LECTURES

By

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On

TENSOR ANALYSIS AND DIFFERENTIAL GEOMETRY

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TABLE OF CONTENTS

Part I

Page

CHAPTER I

TENSOR ALGEBRA

1. The n-dimensional point space and coordinate transformations.

Coordinate Systems. At the start of this work we will be concerned only with local properties of our space. Hence we only need a portion of the space coverable by a single coordinate system, which we define in the following manner. Let $\mathcal{T}$ be a point set and E_n the n-dimensional number space. $\mathcal{T}$ will be said to be an n-dimensional space if there exists a one-to-one correspondence between the points of $\mathcal{T}$ and the points of a subset S of E_n where S is the interior of an n-dimensional topological sphere. Any such one-to-one correspondence will be called a coordinate system of $\mathcal{T}$.

m-Systems. Let $\varphi(P)$ be any real point function defined over $\mathcal{T}$. Then, in any one coordinate system, $\varphi(P)$ is a function $\varphi(x_1, \ldots, x_n)$ of the coordinates (x). Up to this point there is no notion of continuity or differentiability of a function $\varphi(P)$ defined over $\mathcal{T}$. We now define, in the following manner, functions $\varphi(P)$ of class C^ρ, $\rho = 0, \ldots, m$, for a given $m \geq 1$. We chose a definite coordinate system (x). A function $\varphi(P)$ defined over $\mathcal{T}$ is said to be of class C^ρ in $\mathcal{T}$, $0 \leq \rho \leq m$, if and only if $\varphi(P)$ as $\varphi(x_1, \ldots, x_n)$ in the (x) system is of class C^ρ *. According

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* A function of real variables is said to be of class C^t if it possesses all continuous derivatives up to and including those of order t. A continuous function is said to be of class C^0.

to this definition the following property holds: the totality of functions $\varphi(P)$ of class C^{ρ} in γ coincides with the totality of functions $\varphi(x_1, \ldots, x_n)$ of class C^{ρ} in the (x) system. Any coordinate system which possesses the above property will be called an m-system. Thus an m-system (x') has the property that any function of class C^{ρ} in γ, and only such a function, is of class C^{ρ} as a function in the (x') system, $0 \leq \rho \leq m$.

Consider any two m-systems (x) and (x'). Since (x) and (x') are both in one-to-one correspondence with γ they are in one-to-one correspondence with each other, and we can write

$$(1.1) \qquad \begin{aligned} x'_i &= x'_i(x_1, \ldots, x_n) = x'_i(x) \\ x_i &= x_i(x'_1, \ldots, x'_n) = x_i(x') \end{aligned}$$

where x'_i and $x_i(x')$ are single-valued functions. From (1.1) we get the identity

$$(1.2) \qquad x'_i \equiv x'_i(x(x')).$$

The x'_i as functions of the x'_i are of class C^m in the (x') system, and hence of class C^m in the (x) system, and likewise the x_i are of class C^m in the (x') system. Therefore, since $m \geq 1$, we can differentiate (1.2) getting

$$(1.3) \qquad \delta_i^k = \frac{\partial x'_i}{\partial x_\ell} \cdot \frac{\partial x_\ell}{\partial x'_k}.$$

Hence

$$1 = \left|\frac{\partial x'_i}{\partial x_\ell}\right| \cdot \left|\frac{\partial x_\ell}{\partial x'_k}\right|$$

and, since the determinants are continuous,

$$\left|\frac{\partial x'_i}{\partial x_\ell}\right| \neq 0, \quad \left|\frac{\partial x_\ell}{\partial x'_k}\right| \neq 0,$$

i.e. the Jacobians of the transformations (1.1) exist and are different from zero.

We might note here that the transformations between m-systems do not form a group, since we have no rule of combination of the transformation from (x) to (x') and the one from (x") to (x"').

2. Scalars and Vectors.

Scalar. A scalar at a point P is defined to be a real number φ assigned to P. Using a particular coordinate system, the scalar might be written as

$$(x_1, \dots, x_n, \varphi) \equiv (x, \varphi),$$

emphasizing the fact that n + 1 numbers are necessary to determine the scalar at a point. If $(x'_1, \dots, x'_n, \varphi')$ is the same scalar in another coordinate system, the law of transformation is obviously

$$x'_i = x'_i(x), \quad \varphi' = \varphi . \tag{2.1}$$

Contravariant Vector. By a contravariant vector at a point $P(x_1, \dots, x_m)$ is meant a set of 2n numbers in each coordinate system, $(x_1 \dots, x_n, \lambda^1, \dots, \lambda^n) = (x, \lambda^i)$, which transform from any one coordinate system (x) into any other system (x')

according to the law

$$x'_i = x'_i(x), \quad \lambda'^i = \frac{\partial x'_i}{\partial x_k} \lambda^k \tag{2.2}$$

The quantities $\lambda^1, \dots, \lambda^n$ which occur in (x, λ^i) are called the components of the vector (with respect to the x coordinate system). We shall usually write λ^i as the designation for the vector (x, λ^i). An analogous designation will apply to terms to be defined later. One can easily show that if one starts with the components of a contravariant vector (x, λ^i) in the (x) system, and transforms from the (x) system to an (x') system and from the (x') system to an (x'') system, the components λ''^i resulting are the components one would obtain by transforming directly from the (x) system to the (x'') system. Thus we see that the definition of contravariant vectors is free from contradictions.

If the components of a contravariant vector are zero at a point in a particular coordinate system, i.e., if they are $(x_1, \ldots, x_n, 0, \ldots, 0)$ then by (2.2) they are zero at the point in all coordinate systems. This vector will be called the contravariant zero vector at the point $(x_1, \ldots, x_n)$. If (x, λ^i) and (x, μ^i) are contravariant vectors, $(x, \lambda^i + \mu^i)$ is also; if (x, λ^i) is a contravariant vector and σ is a scalar, $(x, \sigma\lambda^i)$ is a contravariant vector. Also $(x, \lambda^i) + (x, \mu^i) = (x, \mu^i) + (x, \lambda^i)$, and the negative $(x, -\lambda^i)$ of a vector (x, λ^i) exists. These results show that the set of all contravariant vectors at a point form a linear vector space.

A set of contravariant vectors $\lambda^i, \mu^i, \cdots, \nu^i$ at a point will be said to be linearly dependent if a set of scalars a, b, ..., c not all zero exists such that the following relation between the components of the vectors,

$$a\lambda^i + b\mu^i + \cdots + c\nu^i = 0,$$

holds for all i. Otherwise the set of vectors will be said to be independent. Notice that, since the components of the contravariant zero vector are unchanged by coordinate transformations, the notion of linear dependence or independence does not depend on the choice of coordinate system. In $\mathcal{P}$ there are n and only n linearly independent contravariant vectors at a point. Take in a particular coordinate system, n contravariant vectors $\underset{\alpha}{\lambda}^i$ at a point such that, for their components $\underset{\alpha}{\lambda}^i$,

$$\begin{vmatrix} \underset{1}{\lambda}^1 & \cdots & \underset{1}{\lambda}^n \\ \vdots & & \vdots \\ \underset{n}{\lambda}^1 & \cdots & \underset{n}{\lambda}^n \end{vmatrix} \neq 0.$$

Then $a_\alpha \underset{\alpha}{\lambda}^i = 0$ if and only if $a_\alpha = 0$, and the n contravariant vectors $\underset{\alpha}{\lambda}^i$ are linearly independent. Consider any other contravariant vector η^i at the same point. Since $|\underset{\alpha}{\lambda}^i| \neq 0$, the set of equations

$$a_\alpha \underset{\alpha}{\lambda}^i = \eta^i$$

has a unique solution a_α, and η^i is linearly dependent on the $\underset{\alpha}{\lambda}^i$. Any such system of n linearly independent contravariant vectors at a point is called a contravariant n-Bein at the point.

Covariant Vector. By a covariant vector at a point P is meant a set of 2n numbers in each coordinate system,

$$(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n) \equiv (x, \lambda_i),$$

which transform from any one coordinate system (x) into any other (x') according to the law

$$x'_i = x'_i(x), \quad \lambda'_i = \frac{\partial x_j}{\partial x'_i} \lambda_j . \tag{2.3}$$

As in the case of contravariant vectors we can show 1) that the definition of covariant vector is free from contradiction, 2) that at each point of $\mathcal{R}$ the totality of covariant vectors forms a linear vector space, 3) that linear dependence or independence is unaffected by coordinate transformations, and 4) that at each point of $\mathcal{R}$ there are n and only n linearly independent covariant vectors, any such set of n covariant vectors at a point being called a covariant n-Bein.

An alternative definition of covariant vector is contained in the following. Consider $a_i \lambda^i$, where λ^i are the components of an arbitrary contravariant vector λ^i, and let the a_i transform in such a way that $a_i \lambda^i$ is an invariant, i.e.,

$$a_i \lambda^i = a'_i \lambda'^i.$$

Then

$$a_K \lambda^K = a'_i \frac{\partial x'_i}{\partial x_K} \lambda^K$$

for any λ^i; hence

$$a_K = a'_i \frac{\partial x'_i}{\partial x_K}$$

so that the a_K are the components of a covariant vector. The invariant form $a_i \lambda^i$ is called the inner product of the covariant vector (x, a_i) and the contravariant vector (x, λ^i).

Adjoint Beins. To any particular (contravariant or covariant) n-Bein at a point we can assign a particular (covariant or contravariant) n-Bein at the same point in the following manner. Let $\lambda_{\alpha i}$ be a covariant n-Bein at a point. There exists a contravariant n-Bein μ_β^i such that

(2.4) $$\lambda_{\alpha i}\, \mu_{\beta}^{i} = \delta_{\alpha\beta}$$ (Kronecker deltas)

for, if (2.4) is considered in a particular coordinate system, for each β ~~we~~ can solve the n equations uniquely for μ_{β}^{i} since $|\lambda_{\alpha i}| \neq 0$. By this a set of n contravariant vectors μ_{β}^{i} (at that point) is defined. The so defined set μ_{β}^{i} will solve (2,4) in any coordinate system (in a unique wqy) because (2,4) are invariant relations. Hence the solutions of (2,4), μ_{β}^{i}, form a contravariant n-Bein, $|\mu_{\beta}^{i}| \neq 0$. According to the lemma to be proved immediately the defining relationship (2,4) may be written as

(2.4') $$\lambda_{\alpha i}\, \mu_{\alpha}^{k} = \delta_{ik}$$

Any pair of n-Beins at a point satisfying (2.4) or (2.4') will be called adjoint n-Beins.

Lemma: If $a_{iK}\, b_{jK} = \delta_{ij}$

then $a_{Ki}\, b_{Kj} = \delta_{ij}$.

Proof: Let $a_{Ki}\, b_{Kj} = c_{ij}$.

Then

$$b_{\ell i}\, a_{Ki}\, b_{Kj} = c_{ij}\, b_{\ell i},$$

$$\delta_{K\ell}\, b_{Kj} = c_{ij}\, b_{\ell i};$$

$$b_{\ell j} = c_{ij}\, b_{\ell i};$$

$$\delta_{ij}\, b_{\ell i} = c_{ij}\, b_{\ell i},$$

$$(\delta_{ij} - c_{ij})\, b_{\ell i} = 0.$$

By hypothesis $|b_{\ell i}| \neq 0$; hence

$$c_{ij} = \delta_{ij}.$$ Q. E. D.

3. Tensors.

By a tensor $T^{i,j\ldots l}_{m,n\ldots p}$ at a point P we mean the totality of coefficients of an invariant multilinear form

$$(3.1) \qquad T^{ij\ldots l}_{mn\ldots p}\lambda_i\mu_j\ldots\nu_l\rho^m\sigma^n\ldots\tau^p$$

in several arbitrary co- and contravariant vectors at P. The upper indices will be called contravariant the lower covariant. As an example, we see that the Kronecker delta, δ^j_i, is a tensor at any point of $\mathcal{V}$; for

$$\delta^j_i\rho_j\sigma^i = \rho_i\sigma^i,$$

which is an invariant form. Another example of a tensor is the product of a number of contravariant and covariant vectors. The order of a tensor is the sum of the number of contravariant and the number of covariant indices; two tensors are said to be of the same kind if they have the same number of contravariant and the same number of covariant indices.

We can obtain the law of transformation for components of a tensor from our definition, in the following manner. Let B^i_K be the components of a tensor, λ^i and ρ_i the components of contravariant and covariant vectors in the (x) system, and let $\bar{B}^i_K$, $\bar{\lambda}^i$ and $\bar{\rho}_i$ be the components of the same entities in the $(\bar{x})$ system). Then according to our definition of a tensor,

$$B^i_K\lambda^K\rho_i = \bar{B}^i_K\bar{\lambda}^K\bar{\rho}_i$$

Since $\bar{\lambda}^K = \frac{\partial\bar{x}_K}{\partial x_j}\lambda^j$ and $\bar{\rho}_i = \frac{\partial x_l}{\partial\bar{x}_i}\rho_l$ we have

$$\left(B_j^{\ell} - \overline{B}_k^{\,i} \frac{\partial \overline{x}_k}{\partial x_j} \frac{\partial x_\ell}{\partial \overline{x}_i}\right) \lambda^i \rho_\ell = 0$$

for arbitrary λ and ρ. Hence we have

$$B_j^{\ell} = \overline{B}_k^{\,i} \frac{\partial \overline{x}_k}{\partial x_j} \frac{\partial x_\ell}{\partial \overline{x}_i},$$

the law of transformation of a tensor of the second order. The law of transformation of a general tensor is

$$T^{i_1 \dots i_r}_{j_1 \dots j_s} = T^{k_1 \dots k_r}_{\ell_1 \dots \ell_s} \frac{\partial \overline{x}_{\ell_1}}{\partial x_{j_1}} \dots \frac{\partial \overline{x}_{\ell_s}}{\partial x_{j_s}} \frac{\partial x_{i_1}}{\partial \overline{x}_{k_1}} \dots \frac{\partial x_{i_r}}{\partial \overline{x}_{k_r}}. \tag{3.2}$$

An alternative definition of tensor may be given in terms of this law of transformation.

The general tensor cannot be written as the product of a set of contravariant and covariant vectors, but it can be written as the sum of such products in a way indicated by the following example. Let B_i^k be a general tensor of order two, and let $\underset{\beta}{\lambda}{}^i$ and $\underset{\alpha}{\mu}{}_k$ be two n-Beins at the point in question. Consider

$$B_K^i = b_{\alpha\beta}\, \underset{\alpha}{\mu}{}_K \underset{\beta}{\lambda}{}^i \tag{3.3}$$

where the $b_{\alpha\beta}$ are scalars to be determined by (3.3). (3.3) can be rewritten as

$$\left\{ \begin{aligned} b_{\alpha\beta}\, \underset{\alpha}{\mu}{}_K &= C_{\beta K} \\ C_{\beta K}\, \underset{\beta}{\lambda}{}^i &= B_K^i \end{aligned} \right. \tag{3.3'}$$

Since $|\underset{\beta}{\lambda}{}^i| \neq 0$, the second set of (3.3') can be uniquely solved for $C_{\beta K}$; since $|\underset{\alpha}{\mu}{}_K| \neq 0$, the first set of (3.3') can be uniquely solved for $b_{\alpha\beta}$. This process has been carried out in a particular m system, but because of the invariance of (3.3) and (3.3') the results hold in all m systems. In particular, if $\underset{\alpha}{\lambda}^{i}$ and $\underset{\beta}{\mu}_{K}$ are adjoint n-Beins, we have, using (2.4),

$$b_{\alpha\beta} = B^{i}_{K} \underset{\alpha}{\lambda}^{K} \underset{\beta}{\mu}_{i} .$$

<u>The following are the fundamental algebraic operations with tensors.</u>

1. <u>Addition.</u> If $T^{i_1 \ldots i_q}_{K_1 \ldots K_r}$ and $U^{i_1 \ldots i_q}_{K_1 \ldots K_r}$ are tensors of the same kind at the same point, $T^{i_1 \ldots i_q}_{K_1 \ldots K_r} + U^{i_1 \ldots i_q}_{K_1 \ldots K_r}$ is a tensor of the same kind at that point. The indicated addition means the addition of corresponding components in the same m-system.

2. <u>Multiplication.</u> If $T^{i_1 \ldots i_q}_{K_1 \ldots K_r}$ and $S^{j_1 \ldots j_s}_{h_1 \ldots h_p}$ are two tensors at the same point, $R^{i_1 \ldots i_q j_1 \ldots j_s}_{K_1 \ldots K_r h_1 \ldots h_p} = T^{i_1 \ldots i_q}_{K_1 \ldots K_r} S^{j_1 \ldots j_s}_{h_1 \ldots h_p}$ is a tensor at that point.

3. <u>Contraction.</u> If $T^{i_1 \ldots i_s}_{j_1 \ldots j_r}$ is a mixed tensor then

$$\overline{T}^{i_1 \ldots i_{m-1} i_{m+1} \ldots i_s}_{j_1 \ldots j_{p-1} j_{p+1} \ldots j_r} = T^{i_1 \ldots i_{m-1} K i_{m+1} \ldots i_s}_{j_1 \ldots j_{p-1} K j_{p+1} \ldots j_r}$$ is a tensor too.

This is easily seen by representing the tensor by means of a Bein and its adjoint.

4. <u>Fixed permutation operation.</u> Let $t_{i_1 \ldots i_r}$ be covariant (or contravariant) tensor and put

$$\overline{t}_{i_1 \ldots i_r} = t_{i_{\alpha_1} \ldots i_{\alpha_r}}$$

where $\alpha_1 \ldots \alpha_r$ is a fixed permutation of 1, ..., r. Then

$\bar{t}_{i_1 \ldots i_r}$ is a term of the same kind as $t_{i_1 \ldots i_r}$.

Taking for $\alpha_1, \cdots, \alpha_r$ all permutations of 1, ... , r we get n! tensors of the same kind. The sum of them is a symmetric tensor. The same sum with a negative sign attached to those members of the sum produced by an odd permutation is an anti-symmetric tensor.

4. Vector spaces at a point P.

Definition. A set of vectors (of covariant or contravariant kind) at a point P form a linear vector space at P if the set is closed under addition, and multiplication by scalars.

We have already seen the existence of an n-dimensional vector space at a point P. We can consider vector subspaces at P of lower dimensionality. As an example, all vectors μ_i which are solutions of $t^{iK}\mu_i = 0$, t^{iK} being a given tensor, form a vector space. If $\| t^{iK} \|$ is of rank r the vector space thus defined is of dimension n - r. As another example, all vectors ρ^K such that $t^{iK}\sigma_i = \rho^K$, t^{iK} being a given tensor and σ_i an arbitrary vector, form a vector space.

Consider a vector space V at P. V has a dimension less than or equal to n. For take a vector λ_1 from V. Either V equals the totality of vectors $\alpha_1\lambda_1$ or there exists another vector λ_2 in V which is not in the totality $\alpha_1\lambda_1$. Then as before we can consider the totality of vectors $\alpha_1\lambda_1 + \alpha_2\lambda_2$ and continue, arriving after r steps, $r \leq n$, at a stage where V equals the totality of vectors $\alpha_1 \lambda_1 + \cdots + \alpha_n \lambda_n$. Then V is of dimension $r \leqq n$; such a V of dimension r may be written as $V^{(r)}$ if contravariant, $V_{(r)}$ if covariant. It can easily be shown that the dimensionality of a vector space is unique.

Let $V^{(r)}$ be an r-dimensional contravariant vector space at a point P. The totality of all covariant vectors ρ normal to each vector of $V^{(r)}$ i.e. $\rho_i \lambda^i = 0$; $\lambda \in V^{(r)}$, form a covariant vector space $V_{(n-r)}$ of dimensionality n - r. $V_{(n-r)}$ will be said to be normal to $V^{(r)}$; conversely, to each covariant vector space $V_{(r)}$ there exists a normal contravariant vector space $V^{(n-r)}$.

5. Canonical Representation of a Symmetric Tensor t_{iK} of Rank r.

In the following work we bear in mind that only vectors or tensors at a fixed point P of $\mathcal{V}$ will be considered. We start with the following

Lemma. Let $\underset{\alpha}{\lambda}{}^i$ and $\underset{\alpha}{\mu}{}_i$, $\alpha = 1, \ldots, t$, be two sets of t vectors such that

(5.1)
$$\begin{cases} \underset{\alpha}{\lambda}{}^i \underset{\beta}{\mu}{}_i = 0, & \alpha \neq \beta, \ \alpha, \beta = 1, \ldots, t \\ \underset{\alpha}{\lambda}{}^i \underset{\alpha}{\mu}{}_i \neq 0, & \alpha \text{ not summed.} \end{cases}$$

Then each set of t vectors are independent.

If we assume $A_\alpha \underset{\alpha}{\lambda}{}^i = 0$ and multiply both sides of that equation by $\underset{\beta}{\mu}{}_i$ we get, using (5.1), $A_\alpha = 0$. In the same way $B_\alpha \underset{\alpha}{\mu}{}_i = 0$ leads to $B_\alpha = 0$, and the lemma is proved.

As a consequence of this the vector space $\{\xi^i\}^{(n-t)}$ defined by

(5.2) $$\zeta^i \underset{\beta}{\mu}_i = 0, \quad \beta = 1, \cdots, t,$$

has dimension n - t. The vector spaces $\{\zeta^i\}^{(n-t)}$ of dimension n - t and $\{\underset{1}{\lambda}^i, \cdots, \underset{t}{\lambda}^i\}^{(t)}$ of dimension t have no vector in common other than the zero vector, for if $\zeta^i = A_\alpha \underset{\alpha}{\lambda}^i$, $\zeta^i \varepsilon \{\zeta^i\}^{(n-t)}$ then $A_\alpha \underset{\alpha}{\lambda}^i \underset{\beta}{\mu}_i = 0$, $\beta = 1, \ldots, t$, and hence $A_\alpha = 0$, $\alpha = 1, \ldots, t$. Hence any vector (at P) is expressible as a sum of a vector of $\{\zeta^i\}^{(n-t)}$ and a vector of $\{\underset{1}{\lambda}^i, \cdots, \underset{t}{\lambda}^i\}^{(t)}$.

Now let a symmetric tensor t_{iK} of rank r be given. By the rank of a tensor t_{iK} is meant the rank of $\| t_{iK} \|$. The vector space $\{\theta^i\}^{(n-r)}$ defined by $t_{iK}\theta^K = 0$ is of dimension n - r. (This proves the invariance of the rank of a tensor.) Assume we have constructed t < r contravariant vectors $\underset{\alpha}{\lambda}^i$, $\alpha = 1, \ldots, t$, such that

(5.3) $$\begin{cases} t_{iK} \underset{\alpha}{\lambda}^i \underset{\beta}{\lambda}^K = 0, & \alpha \neq \beta, \ \alpha, \beta = 1, \cdots, t \\ t_{iK} \underset{\alpha}{\lambda}^i \underset{\alpha}{\lambda}^K \neq 0, & \alpha \text{ not summed.} \end{cases}$$

Let

$$\underset{\alpha}{\mu}_i = t_{iK} \underset{\alpha}{\lambda}^K .$$

According to the lemma the t vectors $\underset{\alpha}{\mu}_i$ and the t vectors $\underset{\alpha}{\lambda}^i$ are both linearly independent. The equations

$$\underset{\alpha}{\mu}_i \zeta^i = 0 \quad (t_{iK} \underset{\alpha}{\lambda}^K \zeta^i = 0) \quad \alpha = 1, \ldots, t,$$

define an n - t dimensional vector space $\{\zeta^i\}^{(n-t)}$ which has no vector in common with the vector space $\{\underset{1}{\lambda}^i, \cdots, \underset{t}{\lambda}^i\}^{(t)}$. Hence each vector γ at P can be represented by

(5.4) $$\gamma^i = \zeta^i + A_\alpha \lambda^i, \quad \zeta^i \varepsilon \{\zeta^i\}^{(n-t)}.$$

Now obviously

$$(5.5) \qquad \{\xi^i\}^{(n-t)} \supset \{\theta^i\}^{(n-r)}$$

because $\mu_{\alpha i}\theta^i = 0$ for $\theta^i \in \{\theta^i\}^{(n-r)}$. We now show that there exists a vector $\mu^i \in \{\xi^i\}^{(n-t)}$ such that $t_{iK}\mu^i\mu^K \neq 0$. If this were not the case,

$$t_{iK}(\mu^i + \nu^i)(\mu^K + \nu^K) = 0$$

for all $\mu^i, \nu^i \in \{\xi^i\}^{(n-t)}$, which would yield

$$t_{iK}\mu^i\nu^K = 0.$$

But this last equation can be written as

$$t_{iK}\,\mu^i(\nu^K + A_\alpha \lambda_\alpha^K) = 0$$

and hence, since $\nu^K + A_\alpha \lambda_\alpha^K$ is a general vector of γ^i we have

$$t_{iK}\mu^i = 0, \quad \mu^i \in \{\xi^i\}^{(n-t)},$$

and

$$(5.6) \qquad \{\xi^i\}^{(n-t)} \subset \{\theta^i\}^{(n-r)}.$$

(5.5) and (5.6) give

$$\{\xi^i\}^{(n-t)} \equiv \{\theta^i\}^{(n-r)}$$

and hence $n - t = n - r$, $t = r$, in contradiction to the assumption $t < r$. Thus there exists a vector $\lambda_{t+1}^i \in \{\xi^i\}^{(n-t)}$ such that

$$t_{iK}\,\lambda_{t+1}^i\,\lambda_{t+1}^K \neq 0,$$

and we have proved (5.3) for λ_α^i, $\alpha = 1, \dots, t+1$.*

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* This method of obtaining a canonical representation is a constructive method and not a mere existence proof. For this step we show here how to construct a vector $\rho^i \in \{\xi^i\}^{(n-t)}$ such that $t_{iK}\rho^i\rho^K \neq 0$. Take a vector μ_i such that

$\mu_i \in \{\xi^i\}^{(n-t)}$ but $\mu_i \notin \{\theta^i\}^{(n-r)}$. Then $t_{iK}\mu^i \neq 0$, and there exists a vector σ^K such that $t_{iK}\mu^i\sigma^K \neq 0$. Split σ^K into the sum of a vector of $\{\xi^i\}^{(n-t)}$ and a vector of $\{\underset{1}{\lambda}^i, \ldots, \underset{t}{\lambda}^i\}^{(t)}$: $\sigma^K = \nu^K + \underset{\alpha}{A}\underset{\alpha}{\lambda}^K$, $\alpha = 1, \cdots, t$. Then $t_{iK}\mu^i\nu^K \neq 0$. Hence either $t_{iK}\mu^i\mu^K$ or $t_{iK}\nu^i\nu^K$ is different from zero or if both are zero, $t_{iK}(\mu^i+\nu^i)(\mu^K+\nu^K) \neq 0$. We can then take ρ^i equal to μ^i, ν^i or $\mu^i+\nu^i$ according to which of the above possibilities are true.

We can proceed in this manner until we get an r-Bein $\underset{\alpha}{\lambda}^i$ for which (5.3) holds. Then if $\{\xi^i\}^{(n-r)}$ is defined by $\underset{\alpha}{\mu}_i\xi^i = 0$, $\alpha = 1, \ldots, r$, we have

$$\{\xi^i\}^{(n-r)} \equiv \{\theta^i\}^{(n-r)}$$

Of course then $t_{iK}\mu^i\mu^K = 0$ for any $\mu^i \in \{\xi^i\}^{(n-r)}$. Again the two vector spaces $\{\xi^i\}^{(n-r)}$ and $\{\underset{1}{\lambda}^i, \ldots, \underset{r}{\lambda}^i\}^{(r)}$ have no vector in common other than the zero vector, and hence we can complete the r-Bein $\underset{\alpha}{\lambda}^i$, $\alpha = 1, \ldots, r$, by adding to it $n - r$ independent vectors of $\{\xi^i\}^{(n-r)} \equiv \{\theta^i\}^{(n-r)}$ to form an n-Bein. For this n-Bein $\underset{\alpha}{\lambda}^i$, $\alpha = 1, \ldots, n$, (5.3) holds for $\alpha, \beta = 1, \ldots, r$, but $t_{iK}\underset{\alpha}{\lambda}^i\underset{\beta}{\lambda}^K = 0$ if α or $\beta > r$.

Let $\underset{\alpha}{\nu}_i$ be the adjoint Bein to $\underset{\alpha}{\lambda}^i$. Then from

(5.7) $$t_{iK} = (t_{pq}\underset{\alpha}{\lambda}^p\underset{\beta}{\lambda}^q)\underset{\alpha}{\nu}_i\underset{\beta}{\nu}_K$$

we get

$$\begin{cases} t_{iK} = t_\alpha \underset{\alpha}{\nu}_i \underset{\alpha}{\nu}_K, & \alpha = 1, \cdots, r, \quad \text{(not summed on } \alpha) \\ t_\alpha = t_{iK}\underset{\alpha}{\lambda}^i\underset{\alpha}{\lambda}^K, & \alpha = 1, \cdots, r, \quad (\text{" " " "}) \end{cases}$$

By multiplying the vectors $\underset{\alpha}{\nu}_i$ by scalars we obtain

$$(5.8) \qquad t_{iK} = \pm \underset{\alpha}{\gamma_i} \underset{\alpha}{\gamma_K} \qquad \alpha = 1, \dots, r \text{ (not summed on } \alpha\text{)},$$

the so-called canonical form for symmetric covariant tensors of the second order.

If t_{iK} is of class C^ρ, $\rho \leq m-1$ in some neighborhood $\mathcal{U}(P)$ we can find vectors $\underset{\alpha}{\gamma_i}$ (in (5.8)) of class C^ρ in $\widetilde{\mathcal{U}} \subset \mathcal{U}(P)$. We shall first assume that the $\underset{\alpha}{\lambda^i}$, $\alpha = 1, \dots, t$, are of class C^ρ in a $\mathcal{U}' \subset \mathcal{U}(P)$ and then prove that in a $\mathcal{U}'' \subset \mathcal{U}'$ the $\underset{t+1}{\lambda^i}$ is also of class C^ρ. The $\underset{\alpha}{\mu_i}$ in

$$(5.9) \qquad \underset{\alpha}{\mu_i} \zeta^i = 0, \quad \alpha = 1, \dots, t.$$

are class C^ρ in $\mathcal{U}'$. Let $|\underset{\alpha}{\mu_i}| \neq 0$, $\alpha = 1, \dots, t$, $i = 1, \dots, t$, in a $\mathcal{U}'' \subset \mathcal{U}'$. We can then solve (5.9), expressing $\zeta^1, \dots, \zeta^t$ linearly in terms of $\zeta^{t+1}, \dots, \zeta^n$ with coefficients of class C^ρ in $\mathcal{U}''$. But at P

$$t_{iK} \underset{t+1}{\lambda^i} \underset{t+1}{\lambda^K} \neq 0$$

and in $\mathcal{U}''$

$$(5.10) \qquad \underset{t+1}{\lambda^i} = a^i_j \underset{t+1}{\lambda^j}, \qquad i = 1, \dots, t, \; j = t+1, \dots, n,$$

where the coefficients a^i_j are of class C^ρ in $\mathcal{U}''$. Then, for a $\mathcal{U}''' \subset \mathcal{U}''$, the vector field given in $\mathcal{U}'''$ by (5.10) for constant $\underset{t+1}{\lambda^{t+1}}, \dots, \underset{t+1}{\lambda^n}$ will have the property

$$t_{iK} \underset{t+1}{\lambda^i} \underset{t+1}{\lambda^K} \neq 0,$$

together with $\underset{\alpha}{\mu_i} \underset{t+1}{\lambda^i} = 0$, $\alpha = 1, \dots, t$.

Hence there will exist a $\widetilde{\mathfrak{V}}(P)$ such that in it all λ_{α}^{i}, $\alpha = 1, \ldots, r$, are of class C^{ρ}. In the same way the $n - r$ vectors λ_{α}^{i}, $\alpha = r + 1, \ldots, n$, can be chosen such that in a $\widetilde{\widetilde{\mathfrak{V}}} \subset \widetilde{\mathfrak{V}}$ they are of class C^{ρ}. Thus the n-B ein λ_{α}^{i}, $\alpha = 1, \ldots, n$, and accordingly the adjoint n-Bein $\chi_{\alpha i}$, will be of class C^{ρ} in $\widetilde{\widetilde{\mathfrak{V}}}$.

<u>Conversely, if</u>

$$t_{iK} = \pm \mu_{\alpha i} \mu_{\alpha K}, \qquad \alpha = 1, \ldots, r,$$

<u>where</u> $\mu_{\alpha i}$ <u>is an r-Bein, then</u> t_{iK} <u>is of rank</u> r.

Proof. The system of equations $t_{iK}\rho^{K} = 0$ is equivalent to the system of equations $\mu_{\alpha K}\rho^{K} = 0$, $\alpha = 1, \ldots, r$. Since the latter has $n - r$ independent solutions, the former must also have $n - r$ independent solutions. Thus t_{iK} is of rank r.

If we have any two canonical representations of t_{iK}, $t_{iK} = \pm \mu_{\alpha i}\mu_{\alpha K}$ and $t_{iK} = \pm \mu'_{\alpha i}\mu'_{\alpha K}$, the equations

$$\mu_{\alpha K}\rho^{K} = 0 \quad \text{and}$$
$$\mu'_{\alpha K}\rho^{K} = 0$$

define the same vector space $\{\rho^{K}\}$. This follows from the above proof, since, according to it, $\{\rho^{K}\}$ is also given by $t_{iK}\rho^{K} = 0$. Hence we have

$$\{\mu_{1 i}, \ldots, \mu_{n i}\} \equiv \{\mu'_{1 i}, \ldots, \mu'_{n i}\}$$

since their normal spaces are equal. We than have

$$\mu'_{\alpha i} = A_{\alpha\beta}\,\mu_{\beta i}, \quad |A_{\alpha\beta}| \neq 0. \tag{5.11}$$

6. <u>Definite, semi-definite and indefinite symmetric tensors of the second order.</u>

If $t_{iK}\rho^{i}\rho^{K} \geq 0$, for arbitrary ρ^{i} then the canonical representation of t_{iK} takes the form

$$(6.1) \qquad t_{iK} = \underset{\alpha}{\gamma}_i \underset{\alpha}{\gamma}_K , \qquad \alpha = 1, \ldots , r.$$

Conversely, if t_{iK} has the representation (6.1) then

$$t_{iK}\rho^i\rho^K = \sum (\underset{\alpha}{\gamma}_i \rho^i)^2 \geq 0,$$

and indeed the equality holds only if $\underset{\alpha}{\gamma}_i \rho^i = 0$, $\alpha = 1, \ldots , r$.

A symmetric tensor t_{iK} for which $t_{iK}\rho^i\rho^K > 0$ (< 0) if $\rho^i \neq 0$ is called positive (negative) definite. A positive (negative) definite tensor is always of rank n.

A symmetric tensor t_{iK} for which $t_{iK}\rho^i\rho^K \geqq 0$ ($\leqq 0$), where the equality may hold for a $\rho^i \neq 0$, is called positive (negative) semi-definite. A semi-definite tensor will be of rank $r < n$.

Any other symmetric tensor t_{iK} will be called indefinite. In a canonical representation of an indefinite tensor, both plus and minus signs will occur as coefficients.

CHAPTER II

Tensor Analysis

7. Covariant Differentials of Vectors.

In the discussion of tensor algebra we were chiefly concerned with tensors which were defined at a single point, P. We now shift our point of view and consider quantities which are defined as functions of the coordinates, x, of our space. The set of vectors $\lambda^i(x)$ which are defined for the various points of our space is called a contravariant vector field. Similarly we may define covariant vector fields and tensor fields. For convenience these will simply be called vectors and tensors.

For the present we consider a curve $x^i(s)$, $[0 \leq s \leq 1]$ where $x^i(s)$ are continuous functions having continuous first derivatives; i.e. they are of class C^1. Along this curve we have defined a one-parameter family of vectors, $\lambda^i(s)$, where $\lambda^i(s)$ are also functions of class C^1. Now the law of transformation of this vector field is

$$\bar{\lambda}^i = \frac{\partial \bar{x}^i}{\partial x^k} \lambda^k \tag{7.1}$$

where the λ's and $\bar{\lambda}$'s are understood to be functions of s. In order that $\bar{\lambda}^i$ may be of class C^1 as a consequence of our assumption that λ^i is of class C^1, it is necessary to assume the existence of $\dfrac{\partial^2 \bar{x}^i}{\partial x^k \partial x^q}$. That means that in defining our m-systems we must take $m \geq 2$. It is clear that this assumption is all that is required to enable us to speak of vectors and tensors of class C^1.

The differential of λ^i in the x coordinate system may be defined by

$$d\lambda^i = \frac{\partial \lambda^i}{\partial s}\, ds \tag{7.2}$$

where dx^k are the components of a displacement along the curve. From (7.1) we see that its components in the x system are given by:

$$d\bar{\lambda}^i = \frac{\partial \bar{x}^i}{\partial x^k}\, dx^k + \frac{\partial^2 \bar{x}^i}{\partial x^k \partial x^l}\, dx^k dx^l . \tag{7.3}$$

Because of the final term, this is not the law of transformation of a vector under general transformations of coordinates. Only for linear transformations does this term vanish, giving the tensor relation.

It is highly desirable, therefore, that a new type of differential be defined such that the differential of a vector is again a vector. In order to accomplish this we must add some new element of structure to our space. This may be done in many different ways. We choose the following method because of its simplicity, and shall give a more logical procedure in a few paragraphs. To start with we introduce into our space a field of independent covariant vectors, $\underset{\alpha}{\mu}_i(x)$, ($i, \alpha = 1, \dots, n$) whose components are of class C'. For example we can choose $\underset{\alpha}{\mu}_i = \delta_{i\alpha}$ in the x coordinate system. This is a constant Bein in the given coordinate system, but is not necessarily so in any other systems. With respect to $\underset{\alpha}{\mu}_i$ we define the adjoint Bein (see), $\underset{\alpha}{\xi}^i(x)$. We observe that this definition implies that the $\underset{\alpha}{\xi}^i$ are also functions of class C^1.

In terms of this Bein we can represent our given vector field $\lambda^i(s)$ by the equation

$$\lambda^i(s) = \underset{\alpha}{\lambda}(s)\, \underset{\alpha}{\xi}^i(s), \tag{7.4}$$

where $\underset{\alpha}{\lambda}(s)$ are scalars of class C^1 defined by

(7.5) $$\underset{\alpha}{\lambda}(s) = \lambda^i(s)\,\underset{\alpha}{\mu}_i(s)\,.$$

From (7.5) we may define another set of scalars, namely $d\underset{\alpha}{\lambda}(s)$ as the ordinary differentials of the $\underset{\alpha}{\lambda}(s)$:

(7.6) $$d\underset{\alpha}{\lambda} = d\lambda^i\,\underset{\alpha}{\mu}_i + \lambda^i\,\frac{\partial \underset{\alpha}{\mu}_i}{\partial x^j}\,dx^j.$$

where the λ's and μ's are understood to be functions of s. This enables us to define our new type of differential.

<u>Definition: The covariant differential</u> $\delta\lambda^i$ <u>is defined to be</u>

(7.7) $$\delta\lambda^i \equiv (d\underset{\alpha}{\lambda})\,\underset{\alpha}{\xi}^i$$

Substituting into (7.7) from (7.6) we have that

(7.8) $$\delta\lambda^i \equiv d\lambda^i + \underset{\alpha}{\xi}^i\,\frac{\partial \underset{\alpha}{\mu}_k}{\partial x^j}\,\lambda^k\,dx^j.$$

From the fact that $\underset{\alpha}{\xi}^i\,\underset{\alpha}{\mu}_k = \delta^i_k$, we see that

(7.9) $$\underset{\alpha}{\xi}^i\,\frac{\partial \underset{\alpha}{\mu}_k}{\partial x^j} = -\frac{\partial \underset{\alpha}{\xi}^i}{\partial x^j}\,\underset{\alpha}{\mu}_k.$$

We define new quantities Γ^i_{kj} by the relation

(7.10) $$\Gamma^i_{kj} = \underset{\alpha}{\xi}^i\,\frac{\partial \underset{\alpha}{\mu}_k}{\partial x^j} = -\frac{\partial \underset{\alpha}{\xi}^i}{\partial x^j}\,\underset{\alpha}{\mu}_k.$$

And hence we write (7.8) in the final form,

(7.11) $$\delta\lambda^i \equiv d\lambda^i + \Gamma^i_{kj}\,\lambda^k\,dx^j.$$

<u>Alternative Definition of</u> $\delta\lambda^i$. The Bein which was

originally introduced appears in (7.11) only in the expression, Γ^i_{kj}. Hence it is a natural generalization of our definition to define the covariant differential by (7.11) where Γ^i_{kj} are now certain functions of the coordinates whose law of transformation is yet undetermined. The Γ^i_{jk} are subjected to the condition that $\mathcal{D}\lambda^i$ must be a contravariant vector for λ^i of class C^1 but are otherwise arbitrary. This condition has for its analytical expression:

$$(7.12)\quad d\bar\lambda^i + \bar\Gamma^i_{kj}\,\bar\lambda^k d\bar x^j = \frac{\partial \bar x^i}{\partial x^r}\left(d\lambda^r + \Gamma^r_{pq}\,\lambda^p dx^q\right).$$

Substituting for $d\bar\lambda^i$ from (7.3), for λ^k from (7.1); and for $d\bar x^j$ from $d\bar x^j = \dfrac{\partial \bar x^j}{\partial x^q}\,dx^q$, we have that:

$$\frac{\partial \bar x^i}{\partial x^r}\,d\lambda^r + \frac{\partial^2 \bar x^i}{\partial x^p \partial x^q}\,\lambda^p \lambda^q + \bar\Gamma^i_{kj}\,\frac{\partial \bar x^k}{\partial x^p}\,\lambda^p\,\frac{\partial \bar x^j}{\partial x^q}\,dx^q$$

$$= \frac{\partial \bar x^i}{\partial x^r}\left(d\lambda^r + \Gamma^r_{pq}\,\lambda^p dx^q\right), \text{ or}$$

$$(7.13)\quad \left\{\frac{\partial^2 \bar x^i}{\partial x^p \partial x^q} + \bar\Gamma^i_{kj}\,\frac{\partial \bar x^k}{\partial x^p}\,\frac{\partial \bar x^j}{\partial x^q} - \Gamma^r_{pq}\,\frac{\partial \bar x^i}{\partial x^r}\right\}\lambda^p dx^q = 0.$$

And since λ^p and dx^q are quite arbitrary, the law of transformation of the Γ's is:

$$(7.14)\quad \frac{\partial^2 \bar x^i}{\partial x^p \partial x^q} + \bar\Gamma^i_{kj}(\bar x)\,\frac{\partial \bar x^k}{\partial x^p}\,\frac{\partial \bar x^j}{\partial x^q} = \Gamma^r_{pq}(x)\,\frac{\partial \bar x^i}{\partial x^r}.$$

We may take the Γ's to be arbitrary functions in one coordinate system, and define their components in any other system by (7.14). Then (7.11) defines a "covariant differential" which is a contravariant vector.

Geometric Objects. The second method of defining Γ leads us to the notion of a geometric object at a point P. If 1) a set of numbers is defined in one coordinate system, 2) corresponding sets of numbers are defined in any other system by some definite law of transformation, and 3) this law is transitive [i.e. the transformation $x \rightarrow y \rightarrow z$ gives the same result as $x \rightarrow z$], the totality of the numbers so defined is called a geometric object at P. Vectors and tensors are obvious examples of such objects, and it is easy to see that the Γ's given by our second definition are also objects. In this case (7.14) is the law of transformation and it is evidently a transitive one. Hence we may speak of Γ-objects.

A shorter definition of a geometric object is the following. If in any coordinate system a certain set of numbers is given with a definite law of transformation between any two coordinate systems and if their transformation law is consistent, then the set of numbers define a geometric object.

The question naturally arises as to the relation of the two definitions of the covariant differential. From (7.10) the law of transformation can be computed for the Γ^i_{jk} defined by means of the Bein. This law actually turns out to be (7.14), and consequently the Γ^i_{jk} so defined are the components of a Γ-object. The second definition of the differential therefore includes the first. It is possible to show by a method to be given later that the second definition is in fact more general than the first. Hereafter the covariant differential will be taken as defined by the second method.

The Γ-objects have a number of properties in addition to their law of transformation which distinguish them from tensors. In particular it has no meaning to speak of a zero Γ-object. For even if all the components of a Γ-object are zero in one coordinate system, it does not follow that they are zero in all coordinate systems. Furthermore suppose that we have found two objects, Γ^i_{jk} and $\tilde{\Gamma}^i_{jk}$. Subtracting (7.11) from

$$\delta\tilde{\lambda}^i \equiv d\lambda^i + \tilde{\Gamma}^i_{jk}\lambda^j dx^k, \tag{7.15}$$

we see that

$$(\tilde{\Gamma}^i_{jk} - \Gamma^i_{jk})\lambda^j dx^k = \delta\tilde{\lambda}^i - \delta\lambda^i, \text{ a vector.} \tag{7.16}$$

<u>Hence it follows that</u> $(\tilde{\Gamma}^i_{jk} - \Gamma^i_{jk})$ <u>is a tensor.</u> <u>This result shows that the most general</u> Γ<u>-object is obtained from a particular</u> Γ^i_{jk} <u>by adding a completely arbitrary</u> T^i_{jk}. From this fact and the existence of a particular Γ (derived from the Bein) it follows that there can be no contradiction in the transformations laws defining Γ objects. It also follows from this that if a Γ-object is symmetric in its lower indices in one coordinate system, it is symmetric in all systems. Thus we may speak of a "symmetric Γ-object".

Further characteristics of the algebra of Γ-objects are 1) addition of Γ^i_{jk} and $\tilde{\Gamma}^i_{jk}$ does not give a Γ-object; 2) $\sigma\Gamma^i_{jk}$ is not a Γ-object, where σ is a scalar; 3) $a\Gamma^i_{jk} + b\tilde{\Gamma}^i_{jk}$ is a Γ-object if $a + b = 1$; 4) if Γ^i_{pq} is a Γ-object, so also is Γ^i_{qp}. We can write, moreover,

$$\Gamma^i_{pq} = \frac{\Gamma^i_{pq} + \Gamma^i_{qp}}{2} + \frac{\Gamma^i_{pq} - \Gamma^i_{qp}}{2}. \tag{7.17}$$

The first term on the right is a symmetric Γ-object and the second term is a tensor. Hence any Γ-object is the sum of a symmetric Γ-object and a tensor of the third order.

Theorem: Corresponding to any point P there exists a coordinate system in which the components of a given Γ-object are all zero at P.

Proof: Let P have coordinates x_o^i in a given x coordinate system. The required transformation is asserted to be

$$(7.18) \qquad \bar{x}^i = (x^i - x_o^i) + \tfrac{1}{2}\Gamma^i_{pq}(x)\,(x^p - x_o^p)(x^q - x_o^q).$$

First we note that the Jacobian $J \equiv \left|\frac{\partial \bar{x}}{\partial x}\right| = 1$ at P. Consequently there exists a neighborhood of P within which (7.18) defines a coordinate transformation. Furthermore it follows that $\bar{\Gamma}^i_{pq}$ are zero at P. For substitute (7.18) into (7.14) and evaluate at P noting that at P

$$\left.\frac{\partial^2 \bar{x}^i}{\partial x^p \partial x^q}\right)_P = \Gamma^i_{pq} \quad ; \quad \left.\frac{\partial \bar{x}^i}{\partial x^r}\right)_P = \delta^i_r$$

The result is that $\left.\bar{\Gamma}^i_{pq}\right)_P = 0.$

§ 8. Covariant Differentials of Tensors.

Definition. The covariant differential, $\mathcal{D}T$, of a tensor T is defined by an operator $\mathcal{D}$ which satisfies the following requirements:

I. $\mathcal{D}T$ is a tensor of the same kind as T.

II. $\mathcal{D}\sigma = d\sigma$, if σ is any scalar.

III. $\mathcal{D}\lambda^i = d\lambda^i + \Gamma^i_{jk}\lambda^j dx^k$, where λ^i is any contravariant vector.

IV. $\mathcal{D}(T + V) = \mathcal{D}(T) + \mathcal{D}(V)$, where T and V are any two tensors of the same kind.

V. $\vartheta(TV) = (\vartheta T)V + T(\vartheta V)$, where T and V are any two tensors.

VI. $\sum_{i=k}(\vartheta T^i_k) = \vartheta(\sum_{i=k} T^i_k)$, where i and k are any two indices, one contravariant and the other comariant. This may be read to mean that the operator ϑ and the operation of contraction are commutative.

On the basis of these properties, we shall now find a unique expression for ϑ T. First we consider a given covariant vector μ_i and observe that $\lambda^i\mu_i$ is a scalar for every choice of λ^i. Hence as a result of II

(8.1) $\vartheta(\lambda^i\mu_i) = d(\lambda^i\mu_i)$

Because of V and VI this may be written as:

(8.2) $(\vartheta\lambda^i)\mu_i + \lambda^i(\vartheta\mu_i) = (d\lambda^i)\mu_i + \lambda^i(d\mu_i)$.

From this it follows that $\lambda^i\vartheta\mu_i$ is an invariant for all contravariant vectors λ^i and hence $\vartheta\mu_i$ (if it exists) is a vector. Substituting for $\vartheta\lambda^i$ from III we have:

(8.3) $(d\lambda^i)\mu_i + \Gamma^i_{jk}\lambda^j\mu_i\, dx^k + \lambda^i(\vartheta\mu_i) = (d\lambda^i)\mu_i + \lambda^i(d\mu_i$

or

(8.4) $\lambda^i(\vartheta\mu_i - d\mu_i + \Gamma^l_{ik}\mu_l\, dx^k) = 0$.

Because this holds for any vector λ^i, it follows that

(8.5) $\vartheta\mu_i = d\mu_i - \Gamma^l_{ik}\mu_l\, dx^k$.

Going back with the expression for $\vartheta\mu_i$ given by (8.5) we obtain (8.4), (8.3), and (8.2) thus showing the vector quality of $\vartheta\mu_i$ as defined by (8.3).

It is immediately evident that ϑ as applied to μ_i satisfies conditions I to V, and that VI has no meaning in this case.

To extend this result to a tensor of any kind, let us represent an arbitrary tensor by $T^{i_1 \cdots}_{k_1 \cdots}$ where i_1 is a typical contravariant index and k_1 is a typical covariant index. By making use of an n-Bein λ^{i}_{α} and an n-Bein $\mu_{\beta i}$, we can express our tensor in the form:

$$T^{i_1 \cdots}_{k_1 \cdots} = T_{\alpha_1 \cdots \beta_1 \cdots} \lambda^{i_1}_{\alpha_1} \cdots\cdots \mu_{\beta_1 k_1} \cdots\cdots \tag{8.6}$$

where $T_{\alpha_1 \cdots \beta_1 \cdots}$ are scalars. Now apply ϑ to (8.6) and use assumptions II, III, IV, and V and equation (8.5). The result is:

$$\begin{aligned} \vartheta T^{i_1 \cdots}_{k_1 \cdots} &= d_x T^{i_1 \cdots}_{k_1 \cdots} + T^{\ell_1 \cdots}_{k_1 \cdots} \Gamma^{i_1}_{\ell_1 j} d_x{}^j + \cdots\cdots \\ &\quad - T^{i_1 \cdots}_{\ell_1 \cdots} \Gamma^{\ell_1}_{k_1 j} d_x{}^j \cdots\cdots \end{aligned} \tag{8.7}$$

In the right hand side of (8.7) there is a term like the second one corresponding to each contravariant index in $T^{i_1 \cdots}_{k_1 \cdots}$, and a term with a minus sign like the last term written above for each covariant index in $T^{i_1 \cdots}_{k_1 \cdots}$.

Careful consideration shows that all the conditions I-VI are actually satisfied by the definition (8.7). First we observe that I is satisfied, i.e. ϑ T is a tensor. Then at any point P we choose a coordinate system in which the Γ^{i}_{jk} at that point are zero. In this coordinate system we see that ϑ T = dT at P. Hence II-VI are satisfied at P because of the rules governing ordinary differentials. But since ϑ T are tensors, these relations II-VI must hold at P in any coordinate system, so that our definition fulfills all our requirements.

§ 9. Covariant Derivative.

Suppose now that T^i_k are defined throughout a neighborhood $\mathcal{U}(P)$ of our space as functions of x. Then

$$d\, T^i_k = \frac{\partial T^i_k}{\partial x^q}\, dx^q. \tag{9.1}$$

Combining (9.1) with (8.7) we have that

$$\delta T^i_k = A^i_{kq}\, dx^2 \text{ where} \tag{9.2}$$

$$A^i_{kq} = \frac{\partial T^i_k}{\partial x^q} + (\Gamma^i_{pq} T^p_k - \Gamma^p_{kq} T^i_p). \tag{9.3}$$

From (9.2) we see that A^i_{kq} are the components of a tensor. We shall write this as $T^i_{k;q}$ and call it the covariant derivative of T^i_k. Covariant differentiation will regularly be denoted by placing a semicolon before the index of differentiation.

CHAPTER III

Riemann Spaces.

§ 10. Introduction.

We have previously introduced two kinds of structure by means of which it was possible to define covariant differentiation. The first of these was the construction of an n-Bein; and the resulting space has been called a "space of absolute parallelism". This space was of importance in an attempt to establish a unified field theory. The second kind of structure was given by a Γ-object. A space in which a Γ-object is defined is called a space with an "affine connection".

The most important space with a structure is the Riemann space. Here there is defined in each point a symmetric positive definite tensor g_{ik} of class C^{ρ} where $\rho \geq 1$. Some of the following discussion will hold even if g_{ik} is not positive definite provided the determinant $g_{ik} \neq 0$. We shall not consider this case.

In order to define covariant differentiation in this space we first must define a Γ-object if such an object exists. We require this object to satisfy:

1) $g_{ik;r} = 0$

2) $\varphi_{;i;j} = \varphi_{;j;i}$ for every scalar φ.

This second assumption is equivalent to the requirement that the Γ-object be symmetric in its lower indices. We now consider a definite coordinate system and write the first assumption in the form:

(10.1) $$\frac{\partial g_{ik}}{\partial x^r} = \Gamma^p_{ir}\, g_{pk} + \Gamma^p_{kr}\, g_{pi}.$$

We may also write:

(10.2) $$\frac{\partial g_{kr}}{\partial x^i} = \Gamma^p_{ki}\, g_{pr} + \Gamma^p_{ri}\, g_{pk}, \text{ and}$$

(10.3) $$\frac{\partial g_{ri}}{\partial x^k} = \Gamma^p_{rk}\, g_{pi} + \Gamma^p_{ik}\, g_{pr}.$$

Taking (10.1) + (10.2) - (10.3) and using the symmetry of the Γ's, we have

(10.4) $$\Gamma^p_{ir}\, g_{pk} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^r} + \frac{\partial g_{kr}}{\partial x^i} - \frac{\partial g_{ri}}{\partial x^k} \right).$$

The right hand sides are often denoted by $[ri,k]$, which are called the Christoffel symbols of the first kind. Finally we can solve (10.4) for Γ^p_{rk} and write:

(10.5) $$\Gamma^q_{ir} = \frac{1}{2}\, g^{qk}\, [ri,k] = \left\{ {q \atop i\ r} \right\}$$

where $\left\{ {q \atop i\ r} \right\}$ are the Christoffel symbols of the second kind. We observe that the Γ^q_{ir} given by (10.5) actually satisfy (10.1), and that these quantities are uniquely determined by (10.1) for the given coordinate system. Let their components in any other coordinate system be given by (7.14), thus defining a Γ-object. Since (10.1), respectively $g_{ik;r} = 0$, are invariant equations, they will be satisfied by the Γ's so defined in any coordinate system. Since the solution of (10.1) is unique in any coordinate system, our definition of the Γ-object does not depend on the original system chosen for our definition.

<u>Raising and Lowering Indices.</u> Because of the relations

$$\lambda_i = g_{ik} \lambda^k ; \quad \lambda^k = g^{ki} \lambda_i, \tag{10.6}$$

there is a one-to-one correspondence between covariant and contravariant vectors in a Riemann space. Hence we may speak of a "vector" and of its "covariant" and "contravariant" components. The process illustrated in the first of (10.6) is called "lowering an index" (the index i), and the second of (10.6) illustrates "raising an index". This process may be applied to all orders of tensors; for example in

$$T^i_{jk} = g_{j\ell} T^\ell_k \tag{10.7}$$

the index j has been lowered. Care should be taken to specify where the lowered index is to appear. Thus we might have written (10.7) as

$$T_{kj} = g_{j\ell} T^\ell_k . \tag{10.8}$$

Either of these is correct, but not both at once. A more careful notation would write (10.7) as:

$$T_{jk} = g_{j\ell} T^{\ell}_{\cdot k} \tag{10.9}$$

a dot indicating the position to which the index is to be lowered.

§ 11. <u>Metric Properties.</u>

<u>Length.</u> The length of a vector is defined to be

$L = +\sqrt{g_{ik}\lambda^i\lambda^k} \equiv \sqrt{(\lambda\lambda)}$. It is always real since g_{ir} is positive definite.

Angle. The angle between two vectors λ and μ at the same point is defined to be:

$$(11.1) \quad \measuredangle \lambda\mu = \cos^{-1}\left|\frac{g_{ik}\lambda^i\mu^k}{\sqrt{\lambda\lambda}\sqrt{\mu\mu}}\right| \equiv \cos^{-1}\left|\frac{(\lambda\mu)}{\sqrt{\lambda\lambda}\sqrt{\mu\mu}}\right|$$

In order for this definition to have meaning we must show that $\left|\frac{(\lambda\mu)}{\sqrt{\lambda\lambda}\sqrt{\mu\mu}}\right| \leq 1$. We suppose λ and μ to be independent, hence for all α and β not both zero

$$\alpha\lambda^i + \beta\mu^i \neq 0 \qquad \text{so}$$

$$g_{ik}(\alpha\lambda^i + \beta\mu^i)(\alpha\lambda^k + \beta\mu^k) > 0 \qquad \text{, or}$$

$$\alpha^2(\lambda\lambda) + 2\alpha\beta(\lambda\mu) + \beta^2(\mu\mu) > 0.$$

In order that this be the case, the discriminant of the above form must be negative. This gives the desired result.

From this definition it follows that two vectors are normal if $g_{ik}\lambda^i\mu^k = 0$. This agrees with our previous definition which required that $\lambda_i\mu^i = 0$.

