}$ are the Christoffels' symbols

$$\Gamma^i_{jk} = \tfrac{1}{2}\sum_a\left(\frac{\partial g_{aj}}{\partial x^k} + \frac{\partial g_{ak}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^a}\right)g^{ia}$$

Let us recall that the covariant derivatives of a sum and of a product obey to the same rules as ordinary derivatives, and that the covariant derivatives of $g_{ij}$, $g^{ij}$, $\delta^{j_1\ldots j_p}_{i_1\ldots i_p}$ and $e_{i_1\ldots i_n}$ are all identical to zero.

We shall denote by $\{a_{i_1\ldots i_p}\}$ the p-form associated with the p-vector $a_{i_1\ldots i_p}$, i.e. the p-form whose general coefficient is $a_{i_1\ldots i_p}$.

On account of the symmetry of the Christoffel symbols $\Gamma^i_{jk} = \Gamma^i_{kj}$, in the formula of §2 giving the differential of a form, the ordinary derivatives can be replaced by the covariant derivatives without changing anything, and we get

$$d\ \{a_{i_1\ldots i_p}\} = \Big\{ \sum_{\substack{j_1<\ldots< j_p \\ j}} \delta^{jj_1\ldots j_p}_{i_1\ldots i_{p+1}}\ a_{j_1\ldots j_p,j} \Big\}$$

Now, on account of the rules we have recalled for covariant differentiation and of the formula of §13 giving the adjoint, we see that the coefficients of $\delta\ \{a_{i_1\ldots i_p}\}$ will also be linear combinations of the $a_{i_1\ldots i_p,i}$, and that the coefficients of $d\delta\{a_{i_1\ldots i_p}\}$ and $\delta d\ \{a_{i_1\ldots i_p}\}$ will be linear combinations of the second covariant derivatives $a_{i_1\ldots i_p,i,j}$.

By direct calculation, we have

$$\delta\ \{a_{i_1\ldots i_p}\} = \Big\{ - \sum_{\substack{j_1<\ldots< j_p \\ \ell,i}} \delta^{j_1\ldots j_p}_{\ell i_1\ldots i_{p-1}}\ g^{\ell i}\ a_{j_1\ldots j_p,i} \Big\}$$

$$d\delta\{a_{i_1\ldots i_p}\} = \Big\{ - \sum_{\substack{j_1<\ldots<j_p \\ \ell_1<\ldots<\ell_{p-1} \\ i,j,\ell}} \delta^{i\ell_1\ldots\ell_{p-1}}_{i_1\ \ldots\ i_p}\ \delta^{j_1\ldots j_p}_{\ell\ell_1\ldots\ell_{p-1}}\ g^{\ell j} a_{j_1\ldots j_p,j,i} \Big\}$$

$$\delta d\left\{a_{i_1\cdots i_p}\right\} = \left\{-\sum_{\substack{j_1<\cdots<j_p\\ i,j,\ell}} \delta^{ij_1\cdots j_p}_{\ell i_1\cdots i_p}\, g^{\ell j}\, a_{j_1\cdots j_p,i,j}\right\}$$

From the above, using the identity

$$\sum_{\ell_1<\cdots<\ell_p} \delta^{i\,\ell_1\cdots\,\ell_{p-1}}_{j_1\,\cdots\,j_p}\, \delta^{i_1\cdots\, i_p}_{\ell\,\ell_1\cdots\,\ell_{p-1}} = \delta^i_\ell\, \delta^{i_1\cdots i_p}_{j_1\cdots j_p} - \delta^{i\,i_1\cdots i_p}_{\ell\, j_1\cdots j_p} ,$$

we obtain for $\Delta = d\delta + \delta d$ the following expression

$$\Delta\left\{a_{i_1\cdots i_p}\right\} = \left\{-\sum_{i,j} g^{ij} a_{i_1\cdots i_p,i,j} + \sum_{\substack{j_1<\cdots<j_p\\ i,j,\ell}} \delta^{ij_1\cdots j_p}_{\ell i_1\cdots i_p} g^{\ell j}(a_{j_1\cdots j_p,j,i} - a_{j_1\cdots j_p,i,j})\right\}$$

Now, on account of Ricci's formulae, the differences $a_{j_1\cdots j_p,j,i} - a_{j_1\cdots j_p,i,j}$ are linear combinations of the $a_{i_1\cdots i_p}$ themselves, and we finally obtain for $\Delta\left\{a_{i_1,\cdots i_p}\right\}$ the expression

$$\left\{-\sum_{i,j} g^{ij} a_{i_1\cdots i_p,i,j} + \sum_{\substack{j_1<\cdots<j_p\\ i,j,\ell,h}} \sum_{\mu=1}^{p} \delta^{ij_1\cdots j_p}_{\ell i_1\cdots i_p} g^{\ell j} R^h{}_{j_\mu j i}\, a_{j_1\cdots j_{\mu-1} h j_{\mu+1}\cdots j_p}\right\}$$

where $R^h{}_{ijk}$ is the curvature tensor.

If the space is Euclidean, the curvature tensor vanishes, and if the coordinates are rectangular, we have $g^{ij} = \delta^j_i$ and

$$\Delta\left\{a_{i_1\cdots i_p}\right\} = \left\{-\sum_i \frac{\partial^2 a_{i_1\cdots i_p}}{\partial x_i^2}\right\}$$

Except to the sign, $\Delta$ is the Laplacian.

In a Riemann space, for $p = 0$, i.e. for a scalar f, we get the Beltrami operator

$$\Delta f = -\sum_{i,j} g^{ij} f_{i,j}$$

where

$$f_{i,j} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_k \frac{\partial f}{\partial x^k} \Gamma^k_{ij} .$$

The above formulae have been established by L.E.J.Brouwer for Euclidean space and by R.Weitzenböck for Riemann space. See references in: G. de Rham, Remarque au sujet de la théorie des formes différentielles harmoniques, Annales de l'Université de Grenoble, Années 1947 et 1948, p.55.

From the above general expression for $\Delta$, one gets immediately

$$\Delta\left\{f a_{i_1 \ldots i_p}\right\} = f\Delta\left\{a_{i_1 \ldots i_p}\right\} - 2\left\{\sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} a_{i_1 \ldots i_p, j}\right\} + (\Delta f)\left\{a_{i_1 \ldots i_p}\right\} .$$

§16. The Parametrix

The word "Parametrix", introduced by Hilbert, is, in the theory of elliptic differential equations, the name for a rough approximation of an elementary solution. For equations with constant coefficients, it is in general easy to find an elementary solution. In terms of this, as has been shown by E.E.Levi and Hilbert, one can get a parametrix for the corresponding equations with variable coefficients, and finally, the problem of constructing an elementary solution is reduced to solving an integral equation.

Here, we shall first write the expression of the elementary solution for the operator $\Delta$ in Euclidean space. Then we shall get a parametrix by replacing in it the Euclidean distance by the geodesic distance in a Riemann space.

In Euclidean space with rectangular coordinates, let $r(x,y)$ be the distance between x and y,

$$r(x,y) = \sqrt{\sum (x^i - y^i)^2} \, ,$$

and let $s_n$ be the area of the (n-1)dimensional sphere $\sum x_i^2 = 1$.

We set

$$g_o(x,y) = \frac{r^{2-n}(x,y)}{s_n(n-2)} \qquad \text{for } n > 2 \, ,$$

$$g_o(x,y) = \frac{-1}{2\pi} \log r(x,y) \qquad \text{for } n = 2 \, .$$

Then, as is well known from potential theory, if $\varphi(x)$ is a function $C^\infty$ with a compact carrier, the function

$$G\varphi = \int g_o(x,y)\varphi(y)dy \qquad \text{(where dy is the volume element)}$$

is also a function $C^\infty$ and we have

$$G\Delta\varphi = \varphi \qquad \text{and} \qquad \Delta G\varphi = \varphi$$

These equations express the characteristic properties according to which $g_o(x,y)$ is called an elementary solution of degree zero for the operator $\Delta$.

Now, let us consider the double form

$$g_p(x,y) = g_o(x,y) \sum_{i_1 < \dots < i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} . \, dy^{i_1} \wedge \dots \wedge dy^{i_p}$$

and let us set, for every p-form $\alpha(x)$ with a compact carrier,

$$G\alpha = (g_p(x,y), \alpha(y)) = \int g_p(x,y) \wedge * \alpha(y) .$$

From the above result relative to the case where $p = 0$ it follows immediately that, for any p $(0 \leq p \leq n)$, if $\alpha$ is $C^{\infty}$ with a compact carrier, $G\alpha$ is also $C^{\infty}$ and we still have the relations

$$G \triangle \alpha = \alpha \quad \text{and} \quad \triangle G \alpha = \alpha .$$

We shall call $g_p(x,y)$ <u>the elementary solution of degree p in the Euclidean space</u>. We shall now write its expression in an <u>arbitrary</u> coordinate system.

Let us set

$$A(x,y) = -\frac{r^2(x,y)}{2} , \qquad A_{i,j} = \frac{\partial^2 A}{\partial x^i \partial y^j}$$

For fixed values of j and y, the $A_{i,j}$ $(i=1\ldots n)$ are the components of a covariant vector at the point x, and for fixed values of i and x, the $A_{i,j}(j=1\ldots n)$ are the components of a covariant vector at the point y.

In a rectangular coordinate system, $A_{i,j} = \delta_i^j$ . Hence

$$g_o(x,y) \sum_{i,j} A_{i,j}\, dx^i . dy^j$$

is the invariant expression for $g_1(x,y)$. It can be written in any coordinate system as soon as we have the expression of the distance function $r(x,y)$.

Let us now consider the determinants

$$A_{i_1\ldots i_p, j_1\ldots j_p} = \begin{vmatrix} A_{i_1,j_1} \cdots A_{i_1,j_p} \\ \vdots \qquad \vdots \\ A_{i_p,j_1} \cdots A_{i_p,j_p} \end{vmatrix} = \sum_{k_1,\ldots,k_p} \delta_{i_1\ldots i_p}^{k_1\ldots k_p} A_{k_1,j_1} \cdots A_{k_p,j_p}$$

For fixed values of $j_1 \dots j_p$ and y, they are the components of a covariant p-vector at the point x, and for fixed values of $i_1 \dots i_p$ and x, they are the components of a covariant p-vector at the point y.

In a rectangular coordinate system, we have $A_{i_1 \dots i_p, j_1 \dots j_p} = \delta^{j_1 \dots j_p}_{i_1 \dots i_p}$. Hence we can write the invariant expression for $g_p(x,y)$ in the following way:

$$g_p(x,y) = g_o(x,y) \sum_{\substack{i_1 < \dots < i_p \\ j_1 < \dots < j_p}} A_{i_1 \dots i_p, j_1 \dots j_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} . dy^{j_1} \wedge \dots \wedge dy^{j_p} .$$

Let us now consider a <u>Riemann space</u> M and let r(x,y) be the <u>geodesic distance</u> from x to y, i.e. the lower bound of the length of all arcs joining x to y. We know that if x and y are near enough, there is one and only one geodesic arc from x to y whose length is r(x,y). More precisely, to each compact set $K \subset M$ there corresponds a positive number $\eta$ such that:
1) if $r(x,y) < \eta$ and $y \in K$, there is one and only one geodesic arc from x to y whose length is r(x,y); 2) $r^2(x,y)$ is a function $C^\infty$ of x and y for $y \in K$ and $r(x,y) < \eta$. (This will be proved in the next §). If M is compact, we can take K = M.

Let $\rho(x,y)$ be a function $C^\infty$ which depends only on r(x,y), such that $0 \leq \rho(x,y) \leq 1$, $\rho(x,y) = 1$ if $r(x,y) < \frac{\eta}{2}$ and $\rho(x,y) = 0$ if $r(x,y) > \eta$. This function contains $\eta$ as a parameter and can be written in the form $\rho = \sigma(\frac{r}{\eta})$ where $\sigma$ is a once for all determined function of one argument.

Now, we set $A = -\frac{r^2(x,y)}{2}$, we define $A_{i,j}$ and $A_{i_1 \dots i_p, j_1 \dots j_p}$ by the same formulae as in the case of the Euclidean space and we call <u>parametrix of degree p</u> the double form

$$\omega_p(x,y) = \frac{r^{2-n}(x,y)}{s_n(n-2)}\rho(x,y)\sum_{\substack{i_1<\dots<i_p\\ j_1<\dots<j_p}} A_{i_1\dots i_p,j_1\dots j_p}\, dx^{i_1}\wedge\dots\wedge dx^{i_p} dy^{j_1}\wedge\dots\wedge dy^{j_p}.$$

For $p = 0$, of course, the sum has to be replaced by 1.

We have supposed $n > 2$. For $n = 2$, $\frac{r^{2-n}}{n-2}$ has to be replaced by $-\log r$.

When there can be no misunderstanding, we shall write $\omega(x,y)$ instead of $\omega_p(x,y)$.

The parametrix has the following properties.

1) If $\varphi$ is $C^\infty$, $\Omega\varphi = \int \omega(x,y)\wedge *\varphi(y)$ is $C^\infty$.

We must assume, of course, that the integral $\Omega\varphi$ is convergent, and also that $\omega(x,y)$ is $C^\infty$ if $y$ belongs to the carrier of $\varphi$. These two conditions are satisfied if the carrier of $\varphi$ is contained in the compact set $K$ which appeared in the definition of the parameter $\eta$ in $\rho$.

Since $r(x,y) = r(y,x)$, the parametrix is also symmetric and the operator $\Omega$ is selfadjoint:

$$(\Omega\varphi, \psi) = (\varphi, \Omega\psi)$$

2) The form $q(x,y) = -\Delta_x\omega(x,y)$ is $O(r^{2-n})$, i.e. the coefficients of $\frac{1}{r^{2-n}}q(x,y)$ remain bounded when $r$ tends to zero (for $n > 2$; for $n = 2$, $r^{2-n}$ must be replaced by $\log r$).

3) Let us set

$$Q\varphi = \int q(x,y)\wedge *\varphi(y) \quad \text{and} \quad Q'\varphi = \int q(y,x)\wedge *\varphi(y) .$$

If $\varphi$ satisfies the conditions stated in 1) and if its carrier is compact, we have

$$\text{(I)} \qquad \Omega \Delta \varphi = \varphi - Q'\varphi$$

If $\varphi$ satisfies the conditions stated in 1) (its carrier need not be compact), we have

$$\text{(II)} \qquad \Delta \Omega \varphi = \varphi - Q\varphi$$

These properties are proved in: P. Bidal et G. de Rham, Les formes différentielles harmoniques, Commentarii Mathematici Helvetici, Volumen 19 (quoted in the following CMH). Part of the proof can be simplified along the following lines.

