INSTITUTE FOR ADVANCED STUDY

Logical Calculus - P. Bernays

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by

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CONTENTS

Errata

Page 16, line 14, "equality" should be "equivalence".

Page 52, the last sentence of the next-to-last paragraph should read "And so our substitution-rule has for the time being the rôle of a derived rule."

LOGICAL CALCULUS

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1. Introduction of the logical symbols

The purpose of these lectures is first to introduce the usual elementary logical calculus and at the same time to make the first steps towards the exposition of some general logical theorems, the precise statement of which refers to this calculus.

These theorems are obtained by the meta-mathematical method which in the domain of theoretical logic has been more successful than in the prosecution of the aim for which it was intended by Hilbert.

By speaking of the usual elementary logical calculus I mean the calculus which was at first established in somewhat different ways by Frege, Peano, Schroeder, Russell-Whitehead. In the elementary part, which we shall be considering, the different sorts of the calculus come out nearly the same, so that there results a formalism which is independent of the initial conventions and which is delimited, from the mathematical point of view, in a rather natural way.

Another naturally delimited portion of logic is the combinatorial logic whose investigation was begun by Mr. Schönfinkel, continued and brought to a certain conclusion by H. B. Curry, and which has now developed by the researches of Professor Church, together with Mr. Rosser and Mr. Kleene, to a general theory of constructive functionality.

The calculus on which I have to speak is adapted to the axiomatic treatment of theories now applied in almost all domains of theoretical research, especially in mathematics.

So a natural way of introducing this calculus will be to start with an example of an axiomatic theory, and to examine how the symbolic treatment can be made,- first of all with regard to the symbolic representation of propositions.

Let us consider the theory of groups. We have here

1. a domain of individuals, the "elements" of the group. The reference to this domain is formally given by a kind of variables

$$a, b, c, \dots$$

With the concept of element the notion of identity is immediately connected. This notion generally is regarded as belonging to the logic and therefore not to be specific to the axiomatic system. To denote the identity we use the ordinary mathematical symbol of equality =. (We do not distinguish between "identity" and "equality".)

2. We have an operation of compounding two elements "a b". This has the character of a binary mathematical function, that is, a function which with each ordered pair of elements associates again an element. The properties of this function are expressed by the following axioms:

1) $a(bc) = (ab)c$ (law of associativity).

2) "To each a and b there is an x such that $ax = b$"
To each a and b there is an x such that $xa = b$

To formulate this, we need a symbol for existence: (Ex). Using this symbol, we have: $(Ex)(ax = b)$, $(Ex)(xa = b)$.

Let us consider here the role of the variables. In the formula

$$a(bc) = (ab)c$$

we express, by means of the variables, the generality; the formula is to be understood in such a way that arbitrary elements can be taken for a, b, c.

The variable connected with the symbol E of existence has quite another character. It is only required to denote the places on which the existential-symbol operates. Its role corresponds to that of a summation-index or an integration variable. Such a variable of the character of an integration-variable shall be called a bound variable.

And to make the reading of logical formulas easier, we shall denote bound variables by other letters than free variables.

It seems very unsymmetrical that generality shall be expressed by free variables, existence by means of bound variables. Indeed that is not true: there is not always the possibility of expressing the generality by free variables, but only in the case that the generality is extended over a whole formula. That will be clear at once by taking an instance.

The formula

$$(Ex)(ax = a)$$

expresses that to every element a there exists an x such that a compounded with x gives again a. This statement is contained, as a special case, in the assertion of the axiom

$$(Ex)(ax = b)$$

But there is a stronger statement (which, as we shall see later on, holds also), namely that there exists an x such that for every a the composition of a with x gives again a.

The difference between the two statements is that in the first the element x may depend on a, whereas in the second it is required that the element x is the same for all a.

To express the second statement by a formula, we must apply a symbol of generality with a bound variable. In analogy to the symbol (Ex) one might take for the generality the symbol (Ax), but it has become usual to write simply (x).

In the present case we have still to observe that the bound variable belonging to the symbol of generality must be distinguished from the variable belonging to the existence-symbol. So we come to a formula

$$(Ex)(y)(yx = y)$$

(As is seen in this case, the process of binding a free variable by a sign of generality generally changes the meaning of a formally expressed proposition. Thus the formula

$$(Ex)(x = a)$$

expresses the fact that to an element a there exists always an x equal to it (namely the element a itself). But the formula

$$(Ex)(y)(x = y)$$

expresses the statement that there is an x that is equal to all y, which holds only in the case that the group contains but one element.)

It may be remarked that for the use of the bound variables we have quite the same conventions as in the mathematics for the integration-variables:

1) Each bound variable has a scope over which it is extended. (This scope is generally denoted by brackets.)
2) It shall be avoided that in the scope of a bound variable w there occurs the symbol (w) or (Ew).
3) A bound variable can be replaced throughout its scope by another bound variable (under the restriction 2)).

I wrote w to denote an arbitrary bound variable. This is a sign which doesn't belong to the formalixm itself. To distinguish such letters, which denote expressions of our formalism, from the variables of the formalism, I shall write them as Roman letters, whereas the variables shall be italics.--

The formalization of our system of axioms suggests the following remark: Instead of the binary mathematical function ab we can introduce a ternary predicate:

"a compound with b gives c ",

for which we may introduce the symbol $\Gamma(a, b, c)$

(The term "predicate" shall be used in the following to signify not only properties of one thing, but also relations between two or more things. We are speaking of "uninary", "binary", "ternary", ..., predicates. A predicate defines a logical function,- that is, a function with the values "true" or "false". So for instance $\Gamma(a,b,c)$ defines the logical function of three arguments a, b, c (ranging over our domain of elements), which has the value "true" or "false" according as $\Gamma(a, b, c)$ holds or not.)

The question now is how to translate the conditions for the function ab into axioms for $\Gamma(a, b, c)$. We must first express that c is uniquely determined by a and b. This statement can be divided into two parts:

"To each a, b there exists an x such that $\Gamma(a, b, x)$ ",

which is expressed by the formula

$$(E\,x)\ \Gamma(a, b, x)$$

"If $\Gamma(a, b, c)$ and $\Gamma(a, b, d)$ then $c = d$."

To express this formally, we introduce the symbol $\longrightarrow$ for "if -- then", ("Implication"), and the symbol $\&$ for "and" ("Conjunction"). With these symbols the last condition is expressed by the formula

$$\Gamma(a, b, c)\ \&\ \Gamma(a, b, d) \longrightarrow c = d.$$

(To spare brackets we agree that the implication-symbol gives a stronger separation of expressions than the conjunction-symbol, provided that no brackets are applied.)

Now our former axioms for ab must be transformed into axioms for $\Gamma(a,b,c)$.

The existential axioms give immediately

$$(Ex)\,\Gamma(a,x,b), \qquad (Ex)\,\Gamma(x,a,b)$$

The translation of the law of associativity is a little more complicated; we get

$$\Big(\Gamma(b,c,d)\;\&\;\Gamma(a,b,e)\Big)\;\&\;\Gamma(a,d,r)\rightarrow\Gamma(e,c,r)$$

We may observe here that by introducing the predicate $\Gamma(a,b,c)$ instead of the function ab we are obliged to add two axioms. These axioms express the necessary and sufficient condition for solving the relation $\Gamma(a,b,c)$ for c by an equation $c = ab$

(They may be denoted as the "uniqueness axioms" for $\Gamma(a,b,c)$ in relation to c.)

Let us briefly consider still another simple system of axioms: the axioms for quantities. Here we have again a sort of composition; we denote it by $+$.

The axioms for it are:

$$a+b=b+a \qquad \text{(law of commutativity)},$$

$$a+(b+c)=(a+b)+c \qquad \text{(law of associativity)},$$

$$a+b\neq a,$$

and still one axiom expressing the possibility of subtracting the one of two different quantities from the other. To formalize this last axiom we need a symbol for disjunction: $\vee$ ("or"); it shall correspond to the meaning that at least one of two possibilities holds. Then our axiom is:

$$a\neq b\longrightarrow(Ex)(a+x=b)\vee(Ex)(b+x=a).$$

(We agree again that the implication-symbol gives a stronger separation than the disjunction-symbol).

The symbol $\neq$ is here regarded as giving an abbreviated expression for the negation of an equality. As the general symbol for the negation we take the bar extended over the expression to be negated, using at once the following conventions:

$$\overline{a = b} \text{ may be replaced by } a \neq b,$$

$$\overline{(w)B(w)} \text{ may be replaced by } \overline{(w)}B(w),$$

$$\overline{(Ew)B(w)} \text{ may be replaced by } \overline{(Ew)}B(w).$$

($B(w)$ denotes an expression containing w.)

The symbols (w) and (Ew) are called the "all-symbol" and the "existence-symbol"; both are called "quantifiers" in accordance with the classical terminology, using the term "quantity" for the distinction between general and particular propositions.

As an example of an explicit formal definition we may take the definition of $a < b$, which is given by the expression $(Ex)(a + x = b)$. Using "$a < b$" as an abbreviation for this expression we can write the last of the axioms for quantities: $a \neq b \longrightarrow a < b \vee b < a$. As an exercise it may be suggested to transform the axioms for quantities by introducing a ternary predicate instead of the binary function $+$.

In the course of the preceding considerations we have been led to introduce already all the logical symbols which are used for the elementary logical calculus.

2. Rules of the Calculus, first form

It is now our task to formalize mathematical proofs. For that purpose two things are required:

1. to establish the rules which regulate the handling of the logical symbols

$$\longrightarrow,\ \&,\ \vee,\ \overline{},\ (w),\ (Ew),$$

2. to formalize the properties of identity.

We shall do this by a method dealing with the concept of assumption that has been established by G. Gentzen ("Untersuchungen über das logische Schliessen", Math. Zeitschrift 1934, Vol. 39), and also by St. Jaskowski "On the rules of suppositions in formal logic", Studia Logica, Nr. 1, 1934). (Jaskowski followed a suggestion given by Lukasiewicz.)

I shall give here the method of Gentzen. His heuristic idea is that to each of the six logical symbols two rules belong, the one introducing the symbol, the other eliminating it. (A little modification will be necessary for the negation.)

To formulate the rules, we begin by defining the notions of term and formula. For this we have to remember that our calculus applies to a mathematical theory in which we have a domain of things (individuals) over which free variables

$$a, b, c, \ldots$$

range; furthermore we may use, in formalizing the theory:

symbols for individuals,
" " mathematical functions,
" " predicates;

the symbol of equality = is a predicate-symbol.

The free variables and the symbols for individuals are the primary terms, and generally a term is an expression which is either a primary term or composed of primary terms by means of the symbols of mathematical functions.

A prime-formula consists of a predicate-symbol with terms as its arguments. Generally a formula is an expression which is either a prime-formula or is formed of one or more prime-formulas by means of the logical operations:

conjunction (&), disjunction ($\vee$), implication ($\rightarrow$), negation ($\bar{\ }$), generality ((w)), existence ((Ew)).

By applying the operations (w), (Ew) to a formula A(c) containing the free variable c and not containing w, we get the formulas (w)A(w), (Ew)A(w), w standing in A(w) instead of c in A(c).

Concerning the bound variables belonging to the quantifiers (w), (Ew), we have the same agreements as in mathematics for the summation-indices and the integration-variables, that a bound variable can be replaced throughout its scope; (that is: the scope of the binding quantifier) by another bound variable, and that in the scope of (w) or (Ew) neither (w) nor (Ew) shall occur.

Now the essential rules are the schemata for the introduction and elimination of the logical symbols. We have the following rules:

	Introduction	Elimination
Implication	$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \rightarrow B}$	$\dfrac{\begin{matrix}A\\ A \rightarrow B\end{matrix}}{B}$
Conjunction	$\dfrac{A, B}{A \mathbin{\&} B}$	$\dfrac{A \mathbin{\&} B}{A}$ $\qquad$ $\dfrac{A \mathbin{\&} B}{B}$
Disjunction	$\dfrac{A}{A \vee B}$, $\dfrac{B}{A \vee B}$	$\dfrac{A \vee B \qquad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}$
Generality	$\dfrac{A(b)}{(w)A(w)}$	$\dfrac{(w)A(w)}{A(t)}$
Existence	$\dfrac{A(t)}{(Ew)A(w)}$	$\dfrac{(Ew)A(w) \qquad \begin{matrix}[A(b)]\\ \vdots \\ C\end{matrix}}{C}$
Negation	$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix} \quad \begin{matrix}[A]\\ \vdots \\ \bar{B}\end{matrix}}{\bar{A}}$	$\dfrac{\bar{\bar{A}}}{A}$

Explanatory remarks:

w denotes a bound variable, b a free variable, t a term.

The notation $\begin{matrix}\lfloor A \rfloor \\ B\end{matrix}$ means that under the assumption A the formula B is to be derived (no matter if the assumption is really used). Such a process of obtaining a formula under an assumption may be called a "subsidiary deduction". It is to be observed that, within a subsidiary deduction made under a certain assumption, new assumptions can be introduced.

The bar means that under the conditions denoted above it, the formula denoted below it can be taken as deduced.

In the cases of a schema with one or two subsidiary deductions the application of the schema has the effect of "discharging" the assumption-formulas denoted in the schema, so that the availability of the resulting formula is independent of these assumptions.

So in the Implication-Introduction and the Negation-Introduction the assumption A, in the Disjunction-Elimination the assumptions A, B, in the Existence-Elimination the assumption A(b), is discharged.

Restrictive conditions:

In the Generality-Introduction the variable b is not allowed to occur in A(w). (That means: the replacement of b by w must be made for all occurrences of b in A(b).)

In the Existence-Elimination the variable b is not allowed to occur either in A(w) or in C.

Condition for the subsidiary deductions: If a subsidiary deduction contains a Generality-Introduction or an Existence-Elimination, then the variable b of this schema is not permitted to occur in the assumption-formula.

We still need the rules for operating with the equality-symbol. A natural way is to take the axiom

$$a = a$$

and the schema

$$\frac{r = s, \ C(r)}{C(s)}$$

which means that for a particular occurrence of the term r in a formula the term s can be substituted, if we have r = s. (The schema formalizes the principle: "If r is equal to s, then r can be replaced by s").

To become better acquainted with our system of rules and to see how it works, it will be desirable to consider some examples of formal deductions. For this purpose it will be obvious to choose examples from the axiomatic theories which we have already considered: the theory of groups and the theory of quantities.

Let us take first an example of the theory of quantities. Here we have the axioms

a) $$a + b = b + a$$

b) $$a + (b + c) = (a + b) + c$$

c) $$a + b \neq a$$

d) $$a \neq b \rightarrow (Ex)(a + x = b) \vee (Ex)(b + x = a)$$

and the definition

$$a < b : \quad (Ex)(a + x = b)$$

which allows us to write the last axiom in the abbreviated manner

d') $$a \neq b \rightarrow a < b \vee b < a$$

The formula to be deduced is

$$a + b = a + c \rightarrow b = c$$

Preliminary remarks:

1. The system of our rules contains no rule of substitution for the free variables. But we can, by means of these rules, perform such substitutions. Indeed, from a formula A(b) with a free variable b, we can pass (by a Generality-Introduction) to the formula (w)A(w), (A(w) containing w in all places in which A(b) contains b), and from (w)A(w) we get (by a Generality-Elimination) A(t), t being an arbitrary term substituted for each occurrence of w in A(w). The effect of these processes is to substitute the term t for the variable b. In the same way, from a formula A(b, c) with free variables b, c we get, for each pair of terms r, s, the formula A(r, s), r and s replacing respectively b and c for all occurrences of these variables.

To abbreviate the exhibition of proofs we shall immediately perform substitutions of terms for free variables and also use the axioms in such a way that we immediately replace each of the free variables occurring by some term.

2. If we have r = s, r and s being terms, we get from the formula

$$r = r$$

(which is obtained from the axiom $a = a$), by means of the schema of equality,

$$s = r.$$

For an abbreviated exhibition of a proof we shall immediately perform the passing from r = s to s = r.

The remarks 1 and 2 are instances of "derived rules". A derived rule summarizes a process of passing by means of our rules from one or more formulas of a certain structure to another formula; after this derived rule is established the process in question need not be carried out explicitly in each case occurring in the exhibition of a formal proof. So a derived rule can be used in the same way as a proper rule of our system, the domain of deducible formulas not being thereby extended.

Using the two derived rules resulting from the remarks 1 and 2, we can give the proof of the formula

$$a+b=a+c \rightarrow b=c$$

in the following form (the assumption-formulas we have to introduce are numbered 1), 2), ..., these numbers being placed at the left of the first occurrence of the assumption-formula in question):

1) $a+b=a+c$

2) $b \neq c$

$$\frac{b \neq c \rightarrow b<c \vee c<b}{b<c \vee c<b}$$ (by axiom d') (Impl. - Elim.)

3) $b<c$

$$\frac{a+(b+d)=(a+b)+d}{a+(b+d)=(a+c)+d}$$ (by axiom b)) (by the equality-schema, using 1))

4) $b+d=c$

$$a+c=(a+c)+d$$
$$(a+c)+d=a+c$$ (by the equality rules)

$$(a+c)+d \neq a+c$$ (by axiom c))

$$a+c \neq a+c$$ (by the equality schema)

$$\frac{(Ex)(b+x)=c}{a+c \neq a+c}$$ (by 3) and Def. of <) (Exist.-Elim., 4) is discharged)

So we have:

$$\begin{array}{c} b<c \\ \vdots \\ a+c \neq a+c \end{array}$$

In the same way we get from

5) $$\dfrac{\dfrac{\ulcorner c<b \urcorner}{a+b \neq a+b}}{a+c \neq a+c}$$ (by the equality schema, using 1)).

$$\dfrac{b<c \vee c<b \qquad \begin{matrix}\ulcorner b<c \urcorner \\ \vdots \\ a+c \neq a+c\end{matrix} \qquad \begin{matrix}\ulcorner c<b \urcorner \\ \vdots \\ a+c \neq a+c\end{matrix}}{a+c \neq a+c}$$ (Disj.-Elim., 3), 5) are discharged)

Now we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\ulcorner b \neq c \urcorner \\ \vdots \\ a+c \neq a+c\end{matrix} \qquad \begin{matrix}\ulcorner b \neq c \urcorner \\ \vdots \\ a+c = a+c\end{matrix}}{\overline{\overline{b=c}}}}{b=c}}{}}{a+b=a+c \rightarrow b=c}$$

(equality axiom)

(Negation-Intr., 2) is discharged)

(Neg.-Elim.)

(Impl.-Elim., 1) is discharged).

The second example of a formal proof made by our rules shall consist in deducing from the axioms of groups

a) $$a(bc) = (ab)c$$

b) $$(Ex)\ ax = b$$

c) $$(Ex)\ xa = b$$

the formula

$$(Ey)(x)(xy = x).$$

by which the theorem of the existence of a right-side unit-element is formalized.

1) $$\dfrac{\dfrac{\ulcorner ab = a \urcorner}{(ca)b = c(ab)}}{ca = (ca)b}$$ (from axiom a)) (by the equality-schema)

2) $$\dfrac{\ulcorner ca = d \urcorner}{(ca)b = d}$$ (by the equality-schema)

$$\dfrac{(Ex)(xa = d) \text{ (from axiom c))} \qquad db = d(\text{.. " "})}{db = d}$$

(Exist.-Elim., 2) is discharged; the var. b of the schema is here c)

$$\dfrac{\dfrac{(x)(xb = x)}{(Ey)(x)(xy = x)}\;\text{(Gener.-Intr.)}}{}\;\text{(Exist.-Intr.)}$$

So we have $\lfloor ab = a \rfloor$

$$\vdots$$

$$\dfrac{(Ex)(ax = a) \quad \text{(from Ax. b))} \quad (Ey)(x)(xy = x)}{(Ey)(x)(xy = x)} \quad \text{(Exist.-Elim., 1) is discharged)}$$

3. Derived rules, especially equivalences

I want to give now a more general survey of the formalism which results from our system of rules. The natural and almost indispensable means to this end is to establish some derived rules. Most of them will be rules for transformation of formulas, corresponding to logical equivalences.

Let us first consider how "equivalence" shall be defined in our logical calculus. One would think that the suitable definition of equivalence consists in defining A equivalent to B if B can be deduced from A and A from B.

This conception of equivalence (we may use for it the expression "equal deducibility") is surely important, and it would be quite sufficient if we had not to deal with free variables.

The use of free variables is advantageous for the calculus; on the other hand it requires some attention, especially with regard to the following: A formula A(c) with a free variable c is to be interpreted in the same way as (x)A(x), provided that c does not occur in A(x). Correspondingly, $\overline{A(c)}$ is to be interpreted as representing the same proposition as $(x)\overline{A(x)}$, $A(c) \longrightarrow B$ (if c is not contained in B) as representing the same proposition as $(x)(A(x) \longrightarrow B)$, and $A(c) \vee B(c)$ the same proposition as $(x)(A(x) \vee B(x))$. On the other hand, each of the formulas $\overline{A(c)}$, $A(c) \longrightarrow B$, $A(c) \vee B(c)$ has $A(c)$ as its part; but if one would replace here $A(c)$ by $(x)A(x)$, one would not agree with our previous interpretation.

We have here an example of two formulas, $A(c)$, $(x)A(x)$, which are of equal deducibility, but are not equivalent in the full sense: that is, we cannot replace the one by the other everywhere.

We are therefore led to seek a definition of equivalence in a stricter sense. Such a definition can be given in the following way:

A is said to be equivalent to B if the formula $(A \longrightarrow B) \& (B \longrightarrow A)$ (it may be abbreviated ty "$A \sim B$") is deducible, or -- what comes to the same thing -- if the formulas $A \longrightarrow B$, $B \longrightarrow A$ are both deducible.

Remark: This definition is not meant to give us a general method of deciding whether two arbitrary given formulas are equivalent.

In the case of formulas without free variables, equality (according to this definition) is no other than equal deducibility.

Indeed, if from A we get B, and from B we get A, and the two formulas contain no free variables, there is no restriction on the application of the implication-introduction rule, and we get $A \longrightarrow B$, $B \longrightarrow A$.

The inverse holds for quite arbitrary formulas: if $A \longrightarrow B$, $B \longrightarrow A$ can be deduced, then A, B are of equal deducibility.

To justify our definition of equivalence, we have now to show:

1. Equivalence has the formal properties: each formula is equivalent to itself; if two formulas are equivalent to a third, they are equivalent to another.

2. If A is equivalent to B, then for each occurrence of A in a formula it can -- by means of our formal rules -- be replaced by B.

1 results from the facts that $A \to A$ is always deducible, and that from $A \to B$, $B \to C$ we get $A \to C$ ("Syllogism"). Indeed

$$\dfrac{\dfrac{\llcorner A \lrcorner \qquad A \to B}{B} \qquad B \to C}{C}$$

so we have

$$\begin{array}{c} \llcorner A \lrcorner \\ \vdots \\ C \end{array}, \qquad A \to C$$

To prove 2 we have to show that from equivalent formulas we get again equivalent formulas by applying to each the same logical operation. Thus we have to show:

a) if we have $A \sim B$, then we have also

$$(A \to C) \sim (B \to C), \qquad (C \to A) \sim (C \to B)$$

$$A \,\&\, C \sim B \,\&\, C, \qquad C \,\&\, A \sim C \,\&\, B$$

$$A \vee C \sim B \vee C, \qquad C \vee A \sim C \vee B$$

$$\overline{A} \sim \overline{B}$$

b) if we have $A(c) \sim B(c)$, we have also

$$(w)A(w) \sim (w)B(w)$$

$$(Ew)A(w) \sim (Ew)B(w)$$

provided that c does not occur in $A(w)$, $B(w)$.

To prove a) it is enough to show: If we have $A \to B$, then we have also

1) $$(B \to C) \to (A \to C), \qquad (C \to A) \to (C \to B)$$

2) $$A \,\&\, C \to B \,\&\, C$$

$$A \vee C \to B \vee C$$

and generally we have

$$A \,\&\, B \to B \,\&\, A, \qquad A \vee B \to B \vee A.$$

3) If we have $A \to B$, we also have $\overline{B} \to \overline{A}$.

1) results from the syllogism, to which an Implication-Introduction is

to be added.

2) is easily shown

3) is the rule of "contraposition"; it results in the following way: We have $A \to B$ and the result follows from the deduction

$$\cfrac{[\overline{B}] \qquad \cfrac{\cfrac{[A] \quad A \to B}{B} \qquad \begin{matrix}[A] \\ \overline{B}\end{matrix}}{\overline{A}}}{\overline{B} \to \overline{A}}$$

(assumption A discharged by the Negation-Intr.)

To prove b) we need only to show: If we have $A(c) \to B(c)$, and c does not occur in $A(w)$, $B(w)$, then we have also

$$(w)A(w) \to (w)B(w)$$

$$(Ew)A(w) \to (Ew)B(w).$$

Indeed:

$$\cfrac{\cfrac{\cfrac{[(w)A(w)]}{A(c)} \quad , \; A(c) \to B(c)}{B(c)}}{(w)B(w)}$$

$$(w)A(w) \to (w)B(w)$$

$$\cfrac{[(Ew)A(w)] \qquad \begin{matrix}\cfrac{[A(c)] \quad , \; A(c) \to B(c)}{B(c)} \\ (Ew)B(w)\end{matrix}}{(Ew)B(w)}$$

$$(Ew)A(w) \to (Ew)B(w).$$

Now it is the question what transformations can be performed by equivalences. We state some essential póssibilities of such transformations.