Normalized Beins. Consider now an r-dimensional vector space, V_r. We can always span this by a "normalized Bein", i.e. a set of r unit, mutually orthogonal vectors. First choose one vector, say λ^i. Divide it by its length and write

$$\underset{1}{\lambda}{}^i = \frac{\lambda^i}{\sqrt{g_{ab}\lambda^a\lambda^b}}$$

This is the first vector of our Bein. If $r = 1$, we have finished; otherwise there exists a second vector μ^i independent of $\underset{1}{\lambda}{}^i$.

Consider

$$g_{ik}(\sigma \underset{1}{\lambda}^i + \mu^i)\underset{1}{\lambda}^k = 0$$

where we are looking for σ such that $(\sigma \underset{1}{\lambda}^i + \mu^i)$ is normal to $\underset{1}{\lambda}^i$. The result obviously is that

$$\sigma = -g_{ik}\mu^i \underset{1}{\lambda}^k.$$

This defines $\nu^i = (\sigma \underset{1}{\lambda}^i + \mu^i)$ which is not a zero vector. Dividing it by its length we have $\underset{2}{\lambda}^i$. If r = 2 we have finished; otherwise there exists a third vector ν^i in V_r independent of $\underset{1}{\lambda}^i, \underset{2}{\lambda}^i$

Next consider

$$g_{ik}(\sigma \underset{1}{\lambda}^i + \rho \underset{2}{\lambda}^i + \nu^i)\underset{1}{\lambda}^k = 0 \rightarrow \sigma + g_{ik}\nu^i \underset{1}{\lambda}^k = 0$$
$$g_{ik}(\sigma \underset{1}{\lambda}^i + \rho \underset{2}{\lambda}^i + \nu^i)\underset{2}{\lambda}^k = 0 \rightarrow \rho + g_{ik}\nu^i \underset{2}{\lambda}^k = 0$$

and proceed with $\sigma \underset{1}{\lambda}^i + \rho \underset{2}{\lambda}^i + \nu^i$ as before to define $\underset{3}{\lambda}^i$. Eventually exactly r vectors will be determined, and these form the required r-Bein. (Bein: because any set of normalized vectors are independent vectors).

We can do this same thing in the space V_{n-r} which is normal to V_r. The result is that the combined normalized Beins form a normalized n-Bein which spans the entire space. We can represent the metric tensor by means of this Bein thus:

(11.2) $$g_{ik} = g_{\alpha\beta} \underset{\alpha}{\lambda}_i \underset{\beta}{\lambda}_k$$

where

(11.3) $$g_{\alpha\beta} = g_{ik} \underset{\alpha}{\lambda}^i \underset{\beta}{\lambda}^k$$

since the adjoint of a normalized Bein is the Bein itself. But from (11.3) we see that $g_{\alpha\beta} = \delta_{\alpha\beta}$, and so

(11.4) $$g_{ik} = \underset{\alpha}{\lambda}_i \underset{\alpha}{\lambda}_k$$

Thus we see that the problem of obtaining a normalized n-Bein is equivalent to that of obtaining the canonical representation of g_{ik}.

Projection Tensors and Projections: We may write (11.4) in the form:

(11.5) $$g_{ik} = \sum_{\alpha=1}^{r} \underset{\alpha}{\lambda}_i \underset{\alpha}{\lambda}_k + \sum_{\alpha=r+1}^{n} \underset{\alpha}{\lambda}_i \underset{\alpha}{\lambda}_k$$

where the first summation is over the Bein which spans V_r and the second summation refers to V_{n-r}. The projection tensor with respace to V_r is defined to be $p_{ik} = \sum_{\alpha=1}^{r} \underset{\alpha}{\lambda}_i \underset{\alpha}{\lambda}_k$; and the projection tensor with respect to V_{n-r} is defined to be $q_{ik} = \sum_{\alpha=r+1}^{n} \underset{\alpha}{\lambda}_i \underset{\alpha}{\lambda}_k$. Hence

(11.6) $$g_{ik} = p_{ik} + q_{ik}.$$

From this it follows that p_{ik} is independent of our choice of normalized Bein spanning V_r. For suppose $\tilde{p}_{ik}$ is defined by another B ein in V_r, the Bein in V_{n-r} remaining the same. Then

$$g_{ik} = \tilde{p}_{ik} + q_{ik}$$

so that $$\tilde{p}_{ik} = p_{ik}.$$

$\underline{\lambda}^i$ is called the projection of λ^i in V_r provided that

1) If $\lambda^i - \underline{\lambda}^i = \nu^i$, then ν^i is normal to V_r.

2) $\underline{\lambda}^i \subset V_r$.

If such a vector $\underline{\lambda}^i$ exists it is unique, for suppose

$$\lambda^i = \underline{\lambda}^i + \nu^i$$ where ν^i is normal to V_r and

also $$\lambda^i = \underline{\lambda}^{i\prime} + \nu^{\prime i}$$ where $\nu^{\prime i}$ is normal to V_r

give two projections. Then

$$(\underline{\lambda}^{i} - \underline{\lambda}^{i'}) = (\nu'^{i} - \nu^{i})$$

Multiplying by $g_{ik}(\underline{\lambda}^{k} - \underline{\lambda}^{k'})$ and summing we have:

$$g_{ik}(\underline{\lambda}^{i} - \underline{\lambda}^{i'})(\underline{\lambda}^{k} - \underline{\lambda}'^{k}) = 0$$

since $\underline{\lambda}^{i}$ and $\underline{\lambda}^{i'}$ lie in V_r and ν^{i} and ν'^{i} are normal to V_r. From the assumption that g_{ik} is positive definite it follows that $\underline{\lambda}^{i} = \underline{\lambda}^{i'}$, i.e. the projection is unique. A unique projection does not exist for an indefinite metric tensor.

We now show that such a projection exists. For multiplying (11.6) by λ^{k} we have

$$\lambda_{i} = p_{ik}\lambda^{k} + q_{ik}\lambda^{k}. \tag{11.7}$$

Now $p_{ik}\lambda^{k} = \underset{\alpha}{\lambda}_{i}(\underset{\alpha}{\lambda}_{k}\lambda^{k})$, and consequently is a vector of V_r. Similarly $q_{ik}\lambda^{k}$ is a vector of V_{n-r}. The result is that $\underline{\lambda}_{i} = p_{ik}\lambda^{k}$; $\nu_{i} = q_{ik}\lambda^{k}$.

If a vector space is defined at each point of a curve or or a neighborhood, the totality of the vector spaces is called a <u>vector space of class C^{ρ}</u> provided that at each point the vectors are linearly dependent on a Bein, each vector of which is a function of class C^{ρ} in the neighborhood. By considering the discussion given above it is clear that a vector space of class C^{ρ} can be spanned by a <u>normalized</u> Bein of class C^{ρ}. And then from (11.5) and (11.6) it follows that the <u>projection tensor</u> of this vector space is a function <u>of class</u> C^{ρ}.

12. Special Tensors and Invariants.

First we consider a set of r vectors λ^i_α ($\alpha = 1, \ldots, r$) and write $g_{ik}\lambda^i_\alpha \lambda^k_\beta = (\alpha\beta)$. We observe that $(\alpha\beta)$ is a scalar, and hence that the determinant $|(\alpha\beta)|$ is one also. Now if the λ^i_α are independent vectors, it follows that $|(\alpha\beta)|$ is positive definite. For take any numbers ν_α (not all zero) and form

$$(12.1) \qquad (\alpha\beta)\nu^\alpha\nu^\beta = g_{ik}\lambda^i_\alpha\lambda^k_\beta\nu^\alpha\nu^\beta$$
$$= g_{ik}(\lambda^i_\alpha\nu^\alpha)(\lambda^k_\beta\nu^\beta).$$

But this is greater than zero since $|g_{ik}|$ is positive definite and since $(\lambda^i_\alpha\nu^\alpha)$ are not all zero because of the independence of the λ^i_α. Since this holds for any ν^α (not all zero), it follows that $|(\alpha\beta)|$ is positive definite. Hence we can write $(\alpha\beta) = \xi_{\gamma\alpha}\xi_{\gamma\beta}$ ($\gamma = 1, \ldots, r$). Taking the determinants of both sides we have that $|(\alpha\beta)| = |\xi_{\gamma\alpha}|^2 > 0$. Thus the determinant of any positive definite form is greater than zero, and in particular this holds for $g = |g_{\alpha\beta}|$.

If, however, the λ^i_α are not independent, there exist ν^α (not all zero) such that $\lambda^i_\alpha\nu^\alpha = 0$. For these ν^α it follows that

$$(12.2) \qquad (\alpha\beta)\nu^\alpha = 0$$

and hence that $|(\alpha\beta)| = 0$.

Therefore $|(\alpha\beta)| \geq 0$, the equality holding when and only when the λ^i_α are dependent. The square root of the value of this determinant is called the volume of the "parallelopiped" defined by the λ^i_α.

The η tensor: Let $\eta_{i_1 i_2 \ldots i_n}$ be defined to be + 1 if the indices $i_1, \ldots, i_n$ are an even permutation of the natural numbers

1, ..., n, to be − 1 if $i_1, \ldots, i_n$ are an odd permutation of the natural numbers, and to be zero in any other case. Then it follows that

$$(12.3)\quad \eta_{i_1 \ldots i_n} = \eta_{k_1 \ldots k_n} \frac{\partial x^{k_1}}{\partial \bar{x}^{i_1}} \cdots \frac{\partial x^{k_n}}{\partial \bar{x}^{i_n}} \left| \frac{\partial \bar{x}}{\partial x} \right|$$

where $\left| \frac{\partial \bar{x}}{\partial x} \right|$ is the Jacobian of $\bar{x}$ with respect to x. Also we have that

$$\bar{g} = g \left| \frac{\partial x}{\partial \bar{x}} \right|^2$$

If we assume that $\left| \frac{\partial x}{\partial \bar{x}} \right| > 0$, this shows that

$$(12.4)\quad +\sqrt{\bar{g}} = +\sqrt{g} \left| \frac{\partial x}{\partial \bar{x}} \right|$$

From (12.3) and (12.4) we have that:

$$(12.5)\quad +\sqrt{\bar{g}}\, \eta_{i_1 \ldots i_n} = +\sqrt{g}\, \eta_{k_1 \ldots k_n} \frac{\partial x^{i_1}}{\partial \bar{x}^{k_1}} \cdots \frac{\partial x^{i_n}}{\partial \bar{x}^{k_n}}$$

which shows that $+\sqrt{g}\, \eta_{k_1 \ldots k_n}$ is a tensor for transformations of coordinates with positive Jacobian.

Orientation: This restriction on the sign of the Jacobian suggests that we separate our class of m-systems into two sub-classes. The m-systems of each subclass are to transform into each other by means of transformations with positive Jacobians, but the transformation from any system of the first class to any system of the second class or vice versa is to have a negative Jacobian. It is evident that this subdivision of our original class of m-systems is exhaustive and unique.

A point set which is endowed with the set of m-systems belonging to either of these sub-classes is called an "oriented space". We may say that the first sub-class gives a positive orientation, and the second sub-class gives a negative orientation, but these terms are merely relative and might equally well have been applied in the reverse order.

In an oriented space let us consider an n-Bein $\underset{\alpha}{\lambda}^{i}$. Then we have:

$$\underset{\alpha}{\bar{\lambda}}^{i} = \underset{\alpha}{\lambda}^{i} \frac{\partial \bar{x}^{i}}{\partial x^{i}}$$

or that

$$\left| \underset{\alpha}{\bar{\lambda}}^{i} \right| = \left| \underset{\alpha}{\lambda}^{i} \right| \cdot \left| \frac{\partial \bar{x}}{\partial x} \right|.$$

Since the space is oriented we see that the sign of the determinant of an n-Bein is unchanged under allowable coordinate transformations.

There is no question concerning the possibility of orienting a space covered by a single coordinate neighborhood, for we have just shown how to do it. However, we shall have occasion to consider spaces which are covered by a number of neighborhoods. Each of these neighborhoods can be oriented in the manner we have described. The whole space will be called "orientable" if the orientations can be so chosen that in the intersection of two neighborhoods the transformations from the coordinate systems of one neighborhood to the coordinate systems of another neighborhood have positive Jacobians. Otherwise the space is non-orientable.

Length of a curve. Consider a curve of class C^1: $x^i(t)$; $0 \leq t \leq 1$. Then the length of this curve from a to b (for $0 \leq a < b \leq 1$) is:

$$L = \int_a^b + \sqrt{g_{ik} \frac{dx^i}{dt} \frac{dx^k}{dt}}\, dt.$$

The number so defined is an invariant of the curve and the set of allowable parameters.

Volume of a space. The volume of a Riemann space, R_n, is defined to be

$$V = \int_{R_n} + \sqrt{g}\ dx^1 \ldots dx^n .$$

In any other coordinate system the volume is

$$\overline{V} = \int_{R_n} + \sqrt{\overline{g}}\ d\overline{x}^1 \ldots d\overline{x}^n .$$

It can be proved that $V = \overline{V}$; i.e. the volume is an invariant of the space, but this proof is too lengthy to be given here.

13. Subspaces.

The n-dimensional space $\mathcal{V}$ with its associated m-systems [cf. § 1] is a topological space: its neighborhoods are just the maps of the neighborhoods in S. (We recall that S is the topological sphere of E_n which corresponds to $\mathcal{V}$, thus defining a coordinate system).

Any subset $\mathfrak{M}$ of $\mathcal{V}$ may be made a topological space by taking its neighborhoods to be the intersections with $\mathfrak{M}$ of the neighborhoods of $\mathcal{V}$. Such a subset $\mathfrak{M}_r$ of $\mathcal{V}$ will be called a surface element (subspace) of dimension r provided it satisfies certain further requirements:

I. Dimensionality: $\mathfrak{M}_r$ is homeomorphic with the interior of an r-dimensional topological sphere in the Euclidean r-space.

II. m_1-systems: by means of this mapping, a class of allowable coordinate systems $y_1, \ldots, y_r$ (m_1-systems, $1 \leqq m_1 \leqq m$) are introduced into $\mathfrak{M}_r$, as was done for $\mathcal{V}$ in § 1. That is, $\varphi(P)$ will be of class C^ℓ at a point P of $\mathfrak{M}_r$ provided it is of class C^ℓ as a function $\varphi(y_1, \ldots, y_r)$ of the y's of some selected coordinate system, etc.

These coordinates y will hereafter be known as parameters, to distinguish them from the coordinates $(x_1, \ldots, x_n)$ of the same points considered as elements of γ. Because of II, we can speak of functions on $\mathcal{M}_r$ of class C^ρ in the parameters $[\rho \leq m_1]$. Now any function on γ, insofar as it is defined at points of $\mathcal{M}_r$, is a function of the y's as well as of the x's. It is therefore desirable to impose some regularity on the connection between the parameter and coordinate systems. This is done by a final postulate on

III Parameter-coordinate relations: a function of class C^ρ in the x's of γ at a point P of $\mathcal{M}_r$ shall be a function of class C^ρ in the y's of $\mathcal{M}_r$, unless $\rho > m_1$, in which case it shall be of class C^{m_1}.

In particular the coordinates x of γ are functions $x_i(x)$ of class C^m at each point of $\mathcal{M}_r$. By virtue of the correspondence

$$(y_1, \ldots, y\ \ ) \longleftrightarrow P(\mathcal{M}_r) \longrightarrow (x_1, \ldots, x_n)$$

we may consider them as functions $x_i(y)$ over $\mathcal{M}_r$. Because of III, then, the functions

$$x_i = x_i(y_1, \ldots, y_r) \tag{13.1}$$

(which give the so-called parametric representation of $\mathcal{M}_r$) are of class C^{m_1}. But conversely, if the $x_i(y)$ are of class C^{m_1}, the postulate is surely satisfied. Hence III is equivalent to the requirement that the x's on $\mathcal{M}_r$ shall be of class C^{m_1} in the y's.

Theorem: The rank $\mathcal{R}_P$ of the matrix $\left\|\frac{\partial x^i}{\partial y^\alpha}\right\|$* at a given point P of $\mathfrak{M}_r$ is independent of any special choice of coordinates (x) or parameters (y).

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*In this section Greek indices have the range 1 ... r , and Latin the range 1, ... , n, unless otherwise indicated.

This follows directly from the equations

$$\frac{\partial \bar{x}^i}{\partial \bar{y}^\alpha} = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^k}{\partial y^\beta}\frac{\partial y^\beta}{\partial \bar{y}^\alpha} \qquad\qquad \frac{\partial x^i}{\partial y^\alpha} = \frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial \bar{x}^k}{\partial \bar{y}^\beta}\frac{\partial \bar{y}^\beta}{\partial y^\alpha}$$

for since the rank of a product of matrices cannot exceed the rank of any factor, we have simultaneously $\bar{\mathcal{R}} \leq \mathcal{R}$, $\mathcal{R} \leq \bar{\mathcal{R}}$.

In addition to the rank $\mathcal{R}$ of the Jacobian just considered, there are other properties of $\mathfrak{M}_r$ which are not dependent on the choice of parameter and coordinate systems and which therefore have a geometric meaning. For example, at a point P of $\mathfrak{M}_r$ we have tensors of Υ such as λ^i (which behave like scalars under parameter transformations). Similarly we have tensors of $\mathfrak{M}_r$ such as v_α (which behave like scalars under coordinate transformations). We also have products $\lambda^i v_\alpha$ of vectors of Υ by vectors of $\mathfrak{M}_r$, i.e., geometric objects which are covariant vectors for transformations of parameters, and contravariant vectors for coordinate transformations.

<u>Tangent vectors.</u> A particularly important example of this latter kind of geometric object is afforded by the quantities $\frac{\partial x^i}{\partial y^\alpha}$, whose law of transformation is

$$(13.2) \qquad \frac{\partial \bar{x}^i}{\partial y^\alpha} = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^k}{\partial y^\alpha} \qquad\qquad \frac{\partial x^i}{\partial \bar{y}^\alpha} = \frac{\partial y^\beta}{\partial \bar{y}^\alpha}\frac{\partial x^i}{\partial y^\beta}$$

For a fixed set of y's, they are seen to be Υ contravariant vectors $\frac{\partial x^i}{\partial y^1}$; $\frac{\partial x^i}{\partial y^2}$; . . . ; $\frac{\partial x^i}{\partial y^r}$ in γ, defined at each point P of $\mathfrak{M}_r$. We call them <u>contravariant tangent vectors</u> to the subspace at P.

Consider now the vector space $\left\{\frac{\partial x^i}{\partial y^\alpha}\right\}_P$. From the second set of equations (13.2) and from the corresponding relations

$$\frac{\partial x^i}{\partial y^\alpha} = \frac{\partial \bar{y}^\beta}{\partial y^\alpha}\frac{\partial x^i}{\partial \bar{y}^\beta}$$

it is clear that $\left\{\frac{\partial x^i}{\partial y^\alpha}\right\} \cong \left\{\frac{\partial x^i}{\partial \bar{y}^\alpha}\right\}$, or in other words, that <u>the tangent vector space at P is independent of the parameterization</u> of $\mathfrak{M}_r$. This affords a second proof that the dimensionality of this vector space, namely, the rank $\mathcal{R}_P \left\|\frac{\partial x^i}{\partial y^\alpha}\right\|$, is not affected by transformations of coordinates or parameters. But before saying anything more precise about this dimensionality we must distinguish between two types of points of $\mathfrak{M}_r$.

<u>Regular and singular points</u>. Let $P_{(y)}$ be a point of $\mathfrak{M}_r$ and let the rank $\mathcal{R}_P$ be <u>s</u>. Because of the continuity of $\frac{\partial x^i}{\partial y^\alpha}$, an s-rowed minor of $\left\|\frac{\partial x^i}{\partial y^\alpha}\right\|$ which does not vanish at $P_{(y)}$ remains different from zero throughout some neighborhood $\mathcal{U}(y) \supset P(y)$. Thus the rank at any point of this $\mathcal{U}(y)$ is certainly not less than s, though it may be greater. We make the following definition:

<u>A point</u> <u>P</u> $\varepsilon\, \mathfrak{M}_r$ <u>is regular</u> if and only if there exists a neighborhood $\mathcal{U}$ (P) throughout thich the <u>rank</u> of $\left\|\frac{\partial x^i}{\partial y^\alpha}\right\|$ is <u>constant</u>; if no such neighborhood exists, then P is singular.

Since the rank $\mathcal{R}$ is invariant, this definition does not depend on any particular system of coordinates or parameters.

<u>The set of regular points of</u> $\mathfrak{M}_r$ <u>is open</u> (with respect to $\mathfrak{M}_r$). For if P is a regular point, and $\mathcal{U}$ (P) is a neighborhood in which $\mathcal{R}$ has the constant value t, then any point

$P_1 \varepsilon \mathcal{U}(P)$ has a neighborhood $\tilde{\mathcal{U}}(P_1) \subset \mathcal{U}(P)$ throughout which $\mathcal{R} = t$. Thus every point of $\mathcal{U}(P)$ is regular.

On the other hand, the set of singular points of $\mathfrak{M}_r$ contains no open set; more than this, the set of singular points is nowhere dense in $\mathfrak{M}_r$. We first observe that if at a point of $\mathfrak{M}_r$ the rank $\mathcal{R}$ has its maximum value r, then that point cannot be singular. Now suppose that Q is singular, and that $\mathcal{R}_Q = S$. Any neighborhood $\mathcal{U}(Q)$ must contain a point Q_1 for which $\mathcal{R}_{Q_1} \geqq S + 1$. Either Q_1 is regular, or any $\mathcal{U}(Q_1) \subset \mathcal{U}(Q)$ contains a Q_2 for which $\mathcal{R}_{Q_2} \geqq S + 2$. Proceeding in this way, we must ultimately find a point $P = Q_K \varepsilon \mathcal{U}(Q)$ which is regular; for the rank $\mathcal{R}_{Q_i}$ increases at each step, and yet must remain $\leqq r$. Hence any neighborhood of a singular point contains a regular point and therefore an open set of regular points.

The following two examples may serve to illustrate the notion of regular and singular points of subspaces

(a) $$\mathfrak{M}'_r: \quad \begin{array}{ll} x_i = y_i^3 & i = 1, \ldots r, \\ x_K = 0 & K = r+1, \ldots, n \end{array} \quad |y_i| < 1$$

All points of the "planes" $y_i = 0$ are evidently singular.

(b) $$\mathfrak{M}''_r: \quad \begin{array}{ll} x_i = z_i & i = 1, \ldots, r \\ x_K = 0 & K = r+1, \ldots, n \end{array} \quad |z_i| < 1$$

There are no singular points.

Note that both $\mathfrak{M}'_r$ and $\mathfrak{M}''_r$ contain exactly the same points of $\mathcal{T}$, but that they are different subspaces, in the sense that no allowable transformation of parameters can send one into the other. The class of parameter systems (m_1-systems) associated with $\mathfrak{M}_r'$ is thus completely distinct from that associated with $\mathfrak{M}_r''$; for the regularity or singularity of any point is a notion which is independent of the choice of parameters within a particular class of m_1-systems. There is of course the transformation

$$z_\alpha = y_\alpha^3 \quad ; \quad y_\alpha = \sqrt[3]{z_\alpha} \qquad |y_\alpha| \, , |z_\alpha| < 1$$

between $\mathfrak{M}_r'$ and $\mathfrak{M}_r''$; but this is easily seen to be irregular at all singular points of $\mathfrak{M}_r'$.

We are now in a position to prove the

<u>Theorem:</u> <u>The tangent vector space at a regular point</u> P <u>of</u> $\mathfrak{M}_r$ <u>has the dimension</u> r .

To do this, we need only show that <u>if P is regular, then</u> $\mathcal{R}_P \left\| \frac{\partial x^i}{\partial y^\alpha} \right\| = r$. Suppose therefore that P_0 is regular, and that $\mathcal{R}_{P_0} = t$. After renumbering, if necessary, the x's and y's, we may assume that

$$\left| \frac{\partial x^i}{\partial y^\alpha} \right| \neq 0 \qquad i, \alpha = 1, \ldots, t$$

throughout some nbd $\mathcal{U}'(P_0)$. Consider the functions

$$\begin{aligned} \tilde{y}_i &= x_i(y_1, \ldots, y_r) \qquad & i &= 1, \ldots, t \\ \tilde{y}_\kappa &= y_\kappa & \kappa &= t+1, \ldots r, \end{aligned} \tag{13.3}$$

where the $x_i(y_1, \ldots, y_r)$ are those in (13.2). The Jacobian $\left| \frac{\partial \tilde{y}^\alpha}{\partial y^\beta} \right|$ is different from zero throughout $\mathcal{U}'(P_0)$. The Dini theorem then assures us of the existence of a neighborhood $\mathcal{U}(P_0) \subset \mathcal{U}'(P_0)$ throughout which (13.3) defines an allowable transformation of parameters. That is, if we designate by $\mathcal{U}(y_0)$ the topological sphere in the arithmetic y space of which $\mathcal{V}(P_0)$ is the map, and by $\mathcal{V}(\tilde{y}_0)$ the corresponding neighborhood in the $\tilde{y}$ space, we are certain that

$$\mathcal{V}(P_0) \longleftrightarrow \mathcal{V}(y_0) \leftrightarrow \mathcal{V}(\tilde{y}_0) .$$

In $\mathcal{V}(P_0)$ the new parameters $\tilde{y}$ are also an m_I-system.

Introducing the notation $\tilde{x}_i$ for the space coordinates of a point of $\mathcal{V}(P_0)$ expressed in terms of the parameters y, i.e.,

$$\tilde{x}_i(\tilde{y}) = x_i(y(\tilde{y})), \tag{13.4}$$

we have $\mathcal{R}\left\|\frac{\partial \tilde{x}^i}{\partial \tilde{y}^\alpha}\right\| = t$. For this rank is invariant under transformations of parameters. Hence any (t+1)-rowed minor of $\left\|\frac{\partial \tilde{x}^i}{\partial \tilde{y}^\alpha}\right\|$ vanishes throughout $\mathcal{V}(P_0)$. Consequently the equations

$$\begin{vmatrix} \frac{\partial \varphi}{\partial \tilde{y}^\kappa} & \frac{\partial \varphi}{\partial \tilde{y}^1} & \cdots & \frac{\partial \varphi}{\partial \tilde{y}^t} \\ \frac{\partial \tilde{x}^1}{\partial \tilde{y}^\kappa} & \frac{\partial \tilde{x}^1}{\partial \tilde{y}^1} & \cdots & \frac{\partial \tilde{x}^1}{\partial \tilde{y}^t} \\ \vdots & \vdots & & \\ \frac{\partial \tilde{x}^t}{\partial \tilde{y}^\kappa} & \frac{\partial \tilde{x}^t}{\partial \tilde{y}^1} & \cdots & \frac{\partial \tilde{x}^t}{\partial \tilde{y}^t} \end{vmatrix} = 0 \qquad \kappa = t+1, \ldots, r \tag{13.5}$$

are satisfied by $\varphi = \tilde{x}_j$ for $j = 1, \ldots, n$.

But since (13.3) has an inverse, we have

$$\tilde{y}_i = x_i(y_1(\tilde{y}), \ldots, y_r(\tilde{y})) = \tilde{x}_i(\tilde{y}) \qquad i = 1, \ldots, t$$

so that $\frac{\partial \tilde{x}^i}{\partial \tilde{y}^\rho} = \delta^i_\rho$ $[i = 1, \ldots, t ; \rho = 1, \ldots, r]$ and equations (13.5) are actually

$$\frac{\partial \varphi}{\partial \tilde{y}^\kappa} = 0 \qquad \kappa = t+1, \ldots, r .$$

From this we conclude that each $x_j(y)$ is a function of at most $y_1, \dots, y_t$.

Unless $t = r$, we are thus led to a contradiction. The homeomorphism $\mathcal{U}(\tilde{y}_0) \longleftrightarrow \mathcal{U}(P_0)$ implies that distinct points $(\tilde{y}_1, \dots, \tilde{y}_r)$ in $\mathcal{U}(\tilde{y}_0)$ are mapped into distinct points P of $\mathcal{U}(P_0)$. Yet if $t < r$, it is clear that the <u>distinct points</u> $(y_1, \dots, y_{r-1}, y_r^*)$, $(y_1, \dots, y_{r-1}, y_r^{**})$ must give rise to the same set of coordinate values (x) in (13.4), and so to the <u>same point</u> in $\mathcal{U}(P_0)$.

Thus result, together with the observation that if $\mathcal{R}_P$ is r then P is regular, enables us to restate our previous <u>definition of regular and singular points</u>:

A point P of $\mathcal{M}_r$ <u>is regular</u> if and only if $\mathcal{R}\left\|\frac{\partial x^i}{\partial y^\alpha}\right\|_P = r$.

A point P of is <u>singular</u> if and only if $\mathcal{R}\left\|\frac{\partial x^i}{\partial y^\alpha}\right\|_P < r$.

<u>Note:</u> the definition of surface element includes the cases $r = 1$ (curves) and $r = n$ (open sets in $\mathcal{V}$).

Subspaces given parametrically.

If a subset Σ of γ is defined by equations of the form

(13.6) $$x_i = x_i(y_1, \ldots, y_r)$$

(where the $x_i(y)$ are single-valued functions of class C^ρ $[\rho \geq 1]$ and the y's range over some domain Δ in E_r), it does not follow that Σ is an $\mathfrak{M}_r$-element, even if Δ is a topological sphere and $\mathcal{R}\left\|\frac{\partial x^i}{\partial y^\alpha}\right\| = r$ throughout the domain. For the essential 1-1 continuous correspondence between Σ and Δ will in general break down in various regions.

We will now show, however, that <u>if at a point</u> $P_o(y_o)$* <u>of</u>

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* The notation $P_o(y)$ means that we have chosen a particular one of the several images that P_o may have in Δ.

Σ <u>the rank of</u> $\left\|\frac{\partial x^i}{\partial y^\alpha}\right\|$ <u>is r, there is a subset</u> $\mathfrak{M}$ <u>of</u> Σ, <u>containing</u> P_o, <u>which is a</u> (regular) $\mathfrak{M}_r$ <u>-element</u>. Consider the functions

$$\tilde{y}_\alpha = x_\alpha(y_1, \ldots, y_r) \qquad \alpha = 1, \ldots, r$$

where the x's of (13.6) have been renumbered in such a way as to have $\left|\frac{\partial x^\alpha}{\partial y^\beta}\right| \neq 0$. By the Dini theorem this establishes a homeomorphism between a certain sphere-neighborhood $\mathfrak{U}(y_o)$ in the y-space, and a neighborhood $\mathfrak{U}(\tilde{y}_o)$ in the $\tilde{y}$-space. We now have a set of functions

(13.7) $$\begin{aligned} x_\alpha &= \tilde{y}_\alpha & \alpha &= 1, \ldots, r \\ x_k &= x_k(y(\tilde{y})) & k &= r+1, \ldots, n \end{aligned}$$

which map $\mathfrak{U}(y_o)$ into a subset $\mathfrak{M} \ni P_o$ of Σ. Since the x's are single-valued continuous functions of the y's, and the y's are single-valued continuous functions of the $\tilde{y}$'s, it follows that

$$\mathfrak{V}(\tilde{y}_0) \longrightarrow \mathfrak{M}.$$

But it is clear from the first of (13.7) that the x's of any point $P \in \mathfrak{M}$ uniquely determine all the $\tilde{y}$'s; hence

$$\mathfrak{M} \longrightarrow \mathfrak{V}(\tilde{y}_0)$$

and finally

$$\mathfrak{M} \longleftrightarrow \mathfrak{V}(\tilde{y}_0) \longleftrightarrow \mathfrak{V}(y_0).$$

The surface element thus defined is said to be a <u>regular $\mathfrak{M}_r$-element</u>; for all its points are regular.

<u>Subspaces given implicitly</u>. Let Σ be an open point set (of $\mathfrak{S}$) over which there is defined a system of n - r scalars

$$\varphi_\kappa(x_1, \ldots, x_n) \qquad \kappa = r+1, \ldots, n.$$

In Σ let the φ's be of class c^ρ $[1 \leq \rho \leq m]$; and let the rank of $\left\|\frac{\partial \varphi_\kappa}{\partial x^i}\right\|$ be n - r throughout. If P^* (of coordinates x_i^*) is a point of Σ, we can define a subset $\mathfrak{M} \supset P^*$ by the n - r equations

$$(13.8) \qquad \mathfrak{M}: \quad F_\kappa(x_1, \ldots, x_n) \equiv \varphi_\kappa(x) - \varphi_\kappa(x^*) = 0.$$

Under the conditions just stated, we have the

<u>Theorem:</u> <u>Each point $P_0(x_0)$ of $\mathfrak{M}$ has a neighborhood (in $\mathfrak{M}$) which can be so parameterized as to become a regular element of class c^ρ.</u>

<u>Proof:</u> We may assume that at $P_0(x_0)$

$$\left|\frac{\partial F_\kappa}{\partial x_j}\right| \neq 0 \qquad j, \kappa = r+1, \ldots, n.$$

Then the Jacobian of the transformation defined by means of r additional functions F_i, namely

$$(13.9)\qquad \begin{aligned} F_i &= x_i & i &= 1, \ldots, r \\ F_\kappa &= F_\kappa (x_1, \ldots, x_n) & \kappa &= r+1, \ldots, n \end{aligned}$$

is different from zero at $P_o(x_o)$. Consequently there exists a neighborhood* $U(x_o)$ and a neighborhood $U(F_o)$ $(\underset{o}{F}_1 = \underset{o}{x}_1; \ldots; \underset{o}{F}_r = \underset{o}{x}_r; \underset{o}{F}_{r+1} = 0 ; \ldots ; \underset{o}{F}_n = 0)$ which are mapped homeomorphically on one

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* We shall use $U(P)$, $U(x)$ to designate neighborhoods of $\mathcal{V}$ or of the arithmetic n-space, and $\mathfrak{U}(P)$, $\mathfrak{U}(y)$ to designate neighborhoods of $\mathfrak{M}$ or of E_r. By definition $\mathfrak{U}(P) = U(P) \cap \mathfrak{M}$.

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another by (13.9) and its inverse

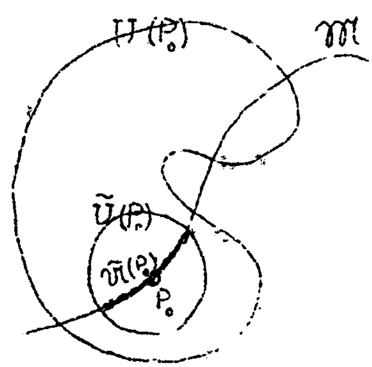
$$\begin{aligned} x_i &= F_i & i &= 1, \ldots, r \\ x_\kappa &= x_\kappa (F_1, \ldots, F_n) & \kappa &= r+1, \ldots, n . \end{aligned}$$

In the corresponding neighborhood $U(P_o)$ of $\mathcal{V}$ we now have F-coordinates as well as x-coordinates:

$$U(P_o) \longleftrightarrow U(x_o) \longleftrightarrow U(F_o),$$

and the F-coordinates are also a ρ -system. Passing to $\mathfrak{M}$, we have a neighborhood $\mathfrak{U}(P_o)$ defined by the relations

$$\mathfrak{U}(P_o): \quad \begin{cases} (F_1, \ldots, F_n) \in U(F_o) \\ F_\kappa = 0 \qquad \kappa = r+1, \ldots, n \end{cases}$$



Of course, this $\mathfrak{U}(P_o)$ may not be the interior of a topological sphere. But a $\tilde{U}(F_o) \subset U(F_o)$ can always be found, the intersection of whose map with $\mathfrak{M}$ has the desired property. For example, there exists an ϵ such that $\tilde{U}(F)$ defined by

$$\tilde{U}(\underset{o}{F}): \quad \sum_{\nu=1}^{n} (F_\nu - \underset{o}{F}_\nu)^2 < \epsilon$$

lies entirely within $U(\underset{o}{F})$. The corresponding $\tilde{\mathfrak{U}}(\underset{o}{P}) = \tilde{U}(\underset{o}{P}) \cap \mathfrak{M}$ can then be parameterized by

$$\tilde{\mathfrak{U}}(\underset{o}{P}): \begin{cases} F_\alpha = y_\alpha & \alpha = 1, \dots, r \\ F_K = 0 & K = r+1, \dots, n \\ y_\alpha \in \tilde{\mathfrak{U}}(y_o): \sum (y_\alpha - \underset{o}{F}_\alpha)^2 < \epsilon \end{cases}$$

Changing back to the x-coordinates (13.9), we have

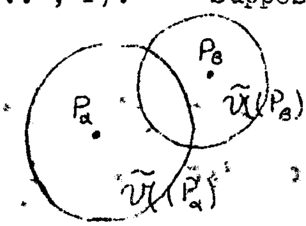
$$\text{(13.10)} \qquad \tilde{\mathfrak{U}}(P_o): \begin{cases} x_\alpha = y_\alpha & \alpha = 1, \dots, r \\ x_K = x_K(y_1, \dots, y_r, 0, \dots, 0) \end{cases}$$

This is easily seen to be a <u>regular $\mathfrak{M}_r$-element of class</u> C^ℓ.

As is clear from the form of (13.10), we might have taken the parameters y_α to be some r of the original x's, say $x_{\alpha_1}, \dots, x_{\alpha_r}$, subject only to the restriction that at P_o

$$\left| \frac{\partial \varphi_k}{\partial x_{\alpha_j}} \right| \neq 0 \qquad j, k = r+1, \dots, n.$$

Suppose then that we have any two points P_α, P_β of $\mathfrak{M}$, whose $\mathfrak{M}_r$ neighborhoods $\tilde{\mathfrak{U}}(P_\alpha)$, $\tilde{\mathfrak{U}}(P_\beta)$ are described by the respective sets of parameters (x_{α_i}) and (x_{β_i}) $(i = 1, \dots, r)$. Suppose further that the two neighborhoods intersect.



Then in the intersection, any x_{α_t} is a function of $x_{\beta_1}, \dots, x_{\beta_r}$ of class C^ℓ (for this is true of all the x's), and conversely. Hence: <u>The transformation of parameters in the intersection of any two of the above $\mathfrak{M}_r$-neighborhoods is of class</u> C^ℓ.

<u>Pseudo-regular transformations.</u> Let P_o be a singular point of an $\mathfrak{M}_r$-element of class C^{m_1}. Any transformation of parameters (1-1 and bicontinuous by definition) which is defined throughout some

neighborhood $\mathfrak{V}(P_0)$ in $\mathfrak{M}_r$ will be called pseudo-regular, provided the functions

$$y = y(\tilde{y}) \qquad \tilde{y} = \tilde{y}(y)$$

are of class C^{m_1} at all regular points of $\mathfrak{V}(P_0)$.

It may be possible to find a pseudo-regular transformation which will transform an $\mathfrak{M}_r$-element containing P_0 into an $\mathfrak{M}_r'$ (necessarily different from $\mathfrak{M}_r$) for which P_0 is regular. In this case P_0 is said to be a non-essential singularity of $\mathfrak{M}_r$.

If P is any regular point of $\mathfrak{V}(P_0)$, there is a neighborhood $\mathfrak{V}(P) \subset \mathfrak{V}(P_0)$ which is a regular $\mathfrak{M}_r$ element. The character of such a surface element is unaltered by a pseudo-regular transformation.

14. Vectors defined on subspaces.

Tangent vectors. Let P be a regular point of an $\mathfrak{M}_r$ in a general space $\mathfrak{X}$. At P we then have an r-dimensional tangent vector space $\mathfrak{T}: \left\{\frac{\partial x^i}{\partial y^\alpha}\right\}$ whose elements

$$\lambda^i = \frac{\partial x^i}{\partial y^\alpha} \lambda^\alpha \tag{14.1}$$

we shall call (contravariant) tangent vectors (in $\mathfrak{X}$) of the $\mathfrak{M}_r$.

The r quantities λ^α appearing in (14.1) will hereafter be known as components of the tangent vector λ^i with respect to $\mathfrak{M}_r$.

Under coordinate transformations these λ^α behave like scalars, for the λ^i and $\frac{\partial x^i}{\partial y^\alpha}$ are contravariant vectors in $\mathfrak{X}$. But under parameter transformations the λ^α are contravariant vectors of $\mathfrak{M}_r$, since the λ^i are scalars and $\frac{\partial x^i}{\partial y^\alpha}$ are covariant vectors under such transformations.

The following direct demonstration of the transformation law of the λ^α brings out a feature which will be of importance later on when we come to discuss singular points.

If we define $\bar{\lambda}^\alpha$ in any $\bar{y}$-system in terms of their values (14.1) in some fixed system, i.e.,

$$\bar{\lambda}^\alpha = \frac{\partial \bar{y}^\alpha}{\partial y^\beta} \lambda^\beta$$

we have a set of $\bar{\lambda}$'s satisfying the requirements

$$\lambda^i = \frac{\partial x^i}{\partial y^\beta} \lambda^\beta = \frac{\partial x^i}{\partial \bar{y}^\alpha} \bar{\lambda}^\alpha$$

but since $\left\| \frac{\partial x^i}{\partial y^\alpha} \right\|$ is of rank r, the λ^α in each parameter system are uniquely determined. A similar argument shows the scalar character of the λ^α under coordinate transformations; we see at once that the definition: $\bar{\lambda}^\alpha = \lambda^\alpha$ in all coordinates systems, is consistent and unique.

Projection. To each covariant vector μ_i of γ, defined at P $\varepsilon\, \mathfrak{M}_r$ there corresponds a covariant vector μ_α of $\mathfrak{M}_r$:

$$\frac{\partial x^i}{\partial y^\alpha} \mu_i = \mu_\alpha \tag{14.2}$$

(Note that this holds for any covariant μ_i, whereas the correspondence $\lambda^i \to \lambda^\alpha$ discussed in the preceding paragraph is confined to contravariant tangent vectors).

The μ_α derived from μ_i by means of (14.2) will be called components of the projection of μ_i on the tangent-space $\mathcal{T}$ of $\mathfrak{M}_r$ at P. For we shall see that when γ is a Riemann space, this definition agrees with that previously given for the projection of a vector upon a vector space.

The relations (14.1) and (14.2) are the only ones of that type existing between vectors of $\mathcal{T}$ and vectors of $\mathcal{M}_r$ at points $P \in \mathcal{M}_r$, unless $\mathcal{T}$ has additional structure.

15. Subspaces of a Riemann space $\mathcal{R}_n$:

Tensors of $\mathcal{M}_r$ and $\mathcal{R}_n$ at regular points. Let λ^i, ν^j be two vectors of $\mathcal{T}$ at P. ($\mathcal{T}$ is r-dimensional). Their components in $\mathcal{M}_r$ are given by

$$\lambda^i = \frac{\partial x^i}{\partial y^\alpha}\lambda^\alpha \qquad \nu^j = \frac{\partial x^j}{\partial y^\beta}\nu^\beta \tag{15.1}$$

The combination

$$g_{ij}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta} = y_{\alpha\beta} \tag{15.2}$$

defines in $\mathcal{M}_r$ a symmetric covariant tensor of second order.

At a regular point, $y_{\alpha\beta}$ is positive definite. For it is clear that

$$g_{ij}\lambda^i\nu^j = y_{\alpha\beta}\lambda^\alpha\nu^\beta \tag{15.3}$$

whence

$$y_{\alpha\beta}\lambda^\alpha\lambda^\beta \;\; (= g_{ij}\lambda^i\lambda^j) \geq 0$$

The equality holds only for $\lambda^i = 0$; but in this case we have, from (15.1) and from the fact that the rank of $\left\| \frac{\partial x^i}{\partial y^\alpha} \right\|$ is r, that $\lambda^\alpha = 0$.

Thus a regular $\mathcal{M}_r$ becomes a Riemann space $\mathcal{R}_r$ of class C^{m_1}, whose fundamental metric tensor is given by (15.2); that is, a Riemann space induces a Riemann metric in its sub-spaces.

Moreover we see from (15.3) that tangent vectors can be measured equally well by their $\mathcal{M}_r$ components and the metric tensor $\gamma_{\alpha\beta}$.

Lowering indices in $\mathcal{R}_n$ and in $\mathcal{M}_r$ produces the covariant components λ_i, λ_α of tangent vectors. These are related by equations obtainable from (15.1) by multiplication with $g_{ij}\dfrac{\partial x^j}{\partial y^\beta}$, namely

$$\frac{\partial x^j}{\partial y^\beta}\lambda_j = \gamma_{\alpha\beta}\lambda^\alpha = \lambda_\beta \tag{15.4}$$

Now (15.4) will define covariant components λ_β in $\mathcal{M}_r$, quite independently of whether λ_j are covariant components of a tangent vector or not. Suppose then that π_i is such an arbitrary vector, and that its $\mathcal{M}_r$-components are π_α. It is natural to ask what tangent vector p^i has the contravariant $\mathcal{M}_r$-components $\pi^\beta = \pi_\alpha \gamma^{\alpha\beta}$. We shall show that p^i is the projection of π^i upon the tangent space $\mathcal{T}$. For by hypothesis we have

$$\pi_\alpha = \pi_i\frac{\partial x^i}{\partial y^\alpha}\ ;\ \pi^\beta = \gamma^{\alpha\beta}\pi_\alpha\ ;\ p^j = \frac{\partial x^j}{\partial y^\beta}\pi^\beta \tag{15.5}$$

The first of these equations can be written

$$g_{ij}\frac{\partial x^i}{\partial y^\alpha}\pi^j = \gamma_{\alpha\beta}\pi^\beta = g_{ij}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}\pi^\beta$$

Hence

$$g_{ij}\frac{\partial x^i}{\partial y^\alpha}(\pi^j - p^j) = 0$$

Therefore $\pi^j - p^j$ is normal to $\mathcal{T}$, so that p^j is the projection of π^j (or π_j) in $\mathcal{T}$. This justifies the terminology used in connection with (14.2).

We shall make use of this result to obtain a formula for the projection tensor p^{ik} of the tangent space $\mathcal{T}$. For from the characteristic property of p^{ik} we have

$$p^i = \frac{\partial x^i}{\partial y^\beta}\pi^\beta = p^{ik}\,\pi_k$$

so that

$$p^{ik}\pi_k = \frac{\partial x^i}{\partial y^\beta}\,\pi_\alpha\, y^{\alpha\beta} = \frac{\partial x^i}{\partial y^\beta}\frac{\partial x^k}{\partial y^\alpha}\, y^{\alpha\beta}\,\pi_k \qquad \text{(by 15.5)}$$

for arbitrary π_k. Hence

$$p^{ik} = \frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^k}{\partial y^\beta}\, y^{\alpha\beta} \tag{15.6}$$

The formula (15.6) for p^{ik} is of course obtainable directly from the definition (§ 11) of the projection tensor of $\mathcal{T}$. Let us represent $y^{\alpha\beta}$ by a normalized r-Bein $\underset{\nu}{\lambda}^\alpha$ ($\nu = 1, \ldots, r$) at a point $P \in \mathfrak{M}_r$:

$$y^{\alpha\beta} = \underset{\nu}{\lambda}^\alpha\,\underset{\nu}{\lambda}^\beta$$

Then the corresponding tangent vectors

$$\underset{\nu}{\lambda}^i = \frac{\partial x^i}{\partial y^\alpha}\,\underset{\nu}{\lambda}^\alpha$$

are a normalized r-Bein in $\mathcal{R}_n$ (15.3) spanning $\mathcal{T}$. Hence by definition of the projection tensor we have

$$p^{ik} = \underset{\nu}{\lambda}^i\,\underset{\nu}{\lambda}^k = \frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^k}{\partial y^\beta}\,\underset{\nu}{\lambda}^\alpha\,\underset{\nu}{\lambda}^\beta = \frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^k}{\partial y^\beta}\, y^{\alpha\beta}$$

Tensors at singular points. If P is a singular point of $\mathfrak{M}_r$, a tangent vector λ^i at P will (by definition) still have the representation (14.1)

$$\lambda^i = \frac{\partial x^i}{\partial y^\alpha}\,\lambda^\alpha$$

But the $\mathfrak{M}_r$-components λ^α are now determined only to within an additive ϑ^α, satisfying

$$\frac{\partial x^i}{\partial y^\alpha}\vartheta^\alpha = 0 .$$

Consequently (14.1) no longer imposes a definite transformation law upon the λ^α under change of parameters. However, if we define λ^α to be a contravariant vector of $\mathfrak{M}_r$, this definition will at least be consistent with (14.1).

At a singular point the tensor $y_{\alpha\beta} = g_{ij}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}$ is only <u>positive semidefinite</u>. Its rank t is equal to that of $\left\|\frac{\partial x^i}{\partial y^\alpha}\right\|$. For we have

$$\frac{\partial x^i}{\partial y^\alpha}\lambda^\alpha = 0 \longrightarrow y_{\alpha\beta}\lambda^\alpha = 0$$

and

$$y_{\alpha\beta}\lambda^\alpha = 0 \longrightarrow y_{\alpha\beta}\lambda^\alpha\lambda^\beta = 0 \longrightarrow g_{ij}\frac{\partial x^i}{\partial y^\alpha}\lambda^\alpha\frac{\partial x^j}{\partial y^\beta}\lambda^\beta = 0$$

$$\longrightarrow \frac{\partial x^i}{\partial y^\alpha}\lambda^\alpha = 0$$

A covariant surface vector μ_α at a singular point may have no corresponding space-vector μ_i, and may have no contravariant $\mathfrak{M}_r$-components μ^α. In fact, neither of the sets of relations

$$\mu_K\frac{\partial x^K}{\partial y^\alpha} = \mu_\alpha \qquad y_{\alpha\beta}\mu^\beta = \mu_\alpha$$

is consistent unless

$$\mu_\alpha\vartheta^\alpha = 0 .$$

It is clear from the above remarks that at a singular point we cannot define a tensor $y^{\alpha\beta}$ in such a way that $y^{\alpha\epsilon}y_{\epsilon\beta} = \delta^\alpha_\beta$. We can however obtain a set of quantities $y^{\alpha\beta}$ having many of the desired properties by using the projection tensor of $\mathcal{T}$:

$$p^{ik} = \underset{\tau}{\lambda}^{i} \underset{\tau}{\lambda}^{k} \qquad (\tau = 1, \ldots, t)$$

($\underset{\tau}{\lambda}^{i}$ a normalized t-Bein spanning $\mathcal{T}$). Since each $\underset{\tau}{\lambda}^{i}$ is expressible as a linear combination of $\frac{\partial x^i}{\partial y^\alpha}$, the equations

$$p^{ik} = \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^k}{\partial y^\beta} y^{\alpha\beta} \tag{15.6}$$

are surely consistent. These equations will not, however, define $y^{\alpha\beta}$ uniquely; for if $y_0^{\alpha\beta}$ satisfies (15.6), so also does

$$y^{\alpha\beta} = y_0^{\alpha\beta} + \underset{\nu}{\sigma}^{\alpha} \underset{\nu}{\vartheta}^{\beta} \qquad (\nu = t+1, \ldots, r)$$

To find the relation between the covariant and contravariant y's, we multiply both sides of (15.6) by the tangent vector $g_{ij}\frac{\partial x^j}{\partial y^\epsilon}$:

$$g_{ij} p^{ik} \frac{\partial x^j}{\partial y^\epsilon} = \frac{\partial x^k}{\partial y^\epsilon} = y_{\alpha\epsilon} y^{\alpha\beta} \frac{\partial x^k}{\partial y^\beta}$$

or

$$\frac{\partial x^k}{\partial y^\beta} \left(y_{\alpha\epsilon} y^{\alpha\beta} - \delta^\beta_\epsilon \right) = 0$$

whence

$$y_{\alpha\epsilon} y^{\alpha\beta} = \delta^\beta_\epsilon + \underset{\nu}{\sigma}_\epsilon \underset{\nu}{\vartheta}^{\beta} \quad , \nu = t+1, \ldots, r .$$

Thus multiplication of $\lambda_\alpha = y_{\alpha\beta}\lambda^\beta$ by $y^{\epsilon\alpha}$ yields

$$y^{\epsilon\alpha}\lambda_\alpha = \left(\delta^\epsilon_\beta + \underset{\nu}{\sigma}_\beta \underset{\nu}{\vartheta}^{\epsilon}\right)\lambda^\beta = \lambda^\epsilon + \vartheta'^\epsilon$$

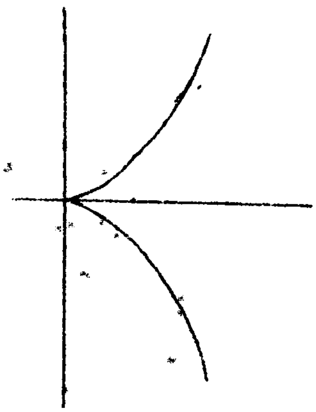
while multiplication of λ^ϵ (plus any ϑ'^ϵ) by $y_{\beta\epsilon}$ leads to the unique result

$$y_{\beta\epsilon}\lambda^\epsilon = y_{\beta\epsilon}(\lambda^\epsilon + \vartheta'^\epsilon) = y_{\beta\epsilon} y^{\epsilon\alpha}\lambda_\alpha = \left(\delta^\alpha_\beta + \underset{\nu}{\sigma}_\beta \underset{\nu}{\vartheta}^{\alpha}\right)\lambda_\alpha$$
$$= \lambda_\beta$$

Existence of essential singularities.

That an $\mathfrak{M}_r$ element may have a singularity which is not removable by a pseudo-regular transformation is shown by the following example:

$$x = t^2 \qquad y = t^3 \qquad |t| < 1 .$$



This is an $\mathfrak{M}_r$, regular at every point but (0,0), as is easily verified.

Assume that a paramet rization

$$x = x(u)$$
$$y = y(u)$$

has been found such that (0,0) and a neighborhood of (0,0) is regular; then both $\frac{dx}{du}$ and $\frac{dy}{du}$ must be continuous at (0,0), and at least one of them must be different from zero. If $\left.\frac{dx}{du}\right]_{(0,0)} \neq 0$, the equation $x = x(u)$ can be solved for $u = u(x)$, i.e., there is a neighborhood of $x = 0$ in the one-dimensional x-space which is homeomorphic with a neighborhood of $u = u_0$ in the u-space. The parametrization

$$x = x$$
$$y = y(u(x)) = f(x)$$

is then allowable in a neighborhood of (0,0), and leaves this point regular. Similarly if $\left.\frac{dy}{du}\right]_{(0,0)} \neq 0$, the parameterization

$$x = g(y)$$
$$y = y$$

is allowable.

But either of these two cases leads to a contradiction. For the first implies

$$x = x$$

$$y = x^{3/2}$$

which is plainly not 1-1 in a neighborhood of (0,0) and the second gives

$$x = y^{2/3}$$

$$y = y$$

which has a discontinuous derivative $\frac{dx}{dy}$ at (0,0).

Hence the assumption that there exists a transformation $t = t(u)$, $u = u(t)$ leading to a regular parametrization must be in error.

This argument applies equally well to an $\mathfrak{M}_r$ element of any dimensionality. In general, to prove that a point $P \in \mathfrak{M}_r$ is an essential singularity it is sufficient to show that no neighborhood $\mathfrak{N}(P)$ has a regular parametric representation in terms of any r of the space-coordinates.

16. The extended absolute differential.

We have already noted the possibility of introducing mixed tensors at a point P of a subspace $\mathfrak{M}_r$. Any such tensor can be regarded as the set of coefficients of an invariant multilinear form which may have both space and surface vectors as its arguments:

$$T^{\alpha i}_{\beta j k} : \qquad \varphi = T^{\alpha i}_{\beta j k}\, \mu_\alpha \nu^\beta \rho_i \sigma^j \tau^k$$

(We do not exclude the case where only Latin or only Greek indices appear). From this representation it is clear that the rules for addition, multiplication, and contraction (p. 10) hold equally well for these more general tensors. (Two mixed tensors are of the same kind if they are alike in the number and position of their Latin and Greek indices separately; and contraction is permitted only on indices having the same range.)

Mixed tensors can always be expressed uniquely in terms of a pair of independent Beins, the first $\underset{t}{\lambda}^i, \underset{t}{\lambda}_i$ (t = 1, ... , n) spanning the enveloping space, and the second $\underset{\nu}{\omega}^\alpha, \underset{\nu}{\omega}_\alpha$ (ν = 1, ... , r) spanning the subspace.

If $\mathfrak{M}_r$ is a <u>regular</u> subspace of class C^{m_1} $[m_1 \geq 2]$ lying in a Riemann space $\mathfrak{R}_m$, the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ formed from the y's exist and are continuous. Let $T^{\alpha i \dots}_{\beta \dots}$ be a tensor (of class C') defined along a curve $y^\alpha = y^\alpha(t)$ (of class C') in $\mathfrak{M}_r$. By adjoining to I-VI of § 8 the further postulate

$$\text{III}' \qquad \delta\lambda^\alpha = d\lambda^\alpha + \Gamma^\alpha_{\mu\nu}\lambda^\mu dy^\nu$$

and by making use of the Bein-representation of $T^{\dots}_{\dots}$ we obtain, in much the same way as before, a unique expression for $\delta T^{\dots}_{\dots}$ satisfying all the postulates of the extended set. The general method of formation of the <u>absolute differential of a mixed tensor</u> is indicated by the following example:

$$\delta\eta^i_\alpha = d\eta^i_\alpha + \Gamma^i_{jk}\eta^j_\alpha dx^k - \Gamma^\mu_{\alpha\nu}\eta^i_\mu dy^\nu$$

<u>Covariant derivative</u>. If η^i_α is defined throughout an entire neighborhood $\mathfrak{U}$(P), we may write

$$\delta\eta^i_\alpha = \left(\frac{\partial\eta^i_\alpha}{\partial y^\epsilon} + \Gamma^i_{jk}\eta^j_\alpha\frac{\partial x^k}{\partial y^\epsilon} - \Gamma^\mu_{\alpha\epsilon}\eta^i_\mu\right)dy^\epsilon$$

Since dy^ϵ is an arbitrary vector, its coefficient is seen to be a tensor $\eta^i_{\alpha;\epsilon}$; we shall call this coefficient the <u>covariant derivative</u> $\frac{\delta\eta^i_\alpha}{\delta y^\epsilon}$. For a tensor T_{ij}, of class C' (only Latin indices) throughout a neighborhood U(P) of $\mathfrak{R}_m$, the covariant derivative $\frac{\delta T_{ij}}{\delta x^k}$ has already been defined in § 9. However, suppose that U(P) contains a neighborhood $\mathfrak{U}$(P) of $\mathfrak{M}_r$. In $\mathfrak{U}$(P) we can write

$$\mathfrak{D} T_{ij} = \left(\frac{\partial T_{ij}}{\partial x^k} - \Gamma^p_{ik} T_{pj} - \Gamma^q_{jk} T_{iq} \right) \frac{\partial x^k}{\partial y^\epsilon} dy^\epsilon$$
$$= \frac{\mathfrak{D} T_{ij}}{\mathfrak{D} x^k} \frac{\partial x^k}{\partial y^\epsilon} \cdot dy^\epsilon .$$

By definition, then, of $\dfrac{\mathfrak{D}}{\mathfrak{D} y^\epsilon}$, we have

$$\frac{\mathfrak{D} T_{ij}}{\mathfrak{D} y^\epsilon} = T_{ij;\epsilon} = \frac{\mathfrak{D} T_{ij}}{\mathfrak{D} x^k} \frac{\partial x^k}{\partial y^\epsilon}$$

Setting $T_{ij} = g_{ij}$ leads to the result that

(16.1)
$$\frac{\mathfrak{D} g_{ij}}{\mathfrak{D} y^\epsilon} = 0$$

<u>Theorem:</u> If $\eta^i_\alpha = \dfrac{\partial x^i}{\partial y^\alpha}$, then $\eta^i_{\alpha;\beta} = \eta^i_{\beta;\alpha}$. This follows immediately from

$$\frac{\mathfrak{D} \eta^i_\alpha}{\mathfrak{D} y^\beta} = \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} + \Gamma^i_{jk} \frac{\partial x^j}{\partial y^\alpha} \frac{\partial x^k}{\partial y^\beta} - \Gamma^\mu_{\alpha\beta} \frac{\partial x^i}{\partial y^\mu}$$

and from the symmetry of the Γ's.