Write $\dot{x}^i = \frac{dx^i}{dt}$, $\ddot{x}^i = \frac{d^2x^i}{dt^2}$, and consider the differential equations of the geodesic, or, considering t as the time, of a moving particle on the Riemann space

$$\ddot{x}^i + \sum_{j,\ell} \Gamma^i_{j\ell}\, \dot{x}^j \dot{x}^\ell = 0$$

If x and y are two given points near enough to each other, there is a well determined solution x(t) of these equations such that x(0) = x and x(1) = y. Let $\xi = \dot{x}(0)$ and $-\eta = \dot{x}(1)$ be the initial and final velocities of the particle. $\xi$ is a vector at x, $\eta$ a vector at y, their components are functions $C^\infty$ of x and y, and since

$$r^2(x,y) = \sum_i \xi_i \xi^i = \sum_i \eta_i \eta^i$$

we see that $r^2(x,y)$ is $C^\infty$ (for x and y near enough).

By considering the variation of the integral $r(x,y) = \int ds$ taken along the geodesic from x to y, we get

$$dA = -r(x,y)dr(x,y) = \sum_i \xi_i dx^i + \sum_j \eta_j dy^j$$

and consequently

$$\frac{\partial A}{\partial x^i} = \xi_i \, , \quad \frac{\partial A}{\partial y^j} = \eta_j \, , \quad A_{i,j} = \frac{\partial \xi_i}{\partial y^j} = \frac{\partial \eta_j}{\partial x^i}$$

Now, to prove 1), one remarks that $\frac{\partial A}{\partial x^i} + \frac{\partial A}{\partial y^i} = \xi_i + \eta_i = O(r^2)$ and one follows CMH p.32-34.

To prove 2), it is convenient to use $\eta^1 \ldots \eta^n$ as coordinates of x while y is considered as fixed. These are the normal coordinates of x with origin at y. See: O. Veblen, Invariants of Quadratic Differential Forms, Chapter VI. One shows that the covariant derivative of the vector $A_{i,j}$ ($i = 1 \ldots n$, j and y fixed) vanishes at the point y and that $\Delta r^{2-n} = O(r^{2-n})$. Then, if we consider y and the products $dy^{i_1} \wedge \ldots \wedge dy^{i_p}$ as fixed, we can write $\omega(x,y) = r^{2-n} \{a_{i_1 \ldots i_p}\}$ and we apply the last formula of §15, which gives the proof of 2). Besides, for our actual purpose, it would be sufficient to prove that $q(x,y) = O(r^{1-n})$ and this is still easier.

To prove 3), one proves Formula I as in CMH p.27-31. Formula II then follows from it, on account of the fact that $\Delta \Omega$ is adjoint to $\Omega \Delta$, i.e. that, if $\varphi$ has a compact carrier, $(\Omega \Delta \varphi, \psi) = (\varphi, \Delta \Omega \psi)$.

The parametrix which is considered here was introduced by H.Kneser and has been used by Hodge and by H. Weyl in proving the existence theorem for harmonic integrals. See: H. Weyl, On Hodges' Theory of Harmonic Integrals, Annals of Mathematics, Vol. 44, 1943, p.1. It is used in a different way , for the same purpose, in CMH and in: G. de Rham, Sur la théorie des formes différentielles harmoniques, Annales de l'Université de Grenoble, Année 1946, p. 135. We shall follow this last paper in §18. In §17, we shall use the parametrix to construct an elementary solution.

## §17. The Equation $\Delta\mu = \beta$ in a Small Domain. Elementary Solution.

Let us consider a domain D in the Riemann space M. D can itself be considered as a Riemann space and we can apply to it the formulae I and II, in which the integrals defining $\Omega$, Q and Q' have to be extended to D instead of to M.

Now, in order to find a solution $\mu$ of the equation $\Delta\mu = \beta$, where $\beta$ is a given form in D, according to E.E.Levis' method, we set $\mu = \Omega\xi$. On account of Formula II, we get for $\xi$ the integral equation

$$\xi - Q\xi = \beta.$$

We shall show that, if D is small enough, this integral equation can be solved by the Liouville - Neumann iteration process, based upon the identity

$$(1 - Q)(1 + Q + \dots + Q^{m-1}) = 1 - Q^m$$

Suppose D is contained in the domain of the coordinate system $x^1 \dots x^n$ and let us denote by $|\varphi|$ the upper bound of the moduli of all coefficients of the form $\varphi$ in D. We have an inequality of the form

$$|Q\varphi| \leq k|\varphi|$$

in which k depends only on D and not on $\varphi$. If D is small enough, we have $k < 1$, and, since $|Q^m\varphi| \leq k^m|\varphi|$, the series

$$\xi = \beta + Q\beta + \dots + Q^h\beta + \dots$$

is uniformly convergent and, on account of the above identity, gives the solution of our integral equation.

Let us set $\xi = \beta + P\beta$. The operator P is an integral operator whose kernel is

$$p(x,y) = q(x,y) + Qq(x,y) + \dots + Q^h q(x,y) + \dots .$$

The solution $\mu$ of our differential equation is given by

$$\mu = \Omega\xi = (\Omega + \Omega P)\beta .$$

Since the kernel of the integral operator $\Omega P$ is $O(r^{4-n})$ (See CMH, p.46-47), we see that the kernel $\gamma(x,y)$ of the operator $\Gamma = \Omega + \Omega P$ has essentially the same singularity as $\omega(x,y)$.

Thus, for each form $\beta$ in D, we have

$$\Delta\Gamma\beta = \beta \quad \text{i.e.} \quad \Delta\int\gamma(x,y)\wedge{*}\beta(y) = \beta(x).$$

On the other hand, in the same way as in Formula II §16, we get

$$\Delta\Gamma\beta = \beta(x) + \int\Delta_x\gamma(x,y)\wedge{*}\varphi(y)$$

and consequently $\Delta_x\gamma(x,y) = 0$ for $x \neq y$. Hence, the double form $\gamma(x,y)$ is an elementary solution.

The existence of an elementary solution corresponding to the operator $\Delta$ was first proved by K. Kodaira (loc.cit.) in the case of an analytical Riemann Space, with the help of a method of Hadamard.

Now, the argument used in §11 can be applied and we get the following

Theorem. If $\Delta T$ is $C^\infty$ in a domain D, T is also $C^\infty$ in D .

In particular, a harmonic current is $C^\infty$, i.e. equal to a form $C^\infty$, which of course has to be harmonic.

## §18. The Equation $\Delta\mu = \beta$ on a Compact Space.

Let $\beta$ be a form on the compact space M. If there exists a form $\mu$ such that $\Delta\mu = \beta$, we have, for each harmonic form $\varphi$,

$$(\beta, \varphi) = (\Delta\mu, \varphi) = (\mu, \Delta\varphi) = 0 .$$

Thus, for the existence of a form $\mu$ such that $\Delta\mu = \beta$, it is necessary that $\beta$ be orthogonal to all harmonic forms. We shall now prove that this condition is also sufficient.

A harmonic form $\varphi$, since it satisfies $\Delta\varphi = 0$, also satisfies $\Omega\Delta\varphi = 0$, or, on account of Formula I, $\varphi - Q'\varphi = 0$. Now, according to the theory of Fredholm, this equation has only a finite number of linearly independent solutions, which form a linear space E. Consequently, there are only a finite number of linearly independent harmonic forms. They constitute a linear subspace E' of E. Let E" be the orthogonal complement of E' in E.

If we try to solve the equation $\Delta\mu = \beta$ by setting $\mu = \Omega\xi$, we get as in §17 the integral equation $\xi - Q\xi = \beta$, and according to the theory of Fredholm, this equation can be solved if, and only if, $\beta$ is orthogonal to all solutions of the associated homogeneous equation $\varphi - Q'\varphi = 0$, i.e. if $\beta$ is orthogonal to E. But $\beta$ is supposed to be orthogonal to E' and it may not be orthogonal to E", so that our integral equation may be insoluble. Nevertheless, we can solve our differential equation in the following way.

If $\phi_1 \in E''$ and $\phi_1 \neq 0$, $(\Delta^2\phi_1,\phi)$ cannot vanish for each $\phi \in E''$, because $(\Delta^2\phi_1,\phi_1) = (\Delta\phi_1,\Delta\phi_1)$ and this is different from zero, otherwise we would have $\Delta\phi_1 = 0$, which is impossible since $\phi_1$ is orthogonal to $E'$. In this way, to each $\phi_1 \in E''$ there corresponds a linear function $(\Delta^2\phi_1,\phi)$ defined on $E''$, which cannot vanish identically unless $\phi_1 = 0$. Since this correspondence is linear and since the dimensionality of $E''$ is finite, it follows from the above that, conversely, each linear function on $E''$ can be represented by $(\Delta^2\phi_1,\phi)$ with a convenient $\phi_1$.

In particular, there is a $\phi_1$ such that $(\beta,\phi) = (\Delta^2\phi_1,\phi)$ for each $\phi \in E''$. Then $\beta - \Delta^2\phi_1$ is orthogonal to $E''$, and also to $E'$ because $\beta$ and $\Delta^2\phi_1$ are. Consequently, the integral equation $\xi - Q\xi = \beta - \Delta^2\phi_1$ can be solved and $\mu = \Omega\xi + \Delta\phi_1$ satisfies our differential equation $\Delta\mu = \beta$.

Thus, we have proved the following

Theorem. On a compact Riemann Space, there are only a finite number of linearly independent harmonic forms, and the equation $\Delta\mu = \beta$ is possible if and only if $\beta$ is orthogonal to all harmonic forms.

## §19. The Operators H and G on a Compact Space.

To each current T, on a compact Riemann space, we can associate a well determined harmonic form $h_1$ such that $(T,h) = (h_1,h)$ for each harmonic form h. We shall call it the harmonic part of T and we shall write $h_1 = HT$.

Clearly, T is an harmonic form if and only if $T = HT$. The linear operator H is a projector: It is self adjoint, $(HT,S) = (HT,HS) = (T,HS)$, and $H^2 = H$. It is orthogonal to d and to $\delta$,

$$dH = Hd = 0 , \qquad \delta H = H\delta = 0 .$$

As a matter of fact, $dHT = 0$ and $\delta HT = 0$ because $HT$ is closed and coclosed, $HdT = 0$ because $(HdT,h) = (dT,h) = (T,\delta h) = 0$ for each harmonic form $h$ and $H\delta T = 0$ because $(H\delta T,h) = (\delta T,h) = (T,dh) = 0$.

Moreover, the operator $H$ is permutable with $*$,

$$*H = H*$$

because $(H*T,*h) = (*T,*h) = (T,h) = (HT,h) = (*HT,*h)$.

Now, according to the theorem of §18, the equation $\Delta\mu = \varphi - H\varphi$ always has a solution $\mu$, and there is only one solution which is orthogonal to all harmonic forms, i.e. such that $H\mu = 0$. We shall denote this solution by $G\varphi$. The relation $\mu = G\varphi$ is thus equivalent to the two equations $\Delta\mu = \varphi - H\varphi$ and $H\mu = 0$.

The linear operator $G$ is permutable with $d$, $\delta$ and $*$, and is self adjoint.

This follows immediately from the definition of $G$ on account of the fact that $\Delta$ and $H$ are permutable with $d$, $\delta$ and $*$ and are self adjoint.

Hence, $G$ is also permutable with $\Delta$.

For any current $T$, we now define $GT$ by setting

$$(GT,\varphi) = (T,G\varphi).$$

From this definition, it follows that $(\Delta GT + HT,\varphi) =$ $= (T,G\Delta\varphi + H\varphi) = (T,\varphi)$ for each form $\varphi$ and consequently $\Delta GT = T - HT$ for each current $T$. In the same way, we see that $HGT = 0$. Thus we have the following generalization of the theorem of §18.

Theorem I. To each current T, there is one and only one current S which satisfies the equations

$$\triangle S = T - HT, \quad HS = 0.$$

This is the current $S = GT$. If T is $C^\infty$ in D, GT is also $C^\infty$ in D.

The last statement follows from the theorem of §17.

For any current T, let us set $T_1 = d\delta GT$, $T_2 = \delta d\, GT$, $T_3 = HT$. We have the decomposition formula

$$T = T_1 + T_2 + T_3.$$

If T is closed, $T_2 = 0$ because $T_2 = \delta GdT$. Conversely, if $T_2 = 0$, T is closed because $T_1$ and $T_3$ are always closed. As $T_1$ bounds $\delta GT$, we see that a closed current is homologous to its harmonic part.

If T is homologous to zero, $T = dU$, its harmonic part $T_3 = HT = HdU = 0$ because $Hd = 0$. Then T bounds the current $\delta GT$, which is cohomologous to zero. This is the only current cohomologous to zero and bounded by T, because if U were another one, $U - \delta GT$ would be closed and cohomologous to zero, consequently harmonic and, on account of $H\delta = 0$, identical to zero.

Let us further remark that, if T is $C^\infty$ in a domain D, as GT is $C^\infty$ in D, dGT, $\delta GT$, $T_1$ and $T_2$ will also be $C^\infty$ in D.

We have thus proved the following

Theorem II. 1) Each closed current T is homologous to a harmonic form, which is the harmonic part HT of T.

2) A current T which is homologous to zero bounds one and only one current U which is cohomologous to zero, and this current $U = \delta GT$ is $C^\infty$ in every domain in which T is $C^\infty$.

3) A closed current $T$ is homologous to zero if, and only if, its harmonic part $HT$ vanishes.

Let us remark that 1) and 2) contain Theorem A of §12 for compact manifolds. Theorem C of §12 follows from 3), because if $T[\varphi] = 0$ for each closed $\varphi$, $T[*h] = (T,h) = 0$ for each coclosed $h$, in particular for each harmonic $h$ and this means that $HT = 0$.

Of course, the integral equation method does not give a means for constructing chains and we cannot hope to prove Theorem B of §12 in this way. Nevertheless, in §20, with the help of the Poincaré Duality Theorem, we shall get part 2 of this Theorem B.

We shall here apply the above results to some general problems of construction of harmonic forms and currents.

<u>Problem 1.</u> Construction of a harmonic form with given periods relative to $r$ given linearly independent cycles.

Let $c_1,\ldots,c_r$ be $r$ cycles (i.e. closed chains), of the same dimension and of the same kind, no linear combination of which bounds a current. Then $Hc_1,\ldots,Hc_r$ are $r$ linearly independent harmonic forms and if $p_1,\ldots,p_r$ are $r$ arbitrary given numbers, we can always find a harmonic form $h$ such that $(Hc_i,h) = p_i (i=1,\ldots,r)$. The form $*h$ is harmonic and, since

$$\int_{c_i} *h = (c_i,h) = (Hc_i,h),$$

it has the given periods. Thus the theorem of Hodge, which asserts precisely the existence of such a form, is proved.