1) Elimination of implication: For arbitrary formulas A, B, the formula

$$(A \to B) \sim \overline{A} \vee B$$

is deducible. We have to deduce $\overline{A} \vee B \to (A \to B)$, $(A \to B) \to \overline{A} \vee B$. Let us make two preliminary statements. From A, $\overline{A}$ we get B. For with A, $\overline{A}$ we can deduce

$$\dfrac{\dfrac{\begin{matrix}\lfloor\bar B\rfloor & \lfloor\bar B\rfloor\\ A & \bar A\end{matrix}}{\bar{\bar B}}}{B} \qquad \text{(Assumption } \bar B \text{ discharged)}$$

For every formula A we can deduce $A \vee \bar A$, in the following way:

$$\lfloor\overline{A \vee \bar A}\rfloor$$

$$\dfrac{\dfrac{\begin{matrix}\dfrac{\lfloor A\rfloor}{A \vee \bar A} & \dfrac{\lfloor A\rfloor}{\overline{A \vee \bar A}}\end{matrix}}{\bar A} \qquad \dfrac{\begin{matrix}\dfrac{\lfloor\bar A\rfloor}{A \vee \bar A} & \dfrac{\lfloor\bar A\rfloor}{\overline{A \vee \bar A}}\end{matrix}}{\bar{\bar A}}}{\dfrac{\overline{\overline{A \vee \bar A}}}{A \vee \bar A}}$$

Remark: The two formal possibilities correspond to the two principles of contradiction and of excluded middle, the first appearing in the form that if we get two contradictory formulas then every formula is deducible.

Now we make the two required deductions:

$$\dfrac{\lfloor\bar A \vee B\rfloor \quad \dfrac{\dfrac{\lfloor\bar A\rfloor, \lfloor A\rfloor}{B}\ \text{(Ass. A disch.)}}{\bar A \vee B \quad A \to B} \qquad \dfrac{\dfrac{\lfloor B\rfloor, \lfloor A\rfloor}{B}\ \text{(Ass. A disch.)}}{A \to B}}{A \to B}$$

$$\bar A \vee B \to (A \to B)$$

Remark: On the way we have deduced:

$$\bar A \to (A \to B)$$

$$B \to (A \to B)$$

The formula $(A \to B) \to \bar A \vee B$ is deducible as follows:

$$\lfloor A \to B\rfloor$$

$$\dfrac{A \vee \bar A \qquad \dfrac{\dfrac{\lfloor A\rfloor, A \to B}{B}}{\bar A \vee B} \qquad \dfrac{\lfloor\bar A\rfloor}{\bar A \vee B}}{\bar A \vee B}$$

$$(A \to B) \to \bar A \vee B.$$

2) The negations can be reduced to the prime-formulas. This reduction results from the following equivalences:

$$\overline{\overline{A}} \sim A, \qquad \overline{A \& B} \sim \overline{A} \vee \overline{B} \qquad \overline{A \vee B} \sim \overline{A} \& \overline{B},$$

$$\overline{(w)}A(w) \sim (Ew)\overline{A(w)}, \quad \overline{(Ew)}A(w) \sim (w)\overline{A(w)},$$

the first three being provable for arbitrary formulas A, B, the last two for every expression A(w) derived from a formula A(c) by replacing the free variable c by a bound variable w not previously occurring in it. $\overline{\overline{A}} \to A$ is immediate. $A \to \overline{\overline{A}}$ is got by the deduction:

$$\dfrac{\dfrac{[A] \quad \dfrac{[\overline{A}]}{A} \quad \dfrac{[\overline{A}]}{\overline{A}}}{\overline{\overline{A}}}}{A \to \overline{\overline{A}}}$$

<u>Application</u>: From $\overline{A} \to B$ we get by contraposition $\overline{B} \to \overline{\overline{A}}$; here $\overline{\overline{A}}$ can be replaced by A; so we get $\overline{B} \to A$.

In the same way we see that from $A \to \overline{B}$ one gets $B \to \overline{A}$ and from $\overline{A} \to \overline{B}$ one gets $B \to A$. So we have four forms of generalized contraposition.

Deduction of $\overline{A} \vee \overline{B} \to \overline{A \& B}$:

$$\begin{matrix}\dfrac{\begin{matrix} A \\ \\ \\ \\ [\overline{A} \vee \overline{B}] \end{matrix} \qquad \begin{matrix} A \& B \to A \\ \overline{A} \to \overline{A \& B} \\ [\overline{A}] \\ \vdots \\ \overline{A \& B} \end{matrix} \qquad \begin{matrix} A \& B \to B \\ \overline{B} \to \overline{A \& B} \\ \overline{B} \\ \vdots \\ \overline{A \& B} \end{matrix}}{\overline{A \& B}} \\ \overline{A} \vee \overline{B} \to \overline{A \& B} \end{matrix}$$

Deduction of $\overline{A \& B} \to \overline{A} \vee \overline{B}$

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix} \overline{A} \to \overline{A} \vee \overline{B} \\ \overline{\overline{A} \vee \overline{\overline{A}}} \to A \\ [\overline{\overline{A} \vee \overline{\overline{B}}}] \end{matrix}}{A} \qquad \dfrac{\begin{matrix} \overline{B} \to \overline{A} \vee \overline{B} \\ \overline{\overline{A} \vee \overline{\overline{B}}} \to B \\ [\overline{\overline{A} \vee \overline{B}}] \end{matrix}}{B}}{A \& B} \\ \overline{\overline{A} \vee \overline{\overline{B}}} \to A \& B \\ \overline{A \& B} \to \overline{A} \vee \overline{B} \end{matrix}$$

At once we get for arbitrary formulas A, B:

$$A \,\&\, B \sim \overline{\bar{A} \vee \bar{B}}$$
$$\bar{A} \,\&\, \bar{B} \sim \overline{A \vee B}$$
$$\overline{\bar{A} \,\&\, \bar{\bar{B}}} \sim A \vee B$$

Thus conjunction is expressible by disjunction and negation, and disjunction is expressible by conjunction and negation. Furthermore we get:

$$A \vee B \sim (\bar{A} \to B) \qquad (A \to B) \sim \overline{A \,\&\, \bar{B}}$$
$$A \,\&\, B \sim A \to \bar{\bar{B}}$$

Now to obtain the equivalences for $(\overline{w})$ and $(\overline{Ew})$ we first introduce the following derived rules:

(α) From $F \to G(a)$

we get $F \to (w)\,G(w)$ if a does not occur in F nor in G(w)

(β) From $G(a) \to F$

we get $(Ew)\,G(w) \to F$ " " " " " " " " " "

Indeed

$$\cfrac{\cfrac{\lfloor F \rfloor \qquad F \to G(a)}{G(a)}}{(w)\,G(w)} \qquad\qquad \cfrac{\cfrac{\lfloor E(w)\,G(w) \rfloor}{E(w)\,G(w)} \qquad \cfrac{\lfloor G(a) \rfloor \qquad G(a) \to F}{F}}{F}$$

(<u>Remark</u>: We can also infer in the inverse direction: Indeed, if we have F $\to$ (w) (w), we can deduce

$$\lfloor F \rfloor \qquad \cfrac{\cfrac{F \to (w)\,G(w)}{(w)\,G(w)}}{G(a)}$$

and if we have (Ew)G(w) $\to$ F, we can deduce

$$\cfrac{\cfrac{\lfloor G(a) \rfloor}{(Ew)\,G(w)} \qquad E(w)\,G(w) \to F}{F}$$

Here we do not need a supposition about the occurrence of a .)

Applying the rules (α), (β) we have the following deductions:

$$\frac{(w)\,A(w)}{A(c)}$$ (The variable c can be chosen as not occurring in A(w).)

$$\frac{\begin{array}{c}(w)\,A(w) \to A(c)\\ \overline{A(c)} \to \overline{(w)}\,A(w)\end{array}}{}$$ (Contraposition)

(1) $$(Ew)\,\overline{A(w)} \to \overline{(w)}\,A(w)$$ (Rule (β))

$$(w)\,A(w) \to \overline{(Ew)}\,\overline{A(w)}$$ (Contraposition)

(2) $$(w)\,\overline{A(w)} \to \overline{(Ew)}\,A(w)$$ (By application of the former formula on $\overline{A(w)}$ and canceling the double negation.)

$$\frac{A(c)}{(Ew)\,A(w)}$$

$$A(c) \to (Ew)\,A(w)$$

$$\overline{(Ew)}\,A(w) \to \overline{A(c)}$$ (Contraposition)

(3) $$\overline{(Ew)}\,A(w) \to (w)\,\overline{A(w)}$$ (Rule (α))

$$\overline{(w)}\,\overline{A(w)} \to (Ew)\,A(w)$$ (Contraposition)

(4) $$\overline{(w)}\,A(w) \to (Ew)\,\overline{A(w)}$$ (Application of the former formula on $\overline{A(w)}$.)

From the formulas (1), (2), (3), (4) the two required equivalences follow. At once we get (by taking on both sides of these equivalences the negation and canceling the double negations):

$$(w)\,A(w) \sim \overline{(Ew)}\,\overline{A(w)}, \qquad (Ew)\,A(w) \sim \overline{(w)}\,\overline{A(w)}$$

So we could eliminate one of the symbols (w), (Ew). By the iterated application of the five equivalences 2) every formula which contains no implication can be transformed into an equivalent formula (which also contains no implication), in which the negation is reduced to the prime-formulas. <u>For a formula of this kind one gets the negation by interchanging & with v , (w) with (Ew) and a prime-formula with its negation.</u>

We have here a sort of <u>duality</u>, similar to that in projective geometry. And, in a way analogous to that of projective geometry, the duality is effected by extending the elementary Euclidean geometry (that is, by adjoining to it the ideal elements); the duality in the logical calculus comes about by extending the usual method of logical reasoning.

There is the following general way of using the duality: Given an equivalence:

$$F \sim G,$$

where the parts, of which F and G are composed by means of the logical operations, are left arbitrary (eventually with the indication of some arguments); among the logical operations in F, G implication may not occur, and negation may be only applied directly to some of the arbitrary parts. From this stated equivalence we get first

$$\overline{F} \sim \overline{G};$$

using the equivalences 2), we can here make the negations apply directly to the arbitrary component formulas, and in the resulting equivalence we can replace the arbitrary formulas by their negations (because the negation of an arbitrary formula is a case of a formula); finally, the double negations can be canceled.

But the equivalence we get in this way is to be obtained from the equivalence $F \sim G$ by replacing every conjunction by a disjunction and inversely, and every generality by an existence and inversely.

Thus this interchanging can be performed on every equivalence $F \sim G$, which is of the kind indicated above, and we may speak of it as an application of the duality.

A corresponding duality rule applies to a statement on deducible implications

$$F \rightarrow G$$

where F, G are of the same form as before. Here we can pass at first to

$$\overline{G} \rightarrow \overline{F}$$

and then perform the same processes as in the case of the equivalence.

Thus from a statement $F \rightarrow G$ by means of conjunction, disjunction, generality, existence, in which F, G are composed of parts which are left arbitrafy (eventually with some indicated arguments), containing no implication and the negation only applied on some arbitrary parts, we can pass to the implication deriving from

$$G \rightarrow F$$

by interchanging the conjunction with the disjunction, the generality with the existence.

Instances: As we stated, for every A, B, the implications

$$A \,\&\, B \rightarrow B \,\&\, A, \qquad A \vee B \rightarrow B \vee A$$

are deducible. Using the duality, we can immediately pass from

$$A \,\&\, B \rightarrow B \,\&\, A$$

(where A, B are arbitrary) to

$$B \vee A \rightarrow A \vee B$$

We could also proceed in this way to infer from

$$A \,\&\, B \rightarrow B \,\&\, A$$

which (because of the arbitrariness of A, B) gives at once

$$B \,\&\, A \rightarrow A \,\&\, B$$

the equivalence

$$A \,\&\, B \sim B \,\&\, A$$

and from that, by the duality, the equivalence

$$A \vee B \sim B \vee A$$

Another example: By the deducibility of $B \vee \overline{B}$ we get for every A, B, C the equivalences:

$$A \,\&\, (B \vee \overline{B}) \sim A$$

$$A \,\&\, ((B \vee \overline{B}) \vee C) \sim A$$

Applying on these our rule of duality we get at once:

$$A \vee (B \,\&\, \bar{B}) \sim A,$$

$$A \vee ((B \,\&\, \bar{B}) \,\&\, C) \sim A.$$

Further instances are contained in what follows.

We have now to consider the third kind of reductions, concerning the operations $\&$, $\vee$, (w), (Ew).

3) The following equivalences are meant to be deducible for every formula A, B, C respectively. In all these cases we have pairs of equivalences the one of which is obtained from the other by using the duality, so that it is sufficient to verify one of them.

a) Algebraic equivalences for $\&$, $\vee$

$$A \,\&\, A \sim A \quad , \qquad A \vee A \sim A.$$

The commutativity

$$A \,\&\, B \sim B \,\&\, A \qquad A \vee B \sim B \vee A$$

has already been stated.

Associativity:

$$A \,\&\, (B \,\&\, C) \sim (A \,\&\, B) \,\&\, C, \qquad A \vee (B \vee C) \sim (A \vee B) \vee C$$

(The deduction of the first equivalence is rather easy.) According to the Associativity of conjunction and disjunction we can speak of conjunctions and disjunctions with several members, omitting the brackets.

Distributivity:

$$(A \,\&\, B) \vee C \sim (A \vee C) \,\&\, (B \vee C)$$

$$(A \vee B) \,\&\, C \sim (A \,\&\, C) \vee (B \,\&\, C)$$

We verify the second equivalence by deducing the two implications

$$(A \vee B) \,\&\, C \longrightarrow (A \,\&\, C) \vee (B \,\&\, C)$$

$$(A \,\&\, C) \vee (B \,\&\, C) \longrightarrow (A \vee B) \,\&\, C$$

that is done in the following way:

$$\frac{(A \vee B)\,\&\,C}{A \vee B, \quad C} \qquad\qquad \frac{A\,,\ C}{A\,\&\,C} \qquad \frac{B\,,\ C}{B\,\&\,C}$$

$$\frac{A \vee B \qquad\qquad (A\,\&\,C) \vee (B\,\&\,C) \qquad (A\,\&\,C) \vee (B\,\&\,C)}{(A\,\&\,C) \vee (B\,\&\,C)}$$

$$(A \vee B)\,\&\,C \longrightarrow (A\,\&\,C) \vee (B\,\&\,C)$$

$$(A\,\&\,C) \vee (B\,\&\,C) \qquad\qquad \frac{\dfrac{A\,\&\,C}{A \qquad C}}{A \vee B} \qquad \frac{\dfrac{B\,\&\,C}{B \qquad C}}{A \vee B}$$

$$\frac{(A\,\&\,C) \vee (B\,\&\,C) \qquad\qquad (A \vee B)\,\&\,C \qquad (A \vee B)\,\&\,C}{(A \vee B)\,\&\,C}$$

$$(A\,\&\,C) \vee (B\,\&\,C) \longrightarrow (A \vee B)\,\&\,C$$

<u>Consequences</u> of the equivalences:

a) Every propositional expression composed of $A_1, \cdots, A_k$ by means of $\&, \vee, \rightarrow, \overline{}$, can be brought to a <u>conjunctive normal-form</u>, that is a conjunction of disjunctions each member of a disjunction being one of the formulas $A_1, \cdots, A_k$ or its negation. Correspondingly we can obtain a <u>disjunctive normal-form</u>, which is the dual to the conjunctive normal-form. These normal forms can be used to recognize the equivalence of expressions. For instance the expressions

$$A\,\&\,B \rightarrow C, \quad A \rightarrow (B \rightarrow C), \quad B \rightarrow (A \rightarrow C)$$

have respectively the conjunctive normal-forms:

$$(\bar{A} \vee \bar{B}) \vee C, \qquad \bar{A} \vee (\bar{B} \vee C), \qquad \bar{B} \vee (\bar{A} \vee C)$$

which can be transformed into another by virtue of the associativity and commutativity of the disjunction. So we have the equivalences

$$(A\,\&\,B \rightarrow C) \sim (A \rightarrow (B \rightarrow C))$$
$$(A \rightarrow (B \rightarrow C)) \sim (B \rightarrow (A \rightarrow C))$$

(The passing from $A\ \&\ B \rightarrow C$ to $A \rightarrow (B \rightarrow C)$ is called "exportation"; the inverse operation is "importation".)

b) Equivalences for (w), (Ew):

B may not contain w; then

$$(w)(A(w) \& B) \sim (w)A(w) \& B$$
$$(Ew)(A(w) \vee B) \sim (Ew)A(w) \vee B$$
$$(w)(A(w) \vee B) \sim (w)A(w) \vee B$$
$$(Ew)(A(w) \& B) \sim (Ew)A(w) \& B$$
$$(w)(A(w) \& B(w)) \sim (w)A(w) \& (w)B(w)$$
$$(Ew)(A(w) \vee B(w)) \sim (Ew)A(w) \vee (Ew)B(w)$$
$$(u)(v)A(u,v) \sim (v)(u)A(u,v)$$
$$(Eu)(Ev)A(u,v) \sim (Ev)(Eu)A(u,v).$$

If one considers generality as a conjunction extended over the domain of individuals, and the existence-operator as a disjunction over this domain, then, for a finite domain of individuals these equivalences b) are consequences of the equivalences a). By means of the two first of the equivalences b) we can transform every formula in a "prenex" formula, that is, a formula in which the quantifiers are all at the beginning and their scope extends to the end of the formula.

Example: A(x) may not contain y, B(y) not contain x; then first

$$(x)A(x) \& (Ey)B(y) \sim Ey((x)A(x) \& B(y));$$

furthermore, we have

$$(x)A(x) \& B(c) \sim (x)(A(x) \& B(c)),$$

and from this we get

$$(Ey)((x)A(x) \& B(y)) \sim (Ey)(x)(A(x) \& B(y));$$

thus in the end we obtain

$$(x)A(x) \& (Ey)B(y) \sim (Ey)(x)(A(x) \& B(y)).$$

It may be observed that we generally can proceed in the way we have proceeded in the second step of this example, by using the general possibility of passing from an equivalence

$$A(c) \sim B(c)$$

(not containing the variable w) to

$$(w)\, A(w) \sim (w)\, B(w),$$

and to

$$(Ew)\, A(w) \sim (Ew)\, B(w),$$

thus performing an equivalent transformation within the scope of a quantifier (w), (Ew), quite as if a free variable stood in place of the bound variable w.

From this remark it follows in particular that every formula can be transformed into a prenex formula in which the expression in the scope of the quantifiers (or, if there are no quantifiers, the whole formula) is a conjunctive normal form, or, if we prefer it, a disjunctive normal form.

Remark: The equivalences 3), b) can also be used for a process inverse to that of getting a prenex formula, namely, to distribute quantifiers on different members. For instance, a formula

$$(x)\,(Ey)\big((A(x) \vee B(y)) \vee C(x)\big)$$

can be transformed by these equivalences first into

$$(x)\big((Ey)\,(A(x) \vee B(y))\big) \;\&\; (x)\, C(x),$$

and further into

$$\big((x)\, A(x) \vee (Ey)\, B(y)\big) \;\&\; (x)\, C(x).$$

There are generally many different ways to get a prenex formula equivalent to a given formula, and also different possibilities of getting a conjunctive or a disjunctive normal form.

For the conjunctive normal form we may take the following instance:

$$(\bar{A} \vee B) \;\&\; (\bar{B} \vee C) \;\&\; (\bar{C} \vee A)$$

is equivalent to

$$(A \vee \bar{B}) \;\&\; (B \vee \bar{C}) \;\&\; (C \vee \bar{A})$$

Indeed, if we make the distributive development of

$$(\bar{A} \vee B) \,\&\, (\bar{B} \vee C) \,\&\, (\bar{C} \vee A),$$

we can omit, by the equivalence

$$A \vee ((B \,\&\, \bar{B}) \,\&\, C) \sim A,$$

every member of which contains two conjunctive members, of which the one is the negation of the other. So we obtain:

$$(\bar{A} \,\&\, \bar{B} \,\&\, \bar{C}) \vee (B \,\&\, C \,\&\, A)$$

which is equivalent to $(A \,\&\, B \,\&\, C) \vee (\bar{A} \,\&\, \bar{B} \,\&\, \bar{C})$

Now if we examine the two given conjunctive normal forms, we see that the second is got from the first by interchanging A with C and performing then equivalent transformations; but this rearrangement doesn't alter the formula $(A \,\&\, B \,\&\, C) \vee (\bar{A} \,\&\, \bar{B} \,\&\, \bar{C})$ except as to the order of members; so the initial two formulas are equivalent.

The disjunctive normal form we meet here, namely

$$(A \,\&\, B \,\&\, C) \vee (\bar{A} \,\&\, \bar{B} \,\&\, \bar{C})$$

has a special character: each disjunction-member contains exactly once every component A, B, C, and no two disjunction-members are composed of exactly the same conjunction members.

A disjunctive normal-form of this kind may be called a <u>principal disjunctive normal-form.</u> This notion refers to a certain finite set of components $A_1, \ldots, A_k$.

Every principal normal-form which is formed with the components $A_1, \ldots, A_k$ can be got by performing at first the distributive development of the expression

$$(A_1 \vee \bar{A}_1) \,\&\, (A_2 \vee \bar{A}_2) \,\&\, \cdots \,\&\, (A_k \vee \bar{A}_k)$$

and omitting eventually some of the disjunction-members. (Indeed every disjunction-member of a principal disjunctive normal form composed out of $A_1, \ldots, A_k$ is also a disjunction-member of the distributive development of the indicated expression.) Now every disjunctive normal-form can be transformed, by using the equivalences

$$A \,\&\, (B \vee \bar{B}) \sim A \qquad A \vee A \sim A$$
$$A \vee (B \,\&\, \bar{B} \,\&\, C) \sim A \qquad A \,\&\, A \sim A$$
$$A \vee (B \,\&\, \bar{B}) \sim A,$$

either into a principal disjunctive normal-form, or into a formula $A \,\&\, \bar{A}$. For instance, if we have

$$(A \,\&\, B) \vee C,$$

C can be replaced by $(C \,\&\, (A \vee \bar{A})) \,\&\, (B \vee \bar{B})$, the distributive development of which gives

$$(C \,\&\, A \,\&\, B) \vee (C \,\&\, \bar{A} \,\&\, B) \vee (C \,\&\, A \,\&\, \bar{B}) \vee (C \,\&\, \bar{A} \,\&\, \bar{B}).$$

$A \,\&\, B$ can be replaced similarly by

$$(A \,\&\, B \,\&\, C) \vee (A \,\&\, B \,\&\, \bar{C}).$$

Since the member $A \,\&\, B \,\&\, C$ now occurring twice in the disjunction can be canceled one time, we get

$$(A \,\&\, B \,\&\, C) \vee (A \,\&\, B \,\&\, \bar{C}) \vee (A \,\&\, \bar{B} \,\&\, C) \vee (\bar{A} \,\&\, B \,\&\, C) \vee (\bar{A} \,\&\, \bar{B} \,\&\, C).$$

Corresponding to the principal disjunctive normal-form we can define a principal conjunctive normal-form, and we have the fact that every conjunctive normal-form can either be transformed into an equivalent principal conjunctive normal-form or be transformed into a formula $A \vee \bar{A}$. Since every expression formed of some components by the operations $\rightarrow$, $\&$, $\vee$, $\bar{\ }$, can be transformed into a conjunctive as well as into a disjunctive normal form, we have the result that every such expression can be transformed:

either into a principal conjunctive normal form or into a formula $A \vee \bar{A}$, and

either into a principal disjunctive normal form or into a formula $A \,\&\, \bar{A}$.

As to the different possibilities which enter into the processes of transforming a formula into a prenex formula, we may consider the following example:

$$(x)(Ey)\,A(x,y)\ \&\ (x)\,B(x) \longrightarrow (x)(Ey)\,C(x,y), \quad \text{or}$$

$$(Ex)(y)\,\overline{A(x,y)}\ \vee\ (Ex)\,\overline{B(x)}\ \vee\ (x)(Ey)\,C(x,y).$$

Here you can at first change the variables:

$$(Ex)(y)\,\overline{A(x,y)}\ \vee\ (Ez)\,\overline{B(z)}\ \vee\ (u)(Ev)\,C(u,v).$$

Now according to the order in which one applies the equivalences

$$A\ \&\ (w)\,B(w) \sim (w)(A\ \&\ B(w))$$
$$A\ \&\ (Ew)\,B(w) \sim (Ew)(A\ \&\ B(w))$$
$$A \vee (w)\,B(w) \sim (w)(A \vee B(w))$$
$$A \vee (Ew)\,B(w) \sim (Ew)(A \vee B(w))$$

one can attain many different orders of the quantifiers, for instance

$$(Ex)(y)(Ez)(u)(Ev),\quad (Ex)(y)(u)(Ez)(Ev),\quad (u)(Ez)(Ex)(Ev)(y), \cdots$$

The only restriction is that (Ex) comes before (y) and (u) before (Ev), so there are thirty possibilities. But we can proceed still in other ways, by applying, together with the equivalences just indicated, also the equivalence

$$(Ew)\,P(w) \vee (Ew)\,Q(w) \sim (Ew)(P(w) \vee Q(w))$$

In our example it is possible to apply this equivalence either to two of the disjunctive members or to all three. To do the latter, we have to replace first the variables z and v both by x. The result of applying the equivalences is then

$$(u)(Ex)(y)\left(\overline{A(x,y)} \vee \overline{B(x)} \vee C(u,x)\right).$$

It may be remembered that generally it is not allowed to permute quantifiers of different kinds, nor to contract several quantifiers of the same kind into one. We have equivalances

$$(x)(y)\,A(x,y) \sim (y)(x)\,A(x,y)$$

and the corresponding equivalence for the existential quantifier, but (x)(Ey)A(x, y) is not generally equivalent to (Ey)(x)A(x, y), nor (x)(y)A(x,y) to (x)A(x, x), nor (Ex)(Ey)A(x, y) to (Ex)A(x, x). Only the implications are deducible:

$$(Ey)(x)\, A(x,y) \longrightarrow (x)(Ey)\, A(x,y)$$
$$(x)(y)\, A(x,y) \longrightarrow (x)\, A(x,x)$$
$$(Ex)\, A(x,x) \longrightarrow (Ex)(Ey)\, A(x,y).$$

Finally, the following two equivalences concerning the equality may be mentioned:

$$A(c) \sim (w)(w = c \longrightarrow A(w))$$
$$A(c) \sim (Ew)(w = c \ \& \ A(w))$$

To get the first, we have to deduce the implications

$$A(c) \longrightarrow (w)(w = c \longrightarrow A(w)), \qquad (w)(w = c \longrightarrow A(w)) \longrightarrow A(c).$$

That is to be done in the following way:

$$\frac{A(c) \qquad b = c}{A(b)}$$ (b a variable not occurring in A(c)
(by the equality schema)

$$b = c \longrightarrow A(b)$$ (the assumption b = c is discharged)

$$(w)(w = c \longrightarrow A(w))$$ (Generality-Introduction)

$$A(c) \longrightarrow (w)(w = c \longrightarrow A(w))$$ (the assumption A(c) is discharged)

$$\cfrac{\cfrac{(w)(w = c \longrightarrow A(w))}{c = c \longrightarrow A(c)} \qquad c = c}{A(c)}$$ (Generality-Elimination)
(from the equality-axiom)

$$(w)(w = c \longrightarrow A(w)) \longrightarrow A(c)$$ (the assumption is discharged).