<u>Theorem:</u> The (space) vector $\eta^i_{\alpha;\beta}$ is normal to the tangent space $\left\{ \dfrac{\partial x^i}{\partial y^\alpha} \right\}$.

For since $g_{ij;\epsilon} = \gamma_{\alpha\beta;\epsilon} = 0$, differentiation of $g_{ij} \dfrac{\partial x^i}{\partial y^\alpha} \dfrac{\partial x^j}{\partial y^\beta} = \gamma_{\alpha\beta}$ gives

(16.2)
$$g_{ij} \eta^i_{\alpha;\epsilon} \frac{\partial x^j}{\partial y^\beta} + g_{ij} \eta^j_{\beta;\epsilon} \frac{\partial x^i}{\partial y^\alpha} = 0 .$$

Set $g_{ij} \eta^i_{\alpha;\epsilon} \dfrac{\partial x^j}{\partial y^\beta} = \sigma_{\alpha\epsilon\beta}$. Then (16.2) becomes

$$\sigma_{\alpha\epsilon\beta} = -\sigma_{\beta\epsilon\alpha}$$

And from the preceding theorem

$$\sigma_{\mu\nu\tau} = \sigma_{\nu\mu\tau}$$

Thus

$$\sigma_{\alpha\epsilon\beta} = -\sigma_{\beta\epsilon\alpha} = -\sigma_{\epsilon\beta\alpha}$$

that is, a cyclic permutation of the three indices produces a change in sign. Consequently

$$\sigma_{\alpha\epsilon\beta} = -\sigma_{\epsilon\beta\alpha} = \sigma_{\beta\alpha\epsilon} = -\sigma_{\alpha\epsilon\beta}$$

so that

$$g_{ij}\, \eta^{i}_{\alpha;\epsilon} \frac{\partial x^j}{\partial y^\beta} = 0$$

Consider now a tangent vector λ^i of class C' defined along a curve of class C' in $\mathfrak{M}_r$:

$$\lambda^i = \frac{\partial x^i}{\partial y^\alpha} \lambda^\alpha$$

Taking the absolute differential of both sides, we find that

$$\delta \lambda^i = \frac{\partial x^i}{\partial y^\alpha} \delta \lambda^\alpha + \eta^{i}_{\alpha;\beta} \lambda^\alpha dy^\beta \tag{16.3}$$

The first term on the right is again a tangent vector, while the second is normal to $\mathcal{T}$. Thus $\delta\lambda^i$ will not in general belong to the tangent space.* But with respect to the projection $\underline{\delta}\lambda^i$ we have the

<u>Theorem:</u> For any (differentiable) vector λ^α of $\mathfrak{M}_r$ the components of the surface differential $\delta\lambda^\alpha$ are given by the surface components of the projection upon $\mathcal{T}$ of the space differential $\delta\lambda^i$, where λ^i is the corresponding tangent vector. In brief,

$$\underline{\delta}\lambda^i = \frac{\partial x^i}{\partial y^\alpha} \delta \lambda^\alpha$$

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* This shows that we must distinguish between the space differential, $\delta\lambda^i$, and the surface differential $\delta\lambda^\alpha$ of the corresponding $\mathfrak{M}_r$-vector. However, since $g_{ij;k} = g^{ij}_{\ \ ;k} = g^{i}_{j;k} = 0$, there is no corresponding distinction between differentials of covariant and contravariant representations of the same vector; i.e. if $\delta\lambda^i = \rho^i$, then $\delta\lambda_i = \rho_i$.

This theorem leads at once to a certain generalization of the concept of absolute differential. For let V_t be any t-dimensional vector space of class C' defined along some $\mathfrak{M}_r$ in $\mathcal{R}_n$. The tensors of V_t will be sums of products of vectors of V_t. It is easily seen that the set of tensors of V_t is closed under the fundamental algebraic operations of p. 10. But the differential $\delta T^{\cdots}_{\cdots}$ of a differentiable tensor of V_t will not in general belong to V_t. However, if we take the projection* of $\delta T^{\cdots}_{\cdots}$ on V_t, we have a differentiation process under which V_t is closed.

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* If $S^{i\ldots}_{jk\ldots}$ is a tensor at $P \in \mathfrak{M}_r$ (S need not belong to V_t) then as far as Latin indices are concerned it has an n-Bein representation

$$S^{i\ldots}_{jk\ldots} = \overset{S}{(tuv\ldots)}\, \nu^{i}_{(t)}\, \mu_{(u)j}\, \mu_{(v)k} \ldots$$

Then by definition

$$\underline{S}^{i\ldots}_{jk\ldots} = \overset{S}{(tuv\ldots)}\, \underline{\nu^{i}_{(t)}}\, \underline{\mu_{(u)j}}\, \underline{\mu_{(v)k}} \ldots$$

The definition does not depend on the choice of the n-Bein.

As examples of such vector spaces we have the <u>normal</u> and <u>osculating spaces of an</u> $\mathfrak{M}_r$. We have seen that $\eta^{i}_{\alpha;\beta} = \frac{\delta}{\delta y^\beta}\left(\frac{\partial x^i}{\partial y^\alpha}\right)$, when considered as a space vector, is normal to $\mathcal{T}$ at a point $P \in \mathfrak{M}_r$. We thus have a vector space $\{\eta^{i}_{\alpha;\beta}\}$ (which does not depend on the choice of parameters in $\mathfrak{M}_r$) which we shall call the <u>first normal space</u> of $\mathfrak{M}_r$ at P.

Repeated absolute differentiation gives rise to new invariant vector spaces $\{\eta^{i}_{\alpha;\beta;\gamma}\}\ldots$ etc. Let us adopt the notation $\mathcal{T}_1 = \{\eta^{i}_{\alpha}\}$, $\mathcal{T}_{12} = \{\eta^{i}_{\alpha}, \eta^{i}_{\alpha;\beta}\}$, etc. The space $\mathcal{T}_{1\ldots K+1}$ will be called the <u>K-th osculating space</u> of $\mathfrak{M}_r$ at P.

Now $\eta^i_{|\alpha;\beta;\gamma}$ is in general, neither symmetric in α, β, γ nor normal to $\mathcal{T}_{12} = \{\eta^i_{|\alpha}, \eta^i_{|\alpha;\beta}\}$. However

$$\mathcal{T}_{123} = \{\eta^i_\alpha, \eta^i_{\alpha\beta}, \eta^i_{\alpha\beta\gamma}\} \supset \mathcal{T}_{12} .$$

The set of vectors of $\mathcal{T}_{123}$ which are normal to $\mathcal{T}_{12}$ will clearly be a vector space. It will be called the <u>second normal space</u> $\mathcal{T}_3$.

In general the space of vectors of $\mathcal{T}_{1\ldots k+1}$ which are normal to $\mathcal{T}_{1\ldots k}$ defines the K-th normal space $\mathcal{T}_{k+1}$. $[\mathcal{T}_1 = \{\eta^i_\alpha\}$, $\mathcal{T}_2 = \{\eta^i_{\alpha,\beta}\}]$

17. <u>The Frenet formulas.</u>

We now develop some of the properties of a one-dimensional subspace of $\mathcal{R}_n$ - a curve. The analogous theorems for subspaces of higher dimension will not be given here (cf. W. Mayer, Trans. Amer. Math. Soc. <u>38</u> (1935), p. 267).

We suppose that $m \geq 2$ (the space $\mathcal{R}_n$ is of class at least 2), and that the positive definite metric tensor $g_{ij}(x)$ is of class C^1. Let

$$x_i = x_i(t), \qquad a < t < b$$

represent a regular $\mathfrak{M}_1$ of class C^2 (given parametrically), that is, a curve for which d^2x_i/dt^2 is continuous and $dx_i/dt \neq 0$.

If we make a change of parameter $s = s(t)$, the quantity

$$g_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt}$$

(the metric tensor $\gamma_{\alpha\beta} \equiv \gamma_{11}$ of the subspace; cf. (15.2)) is multiplied by $(dt/ds)^2$. Hence to find a parameter s for which

$$\boxed{g_{ij} \frac{dx_i}{ds} \frac{dx_j}{ds} = 1} \tag{17.1}$$

we must set

$$\frac{ds}{dt} = \sqrt{g_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt}}$$

(or else the negative of the radical may be taken, unless the curve is oriented). Thus for any $a < t_o < b$

$$(17.2) \qquad s(t) = \int_{t_0}^{t} \sqrt{g_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt}}\, dt + \text{const.},$$

and so $s(t)$ is determined up to an additive constant and a choice of sign. Conversely, since ds/dt has a constant sign and d^2s/dt^2 is continuous for $a < t < b$, $s = s(t)$ is an allowable change of parameter. Finally we observe that the set of "arc-length parameters" (cf. 12) defined by (17.2) is independent of the particular parameter t, because arc-length is invariant under change of parameter. Our result is that (17.1) is characteristic for arc-length parameters.

We assume now that the curve

$$x_i = x_i(s), \qquad a_1 < s < b_1$$

is referred to an arc-length parameter s. Of course the $x_i(s)$ are still of class C^2. We denote the tangent vector by

$$\underset{1}{\xi^i} = \frac{dx_i}{ds} .$$

Then $\underset{1}{\xi^i}(s)$ is of class C^1, and by (17.1) $\underset{1}{\xi}$ is a unit vector. In successive steps $n - 1$ other unit vectors $\underset{2}{\xi^i}(s), \ldots, \underset{n}{\xi^i}(s)$ will be constructed - if certain conditions are fulfilled - forming together with $\underset{1}{\xi^i}$ a normalized n-Bein.

By absolute differentiation with respect to s we obtain from $\underset{1}{\xi^i}$ the continuous vector

(17.3) $$\underset{2}{\tilde{\xi}}{}^i = \frac{\delta \xi^i}{\delta \lambda} = \frac{d^2x_i}{ds^2} + \left\{ {p \atop} {i \atop} {q \atop} \right\} \frac{dx_p}{ds} \frac{dx_q}{ds} .$$

Differentiating (17.1) we have

$$g_{ij} \underset{1}{\xi}{}^i \frac{\delta \underset{1}{\xi}{}^j}{\delta \lambda} = 0 ,$$

so that $\underset{2}{\tilde{\xi}}{}^i$ is normal to $\underset{1}{\xi}{}^i$ (this was also proved in the preceding section).

The relation

$$\frac{1}{\rho_1} = \pm \sqrt{g_{ij} \underset{2}{\tilde{\xi}}{}^i \underset{2}{\tilde{\xi}}{}^j}$$

determines a scalar $\frac{1}{\rho_1}$ (s), the "first curvature", as plus or minus the length of $\underset{2}{\tilde{\xi}}{}^i$. That is, the sign of $1/\rho_1$ is as yet undetermined, except of course where $1/\rho_1 = 0$. At this point we exclude from consideration the set of (closed) sub-intervals of the parameter-interval $a_1 < s < b_1$ throughout which $1/\rho_1$ vanishes identically.

The equation

(17.4) $$\underset{2}{\tilde{\xi}}{}^i = \frac{1}{\rho_1} \underset{2}{\xi}{}^i$$

defines a unit vector $\underset{2}{\xi}{}^i$ up to sign, wherever $1/\rho_1 \neq 0$ (assuming that such points exist). If $1/\rho_1 = 0$, $\underset{2}{\xi}{}^i$ is as yet completely undetermined.

Suppose that in a certain sub-interval $a_1' < s < b_1'$ of the original parameter-interval it is possible, by making one of the two choices for $\underset{2}{\xi}{}^i$ where $1/\rho_1 \neq 0$ and making any choice at all of $\underset{2}{\xi}{}^i$ as a unit vector where $1/\rho_1 = 0$, to define $\underset{2}{\xi}{}^i$ (s) so as to be a continuous function of s.* (For example, if $1/\rho_1 \neq 0$ at a certain point

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* If such a continuous function $\underset{2}{\xi}{}^i$ (s) exists, it is unique except for a change of sign. This follows from the fact that if $\lim_{s \to s_0} \underset{2}{\xi}{}^i$ (s)

exists, where s approaches s_0 through values for which $1/\rho_1 \neq 0$ and either choice is made for $\underset{2}{\xi}{}^i$ at each of these values, then the limit is unique up to sign.

---

such a sub-interval can obviously be constructed about that point.) Then $\frac{1}{\rho_1}(s)$, with its sign fixed now by (17.4), will also be continuous for $a_1' < s < b_1'$, since it can be expressed in a neighborhood of each value of s as the quotient of a continuous component of $\underset{2}{\tilde{\xi}}{}^i$ by a continuous, non-vanishing component of $\underset{2}{\xi}{}^i$.

In the general case the "first normal" $\underset{2}{\xi}{}^i(s)$ does not possess a derivative, and our construction stops here. But if we assume that the given curve is such that $\underset{2}{\xi}{}^i(s)$ is of class C^1 in some interval $a_2 < s < b_2$, we can proceed to the next step.

Differentiating $\underset{2}{\xi}{}^i(s)$ absolutely we obtain a continuous vector $\delta\underset{2}{\xi}{}^i/\delta s$. Again, because $\underset{2}{\xi}{}^i$ is a unit vector, $\delta\underset{2}{\xi}{}^i/\delta s$ is normal to $\underset{2}{\xi}{}^i$, but it need not be normal to $\underset{1}{\xi}{}^i$. Let $\underset{3}{\tilde{\xi}}{}^i$ denote the projection (§ 11) of $\delta\underset{2}{\xi}{}^i/\delta s$ onto the normal space of $\underset{1}{\xi}{}^i$ and $\underset{2}{\xi}{}^i$. Then $\delta\underset{2}{\xi}{}^i/\delta s$ is the sum of $\underset{3}{\tilde{\xi}}{}^i$ and a vector in the space spanned by $\underset{1}{\xi}$ and $\underset{2}{\xi}$; that is,

$$(17.5) \qquad \frac{\delta\underset{2}{\xi}{}^i}{\delta s} = A_1\underset{1}{\xi}{}^i + A_2\underset{2}{\xi}{}^i + \underset{3}{\tilde{\xi}}{}^i .$$

Multiplying by $\underset{2}{\xi}{}^i$ and summing with respect to i, and using the fact that $\underset{2}{\xi}{}^i$ is normal to $\delta\underset{2}{\xi}{}^i/\delta s$, $\underset{1}{\xi}{}^i$ and $\underset{3}{\tilde{\xi}}{}^i$, we find that $A_2 = 0$. Similarly if we multiply by $\underset{1}{\xi}{}_i$ and sum we obtain

$$A_1 = \underset{1}{\xi}_i \frac{\partial \underset{2}{\xi}^i}{\partial s} \qquad (17.6)$$

Differentiation of the identity $\underset{1}{\xi}_i \underset{2}{\xi}^i = 0$ yields

$$\underset{1}{\xi}_i \frac{\partial \underset{2}{\xi}^i}{\partial s} = - \frac{\partial \underset{1}{\xi}_i}{\partial s} \underset{2}{\xi}^i$$

Since $\partial \underset{1}{\xi}_i / \partial s = \underset{2}{\tilde{\xi}}_i = \underset{2}{\xi}_i / \rho_1$, (17.6) becomes

$$A_1 = - \frac{1}{\rho_1} \underset{2}{\xi}_i \underset{2}{\xi}^i = - \frac{1}{\rho_1}$$

Consequently (17.5) reduces to

$$\frac{\partial \underset{2}{\xi}^i}{\partial s} = - \frac{1}{\rho_1} \underset{1}{\xi}^i + \underset{3}{\tilde{\xi}}^i \qquad (17.7)$$

The last relation shows that $\underset{3}{\tilde{\xi}}^i$ (s) is continuous throughout the interval $a_2 < s < b_2$. It is normal to $\underset{1}{\xi}^i$ and $\underset{2}{\xi}^i$, but it need not be a unit vector. We therefore normalize again by setting

$$\underset{3}{\tilde{\xi}}^i = \frac{1}{\rho_2} \underset{3}{\xi}^i \qquad (17.8)$$

where $\underset{3}{\xi}^i$ is a unit vector and

$$\frac{1}{\rho_2} = \pm \sqrt{g_{ij} \underset{3}{\tilde{\xi}}^i \underset{3}{\tilde{\xi}}^j}$$

Here we exclude all sub-intervals of $a_2 < s < b_2$ throughout which $1/\rho_2$ vanishes identically.

Once more we suppose that by making one of the two possible choices for $\underset{3}{\xi}^i$ where $1/\rho_2 \neq 0$ and any choice where $1/\rho_2 = 0$, it is possible to define $\underset{3}{\xi}^i$ (s) as a continuous function of s in a certain sub-interval $a_2' < s < b_2'$. Then (17.8) will determine $\frac{1}{\rho_2}$ (s) as a continuous function in the same interval. Beyond this we cannot go in general. We must make the fresh assumption that $\underset{3}{\xi}^i$ (s) is of class C^1; then we can continue as before.

The equations (17.4) and (17.7), or rather

(17.9) $$\frac{\delta \xi_1^i}{\delta s} = \frac{1}{\rho_1} \xi_2^i$$ and

(17.10) $$\frac{\delta \xi_2^i}{\delta s} = -\frac{1}{\rho_1}\xi_1^i + \frac{1}{\rho_2}\xi_3^i ,$$

are the first two Frenet formulas. The general formula can be written as

(17.11) $$\boxed{\frac{\delta \xi_\alpha^i}{\delta s} = -\frac{1}{\rho_{\alpha-1}}\xi_{\alpha-1}^i + \frac{1}{\rho_\alpha}\xi_{\alpha+1}^i} \quad \alpha = 1, 2, \ldots, n,$$

with the convention that $1/\rho_0 = 1/\rho_n = 0$ (i.e., when $\alpha = 1$ the first term, and when $\alpha = n$ the last term, is to be omitted). The vectors $\xi_1^i, \xi_2^i, \xi_3^i, \ldots \xi_n^i$ (the tangent, the first normal, the second normal, ..., the $(n-1)^{st}$ normal) constitute a normalized n-Bein, the "moving polyhedral"; and the scalars $1/\rho_1$, $1/\rho_2$, ..., $1/\rho_{n-1}$ are called the first curvature, the second curvature, ..., the $(n-1)^{st}$ curvature of the given curve. The formulas serve to express the absolute derivatives of the ξ_α^i with respect to arc-length in terms of the vectors of the n-Bein. Of course not all of the n formulas will hold for a given curve unless our successive assumptions with respect to the differentiability of the $\xi_\alpha^i(s)$ and the non-vanishing of the curvatures are satisfied.

The theorem which we are proving may be stated as follows. Suppose that we have a normalized k-Bein, $k \leq n$, of vectors $\xi_1^i(s)$, ... , $\xi_k^i(s)$, and a set of $k - 1$ scalars $1/\rho_1$, ... , $1/\rho_{k-1}$, satisfying (17.11) for $\alpha = 1, \ldots, k-1$. The vectors are to be of class C^1 and the scalars are to be continuous, in an interval $a_k < s < b_k$. Then we can construct a continuous unit vector ξ_{k+1}^i normal to all the preceding vectors, and a continuous scalar $1/\rho_k$, which will satisfy

(17.11) for $\alpha = k$. The new vector and scalar will be defined in sub-intervals $a'_k < s < b'_k$ specified below. (If $k = n$ we merely show that (17.11) holds also for $\alpha = n$).

Proof: The absolute derivative $\delta \underset{k}{\xi}^i / \delta s$ is continuous throughout the given interval. Let $\underset{k+1}{\tilde{\xi}}^i$ represent its projection on the normal space of $\underset{1}{\xi}^i, \dots, \underset{k}{\xi}^i$ (if $k = n$ no normal space and no projection exist). Then there is a representation of the form

$$\frac{\delta \underset{k}{\xi}^i}{\delta s} = \underset{1}{A} \underset{1}{\xi}^i + \dots + \underset{k}{A} \underset{k}{\xi}^i + \underset{k+1}{\tilde{\xi}}^i .$$

If we multiply by $\underset{\alpha}{\xi}_i$ ($\alpha = 1, \dots, k$) and sum, we obtain

$$\underset{\alpha}{A} = \underset{\alpha}{\xi}_i \frac{\delta \underset{k}{\xi}^i}{\delta s} .$$

In virtue of the identities $\underset{\alpha}{\xi}_i \underset{k}{\xi}^i = \delta_{\alpha k}$,

$$\underset{\alpha}{\xi}_i \frac{\delta \underset{k}{\xi}^i}{\delta s} = - \frac{\delta \underset{\alpha}{\xi}_i}{\delta s} \underset{k}{\xi}^i .$$

For $\alpha = k$ this shows that $\underset{k}{A} = 0$. By the α^{th} Frenet formula ($\alpha = 1, \dots, k-2$) $\delta \underset{\alpha}{\xi}_i / \delta s$ is a linear sum of $\underset{1}{\xi}_i, \dots, \underset{k-1}{\xi}_i$, and so $\underset{1}{A} = \dots = \underset{k-2}{A} = 0$. Finally,

$$\underset{k-1}{A} = - \frac{\delta \underset{k-1}{\xi}^i}{\delta s} \underset{k}{\xi}_i = - \frac{1}{\rho_{k-1}} .$$

Consequently

$$\frac{\delta \underset{k}{\xi}^i}{\delta s} = - \frac{1}{\rho_{k-1}} \underset{k-1}{\xi}^i + \underset{k+1}{\tilde{\xi}}^i ,$$

so that $\underset{k+1}{\tilde{\xi}}^i$ (s) is continuous.

Let

$$\underset{k+1}{\tilde{\xi}}^i = \frac{1}{\rho_k} \underset{k+1}{\xi}^i ,$$

where $\underset{k+1}{\xi}^i$ is a unit vector. Let $a'_k < s < b'_k$ be any sub-interval no segment of which is filled by zeros of $1/\rho_k$ and in which $\underset{k+1}{\xi}^i$ can be defined so as to be continuous. Then $1/\rho_k$ is continuous, and the k^{th} Frenet formula holds.

As for a sub-interval in which $1/\rho_k \equiv 0$, (17.11) is correct with $\alpha = k$ if we leave off the last term. The Frenet equations of a curve whose k^{th} curvature vanishes identically are k in number instead of n, and the moving polyhedral is a k-Bein instead of an n-Bein.

The theorem just proved applies to perhaps every curve for which the Frenet formulas can have significance, but for a given curve it would be rather hard to decide from the theorem just how many of the formulas hold along its various arcs. We now present a shorter derivation of the formulas which is good for a more restricted but more simply defined class of curves.

<u>Theorem</u>: Let the class m of the space be at least 2, and let g_{ij} be of class C^1. Let $x_i = x_i(s)$, $a < s < b$ be a regular curve of class C^2, referred to an arc-length parameter s. Suppose that the successive absolute derivatives

$$(17.12) \qquad \frac{\delta x_i}{\delta s} = \frac{dx_i}{ds}, \quad \frac{\delta^2 x_i}{\delta s^2} = \frac{\delta}{\delta s}\left(\frac{\delta x_i}{\delta s}\right), \quad \dots, \quad \frac{\delta^{k+1} x_i}{\delta s^{k+1}}$$

exist and are continuous for $a < s < b$, where k is some fixed integer $\leq n$. Let the first k of these derivatives be linearly independent throughout the interval, but assume that the $(k+1)^{st}$ derivative is linearly dependent on the others in the whole interval (if $k = n$ this is necessarily the case). Then there will exist a normalized k-Bein $\xi^i_1(s), \dots, \xi^i_k(s)$ (where $\xi^i_l = \delta^l x_i/\delta s^l$) and a set of scalars $\frac{1}{\rho_1}(s), \dots, \frac{1}{\rho_{k-1}}(s)$ which satisfy equations (17.11) for $\alpha = 1, \dots, k$ (the final term in the k^{th} equation being omitted). The vectors are of class C^1 and the scalars are continuous for $a < s < b$.

(It should not be supposed that d^3x_i/ds^3, for example, must exist because $\frac{d}{ds}\left(\frac{\delta^2 x_i}{\delta s^2}\right)$ exists — indeed a curve cannot be of class C^3 in a space of class 2. The formula for $\delta^2 x_i/\delta s^2$ was given in (17.3), and it is obvious that a sum can possess a derivative when neither summand does.)

As on p. 63, let $\mathfrak{J}_1$ denote the one-dimensional vector space $\{\delta x_i/\delta s\}$ spanned by the tangent, $\mathfrak{J}_{12}$ the two-dimensional osculating space $\{\delta x_i/\delta s, \delta^2 x_i/\delta s^2\}$, ... , $\mathfrak{J}_{12\cdots k}$ the k-dimensional osculating space $\{\delta x_i/\delta s, \ldots, \delta^k x_i/\delta s^k\}$. Evidently each of these is a proper subspace of the following ones, and the last one contains $\delta^{k+1} x_i/\delta s^{k+1}$.

Let $\underset{1}{\xi^i} \equiv \delta x_i/\delta s$, and let $\underset{\alpha}{\xi^i}$ be the unit vector in the direction determined by projecting $\delta^\alpha x_i/\delta s^\alpha$ onto the normal space of $\mathfrak{J}_{1\cdots\alpha-1}$, $\alpha = 2, \ldots, k$ (the projection does not vanish, as $\delta^\alpha x_i/\delta s^\alpha$ is not contained in $\mathfrak{J}_{1\cdots\alpha-1}$). Since $\delta^\alpha x_i/\delta s^\alpha$, a vector in $\mathfrak{J}_{1\cdots\alpha}$, is the sum of its projection onto $\mathfrak{J}_{1\cdots\alpha-1} (\subset \mathfrak{J}_{1\cdots\alpha})$ and its projection onto the normal space of $\mathfrak{J}_{1\cdots\alpha-1}$, the latter projection must lie in $\mathfrak{J}_{1\cdots\alpha}$. Thus $\underset{1}{\xi^i}, \ldots, \underset{\alpha}{\xi^i}$ constitute a normalized α-Bein spanning $\mathfrak{J}_{1\cdots\alpha}$, $\alpha = 1, \ldots, k$.

The vector space $\mathfrak{J}_{1\cdots\alpha}$ is of class C^1 (§ 11). Hence its projection tensor, and so the projection tensor of its normal space, are of the same class. By hypothesis $\delta^\alpha x_i/\delta s^\alpha$ $(\alpha = 1, \ldots, k)$ must be of class C^1. It follows that the projection of $\delta^\alpha x_i/\delta s^\alpha$ onto the normal space of $\mathfrak{J}_{1\cdots\alpha-1}$, and therefore the unit vector $\underset{\alpha}{\xi^i}$

determined by that (non-vanishing) projection, are of class C^1.

Now ξ_α^i, as a vector of $\mathcal{J}_{1\ldots\alpha}$, is a linear combination of the basis $\partial x_i/\partial s$, ..., $\partial^\alpha x_i/\partial s^\alpha$, with coefficients necessarily of class C^1 (this is seen by solving for the coefficients). Differentiating, we find that $\partial \xi_\alpha^i/\partial s$ is a linear combination of $\partial x_i/\partial s$, ..., $\partial^{\alpha+1} x_i/\partial s^{\alpha+1}$, and so it lies in $\mathcal{J}_{1\ldots\alpha+1}$ (or for $\alpha = k$, in $\mathcal{J}_{1\ldots\alpha}$). Hence

$$\frac{\partial \xi_\alpha^i}{\partial s} = \sum_{\beta=1}^{k} (\alpha\beta)\, \xi_\beta^i, \qquad \alpha = 1, \ldots, k, \tag{17.13}$$

where the coefficients $(\alpha\beta)$ are necessarily continuous and $(\alpha\beta) = 0$ if $\beta > \alpha + 1$. It remains to be shown that the coefficient matrix $\|(\alpha\beta)\|$ has the special Frenet form.

Since the ξ_α^i form a normalized k-Bein,

$$\xi_\alpha^i\, \xi_{\beta i} = \delta_{\alpha\beta}. \tag{17.14}$$

Solving for $(\alpha\beta)$ in (17.13) we find

$$(\alpha\beta) = \frac{\partial \xi_\alpha^i}{\partial s}\, \xi_{\beta i}.$$

From (17.14) we have

$$\frac{\partial \xi_\alpha^i}{\partial s}\, \xi_{\beta i} = -\xi_\alpha^i\, \frac{\partial \xi_{\beta i}}{\partial s} = -\frac{\partial \xi_\beta^i}{\partial s}\, \xi_{\alpha i}.$$

Thus $(\alpha\beta) = -(\beta\alpha)$, and the matrix $\|(\alpha\beta)\|$ is skew-symmetric.

As $(\alpha\beta) = 0$ for $\beta > \alpha + 1$, we must have $(\alpha\beta) = 0$ for $\alpha > \beta + 1$, that is, for $\beta < \alpha - 1$. By the skew symmetry, $(\alpha\beta) = 0$ if $\alpha = \beta$. Hence the only coefficients not proved to vanish are (12), (23), ... , (k-1 k) and their negatives (21), ... , (k k-1). If we write $1/\rho_\alpha$ for (α α+1), α = 1, ... , k-1, we have the required relations (17.11). This completes the proof.

We may add that no one of the curvatures $1/\rho_1$, ... , $1/\rho_{k-1}$ vanishes at any point (while $1/\rho_k \equiv 0$). For the coefficient of $\delta^\alpha x_i/\delta s^\alpha$ in the expression for $\underset{\alpha}{\xi}{}^i$ (which is also the coefficient of $\delta^{\alpha+1} x_i/\delta s^{\alpha+1}$ in the expression for $\delta \underset{\alpha}{\xi}{}^i/\delta s$) is not zero, since $\underset{\alpha}{\xi}{}^i$ is not in $\mathcal{J}_{1 \dots \alpha-1}$. Hence $\delta \underset{\alpha}{\xi}{}^i/\delta s$ is not in $\mathcal{J}_{1 \dots \alpha}$, and ($\alpha$ α+1) $\neq 0$ for $\alpha < k$.

The vectors $\underset{\alpha}{\xi}{}^i$ and the scalars $1/\rho_\alpha$ which we have constructed are unique, in the sense that any other vectors and scalars satisfying the same Frenet relations and having $\underset{1}{\xi}{}^i = \pm\, dx_i/ds$ would be identical with these except for certain changes of sign (for instance, if $\underset{\alpha}{\xi}{}^i$, $1/\rho_{\alpha-1}$ and $1/\rho_\alpha$ are replaced by their negatives the equations are still satisfied). This can be shown by going through the equations in order. Also, if another arc-length parameter $\tilde{s} = \pm\, s$ + const. had been used, the quantities obtained would still have been the same up to sign.

In a space of class at least n+1, with g_{ij} of class C^n, the absolute derivatives (17.12) will certainly exist and be continuous if the curve $x_i(s)$ is of class C^{n+1}. Indeed $\delta^\alpha x_i/\delta s^\alpha$ will be of class $C^{(n+1)-\alpha}$, and if the other hypotheses of the theorem are satisfied the proof shows that $\underset{\alpha}{\xi}{}^i$ will be of class $C^{n-\alpha+1}$ also, α = 1, ... , k (as a matter of fact $\underset{k}{\xi}{}^i$ is of class C^{n-k+2} !). The curvature $1/\rho_\alpha$ will be of class $C^{n-\alpha}$, α = 1, ... , k-1. These statements remain true if n is replaced by any integer not smaller than k.

18. Determination of a curve by its curvatures and initial polyhedral.

We have seen that any curve in the space R_n satisfying certain general conditions determines, at least up to sign, a set of k-1 continuous curvatures, where k is some integer $\leq$ n. We shall prove the following converse.

Theorem: Let the space be of class at least 3, and let g_{ij} be of class C^2. Suppose that

$$(18.1)\qquad \frac{1}{\rho_1}(s), \ldots, \frac{1}{\rho_{k-1}}(s), \qquad a < s < b$$

are any continuous functions no one of which vanishes identically in any sub-interval of $a < s < b$. Let x_i^o be a given point, let $\xi^i_{1\,o}, \ldots, \xi^i_{k\,o}$ be any normalized k-Bein at x_i^o, and take any $a < s_o < b$. Then there exists a curve

$$(18.2)\qquad x_i = x_i(s)$$

of class C^2, having s as an arc-length parameter, and defined in some sub-interval of $a < s < b$ about s_o, which has the functions (18.1) as curvatures, which passes through x_i^o for $s = s_o$, and for which the moving polyhedral takes the position $\xi^i_{\alpha\,o}$ at $s = s_o$. Any other curve with these properties will coincide with (18.2) in the common interval of definition.

Proof: Consider the system of (1+k)n differential equations

$$(18.3)\qquad \left\{\begin{aligned} &\frac{dx_i}{ds} = \xi^i_1, \\ &\frac{d\xi^i_1}{ds} = -\left\{{i \atop p\,q}\right\}\xi^p_1\xi^q_1 + \frac{1}{\rho_1}\xi^i_2, \\ &\frac{d\xi^i_2}{ds} = -\left\{{i \atop p\,q}\right\}\xi^p_2\xi^q_1 - \frac{1}{\rho_1}\xi^i_1 + \frac{1}{\rho_2}\xi^i_3, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ &\frac{d\xi^i_k}{ds} = -\left\{{i \atop p\,q}\right\}\xi^p_k\xi^q_1 - \frac{1}{\rho_{k-1}}\xi^i_{k-1}, \end{aligned}\right.$$

in the (1+k)n unknowns $x^i(s)$, $\xi_1^i(s)$, ..., $\xi_k^i(s)$. This system is in solved, or normal form (we have been careful to write ξ_1^q instead of dx_q/ds). The right members are continuous for any x_i in the topological sphere which is the coordinate system, any ξ_α^i whatever, and any $a < s < b$. They are of class C^1 in the ξ_α^i and the x_i, because the g_{ij} are of class C^2. Consequently we can apply the fundamental existence theorem for such a system of differential equations.

According to this theorem there exists a set of functions $x_i(s)$, $\xi_\alpha^i(s)$ of class C^1 satisfying the equations, which are defined in some sub-interval of $a < s < b$ about s_0, and for which $x_i(s_0) = x_i^0$ and $\xi_\alpha^i(s_0) = \xi_{\alpha\cdot 0}^i$. Any two sets of such functions are identical in their common interval of definition. (We note that the special form of the first equation implies that $x_i(s)$ is of class C^2).

The existence theorem applies to a fixed coordinate system. But if we now assign to the g_{ij}, x_i, ξ_α^i, $1/\rho_\alpha$ and s their usual tensor behavior, the solutions $x_i(s)$, $\xi_\alpha^i(s)$ in one coordinate system will transform under change of coordinates into the solutions in any other. For if we replace ξ_1^q by dx_q/ds and transpose, equations (18.3) become $dx_i/ds = \xi_1^i$ followed by the Frenet equations, and each equation then has the invariant form, <u>vector</u> = <u>vector</u>.

If we knew that the $\xi_\alpha^i(s)$ form a normalized k-Bein all along the curve $x_i(s)$, not merely at s_0, we would now have the theorem. For then s is an arc-length parameter since dx_i/ds is a unit vector, and the other conclusions can be drawn without difficulty.

Hence we must show that the scalars

$$(18.4)\qquad \underset{\alpha\beta}{T}(s) \equiv \underset{\alpha}{\xi}^i(s)\,\underset{\beta}{\xi}_i(s)$$

keep the values $\underset{\alpha\beta}{\delta}$ which they have when $s = s_0$. Calculating from (17.11) we find that

$$\frac{d\underset{\alpha\beta}{T}}{ds} = \frac{\delta \underset{\alpha\beta}{T}}{\delta s} = \frac{\delta \underset{\alpha}{\xi}^i}{\delta s}\underset{\beta}{\xi}_i + \underset{\alpha}{\xi}_i \frac{\delta \underset{\beta}{\xi}^i}{\delta s} =$$

$$\left(-\frac{1}{\rho_{\alpha-1}}\underset{\alpha-1}{\xi}^i + \frac{1}{\rho_\alpha}\underset{\alpha+1}{\xi}^i\right)\underset{\beta}{\xi}_i + \left(-\frac{1}{\rho_{\beta-1}}\underset{\beta-1}{\xi}^i + \frac{1}{\rho_\beta}\underset{\beta+1}{\xi}^i\right)\underset{\alpha}{\xi}_i .$$

Thus

$$(18.5)\qquad \frac{d\underset{\alpha\beta}{T}}{ds} = -\frac{1}{\rho_{\alpha-1}}\underset{\alpha-1\,\beta}{T} + \frac{1}{\rho_\alpha}\underset{\alpha+1\,\beta}{T} - \frac{1}{\rho_{\beta-1}}\underset{\beta-1\,\alpha}{T} + \frac{1}{\rho_\beta}\underset{\beta+1\,\alpha}{T} .$$

If we think of the $\underset{\alpha\beta}{T}$ as independent variables, (18.5) is again a system of differential equations in normal form, and the right members satisfy the conditions of the existence theorem for $a < s < b$ and $\underset{\alpha\beta}{T}$ arbitrary. Now it is easy to verify that $\underset{\alpha\beta}{T}(s) = \underset{\alpha\beta}{\delta}$, $a < s < b$ is a solution of (18.5); for

$$0 = -\frac{1}{\rho_{\alpha-1}}\underset{\alpha-1\,\beta}{\delta} + \frac{1}{\rho_\alpha}\underset{\alpha+1\,\beta}{\delta} - \frac{1}{\rho_{\beta-1}}\underset{\beta-1\,\alpha}{\delta} + \frac{1}{\rho_\beta}\underset{\beta+1\,\alpha}{\delta} ,$$

$\alpha, \beta = 1, \ldots, k$. But then the existence theorem asserts that the solution (18.4) must equal $\underset{\alpha\beta}{\delta}$ as far as it is defined, and this is what we had to prove.

The totality of curves in R_n obtained by integration of the Frenet equations for all possible choices of the curvatures $1/\rho_\alpha$ in (18.1), will be found to coincide precisely with the totality of curves for which the Frenet equations can be derived according to the first procedure explained in § 17. In other words, our first derivation of the Frenet equations and the result of the present section are a theorem and its converse, at least when g_{ij} is of class C^2.

19. Coordinate spaces.

Before going further we wish to define a class of spaces of a more general character than the space $\mathcal{T}_n$, to which almost all of the preceding work applies without much change. By definition $\mathcal{T}_n$ was homeomorphic to a euclidean sphere, but a "coordinate space" (we still call it $\mathcal{T}_n$) can have a fairly arbitrary topological structure.

To begin with, $\mathcal{T}_n$ is a (connected) Hausdorff space. That is, $\mathcal{T}_n$ is a set of elements called points, and to each point there correspond certain sets of points called neighborhoods satisfying the well-known Hausdorff axioms. In effect, the axioms enable us to define open sets, limit points, continuity and similar notions of point set theory.

Let P be any point of $\mathcal{T}_n$. We assume that there exists a certain neighborhood of P, U(P), which is homeomorphic to the interior of a topological sphere in n-dimensional number space E_n. We call U(P) a "coordinate neighborhood" of P, because coordinates $x_1, \dots, x_n$ can be carried over from the points of the topological sphere to the corresponding points of U(P). Of course for each U(P) this can be done in infinitely many ways, depending on the choice of the topological sphere and of the homeomorphism.

Now corresponding to each U(P) in $\mathcal{T}_n$ let a definite choice be made of a topological sphere S in E_n and of a homeomorphism between S and U(P). This assigns a fixed coordinate system to each coordinate neighborhood.

Two coordinate neighborhoods which have common points intersect in an open point set. In the intersection there is defined a coordinate transformation

$$x_i' = x_i'(x_1, \ldots , x_n) \quad \text{or} \quad x_i = x_i(x_1', \ldots , x_n').$$

We assume that it has been possible to make the choice described in the preceding paragraph in such a way that the functions $x_i'(x)$ and $x_i(x')$ in every such transformation are of class C^m, where m is a fixed integer $\geqq 1$. Just as in § 1 it can be shown that this is equivalent to requiring the concept "function of class C^ρ" (defined in an open set) to be independent of particular coordinate neighborhoods for $\rho \leqq m$.

Any homeomorphism between an open set in n-dimensional number space and an open set $\mathfrak{A}$ in γ_n gives rise to a "general coordinate system" in $\mathfrak{A}$. If the transformation of coordinates to every U(P) intersecting $\mathfrak{A}$ is of class C^{m_1} both ways, where $m_1 \leqq m$, we speak of an "m_1-system" in $\mathfrak{A}$. If $m_1 = m$ we have an m-system or a "<u>coordinate system</u>". Again, the m_1-systems can be characterized by the property of preserving class for functions of class C^ρ, $\rho \leqq m_1$.

The coordinate space or "coordinate manifold" γ_n is said to be an "m-space" or to be "of class m". Evidently a single Hausdorff space may give rise to infinitely many distinct m-spaces (distinct in the sense of having different totalities of coordinate systems).

20. The Riemannian Space with Congruence Transformation.

In a Riemannian space R_n of class C^m let us consider a point transformation

$$P \longleftrightarrow P'.$$

This may be written as

$$x_i' = x_i'(x_1, \ldots, x_n), \; x_i = x_i(x_1', \ldots, x_n'), \; i = 1, \ldots, n \tag{20.1}$$

where x and x' are coordinates in coordinate neighborhoods $\mathcal{U}(P)$ and $\mathcal{U}(P')$ respectively. This transformation will be called a congruence if the metric tensor g_{ik} transforms under it according to the tensor law of transformation for a change of coordinates, i.e. if

$$\left\{ \begin{aligned} g'_{ik}(x') &= \frac{\partial x_p}{\partial x_i'} \frac{\partial x_q}{\partial x_k'} g_{pq}(x) \\ g_{ik}(x) &= \frac{\partial x_p'}{\partial x_i} \frac{\partial x_q'}{\partial x_k} g'_{pq}(x') . \end{aligned} \right. \tag{20.2}$$

By a geometric figure is meant a point set in R_n with a set of tensors defined at its points. If F is a geometric figure in $\mathcal{U}(P)$ we then transform it into a corresponding figure F' in $\mathcal{U}(P')$ under (20.1) in such a manner that the components of the tensors associated with F will transform under the congruence transformation (20.1) as if it were a coordinate transformation. But according to (20.2) the metric tensor g_{ik} will also transform as a tensor under congruence transformation. Hence any invariant of F considered as a simultaneous invariant of the figure and the metrical tensor will equal the corresponding invariant of F'. When this last statement is true F and F' will be called congruent.

Theorem. If any geometric figure F and its transform F' are congruent (in the above sense) then equations (20.2) hold.

To show this, take two contravariant vectors λ^i and μ^i as functions of x. Then

$$g_{ik}(x)\,\lambda^i\mu^k = g'_{ik}(x')\,\lambda'^i\mu'^k$$

But

$$g_{ik}(x)\,\lambda^i\mu^k = g_{ik}(x)\frac{\partial x_i}{\partial x'_p}\lambda'^p\frac{\partial x_k}{\partial x'_q}\mu'^q.$$

Hence

$$g'_{ik}(x') = \frac{\partial x_i}{\partial x'_p}\frac{\partial x_k}{\partial x'_q}\,g_{pq}(x).$$

Obviously the set of all congruence transformations form a group which is called the group of motions. In general this group need have only one element, the identity.

Remark. In some cases we consider only transformations which transform some particular subspace of the R_n into a congruent subspace. As an example, consider the Euclidean torus, and the group of all rotations about a point P of it for a restricted neighborhood of it. There exists no rotation for the whole space.

21. Euclidean Space.

The Euclidean n-space E_n is defined to be a Riemannian n-space R_n which can be covered by a single coordinate system (x), $-\infty < x_i < +\infty$, in which $g_{ik} = \delta_{ik}$. Such a coordinate system is called a rectangular Cartesian system. Not only one such system will exist in an E_n but a set of them; the transformation between any two such systems, (x) and (x') being

$$x'_i = a_{ik}\,x_k + b_i, \quad a_{ik}\,a_{ij} = \delta_{kj}. \tag{21.1}$$

In fact,

$$g_{ik}(x)\frac{\partial x_i}{\partial x'_p}\frac{\partial x_k}{\partial x'_q} = \delta_{ik}a_{ip}a_{kq} = a_{ip}a_{iq} = \delta_{iq} = g_{iq}(x').$$

Obviously (21.1) may be considered as a congruence transformation, and thus under it any geometric figure is transformed into a congruent one.

Theorem. The group of congruence transformations (21.1) contains one and only one transformation which transforms a given normalized n-Bein at a point P into a given normalized n-Bein at a point Q.

We have to solve (21.1) and

(21.1') $$\overline{\underset{\alpha}{\xi}}_i = a_{ik}\underset{\alpha}{\xi}_k ,^{*} \text{ where}$$

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* No distinction need be made between co- and contra-variant indices when dealing with rectangular Cartesian coordinates since $\xi_i = \delta_{ik}\xi^k$.

$$\underset{\alpha}{\xi}^i \underset{\beta}{\xi}_i = \delta_{\alpha\beta} , \quad \overline{\underset{\alpha}{\xi}}^i \overline{\underset{\beta}{\xi}}_i = \delta_{\alpha\beta} .$$

Multiplying (21.1') by $\underset{\alpha}{\xi}_j$ we get

(21.2) $$a_{ij} = \overline{\underset{\alpha}{\xi}}_i \underset{\alpha}{\xi}_j ;$$

from (21.1) and the fact that P transforms into Q, b_i can be found. From (21.2) we get

$$a_{ij}a_{kj} = \overline{\underset{\alpha}{\xi}}_i \overline{\underset{\beta}{\xi}}_k \underset{\alpha}{\xi}_j \underset{\beta}{\xi}_j = \overline{\underset{\alpha}{\xi}}_i \overline{\underset{\alpha}{\xi}}_k = \delta_{ik} ;$$

thus the transformation obtained is a congruence transformation.

Theorem. By a given set of curvatures $\frac{1}{\underset{\alpha}{\rho}(s)}$, $\alpha = 1, \dots k$, functions of s, an arc is given up to a congruence.

Proof. Let C and C' be two arcs containing points P and P' respectively, which have identical curvatures as functions of the arc length s;

$$C; \quad x_i(s), \qquad 0 \leqq s \leqq a,$$

$$C': \quad x'_i(s), \qquad 0 \leqq s \leqq a,$$

$$x_i(0) = P, \quad x'_i(0) = P'.$$

By the preceeding theorem a congruence (3) will exist transforming the k-Bein $\underset{\alpha}{\xi}_i$ at P into the corresponding $\underset{\alpha}{\xi}'_i$ at P'. (If $k < n$ there will exist an infinity of such congruences). This congruence will transform C into an arc C" having the same $\frac{1}{\rho_\alpha(s)}$ and the same initial values at P' as C'; by a preceeding result C' and C" will coincide.

<u>Theorem</u>. If $\frac{1}{\rho_k(s)} = 0$ the arc given by the $\frac{1}{\rho_\alpha(s)}$, $\alpha = 1, \ldots, k$, will lie in a k-plane, E_k. By a k-plane is meant a k-dimensional subspace of E_n given by linear equations in the rectangular Cartesian coordinates of E_n. First we integrate the Frenet equations in an E_k using rectangular Cartesian coordinates to get a curve $x_i(s)$, $\underset{\alpha}{\xi}^i(s)$, $\alpha = 1, \ldots, k$, $i = 1, \ldots, k$. Then $x_i(s)$, $\underset{\alpha}{\xi}^i(s)$, $\alpha = 1, \ldots, k$, $i = 1, \ldots, n$, where $x_i = 0$, $\underset{\alpha}{\xi}^i = 0$ for $i = k+1, \ldots, n$, will satisfy the Frenet equations

$$\frac{d\underset{\alpha}{\xi}^i}{ds} = -\frac{1}{\rho_{\alpha-1}}\underset{\alpha-1}{\xi}^i + \frac{1}{\rho_\alpha}\underset{\alpha+1}{\xi}^i, \quad \alpha = 1, \cdots, k; \ i = 1, \cdots, n,$$

$$\frac{1}{\rho_0} = \frac{1}{\rho_k} = 0,$$

in the E_n. Since $\sum_1^k \underset{\alpha}{\xi}^i \underset{\beta}{\xi}^i = \sum_1^n \underset{\alpha}{\xi}^i \underset{\beta}{\xi}^i = \delta_{\alpha\beta}$, $\underset{\alpha}{\xi}^i$, $i = 1, \cdots, n$, is a normalized k-Bein and hence the $\frac{1}{\rho_\alpha(s)}$ are the curvatures of the arc in E_n. Thus there exists in E_n a curve with $\frac{1}{\rho_\alpha(s)}$, $\alpha = 1, \ldots, k$, as curvatures which lie in a k-plane E_k. Any other curve $\tilde{C}$ having these curvatures can be obtained by a congruence transformation. Such a congruence transformation will transform the E_k in which the first curve lies into a k-plane in which $\tilde{C}$ will lie, and the theorem is proved.

Remark. The theorems of this section are true for spaces of constant curvature, but because of the lack of rectangular Cartesian coordinate systems, the proofs are more intricate.

CHAPTER IV

PARALLEL DISPLACEMENT.

22. Parallel Displacement of Tensors in a Riemannian Space.

Let $T(t)$ be any tensor defined along an arc $C(t)$ given by

$$x_i = x_i(t), \qquad a \leqq t \leqq b.$$

Let $T(t)$ and $C(t)$ both be of class C^{ρ}, $\rho \geqq 1$, and $g_{ik}(x)$ be of class C^{σ}, $\sigma \geqq 1$. Then $\mathcal{D} T$, written symbolically as

$$\mathcal{D} T = d\,T \pm \Gamma\, T\, dx,$$

is a tensor of the same kind as T which is defined along $C(t)$ and is of class C^{τ}, τ being the smaller of $\rho - 1$ and $\sigma - 1$. If $\mathcal{D} T = 0$, T is said to be a tensor which is parallel-displaced along $C(t)$.

The system $\mathcal{D} T = 0$, i.e.

$$\frac{dT}{dt} = \pm \Gamma\, T\, \frac{dx}{dt}, \tag{22.1}$$

is a system of ordinary differential equations in the normal form, with T as unknowns. Because the Lipschitz condition holds we can state that there exists one and only one solution $T(t)$ with given initial values $T(a)$ for $a \leqq t \leqq b$. Hence the

Theorem. If a tensor $T(a)$ is given at some point a of a curve $C(t)$ of class $\geqq 1$, there will exist a unique parallel-displaced tensor along $C(t)$ having $T(a)$ as initial values.

Since $\mathcal{D} T = 0$ is invariant under coordinate transformations the set of coordinate systems chosen to cover $C(t)$ is immaterial.

As examples of parallel-displaced tensors we mention the following: g_{ik}; g^{ik}; $g^i_k = \delta^i_k$; φ, a constant scalar; any tensor identically zero; and, as will be seen later, $\sqrt{g}\,\eta_{i_1 \ldots i_n}$ in an oriented R_n.

The following statements are immediately obvious:

a) If T and S are tensors of the same kind which are parallel-displaced along C(t), T + S is such a tensor.

b) If T and S are any two parallel-displaced tensors along C(t) then T S is such a tensor.

c) If $T^{i\,\cdots}_{j\,\cdots}$ is a parallel-displaced tensor along C(t) then the contracted tensor $T^{i\,\cdots}_{i\,\cdots}$ is such a tensor.

Let $T_1(t), \ldots, T_r(t)$ be r tensors of the same kind which are parallel-displaced along C(t) and let $a_\alpha T_\alpha(t_o) = 0$ for some t_o and some set a_α not all zero. From (a), (b) and the fact that $\delta a_\alpha = 0$ for constant a_α, we see that $a_\alpha T_\alpha(t)$ with constant a_α is a parallel-displaced tensor along C(t) which is the zero tensor at t_o. Since the zero tensor along C(t) is a parallel-displaced tensor and since it and $a_\alpha T_\alpha(t)$ have the same initial values at t_o, the two are identical and

$$a_\alpha T_\alpha(t) = 0$$

along C(t). Hence the

Theorem. <u>If parallel-displaced tensors of the same kind are linearly dependent at some point they are linearly dependent at all points.</u>

Let λ^i and μ^i be parallel-displaced tensors along a curve C(t). Since g_{ik} is parallel-displaced, $\varphi = g_{ik}\lambda^i\mu^k$ is parallel-displaced; hence $d\varphi = 0$ and φ is constant. Thus <u>lengths and angles of parallel displaced vectors are constant.</u>

Because of this, a normalized n-Bein will remain a normalized n-Bein under parallel displacement. Let $\underset{\alpha}{\lambda}_i(t), \alpha = 1,\ldots,n,$

be such a parallel-displaced normalized n*bein along $C(t)$ such that $\left|\underset{\alpha}{\lambda}{}_i(t_0)\right| > 0$ for some t_0. Since $\left|\underset{\alpha}{\lambda}{}_i(t)\right|$ is constant, $\left|\underset{\alpha}{\lambda}{}_i(t)\right| > 0$ along $C(t)$. We have previously seen that, in an oriented R_n,

$$\sqrt{g}\,\eta_{i_1\cdots i_n} = \begin{vmatrix} \underset{1}{\lambda}{}_{i_1} & \cdots & \underset{1}{\lambda}{}_{i_n} \\ \vdots & & \vdots \\ \underset{n}{\lambda}{}_{i_1} & \cdots & \underset{n}{\lambda}{}_{i_n} \end{vmatrix};$$

hence $\sqrt{g}\,\eta_{i_1 \cdots i_n}$ is parallel-displaced along $C(t)$.

Let $T^{i\cdots}_{j\cdots}$ be a parallel-displaced tensor along $C(t)$. At t_0 it can be expressed in terms of the above parallel-displaced n-Bein $\underset{\alpha}{\lambda}{}^i(t)$ and its adjoint n-Bein $\underset{\alpha}{\lambda}{}_i(t)$:

$$T^{i\cdots}_{j\cdots} = {}_{\alpha\cdots}T_{\beta\cdots}\, \underset{\alpha}{\lambda}{}^{i}\cdots \underset{\beta}{\lambda}{}_j\cdots \qquad \text{at } t_0. \tag{22.2}$$

The right side of this equation, for constant ${}_{\alpha\cdots}T_{\beta\cdots}$, will be a parallel-displaced tensor of the same kind as $T^{i\cdots}_{j\cdots}$ with the same initial values at t_0. Thus (22.2) holds, with constant ${}_{\alpha\cdots}T_{\beta\cdots}$, for any t. By (22.2), then, the study of parallel-displaced tensors reduces to that of parallel-displaced vectors. For vectors λ^i and λ_i the equations for parallel displacement are

$$\begin{aligned} d\lambda^i &= -\Gamma^i_{pq}\,\lambda^p\, dx_q \\ d\lambda_i &= \Gamma^p_{iq}\,\lambda_p\, dx_q . \end{aligned} \tag{22.3}$$

23. The curvature tensor.

Take a $\mathfrak{M}_2$ element lying in one coordinate system, which may be defined by

$$x_i = x_i(t, \epsilon).$$

Let a contravariant vector $\lambda^i(t, \epsilon)$ be defined at each point of $\mathfrak{M}_2$. Let $\frac{\partial x_i}{\partial t}, \frac{\partial x_i}{\partial \epsilon}, \frac{\partial^2 x_i}{\partial t \partial \epsilon}, \frac{\partial \lambda^i}{\partial t}, \frac{\partial \lambda^i}{\partial \epsilon}$, and $\frac{\partial^2 \lambda^i}{\partial t \partial \epsilon}$ exist and be continuous. Then

$$\frac{\delta \lambda^i}{\delta t} = \frac{\partial \lambda}{\partial t} + \Gamma^i_{pq} \lambda^p \frac{\partial x_q}{\partial t} \qquad \text{and}$$

$$\frac{\delta}{\delta \epsilon}\left(\frac{\delta \lambda^i}{\delta t}\right) = \frac{\partial}{\partial \epsilon}\left(\frac{\partial \lambda^i}{\partial t} + \Gamma^i_{pq}\lambda^p \frac{\partial x_q}{\partial t}\right) + \Gamma^i_{rs}\left(\frac{\partial \lambda^r}{\partial t} + \Gamma^r_{lm}\lambda^l \frac{\partial x_m}{\partial t}\right)\frac{\partial x_s}{\partial \epsilon}$$

$$= \frac{\partial^2 \lambda^i}{\partial \epsilon \partial t} + \frac{\partial \Gamma^i_{pq}}{\partial x_k}\frac{\partial x_k}{\partial \epsilon}\lambda^p \frac{\partial x_q}{\partial t} + \Gamma^i_{pq}\frac{\partial \lambda^p}{\partial \epsilon}\frac{\partial x_q}{\partial t}$$

$$+ \Gamma^i_{pq}\lambda^p \frac{\partial^2 x_q}{\partial t \partial \epsilon} + \Gamma^i_{rs}\frac{\partial \lambda^r}{\partial t}\frac{\partial x_s}{\partial \epsilon} + \Gamma^i_{rs}\Gamma^r_{lm}\lambda^l \frac{\partial x_m}{\partial t}\frac{\partial x_s}{\partial \epsilon};$$

Hence

$$\frac{\delta}{\delta \epsilon}\frac{\delta \lambda^i}{\delta t} - \frac{\delta}{\delta t}\frac{\delta \lambda^i}{\delta \epsilon} = R^i_{pkq}\lambda^p \frac{\partial x_k}{\partial \epsilon}\frac{\partial x_q}{\partial t} \tag{23.1}$$

where

$$R^i_{pkq} = \frac{\partial \Gamma^i_{pq}}{\partial x_k} - \frac{\partial \Gamma^i_{pk}}{\partial x_q} + \Gamma^i_{rk}\Gamma^r_{pq} - \Gamma^i_{rq}\Gamma^r_{pk}. \tag{23.2}$$

We can always find an $\mathfrak{M}_2$ element and a vector $\lambda^p(t, \epsilon)$ such that $\frac{\partial x_p}{\partial t}$, $\frac{\partial x_p}{\partial \epsilon}$ and λ^p assume arbitrary values. Hence, since the left side of (23.1) is a tensor, R^i_{pkq}, is a tensor, called the curvature tensor.