Let $R'_p$ be the maximal number of closed currents of degree p and of odd kind, no combination of which is homologous to zero. According to part 3 of Theorem II, $R'_p$ is also the maximal number of linearly independent harmonic forms of degree p and of odd kind, and, according to the theorem of §18, it is finite. Moreover, if $R''_p$ is the corresponding number for those of even kind and n the dimension of our compact space, on account of the correspondence $h \longrightarrow *h$ between harmonic forms of odd and of even kinds, we have $R'_p = R''_{n-p}$. In the case of an orientable space, $R'_p = R''_p = R_p$, we get the relation $R_p = R_{n-p}$ between Betti numbers which is an essential part of the Poincaré Duality Theorem.

Let us now consider two problems of constructing currents with given singularities.

Problem 2. Let E be a closed set of points in M, $\{U_i\}$ $(i=1,2,\ldots)$ an open covering of E and $T_i$ a current defined in $U_i$ such that, if CE is the complement of E, $T_i$ is harmonic in $U_i \cap CE$ and $T_i - T_j$ is harmonic in $U_i \cap U_j$ (for every i and j).

To be found is a current T, harmonic in CE, such that $T - T_i$ is harmonic in $U_i$ (for every i).

To solve this problem, we remark that there is a well determined current S such that $S = 0$ in CE and $S = \triangle T_i$ in $U_i$, for every i. Then the required current T, if it exists, is a solution of the equation

$$\triangle T = S$$

The condition of solubility is clearly $HS = 0$ and if it is satisfied a solution is given by $T = GS$.

Problem 3. With the same notations as in Problem 2, we suppose now that $T_i$ is closed and coclosed in $U_i \cap CE$ and that $T_i - T_j$ is closed and coclosed in $U_i \cap U_j$ (for every i and j).

To be found is a current T, closed and coclosed in CE, such that $T - T_i$ is closed and coclosed in $U_i$ (for every i).

There are two well determined currents $S_1$ and $S_2$, such that $S_1 = S_2 = 0$ in CE and $S_1 = dT_i$, $S_2 = \delta T_i$ in $U_i$. Then the required current T, if it exists, is a solution of the equations

$$dT = S_1, \qquad \delta T = S_2 .$$

The conditions of solubility are that $S_1$ be homologous to zero and $S_2$ cohomologous to zero, or, what is the same, $HS_1 = HS_2 = 0$. If these conditions are satisfied, a solution is given by $T = GdS_2 + G\delta S_1$.

This problem can also be reduced to the preceding one by setting $S = dS_2 + \delta S_1$.

The construction of harmonic functions and harmonic differentials with given singularities or periods on a Riemann surface and the analogous problems on a Riemann space, considered and solved by K. Kodaira, are particular cases of the above general problems.

## §20. Extension of the Scalar Product. Kronecker Index.

Up to now, the scalar product (S,T) has only been defined in the case where one of the two currents S and T is $C^\infty$. We shall now extend its definition to some other cases.

In the first place, if the carriers of S and T don't meet, we set $(S,T) = 0$.

Now let us suppose that $S = S' + S''$ and $T = T' + T''$, where $S''$ and $T''$ are $C^\infty$ while the carriers of $S'$ and $T'$ don't meet. Then we set

$$(S,T) = (S',T') + (S',T'') + (S'',T') + (S'',T'')$$

where of course $(S',T') = 0$ according to the above convention.

It is easy to see that the value thus obtained for $(S,T)$ is independent of the particular splittings of $S$ and $T$ into $S' + S''$ and $T' + T''$. But under what conditions are such splittings possible?

Let us consider the smallest closed set of points outside of which a distribution is $C^\infty$ and let us call it the <u>singular set</u> of the distribution. It is clear that, in the above splittings, $S$ and $S'$ have the same singular set, also $T$ and $T'$, and consequently the singular sets of $S$ and $T$ don't meet. Conversely, if the singular sets of $S$ and $T$ don't meet, we can find two functions $C^\infty$, $f$ and $g$, whose carriers don't meet, such that $f = 1$ in a neighborhood of the singular set of $S$ and $g = 1$ in a neighborhood of the singular set of $T$, and we get splittings satisfying to the above conditions by setting $S' = fS$, $T' = gT$, $S'' = S - S'$ and $T'' = T - T'$.

Hence, <u>the scalar product</u> $(S,T)$ <u>is defined by the above convention in the case where the singular sets of S and T don't meet.</u>

We can go further by using the decomposition formula of §19. Let us first remark that, if $T = T_1 + T_2 + T_3$ is the decomposition of $T$, the singular set of $T_1$ is identical to the singular set of $\delta T$, because $T_1 = dG\delta T$ and $\delta T = \delta T_1$, and the singular set of $T_2$ is identical to the singular set of $dT$ for a similar reason.

<u>Now, in the case where neither the carriers of $\delta S$ and $\delta T$ nor those of $dS$ and $dT$ meet, we can define</u> $(S,T)$ <u>by setting</u>

$$(S,T) = (S_1,T_1) + (S_2,T_2) + (S_3,T_3) ;$$

On account of our preceding remark, each scalar product in the right hand side is well determined, and it is easy to verify that in the case where the carriers of S and T don't meet and consequently (S,T) has already been defined, this formula is correct.

It is also easy to verify that the following formulae are still valid

$$(dS,T) = (S,\delta T), \quad (S,\Delta T) = (\Delta S,T), \quad (S,GT) = (GS,T),$$

$$(HS,T) = (S,HT), \quad (S_i,T) = (S_i,T_i) = (S,T_i),$$

and the defining formula is equivalent to

$$(S,T) = (G\delta S,\delta T) + (GdS,dT) + (HS,T) .$$

Let us now consider two chains of different kinds and of complementary dimensions, $c^p$ and $c^{n-p}$, and let us suppose that neither of them meets the boundary of the other. The conditions of our definition are satisfied for $S = *c^p$ and $T = c^{n-p}$, because the carrier of $\delta S = \pm * dc^p$ does not meet the carrier of T and the carrier of $dT = dc^{n-p}$ does not meet the carrier of S. Hence, $(*c^p,c^{n-p})$ is well determined.

The number $(*c^p,c^{n-p})$ has the following properties.

1) It is a bilinear function of $c^p$ and $c^{n-p}$ which is equal to zero if $c^p$ and $c^{n-p}$ don't meet.

2) For $p = 0$, if $c^p$ is a 0-simplex of odd kind consisting in a point $x_o$ with the coefficient +1 and if $c^{n-p}$ is the whole manifold M which, considered as a current, is equal to the function 1, since $(*x_o,1) = (x_o,*1) = 1[x_o] = 1$, we have

$$(*x_o,M) = 1 .$$

3) On account of the formulae $(\delta S,T) = (S,dT)$ and $\delta *c^p=(-1)^{n-p+1}*dc^p$, we have

$$(*dc^p,c^{n-p+1}) = (-1)^{n-p+1}(*c^p,dc^{n-p+1}) .$$

Now, the Kronecker index $I(c^p,c^{n-p})$ or algebraic number of intersections of the two chains $c^p$ and $c^{n-p}$, considered in Topology, has the same properties. Since they are characteristic (for $p = 0$ this is immediate and on account of 3) the case of any $p > 0$ is reduced to $p-1$), we must have

$$(*c^p,c^{n-p}) = I(c^p,c^{n-p}) .$$

Now, Poincaré's Duality Theorem asserts that a closed chain $c^p$ bounds a chain if $I(c^p,c^{n-p}) = 0$ for each closed chain $c^{n-p}$.

Part 2 of Theorem B of §12 can be deduced from this as follows. Suppose $c^p$ bounds a current; then $S = *c^p$ is cohomologous to zero, consequently $S = S_2$ and if $c^{n-p}= T$ is closed, $T_2= 0$ and $(S,T) = 0$, i.e., $I(c^p,c^{n-p}) = 0$. Thus the Poincaré condition is satisfied and $c^p$ must bound a chain.

## §21. The Kernels of the Operators H and G. Explicit Formula for the Kronecker Index.

Let us consider a contravariant p-vector at a fixed point y in the compact Riemann space M, with components $Y^{i_1\ldots i_p}$. It determines a current Y defined by

$$(Y,\varphi) = \sum_{i_1<\ldots< i_p} a_{i_1\ldots i_p}(y)\, Y^{i_1\ldots i_p}$$

where $\varphi(x) = \sum_{i_1<\ldots< i_p} a_{i_1\ldots i_p}(x)\, dx^{i_1}\wedge\ldots\wedge dx^{i_p}$ .

Let us replace $Y^{i_1 \ldots i_p}$ by $dy^{i_1} \wedge \ldots \wedge dy^{i_p}$. We have

$$(Y(x), \varphi(x)) = \varphi(y).$$

The carrier of Y is the single point y.

Suppose $h_1, \ldots, h_r$ is an orthonormal basis for the harmonic forms of degree p, $(h_i, h_j) = \delta_i^j$, and let us set

$$h(x,y) = \sum_i h_i(x)\, h_i(y) .$$

For fixed y, h(x,y) is harmonic in x, and for each harmonic form $\varphi(x)$ we have

$$\varphi(y) = (h(x,y), \varphi(x)) = (Y(x), H\varphi(x)) = (HY(x), \varphi(x)).$$

This means that <u>the form</u> h(x,y) <u>is the harmonic part of the current Y.</u>

Moreover, for <u>each</u> form $\varphi$, also not harmonic, we have

$$H\varphi(y) = (Y(x), H\varphi(x)) = (HY(x), \varphi(x)) = (h(x,y), \varphi(x))$$

or, on account of the symmetry of h(x,y),

$$H\varphi(x) = \int h(x,y) \wedge * \varphi(y)$$

We see that h(x,y) = HY(x) <u>is the kernel of the operator H.</u> There is one such a kernel $h_p(x,y)$ for each degree p $(0 \leq p \leq n)$ and for each kind (even or odd).

Let us now consider the current GY(x). Since $\Delta GY = Y - HY$, GY is $C^\infty$ outside the point y, i.e. GY is equal to a form g(x,y) which satisfies the equation $\Delta_x g(x,y) = -h(x,y)$ for $x \neq y$.

In a neighborhood D of y, consider an elementary solution $\gamma(x,y)$ and the current $\Gamma(x)$ equal in D to $\gamma(x,y)$, where Y is fixed. From the properties of the elementary solution it follows immediately that $\Delta\Gamma(x) = Y(x)$ in D. Consequently, $\Delta(GY - \Gamma) = -HY$ in D, $GY - \Gamma$ is $C^{\infty}$ in D. $g(x,y)$ has the same singularity as $\gamma(x,y)$ for $x = y$ and $GY = g(x,y)$ in the whole space M.

Moreover, for each form $\varphi$, we have

$$G\varphi(y) = (Y(x), G\varphi(x)) = (GY(x), \varphi(x)) = \int g(x,y) \wedge * \varphi(x) \ .$$

We see that $g(x,y) = GY(x)$ is the kernel of the operator G.

This double form $g(x,y)$ is called the Green's form. As a matter of fact, we have one Green's form $g_p(x,y)$ for each degree p $(0 \leq p \leq n)$ and for each kind (even or odd). It has the following properties.

1) It is symmetric, $g(x,y) = g(y,x)$, because G is selfadjoint.

2) Since $G\delta = \delta G$, $(g_{p-1}(x,y), \delta\varphi(y)) = (\delta_x g_p(x,y), \varphi(y))$, and, as $(g_{p-1}(x,y), \delta\varphi(y)) = (d_y g_{p-1}(x,y), \varphi(y))$, for any form $\varphi$, we have

$$\delta_x g_{p-1}(x,y) = d_y g_p(x,y) \ .$$

Consequently, $d_x \delta_x g_p(x,y) = d_x d_y g_p(x,y)$ which form is obviously symmetric:

$$d_x \delta_x g_p(x,y) = d_y \delta_y g_p(x,y).$$

3) As already noticed, $\Delta_x g_p(x,y) = -h_p(x,y)$, i.e.

$$d_x \delta_x g_p(x,y) = -\delta_x d_x g_p(x,y) - h_p(x,y) \qquad \text{for } x \neq y.$$

4) From the relation $*G = G*$ there follows

$$g_p(\overset{*}{x},y) = (-1)^{pn+p} g_{n-p}(x,\overset{*}{y})$$

where $g(\overset{*}{x},y)$ means the adjoint with respect to x of $g(x,y)$ and $g(x,\overset{*}{y})$ the adjoint with respect to y. In this relation, $g_p(x,y)$ and $g_{n-p}(x,y)$ are the Green's forms of degrees p and n-p and of different kinds. We have further

$$g_p(\overset{*}{x},\overset{*}{y}) = g_{n-p}(x,y)$$

and we see that the Green's form of degree n-p is the adjoint, with respect to x and to y, of the Green's form of degree p and of different kind.

We shall now write an explicit formula for the Kronecker Index, with the help of the form $d_x\delta_x g_p(x,\overset{*}{y})$.

The defining formula for (S,T) given in §20 can be written as follows:

$$(S,T) = (d\delta GS,T) + (S,\delta dGT + HT).$$

Replacing S by $*c^p$ and T by $c^{n-p}$, we get

$$(*c^p,c^{n-p}) = (d\delta{*}Gc^p,c^{n-p}) + (*c^p,\delta dGc^{n-p} + Hc^{n-p}).$$

Outside of $c^p$, the current $Gc^p$ is equal to the following form, in which the variable point is y,

$$Gc^p(y) = (Y,Gc^p) = (c^p,GY) = \int_{c^p_x} g_{n-p}(\overset{*}{x},y)$$

where $c^p_x$ means that the integral is to be taken over $c^p$ with respect to x. We have also, outside of $c^p$,

$$d\delta{*}Gc^p = \int_{c^p_x} d_y\delta_y g_{n-p}(\overset{*}{x},\overset{*}{y}) = \int_{c^p_x} d_x\delta_x g_p(x,y)$$

on account of the above properties 2) and 4). But the singular set of $d\delta{*}Gc^p = *\delta Gdc^p$ is contained in the boundary $Bc^p = \pm dc^p$, and this form is $C^\infty$

outside of $Bc^p$, as we can verify by changing the last integral into the integral of $\delta_x g_p(x,y)$ over $Bc^p$. Consequently, as $c^{n-p}$ does not meet $Bc^p$, we have

$$(1) \qquad (d\delta * Gc^p, c^{n-p}) = \int_{c_y^{n-p}} \int_{c_x^p} d_x \delta_x g_p(x, \overset{*}{y}) .$$

By interchanging p and n-p, x and y, we have a similar expression for $(d\delta * Gc^{n-p}, c^p)$. Since this is equal to $(-1)^{pn+p}(*c^p, \delta d * Gc^{n-p})$ and on account of the relation

$$d_y \delta_y g_{n-p}(y, \overset{*}{x}) = (-1)^{pn+p} \delta_x d_x g_p(x, \overset{*}{y})$$

which follows from the above properties 1), 2) and 4), we get

$$(2) \qquad (*c^p, \delta d G c^{n-p}) = \int_{c_x^p} \int_{c_y^{n-p}} \delta_x d_x g_p(x, \overset{*}{y})$$

In a similar way, we get

$$(3) \qquad (*c^p, Hc^{n-p}) = \int_{c_x^p} \int_{c_y^{n-p}} h_p(x, \overset{*}{y}) .$$

By adding (1), (2) and (3), we have an explicit formula for the Kronecker index. If $c^p$ is closed, (1) vanishes; if $c^{n-p}$ is closed, (2) vanishes; if both $c^p$ and $c^{n-p}$ are closed, the Kronecker index is equal to (3).