To the equivalence

$$A(c) \sim (w)(w = c \longrightarrow A(w))$$

thus to be obtained the other indicated equivalence is the dual one. Indeed if we apply the first to $\overline{A(c)}$ and express the implication by disjunction and negation, we get $\overline{A(c)} \sim (w)(\overline{w = c} \vee \overline{A(w)})$

now taking on both sides the negation and applying the rules for transforming negations, we obtain $A(c) \sim (Ew)(w = c \ \& \ A(w))$

which is the stated equivalence.

4. The Propositional Calculus

We have considered the possibilities of deduction in our calculus, especially regarding the equivalences; now we are prepared to treat the systematical questions.

For that purpose we restrict ourselves at first to the "propositional calculus", that is, the calculus which we obtain from our whole calculus by omitting the rules concerning generality and existence.

In this calculus we have to consider the formulas with regard only to their composition by the operations $\rightarrow$, $\&$, $\vee$, $\overline{}$.

To make this point of view explicit, we introduce a special kind of letters (we may take capital italics), which are to be regarded as prime-formulas. A formula consisting of such a letter or composed of such letters by means of the operations $\rightarrow$, $\&$, $\vee$, $\overline{}$, shall be named a "<u>letter-formula</u>".

It will be sufficient to make all deductions of the propositional calculus only with letter formulas. Indeed if we have a deduction made by the introduction -- and elimination -- schemats for $\rightarrow$, $\&$, $\vee$, $\overline{}$ only, we may collect all those expressions occurring in the deduction which have the property of a formula, but are not of any of the forms

$$A \rightarrow B, \quad A \& B, \quad A \vee B, \quad \overline{A}$$

and are not contained in the scope of a quantifier. If now we replace each of these expressions by a letter (capital italic), equal expressions by the same letter, different ones by different letters, then the applications of the schemata remain valid. Thus we get a deduction made only with letter-formulas, differing from the given deduction only by the replacement of some formulas by single letters.

So the whole propositional calculus can be represented by deductions on letter-formulas only.

Our rules here are the schemata of introduction and elimination for $\rightarrow$, $\&$, $\vee$, $\overline{}$ in which for the arbitrary formulas only letter-formulas are to be taken.

In the deductions performed in this way, we have no restrictions on the assumption-formulas. So if from some formulas

$$A_1, \ldots, A_k$$

a formula B can be deduced, then, by applications of the Implic.-Introd.-Schema, we get

$$A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_k \rightarrow B) \ldots).$$

If two formulas are of equal deducibility they are equivalent. (Notice, that we have no rule of substitution in our calculus!)

Now the system of the deducible letter-formulas and the system of the letter-equivalences can be characterized in a very simple manner, so that the performing of the deductions becomes quite superfluous.

This characterization is accomplished by a method of <u>valuation</u>:

To each of the letters we assign one of the values, +, - ("true", "false"). Furthermore we agree that

$$
\begin{array}{llll}
+ \;\&\; + = + & + \vee + = + & + \rightarrow + = + & \overline{+} = - \\
+ \;\&\; - = - & + \vee - = + & + \rightarrow - = - & \overline{-} = + \\
- \;\&\; + = - & - \vee + = + & - \rightarrow + = + & \\
- \;\&\; - = - & - \vee - = - & - \rightarrow - = + &
\end{array}
$$

By these conventions, for a given letter-formula to each valuation of the letters belongs a value of the formula. Thus the letter formulas represent "truth-functions", that is: functions with a finite number of arguments ranging over (+, -), every value being + or -. There are $2^{(2^k)}$ different truth-functions with k arguments, since the number of possible valuations of the arguments is 2^k.

Our first statement now is that each of these truth-functions can be represented by a letter-formula. In fact, to each valuation of the arguments $\mathcal{A}_1, \ldots, \mathcal{A}_k$ there belongs an expression

$$G_1 \,\&\, G_2 \,\&\, \ldots \,\&\, G_k,$$

in which

$$\left.\begin{array}{l} G_i \text{ is } \mathcal{A}_i, \text{ if } \mathcal{A}_i \text{ has the value } + \\ \text{" " } \overline{\mathcal{A}}_i, \text{ " " " " " } - \end{array}\right\} \quad (i = 1, \ldots, k).$$

This expression has the value + for this valuation of $\mathcal{A}_1, \ldots, \mathcal{A}_k$. For each other of the 2^k possible valuations it has the value −.

Now a truth function is defined by making correspond to each of the 2^k valuations one of the two values $\pm$.

If we take for a given truth function Φ all the valuations of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ for which it yields +, we can join the corresponding conjunctions $G_1, \ldots, G_r$ in a disjunction

$$G_1 \vee \ldots \vee G_r.$$

So we get a letter-formula which is a principal disjunctive normal form and which has the value + for exactly the same valuations of $\mathcal{A}_1, \ldots, \mathcal{A}_k$, for which Φ has the value +. There is only one exception, in the case that Φ has always the value −; but then Φ is represented by $\mathcal{A}_1 \,\&\, \overline{\mathcal{A}}_1$.

So the $2^{(2^k)} - 1$ different principal disjunctive normal forms which can be formed out of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ -- ("different" in the sense that they do not differ only by the order of disjunction members or of conjunction members) --, together with $\mathcal{A}_1 \,\&\, \overline{\mathcal{A}}_1$, represent all truth functions and each of them but once.

This statement remind us of our former result about the principal disjunctive normal form, saying that every formula formed out of some components $A_1, \ldots, A_k$ by means of the operations $\rightarrow$, $\&$, $\vee$, $\overline{}$ is equivalent to a principal disjunctive normal form composed of $A_1, \ldots, A_k$ or to $A_1 \,\&\, \overline{A}_1$. To bring the two statements into connection, we have to show that equivalent letter formulas represent the same truth function, or, in other words, that if A, B are letter-formulas such that $A \rightarrow B$, $B \rightarrow A$ are deducible, then A represents the same truth function as B.

This will be a consequence of the following more general statement: If from the letter-formulas $A_1, \ldots, A_k$ the letter-formula B can be deduced by our calculus, then for each valuation of the letters occurring in $A_1, \ldots, A_k$, B, for which $A_1, \ldots, A_k$ yield +, B also yields +. Especially if B is deducible without an assumption, then for each valuation B yields +.

We prove this by an intuitive induction on the number of applications of our rules.

A deduction without application of a rule is one in which B is one of the formulas $A_1, \ldots, A_k$. For this case our assertion is obvious.

Now we suppose that our statement holds for deductions with less than n applications of the rules, and we take a deduction of B from $A_1, \ldots, A_k$ made by n applications, the last leading to B.

If we now look at the eight rules in question, it is easy to infer in each case, from our assumption made on n, that our assertion holds:

Let B have the form $P \,\&\, Q$ and let the final schema be the Conj.-Introd. $\dfrac{P, Q}{P \,\&\, Q}$. Here P, Q are secured by less than n applications of our rules; thus for each valuation of the letters, for which $A_1, \ldots, A_k$ yield +, P yields +, Q yields +; therefore P Q also yields +.

Let B be obtained by the Disj.-Elim. schema $\dfrac{P \vee Q \quad \begin{matrix}[P]\\ \vdots \\ B\end{matrix} \quad \begin{matrix}[Q]\\ \vdots \\ B\end{matrix}}{B}$. Here $P \vee Q$ is obtained by less than n applications of our rules; hence for each valuation of the letters for which $A_1, \ldots, A_k$ yield +, $P \vee Q$ yields +; thus for each of these valuations either P yields + or Q yields +. Furthermore the subsidiary deductions are made by less than n applications of our rules. Hence if P, $A_1, \ldots, A_k$ yields +, then also B yields +, and also if Q, $A_1, \ldots, A_k$ yields +, then B yields +. Therefore our statement is right. Let B be obtained by the Impl.-Elim. schema $\dfrac{A,\ A \to B}{B}$. Here A, $A \to B$ are obtained by less than n applications of our rules; so for each valuation of the letters, for which $A_1, \ldots, A_k$ yield +, A yields + and $A \to B$ yields +; so B must yield +.

Let B be obtained by the schema

$$\dfrac{\begin{matrix}[P]\\ \vdots \\ Q\end{matrix}}{P \to Q}$$

, where $P \to Q$ is the formula B. Since the deduction of Q from P, $A_1, \ldots, A_k$ is made by less than n applications of our rules, it follows from our assumption that, for each valuation for which $A_1, \ldots, A_k$, P yield +, Q also yields +; at once $P \to Q$ then yields +. On the other hand, if P yields −, then $P \to Q$ yields +. Thus our statement is right for this schema.

Let B be obtained by the schema

$$\dfrac{\begin{matrix}[P]\\ \vdots \\ Q\end{matrix} \quad \begin{matrix}[P]\\ \vdots \\ \overline{Q}\end{matrix}}{\overline{P}}$$

where $\overline{P}$ is the formula B. Since the subsidiary deductions are made with less than n applications of our rules, it follows that, if for a valuation for which $A_1, \ldots, A_k$ yield +, P also were to yield +, then for this valuation Q and $\overline{Q}$ would yield +, and thus Q would yield both − and +; but that surely cannot be (because the value of Q for a given valuation is uniquely determined). Hence for every valuation for which $A_1, \ldots, A_k$ yield +, P must yield −, and therefore $\overline{P}$, that is B, yield +.

For the three remaining schemata (Conjunct.-Elim., Disjunct.-Introd., Negation-Elim.) the discussion is rather easily made, using the facts that for a valuation for which

$P \,\&\, Q$ yields +, also P, Q both yield +
P " ", " $P \vee Q$ yields +
Q " ", " P Q " "
$\bar{\bar{P}}$ " ", " P " "

Thus our theorem is proved.

Let us now take the consequences of it. We have first the result that equivalent letter-formulas, each composed of some of the letters $\mathcal{A}_1, \ldots, \mathcal{A}_k$, represent the same truth function of $\mathcal{A}_1, \ldots, \mathcal{A}_k$. For, if A is equivalent to B, then from A we can deduce B and from B deduce A; hence, according to our theorem, for every valuation of the letters $\mathcal{A}_1, \ldots, \mathcal{A}_k$, for which A yields +, B also does, and inversely. Thus A, B have the same value for every valuation.

Also the inverse holds: if a letter-formula A represents the same truth-function of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ as B -- (it is not required that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ all occur in A, nor in B) -- then A is equivalent to B. For let $N_1, \ldots, N_{2^{2^k}-1}$ be different principal disjunctive normal forms composed of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ ("different" in the sharper sense, as defined above) and $N_{2^{2^k}}$ the formula $A_1 \,\&\, \bar{A}_1$. According to our former theorem (proved in §3), there is a formula A* equivalent to A, being one of the formulas $N_1, \ldots, N_{2^{2^k}}$ and a formula B* equivalent to B, being also one of these 2^{2^k} formulas. From the equivalence of A to A* and of B to B* it follows that A* represents the same truth-function of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ as A does, and B* the same as B. Therefore, since A, B are assumed to represent the same truth-function, A* and B* must represent the same truth-function. On the other hand the truth functions represented respectively by $N_1, \ldots, N_{2^{2^k}}$ are different from one another, as we found. Hence A* must be the same formula as B*, and therefore A, B, since they are both equivalent to A*, are equivalent to one another.

Thus two formulas A, B are equivalent when and only when they represent the same truth-function (with respect to the letter-arguments occurring in at least one of them).

Our proved theorem includes the statement that a deducible letter-formula B has for each valuation of the letters occurring in it the value +. Such a letter-formula, which for every valuation yields +, may be called a "+-formula".

We have then the statement that every deducible letter-formula is a +-formula. Here again the converse holds: every +-formula is deducible. For if B is a +-formula, it represents the same truth-function as $\mathcal{A} \vee \overline{\mathcal{A}}$ (considered as function of $\mathcal{A}$ and the letters occurring in it). Hence B is equi valent to $\mathcal{A} \vee \overline{\mathcal{A}}$ and therefore deducible from $\mathcal{A} \vee \overline{\mathcal{A}}$ But this formula is itself deducible, and so B is deducible.

As a consequence of the fact that every deducible letter-formula is a +-formula, we state that a letter-formula and its negation cannot be both deducible. For the negation of a +-formula has always the value -.

A simple criterion for the deducibility of a letter-formula is connected with conjunctive normal form. From our considerations of §3 it results that using the equivalences eliminating the implication, the negation of conjunctions and of disjunctions and the double negation, furthermore using the distributive law

$$A \vee (B \,\&\, G) \sim (A \vee B) \,\&\, (A \vee G),$$

every letter-formula can be transformed to an equivalent conjunctive normal form

$$\Delta_1 \,\&\, \cdots \,\&\, \Delta_s$$

where every Δ_i $(i = 1, \ldots, s)$ is a disjunction of letters and negated letters.

Now the necessary and sufficient condition for the deducibility of a letter formula is the deducibility of one of its equivalent conjunctive normal forms. For the deducibility of a conjunctive normal form

$$\Delta_1 \& \cdots \& \Delta_s$$

the necessary and sufficient condition is that every Δ_i $(i = 1, \ldots, s)$ is deducible, or also that every Δ_i is a +-formula. But a disjunction Δ_i of letters and negated letters is a +-formula when and only when there is at least one letter which itself as well as its negation is a disjunction-member of Δ_i. (For otherwise the values of the letters can be so chosen as to give Δ_i the value –.) Thus a conjunctive normal form $\Delta_1 \& \ldots \& \Delta_s$ is deducible when and only when every disjunction Δ_i contains two members, the one of which is the negation of the other. This criterion is useful for deciding practically on deducibility of given letter-formulas.

The main point of our results is that in the propositional calculus with letter-formulas every question concerning the equivalence of two formulas or the deducibility of a formula can be decided according to a general method of computation. Thus the performing of deductions becomes superfluous.

We have here a similar situation to that in elementary geometry, where the inventing of special proving methods for the single theorems becomes on principle superfluous as soon as the analytical method is recognized to be applicable; indeed by this method every question concerning an incidence or a betweenness or a congruence in a configuration composed of a given finite number of elements, can be decided in a way prescribed in advance.

The way of establishing the propositional calculus we considered is not the usual one.

In the usual propositional calculus one does not introduce assumption formulas, and most of the schemata are replaced by starting-formulas which have the role of formalized axioms, their application being made by means of a substitution-rule.

The passage from our calculus with the assumption-formulas to this more usual form of the propositional calculus can be performed by interpolating a third form of the calculus, which is advantageous for the purposes of metamathematics in so far as it requires neither the operating with assumption-formulas nor a rule of substitution. Here one has to deal with formula-schemata, a formula-schema being a rule saying that every formula of a certain kind of composition can be taken as a starting-formula.

(The using of formula-schemats instead of formal axioms for the purposes of metamathematics was suggested by Professor von Neumann in his paper "Zur Hilbertschen Beweistheorie", Math. Zeitschrift, vol. 26, 1927.)

Let us consider how we can pass from our propositional calculus to a calculus without assumption-formulas, in which all the schemata with exception of the Implication-Elim. $\dfrac{A,\ A \rightarrow B}{B}$ are replaced by formula-schemata. The replacement is to be understood in the sense that the replacing schemata give the same possibilities of deduction as the replaced schemata. Thus the schema $\dfrac{A \mathbin{\&} B}{A}$ can be replaced by -- we may also say "is equivalent to" -- the formula-schema $A \mathbin{\&} B \rightarrow A$, saying that every formula of the form $A \mathbin{\&} B \rightarrow A$ can be taken as a starting formula. Indeed the "equivalence" of the two schemata holds in virtue of the Implic.-Elim.-schema, as is immediately seen.

The Negat.-Introd.-schema $\dfrac{\begin{matrix}\lfloor A \rfloor & \lfloor A \rfloor \\ \vdots & \vdots \\ B & \bar{B}\end{matrix}}{\bar{A}}$ can be replaced by the schema $\dfrac{A \rightarrow B,\ A \rightarrow \bar{B}}{\bar{A}}$; for by means of the formulas $A \rightarrow B$, $A \rightarrow \bar{B}$ and the Implic.-Elim. we can from A deduce B and also $\bar{B}$; and if from A we can deduce the formulas B, $\bar{B}$ we get by the Implic.-Introd. $A \rightarrow B$ and also $A \rightarrow \bar{B}$. The schema $\dfrac{A \rightarrow B,\ A \rightarrow \bar{B}}{\bar{A}}$ again can be replaced by the formula-schema

$$(A \rightarrow B) \rightarrow ((A \rightarrow \bar{B}) \rightarrow \bar{A}).$$

So this formula-schema is equivalent to the Negation-Introd.-schema by virtue of the two implication-schemata.

In this way we see that the schemata for conjunction, disjunction, negation can be replaced by the following formula-schemata:

$$A \,\&\, B \rightarrow A, \quad A \,\&\, B \rightarrow B, \quad A \rightarrow (B \rightarrow A \,\&\, B)$$

$$A \rightarrow A \vee B, \quad B \rightarrow A \vee B, \quad (A \rightarrow G) \rightarrow ((B \rightarrow G) \rightarrow (A \vee B \rightarrow G))$$

$$(A \rightarrow B) \rightarrow ((A \rightarrow \bar{B}) \rightarrow A), \quad \bar{\bar{A}} \rightarrow A.$$

Besides we have still the two implication-schemata. Now we are to show that in this modified system the Implic.-Introd.-schema $\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \rightarrow B}$ can be replaced by the two formula-schemata

$$A \rightarrow (B \rightarrow A), \quad (A \rightarrow (B \rightarrow G)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow G)).$$

First it is easily seen that these two schemata can be derived from the two implication-schemata. On the other hand, if we introduce the two formula-schemata in our system instead of the Implic.-Introd.-schema, this schema becomes a derived rule. To prove this, we begin with the following statements:

1) From a formula F we get $P \rightarrow F$ by applying the schema $A \rightarrow (B \rightarrow A)$ and the Impl.-Elim.-schema.

2) From $P \rightarrow F$, $P \rightarrow (F \rightarrow G)$ we get $P \rightarrow G$ by applying the schema $(A \rightarrow (B \rightarrow G)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow G))$ and the Impl.-Elim.-schema.

3) Every formula of the form $P \rightarrow P$ is deducible by means of the two introduced formula-schemata and the Impl.-Elim.-schema. For from the two formula-schemata we get, for an arbitrary formula P, the formulas

$$P \rightarrow ((P \rightarrow P) \rightarrow P)$$

$$(P \rightarrow ((P \rightarrow P) \rightarrow P)) \rightarrow ((P \rightarrow (P \rightarrow P)) \rightarrow (P \rightarrow P))$$

$$P \rightarrow (P \rightarrow P);$$

by the Impl.-Elim.-schema the two first of them give

$$(P \rightarrow (P \rightarrow P)) \rightarrow (P \rightarrow P)$$

and this formula, together with $P \rightarrow (P \rightarrow P)$, gives

$$P \rightarrow P.$$

Now to recognize that the Implic.-Introd.-schema becomes a derived rule by the introduction of the two formula schemata, it is sufficient to state that, in every deduction made in our modified system with the application of the Impl.-Introd.-schema the first application of this schema can be eliminated by means of the two adjoined formula-schemata.

Let

$$\frac{\begin{matrix}\llcorner P \lrcorner\\ \vdots\\ Q\end{matrix}}{P \rightarrow Q}$$

be such a first application. Then the subsidiary deduction of Q from P is made by means of the formula-schemata and the Impl.-Elim.-schema only.

Now we show that for every formula G which we get on the way of this subsidiary deduction (by means of the assumption P) the corresponding formula $P \rightarrow G$ is deducible without an assumption by means of the formula schemata and the Impl.-Elim.-schema. Indeed this holds for every starting formula taken from a formula-schema, according to our statement 1); furthermore for the assumption formula P, according to 3), and if it holds for the premises F, $F \rightarrow G$ in an application of the Impl.-Elim.-schema, it holds also for the formula G, according to 2). So by an intuitive induction we find it to hold for all formulas in question, especially for the formula Q. But that means that we can get the formula $P \rightarrow Q$ by the formula-schemata and the Impl.-Elim.-schema in the way that we first adjoin to every formula of the subsidiary deduction the formula P as an antecedent of an implication, and afterwards interpolate the deductions by which first the formula $P \rightarrow P$ is to be got; furthermore, the new starting formulas (with the antecedent $P \rightarrow$) are to be got from the former starting formulas; and thirdly, the antecedent P is carried through the passages made according to the Impl.-Elim.-schema.

Thus the possibility of replacing the Impl.-Introd.-schema by the two adjoined formula-schemata is proved, and so we come to a system of rules consisting of the ten formula-schemata

$$A \to (B \to A), \quad (A \to (B \to G)) \to ((A \to B) \to (A \to G))$$

$$A \,\&\, B \to A, \; A \,\&\, B \to B, \quad A \to (B \to A \,\&\, B)$$

$$A \to A \vee B, \quad B \to A \vee B, \quad (A \to G) \to ((B \to G) \to (A \vee B \to G))$$

$$(A \to B) \to ((A \to \bar{B}) \to \bar{A}), \quad \bar{\bar{A}} \to A$$

and the schema $\frac{A,\ A \to B}{B}$,(which in the calculus without assumption-formulas may be called simply the "implication-schema").

As to the independence of the schemata, there is only one undecided point, namely the question of the independence of the schema $A \to (B \to A)$; but if we take instead of the schema $A \to (B \to A \,\&\, B)$, the schema

$$(A \to B) \to ((A \to G) \to (A \to B \,\&\, G))$$

(by which it can be replaced in the frame of the system), then also the independence of $A \to (B \to A)$ guaranteed.

Another remark referring to the formula-schemata is that the schema

$$(A \to B) \to ((A \to \bar{B}) \to \bar{A})$$

can be replaced by the simpler one

$$(A \to \bar{B}) \to (B \to \bar{A}).$$

The application of the new system of rules depends, in the same way as our former system of the introduction- and elimination-rules, on the concept of a formula. For the propositional calculus with letter-formulas we have to restrict the notion of formula to letter-formulas.

The rules of our former system are derived rules with respect to the new system; only the Impl.-Elim.-schema is a fundamental rule in the new system.

Now to come to the usual form of the propositional calculus, we state another derived rule: From a deduced formula we get again a deducible formula if we replace a letter occurring in it for each of its occurrences by the same formula F. Indeed in the place where that letter has been introduced we can take instead of it the formula F, since, according to our rules, no property of the letter can have been used other than that it is a formula.

Thus the system of deducible formulas remains unchanged if we introduce a rule of substitution, allowing us to pass from a formula A containing a letter L to a formula we get from A by replacing L for each of its occurrences by the same formula F; such a replacement is called a substitution, and the formula arising in the described way may be denoted (according to the notation of Professor Church) by $S^{L}_{F}A\Big|$.

The introduction of the rule of substitution in the propositional calculus enables us to replace the formula-schemata by corresponding formulas (formal axioms). To write them down, we simply have to put instead of the Roman letters denoting arbitrary formulas, the corresponding italics. The ten formulas we get in this way, together with the implication-schema and the rule of substitution constitute a system of the propositional calculus in the usual form. (For the pure propositional calculus we have to restrict the notion of formula in the rule of substitution to letter-formulas.)

The system of formal axioms can be chosen in many different ways. A great deal of research, containing many interesting remarks, has been done about the possible axiomatics for the propositional calculus, and also for partial-systems of it. (Literature: Lukasiewicz and Tarski, "Untersuchungen über den Aussagenkalkül", C. R. Soc. Sci. Varsovie, vol. 23, 1930; Hilbert-Bernays, "Grundlagen der Mathematik I", Berlin, Springer, 1934, pp. 64-82.)

A complication arising from the rule of substitution is that our former schema $\dfrac{\begin{matrix}\lfloor A \rfloor\\ \vdots\\ B\end{matrix}}{A \to B}$ is no longer a derived rule, unless we make a restriction on it. Indeed, if we want to apply again the method by which we proved that the schema is valid (as a derived rule) for the calculus with the formula schemata, we have to show that from a subsidiary deduction $\begin{matrix}\lfloor P\\ \vdots\\ Q\end{matrix}$, made with the assumption P, we get a deduction of $P \to Q$ without an assumption by adjoining everywhere the antecedent

P. In order to extend this proof to the calculus with the substitution-rule we would have to show that the antecedent P can be carried not only through the steps according to the implication-schema, but also through those according to the substitution-rule. Thus it would be required that if a substitution made for a letter L leads from F to G, then we also can pass from P$\rightarrow$F to P$\rightarrow$G. But this is not always true (as may be seen from the case where F and P both consist only of the letter $\mathfrak{A}$ and G is $\overline{\mathfrak{A}}$). However, in case P does not contain the letter L, the substitution which leads from F to G leads also from P$\rightarrow$F to P$\rightarrow$G.

Thus we have the result: If in the propositional calculus with formal axioms and the substitution-rule a formula B can be deduced from an assumption-formula A in such a way that no substitution is performed on a letter occurring in A, then A$\rightarrow$B is deducible without an assumption.