24. Fields of Parallel Vectors in Non-analytic Manifolds in the Large.

This section will consist of a reprint of a forthcoming paper by W. Mayer and T. Y. Thomas which bears the same title as does this section.

Let $\mathfrak{M}$ be a coordinate manifold of class C^r. We assume $\mathfrak{M}$ to be connected in the topological sense and to admit an affine connection L with components $L^{\alpha}_{\beta\gamma}$ which are of class C^s as functions of the coordinates of $\mathfrak{M}$. Evidently we must have $s \leq r - 2$ from the equations of transformation of the components of the connection L.

In the following paper we shall consider the question of the existence of fields of parallel vectors defined over open point sets in $\mathfrak{M}$. This problem has been previously discussed under the analytic hypothesis by T. Y. Thomas, <u>Fields of Parallel Vectors in the Large,</u> Compositio Mathematica, 3, 1936, pp. 453-468. Indeed one of our objects will be to exhibit the essential differences between the analytic and non-analytic cases and to indicate why the algebraic characterization obtained under the analytic hypothesis can probably not be extended to the case of coordinate manifolds of class C^r.

24.1. It will be sufficient to assume for our purpose that the above integer s is $\geq n + 1$ where n is the dimensionality of $\mathfrak{M}$.

Analytically we are concerned with the existence of solutions of the system of linear partial differential equations which define the field of parallel vectors in $\mathfrak{M}$. But our discussion will apply likewise to any invariantive system of linear equations in $\mathfrak{M}$ with the x coordinates of $\mathfrak{M}$ as independent variables and with unknowns which may be scalars, the components of tensors, etc. In other words our methods are representative of the treatment to be applied to this general category of differential equations defined over the manifold $\mathfrak{M}$. We shall define regular points in the following manner. Consider the set of equations

$$(E_0) \qquad \xi^{\mu} B^{\alpha}_{\mu\beta\gamma} = 0$$

$$(E_1) \qquad \xi^{\mu} B^{\alpha}_{\mu\beta\gamma,\delta_1} = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(E_n) \qquad \xi^{\mu} B^{\alpha}_{\mu\beta\gamma,\delta_1,\ldots,\delta_n} = 0$$

where the B's denote the components of the curvature tensor and its successive covariant derivatives. A point P of $\mathfrak{M}$ will be said to be regular with respect to the system $E_0 + \ldots + E_t$ where $t \leqq n-1$ if there exists a neighborhood U(P) in which the rank of the matrix M of the B's of this system is constant. All other points of $\mathfrak{M}$ will be said to be singular with respect to this system. By this definition it is obvious that the regular points form an open point set and it is easily seen that the singular points are no where dense. To prove this last statement we observe that if P is a singular point any neighborhood U(P) contains a point Q such that the rank of M at Q is greater than the rank of M at P. In fact there exists a neighborhood $U'(P) \subset U(P)$ in which the rank of M is at least equal to its rank at P. If U'(P) did not contain a point Q_1 at which the rank of M is greater than the rank of M at P then P by definition would be a regular point. If Q_1 is a regular point the proof is complete. If Q_1 is a singular point then by the above argument U(P) considered as a neighborhood of Q_1 will contain a point Q_2 such that the rank of M at Q_2 is greater than the rank of M at Q_1. Continuing we obtain a finite sequence of point Q_1, Q_2, Q_3, ... in U(P) such that the rank of M at at Q_α is greater than the rank of this matrix at $Q_{\alpha-1}$. This sequence must contain a first point Q_m which is regular since a point at which the rank of M is n is necessarily regular. Since Q_m is regular there exists a neighborhood $U(Q_m) \subset U(P)$ which consists entirely of regular points. Hence the singular points are nowhere dense.

Let us denote by R_t the set of points regular with respect to the system $E_0 + \ldots + E_t$ for $t = 0, \ldots, n-1$ and by S_t the corresponding sets of singular points. The points of the intersection

$$\mathcal{R} = R_0 \cap R_1 \cap \ldots \cap R_{n-1}$$

will merely be said to be regular. A point of $\mathfrak{M}$ not in the set R will be said to be <u>singular</u>. Denoting by S the set of singular points in $\mathfrak{M}$ this set is the logical sum

$$S = S_0 + S_1 + \ldots + S_{n-1}.$$

Since the intersection of a finite number of open point sets is open it follows that $\mathcal{R}$ is an open point set. Also S is nowhere dense since it is the sum of a finite number of sets each of which is nowhere dense. To prove this let P be any point of S and U(P) any neighborhood of P. Then there is a neighborhood $U_0 \subset U(P)$ composed entirely of points of R_0 since S_0 is nowhere dense. Again there is a neighborhood $U_1 \subset U_0$ such that U_1 contains only points of R_1 and so only points of R_0. Finally we get a neighborhood U_{n-1} containing only points of the intersection of $R_0, R_1, \ldots, R_{n-1}$, i.e. of R.

24.2. We shall now show that if P is a regular point in $\mathfrak{M}$ the set of equations E_n is linearly dependent on the set $E_0 + \ldots + E_{n-1}$ at P and in fact that there exists a neighborhood U(P) of regular points in $\mathfrak{M}$ in which one can find equations with continuous coefficients expressing this dependence.

Case I. If all the coefficients B of the set of equations E_o are zero at the point P then these coefficients vanish identically in some neighborhood U(P) since P is a regular point in $\mathfrak{M}$. Hence in U(P) the coefficients B of all the equations $E_1, \ldots, E_n$ will vanish and the above statement is therefore valid.

Case II. Let E_t for t = 1, ... , n-1 be the first set of equations possessing the property that it is linearly dependent on the set $E_o + \ldots + E_{t-1}$ at P. Then the ranks of the matrices of the systems $E_o + \ldots + E_{t-1}$ and $E_- + \ldots + E_t$ will be equal at P. Call this rank r. Since P is a regular point there will be a neighborhood $U_1(P)$ in which the matrices of the above systems have the constant rank r. Any r independent equations of the system $E_o + \ldots + E_{t-1}$ at P will be independent in some neighborhood $U(P) \subset U_1(P)$ and in this neighborhood U(P) we can express the set E_t linearly in terms of the above independent equations with coefficients which have the same properties of continuity and differentiability as the coefficients of the set E_t. Hence we can find tensor relations of the form[1)]

---

1) These equations may be taken to represent the dependence of the set E_t on the above r independent equations of the system $E_o + \ldots + E_{t-1}$ in the coordinate system under consideration, those A's which do not correspond to these r independent equations having the value zero. To obtain these relations in any coordinate system we have merely to transform the A's as the components of tensors as indicated by their indices.

$$B^{\alpha}_{\mu\beta\gamma,\delta_1\cdots\delta_t} = \sum_{s=0}^{s=t} A^{pq r_1\cdots r_s}_{h\beta\gamma\delta_1\cdots\delta_t} B^{h}_{\mu pq,r_1,\cdots,r_s} \tag{24.1}$$

valid in U(P) and having coefficients A continuous and with continuous partial derivatives to the order n - t inclusive. By covariant differentiation of these relations we see that each of the sets of equations $E_{t+1}, \ldots, E_n$ can be expressed linearly in U(P) in terms of the set $E_o + \ldots + E_{t-1}$ with continuous coefficients.

Case III. The set $E_o + \ldots + E_{n-1}$ contains n independent equations at P. Then the set E_n is evidently dependent on the set $E_o + \ldots + E_{n-1}$ at P. Since P is a regular point the rank of the matrix of the set $E_o + \ldots + E_{n-1}$ will be n in some neighborhood $U_1(P)$. As in case II the set E_n will be linearly dependent on the set $E_o + \ldots + E_{n-1}$ in some neighborhood $U(P) \subset U_1(P)$ and the coefficients of the equations expressing this dependence will be continuous functions of the coördinates in this neighborhood.

Since one of the above three cases must occur we see that for any regular point P in $\mathfrak{M}$ there exists a neighborhood U(P) in which the equations (24.1) for t = n are valid with continuous coefficients A. This neighborhood U(P) can of course be taken to be a neighborhood composed entirely of regular points in $\mathfrak{M}$.

In the following the above property of regular points in $\mathfrak{M}$ is the only one of which use will be made. A singular point in $\mathfrak{M}$ which also possesses this property will be called a non-essential singular point in $\mathfrak{M}$. All other singular points will be called essential singular points. Obviously the set composed of all regular and non-essential singular points in $\mathcal{M}$ is open and its complement, i.e. the essential singular points in $\mathcal{M}$ are nowhere dense.

24.3. Consider the open set R of regular and non-essential singular points in $\mathcal{M}$. By the component of a point P of R we mean the greatest open connected point set in R which contains the point P. We denote such a component by K(P). If $Q \subset K(P)$ then obviously $K(Q) = K(P)$. Thus the set R is divided into a finite or infinite number of components K(P) with boundaries composed of essential singular points.

Let C(t) for $0 \leqq t \leqq 1$ be a continuous arc (continuous map of the unit interval) in a particular component K(P). Along this arc the set of equations E_n can be represented linearly in terms of the set $E_0 + \dots + E_{n-1}$ with coefficients (components of tensors A) which are continuous functions of t in the interval $0 \leqq t \leqq 1$ (irrespective of coordinate transformations). To prove this take any value of $t = t'$ which will then correspond to a point C(t') of the arc. Since C(t') is a regular or non-essential singular point in $\mathcal{M}$ there exists a neighborhood $U \subset R$ in which the equations

$$(24.2) \qquad B^{\alpha}_{\mu\beta\gamma,\delta_1,\dots,\delta_n} = \sum_{s=0}^{n-1} A^{pq r_1\cdots r_s}_{h\beta\gamma\delta_1\cdots\delta_n} B^{h}_{\mu pq,r_1\cdots r_s}$$

are valid with A's which are continuous functions of the coordinates. From the fact that the arc is a continuous map of the unit interval there will be some t-interval containing t' whose map lies entirely in the neighborhood U and for this t-interval equations (24.2) will hold with A's which are continuous functions of t. Corresponding to any value of $t = t'$ such that $0 \leqq t' \leqq 1$ there exists an interval containing t in which the above statement is true. The whole interval $0 \leqq t \leqq 1$ can now be covered by a finite number of the above t-intervals $N_1, \dots, N_m$ corresponding to increasing values of the variable t. We shall now construct from these m representations (24.2) of the equations E_n a single continuous representation valid for the entire interval $0 \leqq t \leqq 1$ as above stated.

Consider two successive t-intervals N_p and N_{p+1} and let t_p and t_{p+1} where $t_{p+1} > t_p$ be two values of the variable t lying in the intersection $N_p \cap N_{p+1}$. Obviously the entire interval $t_p \leqq t \leqq t_{p+1}$ lies in the intersection. We shall define a representation (24.2) of E_n in this intersection which will continue in a continuous manner the representation of E_n in N_p from the right and the representation of E_n in N_{p+1} from the left. Denote briefly by $B_n = \sum A B$ and $B_n = \sum \bar{A} B$ the above representations in the intersection $N_p \cap N_{p+1}$ with respect to a single coordinate system covering this intersection. If $F(t)$ is any continuous function of t in the interval $t_p \leqq t \leqq t_{p+1}$ then $B_n = \sum [(1-F)A + F\bar{A}]B$ will give a (continuous) representation of E_n in the interval. We have now merely to choose $F(t_p) = 0$ and $F(t_{p+1}) = 1$ to obtain the above continuation of the representation of E_n. By proceeding along the intervals $N_1, \dots, N_m$ is succession we obtain the desired representation for the arc $C(t)$.

24.4. We shall now show that any solution vector ξ of the system $E_0 + \dots + E_{n-1}$ at any point Q of a particular component K(P) will result in a solution vector ξ' at any other point Q' of this component by parallel displacement of the vector ξ at Q along an arc of class C^1 lying in K(P). Let C(t) where $0 \leq t \leq 1$ be the arc joining Q to Q' so that C(0) = Q and C(1) = Q'. By this parallel displacement we will obtain a vector $\xi(t)$ on the arc C with components $\xi^\alpha(t)$ of class C'. On C(t) put

$$
(24.3) \quad \begin{cases} S^\alpha_{\beta\gamma} = \xi^\mu B^\alpha_{\mu\beta\gamma}, \\ S^\alpha_{\beta\gamma\delta_1} = \xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1}, \\ - \quad - \quad - \quad - \quad - \quad - \\ S^\alpha_{\beta\gamma\delta_1\cdots\delta_{n-1}} = \xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1,\cdots,\delta_{n-1}} . \end{cases}
$$

By invariant differentiation of (24.3) with respect to t we obtain

$$
\frac{\delta S^\alpha_{\beta\gamma}}{\delta t} = S^\alpha_{\beta\gamma\delta_1} \frac{dx^{\delta_1}}{dt},
$$

$$
\frac{\delta S^\alpha_{\beta\gamma\delta_1}}{\delta t} = S^\alpha_{\beta\gamma\delta_1\delta_2} \frac{dx^{\delta_2}}{dt},
$$

$$
- \quad - \quad - \quad - \quad - \quad - \quad -
$$

$$
- \quad - \quad - \quad - \quad - \quad - \quad -
$$

$$
\frac{\delta S^\alpha_{\beta\gamma\delta_1\cdots\delta_{n-1}}}{\delta t} = \sum_{s=0}^{n-1} A^{\alpha pq r_1\cdots r_s}_{h\beta\gamma\delta_1\cdots\delta_n} S^h_{pq r_1\cdots r_s} \frac{dx^{\delta_n}}{dt},
$$

where use has been made of the continuous representation (24.2) of E_n along the entire arc C(t) in writing the last set of these equations. Since the left members of (24.3) and $S \equiv 0$ are solutions of the above system having the same initial values it follows that these two solutions are identical (uniqueness theorem).

Due to the fact that the property of dependence or independence of vectors is invariant under parallel displacement it follows from the above result that the rank of the matrix of the system $E_o + \dots + E_{n-1}$ is constant in each component $K(P)$. As a consequence a non-essential singular point is a regular point with respect to the system $E_o + \dots + E_{n-1}$. We shall now prove conversely that if P is a regular point with respect to the system $E_o + \dots + E_{n-1}$ then P is either a regular point or a non-essential singular point in $\mathcal{M}$. By hypothesis the rank of matrix M of the system $E_o + \dots + E_{n-1}$ is constant in some neighborhood $U_1(P)$. Let the rank of M be r in $U_1(P)$. Then there exists r independent equations $E_{\alpha_1}, \dots, E_{\alpha_r}$ in the system $E_o + \dots + E_{n-1}$ such that an rth ordered determinant D formed from the coefficients of these r equations will not vanish in some neighborhood $U(P) \subset U_1(P)$. At any regular point Q in U(P) the system E_n can be expressed in terms of these r independent equations by means of a definite (i.e. the same for all points Q) set of equations

$$E_n(Q) = \sum_{i=1}^{r} \frac{D^{\alpha_i}(Q)}{D(Q)} E_{\alpha_i}(Q)$$

with coefficients which are rational functions of the coefficients of the equations E_{α_i} and E_n and having denominators depending only on the above determinant D. Since any point in U(P) is a limit of regular points Q it follows that the above equations hold for all points in U(P). This proves the above statement.

To sum up we now have the following result: <u>Any regular or non-essential singular point is a regular point with</u>

respect to the system $E_0 + \ldots + E_{n-1}$ and conversely any point which is regular with respect to this system is either a regular or non-essential singular point. In other words the set of regular and non-essential singular points is identical with the set of regular points with respect to the system $E_0 + \ldots + E_{n-1}$.

As a by-product of the above we obtain the further result that the vector spaces defined at the points of K(P) by the solutions of the system $E_0 + \ldots + E_{n-1}$ are parallel in the sense that the vector space at any point of K(P) is carried into the vector space at any other point by parallel displacement along any arc C(t) joining these points. In particular if the rank of the matrix of the system $E_0 + \ldots + E_{n-1}$ is n-1 at any point of K(P) parallel displacement of the solution vector ξ of $E_0 + \ldots + E_{n-1}$ at a point $Q \subset K(P)$ to any other point of this component will result in a solution vector of this system which is determined to within a factor depending on the arc of displacement. Under this latter condition a single field of parallel vectors will exist in the component K(P) for the case of a Riemann space since length of a vector is then invariant under parallel displacement.

24.5. Let $P(t, \epsilon)$ be the continuous map in a component K(P) of the unit square $0 \leqq t \leqq 1$, $0 \leqq \epsilon \leqq 1$ such that $P(0, \epsilon)$ and $P(1, \epsilon)$ are fixed points Q and Q' for all values of ϵ, i.e. each ϵ-arc joins the points Q and Q'. We assume that the local representations $x^{\alpha}(t, \epsilon)$ of this map with respect to any coordinate system have the following continuous derivatives:

$$\frac{\partial x^\alpha}{\partial t}, \frac{\partial x^\alpha}{\partial \epsilon}, \frac{\partial^2 x^\alpha}{\partial t \partial \epsilon} \left(= \frac{\partial^2 x^\alpha}{\partial \epsilon \partial t}\right).$$

Hence $\frac{\partial x^\alpha}{\partial \epsilon} = 0$ at Q and Q'. By parallel displacement of an arbitrary (but fixed) vector ξ at Q along ϵ-arcs of the above map P(t, ϵ) we obtain a vector distribution ξ(t, ϵ) defined in the unit square such that the components ξ^α(t, ϵ) with respect to any local x coordinate system are continuous and have the following continuous derivatives

$$\frac{\partial \xi^\alpha}{\partial t}, \frac{\partial \xi^\alpha}{\partial \epsilon}, \frac{\partial^2 \xi^\alpha}{\partial t \partial \epsilon} \left(= \frac{\partial^2 \xi^\alpha}{\partial \epsilon \partial t}\right)$$

as follows from the existence theorem for differential equations. By using only the above derivatives we can deduce the following invariant relations

$$\frac{\delta}{\delta t}\left(\frac{\delta \xi^\alpha}{\delta \epsilon}\right) - \frac{\delta}{\delta \epsilon}\left(\frac{\delta \xi^\alpha}{\delta t}\right) = B^\alpha_{\beta\gamma\delta}\, \xi^\beta \frac{\partial x^\gamma}{\partial \epsilon} \frac{\partial x^\delta}{\partial t}.$$

We now observe that the second term in the left member vanishes since $\delta \xi^\alpha / \delta t$ is equal to zero by the parallel displacement. A necessary condition for the existence of a field of parallel vectors ξ (x) in the component Γ(P) is that the rank of the system $E_0 + \ldots + E_{n-1}$ be less than n in this component. Assuming such a rank for this system let us choose the initial values of the components of the above vector ξ at Q to be a non-trivial solution of the system $E_0 + \ldots + E_{n-1}$. Then by the result of § 24.4 the right members of the above relations will vanish along all ϵ-arcs. Hence these relations reduce to

$$\text{(24.4)} \quad \frac{\delta}{\delta t}\left(\frac{\delta \xi^\alpha}{\delta \epsilon}\right) = \frac{\partial}{\partial t}\left(\frac{\delta \xi^\alpha}{\delta \epsilon}\right) + L^\alpha_{\beta\gamma} \frac{\delta \xi^\beta}{\delta \epsilon} \frac{\partial x^\gamma}{\partial t} = 0.$$

Since

$$(24.5) \qquad \frac{\delta \xi^\alpha}{\delta \epsilon} = \frac{\partial \xi^\alpha}{\partial \epsilon} + L^\alpha_{\beta\gamma} \xi^\beta \frac{\partial x^\gamma}{\partial \epsilon}$$

and since $\frac{\partial \xi^\alpha}{\partial \epsilon}$ and $\frac{\partial x^\gamma}{\partial \epsilon}$ both vanish at Q we have that $\delta \xi^\alpha / \delta \epsilon$ is equal to zero at the point Q. Since $\frac{\delta \xi^\alpha}{\delta \epsilon} = 0$ is a solution of (24.4) having the same initial values it follows from the existence theorem for ordinary differential equations that $\frac{\delta \xi^\alpha}{\delta \epsilon}$ vanishes along any ϵ-arc. Hence in particular these derivatives vanish at the point Q'. Then from (24.5) it follows that $\frac{\partial \xi^\alpha}{\partial \epsilon}$ vanishes at Q' because $\frac{\partial x^\gamma}{\partial \epsilon}$ vanishes at this point. <u>We have now proved that we arrive at the same vector ξ' at the point Q' by parallel displacement of any solution vector ξ of the system $E_0 + \ldots + E_{n-1}$ at Q along any ϵ-arc of the map $P(t, \epsilon)$.</u>

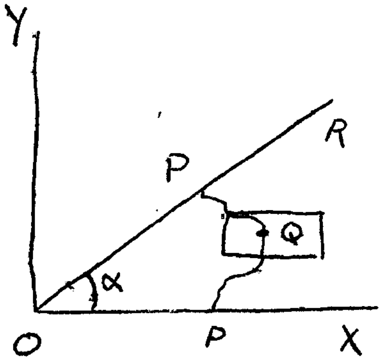
24.6. As a consequence of the above result it will follow that in any connected and simply connected open point set O contained in any component K(P) the parallel displacement of any solution vector ξ of the system $E_0 + \ldots + E_{n-1}$ at a point Q to any other point Q' of O will be independent of the path of the displacement and hence will give rise to a field of parallel vectors $\xi(x)$ in O. Obviously the class of the components of the vectors $\xi(x)$ is one greater than that of the components of the connection L.

A necessary condition for the existence of a field of parallel vectors $\xi(x)$ over $\mathcal{M}$ is that the system $E_0+\ldots+E_{n-1}$ shall possess a non-trivial solution at any point of $\mathcal{M}$ and this condition can be expressed by the vanishing of the resultant system R_1 of the equations $E_0 + \ldots + E_{n-1}$ over $\mathfrak{M}$. If now conversely $R_1 = 0$ over $\mathfrak{M}$ a field of parallel vectors will exist in the open point sets 0 in any component K(P). In particular if all the components K(P) in $\mathfrak{M}$ are simply connected a field of parallel vectors will exist in each of these components, but it may not be possible to choose these fields so that discontinuities will not arise at the essential singular points in $\mathfrak{M}$, i.e. at the boundaries of the various components K(P). Whether or not the space $\mathfrak{M}$ is itself simply connected appears to be without especial significance in this connection. Here arises one of the essential differences in the problem of characterizing spaces admitting a field of parallel vectors under the non-analytic and analytic hypotheses. For in the analytic case the condition $R_1 = 0$ is both necessary and sufficient for the existence of a (continuous) field of parallel vectors over a simply connected space (Thomas, loc. cit.). Thus it appears that in a space $\mathfrak{M}$ of class C^r the various components K(P) in play the same role as that of the entire space in the analytic case.

An investigation of the problem of characterizing spaces of class C^r admitting one or more continuous fields of parallel vectors which thus involves the construction of necessary and sufficient conditions for the removal of possible boundary discontinuities would be of interest but will not be considered here. In this connection it may be observed that if R_2 represents the set of all minors of order n-1 which can be formed from the matrix of the coefficients of the system

$E_0 + \dots + E_{n-1}$ then $R_1 = 0$, $R_2 \neq 0$, i.e. at least one of the minors of the set R_2 does not vanish at any point, over $\mathfrak{M}$ is a set of invariant conditions which are sufficient to insure the non-existence of essential singular points in $\mathfrak{M}$. Under these conditions there will exist only one component K(P) and if $\mathfrak{M}$ is further more simply connected there will exist a field of parallel vectors in $\mathfrak{M}$. Analogous conditions can of course be given for the existence of more than one field of parallel vectors, in $\mathfrak{M}$.

25. <u>Remarks on the above paper.</u> 1. We shall now give an example which shows what may happen if we consider a space (or a single component of one) which is not simply connected. Let us consider that portion of the Euclidean plane bounded by the rays OX and OR but excluding the origin O. The polar coordinates



$$x = r \cos \varphi; \quad 0 \leq \varphi \leq \alpha$$
$$y = r \sin \varphi; \quad 0 < r$$

can be used to describe this space. Also let us identify the points (φ, r) and $(\varphi + \alpha, r)$; i.e. identify each point on O X with that point on O R, which is at the same distance from the origin. We are thus really considering one nappe of a cone with the vertex removed, - a space which is not simply connected.

It is clear that any point of this space is contained in some neighborhood in which we can introduce rectangular cartesian coordinates, for which we have $g_{ik} = \delta_{ik}$. Thus we see that in any such neighborhood there are two independent parallel vector fields.

Now consider any point Q and the vector λ^i at Q where components are (1,0) in the local cartesian coördinate system at that point. If we displace this vector by parallel displacement along any closed curve not passing entirely around the cone, it is clear that the final vector again has the components (1,0) in the original coordinate system. For a curve, however, such as Q P Q (which cannot be shrunk to a point) the result of this displacement is the vector ($\cos \alpha$, $-\sin \alpha$). Thus we see that parallel displacement is not necessarily unique in non-simply connected spaces even when the curvature tensor is identically zero.

2. As a further remark we recall that in the treatment of the above paper it was necessary to assume that the g_{ik} had derivatives to an order at least as high as the n+2nd. The problem of parallel fields, however, can be stated when the g_{ik} are merely of class C^1. Hence there remains open the question of the existence of a parallel field of vectors under this less restrictive hypothesis. Such a treatment would involve entirely new methods, and in particular could not make use of the curvature tensor.

26. Locally Euclidean (Flat) Spaces. A space $\mathcal{V}$ will be called "locally Euclidean" or "flat" when $R_{ijk\ell} = 0$ at all points of $\mathcal{V}$.

A consequence of the definition is that the matrix of $E_0 + \dots + E_{n-1}$ (§ 24) is of rank zero in $\mathcal{V}$ and hence that there exist n fields of parallel vectors in any simply connected portion of it. We now prove the

Theorem: To every point P of a locally Euclidean space, $\mathcal{R}$, ($m \geq 3$) there corresponds a neighborhood $U(P) \subset \mathcal{R}$ within which there can be introduced rectangular cartesian co-ordinates.

Proof: Since the space is locally Euclidean, we can choose a normalized n-Bein in any simply connected region containing P for which

$$(26.1) \quad \frac{\delta \lambda_{(\alpha)}{}^{i}}{\delta x_k} \equiv \frac{\partial \lambda_{(\alpha)}{}^{i}}{\partial x_k} - \Gamma^{p}_{ik} \lambda_{(\alpha)p} = 0.$$

Furthermore these $\lambda_{(\alpha)i}$ are of the same C-class as the g_{ik}, which we now assume to be of class C^2. From the symmetry of $\Gamma^{\cdot p}_{ik}$ in (26.1) it follows that

$$(26.2) \quad \frac{\partial \lambda_{(\alpha)i}}{\partial x^k} = \frac{\partial \lambda_{(\alpha)k}}{\partial x^i}.$$

Consequently the equations

$$(26.3) \quad \frac{\partial \varphi^{\alpha}}{\partial x_i} = \lambda_{(\alpha)}{}^{i}(x) \qquad (i, \alpha = 1, \ldots, n)$$

are integrable in the above simply connected region $U(P) \subset \mathcal{R}$. We now may introduce the coordinate transformation $\bar{x}^{\alpha} = \varphi^{\alpha}(x^1 \ldots x^n)$ in U(P); for it is clear that the Jacobian

$$\left|\frac{\partial \bar{x}}{\partial x}\right| = \left|\lambda_{(\alpha)}{}^{i}\right| \neq 0 \text{ in } U(P).$$

Under this transformation our normalized n-Bein becomes:

$$(26.4)\qquad \bar{\lambda}^{i}_{(\alpha)} = \frac{\partial \bar{x}^{i}}{\partial x^{k}} \lambda^{k}_{(\alpha)} = \lambda_{(i)k} \lambda^{k}_{(\alpha)} = \delta_{i\alpha} .$$

Therefore the new metric tensor is

$$(26.5)\qquad \bar{g}^{ik} = \bar{\lambda}^{i}_{(\alpha)} \bar{\lambda}^{k}_{(\alpha)} = \delta^{ik}$$

which shows that the x system is rectangular Cartesian. We note that $\bar{x}_{\alpha} = \varphi_{\alpha}(x_1 \ldots x_n)$ are of class C^3 since $\lambda_{(\alpha)i}$ are of class C^2. Hence this process will in general, send m-systems into m-systems if g_{ik} is of class m - 1. Furthermore the restriction of this theorem to neighborhoods U(P) is essential; for an entire cylinder (locally Euclidean) can not be covered by a single cartesian coordinate system.

27. <u>The Tensor of Parallelism:</u>. We shall now consider the integration of the equations:

$$(27.1)\qquad \delta \lambda^{i} = 0$$

along an arc C_{PQ} of class C^1. (The argument also holds for a continuous arc composed of a finite number of pieces, each of class C^1.) We assume m = 2 and that the g_{ik} are of class C^1 throughout the space. First we consider a single coordinate system and put $\Gamma^{i}_{jk} = -\Lambda^{i}_{jk}$. Then we may write (27.1) as:

$$(27.2)\qquad d\lambda^{i} = \Lambda^{i}_{jk} \lambda^{j} dx^{k} ,$$

or

$$(27.3) \qquad \lambda^i(Q) = \lambda^i(P) + \int_P^Q \Lambda^i_{jk}(R)\, \lambda^j(R)\, dx^k(R)$$

where R lies between P and Q. But for $\lambda^i(R)$ an analogous formula exists, and proceeding step by step in this fashion we obtain:

$$(27.4) \qquad \lambda^i(Q) = \lambda^j(P)\, \varphi^i_j(C_{PQ})$$

where

$$(27.5) \qquad \varphi^i_j(C_{PQ}) = \delta^i_j + \int_P^Q \Lambda^i_{jk}\, dx^k + \int_P^Q \Lambda^i_{j_1 k}\, dx^k \int_P^{Q'} \Lambda^{j_1}_{jk_1}\, dx^{k_1}$$

$$+ \int_P^Q \Lambda^i_{j_1 k}\, dx^k \int_P^{Q'} \Lambda^{j_1}_{j_2 k_1}\, dx^{k_1} \int_P^{Q''} \Lambda^{j_2}_{jk_2}\, dx^{k_2} + \dots$$

It can be shown that this series converges uniformly along C_{PQ}. Hence we may multiply (27.5) by $\Lambda^p_{iq}\, dx^q$ and integrate term by term. We obtain:

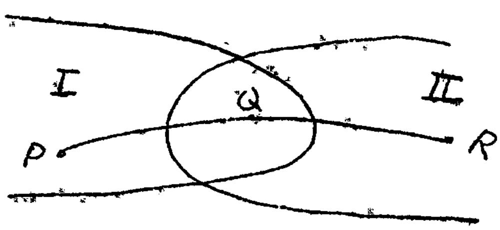
$$(27.6) \qquad \int_P^Q \varphi^i_j(C_{PR})\, \Lambda^p_{iq}(R)\, dx^q = \varphi^p_j(C_{PQ}) - \delta^p_j .$$

Multiplying by $\lambda^j(P)$ and using (27.4) we obtain:

$$(27.7) \qquad \int_P^Q \lambda^i(R)\, \Lambda^p_{iq}(R)\, dx^q = \lambda^p(Q) - \lambda^p(P)$$

which shows that (27.4) does actually represent the solution of (27.3) and of (27.2).

We now consider the case where the arc lies in a number of coordinate systems. For example



take $\widehat{PQR}$ where $\widehat{PQ}$ lies in the coordinate system I and $\widehat{QR}$ in the system II. Then we have

$$\lambda^i(Q) = \lambda^j(P)\ \varphi^{\,i}_{\,j}(C_{PQ})$$

in the system I. And also that

$$\bar{\lambda}^k(R) = \bar{\lambda}^h(Q)\ \bar{\varphi}^{\,k}_{\,h}(C_{QR})$$

in the system II. Now

$$\bar{\lambda}^h(Q) = \lambda^i(Q)\left(\frac{\partial \bar{x}^h}{\partial x^i}\right)_Q$$

and so we have that

$$(27.8)\qquad \bar{\lambda}^k(R) = \lambda^j(P)\ \varphi^{\,i}_{\,j}(C_{PQ})\left(\frac{\partial \bar{x}^h}{\partial x^i}\right)_Q \bar{\varphi}^{\,k}_{\,h}(C_{QR})$$

$$= \lambda^j(P)\ \varphi'^{\,k}_{\,j}(C_{PR}).$$

From this discussion we see that $\varphi^{\,k}_{\,j}(C_{PQ})$ is a geometric object defined as a function of the arc, that it is a contravariant vector at Q and a covariant vector at P and of scalar character for all other points of C_{PQ}. It may thus be said that it transforms like a product $\rho^k(Q)\ \sigma_j(P)$ under coordinate transformations in U(P) and U(Q) respectively.

In this treatment we have derived $\varphi^{\,k}_{\,j}(C_{PQ})$ from a Riemann metric, but now we can dispense with this metric and merely introduce tensors $\varphi^{\,k}_{\,j}(C_{PQ})$ having suitable properties. By this means we introduce a certain structure into the space; i.e. it becomes a space with parallelism. Among the axioms which such a $\varphi^{\,k}_{\,j}(C_{PQ})$ must satisfy are the following:

1) To any vector λ^j at P, $\lambda^j\ \varphi^{\,k}_{\,j}(C_{PQ})$ defines a vector at Q. Thus the sum of two vectors transforms into the sum of their transforms.

2) Transitivity:

$$\varphi_k^{\,j}(C_{PR})\,\varphi_j^{\,i}(C_{RQ}) = \varphi_k^{\,i}(C_{PQ}).$$

3) $\varphi_k^{\,j}(C_{PQ})\,\varphi_j^{\,i}(C_{QP}) = \delta_k^i$. This last implies that $|\varphi_j^{\,i}| \neq 0$.

If we consider curves C_{PQ} which may be differentiated at least once with respect to their parameters, and also vectors of class C^1, this theory may be somewhat elaborated. Let P be a fixed point and R(t) be a varialbe point on a given curve C_{PQ}. Then (27.4) becomes

$$\lambda^i(R(t)) = \varphi_k^{\,i}(C_{PR(t)})\,\lambda^k(P)\,. \tag{27.9}$$

Differentiating (27.9) we obtain:

$$\frac{d\lambda^i(R(t))}{dt} = \frac{d\varphi_k^{\,i}(C_{PR(t)})}{dt}\,\lambda^k(P) = \Gamma_k^i(t)\,\lambda^k(P) \tag{27.10}$$

where $\Gamma_k^i(t)$ are thus defined. These Γ_k^i are functions of the curve and of t. Again we may abstract and assume only that such functions Γ are given. Then as for (27.2) a series like (27.5) may be constructed which defines a $\varphi_k^{\,i}(C_{PQ})$ such that (27.9) is an integral of (27.10). The Γ's thus determine the same structure in the space as the $\varphi_k^{\,i}$'s do.

§ 28. Extremals. An extremal of a space, R_n, is a curve $x^i(t)$ for which:

$$\frac{\delta^2 x^i}{\delta t^2} = 0 . \tag{28.1}$$

Since the tangent vector to the curve, $\xi^i = \frac{dx^i}{dt}$, we may write (28.1) as

$$\frac{dx^i}{dt} = \xi^i ; \quad \frac{\delta \xi^i}{\delta t} = 0 . \tag{28.2}$$

Thus an extremal is a curve which may be so parameterized that its tangent is parallel along the curve. A first integral of (28.1) is

$$g_{ik} \frac{dx^i}{dt} \frac{dx^k}{dt} = \text{const.}, \text{ for} \tag{28.3}$$

$$\frac{\delta}{\delta t} \left(g_{ik} \frac{dx^i}{dt} \frac{dx^k}{dt}\right) = 0.$$

If this constant is unity, then the parameter gives the arc length of the curve. Also for definite t-intervals there is a unique extremal passing through a given point of the space in a given direction.

Now let us consider a regular $\mathcal{M}_r$ element imbedded in R_n whose parametric equations are $x^i = x^i (y^1 \ldots\ldots y^r)$, the functions being of class C^2. A curve $y^\alpha (t)$ is also a curve $x^i(t)$ in the R_n, and we have the relation

$$\frac{dx^i}{dt} = \frac{\partial x^i}{\partial y^r} \frac{dy^r}{dt} . \tag{28.4}$$

Differentiating (28.4) we have:

$$\frac{\delta^2 x^i}{\delta t^2} = \frac{\partial x^i}{\partial y^\alpha} \frac{\delta^2 y^\alpha}{\delta t^2} + \eta^i_{\alpha\beta} \frac{dy^\alpha}{dt} \frac{dy^\beta}{dt} \tag{28.5}$$

where we have written $\eta^i_{\alpha\beta} = \frac{\delta^2 x^i}{\partial y^\alpha \partial y^\beta}$. (Compare § 16). It was previously demonstrated that, for fixed valued of α and β, $\eta^i_{\alpha\beta}$ are the components of a contravariant vector which is normal to $\mathcal{M}_r$.

Let $\frac{1}{\rho_1}$ be the first curvature of the curve in the R_n and let $\underset{2}{\xi}^i$ be the first unit normal. Also let $\frac{1}{\sigma_1}$ and $\underset{2}{\eta}^\alpha$ be the corresponding quantities in the $\mathcal{M}_r$. Then (28.5) becomes:

$$(28.6) \qquad \frac{\underset{(2)}{\xi}^i}{\rho_1} = \frac{\partial x^i}{\partial y^\alpha}\frac{\underset{(2)}{\eta}^\alpha}{\sigma_1} + \eta^i_{\alpha\beta}\frac{dy^\alpha}{dt}\frac{dy^\beta}{dt}.$$

We now define an extremal of $\mathcal{M}_r$ to be a curve for which $\frac{\delta^2 y^\alpha}{\delta t^2} = 0$; i.e. for which $\frac{1}{\sigma_1} = 0$. From (28.6) it follows that for an extremal of $\mathcal{M}_r$, the first normal to the curve (in R_n) is normal to $\mathcal{M}_r$. Also if a curve of $\mathcal{M}_r$ has its first normal (in R_n) normal to $\mathcal{M}_r$, then the curve is an extremal of $\mathcal{M}_r$.

§ 29. Geodesic Subspaces; "Planes". Definition I. A subspace is said to be geodesic at a point P if every curve of the subspace passing through P and having $\frac{\delta^2 y^\alpha}{\delta t^2} = 0$ at P also has $\frac{\delta^2 x^i}{\delta t^2} = 0$ at P.

From this definition it follows from (28.5) that $\eta^i_{\alpha\beta} = 0$ at P when the subspace is geodesic at P. The converse also holds, and hence we have: Definition II: A subspace is said to be geodesic at a point P if $\eta^i_{\alpha\beta} = 0$ at P.

Planes. Definition I. A subspace which is geodesic at each point is called a "plane". This definition is equivalent to the following one:

Definition II. A subspace containing any extremal of which it contains an element (point and direction) is a plane.

To prove the equivalence of these two definitions we first suppose that we have a plane according to Definition I; i.e. $\eta^{i}_{\alpha\beta} = 0$ in the subspace. Then (28.5) shows that any extremal of the subspace is an extremal of the R_n. But since there is a unique extremal in R_n at any point in a given direction, we see that Definition II is fulfilled.

Conversely we suppose that Definition II is fulfilled. Then (28.5) holds for any extremal of the R_n which contains an element of the plane. For such a curve each term on the right hand side of (28.5) must vanish; and since this is true for every such extremal, it follows that $\eta^{i}_{\alpha\beta} = 0$ in the subspace.

We now discuss a third approach to the subject of planes. Suppose that a vector λ^α is defined at a point P of a plane. Then it satisfies

$$\lambda^i = \frac{\partial x^i}{\partial y^\alpha}\lambda^\alpha \tag{29.1}$$

Now displace this vector along a curve of the plane by means of parallel displacement with respect to the plane. This shows that

$$\delta\lambda^i = \frac{\partial x^i}{\partial y^\alpha}\delta\lambda^\alpha + \eta^{i}_{\alpha\beta}\lambda^\alpha dy^\beta = 0 \tag{29.2}$$

Hence the displacement is also parallel with respect to the R_n. But since this latter displacement is unique, our plane has the property that any vector of it remains a vector of it after parallel displacement with respect to R_n. Hence we make:

<u>Definition III:</u> If a subspace has the property that any vector of it remains such a vector after parallel displacement with respect to R_n, then the subspace is a plane.

To show that this definition includes definitions I and II, we note first that it permits us to apply (29.2) for displacements where $\delta\lambda^i = 0$. Since this holds for all such displacements, it follows that $\eta^i_{\alpha\beta} = 0$; i.e. I is fulfilled.

From (29.2) we observe that a vector is displaced parallelly with respect to $\mathcal{M}_r$ if and only if the growth, $\delta\lambda^i$, is normal to $\mathcal{M}_r$. Suppose then that we have two subspaces $\mathcal{M}_r$ and $\mathcal{M}_r'$ which are tangent along a curve. Since an increase $\delta\lambda^i$ which is normal to one of these is normal to the other also, parallel displacement along the curve is the same for the two spaces. If we consider ordinary two dimensional surfaces, let one of these tangent surfaces be a "developable" surface. Then it may be "rolled out" on to a Euclidean plane and parallel vectors along the curve become parallel in the Euclidean sense. This was Levi-Civita's approach to the problem of parallelism.

As further remarks concerning planes, we state the following. If, for any point of an R_n, and for any r-element, the geodesic subspace is a plane, the space R_n is of constant curvature. A space for which this property holds for a single point is called a Schur space. A discussion of these spaces is given in Mayer "Lehrbuck", Chapter VIII.

CHAPTER VI

Normal Coordinates.

§ 30. Let $\mathfrak{X}_n$ be a coordinate manifold in which there is defined a symmetric Γ -object (§ 7). To simplify the discussion we shall assume the Γ 's analytic.

The condition that a vector ξ^i be parallel-displaced along an arc $x^i = x^i(t)$ is

$$\frac{d\xi^i}{dt} + \xi^j \Gamma^i_{jk} \frac{dx^k}{dt} = 0.$$

If the arc has the property that its tangent vector $\frac{dx^i}{dt}$ is parallel-displaced along it, i.e., if

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \ , \tag{30.1}$$

then it is called a "path". We seek coordinates* in terms of which the

---

* These will be "affine normal coordinates"; c.f. T. Y. Thomas: "Differential Invariants of Generalized Spaces", §§ 3, 29.

---

equations of such paths are linear in the parameter.

Corresponding to any initial point $P[x = x_o]$ and any initial direction $\frac{dx^i}{dt} = \xi^i_o$ there exists a unique solution of (30.1), which may be obtained, by successive differentiation and substitution, in the form

$$x^i = x^i_o + \xi^i_o t - \frac{1}{2!}\Gamma^i_{jk}(P)\xi^j_o \xi^k_o t^2 - \frac{1}{3!}\Gamma^i_{jk\ell}(P)\xi^j_o \xi^k_o \xi^\ell_o t^3 - \dots \ . \tag{30.2}$$

[The series (30.2) can be shown to converge in some $\mathfrak{U}(x_o)$ provided the expansion of the Γ 's possess a dominant function.]

Let us now set $\xi^i_0 t = y^i$. We obtain

$$x^i = x^i_0 + y^i - \frac{1}{2!}\Gamma^i_{jk}(P)\, y^j y^k - \frac{1}{3!}\Gamma^i_{jk\ell}(P)\; y^j y^k y^\ell - \dots . \tag{30.3}$$

Since $\left.\frac{\partial x^i}{\partial y^j}\right]_P = \delta^i_j$, the relation (30.3) may be inverted, giving rise to an allowable transformation of coordinates throughout some $\mathcal{W}(x_0)$. The curves

$$y^i = \xi^i_0\, t$$

are the paths (30.2) referred to y coordinates, a fact which becomes evident on changing the y's to x's by means of (30.3).

The y coordinates thus obtained will be called normal coordinates. They are completely determined by the x system from which they are derived and by the point $P(x_0)$, their origin.

<u>Theorem:</u> For any point P there exists a neighborhood $\mathcal{U}(P)$ such that P can be joined to any $Q \in \mathcal{U}(P)$ by one and only one path lying within the neighborhood.

We have only to set up normal coordinates y at P. Any "cube" $|y^i| < a$ which is contained in the region covered by the normal coordinate system will then serve as the required neighborhood. For if Q has coordinates $y^i = q^i$, then the (unique) path

$$y^i = q^i\, t \qquad 0 \leq t \leq 1$$

clearly satisfies the conditions.

We next investigate the effect on the normal coordinates of transforming the underlying system x. Consider a path through P having the initial direction represented by ξ^i in the x system and by $\bar{\xi}^j$ in the $\bar{x}$ system. In the corresponding normal coordinate systems $y(P,x)$, $\bar{y}(P,\bar{x})$ this path has the equations

$$y^i = \xi^i t, \quad \bar{y}^j = \bar{\xi}^j t$$

respectively. Now

$$\bar{\xi}^j = \frac{\partial \bar{x}^j}{\partial x^i}\Big]_P \xi^i$$

so that for each point of this path

$$\bar{y}^j = \frac{\partial \bar{x}^j}{\partial x^i}\Big]_P \xi^i t = \frac{\partial \bar{x}^j}{\partial x^i}\Big]_P y^i .$$

But since every point [in a suitably restricted neighborhood of P] lies on some path through P, the relation

(30.4) $$\bar{y}^j = a^j_i y^i \qquad \left(a^j_i = \frac{\partial \bar{x}^j}{\partial x^i}\Big]_P\right)$$

must hold in an entire neighborhood of P.

Suppose that under transformation to normal coordinates the Γ's of the x system become C's. In the y system the paths through the origin are solutions of

$$\frac{d^2y^i}{dt^2} + C^i_{jk} \frac{dy^j}{dt} \frac{dy^k}{dt} = 0 .$$

Since $y^i = \xi^i t$ is such a path, we see at once that $C^i_{jk} \xi^j \xi^k = 0$, and consequently $C^i_{jk} y^j y^k = 0$, along any path. This relation must therefore hold identically in y, for each point in some neighborhood lies on a path.

<u>Theorem</u>: A necessary and sufficient condition that the y's be normal coordinates is that

(30.5) $$C^i_{jk} y^j y^k = 0 .$$

The necessity of the condition has just been demonstrated. To prove the condition sufficient we shall show that when (30.5) holds there exists a system x for which the y's are normal coordinates. Consider the differential equations

(30.6) $$C^i_{jk}(y) \frac{\partial x^p}{\partial y^i} = \frac{\partial^2 x^p}{\partial y^j \partial y^k} + \Gamma^p_{qr}(x) \frac{\partial x^q}{\partial y^j} \frac{\partial x^r}{\partial y^k}$$

where the Γ 's are arbitrary (analytic) functions of their arguments. On multiplication by $y^j y^k$ the left member vanishes by hypothesis, so that we have

$$\left(\frac{\partial^2 x^p}{\partial y^j \partial y^k} + \Gamma^p_{qr} \frac{\partial x^q}{\partial y^j} \frac{\partial x^r}{\partial y^k} \right) y^j y^k = 0$$

Under the initial conditions $x^i = p^i$, $\frac{\partial x^i}{\partial y^q} = \delta^i_q$, $y = 0$, the series solution is found to be

$$x^i = p^i + y^i - \frac{1}{2!} \Gamma^i_{jk}(P)\, y^j y^k - \dots ,$$

which is identical with (30.3), the relations which we have taken as defining normal coordinates.

§ 31. Alternative treatment. We may take as our definition of normal coordinates at P with respect to a given x system the inverse functions y(x) of the functions x(y) determined by the equations

$$\text{(31.1)} \qquad \left(\frac{\partial^2 x^i}{\partial y^p \partial y^q} + \Gamma^i_{jk}(x) \frac{\partial x^j}{\partial y^p} \frac{\partial x^k}{\partial y^q} \right) y^p y^q = 0$$

$$x^i = p^i, \quad \frac{\partial x^i}{\partial y^q} = \delta^i_q, \quad y = 0$$

For as we have just seen, the relation between the y's and the x's must be of the form (30.3), so that the equations of the paths in the y system become $y^i = \xi^i t$.

Moreover, transforming the x's into $\bar{x}$'s sends the corresponding y's into $\bar{y}$'s standing in the relation (30.4). For we shall have

$\Gamma(x) \longrightarrow C(y)$ $\qquad\qquad \bar{\Gamma}(\bar{x}) \longrightarrow \bar{C}(\bar{y})$ $\qquad\qquad$ and

$$\text{(31.2)} \qquad C^p_{qr} \frac{\partial \bar{y}^i}{\partial y^p} = \frac{\partial^2 \bar{y}^i}{\partial y^q \partial y^r} + \bar{C}^i_{jk} \frac{\partial \bar{y}^j}{\partial y^q} \frac{\partial \bar{y}^k}{\partial y^r}$$

Since $\left|\frac{\partial x^p}{\partial y^q}\right|_P \neq 0$ we see from (31.1) and the transformation law (30.6) of the Γ's into C's that $C^p_{qr}\, y^q y^r = 0$. Hence (31.2) implies

$$(31.3) \qquad \left(\frac{\partial^2 \bar{y}^i}{\partial y^q \partial y^r} + \bar{C}^i_{jk} \frac{\partial \bar{y}^j}{\partial y^q} \frac{\partial \bar{y}^k}{\partial y^r} \right) y^q y^r = 0$$

We now recall that from the nature of $y(x)$; $\bar{y}(\bar{x})$ we have

$$\left.\frac{\partial \bar{y}^i}{\partial \bar{x}^k}\right]_P = \left.\frac{\partial x^i}{\partial y^k}\right]_P = \delta^i_k$$

so that

$$\left.\frac{\partial \bar{y}^i}{\partial y^q}\right]_P = \left[\frac{\partial \bar{y}^i}{\partial \bar{x}^k} \frac{\partial \bar{x}^k}{\partial x^j} \frac{\partial x^j}{\partial y^q} \right]_P = \left.\frac{\partial \bar{x}^i}{\partial x^q}\right]_P = a^i_q$$

But $y^i = a^i_q\, y^q$ is easily seen to be a solution of (31.3) satisfying the initial conditions

$$\bar{y}^i = 0 - \frac{\partial \bar{y}^i}{\partial y^q} = a^i_q \qquad y^p = 0$$

so that it must be the solution $\bar{y}^i = \bar{y}^i(y)$ resulting from the relations $\bar{y} = \bar{y}(\bar{x})$, $\bar{x} = \bar{x}(x)$, $x = x(y)$.

Finally, the theorem that $\tilde{C}^i_{jk}\, \tilde{y}^j\, \tilde{y}^k = 0$ characterizes the $\tilde{y}$'s as normal coordinates follows from this second definition. For let us determine normal coordinates y for such a system $\tilde{y}$. We must solve

$$\left(\frac{\partial^2 \tilde{y}^i}{\partial y^q \partial y^r} + \tilde{C}^i_{jk} \frac{\partial \tilde{y}^j}{\partial y^q} \frac{\partial \tilde{y}^k}{\partial y^r} \right) y^q y^r = 0$$

$$\tilde{y}^i = 0, \quad \frac{\partial \tilde{y}^i}{\partial y^q} = \delta^i_q \qquad y = 0$$

Under the above hypothesis, $y = \tilde{y}$ is an obvious solution, so that the $\tilde{y}$'s are themselves normal coordinates.

§ 32. Identities for Riemann Spaces.

We now suppose that the Γ's of the foregoing discussion are Christoffel symbols formed from the metric tensor g_{ij}. In normal coordinates let the g's and Γ's become

$$g_{ij} \rightarrow h_{ij}, \quad \Gamma^{i}_{jk} \rightarrow C^{i}_{jk} = \frac{1}{2} h^{i\ell}\left(\frac{\partial h_{\ell j}}{\partial y^k} + \frac{\partial h_{\ell k}}{\partial y^j} - \frac{\partial h_{jk}}{\partial y^\ell}\right)$$

Since $|h^{i\ell}| \neq 0$, the relation $C^{i}_{jk}\, y^j y^k = 0$ then implies

$$\left(2\frac{\partial h_{\ell k}}{\partial y^j} - \frac{\partial h_{jk}}{\partial y^\ell}\right) y^j y^k = 0 \tag{32.1}$$

We know that along a geodesic (path) in a Riemann space the length of the (parallel-displaced) tangent vector is constant:

$$g_{ij}\frac{dx^j}{dt}\frac{dx^k}{dt} = \text{const.}$$

When expressed in normal coordinates, where $y^i = \xi^i t$ along a path, this relation becomes

$$h_{jk}(y)\,\xi^j \xi^k = \text{const.} = h_{jk}(0)\xi^j \xi^k \; .$$

Consequently

$$h_{jk}(y)\, y^j\, y^k \equiv h^{(0)}_{jk}\, y^j\, y^k \tag{32.2}$$

(because each point y is on a path).

We can in fact prove more, namely that

$$h^{(y)}_{jk}\, y^k \equiv h^{(0)}_{jk}\, y^k \tag{32.3}$$

and that this identity characterizes the y's as normal.

For suppose the coordinates are normal. Then differentiation of (32.2) yields

$$\frac{\partial h_{jk}}{\partial y^\ell} y^j y^k + 2h_{j\ell}\, y^j - 2h_{j\ell}^{(0)}\, y^j = 0 \,.$$

Replacing the first term by its value from (32.1) and cancelling the factor 2, we obtain

(32.4)
$$\frac{\partial h_{\ell k}}{\partial y^j} y^j y^k + h_{j\ell}\, y^j - h_{j\ell}^{(0)} y^j = 0 \,.$$

Now along a geodesic consider the expression

$$\frac{\partial h_{\ell k}}{\partial y^j} y^k \xi^j + h_{\ell k}\, \xi^k - h_{\ell k}^{(0)}\, \xi^k = \frac{d}{dt}\left[h_{\ell k}\, y^k - h_{\ell k}^{(0)} y^k\right]$$

The left member vanishes for $t \neq 0$ because of (32.4); but since each of its terms is continuous in y, and hence in t, it vanishes also at $t = 0$. Thus $\varphi = h_{\ell k}\, y^k - h_{\ell k}^{(0)}\, y^k$ is a solution of the differential equation $\frac{d\varphi}{dt} = 0$. And since it evidently satisfies the initial condition $\varphi(0) = 0$, we have by the unicity theorem

$$h_{\ell k}\, y^k = h_{\ell k}^{(0)}\, y^k$$

along any geodesic, and therefore at each point y.

Conversely, suppose that (32.3) holds in some coordinate system y. Then by differentiation we get

$$\frac{\partial h_{\ell k}}{\partial y^j} y^k + h_{\ell j} - h_{\ell j}^{(0)} = 0 \,.$$

Multiplying this in turn by y^j and y^ℓ, and using (32.3), we obtain the respective equations

(32.5)
$$\frac{\partial h_{\ell k}}{\partial y^j} y^k\, y^j = 0 \,,$$

(32.6)
$$\frac{\partial h_{\ell k}}{\partial y^j} y^k y^\ell = 0 \,.$$

Returning to the definition of the C's, we have at once that $C^i_{k\ell}\, y^k y^\ell = 0$, so that the y's are normal coordinates.

Again, <u>either of the identities</u> (32.5), (32.6) <u>characterizes</u>

the y's as normal coordinates. For if the y's are normal then we have both (32.5) and (32.6). Conversely, suppose (32.5) is true for a system y. Take a curve $y^i = \xi^i t$, where the ξ's are constants. Along this curve

$$\frac{d}{dt}\left[h_{\ell k}\xi^k - h_{\ell k}^{(0)}\xi^k \right] = \frac{\partial h_{\ell k}}{\partial y^j}\xi^j \xi^k$$

The right side must vanish for $t \neq 0$ (32.5). By continuity it vanishes also at $t = 0$; so that by the unicity theorem for such differential equations, $h_{\ell k}\xi^k = h_{\ell k}^{(0)}\xi^k$. Hence

$$h_{\ell k} y^k = h_{\ell k}^{(0)} y^k$$

on any curve $y^i = \xi^i t$. But since any point in a suitably chosen neighborhood of $y = 0$ lies on some such curve, we have (32.3) and the coordinates are normal.

Similarly, suppose (32.6) holds. Then along any curve $y^i = \xi^i t$ we have

$$\frac{d}{dt}\left[h_{\ell k}\xi^\ell \xi^k - h_{\ell k}^{(0)}\xi^\ell \xi^k \right] = \frac{\partial h_{\ell k}}{\partial y^j}\xi^\ell \xi^k \xi^j$$

Repetition of the preceding argument leads to the result that

$$h_{\ell k}\, y^\ell \; y^k - h_{\ell k}^{(0)}\, y^\ell \; y^k \equiv 0 .$$

Differentiating as to y^j, we obtain

$$\frac{\partial h_{\ell k}}{\partial y^j} y^\ell y^k + 2h_{\ell j} y^\ell - 2h_{\ell j}^{(0)} y^\ell = 0$$

Since the first term vanishes by hypothesis, we again have (32.3).

A final important, though not characteristic, property of normal coordinates, valid for affine spaces in general, is expressed in the following

Theorem. At the origin of normal coordinates the coefficients of affine connection are zero.

For as we have seen, along any path $C^i_{jk}(q)\xi^j\xi^k = 0$. But at the origin the ξ's are arbitrary, so that the vanishing of the polynomial $C^i_{jk}(o)\xi^j\xi^k$ for all ξ's implies the vanishing of each coefficient.

In a Riemann space we have the

Corollary. At the origin of normal coordinates the first derivatives of the metric tensor vanish.

This follows from the identity

$$\frac{\partial g_{ij}}{\partial x^k} = g_{\ell j}\Gamma^\ell_{ik} + g_{i\ell}\Gamma^\ell_{jk}$$

§ 33. Extension. We are familiar with the process of obtaining new tensors from a given affine tensor T by covariant differentiation. A second method of constructing tensors will not be described + that of extension.*

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* T. Y. Thomas, loc. cit., § 32. Explicit formulas for expressing various extensions in terms of the Γ's will be found in § 33 of that book.