On account of the above property 3), we can transform the sum of (2) and (3) and we eventually get

$$I(c^p, c^{n-p}) = \int_{c_y^{n-p}} \int_{c_x^p} - \int_{c_x^p} \int_{c_y^{n-p}} d_x \delta_x g_p(x, \overset{*}{y})$$

Let us remark that, for $x = y$, the form $d_x\delta_x g_p(x,\overset{*}{y})$ is $O(r^{-n})$; this is the reason why the order of the integrations cannot be changed, except if $c^p$ and $c^{n-p}$ dont meet, in which case $I(c^p,c^{n-p}) = 0$.

This formula (which is in my paper in Annales de Grenoble 1946) is also valid in Euclidean space, if we take for $g_p(x,y)$ the elementary solution considered in §16; it contains as a particular case the Gauss formula for the looping coefficient. It can be generalized to any non-compact Riemann space, as we shall show in the next §.

## §22. Method of Orthogonal Projection.

In an n-dimensional Riemann space M, which will not be assumed to be compact, let us consider all forms $C^\infty$ with a compact carrier, of a given degree and a given kind. They constitute a linear space F.

We shall say that a series of forms $\omega_h \in F$ $(h = 1,2,\ldots)$ is convergent in the mean, if

$$\lim_{m,\ell = \infty} (\omega_m - \omega_\ell, \omega_m - \omega_\ell) = 0 .$$

If this condition is satisfied, there is a well determined current T such that

$$(T,\varphi) = \lim_{h=\infty} (\omega_h, \varphi) \qquad \text{for each} \quad \varphi \in F.$$

Such a current, limit of a series which is convergent in the mean, will be said to be square integrable.

The scalar product of two such currents can be defined by setting $(T,S) = \lim_{h=\infty} (\omega_h, \varphi_h)$, where $\varphi_h$ converges to S in the same manner as $\omega_h$ to T, and it can easily be shown that the linear space $\bar{F}$ of all square integrable currents is a complete Hilbert space. $\bar{F}$ is the closure of F.

Let $F_1$ be the subspace of $F$ consisting of all forms which are differentials of forms $C^\infty$ with compact carriers, let $\bar{F}_1$ be the closure of $F_1$ in $\bar{F}$, $F_2$ the subspace of $F$ consisting of all forms which are codifferentials of forms $C^\infty$ with compact carriers and $\bar{F}_2$ the closure of $F_2$ in $\bar{F}$. On account of the formulae (1) and (2) of §14, a current is closed if and only if it is orthogonal to $F_2$ and it is coclosed if and only if it is orthogonal to $F_1$. Consequently, $\bar{F}_1$ and $\bar{F}_2$ are orthogonal and the orthogonal complement $F_3$ to their direct sum in $\bar{F}$ consists of all square integrable currents which are closed and coclosed.

On account of the completeness of the Hilbert space, we get

Theorem I. Each square integrable current $\alpha$ can be decomposed in a unique way into a sum

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3$$

where $\alpha_1 \in \bar{F}_1$, $\alpha_2 \in \bar{F}_2$, $\alpha_3 \in F_3$.

Moreover, $\alpha_3$ is $C^\infty$ and $\alpha_1$ and $\alpha_2$ are $C^\infty$ in each domain in which $\alpha$ is $C^\infty$.

The last statement follows from the theorem of §17, because $\alpha_3$ is harmonic, $\Delta \alpha_1 = d\delta\alpha$ and $\Delta \alpha_2 = \delta d\alpha$ .

This theorem together with the above proof are due to K. Kodaira (loc. cit.). On the general method used here, see: H. Weyl, Method of Orthogonal Projection in Potential Theory, Duke Math. Journal 7(1940), pp.411-444. Previously, the equivalent Minimum Method was introduced by W.V.D.Hodge in his first papers on the theory of harmonic integrals (see references in his book).

We can now extend this decomposition formula to <u>the class C of all currents T such that</u> $T[\varphi]$ <u>is convergent for every form $\varphi$ which is $C^\infty$ and square integrable.</u>

We have only to set

$$(T_i, \varphi) = (T, \varphi_i) \qquad (i = 1,2,3)$$

where $\varphi = \varphi_1 + \varphi_2 + \varphi_3$ is the decomposition of $\varphi$. Even if the carrier of $\varphi$ is compact, the carrier of $\varphi_i$ need not be compact, but, as $T \in C$, $(T, \varphi_i)$ is always convergent and the definition is correct.

Clearly, $T = T_1 + T_2 + T_3$, $T_1$ and $T_3$ are orthogonal to $F_2$ and consequently closed, $T_2$ and $T_3$ are coclosed, $T_3$ is an harmonic form (closed and coclosed) which is square integrable because $T_3 \in C$ and each form $C^\infty$ belonging to C is square integrable, as can be easily proved. Moreover, in each domain in which T is $C^\infty$, $T_1$ and $T_2$ are also $C^\infty$, because $\Delta T_1 = d\delta T$ and $\Delta T_2 = \delta d T$.

The class C obviously contains all currents with a compact carrier and all square integrable currents. It contains the current Y, defined in §21, and we can decompose Y into

$$Y = Y_1 + Y_2 + Y_3 ,$$

where each component has to be $C^\infty$ outside of y.

In the case of a compact space, as we have seen,

$$Y_1 = d_x \delta_x g(x,y) \quad , \qquad Y_2 = \delta_x d_x g(x,y) \quad , \qquad Y_3 = h(x,y) .$$

In the case of a non-compact space, we have not proved the existence of a Green's form $g(x,y)$, but we have the forms (or currents) $Y_1$, $Y_2$ and $Y_3$.

If $h_1, h_2, \ldots$ is a complete orthonormal system of square integrable closed and coclosed forms, we still have

$$Y_3 = h(x,y) = \sum_i h_i(x)h_i(y).$$

The series may have an infinite number of terms, but it is always convergent, and $h(x,y)$ is still the kernel of the projector H defined by $H\alpha = \alpha_3$.

Let us set $e(x,y) = Y_1$, $f(x,y) = Y_2$. These expressions are symmetric double forms, $C^\infty$ for $x \neq y$, which can be considered as the kernels of the projectors which project T on $T_1$ and $T_2$ respectively, but they are in general not integrable in the neighborhood of $x = y$.

The forms $e(x,y)$, $f(x,y)$ and $h(x,y)$ are nothing else (sign excepted) than the "harmonic fields" $e^{**}(x,\xi)$, $e(x,\xi)$ and $\omega(x,\xi)$ introduced by K. Kodaira (loc.cit. p. 647 and 657). The form $h(x,y)$ is closely related to the Kernel Function introduced by S. Bergmann (see: S. Bergmann, Sur la fonction-noyau d'un domaine et ses applications dans la théorie des transformations pseudo-conformes, Paris 1948, Gauthier-Villars).

As a matter of fact, we have a form $e(x,y) = e_p(x,y)$ for each degree p, and for each kind. Now, the argument of §21 leads in the same way to the formula for the Kronecker index in a non-compact space

$$I(c^p,c^{n-p}) = \int_{c_y^{n-p}} \int_{c_x^{n-p}} e_p(x,\overset{*}{y}) - \int_{c_x^p} \int_{c_y^{n-p}} e_p(x,\overset{*}{y}) \quad .$$

# Chapter IV.

## Homotopy and Smoothing.

### §23. Homotopy and Currents.

Let us consider two manifolds $C^\infty$, M and $M_1$, and a mapping $f_t$ of $M_1$ in M, $x = f_t y$ with $y \in M_1$ and $x \in M$, depending on a parameter t $(0 \leq t \leq 1)$, such that $f_t y$ is $C^\infty$ with respect to y and t together. We shall denote by $M_2$ the topological product of $M_1$ and the interval $0 \leq t \leq 1$ and by f the mapping $x = f(y,t) = f_t y$ of $M_2$ in M.

Suppose $y^1,\ldots,y^m$ are local coordinates in $M_1$. Then $y^1,\ldots,y^m,t$ are local coordinates in $M_2$ and any form in $M_1$, whose coefficients may depend upon t, can also be considered as a form in $M_2$ whose expression does not contain dt. Any form $\alpha$ in $M_2$ can be represented by the expression

$$\alpha = \alpha' + dt \wedge \alpha''$$

where $\alpha'$ and $\alpha''$ are forms in $M_1$ of degrees p and p-1 respectively with coefficients depending upon t, and such a representation is unique.

In particular, if $\varphi$ is a p-form in M, for $\alpha = f^*\varphi$, we have $\alpha' = (f^*\varphi)' = f_t^*\varphi$. Let us set

$$\alpha'' = (f^*\varphi)'' = X\varphi .$$

Then

$$(1) \qquad f^*\varphi = f_t^*\varphi + dt \wedge X\varphi$$

The differentiation operator d in $M_2$ splits into

$$d = d' + dt \wedge \frac{\partial}{\partial t} \qquad \text{where} \qquad d' = \sum_i dy^i \wedge \frac{\partial}{\partial y^i}$$

There follows from (1)

$$d\, f^*\varphi = d' f_t^*\varphi + dt \wedge \left(\frac{\partial}{\partial t} f_t^*\varphi - d'X\varphi\right)$$

and consequently, since $df^*\varphi = f^*d\varphi$, on account of (1) applied to $d\varphi$ instead of $\varphi$,

$$Xd\varphi = (f^*d\varphi)'' = \frac{\partial}{\partial t} f_t^*\varphi \;-\; d'X\varphi$$

If we consider $X\varphi$ as a form in $M_1$ and t as a parameter, we can write $dX\varphi$ for $d'X\varphi$. By integrating with respect to t from 0 to 1 and setting

$$\int_0^1 X\varphi \cdot dt = F^*\varphi \quad ,$$

we eventually get

(2) $$f_1^*\varphi - f_0^*\varphi = d\,F^*\varphi + F^*d\varphi \quad .$$

Clearly, if $\varphi$ is a p-form $C^\infty$ in M, $F^*\varphi$ is a (p-1)-form $C^\infty$ in $M_1$; for $p = 0$, $F^*\varphi = X\varphi = 0$.

Now, if T is a current of dimension p in $M_1$, with a compact carrier, we define the current FT of dimension p+1 in M by

$$FT[\varphi] = T[F^*\varphi] \; .$$

This defining formula can also be applied to a current T with a non compact carrier, provided $f_t$ has the following property: for any compact set $K \subset M$ $f^{-1}K$ is compact; then the carrier of $F^*\varphi$ is compact if that of $\varphi$ is. This holds in particular if $f_t$ is a homeomorphism.

This operator F can also be defined in the following way. Let $T_2$ be the tensor product, in $M_2$, of T and the current 1 in the interval $0 \leq t \leq 1$, defined by

$$T_2[\alpha] = T[\int_0^1 \alpha'' dt] .$$

We have $fT_2[\varphi] = T_2[f^*\varphi] = T[F^*\varphi]$ and consequently $FT = fT_2$ .

Now, on account of the definition of the differential of a current, there follows from (2) the following formula, in which m, n and p are the dimensionalities of $M_1$, M and T respectively

$$(3) \qquad f_1T - f_0T = (-1)^{n-p} dFT - (-1)^{m-p} FdT \quad .$$

In the particular case where T is a chain, this formula is well known in Topology. We can say that the currents $f_1T$ and $f_0T$ are homotopic, and we see that two homotopic closed currents are homologous.

## §24. Smoothing of Currents in Euclidean Space.

In Euclidean space E, let $s_y$ be the translation $s_y x = x + y$ which carries the origin into the point y. Let $\rho(x)$ be a function $C^\infty$, containing a parameter $\varepsilon > 0$, $0 \leq \rho(x)$, with a carrier contained in the sphere of centre 0 and radius $\varepsilon$ , such that, if dx is the volume element and $r(x) = \rho(x)dx$, $\int r(x) = 1$.

Now, to each form $\alpha$ in E we associate the forms

$$R^*_\varepsilon \alpha = \int s^*_y \alpha \cdot r(y) \qquad \text{and} \qquad R_\varepsilon \alpha = \int s_y \alpha \cdot r(y)$$

where, of course, the integration is taken with respect to y. Let us remark that $s_y \alpha = (s_y^{-1})^* \alpha$ , which form is $C^\infty$ if $\alpha$ is, and that

$$R_\varepsilon \alpha [\varphi] = \alpha [R^*_\varepsilon \varphi] .$$

<u>For any current T we define $R_\varepsilon T$ by</u>

$$R_\varepsilon T[\varphi] = T[R_\varepsilon^* \varphi]$$

Clearly the operators $s_y$, $s_y^*$, $R_\varepsilon^*$ and $R_\varepsilon$ are permutable with d.

Let us now consider the translation $s_{ty}$ with a parameter $0 \leq t \leq 1$. For $t = 0$, this is the identical mapping and for $t = 1$ the translation $s_y$. If $F_y$ and $F_y^*$ are the operators associated with $s_{ty}$ in the same way as F and $F^*$ are associated with $f_t$ in §23, we have, according to (2) §23,

$$s_y^* \varphi - \varphi = d\, F_y^* \varphi + F_y^*\, d\varphi \tag{1}$$

Let us now define the operators $A_\varepsilon^*$ and $A_\varepsilon$ by setting

$$A_\varepsilon^* \varphi = \int F_y^* \varphi \cdot r(y)\,, \qquad A_\varepsilon T[\varphi] = T[A_\varepsilon^* \varphi]$$

If we multiply the relation (1) by r(y) and integrate with respect to y, we get

$$R_\varepsilon^* \varphi - \varphi = d\, A_\varepsilon^* \varphi + A_\varepsilon^*\, d\varphi$$

and there follows

$$R_\varepsilon T - T = (-1)^{n-p}(d\, A_\varepsilon T - A_\varepsilon\, d\, T)$$

where n and p are the dimensionalities of E and T respectively.

Clearly the carrier of $R_\varepsilon^* \varphi$ is contained in the $\varepsilon$-neighborhood of the carrier of $\varphi$ (i.e. the set of all points whose distance from the carrier of $\varphi$ is less than $\varepsilon$) and the carrier of $R_\varepsilon T$ is contained in the $\varepsilon$ - neighborhood of the carrier of T.