The complication we have here to deal with is due to the following fact: though the system of deducible formulas has not been changed by the substitution-rule, the possibilities of deductions from assumption-formulas has been extended. For instance: from the formula consisting of the letter $\mathfrak{A}$, taken as an assumption-formula, every formula is to be got by substitution, whereas in the calculus without the substitution-rule from $\mathfrak{A}$, only those formulas can be deduced which have the value + for every valuation for which $\mathfrak{A}$ has the value +.

It is also a consequence of this fact that the propositional calculus with the substitution-rule has the following property of completeness: If we adjoin to the formal axioms any letter-formula not deducible in the calculus, then every letter-formula becomes deducible.

For if F is the adjoined letter-formula and N a conjunctive normal form of it, then F$\rightarrow$N and N$\rightarrow$F are deducible formulas. Therefore, since F is not deducible, N is not deducible. On the other hand N can be deduced from F, and also every conjunction-member of N can be deduced from F. The conjunction-members of N are disjunctions of letters and negated letters. According to our proved criterion, since N is not deducible, there must be at least one such disjunction in N, in which the negated letters are all different from the unnegated ones. Let now Δ be such a disjunction in N, and A an arbitrary letter-formula. If we substitute for each of the unnegated letters in Δ the formula A, and for each of the negated letters in Δ the formula $\overline{A}$, we obtain a disjunction Δ^*, every member of which is either A or A. From this disjunction we easily deduce the formula A. Thus from F we can deduce Δ, from Δ by substitutions Δ^* and from Δ^* the formula A. So indeed every letter-formula can be deduced from F.

5. Different Forms of the Logical Calculus

We now have to extend the systematical considerations to our whole logical calculus. Let us recall what has to be adjoined to extend the propositional calculus to the whole calculus which includes operating with the individual-variables and the quantifiers.

The first extension is that we take instead of the letter-formulas the more general logical formulas. This is done:

1. by taking as prime formulas, besides the letters $\mathcal{A}$, $\mathcal{B}$, ..., also the formulas consisting of such a letter with free variables as arguments (the free variables a, b, ... being the only terms in the pure logical calculus);

2. by adjoining to the logical operations $\rightarrow$, &, $\vee$, $\overline{}$ the process of forming out of a formula A(G) with a free variable G the formula

$$(w)A(w) \text{ or also } (Ew)A(w),$$

w being a bound variable not contained in A(G).

The other extension is that we have to add to the system of rules the four schemata for generality and existence:

	Generality	Existence
Introduction	$\dfrac{A(b)}{(w)A(w)}$	$\dfrac{A(t)}{(Ew)A(w)}$
Elimination	$\dfrac{(w)A(w)}{A(t)}$	$\dfrac{(Ew)A(w) \qquad \begin{matrix}\lfloor A(b) \rfloor \\ \vdots \\ G\end{matrix}}{G}$

To the Generality-Introd. and the Existence-Elim.-schema belong these restrictive conditions: The variable b is not allowed to occur in A(w) nor in G nor in the assumption-formula of any subsidiary deduction of which the application of the schema in question is a part.

(Besides, we have, for all the four schemata, the condition that the variable w does not occur in A(b).)

As we have seen in §3, the Gen.-Introd.- and the Exist.-Elim.-schema can be replaced by the following schemata:

$$(\alpha): \quad \frac{G \longrightarrow A(b)}{G \longrightarrow (w)A(w)}, \qquad (\beta): \quad \frac{A(b) \longrightarrow G}{(Ew)A(w) \longrightarrow G}$$

where the restrictive conditions are the same as before.

We can now again eliminate the assumption formulas. First we can replace the Gener.-Elim.- and the Exist.-Introd.-schema by the two formula-schemata

$$(\gamma): \quad (w)A(w) \longrightarrow A(t), \qquad (\delta): \quad A(t) \longrightarrow (Ew)A(w).$$

The possibility of replacing the four original schemata for generality and existence by the schemata (α), (β), (γ), (δ) arises from the two implication-sehcmata.

But in order to come to a formalism without assumption-formulas, we have to eliminate the Impl.-Introd.-schema, as we did in the propositional calculus.

Thus we have to show that if we take as fundamental rules our ten proponal formula-schemata, the formula-schemata (γ), (δ), the implication-schema $\dfrac{A,\ A \longrightarrow B}{B}$ and the schemata (α), (β), then the schema

$$\frac{\begin{array}{c}\llcorner A \lrcorner \\ \vdots \\ B\end{array}}{A \rightarrow B}$$

becomes a derived rule.

For this purpose it issufficient to extend our corresponding proof for the propositional calculus (cf. §4, pp. 43-43) by showing that a premise P can be carried also through the schemata (α), (β). provided that none of the variables to be taken for b in the occurring applications of these schemata is contained in P. (The last restriction can be made according to the condition prescribed for the application of the schemata (α), (β) in a subsidiary deduction.)

Now this supplementary proof is easily to be given; we have to show that, from a formula

$$P \rightarrow (G \rightarrow A(b))$$

where b is not contained in P, we get

$$P \rightarrow (G \rightarrow (w)A(w))$$

and from a formula

$$P \rightarrow (A(b) \rightarrow G),$$

where again b is not contained in P, we get

$$P \rightarrow ((Ew)A(w) \rightarrow G).$$

Indeed, from

$$P \rightarrow (G \rightarrow A(b))$$

we deduce first, by the propositional calculus,

$$P \,\&\, G \rightarrow A(b)$$

then by the schema (α)

$$P \,\&\, G \rightarrow (w)A(w)$$

and again by the propositional calculus

$$P \rightarrow (G \rightarrow (w)A(w)).$$

And from

$$P \longrightarrow (A(b) \longrightarrow G)$$

we deduce first, by the propositional calculus,

$$A(b) \longrightarrow (P \longrightarrow G)$$

then by the schema (β)

$$(Ew)A(w) \longrightarrow (P \longrightarrow G)$$

and again by the propositional calculus

$$P \longrightarrow ((Ew)A(w) \longrightarrow G).$$

So we have now for our whole calculus a system of rules in which the notion of an assumption-formula does not occur. This new system of rules applies, as well as the original one, not only to the pure logical calculus (in which the only terms are free individual-variables and the only prime-formulas are letters or letters with free individual-variables as arguments), but also to the deductions to be made in a formalized axiomatic theory, where we have the special symbols belonging to the theory (symbols for individuals, predicates, mathematical functions). We then have to deal with the more general concept of a term, as it was defined in §2, p. 8, and as prime formulas we have the predicate-symbols with terms as arguments.

Remark: By the extension of the concept of a term especially the application of the schemata (γ), (δ) becomes more general, because the t here denotes an arbitrary term (whereas the b in the schemata (α), (β) denotes a variable).

There is always the possibility to include the pure logical calculus in the formalism of a special theory, simply by adding the letters without arguments and the letters with terms as arguments to the prime-formulas. This incorporation of the pure logical calculus, though in general dispensable, is required if we now want to pass to a <u>third form of our calculus</u>, by introducing a rule of substitution for the letters corresponding to that of the propositional calculus. This substitution-rule allows first the substitution for a letter $\mathcal{L}$ in a formula F, in the way we have denoted it by $S^{L}_{P}F/$, where now P can be taken to be any formula of our formalism. But also substitutions for letters with arguments are allowed. Such a substitution in the case (for instance) of the letter $\mathcal{A}$ with two arguments is to be indicated in the form

$$S^{\mathcal{A}(b,\ c)}_{P(b,\ c)}F/$$

(P(b, c) and F being formulas), and it has to be performed so that wherever in F an expression $\mathcal{A}(r,\ s)$ occurs, r, s each one being either a term or a bound variable, it has to be replaced by P(r, s).[1)]

1) By this rule of substitution the letters $\mathcal{A}$, $\mathcal{B}$, ... without argument get the rôle of propositional variables and the letters with arguments the rôle of variables of predicates.

A restrictive condition is that we have to avoid "collisions" between bound variables, that means we have to be careful that in the expression arising by a substitution from a formula F no one of the quantifiers (w), (Ew) occurs in the scope of one of these quantifiers with the same variable w.

It may be observed that the same formula can in different ways occur as a formula P(r). For instance, the formula a = a can occur as the formula $P(\mathfrak{a})$ or $Q(\mathfrak{a})$ or $R(\mathfrak{a})$, where

$$\begin{aligned} &P(c) \text{ is } \mathfrak{a} = c \\ &Q(c) \text{ " } c = \mathfrak{a} \\ &R(c) \text{ " } c = c. \end{aligned}$$

By the introduction of the substitution-rule no formula becomes deducible which was not before. To prove this it will be sufficient to show that in a given deduction made with application of the substitution-rule the first occurring substitution can be eliminated.

For instance, let

$$\mathop{S}\limits_{P(c)}^{\mathfrak{A}(c)} F/$$

be the first occurring substitution and F* the formula arising by it from F. Then from the given deduction of F we get a deduction of F* by replacing every expression $\mathfrak{A}(r)$ by the corresponding P(r), only the following accessory measure being required: we have to prevent that by the replacings to be performed some application of the schemata (α), (β) may no longer satisfy the restrictive condition concerning the variable b of the schema. This could occur in case one of the variables taken for b in the formula $G \rightarrow A(b)$ of the schema (α) or in the formula $A(b) \rightarrow G$ of the schema (β) is contained in P(c). But in every such case we can change, throughout the deduction of the formula in question $G \rightarrow A(b)$ or $A(b) \rightarrow G$, the variable b into another one not occurring in P(c) (and also not occurring before in the deduction of F), without disturbing the deduction of F.

So we get a deduction of F* containing no substitution. Besides it we have, in general, to keep the deductions of F, because the formula F or also some preceding formulas may have to be applied in the later part of the given deduction.

In this way every substitution made for a letter with or without arguments can be eliminated. And so our earlier substitution-rule has for the time being the rôle of a derived rule.

We can also add, without extending the scope of deducibility, a rule of substitution for the free individual-variables, which can be represented by the schema

$$\frac{A(b)}{A(t)}$$

to be applied in such a way that the replacement of the variable b by the term t is made for <u>all</u> occurrences of b in A(b).

Indeed this is a derived rule. For from A(b) we get first (by the schema $A \rightarrow (B \rightarrow A)$)

$$(\mathcal{C} \rightarrow \mathcal{C}) \rightarrow A(b),$$

then by the schema (α):

$$(\mathcal{C} \rightarrow \mathcal{C}) \rightarrow (w)A(w)$$

and by (γ)

$$(w)A(w) \rightarrow A(t);$$

from these formulas we get, by the propositional calculus, A(t).

If we now take the two rules of substitution as fundamental rules, we can replace the ten propositional formula-schemata by the corresponding ten formulas and the formula-schemata (γ), (δ):

$$(w)A(w) \rightarrow A(t), \qquad A(t) \rightarrow (Ew)A(w)$$

by the formulas

$$(x)\mathcal{A}(x) \rightarrow \mathcal{A}(a), \qquad \mathcal{A}(a) \rightarrow (Ex)\mathcal{A}(x),$$

provided that we still adjoin a convention allowing us to change a bound variable, within its scope, into another bound variable not occurring before in this scope.

So we come to a formalism in which we have

1. twelve formal axioms, 10 of the propositional calculus

$$\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{A}) \qquad (\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})) \rightarrow ((\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\mathcal{A} \rightarrow \mathcal{C}))$$

$$\mathcal{A} \& \mathcal{B} \rightarrow \mathcal{A} \qquad \mathcal{A} \& \mathcal{B} \rightarrow \mathcal{B} \qquad (\mathcal{A} \rightarrow \mathcal{B}) \rightarrow ((\mathcal{A} \rightarrow \mathcal{C}) \rightarrow (\mathcal{A} \rightarrow \mathcal{B} \& \mathcal{C}))$$

$$\mathcal{A} \rightarrow \mathcal{A} \vee \mathcal{B} \qquad \mathcal{B} \rightarrow \mathcal{A} \vee \mathcal{B} \qquad (\mathcal{A} \rightarrow \mathcal{C}) \rightarrow ((\mathcal{B} \rightarrow \mathcal{C}) \rightarrow (\mathcal{A} \vee \mathcal{B} \rightarrow \mathcal{C}))$$

$$(\mathcal{A} \rightarrow \overline{\mathcal{B}}) \rightarrow (\mathcal{B} \rightarrow \overline{\mathcal{A}}) \qquad \overline{\overline{\mathcal{A}}} \rightarrow \mathcal{A}$$

and two for the quantifiers

$$(x)\mathcal{A}(x) \rightarrow \mathcal{A}(a), \qquad \mathcal{A}(a) \rightarrow (Ex)\mathcal{A}(x);$$

2. three schemata: the implication-schema $\dfrac{A,\ A \rightarrow B}{B}$, and the schemata ($\alpha$), ($\beta$);

3. two substitution-rules, one for the letters and one for the individual-variables, and finally the rule for changing the bound variables.

As an example of a deduction performed within this formalism we may consider the deducing of the formula

$$(x)(y)\mathfrak{A}(x, y) \rightarrow (y)(x)\mathfrak{A}(x, y),$$

which can be done as follows:

(G) $(x)\mathfrak{A}(x) \rightarrow \mathfrak{A}(a)$ (axiom)

(F) $(x)(y)\mathfrak{A}(x, y) \rightarrow (y)\mathfrak{A}(a, y)$ (from the preceding formula (G) by the substitution $S^{\mathfrak{A}(b)}_{(y)\mathfrak{A}(b,y)}(G)|$

(D) $(y)\mathfrak{A}(y) \rightarrow \mathfrak{A}(b)$ (from (G) by the substitution $S^{a}_{b}(G)|$ and changing of x into y)

$(y)\mathfrak{A}(a, y) \rightarrow \mathfrak{A}(a, b)$ (substitut. $S^{\mathfrak{A}(c)}_{\mathfrak{A}(a,c)}(D)|$

$\underline{(x)(y)\mathfrak{A}(x, y) \rightarrow (y)\mathfrak{A}(a, y)}$ (deduced formula (F))

$(x)(y)\mathfrak{A}(x, y) \rightarrow \mathfrak{A}(a, b)$ (from the last two formulas by means of the propositional calculus)

$(x)(y)\mathfrak{A}(x, y) \rightarrow (x)\mathfrak{A}(x, b)$ (from the preceding formula by the schema (α)

$(x)(y)\mathfrak{A}(x, y) \rightarrow (y)(x)\mathfrak{A}(x, y)$ (from the preceding formula by the schema (α)).

We have now three equivalent forms of our calculus, the first with subsidiary deductions, the second with formula-schemata, and the third with substitution-rules. So we can choose for each problem the most suitable form of the calculus.

<u>Remark</u>: Concerning the schema $\dfrac{\begin{matrix}\llcorner A \lrcorner\\ \vdots \\ B\end{matrix}}{A \rightarrow B}$ (which in the first formalism is a fundamental rule), we found that it is in the second formalism a derived rule with the restrictive condition that in every application of one of the schemata (α), (β) occurring in the subsidiary deduction $\begin{matrix}\llcorner A \lrcorner\\ \vdots \\ B\end{matrix}$, the variable to be taken for b has to be different from any variable contained in A.

In order that the schema may also hold, as a derived rule, for the third formalism, we have to adjoin still the condition that every letter or free variable, for which a substitution is to be made in the subsidiary deduction $\begin{matrix}\llcorner A \lrcorner \\ \vdots \\ B\end{matrix}$, is different from any letter or variable contained in A. That this accessory condition (together with the former) is sufficient to make the schema hold in the third formalism, can be shown in a way quite analogous to our proof of the corresponding part for the propositional calculus (cf. §4, pp. 45-46).

The two restrictive conditions can be taken together in the <u>one condition</u> saying that every letter and every free individual-variable contained in A has to be kept freed throughout the subsidiary deduction $\begin{matrix}\llcorner A \lrcorner \\ \vdots \\ B\end{matrix}$.

As to the formalization of the <u>equality</u> it may be remarked that the schema

$$\frac{r = s,\ A(r)}{A(s)}$$

has to be replaced in the formalism with the formula-schemata by the formula-schema

$$r = s \qquad (A(r) \longrightarrow A(s)).$$

This schema again can be replaced in the formalism with the substitution-rules by the formal axiom

$$a = b \longrightarrow (\mathcal{A}(a) \longrightarrow \mathcal{A}(b))$$

(We could already in the formalism with the formula schemata take the schema

$$a = b \qquad (A(a) \qquad A(b))$$

instead of

$$r = s \qquad (A(r) \longrightarrow A(s)),$$

because this more general schema can easily be derived from the former by means of the schemata (α), (γ) and the propositional calculus.)

6. Consistency; k-formulas; cases of deciding on deducibility

The immediate question now concerns the consistency of the pure logical calculus.

We shall prove at once somewhat more than the consistency, namely that every deducible formula of the pure logical calculus, if applied to a domain of k individuals, becomes a formula provable by the propositional calculus,- or what we found to be the same: to a +-formula.

Before explaining this assertion generally, let us consider first a special case. The formula

$$(x)(\mathcal{A}(x) \longrightarrow (Ey)\mathcal{A}(y))$$

can be deduced in our calculus.

To apply this formula on the domain of the k individuals $\alpha_1, \cdots, \alpha_k$ means to replace (Ey) by a disjunction and (x) by a conjunction taken over $\alpha_1, \ldots, \alpha_k$. Thus we get

$$(\mathcal{A}(\alpha_1) \rightarrow \mathcal{A}(\alpha_1) \vee \cdots \vee \mathcal{A}(\alpha_k)) \ \& \cdots \& \ (\mathcal{A}(\alpha_k) \rightarrow \mathcal{A}(\alpha_1) \vee \cdots \vee \mathcal{A}(\alpha_k))$$

If we here replace $\mathcal{A}(\alpha_j)$ by $\mathcal{A}_j$ (for j = 1, ..., k), the resulting formula

$$(\mathcal{A}_1 \rightarrow \mathcal{A}_1 \vee \cdots \vee \mathcal{A}_k) \ \& \cdots \& \ (\mathcal{A}_k \rightarrow \mathcal{A}_1 \vee \cdots \vee \mathcal{A}_k)$$

is a +-formula.

Now our general statement is that from every deducible formula of the pure logical calculus we come to a +-formula, if we first replace each of the different occurring free variables by some of the symbols $\alpha_1, \ldots, \alpha_k$, every generality by a conjunction, every existence by a disjunction, taken over $\alpha_1, \ldots, \alpha_k$, and afterwards replace in the arising formula each of the different prime-formulas containing $\alpha_1, \ldots, \alpha_k$ by letters (capital italics) different from another and from the other occurring letters.

Generally a formula which is transformed into a +-formula by the two indicated processes may be called a "k-formula". Our assertion then is that every deducible formula of the pure logical calculus is, for every finite positive number k, a k-formula.

To prove this we take the logical calculus in the second form, where we have the formula schemata, the implication-schema and the schemata (α), (β). Let k be a finite positive number. We now show:

1. Every formula obtained by a formula-schema is a k-formula.

2. If A and $A \longrightarrow B$ are k-formulas, then also B is a k-formula.

3. If $C \longrightarrow A(b)$ is a k-formula and b is neither contained in C nor in A(w), then also $C \longrightarrow (w)A(w)$ is a k-formula;

if $A(b) \longrightarrow C$ is a k-formula and b is neither contained in C nor in A(w), then also (Ew)A(w) is a k-formula.

1 is obvious for the formula schemata of the propositional calculus but also for the schemata

$$(w)A(w) \longrightarrow A(t), \qquad A(t) \longrightarrow (Ew)A(w)$$

(in the pure logical calculus t can only be a variable).

2 results from the fact that if A and $A \longrightarrow B$ are +-formulas, then also B is a +-formula.

3 results from the possibility of deducing from the formulas

$$P \longrightarrow Q_1, \quad P \longrightarrow Q_2, \ldots, P \longrightarrow Q_k$$

the formula

$$P \longrightarrow Q_1 \& Q_2 \& \ldots \& Q_k$$

and from

$$Q_1 \longrightarrow P, \quad Q_2 \longrightarrow P, \ldots, Q_k \longrightarrow P$$

the formula

$$Q_1 \vee \ldots \vee Q_k \longrightarrow P.$$

Thus indeed every formula deduced by the pure logical calculus must be a k-formula, for every finite positive number k.

From this result we can immediately infer that the pure logical calculus is consistent. For if A is a k-formula, then $\overline{A}$ is surely not a k-formula; so A and $\overline{A}$ cannot be both deducible by the pure logical calculus.

The property of a formula to be a k-formula can also be described in the following way. We first define, what is a "k-valuation" of a formula A of the pure logical calculus, generalizing the concept of valuation we used in the propositional calculus, in the following way: to each letter (capital italic) without argument in A we assign a constant value + or -; to a letter with n arguments we assign a logical function with n arguments ranging over $\alpha_1, \ldots, \alpha_k$, the values of which are + or -, and to each free variable we assign some of the elements $\alpha_1, \ldots, \alpha_k$; the general and existential quantifiers are interpreted as symbols for conjunctions and disjunctions taken over $\alpha_1, \ldots, \alpha_k$. Now the property of A to be a k-formula is that for every k-valuation the value of A, resulting from the definition of $\rightarrow$, $\&$, $\vee$, $^{-}$ as two-valued functions (truth-functions), must be +.

That this characterization of a k-formula comes out to the same as the above definition will easily be seen by applying the definition of a +-formula.

A slight generalization of these considerations, useful for some applications, consists in removing the restriction to formulas of the pure logical calculus. Indeed the notion of a k-formula can easily be extended to formulas which may contain symbols of individuals, predicates, mathematical functions. This is to be done in such a way that the individual-symbols are treated as the free variables, the predicate-symbols as the letters with the same number of arguments and the function-symbols are treated as symbols for arbitrary functions with the given number of arguments ranging over $\alpha_1, \ldots, \alpha_k$ and having their values out of this domain. Thus, to see if a formula is a k-formula in the more general sense, we have to replace first every free variable and every individual-symbol by one of the values $\alpha_1, \ldots, \alpha_k$; after that we assign to each function-symbol the meaning of a function, whose values for the different possible sets of arguments to be got out of the domain $(\alpha_1, \ldots, \alpha_k)$, are chosen in an arbitrary way out of the domain $(\alpha_1, \ldots, \alpha_k)$; having chosen the definition for the different function-symbols we compute the terms; every term then turns out to be one of the values $\alpha_1, \ldots, \alpha_k$; finally we replace the different prime-formulas by different letters. Now the condition is that we must come to a +-formula.

According to this generalized definition of a k-formula it can be shown, quite in the same way as before, that every formula which is deducible by our calculus without auxiliary axioms (it itself, or its deduction, may contain symbols of individuals, predicates, mathematical functions) is, for every finite number k, a k-formula.

The necessary condition for deducibility given by our proved theorem turns out, in some special cases, to be at once a sufficient condition, and so we get, for some kinds of formulas, a method of deciding on the deducibility, being a generalization of the method we have already for the propositional calculus.

A simple example of such a possibility is that of the formulas without bound variables. We consider first the case of a pure logical formula. Let

$$A(a, b, \ldots, r)$$

be such a formula, in which the only occurring individual-variables are the free variables a, b, ..., r; let their number be n. In order to be deducible, this formula must be an n-formula; so

$$A(\alpha_1, \ldots, \alpha_n)$$

must turn out to a +-formula if we replace the prime formulas by different letters. But that means that $A(\alpha_1, \ldots, \alpha_n)$ and also A(a, b, ..., r) is to be got from a +-formula by substituting prime-formulas for the letters. This condition of course is also sufficient for the deducibility of the formula.

So we get the result that a formula without bound variables, in order to be deducible by the pure logical calculus, must be deducible by the propositional schemata alone.

Now we can extend this result to the case that individual-, predicate- function-symbols may occur in the formula to be considered. This is done by the following argument. Let a, b, ..., r now denote the different primary terms (free variables and individual-symbols) in a formula A(a, b, ..., r) containing no bound variable; the number of these primary terms may be n.

Now we want to apply the condition for the deducibility of A(a,b,...,r), saying that for every k-valuation, k being a finite positive integer, the value of this formula must be +. For this purpose we have to assign values to the terms formed out of $\alpha_1, \ldots, \alpha_k$ -- a, ..., r are each to be replaced by some of these symbols -- by means of the function-symbols.

By taking k sufficiently large, we are able to choose the functions so that by the computation of the values different prime-formulas turn out again to different prime-formulas. Indeed, if we take for k the number of all different terms in A(a, ..., r), including the terms which are parts of other terms, then the values of the functions in the domain $\alpha_1, \ldots, \alpha_k$ can be chosen so that different terms in our formula have different values, because for every new term occurring we are free to choose the value in an arbitrary way out of the domain $\alpha_1, \ldots, \alpha_k$ and this domain is large enough for making correspond to every new term a new value.

So the situation becomes quite the same as in the former case, and we again get the result that the formula A(a, ..., r) in order to be deducible by our calculus, must be deducible by the propositional schemata alone.

Let us now consider another case out of the pure logical calculus. We take a formula with no quantifiers other than existential ones, operating over the whole formula, for instance

$$(Ez)A(a, b, z),$$

where $A(a, b, z)$ contains no other individual-variables than the free variables a, b and the bound variable z. If this formula is deducible, it must be a 2-formula; so

$$A(\alpha_1, \alpha_2, \alpha_1) \vee A(\alpha_1, \alpha_2, \alpha_2)$$

must arise from a +-formula by replacing the letters by prime-formulas. On the other hand, if this holds, then

$$A(a, b, a) \vee A(a, b, b)$$

is deducible by the propositional schemata; but then also our considered formula $(Ez)A(a, b, z)$ is deducible. For by the schema (δ) we have

$$A(a, b, a) \rightarrow (Ez)A(a, b, z)$$
$$A(a, b, b) \rightarrow (Ez)A(a, b, z)$$

and from these formulas, together with $A(a, b, a) \vee A(a, b, b)$ we get by the propositional schemata

$$E(z)A(a, b, z).$$

In the same way we generally recognize that a formula

$$(Ey_1) \ldots (Ey_r)A(a, b, \ldots, k, y_1, \ldots, y_r)$$

in which $a, b, \ldots, k, y_1, \ldots, y_r$ are the only individual-variables occurring and the number of the free variables $a, b, \ldots, k$ is m, is deducible when and only when it is an m-formula.