Let a tensor T with components $T^{i\,\dots\,j}_{k\,\dots\,\ell}(x)$ in an x coordinate system become $t^{i\,\dots\,j}_{k\,\dots\,\ell}(y)$ when referred to the corresponding normal coordinate system y erected at a point $P(x_o)$. With each x system and the (fixed) point P we associate the numbers

$$\left(\frac{\partial^r t^{i\ldots j}_{k\ldots l}}{\partial y^{p_1}\ldots\partial y^{p_r}}\right)_P = T^{i\ \ldots\ j}_{k\ \ldots\ l,\ p_1\ \ldots\ p_r}(x_0) \quad .$$

These will form a tensor. For beginning with an $\bar{x}$ system, we are led to a normal coordinate system $\bar{y}$, in which $T^{i\ \ldots\ j}_{k\ \ldots\ l}$ has components $\bar{t}$ related to the t's by the rule

$$\bar{t}^{s\ \ldots\ t}_{u\ \ldots\ v}(\bar{y}) = t^{i\ \ldots\ j}_{k\ \ldots\ l}(y)\ \frac{\partial \bar{y}^s}{\partial y^i}\cdots\frac{\partial y^l}{\partial \bar{y}^v}$$

But because of (30.4) the derivatives on the right are constants.

Therefore

$$\frac{\partial^r \bar{t}^{s\ldots t}_{u\ldots v}}{\partial \bar{y}^{q_1}\ldots\partial \bar{y}^{q_r}} = \frac{\partial^r t^{i\ldots j}_{k\ldots l}}{\partial y^{p_1}\ldots\partial y^{p_r}}\ \frac{\partial \bar{y}^s}{\partial y^i}\cdots\frac{\partial y^l}{\partial \bar{y}^v}\ \frac{\partial y^{p_1}}{\partial \bar{y}^{q_1}}\cdots\frac{\partial y^{p_r}}{\partial \bar{y}^{q_r}}$$

And since $\left.\dfrac{\partial \bar{y}^i}{\partial y^j}\right]_P = \left.\dfrac{\partial \bar{x}^i}{\partial x^j}\right]_P$, we have, on evaluation at P,

$$\bar{T}^{s\ \ldots\ t}_{u\ \ldots\ v,q_1\ \ldots\ q_r}(x_0) = T^{i\ \ldots\ j}_{k\ \ldots\ \ \ p_1\ \ldots\ p_r}(x_0)\ \frac{\partial \bar{x}^s}{\partial x^i}\cdots\frac{\partial x^{p_r}}{\partial \bar{x}^{q_r}}$$

The tensor $T^{i\ \ldots\ j}_{k\ \ldots\ l,\ p_1\ \ldots\ p_r}$, which can be defined as above for each point of the space, is called the <u>r-th extension</u> of $T^{i\ \ldots\ j}_{k\ \ldots\ l}$.

Note that although the first extension of a tensor is the same as its covariant derivative, its higher extensions and higher derivatives are not in general equal. Nor does the first extension of the first extension give the second extension. For example, in a Riemann space the second covariant derivative of g_{ij} vanishes because the first does, but the second extension $g_{ij,k}$ need not be zero. Finally, the r-th extension of a tensor is symmetric in the adjoined indices; the repeated covariant derivative is not.

§ 34. Normal Tensors. In a similar fashion the derivatives of the Γ's, evaluated at the origin of normal coordinates, will serve to define tensors $A^i_{j\,k\,l_1\ldots l_r}$ known as normal tensors:

$$A^i_{jk\,l_1\ldots l_r}(x_0) = \left(\frac{\partial^r C^i_{jk}(y)}{\partial y^{l_1}\ldots\partial y^{l_r}}\right)_P$$

For if $\Gamma(x)$ and $\bar{\Gamma}(\bar{x})$ become $C(y)$ and $\bar{C}(\bar{y})$ in their respective normal coordinate systems, we have

$$\bar{C}^l_{mn}(\bar{y}) = C^i_{jk}(y)\,\frac{\partial \bar{y}^l}{\partial y^i}\,\frac{\partial y^j}{\partial \bar{y}^m}\,\frac{\partial y^k}{\partial \bar{y}^n}$$

the second derivative term in (31.2) vanishing because of (30.4). The derivatives on the right are constants, and the argument just given for the tensor character of extensions applies at once.

35. Replacement. A tensor T is called an affine tensor differential invariant* of order r of its components

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* The notion can be extended; c.f. T. Y. Thomas, loc. cit. §§ 11, 39.

(35.1) $$T^{\cdots}_{\cdots}\left(\Gamma^s_{tn}, \frac{\partial \Gamma^s_{tu}}{\partial x^{p}}, \ldots, \frac{\partial^r \Gamma^s_{tu}}{\partial x^{p_1}\ldots\partial x^{p_r}}\right)$$

are functions of the Γ's and their derivatives, to the r-th order, such that each retains its functional form under transformation of co-ordinates. Transforming (35.1) to normal coordinatds gives

$$T^{i\,\ldots\,j}_{k\,\ldots\,l}\left(C^i_{jk}, \ldots, \frac{\partial^r C^i_{jk}}{\partial y^{l_1}\ldots\partial y^{l_r}}\right) = T^{s\ldots t}_{u\ldots v}\left(\Gamma^i_{jk}, \ldots, \frac{\partial^r \Gamma^i_{jk}}{\partial x^{l_1}\ldots\partial x^{l_r}}\right)\frac{\partial y^i}{\partial x^s}\cdots\frac{\partial\ }{\partial\ }$$

Evaluating at the origin (the derivatives on the right then become δ 's) we have

$$(35.2) \qquad T^{i \cdots j}_{k \cdots l}(0, A^{i}_{jkl_1}, \ldots) = T^{i \cdots j}_{k \cdots l}\left(\Gamma^{i}_{jk}, \frac{\partial \Gamma^{i}_{jk}}{\partial x^{l_1}}, \ldots \right)$$

This proves the

<u>First Replacement Theorem.</u> Any affine tensor differential invariant may be represented in the form (35.2), where the Γ's are replaced by 0; and the derivatives of the Γ's by the corresponding normal tensors.

For example, the curvature tensor (23.2)

$$B^{i}_{jkl} = - R^{i}_{jkl} = \frac{\partial \Gamma^{i}_{jk}}{\partial x^{l}} - \frac{\partial \Gamma^{i}_{jl}}{\partial x^{k}} + \Gamma^{r}_{jk} \Gamma^{i}_{rl} - \Gamma^{r}_{jl} \Gamma^{i}_{rk}$$

may be written

$$B^{i}_{jkl} = A^{i}_{jkl} - A^{i}_{jlk} .$$

A metric tensor differential invariant of order r is defined by the property that its components

$$(35.3) \qquad I^{j \cdots k}_{l \cdots m}\left(g_{ij}, \frac{\partial g_{ij}}{\partial x^{k_1}}, \ldots, \frac{\partial^r g_{ij}}{\partial x^{k_1} \cdots \partial x^{k_r}}\right)$$

retain their functional form under transformation of coordinates. When expressed in normal coordinates and evaluated at the origin, the components (35.3) are equal to

$$(35.4) \qquad I^{j \cdots k}_{l \cdots m}(g_{ij}(0), 0, g_{ij,k_1 k_2}, \ldots, g_{ij,k_1 \ldots k_r}) .$$

This gives us the

<u>Second Replacement Theorem:</u> Any metric tensor differential

invariant may be put in the form (35.4), where the g's are left unaltered, their first derivatives are replaced by 0, and their higher derivatives go over into the corresponding extensions (also called normal tensors).

As a consequence of this theorem we see that there is no metric tensor differential invariant of order one.

36. Complete sets of identities. The normal tensors play an important rôle in the determination of complete sets of identities in the components of tensor invariants. (A set of identities is said to be complete if it furnishes all the algebraic conditions on the components.) As an illustration of the methods which may be employed (cf. T. Y. Thomas, Chap. VI) we shall determine a complete set of identities for the curvature tensor $B^i_{jkl} = -R^i_{jkl}$ of an affine space.

We first derive certain identities of the normal tensor A^i_{jkl}. Since the A's are derived from symmetric Γ's, they must be symmetric in their first two lower indices. Again, the series expansion for the Γ's expressed in normal coordinates gives

$$\Gamma^i_{jk}(x) \longrightarrow C^i_{jk}(y) = C^i_{jk}(0) + \left(\frac{\partial C^i_{jk}}{\partial y^l}\right)_0 y^l + \cdots$$

whence the relation $C^i_{jk}\, y^j\, y^k = 0$ leads to

$$C^i_{jk}\, y^j y^k \equiv 0 \equiv A^i_{jkl}\, y^j y^k y^l + \dots \; .$$

Thus the cyclic sum of the A's on their three lower indices vanishes.

Lemma. The set

(36.1) $$A^i_{jkl} = A^i_{kjl}$$

(36.2) $$A^i_{jkl} + A^i_{ljk} + A^i_{klj} = 0$$

constitutes a complete set of identities in the components of the first normal tensor A.

For corresponding to any set of numbers A^i_{jkl} satisfying the above identities we may construct the functions

$$C^i_{jk}(y) = A^i_{jkl}\, y^l \, .$$

Evidently $C^i_{jk} = C^i_{kj}$, and by (36.2)

$$C^i_{jk}\, y^j\, y^k = 0,$$

The C's will therefore serve as components of a symmetric affine connection in a system of normal coordinates y. For this connection the numbers A^i_{jkl} are seen to be the components of the first normal tensor at the point y = 0. Thus any identity between the components of a general first normal tensor can be satisfied by a set of numbers subject only to conditions (36.1) and (36.2).

If we now use the replacement theorem to express the curvature tensor in the form

(36.3) $$B^i_{jkl} = A^i_{jkl} - A^i_{jlk}$$

we have at once

(36.4) $$B^i_{jkl} = - B^i_{jlk}$$

and

(36.5) $$B^i_{jkl} + B^i_{ljk} + B^i_{klj} = 0$$

the latter being a consequence of (36.2). Moreover we have

$$B^i_{jkl} + B^i_{kjl} = A^i_{jkl} + A^i_{kjl} - A^i_{jlk} - A^i_{klj}$$

$$= 2A^i_{jkl} - (A^i_{ljk} + A^i_{klj})$$

$$= 3A^i_{jkl} .$$

<u>Theorem.</u> The identities (36.4) and (36.5) constitute a complete set for the components of the curvature tensor of an affine space with a symmetric affine connection.

Let $\tilde{B}^i_{jkl}$ be any set of numbers satisfying (36.4) and (36.5). Define numbers $\tilde{A}^i_{jkl}$ by the relation

$$3\tilde{A}^i_{jkl} = \tilde{B}^i_{jkl} + \tilde{B}^i_{kjl} .$$

These $\tilde{A}$'s will satisfy (36.1) and (36.2);

$$3(\tilde{A}^i_{jkl} - \tilde{A}^i_{kjl}) = \tilde{B}^i_{jkl} + \tilde{B}^i_{kjl} - \tilde{B}^i_{kjl} - \tilde{B}^i_{jkl} = 0$$

$$3(\tilde{A}^i_{jkl} + \tilde{A}^i_{ljk} + \tilde{A}^i_{klj}) = \tilde{B}^i_{jkl} + \tilde{B}^i_{ljk} + \tilde{B}^i_{klj} + \tilde{B}^i_{kjl} + \tilde{B}^i_{jlk} + \tilde{B}^i_{lkj} = 0 .$$

On the other hand, the $\tilde{A}$'s uniquely determine the $\tilde{B}$'s from which they were derived:

$$3(\tilde{A}^i_{jkl} - \tilde{A}^i_{jlk}) = \tilde{B}^i_{jkl} + \tilde{B}^i_{kjl} - \tilde{B}^i_{jlk} - \tilde{B}^i_{ljk}$$

$$= \tilde{B}^i_{jkl} + \tilde{B}^i_{jkl} - (\tilde{B}^i_{klj} + \tilde{B}^i_{ljk})$$

$$= 3\,\tilde{B}^i_{jkl} .$$

Consequently we can construct a connection $C^i_{jk} = \tilde{A}^i_{jkl}\, y^l$ for which the $\tilde{B}$'s will be the components of the curvature tensor at $y = 0$. The $\tilde{B}$'s, subject only to (36.4) and (36.5), will therefore satisfy any identity which holds for all curvature tensors derived from a symmetric affine connection.

A similar argument leads to the

<u>Theorem:</u> The identities

$$B_{ijkl} = -B_{jikl} = -B_{ijlk}$$

$$B_{ijkl} + B_{iljk} + B_{iklj} = 0 ,$$

constitute a complete set of identities in the components B_{ijkl} of the curvature tensor of a metric space.

Lectures

by

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on

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PART II. ANALYTIC FUNCTIONS ON A COMPACT SURFACE

1936-1937
2d term

Notes by George Comenetz

The Institute for Advanced Study
and
Princeton University

CONTENTS

CHAPTER VII

Gradient Vector Fields.

37. <u>Condition for a vector of class C' to be a gradient</u>. Let $\mathcal{X}_n$ be an n-dimensional coordinate space of class C^m, with $m \geq 2$ (§19). Consider a covariant vector a_i which is defined and of class C' in an open set D in $\mathcal{X}_n$.

Suppose there exists a scalar φ, defined and of class C^2 in D, whose gradient is a_i:

$$\frac{\partial \varphi}{\partial x_i} = a_i \tag{37.1}$$

Then evidently

$$\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} = 0 \tag{37.2}$$

It is easy to show that for any vector a_i the left member of (37.2), the "curl" of a_i, is a skew-symmetric tensor T_{ij}. Thus if a_i is a gradient, its curl must vanish.

If a_i is the gradient of φ, it is also the gradient of φ + const. Conversely, if a_i is the gradient of φ and of $\tilde{\varphi}$, then $\psi' = \varphi - \tilde{\varphi}$ is constant, provided D is connected. For if P and Q are any two points of D, and if we join them by a curve of class C lying in D, then along this curve ψ' becomes a function of class C' of the parameter t. Since $\partial \psi/\partial x = 0$ in D, we have $d\psi/dt = \partial\psi/\partial x_i \cdot dx_i/dt = 0$ and so $\psi(P) = \psi(Q)$. Thus, if a_i is the gradient of a scalar in a region (a connected open set) D, then the scalar is determined up to an arbitrary additive constant.

Finally we shall show that if D is a simply connected region, and if the curl of a_i vanishes in D, then a_i is the gradient of some scalar φ defined throughout D.

Let P_0 and Q be any two points of D, and let E and E' be any two curves of class C^1 joining P_0 to Q in D. We wish to prove first that

$$(37.3) \qquad \int_{P_0}^{Q} a_i\, x_i'\, dt \qquad\qquad \left(x_i' \equiv \frac{dx_i}{dt}\right)$$

has the same value along E as along E'. (It should be observed that $a_i\, x_i'$ is an invariant, and that (37.3) is invariant under any change of the parameter t.)

We assume (but see § 38) that it is possible to draw a continuous family of curves $P = P(t, \epsilon)$, $0 \leqq t \leqq 1$, $0 \leqq \epsilon \leqq 1$, lying in D and having the following properties:

a) $P(t,0)$ is E, $P(t,1)$ is E'.

b) $P(0, \epsilon) = P_0$, $P(1, \epsilon) = Q$ for each ϵ,

c) The derivatives $\partial x_i(t,\epsilon)/\partial t$, $\partial x_i(t,\epsilon)/\partial \epsilon$, and $\partial^2 x_i(t,\epsilon)/\partial t\, \partial \epsilon$ exist and are continuous.

Then the integral (37.3), taken along the curves of the family, becomes a function $J(\epsilon)$ of ϵ. That is

$$J(\epsilon) = \int_0^1 a_i \left[x(t, \epsilon)\right] \frac{\partial x_i(t,\epsilon)}{\partial t}\, dt\,, \qquad 0 \leqq \epsilon \leqq 1.$$

To prove that $J(\epsilon)$ is constant, we show that $J'(\epsilon)$ exists and is zero for each ϵ. We merely indicate the steps, however, since another proof that a vector is a gradient if its curl vanishes will be given later, in § 40. (It will be seen that we are using a standard calculus of variations procedure here. The integrand is $a_i\, x_i'$, the Euler vector

$$\frac{\partial a_i}{\partial x_j} x_i' - \frac{d}{dt}(a_j) = x_i' \operatorname{curl}_{ij} a = 0,$$

and hence every curve is an extremal.).

Briefly, then, we have

$$J'(\epsilon) = \int_0^1 \frac{\partial}{\partial \epsilon}(a_i\, x_i')\, dt,$$

$$J'(\epsilon) = \int_0^1 \left(\frac{\partial a_i}{\partial x_j}\frac{\partial x_j}{\partial \epsilon}x_i' + a_i\frac{\partial x_i'}{\partial \epsilon}\right) dt.$$

Integrating the second term by parts,

$$J'(\epsilon) = \int_0^1 \frac{\partial a_i}{\partial x_j}\frac{\partial x_j}{\partial \epsilon}\, x_i'\, dt - \int_0^1 \frac{\partial a_i}{\partial x_j}\, x_j'\, \frac{\partial x_i}{\partial \epsilon}dt + \left[a_i\frac{\partial x_i}{\partial \epsilon}\right]_0^1$$

(no other terms appear, because $a_i \cdot \partial x_i/\partial\epsilon$ is an invariant). The bracket term vanishes, since $\partial x_i/\partial \epsilon = 0$ for $t = 0$ and $t = 1$. Hence

$$J'(\epsilon) = \int_0^1 \left(\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i}\right) x_i' \frac{\partial x_j}{\partial \epsilon}\, dt.$$

As curl $a = 0$, we see that $J'(\epsilon) = 0$, and so $J(0) = J(1)$. This means that the value of (37.3) is the same for all curves of class C^1 lying in D and joining P_0 to Q.

Let this common value be denoted by $\varphi(Q)$, P_0 being thought of as fixed. That is,

$$(37.4) \qquad \varphi(Q) = \int_{P_0}^{Q} a_i\, dx_i$$

where we are free to take the integral along any curve of class C . It is easy to show now that φ possesses a gradient, which is a_i.

In fact, for a given i, select a curve which at Q has the direction of the x_i axis. Follow this by a segment of the x_i axis leading from Q to Q'. Then

$$\varphi(Q') - \varphi(Q) = \int_Q^{Q'} a_i\, dx_i \qquad (\not\Sigma_i).$$

By the theorem of the mean,

$$\frac{\varphi(Q') - \varphi(Q)}{x_i(Q') - x_i(Q)} = a_i(Q''),$$

where Q'' is on the x_i axis between Q and Q'. Taking the limit as $Q' \to Q$, we obtain (37.1).

We have now completed a proof of the following theorem.

<u>Theorem</u>: In order for a covariant vector a_i, defined and of class C' in a simply connected region D, to be the gradient of a scalar φ of class C^2 in D, it is necessary and sufficient that

$$\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} = 0$$

at each point of D. If this condition is satisfied φ is determined up to an arbitrary additive constant.

We may add that in a simply connected region D, curl a = 0 is necessary and sufficient for $\int_{P_0}^{Q} a_i \, dx_i$ to be independent of the curve of class C^1 joining P_0 to Q in D. Indeed in any region the two problems, when is a_i a gradient, and when is $\int a_i \, dx_i$ independent of the path, are completely equivalent, even if a_i is merely assumed to be continuous. For if a_i is the gradient of φ, then

$$\int_{P_0}^{Q} a_i \, dx_i = \int_{P_0}^{Q} \frac{\partial \varphi}{\partial x_i} \, dx_i = \int_{P_0}^{Q} d\varphi = \varphi(Q) - \varphi(P_0) \tag{37.5}$$

for any curve from P_0 to Q. That is, if a_i is a gradient it is the gradient of $\int a_i \, dx_i$, the integral being independent of the path. Conversely, if the integral depends only on the endpoints, we see as in the proof of the above theorem that its value (P_0 being fixed) is a point function whose gradient is a_i.

38. <u>A modified proof</u>. There is one objection to our proof of the theorem above: the family $P(t,\epsilon)$ was assumed differentiable in ϵ), whereas

the statement that D is simply connected merely implies that some <u>continuous</u> family $P(t,\epsilon)$ exists satisfying a) and b). No doubt differentiable families also exist, but this seems never to have been proved. Therefore we shall modify our method so as to avoid the assumption.

Given a fixed point P_0, let $U(P_0)$ be a coordinate system covering P_0 which is the image of a convex region in n-dimensional number space. For instance if P_0 has coordinates $x_i{}^0$, then $\sum_i (x_i - x_i{}^0)^2 < r^2$ defines such a $U(P_0)$ when r is sufficiently small.

If Q is any point in $U(P_0)$, and if E and E' are any two curves of class C^1 lying in $U(P_0)$ and joining P_0 to Q, a family $P(t,\epsilon)$ having the properties a), b), c) of §37 can be drawn in $U(P_0)$. In fact, let E and E' be given by $x_i = x_{1i}(t)$ and $x_i = x_{2i}(t)$, $0 \leqq t \leqq 1$ respectively. Then

$$x_i(t,\epsilon) \equiv (1-\epsilon)\, x_{1i}(t) + \epsilon\, x_{2i}(t) \quad , \quad 0 \leqq t \leqq 1, \quad 0 \leqq \epsilon \leqq 1$$

describes such a family.

The theorem of the previous section applies without question to $U(P_0)$: if curl $a = 0$ in D, there exist scalars φ defined throughout $U(P_0)$, having a_i as gradient there. In fact we have seen that the formula for these scalars is

$$(38.1) \qquad \varphi(Q) = \int_{P_0}^{Q} a_i\, dx_i + \text{const.} ,$$

the integral being taken along any curve in $U(P_0)$ from P_0 to Q. We must show that these locally defined scalars φ can be pieced together so as to form a one-valued point function throughout D.

The fact that this is possible depends on two properties of the local scalars φ :

A) In each $U(P_0)$ the function $\varphi(Q)$ is uniquely determined when its value at any single point of $U(P_0)$ is arbitrarily assigned. This is obvious from (38.1).

B) Suppose that a $U(P_0)$ and a $U'(P'_0)$ have a point R in common. Denote by $[U \cap U']_R$ the component (maximal connected subset) of the intersection $U \cap U'$ which contains R; this is a region consisting of those points of the intersection that can be reached from R by continuous arcs lying in the intersection. Then if a scalar φ corresponding to U and a scalar φ' corresponding to U' have the same value at R, they will have the same value at every point of $[U \cap U']_R$. For if S is any other point in the component and E is an arc of class C^1 joining R to S and lying in the intersection,

$$\varphi(S) - \varphi(R) = \varphi'(S) - \varphi'(R) = \int_{E} a_i \, dx_i ,$$

and so $\varphi(S) = \varphi'(S)$.

Using properties A and B we can show that there exists a scalar $\tilde{\varphi}$ defined throughout D, which in any $U(P_0)$ coincides with some one of the φ's corresponding to that $U(P_0)$. Then a_i will of course be the gradient of $\tilde{\varphi}$ at every point of D, and this will fully establish the theorem of the previous section.

As the method to be used applies just as well in certain other situations that will arise later, we shall present in the next section a more general "monodromie" theorem than the one actually needed here. The following lemma will be necessary.

<u>Lemma</u>: Let E, given by $P = P(t)$, $0 \leqq t \leqq 1$, be a continuous curve in a Hausdorff space. Suppose that to each point P of E there corresponds some open set U(P) containing P. Then a finite number $U_1, \dots, U_p$ of the sets U, and a set of intervals $a_i \leqq t \leqq b_i$ $(i = 1, \dots, p;\ a_1 = 0,\ b_p = 1)$ having $a_1 < a_2$, $a_{j+1} < b_j < a_{j+2}$ $(j = 1, \dots, p-2)$ and $a_p < b_{p-1} < b_p$, can be found such that the arc of E corresponding to the ith interval lies in U_i.

Proof: For each value of t let a U be selected containing P(t). Then for each t the image of some closed interval containing t in its interior still lies in U, because E is continuous (here and in the following we omit the modified statement necessary if $t = 0$ or 1). A finite number of the open intervals derived from these by dropping the endpoints suffices to cover $[0, 1]$, and hence the corresponding closed intervals cover $[0, 1]$ in such a way that each t is interior to some one of them. The finite number of U's in which the images of these closed intervals lie obviously covers E. (The argument shows, indeed, that the continuous image of any bicompact space on a Hausdorff space is bicompact (i.e., possesses the Heine-Borel property).)

Let $a_1 \leq t \leq b_1$ be a longest interval containing $a_1 = 0$, out of the finite number. Next choose as $a_2 \leq t \leq b_2$ one of the intervals containing b_1 in its interior, for which b_2 is a maximum; then $a_2 > a_1$, or else the first interval would not have been a longest one. Next take $a_3 \leq t \leq b_3$ with $a_3 < b_2 < b_3$ and b_3 a maximum; as before, $a_3 \geq b_1$. If $a_3 = b_1$ we shorten the new interval, moving a_3 to the right so that $b_1 < a_3 < b_2$. Continuing in this way, we eventually reach a $b_p = 1$. Moving a_p if necessary, we have the required construction.

39. A general monodromie theorem. We consider a Hausdorff space H which is locally arcwise connected (i.e., each neighbourhood of a point P contains some neighbourhood of P any two points of which can be joined by a continuous arc lying in the former neighbourhood). A region in H is easily shown to be arcwise connected.

Theorem: Let Φ(P) be a many-valued point function defined in a simply connected region D in H. Suppose that each point P_0 of D lies in a certain region $U(P_0)$ in D, in which the function Φ(P) is separated into a set of single-valued functions $\varphi(P)$ each defined throughout $U(P_0)$. That is:

A) Given any value of Φ at any point in $U(P_0)$, there is just one of the φ's of U which takes on that value at that point. The value of a φ at any point is one of the values of Φ at that point.

Further, suppose the functions φ can be <u>continued</u> <u>locally</u>, in this sense:

B) If two distinct regions, U and U', have a point R in common, and if a φ of U and a φ' of U' have the same value at R, they have the same value at every point of the component of the intersection of U and U' which contains R.

Then the conclusion is that the functions φ can be continued so as to produce a separation of Φ into one-valued functions all over D. In other words, there exist one-valued functions $\breve{\varphi}(P)$, defined throughout D, which in each $U(P_0)$ coincide with some one of the φ's corresponding to that $U(P_0)$. Such a $\breve{\varphi}(P)$ is uniquely determined if its value is assigned at one point as any one of the values of Φ at that point.

(The component of $U \cap U'$ containing R is readily shown to be the region consisting of those points of $U \cap U'$ which can be joined to R by arcs in $U \cap U'$).

Proof: Let P_0 be any fixed point in D, and let $\breve{\varphi}(P_0)$ be any one of the values $\Phi(P_0)$. Draw any continuous curve E in D, given by $P(t)$, $0 \leqq t \leqq 1$, from P_0 to an arbitrary point Q in D.

Cover E by a finite number, $U_1, \ldots, U_p$, of the given regions $U(P)$, in the manner described in the lemma. Select any p-1 parameter values $t_1, \ldots, t_{p-1}$ such that $a_{j+1} \leqq t_j \leqq b_j$ $(j = 1, \ldots, p-1)$.

Let $\varphi_1(P)$ be that φ of U_1 whose value at P_0 is $\breve{\varphi}(P_0)$. Next let $\varphi_2(P)$ be the φ of U_2 whose value at the "transition point" $P_1 \equiv P(t_1)$ is $\varphi_1(P_1)$. Then φ_3 is the φ of U_3 for which $\varphi_3(P_2) = \varphi_2(P_2)$, and so on to $\varphi_p(P)$ defined in U_p. We set $\tilde{\varphi}(Q)$ equal to $\varphi_p(Q)$, and we must show that this determination of $\tilde{\varphi}$ does not depend on the arbitrary elements in the construction.

For this purpose we define a one-valued function $f(t)$, $0 \leqq t \leqq 1$, as follows:

$$f(t) = \varphi_i\left[P(t)\right] \quad , a_i \leqq t \leqq b_i \qquad (i = 1, \ldots, p).$$

Of course $f(t)$ is defined in two ways when $a_{j+1} \leqq t \leqq b_j$ $(j = 1, \ldots, p-1)$, but there is no conflict, since the image of this interval lies in the component of $U_j \cap U_{j+1}$ containing P_j. As φ_j and φ_{j+1} agree at P_j they agree at every point of the component, by hypothesis B).

Now suppose that we have another determination of $\tilde{\varphi}(Q)$ along the same curve E by means of regions $U'_1, \ldots, U'_q$, intervals $[a'_k, b'_k]$ $(k = 1, \ldots, q)$, etc. We must show that $f'(1) = f(1)$.

We have $f'(0) = f(0)$. If f and f' are not equal for all values of t, there exists a cut value $0 \leqq T \leqq 1$ such that $f(t) = f'(t)$ for $0 \leqq t < T$, but $f(t) \neq f'(t)$ for some $T \leqq t \leqq T + \epsilon$, ϵ being arbitrarily small.

Now T is an interior point of some $[a_i, b_i]$ and also of some $[a'_k, b'_k]$, and $P(T)$ is an interior point of U_i and of U'_k. Let $\alpha < t < \beta$ be an open interval lying in $[a_i, b_i]$ and in $[a'_k, b'_k]$ and containing T. Its image lies in the component of $U_i \cap U'_k$ containing $P(T)$. As φ_i and φ'_k agree for $\alpha < t < T$, they agree also for $T \leqq t < \beta$; hence $f = f'$ in the latter interval, and this is a contradiction. Therefore $f'(1) = f(1)$.

Finally, $\tilde{\varphi}(Q)$ must be proved to be independent of the curve E. Let E', given by $P(t)$, $0 \leqq t \leqq 1$, be another curve from P_0 to Q lying in D. Because D is simply connected there exists a continuous family of curves $P(t, \epsilon)$ in D, $0 \leqq t \leqq 1$, $0 \leqq \epsilon \leqq 1$, where $P(t, 0)$ is E, $P(t, 1)$ is E', $P(0, \epsilon)$ is P_0, and $P(1,\epsilon)$ is Q. For the curves of this family $\tilde{p}(Q)$ is apparently a function of ϵ, $\tilde{\varphi}(Q, \epsilon)$. Our object is to prove that $\tilde{\varphi}(Q, \epsilon)$ is constant with respect to ϵ.

Let ϵ_0 be any fixed value of ϵ, and denote the corresponding curve by E^0. Let $U_1, \dots, U_p$ be any one of the sets of regions covering E^0 that we have used to define $\tilde{\varphi}(Q, \epsilon_0)$. We denote by V_j the component of $U_j \cap U_{j+1}$ which contains the arc $P(t, \epsilon_0)$, $a_{j+1} \leq t \leq b_j$ $(j = 1, \dots, p-1)$.

Let an $\eta > 0$ be taken so small that for every $\epsilon_0 - \eta < \epsilon < \epsilon_0 + \eta$ the point $P(t, \epsilon)$ lies in the above U_i when $a_i \leq t \leq b_i$ $(i = 1, \dots, p)$, and in V_j when $a_{j+1} \leq t \leq b_j$ $(j = 1, \dots, p-1)$. Such η's exist because $P(t, \epsilon)$ is continuous and p is finite.

The same regions U_i and t-intervals $[a_i, b_i]$ can now be used to determine $\tilde{\varphi}(Q, \epsilon'')$ along any curve E'' of the family for which $\epsilon_0 - \eta < \epsilon'' < \epsilon_0 + \eta$. Let $P_j \equiv P(t_j, \epsilon_0)$, $a_{j+1} \leq t_j \leq b_j$ $(j = 1, \dots, p-1)$ be a sequence of transition points along E^0, and let $P_1'', \dots, P_{p-1}''$ be similar points along E''. Then P_j and P_j'' both lie in V_j.

The functions φ_1 and φ_1'' of U_1, with which our construction starts along E^0 and E'' respectively, are the same, as both are determined by the given $\tilde{\varphi}(P_0)$. Next we see that φ_2'' and φ_2 of U_2 are the same; for φ_2'' agrees with φ_1 (or φ_1'') at P_1'', hence throughout V_1, hence at P_1. Thus φ_2 and φ_2 have the same value at P_1 and therefore coincide.

Evidently we can repeat this argument until we have the result that $\varphi_p'' \equiv \varphi_p$. This shows that $\tilde{\varphi}(Q, \epsilon_0) = \tilde{\varphi}(Q, \epsilon)$ for every ϵ in the interval $(\epsilon_0 - \eta, \epsilon_0 + \eta)$. Covering $[0, 1]$ by a finite number of such intervals, we see that $\tilde{\varphi}(Q, 0) = \tilde{\varphi}(Q, 1)$. Consequently $\tilde{\varphi}(Q)$ is the same whether obtained from E or from E', and so our construction yields a single-valued function $\tilde{\varphi}(Q)$ defined throughout D.

It has still to be proved that such a $\tilde{\varphi}$ (P) has the property asserted in the theorem.

Let U be any one of the given regions and let P be any fixed point in U. We say that the φ of U for which $\varphi(P) = \tilde{\varphi}(P)$, will be equal to $\tilde{\varphi}$ at every point Q of U.

To prove this, draw any curve E' from P_0 to P in D. Because U is connected, a second curve E", lying in U, can be drawn from P to Q. Let the curve E' + E", consisting of E' followed by E", be called E.

Let $U_1, \ldots, U_{p-1}$ be, as usual, a chain of regions covering E'. It is easily seen that $U_1, \ldots, U_{p-1}, U_p \equiv U$ can be used to cover E in the required fashion. Let $P_1, \ldots, P_{p-2}$ be the transition points along E'; the remaining transition point, from U_{p-1} to U, can be taken to be $P_{p-1} \equiv P$.

At P_j $(j = 1, \ldots, p-1)$ the value of φ_j gives the value of φ_{j+1}, by our construction; and it also gives the value of $\tilde{\varphi}$, as we see by thinking of the curve as terminating at P_j. Hence $\tilde{\varphi}(P) = \varphi_p(P)$. This implies that φ_p is the φ of U defined above. By definition, $\tilde{\varphi}$ (Q) is given by $\varphi(Q)$, and this is what we had to show.

As to the uniqueness of $\tilde{\varphi}$: let $\tilde{\varphi}'$ be any function in D which in each U coincides with some φ of that U. Taking any P_0 in D, let $\tilde{\varphi}$ be the function which our construction provides if we start at P_0 with the value $\tilde{\varphi}'(P_0)$. We assert that $\tilde{\varphi}$ and $\tilde{\varphi}'$ are equal at every point Q in D. In fact, considering the construction for $\tilde{\varphi}$ (Q), we see at the first step that $\tilde{\varphi}$ and $\tilde{\varphi}'$ coincide in U_1, since both must be equal to φ_1 there. In particular, $\tilde{\varphi}(P_1) = \tilde{\varphi}'(P_1)$. Hence φ_2 must be the φ of U_2 which $\tilde{\varphi}$ and $\tilde{\varphi}'$ reduce to in U_2. Therefore $\tilde{\varphi}(P_2) = \tilde{\varphi}'(P_2)$. At the last step we find that $\tilde{\varphi}(Q) = \tilde{\varphi}'(Q)$, q. e. d. This completes the proof of the theorem.

It may be emphasized that nothing has been assumed about the character of the range of the many-valued function $\Phi(P)$. In fact the values of Φ need not be real numbers, but can be elements of any set whatever. The only restriction on Φ, it will be found, is that the cardinal number of its values at one point of D must be the same as at any other point. In the application in § 38, Φ would be the function which assumes every real value at every point of D.

The one-valued functions φ may not actually be given, but some construction or operation may be defined such that when a value of Φ is selected at a point, a function φ is uniquely determined in any one of the regions U containing the point. For example, the determination of φ may depend on solving a system of differential equations, the choice of a value of Φ corresponding to a choice of initial conditions.

40. Condition for a continuous vector to be a gradient.

We return to the coordinate space $\mathcal{X}_n$ of § 37, but now with $m \geqq 1$. Let a_i be a covariant vector which is continuous, but need not possess a derivative, in a simply connected region D. The problem can still be proposed, to secure necessary and sufficient conditions for a_i to be the gradient of some scalar φ of class C^1 defined throughout D, but of course the answer can no longer be given in terms of curl a.

At the end of § 37 we showed that a_i is a gradient if, and only if,

$$(40.1) \qquad \int_P^Q a_i \, dx_i$$

has the same value for all curves of class C^1 lying in D and joining any two given points P and Q. Instead of C^1 we can say D^1 (a curve of class D^1 is a continuous curve made up of a finite sequence of closed arcs of class C^1), for the curves of class D^1 include those of class C^1, and on the other hand the argument based on (37.5) applies with a simple modification to a curve of class D^1.

Let x_i^0 be coordinates of a point P_0 of D in some coordinate system. A region in D defined by

$$\left| x_i - x_i^0 \right| < b$$

will be called a coordinate cube about P_0, with the symbol $B(P_0)$.

If for every cube B in D the integral (40.1) has a constant value for all the curves lying in B and joining any two given points of B, a_i will be a gradient in each B and hence in D. This is established by the reasoning of §§ 38 and 39.

Let P and Q be two points in a cube B, and let $i_1, \ldots, i_n$ be any permutation of the integers $1, \ldots, n$. Consider the curve of class D^1 lying in B and joining P to Q, which is obtained by starting at P and first varying x_{i_1} alone, from its value at P to its value at Q; then varying x_{i_2} similarly; and so on to x_{i_n}. There are n! (or fewer) of these "broken lines" between P and Q.

If the value of the integral (40.1) is the same for all of the n! broken lines between any two given points in B, a_i will be a gradient in B. For the point function $\varphi(Q)$ defined by the common value of these n! integrals has the gradient a_i. Indeed it will be found that in the argument following (37.4) no more freedom in the choice of curves is necessary than these n! broken lines afford. (As a matter of fact the n broken lines corresponding to the n cyclic permutations of 1, ... , n would do just as well.) Of course once a_i is known to be a gradient in B, it follows that $\int a_i\, dx_i$ is independent of the path for <u>all</u> curves in B.

Let r, s be any two distinct integers from 1 to n, and let P and Q be two points which differ only in the coordinates x_r and x_s. If for every two such points the integral is the same along either of the two broken lines joining them, it will have a constant value along the n! broken lines between any two points. For if $i_1, \dots, i_p, i_{p+1}, \dots, i_n$ is the permutation describing one of the latter lines, it is clear that the integral does not change when we pass to the line described by $i_1, \dots, i_{p+1}, i_p, \dots, i_n$. And it is easy to show that any permutation of $1, \dots, n$ can be turned into any other (e.g., into $1, \dots, n$) by successive transpositions of adjacent numbers.

Let P and Q be two points that differ only in x_r and x_s. The closed curve which is obtained by following one of the broken lines from P to Q, and then the other one back from Q to P, will be called a "coordinate rectangle", or if $|x_r(P) - x_r(Q)| = |x_s(P) - x_s(Q)|$, a "coordinate square". Our result so far can be stated in this way: a_i is a gradient in D if, and only if, $\int a_i \, dx_i$ vanishes around every coordinate rectangle in every cube in D. (At each step the condition for a_i to be a gradient is being replaced by a weaker one, hence the necessity of the conditions is obvious.)

If the integral vanishes around squares, it vanishes around any rectangle. First, this is true for rational rectangles (i.e., rectangles whose sides are commensurable, "sides" referring to the absolute values of the x_r and x_s coordinate differences). Such rectangles can be cut up into squares by suitable x_r and x_s coordinate lines, and it is seen in the usual way that the integral around the rectangle is a sum of integrals around the component squares. The assertion then follows by a simple argument for irrational rectangles also, since they can be approximated by rational ones.

It will be convenient to speak of the points of a cube as though they were identical with their image points in number-space.

Let P be a point lying in or on the boundary of a square of side c ("square" here refers to the two-dimensional object). If for each point P

$$\frac{\int a_i \, dx_i}{c^2} \longrightarrow 0 \quad \text{as } c \rightarrow 0, \tag{40.2}$$

then the integral will vanish around any square. It is understood that the square of side c is to be permitted to shrink down to P in any way, so long as P never lies outside of it. The integral in (40.2) is taken around the square of side c.

To prove this, consider a given square of side c_1, and let I_1 be the value of the integral around it. Divide the square into four equal squares, each of side $c_2 = c_1 / 2$, by the two lines joining the midpoints of opposite edges. Let I_2 be the value of a numerically largest of the four integrals around these four squares. Since I_1 equals a sum of integrals around the smaller squares,

$$|I_1| \leqq 4 \, |I_2| ,$$

and consequently

$$\frac{|I_1|}{{c_1}^2} \leqq \frac{|I_2|}{{c_2}^2} .$$

That is, the absolute value of the quotient of integral by area does not decrease when we pass from the larger square to the smaller one.

Evidently we can form a nested sequence of squares converging to a point P which lies in all of them, such that the absolute value of the above quotient is never less than $|I_1| / {c_1}^2$. Since we are supposing that (40.2) holds, I_1 must be zero.

What we have proved is this:

<u>Theorem</u>: In order for a covariant vector a_i which is continuous in

a simply connected region D to be the gradient of a scalar of class C^1 in D, it is necessary and sufficient that for any sequence of coordinate squares lying in one plane and converging to a point contained in all of them, the quotient of $\int a_i \, dx_i$ around a square by the area of the square should tend to zero. If this condition holds, the scalar is determined up to an arbitrary additive constant.

The condition need only be verified in some definite set of coordinate cubes covering D, as the proof shows. (It may actually be an invariant condition at each point, but this does not seem to be obvious.)

Let the r^{th} and s^{th} coordinates of the corners of a square be, in order,

$$(x_r \ , x_s) \ , \ (x_r + c, x_s) \ , \ (x_r + c, x_s + c) \ , \ (x_r \ , x_s + c).$$

The quotient in (40.2) can be written (only the r^{th} and s^{th} coordinates being given explicitly) as

$$\frac{1}{c^2}\int_0^c \left[a_r(x_r+t, x_s) - a_r(x_r+t, x_s+c) - a_s(x_r \ , x_s+t) + a_s(x_r+c, x_s+t)\right] dt.$$

By the theorem of the mean, this is

$$\frac{a_r(x_r+\theta, x_s) - a_r(x_r+\theta, x_s+c) - a_s(x_r \ , x_s+\theta) + a_s(x_r+c, x_s+\theta)}{c},$$

(40.3)

where $0 < \theta < c$.

If the first partial derivatives of a_i exist ($m \geqq 2$), (40.3) is a difference quotient of a differentiable function of t:

$$a_s(x_r + t, x_s+\theta) - a_r(x_r+\theta, x_s+t), \qquad 0 \leqq t \leqq c \qquad (\theta \text{ fixed}).$$

Hence it equals

$$(40.4) \qquad \frac{\partial a_s(x_r+\theta', x_s+\theta)}{\partial x_r} - \frac{\partial a_r(x_r+\theta, x_s+\theta')}{\partial x_s},$$

where $0 < \theta' < c$.

If a_i is of class C^1, the limit of (40.4) as $c \to 0$ and $(x_r, x_s) \to (x_r^0, x_s^0)$, exists and equals

$$\frac{\partial a_s}{\partial x_r} - \frac{\partial a_r}{\partial x_s}$$

evaluated at x^0. Therefore in this case what we have is a new proof of the theorem of § 37.

It can be shown that if the first derivatives of a_i exist and curl $a = 0$, a_i is a gradient. In fact somewhat weaker conditions suffice (see Saks, Théorie de l'Intégrale, p.243).

The method of this section can be extended to treat completely integrable systems of linear differential equations, and to yield a generalization of Stokes' theorem.

CHAPTER VIII

ANALYTIC FUNCTIONS IN TWO DIMENSIONS

41. Conformal Riemann Space.

Let $\tilde{\gamma}_n$ be an n-dimensional coordinate manifold of class C^m, $m \geqq 1$. Suppose that at each point of $\tilde{\gamma}_n$ there is given a pencil of positive definite symmetric tensors

(41.1) $$\{\lambda\, g_{jk}\}$$

depending on an arbitrary positive scalar factor λ. An $\tilde{\gamma}_n$ with this structure is called a "conformal Riemann space", or simply a "conformal space". We denote it by R.

The parameter λ should be thought of as independently arbitrary at each point P of R. If we make a definite choice of a one-valued point function $\lambda(P)$ in R, not necessarily continuous, (41.1) defines a single tensor $\lambda(P)\, g_{jk}(P)$ throughout R. If this is interpreted as a metric tensor, R becomes an ordinary Riemann space (perhaps with a discontinuous metric). Thus a conformal space might also be defined as a class of Riemann spaces on a single coordinate manifold, the ratio of any two of whose metric tensors (compared in the same coordinate systems) is an arbitrary positive scalar. Otherwise stated, a conformal space is a Riemann space whose metric tensor is given only up to a positive factor. For instance, the ratios g_{jk} / g_{11} might be assigned in some definite set of coordinate systems (note that $g_{11} \neq 0$).

Consider an oriented conformal space R; that is, one in which the Jacobian of coordinate transformations is always positive. Let g_{jk} be any one of the tensors (41.1) at a point of R, and write g for the determinant $|g_{jk}|$. If we replace g_{jk} by $\lambda\, g_{jk}$ $(\lambda > 0)$, g is multiplied by λ^n. It follows that

$$G_{jk} \equiv \frac{g_{jk}}{\sqrt[n]{g}} \tag{41.2}$$

(where $\sqrt[n]{g}$ is the positive n^{th} root of the positive quantity g) is the same for all the tensors of the pencil (41.1). This object G_{jk}, which is uniquely determined at each point of the conformal space, is termed the "fundamental conformal tensor" of the space.

The conformal tensor G_{jk} is a relative tensor of weight $-2/n$; that is, its transformations law is

$$\overline{G}_{jk} = G_{h\ell} \frac{\partial x_h}{\partial \overline{x}_j} \frac{\partial x_\ell}{\partial \overline{x}_k} \left| \frac{\partial x}{\partial \overline{x}} \right|^{-\frac{2}{n}}$$

This follows from the fact that g is a relative scalar of weight 2.

(By definition, a "relative tensor" of weight p obeys the ordinary tensor transformation law, but with the (positive) p^{th} power of the Jacobian of the old variables with respect to the new ones inserted as a factor in the right member. This law is consistent. In a non-oriented space further definition would be necessary when p is not an integer or a fraction with given odd denominator, but we shall not be concerned with that case.)

It is evident from (41.2) that the determinant $G \equiv |G_{jk}|$ is equal to $+1$, and that G_{jk} is positive definite.

Conversely, consider an oriented coordinate manifold in which there is given a positive definite relative tensor G_{jk} of weight $-2/n$ and determinant 1. Take an arbitrary positive relative scalar μ of weight $2/n$. The product

$$g_{jk} \equiv \mu G_{jk} \tag{41.3}$$

is a positive definite tensor defined up to an arbitrary positive scalar factor (the ratio of two μ's). Moreover,

$$\frac{g_{jk}}{\sqrt[n]{g}} = \frac{\mu G_{jk}}{\sqrt[n]{\mu^n G}} = G_{jk}$$

since $G = 1$. Consequently an oriented conformal space can equally well be defined in terms of its pencil of metric tensors λg_{jk} or its conformal tensor G_{jk}. Either one determines the other, by means of the inverse relations (41.2) and (41.3).

For any one of the metric tensors g_{jk} there is the corresponding contravariant g^{jk} defined by

$$g^{jk} g_{k\ell} = \delta^j_\ell .$$

If g_{jk} is multiplied by λ, g^{jk} will be multiplied by $1/\lambda$. For G_{jk} there is the contravariant relative tensor G^{jk} of weight $2/n$, defined either by

$$G^{jk} G_{k\ell} = \delta^j_\ell \tag{41.4}$$

or by

$$G^{jk} = g^{jk} \sqrt[n]{g} . \tag{41.5}$$

In a Riemann space we can speak merely of a vector without specifying whether a covariant or contravariant vector is meant, since the covariant components of the vector determine uniquely the contravariant components and conversely. But in a conformal space the covariant (contravariant) components of a given contravariant (covariant) vector change by the same factor λ $(1/\lambda)$ as the covariant (contravariant) components of the tensor g_{jk}. The same rule holds for the lowering or raising of an index on any tensor.

The conformal invariants of a geometrical configuration are simultaneous invariants of the tensor g_{jk} and the set of tensors associated with the configuration, possessing the additional property of remaining unchanged by the introduction of the factor λ (we do not need an exact definition here). A conformal invariant is at the same time a metric invariant in each one of the Riemann spaces obtained from the conformal space by making a choice of a g_{jk}. For example, the cosine of the angle between two contravariant vectors is a conformal invariant; on the other hand the length of a non-zero vector is a Riemann invariant but not a conformal invariant.

If we are given a Riemann space, we can associate with it a conformal space simply by multiplying the given metric tensor by the arbitrary factor λ. By a conformal invariant of the Riemann space is understood a conformal invariant of the associated conformal space.

<u>Conformal mapping of n-dimensional Riemann spaces</u>. Let $P \leftrightarrow P'$ be a correspondence of class C^1 between an open set A in a Riemann space R and an open set A' in a Riemann space R'. Take any pair of corresponding points P and P', let U be a neighbourhood of P in A covered by coordinates x, and

denote its image in A' by U'. Then the same coordinates x can be introduced into U'; we may call them x' there, so that the correspondence between U and U' is represented by $x = x'$. Let the components of the metric tensor of R in U(x) be given by $g_{jk}(x)$, and let $g'_{jk}(x')$ be the metric tensor of R' expressed in U'(x'). Then if a relation of the form

$$g_{jk}(x) = \lambda(x')\, g'_{jk}(x') \tag{41.6}$$

always holds at corresponding points P and P', A and A' are said to be mapped conformally on each other. If $\lambda \equiv 1$ we have a congruence, or isometric correspondence.

A geometrical object (x, T) defined at P is said to be mapped into that object of the same kind (x', T') at P' for which the components in the x' coordinates are the same as those of (x, T) in the x coordinates. Under a conformal mapping conformal invariants are not changed in value. In particular, orthogonal vectors remain orthogonal. We shall show conversely that <u>if the correspondence between A and A' is such that every pair of orthogonal vectors at a point of A are carried into orthogonal vectors in A', it is a conformal mapping</u>.

Proof: By definition, the image of a vector at P whose components in the x coordinates are a^j, is a vector at P' having the same components a^j in the x' coordinates. By our assumption, $g_{jk}\, a^j\, b^k = 0$ implies $g'_{jk}\, a^j\, b^k = 0$, where g_{jk} and g'_{jk} are the components of the two metric tensors in the x and x' coordinates at P and P' respectively. Holding a^j fixed and varying b^k, we find easily that

$$g'_{jk}\, a^j = \lambda(a)\, g_{jk}\, a^j. \tag{41.7}$$

Let c^j be any vector at P not orthogonal to a^j. By the previous step

$$g'_{jk}\, c^j = \lambda(c)\; g_{jk}\, c^j.$$

Multiplying the first of these relations by c^k and the second by a^k and subtracting we have

$$\left[\lambda(a) - \lambda(c)\right]\; g_{jk}\, a^j\, c^k = 0.$$

Hence $\lambda(a) = \lambda(c)$. As (41.7) now holds with the same factor λ for a^j and for any vector not orthogonal to a^j, it holds for some n independent vectors. Consequently $g'_{jk} = \lambda\, g_{jk}$, and the mapping between A and A' is conformal.

Complex Tensors.

In any coordinate space, a geometrical object whose components are taken from the field of complex numbers and which obeys the ordinary tensor transformation law is called a complex tensor. (It is understood that all coordinate systems are still to be real.) The operations of addition, multiplication, contraction, and index permutation apply to complex tensors just as to real ones. The linear vector space spanned by a number of complex vectors at a point consists of all linear combinations of those vectors with complex scalar coefficients. In a space with a (real) affine connection, the invariant differential ϑT of a complex tensor T is defined by the same formula as for real tensors. Many other properties of real tensors carry over in the same immediate way to complex tensors. We shall use such properties below without mentioning them explicitly unless they do not obviously hold.

The real and imaginary parts of a complex tensor are real tensors of the same kind, because coordinate transformations are real. The invariant differential of a complex tensor can be obtained by differentiating invariantly the real and imaginary parts separately; this depends on the fact that the affine connection is assumed to be real.

A tensor will be understood to be real unless it is stated explicitly that it is complex.

42. A-vectors.

In an oriented 2-dimensional Riemann space, there exists a skew-symmetric tensor η having the covariant and contravariant components

$$\eta_{\alpha\beta} = \sqrt{g}\,\epsilon_{\alpha\beta} \quad \text{and} \quad \eta^{\alpha\beta} = \frac{\epsilon^{\alpha\beta}}{\sqrt{g}} \tag{42.1}$$

respectively, where $g \equiv |g_{\alpha\beta}|$ and

$$\epsilon_{11} = \epsilon_{22} = \epsilon^{11} = \epsilon^{22} = 0,$$
$$\epsilon_{12} = -\epsilon_{21} = \epsilon^{12} = -\epsilon^{21} = 1.$$

That $\eta_{\alpha\beta}$ is a tensor follows from the fact that g is a relative scalar of weight 2, and from the easily verified identity

$$\epsilon_{\gamma\delta}\frac{\partial x_\gamma}{\partial \bar{x}_\alpha}\frac{\partial x_\delta}{\partial \bar{x}_\beta} = \epsilon_{\alpha\beta}\left|\frac{\partial x}{\partial \bar{x}}\right|$$

which shows that $\epsilon_{\alpha\beta}$ is a relative tensor of weight -1. Similarly $\epsilon^{\alpha\beta}$ (with the same numerical components as $\epsilon_{\alpha\beta}$) is a relative tensor of weight $+1$, and hence $\eta^{\alpha\beta}$ is a tensor. From the relation

$$\epsilon^{\gamma\delta} g_{\alpha\gamma}\, g_{\beta\delta} = \epsilon_{\alpha\beta}\, g \quad ,$$

which may be verified just as above, it follows that $\eta^{\alpha\beta}$ and $\eta_{\alpha\beta}$ are in fact the contravariant and covariant components of a single η tensor. Finally, it is easy to check that these components satisfy the equation

$$\eta_{\alpha\gamma}\,\eta^{\beta\gamma} = \delta_\alpha^\beta \quad . \tag{42.2}$$

From now on we deal with an <u>oriented</u> <u>2-dimensional</u> <u>conformal</u> <u>Riemann</u> <u>space</u> <u>R</u> (§ 41). For the purposes of this and the next section, R need only be of class 1.

Corresponding to each of the ordinary Riemann spaces associated with R there is an η tensor. When $g_{\alpha\beta}$ is changed to $\lambda\, g_{\alpha\beta}$, $\eta_{\alpha\beta}$ and $\eta^{\alpha\beta}$

are multiplied by λ and $1/\lambda$ respectively.

Consider the equation

$$(42.3) \qquad (\eta^{\alpha\beta} + k\, g^{\alpha\beta})\,\xi_\alpha = 0$$

for the determination of the complex scalar k and the complex covariant vector ξ_α. This equation is <u>conformal</u>, that is, invariant under replacement of $g_{\alpha\beta}$ by $\lambda\, g_{\alpha\beta}$. The condition on k that there exist a non-zero vector ξ satisfying (42.3) is that the determinant $|\eta^{\alpha\beta} + k\, g^{\alpha\beta}| = 0$. Expanded, this condition becomes

$$\begin{vmatrix} k\, g^{11} & \sqrt{1/g} + k\, g^{12} \\ -\sqrt{1/g} + k\, g^{21} & k\, g^{22} \end{vmatrix} = \frac{k^2+1}{g} = 0.$$

Hence $k^2 = -1$, so that k is one of the two complex numbers $\pm\, i$.

Let $k = +i$. Then the matrix of the system (42.3) is of rank 1, and these equations define at any point of the space R a one-dimensional complex vector space $\{\xi_\alpha\}$. <u>We call the vectors of the space</u> $\{\xi_\alpha\}$, "<u>A-vectors</u>". Thus by definition an A-vector is a covariant complex vector obeying

$$(42.4) \qquad (\eta^{\alpha\beta} + i\, g^{\alpha\beta})\,\xi_\alpha = 0 \quad .$$

If ξ and ζ are any two A-vectors at a point, they both satisfy (42.4). Multiplying the equation for ξ by ζ_β and the equation for ζ by ξ_β and adding, we obtain

$$(42.5) \qquad g^{\alpha\beta}\,\xi_\alpha\,\zeta_\beta = 0,$$

since $\eta^{\alpha\beta}$ is skew-symmetric. Hence any two A-vectors at a point are orthogonal. In particular, an A-vector is orthogonal to itself.

If we set $\xi = \zeta$, we have

$$(42.6) \qquad g^{\alpha\beta}\,\xi_\alpha\,\xi_\beta = 0.$$

An A-vector is thus of zero length, or "isotropic", and cannot be real unless it is zero. Putting $\xi_\alpha = \mu_\alpha + i\,\nu_\alpha$ in (42.4), where μ and ν are real, we find

$$(42.7) \qquad \begin{aligned} \eta^{\alpha\beta}\mu_\alpha - g^{\alpha\beta}\nu_\alpha = 0, \\ \eta^{\alpha\beta}\nu_\alpha + g^{\alpha\beta}\mu_\alpha = 0 \end{aligned}$$

as the conditions for an A-vector.

These two equations are not independent. If we solve either of them for μ we obtain the one equation

$$(42.8) \qquad \mu_\alpha = \eta_{\alpha\beta}\, g^{\beta\gamma}\nu_\gamma$$

(in solving the first we use (42.2), and for the second we note that $g_{\gamma\beta}\eta^{\alpha\beta} = \eta^\alpha{}_\gamma = g^{\alpha\beta}\eta_{\beta\gamma}$). Any ordered pair of real covariant vectors μ and ν at a point which satisfy (42.8) will be said to be "conjugate $[\mu,\nu]$". Hence a complex vector $\xi_\alpha = \mu_\alpha + i\nu_\alpha$ is an A-vector if, and only if, the real vectors μ and ν are conjugate $[\mu,\nu]$.

Another important set of conditions for an A-vector is derived as follows. Put $\xi = \mu + i\nu$ in (42.6). This equation breaks up into the first two of the equations

$$(42.9) \qquad \begin{aligned} &(a) \quad g^{\alpha\beta}\mu_\alpha\mu_\beta = g^{\alpha\beta}\nu_\alpha\nu_\beta \quad , \\ &(b) \quad g^{\alpha\beta}\mu_\alpha\nu_\beta = 0 \quad , \\ &(c) \quad \mu_1\nu_2 - \mu_2\nu_1 \geqq 0 \quad . \end{aligned}$$

The third is found from (42.7) by multiplying by $\sqrt{g}\,\nu_\beta$:

$$(42.10) \qquad \epsilon^{\alpha\beta}\mu_\alpha\nu_\beta = \mu_1\nu_2 - \mu_2\nu_1 = \sqrt{g}\; g^{\alpha\beta}\nu_\alpha\nu_\beta \geqq 0 .$$

Thus a conjugate pair of vectors must obey (42.9). Furthermore, (42.10) shows that the inequality sign holds in (c) unless $\nu = 0$, in which case $\mu = 0$ also, in view of (a).

