C A L C U L U S   O F   V A R I A T I O N S

by

WALTHER MAYER

(Notes by R. E. Traber)

The Institute for Advanced Study

1935-36

TABLE OF CONTENTS

Page

Chapter I. Introduction -- 1

§1. Definition of the Space -- 1
§2. The Fundamental Metric Function and the Definition of Distance ----- 3
§3. Hilbert Curves and "Generalized" Arc Length ---------------------- 9

Chapter II. Local Properties of a Minimizing Arc ------------------------ 22

§4. First Variation and the Euler Vector ---------------------------- 29
§5. A Second Method of Treating the Euler Vector --------------------- 29
§6. Remarks on the Euler Vector ------------------------------------- 33
§7. The Normal Form of Euler's Equation, and its Solutions ------------ 35
§8. Existence of Normal Coordinates ---------------------------------- 42
§9. Properties of Normal Coordinate Neighborhoods -------------------- 46
§10. Largest Spherical Normal Coordinate Neighborhoods ----------------- 50
§11. An Additional Assumption -- 56
§12. Characterization of Normal Coordinates ---------------------------- 59
§13. The Weierstrass $\mathcal{E}$ -Function ---------------------------------- 62
§14. Minimizing Arcs in Homotopic Classes of Curves ------------------- 65

Chapter III. The Notion of the Field and the Weierstrass $\mathcal{E}$ -Function ------ 71

§15. Essentially Positive Weierstrass $\mathcal{E}$ -Functions ------------------ 71
§16. The Notion of the Field -- 73
§17. Fields of Extremals Obtained from the Jacobi-Hamilton Normal Form - 82

Chapter IV. Minimum Points and Conjugate Points ------------------------ 85

§18. Definition of Absolute Minimum Point and Conjugate Point, and Some Properties of the Absolute Minimum Point ------------------------ 85
§19. Definition of Relative Minimum Point and First Double Point ------- 90
§20. Theorem on Relative Minimum Points ------------------------------- 92
§21. The Second Variation --- 95
§22. Formulas for the Second Variation -------------------------------- 97
§23. Characterization of Conjugate Points through Solutions of the Jacobi Differential Equation ------------------------------------- 101
§24. Completion of the Proof of the Theorem of §20 --------------------- 106
§25. Further Properties of the Conjugate Points and the Absolute Minimum Points --- 109
§26. The Locus of the Absolute Minimum Points and its Properties ------- 114

*

CALCULUS OF VARIATIONS

by

WALTHER MAYER

Chapter I. Introduction

1. Consider as the fundamental space of this course a point set which is a Hausdorff space, that is, which satisfies the following four Hausdorff axioms:

(1.1) There exists a set of points and a set of neighborhoods such that to every point p there corresponds one or more neighborhoods $\mathfrak{U}(p)$, each $\mathfrak{U}(p)$ containing p.

(1.2) If $\mathfrak{U}(p)$ and $\mathfrak{U}'(p)$ are any two neighborhoods of p, there exists a third neighborhood $\mathfrak{U}''(p)$ of p such that $\mathfrak{U}''(p)$ is contained in the intersection of $\mathfrak{U}(p)$ and $\mathfrak{U}'(p)$.

(1.3) If a point q is contained in a neighborhood of p, $\mathfrak{U}(p)$, then there exists a neighborhood of q, $\mathfrak{U}(q)$, such that $\mathfrak{U}(q)$ is contained in $\mathfrak{U}(p)$.

(1.4) If p is different from q, there exist neighborhoods $\mathfrak{U}(p)$ and $\mathfrak{U}(q)$ of p and q respectively without common points.

Assume that to each point p of this space there exists a particular neighborhood and a particular n-dimensional sphere such that a definite homeomorphism exists between them. To each point of this neighborhood assign the coordinates of the corresponding point of the n-sphere determined by the given homeomorphism. Such a neighborhood $\mathfrak{U}(p)$ will be called a coordinate neighborhood. A further assumption will be that the space is such that the transformation from the coordinate system of one coordinate neighborhood to that of an overlapping one will be regular. By a regular transformation is meant a continuous transformation with continuous derivatives up to and including those of the m^{th} order, $m \geq 1$, where m is sufficiently large to satisfy all of the proofs. For the present m will be taken equal to 1, and only first derivatives will be assumed. Since the first derivatives exist,

$$(1.5) \qquad \frac{\partial x'^i}{\partial x^j}\frac{\partial x^j}{\partial x'^\ell} = \delta^i_\ell \qquad \text{and}$$

$$(1.5') \qquad \left|\frac{\partial x'^i}{\partial x^j}\right| \cdot \left|\frac{\partial x^j}{\partial x'^\ell}\right| = 1$$

and hence the Jacobian will be different from zero. Such a space as has been described will be known throughout these notes as a manifold.

By a connected manifold is meant a manifold which cannot be separated into two mutually exclusive manifolds. Assume the space to be a connected manifold. Then any two points of the space can be joined by a continuous curve with continuous first derivatives at all but a finite number of points,* which will be called an admissible curve. For take a point P in a connected manifold $\mathfrak{M}$ and consider the set S of all points Q that can be joined to P by an admissible curve. S is open, since any Q that can be joined to P by an admissible curve lies in a coordinate neighborhood whose points can be thus joined to Q and hence, since the sum of two admissible curves with a common end point is an admissible curve, can be joined to P. S is closed relative to $\mathfrak{M}$ for if R is a limit point in $\mathfrak{M}$ of a sequence of points Q_i that can be joined to P, then there are an infinite number of Q_i in a coordinate neighborhood of R, and R can be joined to any one of them and hence to P. The set E of all points that cannot be joined to P is also both open and relatively closed since it is the complement of S. All points of either S or E are in coordinate neighborhoods. Hence both S and E are manifolds. But S is non-vacuous since P is in

* The right hand derivative must exist and be finite at all points but the right end point; the left hand one at all but the left end point.

it. Therefore E is vacuous since $\mathfrak{M}$ is connected. Hence, by this argument it is shown that any two points of a connected manifold can be joined by an admissible curve. If the regular transformations between coordinate neighborhoods have continuous derivatives of order m > 1, then the curve can be made to have continuous derivatives of order m at all but a finite number of points.

2. Let there be given a function $F(x_1, \ldots, x_n, \lambda^1, \ldots, \lambda^n) \equiv F(x, \lambda)$ of a contravariant vector $(x_1, \ldots, x_n, \lambda^1, \ldots, \lambda^n)$ which satisfies the following conditions:

(2.1) $F(x, c\lambda) = cF(x, \lambda)$, $c > 0$, i.e. $F(x, \lambda)$ is positive homogeneous of the first degree in λ;

(2.2) $F(x, \lambda) \geqq 0$, $= 0$ if and only if $\lambda = 0$, i.e. $F(x, \lambda)$ is positive definite in λ;

(2.3) $F(x, \lambda)$ is continuous in the 2n variables x, λ for x in a coordinate neighborhood and for any λ.

Let x(t), a monotonic increasing function of t, be a continuous mapping of a straight line segment on to an admissible curve joining two given points P and Q such that $x(t_o) = P$, $x(t_1) = Q$. At each point of the curve, except perhaps at a finite number, there is given the set of values $x_1, \ldots, x_n$, $\frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt}$. Hence for each admissible curve C_P^Q there exists an $\mathcal{J}(C_P^Q) = \int_{C_P^Q} F(x, \frac{dx}{dt})\, dt$ attaching a real finite non-negative number, to each admissible curve connecting P to Q. $\mathcal{J}(C_P^Q)$, for a definite oriented curve C_P^Q, is independent of the parameterization. For set $t = t(s)$. Then if the parameter s is to give the same orientation

$$\therefore \quad \frac{ds}{dt} > 0.$$

$$\int_{C_P^Q} F\left(x, \frac{dx}{dt}\right) dt = \int_{C_P^Q} F\left(x, \frac{dx}{ds}\cdot\frac{ds}{dt}\right) dt$$

$$= \int_{C_P^Q} F\left(x, \frac{dx}{ds}\right) ds \qquad \text{by } (2.1)$$

Call $\mathcal{J}(C_P^Q)$ the arclength of the curve C_P^Q The question naturally arises whether there is a curve connecting P to Q whose arclength is either greatest or least. There is obviously none whose arclength is greatest, for the arclength of any curve can be surpassed by adding a closed circuit to the curve.

Since for admissible curves connecting P to Q the arclengths are finite and since all arclengths of curves are non-negative, there exists a finite positive greatest lower bound to the arclengths of the curves joining P to Q. Call this greatest lower bound the distance from P to Q, written $\overline{PQ}$. Due to the fact that only positive homogeneity is assumed, $\overline{PQ} \neq \overline{QP}$ in general. In fact there is no simple relation between them. However, the other two distance properties hold.

(2.4) $\overline{PQ} \geqq 0, = 0$ if and only if $P = Q$

(2.5) $\overline{PQ} + \overline{QR} \geqq \overline{PR}$

The proofs will follow. First a lemma is necessary.

Lemma. Given a closed, compact point set $\mathcal{M}$ lying entirely in a definite coördinate neighborhood $\mathcal{U}$, and an $x \in \mathcal{M}$. Then

2.6 $\quad A \max \mu \geqq F(x, \mu) \geqq B \max \mu$

where $\infty > A \geqq B > 0$ and $\max \mu$ = greatest of $|\mu^1|, \cdots, |\mu^n|$.

Proof. Consider $F(x, \lambda)$ where $x \in \mathcal{M}$ and $\max \lambda = 1$. Then $F(x, \lambda)$ is a continuous function of a compact closed point set and hence assumes its least upper bound A and its greatest lower bound B. That is

$$A \geqq F(x, \lambda) \geqq B.$$

Since $F(x, \lambda)$ assumes its greatest lower bound B and since $\lambda \neq 0$ for max $\lambda = 1$, we have by (2.2) that $B > 0$. Hence

$$Ac \geqq F(x, c\lambda) \geqq cB > 0; \quad c > 0.$$

Set $c\lambda = \mu$. Max $\mu = c$.

$$\therefore \quad A \max \mu \geqq F(x, \mu) \geqq B \max \mu \; ; \quad B > 0.$$

The proof of (2.4), the first axiom of distance, follows. Since $F(x, \lambda)$ is positive definite, (2.2), $\int_{C_P^Q} F(x, \frac{dx}{dt})dt \geqq 0$ and their greatest lower bound for all C_P^Q, $\overline{PQ} \geqq 0$. Proof that $\overline{PP} = 0$ will follow later. In order to show that $\overline{PQ} \neq 0$ if $P \neq Q$, take a $\mathcal{U}_1(P)$ excluding Q and a $\mathcal{U}(P)$ such that the closure of $\mathcal{U}(P)$, $\overline{\mathcal{U}}(P)$, is compact and contained in $\mathcal{U}_1(P)$. By the lemma, if $x \in \overline{\mathcal{U}}$, $F(x, \lambda) \geqq B \max \lambda$, $B > 0$. Let $\Delta_i R = x_i(R) - x_i(P)$, R being a point on the boundary of $\overline{\mathcal{U}}(P)$, and let σ be the greatest lower bound of max $\Delta_i(R)$. Then

$$\max \Delta_i(R) \geqq \sigma > 0.$$

$\sigma > 0$ since P is an interior point of $\overline{\mathcal{U}}(P)$. Connect P to Q with an admissible curve. There will be a first point R at which the curve will intersect the boundary of $\overline{\mathcal{U}}(P)$. Then

$$\begin{aligned}
\int_P^Q F(x, \tfrac{dx}{dt})dt &\geqq \int_P^R F(x, \tfrac{dx}{dt})dt \\
&\geqq B\int_P^R (\max \tfrac{dx}{dt})dt \\
&\geqq B\int_P^R \left|\tfrac{dx^i}{dt}\right| dt \quad \text{for all } i, \\
&\geqq B\left|\int_P^R \tfrac{dx^i}{dt}\, dt\right| \quad \text{for all } i, \\
&\geqq B\,|x_i(R) - x_i(P)| \quad \text{for all } i, \\
&\geqq B \max \Delta_i R \geqq B\sigma > 0.
\end{aligned}$$

$$\therefore \quad \overline{PQ} \geqq B\sigma > 0, \text{ and}$$
$$\overline{PQ} \neq 0 \text{ if } P \neq Q.$$

Note that this same construction shows that all points Q' such that $\overline{PQ'} < B\sigma$ are inside of $\mathcal{U}(P)$, i.e. for every neighborhood $\mathcal{U}(P)$ there exists a $\rho > 0$ such that all points Q' whose distance $\overline{PQ'} < \rho$ form a set $\mathcal{U}_\rho(P)$ lying in $\mathcal{U}(P)$. A similar proof would show that for every neighborhood $\mathcal{U}(P)$ there exists a $\rho > 0$ such that all points Q' whose distance $\overline{Q'P} < \rho$ form a set $\widetilde{\mathcal{U}}_\rho(P)$ lying in $\mathcal{U}(P)$. (See footnote page 7).

The proof of (2.5), the triangle axiom of distance, now follows. Consider any three points P, Q and R. Suppose that the triangle axiom were not true and $\overline{PR} = \overline{PQ} + \overline{QR} + \delta$, $\delta > 0$. There exist curves C_P^Q and C_Q^R such that

$$\mathcal{J}(C_P^Q) < \overline{PQ} + \delta/2,$$

$$\mathcal{J}(C_Q^R) < \overline{QR} + \delta/2.$$

Take as a curve connecting P to R the curve $C_P^R = C_P^Q + C_Q^R$. Then by definition

$$\overline{PR} \leqq \mathcal{J}(C_P^R) = \mathcal{J}(C_P^Q) + \mathcal{J}(C_Q^R) < \overline{PQ} + \overline{QR} + \delta.$$

$$\therefore \quad \overline{PR} \leqq \overline{PQ} + \overline{QR}.$$

A proof of simultaneous continuity of the distance $\overline{PQ}$ with regard to both P and Q will now be given. Consider any two points P and Q and any neighborhoods $\mathcal{U}(P)$ and $\mathcal{U}(Q)$. For any $P_1 \in \mathcal{U}(P)$ and $Q_1 \in \mathcal{U}(Q)$

$$\overline{P_1Q_1} \leqq \overline{P_1P} + \overline{PQ_1}, \text{ and hence}$$

$$\overline{P_1Q_1} - \overline{PQ_1} \leqq \overline{P_1P}.$$

Interchanging the roles of P and P_1,

$$\overline{PQ_1} - \overline{P_1Q_1} \leqq \overline{PP_1}, \text{ and}$$

$$|\overline{P_1Q_1} - \overline{PQ_1}| \leqq \widetilde{PP_1}, \text{ the greater of } \overline{PP_1} \text{ and } \overline{P_1P}.$$

Similarly

$$\overline{PQ_1} \leqq \overline{PQ} + \overline{QQ_1},$$

$$\overline{PQ_1} - \overline{PQ} \leqq \overline{QQ_1},$$

$$\overline{PQ} - \overline{PQ_1} \leqq \overline{Q_1Q},$$

$$|\overline{PQ} - \overline{PQ_1}| \leqq \widetilde{QQ_1}.$$

$$(2.7) \quad \therefore \quad |\overline{P_1Q_1} - \overline{PQ}| \leqq \widetilde{PP_1} + \widetilde{QQ_1}.$$

Now use the lemma as follows. Take a coordinate neighborhood of P, $\mathcal{U}(P)$, and in this coordinate system take a bounded closed sphere S with P as center and let $P_1 \in S$. Let $x_i(P)$ and $x_i(P_1)$ be the coordinates of P and P_1. Set

$$x_i(t) = x_i(P) + t\left[x_i(P) - x_i(P_1)\right]$$

which will be called a straight line joining P to P_1. Write

$$x_i(P) - x_i(P_1) = \Delta_i.$$

For $t = 0$, $x_i(t) = P$; for $t = 1$, $x_i(t) = P_1$. Since S is a closed compact set the lemma applies and

$$F(x, \lambda) \leqq A \max \lambda \qquad \text{when } x \in S.$$

Now

$$\begin{aligned} \overline{P_1P} &\leqq \int_0^1 F\left(x, \frac{dx}{dt}\right) dt, \\ &\leqq A \int_0^1 (\max \Delta_i)\, dt \quad \text{since } \frac{dx'}{dt} = \Delta_i, \\ &\leqq A \max \Delta_i . \end{aligned}$$

Hence for any $\epsilon > 0$ there is an S such that for all $P_1 \in S$, $\overline{PP_1} < \epsilon$. Similarly $\overline{P_1P}$,* $\overline{QQ_1}$ and $\overline{Q_1Q}$ * can be made small at pleasure. Therefore $\widetilde{PP_1}$ and

---

* Consider any curve C_P^Q joining P to Q and parameterized by $p(t)$ where $p(t_0) = P$, $p(t_1) = Q$. Now when traversing the curve in the opposite sense it is parameterized by $p(-t)$, t running from $-t_2$ to $-t_1$, and the new parameterization is given by $\tau = -t$ so that $\frac{d\tau}{dt} = -1$. Then

$$\int_{-t_2}^{-t_1} F\left(x(-t), \frac{dx(-t)}{dt}\right) dt = -\int_{\tau_2}^{\tau_1} F\left(x(\tau), -\frac{dx(\tau)}{d\tau}\right) d\tau = \int_{\tau_1}^{\tau_2} F\left(x, -\frac{dx}{d\tau}\right) d\tau .$$

Hence arclength, with a metric $F(x, \lambda)$, for the curve C_P^Q is equal to arc-length, with $F(\chi, -\lambda)$ as metric, for the curve C_Q^P. Similarly for distance, $\overline{PQ}$ with a metric $F(\chi, \lambda)$ is equal to $\overline{QP}$ with a metric $F(\chi, -\lambda)$ and any proof of properties of a distance $\overline{PQ}$ includes the proof of the corresponding properties of the distance $\overline{QP}$ since $F(\chi, -\lambda)$ possesses the same three properties (2.1), (2.2), (2.3), as $F(\chi, \lambda)$. However, note that since the metric function is fixed for a given problem this does not imply that the distances $\overline{PQ}$ and $\overline{QP}$ are equal.