We shall now prove that $R_\varepsilon T$ is $C^\infty$. Suppose first that T is of degree zero. We have

$$R^*_\varepsilon \varphi(x) = \int \rho(y)\varphi(x+y)dy = \int \rho(y-x)\varphi(y)dy .$$

According to the definition of $R_\varepsilon T$ and to Theorem II, §10, there follows,

$$R_\varepsilon T(x)[\varphi(x)dx] = T(x)dx\,\varphi(y)dy[\rho(y-x)] = \varphi(y)dy\ T(x)dx[\rho(y-x)] .$$

If we set $T(x)dx[\rho(y-x)] = g(y)$, we have

$$R_\varepsilon T(x)[\varphi(x)dx] = g(y)[\varphi(y)dy]$$

and this means that $R_\varepsilon T(x) = g(x)$, which function is $C^\infty$ according to Theorem I §10.

The case where T is of degree > 0 follows from the above, because if S is any coefficient of the symbolic form associated with T with respect to rectangular coordinates, $R_\varepsilon S$ is the corresponding coefficient of the symbolic form associated with $R_\varepsilon T$.

The defining formula of $R^*_\varepsilon$ shows that, for $\varepsilon \to 0$, $R^*_\varepsilon \varphi \to 0$. Moreover, since $\frac{\partial}{\partial x^i} R^*\varphi = R^* \frac{\partial \varphi}{\partial x^i}$ in rectangular coordinates, for $\varepsilon \to 0$, each derivative of $R^*_\varepsilon \varphi$ tends to zero. Consequently, on account of the continuity of the current T, $R_\varepsilon T[\varphi] = T[R^*_\varepsilon \varphi] \to 0$ for $\varepsilon \to 0$. We see that, for $\varepsilon \to 0$, $R_\varepsilon T$ converges towards T.

From the defining formulae of $A^*_\varepsilon$ and $A_\varepsilon$, it follows that the carrier of $A_\varepsilon T$ is contained in the $\varepsilon$-neighborhood of the carrier of T. In general, $A_\varepsilon T$ is not $C^\infty$. But if T is $C^\infty$, $A_\varepsilon T$ is also $C^\infty$. This can be shown in the following way. If $\alpha$ is a p-form, a direct calculation shows that $F_y \alpha = (-1)^{p+1} F_{-y} \alpha$, which form is $C^\infty$ if $\alpha$ is $C^\infty$, and consequently $A_\varepsilon \alpha$ is also $C^\infty$.

Thus, we have proved the following

Theorem. In Euclidean space E, for any $\varepsilon > 0$ there are operators $R_\varepsilon$ and $A_\varepsilon$ with the following properties.

1) If T is a current of dimension p in E, $R_\varepsilon T$ and $A_\varepsilon T$ are currents of dimensions p and p+1 respectively, whose carriers are contained in the $\varepsilon$ - neighborhood of the carrier of T, which satisfy

$$R_\varepsilon T - T = (-1)^{n-p}(d\, A_\varepsilon T - A_\varepsilon d\, T)$$

2) $R_\varepsilon T$ is $C^\infty$ and $R_\varepsilon T \longrightarrow T$ for $\varepsilon \longrightarrow 0$.

3) If T is $C^\infty$, $A_\varepsilon T$ is $C^\infty$.

## §25. Smoothing in a Manifold.

We shall extend the result of §24 to a general manifold M and sketch a few applications.

Let M be an n-dimensional manifold $C^\infty$. According to a well known theorem of Whitney, M can be regularly imbedded in a Euclidean space E of dimensionality n+m (m can be assumed $\leq$ 2n+1). At every point $x \in M$ there is a well determined m-dimensional normal plane $N_x$ to M. We shall assume that there is a constant $\eta > 0$ such that for $x' \neq x$ the distances from x and from x' of any point of $N_x \cap N_{x'}$ are greater than $\eta$. As a matter of fact, the existence of a positive function $\eta = \eta(x)$ with this property follows from the theorem of Whitney, but we assume in addition that this function has a positive lower bound.

Let D and D' be the sets of points of E whose distances from M are less than $\eta$ and $\eta/2$ respectively. To each point $y \in D$ there corresponds a point $x = Py \in M$, which will be said to be the projection of y, such that $y \in N_x$ and the distance from x to y is less than $\eta$. Clearly P is a mapping $C^\infty$ of D in M.

Let us denote by I the identical mapping of M in E. The product PI is the identical mapping of M onto itself. If M is orientable, the mappings P and I are orientable and we shall choose orientations such that the resulting orientation for PI is the natural one. If M is not orientable, P and I are not orientable, but we can always orient them in any neighborhood of M and the corresponding set of D and we shall suppose the above condition is satisfied. For the following definitions of $R_\varepsilon$ and $A_\varepsilon$, this will be sufficient.

If T is a current in E, whose carrier is contained in D', we define the projection PT of T in M by

$$PT[\varphi] = T[P^*\varphi] .$$

In this definition, $\varphi$ is a form in M with a compact carrier, $P^*\varphi$ is a form in D whose carrier is not compact, but the intersection of the carriers of T and $P^*\varphi$ is compact and consequently $T[P^*\varphi]$ is always convergent and our definition is correct. The following lemma states that PT is $C^\infty$ if T is $C^\infty$.

Lemma. The projection $P\alpha$ in M of a form $C^\infty$ $\alpha$ whose carrier is contained in D' is a form $C^\infty$ in M.

Proof: We can introduce local coordinates $x^1,\dots,x^n,y^1,\dots,y^m$ in D, such that $x^1,\dots,x^n$ are local coordinates in M and are constant in $N_x$ and that $y^1,\dots,y^m$ are rectangular coordinates in $N_x$. Let us set

$$\omega = dy^1 \wedge \dots \wedge dy^m .$$

The form $\alpha$ can be represented by the expression

$$\alpha = \alpha_1 + \omega \wedge \beta$$

where $\beta$ is a form of degree zero with respect to the $dy^i$ and $\alpha_1$ is a sum of terms of degrees less than m with respect to the $dy^i$. Such a representation is unique.

Let n+m-p be the degree of $\alpha$. If $\varphi$ is a p-form in M, $P^*\varphi$ is of degree zero with respect to the $dy^i$ and we have

$$\alpha \wedge P^*\varphi = \omega \wedge \beta \wedge P^*\varphi$$

and consequently

$$P\alpha[\varphi] = \alpha[P^*\varphi] = \int \omega \wedge \beta \wedge P^*\varphi$$

The integration has to be taken over D' and can be decomposed into an integration over $N_x$ and an integration on M. If we set

$$\int_{N_x} \omega \wedge \beta = \beta_1 \text{, we have} \qquad P\alpha[\varphi] = \int \beta_1 \wedge \varphi$$

and this means that $P\alpha = \beta_1$, which form is $C^\infty$ in M.

If T is a current in M, IT is the current in D defined by $IT[\alpha] = T[I^*\alpha]$. Clearly PIT = T for any current T in M and $I^*P^*\varphi = \varphi$ for any form $\varphi$ in M.

Now let us denote by $R'_\varepsilon$ and $A'_\varepsilon$ the operators relative to the Euclidean space E which are respectively equal to the operators $R_{\varepsilon/2}$ and $A_{\varepsilon/2}$

defined in §24. If $\varepsilon < \eta$, for any current T in M the carriers of $R'_\varepsilon IT$ and $A'_\varepsilon IT$ are contained in D', because D' is the $\eta/2$ - neighborhood of M and the carrier of IT is in M. Then, if we set

$$(1) \qquad R_\varepsilon = PR'_\varepsilon I \quad \text{and} \quad A_\varepsilon = PA'_\varepsilon I ,$$

it follows from the above and from §24 that these operators $R_\varepsilon$ and $A_\varepsilon$ have with respect to M the properties 1) and 2) of the theorem of §24.

The property 3) does not follow, because IT is never $C^\infty$ in E, except if T = 0. But by slightly modifying our definition, we shall get operators which have the three properties 1), 2) and 3).

Let $\sigma(y)$ be a function $C^\infty$ in E such that: the value of this function depends only on the distance of y from M, $\sigma(y) = 0$ whenever this distance is greater than $\varepsilon/2$ (as is the case outside of D' since $\varepsilon < \eta$), $\sigma(y) \geq 0$ everywhere, and

$$\int_{N_x} \sigma\omega = 1$$

For any form $\alpha$ in E and for any current T in M, we set

$$I^*_\varepsilon \alpha = \int_{N_x} \sigma\omega \wedge \alpha \quad \text{and} \quad I_\varepsilon T[\alpha] = T[I^*_\varepsilon \alpha] .$$

The first formula means that each coefficient of the form $I^*_\varepsilon \alpha$ in M, expressed with the help of the local coordinates $x^1,\dots,x^n$, is equal to the integral over $N_x$ (i.e. with respect to y) of the corresponding coefficient of the form $\alpha$ expressed in D with the help of the coordinates $x^1,\dots,x^n,y^1,\dots,y^m$, multiplied by $\sigma\omega$.

For any form $\varphi$ in M and for any current T in M, we have $I_{\varepsilon}^{*}P^{*}\varphi = \varphi$ and $PI_{\varepsilon}T = T$. Moreover,

$$I_{\varepsilon}\varphi[\alpha] = \varphi[I_{\varepsilon}^{*}\alpha] = \int_{M}\varphi\wedge\int_{N_x}\sigma\omega\wedge\alpha = \int P^{*}\varphi\wedge\sigma\omega\wedge\alpha$$

This means that $I_{\varepsilon}\varphi = P^{*}\varphi\wedge\sigma\omega$, which form is $C^{\infty}$. Hence, if T is $C^{\infty}$ in M, $I_{\varepsilon}T$ is $C^{\infty}$ in E.

Let us now set, instead of (1),

(2) $$R_{\varepsilon} = PR'_{\varepsilon}I_{\varepsilon} \quad \text{and} \quad A_{\varepsilon} = PA'_{\varepsilon}I_{\varepsilon}$$

we see that these operators $R_{\varepsilon}$ and $A_{\varepsilon}$ have with respect to M the three properties 1), 2) and 3) of the theorem of §24. Thus, <u>the theorem of §24 holds for the manifold M instead of for E.</u>

As first application, we shall prove <u>Theorem A</u> of §12.

Suppose the current T is closed. Then $R_{\varepsilon}T - T = \pm dA_{\varepsilon}T$ and T is homologous to $R_{\varepsilon}T$, which current is $C^{\infty}$. If the carrier of T is compact, the homology is a compact one, because the carrier of $A_{\varepsilon}T$ is compact.

Suppose the form $\alpha$ bounds a current, $\alpha = dS$. Then, since $R_{\varepsilon}\alpha = dR_{\varepsilon}S$ and $R_{\varepsilon}\alpha - \alpha = \pm dA_{\varepsilon}\alpha$, $\alpha$ bounds $-R_{\varepsilon}S \mp A_{\varepsilon}\alpha$, which current is $C^{\infty}$ if $\alpha$ is $C^{\infty}$.

As second application, let us consider the n-dimensional Euclidean space E. Let $f_t x$ $(0 \leq t \leq 1)$ be a retraction of E into a point z, i.e. a mapping $C^{\infty}$ with respect to x and t of E in itself, such that $f_0 x = x$ (identical mapping) and $f_1 x = z$ (constant mapping). If $\varphi$ is a closed form of degree $p > 0$, since $f_0^{*}\varphi = \varphi$ and $f_1^{*}\varphi = 0$, the formula (2) §23 gives $\varphi = -dF^{*}\varphi$ and shows that $\varphi$ is homologous to zero; for $p = 0$, this formula gives $\varphi(x) = \varphi(z)$ and shows that $\varphi$ is a constant, which fact is evident.

On account of Theorem A just proved above, it follows that each closed current of degree $> 0$ in E is homologous to zero and that each closed current of degree zero is equal to a constant function. This last statement means that a distribution all of whose derivatives of the first order vanish is equal to a constant function.

In a similar way, by using Formula (3) §23, we get: each closed current of dimension $d > 0$ with a compact carrier in E is compact homologous to zero; a current of dimension zero with a compact carrier is compact homologous to a multiple of a 0-simplex.

On account of Theorem A, this contains as a special case the following lemma that I used in my Thesis: a closed form of degree $p < n$ with a compact carrier in E is the differential of a form of degree $p-1$ with a compact carrier; for $p = n$ the same is true if and only if the integral of the form over E is equal to zero.

## §26. Additional References to Chapters I and II.

The concept of chains of even kind was introduced under the name "champs de seconde espèce" in my paper: "Sur la théorie des intersections et les intégrales multiples" (Commentarii Mathematici Helvetici, vol. 4, p.151-157) and also, in another form, in my Thesis:"Sur l'analysis situs des variétés à n dimensions" (Journal de Mathématiques pures et appliquées, 1931, p. 115-200).

The concept of current was introduced, in a less precise and less general form than here, in my papers:"Relations entre la Topologie et la

Théorie des intégrales multiples" (L'Enseignement Mathématique, 1936, p. 213-228) and "Über mehrfache Integrale" (Abhandlungen aus dem Mathematischen Seminar der Hansischen Universität, Bd.12 (1938), p.313-339). For the development of the concept given here, I used the ideas introduced by Mr. Laurent Schwartz in his paper "Généralisation de la notion de fonction, de dérivation, de transformation de Fourier et applications mathématiques et physiques" (Annales de l'Université de Grenoble, Année 1945, p.57-74). The distributions there introduced are at the same time currents of degree n; in the n-dimensional Euclidean space, it is possible to identify a current of degree zero and the adjoint of degree n.

A book by L. Schwartz, "Théorie des distributions, Tome I" (Paris, Hermann et C$^{ie}$, 1950) has just appeared since the writing of these lectures. It contains the two theorems of our §10, together with their proofs, the ideas for which are already to be found in the paper referred to above. The book also contains the theorem of §12, with a proof very similar to ours; the main idea of this proof, which consists in using a modified elementary solution, is already in the papers of K. Kodaira.

# Part II.

By

Kunihiko Kodaira

## Chapter V.

## Differential Forms on Complex Analytic Manifolds.

### §27. Complex Analytic Manifolds with Kählerian Metrics.

A complex analytic manifold $M^{2n}$ of complex dimension n is, by definition, a topological 2n-dimensional manifold with a complex analytic structure. The concept of a complex analytic structure can be defined, in the same way as in the definition of a $C^{\infty}$ structure (see §1), by means of the concept of regular analytic functions in a neighborhood of a point and of the following two axioms:

Axiom 1. $f(q)$ being an arbitrary function defined in a neighborhood U of a point in $M^{2n}$, $f(q)$ is either regular analytic in U or it is not.