Conversely, let μ and ν be two real vectors obeying (42.9). By equation (b),

$$(\mu_\alpha)\,\nu^\alpha = 0.$$

Now

$$(\eta_{\alpha\beta}\nu^\beta)\,\nu^\alpha \equiv 0.$$

Assuming that $\nu \neq 0$, it follows that $\mu_\alpha = k\,\eta_{\alpha\beta}\nu^\beta$ (if $\nu = 0, \mu = 0$).

Substituting in (a) for μ, we have

$$k^2 g^{\alpha\beta}\eta_{\alpha\gamma}\nu^\gamma\eta_{\beta\delta}\nu^\delta = k^2\eta^{\beta\gamma}\nu_\gamma\eta_{\beta\delta}\nu^\delta = k^2\nu_\gamma\nu^\gamma = \nu_\alpha\nu^\alpha,$$

so that $k^2 = 1$. Finally, from (c),

$$\epsilon^{\alpha\beta}\mu_\alpha\nu_\beta = \sqrt{g}\,\eta^{\alpha\beta}k\,\eta_{\alpha\gamma}\nu^\gamma\nu_\beta = k\sqrt{g}\,\nu^\beta\nu_\beta \geqq 0.$$

Hence $k \geqq 0$, which means that $k = 1$. Thus $\mu_\alpha = \eta_{\alpha\beta}\nu^\beta$, and the two vectors are conjugate $[\mu, \nu]$.

The results of this section can be summarized as follows: _any one of the three conditions_ (42.4), (42.8), _and_ (42.9) _is necessary and sufficient for the complex covariant vector_ $\xi_\alpha = \mu_\alpha + i\nu_\alpha$ _to be an A-vector, or what is the same thing, for the real covariant vectors_ μ _and_ ν _to be conjugate_ $[\mu, \nu]$.

Condition (42.3) can be thought of as stating a sort of principal axes problem for the two tensors $\eta^{\alpha\beta}$ and $g^{\alpha\beta}$. From (42.9), we see that the vectors of a conjugate pair have equal lengths and are orthogonal.

The A-vectors are one of the two families of null vectors at a point, selected by the choice of the root $k = +i$ in (42.3).

43. A-functions and analytic functions.

Let $f = \varphi + i\psi$ be a complex scalar which is defined and of class C^1 in some open set D in the conformal space R. The real scalars φ and ψ are then of class C^1 in D also.

The complex scalar f is said to be an "_A-function_" in D provided the gradient of f is an A-vector at every point of D. Hence an A-function f is characterized by the equation

$$(\eta^{\alpha\beta} + i g^{\alpha\beta})\frac{\partial f}{\partial x_\alpha} = 0. \tag{43.1}$$

If $f = \varphi + i\psi$, $\dfrac{\partial f}{\partial x_\alpha} = \dfrac{\partial \varphi}{\partial x_\alpha} + i\dfrac{\partial \psi}{\partial x_\alpha}$. Hence f is an A-function provided the gradients of its real and imaginary parts are conjugate $[\text{grad}\,\varphi, \text{grad}\,\psi]$. Thus the condition for an A-function may be given as

(43.2) $$\frac{\partial \varphi}{\partial x_\alpha} = \eta_{\alpha\beta}\, g^{\beta\gamma} \frac{\partial \psi}{\partial x_\gamma} .$$

These are the generalized Cauchy-Riemann equations.

As we have seen, another form of the condition for $f = \varphi + i\psi$ to be an A-function is

(43.3) $$g^{\alpha\beta} \frac{\partial \varphi}{\partial x_\alpha} \frac{\partial \varphi}{\partial x_\beta} = g^{\alpha\beta} \frac{\partial \psi}{\partial x_\alpha} \frac{\partial \psi}{\partial x_\beta} , \qquad g^{\alpha\beta} \frac{\partial \varphi}{\partial x_\alpha} \frac{\partial \psi}{\partial x_\beta} = 0 ,$$

(43.4) $$\frac{\partial \varphi}{\partial x_1} \frac{\partial \psi}{\partial x_2} - \frac{\partial \varphi}{\partial x_2} \frac{\partial \psi}{\partial x_1} \geqq 0 .$$

It was observed that the inequality sign applies in the third relation, except when grad $f = 0$.

A coordinate system in an open set D will be called "canonical" (or isothermal) if at each point of D the components, in that coordinate system, of the metric tensors satisfy the relations

(43.5) $$g_{11} = g_{22} \text{ and } g_{12} = 0 \quad (\text{or } g^{11} - g^{22} = g^{12} = 0).$$

Evidently this holds for all the tensors of the pencil (41.1) if it holds for one of them; that is, it is a conformal condition.

Let φ and ψ be two real scalars of class C^1 in D. Suppose that

(43.6) $$\frac{\partial \varphi}{\partial x_1} \frac{\partial \psi}{\partial x_2} - \frac{\partial \varphi}{\partial x_2} \frac{\partial \psi}{\partial x_1} > 0$$

throughout D, and furthermore that to distinct points in D there correspond distinct points φ, ψ in the number plane. Under these circumstances φ and ψ, in that order, can be introduced as coordinates in D.

Theorem I: Let φ and ψ be real scalars of class C^1 in an open set D, which can be introduced in the given order as coordinates in D. *The scalars φ and ψ will be canonical coordinates if, and only if, $\varphi + i\psi$ is an A-function in D.*

Proof: Since (43.6) holds by assumption, (43.3) becomes the condition for $\varphi + i\psi$ to be an A-function. But (43.3) is exactly the condition that when we transform from any coordinates x_1, x_2 to the coordinates φ, ψ, the new components of the metric tensors shall have the form (43.5). This is obvious from the transformation law of a $g^{\alpha\beta}$. Hence the metric tensors assume the canonical form in the φ, ψ coordinate system if, and only if, $\phi + i\psi$ is an A-function.

An A-function $f = \varphi + i\psi$ such that φ and ψ can be introduced in the given order as coordinates in an open set D, will be called a "<u>complex coordinate</u>" in D. The gradient of f then cannot vanish at any point of D, by (43.6). The real and imaginary parts of a complex coordinate are canonical coordinates.

<u>Theorem II</u>: The totality of <u>A-functions</u> defined in an open set D admitting a complex coordinate, coincides with the totality of <u>analytic functions of the complex coordinate in D</u>.

Proof: Let $z = x_1 + i\,x_2$ be a complex coordinate in D. Then x_1 and x_2 are canonical coordinates in D, so that (43.5) holds. The condition (43.2) for a complex scalar $f = \varphi + i\psi$ of class C^1 in D to be an A-function, therefore reduces in these coordinates to

$$(43.7) \qquad \frac{\partial\varphi}{\partial x_1} = \frac{\partial\psi}{\partial x_2}\,, \qquad \frac{\partial\varphi}{\partial x_2} = -\frac{\partial\psi}{\partial x_1}$$

(since now $\sqrt{g} = g_{11} = 1/g^{11}$). But here we have the ordinary Cauchy-Riemann equations. These, as is well known, express the condition for an f of class C^1 to be an ordinary analytic function of the complex variable z in the open set D (or rather in the image of D in the x_1, x_2 number plane). Consequently a complex scalar f which is an A-function in D, when expressed in terms of a complex coordinate z, is the same thing as an analytic function of z.

<u>Corollary</u>: Let x_1, x_2 be canonical coordinates in an open set D, and let x_1', x_2' be any coordinates in D. Then x_1', x_2' will be canonical coordinates if, and only if, $x_1' + i\,x_2'$ is an analytic function of $x_1 + i\,x_2$ in D.

In fact, the condition for x_1', x_2' to be canonical coordinates is that $x_1' + i x_2'$ shall be an A-function, according to theorem I. By theorem II, this is the same as saying that $x_1' + i x_2'$ is an analytic function of the complex coordinate $x_1 + i x_2$.

<u>Theorem III</u>: If the gradient of an A-function f does not vanish at a point P, there exists some neighbourhood U(P) in which f can be introduced as a complex coordinate.

Proof: Since grad $f(P) \neq 0$, (43.6) holds at P, by the remark following (43.4). Hence ϕ, ψ can be introduced as coordinates in some U(P), in virtue of the implicit function theorem. By definition, f is then a complex coordinate in U(P).

The connection between A-functions and analytic functions may be stated as follows. An A-function is any complex point function whose gradient is a certain kind of complex vector (A-vector) -- this definition is independent of particular coordinates. An A-function becomes an analytic function when it is expressed in terms of another A-function as coordinate. That is, the relation between two A-functions (the second of which, say, can be used as a coordinate) is that the first is an analytic function of the second. In the case of the ordinary complex plane with conformal euclidean metric, it would be correct to say that the property of being an A-function is characteristic for an analytic function (in the ordinary sense) when expressed in general coordinates of class C^1.

Let f and z be two A-functions, and suppose that grad $z \neq 0$ at a certain point P. The gradients at P of f and z are A-vectors. As the A-vectors at a point form a one-dimensional complex vector space,

$$\frac{\partial f}{\partial x_\alpha} = \chi \frac{\partial z}{\partial x_\alpha} \qquad \text{at } P,$$

where χ is a complex scalar. The interpretation of χ is readily found by introducing z as a complex coordinate in a neighbourhood of P. The above equation then becomes

$$\frac{\partial f}{\partial x_1} = \chi \quad , \quad \frac{\partial f}{\partial x_2} = i\chi \ .$$

Either of these relations states that

$$\chi = \frac{df}{dz} \qquad \text{at } P$$

($df/dz = \partial f/\partial x_1 = \partial f/\partial i x_2$); that is, the factor χ is the derivative of the analytic function f(z) of the complex variable $z = x_1 + i\, x_2$. Thus the derivative of the analytic function f(z) appears in general coordinates as the ratio of the gradients of two A-functions.

44. Conformal Spaces and Analytic Manifolds.

From now on the oriented two-dimensional conformal space R will be assumed to be of class at least 2. The conformal tensor

(44.1) $$G_{\alpha\beta} = \frac{g_{\alpha\beta}}{\sqrt{g_{11} g_{22} - g_{12}^2}}$$

is to be of class C^1. (Spaces of class 1 could be also considered, under the assumption that $G_{\alpha\beta}$ is of class C^1 in some coordinate neighbourhood of each point.) In terms of the conformal tensor the condition (43.5) for a canonical coordinate system becomes

(44.2) $$G_{\alpha\beta} = \delta_{\alpha\beta} \ .$$

An A-function was defined as a complex scalar of class C^1 with an A-vector gradient. Hence when an A-function is introduced as a complex coordinate, the transformations to the allowable coordinate systems given in the definition of the space R will be of class C^1, but need not be of class C^2. This does not mean that a complex coordinate cannot be used in a space of class 2, but only that in using such a coordinate we must keep in mind that properties which depend on second order differentiability of coordinate transformations may be lost in transferring between the complex coordinate system and the allowable coordinate systems.

For example, the conformal tensor is of class C^1 in the allowable coordinate systems, but after a general coordinate transformation of class C^1 it would be merely continuous. As it happens, in this particular case no differentiability is lost with a complex coordinate; on the contrary, the components of the conformal tensor then reduce to constants, therefore actually to analytic functions. For we have seen that with a complex coordinate the metric tensors take the canonical form, which means that (44.2) holds.

So far nothing has been said about the existence of non-constant A-functions. For this we shall refer to a paper of L. Lichtenstein, "Konforme Abbildung nichtanalytischer Flächenstücke" (Abh. K. Preuss. Akad. Wiss., 1911, pp. 3 - 49; see also Bull. Acad. Sci. Cracovie, 1916, pp. 192 - 217, and the references there to E. E. Levi and A. Korn). In this paper it is shown that under the present hypotheses there exists for each point P of the space R a neighbourhood U(P) in which canonical coordinates φ, ψ of class C^1 can be introduced. Then $\varphi + i\psi$ is an A-function with non-vanishing gradient which can be used as a complex coordinate. We may therefore state this theorem:

<u>Local Existence Theorem</u>: Let R be an oriented two-dimensional conformal space of class $\geqq 2$, in which the conformal tensor $G_{\alpha\beta}$ is of class C^1. <u>Any point P of R has a neighbourhood U(P) admitting a complex coordinate z</u>. The transformations from the allowable coordinate systems of R to a complex coordinate system are of class C^1.

(As a matter of fact Lichtenstein's result is that if $G_{\alpha\beta}$ is continuous and satisfies a Lipschitz condition or merely a Hölder condition in some coordinate neighbourhood of each point, then there exist canonical coordinates of class C^1 whose derivatives satisfy a Lipschitz or Hölder condition respectively. Perhaps under our stronger hypothesis that $G_{\alpha\beta}$ is of class C^1 it can be proved that the canonical coordinates are of class C^2.

The local existence theorem can be shown to be equivalent to this statement: in a two-dimensional Riemann space of class $\geqq 2$, for which the ratios $g_{\alpha\beta} / \sqrt{g}$ are of class C^1, every point has a neighbourhood which can be mapped conformally (§41) on an open set in the euclidean plane.)

If two of the complex coordinate neighbourhoods in R intersect, then in the open set D which is the intersection either complex coordinate is an analytic function of the other. This follows from the corollary in § 43.

A coordinate manifold of two dimensions is said to be "analytic in the strict sense" if, when two of its allowable coordinate neighbourhoods $U(x)$ and $\overline{U}(\overline{x})$ intersect, the coordinate transformation is such that $x_1 + ix_2$ is an analytic function of the complex variable $\overline{x}_1 + i\overline{x}_2$ (and $\overline{x}_1 + i\overline{x}_2$ is an analytic function of $x_1 + ix_2$) in the intersection. (This of course is more than requiring x_1 and x_2 to be real analytic functions of $\overline{x}_1$, $\overline{x}_2$, which is the condition for a manifold analytic in the ordinary sense.)

Theorem: By taking the complex coordinate neighbourhoods covering the space R as new allowable coordinates, we can obtain from R a coordinate manifold analytic in the strict sense, which we may call S. In S the conformal tensor $G_{\alpha\beta}$ has the constant components $\delta_{\alpha\beta}$. An A-function in R is characterized in S as a complex scalar which is an analytic function of each complex coordinate in its domain of definition.

The last sentence is a consequence of theorem II, § 43. The other statements have already been proved.

A conformal space R thus gives rise to an analytic manifold S. Conversely, any analytic manifold is associated in this way with some conformal space:

Conformal Metrization Theorem: Let S be any manifold analytic in the strict sense. Then S is necessarily oriented, and if we define the conformal tensor $G_{\alpha\beta}$ to have the constant components $\delta_{\alpha\beta}$, S becomes a conformal space of the type R considered above. Moreover, the analytic manifold obtainable from this conformal space according to the previous theorem, is S itself.

Proof: It is readily shown from the ordinary Cauchy-Riemann equations that the Jacobian of each coordinate transformation is positive; hence S is oriented. The definition $G_{\alpha\beta} = \delta_{\alpha\beta}$ is consistent; this again follows from the Cauchy-Riemann equations by substitution in the transformation law of $G_{\alpha\beta}$ on page 146, with $n = 2$ (or else we can argue that analytic coordinate transformations preserve canonical form in a metric tensor). Therefore S does become a conformal space. Finally, the allowable coordinate systems of S are evidently canonical. Hence by theorems I and II, § 43, they are the complex coordinate systems of S. This proves the final assertion in the theorem.

Briefly stated, what we have shown is that the study of A-functions in a conformal space R can be reduced to the study of analytic functions in an analytic manifold S. If one were to take the subject of analytic functions in an analytic manifold as a starting point, the above work would be interesting partly as a solution of the problem of characterizing the analytic functions when they are expressed in general coordinates of class C^1.

Ordinary function theory begins by studying analytic functions in a special analytic manifold, a sphere covered by two coordinate systems, z and ζ. The z-coordinate is carried onto the sphere by the usual stereographic projection from the complex plane; it covers the entire sphere except for the north pole. The ζ-coordinate covers the ~~entire~~ sphere except for the south pole. It is defined by $\zeta = 1/z$ in the intersection of the two coordinate neighbourhoods, and by $\zeta = 0$ at the north pole.

45. The Laplace Equation.

Let S be the analytic manifold associated with the given conformal space R, as described above. That is, the allowable coordinates of S are the complex coordinates of R, and in S $G_{\alpha\beta} = \delta_{\alpha\beta}$. So far as the rest of this chapter is concerned, we can work just as well with any coordinates in S which are of class C^2 with respect to the allowable coordinates of S. We shall use the symbol S' for the space of class 2 derived from S by allowing these coordinate transformations of class C^2. Then in S', $G_{\alpha\beta}$ is of class C^1 and A-functions are of class C^2. What follows in this chapter is understood to be with reference to S'.

Let f be an A-function defined in an open set D. At each point of D, f obeys (43.1), which can be written as

(45.1) $$(\epsilon^{\alpha\beta} + i\, G^{\alpha\beta}) \frac{\partial f}{\partial x_\alpha} = 0$$

in view of the definitions (42.1) and (41.5). (For any covariant vector ξ_α, $(\epsilon^{\alpha\beta} + i\, G^{\alpha\beta})\xi_\alpha$ is a contravariant relative vector of weight 1.)

Differentiating (45.1) with respect to x_β and using the fact that $\epsilon^{\alpha\beta}$ is constant and skew-symmetric, we find that

(45.2) $$\frac{\partial}{\partial x_\alpha}\left(G^{\alpha\beta} \frac{\partial f}{\partial x_\alpha} \right) = 0 .$$

This is the generalized Laplace equation. In a complex coordinate system $G^{\alpha\beta} = \delta^{\alpha\beta}$, and then we have the ordinary Laplace equation

(45.3) $$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} = 0 .$$

(For any covariant vector ξ_α of class C^1, $\frac{\partial}{\partial x_\alpha}(G^{\alpha\beta}\xi_\beta)$ is a relative scalar of weight 1. This can be proved directly from the transformation law of the relative vector $G^{\alpha\beta}\xi_\beta$. A shorter proof can be given using the known expression $\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_\alpha}(\sqrt{g}\,\xi^\alpha)$ for the divergence $\xi^\alpha_{;\alpha}$ of the vector $\xi^\alpha = g^{\alpha\beta}\xi_\beta$ with respect to a metric tensor $g_{\alpha\beta}$ which is locally of class C^1.)

A real (or complex) point function which is defined and of class C^2 (as to S') in an open set D, and which obeys the generalized Laplace equation at every point of D, is called a real (or complex) "<u>potential function</u>". A complex potential function is nothing more than the sum of a real potential function and i times another real potential function.

We saw above that an A-function $f = \varphi + i\psi$ is a complex potential function. Hence φ and ψ are real potential functions. In particular, we can say that in an arbitrary open set D, the imaginary part of an A-function is a real potential function. When D is connected and simply connected, there is the following converse.

<u>Theorem</u>: Let ψ be a real potential function defined in a simply connected region D. There exists in D another real potential function φ such that $\varphi + i\psi$ is an A-function in D. This φ is unique except for an additive constant.

Proof: By definition, ψ is of class C^2. Hence the covariant vector

$$a_\alpha \equiv \eta_{\alpha\beta}\, g^{\beta\gamma} \frac{\partial \psi}{\partial x_\gamma} = \epsilon_{\alpha\beta}\, G^{\beta\gamma} \frac{\partial \psi}{\partial x_\gamma} \tag{45.4}$$

is of class C^1 in D. According to the generalized Cauchy-Riemann equations (43.2), what we have to find is a real scalar φ of class C^1 (in fact, C^2) having the property that $\partial\varphi/\partial x_\alpha = a_\alpha$. That is, the condition on φ is that its gradient shall be a_α. Now

$$\frac{\partial a_\alpha}{\partial x_\delta} = \epsilon_{\alpha\beta} \frac{\partial}{\partial x_\delta}\left(G^{\beta\gamma} \frac{\partial \psi}{\partial x_\gamma}\right).$$

Multiplying by $\epsilon^{\alpha\delta}$,

$$\epsilon^{\alpha\delta} \frac{\partial a_\alpha}{\partial x_\delta} = \frac{\partial a_1}{\partial x_2} - \frac{\partial a_2}{\partial x_1} = \frac{\partial}{\partial x_\beta}\left(G^{\beta\gamma} \frac{\partial \psi}{\partial x_\gamma}\right) = 0,$$

since ψ is a potential function. Thus curl a vanishes in D. Using the theorem of § 37 we conclude that there exists in D a function φ of class C^2, unique up to an additive constant, having the gradient a_α as required.

In a simply connected region, therefore, a real potential function is the same thing as the imaginary part of an A-function. Instead of imaginary part we could also say real part, for the imaginary part of an A-function f is the real part of the A-function -if, and the real part of f is the imaginary part of if.

Let ψ be a given real potential function in a given simply connected region D. The A-function $f = \varphi + i\psi$ in D which has ψ for its imaginary part is determined by the formula

$$\varphi(Q) = \int_{P_0}^{Q} \epsilon_{\alpha\beta}\, G^{\beta\gamma} \frac{\partial\psi}{\partial x_\gamma}\, dx_\alpha + \text{const.}, \tag{45.5}$$

where P_0 is any fixed point and Q is a variable point in D, and where the integral is taken along any curve of class C^1 (or D^1, by § 40) from P_0 to Q in D. For it was proved in § 37 that $\int_{P_0}^{Q} a_\alpha dx_\alpha$ + const. is the function whose gradient is a_α, when a_α is a gradient. In the analytic manifold S, in which $G^{\alpha\beta} = \delta^{\alpha\beta}$, (45.5) becomes

$$\varphi(Q) = \int_{P_0}^{Q} \left(\frac{\partial\psi}{\partial x_2} dx_1 - \frac{\partial\psi}{\partial x_1} dx_2 \right) + \text{const.} \tag{45.6}$$

46. Two Integral Theorems.

In this section we consider an open set U in the space S' covered by a single coordinate system x_1, x_2. Our first object will be to give a statement of Gauss' theorem, without proof, in the form in which we shall use it.

In the U(x) coordinates $G_{\alpha\beta}$ is of class C^1. Hence we can obtain a metric tensor $g_{\alpha\beta}$ which is likewise of class C^1 in these coordinates, for instance by giving the relative scalar μ in (41.3) the constant value 1 in U(x). Such a $g_{\alpha\beta}$ is to be understood in the discussion below. (We can think of U as a Riemann space of class 2 with a metric tensor of class C^1.)

Let E, given by $x_\alpha = x_\alpha(t)$, $0 \leqq t \leqq 1$, be a simple closed curve of class D^1 in U such that the tangent vector dx_α/dt is never zero. Let u be the open set in the x_1, x_2 number plane which has U as its image, and denote by e the pre-image of E in u. We assume that the interior of e lies in u. The image in U of the interior of e may be called the interior of E (with respect to the fixed coordinate system U(x)).

Let P be any point on an arc $\mathcal{E}$ of E of class C^1. There are two unit normal vectors to $\mathcal{E}$ at P, which are negatives of each other. Consider an arbitrary oriented curve F of class C^1, which starts at P with a non-zero tangent vector and is normal to $\mathcal{E}$ at P. The unit tangent vector to F at P is one of the two normals to $\mathcal{E}$. Now it can be shown that if P is not an endpoint of $\mathcal{E}$, just one of these normals, the "<u>inward normal</u>", has the following property: for every F which has this normal as its initial tangent, some initial arc of F lies entirely (except for P) in the interior of E. The "<u>outward normal</u>", which will be denoted by n^α, is the negative of the inward normal. Some initial arc of any F whose initial unit tangent is n^α, lies in the "exterior" of E (i.e., in the complement in S' of E plus its interior).

Thus at each inner point P(t) on $\mathcal{E}$ there is defined an outward normal $n^\alpha(t)$. This unit vector $n^\alpha(t)$ can be shown to be continuous in t at every inner point of $\mathcal{E}$. Moreover, it can be proved that at either endpoint of $\mathcal{E}$, $n^\alpha(t)$ has a limit which is one of the normals to $\mathcal{E}$ — we define it to be the "outward normal" at that endpoint.

A particular one of the two unit vectors normal to n^α — we call it s^α — is now selected by requiring $s_\alpha - in_\alpha$ (or $n_\alpha + is_\alpha$) to be an A-vector. That is, s^α is determined by

$$s^\alpha = -\eta^{\alpha\beta} g_{\beta\gamma} n^\gamma. \tag{46.1}$$

This relation shows that $s^\alpha(t)$ is continuous along each arc $\mathcal{E}$ of E of class C^1.

As s^α is a tangent vector to the curve E, we have

$$s^\alpha(t) = c(t)\ \frac{dx_\alpha(t)}{dt}.$$

Along each $\mathcal{E}$ arc the factor c(t), being the ratio of two continuous non-zero vectors, is itself continuous and not zero. Hence it does not change sign on any one $\mathcal{E}$ arc. It can be proved that c(t) actually has a constant sign all along the curve E (this is immediate if E is of class C^1, or if E has just one corner; if E has more than one corner, an argument can be given which depends on rounding off the corners and then using the result for the curve of class C^1 which is thereby obtained).

The function

$$s(t) \equiv \int_0^t \frac{d\tau}{c(\tau)}, \quad 0 \leqq t \leqq 1,$$

is continuous and monotone along E, and of class C^1 on each $\mathcal{E}$ arc. Hence s is an allowable parameter for the curve E. We have

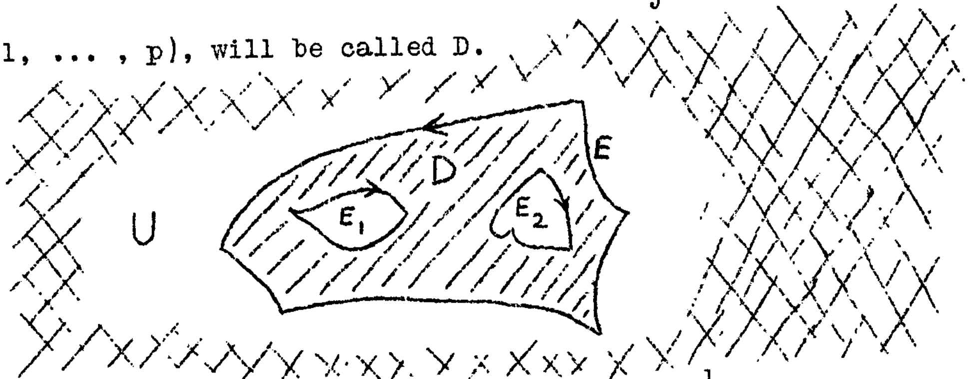
$$\frac{dx_\alpha}{ds} = \frac{dx_\alpha}{dt} \Big/ \frac{ds}{dt} = c\,\frac{dx_\alpha}{dt} = s^\alpha .$$

That is, the introduction of s as parameter makes s^α the tangent vector to E. As s^α is a unit vector, s is an arc-length parameter.

The result can be stated in this way: when the arc-length parameter s is introduced on E as described above, the tangent vector $s^\alpha = dx_\alpha/ds$ is a unit vector, and the unit vector n^α which is the outward normal is determined by the property that $n_\alpha + is_\alpha$ is an A-vector.

The form of Gauss' theorem (or the divergence theorem) which we shall need is as follows.

Let $E_1, \dots, E_p$ be a finite number (perhaps zero) of non-intersecting simple closed curves of class D^1 lying in the interior of E, such that no one of them is in the interior of any other. The region which is the interior of E minus the curves E_j and their interiors $(j = 1, \dots, p)$, will be called D.



Let ξ_α be a vector which is of class C^1 in some open set containing D and its boundary. If we take the covariant derivative $\xi^\alpha{}_{,\beta}$ of ξ^α with respect to the metric tensor $g_{\alpha\beta}$, and then form the scalar $\xi^\alpha{}_{,\alpha}$, we have the "divergence" of ξ, abbreviated div ξ. It is continuous in D, and it has the well-known expression

$$\operatorname{div} \xi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_\alpha} (\sqrt{g}\, \xi^\alpha) .$$

The theorem is that

$$(46.2) \qquad \iint_D \operatorname{div} \xi \cdot \sqrt{g}\, dx_1\, dx_2 = \int_E \xi^\alpha n_\alpha ds - \sum_{j=1}^{p} \int_{E_j} \xi^\alpha n_\alpha ds.$$

In words: the integral of the divergence of ξ over the area of D equals the integral of the outward component of ξ (outward from D, that is) along the length of the boundary of D. It is understood that for each E_j, n^α is the corresponding outward normal and s is the associated arc-length parameter as previously explained. The integrals on the right are taken in the sense of increasing s.

Instead of (46.2) we can write

$$\iint_D \frac{\partial}{\partial x_\alpha}(\sqrt{g}\ \xi^\alpha)\, dx_1\, dx_2 = \int_B \xi^\alpha n_\alpha\, ds, \tag{46.3}$$

where B is the boundary of D, with the proviso that minus signs are to be placed before the integrals along the curves E_j (if there are any).

The theorem has been stated for a real vector ξ, but it is also true for a complex vector, since it holds for the real and imaginary parts separately.

(It may be recalled that the proof is given first for a triangle whose sides are straight lines in u(x), by direct integration; then for a simple closed polygon, which is divided into triangles by interior diagonals; then for a single curve E by approximating it with simple closed polygons; and finally, when there are the inner curves E_j, by means of cuts which reduce the problem to the previous case.)

Let f and h be two A-functions defined in some open set which contains D and the curves E and E_j which are the boundary B of D. An A-function is of class C^2 in U(x). Hence the complex vector $f\dfrac{\partial h}{\partial x_\alpha}$ is of class C^1, and so it is a vector ξ_α to which Gauss' theorem applies.

The divergence of $f\dfrac{\partial h}{\partial x_\alpha}$ vanishes at every point of D. For

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_\alpha}\left(\sqrt{g}\ \ g^{\alpha\beta} f \frac{\partial h}{\partial x_\beta}\right) \equiv \tag{46.4}$$

$$g^{\alpha\beta}\frac{\partial f}{\partial x_\alpha}\frac{\partial h}{\partial x_\beta} + \frac{f}{\sqrt{g}}\frac{\partial}{\partial x_\alpha}\left(\sqrt{g}\ \ g^{\alpha\beta}\frac{\partial h}{\partial x_\beta}\right).$$

The first term on the right is zero because any two A-vectors at a point are perpendicular to each other, and the second term is zero because the A-function h satisfies the Laplace equation (45.2).

By (46.3),

$$\int_B g^{\alpha\beta} f \frac{\partial h}{\partial x_\beta} n_\alpha \, ds = \int_B f \frac{\partial h}{\partial x_\alpha} n^\alpha \, ds = 0.$$

Now by definition of s, $n_\alpha + is_\alpha$ is an A-vector. As $\partial h/\partial x_\alpha$ is also an A-vector, we have

$$(46.5) \qquad \frac{\partial h}{\partial x_\alpha}(n^\alpha + is^\alpha) = \frac{\partial h}{\partial x_\alpha} n^\alpha + i \frac{\partial h}{\partial x_\alpha} s^\alpha = 0.$$

Thus

$$(46.6) \qquad \int_B f \frac{\partial h}{\partial x_\alpha} s^\alpha \, ds = 0.$$

But s^α is the tangent vector dx_α/ds along the curves E and E_j. Consequently the result takes the simple form

$$(46.7) \qquad \int_B f \, dh = 0.$$

Special parameters s were used on the curves E and E_j to derive this formula. But the value of $\int f \, dh$ around any one of the curves does not depend on what parameter is used (provided the integration is always performed in the sense determined by the s parameter). Hence (46.7) holds with any allowable parameters describing the curves of the (sensed) boundary B. It can easily be shown that instead of interpreting dh as $\frac{dh}{dt}\,dt$, we may think of $\int f \, dh$ as the limit of a sum of products $f \cdot \Delta h$ in the usual fashion of complex integration.

What we have proved is essentially a form of the Cauchy integral theorem, at least when U(x) can be assumed to be a complex coordinate neighbourhood, with the coordinate $z = x_1 + ix_2$. For f(z) and h(z) are then simply any two functions which are analytic in D and on its boundary. If in particular we take h(z) to be z itself, our theorem states that $\int_B f(z)\, dz = 0$ for any analytic f(z), and this is exactly the Cauchy integral theorem for the closed region D+B (or rather for its pre-image in the z-plane). On the other hand, the present theorem is in reality not more general than Cauchy's. For dh/dz, and hence f dh/dz, is an analytic function of z. Consequently, by the Cauchy theorem, $\int_B (f \frac{dh}{dz})\, dz = 0$. This is the same as (46.7), since $\frac{dh}{dt} = \frac{dh}{dz}\frac{dz}{dt}$.

Of course we might have taken Cauchy's theorem for granted and then derived (46.7) as indicated above, for the case that U(x) is a complex coordinate neighbourhood. This would really have been enough for our purposes.

By making use of the theorem of § 37, we can show very easily that if f and h are two A-functions in a simply connected region not necessarily lying in one coordinate system, then $\int f\, dh$ vanishes around any single closed curve of class D^1 in the region, even if the curve has double points. For $f \frac{\partial h}{\partial x_\alpha}$ is a covariant vector a_α of class C^1, and

$$\epsilon^{\alpha\beta}\frac{\partial a_\alpha}{\partial x_\beta} = f\,\epsilon^{\alpha\beta}\frac{\partial^2 h}{\partial x_\alpha \partial x_\beta} + \epsilon^{\alpha\beta}\frac{\partial h}{\partial x_\alpha}\frac{\partial f}{\partial x_\beta} = \epsilon^{\alpha\beta}\frac{\partial h}{\partial x_\alpha}\frac{\partial f}{\partial x_\beta} = 0$$

since $\partial h/\partial x_\alpha$ and $\partial f/\partial x_\alpha$, being A-vectors, are proportional at each point. That is, $\frac{\partial a_1}{\partial x_2} - \frac{\partial a_2}{\partial x_1} = 0$ in the region. Now the theorem of § 37 was proved for a real vector a_i, but it follows that it is true for a complex vector as well — we have only to apply the theorem to the real and imaginary parts of the complex vector separately, to see this (the scalar φ and the arbitrary constant mentioned in the theorem are of course now complex). Hence a_α is the gradient of some complex scalar F (and we may add that F is an A-function, since its gradient $f\frac{\partial h}{\partial x_\alpha}$ is an A-vector). Around any closed curve of class D^1 in the region

$$\int f\, dh = \int f \frac{\partial h}{\partial x_\alpha} dx_\alpha = \int \frac{\partial F}{\partial x_\alpha} dx_\alpha = \int dF = 0,$$

q.e.d.

The result expressed in (46.7) might now be derived without any appeal to Gauss' theorem, by making cuts in the usual way. More essential use will be made of Gauss' theorem to obtain the next result, which is closely related to a formula of Green.

Returning to the region D bounded by B (the curves E and E_j), we consider an A-function $f = \varphi + i\psi$ which is defined in an open set containing $D + B$. The real vector $\varphi \frac{\partial \varphi}{\partial x_\alpha}$ is of class C^1. To calculate its divergence we may replace both f and h by φ in the identity (46.4). Since φ is a potential function, the second term on the right vanishes, and so

$$\operatorname{div} g^{\alpha\beta} \varphi \frac{\partial \varphi}{\partial x_\beta} = g^{\alpha\beta} \frac{\partial \varphi}{\partial x_\alpha} \frac{\partial \varphi}{\partial x_\beta} .$$

By Gauss' theorem (46.2),

$$\iint_D g^{\alpha\beta} \frac{\partial \varphi}{\partial x_\alpha} \frac{\partial \varphi}{\partial x_\beta} \sqrt{g}\, dx_1\, dx_2 = \int_B g^{\alpha\beta} \varphi \frac{\partial \varphi}{\partial x_\beta} n_\alpha\, ds .$$

We write this as

$$(46.8) \qquad \int_B \varphi \frac{\partial \varphi}{\partial x_\alpha} n^\alpha\, ds = \iint_D g^{\alpha\beta} \frac{\partial \varphi}{\partial x_\alpha} \frac{\partial \varphi}{\partial x_\beta}\, da$$

where $da = \sqrt{g}\, dx_1\, dx_2$, to emphasize that we are dealing with an integral over an area.

Just as in (46.5) — (46.7) we have

$$\left(\frac{\partial \varphi}{\partial x_\alpha} + i \frac{\partial \psi}{\partial x_\alpha}\right)\left(n^\alpha + i s^\alpha\right) = \left(\frac{\partial \varphi}{\partial x_\alpha} n^\alpha - \frac{\partial \psi}{\partial x_\alpha} s^\alpha\right) + i\left(\frac{\partial \varphi}{\partial x_\alpha} s^\alpha + \frac{\partial \psi}{\partial x_\alpha} n^\alpha\right) = 0,$$

$$\frac{\partial \varphi}{\partial x_\alpha} n^\alpha = \frac{\partial \psi}{\partial x_\alpha} s^\alpha ,$$

$$\frac{\partial \varphi}{\partial x_\alpha} n^\alpha\, ds = \frac{\partial \psi}{\partial x_\alpha} \frac{dx_\alpha}{ds}\, ds = d\psi .$$

Consequently the result becomes

$$(46.9) \qquad \int_B \varphi\, d\psi = \iint_D g^{\alpha\beta} \frac{\partial \varphi}{\partial x_\alpha} \frac{\partial \varphi}{\partial x_\beta}\, da .$$

(Since $\psi - i\varphi$ is also an A-function, φ and ψ might be replaced by ψ and $-\varphi$ respectively in this formula.)

As to the interpretation of $\int_B \varphi\, d\psi$, the paragraph following (46.7) can be repeated here almost word for word.

If we had started with the real potential function φ alone, not necessarily given as the real part of an A-function, (46.8) would have been the final result. Of course if φ is defined in some simply connected region containing $D + B$, a conjugate potential ψ will exist by the theorem of the previous section, and then (46.9) holds as before.

Since the integrand in the right member of (46.9) is non-negative, the double integral vanishes only if the integrand is zero throughout D. This requires that grad $\varphi = 0$, or that φ is constant in D and hence in $D + B$. If φ is constant, so is ψ and therefore also f; this follows from (43.2) with φ and $-\psi$ interchanged. Consequently

(46.10) $$\int_B \varphi \, d\psi \geq 0,$$ <u>and equal to 0 if, and only if, $f = \varphi + i\psi$ is constant in D and on B.</u>

A similar conclusion can be drawn for the left member of (46.8) even when ψ does not exist; the condition for vanishing then is that $\varphi =$ const. in $D + B$. In particular we can say that a potential function in an open set containing $D + B$, if it vanishes on B, must vanish throughout D.

<u>Summary</u>: We have a region D bounded by a set B of simple closed curves of class D^1, E and E_j, which are oriented by a definite process that has been explained above. The entire figure lies in a single coordinate system U(x) of the space S', as illustrated on page 167. For a pair of A-functions f and h defined in an open set containing $D + B$, we have the result that $\int_B f \, dh = 0$. For the real and imaginary parts φ and ψ of a single A-function in such an open set there is the formula (46.9) for $\int_B \varphi \, d\psi$; from this (46.10) is deduced.

The first result is obviously conformal, since it does not depend on a $g_{\alpha\beta}$ altogether. The second, when written as

$$\int_B \varphi \, d\psi = \iint_D G^{\alpha\beta} \frac{\partial \varphi}{\partial x_\alpha} \frac{\partial \varphi}{\partial x_\beta} \, dx_1 dx_2 ,$$

is seen to be conformal also.

(It is interesting that Gauss' theorem is really conformal, provided we state it for a covariant vector ξ_α:

$$\iint_D \frac{\partial}{\partial x_\alpha}\left(G^{\alpha\beta}\xi_\beta\right) dx_1 dx_2 = \int_B \xi_\alpha N^\alpha \sqrt{\frac{G_{\alpha\beta}\frac{dx_\alpha}{dt}\frac{dx_\beta}{dt}}{G_{\alpha\beta}N^\alpha N^\beta}}\, dt.$$

To avoid mentioning unit vectors and arc-length, we have replaced n^α by any vector N^α having the same direction, and the s parameters by general parameters t. The integrals on the right are to be taken in the sense of increasing t.)

The theorem about $\int f\, dh$ has meaning and is true in any coordinate systems covering D + B, even if they are only of class C^1. Thus it is a theorem in the original space R, as well as in S'. As for (46.9), its left member may likewise be evaluated in any coordinates. The right member has a value which is independent of coordinates, at any rate so far as single coordinate systems are concerned which cover the entire region D. But (46.10) holds with the same generality as (46.7).

Probably the results can be extended, with suitable definitions, to any region having the shape shown in the figure on page 167, even if it does not lie in one coordinate system.

CHAPTER IX

RATIONAL FUNCTIONS

47. Triangulation of the space.

At this point we assume that our two-dimensional manifold is connected, and that it is bicompact (i.e., given any infinite number of open sets covering the space, there exists some finite selection from them which still covers the space). It is then necessarily compact; for each point has some compact neighborhood, and because of the assumption of bicompactness the entire space is the sum of finitely many of these neighborhoods and hence is compact. It is not hard to show that requiring the space to be bicompact is equivalent to requiring it to be compact and separable.

From here on we operate only with the complex coordinate systems; that is, we restrict ourselves to the space S as defined in the second theorem of §44. The remainder of our work, therefore, may be said to be concerned with an arbitrary connected, bicompact, two-dimensional coordinate manifold which is analytic in the strict sense. We may assume that the complex coordinate neighborhoods are connected. They will be called simply "coordinate neighborhoods" henceforth, and the space S will be referred to as a "surface".

If U and U' are any two coordinate neighborhoods, they can be joined by a "chain of neighborhoods". That is, there exists a finite sequence $U_1, \ldots, U_n$ of coordinate neighborhoods, with $U_1 \equiv U$ and $U_n \equiv U'$, such that the intersection of any two consecutive neighborhoods of the chain is not empty. If this were not the case, we could form the set A of all neighborhoods joinable to U_1 by chains, and the set B of all other neighborhoods. The points of A and B would constitute a pair of disjunct open sets whose sum would be the whole surface S. This is impossible, since S is connected.

The surface S can be triangulated. We shall say what this means and indicate the proof.

A "triangle" in S is a closed point set which is homeomorphic in a given way to some euclidean triangle. The terms "interior", "edge", "vertex", etc., are defined for the triangle in S, by means of the homeomorphism, from the corresponding parts of the euclidean triangle.

Suppose that we have a system of triangles on S with the following properties:

a) Each point of the surface belongs to at least one, and at most a finite number, of triangles. More definitely: each point appears in at least one triangle, either as an interior point, as an inner point of an edge, or as a vertex. If a point appears in a certain one of these three categories in one triangle, it does not appear in a different category in another triangle. An interior point belongs to just one triangle, an inner point of an edge to just two triangles, and a vertex to three or more triangles.

b) If two triangles intersect, their intersection is either a single edge or a single vertex.

Under these circumstances the system of triangles is said to be a "triangulation" of S.

The set of triangles having a given point P as a vertex can be arranged in cyclic order in such a way that each one has an edge (ending at P) in common with the following one. This can be proved from the fact that S is two-dimensional.

It can be shown that each point of the surface has some neighborhood U which is contained in a finite number of triangles. The total number of triangles in a triangulation is therefore finite; if it were not, we could select a single interior point from each triangle, and the resulting infinite point set would have a limit point since S is compact. This is impossible, because at most a finite number of points of the set could enter the above-mentioned neighborhood U of the limit point.

To obtain a triangulation we proceed as follows. Each point P of the surface is contained in some coordinate neighborhood U, described by a complex coordinate z. In U we draw any simple closed analytic curve E with non-vanishing tangent vector, whose interior lies in U and contains P. (That the interior of E is in U, is to be understood with reference to the z-plane, just as on page 165. We may for instance take E as the image of a sufficiently small circle in the z-plane about z(P).)

Thus to each point P of the surface S there corresponds a curve E. The interiors of all the curves E obviously cover S. Since S is bicompact, a finite number of these interiors can be found still covering S. Let E_j ($j = 1, \ldots, q$) be curves which have these interiors.

It can be proved that two analytic curves such as we have used, intersect at most a finite number of times, if they do not coincide. Hence two distinct E_j's either have a finite number of points in common or do not intersect.

It is possible to show that those arcs of $E_2, \ldots, E_q$ which have points interior to E_1, cut the interior of E_1 into a finite number of regions. Some of these regions may not be simply connected, or may not be bounded by a simple closed curve. By drawing suitable simple arcs across such regions, we can derive a further subdivision of E_1 and its interior into (closed) regions each of which is homeomorphic to a circle plus its interior. The auxiliary arcs drawn here and below can be assumed to be of class D^1 with non-vanishing tangent.

Next we consider that part of the interior of E_2 which is not interior to E_1. This, it can be shown, consists of a finite number of regions each bounded by arcs of E_1 or E_2 or both. Each such region can be treated in the same way as the interior of E_1, so that it is subdivided into polygonal regions (homeomorphs of closed circles). Then the part of E_3 not previously considered is cut up in a similar way, and so on to E_q. The final step is to select a single point P in the interior of each polygonal region, and by drawing arcs from P to suitable points on the boundary of the region, to effect a subdivision of S into triangles as required.

What we obtain in this manner is in fact a "triangulation of class D^1". That is, each triangle lies in some one coordinate system, and its three edges form a simple closed curve of class D^1 with a non-zero tangent vector.

By the process explained at the beginning of §46, an orientation can be assigned to the boundary of each triangle. Moreover, if T_1 and T_2 are the boundaries of two triangles which have an edge $\mathcal{E}$ in common, the orientations of T_1 and T_2 along $\mathcal{E}$ will be opposite. For consider an inner point P of $\mathcal{E}$ at which $\mathcal{E}$ does not have a corner, and let F be an oriented curve starting at P with a non-zero tangent vector N^α which is normal to $\mathcal{E}$. If some initial arc of F lies in the interior of one of the triangles, that arc has no point interior to the other triangle, since no point of the surface is interior to two triangles. Consequently if N^α has the direction of the inward normal to one triangle, it has the direction of the outward normal to the other. The tangent vectors of T_1 and T_2 are therefore opposite at P, and hence along $\mathcal{E}$.

This result shows that our oriented surface is at the same time an orientable manifold in the topological sense. It may be recalled that the steps involved were these: to orient the triangles we employed A-vectors; the latter were defined by means of the η tensor, whose existence depends on the fact that all coordinate transformations have positive Jacobians.

Given any finite set of points on the surface S, the triangulation can be made in such a way that all of them are interior points of the triangles. For every curve in the construction can be drawn so as to avoid an assigned finite number of points.

If T and T' are two triangles of a triangulation, a "<u>chain of triangles</u>" $T \equiv T_1, T_2, \ldots, T_n \equiv T'$ can be found such that each member of the chain has at least a vertex in common with the following one. For if we take all the triangles that can be reached from T by chains, their points may be shown to form an open set. Similarly the points of the triangles not reachable from T, if there were any such, would form an open set disjunct from the previous one. As S is connected, this cannot happen, and so the second set is empty.

48. Definition of rational functions.

An A-function on the surface S, it will be remembered, can be defined as a complex point function which is analytic in terms of the complex coordinates — or "local parameters", as they are sometimes called.

We shall be interested mainly in A-functions whose domain of existence is as large as possible. The maximum domain however, which would be the whole surface S, is ruled out by the following theorem, which in the case of the complex sphere is Liouville's:

Theorem: An A-function which is defined over the entire surface is a constant.

Proof: Let $f = \varphi + i\psi$ be such an A-function. Consider $\int \varphi\, d\psi$, taken once around each triangle of a triangulation of S, the triangles being oriented as at the end of the preceding section. The total integral is zero, since in evaluating it we integrate just twice along each edge of the triangulation, once in either direction. On the other hand, (46.10) holds for the integral around each one of the triangles (D representing the interior of the triangle and B its boundary). It follows that the integral around each triangle vanishes, and hence f is constant throughout any one triangle. Then f must be constant on any chain of triangles, and consequently on the entire surface.

Another proof can be given which uses more of plane function theory, but does not depend on Gauss's theorem. As S is compact, the absolute value of f, which is a continuous real function, reaches a maximum at some point P. Let U be a coordinate neighborhood of P. It is well known that a function which is analytic in a region in the plane, cannot have a maximum modulus at any point of the region unless it is constant. Hence f is constant in U. Next, if a function analytic in a plane region is constant throughout a neighborhood of some point in the region, it is constant in the whole region. Therefore f is constant on any chain of neighborhoods starting from U. Thus f is constant on S.

It may be remarked here that a similar result holds for potential functions:

Theorem: A real potential function which is defined over the entire surface is a constant.

Either of the two proofs just given applies here with little change. (To use the method of the first proof, we would have to note that by the sentence following (46.10) we have

$$\int_B \varphi \frac{\partial \varphi}{\partial x_\alpha} N^\alpha \sqrt{\frac{G_{\alpha\beta}\frac{dx_\alpha}{dt}\frac{dx_\beta}{dt}}{G_{\alpha\beta}N^\alpha N^\beta}}\, dt \geqq 0,$$

and 0 only if φ is constant in D+B.

Here N^α is any vector in the direction of n^α, and the integrals are taken in the sense of increasing t. This form of the left member of (46.8) is clearly independent of special metric tensors and arc-length parameters.)

The domains of analyticity of the A-functions to be dealt with below will be the entire surface S minus a set of isolated points. Instead of "set of isolated points" we can say "finite set of points" without really losing any generality, for an isolated set on the compact space S whose complement in S is open (as a domain of analyticity is by definition) is necessarily finite. Functions of this sort do exist at least when S is the complex sphere, as we know.

It is necessary first to consider the behavior of a function which is analytic about an isolated point.

Let f be a function which is given as analytic in some region D, and suppose that D includes a deleted neighborhood of a certain point P (that is, a neighborhood of the point minus the point itself). Let U be any coordinate neighborhood of P, with a complex coordinate z. Let z = 0, say, correspond to P. (When a particular point is under consideration, we shall usually assume for simplicity's sake that it has the coordinate 0 in whatever coordinate neighborhoods we use. This is to be understood in all that follows.)

Expressed in terms of z, f is an ordinary analytic function f(z) in the intersection of D and U. Consequently f has a Laurent expansion

$$f(z) = \dots + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + \dots, \tag{48.1}$$

which is valid in some deleted neighborhood of z = 0. According as the number of negative powers of z actually present in (48.1) is infinite, finite, or zero, f(z) is said to have an essential singularity, a pole of order m (where z^{-m} is the lowest power of z that appears in (48.1)), or a removable singularity at z = 0.

Now which one of these types of behavior f exhibits at P, does not depend on the particular coordinate system U(z). For if we take another coordinate system U' with a complex coordinate z', the transformation of coordinates from z' to z is represented in some neighborhood of P by a power series

$$z = c_1 z' + c_2 z'^2 + \dots, \tag{48.2}$$

with c_1 distinct from zero. In the corresponding deleted neighborhood, 1/z is an analytic function of z'. Since

$$\frac{1}{z} = \frac{1}{z'} \cdot \frac{1}{c_1 + c_2 z' + \dots},$$

we see that $1/z$ has a Laurent expansion of the form

$$(48.3) \qquad z^{-1} = d_{-1}z'^{-1} + d_o + d_1 z' + \dots,$$

where $d_{-1} = 1/c_1 \neq 0$. The Laurent series for f in terms of z' can be obtained by substituting (48.2) and (48.3) into (48.1). It is clear that if a lowest power of z appears in (48.1), the same power of z' will be the lowest in the series for f(z'). Moreover if the series for f(z') has a lowest term, then (48.1) must have one, by the same argument applied to the inverse transformation from z to z'. Hence if (48.1) has infinitely many negative powers, so has the f(z') series. Thus f(z') has the same kind of singularity at P as f(z).

This fact gives us the right to speak of an A-function f as having an essential singularity, a pole of order m, or a removable singularity at a point, without regard to what coordinate system the function is expressed in.

If f has a pole of order m at P, its Laurent series in terms of a coordinate z has the form

$$(48.4) \qquad f(z) = \frac{a_{-m}}{z^m} + \dots + a_o + a_1 z + \dots .$$

It may be remarked that the value of a_{-m} varies with the coordinate system. Only the non-vanishing of a_{-m} has significance.

Suppose that, contrary to the above, f is given as analytic at P. Let its power series expansion about P be

$$(48.5) \qquad f(z) = a_o + a_1 z + a_2 z^2 + \dots,$$

where of course a_o is f(P). Assuming that f is not constant, there will be a first non-zero coefficient, a_m, after a_o in (48.5). The A-function f is said to have an a_o-place, or simply an a_o, of order m at P. Just as above, using (48.2), we see that the positive integer m does not depend on the particular coordinate z.

At a pole an A-function is said to have the value ∞. If the pole is of order m, we speak of an ∞ of order m.

By the "number of times" an A-function assumes a given value a_0 (which may be ∞) is meant the sum of the orders to which it assumes the value at all its a_0-places.

The class of functions that we shall study are the "rational functions" on the surface S. A rational function is defined as an A-function which is analytic on the entire surface, except for poles — necessarily then, except for a finite number of poles. When S is the complex sphere, these are the ordinary rational functions.

By the "order" of a rational function is understood the number of times it assumes the value ∞. From the first theorem of this section it follows that a rational function which never equals ∞ is constant, in which case it is said to have the order 0.

No value can be assumed infinitely often by a non-constant rational function. For if f is rational, and if it equals a_0 an infinite number of times, it equals a_0 at an infinite number of points. These points have a limit point P which is not a pole, since in the neighborhood of a pole the modulus of f tends to infinity. Thus f is analytic at P, hence constant in a neighborhood of P, hence constant over S by an easy application of the chain of neighborhoods argument.

However, there is the following much stronger result:

Theorem: A non-constant rational function of order n assumes each complex value exactly n times.

Proof: Let f be of order $n > 0$. By definition, f equals ∞ n times. It will suffice to prove that f equals zero n times. For then if a_o is any complex number, the function $f-a_o$ will still be a rational function of order n, and will likewise equal zero n times. That is, f will equal a_o n times.

Let a triangulation of class D^1 be constructed, no edge of which passes through any one of the finite number of poles and zeros of f. Consider a typical triangle T, lying in a coordinate system U(z). Let P_j ($j = 1, \ldots, m$) be those poles and zeros of f which are contained in T.

Referring now to the coordinate system U(z), we draw m mutually exterior circles, $E_1, \ldots, E_m$, about the points P_j as centers, each circle lying in the interior of T. Let the circle E_1, for example, be given by

$$x_1 = r \cos t, \qquad x_2 = r \sin t, \qquad 0 \leqq t \leqq 2\pi , \tag{48.6}$$

where $z = x_1 + ix_2$ and $r > 0$. The region interior to T and exterior to the circles is of the kind discussed in §46. As a metric tensor in U(z) we can take $g_{\alpha\beta} = \delta_{\alpha\beta}$

The curve

$$x_1 = r + t', \qquad x_2 = 0, \qquad 0 \leqq t'$$

is normal to E_1 at the point $t' = 0$, and it may be shown that it is initially exterior to E_1. Hence the unit vector (1, 0), which is the initial tangent to this curve, is the outward normal n^α to the circle at the point (r, 0). The unit tangent s^α to E_1 at this point determined by (46.1), is the vector (0,1). From this it follows that the circle (48.6) must be oriented in the sense of increasing t to have the sense given by the s parameter of §46.

We now apply the theorem expressed in (46.7), replacing the f there by the reciprocal of the present f, and h by the present f. Thus we have

$$\int_{\beta} \frac{df}{f} = \int_{\beta} \frac{df/dz}{f}\, dz = 0, \tag{48.7}$$

where B consists of the boundary E of the triangle T and the circles E_j, these m+1 curves being oriented according to the s parameters of §46.

Suppose first that f has a zero of order p at P_1. Then we can set

$$f = z^p \cdot h(z),$$

where h(z) is analytic and distinct from zero in and on the circle E_1. Substituting for f, we find that

$$\int_{C_1} \frac{df}{f} = \int_{C_1} \frac{p}{z}\, dz + \int_{C_1} \frac{dh}{h}.$$

The second term on the right vanishes, by (46.7) as applied to the integrand 1/h in the region interior to E_1. As for the first term, we have

$$\int_{C_1} \frac{dz}{z} = \int_0^{2\pi} \frac{-\sin t + i \cos t}{\cos t + i \sin t}\, dt = i \int_0^{2\pi} dt = 2\pi i.$$

Consequently

$$\int_{C_1} \frac{df}{f} = 2\pi p i.$$

Similarly, if f has a pole of order q at P_1,

$$\int_{C_1} \frac{df}{f} = -2\pi q i.$$

Thus we have the result that

$$\int_B \frac{df}{f} = \int_E \frac{df}{f} - 2\pi i\left(\sum p - \sum q\right) = 0,$$

where $\sum p$ is the number of times that f = 0, and $\sum q$ is the number of times that $f = \infty$, in the triangle T. A similar equation holds for each triangle of the triangulation. If we add the equations for all the triangles, the integrals along the edges E cancel. Thus we find that the number of times that f = 0 over the surface S is the same as the number of times that $f = \infty$, and this was what we had to prove.

A point P at which a rational function f assumes a finite value more than once, is characterized by the fact that df/dz vanishes at P, z being an arbitrary local parameter. This is evident from the power series expansion of f about P, since the coefficient of the first power of z in the series is the value of df/dz at P.

In the case of a non-constant rational function f, there cannot be more than a finite number of such points P. Otherwise they would have a limit point P_0. Let U(z) be some coordinate neighborhood of P_0. In U(z), df/dz is analytic except perhaps for poles. As df/dz vanishes at an infinite set of points having a limit point in U(z), it must vanish throughout U(z). Hence f must be constant in U(z). This is impossible, since f cannot assume a value infinitely often.

A point at which a non-constant rational function assumes a value (which may be ∞) m times, where $m > 1$, is called a "critical point of order m-1" of the function. It follows from what we have just seen that such a function has at most a finite number of critical points.

49. The separation of n-valued functions into continuous one-valued functions.

We shall need the following general theorem:

Theorem I: Let H be a connected point set in a Hausdorff space H', and let K be any Hausdorff space. Let F be an n-valued function defined over H, whose values are points of K; that is, to each point of H, F is to make correspond n distinct points of K. Suppose that it is possible to separate F into n one-valued continuous functions, $F_1, \dots, F_n$, each of which is defined throughout H. Then this can be done in only one way.

Proof: We may consider H as a Hausdorff space in itself, with neighborhoods which are the intersections of H with the neighborhoods of H'. The space H is connected, and continuity of a function in the space H is equivalent to continuity in H considered as a point set in H'.