$\widetilde{QQ}$ can be made small at pleasure. Then recalling (2.7), for every $\epsilon > 0$ there is an $S(P)$ and an $S(Q)$ such that for all $P_1 \epsilon\ S(P)$ and all $Q_1 \epsilon\ S(Q)$

$$|\overline{P_1 Q_1} - \overline{PQ}| < \epsilon$$

Hence the distance $\overline{PQ}$ is simultaneously continuous with regard to both P and Q. There is no knowledge concerning the existence of derivatives of $\overline{PQ}$ with regard to P or Q under the present assumptions.

Using this theorem the remaining part of the first axiom of distance (2.4), that $\overline{PP} = 0$ can be proved. For

$$\overline{PP} \leqq \overline{PQ} + \overline{QP}$$

But $\overline{PQ}$ and $\overline{QP}$ approach zero as Q approaches P, while $\overline{PP}$ is constant.

$$\therefore \quad \overline{PP} = 0.$$

Previously we defined the set $\mathcal{U}_\rho(P)$. It can be easily shown that $\mathcal{U}_\rho(P)$ is an open set, by showing that $C(\mathcal{U}_\rho(P))$, its complement, is closed. Its complement is by definition the set of all points Q such that $\overline{PQ} \geqq \rho$. Now consider any sequence of points Q_i having a limit point Q and such that $Q_i \epsilon\ C(\mathcal{U}_\rho(P))$. Now $\overline{PQ_i}$ is a continuous function of Q_i; hence

$$\lim_{i \to \infty} \overline{PQ_i} = \overline{PQ}$$

But $\overline{PQ_i} \geqq \rho$ Therefore $\overline{PQ} \geqq \rho$, and $Q \in C(\mathcal{U}_\rho(P))$. Therefore $C(\mathcal{U}_\rho(P))$ is closed and $\mathcal{U}_\rho(P)$ is open. We shall call $\mathcal{U}_\rho(P)$ a metric neighborhood or a ρ-neighborhood. It has been shown that for any neighborhood $\mathcal{U}(P)$ there exists a ρ such that $\mathcal{U}_\rho(P)$ is contained in $\mathcal{U}(P)$. Conversely, for any $\mathcal{U}_\rho(P)$ there exists a $\mathcal{U}(P)$ contained in it, namely itself. Therefore the two systems of neighborhoods are equivalent and the system $\mathcal{U}_\rho(P)$ of metric neighborhoods can be used without changing the topology of the manifold.

The two main problems can be stated as follows:

I. Given two points P and Q. Is there an admissible curve connecting P to Q whose arclength is equal to $\overline{PQ}$? This is the problem of the absolute minimum.

II. Take any connected open subset of the manifold. Is there an admissible curve lying entirely in the subset connecting two given points of the subset, P and Q, whose arclength is the greatest lower bound of the arclength of all such curves? This is the problem of the relative minimum.

The absolute problem for a subspace is the relative problem for the space. It might also be noted that the solution of both problems depends upon the order of traversing the curve connecting P and Q.

3. Another assumption will be placed on the space.

(3.1) Any infinite set of points $\{Q\}$ such that there exists a point P so that $\overline{PQ}$ are bounded will have an accumulation point in the space. A space having this property will be called almost compact.

The so-called Hilbert curve will now be constructed. Roughly, the procedure will be to show the existence of a mid-point between two given points P and Q, and in general 2^p-1 points dividing $\overline{PQ}$ into 2^p equal parts for all p.

The limit points will then be added to form a continuous curve. First the following theorem will be needed.

THEOREM. For any two points P and Q in the manifold there exists a point R in the manifold such that

$$\overline{PR} = \overline{RQ} = \tfrac{1}{2}\overline{PQ}.$$

Proof. Set $\overline{PQ} = \rho$ Construct a point set $\mathcal{P}_{\rho/2}(P)$ of all points R such that $\overline{PR} = \rho/2$. $\mathcal{P}_{\rho/2}(P)$ is not vacuous, for connect P to Q with any continuous curve $p(t)$. $\overline{Pp(t)}$ is a continuous function of $p(t)$ which is a continuous function of t. Hence $\overline{Pp(t)}$ is a continuous function of t assuming the values 0 and ρ. Therefore $\overline{Pp(t)}$ assumes for some point R_0 the value $\rho/2$ since a continuous function defined over a closed compact set assumes all values between any two particular values that it assumes. Hence there is a point R_0 in $\mathcal{P}_{\rho/2}(P)$. $\mathcal{P}_{\rho/2}(P)$ is closed, as follows. Given any sequence of points S_n such that $\overline{PS_n} = \rho/2$ and approaching a limit S. Since $\overline{PS_n}$ is a continuous function of S_n,

$$\lim_{n\to\infty} \overline{PS_n} = \overline{PS}.$$

$$\therefore \quad \overline{PS} = \rho/2$$

and S is a point of $\mathcal{P}_{\rho/2}(P)$. Notice that the assumption (3.1) is necessary in this proof to show compactness. Accordingly $\mathcal{P}_{\rho/2}(P)$ is a non-vacuous, closed, compact point set. Take any point R' on $\mathcal{P}_{\rho/2}(P)$ and consider $\overline{R'Q}$. $\overline{R'Q}$ is a continuous function of R' defined over a closed, compact point set, and hence assumes its minimum σ for some point R. Now

$$\overline{PR} + \overline{RQ} \geqq \overline{PQ}.$$

Substituting,

$$\rho/2 + \sigma \geqq \rho$$

$$\therefore \quad \sigma \geqq \rho/2$$

On the other hand, take any admissible curve joining P to Q and form

$$\int_P^Q = \int_P^{P'} + \int_{P'}^Q$$

where P' is a point on the curve such that $\overline{PP'} = \rho/2$. The existence of such a point on any continuous curve through P has already been proved. .. Then

$$\int_P^Q \geqq \overline{PP'} + \overline{P'Q},$$
$$\geqq \rho/2 + \sigma, \text{ since } P' \in \gamma^{\circ}_{\rho/2}(P).$$
$$\therefore \quad \overline{PQ} = \rho \geqq \rho/2 + \sigma \text{ and}$$
$$\sigma \leqq \rho/2.$$
$$\therefore \quad \sigma = \rho/2 \text{ and the proof of the theorem is complete.}$$

Note that the definition of distance in terms of the greatest lower bound of an integral was used. This or some equivalent assumption is necessary for the proof of the theorem. The existence only, not the uniqueness, of the point R has been shown. In fact there might be more than one, or even infinitely many. Hence the construction does not give a unique Hilbert curve.

Consider the Euclidean line segment (0, 1) and let 0 correspond to P and 1 to Q, and $\frac{1}{2}$ to $P(\frac{1}{2})$, a midpoint of P and Q. The midpoint of P and $P(\frac{1}{2})$ and that of $P(\frac{1}{2})$ and Q are constructed, and are made to correspond to 1/4 and 3/4 respectively. This process is continued indefinitely. In the p^{th} step there are 2^p+1 points $P(\frac{n}{2^p})$, $0 \leqq n \leqq 2^p$, $P(0) = P$ and $P(1) = Q$, such that the distance between any two consecutive points is $\frac{1}{2^p}\cdot\rho$. To $P(\frac{n}{2^p})$ let correspond $\frac{n}{2^p}$ in the line segment (0, 1). In the p^{th} step, let us temporarily denote $P(\frac{n}{2^p})$ by P_n. Then

$$\overline{P_iP_k} = \frac{k-i}{2^p}\cdot\rho, \quad k \geqq i.$$

For
$$\overline{P_iP_k} \leqq \overline{P_iP_{i+1}} + \overline{P_{i+1}P_{i+2}} + \dots + \overline{P_{k-1}P_k}$$
$$\leqq \rho/2 \quad (k - i)$$

Now
$$\overline{PQ} \leqq \overline{P_oP_i} + \overline{P_iP_k} + \overline{P_kP_{2^p}}$$
$$\leqq \rho\left[\frac{i}{2^p} + \frac{k-i}{2^p} + \frac{2^p-k}{2^p}\right] = \rho.$$

But now $\overline{PQ} = \rho$, so $\overline{P_iP_k}$ must equal $\frac{k-i}{2^p}\rho$, since otherwise the contradiction $\rho < \rho$ arises. That is, for any two points α and β of the set of points of the space obtained by this process of taking midpoints

$$(3.2)\qquad \overline{P(\alpha)P(\beta)} = (\beta-\alpha)\rho, \quad \beta \geq \alpha,$$

since α and β are both points of some p^{th} step.

A mapping function p(t) defined over the set $t = \frac{n}{2^p}$ now exists. On the segment (0, 1) take a sequence $\alpha_n' \to a$ and any other sequence $\beta_n' \to a$, where α_n' and β_n' are of the form $\frac{n}{2^p}$, but a is any real number. Under the assumption (3.1) the sequence $P(\alpha_n')$ have an accumulation point P'. Pick then from α_n' a subsequence α_n such that $P(\alpha_n)$ have the limit point P'. Also pick a subsequence β_n of β_n' such that $P(\beta_n)$ have a limit point P". Now $\beta_n - \alpha_n$ is either greater than, equal to, or less than zero, and, to be sure, it falls into at least one of these three categories for an infinity of n; say for instance that $\beta_n - \alpha_n > 0$ for an infinity of n. Then for this subsequence

$$\overline{P(\alpha_n)P(\beta_n)} = (\beta_n - \alpha_n)\rho.$$

Now

$\lim_{n\to\infty} P(\alpha_n) = P'$ and $\lim_{n\to\infty} P(\beta_n) = P''$ for the subsequence $\beta_n - \alpha_n > 0$ since if a sequence converges to a limit any subsequence converges to the same limit. Therefore

$$\lim_{n\to\infty} \overline{P(\alpha_n)P(\beta_n)} = \overline{P'P''} = (a-a)\rho = 0.$$

By (2.4), $\overline{P'P''} = 0$ implies P' = P". If the original set $P(\alpha_n')$ had more than one accumulation point, a subsequence of $P(\alpha_n')$ could be picked converging to each, so that by the above all accumulation points of $P(\alpha_n')$ coincide. In general then, given any two sequences α_n and β_n converging to a, α_n and β_n being of the form $\frac{m}{2^p}$ while a is any real number, then

$$\lim_{n\to\infty} P(\alpha_n) = \lim_{n\to\infty} P(\beta_n).$$

The definition of P(t) will be enlarged as follows. Take any sequence $\alpha_n \to t$, α_n of the form $\frac{m}{2^p}$, and set $P(t) = \lim_{n\to\infty} P(\alpha_n)$. If t is a number of the form $\frac{m}{2^p}$ this does not conflict with the previous definition of P(t) as can be seen by taking the sequence $\alpha_n = t$. For any t_1 and t_2 now,

$t_2 \geqq t_1$,

$$(3.2') \qquad \overline{P(t_1)P(t_2)} = (t_2 - t_1)\rho$$

as is seen by taking the limit of $\overline{P(t_{n_1})P(t_{n_2})} = (t_{n_1} - t_{n_2})\rho$ for two sequences of points t_{n_1} and t_{n_2} approaching t_1 and t_2 respectively.

To show that the curve $P(t)$ is continuous, consider any t and any sequence t_n approaching it. Now

$$\overline{P(t)P(t_n)} = (t_n - t)\rho \qquad \text{for } t_n > t,$$
$$\overline{P(t)P(t_n)} = 0 \qquad \text{for } t_n = t, \text{ and}$$
$$\overline{P(t_n)P(t)} = (t - t_n)\rho \qquad \text{for } t_n < t.$$

Each of these three sequences of distances, if it exists, has the limit zero. Hence the sequence consisting of the totality of them has the limit zero, and $P(t)$ is continuous.

$P(t_1) = P(t_2)$ if and only if $t_1 = t_2$, and the mapping $P = P(t)$ is one to one. Hence since $P(t)$ is continuous over a compact set, the mapping is bicontinuous. This curve, the so-called Hilbert curve, is a simple arc joining P to Q. It has the property that given any three points on it, A, B, and C, in the order P, A, B, C, Q, then

$$(3.3) \qquad \overline{AC} = \overline{AB} + \overline{BC}$$

which follows immediately out of the distance formula. This Hilbert curve, however, does not necessarily under the present assumptions give the distance $\overline{PQ}$ if used as the path of integration for two reasons: first, the curve may not possess a derivative; and second, if the first derivatives exist the integral may not minimize the arclength.

There are other methods by which a Hilbert curve can be constructed. First, from an infinite sequence of curves C_n such that $\lim_{n \to \infty} \int_{C_n} = \overline{PQ}$ a subsequence C'_n can be picked converging to a limit curve, which is a Hilbert curve*

* Bolza, O. Vorlesungen über Variationsrechnung, B.G.Teubner, Leipzig, 1909, Chap. IX.

A second method is similar to the original one. Let us call R a between point of P and Q if $\overline{PR} + \overline{RQ} = \overline{PQ}$. It can be shown that there exists a between point R such that $\overline{PR} = \alpha\rho$, $0 \leqq \alpha \leqq 1$, $\rho = \overline{PQ}$. Then the original method could be carried through picking other between points than the mid-point and getting as before an everywhere dense point set. Neither method will be considered in detail.

The characteristic property of a Hilbert curve is the distance formula

$$\overline{AB} + \overline{BC} = \overline{AC} \tag{3.3}$$

for all A, B, C on the curve in the order of orientation. For if for any continuous curve $P(t)$, $0 \leqq t \leqq 1$, we have

$$\overline{P(t_1)P(t_2)} + \overline{P(t_2)P(t_3)} = \overline{P(t_1)P(t_3)}$$

for any three $0 \leqq t_1 \leqq t_2 \leqq t_3 \leqq 1$, then the curve $P(t)$ is a Hilbert curve and can be constructed by the mid-point method. Let $P(0) = P$, and set $s = \overline{PP(t)}$. s is a continuous increasing function of t running from 0 to $\rho = \overline{PP(t)}$, and therefore there must be a $\bar{t}$ such that $\overline{PP(\bar{t})} = \rho/2$. Then

$$\overline{PP(\bar{t})} + \overline{P(\bar{t})P(1)} = \overline{PP(1)},$$

$$\rho/2 + \overline{P(\bar{t})P(1)} = \rho \text{, and}$$

$$\overline{P(\bar{t})P(1)} = \rho/2.$$

Thus $P(\bar{t})$ is a mid-point on the curve. The process obviously can be continued indefinitely and a Hilbert curve constructed. Since $P(s) = P(t(s))$ and the Hilbert curve just constructed coincide on a set of points whose parameters s are of the form $\frac{m}{2^p}$ and hence are everywhere dense, and since both curves are continuous it is seen that P(t) and the Hilbert curve are identically equal.

A minimizing arc M_P^Q, i.e. an arc such that $\int_{M_P^Q} = \overline{PQ}$, obviously has the property that every sub-arc of M_P^Q is itself a minimizing arc between its end points. On a minimizing arc, take any three points A, B, C in the order of orientation. Then

$$\overline{AB} = \int_A^B, \ \overline{BC} = \int_B^C, \ \overline{AC} = \int_A^C, \text{ and thus}$$

$$\overline{AB} + \overline{BC} = \overline{AC}. \tag{3.3}$$

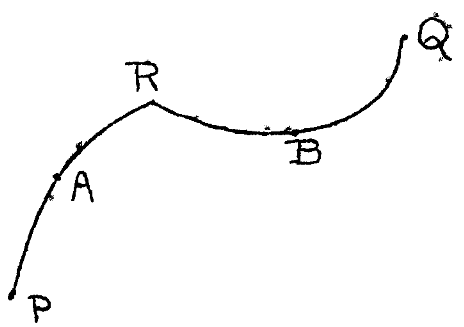
This is the characteristic property of a Hilbert curve and we see that any minimizing arc is a Hilbert curve. It will be shown later under <u>more</u> assumptions that any Hilbert curve is a minimizing arc. However this is not true under the <u>present</u> assumptions as a later example will show.

THEOREM. Given an ordered pair P, Q and a between point R of P and Q. Then there exists a Hilbert curve joining P to Q and passing through R.