Axiom 2. For every point $q_o$ in $M^{2n}$, there exists a neighborhood U of $q_o$ and n complex-valued functions $z^1(q), \ldots, z^n(q)$ defined in U such that

a) the mapping $q \longrightarrow (z^1(q),\ldots,z^n(q))$ is a topological mapping of U on an open subset of the space of n complex variables, and that

b) an arbitrary function $f(q)$ defined in $V \subseteq U$ is regular analytic in V if and only if $f(q) = f(z^1(q),\ldots, z^n(q))$ is a regular analytic function of n complex variables $z^1 = z^1(q),\ldots,z^n = z^n(q)$, where V is an arbitrary open subset of U.

In this Chapter we consider a complex analytic manifold $M^{2n}$ which is compact. The functions $z^1,\dots,z^n$ appearing in Axiom 2b) are called local analytic coordinates in U. Axiom 2 asserts the existence of local analytic coordinates in a neighborhood of every point of $M^{2n}$. Putting

$$(1) \qquad z^\alpha = x^{2\alpha-1} + \sqrt{-1}\, x^{2\alpha}, \qquad (\alpha = 1,2,\dots,n),$$

we introduce real local coordinates $x^1,x^2,\dots,x^{2n}$ on $M^{2n}$, which clearly determine a $C^\infty$ structure on $M^{2n}$. Thus $M^{2n}$ is a manifold $C^\infty$; moreover $M^{2n}$ is orientable. Indeed, for an arbitrary transformation of analytic coordinates:

$$z^\alpha = x^{2\alpha-1} + \sqrt{-1}\, x^{2\alpha} \longrightarrow \tilde{z}^\alpha = \tilde{x}^{2\alpha-1} + \sqrt{-1}\, \tilde{x}^{2\alpha},$$

we have

$$\det\left(\frac{\partial \tilde{x}^k}{\partial x^i}\right) = \left|\det\left(\frac{\partial \tilde{z}^\beta}{\partial z^\alpha}\right)\right|^2 > 0 ,$$

as one readily verifies by using the relations:

$$\frac{\partial \tilde{x}^{2\beta-1}}{\partial x^{2\alpha-1}} = \frac{\partial \tilde{x}^{2\beta}}{\partial x^{2\alpha}} = \mathrm{Re}\,\frac{\partial \tilde{z}^\beta}{\partial z^\alpha}, \quad \frac{\partial \tilde{x}^{2\beta}}{\partial x^{2\alpha-1}} = -\frac{\partial \tilde{x}^{2\beta-1}}{\partial x^{2\alpha}} = J\,\frac{\partial \tilde{z}^\beta}{\partial z^\alpha} .$$

We choose the orientation $\epsilon$ of $M^{2n}$ so that $\epsilon = +1$ with respect to the system of coordinates $x^1,x^2,\dots,x^{2n}$ introduced in (1). This orientation will be called the natural orientation of $M^{2n}$; it is determined uniquely by the analytic structure.

Now, supposing the conjugate variables $\bar{z}^\alpha = x^{2\alpha-1} - \sqrt{-1}\, x^{2\alpha}$ to be formally independent of $z^\alpha = x^{2\alpha-1} + \sqrt{-1}\, x^{2\alpha}$, we write an arbitrary function $f(x^1,x^2,\dots,x^{2n})$ of $x^1,\dots,x^{2n}$ as $f(z^1,\dots,z^n, \bar{z}^1,\dots,\bar{z}^n)$ and define the partial derivatives $\partial f/\partial z^\alpha$, $\partial f/\partial \bar{z}^\alpha$ of $f$ as follows:

$$\frac{\partial f}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial f}{\partial x^{2\alpha-1}} - \sqrt{-1}\;\frac{\partial f}{\partial x^{2\alpha}}\right),$$

$$(\alpha = 1, 2, \ldots, n).$$

$$\frac{\partial f}{\partial \bar{z}^\alpha} = \frac{1}{2}\left(\frac{\partial f}{\partial x^{2\alpha-1}} + \sqrt{-1}\;\frac{\partial f}{\partial x^{2\alpha}}\right).$$

Then an arbitrary function $f(z, \bar{z})$ of $C^1$ is regular analytic with respect to $z^1, \ldots, z^n$ if and only if $\frac{\partial f}{\partial \bar{z}^\alpha} = 0 \; (\alpha = 1, \ldots, n)$.[1)]

A positive definite Hermitian metric

$$ds^2 = \sum_{\alpha,\beta=1}^{n} 2g_{\alpha\dot{\beta}}\, dz^\alpha\, d\bar{z}^\beta, \qquad (g_{\beta\dot{\alpha}} = \bar{g}_{\alpha\dot{\beta}})$$

will be called a Kählerian metric[2)], if $g_{\alpha\dot{\beta}}$ satisfy the partial differential equations

$$(2) \qquad \frac{\partial g_{\gamma\dot{\beta}}}{\partial z^\alpha} - \frac{\partial g_{\alpha\dot{\beta}}}{\partial z^\gamma} = 0, \qquad \frac{\partial g_{\alpha\dot{\gamma}}}{\partial \bar{z}^\beta} - \frac{\partial g_{\alpha\dot{\beta}}}{\partial \bar{z}^\gamma} = 0.$$

Now we assume that <u>a Kählerian metric</u> $ds^2 = \sum 2g_{\alpha\dot{\beta}}\, dz^\alpha\, d\bar{z}^\beta$ <u>is defined on</u> $M^{2n}$. By means of the system of real local coordinates, $ds^2$ can be written as

$$ds^2 = \sum_{j,k=1}^{2n} g_{jk}\, dx^j dx^k, \qquad (g_{jk} = g_{kj});$$

thus $M^{2n}$ is a 2n-dimensional orientable Riemannian manifold with a positive definite metric. Hence the whole theory of differential forms expounded in the previous chapters can be applied to $M^{2n}$; we can consider differential forms

1) See, for example, Bochner and Martin [2], pp.36-40.
2) Kähler [5].

(3) $$\varphi^p = \sum_{i_1<\cdots<i_p} \varphi_{i_1\cdots i_p}\, dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

defined on $M^{2n}$, the derived form $d\varphi^p$, the dual form $*\varphi^p$, etc., where the coefficients $\varphi_{i_1\cdots i_{p_2}}$ may be complex valued functions. Since our system of local coordinates $x^1, x^2, \ldots, x^{2n}$ defined by (1) has always the positive orientation, it is not necessary to distinguish differential forms into even and odd kinds (see §3).

Inserting $x^{2\alpha-1} = \frac{1}{2}(z^\alpha + \bar{z}^\alpha)$, $x^{2\alpha} = \frac{1}{2\sqrt{-1}}(z^\alpha - \bar{z}^\alpha)$ into (3), we can rewrite $\varphi^p$ as

$$\varphi^p = \sum_{r+s=p} \sum_{\substack{\alpha_1<\cdots<\alpha_r \\ \beta_1<\cdots<\beta_s}} \varphi_{\alpha_1\cdots\alpha_r\dot\beta_1\cdots\dot\beta_s}\, dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_r}\wedge d\bar{z}^{\beta_1}\wedge\cdots\wedge d\bar{z}^{\beta_s}.$$

Now we introduce two operators $\Lambda$ and $C$ operating on $\varphi^p$ as follows[3]:

$$\Lambda\varphi^p = \sum_{r+s=p} \sum_{\substack{\alpha_1<\cdots<\alpha_{r-1} \\ \beta_1<\cdots<\beta_{s-1}}} \sqrt{-1}\,(-1)^{r-1} g^{\alpha\dot\beta}\varphi_{\alpha\alpha_1\cdots\alpha_{r-1}\dot\beta\dot\beta_1\cdots\dot\beta_{s-1}}\, dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_{r-1}}\wedge d\bar{z}^{\beta_1}\wedge\cdots\wedge d\bar{z}^{\beta_{s-1}},$$

$$C\varphi^p = \sum_{r+s=p} \sum_{\substack{\alpha_1<\cdots<\alpha_r \\ \beta_1<\cdots<\beta_s}} (\sqrt{-1})^{r-s}\varphi_{\alpha_1\cdots\alpha_r\dot\beta_1\cdots\dot\beta_s}\, dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_r}\wedge d\bar{z}^{\beta_1}\wedge\cdots\wedge d\bar{z}^{\beta_s},$$

where $g^{\alpha\dot\beta}$ are the quantities defined by the equations $\sum_\gamma g^{\alpha\dot\gamma} g_{\beta\dot\gamma} = \delta^\alpha_\beta$

3) These two operators were introduced by W.V.D.Hodge. See Hodge [4], p.171.
The definitions of $\Lambda$ and C employed here are due to A.Weil. Cf.Weil [8].

(in case $p = 0$ or $p = 1$, we have to put $\Lambda\varphi^p = 0$). It can be easily verified that $\Lambda$ and $C$ are invariant under an arbitrary analytic transformation of local coordinates $z^1, \ldots, z^n$. Furthermore we have the following formulae[4]:

(4) $$C\Delta = \Delta C,$$

(5) $$\Lambda\Delta = \Delta\Lambda,$$

(6) $$CC\varphi^p = (-1)^p\varphi^p,$$

(7) $$\Lambda d - d\Lambda = C^{-1}\delta C.$$

By means of the 2-form

$$\omega = \sqrt{-1}\sum_{\alpha,\beta=1}^{n} g_{\alpha\dot{\beta}}\, dz^\alpha \wedge d\bar{z}^\beta$$

associated with $ds^2 = \sum 2g_{\alpha\dot{\beta}}\, dz^\alpha\, d\bar{z}^\beta$, $\Lambda\varphi^p$ can be written as

(8) $$\Lambda\varphi^p = (-1)^{p-1} * (\omega \wedge * \varphi^p).$$ [5]

Hence we have

$$(-1)^{p-1}\Lambda\varphi^p[\psi] = \int_{M^{2n}} *(\omega\wedge *\varphi^p)\wedge\psi =$$

$$= \int_{M^{2n}} \varphi^p \wedge *(\omega\wedge *\psi) = (-1)^{p-1}\varphi^p[\Lambda\psi],$$

or

(9) $$\Lambda\varphi^p[\psi] = \varphi^p[\Lambda\psi],$$

$\psi$ being an arbitrary $(2n-p+2)$-form. Again we get readily

(10) $$C\varphi^p[\psi] = (-1)^p\varphi^p[C\psi],$$

where $\psi$ is an arbitrary $(2n-p)$-form. These two formulae (9), (10) lead us

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4) The formulae (4), (5), (6), (7) are to be found in Eckmann and Guggenheimer [3]. Cf. also Weil [8]; Hodge [4], pp.165-168, p.171, p.175. It should be noted here that Eckmann-Guggenheimer's operators $C$, $\Lambda$ are not exactly the same as our $C$, $\Lambda$.

5) Weil [9], p.111; Eckmann and Guggenheimer [3], p.489.

to define $\Lambda T$, $CT$ for an arbitrary current of the degree p as follows;

(11) $$\Lambda T[\psi] = T[\Lambda\psi] ,$$

(12) $$CT[\psi] = (-1)^p T[C\psi] .$$

It is to be noted here that the 2-form $\omega$ defined above is a harmonic form[6] on $M^{2n}$. Indeed, as one readily infers, the condition (2) is equivalent to

(2)' $$d\,\omega = 0 ,$$

while, since $\Lambda\,\omega = -n$, we get, using (7),

$$\delta\,\omega = C(\Lambda d - d\Lambda)\,\omega = 0 .$$

By an exact regular analytic differential in an open subdomain $\Omega \subseteq M^{2n}$ we shall mean a 1-form $\varphi$ of the type $\varphi = \sum_{\alpha=1}^{n} \varphi_\alpha\, dz^\alpha$ defined in $\Omega$ satisfying $d\,\varphi = 0$ whose coefficients $\varphi_\alpha$ are regular analytic in $\Omega$. The integral

$$\Phi(z) = \int_{z_0}^{z} \varphi$$

of the exact regular analytic differential $\varphi$ is a (many valued) regular analytic function in $\Omega$ and $\varphi$ coincides with its differential $d\,\Phi$.

Lemma 1. If a current T of the degree 1 defined in $\Omega \subseteq M^{2n}$ satisfies $CT = \sqrt{-1}\,T$, then we have

$$\delta\,T = -\sqrt{-1}\,\Lambda d\,T .$$

Proof: We get, using (7)

$$\delta\,T = -\sqrt{-1}\,\delta\,C\,T = -\sqrt{-1}\,C\Lambda d\,T = -\sqrt{-1}\,\Lambda d\,T, \quad \text{q.e.d.}$$

6) Cf. Hodge [5], pp.168-171.

Lemma 2. If a current T of the degree 1 defined in an open subset $\Omega \subseteq M^{2n}$ satisfies $CT = \sqrt{-1}\, T$ and $dT = 0$, then we have $\delta T = 0$ and T is an exact regular analytic differential in $\Omega$ .

Proof: By Lemma 1, T satisfies $\delta T = 0$ in $\Omega$ . Thus T is a harmonic current and therefore, by virtue of a theorem in §17, T is a harmonic form; moreover, since $CT = \sqrt{-1}\, T$, T must be a form of the type: $T = \sum \varphi_\alpha \, dz^\alpha$. Now we have

$$\sum_{\alpha,\beta=1}^{n} \frac{\partial \varphi_\alpha}{\partial \bar{z}^\beta} \cdot d\bar{z}^\beta \wedge dz^\alpha + \sum_{\alpha,\beta=1}^{n} \frac{\partial \varphi_\alpha}{\partial z^\beta} \cdot dz^\beta \wedge dz^\alpha = dT = 0 .$$

This implies $\partial \varphi_\alpha / \partial \bar{z}^\beta = 0$ $(\alpha, \beta = 1,\ldots,n)$, proving that $\varphi_\alpha$ are regular analytic functions in $\Omega$ , q.e.d.

In view of the above lemma, we introduce the following

Definition. A current T of degree 1 defined in an open subset $\Omega \subseteq M^{2n}$ satisfying $CT = \sqrt{-1}\, T$ is called an analytic current, if dT vanishes in $\Omega$ except for a nowhere dense closed subset E of $\Omega$ ; then T is said to be regular in $\Omega - E$ and singular on E.

By virtue of Lemma 2, the analytic current T mentioned above coincides in $\Omega - E$ with an exact regular analytic differential. Now, let us consider the following problem, which is similar to Problem 3 in §19:

Problem. Assume that, for every point q of $M^{2n}$, an analytic current $T_q$ defined in some neighborhood U(q) of q is given so that each $T_q$ is regular in U(q) - E and that each $T_q - T_r$ is regular in U(q) - U(r).* Then, does there exist an analytic current T defined in $M^{2n}$ such that $T - T_q$ is regular in U(q) for every point q of $M^{2n}$ ?

* E is a nowhere dense compact subset of $M^{2n}$.