We have to show that if F is separated into n one-valued continuous functions F_1', ..., F_n', then these are identical in some order with $F_1, \ldots, F_n$.

Let H_1 denote the set of points of H at which F_1' has the same value as F_1. We shall prove that H_1 is both open and closed. As H is connected, this means that H_1 either coincides with H or is the null set. Evidently this amounts to saying that if one of the primed functions agrees with one of the unprimed functions at a single point of H, the two functions are identical. As the two sets of functions do agree in some order at any selected point of H, they must agree in that order throughout H, and the theorem will be established.

That H_1 is closed is immediate: if two continuous functions agree at a set of points, they agree at any limit of those points, because the value of either function at the limit is determined by the common values at the points approaching the limit.

It remains to be shown that H_1 is open. Let P be a point of H_1, and let Q_1, ..., Q_n be the n distinct points of K which are the images of P under F_1, ..., F_n respectively. Let V_1, ..., V_n be neighborhoods of Q_1, ..., Q_n respectively, such that V_1 does not intersect any one of V_2, ..., V_n.

Since F_1 is continuous, there exists a neighborhood U_1 of P whose image under F_1 lies in V_1. Similarly there exist neighborhoods of P which we call U_2, ..., U_n, and U_1', such that $F_2(U_2) \subset V_2$, ..., $F_n(U_n) \subset V_n$, and finally $F_1'(U_1') \subset V_1$ (note that $F_1'(P) = Q_1$, because P is in H_1). Let U be a neighborhood of P contained in all of the neighborhoods U_1, ..., U_n, U_1'. Of course $F_1(U) \subset V_1$, ..., $F_n(U) \subset V_n$, and $F_1'(U) \subset V_1$. We assert that U is in H_1, which will mean that H_1 is open.

For let P' be any point of U, and let $Q'_1, \ldots, Q'_n$ be the images of P' under $F_1, \ldots, F_n$ respectively. Then $Q'_1, \ldots, Q'_n$ lie in $V_1, \ldots, V_n$ respectively. As V_1 has no point in common with any of the other V's, Q'_1 is the only one of the points Q' which lies in V_1. Now the image of P' under F'_1 must be one of the points Q', and it must lie in V_1. This identifies the image as Q'_1. Thus F'_1 and F_1 have the same value at P', and so P' is in H_1; that is, U is in H_1. This completes the proof of the theorem.

By combining this result with that of §39, we obtain a theorem which is fundamental for our purposes:

Theorem II: Let H be a Hausdorff space which is connected, locally arcwise connected, and simply connected. Let F be an n-valued function defined over H, whose values are points in a Hausdorff space K. Suppose that each point P of H lies in a region V(P) in which F can be separated into n one-valued continuous functions. Then F can be separated into n one-valued continuous functions each defined throughout H. The separation of F into such functions in H is unique.

Proof: Since H is connected, the separation of F into continuous functions in H, if it exists, is unique as a consequence of Theorem I. (In fact, the separation in any one of the regions V is unique, for the same reason.) Therefore we have only to prove that a separation of F throughout H is possible.

Considering the statement of the theorem in §39, we see that the one thing to be settled is that the separation of F in the regions V can be continued locally.

Let V and V' be an intersecting pair of these regions, and let W be any component of their intersection. The separation of F into n one-valued functions in V, and the corresponding separation of F in V', must both produce the same separation of F in the component W. This again follows from Theorem I, as W is a connected point set and all the one-valued functions in question are continuous in W. Hence the separation of F can be continued locally, and the theorem is proved.

Briefly, the point has been this: in the theorem of §39 as applied to a finitely many-valued function Φ, if we suppose that the one-valued functions into which Φ is locally separated are <u>continuous</u>, it is superfluous to assume that they can be continued locally.

50. <u>Mapping of the surface S onto a sphere</u>.

Assuming that there exists a non-constant rational function f of order n on the given surface S, we are going to show that S can be represented as a Riemann surface of n leaves.

Let Σ denote the ordinary complex sphere. Instead of calling the two coordinates on the sphere z and ζ, as we did at the end of §44, we shall use the letters f (which therefore has two meanings now) and φ.

The function f sets up a mapping of S onto Σ, as follows: to a point of S at which f has a finite value, we make correspond the point of Σ at which the coordinate f has that value. To a pole of f on S, we make correspond the point $\varphi = 0$ (the north pole) of Σ. Thus we have a mapping of the entire surface S onto the entire sphere Σ.

The mapping is n to 1 (that is, each point of Σ has n distinct points of S as pre-images) except for a finite number of points of Σ, the images of the critical points of f on S. These points of Σ will be denoted by π_j ($j = 1, \ldots, N$). Each point π_j has fewer than n pre-images on S, including at least one critical point of the function f.

If we take a point of S at which f is finite, the mapping can be expressed in some neighborhood of the point by a power series such as (48.5), in which z is some local parameter on S. As for a point where the function equals ∞, the mapping can be expressed in some deleted neighborhood by a Laurent series. But to give the mapping in a complete neighborhood of such a point we must use the coordinate φ on the sphere, instead of the coordinate f. The function $\varphi(z)$, defined by $\varphi = 1/f$ when $f(z)$ is finite but not 0, and by $\varphi = 0$ when $f(z) = \infty$, is readily proved to be analytic, with a zero of order p where $f(z)$ has a pole of order p. Thus the mapping of S onto Σ is analytic at every point of S.

Of course the coordinate φ can be used to describe the mapping, just as well as f, whenever the image point is not the south pole of the sphere. For that matter, any allowable complex coordinate could be used, throughout its domain of definition, on the analytic manifold which is the sphere Σ. However, we shall prefer the coordinate f except when the image point is the north pole, in which case φ may be employed. This emphasizes the fact that we are thinking of the complex-valued rational function $f(z)$, rather than of the analytic mapping of the surface S onto the sphere Σ which is induced by $f(z)$, although the two concepts are equivalent.

Going back to the surface S for a moment, we consider a point P which is not a pole or a critical point of the function f. Then df/dz does not vanish at P, or what is the same thing, the gradient of the A-function f does not vanish at P. Referring to Theorem III of §43, we see that f can be introduced as a local parameter in some neighborhood of P.

If P is a pole but not a critical point, the function $\varphi(z)$ defined above is analytic at P, with $d\varphi/dz \neq 0$. Hence $1/f$ (which symbol represents 0 at a pole of f) can be introduced as a local parameter in some neighborhood of P.

Now let P be a critical point of order m-1, but not a pole. Then in some neighborhood of P we have

(50.1) $$f(z) = a_o + a_m z^m + a_{m+1} z^{m+1} + \dots,$$

where $a_m \neq 0$. Hence

$$f - a_o = z^m(a_m + a_{m+1}z + \dots).$$

Let

$$w(z) \equiv a_m + a_{m+1}z + \dots,$$

so that $w(o) = a_m \neq 0$. Consider w as an independent complex variable. There exists a one-valued function of w, $\sqrt[m]{w}$, which is analytic in a neighborhood of $w = a_m$ and whose m^{th} power equals w.

Since $\sqrt[m]{w}$ is analytic in w about $w = a_m$, and since w is analytic in z about $z = 0$, $\sqrt[m]{w(z)}$ is analytic in z about $z = 0$. Let

$$\sqrt[m]{f-a_o} \equiv z\sqrt[m]{w(z)}\,.$$

This function of z is analytic in a neighborhood of $z = 0$, and its m^{th} power equals $f - a_o$. Its derivative does not vanish at $z = 0$, since $\sqrt[m]{w(0)} \neq 0$. Consequently $\sqrt[m]{f-a_o}$ can be introduced as a local parameter in some neighborhood of the critical point P. If we call this local parameter z', we have

(50.2) $$f(z') = a_o + z'^m$$

instead of (50.1).

Finally, let P be a pole of order $m > 1$. Then $\sqrt[m]{1/f}$ can be introduced as a local parameter in some neighborhood of P. This follows from the previous case if we consider $\varphi(z)$ instead of $f(z)$ and recall that $\varphi(z)$ has a zero of order m at P.

Thus we have shown that about any point P of the surface S there exists a local parameter z' (depending on P) in terms of which the mapping of S onto $\sum$ becomes particularly simple. If P is not a critical point, the mapping is given in some neighborhood of P by

(50.3) $$f = z'$$

when P is not a pole, and by

$$\varphi = z' \tag{50.4}$$

when P is a pole. If P is a critical point of order m-1 the mapping is given by

$$f = a_0 + z'^m \tag{50.5}$$

when P is not a pole, and by

$$\varphi = z'^m \tag{50.6}$$

when P is a pole. Except in (50.3), z' has the value 0 at P.

It is obvious from (50.3) and (50.4) that any non-critical point on S has a neighborhood — in fact, any neighborhoos in which the coordinate z' can be used — which is homeomorphic to its image on Σ.

51. The space S as a Riemann surface.

The inverse of the mapping of S onto Σ will be denoted by f^{-1}. Under f^{-1}, each point of Σ which is not one of the points $\pi_1, \ldots, \pi_N$ goes into n distinct points on S, no one of the latter being a critical point. Each of the π_j goes into fewer than n points, including one or more critical points.

The symbol $\Sigma - \pi$ will be used for the space obtained from the sphere Σ by deleting from it the points $\pi_1, \ldots, \pi_N$.

Let Q be any point of $\Sigma - \pi$, and let P_k $(k = 1, \ldots, n)$ be the n non-critical points of S that correspond to Q. We have seen that under the mapping of S onto Σ, each P_k has a neighborhood U_k which goes homeomorphically into its image, a neighborhood V_k of Q. We take the U_k so that no two of them intersect.

Let V be a neighborhood of Q contained in $\Sigma - \pi$ and lying in all of the V_k. The image of V in U_k, under the homeomorphism between V_k and U_k, will again be called U_k. Thus each point Q in $\sum - \pi$ has a neighborhood V(Q) whose image under f^{-1} consists of n disjunct neighborhoods, the new U_k, to each of which V(Q) is homeomorphic. Therefore in any such V neighborhood the n-valued function f^{-1} is separated into n one-valued continuous functions by the n homeomorphisms.

Suppose now that N, the number of the points π_j, is zero or one. Then the space $\sum - \pi$ is simply connected. Hence Theorem II of §49 applies, and we conclude that f^{-1} can be separated into n one-valued continuous functions throughout the entire space $\sum - \pi$. These one-valued functions, each defined over $\sum - \pi$, will be called $F_1, \ldots, F_n$.

Each of the functions F_k $(k = 1, \ldots, n)$ maps the space $\sum - \pi$ homeomorphically onto a portion of the surface S. The part of S which is the image of $\sum - \pi$ under F_k will be called S_k.

The relationship which F_k sets up between $\sum - \pi$ and S_k is an analytic homeomorphism; that is, it is expressed about any pair of corresponding points by setting a local parameter on one surface equal to an analytic function of a local parameter on the other. This is evident from (50.3) and (50.4). It follows that the local parameters on $\sum - \pi$ can be introduced into S_k, and vice versa. For instance, f can be taken as a local parameter throughout S_k (except perhaps at a single point of S_k, where the function f may have a simple pole. This happens when the north pole of the sphere is not one of the points π_j, and therefore lies in $\sum - \pi$).

As the functions F_k constitute a separation of f^{-1} over $\sum - \pi$, the situation is this: the entire surface S, except for a finite number of points, appears as the sum of the n regions S_k, each of which is analytically homeomorphic to $\sum - \pi$. The exceptions on S are the critical points and all other points at which the value of f is the same as at some critical point.

In other words, if the points which correspond to the π_j are omitted from S, what remains is divided into n regions in each of which the range of f covers $\Sigma - \pi$ exactly once.

We have said that the above holds for $N < 2$. Suppose first that $N = 0$, so that f has no critical points, and $\Sigma - \pi$ is simply the sphere Σ. Then n must be 1. Otherwise the entire surface S would be the sum of two or more regions each homeomorphic to a sphere, and therefore each both open and closed. This is not possible, since S is connected.

Thus when $N = 0$, S is topologically equivalent to a sphere, and f is of order 1; that is, f assumes each complex value, including ∞ exactly once on S.

Two analytic manifolds are said to be "essentially identical" if an analytic homeomorphism can be set up between them; for then they are isomorphic with regard to topological structure and with regard to totality of coordinate systems. Our result may be stated in this way: if there exists on the surface S a non-constant rational function f which has no critical points, then S is essentially identical with the ordinary complex z-sphere, and the analytic homeomorphism between S and the sphere can be made in such a way that f appears on the sphere simply as the function z.

The condition that f shall have no critical points is equivalent to the condition that f shall be of order 1. We have seen that the first condition implies the second, and conversely, if f is of order 1 it can have no critical points, since at a critical point some value is assumed at least twice.

The case $N = 1$ is not possible. For let $P_1, \ldots, P_r$ $(r < n)$ be the points of S which correspond to π_1, and suppose, for instance, that P_1 is a critical point of order 1, while $P_2, \ldots, P_r$ are not critical points. We can find r disjunct neighborhoods, $U_1, \ldots, U_r$, of these respective points on S, all going into a single neighborhood V of π_1 in the following way: $U_2, \ldots, U_r$ are homeomorphic to V, and U_1 is a circular neighborhood of P_1 with a coordinate z' such that the mapping of U_1 onto V is given by

$$f = \pi_1 + z'^2 \tag{51.1}$$

(or if $\pi_1 = \infty$, the form (50.6) would be used). Now f^{-1} is separated into n homeomorphisms throughout $\Sigma - \pi_1$, hence in $V - \pi_1$. This implies that the mapping of $U_1 - P_1$ onto $V - \pi_1$, described by (51.1), can be separated into two homeomorphisms. The familiar properties of a mapping of the simple type (51.1) show that this cannot happen.

Finally we outline the treatment of the general case, in which $N > 1$. Let C be an oriented simple arc of class C^1 joining the points $\pi_1, \ldots, \pi_N$ in order.

Every sufficiently small circular neighborhood of an inner point of C is divided into two parts by C, one part on each side of C. Let one side of C be called the right side, and the other the left. Denote by Σ' the topological space which is obtained from $\Sigma - \pi$ by changing the neighborhoods of points of C to be those parts of the small circular neighborhoods referred to above, which are on the right side of C or on C, but not on the left side of C. Then Σ' is simply connected, and in fact all the conditions of Theorem II, §49, are satisfied. As before, it follows that f^{-1} separates into n homeomorphisms throughout Σ', which we call $F_1, \ldots, F_n$.

Now the separation of f^{-1} by the F_k, along any arc of C between two successive points π_ν is the only separation of f^{-1} into continuous functions which is possible along such an arc. This follows from Theorem I, §49, as applied to the arc considered as a space in itself. Furthermore, if we had defined Σ' by rejecting the right halves of neighborhoods of points on C instead of the left halves, the consequent separation of f^{-1} would have agreed with one actually obtained, throughout $\Sigma - C$ — again by Theorem I, §49. From these two remarks we draw the conclusion that along any given arc of C between two successive points π each of the F_k possesses boundary values with respect to approach from the left, which are the values of the same or another F_k along the whole of that arc.

Let $C_1, \ldots, C_{N-1}$ denote the successive arcs into which C is divided by the points π. To each F_k there corresponds another one of the F_k, namely, that which has the values of the former as boundary values along C_j. Thus with each C_j there is associated a permutation p_j of the subscripts $1, \ldots, n$ of the F_k.

The product $p_j p_{j+1}^{-1}$ is a single permutation which we may suppose separated into cycles.

Let $\Sigma_1, \ldots, \Sigma_n$ be n copies of the sphere Σ. We define a single space Σ'' to be the sum of the Σ_k with the following modifications: a point on Σ_k which is on C_j has as neighborhoods the neighborhoods which it would have on Σ', together with what would be the left halves of these neighborhoods, but on Σ_{k_1} instead of on Σ_k, where k_1 is derived from k by p_j. A point π_j on Σ_k is identified with the similar point π_j on $\Sigma_\ell, \Sigma_m, \ldots$, where $(k, \ell, m, \ldots)$ is one of the cycles mentioned above. A single neighborhood of π_j consists of a sufficiently small circle about π_j, counted on $\Sigma_k, \Sigma_\ell, \Sigma_m, \ldots$, with its left half, however, replaced by the left half of the same circle on $\Sigma_{k_1}, \Sigma_{\ell_1}, \Sigma_{m_1}, \ldots$.

By means of F_k the points of Σ'' which came from $\Sigma_k - \pi$ are mapped onto S. The mapping is a homeomorphism. It can be extended so as to include the points of Σ'' which came from the π's and still be continuous, in just one way. The extended mapping is a homeomorphism of the whole of Σ'' onto the whole of S.

The local parameter f, or where necessary 1/f, or $\sqrt[m]{f-a_0}$, or $\sqrt[m]{1/f}$, can be introduced onto Σ'' from S by the homeomorphism. Then Σ'' is an analytic manifold which is essentially identical with S.

Now the construction of Σ'' was simply that of an ordinary n-leaved Riemann surface of an algebraic function over the complex sphere Σ. Consequently if on the surface S a rational function f of order n can be found, then S can be identified with an n-leaved Riemann surface over the sphere of complex numbers f. We may think of S, if we wish, as wrapped around the f-sphere Σ so as to have the shape Σ'', in such a way that the n (or fewer) points of S at which the function f assumes a certain value f, all lie above the point of Σ corresponding to that numerical value f. Above each point π on Σ there would lie at least one critical point of the function f, of some order m-1, and such a critical point would appear on Σ'' as a winding-point of m leaves.

If F is any algebraic function on the ordinary complex sphere of z, the Riemann surface of F, with z or 1/z or an appropriate root of $z-a_0$ or 1/z as local parameters, is an example of the type of surface S we have been discussing. On this surface F and z are examples of rational functions in our sense.

52. Algebraic relations.

Let f and h be any two rational functions on S, the order of f being n. Let $P_1, \ldots, P_n$ be the n points of S at which f takes on a given value, and let $h_1, \ldots, h_n$ be the values of the function h at these points. It is understood that in the case of a critical point of order m-1, m of the P's stand for that point, and the corresponding m values of h are identical.

Lemma: Any symmetric polynomial $P(h_1, \ldots, h_n)$ is a rational function of f on the sphere Σ.

(A rational function on the sphere is the same as a rational function in the ordinary sense, namely, a quotient of polynomials.)

Since P is symmetric, its value is determined by the value of f, and hence it is a one-valued function of f. Let a_1 be any regular point on the sphere, but not the infinite point. The h_j will be given by Laurent series in $f-a_1$ in some neighborhood of a_1, the series having at most a finite number of terms with negative exponents. This is because $f-a_1$ is a local parameter in neighborhoods of $P_1, \ldots, P_n$. Hence P is given by a Laurent series in $f-a_1$ in the neighborhood of a_1.

Next let a_1 be a non-regular finite point; that is, a point on the sphere above which Σ'' has at least one winding point. Suppose for example that there are two winding points above a_1, of orders r-1 and s-1, where $r+s=n$. Let z_1 and z_2 be local parameters about these winding points, such that f is given about them respectively by $f = a_1 + z_1^r$ and $f = a_1 + z_2^s$. Then in the z_1 neighborhood h is given by a Laurent series $h = L_1(z_1)$ with at most a finite number of negative exponents. Similarly $h = L_2(z_2)$ in the z_2 neighborhood. Let ϵ_1 and ϵ_2 be primitive <u>r</u>th and <u>s</u>th roots of unity respectively, say $e^{2\pi i/r}$ and $e^{2\pi i/s}$. In the different points $z_1, \epsilon_1 z_1, \ldots, \epsilon_1^{r-1} z_1$ of the z_1 neighborhood f has the same value, and hence corresponding values of h will be given by the series

$$h_1 = L_1(z_1),\ h_2 = L_1(\epsilon_1 z_1),\ \ldots,\ h_r = L_1(\epsilon_1^{r-1} z_1).$$

Similarly we have

$$h_{r+1} = L_2(z_2),\ h_{r+2} = L_2(\epsilon_2 z_2),\ \ldots,\ h_n = L_2(\epsilon_2^{s-1} z_2)$$

at the s points in the z_2 neighborhood at which f has the value $a_1+z_2^s$.

Let t be the least common multiple of r and s. Put $\sigma^t = z_1^r = z_2^s$ (thus $\sigma^t = f - a_1$). Then $z_1 = \sigma^{t/r}$ and $z_2 = \sigma^{t/s}$, and $h_1, \ldots, h_n$ are all equal to Laurent series in σ. Let ϵ be $e^{2\pi i/t}$. If we substitute $\epsilon\sigma$ for σ then z_1 is multiplied by $\epsilon^{t/r}$ and z_2 by $\epsilon^{t/s}$, or z_1 is replaced by

$\epsilon_1 z_1$ and z_2 by $\epsilon_2 z_2$. Also, $f-a_1$ is unchanged. But the change of z_1 into $\epsilon_1 z_1$ merely permutes the $h_1, \ldots, h_r$, and the change of z_2 to $\epsilon_2 z_2$ permutes $h_{r+1}, \ldots, h_n$. Now $P(h_1, \ldots, h_n)$ has a Laurent expansion in σ with at most finitely many negative exponents, say $\sum b_j \sigma^j$. Since P is symmetric, this series remains unchanged in value when σ is replaced by $\epsilon\sigma$. Hence $b_j \epsilon^j = b_j$ for each fixed j, so that $b_j = 0$ when j is not a multiple of t. That is, P equals a Laurent series in powers of σ^t, or in powers of $f-a_1$.

Similarly we could prove that P has a Laurent expansion in powers of $1/f$ about the point $a_1 = \infty$ on the f-sphere, with a finite number of negative powers.

Hence $P(h_1, \ldots, h_n)$ is a function of f on the sphere having only poles as singularities, and therefore P is a rational function of the complex variable f, as the lemma asserted.

Let R(f) denote the field of all rational functions of the complex variable f.

<u>Theorem I:</u> Let f and h be any two non-constant rational functions on the surface S, the order of f being n. There exists a polynomial P(H, f) in the letter H with coefficients in the field R(f), irreducible in R(f), such that P(h, f) = 0 if, and only if, h and f are a pair of values of these functions at a point of S. The degree of the P(H, f) obtained is a factor of n.

As before, let $h_1, \ldots, h_n$ be the values of h at the n or fewer points at which f has a value f. A polynomial Q(H, f) which vanishes for pairs of values of h, f corresponding to points of the space and only for such pairs is defined by

$$(52.1) \qquad Q(H, f) \equiv (H-h_1)(H-h_2) \ldots (H-h_n).$$

This follows from the lemma just proved, since the coefficients of Q are symmetric polynomials in $h_1, \ldots, h_n$.

Let Q have the decomposition

$$Q(H, f) = P_1P_2 \dots P_N$$

into irreducible factors $P_j(H, f)$ which we may assume to have leading coefficients unity. For all but a finite number of values of f, the coefficients of the polynomials $P_1, \dots, P_N$ are finite. Using only values of f for which these coefficients are finite, it is possible to find for each fixed j an infinite set of number pairs h, f which cause $P_j(h, f)$ to vanish. (For we can solve $P_j(H, f) = 0$ for H.) Each such number pair makes Q vanish and therefore corresponds to at least one point of S. Thus P_j vanishes at an infinite number of points of S. Now $P_j(h, f)$ is a rational function of position on the space S (the rational functions on S form a field). Hence $P_j(h, f)$ vanishes identically on S, and consequently it vanishes for every pair of values h, f corresponding to a point of S.

Let P_k and P_m be any two of the P_j's. Their greatest common divisor (P_kP_m) obeys a linear relation

(52.2) $$(P_kP_m) = AP_k + BP_m,$$

where A and B are some polynomials in H with coefficients in $R(f)$. As P_k and P_m are irreducible and have leading coefficients unity, they are either equal or relatively prime. Therefore if they are not equal, (P_kP_m) is simply a rational function of f. Since P_k and P_m vanish together for infinitely many pairs of values of h, f, (P_kP_m) vanishes for infinitely many values of f. But then (P_kP_m) must be identically zero, which it is not. Hence $P_k = P_m$.

This shows that

$$Q(H, f) = [P_1(H, f)]^N,$$

and if r is the degree of P_1, $n = rN$. Thus P_1 is a polynomial $P(H, f)$ with the properties described in the theorem. That is, the polynomial Q formed in (52.1) is either prime or a power of a prime; and P can be taken to be Q, or if Q is not primo, to bo the primo factor of Q.

If for a single value of f the numbers $h_1, \ldots, h_n$ are all distinct, Q must be prime. Conversely, if Q is prime the $h_1, \ldots, h_n$ are all distinct except for at most a finite number of values of f. For if Q has a multiple root for a certain value of f, the greatest common divisor of Q and dQ/dH vanishes for that f and that root. But the greatest common divisor is simply a rational function of f, and vanishes for at most finitely many f's. Hence if Q is prime, $h_1, \ldots, h_n$ are almost always distinct.

If f and h are such that Q(h, f) is irreducible, h is said to be <u>primitive</u> with respect to f (we need not write H instead of h any longer to make it clear when we are thinking of Q as a rational combination of two independent complex variables). This happens for example when the order n of f, which is the degree of Q in h, is a prime number.

In view of what was said above, we see that the condition for h to be primitive with respect to f is that the n values of h corresponding to a given value of f shall all be distinct, except for a finite number of values of f. Thus if f is primitive with respect to h, and if we exclude a finite number of points of the surface S, then to distinct points out of those which remain, there correspond distinct number pairs (h, f). Therefore h is primitive with respect to f. This justifies us in speaking of f and h as a "primitive pair".

<u>Theorem II:</u> If f and h are a primitive pair of rational functions on the surface S, then any rational function on S is equal to a rational combination (in the ordinary sense) of f and h.

Let f be of order n. A set of n rational functions $h^{(1)}, \ldots, h^{(n)}$ are said to be a <u>basis</u> with respect to f if the determinant

$$D \equiv \begin{vmatrix} h_1^{(1)} & \ldots & h_n^{(1)} \\ \vdots & \ldots & \vdots \\ h_1^{(n)} & \ldots & h_n^{(n)} \end{vmatrix}$$

is different from zero for at least one value of f.

The trace T(H) of any rational function H with respect to f, is defined to be $H_1 + \dots + H_n$. By the lemma proved at the beginning of this section, T(H) is rational in f.

The square of D is given by

$$D^2 = \left| T(h^{(j)}h^{(k)}) \right|.$$

As each trace is a rational function on S, D^2 is also. Therefore if D^2 vanishes for infinitely many f's, it vanishes identically. Hence if $h^{(1)},\dots,h^{(n)}$ are a basis with respect to f, D vanishes for at most a finite number of values of f.

A rational function H such that $T(Hh^{(j)}) = 0$ for $j = 1, \dots, n$, is identically zero. For if

$$H_1 h_1^{(j)} + \dots + H_n h_n^{(j)} = 0 \qquad (j = 1, \dots, n),$$

then $H_1 = \dots = H_n = 0$ whenever $D \neq 0$. As the $h^{(j)}$ form a basis, H vanishes for almost all values of f. The rational function H, vanishing infinitely often, must be zero.

Now suppose that g is an arbitrary rational function on S, and that $h^{(1)}, \dots, h^{(n)}$ are a basis with respect to f. We shall show that g equals a linear combination of the $h^{(j)}$ with coefficients which are rational in f:

$$g = R^{(1)}(f)h^{(1)} + \dots + R^{(n)}(f)h^{(n)}. \tag{52.3}$$

In fact, the $R^{(j)}$ are determined if we multiply by $h^{(k)}$,

$$gh^{(k)} = \sum_j R^{(j)}h^{(j)}h^{(k)},$$

and take traces:

$$T(gh^{(k)}) = \sum_j R^{(j)}T(h^{(j)}h^{(k)}) \qquad (k = 1, \dots, n). \tag{52.4}$$

(The n values of $R^{(j)}$ corresponding to a given value of f are all equal because $R^{(j)}$ is a function of f.) Since $D^2 \neq 0$ the last set of equations can be solved for the $R^{(j)}$, and the latter will be found as rational functions of f.

because all the traces which appear in (52.4) are rational in f. Finally, the $R^{(j)}(f)$ thus found will obey (52.3). For

$$T[(g - \sum_{j} R^{(j)}h^{(j)})h^{(k)}] = 0 \qquad (k = 1, \ldots, n),$$

and as noted above, this means that

$$g - \sum_{j} R^{(j)}h^{(j)} = 0.$$

To prove Theorem II it suffices to show that a set of n rational functions of h can be found which form a basis with respect to f. The first n powers of h are such a set: $h^{(1)} = 1$, $h^{(2)} = h$, ..., $h^{(n)} = h^{n-1}$. For in this case D is the Vandermonde determinant, which equals $\prod (h_j - h_k)$, $(j > k;\ j,k = 1, \ldots, n)$. Since by assumption f and h are a primitive pair, D does not vanish identically. Therefore 1, h, ..., h^{n-1} are a basis, and the theorem is proved. In fact we have shown that any rational function on S may be expressed as a polynomial in h of degree less than the order of f, with coefficients which are rational in f.

Of course the expression

$$g = R^{(1)} + R^{(2)}h + \ldots + R^{(n)}h^{n-1}$$

which we have obtained for g is not unique. By means of Q(h, f) = 0 it may be transformed in different ways.

The final theorem in the group is this:

Theorem III: Let f and h be any two non-constant rational functions on the surface S. There exists a polynomial F(h, f), irreducible in the pair of variables h, f, which vanishes over S. The degree of F in h divides the order of f, and the degree of F in f divides the order of h. Any irreducible polynomial in h and f which vanishes over S is equal to F up to a constant factor.

Consider the polynomial in h, P(h, f), which may be obtained according to Theorem I:

$$\text{(52.5)} \qquad P(h, f) = h^r + \frac{A_{r-1}}{B_{r-1}}h^{r-1} + \ldots + \frac{A_0}{B_0},$$

where the A_j and B_j are polynomials in f, $\bar{P}$ is irreducible in R(f), P vanishes over S, and r divides the order of f. We may suppose that A_j, B_j are relatively prime.

Denote the least common multiple of $B_{r-1}, \ldots, B_0$ by B. Then the polynomials in f:

$$B/B_{r-1}, \ldots, B/B_0$$

are relatively prime.

Now

$$BP = Bh^r + A'_{r-1}h^{r-1} + \ldots + A'_0, \tag{52.6}$$

where $A'_j = A_jB/B_j$, is a polynomial in h and f. This polynomial is irreducible in its two variables. For if BP can be factored, not both of the factors can contain h; otherwise dividing by B, we should have a factoring of $\bar{P}$ with respect to h, whereas $\bar{P}$ is irreducible as a polynomial in h.

The other possibility is that BP equals a polynomial in f times a polynomial in f and h. In that case the coefficients B, $A'_{r-1}, \ldots, A'_0$ in (52.6) have a common factor. That is, B and the A_jB/B_j have a common factor. As the B/B_j are relatively prime, at least one of them, say B/B_1 does not have that factor. Then the factor divides A_1. As A_1 is prime to B_1, the factor does not divide B_1. But it divides B, and therefore also B/B_1. This is contrary to the supposition. Hence BP is irreducible.

Since P vanishes over S, and since BP is a rational function which vanishes at any rate where B does not have a pole, BP also vanishes over S.

To show that BP can be taken as the F(h, f) in the theorem, we need only prove that the degree of BP in f divides the order of h.

If $\bar{P}$ is an irreducible polynomial in h and f, and if we arrange $\bar{P}$ according to powers of h:

$$\bar{P} = C_th^t + \ldots + C_0,$$

then $\overline{P}/C_t$ is irreducible as a polynomial in h with coefficients in R(f). For suppose that

$$\overline{P}/C_t = Q_1Q_2,$$

where Q_1 and Q_2 are polynomials in h of lower degree than $\overline{P}$. Let U_1 and U_2 be polynomials in f which are common multiples of the denominators of the coefficients in Q_1 and Q_2 respectively. Then in

$$U_1U_2\overline{P} = C_t(U_1Q_1)(U_2Q_2)$$

there appear six polynomials. As $\overline{P}$ is irreducible it must divide one of the three factors on the right. Since all of the latter are of lower degree in h than $\overline{P}$, this is impossible.

Let BP be arranged according to powers of f, and let B'(h) be the coefficient of the highest power of f in BP. By the above result BP/B' is irreducible as a polynomial in f. Furthermore it vanishes over S, and its highest coefficient is unity.

By Theorem I there exists a polynomial T(f, h) in f, irreducible in R(h), vanishing over S, of a degree dividing the order of h, and with leading coefficient unity. Since BP/B' and T are both irreducible, zero over S, and of leading coefficient unity, they are identical. This is seen by exactly the same argument as the one based on (52.2), with BP/B', T instead of P_k, P_m. Hence the degree of BP in f divides the order of h.

In the same way it is seen that if any polynomial with the properties described in the last sentence of Theorem III is divided by the coefficient of its highest power of f, T is obtained. Consequently the most general polynomial with these properties is found by multiplying T by the least common multiple of the denominators of its coefficients. This disposes of the uniqueness clause in Theorem III and completes the proof of the theorem.

Let f, h be a primitive pair of rational functions on S, f being of order n. We have seen that there exists an irreducible polynomial $Q(h,f)$ of degree n in h, with coefficients in $R(f)$, such that

$$Q(h,f) = 0 \tag{52.7}$$

when, and only when, h,f are a pair of values at a point of S. Except for a finite number of points of S, distinct points have distinct pairs of values f, h. It can be shown that if we start with (52.7) defining h as an algebraic function of f, and construct the n-leaved Riemann surface for h over the f-sphere in the way in which it is ordinarily done, we will obtain precisely the surface S or Σ''. Thus S is really an algebraic Riemann surface (assuming the existence of a primitive pair), as was remarked near the end of §51.

If f alone is given, and not h, it is still possible to find the form of S as a Riemann surface over the f-sphere. This is just what was done in §§50 and 51. Thus the Σ'' found there and the Riemann surface of (52.7) are the same. An implication is that for a given f the Riemann surface is independent of the rational function h, provided h is primitive with respect to f. However, different choices of f can lead to different presentations of S as a Riemann surface.

Suppose that f,h and f',h' are two primitive pairs on S. By Theorem II,

$$\begin{cases} f = R_1(f',h') \\ h = R_2(f',h') \end{cases} \qquad \begin{cases} f' = R_1'(f,h) \\ h' = R_2'(f,h) \end{cases} \tag{52.8}$$

where R_1, R_2, R_1', R_2' are rational combinations of two variables. If in R_1 we replace f', h' by $R_1'(f,h)$, $R_2'(f,h)$, the result is a rational expression in f and h. As this expression equals f for the infinite number of pairs of values f,h corresponding to the points of S, it equals f identically. Thus R_1', R_2' are the inverse expressions to R_1, R_2, and conversely. Hence (52.8) is a birational transformation in the plane of two complex variables, carrying the algebraic curve represented by (52.7) into the algebraic curve $Q'(h',f') = 0$, where Q' is the irreducible polynomial satisfied by h', f' on S according to Theorem I.

The ordinary development of this entire theory is the reverse of that explained in the present chapter. The usual starting point is an irreducible polynomial $F(h,f)$ in the two complex variables h,f, of some degree n in h. For each value of f there are in general n distinct values of h. Locally on the f-sphere these values are separated into n continuous functions. A Riemann surface Σ'' is constructed on which these separated functions join into a single one-valued continuous function h (with poles). About each point of Σ'' this function has a Laurent expansion with finitely many negative powers, in terms of f, or exceptionally in terms of $1/f$ or a root of f or $1/f$. If we think of f or $1/f$ or the appropriate root as local parameters on Σ'' we arrive at an analytic manifold on which f, h are a primitive pair of rational functions. That is, the situation at the end is that which we took in this chapter as given to begin with, and vice versa.

CHAPTER X

INTEGRALS

53. Topology of the Surface S

It will be necessary in the remainder of our work to take into account the special topological form of the connected, oriented, bicompact surface S. With respect to this we merely state the well-known result, referring for the proof for instance to Seifert-Threlfall's Topology.

The topology of S is determined by the value of a non-negative integer p, the genus. If $p = 0$, S is homeomorphic to a sphere. If $p > 0$, S is homeomorphic to the interior and boundary of a plane polygon Π of 4p sides, whose perimeter Γ has the following identifications of points.

First, all of the 4p vertices represent a single point P. Second, the sides are labeled in order $a_1, b_1, -a_1, -b_1$; $a_2, b_2, -a_2, -b_2$; ...; $a_p, b_p, -a_p, -b_p$. The points of a_1, taken in their order with respect to the positive sense of description of Γ are identified each with one point of $-a_1$, the latter being taken in order in the negative sense along $-a_1$. The notation $a_1, -a_1$ indicates this. Similarly, as a point travels clockwise along b_1, the same point travels counterclockwise along $-b_1$; and likewise $a_2, -a_2$; ...; $b_p, -b_p$ are "opposite pairs". Each opposite pair can be thought of as a cut in the surface S, starting and ending at the same point P. When these 2p cuts are made, S opens up into the form of the polygon Π.

Given a simple arc on S, the 2p cuts can always be made in such a way that no one of them intersects the arc. Furthermore, the cuts may be assumed to be curves of class D^1 on S.

The genus can be calculated from a triangulation of S by the formula

$$\alpha_0 - \alpha_1 + \alpha_2 = 2 - 2p, \tag{53.1}$$

where $\alpha_0, \alpha_1, \alpha_2$ are respectively the number of vertices, edges, and triangles in the triangulation. The genus is independent of the particular triangulation (we shall have a proof of this fact in §58). Homeomorphic surfaces have the same genus, and conversely.

We can now derive very easily an interesting relation between the order n of a non-constant rational function f on S, the genus p of S, and the sum w of the orders of all the critical points of f (or of all the winding points of Σ'').

Consider S in the form of the Riemann surface Σ'' with respect to f, which is constructed out of n spheres over the sphere Σ as explained at the end of §51. Take any triangulation of Σ such that all of the points π on Σ appear as vertices (the points π are those above which the winding points of Σ'' are situated). For this triangulation $\alpha_0 - \alpha_1 + \alpha_2 = 2$, since the genus of the sphere Σ is 0.

Repeat this triangulation on each of the n spheres from which Σ'' is made. The result can be seen to be a triangulation of Σ''. Now each vertex, edge, and triangle on Σ appears n times on Σ'', once for each sphere, with this exception: a vertex π on Σ above which there is a winding point of order m-1 (i.e., of m leaves), is identified on m of the spheres and therefore gives rise to one vertex on Σ'' instead of m. That is, a vertex on Σ below a winding point of order m-1 goes into m-1 fewer vertices on Σ'' than if there were no winding point. Consequently the value of $\alpha_0 - \alpha_1 + \alpha_2$ for Σ'' is n times the value for Σ, or 2n, less the sum of the orders of the winding points of Σ'', or w. By (53.1),

$$2 - 2p = 2n - w,$$

or

$$w = 2(n + p - 1).$$

The formula shows that w is always even, and that it is the same for all functions of a given order on a surface of given genus.

54. An Existence Theorem

The truth of the following theorem on the existence of potential functions defined over the entire surface except for prescribed singularities, will be assumed. For the proof reference can be made to C. Neumann's Lectures on Abelian Integrals, second edition, or to Weyl's The Idea of the Riemann Surface.

Consider a complex coordinate neighborhood on S, covered by a coordinate z. Let C be a simply connected region which together with its boundary lies in this z neighborhood. Let M be a closed point set lying in the region C. In our work M will be either a single point or a simple arc.

Let F(z) be a function of z which is analytic in C-M and which has a certain kind of singularity at M. For us this will mean, when M is a point, that F(z) has a pole of some order at M. When M is a simple arc, with end-points a_1 and a_2, F(z) will be some branch of $\log[(z-a_1)/(z-a_2)]$. (The latter function does have distinct branches in the z plane when a simple arc joining a_1 to a_2 is removed.)

The imaginary part ψ_1 of F(z) is a real potential function in C-M. The existence theorem states that a real potential function ψ can be found which is defined over the entire surface S except at M, and which within C-M differs from ψ_1 only by a real potential function ψ_2 which has a removable singularity along M (i.e., ψ_2 is a potential function in C-M having limiting values along M such that if the latter are used to extend the definition of ψ_2 to M also, ψ_2 becomes a potential function in the whole region C). In other words, if the given ψ_1, which is a potential function in C-M, is modified by the addition of a suitable ψ_2, the latter being a potential function throughout C, then the sum $\psi_1 + \psi_2$ can be extended to be a potential function ψ over the whole of S-M. Or another way: if a potential function is given locally with a singularity of a certain type, a second potential function can be found which has the same singularity but is defined over the entire surface, and which differs from the first, where they are both defined, only by a potential function without singularities. Or most briefly: <u>a singularity of a certain type being assigned, there exists a real potential function ψ having that singularity and no other on S.</u>

Both ψ and ψ_2 are uniquely determined up to the same additive constant once ψ_1 is given. For if we find other functions ψ' and ψ_2' with the same properties as ψ and ψ_2, then

$$\psi = \psi_1 + \psi_2, \qquad \psi' = \psi_1 + \psi_2'$$

within C-M, and consequently

$$\psi - \psi' = \psi_2 - \psi_2'$$

in the same region. Thus $\psi - \psi'$ can be extended to be a potential function in C. But $\psi - \psi'$ is then a potential function over all of S. By the second theorem of §48 $\psi - \psi'$ is constant; and $\psi_2 - \psi_2'$ evidently equals the same constant.

The existence theorem applies in a similar way to the real part of F(z).

55. Definition of Integrals

To explain the sense in which the term "integral" will usually be used below, we may as well go back to a more general space of the sort contemplated in §37, since the definition applies quite directly in any such space.

Let a_j be a real or complex covariant vector of class C^1 in a region D of an n-dimensional space, such that curl $a_j = 0$ in D. Let D' stand for an arbitrary simply connected sub-region of D. By the theorem of §37, a_j is the gradient of some scalar point function Φ in D', so that

$$\partial \Phi / \partial x_j = a_j \, . \tag{55.1}$$

Also, Φ is uniquely determined up to an additive constant by the formula

$$\Phi(Q) = \int_{P_0}^{Q} a_j dx_j + \text{const.}, \tag{55.2}$$

where P_0 is a fixed point in D' and Q is a variable point. We can say that Φ is determined when its value is assigned arbitrarily at a single point of D'.

If on the other hand D' is a non-simply connected sub-region of D, a scalar Φ with the above properties may or may not exist.

Now the scalar Φ is called an <u>integral</u> in the given region D. Of course Φ only becomes a determinate scalar point function after a sub-region D', and a value for Φ at some point P_o of D', have been selected arbitrarily (and perhaps not even then, if D' is not simply connected). But no matter what choice is made for D' and for $\Phi(P_o)$, the gradient of Φ at any given point is always the same, being the value of a_j at that point. Thus the infinitely many different scalars represented by the letter Φ have in common the gradient a_j.

The essential thing about an integral is perhaps its gradient, rather than the scalars which have that gradient. But it is convenient to have the name "integral" for the set of scalars.

Let a_j, a_j', ... be a finite number of gradients (i.e., vectors with vanishing curl) in a region D, and let c, c', ... be an equal number of complex coefficients. The linear combination

$$A_j \equiv ca_j + c'a_j' + \ldots$$

is again a gradient in D. If F, Φ, Φ', ... are the integrals corresponding respectively to A_j, a_j, a_j', ..., we write

$$F = c\,\Phi + c'\,\Phi' + \ldots .$$

In any sub-region D' in which Φ, Φ', ... have determinations as one-valued scalars, F also has such a determination, and the last relation holds to within an additive constant. This explains what it means to say that a linear combination, with complex coefficients, of integrals in a given region is again an integral in the same region.

Consider again a real potential function ψ which is defined on the whole surface S except at a closed point set M, such as was discussed in the last section. According to §45, in any simply connected sub-region of S-M there exists another real potential function φ which is conjugate to ψ, so that $\varphi + i\psi$ is an analytic function. The conjugate potential φ is determined by the formula (45.6), and its gradient is given by the Cauchy-Riemann equations (43.7). Evidently in our present terminology φ is an integral in the region S-M. We may add that $\varphi + i\psi$ is a complex integral L in the same region. In any simply connected sub-region of S-M we have the formula

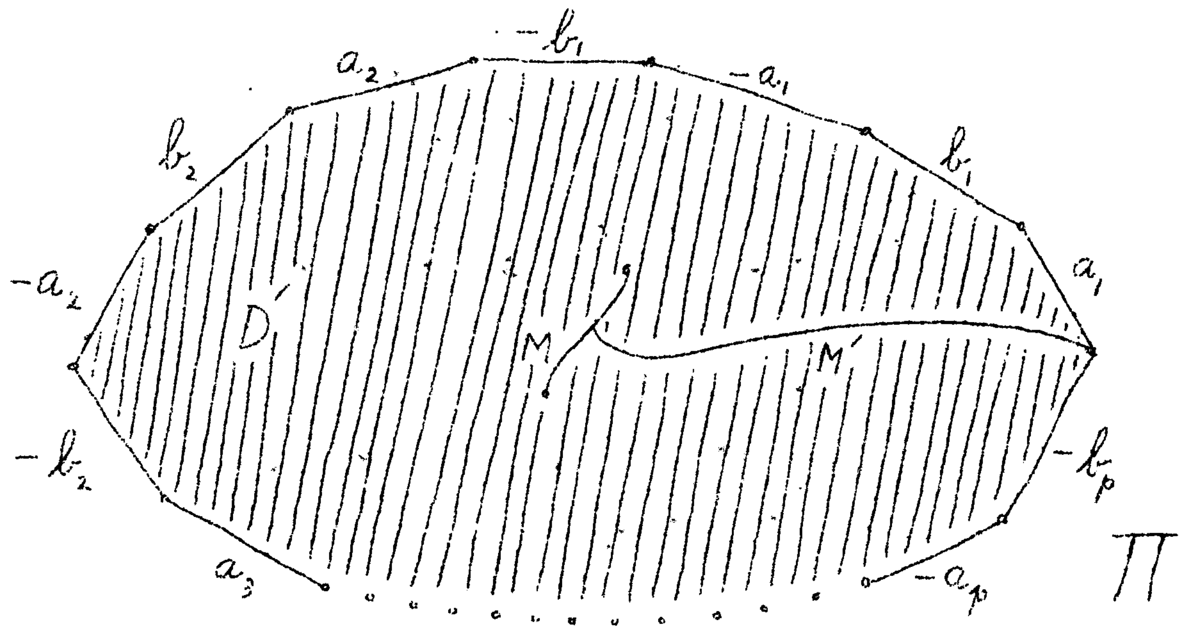
$$(55.3)\quad L(Q) = \int_{P_0}^{Q} \frac{\partial L}{\partial x_j}\,dx_j + \text{const.} = \int_{P_0}^{Q} \left(\frac{\partial \varphi}{\partial x_j} + i\frac{\partial \psi}{\partial x_j}\right)dx_j + \text{const.}$$

as in (55.2), and it has already been mentioned that

$$(55.4)\quad \partial\varphi/\partial x_1 = \partial\psi/\partial x_2\,, \quad \partial\varphi/\partial x_2 = -\partial\psi/\partial x_1\,.$$

Let S be represented as a polygon Π with a boundary Γ. (If the genus of S is zero, no polygonal representation is necessary; we may think of Π as identical with S, and of Γ as the null set. The modifications which are necessary in the following discussion when the genus is zero will not be mentioned when they are easy to supply.)

As noted in §53, we can assume that Γ and M do not intersect, so that M is in the interior of Π. Let M' be a simple arc of class D^1 which lies, except for its endpoints, in the interior of Π-M, and which has one endpoint on M and the other at the first vertex of Γ (i.e., the vertex which lies between the two sides $-b_p$ and a_1). The simply connected region on S obtained by deleting Γ, M and M' from S, will be called D'.



In the region D' the integral L is an analytic function with the imaginary part ψ, uniquely determined except for an additive constant. Now L is given by the integral of $\partial L/\partial x_j \cdot dx_j$, as stated in (55.3). Because of (55.4) and the fact that ψ is a potential function throughout S-M, the gradient $\partial L/\partial x_j$ is continuous in S-M, and in particular on Γ. Thus the integrand for L is continuous not only in D', but also on the part of the boundary of D' which is Γ. From this it can be shown easily that L has a unique limiting value at each point of the boundary Γ, and that these boundary values are continuous. The only exception is the first vertex of Γ. There L has two boundary values, corresponding to approach from the two sides of M'. It is understood that we are now thinking of Π as a plane polygon without identifications of points of its boundary.

The values of L along Γ may be calculated by means of the line integral (55.3) taken along Γ as well as along curves in D'.

The value of L at a point of a_1 minus the value of L at the corresponding point of $-a_1$ is called the jump of L across a_1 or $-a_1$, or across the cut on S which a_1 and $-a_1$ represent. The amount of the jump does not depend on the particular point selected on a_1 and $-a_1$; for if another point is taken, the difference of the values of L at the two points on a_1 is equal to the difference at the two points on $-a_1$. This is because the two differences are given by the integral (55.3) along a_1 between the two points on a_1, and along $-a_1$ between the two points on $-a_1$, respectively; and the integrand is the same in both cases, as the gradient $\partial L/\partial x_j$ is known to be one-valued throughout S-M.

Thus the jump of L across a_1 may be calculated by (55.3), or as we may write it, by $\int dL$, taken between any two corresponding points on a_1 and $-a_1$. Let us take the two points which are the endpoints of b_1, and integrate along b_1. Then we see that the jump of L across a_1 equals $\int dL$ taken clockwise along b_1. If we adopt the convention that integration along Γ is to be performed counterclockwise, we have the formula

(55.5) $$\text{Jump of L across } a_\nu = -\int dL \text{ along } b_\nu \;,$$

where $\nu = 1, \ldots, p$. For the above evidently holds for any one of the a_ν sides of Γ.

In a similar way, but making use of the fact that $\int dL$ along $-a_\nu$ is equal to $-\int dL$ along a_ν, we find that

(55.6) $$\text{Jump of L across } b_\nu = \int dL \text{ along } a_\nu \;.$$

Thus the 2p jumps or <u>periods</u> of the integral L may be evaluated by means of $\int dL$ taken along the 2p cuts in the surface S. The jump across an a_ν cut is found by integrating along the corresponding b_ν cut, and vice versa.

Just as above it may be shown that L has continuous boundary values along each side of the curve M', the endpoint of M' which is on M excepted. Furthermore L has a jump across M' which is constant all along M'.

We have been speaking of L as an integral in S-M derived from a real potential function ψ by means of (55.3). But even if L is any integral whatever in S-M, real or complex, all the remarks that we have made about the existence of definite jumps across the a_ν, the b_ν, and M' still hold. In fact our arguments were applicable to any such integral.

Let G and H be any two real or complex integrals in S-M, and consider their determinations (up to additive constants) as scalars in D'. We wish to evaluate $\int GdH$ taken around Γ. At the first vertex of Γ, G may have two distinct values, and H likewise, but that will not affect the value of $\int GdH$.

To begin with,

$$(55.7) \qquad \int_\Gamma G\,dH = \sum_{\nu=1}^{p}\left(\int_{a_\nu} + \int_{b_\nu} + \int_{-a_\nu} + \int_{-b_\nu}\right),$$

GdH being supplied behind the integral signs. Now

$$(55.8) \qquad \int_{a_1} + \int_{-a_1} = \int_{a_1} - \int_{-(-a_1)},$$

where the last integration is meant to be performed clockwise along $-a_1$. Let G^+ and G^- represent the values of G at corresponding points of a_1 and $-a_1$ respectively. As the gradient of H is the same at corresponding points of a_1 and $-a_1$, we may replace the second member of (55.8) by

$$\int_{a_1}(G^+ - G^-)dH.$$

But $G^+ - G^-$ is the constant jump of G across a_1, and by (55.5) it equals $-\int_{b_1} dG$. Consequently the above expression equals the product

$$-\int_{a_1} dH \cdot \int_{b_1} dG.$$

In the same way we find that

$$\int_{b_1} + \int_{-b_1} = \int_{b_1} dH \int_{a_1} dG.$$

Hence by (55.7),

$$\int_{\Gamma} GdH = \sum_{\nu=1}^{p} \left(\int_{a_\nu} dG \int_{b_\nu} dH - \int_{a_\nu} dH \int_{b_\nu} dG \right). \tag{55.9}$$

This formula shows that for any two integrals G and H in S-M, the value of $\int GdH$ around Γ is determined by the values of the periods of G and H, and in fact is equal to the above simple quadratic polynomial in the periods.

So far as the discussion has gone, the periods or jumps of integrals in S-M may depend on the particular choice of the cuts a_ν and b_ν and of the curve M'.

56. Integrals of the First and Second Kinds

We shall go back now to the analytic function F(z) introduced in §54. It has already been remarked that the point set M can be supposed to lie in the interior of the polygon Π. Restricting if necessary the simply connected region C which contains M, we may assume that it too lies in the interior of Π In addition, it can be shown that C may be so restricted that its boundary is met by M' in only one point. Then the region C-M-M', consisting of the points of C which are not on M or on the part of M' lying in C, is connected.

The real potential function in C-M which is the imaginary part of F(z) was called ψ_1. It was stated that a second real potential function ψ_2 exists in C, such that $\psi_1 + \psi_2$ can be extended to be a potential function ψ in S-M. Then certainly $\psi = \psi_1 + \psi_2$ in C-M-M'.

Now the integral L, or rather its determination as a scalar in the region D', is an analytic function in D' having the imaginary part ψ. In particular, $L(z)$ is analytic in C-M-M' with the imaginary part ψ.

The region C is simply connected, and ψ_2 is a potential function in C. By the theorem of §45 there exists in C an analytic function $F_2(z)$ having ψ_2 as its imaginary part.

The function $F(z) + F_2(z)$ is analytic in C-M, and hence in C-M-M', and its imaginary part is $\psi_1 + \psi_2$ But the same was said for $L(z)$, since $\psi_1 + \psi_2 = \psi$ in C-M-M'. Thus the analytic functions $L(z)$ and $F(z) + F_2(z)$ have the same imaginary part in the connected region C-M-M'. In view of the Cauchy-Riemann equations, their real parts have the same gradient in C-M-M', and hence differ at most by a constant. Therefore

$$L(z) = F(z) + F_2(z) + \text{const.} \tag{56.1}$$

in C-M-M'.

The right member of (56.1) is in fact analytic throughout C-M. It follows that the jump of L along M' is zero. Consequently the definition of L can be extended to all the points of M' except the endpoints, in such a way that L becomes analytic along M'.

We have now reached this conclusion: when the integral L is derived by starting from an analytic function $F(z)$, L has a determination as an analytic scalar throughout the doubly connected region obtained by subtracting M from the interior of Π. Going back to the surface S, this means that L is represented by a one-valued analytic function all over S-M, except that along the 2p cuts a_ν and b_ν the function is two-valued, with definite jumps.

In (56.1) $F_2(z)$ is regular in C and therefore on M. Consequently L has the same type of singularity on M as the given analytic function $F(z)$ has.

Suppose that instead of $F(z)$ we start with another function $F(z) + \tilde{F}(z)$, where $\tilde{F}(z)$ is analytic throughout C. Let $\tilde{\psi}$ be the imaginary part of $\tilde{F}$. We know that to the imaginary part ψ_1 of F a certain ψ_2 can be added such that $\psi_1 + \psi_2$ can be extended into a function ψ in S-M. Hence to the imaginary part $\psi_1 + \tilde{\psi}$ of $F + \tilde{F}$ the function $\psi_2 - \tilde{\psi}$ can be added such that the sum $(\psi_1 + \tilde{\psi}) + (\psi_2 - \tilde{\psi})$ or $\psi_1 + \psi_2$ can be extended into the same ψ in S-M. As noted at the end of §54, the process of extending the imaginary part of an analytic function into S-M, when possible, is essentially unique.

Therefore if a certain F(z) gives rise to an integral L, and if F is modified by the addition of any function analytic in C, the modified F will give rise to exactly the same integral L.

By definition the imaginary part of L differed by at most a constant from ψ, which was a one-valued potential function over S-M. Hence the jumps or periods of L are 2p real numbers.

By extending the real part instead of the imaginary part of F(z) it is possible to construct a different integral L^* in S-M which has 2p imaginary jumps. Just as in (56.1) we have

(56.2) $$L^*(z) = F(z) + F_2^*(z) + \text{const.}$$

in C-M, where F_2^* is analytic throughout C.

The difference $L - L^*$ is an integral in S-M. From (56.1) and (56.2) it is evident that this integral (or its gradient) has only a removable singularity along M. Hence $L-L^*$ extends into an integral which has no singularities over the entire surface S. Such an integral is said to be <u>of the first kind.</u> Of course not all of its jumps can be zero unless it is constant. In fact, not all of its jumps can be real; otherwise its imaginary part, and hence the whole integral, would be constant. Similarly an integral of the first kind whose jumps are all imaginary is merely a constant. In the present case, as L and L^* have real and imaginary jumps respectively, $L-L^*$ has complex jumps, and it is not constant unless the jumps of L or of L^* are all zero (which they may be, however).

On a surface of genus zero the only integrals of first kind are the constants. For as the whole surface is simply connected, any such integral has a determination as an analytic function on the whole surface, and hence must be constant.

Suppose that the point set M is a single point, with the coordinate $z = 0$, and that $F(z)$ equals a_γ/z^γ, γ being a positive integer and a_γ a complex number. The corresponding integral L is called an elementary integral of the second kind, and is denoted by $L^\gamma(Q; M, a_\gamma)$, where Q is a general point on S. It has just one singularity on the entire surface S: a pole of order γ at the point M. Its jumps are all real.

An integral which has one or more poles but no other singularities on the surface S, is said to be <u>of the second kind</u>. If its jumps are zero, it is a rational function (plus an arbitrary constant).

(If an integral when determined in a certain region has a pole at a certain point Q_1 of the region, it will have a pole of the same kind at Q_1 when determined in any other region containing Q_1. This is because in any neighborhood of Q_1, any two determinations of the integral differ by a constant.)

Let L be any integral of the second kind having a pole of order r at M, but no other poles. By the "principal part" of L with respect to the coordinate z is understood, as usual, the terms with negative powers of z in the Laurent series for L about the point M. Evidently by forming a suitable linear combination of the elementary integrals $L^1(Q; M, a_1)$, ..., $L^r(Q; M, a_r)$, we can obtain an integral of the second kind having the same principal part at M that the given L has. Then the difference between L and the linear combination is an integral of the first kind. That is, the most general integral of the second kind having a pole only at M is a linear combination of elementary integrals of the second kind with poles at M, plus an integral of the first kind. If L has only real jumps, it actually equals a linear combination of elementary integrals. For since $L^1, \ldots, L^r$ also have only real jumps, the integral of first kind must in this case be constant.