Proof. Take a Hilbert curve joining P to R and one joining R to Q. It will be shown that the curve H consisting of the sum of these two curves is a Hilbert curve joining P to Q. To show this it will suffice to show that for any three points on the curve A, C, B in the order of orientation,

$$\overline{AC} + \overline{CB} = \overline{AB}. \tag{3.3}$$

If A, C and B are all three on the arc PR or all on the arc RQ (3.3) holds by assumption. Consider now the case where C = R. Then



$$\overline{PA} = \overline{PR} - \overline{AR} \quad \text{and}$$

$$\overline{BQ} = \overline{RQ} - \overline{RB}.$$

$$\overline{AB} \leqq \overline{AR} + \overline{RB} \quad \text{by (2.5), and}$$

$$\overline{PQ} \leqq \overline{PA} + \overline{AB} + \overline{BQ} \text{ by the same,}$$

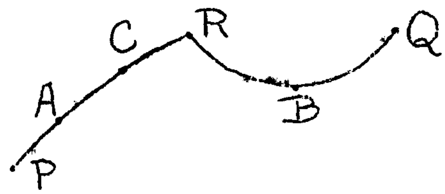
$$\leqq \overline{PR} - \overline{AR} + \overline{AB} + \overline{RQ} - \overline{RB}$$

$$\leqq \overline{PQ} + \overline{AB} - \overline{AR} - \overline{RB} \quad \text{since}$$

$$\overline{PR} + \overline{RQ} = \overline{PQ} \quad \text{by definition of } R.$$

$$\therefore \quad \overline{AB} \geqq \overline{AR} + \overline{RB}$$

$$\therefore \quad \overline{AB} = \overline{AR} + \overline{RB}.$$



Now let C be any point on the arc AR. If C lies on the arc RB a similar proof holds. By the above

$$\overline{AB} = \overline{AR} + \overline{RB}$$

But

$$\overline{AR} = \overline{AC} + \overline{CR} \quad \text{since PR is a Hilbert curve.}$$

Likewise

$$\overline{CB} = \overline{CR} + \overline{RB}.$$

Subtracting,

$$\overline{AC} + \overline{CB} = \overline{AR} + \overline{RB}$$

$$= \overline{AB}.$$

Therefore (3.3) holds on H which hence is a Hilbert curve joining P to Q and passing through R.

If there is a unique Hilbert curve H_P^Q joining P to Q, then H_P^Q is displaced continuously as P and Q are varied. More precisely, take a sequence of points $P_n \to P$ and a sequence $Q_n \to Q$, and join P_n to Q_n by any Hilbert curve $H_{P_n}^{Q_n}$ Take any point R_n on $H_{P_n}^{Q_n}$. Then

$$\overline{PR_n} \leqq \overline{PP_n} + \overline{P_nR_n} \quad \text{by (2.5),}$$

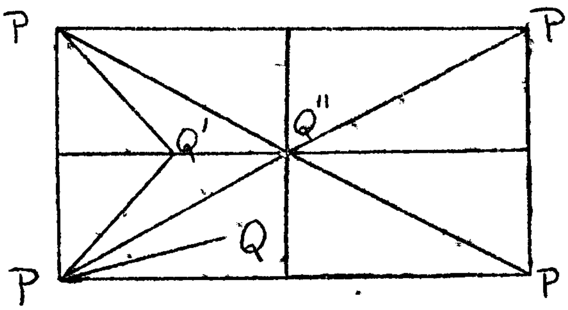
and since $\overline{PP_n}$ and $\overline{P_nR_n}$ are both bounded, $\overline{PR_n}$ is bounded and by (3.1) R_n will have an accumulation point. Take a subsequence R_n' of R_n such that R_n' has a limit point R.

$$\overline{P_nR_n'} + \overline{R_n'Q_n} = \overline{P_nQ_n}$$

and due to continuity we have in the limit

$$\overline{PR} + \overline{RQ} = \overline{PQ}.$$

But H_P^Q is unique and therefore all between points of P and Q lie on it. Therefore R is a point of H_P^Q, and indeed all accumulation points of R_n lie on H_P^Q. It might be noted that if $\overline{P_nR_n} = \alpha_n \overline{P_nQ_n}$ and if $\alpha_n \rightarrow \alpha$ then $R_n \rightarrow R$ such that $\overline{PR} = \alpha\,\overline{PQ}$.



An example showing the importance of the uniqueness of the Hilbert curve in the above proof is a torus, indicated in the diagram in the common manner as a rectangle, with a Euclidean metric on the rectangle. For all points such as Q not on the central cross the Hilbert curve is unique and is displaced continuously with Q, but for points such as Q' or Q" on the central cross there are two or four Hilbert curves as indicated, and as Q' moves off the central cross to one side or the other, the Hilbert curve is obviously displaced discontinuously.

Consider now a Hilbert curve joining two given points P and Q. If there exists a point R such that Q is a between point of P and R, then by a previous argument the original Hilbert curve plus any Hilbert curve joining Q to R would constitute a Hilbert curve joining P to R. Let the Hilbert curve be extended in this manner as far as possible. Take a new parameter

$$S_B = PB; \quad \text{then}$$

$$\overrightarrow{BC} = S_C - S_B \quad \text{by (3.3)} \tag{3.4}$$

There are three apparent possibilities for the domain of s:

I. $0 \leqq s < K$, K being a constant;

II. $0 \leqq s \leqq K$, K " " " ;

III. $0 \leqq s < \infty$

The first case however is impossible for, since (3.3) holds for all points with parameter $s < K$, we could use a limiting process dimilar to that used in the construction of a Hilbert curve and extend the Hilbert curve to include a point whose parameter would be K.

In the second and third cases the Hilbert curve is closed. For if a sequence of points P_n on the Hilbert curve has a limit point P, then it has been shown that

$$\lim_{n\to\infty} \widetilde{PP_n} = 0, \ \widetilde{PP_n} \text{ being the greater of } \overline{PP_n} \text{ and } \overline{P_nP},$$

and hence $\lim_{n\to\infty} \widetilde{P_nP_{n+m}} = 0$ for all m. By (3.4) then $|S_n - S_{n+m}| \to 0$ as n and the S_n, forming a Cauchy sequence, have a limit point S. Since then $|S - S_n| \to 0$ as $n \to \infty$, we have that $\lim_{n\to\infty} \widetilde{P_nP(S)} = 0$ and the sequence P_n has $P(S)$ as a limit point. Therefore $P = P(S)$ and is a point on the Hilbert curve, proving that the Hilbert curve is closed. In the second case the last point on the curve will be called the minimum point. For a compact space it is easily seen that only case II can occur.

Arc length of a curve has been defined only if the curve is admissible. A "generalized" arc length, as distinguished from "real" arc length, can be defined for all continuous curves in the following way. Let the curve be given by $P = P(t)$, $0 \leqq t \leqq 1$, $P(t)$ being a continuous function of t. Take some subdivision of (0, 1) with n+1 points $t_0 = 0, t_1, \ldots, t_n = 1$. Form

$$\sum_{i=0}^{n} \overline{P(t_i)P(t_{i+1})} = S_n \geqq \overline{P(0)P(1)}.$$

If the subdivision is made finer by further subdivisions such that to a given $\epsilon > 0$ there exists an N so that

$$t_{i+1} - t_i < \epsilon, \quad i = 0, 1, \cdots, n$$

if $n > N$, then S_n always increases and tends therefore to a limit S finite or infinite. S will be called the "generalized" arc length of the curve P(t). S can be shown to be independent of this method of subdivision. Take any two kinds of subdivision, form the S_n and S'_m respectively for the n^{th} and m^{th} step, and let $\lim_{n \to \infty} S_n = a$, $\lim_{m \to \infty} S'_m = b$. If, to be definite, $b > a$, then there exists an ϵ such that $b - \epsilon > a$. For some m, $S'_m > b - \epsilon/2$. Now take an n so large that there exist points Q_i of S_n so near to the $P'(t_i)$ of S'_m that

$$\left| S'_m - \sum_{i=0}^{n-1} \overline{Q_i Q_{i+1}} \right| < \epsilon/2 .$$

This is possible because P(t) is continuous and $\overline{PQ}$ is a continuous function of both P and Q. But $\sum_{i=0}^{n-1} \overline{Q_i Q_{i+1}} < S_n < a$. Hence

$$S'_m + \delta = \sum_{i=0}^{n-1} \overline{Q_i Q_{i+1}} < a, \quad |\delta| < \epsilon/2, \quad \text{and}$$

$$b - \epsilon/2 + \delta < S'_m + \delta < a .$$

$$\therefore a > b - \epsilon$$

and a contradiction is reached unless a = b. This proof will hold with an obvious rewording if b is infinite. It might be noted that the S thus defined is the least upper bound of all S_n for all possible subdivisions. If the "generalized" arc length S of an admissible curve is obtained, in each step

$$\overline{P(t_i)P(t_{i+1})} \leqq \int_{C\,P(t_i)}^{P(t_{i+1})}$$

and hence

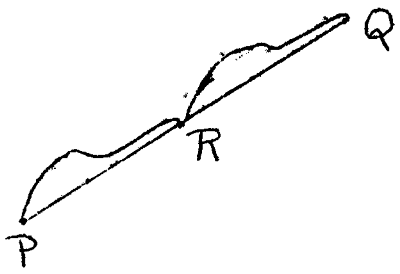
$$S(C) \leqq \mathcal{J}(C)$$

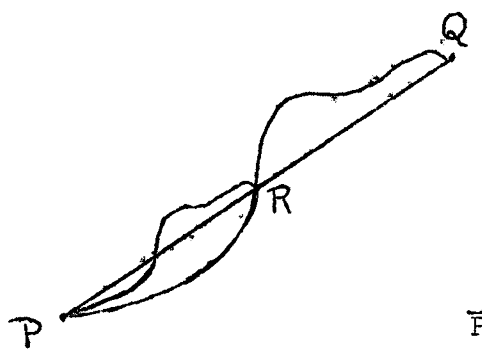
However, since the Hilbert curve is obviously a minimizing curve for "generalized" arc length, the greatest lower bound of the "generalized" arc length of all continuous curves joining P to Q is equal to $\overline{PQ}$.

The following interesting example by Dr. Comenetz shows among other things that curves can exist whose "generalized" arc lengths are actually less than their "real" arc length. Take a 2-dimensional space with the form

$$F(x, x') = F(x, y, x', y') = \sqrt[4]{x'^4 - x'^2 y'^2 + y'^4} .$$

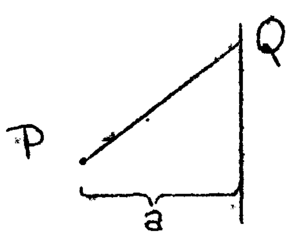
A brief check will reveal that the form satisfies all of the assumptions (2.1), (2.2), (2.3), and in fact gives a symmetric distance between any two points. In this space any Euclidean straight line is a Hilbert curve. For, consider a straight line from P to Q. Take R, the Euclidean mid-point of P and Q. For every curve joining P to R there is a congruent curve joining R to Q, and since the form does not depend on the coordinates x, y, the integral along the first curve equals that along the second. Since this is true for every curve joining P to R, we have that $\overline{PR} = \overline{RQ}$. Now for every curve $x_i = x_i(t)$, $y_i = y_i(t)$ joining P to R, if P be taken as the coordinate origin, there is a curve $x_i = 2x_i(t)$, $y_i = 2y_i(t)$ joining P to Q and the integral along the first is half of the integral along the second. Hence



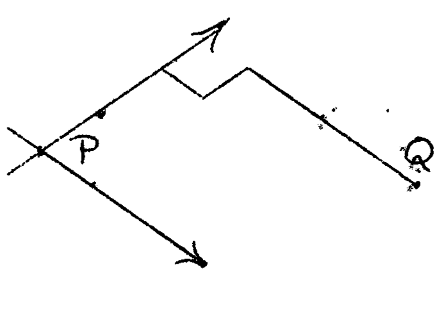


$$\overline{PR} = \tfrac{1}{2}\overline{PQ} = \overline{RQ},$$

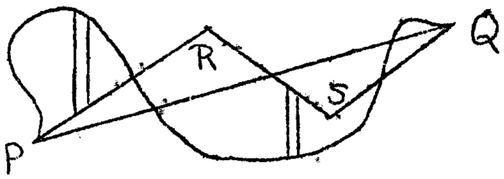
and R is the mid-point in the sense of this metric, of P and Q. The construction for a Hilbert curve can be continued and it is seen that the Euclidean straight line joining P to Q is a Hilbert curve.



Consider any point P and a line parallel to the y-axis at a distance a from it, and join P to all points Q on the line by straight lines It is a matter of simple calculation to verify that the slopes α giving the least integral are $\alpha = \pm\frac{1}{\sqrt{2}}$ and are independent of P or a.

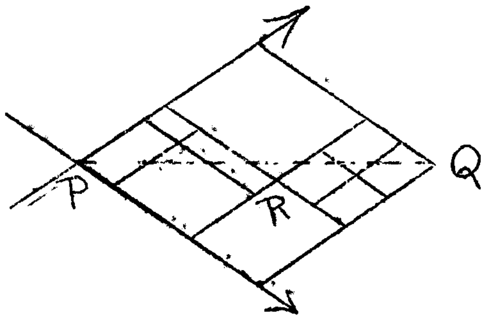


Consider a point P and construct through P the two straight lines of slopes $\pm\frac{1}{\sqrt{2}}$ If Q is in the constructed angle it can be joined to P by a curve consisting only of straight lines of the slopes $\pm\frac{1}{\sqrt{2}}$ and such that it is a single-valued function of x. Comparison of corresponding infinitesimals of such a broken line and those of any other curve from P to Q, shows that the broken line gives the minimum value to the integral and hence



$$\overline{PQ} = \int_{C_P^R} + \int_{C_R^S} + \int_{C_S^Q} .$$

This broken line PRSQ being a minimizing arc is a Hilbert curve, while the straight line PQ is a Hilbert curve whose integral is larger. But for Hilbert curves due to (3.3), their characteristic property, the "generalized" arc length is equal to the distance between the end points. Therefore there are curves whose "generalized" arc length is less than their "real" arc length.



Hilbert curves other than those noted above can be constructed in this method. If Q lies in the basic angle constructed at P, the large parallelogram with P and Q as vertices as in the diagram, can be constructed. Any point R in the parallelogram is a between point and can be taken as a point on the Hilbert curve. In the same manner, between points of P and R, and R and Q can be picked and the method continued constructing a Hilbert curve.

It might be noted that this example shows that different Hilbert curves may have arcs in common

and that Hilbert curves are not even unique in the small, and that there are Hilbert curves with continuous derivatives not being solutions of the variation problem.

Chapter II. Local Properties of a Minimizing Arc.

4. The following standard notation will be adopted. By a curve of class C^m is meant a curve which is continuous and has continuous derivatives up to and including those of the m^{th} order, and at any point not all of the first derivatives vanish. By a curve of class D^m is meant one which can be divided into a finite number of parts, the closure of each part being a curve of class C^m.

A Curve M_P^Q joining P to Q is said to be a relative minimizing arc if its arc length is not larger than that of any curve in a neighborhood of M_P^Q. Any sub-arc of such a curve M_P^Q is also a relative minimizing arc and hence if M_P^Q is of class D^m we can, in the investigation of local properties, consider sub-arcs of class C^m which are also relative minimizing arcs. Indeed, these sub-arcs can be chosen so small that they lie entirely in one definite coordinate neighborhood $\mathcal{U}$.

At this point let us add the following assumption.

(4.1) <u>$F(x, \lambda)$ has continuous partial derivatives up to and including those of the fourth order. In this chapter we do not assume that $F(x, \lambda)$ is positive homogeneous in λ. If $F(x, \lambda)$ is positive homogeneous in λ, then the existence and continuity of the partial derivatives will not be assumed at $(x, 0)$ (zero vectors).</u>

The existence of continuous derivatives of the second order only will be needed immediately, but those of the fourth order will be needed eventually. If $F(x, \lambda)$ is homogeneous of the 1^{st} degree in λ, $\frac{\partial F}{\partial \lambda_i} = F_i$ is homogeneous of the 0^{th} degree in λ, i.e.

$$F_i(x, c\lambda) = F_i(x, \lambda)$$

and F_i would have to be independent of λ if continuity were required at the points (x, 0). Since positive homogeneity of $F(x, \lambda)$ is not assumed in this chapter, the integral along a curve, and hence the minimizing arcs, will be dependent on parameterization. Such a curve whose integral depends upon the parameter is called a parameter curve.

We take now a continuous curve C_o joining P to Q of class C^1, lying in one coordinate neighborhood and giving a relative minimum.

To find now necessary conditions that such a curve C_o be a relative minimum, consider the following family of curves C_ϵ,

$$x_i = x_i(t, \epsilon), \quad \begin{array}{c} -\eta \leqq \epsilon \leqq \eta \\ t_o \leqq t \leqq t_1 \end{array},$$

$C_o = x_i(t, 0)$ being the curve investigated. Let $x_i(t, \epsilon)$ be a continuous function of both t and ϵ, and assume that $\frac{\partial x_i}{\partial t}$, $\frac{\partial x_i}{\partial \epsilon}$ and $\frac{\partial^2 x_i}{\partial t \partial \epsilon}$ are continuous; then $\frac{\partial^2 x_i}{\partial \epsilon \partial t}$ exists and is equal to $\frac{\partial^2 x_i}{\partial t \partial \epsilon}$. Also let

$$x_i(t_o, \epsilon) = x_i(P)$$
$$x_i(t_1, \epsilon) = x_i(Q).$$

Then $\frac{\partial x_i}{\partial \epsilon} = \eta^i$ a covariant vector, and the last two conditions require that η^i vanish at P and Q.