This problem can be solved by the method of de Rham (see §19). By hypothesis, the currents $T_q$ are given so that

(13) $$CT_q = \sqrt{-1}\, T_q\,, \qquad \text{in } U(q)\,,$$

(14) $$dT_q = 0\,, \qquad \text{in } U(q) - E\,,$$

(15) $$d(T_q - T_r) = 0\,, \qquad \text{in } U(q) \cap U(r).$$

The last condition (15) implies that

(16) $$Z = dT_q$$

is a well determined current in $M^{2n}$ satisfying $dZ = 0$, which will be called the polar cycle[7] of the system of the currents $T_q$. Thus our problem reduces to that of solving the simultaneous linear equations

(17) $$CT = \sqrt{-1}\, T\,, \qquad \text{in } M^{2n}\,,$$

(18) $$dT = Z\,, \qquad \text{in } M^{2n}$$

for the unknown current T of degree 1.

To do this, we introduce the operators G and H defined in §19. Then it follows from (4) and (5) that C and $\Lambda$ are commutative with G and H. Now, by Lemma 1, we get

$$\delta T_q = -\sqrt{-1}\, \Lambda\, dT_q = -\sqrt{-1}\, \Lambda\, Z$$

and therefore

$$\Delta T_q = \delta Z - \sqrt{-1}\, d \Lambda Z\,.$$

Combined with (13), this yields

7) Cf. Weil [8], p. 114.

$$C(\delta Z - \sqrt{-1}\, d\Lambda Z) = \sqrt{-1}\,(\delta Z - \sqrt{-1}\, d\Lambda Z) .$$

Hence, putting

$$T = G(\delta Z - \sqrt{-1}\, d\Lambda Z), \tag{19}$$

we infer that the current T thus defined satisfies (17), while we get readily $dT = Gd\delta Z = G\Delta Z$ or

$$dT = Z - HZ . \tag{20}$$

Comparing this with (18), we infer therefore that the above problem has (at least) one solution T if and only if $Z \sim 0$ (homologous to zero) on $M^{2n}$ and, in case $Z \sim 0$, a solution T is given by (19). Obviously the solution is unique up to everywhere regular exact analytic differentials on $M^{2n}$. Thus we conclude:

<u>Theorem.</u> There exists on $M^{2n}$ (at least) one analytic current T such that for every $q \in M^{2n}$ the difference $T - T_q$ is regular in U(q) if and only if the polar cycle Z of the system of currents $T_q$ is homologous to zero; and, in case Z is homologous to zero, such a current T is given by

$$T = \delta G Z - \sqrt{-1}\, d\Lambda G Z + d\Phi , \tag{21}$$

where $d\Phi$ is an arbitrary everywhere regular exact analytic differential.[8)]

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8) Cf. Weil [8]. In order to deduce from this result a theorem of Weil [8] on the existence of meromorphic forms, it is necessary to verify that every meromorphic form can be considered as a current.

## §28. Meromorphic Functions.

We start with the following definitions:

Definition 1. A compact subset $\Gamma$ of $M^{2n}$ is called a (n-1)-dimensional analytic subvariety of $M^{2n}$, if, for every point $q \in M^{2n}$, there exists a regular analytic function $f_q(z)$ defined in some neighborhood $U(q)$ of $q$ such that $\Gamma \cap U(q) = \{z \mid f_q(z) = 0\}$. $f_q(z) = 0$ is called a local equation of $\Gamma$ at $q$ (in case $q \notin \Gamma$ we have to put $f_q(z) \equiv 1$).

Definition 2. A local equation $f_q(z) = 0$ of $\Gamma$ at $q$ is said to be minimal, if, for every local equation $h_q(z) = 0$ of $\Gamma$ at $q$, the ratio $h_q(z)/f_q(z)$ is regular analytic in some neighborhood $U(q)$.

Let us call a function $h_q(z)$ regular analytic at $q$ if it is defined in some neighborhood of $q$ and is regular analytic there. For each $q$ the set of all functions regular analytic at $q$ forms a ring $\mathcal{O}_q$ without null divisors. A member $h_q(z)$ of $\mathcal{O}_q$ is a unit if and only if $h_q(q) \neq 0$. It is known that every $h_q \in \mathcal{O}_q$ can be decomposed into a product $h_q = u_q \cdot \prod_k h_{qk}$ of irreducible factors $h_{qk}$ where the $h_{qk}$ are uniquely determined up to multiplication by a unit $u_q$.[9] It follows that the minimal local equation mentioned in Definition 2 exists.

Definition 3. $\Gamma$ is said to be reducible, if $\Gamma$ can be represented as a sum: $\Gamma = \Gamma' \cup \Gamma''$ of two (n-1)-dimensional analytic proper subvarieties $\Gamma' \subsetneqq \Gamma$, $\Gamma'' \subsetneqq \Gamma$ of $\Gamma$; otherwise $\Gamma$ is said to be irreducible.

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9) See Bochner and Martin [2], Chap. IX.

As is well known, every reducible (n-1)-dimensional subvariety can be decomposed uniquely into a sum of a finite number of irreducible (n-1)-dimensional subvarieties.

Suppose $\Gamma$ to be irreducible and denote for every $q \in M^{2n}$ the minimal local equation of $\Gamma$ at q by $f_q(z) = 0$. Then, a point $q \in \Gamma$ is called a simple point of $\Gamma$, if one of the partial derivatives $\partial f_q(z)/\partial z^\alpha$ $(\alpha = 1,\dots,n)$ does not vanish at q; otherwise q is called a singular point of $\Gamma$. The set S of all singular points of $\Gamma$ constitutes an anlytic subvariety of $\Gamma$ with the complex dimension $\leq n-2$, which will be called the singular locus of $\Gamma$. For every simple point q of $\Gamma$, we can find in a neighborhood U(q) of q a parametric representation

$$z^\alpha = z^\alpha(t^1,\dots,t^{n-1}), \qquad (\alpha = 1,2,\dots,n)$$

of $\Gamma$, where $z^\alpha(t)$ are regular analytic functions of n-1 complex parameters $t^1,\dots,t^{n-1}$ (defined in some open domain in the complex t-space); thus $\Gamma - S$ is an open analytic manifold of complex dimension n-1. Moreover $\Gamma - S$ is a connected set. $\Gamma$ is therefore an orientable (2n-2)-dimensional pseudomanifold with respect to the natural orientation and thus $\Gamma$ is a (2n-2)-cycle. Further, it can be easily verified that, for an arbitrary (2n-2)-form $\psi$ of $C^\infty$, the integral $\Gamma[\psi] = \int_\Gamma \psi$ converges absolutely; thus $\Gamma$ can be considered as a current $\Gamma[\psi]$. The current $\Gamma$ satisfies

$$C\Gamma = \Gamma. \tag{22}$$

In fact, we have

$$\Gamma[C\psi] = \sum_{r+s=2n-2} (\sqrt{-1})^{r-s} \int_\Gamma \sum \psi_{\alpha_1\dots\alpha_r\dot\beta_1\dots\dot\beta_s} dz^{\alpha_1}\wedge\dots\wedge dz^{\alpha_r}\wedge d\bar z^{\beta_1}\wedge\dots$$
$$\dots\wedge d\bar z^{\beta_s}.$$

Inserting $z^\alpha = z^\alpha(t^1,\dots,t^{n-1})$ in this expression, we get readily $\Gamma[C\psi] = \Gamma[\psi]$, proving (22).

Now, let $q$ be an arbitrary point of $M^{2n}$ and put

$$\tau_q = \frac{1}{2\pi}\ d\log f_q(z) = \frac{1}{2\pi}\ \frac{df_q(z)}{f_q(z)}\ ,$$

$f_q(z) = 0$ being the minimal local equation of $\Gamma$ at $q$. Then, for an arbitrary $(2n-1)$-form $\psi$ of $C^\infty$ whose carrier is contained in $U(q)$, the integral

$$\tau_q[\psi] = \int_{U(q)} \tau_q \wedge \psi$$

converges absolutely; thus $\tau_q$ can be considered as a current $\tau_q[\psi]$ defined in $U(q)$.

Lemma 3. As a current defined in $U(q)$, $\tau_q$ satisfies

(23) $$C\tau_q = \sqrt{-1}\ \tau_q\ ,$$

(24) $$d\tau_q = \sqrt{-1}\ \Gamma.$$

Proof. By a suitable choice of the system of local coordinates $w, z^2,\dots,z^n$ whose origin is the point $q$ (where we denote the first coordinate by $w$ instead of $z^1$), $f_q(w,z) = f_q(w,z^2,\dots,z^n)$ can be written as

(25) $$f_q(w,z) = u(w,z)\cdot\{w^m + A_1(z)w^{m-1} + \dots + A_m(z)\}\ .$$ [10)]

where $u(w,z)$ is a regular analytic function of $w,z^2,\dots,z^n$ with $u(0,0) \neq 0$ and $A_k(z)$ $(k=1,2,\dots,m)$ are regular analytic functions of $z^2,\dots,z^n$ such that

10) Bochner and Martin [2], pp.188-190; see also Behnke and Thullen [1], pp.56-58.

$A_k(0) = 0$. Putting

$$F(w,z) = w^m + A_1(z)w^{m-1} + \dots + A_m(z),$$

we have therefore

$$\tau_q = \frac{1}{2\pi} d \log u + \frac{1}{2\pi} d \log F .$$

Obviously we have $d * d \log u = 0$; hence it is sufficient for our purpose to show that

$$d * d \log F = 2\pi \sqrt{-1}\ \Gamma$$

or

(26) $$\int_{U(q)} d \log F \wedge d\,\psi = 2\pi \sqrt{-1} \int_{\Gamma} \psi ,$$

where $\psi$ is an arbitrary (2n-2)-form of $C^{\infty}$ whose carrier is contained in $U(q)$. First consider the case in which $\psi$ is a form of the type

(27) $$\psi = \Psi(w,z)\, dz^2 \wedge d\bar{z}^2 \wedge \dots \wedge dz^n \wedge d\bar{z}^n ,$$

$\Psi(w,z)$ being an arbitrary function of $C^{\infty}$ whose carrier is contained in $U(q)$. $F(w,z)$ can be decomposed as

$$F(w,z) = \prod_{j=1}^{m} \{ w - \alpha_j(z) \},$$

where $\alpha_j(z)$ are <u>continuous</u> functions of $z^2, \dots, z^n$. Moreover, since $F(w,z)$ is the <u>minimal</u> local equation of $\Gamma$, $F(w,z)$ and $\partial F(w,z)/\partial w$ have no common factor as polynomials of $w$ and therefore the discriminant $D(z)$ of $F(w,z)$ does not vanish identically; thus

$$\alpha_j(z) \neq \alpha_k(z) \quad (j \neq k) \quad \text{for every } z \text{ with } D(z) \neq 0 .$$

Hence $\Gamma$ is decomposed into the union $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$ of m different "sheets" $\Gamma_j$ each of which is defined by the parametric representation: $w = \alpha_j(z)$. Now we have, for $\psi$ of the type (27),

$$\int d \log F \wedge d\psi = \int \frac{\partial \log F}{\partial w} \cdot \frac{\partial \Psi}{\partial \bar{w}} dw \wedge d\bar{w} \wedge dz^2 \wedge d\bar{z}^2 \wedge \dots \wedge d\bar{z}^n$$

$$= \sum_{j=1}^{m} \int \frac{1}{w - \alpha_j(z)} \cdot \frac{\partial \Psi}{\partial \bar{w}} dw \wedge d\bar{w} \wedge dz^2 \wedge d\bar{z}^2 \wedge \dots \wedge d\bar{z}^n .$$

First we compute this integral with respect to w, $\bar{w}$ for fixed z. Then we have

$$\int \frac{1}{w - \alpha_j} \cdot \frac{\partial \Psi}{\partial \bar{w}} dw \wedge d\bar{w} = \lim_{\varepsilon \to 0} \int_{|w - \alpha_j| \geq \varepsilon} \frac{1}{w - \alpha_j} \cdot \frac{\partial \Psi}{\partial \bar{w}} dw \wedge d\bar{w} =$$

$$= -\lim_{\varepsilon \to 0} \int_{|w - \alpha_j| \geq \varepsilon} d\left\{ \frac{\Psi(w,z)}{w - \alpha_j} dw \right\} = \lim_{\varepsilon \to 0} \oint_{|w - \alpha_j| = \varepsilon} \frac{\Psi(w,z)}{w - \alpha_j} dw =$$

$$= 2\pi\sqrt{-1}\ \Phi(\alpha_j, z).$$

Hence we get

$$\int d \log F \wedge d\psi = 2\pi\sqrt{-1} \sum_{j=1}^{m} \int \Psi(\alpha_j(z), z) dz^2 \wedge d\bar{z}^2 \wedge \dots \wedge d\bar{z}^n =$$

$$= 2\pi\sqrt{-1} \sum_{j=1}^{m} \int_{\Gamma_j} \psi = 2\pi\sqrt{-1} \int_{\Gamma} \psi ,$$

proving (26) for those $\psi$ of the type (27). In order to deduce (26) for general $\psi$, we introduce a new system of coordinates:

$$\tilde{w} = w , \qquad \tilde{z}^{\nu} = z^{\nu} + t_{\nu} w \quad (\nu = 2,3,\dots,n),$$

$t_\nu$ being small complex parameters, and apply the above result to

$$\tilde{\psi} = \Psi \cdot d\tilde{z}^2 \wedge d\bar{\tilde{z}}^2 \wedge \dots \wedge d\bar{\tilde{z}}^n.$$

Then we get

$$\int d \log F \wedge d \tilde{\psi} = 2\pi\sqrt{-1}\int_{\Gamma} \tilde{\psi} .$$

Putting

$$\psi_{11} = \Psi . dz^2 \wedge d\bar{z}^2 \wedge \dots \wedge dz^n \wedge d\bar{z}^n ,$$
$$\psi_{\lambda 1} = \Psi . dz^2 \wedge \dots \wedge d\bar{z}^{\lambda-1} \wedge dw \wedge d\bar{z}^{\lambda} \wedge \dots \wedge d\bar{z}^n ,$$
$$\psi_{1\lambda} = \Psi . dz^2 \wedge \dots \wedge dz^{\lambda} \wedge d\bar{w} \wedge dz^{\lambda+1} \wedge \dots \wedge d\bar{z}^n ,$$
$$\psi_{\lambda\mu} = \Psi . dz^2 \wedge \dots \wedge dw \wedge d\bar{z}^{\lambda} \wedge \dots \wedge dz^{\mu} \wedge d\bar{w} \wedge \dots \wedge d\bar{z}^n ,$$

we have therefore

$$\sum_{\lambda,\mu=1}^{n} t_{\lambda}\bar{t}_{\mu} \int d \log F \wedge d\psi_{\lambda\mu} = \sum t_{\lambda}\bar{t}_{\mu}\, 2\pi\sqrt{-1} \int \psi_{\lambda\mu} ,$$

where $t_1 = 1$. This yields

$$\int d \log F \wedge d\psi_{\lambda\mu} = 2\pi\sqrt{-1}\int \psi_{\lambda\mu} , \quad (\lambda, \mu = 1, 2, \dots, n),$$

proving that (26) is valid for arbitrary $\psi$, q.e.d.