More generally, we see in the same way that an arbitrary integral of the second kind equals a linear combination of elementary integrals of the second kind, formed at the various points where the given integral has poles, plus an integral of the first kind. The latter will be constant, and may be omitted, when the given integral of second kind has only real jumps.

Of course $L^r(Q; M, a_r)$ is an elementary integral relative to the coordinate z. If we used another coordinate z' about the point M, and formed the integral $L'^r(Q; M, a_r')$ having a_r'/z'^r as its principal part at M, we should obtain another integral than $L^r(Q; M, a_r)$. (For when expressed in terms of z, the principal part of L'^r may contain other negative powers of z than the lowest.) However, it follows from what was said just above that L'^r is a linear combination of $L^1, \ldots, L^r$.

Naturally, instead of taking the elementary integrals $L^r(Q; M, a_r)$ with real jumps, we might have used the second construction, extending the real instead of the imaginary part of a_r/z^r over S, to obtain other elementary integrals of second kind, $L^{*r}(Q; M, a_r^*)$, with imaginary jumps. What has been said about the L^r applies with only obvious changes to the L^{*r}.

57. Elementary Integrals of the Third Kind

Suppose finally that the point set M is a simple arc, with endpoints whose z coordinates are a_1 and a_2, and that F(z) is some branch of $\log [(z-a_1)/(z-a_2)]$ in C-M. By our first construction we obtain from F an integral $L_{a_1a_2}$ in S-M with only real jumps across Γ (and by our second construction an integral $L^*_{a_1a_2}$ with only imaginary jumps).

The integral $L_{a_1a_2}$ appears from its definition to depend on the particular coordinate z, but actually it does not. For let z' be another complex coordinate in C, and let a_1' and a_2' be the z' coordinates of the endpoints of M. Then the equation of transformation from z to z' in C may be written as

$$z' - a_1' = (z-a_1)A(z)$$

or as

$$z' - a_2' = (z-a_2)B(z), \tag{57.1}$$

where A(z) and B(z) are analytic in C. Neither A(z) nor B(z) can vanish in C; for if A, for example, vanished when $z = a_1$, we should have $dz'/dz = 0$ for $z = a_1$; and if A vanished for some other value of z we should have two distinct values of z corresponding to the same value a_1' of z'.

Dividing, we have

$$\frac{z'-a_1'}{z'-a_2'} = \frac{z-a_1}{z-a_2}\, G(z),$$

where $G(z) \equiv A(z)/B(z)$ is analytic and never zero throughout C. As C is simply connected, log G(z) can be defined as a one-valued and analytic function in C and hence in C-M. For a proper choice of the branch in defining log G(z) we then have

$$\log \frac{z'-a_1'}{z'-a_2'} = \log\frac{z-a_1}{z-a_2} + \log G(z)$$

in C-M. That is, F(z) as formed in the z coordinate system differs only by a function analytic in C from F(z') as formed in the z' coordinate system. But we saw in the previous section that the integral $L_{a_1a_2}$ derived from F(z) is then identical with the integral $L_{a_1'a_2'}$. Therefore $L_{a_1a_2}$ really does not depend on z.

The function $F(z)$ has a jump of $2\pi i$ in crossing M. But as is well known, the derivative, and hence the gradient, of $F(z)$ may be extended so as to be analytic at every inner point of M. Thus the integral $L_{a_1a_2}$ is analytic on the whole surface S except at the two points a_1 and a_2, where it has logarithmic singularities.

Let M' be another simple arc, joining a_2 to some point a_3', and lying in a region C' covered by a coordinate z'. Suppose that M and M' together form a simple arc N. Let a_2' and a_3' be the z' coordinates of the ends of M'. Let $L_{a_1a_2}$ be the integral considered above, and let $L_{a_2'a_3'}$ be the similar integral derived from $\log[(z'-a_2')/(z'-a_3')]$ in $C'-M'$.

The sum

$$L_{a_1a_3'} \equiv L_{a_1a_2} + L_{a_2'a_3'}$$

is an integral which is analytic over S except at a_1, a_2, and a_3'. This integral has a jump of $2\pi i$ across N (it may be shown that the sense of crossing to obtain this jump is the same along M as along M').

In the intersection of $C-M$ and $C'-M'$, $L_{a_1a_3'}$ is equal to

$$\log\frac{z-a_1}{z-a_2} + \log\frac{z'-a_2'}{z'-a_3'} \tag{57.2}$$

plus a function analytic in $C \cap C'$. The transformation from z to z' is expressed by an equation of the form (57.1), where as before $B(z)$ is analytic and not zero. Consider a neighborhood of a_2 lying in the intersection of $C-M$ and $C'-M'$, which as a result of the removal of M and M' is in two disconnected parts, one on each side of N. In either part

$$\log(z'-a_2') = \log(z-a_2) + \log B(z). \tag{57.3}$$

If we write (57.2) as

$$\log(z-a_1) - \log(z-a_2) + \log(z'-a_2') - \log(z'-a_3')$$

and use (57.3), we see that $L_{a_1 a_3'}$ is equal to

(57.4) $$\log (z-a_1) - \log (z'-a_3')$$

plus a function analytic in the above neighborhood with the arc of N not excluded. But as for (57.4), different determinations of it differing by $2\pi i$ would be used in the two disconnected parts of the neighborhood.

Now each term of (57.4) is analytic at a_2. Thus while different branches of the terms may be used on the two sides of N, the derivatives are one-valued and analytic even on N. Hence the gradient of $L_{a_1 a_3'}$, and the integral itself, actually has no singularity at a_2. At a_1 and a_3', however, it does have logarithmic singularities.

The jumps of $L_{a_1 a_3'}$ across Γ are all real. By using $L^*_{a_1 a_2}$ and $L^*_{a_2' a_3'}$ we would obtain a similar integral with none but imaginary jumps across Γ.

More generally, let a and b be any two points in the interior of the polygon Π and let N be a simple arc joining a to b and lying in the interior of Π. Then we can construct an integral L_{ab} which is regular on the whole surface S except at a and b, where it has logarithmic singularities. This integral has a jump of $2\pi i$ across N, and real jumps across Γ. Similarly, an L^*_{ab} can be constructed with a jump of $2\pi i$ across N and imaginary jumps across Γ. The method would be to separate N into a finite succession of arcs M, M', M", ..., lying in successive regions C, C', C", ... covered by coordinates z, z', z", ... Just as before we would set up integrals in the successive regions and prove that their sum is an L_{ab} or an L^*_{ab} as required.

The integrals L_{ab} and L^*_{ab} are called <u>elementary integrals of the third kind</u>.

Employing the above integrals of third kind, we shall show that non-constant integrals of first kind exist on any surface of genus greater than zero. This fact was not established in the previous section.

Draw a simple closed curve E on the surface S. Let a, b be any two points of E. Call the two simple arcs into which a and b divide E, N_1 and N_2.

Construct as described above the integral of third kind, L_{ab}, which has logarithmic singularities at a and b, and only real jumps except for a jump of $2\pi i$ across N_1. Construct also the similar integral L_{ba} which has the jump of $2\pi i$ across N_2. (The constructions of L_{ab} and of L_{ba} may require the use of different polygons Γ.)

The sum

$$L \equiv L_{ab} + L_{ba}$$

is an integral on S which has no singularities at a and b, and hence no singularities whatever. This may be proved by the method used earlier in this section. Thus L is an integral of first kind. It remains to show that L is not constant.

The imaginary part of L_{ab} has a determination as a potential function on the entire surface, except for a jump of 2π across N_1. This is because L_{ab} had no imaginary jumps other than the one across N_1. Likewise the imaginary part of L_{ba} is a potential function on S except for a jump of 2π across N_2. It follows that the sum of the imaginary parts of L_{ab} and L_{ba}, and hence the imaginary part of L, has a determination as a potential function on S, except for a jump of 2π across the simple closed curve E. (It can be proved that this jump occurs in the same sense, on crossing from one side of E to the other, all along E.)

Thus the imaginary part of L is a one-valued potential function on the surface S minus the simple closed curve E. There are two possibilities:

either E separates S, in which case S-E consists of two disjunct regions; or E does not separate S, and S-E is a single region.

If E separates S, L will in fact be constant. For if we take the above determination of the imaginary part of L and modify it by adding 2π in the proper one of the two regions into which E divides S, the result is a potential function over the whole of S, and hence a constant. Then the gradient of L is zero, and L is constant.

On the other hand, suppose that E is a non-separating curve on S. If L were constant, any determination of its imaginary part in any region on S would be a constant. But the above determination of the imaginary part in S-E is not constant, as it has a jump of 2π across E. Thus L is not constant. This proves that non-constant integrals of first kind exist on surfaces of positive genus.

Consider a polygon Γ formed for a surface S of positive genus. The sides a_1 and $-a_1$ of Γ are represented by a non-separating curve on S. In consequence of the discussion above we see that there exists on S an integral of first kind having a jump of $2\pi i$ across a_1 but no other imaginary jumps on Γ. If we divide by $\pm 2\pi i$ we can say that there exists an integral of first kind with a jump of +1 across a_1 as its only real jump. Of course any one of $a_2, \ldots, a_p, b_1, \ldots, b_p$ might replace a_1 in the last statement.

58. The Linear Space of Integrals of First Kind

Given a surface S, let V be the totality of integrals of first kind on S. Since a linear combination, with complex coefficients, of integrals of first kind is again an integral of first kind, V is a linear space with respect to complex coefficients (or, similarly, with respect to real coefficients).

The constant integral must be considered as the zero element of this space. We shall show that the dimension of V, that is, the maximum number of integrals of first kind which are linearly independent with respect to complex coefficients, is the genus p.

Lemma 1: Let a set $\{w\}$ of elements w be a linear space with respect to complex coefficients. If $\{w\}$ has a finite dimension q with respect to complex coefficients, it has the dimension 2q with respect to real coefficients, and conversely.

Proof: Suppose $\{w\}$ has the dimension q with respect to complex coefficients, and let $w_1, \ldots, w_q$ be a basis. Then $w_1, \ldots, w_q, iw_1, \ldots, iw_q$ are readily seen to be a basis with respect to real coefficients. Hence $\{w\}$ has the dimension 2q with respect to real coefficients. Conversely, if $\{w\}$ has the real dimension 2q, it has a finite complex dimension $q' \leqq 2q$. By the first case, $2q' = 2q$. Hence $\{w\}$ has the complex dimension q.

Let a polygon Γ be constructed for S. Denote by $\tilde{v}_\nu$ $(\nu = 1, \ldots, p)$ an integral of first kind having a jump of +1 across a_ν, but no other real jump. Also, let $\tilde{v}_{p+\nu}$ be an integral of first kind having a jump of +1 across b_ν as its only real jump.

Now $\tilde{v}_1, \ldots, \tilde{v}_{2p}$ are linearly independent with respect to real coefficients. For if $c_1, \ldots, c_{2p}$ are real, the integral

(58.1) $$v \equiv c_1\tilde{v}_1 + \ldots + c_{2p}\tilde{v}_{2p}$$

has a real jump of c_ν across a_ν and a real jump of $c_{p+\nu}$ across b_ν. If the c's are not all zero, v cannot be constant.

Furthermore, let v be any given integral of first kind, and let $c_1, \ldots, c_{2p}$ be the real parts of the jumps of v across $a_1, \ldots, a_p, b_1, \ldots, b_p$ respectively. Then (58.1) holds, because the difference of the two members is an integral of first kind without real jumps, and hence a constant.

Therefore $\tilde{v}_1, \ldots, \tilde{v}_{2p}$ are a basis for V with respect to real coefficients, and the real dimension of V is 2p. By the lemma, the dimension of V with respect to complex coefficients is then p. On any surface, the number of linearly independent integrals of first kind is exactly equal to the genus.

Incidentally, the last statement is a proof of what was remarked in §53, that p does not depend on the particular triangulation of the surface. For the number of independent integrals has nothing to do with any triangulation. A similar proof is implicit in the formula at the end of §53, except that we do not yet know that non-constant rational functions exist on any given surface.

Our next step depends on a less simple lemma:

Lemma 2: An integral of first kind which has no jumps on the a_ν lines (or, the b_ν lines) is constant.

Proof: Let $\varphi + i\psi$ be a determination of the given integral in the polygon Π. Then $\varphi + i\psi$ is an analytic function in Π which may have jumps on the b_ν lines, but none on the a_ν lines. And the potential functions φ and ψ may be considered as determinations of real integrals in Π for whose jumps the same can be said.

We now use (55.9) applied to φ and ψ as G and H. In view of (55.5) and our hypothesis about jumps on the a_ν lines, we see that $\int_\Gamma \varphi\, d\psi = 0$.

Let a triangulation of class D^1 be made in the polygon Π, in such a way that the totality of edges of the triangles includes the whole of Γ. Proceeding as in the proof of the first theorem of §48, we deduce that φ and ψ are constant. The only difference is that here $\int \varphi\, d\psi$ around all the triangles reduces to $\int \varphi\, d\psi$ around Γ, instead of vanishing directly, when account is taken of the cancellations due to integrating back and forth along common edges of adjacent triangles. But since $\int_\Gamma \varphi\, d\psi = 0$, the final conclusion is quite the same here as in §48. Hence the given integral is constant.

Let $v'_1, \ldots, v'_p$ be a basis for the space V (considered from now on as a linear space with respect to complex coefficients). Represent the jump of v'_α ($\alpha = 1, \ldots, p$) across the line a_ν ($\nu = 1, \ldots, p$) by $\sigma'_{\alpha\nu}$, and the jump of v'_α across b_ν by $\tau'_{\alpha\nu}$.

Let

(58.2) $$v = A_1 v'_1 + \ldots + A_p v'_p$$

be a linear combination of the v'_α with complex coefficients A_α. Then the jump of v across a_ν is the same linear combination of the jumps of the v'_α across a_ν, that is,

(58.3) $$A_1 \sigma'_{1\nu} + \ldots + A_p \sigma'_{p\nu};$$

and similarly the jump of v across b_ν is

$$A_1 \tau'_{1\nu} + \ldots + A_p \tau'_{p\nu}.$$

This may be seen directly, or else with the help of formulas (55.5) and (55.6).

A first conclusion is that the determinant $|\sigma'_{\alpha\nu}|$ of the a_ν jumps of a basis v'_α cannot vanish. For if it did, we could find numbers $A_1, \ldots, A_p$ for which (58.3) would vanish for all the values of ν. Using those A's in (58.2) we would have an integral v without jumps on the a_ν lines, and hence by Lemma 2 a constant. This is impossible, because the v'_α are linearly independent. Similarly $|\tau'_{\alpha\nu}|$ cannot vanish.

If we set

$$v_\alpha = \sum_{\beta=1}^{p} A_{\alpha\beta} v'_\beta, \qquad \text{where } |A_{\alpha\beta}| \neq 0,$$

we obtain a new basis $v_1, \ldots, v_p$. For the p integrals v_α are again linearly independent. The jumps of the v_α across a_ν and b_ν are given by

$$\sigma_{\alpha\nu} = \sum A_{\alpha\beta} \sigma'_{\beta\nu} \quad \text{and} \quad \tau_{\alpha\nu} = \sum A_{\alpha\beta} \tau'_{\beta\nu}$$

respectively.

Since $|\sigma'_{\alpha\nu}| \neq 0$, the coefficients $A_{\alpha\beta}$ can be so chosen that

(58.4) $$\sigma_{\alpha\nu} = \delta_{\alpha\nu}.$$

In fact, $\|A_{\alpha\beta}\|$ is simply taken as the inverse of $\|\sigma'_{\alpha\nu}\|$. When the matrix of the a_ν jumps has thus been reduced to the identity, the matrix of the b_ν jumps becomes symmetric, so that

(58.5) $$\tau_{\alpha\nu} = \tau_{\nu\alpha}.$$

We can prove this as follows.

Let v_α and v_β stand for two analytic functions which are determinations of the integrals v_α and v_β in the polygon Π. Let Π be triangulated as in the proof of Lemma 2. If we integrate $v_\alpha\, dv_\beta$ around all the triangles and use the Cauchy integral theorem of §46, we find that

$$\int_\Gamma v_\alpha\, dv_\beta = 0.$$

From (55.9) it follows that

$$0 = \sum_{\nu=1}^{p}\left(\int_{a_\nu} dv_\alpha \int_{b_\nu} dv_\beta - \int_{a_\nu} dv_\beta \int_{b_\nu} dv_\alpha\right).$$

By (55.5) and (55.6),

$$\sum_{\nu=1}^{p}\left(-\tau_{\alpha\nu}\,\sigma_{\beta\nu} + \tau_{\beta\nu}\,\sigma_{\alpha\nu}\right) = 0.$$

Substituting (58.4), we derive (58.5).

Summary: The linear space V of integrals of first kind on a surface of genus p is of dimension p. A basis of p integrals $v_1, \ldots, v_p$ can be selected for V in such a way that the matrix $\|\sigma_{\alpha\nu}\|$ of the a_ν jumps is the unit matrix, and the matrix $\|\tau_{\alpha\nu}\|$ of the b_ν jumps is symmetric and non-singular.

59. Differentials

The sum of an integral of second kind and one of first kind is still an integral of second kind. The sum of an elementary integral of third kind and an integral of first kind is said to be of third kind. An integral of second or third kind is termed "normalized" if it has no jumps on the a_ν lines (of a given polygon Γ).

Let I be an integral of second or third kind, having a jump of A_ν across a_ν . In

$$A_1 v_1 + \dots + A_p v_p$$

we have an integral of first kind with the same a_ν jumps as I. Hence

$$I' \equiv I - \sum A_\alpha v_\alpha$$

is an integral of the same kind as I, having the same singularities but without a_ν jumps. That is, an integral of second or third kind can always be normalized by the subtraction of a suitable integral of first kind.

Consider an integral L which is a linear combination with complex coefficients of a finite number of integrals of first, second and third kinds. Of course L may itself be an integral of one of these kinds, or it may be a rational function (plus an arbitrary constant).

The derivative dL/dz of L with respect to a complex coordinate z is called a "differential". A differential does not have the character of a point function, therefore, but of a covariant vector in a space of one (complex) dimension, and necessarily of a gradient. For if we took another coordinate z' instead of z, we would have

$$\frac{dL}{dz'} = \frac{dL}{dz}\frac{dz}{dz'} \tag{52,1}$$

The differential derived from L is one-valued (as a vector), just as the ordinary gradient of L is. In fact there are the relations

$$\frac{dL}{dz} = \frac{\partial L}{\partial x_1} = \frac{1}{i}\frac{\partial L}{\partial x_2},$$

where $z = x_1 + ix_2$, between the components of the differential and the gradient

By assumption the singularities of L are poles and logarithmic singularities, or sums of these. It follows that dL/dz, when determined in any specific coordinate system z on the surface S, is an analytic function with no singularities other than poles.

To say that a differential has a pole or a zero of a certain order at a certain point, is an invariant statement. This follows from (59.1) if we express by power series the three functions involved (cf. (48.2) and (48.3)), and keep in mind that dz/dz' is analytic and not zero. A finite value of a differential, other than zero, changes with the coordinate system. (Even the number of times the value is assumed may change.)

A differential which is not identically zero has at most a finite number of zeros and poles.

Let L_1 be a second integral like L, but not a constant. From the transformation law (59.1) it is clear that the ratio

$$f = \frac{dL}{dz} \Big/ \frac{dL_1}{dz} \tag{59.2}$$

of the two differentials derived from L and L_1 is an invariant, a complex point function on S. Since each differential is analytic in any given region except for poles, the same can be said for the ratio. Hence f is a rational function on S.

If L and L_1 are linearly dependent, so that $L = cL_1 + \text{const.}$, where c is a constant, then f is constant. Conversely, if f has a constant value c, the gradient of L equals c times the gradient of L_1, and hence $L = cL_1 + \text{const.}$ and the two integrals are linearly dependent.

Now L and L_1 can surely be taken to be linearly independent; for example, L may be of second kind and L_1 of third kind. It follows for the first time that <u>on any given surface S there exist rational functions which are not</u>

constant. This gives content to the discussion of §§48-51. To perform a similar service for part of §52 we shall prove that for any given non-constant rational function f there exists a rational function h such that f,h are a primitive pair.

Let f be of order n. Let f_0 be a value which f assumes at n distinct points $P_1, \ldots, P_n$. We shall construct a rational function h which assumes the value h_j ($j = 1, \ldots, n$) at P_j, the h_j being n distinct but otherwise arbitrary numbers. Our statement will then be proved (see the discussion before Theorem II, §52).

In some neighborhood of P_j, f can be introduced as a complex coordinate, since no P_j is a critical point of f. Construct the elementary integral of second kind $L^{(j)}$ which has as its only singularity a simple pole at P_j with $1/(f-f_0)$ as principal part.

Consider the rational function

$$F^{(j)} \equiv \frac{dL^{(j)}}{dz} \Big/ \frac{df}{dz},$$

the quotient of the differentials of the integral $L^{(j)}$ and the rational function f. In the neighborhood of P_j, if we use f as the coordinate z, we see that

$$F^{(j)} = \frac{dL^{(j)}}{df} = \frac{-1}{(f-f_0)^2} + \cdots\cdots$$

so that $F^{(j)}$ has a double pole at P_j with $-1/(f-f_0)^2$ as principal part. At $P_1, \ldots, P_{j-1}, P_{j+1}, \ldots, P_n$ on the other hand, $F^{(j)}$ is regular, since $L^{(j)}$ is regular and df/dz is not zero.

The desired function h may be defined by

$$h = -\sum_{j=1}^{n} h_j (f-f_0)^2 F^{(j)}.$$

For the j th summand evidently is a rational function which equals $-h_j$ at P_j

and vanishes at all the other P's.

We return to the integral L, the general linear combination of integrals of first, second and third kinds. Let f,h be a primitive pair of rational functions on S.

The ratio of the differentials of L and f is a rational function. Consequently according to Theorem II of §52,

$$\frac{dL}{dz} = R(f,h)\frac{df}{dz},$$

where R(f,h) is some rational combination of f and h.

It follows that

$$L = \int R(f,h)df. \tag{59.3}$$

The integration can be performed along curves in (at least) any simply connected region in which dL/dz has no poles. If we denote the rational function R(f,h) simply by g, we may write instead

$$L = \int g df. \tag{59.4}$$

It is possible to prove conversely that if g and f are any two rational functions on S, f not being constant, then (59.4) defines an integral L which is a linear combination of integrals of first, second and third kinds. Expressing g as a rational combination of f and h, where h is primitive with respect to f, we have the equivalent form (59.3).

Now (59.3) or (59.4) amount to the ordinary form of definition of an "Abelian integral". Hence the three kinds of integrals that we have been discussing, and the linear combinations of them, are the same thing as Abelian integrals.

60. The Riemann-Roch Theorem

We have already proved that there exist non-constant rational functions on any given surface S. Our object now is to obtain some idea of how many such functions there are on the surface. In particular, what freedom is there in assigning the poles?

Let $P_1, \ldots, P_r$ be r distinct points, where $r \geq 1$. Our first question will be: Do there exist rational functions which are analytic at every point of S other than the P_j $(j = 1, \ldots, r)$, and which at each of the P_j either are analytic or have simple poles? This of course is not quite the same thing as asking for rational functions with assigned simple poles.

The set of all rational functions which have no singularities except perhaps simple poles at the P_j, constitute a linear space or a "module" W with respect to complex coefficients. This space contains at least the constant functions. Our answer will be given as a formula for the dimension of W.

Let S be represented as a polygon $\prod$ with the P_j in its interior. Let z_j be a complex coordinate covering a neighborhood of P_j.

Construct an integral of second kind, L_j, which has a simple pole at P_j with principal part $1/z_j$, but no other singularity. We may assume that L_j is normalized, as described at the beginning of §59, so that it has no jumps on the a_ν lines. In the work below, L_j will stand for some specific determination of the integral L_j in the polygon $\prod$.

Consider a rational function f of the space W. Let the principal part of f at P_j be c_j/z_j, it being understood that c_j may be zero.

The difference between f and $\sum c_j L_j$ has no singularities and is therefore equivalent to an integral of first kind. As there are no jumps on the a_ν lines, the latter integral equals some constant k. That is,

$$f = c_1 L_1 + \cdots + c_r L_r + k \qquad (60.1)$$

(cf. the end of §56).

This formula gives us an upper limit for the dimension of W. For since any element of W can be obtained by some choice of the complex parameters $c_1, \ldots, c_r$, k, the dimension of W is at most r+1. However, it can be less than r+1. This is not because distinct sets of values of c_j, k may furnish the same f; indeed, this cannot happen, for $L_1, \ldots, L_r$, 1 are linearly independent, as is readily seen by considering their poles.

The dimension of W can be less than r+1 because, while any choice of values for the parameters yields a normalized integral of second kind with no singularities but simple poles at the P_j (or a constant), it does not necessarily yield a one-valued function. In fact, we can say that r+1 is the dimension of the module of normalized integrals of second kind which have no singularities that are not simple poles at the P_j (more precisely, the dimension is r, for the additive constant k in (60.1) does not change f considered as an integral).

Our problem thus becomes, to determine what condition must be laid on the c_j in order that the integral f in (60.1) shall be equivalent to a one-valued function. In other words, for what c_j will the jumps of f all be zero? As f has no a_ν jumps, this means: for what values of the c_j will the b_ν jumps of f all be zero?

Let the jump of L_j across b_ν be denoted by $\rho_{j\nu}$ ($j = 1, \ldots, r$; $\nu = 1, \ldots, p$). Then the jump of f across b_ν equals $\sum c_j \rho_{j\nu}$, and the condition on the c_j becomes

(60.2) $$c_1 \rho_{1\nu} + \cdots + c_r \rho_{r\nu} = 0 \quad (\nu = 1, \ldots, p).$$

At this point we can already say that if r is greater than the genus p, c_j not all zero can be found satisfying (60.2), and hence there will exist non-constant rational functions with not more than r poles, all simple.

The rank of the matrix $\|\rho_{j\nu}\|$ is some number between 0 and p. Denote it by $p-\tau$, where $0 \leqq \tau \leqq p$. Then (60.2) has exactly $r-(p-\tau)$ linearly independent solutions c_j. Let $c_{\sigma j}$ ($\sigma = 1, \ldots, r-p+\tau$) be such independent solutions.

The $r-p+\tau+1$ rational functions

$$f_\sigma \equiv \sum c_{\sigma j} L_j, \quad f_{r-p+\tau+1} \equiv 1$$

are a basis for the module W. For they are linearly independent, because the L_j and 1 are linearly independent, and likewise the $c_{\sigma j}$. Furthermore, any f in W is of the form (60.1), in which the c_j are expressible linearly in terms of the $c_{\sigma j}$; hence f is expressible linearly in terms of $f_1, \ldots, f_{r-p+\tau+1}$.

We have now proved with Riemann that <u>on a surface of genus p, the linear space W of all rational functions having at most simple poles at r given points as singularities, is of dimension $r-p+\tau+1$, where $0 \leqq \tau \leqq p$.</u>

It might seem from the definition of the integer τ that its value depends on the special choice of the polygon Γ, or of the coordinates z_j about the points P_j, or of the integrals L_j (although actually the L_j are uniquely determined when Γ and the z_j are fixed). But in view of the formula of Riemann, it is evident that τ depends on nothing but the set of m points P_j. An interpretation for τ was added to Riemann's result by Roch, and we shall now develop this.

Let P be an arbitrary point within the polygon Π, and let ζ be a complex coordinate about P. Write L for the normalized integral of second kind having a simple pole at P with $1/\zeta$ as principal part, as its only singularity. Any one of the L_j is an integral of this sort.

At the end of §58 we had established the existence of a set of p integrals of first kind, v_α ($\alpha = 1, \ldots, p$), which formed a basis for the module V of all integrals of first kind on the given surface. We saw that the jump of v_α across a_ν could be assumed to be $\delta_{\alpha\nu}$.

The differential derived from an integral of first kind is called a "differential of first kind". Such a differential is analytic over the whole surface. The differentials of first kind constitute a linear space of dimension p. For an arbitrary integral v of first kind is given by

(60.3) $$v = A_1 v_1 + \cdots + A_p v_p ,$$

where the A's are complex parameters. Differentiating, we find that

(60.4) $$\frac{dv}{dz} = A_1 \frac{dv_1}{dz} + \cdots + A_p \frac{dv_p}{dz} .$$

The dv/dz on the left is an arbitrary differential of first kind. It cannot be zero, because the v which appears in (60.3) is never a constant. Hence the differentials of first kind are a modulo of dimension p, and in fact the differentials of the v_α can sorve as a basis for this module.

Once more, consider a triangulation of class D^1 in the polygon Π such that the whole of the boundary Γ appears among the edges of the triangles. Let the point P be interior to some triangle Δ lying in the ζ coordinate system.

Consider the result of integrating $L\,dv_\alpha$ about all the triangles. On the one hand, this reduces to $\int_\Gamma L dv_\alpha$. On the other hand, $\int L dv_\alpha$ vanishes about every triangle other than Δ, by the Cauchy integral theorem. About Δ we have to evaluate

(60.5) $$\int L \frac{dv_\alpha}{d\zeta} d\zeta .$$

The integrand has a simple pole at $\zeta = 0$, with the residue $[dv_\alpha/d\zeta]_{\zeta = 0}$. It may be shown as in the proof of the last thoorem in §48 that the value in question is $+2\pi i$ times the residue. Hence

$$\int_\Gamma L dv_\alpha = 2\pi i \left[\frac{dv_\alpha}{d\zeta}\right]_{\zeta = 0} .$$

(This relation may seem impossible, since there is an invariant on the left but not on the right. It should be remembered, however, that the relation is true only for a special coordinate ζ in terms of which the principal part of L is $1/\zeta$).

Here we make our third use of formula (55.9):

$$\int_{\Gamma} L\,dv_{\alpha} = \sum_{\nu=1}^{p} \left(\int_{a_\nu} dL \int_{b_\nu} dv_{\alpha} - \int_{a_\nu} dv_{\alpha} \int_{b_\nu} dL \right).$$

Since

$$-\int_{b_\nu} dv_{\alpha} = \delta_{\alpha\nu} \quad \text{and} \quad -\int_{b_\nu} dL = 0$$

(L is normalized), this becomes

$$\int_{\Gamma} L\,dv_{\alpha} = -\int_{a_\alpha} dL.$$

Our conclusion is that

(60.6) $$\int_{a_\alpha} dL = -2\pi i \left[\frac{dv_\alpha}{d\zeta}\right]_{\zeta=0}.$$

Thus the jump of L across b_α is determined by the value of the differential of v_α at $\zeta = 0$.

The above computation holds in particular for each of the integrals L_j. The simple pole of L_j was at the point P_j, the special coordinate ζ was z_j, and the jump of L_j across b_ν was called $\rho_{j\nu}$. Hence

$$\rho_{j\nu} = -2\pi i \left[\frac{dv_\nu}{dz_j}\right]_{z_j=0}.$$

The rank of $\|\rho_{j\nu}\|$ was $p-\tau$. The rank of

(60.7) $$\left\| \left[\frac{dv_\nu}{dz_j}\right]_{z_j=0} \right\| \qquad (\nu = 1,\dots,p;\ j = 1,\dots,r)$$

is therefore $p-\tau$ also, and of course the rank of the transposed matrix is the same.

The linear space of differentials of first kind was of dimension p, and an arbitrary element dv/dz of this space was given by (60.4) in terms of the basis dv_ν/dz. Now the set of differentials of first kind which vanish at all the points $P_1, \dots, P_r$ are a linear subspace (it may be recalled that the vanishing of a differential at a point is invariant, and even the order of vanishing is invariant). The condition for a differential dv/dz, given by the parameters A_ν in (60.4), to vanish at all the P_j is

$$(60.8) \qquad A_1\left[\frac{dv_1}{dz_j}\right]_{z_j=0} + \dots + A_p\left[\frac{dv_p}{dz_j}\right]_{z_j=0} = 0 \quad (j = 1, \dots, r).$$

The matrix of this system of r equations is the transpose of (60.7), and its rank is $p-\tau$. Hence there are $p - (p-\tau)$ or τ linearly independent solutions A_ν.

Roch's interpretation of the integer τ in Riemann's formula has now been derived: <u>τ is the dimension of the linear space of differentials of the first kind which vanish at all of the points $P_1, \dots, P_r$.</u>

So far we have only discussed rational functions which have single poles at certain given points. The more general result is the following.

<u>Riemann-Roch Theorem:</u> On a surface of genus p, let r distinct points $P_1, \dots, P_r$ be selected. Let a positive integer m_j $(j = 1, \dots, r)$ be assigned to the point P_j, and let $m_1 + \dots + m_r = m$. The set of rational functions on the surface which have no singularities at points other than the P_j, and which at P_j have either no singularity or else a pole of order not more than m_j, constitute a linear space of some dimension T. Then

$$(60.9) \qquad T = m - p + \tau + 1, \quad \text{where } 0 \leqq \tau \leqq p.$$

The integer τ is determined by the point set P_j and the corresponding numbers m_j. It may be interpreted as the dimension of the linear space composed of all differentials of the first kind which vanish at every one of the points P_j, and in fact vanish to at least the order m_j at P_j.

The proof of this theorem is very similar to the proof of the special case in which all the m_j are equal to 1, which has already been given. Therefore it will be enough to show in what way the general proof would differ from the special one. This can be sufficiently illustrated in a particular case, say where $m_1 = 2$ and the other m's are all equal to 1.

It is convenient to write $r-1$ for the number of distinct points instead of r. Then the m of the theorem becomes the same as r. Let a point P_r be introduced which is identical with P_1, so that P_1 now appears $m_1 = 2$ times among the P_j.

The linear space W is now the set of all rational functions f which have no singularities except perhaps simple poles at $P_2, \ldots, P_{r-1}$ and a simple or a double pole at P_1.

The integrals $L_1, \ldots, L_{r-1}$ are the same as they were. But for L_r we take the normalized integral of second kind $L_1^{(2)}$ which has a double pole at $P_r = P_1$ with principal part $1/z_1^2$, and no other singularity.

Given an f in W, write c_j/z_j for the principal part of f at P_j $(1 < j < r)$, but $c_1/z_1 + c_r/z_1^2$ for the principal part of f at P_1. Equation (60.1) then holds, and Riemann's part of the theorem follows as before.

The integral L in the next discussion must be allowed to have a pole with principal part $1/\zeta^2$. In that case the integrand in (60.5) has as residue at $\zeta = 0$ the quantity $[d^2v_\alpha/d\zeta^2]_{\zeta=0}$, the coefficient of the linear term in the expansion of $dv_\alpha/d\zeta$ in powers of ζ. Then the first derivative is replaced by the second within the brackets in (60.6).

The matrix (60.7) becomes

$$\left\| \left[\frac{dv_\nu}{dz_1}\right]_{z_1=0} \cdots \left[\frac{dv_\nu}{dz_{r-1}}\right]_{z_{r-1}=0} \left[\frac{d^2v_\nu}{dz_1^2}\right]_{z_1=0} \right\| \quad (\nu = 1, \ldots, p).$$

The condition for a differential dv/dz given by (60.4) to vanish at all the P_j is (60.8) with $j = 1, \ldots, r-1$, and the condition for its vanishing at P_1 to be of second order is

$$A_1\left[\frac{d^2v_1}{dz_1^2}\right]_{z_1=0} + \cdots + A_p\left[\frac{d^2v_p}{dz_1^2}\right]_{z_1=0} = 0.$$

There are τ linearly independent solutions to these equations. Hence τ is the dimension of the module of differentials of first kind which vanish to at least the second order at P_1 and to at least the first order at $P_2, \ldots, P_{r-1}$. The proof of the theorem is now complete.

Perhaps (60.9) is better thought of as a relation between T and τ rather than as a formula for T in terms of τ and the other letters. For τ does not seem intrinsically simpler than T.

61. The Brill-Noether Reciprocity Theorem

This section contains several results connected with the Riemann-Roch theorem.

Let f be a non-constant rational function of some order m. Let $P_1, \ldots, P_m$ be the m points at which f assumes a value a (a point P where f = a p times is understood to be repeated p times in the list, and a similar convention will be used for the zeros of differentials).

The number τ of the Riemann-Roch theorem has a certain value $\tau(a)$ for the set of points $P_1, \ldots, P_m$ (more precisely, for the r distinct points of the set with their associated multiplicities whose sum is m). The value of τ(a) is independent of a. Otherwise stated, the number of linearly independent rational functions having poles at most where f has poles, is the same as the number having poles at most where f has any given finite value.

Let $Q_1, \ldots, Q_m$ be the poles of f. We must show that the number of linearly independent differentials of first kind which vanish at least at $Q_1, \ldots, Q_m$ is the same as for $P_1, \ldots, P_m$.

Take a basis dv_j^*/dz ($j = 1, \ldots, \tau(\infty)$) for the linear space of differentials of first kind vanishing at least at the Q's. Now $(f-a)dv_j^*/dz$ is still a differential of first kind, say $d\tilde{v}_j/dz$. (The corresponding integral of first kind is given by

$$\tilde{v} = \int (f-a)\frac{dv^*}{dz}\,dz + \text{const.}$$

The integrand is analytic even where f has a pole, since dv^*/dz has a zero there of at least the same order.)

The $d\tilde{v}_j/dz$ are linearly independent, and they vanish at $P_1, \ldots, P_m$. Consequently $\tau(a) \geqq \tau(\infty)$. Similarly by taking a basis of $\tau(a)$ differentials vanishing at the P's and multiplying them by $1/(f-a)$, we would prove that $\tau(\infty) \geqq \tau(a)$. Hence $\tau(a) = \tau(\infty)$ as asserted.

Let a differential $dv/dz \neq 0$ of first kind be given, which we may call φ. Suppose that its zeros are at certain points $P_1, \ldots, P_t$, not necessarily distinct (so far as we know up to now, the number t of these points may be 0). Imagine these points separated in any way into two sets, say $P_1, \ldots, P_m$ and $P_{m+1}, \ldots, P_t$. Let T and τ be the numbers referred to in the Riemann-Roch theorem for the point set P_j ($j = 1, \ldots, m$), and let T', τ' be the corresponding numbers for the P_k ($k = m+1, \ldots, t$).

(One, or conceivably both, of the point sets may be vacuous. In that case the two numbers in question are 1 and p respectively.)

Take a basis $\varphi_1, \ldots, \varphi_\tau$ for the linear space of differentials of first kind vanishing at least at the P_j. Let

$$f = \frac{a_1\varphi_1 + \cdots + a_\tau\varphi_\tau}{\varphi}, \tag{61.1}$$

where the a's are complex numbers. Then f is a rational function (cf. (59.2)). Its poles are at most the points P_k; for although the denominator φ vanishes also at the P_j, the numerator vanishes at the P_j likewise.

Conversely, let f be a rational function with poles at most at the P_k. Then $f\varphi$ is a differential of first kind having zeros at the P_j. As $\varphi_1, \ldots, \varphi_\tau$ are a basis for such differentials, there must exist coefficients $a_1, \ldots, a_\tau$ for which (61.1) holds. Thus (61.1) is a formula, in terms of τ parameters $a_1, \ldots, a_\tau$, for the module of rational functions f which are infinite at most at the P_k. This proves that $T' = \tau$. Similarly, $T = \tau'$.

Applying (60.9), we have:

$$\tau' = \tau + m - p + 1,$$
$$\tau = \tau' + (t-m) - p + 1.$$

<u>These two formulas are the statement of the Brill-Noether reciprocity theorem</u> (or either formula alone would be enough).

Adding, we find that

$$t = 2p - 2, \tag{61.2}$$

<u>A differential of first kind on a surface of genus p has exactly 2p-2 zeros.</u>

Another proof of (61.2) is as follows. Let f be some rational function, of order n, having none but simple poles. (If f_1 is a rational function with simple a's, $1/(f_1-a)$ is an f of the desired type.)

The differential φ has no poles and t zeros. The differential df/dz has 2n poles, because a simple pole of f becomes a double pole of df/dz, and a point which is not a pole of f does not become a pole of df/dz. The number of zeros of df/dz is w, the sum of the orders of the critical points of f. For a zero of df/dz comes only from a critical point of f, the order of the zero being the same as that of the critical point; and every critical point of f does yield zeros of df/dz, since the poles of f are not critical points.

The quotient of differentials

$$\frac{\varphi}{df/dz} \equiv F$$

is a rational function. If no zero of φ coincides with a zero of df/dz, the number of poles of F is evidently $\mathcal{W}$, and the number of zeros is $t + 2n$. Since F, as a rational function, has the same number of zeros as it has poles,

$$t + 2n = \mathcal{W}.$$

This equation is still true even if some number of zeros of φ and df/dz coincide, for then that number is subtracted from both sides of the equation.

At the end of §53 we found that

$$2 - 2p = 2n - \mathcal{W}, \tag{61.3}$$

Eliminating n and $\mathcal{W}$ between this and the previous equation, we obtain (61.2).

Otherwise considered, we have here a new proof of (61.3). (It is true that f was required to have simple poles, but the result is easily seen to hold just as well for the arbitrary rational function f_1 from which f was derived. In fact, f_1 has the same n and $\mathcal{W}$ as f.)

On a surface of genus 0, the only differential of first kind is the one which is identically zero. On a surface of positive genus, we shall prove, it is not possible for all of the differentials of first kind to vanish simultaneously at any point.

For suppose that they all do vanish at some point P. Then in particular, all the differentials of the basis $dv_1/dz, \ldots, dv_p/dz$ vanish at P. Construct a normalized integral L of second kind having a simple pole at P with principal part $1/\zeta$ in some coordinate system ζ about P, and no other singularity. Exactly as in the previous section, (60.6) holds. Hence L has no jumps on the b_α lines, and since its a_α jumps are zero also, it is equivalent to a rational function. This function L is of order 1, for it has only one pole. As in §51 we conclude that L has no critical points and therefore the given surface is of genus zero, contrary to hypothesis. The last step might also be made to depend on (61.3), as $n = 1$ and $\mu = 0$ for the function L.

In the case of a surface of genus 0, (60.9) reduces to

$$T = m + 1.$$

Taking $m = 1$, we see that there exist non-constant rational functions of order 1 on the surface. By §51, this proves that any two of our surfaces which have genus zero are essentially identical with the complex sphere, and consequently with each other.

62. Abelian Integrals of First Kind

Let f be a rational function of order n on a surface S. As described in §51, S can be represented through the agency of f as an n-leaved Riemann surface over the f-sphere Σ.

Take two distinct points a and b on Σ and let $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ be the points of S that correspond to them. The P's and Q's will all be distinct unless a or b is a branch point on Σ (that is, unless a or b is below a branch point of the Riemann surface over Σ).

Join a to b by a curve C_{ab} of class D^1 on Σ, no one of whose points, except perhaps a or b, is a branch point. Then those points of S which correspond to the points of C_{ab} can be separated into n curves of class D^1 joining the points P_j $(j = 1, \ldots, n)$ to the points Q_j in some order, let us say P_1 to Q_1, P_2 to Q_2, ..., P_n to Q_n. These n curves on S, which may be called C_j, would be the projections of C_{ab} onto the Riemann surface.

Let L be an arbitrary integral of first kind on S. Consider the value of

(62.1) $$\sum_{j=1}^{n} \int_{C_j} dL.$$

This can be written as

(62.2)
$$\sum_{j=1}^{n} \int_{C_j} \frac{dL/dz}{df/dz}\, df.$$

The integrand, being a ratio of differentials, is a rational function h. Hence (62.2) becomes

$$\sum_{j=1}^{n} \int_{C_j} h\, df.$$

This equals

(62.3)
$$\int_{C_{ab}} \left(\sum_{j=1}^{n} h_j\right) df,$$

the integration being performed along C_{ab} on Σ, and the integrand being the sum of the values of h at the n points of S corresponding to a given point on C_{ab}.

By the lemma at the beginning of §52, the integrand of (62.3) is a rational function of f on the sphere Σ. This integrand cannot have a pole for any value of f; for if we chose b to be the pole and arranged C_{ab} so as to contain no other pole of the integrand, (62.3) would have to be infinite. As it equals (62.1), which is finite because L is an integral of first kind, this is not possible. Hence the rational function $\sum_{j=1}^{n} h_j$ is constant. The constant value must be zero, otherwise we could derive a contradiction by taking b to be the point $f = \infty$ on Σ and proceeding as before.

Our conclusion is that

(62.2)
$$\sum_{j=1}^{n} \int_{C_j} dL = 0$$

for the arbitrary integral L of first kind.

The curves C_j joined the P_j to the Q_j in a specific order, determined by C_{ab}. However, for any assigned order of the P_j and Q_j there exists a set of curves joining them in that order for which (62.2) holds for every L. For example, if it is required to have P_1 joined to Q_2 and P_2 to Q_1, with the other P's and Q's joined in the same order as before, we have only to draw a curve A of class D^1 from Q_1 to Q_2. Then C_1 followed by A is the new C_1, and C_2 followed by the reverse of A is the new C_2, the remaining C_j being the same as before. Obviously (62.2) is still true for the new set of curves C_j, since all that has been introduced is an integration back and forth along A, which cancels out. Of course the new C_j need not correspond to any single curve C_{ab} on the f-sphere.

The result so far is this: if f is a rational function of order n, and if $P_1, \ldots, P_n$ are the n points at which f has some value a, and $Q_1, \ldots, Q_n$ are the points at which f = b, then the P_j can be joined (in any order) to the Q_j by certain curves C_j such that formula (62.2) will hold true for every integral L of first kind.

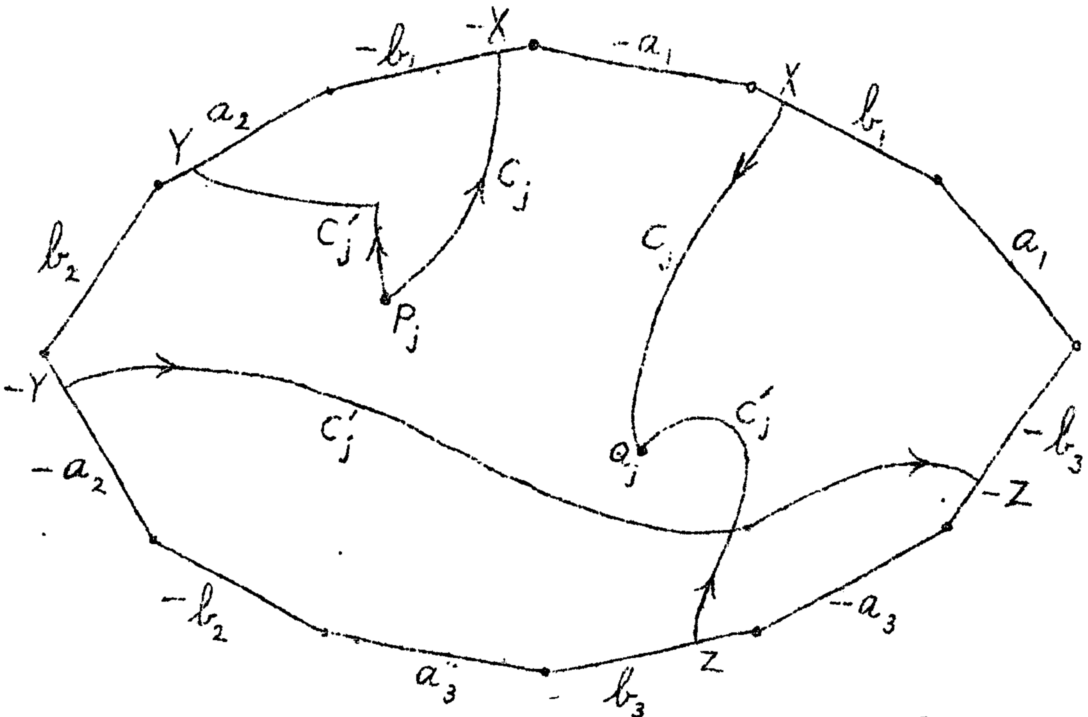
Now let $C_1', \ldots, C_n'$ be n arbitrary curves of class D^1 joining the P_j to the Q_j in any order; let us say again, in the order $P_1, Q_1; \ldots; P_n, Q_n$. Let $C_1, \ldots, C_n$ be curves joining the points in the same order and having the property (62.2).

Represent S by a polygon Γ. We shall make the assumption that all of the C_j and C_j' intersect Γ at most a finite number of times. Take an arbitrary integral L of first kind, and denote its a_ν jumps by σ_ν and its b_ν jumps by τ_ν $(\nu = 1, \ldots, p)$.

The curve C_j', which leads from P_j to Q_j, followed by the reverse of C_j, leading from Q_j back to P_j, is a single closed curve: $C_j' - C_j$. Integrate dL along this curve. In the case shown in the diagram the result would be

$$[L(Y) - L(-X)] + [L(-Z) - L(-Y)] + [L(X) - L(Z)],$$

where L refers now to the values of some



determination of the integral L in the polygon. This equals

$$[(L(X) - L(-X)] + [L(Y) - L(-Y)] - [L(Z) - L(-Z)],$$

or according to the definition of jumps in §55,

$$\tau_1 + \sigma_2 - \tau_3.$$

In the same way it is seen generally that

$$\int_{C_j'} dL - \int_{C_j} dL = \sum_{\nu=1}^{p} (m_{j\nu}\sigma_\nu + n_{j\nu}\tau_\nu) \tag{62.3}$$

($j = 1, \dots, n$), where the $m_{j\nu}$ and $n_{j\nu}$ are integers (positive, negative or zero) depending only on the various curves and not at all on the particular integral L. Of course the σ_ν and τ_ν do depend on L.

Summing (62.3) for the n values of j and taking account of (62.2) we find that

$$\sum_{j=1}^{n} \int_{C_j'} dL = \sum_{\nu=1}^{p} (m_\nu \sigma_\nu + n_\nu \tau_\nu), \tag{62.4}$$

where m_ν is written for $\sum_{j=1}^{n} m_{j\nu}$, and similarly for n_ν. The final result is this:

Theorem: Let f be a rational function of order n, and let $P_1,\dots,P_n$ be the n points at which f takes on some one value, and $Q_1, \dots, Q_n$ the points where f takes on another value. Let n curves $C_1', \dots, C_n'$ of class D^1 be drawn, joining the P_j to the Q_j in any order. Take an arbitrary integral L of first kind, and let σ_ν and τ_ν denote its a_ν and b_ν jumps respectively, across the sides of a polygon Γ. Then a formula of the type (62.4) will hold, in which the 2p integers m_ν and n_ν are independent of the particular integral L.

The m_ν and n_ν do depend on the curves C_j' and on the particular polygon Γ, and they need not be uniquely determined even when the C_j' and Γ are fixed.

The property stated in the theorem is characteristic for sets of (distinct) points P_j and Q_j at which a rational function has fixed values. For the following converse can be proved: if the P_j and Q_j are such that some set of curves C_j' can be drawn for which (62.4) holds for every L, the m_ν and n_ν being independent of L, then a rational function f of order n exists whose zeros and poles are the P_j and Q_j respectively.

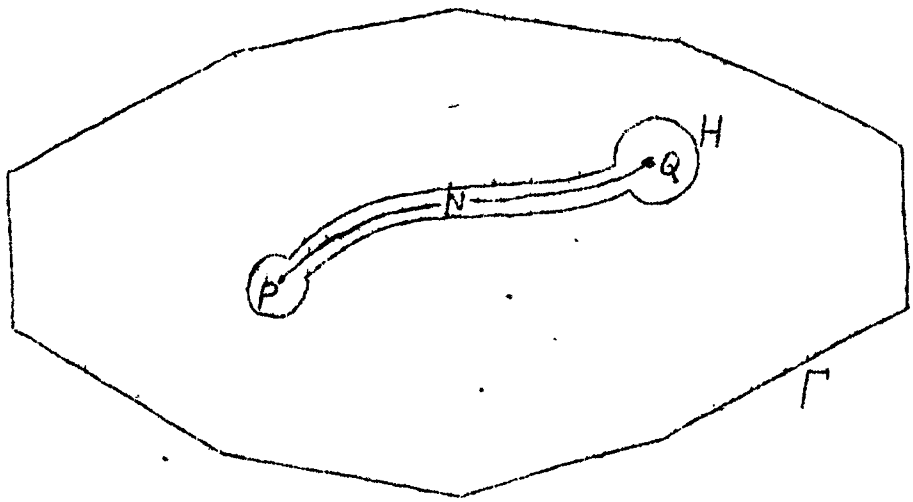
In $(bf+a)/(f+1)$ we would then have a rational function assuming the arbitrary distinct (finite) values a and b at the P_j and Q_j respectively. The condition that some set of C_j' shall satisfy (62.4) is not really weaker than the condition that all sets of curves C_j' shall do so. For it is easy to see by the method of proof of the previous theorem, notably the step from (62.3) to (62.4), that if the property holds for one set of curves it holds for all.

To prove the above converse we first derive a lemma on integrals of third kind. Let P and Q be two distinct points in the interior of a polygon Γ. Construct an integral of third kind (§57), say Π_{PQ}, which is regular except for logarithmic singularities at P and Q. Draw a simple arc N of class

D^1 from P to Q in the interior of Γ. Then Π_{PQ} has a determination as an analytic function in the interior of Γ, N excluded; and across N this determination has a jump of $2\pi i$. Normalize Π_{PQ} (§59) so that it has no a_ν jumps. Its jump across N is then still $2\pi i$.

Once more we make use of the special integrals of first kind, $v_1, \ldots, v_p$, of §58. The jump of v_α ($\alpha = 1, \ldots, p$) across a_ν was simply $\delta_{\alpha\nu}$, and the jump of v_α across b_ν was $\tau_{\alpha\nu}$, where $\tau_{\alpha\nu} = \tau_{\nu\alpha}$.

Draw a simple closed curve H about N and interior to Γ, as shown in the diagram:



Both Π_{PQ} and the v_α are analytic in the region between H and Γ and are continuous on H and on Γ. By using a network of triangles (each in a single complex coordinate system) covering the region in question, and then applying the Cauchy integral theorem, we find that

$$\int_\Gamma \Pi_{PQ} dv_\alpha = \int_H \Pi_{PQ} dv_\alpha \quad . \tag{62.5}$$

Now

$$\int_H \Pi_{PQ} dv_\alpha = +2\pi i \int_P^Q dv_\alpha \; . \tag{62.6}$$

For it may be shown that these two values differ by arbitrarily little. This is proved by shrinking H towards N. The integrations around P and Q yield arbitrarily small quantities, in spite of the logarithmic singularities of Π_{PQ} at P and Q. The integration along the remainder of H is arbitrarily close to the right member of (62.6), because Π_{PQ} has a jump of $2\pi i$ across N and v_α is continuous all along N. (It can be proved that the sign before the $2\pi i$ is correct, the integration about H being in the positive sense.)

Applying (55.9), we may rewrite (62.5) as

$$\sum_{\nu=1}^{p}\left(\int_{a_\nu} d\Pi_{PQ}\int_{b_\nu} dv_\alpha - \int_{a_\nu} dv_\alpha \int_{b_\nu} d\Pi_{PQ}\right) = 2\pi i\int_P^Q dv_\alpha .$$

Since Π_{PQ} is normalized and the jump of v_α across a_ν is $\delta_{\alpha\nu}$ (note (55.5)), this reduces to

$$\int_{a_\alpha} d\Pi_{PQ} = -2\pi i\int_P^Q dv_\alpha . \tag{62.7}$$

Thus we see that the b_α jump of the normalized integral of third kind Π_{PQ} is determined very simply by the difference in values of the special integral of first kind v_α at the two singular points of Π_{PQ}. The values of v_α are understood to mean the values of a determination of v_α in the given polygon. This is the lemma referred to above. It should be compared with the similar formula (60.6) for the b_α jump of a normalized elementary integral of second kind (note the extension of (60.6) indicated near the end of §60).

Returning to the converse theorem, we select a polygon Γ such that all of the P_j and Q_j are in its interior. We join P_j to Q_j by a simple arc N_j of class D^1 lying interior to Γ, and we select the N_j so as to be non-intersecting. As already remarked, (62.4) is obeyed with the N_j in place of the C_j' (perhaps with different integers m_ν and n_ν).

Construct the n normalized integrals of third kind $\Pi_{P_jQ_j}$. From the single Abelian integral

$$J \equiv \sum_{j=1}^{n}\Pi_{P_jQ_j} + 2\pi i\sum_{\nu=1}^{p} n_\nu v_\nu .$$

As the Π's are normalized and the jump of v_α across a_ν is $\delta_{\alpha\nu}$, we see that the jump of J across a_ν is $2\pi i n_\nu$.

As for the jump of J across b_ν:

$$\int_{a_\nu} dJ = \sum_{j=1}^{n} \int_{a_\nu} d\Pi_{P_j Q_j} + 2\pi i \sum_{\alpha=1}^{p} n_\alpha \int_{a_\nu} dv_\alpha .$$

Using (62.7) and the definition of $\tau_{\alpha\nu}$ as the jump of v_α across b_ν, we have

$$\int_{a_\nu} dJ = -2\pi i \sum_{j=1}^{n} \int_{N_j} dv_\nu + 2\pi i \sum_{\alpha=1}^{p} n_\alpha \tau_{\alpha\nu} .$$

Then by (62.4), v_ν being the integral L of first kind,

$$\int_{a_\nu} dJ = -2\pi i \sum_{\alpha=1}^{p} (m_\alpha \delta_{\nu\alpha} + n_\alpha \tau_{\nu\alpha}) + 2\pi i \sum_{\alpha=1}^{p} n_\alpha \tau_{\alpha\nu} .$$

But $\tau_{\nu\alpha} = \tau_{\alpha\nu}$; therefore

$$\int_{a_\nu} dJ = -2\pi i m_\nu .$$

Thus all the jumps of J across the sides of Γ are of the form $2\pi i$ times an integer, and the jump of J across each N_j is $2\pi i$. The only singularities of J are the P_j and Q_j. If z_j is a complex coordinate about P_j (where, say, $z_j = 0$), then in a neighborhood of P_j the difference between J and $\log z_j$ is analytic; and similarly if z_j' is a coordinate about Q_j we know that J equals $-\log z_j'$ plus an analytic function in some neighborhood of Q_j (see §57).

It follows that if we take a determination of J in the region between Γ and the N_j, and form the exponential $f = e^J$, the result will have no jumps at all and hence will be a one-valued function on the surface S. This function f is analytic and distinct from zero at all points other than the P_j and Q_j. At the P_j it vanishes and at the Q_j it has poles. Therefore f is a rational function of order n with the properties that were required.

The above "transcendental" treatment of the theory of analytic functions may be compared with the "algebro-geometric" treatment, which can be found in the notes on S. Lefschetz's lectures on Algebraic Geometry written by M. Richardson and E. D. Tagg, Princeton Mathematical Notes, 1936-1937).