But a formula of this kind is of equal deducibility with the corresponding formula

$$(x_1) \ldots (x_m)(Ey_1) \ldots (Ey_r)A(x_1, \ldots, x_m, y_1, \ldots, y_r)$$

that is obtained from it by replacing the m free variables $a, \ldots, k$ by the bound variables $x_1, \ldots, x_m$ with the corresponding all-symbols at the beginning of the formula.

So the necessary and sufficient condition for the deducibility of a pure logical formula

$$(x_1) \ldots (x_m)(Ey_1) \ldots (Ey_r)A(x_1, \ldots, x_m, y_1, \ldots, y_r),$$

containing no free individual-variable, is that it is an m-formula.

This result can especially be applied to the uninary calculus, that is the part of the logical calculus in which every letter has at most one argument. Indeed the question, if a given formula of the pure uninary calculus is deducible, can be reduced to the case considered just now. I will give here only the general lines of this reduction.

One shows that every formula of the pure uninary calculus can be transformed into a conjunctive normal form, each of the components of which is of one of the forms

1) A_1, $A_1(C)$

2) $(w)(A_1(w) \vee \ldots \vee A_r(w))$

3) $(Ew)(A_1(w) \,\&\, \ldots \,\&\, A_r(w))$,

$A_1, \ldots, A_r$ being each a letter or a negated letter. Passing from this formula by equivalent transformations to a prenex formula (cf. §3, pp. 27, 30-31), one can manage in such a way that one gets all general quantifiers before the existential ones. Finally, if there are some free individual-variables, we pass from the formula we obtained to a formula of equal deducibility by replacing each of these free variables by a bound one and adding the corresponding all-symbols (in some order) at the beginning of the formula.

Thus one comes to a formula

$$(x_1) \ldots (x_m)(Ey_1) \ldots (Ey_r)A(x_1, \ldots, x_m, y_1, \ldots, y_r)$$

without free variables which is deducible when and only when the given uninary formula is deducible.

To come back now from these special cases and to get a more general survey, it will be suitable to state first the following facts about k-formulas. Every (k+1)-formula is also a k-formula. For, every function of $\alpha_1, \ldots, \alpha_k$ with the values $\pm$ defines a special function of $\alpha_1, \ldots, \alpha_{k+1}$ with the values $\pm$, by the prescription that the value shall not change by replacing α_{k+1} by α_k, and therefore, since repetitions of members in a conjunction of disjunction don't influence the final value, every valuation for the domain $\alpha_1, \ldots, \alpha_k$ gives the same value as some valuation for the domain $\alpha_1, \ldots, \alpha_{k+1}$.

On the other side, to every k there exist k-formulas that are not k+1-formulas: so

$$(x)\mathfrak{A}_1(x) \vee (x)(\overline{\mathfrak{A}_1(x)} \vee \mathfrak{A}_2(x)) \vee (x)(\overline{\mathfrak{A}_1(x)} \vee \overline{\mathfrak{A}_2(x)} \vee \mathfrak{A}_3(x)) \vee (x)(\overline{\mathfrak{A}_1(x)} \vee \overline{\mathfrak{A}_2(x)} \vee \overline{\mathfrak{A}_3(x)})$$

is a 3-formula, but not a 4-formula, and in the same way we get for any finite positive number k a k-formula which is not a (k+1)-formula. We may call a formula, which is a k-formula but not a (k+1)-formula, a "proper k-formula".

Thus we come to the following classification:

1. formulas that are not 1-formulas (for instance $(x)\mathfrak{A}(x)$)
2. proper 1-formulas
   2- "

   k- "
3. formulas that, for every finite positive number k, are k-formulas. We have shown that every deducible formula of our calculus belongs to the class 3.

Remark: From the method by which we proved that every formula deducible by our calculus is (for every finite positive number k) a k-formula, we obtain at once the following theorem referring to the calculus with substitution-rules: If we adjoin to our calculus some k-formulas as formal axioms, then every deducible formula is still a k-formula.

On the other hand Mr. Wajsberg has proved: If to the logical calculus with the substitution-rules, a proper k-formula is adjoined as a formal axiom, then every k-formula becomes deducible.

(M. Wajsberg "Untersuchungen über den Funktionenkalkül für endliche Individuenbereiche", Math. Ann. vol. 108, No. 2, 1933.)

Mr. Wajsberg's method of proof consists in reducing the theorem to the case of the uninary calculus.

Now the question arises, if the formulas of the class 3 are all deducible, or if, on the contrary, there are formulas which are k-formulas for every finite positive number k, but still are not deducible.

As we shall see, such formulas exist.

7. Consistency of an Elementary Axiom-System in combination with the logical calculus. Heuristic introduction. First part of the proof.

We were on the point of putting the question whether there are formulas which are k-formulas for every finite positive number k, but nevertheless not deducible by our calculus.

This question is nothing else than that concerning the impossibility of inverting our theorem that every deducible formula is a k-formula for every k. But there is still another relation between this theorem and our question. Namely, this theorem can be formulated in a second form, to which we are led by considering the deducibility of the negation of a formula F. A deduction of $\overline{F}$ may be called a "refutation" of F.

According to our theorem, a necessary condition for the deducibility of $\overline{F}$ (that is for the refutability of F) is, that for every k-valuation $\overline{F}$ has the value +, and so F has the value - ("false"), whatever finite positive number k may be. Now the same can be expressed, by the more usual terms of axiomatics, in the following way: If a formula F can be "satisfied", i.e., made to be true by a finite model with k individuals, where k is suitably chosen, then F cannot be refuted.

By this formulation we see that the consistency our calculus has is not only an interior property of it, but also in agreement with the statements about finite sets.

Now the question we raised can easily be reduced to the other, if the connection between the existence of a model satisfying a formula F and the irrefutability of F that we stated for finite models holds also for infinite models.

For there are formulas which can be satisfied only by an infinite model. Take the formula

(G) $$(x)\overline{\mathfrak{A}(x,x)} \mathbin{\&} (x)(y)(z)(\mathfrak{A}(x,y) \mathbin{\&} \mathfrak{A}(y,z) \rightarrow \mathfrak{A}(x,z)) \mathbin{\&} (x)(Ey)\mathfrak{A}(x,y).$$

One easily sees that it cannot be made to be true for a finite domain of individuals, that is, it cannot be made to give the value + for a k-valuation with a finite positive number k. But it can be satisfied for the infinite domain of the numbers 1, 2, ..., namely by choosing for $\mathfrak{A}(a,b)$ the predicate $a < b$ ("a is less than b").

Now if it holds that a formula which can be satisfied in the domain of numbers is irrefutable, then the negation of the formula G is an instance of a formula which for every finite positive number k is a k-formula but is not deducible. Thus, for answering our question, it would be sufficient to justify the assumption that a formula which can be satisfied for any (finite or infinite) domain cannot be refuted. And if we have no scruple about the infinite case, there is a rather trivial method to prove this.

Indeed, we may define a formula of our calculus to be generally true in the domain of the numbers, if one obtains from it the value + by replacing every free variable by a number, every letter without an agreement by one of the values $\pm$ and every letter with r arguments by a logical function of r number-arguments, using the definitions of $\&$, $\vee$, $\overline{}$, $\rightarrow$ as two-valued functions (truth-functions) and taking (w)A(w) to be + if for every number n, A(n) has the value +, otherwise to be -, and taking (Ew)A(w) to be + if for some number n, A(n) has the value +, otherwise to be -. That means, the formula is generally true in the domain of the numbers if for every valuation with respect to this domain it has the value +. We can also take, instead of the domain of the numbers, an arbitrary domain. Similarly we may say that a formula can be "satisfied" in a domain if there exists a valuation with respect to this domain, for which it has the value +.

We then have the relation: If a formula F is generally true for a domain D, then $\overline{F}$ cannot be satisfied for D, and if F can be satisfied for D, then $\overline{F}$ cannot be generally true.

Furthermore one sees that the property of a formula F to be generally true for the domain D, as well as the property of a formula that it can be satisfied for D, is not changed if instead of D we take a domain D' which is in a one-to-one correspondence with D; for by means of this correspondence, every function defined in D corresponds to a function defined in D', whose values for the corresponding sets of arguments are the same. Thus the reference on the domain D concerns only the cardinal-number of D. If a formula F is generally true for the domain D, then it is generally true for every sub-domain of D, or -- what comes to the same -- if a formula F can be satisfied in a domain D, it can be satisfied in every domain containing D.

This is the set-theoretic aspect of the logical calculus. From this point of view we easily prove the theorem, that every deducible formula is generally true for every domain.

The proof is quite corresponding to that for the finite domains; instead of the properties of conjunction and disjunction we are using now the definition of the values of (w)A(w), (Ew)A(w), according to which it is obvious that from the formula schemata

$$(w)A(w) \longrightarrow A(t), \quad A(t) \longrightarrow (Ew)A(w)$$

only generally true formulas can arise and that the schemata (α), (β) lead from a generally true formula again to such a formula.

Now from the statement that every deducible formula is generally true for every domain we immediately infer that a formula F, which can be satisfied in any domain D, cannot be refuted. Indeed, if F could be refuted, then F would be deducible and therefore generally true for the domain D, and so F could not be satisfied in D.

Thus it results, in particular, that the formula G, which can be satisfied in the domain of the numbers, cannot be refuted.

But this manner of reasoning presupposes the set-theoretic idea of an arbitrary assignment of values $\pm$ to the elements (respectively to the pairs, triples, ... of elements) of a domain, for instance the domain of the numbers; by introducing this idea we are neglecting the question about the possibility of effectively determining the values to be assigned. Even in simple cases this question leads to unsolved problems. For instance, if we take, in the domain of the numbers,

for $\mathfrak{A}(x, y)$	the	predicate	$x < y$
" P(x)	"	"	"x is a prime-number"
" B(x, y)	"	"	$x + 2 = y$,

then the question, if for this valuation the formula

$$(x)(Ey)(Ez)(\mathfrak{A}(x, y) \,\&\, B(y, z) \,\&\, P(y) \,\&\, P(z))$$

has the value +, is an unsolved problem.

Thus the statement that a formula turns out for a given special valuation to the value +, in many cases has no concrete meaning. If we agree to introduce such conceptions, most problems of metamathematics become rather simple. But we turn to a more concrete treatment of the problems.

In particular, for our purpose of showing the formula G to be irrefutable we want an elementary proof. On the other hand, it will be desirable to have a proving method which can easily be extended to similar cases.

To come to such a proof, our first remark is, that for proving the irrefutability of the formula G:

$$\overline{(x)\,\mathcal{A}(x,x)}\ \&\ (x)(y)(z)(\mathcal{A}(x,y)\ \&\ \mathcal{A}(y,z) \rightarrow \mathcal{A}(x,z))\ \&\ (x)(Ey)\mathcal{A}(x,y),$$

it is sufficient to show that the system of axioms

$$(1)\quad \left\{ \begin{array}{c} (x)\ \overline{x < x} \\ (x)(y)(z)(x < y\ \&\ y < z \rightarrow x < z) \\ (x)(Ey)(x < y) \end{array} \right.$$

(containing the predicate-symbol <) together with our logical calculus in the second form (with the formula-schemata) is consistent. For, if G should be deducible, then (after the introduction of the predicate-symbol <) the formula

$$S^{\mathcal{A}(b,c)}_{b<c}\ \overline{G} \Big|$$

would also be deducible, since the substitution-rule for the letters is a derived rule in our calculus.

On the other hand this formula is the negation of the formula

$$(x)\ \overline{x < x}\ \&\ (x)(y)(z)(x < y\ \&\ y < z \rightarrow x < z)\ \&\ (x)(Ey)(x < y),$$

which is the conjunction of the axioms (1) and therefore can be deduced from them. Thus the axioms (1) would lead to a contradiction.

We also easily see that from a deduction leading to a contradiction we should be able to eliminate all occurring letters (capital italics), by replacing every letter without an argument by the formula $(x)(x < x)$, every letter with one argument a by $a < a$, every letter with $(n+1)$ arguments $a_1, \ldots, a_{n+1}$ by $a_1 < a_2 \mathbin{\&} a_1 < a_3 \mathbin{\&} \ldots \mathbin{\&} a_1 < a_{n+1}$.

Thus we can restrict our calculus by the condition that all occurring prime-formulas are of the form $r < s$.

A further remark is that the axioms (1) can be deduced from the axioms

$$(2)\left\{\begin{array}{l} \overline{a < a} \\ a < b \rightarrow (b < c \rightarrow a < c) \\ a < a' \end{array}\right.$$

where the prime in a' is introduced as a <u>function-symbol</u> corresponding to the concept of successor. If the system (2) is consistent (together with the logical calculus) then (1) is consistent; thus it will be sufficient to prove the consistency of the system (2).

The axioms (2) contain no bound variable; they can be interpreted in an elementary way as expressing relations between arbitrary given numbers, put instead of the free variables.

To make use of this fact it will be suitable to represent the numbers in our formalism. For this purpose we introduce the individual-symbol 0. The terms we get from this symbol by means of the function-symbol ' (as for instance 0', 0'''), and also the symbol 0 itself, may be called "numerals".

From the intuitive definition of a numeral it is obvious that if a, b are different numerals, then either the process of forming b is a continuation of the process of forming a or inversely. In the first case we say that $a < b$ is "true" (has the value +), and $b < a$ is "false" (has the value −); in the other case $b < a$ is said to be true, $a < b$ to be false. A formula $a < a$ with a numeral a is defined to be false.

Thus we have a valuation for the "numerical" prime-formulas, that is for the formulas $a < b$ where a, b are numerals. Applying the definitions of the operations $\rightarrow$, &, $\vee$, $\overline{}$ as truth-function, we come to extend the valuation to general "numerical formulas", that is, to formulas which are either numerical prime-formulas or formed out of such formulas with the operations $\rightarrow$, &, $\vee$, $\overline{}$.

By means of this valuation of the numerical formulas the property of the axioms (2) to express elementary number-relations can be formulated more distinctly in the following way: if we replace in the formulas (2) each of the variables a, b, c (for all its occurrences) by some numeral, then the formulas become true formulas.

Now our problem is the following: We have a formalism in which

1. every term is either a numeral or a free variable, or obtained from a free variable by applying (one or more times) the function-symbol '.
2. every prime formula is of the form $r < s$ where r, s are terms.
3. every formula is either a prime-formula or formed out of prime-formulas with the operations $\rightarrow$, &, $\vee$, $\overline{}$, (w), (Ew).
4. the starting formulas are the axioms (2) and the formulas arising from our formula-schemata, the schemata being the 10 propositional ones and furthermore (γ') and (δ).
5. the rules for passing from formulas to others are the implication-schema and the schemata (α), (β).

We have to show that by this formalism no two formulas A and $\overline{A}$ can be deduced.

This statement of consistency can still be replaced by a positive assertion. First we remark that the consistency will be proved as soon as at least one formula is shown not to be deducible. For in case a formula A and also its negation could be deduced from our formalism, then (as we stated in §3, pp. 18-19) every formula of the formalism would be deducible. Thus it is sufficient for our purpose to prove that the formula $0 < 0$ cannot be deduced.

But $0 < 0$ is a false formula, and so the impossibility of deducing it will follow if we show that every deducible numerical formula of our formalism is a true formula. Now we go on to prove this theorem. First we observe that we get an equivalent statement by replacing the axioms (2) by corresponding formula-schemata:

$$\begin{gathered} t < t \\ r < s \rightarrow (s < t \rightarrow r < t) \\ t < t' \end{gathered} \tag{3}$$

where r, s, t are denoting terms.

Our demonstration will consist of two parts: First we restrict the logical calculus by excluding the bound variables and thus applying only the propositional schemata. For this restricted calculus we are to show that every numerical formula which is deducible by it from the formula-schemata (3) is a true formula.

Afterwards we show that from a deduction of a numerical formula made by means of our whole formalism we can eliminate the bound variables and thus obtain a deduction by the restricted calculus. From the two statements together the proof of our theorem results.

The first part of the demonstration is rather easily accomplished. Indeed if we have a deduction of a numerical formula G from the formula-schemata (3) which is made only by means of the propositional schemata and without using bound variables, then we can first replace every occurring free variable by the symbol 0 without disturbing the deduction or changing the formula G.

By this replacement every formula of our deduction becomes numerical. Furthermore the deduction has the following properties:

Every formula is either a starting formula or it is obtained from two preceding formulas by means of the implication-schema $\frac{A,\ A \rightarrow B}{B}$;

Every starting formula is obtained either from one of the 10 propositional formula-schemata by taking for the arbitrary formulas some numerical formulas, or from one of the formula-schemata (3) by taking for the arbitrary terms some numerals.

In the first case the formula is likewise to be obtained from a +-formula, by replacing each letter (for all its occurrences) by some numerical formula; in the second case from one of the axioms (2), by replacing each free variable (for all its occurrences) by some numeral.

But from this we immediately can infer -- according to the characteristic property of the +-formulas and to what we stated about the formulas (2) (cf.p.70) -- that every starting formula of our deduction is a true formula. And, since the implication-schema, if applied to true formulas A, A→B leads again to a true formula, all the formulas of our deduction must be true formulas, especially the formula G.

Thus our theorem will be proved if we can show that from a deduction of a numerical formula performed in our considered formalism (cf. p. 70) we can eliminate the bound variables and thus get a deduction of the same formula by the schemata (3) and the propositional formula-schemata.

We shall prove somewhat more, namely that this possibility of eliminating the bound variables holds not only if the deduced formula is a numerical one, but more generally in case it contains no bound variable.

8. The ϵ -schema. Possibility of Eliminating the ϵ -symbol

We are now to exhibit a method for eliminating the bound variables from a given deduction of a formula M, which itself contains no bound variable. The deduction is assumed to be performed with our formalism containing the ten propositional formula schemata, the implication schemata, the schemata (α), (β), (γ), (δ) and the axioms (2). The formulas of the deduction are assumed to contain no other symbols than

$$\rightarrow, \ \&, \ \vee, \ \overline{}, \ (w), \ (Ew), \ <, \ 0, \ {}',$$

and no other variables than the free and the bound individual-variables.

Many of these assumptions, as we shall see, are really not required for the possibility of the elimination. But there is no harm in keeping them at first, because the possible generalizations of our assumptions will afterwards be rather obvious.

As the result of the elimination we have to get a deduction of M which uses only the propositional schemata and the formula-schemata (3) which correspond to the axioms (2). (That we have to take here the formula-schemata (3) instead of the formulas (2) is in order to get along without adding a substitution-rule for the free variables.)

Our process of elimination consists of several steps. The first of them is the introduction of the <u>Hilbert ϵ-symbol</u>

$$\epsilon_w A(w).$$

This symbol is closely related to the ι-symbol

$$\iota_w A(w)$$

by which the concept "the thing that has the property A" -- which is called a "description" by Whitehead and Russell -- is formalized.

According to the usual and also to the scientific language, an expression like "the thing that has the property A" has a meaning when and only when there is a sole thing of the property A. Corresponding to this notion we can establish the following rule for the ι-symbol:

If $A(c)$ is a formula and the formulas

$$(Ex)A(x), \quad (x)(y)(A(x) \,\&\, A(y) \rightarrow x = y)$$

(x, y being different bound variables) are deducible, then

$$\iota_x A(x)$$

is a term and

$$A[\iota_x A(x)]$$

can be taken as a starting formula.

Explanation: The notation $A[\iota_x A(x)]$ with the square bracket is to indicate that the expression standing as argument may differ from $\iota_x A(x)$ by the changing of some bound variables. Such a changing is compulsory in case that by taking $\iota_x A(x)$ itself as argument a collision between bound variables would arise, "collision" now to be understood in the extended sense that in the scope of a variable w belonging to a quantifier or to a ι-symbol, an expression

$$(w)B(w) \text{ or } (Ew)B(w) \text{ or } \iota_w B(w)$$

with the same variable w occurs.

Remark: Because of the occurrence of the symbol $=$ in the rule for the ι-symbol, the application of this rule presupposes the introduction of the equality-rules.

From the rule for the ι-symbol we come to the schema for the ϵ-symbol by:

first, removing the condition of the deducibility of $(x)(y)(A(x) \,\&\, A(y) \rightarrow x = y)$,

then, taking instead of the condition of the deducibility of $(Ex)A(x)$ an antecedent $(Ex)A(x)$, so that, corresponding to the starting formulas $A[\iota_x A(x)]$, we get the formulas

$$(Ex)A(x) \rightarrow A[\epsilon_x A(x)],$$

and finally, replacing this formula-schema by

$$A(t) \rightarrow A[\epsilon_x A(x)],$$

(the corresponding formulas arising from the two schemata being of equal deducibility).

In this way we get rid of the restrictive conditions we have in the rule for the ι-symbol, and so the rule for the ϵ-symbol is simply the following:

If A(c) is a formula not containing the bound variable x, then $\epsilon_x A(x)$ is a term; and we have the "ϵ-schema":

$$A(t) \rightarrow A[\epsilon_x A(x)],$$

where again the square bracket indicates that some bound variables occurring in $\epsilon_x A(x)$ may be changed into others.

To have an instance for the case where such a change of a bound variable is required to avoid a collision between bound variables, we may take for A(c) the formula

$$(u)(u < c);$$

the corresponding term $\epsilon_x A(x)$ is

$$\epsilon_x (u)(u < x).$$

If now we should put this term in the place of c in $(u)(u < c)$, a collision between bound variables would arise and so we should not get a formula of our formalism.

Thus to apply the ϵ-schema

$$A(t) \rightarrow A[\epsilon_x A(x)]$$

to the formula $(u)(u < c)$, we have first to change the variable u in $\epsilon_x (u)(u < x)$ in another bound variable, say v, and so we get

$$(u)(u < t) \rightarrow (u)(u < \epsilon_x (v)(v < x)),$$

where for t some term can be taken. (If t is of the form $\epsilon_w B(w)$ we have again to take care that it does not contain the variable u.)

According to our rule for the ϵ-symbol, this symbol $\epsilon_x A(x)$ has the rôle of $\iota_x A(x)$ in the case that

$$(Ex)A(x),\ (x)(y)(A(x) \,\&\, A(y) \rightarrow x = y)$$

are deducible formulas; in case that only $(Ex)A(x)$ is deducible, it gives a formalizing of the principle of choice ("Auswahlprinzip") in a specialized form.

Furthermore the quantifiers can be defined by means of the ϵ-symbol. In fact, by the ϵ-schema and the schema (δ) we easily deduce the equivalence

(1) $$(Ex)A(x) \sim A[\epsilon_x A(x)]$$

for any formula $A(c)$ (not containing x).

Applying this equivalence to $\overline{A(c)}$ we get

$$(Ex)\overline{A(x)} \sim \overline{A[\epsilon_x \overline{A(x)}]}$$

and from this, by using the equivalence

$$(x)A(x) \sim \overline{(Ex)}\overline{A(x)},$$

we get

(2) $$(x)A(x) \sim A[\epsilon_x \overline{A(x)}].$$

If now we regard the equivalences (1), (2) as definitions of the quantifiers and eliminate the quantifiers by replacing, according to these definitions, every formula $(Ex)A(x)$ by the corresponding $A[\epsilon_x A(x)]$ (with suitably chosen bound variables) and every formula $(x)A(x)$ by the corresponding $A[\epsilon_x \overline{A(x)}]$, then

1. the schema (δ) becomes identical with the ϵ-schema;
2. the schema (γ) becomes

$$A[\epsilon_w \overline{A(w)}] \rightarrow A(t),$$

which can be obtained by the ϵ-schema as follows:

$$\overline{A(t)} \rightarrow \overline{A[\epsilon_w \overline{A(w)}]} \quad (\epsilon\text{-schema applied to } \overline{A(c)})$$

$$A[\epsilon_w \overline{A(w)}] \rightarrow A(t) \quad \text{(Contraposition)};$$

3. instead of the schemata (β), (α) we have the passage from a formula

$$A(b) \rightarrow C$$

to

$$A[\epsilon_w A(w)] \rightarrow C$$

and from

$$C \rightarrow A(b)$$

to

$$C \rightarrow A[\epsilon_w A(w)],$$

where in both cases b is not contained in C nor in A(w); but both these passages are nothing else than substitutions made for the free variable b.

Thus the quantifiers together with their schemata (α), (β), (γ), (δ) can be eliminated by means of the ϵ-symbol and the ϵ-schema, if we still adjoin the rule of substitution for the free variables, allowing us to pass from a formula A containing the free variable a to

$$S^{a}_{t} A|$$

where t is some term (provided that no collision between bound variables arises).

But also this substitution-rule can be avoided if we take instead of the axioms (2) the corresponding formula-schemata (3). Indeed we can argue here quite as we did in §5 (cf. p. 52) for showing that the substitution-rule for the letters is a derived rule in the calculus with the formula schemata. Let

$$\frac{B(a)}{B(t)}$$

be, in a given deduction, the first application of the substitution rule. In the part of the deduction leading to B(a) -- (this part contains no substitution) -- we replace everywhere the variable a by the term t,- performing at once suitable changes of bound variables, wherever it is required for preventing collisions between bound variables. (We may observe that t can be of the form $\epsilon_w A(w)$ or contain a part of this form.)

By this process we get from the deduction of B(a) a deduction of B(t). For, since the schemata (α), (β) are eliminated, and the axioms (2) have been replaced by the formula-schemata (3), and the deduction of B(a) contains no substitution, this deduction uses only formula-schemata and the implication-schema,-

the formula-schemata being : 1) those of the propositional calculus, 2) the ϵ-schema, 3) the schemata (3).

Now in each of these schemata the variable a can occur only as a part of an arbitrary formula or as an instance of an arbitrary term or as a part of an arbitrary term. Thus all that is done in the deduction of B(a) with the variable a can also be done with the term t, only with the previously mentioned restriction concerning the possible collisions of bound variables.

So we have now a deduction of B(t) without a substitution. To this we add the former deduction of B(a), in order to have at our disposition for the further part of the deduction all formulas we had before. By this procedure one application of the substitution-rule has been eliminated. Going on in the same way we can eliminate all substitutions.