For instance $x_i(t, \epsilon)$ can be taken of the usual form

$$x_i(t, \epsilon) = x_i(t) + \epsilon\, y_i(t),$$

$y_i(t)$ being of class C^1, and $y_i(t_o) = y_i(t_1) = 0$. Note that this linear form for $x_i(t, \epsilon)$ however is dependent on the coordinate system. Each curve C_ϵ of the family is of class C^1 under these assumptions, and there exists an η such that for $|\epsilon| < \eta$ all the curves lie within the neighborhood relative to which C_o is a minimizing arc.

Set up now the integral

$$\mathcal{J}(\epsilon) = \int_{t_0}^{t_1} F(x_i(t,\epsilon), \frac{\partial x_i(t,\epsilon)}{\partial t})\,dt.$$

A necessary condition for C_0 to be a relative minimizing arc is that $\mathcal{J}(0) \leqq \mathcal{J}(\epsilon)$ for all $|\epsilon| < \eta$, and if $\mathcal{J}(\epsilon)$ has a derivative at $\epsilon = 0$, then $\mathcal{J}'(0) = 0$.

Under the assumptions which have been placed on $F(x, \lambda)$ and $x_i(t, \epsilon)$ the derivative exists and the conditions are satisfied for differentiation under the integral sign. Thus we get

$$\mathcal{J}'(\epsilon) = \int_{t_0}^{t_1} \left(\frac{\partial F}{\partial x_i}\frac{\partial x_i}{\partial \epsilon} + \frac{\partial F}{\partial x_i'}\frac{\partial^2 x_i}{\partial t\,\partial \epsilon}\right) dt \tag{4.2}$$

where x_i' are the second n variables of F.

From this point there are two methods of procedure. We will give first the classical way for which it is necessary to assume that C_0 is of class C^2. We can now integrate the second half of the integral (4.2) by parts and get

$$\mathcal{J}'(\epsilon) = \left(\frac{\partial F}{\partial x_i'}\frac{\partial x_i}{\partial \epsilon}\right)_{t_0}^{t_1} + \int_{t_0}^{t_1}\left(\frac{\partial F}{\partial x_i}\frac{\partial x_i}{\partial \epsilon} - \frac{d}{dt}\left(\frac{\partial F}{\partial x_i'}\right)\frac{\partial x_i}{\partial \epsilon}\right)dt$$

and since $\eta^i = \dfrac{\partial x_i}{\partial \epsilon}$ vanishes at t_0 and t_1,

$$\mathcal{J}'(0) = \int_{t_0}^{t_1}\left(\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial x_i'}\right)\frac{\partial x_i}{\partial \epsilon}\,dt \tag{4.3}$$

where now $x_i = x_i(t, 0)$.

<u>$\mathcal{J}'(0)$ is called the first variation of C_0. It obviously depends on the family of curves C_ϵ. If the first variation of C_0 vanishes for all families of comparison curves C_0 is called an extremal.</u> Each minimizing arc is an extremal, but not conversely.

THEOREM. The necessary and sufficient condition that a curve be an extremal is that the "Euler vector"

(4.4) $$\rho_i = \frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial x'_i} \quad .$$

vanishes along the curve.

Proof. (The proof that ρ_i is a covariant vector will follow later.)

Denote $\frac{\partial x_i(t,\epsilon)}{\partial \epsilon}$ by η^i Then (4.3) becomes

(4.3) $$\mathcal{J}'(0) = \int_{t_0}^{t_1} \rho_i \eta^i dt ,$$

when η^i depends upon the family of comparison curves, but ρ_i depends only on C_o.

If $\rho_i = 0$ along C_o then $\mathcal{J}'(0) = 0$ for all η^i, and the sufficiency of the condition is shown.

Let $\mathcal{J}'(0) = 0$ and suppose $\rho_i \neq 0$ at some point $R = x_i(t)$ on C_o. The existence of a family of comparison curves such that $\mathcal{J}'(0) \neq 0$ will now be shown and a contradiction obtained unless ρ_i vanishes on C_o. There exists at $R(t)$ a contravariant vector σ^i such that $\rho_i\sigma^i > 0$ at $R(t)$. But ρ_i is a continuous vector, hence there is an interval about $R(t)$ corresponding to the interval $(t - \tau, t + \tau)$ such that with the above σ^i $\rho_i \sigma^i > 0$ in that interval. Take now as a family of comparison curves

$$x_i(t, \epsilon) = x_i(t) + \epsilon\alpha(t)\sigma^i$$

where $\alpha(t)$ is of class C^1 and such that $\alpha(t) = 0$ outside the interval and $\alpha(t) > 0$ inside of it. Then

$$\eta^i = \frac{\partial x_i}{\partial \epsilon} = \alpha(t)\sigma^i , \quad \text{and}$$

$$\mathcal{J}'(0) = \int_{t-\tau}^{t+\tau} \rho_i \alpha(t) \sigma^i dt > 0$$

and the theorem is proved.

The fact that ρ_i is a covariant vector can be shown in the following manner. Take an l-dimensional sub-manifold S of the entire space R_n defined by

(4.4) $$x_i = x_i(y_1, \ldots, y_l)$$

Take any two points P and Q of S and suppose them to be connected by a curve lying in S,

$$y_\alpha = y_\alpha(t), \quad (\alpha = 1, \cdots, l)$$

This curve considered as a curve in the entire space R_n would be

$$x_i = x_i(y(t)), \quad (i = 1, \ldots, n).$$

Then if we set

$$F\left(x(y), \frac{\partial x_i}{\partial y_\beta}\frac{dy_\beta}{dt}\right) = \mathcal{F}(y, y')$$

$$\begin{aligned}\rho_\alpha &= \frac{\partial \mathcal{F}}{\partial y_\alpha} - \frac{d}{dt}\frac{\partial \mathcal{F}}{\partial y'_\alpha}, \quad (\alpha = 1, \cdots, l)\\ &= \frac{\partial F}{\partial x_i}\frac{\partial x_i}{\partial y_\alpha} + \frac{\partial F}{\partial x'_i}\frac{\partial^2 x_i}{\partial y_\alpha \partial y_\beta}\frac{dy_\beta}{dt}\\ &\quad - \frac{d}{dt}\left(\frac{\partial F}{\partial x'_i}\right)\frac{\partial x_i}{\partial y_\alpha} - \frac{\partial F}{\partial x'_i}\frac{\partial^2 x_i}{\partial y_\alpha \partial y_\beta}\frac{dy_\beta}{dt}.\end{aligned}$$

(4.5) $$\therefore \rho_\alpha = \frac{\partial x_i}{\partial y_\alpha}\rho_i, \quad (\alpha = 1, \cdots, l;\ i = 1, \cdots, n)$$

giving the connection between the Euler vectors of the entire space R_n and S. If $\rho_\alpha = 0$ then the curve is an extremal relative to S and (4.5) states that ρ_i, the Euler vector with regard to R_n, of the curve is normal to S. This proves the

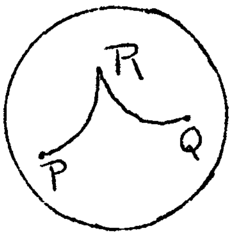
THEOREM. The necessary and sufficient condition that a curve of a submanifold is an extremal of this submanifold is that the Euler vector is normal to the submanifold.

If $\chi = n$ we can consider (4.4) as a coordinate transformation in R_n and since then

(4.6) $$\bar{\rho}_i = \frac{\partial x_j}{\partial \bar{x}_i}\rho_j \qquad (i, j = 1, \ldots, n)$$

we see that ρ_i is indeed a covariant vector.

We will not consider the question of corner conditions. Assume we have a minimizing arc of class D^2 and at a corner R take a coordinate neighborhood.



The curve PRQ is of class D^2 and may be represented by $x_i = x_i(t)$, $t_0 \leqq t \leqq t_1$, where

$$\left.\frac{dx_i}{dt}\right|_{\tau+} \neq \left.\frac{dx_i}{dt}\right|_{\tau-}$$

τ being the parameter of R. Now, as before, take a family

$$x_i(t,\epsilon) = x_i(t) + \epsilon y^i(t), \quad y^i(t_0) = y^i(t_1) = 0$$

and set up

$$J'(\epsilon) = \int_{t_0}^{t_1}\left(\frac{\partial F}{\partial x_i}\frac{\partial x_i}{\partial \epsilon} + \frac{\partial F}{\partial x'_i}\frac{d}{dt}\frac{\partial x_i}{\partial \epsilon}\right)dt$$

Now when integrating by parts, split the interval $(t_0 t_1)$ into intervals $(t_0\,\tau)$ and $(\tau\,t_1)$. Then

$$J'(\epsilon) = \int_{t_0}^{t_1}\left(\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial x'_i}\right)\frac{\partial x_i}{\partial \epsilon}\,dt + \left[\frac{\partial F}{\partial x'_i}\frac{\partial x_i}{\partial \epsilon}\right]_{t_0}^{\tau-} + \left[\frac{\partial F}{\partial x'_i}\frac{\partial x_i}{\partial \epsilon}\right]_{\tau+}^{t_1}$$

$$= \int_{t_0}^{t_1}\rho_i\eta^i\,dt - \left[\frac{\partial F}{\partial x'_i}\eta^i\right]^{\tau+} + \left[\frac{\partial F}{\partial x'_i}\eta^i\right]^{\tau-}$$

But arcs PR and RQ, being subarcs of a minimizing arc, are extremals and $\rho_i = 0$ along them. If now we assume that η^i is continuous we get

$$J'(0) = \eta^i\left[\left(\frac{\partial F}{\partial x'_i}\right)_{\tau-} - \left(\frac{\partial F}{\partial x'_i}\right)_{\tau+}\right] = 0$$

for arbitrary continuous η^i and thus we obtain as a necessary condition for a minimizing arc of class D^2 the Weierstrass-Erdmann corner condition that

$$\left(\frac{\partial F}{\partial x_i'}\right)_{\tau-} = \left(\frac{\partial F}{\partial x_i'}\right)_{\tau+} \tag{4.7}$$

i.e. $\frac{\partial F}{\partial x_i'}$ exists and is continuous at the corners. If a curve of class D^2 is to be an extremal $\rho_i = 0$ and (4.7) holds at all corners.

It can easily be shown that $F_i = \frac{\partial F(x,\lambda)}{\partial \lambda^i}$ is a covariant vector. For let

$$x_i = x_i(\bar{x}_1, \cdots, \bar{x}_n)$$

be a coordinate transformation. Then

$$\lambda^i = \frac{\partial x_i}{\partial \bar{x}_k} \bar{\lambda}^k \qquad \text{and}$$

$$F(x,\lambda) = F\left(x(\bar{x}), \frac{\partial x}{\partial \bar{x}_k}\bar{\lambda}^k\right) = \bar{\mathcal{F}}(\bar{x}, \bar{\lambda}).$$

Therefore we have

$$\frac{\partial \bar{\mathcal{F}}(\bar{x},\bar{\lambda})}{\partial \bar{\lambda}^i} = \frac{\partial F(x,\lambda)}{\partial \lambda^j}\frac{\partial x_j}{\partial \bar{x}_i}$$

and F_i is a covariant vector. In the same way it can be shown that

$\frac{\partial^2 F}{\partial \lambda^i \partial \lambda^j} = \sigma_{ij}$ is a covariant tensor of second rank.

Under the assumption that $F(x, \lambda)$ is positive homogeneous of the first degree in λ we have obviously that F_i is positive homogeneous of the 0^{th} degree in λ. Then using Euler's Theorem on F_i we get

$$\frac{\partial^2 F}{\partial \lambda^i \partial \lambda^j}\lambda^j = 0$$

and hence $|\sigma_{ij}| = 0$ at all points. This is what one would expect, for if $|\sigma_{ij}| \neq 0$ we could solve

$$0 = \rho_i = \frac{\partial F}{\partial x_i} - \frac{\partial^2 F}{\partial x_i' \partial x_k}\frac{dx_k}{dt} - \frac{\partial^2 F}{\partial x_i' \partial x_k'}\frac{d^2 x_k}{dt^2}$$

for $\frac{d^2 x_k}{dt^2}$ and obtain extremals, solutions of $\rho_i = 0$, which depend on the parameterization. But extremals, under the assumption that $F(x, \lambda)$ is positive homogeneous of the first degree in λ, are independent of the parameterization as is to be expected from their definition and also as can easily be shown by direct calculation: Let $\rho_{(t)}^i$ be the Euler vector of a curve for parameter t, and $\rho_{(s)}^i$ for s. Then

$$\rho_{(t)}^i = \frac{\partial F(x, \frac{dx}{dt})}{\partial x_i} - \frac{d}{dt}\frac{\partial F(x, \frac{dx}{dt})}{\partial x'_i}$$

$$= \frac{\partial F(x, \frac{dx}{ds})}{\partial x_i}\frac{ds}{dt} - \frac{d}{ds}\frac{\partial F(x, \frac{dx}{ds})}{\partial x'_i}\frac{ds}{dt}$$

$$(4.8) \qquad = \rho_{(s)}^i \frac{ds}{dt}$$

because $\frac{\partial F}{\partial x_i}$ and $\frac{\partial F}{\partial x'_i}$ are homogeneous of degree 1 and 0 respectively, and if ρ_i vanishes for one parameter it vanishes for all.

5. We have already presented on page 24 and following, one method of handling the integral $J'(0)$. In the following second method we will assume only that $x_i(t)$, the arc under consideration, and $\frac{\partial x_i(t,\epsilon)}{\partial \epsilon}$ are of class D'. We have as before

$$J'(0) = \int_{t_0}^{t_1} \left[\frac{\partial F}{\partial x_i}\frac{\partial x_i}{\partial \epsilon} + \frac{\partial F}{\partial x'_i}\frac{d}{dt}\frac{\partial x_i}{\partial \epsilon}\right]dt = 0$$

and this time we integrate the first part of the integrand by parts, getting

$$J'(0) = \int_{t_0}^{t_1}\left[-\int_{t_0}^{t}\frac{\partial F}{\partial x_i}dt + \frac{\partial F}{\partial x'_i}\right]\frac{d}{dt}\frac{\partial x_i}{\partial \epsilon}dt = 0$$

or

$$(5.1) \qquad J'(0) = \int_{t_0}^{t_1}\left[\frac{\partial F}{\partial x'_i} - \int_{t_0}^{t}\frac{\partial F}{\partial x_i}dt\right]\frac{d\eta^i}{dt}dt = 0$$

for each η^i of class D^1 such that $\eta^i(t_0) = \eta^i(t_1) = 0$. Letting

$$\phi_i = \frac{\partial F}{\partial x'_i} - \int_{t_0}^{t}\frac{\partial F}{\partial x_i}dt$$ we have

$$\int_{t_0}^{t_1} \phi_i \frac{d\eta^i}{dt} dt = \int_{t_0}^{t_1} (\phi_i - c_i) \frac{d\eta^i}{dt} dt = 0$$

c_i being a constant, since η^i vanishes at t_0 and t_1. Now ϕ_i is of class D° so if we set

$$\eta^i = \int_{t_0}^{t_1} (\phi_i - C_i)\, dt$$

η^i is of class D^1 and vanishes at t_0 and t_1 if c_i is taken so that $\eta^i(t_1) = 0$. Then

$$\frac{d\eta^i}{dt} = \phi_i - c_i$$

and

$$\mathcal{J}'(0) = \int_{t_0}^{t_1} \sum_{i=1}^{n} (\phi_i - c_i)^2 dt = 0.$$

$$\therefore \quad (\phi_i - c_i)^2 = 0 \quad \text{and} \quad \phi_i = c_i$$

in each sub-arc in which ϕ_i is of class D°. But c_i is a constant, and hence $\phi_i = c_i$ everywhere. That is

$$(5.2) \qquad \frac{\partial F(x,x')}{\partial x_i'} - \int_{t_0}^{t} \frac{\partial F(x,x')}{\partial x_i} dt = c_i$$

It might be noted that the Weierstrass-Erdmann corner condition can be obtained from (5.2) since continuity of c_i and $\int_{t_0}^{t} \frac{\partial F}{\partial x_i} dt$ insures the continuity of $\frac{\partial F}{\partial x_i'}$. Along a sub-arc of $x_i(t)$ of class C^1, c_i and $\int_{t_0}^{t} \frac{\partial F}{\partial x_i} dt$ both have derivatives and hence $\frac{\partial F}{\partial x_i'}$ has one and

$$(5.3) \qquad \frac{d}{dt} \frac{\partial F}{\partial x_i'} - \frac{\partial F}{\partial x_i} = 0$$

and the Euler equation is again obtained, this time without assuming $x_i(t)$ to be of class C^2. Notice, however, that we are unable to write the Euler equation in the form

$$(5.3') \qquad \frac{\partial F}{\partial x_i} - \frac{\partial^2 F}{\partial x_i' \partial x_j} \frac{dx_j}{dt} - \frac{\partial^2 F}{\partial x_i' \partial x_j'} \frac{d^2 x_j}{dt^2} = 0$$

since the existence of $\frac{d^2 x_i}{dt^2}$ has not been assumed. We will show however that if F(x, x') is not homogeneous of the first degree in x' and $|\sigma_{ij}| \neq 0$, $\sigma_{ij} = \frac{\partial^2 F}{\partial x_i' \partial x_j'}$, an extremal of class C^1 is automatically of class C^2, and if, in the case F(x, x') is homogeneous of the first degree in x', $\|\sigma_{ij}\|$ is of rank n-1, an extremal of class C^1 can be so parameterized that it will be of class C^2.