It is to be noted here that, if $f_q(z) = 0$ is a minimal local equation of $\Gamma$ at q, then, for a sufficiently small neighborhood U(q) of q, $f_q(z) = 0$ represents a minimal local equation of $\Gamma$ at every point $p \in U(q)$.

A (many valued) function F(z) defined on $M^{2n}$ is called a <u>meromorphic</u> function, if, for every point $q \in M^{2n}$, F(z) can be represented in a neighborhood U(q) of q as a ratio $F(z) = h_q(z)/f_q(z)$ of two regular analytic functions $h_q(z)$, $f_q(z)$ defined in U(q). Such a function F(z) is said to be <u>multiplicative</u> if the absolute value $|F(z)|$ is one valued on $M^{2n}$. By an analytic continuation along a closed curve $\gamma$, the multiplicative meromorphic function F(z) is multiplied by a constant factor $\chi(\gamma)$ with $|\chi(\gamma)| = 1$ depending only on the homology class of $\gamma$ on $M^{2n}$.

Suppose that a multiplicative meromorphic function $F(z)$ on $M^{2n}$ is given. Then, assuming that the functions $f_q(z)$ and $h_q(z)$ appearing in the representation $F(z) = h_q(z)/f_q(z)$ have no common divisor in $\mathcal{O}_q$, we denote by $\Gamma_F$ the $(n-1)$-dimensional analytic subvariety defined by the system of local equations $f_q(z) \cdot h_q(z) = 0$. Decompose this subvariety $\Gamma_F$ into a sum: $\Gamma_F = \bigcup_{k=1}^{\ell} \Gamma_k$ of irreducible subvarieties $\Gamma_k (k = 1, 2, \dots, \ell)$ and denote the minimal local equation of each $\Gamma_k$ at $q$ by $f_{kq}(z) = 0$. Then, for every $q \in M^{2n}$, $F(z)$ can be represented in $U(q)$ as

$$(28) \qquad F(z) = u_q(z) \cdot \prod_{k=1}^{\ell} \{ f_{kq}(z) \}^{m_k}, \qquad u_q \text{ is a unit in } \mathcal{O}_q;$$

moreover the integers $m_k$ appearing in this expression are independent of $q$, since, denoting by $S_k$ the singular locus of $\Gamma_k$, each $\Gamma_k - S_k$ is a connected analytic manifold. We associate with the decomposition (28) the $(2n-2)$-cycle

$$D = \sum_{k=1}^{\ell} m_k \Gamma_k ,$$

which will be called the <u>divisor</u> of $F(z)$. The multiplicative meromorphic function is determined uniquely up to a multiplicative constant by its divisor. Incidentally, the subvariety $\Gamma_F$ defined above will be denoted by $|D|$, i.e. $|D| = \bigcup \Gamma_k$.

Now, assume conversely that an integral $(2n-2)$-cycle $D = \sum_{k=1} m_k \Gamma_k$ consisting of irreducible $(n-1)$-dimensional analytic subvarieties $\Gamma_k$ is given. Then, <u>does there exist a multiplicative meromorphic function</u> $F(z)$ <u>having</u> $D$ <u>as its divisor?</u> As an answer of this question, we have

Theorem 1. (Existence Theorem) There exists a multiplicative meromorphic function $F(z)$ with the divisor D if and only if D is homologous to zero and, in case D is homologous to zero $F(z)$ is given by

$$(29) \qquad F(z) = c_o \exp 2\pi \{\Lambda G D + \sqrt{-1} \int_{z_o}^{z} \delta G D\} ,$$

($c_o$ is a constant $\neq 0$)[11].

Proof: Putting

$$\tau_q = \frac{1}{2\pi} \sum_k m_k \, d \log f_{kq} ;$$

$f_{kq}(z) = 0$ being the minimal local equations of $\Gamma_k$ at q, we get a system of the analytic currents $\tau_q$ such that each $\tau_q - \tau_r$ is regular in $U(q) \cap U(r)$. Moreover we have, by Lemma 3,

$$(30) \qquad d\tau_q = \sqrt{-1} \, D ;$$

thus the polar cycle of the system $\{\tau_q\}$ is $\sqrt{-1}\,D$. Now, if a multiplicative meromorphic function F with the divisor D exists, then $\theta = \frac{1}{2\pi} \, d \log F$ is an analytic current such that $\theta - \tau_q$ is regular at each $q \in M^{2n}$; hence, by a theorem in §27, D must be homologous to zero. Assume conversely that D is homologous to zero. Then, by the same theorem,

$$\theta = d\Lambda G D + \sqrt{-1}\, \delta G D$$

is an analytic current such that $\theta - \tau_q$ is regular at each $q \in M^{2n}$. Consequently, putting

$$F(z) = \exp 2\pi \int_{z_o}^{z} \theta = c_o \exp 2\pi \{\Lambda G D + \sqrt{-1} \int_{z_o}^{z} \delta G D\} ,$$

11) The existence of $F(z)$ was proved by A.Weil in a more general form. See Weil [8].

we get a many valued meromorphic function $F(z)$ such that for each $q \in M^{2n}$ the ratio $F(z)/\prod_k \{f_{kq}(z)\}^{m_k}$ is a unit in $\mathcal{O}_q$. The current D is real in the sense that $D[\psi]$ takes a real value for every real form $\psi$, while the operators $\delta$, G, $\Lambda$ transform real currents into real ones. Hence $\Lambda GD$ and $\delta GD$ are both real and therefore

$$|F(z)| = |c_o| \exp 2\pi \Lambda GD ,$$

showing that $|F(z)|$ is univalent on $M^{2n}$. Thus $F(z)$ is a multiplicative meromorphic function with the divisor D, q.e.d.

By using the Green's form $g_{2n-2}(z, \zeta)$ introduced in §21, GD can be written as

$$GD = *g_D(z) , \qquad g_D(z) = \int_D g_{2n-2}(z, \zeta) .$$

Hence we obtain, as a corollary of Theorem 1, the following

Theorem 2. The multiplicative meromorphic function $F(z)$ with the divisor D can be represented as

$$F(z) = c_o \exp 2\pi \left\{ - *(\omega \wedge g_D) - \sqrt{-1} \int_{z_o}^{z} *dg_D \right\} ,$$

where $g_D(z)$ is the integral of the Green's form $g_{2n-2}(z, \zeta)$ over the divisor D.[12)]

It is obvious by (29) that, by an analytic continuation along a closed curve $\gamma$ on $M^{2n}$, $F(z)$ is multiplied by the factor $\chi_D(\gamma) = \exp 2\pi \sqrt{-1} \int_\gamma \delta GD$. Now, denote by Q a $(2n-1)$-chain such that $dQ = D$. Then, as was shown in §20, we have

12) See Kodaira [6].

$$I(Q, \gamma) = (\delta G{*}Q, \delta\gamma) + (H{*}Q, \gamma)\,,$$

yielding immediately

$$I(Q, \gamma) = ({*}\delta GD, \gamma) - (Q, {*}H\gamma) = -\int_{\gamma} \delta GD + \int_{Q} H\gamma\,.$$

Hence we obtain

Theorem 3. By an analytic continuation along a closed curve $\gamma$, the multiplicative meromorphic function $F(z)$ with the divisor $D = dQ$ is multiplied by the factor

$$\chi_D(\gamma) = \exp 2\pi\sqrt{-1}\left\{I(\gamma, Q) + \int_Q H\gamma\right\}.$$

As a corollary of this theorem, we get

Theorem 4. Let $\{\gamma_j \mid j = 1,2,\ldots,b\}$ be a base of integral 1-cycles on $M^{2n}$. A multiplicative meromorphic function with the divisor $D = dQ$ is univalent on $M^{2n}$ if and only if

$$I(\gamma_j, Q) + \int_Q H\gamma_j \equiv 0 \pmod 1, \qquad (j=1,2,\ldots,b).$$

This theorem can be considered as a generalization of Abel's theorem.[13]

The existence theorem proved above can be generalized to the case: $D \not\sim 0$, if we consider a more general class of many valued functions than that of multiplicative ones. To consider such functions, it is convenient to introduce the universal covering manifold $\tilde{M}^{2n}$ of $M^{2n}$, so that every many valued function on $M^{2n}$ can be considered as a univalent function on $\tilde{M}^{2n}$. Now, suppose an arbitrary integral (2n-2)-cycle $D = \sum m_k \Gamma_k$ consisting of irreducible (n-1)-dimensional analytic subvarieties $\Gamma_k$ as given and consider

13) Weyl [10], pp.126-127.

the current

$$\theta = d\Lambda GD + \sqrt{-1}\,\delta GD .$$

Then it follows from the results in §27 and (30) that $\theta$ satisfies

$$C\theta = \sqrt{-1}\,\theta ,$$

$$d\theta = \sqrt{-1}\,(D - HD) ,$$

and that, for every $q \in M^{2n}$, $\theta - \tau_q$ is regular in $U(q)$. Hence, if we can find a 1-form $\tilde{r}$ of $C^{\infty}$ defined in $\tilde{M}^{2n}$ such that

$$(31) \qquad \begin{cases} C\,\tilde{r} = \sqrt{-1}\,\tilde{r} , \\ d\,\tilde{r} = \sqrt{-1}\,HD , \end{cases}$$

then $\theta + \tilde{r}$ is an analytic current in $\tilde{M}^{2n}$ having the same singularity as $\tau_q$ [14] at each $\tilde{q} \in \tilde{M}^{2n}$ and therefore, putting

$$F(\tilde{z}) = \exp 2\pi \int_{\tilde{z}_0}^{\tilde{z}} \{\theta + \tilde{r}\} ,$$

we get a many valued meromorphic function $F(z)$ on $M^{2n}$ with the divisor D. Thus our problem is reduced to the simultaneous equations (31) for the unknown 1-form $\tilde{r}$.

Now we shall show that (31) has a solution $\tilde{r}$ if D can be represented as

$$(32) \qquad D \sim \sum_{j<k} c_{jk}\, Z_j\, Z_k ,$$

where $Z_1, \ldots, Z_j, \ldots$ are (2n-1)-cycles on $M^{2n}$ and each $Z_j Z_k$ denotes the intersection of $Z_j$ and $Z_k$. Let $\{dP_1, dP_2, \ldots, dP_\nu\}$ be a base of everywhere regular exact analytic differentials on $M^{2n}$. The integrals $P_\lambda(\tilde{z}) = \int_{\tilde{z}_0}^{\tilde{z}} dP_\lambda$

14) All functions, currents and subsets on $M^{2n}$ can be considered as functions, currents and subsets on $\tilde{M}^{2n}$.

are obviously univalent regular analytic functions on $\widetilde{M}^{2n}$. As was proved by Hodge[15], the 1-forms $dP_1, \ldots, dP_\nu$, $d\bar{P}_1, \ldots, d\bar{P}_\nu$ constitute a base of the space of all harmonic 1-forms. Consequently, it follows from (32) that HD is represented as a linear combination of $dP_\mu \wedge dP_\lambda$, $d\bar{P}_\mu \wedge d\bar{P}_\lambda$ and $H\{d\bar{P}_\mu \wedge dP_\lambda\}$ [16] $(\lambda, \mu = 1, 2, \ldots, \nu)$; while, since HC = CH, and CD = D by (22), HD and $H\{d\bar{P}_\mu \wedge dP_\lambda\}$ are invariant under the transformation C whereas $dP_\mu \wedge dP_\lambda$ and $d\bar{P}_\mu \wedge d\bar{P}_\lambda$ change their sign by C. Hence we get

$$\sqrt{-1}\, HD = H\Big\{ \sum_{\lambda,\mu=1}^{\nu} a_{\lambda\mu} d\bar{P} \wedge dP_\mu \Big\}.$$

Putting $\eta = \sum a_{\lambda\mu} d\bar{P}_\lambda \wedge dP_\mu$, we get, using $d\eta = 0$,

$$d\delta G\eta = G\Delta\eta = \eta - H\eta = \eta - \sqrt{-1}\, HD.$$

Again, since $CG\eta = GC\eta = G\eta$, we obtain, using (6) and (7)
$c\delta G\eta = -C^{-1}\delta CG\eta = d\Lambda G\eta$, showing that $dC\delta G\eta = 0$.
Hence, putting

$$\tilde{r} = \sum_{\lambda,\mu=1}^{\nu} a_{\lambda\mu}\bar{P}_\lambda\, dP_\mu - (1 + \sqrt{-1}\, C)\delta G\eta, \tag{33}$$

we infer readily that $\tilde{r}$ satisfies (31). Thus we see, under the assumption (32), that

$$F(\tilde{z}) = \exp 2\pi \int_{\tilde{z}_0}^{\tilde{z}} \{\theta + \tilde{r}\}$$

is a many valued meromorphic function with the divisor D, where $\tilde{r}$ is given by (33). It follows from (33) that, for every covering transformation W of

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15) Hodge [4], pp.191-192; Weil [8], pp.111-112.

16) $dP_\mu \wedge dP_\lambda$ and $d\bar{P}_\mu \wedge d\bar{P}_\lambda$ are harmonic forms, whereas $d\bar{P}_\mu \wedge dP_\lambda$ are not necessarily harmonic.

$\tilde{M}^{2n}$ over $M^{2n}$, $F(z)$ is transformed

$$F(W\tilde{z}) = F(\tilde{z})\cdot\exp 2\pi\left\{\nu_0[W] + \sum_{\lambda=1}^{\nu} \nu_\lambda[W]\cdot P_\lambda(\tilde{z})\right\},$$

where $\nu_\lambda[W]$ are constants depending on $W$. A meromorphic function on $\tilde{M}^{2n}$ having such a property will be called a <u>generalized theta function</u> with respect to $M^{2n}$, since it can be considered as a generalization of the classical theta function.[17] We summarize above results in the following

<u>Theorem 5.</u> If an integral (2n-2)-cycle $D = \sum m_k \Gamma_k$ on $M^{2n}$ consisting of irreducible (n-1)-dimensional analytic subvarieties $\Gamma_k$ is homologous to a cycle of the form $\sum_{j<k} c_{jk} Z_j Z_k$, then there exists on $\tilde{M}^{2n}$ a generalized theta function $F(\tilde{z})$ with respect to $M^{2n}$ having $D$ as its divisor.

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17) For the theory of classical theta functions see Siegel [7], Weil [9].

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