Let us now summarize our last results: we are to prove that from a deduction of a formula M containing no bound variables, the deductions being made by means of the logical calculus in the second form (with the formula-schemata) and the axioms (2), the bound variables can be eliminated.

For the purpose of this elimination we can effect the following preparatory reductions: first introducing the ϵ-symbol and the ϵ-schema, then replacing every formula $(Ex)A(x)$ by $A[\epsilon_x A(x)]$

and every formula $(x)A(x)$ by $A[\epsilon_x \overline{A(x)}]$,

then, adding the rule of substitution for the free variables, we can eliminate the quantifiers and the schemata (α), (β), (γ), (δ).

Furthermore we can replace the axioms (2) by the formula-schemata (3) and then eliminate the substitutions, bringing them back to the starting formulas.

By all these processes the formula M at the end of the deduction has not been changed, since it contains no bound variable.

After the reductions have been performed, the schemata of our calculus are only the propositional schemata, the formula-schemata (3) and the ϵ-schema.

In this calculus we have, in particular, the simplification that the schema $\dfrac{\begin{matrix}\lfloor A \rfloor \\ \vdots \\ B\end{matrix}}{A \to B}$, as a derived rule, holds without any restriction.

On the other hand we have a complication with respect to the concept of a term, arising from the possibility of getting from a formula A(c) a term $\epsilon_x A(x)$. Such a term may be called an "ϵ-term".

The problem of our proposed proof is now reduced to that of showing that from a given deduction of a formula M containing no bound variable, which is made by means of our propositional schemata, the formula-schemata (3) and the rule of the ϵ-symbol, the ϵ-symbol can be eliminated.

For this again it is sufficient to show that the applications of the ϵ-schema can be eliminated, because, if no application of the ϵ-schema occurs in a deduction of M, we can replace every ϵ-term by 0 without disturbing the deduction nor changing the formula M.

Thus the whole problem reduces to the elimination of the starting formulas of the form

$$A(t) \to A[\epsilon_x A(x)]$$

arising from the ϵ-schema.

A starting formula of this kind may be called a "critical formula", "belonging to the term $\epsilon_x A(x)$".

To perform the elimination of the critical formulas we have first to consider somewhat more closely the possible structures of ϵ-terms.

Since the ϵ-symbol can be applied to a formula already containing ϵ-terms, we have to deal with various compositions of ϵ-terms. There are two different ways in which an ϵ-term can occur in the formation of another such term:

Let $\epsilon_x B(x)$ occur in the formation of another ϵ-term, say $\epsilon_w A(w)$; then either $\epsilon_x B(x)$ is a part of $A(w)$ or the expression obtained from $\epsilon_x B(x)$ by changing some free variable into the bound variable w is a part of $A(w)$. In the first case we shall say that $\epsilon_x B(x)$ is "placed within $\epsilon_w A(w)$", in the second that the expression arising from $\epsilon_x B(x)$ by the changing of the variable is "embodied in $\epsilon_w A(w)$".

For instance the term

$$\epsilon_x(x' < \epsilon_y(0'' < y))$$

contains the term $\epsilon_y(0'' < y)$ as placed within it, whereas

$$\epsilon_x(x' < \epsilon_y(x'' < y))$$

contains $\epsilon_y(x'' < y)$ as embodied in it. Observe that $\epsilon_y(x'' < y)$ is not a term because of the occurrence of the bound variable x without an ϵ-symbol belonging to it. We may speak in general of "<u>ϵ-expressions</u>" to denote in common the ϵ-terms and the expressions arising from ϵ-terms by changing one or more free variables into bound ones. And we may use the word "embodied" generally where an ϵ-expression is a part of another ϵ-expression $\epsilon_w A(w)$ and contains the bound variable w.

Let us consider what ϵ-expressions occur in the formula we get from

$$(x)(E y)(x < y)$$

eliminating the quantifiers by means of the ϵ-symbol. $(E y)(a < y)$ is to be replaced by $a < \epsilon_y(a < y)$, which may be abbreviated to $B(a)$. Thus $(x)(E y)(x < y)$ is first to be replaced by $(x)B(x)$, and further by $B[\epsilon_x \overline{B(x)}]$, where

$$\epsilon_x \overline{B(x)} \quad \text{is} \quad \epsilon_x(\overline{x < \epsilon_y(x < y)})$$

To form $B[\epsilon_x \overline{B(x)}]$, we have to change the variable y occurring in $\epsilon_x \overline{B(x)}$, in order to avoid a collision between bound variables, into another bound variable, say z.

So we get as the formula replacing $(x)(Ey)(\overline{x<y})$ the following:

$$\epsilon_x(\overline{x<\epsilon_z(x<z)})<\epsilon_y(\epsilon_x(\overline{x<\epsilon_z(x<z)})<y).$$

This formula is already rather complicated, and we see that, for the practical purposes of formalizing, the technique of expressing (w), (Ew) by means of the ϵ - symbol is not advisable.

For considering the different formations of ϵ -terms this example is very suitable. We have here the term

$$\epsilon_y(\epsilon_x(\overline{x<\epsilon_z(x<z)})<y)$$

which contains the term $\epsilon_x\,\overline{x<\epsilon_z(x<z)}$ placed within it, but no ϵ - expression embodied in it. The term $\epsilon_x\,\overline{x<\epsilon_z(x<z)}$ contains $\epsilon_z(x<z)$ as embodied in it but no ϵ -term placed within it.

Referring to the kinds of composition of ϵ -terms, we define the "degree" of an ϵ -term and the "order" of an ϵ -expression.

An ϵ -term containing no ϵ -terms placed within it is of degree 1; an ϵ -term containing at least one ϵ -term placed within it of degree k but none of higher degree, is of degree $k+1$.

An ϵ -expression containing no ϵ -expression embodied in it is of order 1; an ϵ -expression containing at least one ϵ -expression embodied in it of order k but none of higher order, is of order $k+1$. For instance the ϵ -term

$$\epsilon_y(\epsilon_x(\overline{x<\epsilon_z(x<z)})<y)$$

is of degree 2 and of order one, whereas

$$\epsilon_x(\overline{x<\epsilon_z(x<z)})$$

is of degree 1 and of order 2. This example shows at once that an ϵ -term can be of lower order than another placed within it.

A general remark we have to use is that the order of an ϵ -expression is not changed if a term which is a proper part of it is replaced by another term. This follows from the fact that an ϵ -term cannot be embodied in any ϵ -expression.

After these preliminaries we now come to show how the critical formulas can be eliminated from a given deduction made by means of the propositional schemata, the formula-schemata (3) and the ϵ -schema, provided that the deduced formula M contains no ϵ -expression.

To explain the Hilbert idea for this elimination let me first consider the special case where all critical formulas belong to the same ϵ-term $\epsilon_w A(w)$. Let the critical formulas be

$$A(t_1) \rightarrow A[\epsilon_w A(w)]$$
$$\vdots$$
$$A(t_k) \rightarrow A[\epsilon_w A(w)].$$

We first adjoin the formula $A(t_1)$ as an axiom and replace every term equal to $\epsilon_w A(w)$ occurring in the given deduction by t_1.

Remark: For this replacement as well as for those to be made later in the procedure of eliminating the critical formulas, it is to be understood that ϵ -expressions which differ only by some bound variables standing in corresponding places, are considered to be equal and that in performing the replacements we have to be careful to avoid collisions between bound variables by suitable changes of bound variables. The replacement to be performed does not disturb the applications of the propositional schemata nor those of the schemata (3). The critical formulas are changed by it into

$$A_1 \rightarrow A(t_1)$$
$$\vdots$$
$$A_k \rightarrow A(t_1),$$

where A_1, ..., A_k are some formulas we do not need to consider more closely. ($A[\epsilon_w A(w)]$ must change into $A(t_1)$, because $\epsilon_w A(w)$ cannot be equal to a part of $A(c)$.) But these formulas

$$A_1 \rightarrow A(t_1), \; \dots, \; A_k \rightarrow A(t_1)$$

can all be deduced from $A(t_1)$.

So we have a deduction of the formula M by means of the adjoined axiom $A(t_1)$, in which the ϵ-schema is not used. Now, according to the fact that in our formalism the schema $\dfrac{\begin{matrix}\lfloor A \rfloor \\ \vdots \\ B\end{matrix}}{A \to B}$ is a derived rule, from the deduction we have of the formula M by means of $A(t_1)$ we get a deduction of

$$A(t_1) \to M$$

in which neither the axiom $A(t_1)$ nor the ϵ-schema is used.

In quite a corresponding way we get deductions of the formulas

$$A(t_2) \to M, \ \ldots, \ A(t_k) \to M$$

which are all made without an application of the ϵ-schema. The implications

$$A(t_1) \to M, \ \ldots, \ A(t_k) \to M$$

together give us, by the propositional calculus, the formula

((1)) $$A(t_1) \vee \ldots \vee A(t_k) \to M.$$

On the other hand, if we take as an axiom the formula

$$\overline{A(t_1) \vee \ldots \vee A(t_k)}$$

which can be transformed into

$$\overline{A(t_1)} \ \& \ldots \& \ \overline{A(t_k)},$$

we first get the formulas

$$\overline{A(t_1)}, \ \ldots, \ \overline{A(t_k)}.$$

But from $\overline{A(t_j)}$ $(j = 1, \ldots, k)$ we get, by the propositional calculus

$$A(t_j) \to A[\epsilon_w A(w)].$$

(Generally from $\overline{B}$ we can deduce $B \to C$.)

Thus the critical formulas in our given deduction of M become now deduced formulas and we get a deduction of M by means of the axiom $\overline{A(t_1) \vee \ldots \vee A(t_k)}$, in which no application of the ϵ-schema is made. From this deduction again we obtain a deduction of the formula

((2)) $$\overline{A(t_1) \vee \ldots \vee A(t_k)} \to M,$$

which is made without using the axiom $\overline{A(t_1) \vee \dots \vee A(t_k)}$ nor the ϵ-schema.

The formulas ((1)) and ((2)) together lead to M, by the propositional calculus.

So, finally, we come to a deduction of M containing no application of the ϵ-schema. This deduction consists of (k+2) parts. In the j-th part (j = 1, ..., k), leading to the formula $A(t_j) \longrightarrow M$, the ϵ-terms equal to $\epsilon_w A(w)$ have to be replaced by t_j; furthermore the antecedent $A(t_j)$ has to be everywhere adjoined (and to be carried through the applications of the implication-schema). In the (k+1)-th part, leading to ((2)), the antecedent $\overline{A(t_1) \vee \dots \vee A(t_k)}$ has to be everywhere adjoined (but no replacement has to be made). The last part is the deduction of M from the formulas

$$A(t_1) \longrightarrow M, \dots, A(t_k) \longrightarrow M, \quad \overline{A(t_1) \vee \dots \vee A(t_k)} \longrightarrow M$$

by means of the propositional schemata.

Thus in the considered special case the elimination of the critical formulas can be performed.

Now to come to the general case, we have to deal with a deduction which may contain critical formulas belonging to different ϵ-terms. To eliminate all these critical formulas we shall proceed in such a way that we distribute the critical formulas into sets, joining into one set all critical formulas belonging to the same ϵ-term (or to equal ϵ-terms), bring these sets in a certain order, and then apply to each of them, proceeding according to the introduced order, the Hilbert eliminating method.

The difficulty we have to face here is that, by the replacing-processes we have to perform for the elimination of the critical formulas of one set, new sets of critical formulas may arise or, also, what would be still worse, some critical formulas may lose their form as formulas arising from the ϵ-schema.

So we must consider somewhat more closely how a critical formula

$$A(t) \longrightarrow A[\epsilon_w A(W)]$$

can be affected if we replace an ϵ-term $\epsilon_u B(u)$ occurring in it, which is different from $\epsilon_w A(w)$, by a term S.

The classification of the possible cases will be made according to the place where $\epsilon_u B(u)$ occurs in the critical formula. There are the following possibilities:

1. $\epsilon_u B(u)$ can be identical with t **or** a part of t, whereas it does not occur elsewhere at all.
2. It can occur in A(w) and A(t) outside of the argument; in this case it may also occur as a part of t or be t itself, as in 1, but it is not permitted to occur in any other way.
3. It can occur in A(t) containing t as a part or in $A[\epsilon_w A(w)]$ containing $\epsilon_w A(w)$ (or a term equal to it) as a part. Both these kinds of occurrence may happen together, or together with the first there may still be the occurrence in A(t) and A(w) outside of the argument.)

In <u>case 1</u>, after replacing $\epsilon_u B(u)$ by S we get a formula

$$A(r) \longrightarrow A[\epsilon_w A(w)],$$

which is a critical formula still belonging to $\epsilon_w A(w)$.

In <u>case 2</u>, we get a formula

$$A^*(r) \longrightarrow A^*[\epsilon_w A^*(w)],$$

which is a critical formula belonging to $\epsilon_w A^*(w)$. A(w) has the form $C(\epsilon_u B(u), w)$, thus

$$\epsilon_w A(w) \text{ is } \epsilon_w C(\epsilon_u B(u), w)$$

and

$$\epsilon_w A^*(w) \text{ is } \epsilon_w C(s, w).$$

As we see, $\epsilon_u B(u)$ is placed within $\epsilon_w A(w)$, and so $\epsilon_w A(w)$ has a higher degree than $\epsilon_u B(u)$; furthermore $\epsilon_w A^*(w)$ has the same order as $\epsilon_w A(w)$, since it arises from $\epsilon_w A(w)$ by replacing a term which is a proper part of it by another term.

In case 3, the critical formula can possibly lose its characteristic form. But this case can occur only if $\epsilon_w A(w)$ is of higher order than $\epsilon_u B(u)$. In fact, $\epsilon_u B(u)$ must have one of the forms

$$\epsilon_u D(u, t), \qquad \epsilon_u D(u, \epsilon_w A(w)),$$

$A(c)$ must be of the form $H(\epsilon_u D(u, c))$ and $\epsilon_w A(w)$ of the form $\epsilon_w H(\epsilon_u D(u, w))$. But $\epsilon_u D(u, w)$ has the same order as $\epsilon_u D(u, c)$, and also the same order as each of the terms $\epsilon_u D(u, t)$, $\epsilon_u D(u, \epsilon_w A(w))$, one of which is $\epsilon_u B(u)$. Therefore $\epsilon_w A(w)$ has a higher order than $\epsilon_u B(u)$.

Thus we have the following results: The formula

$$A(t) \longrightarrow A[\epsilon_w A(w)]$$

remains a critical formula belonging still to $\epsilon_w A(w)$ if the order and the degree of the term $\epsilon_u B(u)$ to be replaced by another term is at least as high as that of $\epsilon_w A(w)$. And the formula turns to a critical formula belonging to an ϵ-term of the same order as $\epsilon_w A(w)$ if the order of $\epsilon_u B(u)$ is at least as high as that of $\epsilon_w A(w)$.

Using these statements we can proceed in the following way: We bring the different ϵ-terms to which critical formulas belong, into a succession

$$e_1, \ldots, e_m, e_{m+1}, \ldots, \ldots e_n$$

such that always e_{n+1} has at most the same order as e_n, and if e_{n+1} has the same order as e_n, the degree of e_{n+1} is at most the same as the degree of e_n. Let $e_1, \ldots, e_m$ be the terms of the highest order.

Now if we eliminate the critical formulas belonging to e_1, the formulas arising from the critical formulas belonging to $e_2, \ldots, e_m$ by the change of e_1 into other terms are again critical formulas belonging to $e_2, \ldots, e_m$ respectively; and the formulas arising from the critical formulas belonging to $e_{m+1}, \ldots e_n$ are critical formulas belonging respectively to an ϵ-term of the same order. Thus the number of different ϵ-terms of the highest order to which critical formulas belong has become less by one, and there is no ϵ-term to which critical formulas belong of an order not occurring before. Going on in the same way we can eliminate all the critical formulas belonging to ϵ-terms of the highest order (occurring in our original deduction).

So we have a method of diminishing the number of different orders of ϵ-terms to which critical formulas belong, and by repeated applications of it we can reduce this number to zero; that means eliminate all the critical formulas.

This accomplishment of the Hilbert device is due to Wilhelm Ackermann. (Not published.)

9. Extensions of the Result of our Consistency-Proof

Let us now discuss what results we can infer from our method of eliminating the bound variables.

The theorem we proved in the last section says that from a deduction made by our logical calculus in the second form (with the formula-schemata) and the axioms (2) leading to a formula M which contains no bound variable, we can obtain a deduction of the same formula M made only by the propositional schemata and the formula-schemata (3).

From this statement, together with our arguments of §7, we get the result that the axioms (2) together with our logical calculus are consistent, and that in the strict sense every numerical formula which is deducible from the axioms (2) by means of our logical calculus, is a true formula.

As an immediate consequence of this we may note that every formula formed out of the arithmetical symbols $<$, 0, ' and the logical symbols $\longrightarrow$, $\&$,

$\vee$, $\overline{}$, and free individual-variables, which is deducible from the axioms (2) by means of our logical calculus, has the property that by replacing the free variables in it by numerals we obtain a true formula. A formula of this property may be called a "<u>verifiable formula</u>".

That indeed every formula F of the kind described is verifiable follows simply for the reason that every replacement of the free variables in F can be performed by a deduction-process in our logical calculus (cf. §5, pp. 52-53) and therefore a formula obtained from F by replacing the free variables by numerals is deducible, as well as F itself, from the axioms (2) by our logical calculus, so that our result about the deducible numerical formulas applies to it.

There is still another kind of formula to which we can apply our elimination-method. Let M be a formula

$$(Ex_1) \dots (Ex_n) H(x_1, \dots, x_n),$$

containing no other bound variable than $x_1, \dots, x_n$, for which we have a deduction by means of our logical calculus and the axioms (2).

Let us consider the effect of applying to this deduction the method of §8.

First we have to eliminate the quantifiers by means of the ϵ-symbol; hereby the final formula M is changed into a formula

$$H(s_1, \dots, s_n)$$

where $s_1, \dots, s_n$ are certain ϵ-terms. (This is easily shown by an intuitive induction with respect to n.)

At once the schemata (α), (β), (γ), (δ) are to be eliminated and the axioms (2) to be replaced by the schemata (3).

Now the different sets of critical formulas have to be eliminated. Here a little modification of our procedure is required, because the final formula can be changed by the replacements to be performed. Let

$$A(t_1) \longrightarrow A[\epsilon_w Aw], \ldots, A(t_k) \longrightarrow A[\epsilon_w A(w)]$$

be the first set of critical formulas to be eliminated.

Proceeding as before we obtain from the deduction we have of the formula $H(s_1, \ldots, s_n)$ first k deductions, corresponding to the k critical formulas. In the j-th of these deductions we replace everywhere $\epsilon_N A(w)$ by t_j and afterwards adjoin in all formulas the antecedent $A(t_j)$, whereby we get, instead of the k critical formulas to be eliminated, formulas of the form

$$A(t_j) \longrightarrow (A_1 \longrightarrow A(t_j)), \ldots, A(t_j) \longrightarrow (A_k \longrightarrow A(t_j))$$

which are deducible by the propositional schemata. The final formula of this j-th deduction has the form

$$A(t_j) \longrightarrow H(s_1^{(j)}, \ldots, s_n^{(j)}), \qquad ((j))$$

where $s_1^{(j)}, \ldots, s_n^{(j)}$ are some terms which in general will be different from $s_1, \ldots, s_n$.

Furthermore we obtain, by adjoining in all formulas the antecedent

$$\overline{A(t_1) \vee \ldots \vee A(t_k)}$$

(but not performing any replacement), a deduction of

$$\overline{A(t_1) \vee \ldots \vee A(t_k)} \longrightarrow H(s_1, \ldots, s_n). \qquad ((k+1))$$

Now this situation differs from the corresponding one in the procedure of §8 in that in the k+1 formulas ((1)), ..., ((k)), ((k+1)), which we are now obtaining, the second member of the implication is not the same. Denoting the second implication-member in the formula ((i)) by H_i ($i = 1, \ldots, k+1$) we have

$$A(t_1) \longrightarrow H_1, \ldots, A(t_k) \longrightarrow H_k$$

$$\overline{A(t_1) \vee \ldots \vee A(t_k)} \longrightarrow H_{k+1};$$

from these formulas we get by the propositional calculus, using the deducibility of

$$A(t_1) \vee \ldots \vee A(t_k) \vee \overline{A(t_1) \vee \ldots \vee A(t_k)},$$

the formula

$$H_1 \vee \dots \vee H_k \vee H_{k+1},$$

that is

$$H(s_1^{(1)}, \dots, s_n^{(1)}) \vee \dots \vee H(s_1^{(k)}, \dots, s_n^{(k)}) \vee H(s_1, \dots, s_n).$$

Thus by the process of eliminating the first set of critical formulas the final formula $H(s_1, \dots, s_n)$ of the deduction is changed into a disjunction; this change however is not essential since every member of the disjunction has the form

$$H(r_1, \dots, r_n),$$

$r_1, \dots, r_n$ being terms. Now after the elimination of the first set of critical formulas the final formula has the form of a disjunction of members $H(r_1, \dots, r_n)$, which remains unchanged by the further eliminations performed in a way corresponding to the first elimination-process. For if F is the final formula we have before the elimination of some set of critical formulas, then after this elimination the final formula will have the form

$$F_1 \vee F_2 \vee \dots \vee F_h,$$

where F_h is F and the other disjunction-members differ from F only in that some ϵ-terms are replaced by other terms (which may be ϵ-terms or not). But if F is a disjunction of members $H(r_1, \dots, r_n)$, then each F_i $(i = 1, \dots, h)$ is such a disjunction, since an ϵ-term to be replaced in F cannot occur in any other way than being one (or contained in one) of the terms $r_1, \dots, r_n$ in a member $H(r_1, \dots, r_n)$. Thus

$$F_1 \vee \dots \vee F_h$$

is again a disjunction, the members of which have the form $H(r_1, \dots, r_n)$.

After the elimination of all sets of critical formulas has been carried out, we replace all remaining ϵ-terms by zero.

We then have a deduction of a formula M* of the form

$$H(r_1^{(1)}, \dots, r_n^{(1)}) \vee \dots \vee H(r_1^{(g)}, \dots, r_n^{(g)}),$$

containing no ϵ-term and no bound variable whatever; and this deduction is made only by the propositional calculus and the formula-schemata (3).

It may be observed that from the formula M* we come back to the formula M by means of the schema (δ) and the propositional calculus. Indeed, by several applications of the schema (δ) and the syllogism, we get, for i = 1, ..., g

$$H(r_1^{(i)}, \ldots, r_n^{(i)}) \rightarrow (Ex_1) \ldots (Ex_n) H(x_1, \ldots, x_n)$$

that is

$$H(r_1^{(i)}, \ldots, r_n^{(i)}) \rightarrow M,$$

and these formulas together by the propositional calculus give

$$H(r_1^{(1)}, \ldots, r_n^{(1)}) \vee \ldots \vee H(r_1^{(g)}, \ldots, r_n^{(g)}) \rightarrow M,$$

that is

$$M^* \rightarrow M.$$

In case M contains no free variable, the formula M* becomes a numerical formula. According to our consistency-theorem it must be a true formula, and so at least one of the disjunction-members must be a true formula. This member has the form $H(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are numerals.

Thus we have the result: If a formula

$$(Ex_1) \ldots (Ex_n) H(x_1, \ldots, x_n)$$

containing no variables other than $x_1, \ldots, x_n$ is deducible from the axioms (2) by means of our logical calculus (in the second form), then there are numerals

$$a_1, \ldots, a_n,$$

to be found by our method of eliminating the bound variables, such that

$$H(a_1, \ldots, a_n)$$

is a true formula.

Till now we have been confining the applications of our method of elimination to the formalism of the axioms (2) together with the logical calculus in the second form. This formalism is a rather special one and for the purpose of proving it to be consistent we could have got along with a more direct method.

But our results really have a much higher generality. Indeed, as has already been indicated, a great many of the assumptions we made can be replaced by more general ones, and to come to the main applications of our method we have to extend our statements in several respects.

First regarding our procedure of eliminating the bound variables from a deduction, it is obvious that it does not depend on the special form of the axioms (2) nor on the special kind of arithmetical symbols they contain. The only reference to these symbols we made was that after the critical formulas had been eliminated we replaced the remaining ϵ-terms by 0. But instead of that we can replace these ϵ-terms by a free individual-variable.

Moreover, the assumption that no letters (capital italics) occur in the given deduction has not been used; indeed in the frame of the second form of our logical calculus the letters with arguments are to be treated in quite the same way as predicate-symbols with their argument and the letters without argument in the same way as prime-formulas.

Still the restriction to the second form of our logical calculus is not necessary, since the statements we made in §5 about the equivalence of the three forms of the logical calculus hold also if some axioms are added, provided these axioms contain no letters.

Remark: The case of an axiom containing one or more letters has to be excepted, since in passing from the third form of our logical calculus to the second form we have to undo the substitutions made for the letters in the starting formulas (cf. pp. 51-53), and so an axiom in which a letter occurs would in general have to be changed into other formulas. In this case a suitable way of performing the passage from the third form of the calculus to the second would be to let correspond to the axiom in question a formula-schema containing, instead of every letter occurring in the axiom, an indication (by the use of Roman letters) of an arbitrary formula (with the same arguments). But then the difficulty is that the formulas arising from such a schema in general will lose their characteristic form by the replacements to be made in the elimination-process. Thus every case of this kind requires a special discussion, such as we shall soon give for the equality-schema (or -axiom).

Taking together the different extensions, we come to the following elimination-theorem: Let there be given a formalism consisting of the logical calculus in one of the three forms, including the pure logical calculus, individual-, predicate- and function-symbols, and some axioms $A_1, \dots, A_k$ containing no letter and no bound variable; thus the axioms are formed out of terms, predicate-symbols, and the operations $\rightarrow$, $\&$, $\vee$, $\overline{}$.

If by means of this formalism a formula containing no bound variable is deduced, then we can obtain also a deduction of this formula which is made only by the propositional calculus and the formula-schemata $s_1, \dots, s_k$ which we get from the axioms $A_1, \dots, A_k$ by replacing every free variable by the indication of an arbitrary term which is the same for all occurrences of the variable in one of the axioms.