First we assume $|\sigma_{ij}| \neq 0$ and $x_i(t)$ of class C^1. In (5.2) we set unknowns $z_1, \ldots, z_n$ for $x_1', \ldots, x_n'$ in $\frac{\partial F(x,x')}{\partial x_i}$ and obtain

(5.4)
$$g_i(t, z_1, \cdots, z_n) = 0$$

of class C^1 where $\frac{\partial g_i}{\partial x_j'} = \frac{\partial^2 F}{\partial x_i' \partial x_j'}$. Using now a theorem on implicit functions we see that if $\left|\frac{\partial g_i}{\partial x_j'}\right| \neq 0$ at $\tilde{t}, \tilde{z}_1, \ldots \tilde{z}_n$, a solution set of (5.3), there exists an interval $\tilde{t} - \epsilon, \tilde{t} + \epsilon$, and some neighborhood $\mathcal{U}(\tilde{z})$ such that, for any t in the interval, (5.4) has one and only one solution $z_1, \ldots, z_n$ in $\mathcal{U}(\tilde{z})$, $z_i(t)$ being of class C^1. But t, $z_i = x_i'(t)$ is a solution of (5.4) by construction and hence must be unique in a neighborhood; and since $z_i(t)$ is of class C^1 we see that $x_i'(t) = z_i(t)$ has a derivative $x_i''(t)$. Therefore the extremal $x_i(t)$ must be of class C^2.

To prove the second half of the statement above, let us compare the variation problem of F(x, λ) with that of $\mathcal{F} = F^2$ where F(x, λ) is positive homogeneous of degree one in λ. Assume now that $\left|\frac{\partial^2 \mathcal{F}}{\partial x_i' \partial x_j'}\right| \neq 0$. Let

$$\mathcal{J}(\epsilon) = \int_{t_0}^{t_1} F\, dt$$

and

$$\mathcal{j}(\epsilon) = \int_{t_0}^{t_1} \mathcal{F}\, dt$$

Then

$$\mathcal{J}'(0) = \int_{t_0}^{t_1} \frac{\partial F}{\partial \epsilon}\, dt$$

and

$$\mathcal{j}'(0) = 2\int_{t_0}^{t_1} F \frac{\partial F}{\partial \epsilon}\, dt$$

Now if $x_i(t)$ is an extremal of class C^1 of the F problem and if $t = t(s)$, $\frac{dt}{ds} > 0$, then $x_i(t(s))$ is also an extremal. Let s be so chosen that F is constant along the extremal, i.e.

$$F(x, \tfrac{dx}{ds}) = c = F(x, \tfrac{dx}{dt})\frac{dt}{ds}$$

and

$$s = \frac{1}{c}\int_{t_0}^{t_1} F(x, \tfrac{dx}{dt})\,dt + C_1 .$$

Then

$$\mathcal{J}'(0) = 0 = \int_{s_0}^{s_1}\frac{\partial F}{\partial\epsilon}\,ds$$

$$\mathcal{J}'(0) = 2c\int_{s_0}^{s_1}\frac{\partial F}{\partial\epsilon}\,ds = 0$$

and an extremal of the F problem with arc length or a constant times arc length as parameter is an extremal of the F^2 problem. Hence, since $x_i(s)$ is of class C^1 we see that under the assumption $\left|\frac{\partial^2 F^2}{\partial x_i' \partial x_j'}\right| \neq 0$ an extremal of $F(x, \lambda)$ when parameterized by arc length is of class C^2. The above proof holds for F^k, $k \neq 1$, as well as for F^2.

It will now be shown that the assumption $\left|\frac{\partial^2 F^2}{\partial x_i' \partial x_j'}\right| \neq 0$ is equivalent to the assumption that $\left\|\frac{\partial^2 F}{\partial x_i' \partial x_j'}\right\|$ is of rank n-1, and in fact that if $\tau_{ij} = \frac{\partial^2 F^k}{\partial x_i' \partial x_j'}$, $k \neq 1$, and $\sigma_{ij} = \frac{\partial^2 F}{\partial x_i' \partial x_j'}$ then if σ_{ij} is of rank r, τ_{ij} is of rank r+1. We have

$$\tau_{ij} = k(k-1)F^{k-2}\frac{\partial F}{\partial x_i'}\frac{\partial F}{\partial x_j'} + kF^{k-1}\sigma_{ij}$$

Consider the system of n linear equations

$$\tau_{ij}\lambda^j = 0 \tag{5.5}$$

$\|\tau_{ij}\|$ is of rank s if (5.5) contains s linearly independent equations and no more, or if (5.5) has n-s linearly independent solutions λ^i but no more. Also if $\tau'_{ij}\lambda^j = 0$, $i = 1, \ldots, n'$, and (5.5) have the same solutions, then the rank of $\| \tau_{ij} \|$ is equal to the rank of $\| \tau'_{ij} \|$

If (5.5) has a solution λ^i we note from (5.4) that

(5.6) $$0 = k(k-1) F^{k-2} F_i F_j \lambda^j + k F^{k-1} \sigma_{ij} \lambda^j$$

has the same solution λ^i. But if we multiply (5.6) by x'_i we get

$$0 = k(k-1) F^{k-1} F_j \lambda^j$$

by using twice the Euler theorem ($\sigma_{ij} x'_j = 0$, $F_i x'_i = F$), and hence ($F \neq 0$, $k \neq 1$),

(5.7a) $$F_j \lambda^j = 0$$

But then we have

$$k F^{k-1} \sigma_{ij} \lambda^j = 0$$

giving

(5.7b) $$\sigma_{ij} \lambda^j = 0 .$$

Hence any solution of (5.5) is a solution of (5.7) and obviously conversely, and by a previous remark the rank of (5.5) is equal to that of (5.7). But F_j and σ_{ij} are linearly independent as shown in the above proof so if the rank of (5.7b) is r, that of (5.7) and hence that of (5.5) is r+1, and conversely.

We thus complete the proof that if $F(x, \lambda)$ is positive homogeneous of degree one in λ and $\| \sigma_{ij} \|$ is of rank n-1, any extremal when parameterized by arc length is of class C^2.

6. A series of remarks on the properties of Euler vectors will follow. Consider an open, simply connected point set S with $F(x, \lambda)$ homogeneous and of the first degree in λ. A necessary and sufficient condition that the integral from P to Q for all P and Q in this point set S depends only on the endpoints is that

$$J'(0) = 0$$

for each arc connecting P to Q; that is, for curves of class C^2, $\rho_i = 0$. The necessity follows from previous considerations. The sufficiency can easily be seen by imbedding any two curves connecting P to Q in a one-parameter family $x_i(t, \epsilon) = 0$, $t_0 \leq t \leq t_1$, $-\eta \leq \epsilon \leq \eta$, of curves joining P to Q. Then from $\mathcal{J}'(\epsilon) = 0$ it follows that $\mathcal{J}(\epsilon) =$ const. and the integrals along the two curves in question are equal. If

$$0 = \rho_i = \frac{\partial F}{\partial x_i} - \frac{\partial^2 F}{\partial x_i' \partial x_k} x_k' - \frac{\partial^2 F}{\partial x_i' \partial x_k'} x_k''$$

along all curves ρ_i must be identically zero in $x_i\, x_i'$ and x_i'' because one can always construct a curve having these values. Therefore first

$$\frac{\partial^2 F}{\partial x_i' \partial x_k'} = 0$$

and we easily see that because of homogeneity

(6.1) $$F(x, x') = A_k(x) x_k'$$

This gives

$$\rho_i = \frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial x_i'}$$

$$= \frac{\partial A_k}{\partial x_i} x_k' - \frac{d}{dt} A_i$$

$$= \left(\frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k} \right) x_k' = 0$$

for all x and x', and hence

(6.2) $$\frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k}$$

for all points of S. We thus see that the $\int_P^Q F(x, x')\, dt$, F positive homogeneous of first degree in x', is independent of the path from P to Q if and only if F(x, x') is of the form (6.1) and (6.2) holds for all points of S.

Consider a curve $x_i(t)$ of class C^2. We have along this curve

$$\frac{d}{dt} F(x, x') = \frac{\partial F}{\partial x_i} x_i' + \frac{\partial F}{\partial x_i'} x_i''$$

$$= \rho_i x_i + \frac{d}{dt}\left(x_i' \frac{\partial F}{\partial x_i'} \right)$$

and hence

(6.3) $$\rho_i x_i = \frac{d}{dt}\left(F - \frac{\partial F}{\partial x_i'} x_i'\right).$$

If $x_i(t)$ is an extremal ($\rho_i = 0$) then

(6.4) $$F - \frac{\partial F}{\partial x_i'} x_i' = \text{const.}$$

and an integral of the Euler equation is obtained. Notice that (6.4) holds regardless of homogeneity of F. Now let $F(x, x')$ be positive homogeneous of the k^{th} degree in x'. (6.2) becomes

(6.5) $$(1-k)\frac{dF}{dt} = \rho_i x_i$$

<u>If $k \neq 1$, $F(x, x') = \text{const.}$ is an integral of Euler's equation.</u> If $k = 1$, $\rho_i x_i' = 0$ for any curve and ρ_i is always a normal vector to the curve. It can be shown that in the case of a Riemannian space ρ_i is in the direction of the first normal to the curve.

Let us compare the variation problem for $F(x, \lambda)$ and F^k, $F(x, \lambda)$ being positive homogeneous of the first degree in λ. Let $\underset{(1)}{\rho_i}$ and $\underset{(k)}{\rho_i}$ be the Euler vectors for the F and the F^k problems respectively. Then

$$\underset{(k)}{\rho_i} = kF^{k-1}\underset{(1)}{\rho_i} - k(k-1)F^{k-2}\frac{\partial F}{\partial x_i'}\frac{dF}{dt}.$$

If $\underset{(k)}{\rho_i} = 0$, F is constant along the curve and $\frac{dF}{dt} = 0$, giving $\underset{(1)}{\rho_i} = 0$. If $\underset{(1)}{\rho_i} = 0$, arc length can be introduced as parameter making F a constant and $\frac{dF}{dt}$ zero along the curve, and then $\underset{(k)}{\rho_i} = 0$. Thus the extremals of the F^k problem are extremals of the F problem, and all extremals of the F problem can be so parameterized as to make them extremals of the F^k problem.

7. Take now $\mathcal{F} = F^2$ where $F(x, \lambda)$ is positive homogeneous of the first degree in λ.

$$0 = \rho_i = \frac{\partial \mathcal{F}}{\partial x_i} - \frac{\partial^2 \mathcal{F}}{\partial x_i' \partial x_k}\frac{dx_k}{dt} - \frac{\partial^2 \mathcal{F}}{\partial x_i' \partial x_k'}\frac{d^2 x_k}{dt^2}$$

and since $\left|\frac{\partial^2 \mathcal{F}}{\partial x_i' \partial x_k'}\right| \neq 0$ we can solve and get

$$(7.1) \qquad \frac{d^2 x_k}{dt^2} = H_k(x, x')$$

where, since $\frac{\partial^2 \mathcal{F}}{\partial x_i' \partial x_k'}$ is positive homogeneous of the 0^{th} degree in x',
$H_k(x, x')$ is positive homogeneous of the 2^{nd} degree in x'. (7.1) can be written as

$$(7.1') \qquad \begin{cases} \xi_k' = H_k(x, \xi) \\ x_k' = \xi_k \end{cases}$$

Either (7.1) or (7.1') will be called the normal form of the Euler equations.

At this point we will digress to introduce another form of Euler's equations also of importance. Under the same assumptions as above, set

$$(7.2) \qquad v_i = \frac{\partial F(x, x')}{\partial x_i'}$$

Since $\left|\frac{\partial^2 F}{\partial x_i' \partial x_k'}\right| \neq 0$ we can use the implicit function theorem on (7.2) and obtain

$$\frac{dx_i}{dt} = \mathcal{G}(x, v)$$

Now from the Euler equations follows

$$\frac{dv_i}{dt} = \left.\frac{\partial F(x, x')}{\partial x_i}\right]_{x' = \mathcal{G}(x, v)}$$

or

$$\frac{dv_i}{dt} = \frac{\partial F(x, \mathcal{G})}{\partial x_i} - \frac{\partial F}{\partial x_j'} \frac{\partial \mathcal{G}^j}{\partial x_i}.$$

$$(7.3a) \qquad \frac{dv_i}{dt} = \frac{\partial H(x, v)}{\partial x_i}$$

where

$$H(x, v) = F(x, \mathcal{G}) - v_i \mathcal{G}^i(x, v)$$

is positive homogeneous of the 2^{nd} degree in v_i. Again, we have

$$\frac{\partial H}{\partial v_i} = \frac{\partial F}{\partial x'_j}\frac{\partial g^j}{\partial v_i} - g^i - v_j\frac{\partial g^j}{\partial v_i} ,$$

(7.3b) $$\frac{\partial H}{\partial v_i} = + \frac{d x_i}{d t} .$$

(7.3) are the Jacobi-Hamilton equations.

We will return now to (7.1) or (7.1)', the normal form of Euler's equation. In order to derive this form we had to add the following assumption

(7.4) $$|\sigma_{ij}| \neq 0 \qquad \text{where} \qquad \sigma_{ij} = \frac{\partial^2 F^2}{\partial x'_i \partial x'_j} .$$

It is at this point that we will need the derivatives of F of the fourth order assumed in (4.1) in order that $H(x, \xi)$ be of class C^2 for the domain $x \varepsilon \mathcal{U}(x_0)$, where $\mathcal{U}(x_0)$ is any coordinate neighborhood of any point x_0, and ξ arbitrary except for $\xi \neq 0$. Then by a standard theorem on differential equations*

---

* Bolza, Vorlesungen über Variationsrechnung, p 168 ff.

---

there exists for a given x_0, $\xi_0 \neq 0$ and τ a solution element of (7.1)' of class C^2

(7.5) $$\left\{ \begin{aligned} x_i &= \phi_i(t, \tau, x_0, \xi_0) \\ \xi_i &= \frac{d}{dt}\phi_i(t, \tau, x_0, \xi_0) \end{aligned} \right. , \quad \tau - \eta \leq t \leq \tau + \eta ,$$

such that $x_{i_0} = \phi_i(\tau, \tau, x_0, \xi_0)$ and $\xi_{i_0} = \frac{d}{dt}\phi_i(\tau, \tau, x_0, \xi_0)$

with x_i, ξ_i always in the domain $x \varepsilon \mathcal{U}(x_0)$, $\xi \neq 0$. This solution element is uniquely determined by its initial values $\angle\, x_0, \xi_0, \tau$ in the sense that two such elements with the same initial values coincide in their common t interval. In fact the solution element is independent of τ in that the addition of a constant c to the parameter t does not change the fact that the element is a solution of the Euler equation, since $H(x, x')$ is independent of t, but merely changes the initial values to $x_0, \xi_0, \tau + c$.

Take the end point $\chi_i(\tau+\eta)$ of the above solution element and construct, using any coordinate neighborhood of that point, the solution element of (7.1)' with the initial values $\chi_i(\tau+\eta)$, $\xi_i(\tau+\eta)$

$$(7.6)\quad \begin{cases} \tilde{\chi}_i = \phi_i(t, \tau+\eta, \chi(\tau+\eta), \xi(\tau+\eta)) \\ \tilde{\xi}_i = \frac{d}{dt}\phi_i(t, \tau+\eta, \chi(\tau+\eta), \xi(\tau+\eta)) \end{cases}$$

with $\tau+\eta-\eta_1 \leq t \leq \tau+\eta+\eta_1$. Since (7.5) and (7.6) are both solutions of (7.1)' with the same values for $t=\tau+\eta$ they must coincide in their common t intervals; we call (7.6) the extension (7.5). We thus obtain a solution of (7.1)' which will be of class C^2 in t, τ, x_0 and ξ_0 if we substitute in (7.6) for $\chi(\tau+\eta)$, $\xi(\tau+\eta)$ their values obtained from (7.5). This solution does not depend on η, as can easily be shown by the uniqueness property of a solution element. We can obviously continue the solution to the left as well as to the right, and by repeating indefinitely both processes we obtain a solution of class C^2 in the variables $t, \tau, \chi_0, \xi_0 \neq 0$, defined for an always increasing t-interval, which we will later show to extend from $-\infty$ to $+\infty$

We now show that the extremal depends upon t and τ only in the form $t - \tau$. We know that $\phi_i(t-c, \tau, \chi_0, \xi_0)$, $\xi_i(t-c, \tau, \chi_0, \xi_0)$ is a solution of (7.1)' which, for $t = \tau + c$, has the initial values χ_0, ξ_0. Hence

$$\phi_i(t-c, \tau, \chi_0, \xi_0) \equiv \phi_i(t, \tau+c, x_0, \xi_0)$$

and setting $\tau = 0$ and $c = \tau$

$$\phi_i(t, \tau, \chi_0, \xi_0) \equiv \phi_i(t-\tau, 0, \chi_0, \xi_0) .$$

The only property of $H_i(x, \xi)$ used thus far is its independence of t. We can now write our extremals as

$$(7.7)\qquad \begin{cases} x_i(t) = \phi_i(t-t_P, x(P), \xi(P)) \\ \xi_i(t) = \frac{d}{dt}\phi_i(t-t_P, x(P), \xi(P)) \end{cases}$$

where

$$(7.71)\qquad t_P - \omega' \leq t \leq t_P + \omega$$

Evaluating (7.7) for the point Q with parameter t_Q, one obtains

$$x_i(Q) = \phi_i(t_Q - t_P, x(P), \xi(P))$$

$$\xi_i(Q) = \frac{d}{dt}\phi_i(t_Q - t_P, x(P), \xi(P))$$

Then (7.7) and

$$\begin{cases} x_i(t) = \phi_i(t-t_Q, x(Q), \xi(Q)) \\ \xi_i(t) = \frac{d}{dt}\phi_i(t-t_Q, x(Q), \xi(Q)) \end{cases}$$

will coincide in their common t interval since for $t = t_Q$ they assume the same values $x_i(Q)$, $\xi_i(Q)$; that is to say

$$(7.8)\qquad \phi_i(t-t_P, x(P), \xi(P)) \equiv \phi_i(t-t_Q, x(Q), \xi(Q))$$

for any t in the interval (7.71). This is the analytic expression of the fact that an extremal is defined by any point on it and the tangent vector at this point.