Furthermore, if by the given formalism a formula

$$(Ex_1) \dots (Ex_n) H(x_1, \dots, x_n)$$

is deduced, where $x_1, \dots, x_n$ are the only bound variables in $H(x_1, \dots, x_n)$, then we can obtain a deduction, made by the propositional calculus and the formula-schemata $s_1, \dots, s_k$, of a formula

$$H(r_1^{(1)}, \dots, r_n^{(1)}) \vee \dots \vee H(r_1^{(g)}, \dots, r_n^{(g)}),$$

where $r_1^{(i)}, \dots, r_n^{(i)}$ $(i = 1, \dots, g)$ are terms.

Remark: At once our proving method gives a stronger result, namely that the two statements of the elimination-theorem hold also if, besides the quoted means of deduction, the given formalism also contains the ϵ-schema.

Indeed, in our proof it comes out to the same whether we have the -schema from the beginning as belonging to the given formalism, or introduce it in the course of the argument.

From the second statement of the elimination-theorem there is a relatively easy way to obtain the general theorem that Jacques Herbrand proved in his thesis "Recherches sur la théorie de la démonstration" (Paris 1930).

Let me briefly indicate this connection. The Herbrand theorem concerns deducible formulas of the pure logical calculus. As we know, every formula of this calculus is equivalent to a prenex formula (cf. §3, p. 27). So our attention can be restricted to deducible prenex formulas. Moreover we can assume that in the formula to be considered the first quantifier is an existential symbol. For if the formula begins with an all-symbol and thus has the form (x)A(x), we can take instead of it the formula A(c), where c is a free variable not occurring in A(x),--the two formulas being of equal deducibility. By repeating this procedure we either come to a formula without a bound variable or to a prenex formula beginning with an existential quantifier. In the first case we know already that in order for the formula to be deducible it must arise from a + formula by substitutions for the letters. The Herbrand theorem will be a generalization of this result. So we can now assume that the deducible prenex formula F in question begins with an existential quantifier.

The formula F will in general contain some all-symbols. To each such symbol we assign a function-symbol, the arguments of which are the variables belonging to the existential symbols preceding the all symbol; then we cancel the all-symbol and replace the variable belonging to it everywhere by the function symbol assigned to the canceled all-symbol.

According to this prescription a formula of the form

$$(Ex)(u)(Ey)(Ez)(v)\,A(x, u, y, z, v)$$

in which x, u, y, z, v are the only bound variables, has to be replaced,- if $\phi(x)$, $\psi(x, y, z)$ are the function-symbols assigned respectively to (u) and (v) ,- by

$$(Ex)(Ey)(Ez)\,A(x, \phi(x), y, z, \psi(x, y, z))$$

Now the formula F°, by which F is to be replaced in this way, can be deduced from F by the logical calculus, if we adjoin to it the function-symbols assigned to the canceled all-symbols. It will be sufficient to show this for our special example, by deducing the implication

$$(Ex)(u)(Ey)(Ez)(v)\,A(x, u, y, z, v) \longrightarrow (Ex)(Ey)(Ez)\,A(x, \phi(x), y, z, \psi(x, y$$

This deduction will be performed by means of the Schema (γ), the syllogism and the derived schema of passing from a formula

$$A(c) \longrightarrow B(c),$$

where c is a free variable not occurring in A(w), B(w), to

$$(Ew)A(w) \longrightarrow (Ew)B(w).$$

This last process (cf. §3, p. 18), which is here understood as an abbreviated indication of the passage

$$\begin{array}{ll}
A(c) \longrightarrow B(c) & \\
\underline{B(c) \longrightarrow (Ew)B(w)} & \text{(schema } (\delta)) \\
A(c) \longrightarrow (Ew)B(w) & \text{(syllogism)} \\
(Ew)A(w) \longrightarrow (Ew)B(w) & \text{(schema } (\beta))
\end{array}$$

may be quoted as " δ - β -schema".

Still we have to use the fact that, according to the adjunction of the symbols ϕ, ψ to the calculus, ϕ(a) and ψ(a, b, c) are terms, if a, b, c are free variables. Let a, b, c be free variables not occurring in $A(x, u, y, z, v)$; then we have:

$(v)A(a, \Phi(a), b, c, v) \rightarrow A(a, \Phi(a), b, c, \Psi(a, b, c))$ (schema (γ))

$(E\mathfrak{z})(v)A(a, \Phi(a), b, \mathfrak{z}, v) \rightarrow (E\mathfrak{z})A(a, \Phi(a), b, \mathfrak{z}, \Psi(a, b, \mathfrak{z}))$ (δ-β-schema)

$(E\mathfrak{y})(E\mathfrak{z})(v)A(a, \Phi(a), \mathfrak{y}, \mathfrak{z}, v) \rightarrow (E\mathfrak{y})(E\mathfrak{z})A(a, \Phi(a), \mathfrak{y}, \mathfrak{z}, \Psi(a, \mathfrak{y}, \mathfrak{z}))$ (δ-β-schema)

$(u)(E\mathfrak{y})(E\mathfrak{z})(v)A(a, u, \mathfrak{y}, \mathfrak{z}, v) \rightarrow (E\mathfrak{y})(E\mathfrak{z})(v)A(a, \Phi(a), \mathfrak{y}, \mathfrak{z}, v)$ (schema(γ))

$(u)(E\mathfrak{y})(E\mathfrak{z})(v)A(a, u, \mathfrak{y}, \mathfrak{z}, v) \rightarrow (E\mathfrak{y})(E\mathfrak{z})A(a, \Phi(a), \mathfrak{y}, \mathfrak{z}, \Psi(a, \mathfrak{y}, \mathfrak{z}))$ (syllogism)

$(E\mathfrak{x})(u)(E\mathfrak{y})(E\mathfrak{z})(v)A(\mathfrak{x}, u, \mathfrak{y}, \mathfrak{z}, v) \rightarrow (E\mathfrak{x})(E\mathfrak{y})(E\mathfrak{z})A(\mathfrak{x}, \Phi(\mathfrak{x}), \mathfrak{y}, \mathfrak{z}, \Psi(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}))$ (δ-β-schema).

Thus the formula F° as well as F is deducible. (In case F contains no all-symbol, F° is identical with F.) The formula F° being of the form

$$(Ex_1) \dots (Ex_n)H(x_1, \dots, x_n),$$

we can apply to it the second part of our elimination-theorem; this gives the result that we can deduce by the mere propositional calculus -- (we have here no special axioms) -- a disjunction

$$F_1 \vee \dots \vee F_q,$$

every member of which is to be obtained from F° by canceling the existential symbols and replacing the bound variables by terms. By virtue of the connection holding between the formulas F° and F it follows that every disjunction-member F_i ($i = 1, \dots, q$) is also to be obtained from F by canceling the quantifiers and replacing the bound variables by terms; these terms are formed out of free variables and the introduced function-symbols, and the term replacing a variable belonging to an all-symbol (w) consists of the function-symbol assigned to this all-symbol having as arguments the terms which replace the variables belonging to the existential symbols preceding the all-symbol (w) in the formula F.

For instance, in the case just considered, where F has the form

$$(E\mathfrak{x})(u)(E\mathfrak{y})(E\mathfrak{z})(v)A(\mathfrak{x}, u, \mathfrak{y}, \mathfrak{z}, v)$$

and ϕ, ψ are the introduced function symbols, every disjunction-member F_i has the form

$$A(r, \phi(r), s, t, \psi(r, s, t)),$$

r, s, t being some terms formed out of free variables and the function-symbols ϕ, ψ.

Now since

$$F_1 \vee \cdots \vee F_q$$

is deducible by the propositional calculus, it must arise from a +-formula by substitutions made for the letters. Thus the same must hold for a formula F* we get from this disjunction by replacing every term standing in the place of a bound variable in F and containing at least one function-symbol, by a free variable not occurring before, taking for equal terms the same and for different terms different variables.

From the structure of the terms in the disjunction-members F_i ($i = 1, \ldots, q$) we described, it can be concluded -- the proof is not difficult -- that from the formula F* we can get back the formula F by means of the schemata (α), (δ) and the propositional schemata.

So finally we have the result: to every deducible prenex formula F of the pure logical calculus we can find a formula F* of the following properties:

1. F* is a disjunction every member of which arises from F by canceling the quantifiers and replacing the bound variables by free variables (which need not be all different from one another).
2. F* arises from a +-formula by substitutions for the letters.
3. From F* we can deduce F by means of the schemata (α), (δ) and the propositional schemata.

This is an abbreviated statement of the theorem of Herbrand (cf. J. Herbrand "Sur le problème fondamental de la logique mathématique", Compt. rend. de la soc. des sc. ... de Varwovie, Vol. XXIV, 1931, Classe III; p. 31, footnote 1).) A detailed statement would include some conditions on the variables in F^*, deriving from the structure of the terms in the disjunction F_1 ... F_q.

Gerhard Gentzen recently gave a new proof of the Herbrand theorem, obtaining it as a consequence of another general theorem of logic proved by him.

Let us now come back to questions of consistency, for the treatment of which the elimination-method was intended.

To obtain from our elimination-theorem results concerning consistency we have to assume something more on the axioms $A_1, \ldots, A_k$. Indeed the assumption made in the elimination-theorem on these axioms, namely that they contain no letters and no bound variables, surely cannot suffice for the consistency.

The property of the axioms (2) by which we proved their consistency, was that they are verifiable formulas, or, in other words, that they turn into true formulas if we replace the free variables occurring in them by any numerals. Here the concept of a true formula referred to our valuation of the formulas $a < b$ in which a, b are numerals. Besides we required that in the formalism we considered every term not containing a free variable be a numeral.

Removing in an obvious way some inessential peculiarities of this case, we come to the following <u>consistency-theorem:</u> A formalism arising from the pure logical calculus (in one of the three forms) by adding some individual-, function- and predicate-symbols and some axioms $A_1, \ldots, A_k$ containing neither letters (capital italics) nor bound variables, is consistent in the sense that no two formulas A, $\bar{A}$ are both deducible by it, if the following conditions are satisfied: 1. There is a kind of term containing no free variable -- they may be called "N-terms" (as a generalization of "numeral") -- and a <u>valuation</u> related to them, such that every formula consisting of a predicate-symbol with N-terms as arguments is either true or false and it can be decided which of the two holds.

2. In case there are terms containing no free variable without being N-terms, these terms are formed out of N-terms by means of function-symbols and there is a process of computation by which every such term turns into one and only one N-term, which therefore can be called its "value".

3. Each of the axioms $A_1, \ldots, A_k$ is "verifiable" in the sense that after each free variable has been replaced by an N-term, and afterwards the terms which are not N-terms have been computed, the formula we obtain is a true formula according to the valuation of the prime-formulas and the definition of the operations $\longrightarrow$, $\&$, $\vee$, $\overline{}$ as truth-functions.

Moreover, whenever the conditions 1, 2, 3 are satisfied we can conclude: every deducible formula containing no letter and no bound variable is verifiable; and to every deducible formula

$$(Ex_1) \ldots (Ex_n)H(x_1, \ldots, x_n),$$

which contains no variables besides $x_1, \ldots, x_n$ and no letter, we can find (by the process of eliminating the bound variables) some N-terms $a_1, \ldots, a_n$ such that the formula

$$H(a_1, \ldots, a_n),$$

after the terms, which are not N-terms have been computed, becomes a true formula.

Quite the same property continues to hold if to the given formalism the ϵ-schema is adjoined. (Cf. the remark to the elimination-theorem, p. 94.) Thus the conceptual methods which are formalized by the rule of the ϵ-symbol (cf. §8, pp. 75-76) preserve the consistency of a formalism satisfying the conditions of our consistency theorem.

For the application of the consistency-theorem an essential remark is that from the consistency of a formalism $\{F\}$ we can infer the consistency of any formalism $\{G\}$, which is "included in F", in the sense that every deducible formula of G is also a deducible formula of $\{F\}$. By this argument we are able to apply our consistency-theorem indirectly to formalisms which do not themselves satisfy the assumptions of this theorem.

That is the method by which we proceeded in §7 (cf. pp. 68-69) in reducing the question of consistency for the axioms (1) to that for the axioms (2).

We are now going on to apply this method to the equality-schema and to the axiomatic-geometry.

10. Possibility of Replacing the general equality schema by special axioms. Example of axiomatic geometry

Since the question now concerns the application of our consistency-theorem to formalized axiomatic theoreis, we first meet the difficulty that the formalisms to which the statement of this consistency-theorem refers do not contain the formalization of equality.

Indeed, for operations with the equality-symbol in the formal deductions we have the axiom

$$a = a$$

and, according to the form of the logical calculus, either the schema

$$\frac{r = s,\ A(r)}{A(s)}$$

or the schema

$$r = s \rightarrow (A(r) \rightarrow A(s))$$

(instead of which we may also take

$$a = b \rightarrow (A(a) \rightarrow A(b)) \ )$$

or the axiom

$$a = b \rightarrow (\mathcal{A}(a) \rightarrow \mathcal{A}(b))$$

Now neither these schemata nor this axiom belong to the pure logical calculus, nor can this axiom be taken as one of the adjoined axioms $A_1, \ldots, A_k$, since it contains the letter $\mathcal{A}$.

So we cannot immediately apply our consistency-theorem to a formalized theory which contains the equality-formalism, as most of the formalized axiomatic theories do.

But this difficulty can be removed in two ways. One is that we extend our elimination-theorem to formalisms containing the axioms

$$a = a, \qquad a = b \to (\mathcal{A}(a) \to \mathcal{A}(b))$$

-- or instead of the last axiom, one of the corresponding schemata. This indeed can be done by modifying the process of eliminating the ε-expressions. (One has to consider still a second kind of critical formulas which have to be eliminated.)

By this method one obtains a stronger result with respect to formalisms containing the ε-schema.

For the purpose of making the consistency theorem apply to formalisms to which the general schema of equality (or the corresponding axiom) belongs, we can get along with a simpler measure. In fact, as we will show now, for the deductions to be performed in the formalizing of an axiomatic theory the equality-schema can be replaced by a finite set of special equality-axioms.

Let us take again the second form of the logical calculus and the generality-schema in the form

$$r = s \to (A(r) \to A(s)).$$

We first state some simple deducibilities. As an immediate application of the generality-schema, taking for A(d) the formula $d = c$ and for r, s the variables a, b we get

$$a = b \to (a = c \to b = c)$$

This formula, together with the axiom $a = a$ gives, by means of the propositional calculus and the substitution $\mathop{S}\limits_{a}^{c}(a = c \to b = c)$ (the substitution-rule for the free variables being a derived rule in our logical calculus), the formula

$$a = b \to b = a$$

For a function-symbol f with one argument, we get by the equality-schema, taking for $A(\mathfrak{d})$ the formula $f(a) = f(\mathfrak{d})$ and for r, s the variables a, b:

$$a = b \rightarrow (f(a) = f(a) \rightarrow f(a) = f(b))$$

and from this formula, together with

$$f(a) = f(a)$$

(to be got from $a = a$), we obtain by the propositional calculus

$$a = b \rightarrow f(a) = f(b)$$

In quite a similar way we obtain, for a function-symbol f with two arguments, the formula

$$a = b \rightarrow f(a, c) = f(b, c)$$

and also

$$a = b \rightarrow f(c, a) = f(c, b)$$

For a predicate-symbol, as for instance $<$, we immediately get by the equality-schema

$$a = b \rightarrow (a < c \rightarrow b < c)$$
$$a = b \rightarrow (c < a \rightarrow c < b).$$

Thus for every argument of a function-symbol f a formula

$$a = b \rightarrow f(\cdots, a, \cdots) = f(\cdots, b, \cdots),$$

and for every argument of a predicate-symbol P a formula

$$a = b \rightarrow (P(\cdots, a, \cdots) \rightarrow P(\cdots, b, \cdots))$$

is deducible by means of the equality-schema (in the case of the function-symbol using also the axiom $a = a$), where in the places of arguments, besides that of the distinguished argument, stand variables different from a and b and from one another, and for both occurrences of the function-symbol, or the predicate-symbol, the same variables in the places of corresponding arguments except that of the distinguished argument.

Such a formula may be called an "equality-formula belonging to the argument" in question of the function-symbol, or of the predicate-symbol.

Now we are to prove the following equality-theorem: every formula

$$r = s \rightarrow (A(r) \rightarrow A(s))$$

which arises from the equality-schema and which is formed of individual-variables, individual-, function- and predicate-symbols and our six logical symbols, can be deduced, by means of our logical calculus, from the formulas

$$\text{(e)} \quad \begin{cases} a = a \\ a = b \rightarrow (a = c \rightarrow b = c) \end{cases}$$

and a set of equality-formulas

$$(e_1) \quad a = b \rightarrow f(\cdots, a, \cdots) = f(\cdots, b, \cdots)$$

$$(e_2) \quad a = b \rightarrow (P(\cdots, a, \cdots) \rightarrow P(\cdots, b, \cdots))$$

in which there is for every argument of a function- or a predicate-symbol occurring in $A(c)$ one equality-formula belonging to it.

In fact, given a formula

$$r = s \rightarrow (A(r) \rightarrow A(s))$$

of the kind described, it is easily seen, first that for any term $t(c)$ formed out of function-symbols occurring in $A(c)$ and free variables, the formula

$$a = b \rightarrow t(a) = t(b)$$

is deducible from the formulas (e), (e_1) by means of the (derived) rule of substitutions for the free variables and the propositional calculus.

Using furthermore the formulas (e_2) we can deduce, for every prime-formula $Q(c)$ formed out of symbols contained in $A(c)$ and free variables, the formula

$$a = b \rightarrow (Q(a) \rightarrow Q(b))$$

The substitutions occurring in such a deduction are only to be performed on the variables contained in the formulas (e), (e_1), (e_2). (Observe that the substitutions for the free variables have to be effected in our calculus by means of the schemata (α), (γ).

Now we take two free variables p, q not occurring in (e), (e_1), (e_2) and consider the formula

$$p = q \rightarrow (A(p) \rightarrow A(q)).$$

In order to show it to be deducible from the formulas (e), (e_1), (e_2) it will be sufficient to prove that from

$$p = q$$

together with (e), (e_1), (e_2) we can deduce

$$A(p) \rightarrow A(q)$$

in such a way that the variables p, q are kept fixed. For the schema $\dfrac{\begin{matrix} \lfloor A \rfloor \\ \vdots \\ B \end{matrix}}{A \rightarrow B}$ (to be applied with the restrictive condition) is a derived rule in our formalism, as we stated in §§4, 5 (cf. pp. 42-43 and 48-50).

Now this deducibility of $A(p) \rightarrow A(q)$ from p = q and the formulas (e), (e_1), (e_2) results in the following way. For any prime formula $Q(c)$ formed out of symbols occurring in $A(c)$ and free variables, as was stated just now, we can deduce, using (e), (e_1), (e_2),

$$a = b \rightarrow (Q(a) \rightarrow Q(b))$$

and thus also

$$p = q \rightarrow (Q(p) \rightarrow Q(q))$$

and

$$q = p \rightarrow (Q(q) \rightarrow Q(p)).$$

Moreover, from p = q and the formulas (e) we get

$$q = p.$$

So we obtain

$$Q(p) \rightarrow Q(q) \quad \text{and} \quad Q(q) \rightarrow Q(p),$$

which together give

$$Q(p) \sim Q(q).$$

Thus, according to our statements on equivalences in §3 (cf. pp. 16-18 and bottom of 27 to top of 28,- observe that the schemata of the assumption-calculus are derived rules for our calculus!), in every formula which contains $Q(p)$ somewhere as a part, we can replace this part by $Q(q)$; and furthermore, if an expression $B(p)$ arises from $Q(p)$ by putting some bound variables in the place of some free variables different from p, this expression, for any one of its occurrences in a formula, can be replaced by $B(q)$.

Using these possibilities of replacement, which hold for any prime formula $Q(\mathfrak{c})$, we can pass from the deducible formula

$$A(p) \rightarrow A(p)$$

to

$$A(p) \rightarrow A(q).$$

In the deduction thus performed of the formula

$$A(p) \rightarrow A(q)$$

the variables p, q remain fixed. Indeed, since p, q are different from the variables occurring in (e), (e_1), (e_2), we never have to take p or q as the variable b in one of the schemata (α), (β).

So in fact it results that by means of (e), (e_1), (e_2) the formula

$$p = q \rightarrow (A(p) \rightarrow A(q))$$

can be deduced. But from this formula we get

$$r = s \rightarrow (A(r) \rightarrow A(s))$$

by substitutions. This formula thus is deducible from (e), (e_1), (e_2), as we wanted to show.

Now we are to apply the proved equality-theorem to a formalism of an axiomatic theory, to be obtained from the pure logical calculus by adding some individual-, function- and predicate-symbols, the symbol = included; furthermore some formal axioms $F_1, \ldots, F_k$, one of which is the equality-axiom $a = a$ they all containing no letter and no bound variable, and still the equality-schema

$$r = s \longrightarrow (A(r) \longrightarrow A(s)).$$

Our purpose is to extend to such a formalism our consistency-theorem. Since the statements of this theorem concern properties only of such deducible formulas as contain no letters, we can restrict the consideration of deductions to those of formulas containing no letter.

Now taking the logical calculus again in the second form (with the formula-schemata) we can, from a deduction of a formula not containing a letter, eliminate the letters altogether from the deduction.

After this elimination has been performed, every application of the equality-schema satisfies the conditions of our equality-theorem. Thus, according to this theorem the equality-schema can be replaced, for each of the deductions we have to deal with, by a finite set of special equality-axioms consisting of

1. the formula

$$\mathfrak{a} = \mathfrak{b} \longrightarrow (\mathfrak{a} = \mathfrak{c} \longrightarrow \mathfrak{b} = \mathfrak{c})$$

2. for every function-symbol f of our formalism a set of equality-formulas

$$\mathfrak{a} = \mathfrak{b} \longrightarrow f(\ldots, \mathfrak{a}, \ldots) = f(\ldots, \mathfrak{b}, \ldots)$$

belonging to its different arguments

3. for every predicate-symbol P of our formalism a set of equality-formulas

$$\mathfrak{a} = \mathfrak{b} \longrightarrow (P(\ldots, \mathfrak{a}, \ldots) \longrightarrow P(\ldots, \mathfrak{b}, \ldots))$$

belonging to its different arguments.

Remark: It will not always be necessary to assign to every function- and predicate-symbol as many axioms as it has arguments; for by means of the axioms $F_1, \ldots, F_k$ and

$$\mathfrak{a} = \mathfrak{b} \longrightarrow (\mathfrak{a} = \mathfrak{c} \longrightarrow \mathfrak{b} = \mathfrak{c})$$

some of the equality-formulas in question may be deducible from the others.

But by replacing in this way the equality-schema by special equality-axioms we come to a formalism of the kind to which our consistency-theorem applies, the equality-symbol having now quite the same role as the other predicate-symbols.

And so finally it results that for the two statements of our consistency-theorem holding with respect to the original formalism it is sufficient that for the modified formalism a valuation of the prime formulas formed out of the individual-, function- and predicate-symbols can be defined which is in accordance with the conditions 1, 2, 3 of the consistency-theorem.

By the methods now at our disposal we can prove systems of axioms for Euclidean geometry - not including the axioms of continuity - to be consistent.

Remark: As to the Archimedean axiom it would be possible by an additional consideration to include it in the consistency-proof.

An axiomatization of Euclidean geometry especially appropriate for our purpose is that which we obtain by adjoining to the axioms for projective geometry of Professor Veblen ("A system of axioms for geometry"? Trans. Amer. Math. Soc. Vol. 5 (1904), pp. 343-384) the metrical axioms of R. L. Moore ("Sets of metrical hypotheses for geometry", Trans. Amer. Math. Soc., Vol. 9 (1908), pp. 487-512).

For the sake of simplicity we are restricting our attention to plane geometry. Excluding the continuity-axioms, we have to use the statements of J. L. Dorroh ("Concerning a set of metrical hypotheses for geometry", Annals of Math., 2d Ser., Vol. 29 (1928), pp. 229-231). For the parallel-axiom I shall take the Hilbert form of statement. The structure of the axiomatic system to be considered is the following:

We have only one kind of individuals, the "points" -- and two fundamental predicates:

"b lies between a and c", or "the points a, b, c are in the order abc";

notation: $Od(a, b, c)$

and

"the distance of a from b is equal to that of c from d"; notation: $ab \simeq cd$

The identity is considered as belonging to the logic in the same way as it does in the systems of axioms we considered in §1; we take as before the notation $a = b$ for it.

There is also the derived predicate "the points a, b, c are on a straight line"; notation: $St(a, b, c)$; for this we have the explicit definition:

<u>Def.</u> $St(a,b,c)$: $\quad Od(a,b,c) \vee Od(b,c,a) \vee Od(c,a,b) \vee a=b \vee b=c \vee c=a.$

Mathematical functions do not occur.

Now the axioms written down by means of our logical symbolism -- I sometimes shall adjoin an explanatory comment -- are the following:

1) a) $$Od(a,b,c) \rightarrow Od(c,b,a)$$

b) $$Od(a,b,c) \rightarrow \overline{Od(a,c,b)}$$

c) $$\overline{Od(a,b,a)}$$

2) $$a \neq b \;\&\; St(a,b,c) \;\&\; St(a,b,d) \rightarrow St(a,c,d)$$

3) $$(Ex)(Ey)(Ez)\,\overline{St(x,y,z)}$$

4) $$\overline{St(a,b,c)} \;\&\; Od(b,c,d) \;\&\; Od(c,e,a) \rightarrow (Ex)(Od(a,x,b) \;\&\; St(d,e$$

(Triangle-axiom)

5) $$\overline{St(a,b,c)} \rightarrow (Ex)(St(d,e,x) \;\&\; (St(a,b,x) \vee St(a,c,x)))$$

(Parallel-axiom)

6) a) $$ab \simeq ba$$

b) $$ab \simeq ef \;\&\; cd \simeq ef \rightarrow ab \simeq cd$$

7) a) $$a \neq b \;\&\; c \neq d \rightarrow (Ex)(Od(d,c,x) \;\&\; ab \simeq cx)$$

b) $$Od(a,b,c) \;\&\; Od(a,b,d) \;\&\; bc \simeq bd \rightarrow c = d.$$

(Existence and uniqueness of a point in a given distance and direction from a fixed point)

8) $Od(a,b,c)\ \&\ Od(d,e,f)\ \&\ ab \simeq de\ \&\ bc \simeq ef \longrightarrow ac \simeq df$

(Additivity of segments)

9) $\overline{St(a,b,c)}\ \&\ \overline{St(e,f,g)}\ \&\ Od(a,b,d)\ \&\ Od(e,f,h)\ \&\ ab \simeq ef\ \&\ bc \simeq fg\ \&\ ca \simeq eg$
$\longrightarrow cd \simeq gh.$

(Axiom on congruent triangles)

These axioms are sufficient to permit us to deduce from them, by means of the logical calculus, all theorems of plane geometry which follow from the Hilbert axiom-system of geometry (cf. Hilbert, "Grundlagen der Geometrie", 7th ed., 1930) without the continuity-axioms, and which are expressible by relations of collinearity, betweenness and equidistance holding for some specified points.

Remark 1. Theorems concerning polygons with an arbitrary number of sides are not expressible in our formalism.

Remark 2. In order to make provable the existence in our geometry of all constructions with rule and compasses, we should have to add still one axiom, for instance the following proposed by R. L. Moore

10) $Od(a,b,c)\ \&\ \overline{St(a,b,d)} \longrightarrow (Ex)(St(b,d,x)\ \&\ ax \simeq ac).$

In order now to apply our consistency-theorem to the system of the axioms 1)-9), we first have to remember that for this purpose we need to regard only such deductions as are performed -- in the frame of the logical calculus in the second form -- without using letters. (Cf. pp. 105-106.)

So we can apply our equality theorem and, according to it, replace the general equality schema by special equality axioms.

By virtue of the axioms 1 a), b), and 6 a), b), we need here only two equality-formulas for Od and one for $\simeq$ Thus we have the following equality-axioms

$$
((e)) \quad \begin{cases} a = a \\ a = b \to (a = c \to b = c) \\ a = b \to (Od(a, c, d) \to Od(b, c, d)) \\ a = b \to (Od(c, a, d) \to Od(c, b, d)) \\ a = b \to (ac \simeq de \to bc \simeq de) \end{cases}
$$

and by introducing them we place the equality on the same level in our logic as the fundamental predicates Od, $\simeq$.

To the application of our consistency-theorem there is still the obstacle that bound variables occur in our axioms. These bound variables all belong to existential quantifiers. (We were able to avoid general quantifiers in the axioms because every generality occurring in them operates on a whole axiom, so that it can be expressed by means of a free variable.)

To eliminate the existential quantifiers it will be suitable to use, in an heuristical way, the <u>Hilbert</u> ϵ-symbol. As we know, a formula

$$(Ex)A(x)$$

can be expressed, using the ϵ-symbol, by

$$A[\epsilon_x A(x)].$$

Now let, for instance,

$$B \to (Ex)D(x, a, \ldots, k)$$

be an axiom, where B contains no bound variable and x, a, ..., k are all variables occurring in D(x, a, ..., k). Expressing here the existential symbol by means of the ϵ-symbol, we get

$$B \to D(\epsilon_x D(x, a, \ldots, k), a, \ldots, k).$$

Now introducing a new function-symbol g by the explicit definition

$$g(a, \ldots, k) = \epsilon_x D(x, a, \ldots, k)$$

we come to the formula

$$B \to D(g(a, \ldots, k), a, \ldots, k).$$

On the other hand, from this formula the axiom

$$B \longrightarrow (Ex)D(x, a, \ldots, k)$$

can be deduced (by means of the schema (δ) and the syllogism).

By this consideration the following device is suggested: to replace in the axioms every bound variable belonging to an existential quantifier by a new symbol (individual- or function-symbol) having as arguments the free variables contained in the scope of the existential quantifier canceling at once this quantifier.

In this way we get, instead of the axioms 3) and 7a), introducing the individual-symbols α, β, γ and the function-symbol $\Psi(a, b, c, d)$, the following

3') $$\overline{St(\alpha, \beta, \gamma)}$$

7a') $$a \neq b \;\&\; c \neq d \longrightarrow Od(d, c, \Psi(a,b,c,d)) \;\&\; ab \simeq c\Psi(a,b,c,$$

($\Psi(a, b, c, d)$ represents "the point lying in the continuation of the segment dc beyond c, having the same distance from c, as b has from a".)

Concerning the axioms 4), 5) there is the simplification that for replacing them by axioms without quantifier we can get along with only one new function-symbol $\Phi(a, b, c, d)$, representing "the intersection-point of the lines ab and cd", provided that such a one exists.

Here the formal passage can be effected again by means of the ϵ-symbol. In fact, the axiom 5) can first be transformed into

$$\overline{St(a,b,c)} \longrightarrow (Ex)(St(a,b,x) \;\&\; St(d,e,x)) \vee (Ex)(St(a,c,x) \;\&\; St(d,e,x))$$

Now, defining the function-symbol $\Phi(a, b, c, d)$ by

$$\Phi(a,b,c,d) = \epsilon_x(St(a,b,x) \;\&\; St(c,d,x))$$

we get (by means of the logical calculus and the ϵ-schema) from 5) the formula

5') $$\overline{St(a,b,c)} \longrightarrow \Big\{ (St(a,b,\Phi(a,b,d,e)) \;\&\; St(d,e,\Phi(a,b,d,e))) \vee (St(a,c,\Phi(a,c,d,e)) \;\&\; St(d,e,\Phi(a,c,d,e))) \Big\},$$

and from 4), together with 1 a), 1 b), 1 c), 2), and the definition of $St(a, b, c)$ the formula

4') $\overline{St(a,b,c)} \,\&\, Od(b,c,d) \,\&\, Od(c,e,a) \rightarrow Od(a, \varphi(a,b,d,e), b) \,\&\, St(d, e, \varphi(a,b,d,e$

On the other hand, from 4') we come back to 4), from 5') to 5), by our logical calculus. Thus 4') and 5') can be taken as axioms instead of 4) and 5).

Remark: Obviously for the purpose of our proposed consistency-proof we need not adjoin equality-formulas belonging to the introduced function-symbols.

Now the bound variables are eliminated from the axioms and the question now concerns the consistency of the formalism arising from our logical calculus in the second form by applying it to the system of axioms:

1)-3), 4'), 5'), 6), 7a'), 7b), 8), 9)

and the adjoined equality axioms ((e)).

This is a formalism of the kind considered in the consistency theorem. But still we have to take care to satisfy the conditions 1, 2, 3 of this theorem.

So we have to introduce a kind of "N-terms" and to define by means of them a valuation satisfying those conditions.

To this end we start from that <u>real number-field</u> [R] which consists of the numbers we get, beginning with 1, by the 5 operations: addition, subtraction, multiplication, division, and taking the square-root $\sqrt{c}$ of a positive number c.

In this number field [R] the five operations can be constructively performed on expressions formed out of natural numbers (0, 1, 2, ...) by means of the usual symbols for the five operations, with the necessary restriction for the division and for the applying of the square-root; furthermore, the question whether a given expression represents a positive or a negative number or 0, can in every case be effectively decided; and thus also it can be decided whether two given expressions represent the same number or different numbers.

Also the laws of computation can be intuitively seen to hold.

To show this we can argue by an intuitive induction on the number of applications of the operation $\sqrt{\ }$. Indeed it is sufficient to prove the following: Let [F] be a real field, which has the described elementary character with respect to the 4 operations -- addition, subtraction, multiplication, dividion -- so that these operations can be effectively performed (with the usual exception for the division); the questions whether or not a given expression has a value > 0, < 0 or $= 0$, can be effectively decided and the laws of computation can be intuitively seen to hold. Let p be an expression which is known to have a positive value. Then in the domain of expressions

$$a + b \cdot \sqrt{p},$$

where a, b are expressions of [F], the four operations and the relation $<$ can be defined in such a way that the same conditions as before are satisfied. This is to be shown in the usual lines of algebra. Only one thing is to be observed: since we have assumed no more about p than that it is positive, it can happen that $a + b\sqrt{p}$ has the value 0, without a, b both having this value. But no difficulty arises from this fact, because for any given expression $a + b \cdot \sqrt{p}$ we can decide whether it has the value 0; indeed the necessary and sufficient conditions for that are:

$$a^2 - b^2 \cdot p = 0, \quad a \cdot b \leqq 0,$$

and according to our assumption on [F] we can effectively decide whether these relations hold.

Now from the field [R] we go on to the field [G] of complex numbers $s + t\,i$, where s, t are elements of [R]. In [G] again the 4 elementary operations of computation can be constructively performed in the well known manner. Furthermore we have here the operations

$$\overline{s + t\iota} = s - t\iota,$$

$$\eta_\iota(s + t\iota) = t$$

and

$$|s + t\iota| = \begin{cases} s^2 + t^2, & \text{if } s^2 + t^2 > 0 \\ 0 & \text{otherwise.} \end{cases}$$

An element $s + t\iota$ is said to be "real" if $t = 0$, and

it is said to be "positive" if $t = 0$, $s > 0$.

Adjoining now the elements of [G] to our formalism, taking them as the N-terms (in the sense referring to the formulation of the consistency theorem) and replacing α, β, γ, respectively, by the terms

$$0 + 0\iota, \qquad 1 + 0\iota, \qquad 0 + 1\iota,$$

we can define the truth-values of the predicates =, *Od*, $\cong$, and the values of the mathematical functions Φ, Ψ in such a way that the conditions 1, 2, 3 of the consistency-theorem are satisfied.

In fact, denoting by a, b, c, d, elements of [G], we define:

$a = b$ to be true, if and only if $a - b$ has the value 0,

Od (a, b, c) to be true, if and only if $(b-a)\cdot\overline{(c-a)}$ is positive,

$ab \cong cd$ to be true, if and only if $|a-b|$ has the same value as $|c-d|$.

Each of the predicates is defined to be false, if it is not true.

According to these definitions St(a, b, c) is true if and only if $(b-a)\overline{(c-a)}$ is real.

Furthermore, we define:

Φ (a, b, c, d) to have the value $a + \dfrac{\eta_\iota((a-c)\overline{(c-d)})}{\eta_\iota((a-b)\overline{(c-d)})} \cdot (b-a)$ provided $(a-b)\overline{(c-d)}$ is not real, and otherwise to have the value a;

Ψ (a, b, c, d) to have the value $c + \dfrac{|a-b|}{|c-d|} \cdot (c-d)$, if $c-d$ is different from 0, and otherwise to have the value c.

According to these definitions we can compute every term which contains no free variable, and effectively decide for a prime-formula in which every term is an N-term, whether it is true or false. Furthermore, by means of elementary computations it can be seen that all the axioms of our considered system are verifiable; that is, that they turn into true formulas if the free variables are replaced by N-terms and afterwards the terms containing a function-symbol are replaced by their values.

Thus by the consistency-theorem it follows that from our considered formalism no two formulas A, $\overline{A}$ can be deduced. And from this we can infer, by virtue of our preliminary considerations, that our original system of axioms 1)-9) together with the logical calculus in one of the three forms and the corresponding general formalization of equality is consistent.

Remark 1. According to the usual manner of proceeding, the proof of consistency for the axiomatic geometry is already finished as soon as the arithmetic interpretation of the axioms is established. But this manner of reducing the question of consistency for the axiomatic geometry to that of arithmetic is not sufficient for our purpose. Namely we have the following alternative: either the arithmetic is understood in an unrestricted sense, and from this point of view its consistency is not beyond all doubt; or arithmetic is understood as elementary intuitive arithmetic,- then we are not sure that according to the arithmetical interpretation of the fundamental predicates of the geometry, also the <u>reasonings</u> of geometry as they are formalized by the logical calculus, can be translated into arithmetical reasonings.

So from the point of view of elementary evidence there is no immediate inference from the elementary arithmetic interpretation of the geometric axioms to the statement of the consistency, since this statement concerns a property of the axioms not in themselves but in connection with the logical calculus.

The intervening argument, which is required here, is given by our consistency-theorem.

Remark 2. There is no difficulty to include the axiom 10) (cf. p.109) in our consistency-proof; only one function-symbol more has to be introduced and a corresponding arithmetical definition to be given, which is still possible in the frame of the field [G].

Remark 3. Also the non-Euclidean projective and metric geometry can be proved by our method to be consistent.

11. The Number-theoretic Formalism

We are now going on to see what can be concluded from our method of proving consistency with regard to arithmetic, or more precisely: to the number-theoretic formalism.

As is well known, number-theory has been axiomatized by Peano. His axioms are:

1) 0 is a number.
2) If n is a number, then n' is a number.
3) If $a' = b'$, then $a = b$.
4) $n' \neq 0$ for every number n.
5) If $A(c)$ is a predicate such that $A(0)$ holds, and for every number n for which $A(n)$ holds, also $A(n')$ holds, then $A(n)$ holds for every number n.

To formalize these axioms we do not need a special symbol for the concept of number, because in the whole theory we have no individuals other than numbers.

According to this device the two first axioms are formalized already by introducing the symbols 0, $'$; 3) and 4) can be formalized by the formulas

$$a' = b' \longrightarrow a = b$$

$$a' \neq 0$$

and 5) by the schema of "complete induction"

$$\frac{A(0),\ A(n) \rightarrow A(n')}{A(t),}$$

where n denotes a free variable not occurring in A(c) and t an arbitrary term.

The necessity of the restrictive condition on the variable n can be seen, for instance, from the case where A(c) is the formula

$$c = 0 \vee c = n';$$

here A(0) and A(n'), and therefore also $A(n) \rightarrow A(n')$ are deducible by means of the formula a = a; thus without the restriction on the variable n we would come to A(0'), that is

$$0' = 0 \vee 0' = n'$$

and further, by substituting 0' for n, to the formula

$$0' = 0 \vee 0' = 0'',$$

which together with the axioms

$$a' = b' \rightarrow a = b$$

$$a' \neq 0$$

leads to a contradiction.

For operating with equality, as we know, special equality-axioms are sufficient. Here, since we have but the one function-symbol ' and no predicate symbol besides the equality-symbol, only the three equality-axioms

$$a = a, \qquad a = b \rightarrow (a = c \rightarrow b = c), \qquad a = b$$

are required.

One might think that the five axioms

1)) $$a = a$$

2)) $$a = b \rightarrow (b = c \rightarrow a = c)$$

3)) $$a = b \rightarrow a' = b'$$

4)) $$a' = b' \rightarrow a = b$$

5)) $$a' \neq 0$$

together with the schemata of our logical calculus -- let us take it again in the second form -- and the induction-schema

$$\frac{A(0),\ A(n) \rightarrow A(n')}{A(t)}$$

give already the full number-theoretic formalism.

But by considering somewhat more closely the formalism just described one finds that it is a quite narrow and rather trivial one. We are not even able to express with it the relation $a < b$ nor the function $a + b$.

In order to get the formalism of number-theory we have to add the recursive equations

$$\text{(r)} \qquad \begin{array}{ll} a + 0 = a & a \cdot 0 = 0 \\ a + b' = (a+b)' & a \cdot b' = a \cdot b + a \end{array}$$

introducing with them the function symbols for addition and multiplication.

These equations are regarded in Peano's "Formulario Mathematico" as definitions, also in general it is usual to call them "recursive definitions". In some respect surely they have the character of definitions; we shall have to use this fact. But on the other hand we must recognize that they are not definitions in the proper sense, that is abbreviations for some expressions formed by means of symbols introduced before, but really are enlarging the formalism.

At all events, in our formalism we have to adjoin these equations as formal axioms, though we may call them in the usual way "the recursive definitions of sum and product".

There is no need of adding equality-formulas, belonging to the symbols of sum and product, to our axioms, since all the equality-formulas belonging to these symbols are deducible from the recursive equations (r) and the equality-axioms 1)), 2)), 3)) by means of the induction-schema.

Regarding now the formalism obtained by the adjunction of the equations (r), thus consisting of the logical calculus in the second form, the axioms 1))-5)) and (r) and the induction-schema, the question concerns the possibility of applying to it our method of proving consistency.

In the first place, it is easily stated that the formulas 1))-5)) are verifiable if we take for the numerical equations the obvious definition of the truth-values, according to which a numerical equation a = b is true if and only if a is the same numeral as b.

Furthermore, for the functions, sum and product, the recursive equations can be used to assign numerals as values to the terms

$$a + b, \ a \cdot b$$

where a, b are numerals. Indeed, this is nothing else than the very elementary manner of getting the values of the sum and the product of two numerals by iterated applications of the recursive equations (r), the possibility of which is due to the facts that every numeral is either 0 or of the form n', where n is again a numeral, and that the iteration of the process of passing from a numeral n' to n comes to an end and leads finally to 0.

From the process of computing the values of a + b, a • b for numerals, it also results that the recursive equations (r) turn into true formulas if the variables in them are replaced by any numerals, each of them for each of its occurrences by the same numeral, and the resulting sums and products of numerals are replaced by their values.

So with respect to our axioms, the conditions of the consistency-theorem are satisfied, and so at all events we can infer that every formula containing no bound variable which is deducible in our formalism without applying the induction-schema is verifiable.

And the same will still hold if we adjoin to the axioms any verifiable formula.

But if we now attempt to include the induction-schema in our consistency-proof, then we come into difficulties.

One of them arises from the requirement of proving that the schema $\dfrac{\begin{matrix}\lfloor A \rfloor \\ \vdots \\ B\end{matrix}}{A \to B}$ is still a derived rule in our formalism. Of course our method of eliminating the ϵ-schema depends essentially on the presence of this schema. Now to show this schema still to hold after the induction-schema is added, we would have to show that an antecedent can be carried through the induction-schema. But if this antecedent contains the variable n of this schema -- let us denote the antecedent by P(n) -- then the schema

$$\frac{A(0),\ A(n) \to A(n')}{A(t)}$$

turns into

$$\frac{P(n) \to A(0),\ P(n) \to (A(n) \to A(n'))}{P(n) \to A(t)}$$

From the second formula we can deduce

$$(P(n) \to A(n)) \to (P(n) \to A(n')).$$

but still the resulting figure is not an induction-schema.

Another similar difficulty occurs in the process of replacing the -terms. Let e be an ϵ-term contained in a formula A(c), on which the induction-schema is applied, so that this application has the form

$$\frac{B(e,\ 0),\ B(e,\ n) \to B(e,\ n')}{B(e,\ t)};$$

suppose that e has to be replaced by a term s(n), n being the same variable as in the considered induction-schema; then by the replacement this schema turns into

$$\frac{B(s(n),\ 0),\ B(s(n),\ n) \to B(s(n),\ n')}{B(s(n),\ t)}$$

Here again the form of the induction-schema is disturbed.

Thus there is no hope of extending our former proof of the elimination-theorem to the case in which the induction-schema is included.

However, if we restrict the induction-schema by the condition that the formula A(c), to which it applies, contains no bound variable, then in another way we can include this <u>restricted induction-schema</u> in the consistency proof we have for the axioms 1))-5)), (r).

The argument is the following. We consider in a given deduction of a formula F, containing no bound variable, the first application of the induction-schema

$$\frac{B(0),\ B(n) \longrightarrow B(n')}{B(t)}$$

Here the formulas B(0), $B(n) \longrightarrow B(n')$ are deduced without applying the induction-schema. For any numeral k we not get from the second, by the (derived) substitution-rule for the free variables, the formulas

$$B(0) \longrightarrow B(0'),\quad B(0') \longrightarrow B(0''),\ \ldots,\ B(k) \longrightarrow B(k'),$$

which, together with B(0) give, by means of the implication-schema, the formula B(k').

Thus for any numeral m the formula B(m) is deducible from our axioms 1))-5)), (r) by means of the logical calculus. On the other hand, according to our restrictive condition on the induction-schema, it contains no bound variable. Thus, according to what was proved just before, B(m), for any numeral m, is a verifiable formula. And so also B(b) with a free variable b, is verifiable. But from B(b) we get B(t).

Thus the first application of the induction-schema can be replaced by the introduction of the verifiable formula B(b) as an axiom. Still now, according to what we stated before, (cf. p. 120), the system of the axioms has the property that every formula containing no bound variable which is deducible from the axioms by means of the logical calculus (without the induction-schema) is verifiable.

Thus we can apply the same argument we did for the first occurrence of the induction-schema; also for the second occurrence. And we can continue in the same way. At last we come to the result that the final formula F of the considered deduction (which was assumed to contain no bound variable) must be verifiable.

So indeed it is shown that in the "restricted number-theoretic formalism", consisting of the logical calculus, the axioms 1))-5)), (r) and the induction schema, applied only to formulas without bound variables, every deducible formula containing no bound variable is verifiable. This statement includes the consistency of the restricted number-theoretic formalism.

At once our proof gives the sharper result that in every deduction made by the restricted number-theoretic formalism the application of the (restricted) induction-schema can be replaced by the adjunction of some verifiable formulas as axioms.

From this and the last part of our consistency-theorem (cf. p.97) it follows that, if a formula

$$(Ex_1)\ \dots\ (Ex_n)H(x_1,\ \dots,\ x_n),$$

where $x_1, \dots, x_n$ are the only variables, is deducible in the restricted number-theoretical formalism, we can find (by the process of eliminating the bound variables) some numerals $a_1, \dots, a_n$ such that

$$H(a_1,\ \dots,\ a_n),$$

after the computation of the occurring sums and products, becomes a true formula.

A consequence of this result again is that the negation of a formula

$$(x_1)\ \dots\ (x_n)G(x_1,\ \dots,\ x_n),$$

in which $x_1, \ldots, x_n$ are the only variables, cannot be deducible by the restricted number-theoretic formalism if for every set of numerals

$$a_1, \ldots, a_n$$

the formula

$$G(a_1, \ldots, a_n)$$

is deducible by this formalism.

In fact, from the assumption and the proved consistency-property of the number-theoretic formalism it follows that for every set of numerals $a_1, \ldots, a_n$ the formula

$$G(a_1, \ldots, a_n)$$

after the computation of the sums and products must turn into a true formula, and therefore

$$\overline{G(a_1, \ldots, a_n)}$$

turns into a false formula. On the other hand, if the negation of

$$(x_1) \ldots (x_n) G(x_1, \ldots, x_n)$$

were deducible, then also

$$(Ex_1) \ldots (Ex_n)\overline{G(x_1, \ldots, x_n)}$$

would be deducible, and, according to the theorem stated just now, we should be able to find some numerals $a_1, \ldots, a_n$, for which

$$\overline{G(a_1, \ldots, a_n)},$$

after the computation of the occurring sums and products, would become true.

<u>Remark:</u> The property we have proved here to hold for the number-theoretic formalism is a part of what <u>Gödel</u> calls "ω-consistency". It would be the same if the formula

$$(x_1) \ldots . (x_n) G(x_1, \ldots, x_n)$$

should be permitted to contain, besides $x_1, \ldots, x_n$, still other bound variables.

Still the following consequence of our elimination-theorem may be mentioned: every formula deducible by the number-theoretic formalism, which contains no bound variable, is deducible also by the formalism consisting only of the propositional calculus, the restricted induction-schema, and the formula-schemata corresponding to the axioms 1))-5)), (r).

The different theorems which we have obtained here concerning the restricted number-theoretic formalism can immediately be seen, by means of our method of proof, to hold also for some more extended formalisms.

Indeed our method of proof applies still to the following extensions of the number-theoretic formalism:

1. Generalized forms of the restricted induction-schema, as <u>for instance</u>

$$\frac{\begin{array}{c}A(0,\ b)\\ A(n,\ t_1)\ \&\ \ldots\ \&\ A(n,\ t_r) \rightarrow A(n',\ b)\end{array}}{A(t,\ s)}$$

where n, b are different free variables, which are not contained in A(0, 0), whereas t, t_1, ..., t_r are arbitrary terms.

<u>Remark</u>: For the full number-theoretic formalism this schema gives no extension.

2. Replacement of the axioms (r) by schemata for recursive definitions, for instance the definition by "primitive recursions", consisting of a sequence of pairs of recursive equations having each one of the two forms:

$f(0) = r$	$f(a,\ 0) = r(a)$
$f(n') = s(n,\ f(n)$	$f(a,\ n') = s(a,\ nf(a,\ n))$

where f is a function-symbol, n, a are variables,

r is a term containing no variable,

r(a) " " " " " " besides a,

s(n, f(n)) arises (by substitution) from a term s(b, c) containing no variable besides b, c,

s(a, n, f(n)) arises from a term s(b, c, d) containing no variable besides b,c,d,

and r, r(a), s(b, c), s(b, c, d) contain no other symbol than 0, ', and the function-symbols introduced by the preceding recursions.

Still a quite general device, according to Herbrand-Gödel-Kleene, for introducing constructive ("general recursive") definitions of functions can be adopted.

For proving the consistency of the whole number-theoretic formalism stronger methods are required and the finite point of view has to be extended in some way,- as was first indicated by a general result of Kurt Gödel ("Ueber formal unentsch. Sätze ...", Monatsh. Math. Phys., Vol. 38, 1931.)

A proof by reducing the question of consistency for the number-theoretic formalism to that of the Hyting formalism of intuitionistic arithmetic has been given by Kurt Gödel ("Zur intuitionistischen Arithmetik und Zahlentheorie", Ergebnisse eines math. Koll., Heft 4, 1933).

Recently Gerhard Gentzen proved the consistency of the number-theoretic formalism by using a transfinite induction extending to the first Cantor ϵ-number (that is the limit of the sequence ω, ω^{ω}, $\omega^{\omega^{\omega}}$, ...).

("Die Widerspruchsfreiheit der reinen Zahlentheorie", Math. Ann., Vol. 112, 1936. Here also is quoted previous research of W. Ackermann, J. v. Neumann, J. Herbrand, G. Gentzen, concerning the consistency of arithmetic.)