Now we prove the

THEOREM. Every extremal can be extended to infinite length. By this is meant that to any $s \gtrless 0$ there exists a point P on the extremal such that the arc length along the extremal from P_0 to P is equal to s.

Proof. We will show that there is no bound for the parameter t. Let us assume that the extremal is estended to the right for all the parameter values t less than a certain a. Then we will prove that it can be extended to parameter values $t \geqq a$. For simplicity let us set $t_P = 0$. The equation for this extremal then takes the form

$$\begin{cases} x_i(t) = \phi_i(t, x(P), \xi(P)) \\ \xi_i(t) = \frac{d}{dt}\phi_i(t, x(P), \xi(P)) \end{cases} \qquad 0 \leqq t < a$$

From (7.8) we have for t_s in the t interval

$$\phi_i(t, x(P), \xi(P)) \equiv \phi_i(t-t_s, x(S), \xi(S))$$

Take now a sequence t_n in the t interval converging to a. The corresponding points $S_n = x(t_n)$ will have an accumulation point for, since $F(x, x') = k$ along the extremal,

$$\overline{PS_n} \leqq \int_P^{S_n} F(x,x')dt = k t_n < ka$$

and the points S_n satisfy assumption (3.1). It might be noted that this assumption is stronger than is necessary in this case; completeness of the space would suffice. Let R be an accumulation point, and let S_n now be a subsequence converging to R. Take a coordinate neighborhood $\mathcal{U}(R)$ of R; then for some N, and $n > N$, all S_n will lie in $\mathcal{U}(R)$. In $\mathcal{U}(R)$ form $x_i'(S_n)$, the tangent vectors to the extremal at S_n. Pick another subsequence of points, again denoted by S_n, for which $x_i'(S_n) \to \xi_i$. It is easily shown by use of the lemma on p. 4 that ξ_i is finite. Since $F(x(S_n)x'(S_n)) = k$, we have in the limit $F(x(R), \xi) = k$, showing that ξ is not a null vector but lies in the domain for which $H_i(x, \xi)$ is of class C^2. The extremal

$$x_i(t) = \phi_i(t, x(R), \xi), \quad -\eta \leqq t \leqq \eta,$$

can be constructed. It is of class C^2 and assumes the initial values $x(R), \xi$ for $t = 0$.

We have yet to show that this extremal element actually is a continuation of the extremal. To do this we will show that

$$\phi_i(a-\sigma, x(P), \xi(P)) \equiv \phi_i(-\sigma, x(R), \xi(R))$$

for σ sufficiently small. Now

$$\phi_i(t, x(P), \xi(P)) \equiv \phi_i(t-t_n, x(S_n), \xi(S_n)) \quad , 0 \leqq t < a.$$

Set $t = t_n - \sigma$ where $\eta > \sigma > 0$, and σ is so small that $t_n - \sigma$ lies in the $0 \leqq t < a$ interval. Then

$$\phi_i(t_n-\sigma, x(P), \xi(P)) \equiv \phi_i(-\sigma, x(S_n), \xi(S_n))$$

and as $n \to \infty$ we get by continuity

$$\phi_i(a-\sigma, x(P), \xi(P)) \equiv \phi_i(-\sigma, x(R), \xi(R))$$

completing the proof that an extremal can be extended to infinite parameter t and hence to infinite arc length s since $s = \int_0^t F(x, x')dt = k t$, $k > 0$. It is obvious then that it also can be extended to negative infinity.

We will show now that an extremal is determined uniquely by a given <u>direction</u> at a given point, that is to say that all extremals with the initial directions $\alpha \xi_{oi}$, $\alpha > 0$, at x_o coincide as point sets. Let the extremal with initial values x_o, x'_o be written in the form

$$x_i = x_i(t)$$

and make the change of parameter

$$t = c(\tau - \tau_o) + t_o \quad , c > 0.$$

Then

$$x_i(t) = x_i(c(\tau-\tau_o)+t_o) = \bar{x}_i(\tau),$$
$$\bar{x}_i'(\tau) = x_i'(t)\frac{dt}{d\tau} = c\, x_i'(t),$$
$$\bar{x}_i''(\tau) = c\, x_i''(t)\frac{dt}{d\tau} = c^2 x_i''(t) \quad \text{and}$$
$$\bar{x}_i''(\tau) - H_i(\bar{x}(\tau), \bar{x}'(\tau)) =$$
$$c^2\left[x_i''(t) - H_i(x(t), x'(t))\right] \quad ;$$

then if $x_i(t)$ is an extremal, $\tilde{x}_i(\tau)$ is also. But $\tilde{x}(\tau_0) = x_0$, $\tilde{x}'(\tau_0) = c x_0'$, so the initial values (x_0, cx_0'), for c an arbitrary positive constant, give the same extremal with only a change of parameterization.

8. In this section we will consider the question of the existence of normal coordinates. First a series of auxiliary equations will be derived. Take an extremal

$$x_i(s) = \phi_i(s, s_0, x_0, x_0')$$

and make the following change of parameter

$$s = a\bar{s} + b, \qquad a > 0$$

Then

$$\frac{1}{a}\frac{dx_i}{d\bar{s}} = \frac{dx_i}{ds} \qquad \text{and}$$

$$x_i(a\bar{s}+b) = \phi_i\left(a\bar{s}+b, a\bar{s}_0+b, x_0, \frac{1}{a}\left(\frac{dx}{d\bar{s}}\right)_0\right)$$

$$\equiv \phi_i\left(\bar{s}, \bar{s}_0, x_0, \frac{1}{a}\left(\frac{dx}{d\bar{s}}\right)_0\right)$$

since the extremal is uniquely determined by the initial values $\left(x_0, \frac{1}{a}\left(\frac{dx}{d\bar{s}}\right)_0\right)$ for $\bar{s} = \bar{s}_0$. Setting $a\bar{s}_0 + b = 0$ and dropping the bars gives

$$\phi_i\left(a(s-s_0), 0, x_0, \frac{1}{a}\left(\frac{dx}{ds}\right)_0\right) \equiv \phi_i\left(s, s_0, x_0, \left(\frac{dx}{ds}\right)_0\right), \quad a > 0. \tag{8.1}$$

By setting $a = \frac{1}{s-s_0}$ for $s > s_0$ and $a = \frac{1}{s_0-s}$ for $s_0 > s$, one obtains

$$\begin{cases} \phi_i\left(s, s_0, x_0, \left(\frac{dx}{ds}\right)_0\right) \equiv \phi_i\left(1, 0, x_0, (s-s_0)\left(\frac{dx}{ds}\right)_0\right) & s > s_0 \\ \phi_i\left(s, s_0, x_0, \left(\frac{dx}{ds}\right)_0\right) \equiv \phi_i\left(-1, 0, x_0, -(s-s_0)\left(\frac{dx}{ds}\right)_0\right) & s < s_0 \end{cases} \tag{8.2}$$

The first half of (8.2) is the expression for an extremal issuing from P_0; the second half that of an extremal running into P_0. If the distance function is symmetric, the two expressions are the same. Both equations (8.2) hold also for $s = s_0$, because, as later is shown, $\phi_i(s, s_0, x_0, x_0')$ is also continuous for a null vector $(x_0, 0)$. The second half of (8.2) could be derived from the first half and the fact that $\phi_i\left(-\tilde{s}, -\tilde{s}_0, x_0, -\left(\frac{dx}{d\tilde{s}}\right)_0\right)$ is an extremal of the variation problem for $F(x, -x')$ as a metric function. We first have by introducing $\tilde{s} = -s$,

$$\phi_i\left(-\tilde{s}, -\tilde{s}_0, x_0, -\left(\frac{dx}{d\tilde{s}}\right)_0\right) \equiv \phi_i\left(s, s_0, x_0, \left(\frac{dx}{ds}\right)_0\right)$$

and then by the first half of (8.2)

$$\phi_i\left(-\tilde{s}, -\tilde{s}_0, x_0, -\left(\frac{dx}{d\tilde{s}}\right)_0\right) \equiv \phi_i\left(-1, 0, x_0, -(\tilde{s}-\tilde{s}_0)\left(\frac{dx}{d\tilde{s}}\right)_0\right)$$
$$\equiv \phi_i\left(-1, 0, x_0, -(s-s_0)\left(\frac{dx}{ds}\right)_0\right).$$

For convenience we can set

(8.2)' $$\phi_i\left(1, 0, x_0, (s-s_0)\left(\frac{dx}{ds}\right)_0\right) \equiv \phi_i\left(x_0, (s-s_0)\left(\frac{dx}{ds}\right)_0\right), \quad s > s_0, \text{ and}$$

(8.2)" $$\phi_i\left(-1, 0, x_0, -(s-s_0)\left(\frac{dx}{ds}\right)_0\right) \equiv \psi_i\left(x_0, (s-s_0)\left(\frac{dx}{ds}\right)_0\right), \quad s < s_0.$$

If one sets

$$(s-s_0)\left(\frac{dx_i}{ds}\right)_0 = y_i, \quad i = 1, \cdots, n, \qquad s > s_0,$$

the equations of an extremal issuing from P_0 become

(8.3) $$x_i = \phi_i(x_0, y).$$

The next task is to show that (8.3), for a fixed x_0, can be solved for the y's in terms of the x's, and (8.3) can hence be considered as a coordinate transformation in neighborhoods of $x = x_0$ and $y = 0$ respectively. The y's will be called the <u>normal coordinates with P_0 as origin</u>. (8.3) can be solved for the y's in a neighborhood of the point $y_0 = 0$ if ϕ_i are of class C^1 and the Jacobian is different from zero at the point $y_0 = 0$, as we will now show.

In (4.1) derivatives up to those of the fourth order are assumed for $F(x, x')$ at all points (x, x') except $(x, 0)$. Therefore $H_i(x, x')$ has continuous first and second derivatives $\frac{\partial H_i}{\partial x_j}$, $\frac{\partial H_i}{\partial x'_j}$, $\frac{\partial^2 H_i}{\partial x_j \partial x_k}$, $\frac{\partial^2 H_i}{\partial x_j \partial x'_k}$ and $\frac{\partial^2 H_i}{\partial x'_j \partial x'_k}$ at all points except $(x, 0)$. But since $H_i(x, x')$ is positive homogeneous of the second degree in x', $\frac{\partial H_i}{\partial x_j}$ and $\frac{\partial H_i}{\partial x'_j}$ are positive momogeneous of the second and first degrees respectively in x'. Therefore H_i, $\frac{\partial H_i}{\partial x_j}$ and $\frac{\partial H_i}{\partial x'_j}$ are, as is easily shown, automatically continuous at $(x, 0)$ and vanish at these points. Then since

$$\lim_{\Delta_i \to 0} \frac{H_j(\cdots x_i+\Delta_i, \cdots, 0) - H_j(\cdots x_i \cdots, 0)}{\Delta_i} = 0 = \frac{\partial H_j(x, 0)}{\partial x_i} \quad \text{and}$$

$$\lim_{\Delta_i \to 0} \frac{H_j(x, 0, \cdots, \Delta_i, \cdots 0) - H_j(x, 0)}{\Delta_i} = \lim_{\Delta_i \to 0} \frac{\Delta_i^2 [H_j(x, 0, \cdot 1 \cdot 0) - H(x, 0)]}{\Delta_i}$$

$$= 0 = \frac{\partial H_j(x, 0)}{\partial x'_i}$$

$H_i(x, x')$ is of class C' without any restriction as to the points $(x, 0)$ (and indeed of class C^2 except for the points $(x, 0)$). As a consequence the left side of equations (8.2) is of class C^1 in all of its arguments. By definition $\phi_i(x_0, y) = \phi_i(1, 0, x_0, y)$. But we have shown that $\phi_i(s, s_0, x_0, y)$ is of class C^1 for all of its arguments. Hence $\phi_i(x_0, y)$ is of class C^1 without restriction. We might note that $\phi_i(x_0, y)$ is of class C^2 <u>except, in general, at the points $(x_0, 0)$</u>.

Next it must be shown that the Jacobian of (8.3) does not vanish at the point $y = 0$. Consider the equation of the extremal, $x_i = \phi_i(x_0, \xi t)$, and differentiate with regard to t, getting

$$\frac{d\chi_i}{dt} = \frac{\partial \phi_i(\chi_0, y)}{\partial y_k} \cdot \xi_k$$

on the extremal. Going on the extremal to the limiting point x_0 gives

$$\xi_i = \frac{\partial \phi_i(\chi_0, 0)}{\partial y_k} \cdot \xi_k$$

for all ξ_k. Therefore

$$\frac{\partial \phi_i(\chi_0, 0)}{\partial y_k} = \delta^i_k$$

and the Jacobian is equal to 1 at the points $(x_0, 0)$.

We are now able to use the theorem of Dini to solve $x_i = \phi_i(x_0, y)$ in a neighborhood of $y = 0$, getting

$$y_i = \Phi_i(\chi_0, \chi) \tag{8.5}$$

in a corresponding neighborhood of x_0, where Φ_i is of class C^1. (8.3) sets up a correspondence of some point P of our space to every y, but not necessarily a one-to-one correspondence. The above use of the theorem of Dini states that there exists a neighborhood $\mathcal{U}(y_0)$ of $y_0 = 0$ and a neighborhood $\mathcal{U}(x_0)$ of P_0 such that there is a one-to-one correspondence between the points of $\mathcal{U}(y_0)$ and those of $\mathcal{U}(x_0)$ defined by (8.3) or (8.5). The y's so defined in $\mathcal{U}(y_0)$ are called normal coordinates with P_0 as origin. $\phi_i(x_0, y)$ and $\Phi_i(x_0, x)$ are both of class C^1, and since $\phi_i(x_0, y)$ is of class C^2 except for points $(x_0, 0)$ it can be easily verified that $\Phi_i(x_0, x)$ is of class C^2 except for points (x_0, x).

If we have now $\chi_i = \phi_i\left(\chi_0, \left(\frac{d\chi}{dt}\right)_0 t\right)$ then $y_i = \left(\frac{d\chi}{dt}\right)_0 t$ are the normal coordinates at the point P(x). Let us make the change of coordinates $\tilde{x}_i = \tilde{x}_i(x_1, \ldots, x_n)$ in $\mathcal{U}(x_0)$. Then

$$\left(\frac{d\tilde{\chi}}{dt}\right)_0 = \left(\frac{\partial \tilde{\chi}_i}{\partial \chi_k}\right)_0 \left(\frac{d\chi_k}{dt}\right)_0$$

and we have for the new normal coordinates, $\tilde{y}_i = \left(\frac{d\tilde{x}_i}{dt}\right)_0 t$,

$$\tilde{y}_i = \left(\frac{\partial \tilde{x}_i}{\partial x_K}\right)_0 y_K$$

and the normal coordinates transform like a contravariant vector at P_0 under a transformation of the general coordinates. In general, we will call any coordinate system in which the extremals issuing from (or running into) a given point P_0 have the linear form $y_i = \xi_i t$ a normal coordinate system with P_0 as origin.

9. Let us write (8.3) as $P = P(y)$, emphasizing the fact that it is a continuous mapping of the y-space on points of our manifold independent of the coordinates. This mapping is not necessarily one-to-one and may not cover all of our manifold, but we have shown the existence of a normal coordinate neighborhood, that is a neighborhood $\mathcal{U}(y_0)$ of $y_0 = 0$ in the y-space which is in one-to-one continuous correspondence with a neighborhood $\mathcal{U}(P_0)$ in our manifold, such that y maps on $P(y)$.

$P = P(ty)$, $0 \leq t \leq 1$, is the equation of an extremal g connecting P_0 to $P_1 = P(y)$. If $F(x, \frac{dx}{dt}) = k$ along g, then for arc length s from P_0 to P_1 on g we have ($y = \left(\frac{dx}{dt}\right)_0$)

$$s = \int_0^1 F(x, x')\,dt = k = F(x_0, y). \tag{9.1}$$

Let $\mathcal{W}_\rho(P_0)$ denote the point set each point P of which is connected to P_0 by at least one extremal arc of length less than ρ. Then all points of $\mathcal{W}_\rho(P_0)$ are described, although not uniquely, as $P = P(y)$. Using (9.1) we see that if $P \in \mathcal{W}_\rho(P_0)$ there exists at least one set $y_1, \ldots, y_n$ such that $P = P(y)$ and $F(x_0, y) < \rho$, and conversely, showing a correspondence of $y \in \mathcal{W}_\rho(y_0)$ and $P(y) \in \mathcal{W}_\rho(P_0)$ if we denote the y point set $F(x_0, y) < \rho$ by $\mathcal{W}_\rho(y_0)$. We now show that in any neighborhood $\mathcal{U}(y_0)$ of the origin ($y_0 = 0$)

of the y space there lies a $\mathcal{W}_\rho(y_o), \rho > 0$. Suppose this were not so. Then there would exist a set $(y_i)_n$ of elements of the complement, $C(\mathcal{U}(y_o))$, of $\mathcal{U}(y_o)$ such that $F(x_o, y_n) = \sigma_n < s_n$, where $s_n \to 0$ as $n \to \infty$. Then application of (2.6) gives

$$B \max y_n \leq F(x_o, y_n) = \sigma_n < s_n$$

and hence $y_n \to 0$ as $s_n \to 0$. But a contradiction is now apparent since, on the one hand $C(\mathcal{U}(y_o))$ is closed and contains $\lim_{n\to\infty} y_n$, while, on the other hand, $\lim_{n\to\infty} y_n = 0$ and is in $\mathcal{U}(y_o)$. Therefore for every neighborhood $\mathcal{U}(y_o)$ there exists a $\mathcal{W}_\rho(y_o), \rho > 0$, such that $\mathcal{W}_\rho(y_o) \subset \mathcal{U}(y_o)$. Since the complement, $C(\mathcal{W}_\rho(y_o))$ of $\mathcal{W}_\rho(y_o)$ is given by $F(x_o, y) \geq \rho$, and $F(x_o, y)$ is continuous, $C(\mathcal{W}_\rho(y_o))$ is closed and hence $\mathcal{W}_\rho(y_o)$ is open.

We thus see that in every $\mathcal{U}(y_o)$ there lies an open point set $\mathcal{W}_\rho(y_o)$. By application of the correspondence found above between points of neighborhoods of the y-space and our manifold $(\mathcal{U}(y_o) \leftrightarrow \mathcal{U}(P_o))$ the following immediate theorem is obtained.

THEOREM. The set $\mathcal{W}_\rho(P_o)$, for sufficiently small ρ, form an equivalent set of neighborhoods to $\mathcal{U}(P_o)$, i.e. the $\mathcal{W}_\rho(P_o)$, for sufficiently small ρ, are open point sets, and to every $\mathcal{U}(P_o)$ there exists a $\mathcal{W}_\rho(P_o)$ lying within it. This $\mathcal{W}_\rho(P_o)$ is the set corresponding to $\mathcal{W}_\rho(y_o)$

$\mathcal{W}_\rho(P_o)$, for sufficiently small ρ, has the property that if $P \in \mathcal{W}_\rho(P_o)$ there is <u>one and only one</u> extremal of arc length less than ρ connecting P_o to P. At least one extremal exists by definition. If more than one exists there would exist more than one (x_o, y) in $\mathcal{W}_\rho(y_o)$ corresponding to P in $\mathcal{W}_\rho(P_o)$, since only to y of $\mathcal{W}_\rho(y_o)$ corresponds an extremal of arc length $F(x_o, y) < \rho$ joining P_o to P. We might note that if $P \in \mathcal{W}_\rho(P_o)$ there exists an extremal connecting P_o to P of arc length less than ρ. Then $\overline{P_0 P} < \rho$ and $P \in \mathcal{U}_\rho(P_0)$. That is

$$\mathcal{W}_\rho(P_0) \subset \mathcal{U}_\rho(P_0).$$

(8.3) can also be written as

$$P = \Phi(P_0, Y), \quad P_0 = P_0, \tag{9.2}$$

giving a continuous mapping of the product space $\{P\} \times \{Y\}$, where $\{P\}$ denotes our manifold and $\{Y\}$ the n-dimensional number space $(y_1, \ldots, y_n)$, onto the product space $\{P\} \times \{P\}$. The continuity of the above statement is with respect to the neighborhoods of a topological product space, which are the products of the neighborhoods of the component spaces. To the point $(\overline{P}_0, Y_0)$ of $\{P\} \times \{Y\}$, where $Y_0 \equiv 0$ is the origin, $y_1 = 0, \ldots, y_n = 0$, of the y-space, corresponds the point $(\overline{P}_0, \overline{P}_0)$ of $\{P\} \times \{P\}$. Hence there exists, corresponding to the neighborhood $\mathcal{U}(\overline{P}_0) \times \mathcal{U}(\overline{P}_0)$ of $(\overline{P}_0, \overline{P}_0)$, a neighborhood $\mathcal{U}'(\overline{P}_0) \times \mathcal{U}(y_0)$, $(y_0 \equiv 0)$, of $(\overline{P}_0, y_0)$ whose map under (9.2) lies in $\mathcal{U}(\overline{P}_0) \times \mathcal{U}(\overline{P}_0)$:

$$\text{map of } \mathcal{U}'(\overline{P}_0) \times \mathcal{U}(y_0) \subset \mathcal{U}(\overline{P}_0) \times \mathcal{U}(\overline{P}_0). \tag{9.3}$$

Take now a coordinate neighborhood $\mathcal{U}(\overline{x}_0)$ of $\overline{x}_0$, the coordinates of $\overline{P}_0$. From (9.2) and (9.3) we have that the x_{0i} and x_i defined by

$$x_{0i} = x_{0i}, \quad x_i = \phi_i(x_0, y) \tag{9.4}$$

lie in $\mathcal{U}(\overline{x}_0)$ if $x_0 \in \mathcal{U}'(\overline{x}_0)$ and $y \in \mathcal{U}(y_0)$. The functions in (9.4) are of class C^1, and the Jacobian is equal to 1 at $(\overline{x}_0, y_0)$. Hence there exist a neighborhood $\mathcal{U}(\overline{x}_0, y_0)$ and a neighborhood $\mathcal{U}(\overline{x}_0, \overline{x}_0)$ which are in one-to-one correspondence. Indeed $\mathcal{U}(\overline{x}_0, \overline{x}_0)$ can be taken as a cubic neighborhood $(\overline{x}_0)_a \times (\overline{x}_0)_a$, where $(\overline{x}_0)_a$ is $|x_i - \overline{x}_{0i}| < a$, $i = 1, \ldots, n$. Thus to every $x_0 \in (\overline{x}_0)_a$, $x \in (\overline{x}_0)_a$ there corresponds one and only one pair (x_0, y) in $\mathcal{U}(\overline{x}_0, y_0)$ such that $x_i = \phi_i(x_0, y)$, and the extremal $x_i = \phi_i(x_0, yt)$, $0 \leq t < \infty$, joins $x_{0i}(t = 0)$ to $x_i(t = 1)$. This proves the

THEOREM: There exists a neighborhood $(\overline{x}_o)_a$ of $\overline{x}_o$ such that any two of its points can be joined by an extremal arc, and by only one whose corresponding $(x_o, y) \varepsilon \mathcal{U}(\overline{x}_o, y_o)$.

Since $(\overline{x}_o, y_o)$ is an inner point of $\mathcal{U}(\overline{x}_o, y_o)$ there exist a cubic neighborhood $|x_i - \overline{x}_{oi}| < b < a$ of $\overline{x}_o$, denoted by $(\overline{x}_o)_b$, such that for all x in the closure of $(\overline{x}_o)_b$, (x, y_o) lies in $\mathcal{U}(\overline{x}_o, y_o)$. Let $\mathcal{J}$ be the intersection of the closed point set (x_o, y), where $x_o \varepsilon (\overline{x}_o)_b$ and y is arbitrary, and the closed compact point set $\overline{\mathcal{U}(\overline{x}_o, y_o)} - \mathcal{U}(\overline{x}_o, y_o)$, the boundary of $\mathcal{U}(\overline{x}_o, y_o)$. $\mathcal{J}$ is obviously closed and compact; it is non-vacuous since any curve (x_o, yt) $x_o \varepsilon (\overline{x}_o)_b$, y arbitrary, $0 \leqq t < \infty$, is a non-compact point set, and hence cannot lie entirely in the compact point set $\mathcal{U}(\overline{x}_o, y_o)$ but must cross its boundary. Let ρ_1 be the maximum and ρ the minimum, $\rho_1 \geqq \rho$, of $F(x_o, y)$ over the set $\mathcal{J}$. We recall that $F(x_o, y)$ is the arc-length of the extremal arc $x_i = \phi_i(x_o, yt)$, $0 \leqq t \leqq 1$. $\rho > 0$; for $F(x_o, y) = 0$ if and only if $y = 0$, but it has been shown above that $(x_o, y_o = 0)$ are inner points of $\mathcal{U}(\overline{x}_o, y_o)$ and hence not in $\mathcal{J}$. We thus have

$$\rho_1 \geq F(x_o, y) \geq \rho > 0 \quad \text{on } \mathcal{J}. \tag{9.5}$$

If $F(x_o, y) < \rho$ for some (x_o, y), $x_o \varepsilon \overline{(\overline{x}_o)_b}$, then $(x_o, y) \varepsilon \mathcal{U}(\overline{x}_o, y_o)$, for otherwise there would be some (x_o, y) in $C(\mathcal{U}(\overline{x}_o, y_o))$ for which $F(x_o, y) < \rho$. Consider the continuous curve (x_o, yt), $0 \leqq t \leqq 1$, running from an interior to an exterior point of $\mathcal{U}(\overline{x}_o, y_o)$. Then for some $\tau < 1$ $(x_o, y\tau)$ would lie on the boundary of $\mathcal{U}(\overline{x}_o, y_o)$, and hence on $\mathcal{J}$; then

$$F(x_o, y\tau) = \tau F(x_o, y) < \tau\rho < \rho, \quad \tau < 1,$$

showing a contradiction. Hence all points (x_o, y) such that $x_o \varepsilon \overline{(\overline{x}_o)_b}$ and $F(x_o, y) < \rho$ are points of $\mathcal{U}(\overline{x}_o, y_o)$.

THEOREM: If x_o and x are elements of $\overline{(\overline{x}_o)_b}$ then at most one extremal arc of length $< \rho$ joins x_o to X. For to every $(x_o, x) \varepsilon \overline{(\overline{x}_o)_b} \times \overline{(\overline{x}_o)_b}$ corresponds one and only one pair $(x_0, y) \in \mathcal{U}(\bar{x}_0, y_0)$, and for every pair (x_0, y) not in $\mathcal{U}(\bar{x}_0, y_0)$, $F(x_0, y) \geqq \rho$ as shown above. Hence at most one pair (x_0, y) can exist for which $F(x_0, y) < \rho$ and (x_0, y) corresponds to (x_0, x). No point $(x_0, y) \in \mathcal{U}(\bar{x}_0, y_0)$, $x_0 \in (\bar{x}_0)_b$, exists such that $F(x_0, y) > \rho_1$. For if such a point (x_0, y) exists a value $\tau > 1$ can be found for which the extremal (x_0, yt), $0 \leqq t \leqq \infty$, intersects $\mathcal{J}$ and for which $F(x_0, yt) =$ $= \tau F(x_0, y) > \tau \rho_1 > \rho_1$, giving a contradiction. The combination of the two above underlined statements gives the following important

THEOREM: For any $\bar{x}_0$ there exists a neighborhood $(\bar{x}_0)_b$ and two numbers $\rho_1 \geqq \rho > 0$ such that any two points x_0 and x of $(\bar{x}_0)_b$ can always be joined by an extremal arc of length $< \rho_1$ and by at most one of length $< \rho$.

10. We will now prove the following Theorem of Bolza which we will need.

THEOREM:

(1) Let $\mathcal{P}(Y)$ be a metric space and $\mathcal{P}(X)$ be a topological space. Let $X = f(Y)$ be a continuous mapping of $\mathcal{P}(Y)$ on $\mathcal{P}(X)$.

(2) Let C be a closed compact subset of $\mathcal{P}(Y)$ which is in one-to-one correspondence with its map $f(C)$.

(3) Let there exist corresponding to each $Y \in C$ a neighborhood $\mathcal{U}(Y)$ which is in a one-to-one correspondence with its map $f(\mathcal{U}(Y))$.

Then there exists a $\mathcal{U}_\rho(C)$ which is in one-to-one correspondence with its map $f(\mathcal{U}_\rho(C))$. By $\mathcal{U}_\rho(C)$ is meant the set of points P such that there exist points P' of C such that $\overline{PP'} < \rho$; i.e. $\mathcal{U}_\rho(C) = \sum_{Y'} \mathcal{U}_\rho(Y')$, $Y' \in C$.

If the theorem were false there would exist a monotonically decreasing sequence $\rho_n \to 0$ such that there exist two different points Y_n' and Y_n'', elements of $\mathcal{U}_{\rho_n}(C)$, and having the same image, i.e.

(a) $$Y'_n \text{ and } Y''_n \in \mathcal{U}_{\rho_n}(C),$$

(b) $$Y'_n \neq Y''_n ,$$

(c) $$f(Y'_n) = f(Y''_n).$$

From (a) it follows that there exist two points Z'_n and Z''_n both in C and such that

$$\overline{Y'_n Z'_n} < \rho_n , \quad \overline{Y''_n Z''_n} < \rho_n .$$

Since C is compact there exists a subsequence of Z'_n, again called Z'_n, which is convergent to Z'. Take a subsequence of this such that the corresponding Z''_n converge to a Z''. Since C is closed $Z', Z'' \in C$. $Y'_n \to Z'$ and $Y''_n \to Z''$, as can be seen by use of the triangle inequality, and since $f(Y)$ is continuous, $f(Z') = f(Z'')$ by (c). Therefore $Z' = Z''$ by assumption (2) since both are elements of C. But this contradicts assumption (3), because there would exist in each neighborhood of $Z' = Z''$ different points Y'_n and Y''_n with the same map. Thus the theorem is proved.

We make the following application of the Theorem of Bolza. Let $C = \overline{\mathcal{N}_\rho(y_0)}$ have the following two properties:

(α) $$\overline{\mathcal{N}_\rho(P_0)} \longleftrightarrow \overline{\mathcal{N}_\rho(y_0)} ,$$

(β) $$\left|\frac{\partial x}{\partial y}\right| \neq 0 \quad \text{for } y \in \overline{\mathcal{N}_\rho(y_0)} .$$

Then by the theorem there exists some neighborhood $\mathcal{U}(C)$ of C such that $\mathcal{U}(C)$ (the $\mathcal{U}_\rho(C)$ in the Theorem of Bolza) is also in a one-to-one correspondence with its map.

<u>We now show that a $\mathcal{N}_{\rho'}(y_0)$, $\rho' > \rho$, lies in $\mathcal{U}(C)$ such that properties (α) and (β) hold for $\mathcal{N}_{\rho'}(y_0)$;</u> we will call this "extending" $\mathcal{N}_\rho(y_0)$. Let $\rho_n \to \rho$ be a monotonically decreasing sequence. Then if the statement is true (1st) $\mathcal{N}_{\rho_n}(y_0) \subset \mathcal{U}(C)$, and (2nd) $\left|\frac{\partial x}{\partial y}\right| \neq 0$ for

$y \in \mathcal{V}_{\rho_n}(y_0)$ for all n greater than some N. If the first were false there would exist a sequence $y_n \in \mathcal{V}_{\rho_n}(y_0)$ and $y_n \in C(\mathcal{U}(C))$. Hence $\rho < F(x_0, y_n) < \rho_n$. From this and the lemma on page 4 we obtain

$$\rho_n > F(x_0, y_n) \geq B \max |y_n|,$$

and hence the y_n are bounded and have some convergent subsequence again denoted by y_n; $y_n \to y$. Then from our first inequality follows $F(x_0, y) = \rho$, and hence $y \in \overline{\mathcal{V}_\rho(y_0)} = C$. But $y = \lim_{n\to\infty} y_n$ also lies in the closed complement of $\mathcal{U}(C)$, and a contradiction is reached. If (2^{nd}) $\left|\frac{\partial x}{\partial y}\right|$ were not different from zero for $y \in \mathcal{V}_{\rho_n}(y_0)$ for all n greater than some N, there would exist a sequence y_n such that $F(x_0, y_n) < \rho_n$ but $\left|\frac{\partial \phi(x_0, y_n)}{\partial y}\right| = 0$. As before, we get a subsequence $y_n \to y$, and for this y we would have $F(x_0, y) \leqq \rho$ and $\left|\frac{\partial \phi(x_0, y)}{\partial y}\right| = 0$, in contradiction to ($\beta$). We thus see that if properties (α) and (β) hold for the closure of some $\mathcal{V}_\rho(y_0)$ they hold equally well for a $\mathcal{V}_\sigma(y_0)$, $\sigma > \rho$. It is obvious that if properties (α) and (β) hold for $\mathcal{V}_\rho(y_0)$ they hold for $\mathcal{V}_\tau(y_0)$, $\tau < \rho$. A division of all real numbers into two classes A and B is thus obtained with a number ρ in A or B according to whether (α) and (β) hold for $\mathcal{V}_\rho(y_0)$ or not. A number $P = P(\mathcal{P}_0)$ is thus defined by this Dedekind cut, which, by means of the above investigation, belongs to class B. $P = P(\mathcal{P}_0)$ is a function of the point $\mathcal{P}_0$. $\mathcal{V}_{P(\mathcal{P}_0)}(y_0)$ possesses the following properties:

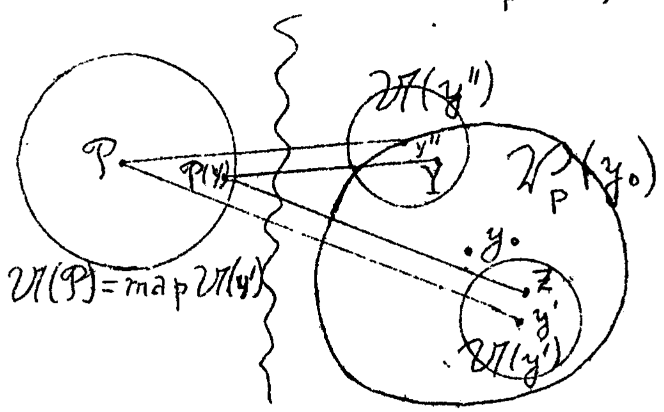
$$(\alpha') \qquad \mathcal{V}_P(y_0) \longleftrightarrow \mathcal{V}_P(\mathcal{P}_0),$$

$$(\beta') \qquad \left|\frac{\partial x}{\partial y}\right| \neq 0 \quad \text{for } y \in \mathcal{V}_P(y_0).$$

If (α') were to fail there would exist $y', y'' \in \mathcal{W}_{\rho(\mathcal{P}_0)}(y_0)$, $y' \neq y''$ and

$\mathcal{P}(y') = \mathcal{P}(y'')$; but y' and y'' lie in some $\mathcal{W}_\sigma(y_0)$, $\sigma < \rho$, for which (α) holds. If (β') were to fail for some y, y would lie in some $\mathcal{W}_\sigma(y_0)$, $\sigma < \rho$, for which (β) holds. Since (α) and (β) do not both hold for $\mathcal{W}_{\rho(\mathcal{P}_0)}(y_0)$ either $\left|\frac{\partial \phi}{\partial y}\right| = 0$ at some point $\mathcal{P}$ (called a conjugate point to $\mathcal{P}_0$ on the extremal $\overline{\mathcal{P}_0\mathcal{P}}$) on the boundary $\overline{\mathcal{W}_\rho(y_0)} - \mathcal{W}_\rho(y_0)$, or there exists a y'' on the boundary and a y' in $\mathcal{W}_\rho(y_0)$ such that $\mathcal{P}(y') = \mathcal{P}(y'')$. In the second case it can be shown that y' is on the boundary too. Assume $y' \in \mathcal{W}_\rho(y_0)$ There exists a neighborhood $\mathcal{U}(y') \subset \mathcal{W}_\rho(y_0)$ in one-to-one correspondence with its map $\mathcal{U}(\mathcal{P})$,

$$\left|\frac{\partial \phi(x_0, y')}{\partial y}\right| \neq 0.$$



$\mathcal{U}(y')$ may be taken so small that $y'' \notin \overline{\mathcal{U}(y')}$. Then there exists a neighborhood $\mathcal{U}(y'')$ of y'' such that $\mathcal{U}(y') \cap \mathcal{U}(y'') = 0$ and ($\mathcal{P}(y)$ being continuous) the map of $\mathcal{U}(y'')$ is contained in $\mathcal{U}(\mathcal{P})$. Let Y be a point in the intersection $\mathcal{U}(y'') \cap \mathcal{W}_\rho(y_0)$. To Y corresponds its map $\mathcal{P}(Y)$ in $\mathcal{U}(\mathcal{P})$ and to $\mathcal{P}(Y)$ corresponds a point $Z \in \mathcal{U}(y')$. But this gives two different points of $\mathcal{W}_\rho(y_0)$ with the same map $\mathcal{P}(Y)$, contradicting (α'). We can collect the above results into a

THEOREM: <u>Either there exists a conjugate point on the boundary $\overline{\mathcal{W}_\rho(y_0)} - \mathcal{W}_\rho(y_0)$ of $\mathcal{W}_\rho(y_0)$ or there exist two different points on the boundary with identical maps</u>, i.e. there exist two extremals of equal length ρ connecting $\mathcal{P}_0$ with $\mathcal{P}(y') = \mathcal{P}(y'')$. <u>Such extremals will be called double extremals.</u>

THEOREM: <u>Let $\mathcal{M}$ be a compact subset of our manifold, and let $\underline{P}$ be the greatest lower bound of $P(\mathcal{P}),\ \mathcal{P}\in\mathcal{M}$; then $\underline{P} > 0$.</u>

Proof. First we show that the greatest lower bound Σ of the lengths of double extremals issuing from $\mathcal{P}$, where $\mathcal{P}\in\mathcal{M}$, is greater than zero. If Σ were equal to zero there would exist a sequence $\overset{\frown}{\mathcal{P}_n Q_n}$, $\mathcal{P}_n\in\mathcal{M}$, of double extremals whose arc-lengths approach zero. Since $\mathcal{M}$ is compact, $\mathcal{P}_n$ can be picked as a convergent sequence, $\mathcal{P}_n \to \mathcal{P}$, and since $\overline{\mathcal{P}_n Q_n} \leq \overline{\mathcal{P}\,\mathcal{P}_n} + \overline{\mathcal{P}_n Q_n}$ we see that $Q_n \to \mathcal{P}$. But this contradicts the last theorem of section 9, according to which there exists a neighborhood of $\mathcal{P}$ which is free from the endpoints of double extremals of lengths less than ρ.

In the second part of the proof we need the function $P'(\mathcal{P}_o)$ defined as the least upper bound of all ρ for which the following property holds

$$(\alpha) \qquad F(x_o,\ y) < \rho \ \text{implies}\ \left| \frac{\partial\phi(x_o, y)}{\partial y} \right| \neq 0,$$

where x_{oi} are the coordinates of $\mathcal{P}_o$. It is obvious that property (α) holds for $P'(\mathcal{P}_o)$; also $P'(\mathcal{P}_o) \geq P(\mathcal{P}_o)$ since we could "extend" $\mathcal{V}_{P'(\mathcal{P}_o)}(y_o)$ unless $\left|\frac{\partial\phi(x_o, y)}{\partial y}\right|$ vanishes at some point of $\mathcal{V}_{P'}(y_o)$ We now show that $P'(\mathcal{P}_o)$ is semicontinuous, i.e. that <u>to every $\epsilon > 0$ there exists a neighborhood $\mathcal{U}(\mathcal{P}_o)$ such that if $\overline{\mathcal{P}}_o$ is a point of $\mathcal{U}(\mathcal{P}_o)$ then $P'(\overline{\mathcal{P}}_o) \geq P'(\mathcal{P}_o) - \epsilon$.</u> If this were false then in each $\mathcal{U}(\mathcal{P}_o)$ there would be a point $\overline{\mathcal{P}}_o$, $(\overline{x}_o)$, for which $P'(\overline{\mathcal{P}}_o) < P'(\mathcal{P}_o) - \epsilon$. For this $\overline{x}_o$ and some y both

$$F(\overline{x}_o, y) < P'(\mathcal{P}_o) - \epsilon \quad \text{and}$$

$$\left| \frac{\partial\phi(\overline{x}_o, y)}{\partial y} \right| = 0$$

would be true. Take such a sequence $\overline{x}_{on} \to \overline{x}_o$. For this and a corresponding set y_n

$$F(\overline{x}_{on}, y_n) < P'(\mathcal{P}_o) - \epsilon \qquad \text{and}$$

$$\left| \frac{\partial \phi(\overline{x}_{on}, y_n)}{\partial y} \right| = 0$$

would both be true. By use of the lemma on page 4 we get

$$B \max |y_n| < P'(\mathcal{P}_o) - \epsilon \;;$$

hence y_n is a bounded set and has some convergent subsequence which we will denote again by y_n. If $y_n \to y$ we obtain in the limit

$$F(\overline{x}_o, y) < P'(\mathcal{P}_o) - \epsilon \qquad \text{and}$$

$$\left| \frac{\partial \phi(\overline{x}_o, y)}{\partial y} \right| = 0$$

in contradiction to the definition of $P'(\mathcal{P}_o)$, thus proving the semicontinuity of $P'(\mathcal{P}_o)$. If $P'(\mathcal{P}_o)$ is infinite it can be shown, in a similar manner, that to each $N > 0$ there exists a $\mathcal{U}(\mathcal{P}_o)$ such that if $\overline{\mathcal{P}}_o \in \mathcal{U}(\mathcal{P}_o)$ then $P'(\overline{\mathcal{P}}_o) > N$.

Now let us go back to our theorem. By definition of $\underline{P}$ there exists a sequence of points $\mathcal{P}_n \in \mathcal{M}$ such that ($\mathcal{M}$ is compact) $\mathcal{P}_n \to \mathcal{P}$ and $P(\mathcal{P}_n) \to \underline{P}$. To a given sufficiently small $\epsilon > 0$ there exists a neighborhood $\mathcal{U}(\mathcal{P})$ such that

$$P'(\overline{\mathcal{P}}) \geq P'(\mathcal{P}) - \epsilon \geq P(\mathcal{P}) - \epsilon$$

for $\overline{\mathcal{P}} \in \mathcal{U}(\mathcal{P})$. Hence for all n greater than some N

$$P'(\mathcal{P}_n) \geq P(\mathcal{P}) - \epsilon .$$

Since the length of a double extremal is always $\geq \Sigma > 0$ we have for all $n > N$

$$P(\mathcal{P}_n) \geq (\Sigma, P(\mathcal{P}) - \epsilon) > 0,$$

if (a, b) denotes the smaller of a and b, and hence, since ϵ is arbitrary and $\underline{P}$ is the greatest lower bound of $P(\mathcal{P})$, where $\mathcal{P} \in \mathcal{M}$,

$$\underline{P} \geq (\Sigma, P(\mathcal{P})) > 0 .$$

We thus complete the proof of the theorem.

11. In addition to our past assumptions (2.1, 2.2, 2.3, 3.1, 4.1) we add the following one.

(11.1) $\mathcal{W}_{P(\mathcal{P}_0)}(\mathcal{P}_0)$ has the property that any extremal arc $E_{\mathcal{P}_0\mathcal{P}}$, $\mathcal{P} \subset \mathcal{W}_{P(\mathcal{P}_0)}(\mathcal{P}_0)$, is a minimizing arc, i.e. the arc-length of $E_{\mathcal{P}_0\mathcal{P}}$ is not greater than that of any other arc from $\mathcal{P}_0$ to $\mathcal{P}$.

(11.1a) $\mathcal{W}_{P(\mathcal{P}_0)}(\mathcal{P}_0)$ has the property that $E_{\mathcal{P}_0\mathcal{P}}$, $\mathcal{P} \subset \mathcal{W}_{P(\mathcal{P}_0)}(\mathcal{P}_0)$, has a shorter arc-length than all arcs from $\mathcal{P}_0$ to $\mathcal{P}$.

According to the definition of distance the arc-length of $E_{\mathcal{P}_0\mathcal{P}} = \overline{\mathcal{P}_0\mathcal{P}}$ under both assumptions, i.e. $E_{\mathcal{P}_0\mathcal{P}}$ is a minimizing arc. That these assumptions are not necessary for the theory is apparent from the fact that it is by no means proved that a minimizing arc be an unbroken extremal (see the example on page 19). A minimizing arc could consist of a finite or infinite number of extremal arcs. In this case the covariant vector $\sigma_i = \frac{\partial F(x,\xi)}{\partial x'_i} - \frac{\partial F(x,\eta)}{\partial x'_i}$ would have to vanish at any corner point A. (ξ and η are the two tangent vectors to the curve at A) We define the invariant

$$\mathcal{E}(x,\xi,\eta) = \sigma_i \xi^i .$$

$\mathcal{E}$, the Weierstrass $\mathcal{E}$-function is a positive homogeneous function of two contravariant vectors ξ and η defined at the same point; it is of the first degree in ξ and of the zero-th degree in η, and vanishes for $\eta = \alpha\xi$. If $\mathcal{E}$ vanishes if and only if $\eta = \alpha\xi$, it is called essentially positive. Later it will be proved that a necessary and sufficient condition for (11.1) to be true is that the Weierstrass $\mathcal{E}$-function be not negative, and that a sufficient condition for (11.1a) is that the $\mathcal{E}$-function be essentially positive. If the $\mathcal{E}$-function is essentially positive, a minimizing arc can have no corners, because the $\mathcal{E}$ -function would vanish at such a corner without $\eta = \alpha \xi$. The important case of a Riemannian metric is a case where the $\mathcal{E}$ -function is essentially positive.

THEOREM: Under assumptions (11.1) or (11.1a)

$$\mathcal{W}_\rho(\mathcal{P}_0) \equiv \mathcal{U}_\rho(\mathcal{P}_0), \quad \rho \leq P(\mathcal{P}_0). \tag{11.2}$$

Proof. a) If $\mathcal{P} \subset \mathcal{W}_\rho(\mathcal{P}_0)$ there exists an extremal $E_{\mathcal{P}_0\mathcal{P}}$ of arc-length less than ρ. Therefore $\overline{\mathcal{P}_0\mathcal{P}} < \rho$, and $\mathcal{P} \in \mathcal{U}_\rho(\mathcal{P}_0)$.
b) If $\mathcal{P} \in \mathcal{U}_\rho(\mathcal{P}_0)$ then $\overline{\mathcal{P}_0\mathcal{P}} < \rho$. Hence there must exist some arc $C_{\mathcal{P}_0}^{\mathcal{P}}$ such that its arc-length $\mathcal{J}(C_{\mathcal{P}_0}^{\mathcal{P}}) < \rho$. But this arc in its entirety, and in particular its endpoint $\mathcal{P}$, lie in $\mathcal{W}_\rho(\mathcal{P}_0)$, for otherwise there would be some point Q both on the curve and on the boundary of $\mathcal{W}_\rho(\mathcal{P}_0)$. But due to the definition of $\mathcal{W}_\rho(\mathcal{P}_0)$, its boundary in normal coordinates is characterized by $F(x_0, y) = \rho$, while

$$\mathcal{J}(C_{\mathcal{P}_0}^{Q}) \geq \mathcal{J}(E_{\mathcal{P}_0}^{Q}) = F(x_0, y) = \rho$$

giving a contradiction.

Consider now a Hilbert arc $H_{\mathcal{P}Q}$; since it is a compact point set the $\underline{P}$, greatest lower bound of $P(\mathcal{P})$, where $\mathcal{P} \in H_{\mathcal{P}Q}$, is greater than zero. Subdivide it into N subarcs by N-1 intermediate points $\mathcal{P}_1, \dots, \mathcal{P}_{N-1}$, $(\mathcal{P}_0 = \mathcal{P}, \mathcal{P}_N = Q)$, such that

$$\overline{\mathcal{P}_i \mathcal{P}_{i+1}} < \underline{P}.$$

Since

$$\mathcal{U}_\rho(\mathcal{P}_i) \equiv \mathcal{W}_\rho(\mathcal{P}_i), \quad \rho \leq P(\mathcal{P}_i),$$

we see that

$$\mathcal{U}_{\underline{P}}(\mathcal{P}_i) \equiv \mathcal{W}_{\underline{P}}(\mathcal{P}_i),$$

and hence $\mathcal{P}_{i+1} \in \mathcal{W}_{\underline{P}}(\mathcal{P}_i)$. Thus there exists, under assumption (11.1), an extremal $E_{\mathcal{P}_i\mathcal{P}_{i+1}}$ of length $\overline{\mathcal{P}_i \mathcal{P}_{i+1}}$ connecting $\mathcal{P}_i$ to $\mathcal{P}_{i+1}$.

The fundamental property of Hilbert arcs gives

$$\overline{\mathcal{P}Q} = \overline{\mathcal{P}\mathcal{P}_1} + \cdots + \overline{\mathcal{P}_{N-1}Q},$$

and since $\overline{\mathcal{P}_i\mathcal{P}_{i+1}} = \mathcal{J}(E_{\mathcal{P}_i\mathcal{P}_{i+1}})$ we get

(11.3)
$$\overline{\mathcal{P}Q} = \sum_{i=0}^{N} \mathcal{J}(E_{\mathcal{P}_i\mathcal{P}_{i+1}}).$$

(11.3) shows that the curve $\sum_{i=0}^{N} E_{\mathcal{P}_i\mathcal{P}_{i+1}}$ is a minimizing arc of class D^2. If the $\mathcal{E}$ -function is essentially positive there can be no corners, and the minimizing arc $\sum_{i=0}^{N} E_{\mathcal{P}_i\mathcal{P}_{i+1}}$ is an extremal arc of class C^2.

Under the assumption (11.1a) we shall prove that each Hilbert arc $H_{\mathcal{P}Q}$ is an extremal arc and minimizing. This is first shown for a subarc H_{RS} of $H_{\mathcal{P}Q}$, for which $\overline{RS} < \underline{P}$, $\underline{P}$ being with respect to $H_{\mathcal{P}Q}$. From this and the fundamental property of Hilbert arcs the above statement to be proved follows. Because $\overline{RS} < \underline{P}$, S lies in $\mathcal{W}_P$ (R) and hence there exists the extremal arc E_{RS} of length $\overline{RS}$. Now each point T of H_{RS} is a point of E_{RS}; if not, let us construct, as above, the minimizing arcs E'_{RT} and E'_{TS} connecting R to T and T to S respectively. Now

$$\overline{RT} = \mathcal{J}(E'_{RT}),$$
$$\overline{TS} = \mathcal{J}(E'_{TS}),$$
$$\overline{RS} = \mathcal{J}(E_{RS}), \quad \text{and}$$
$$\overline{RT} + \overline{TS} = \overline{RS}.$$

Hence

$$\mathcal{J}(E'_{RT} + E'_{TS}) = \mathcal{J}(E_{RS})$$

contradicting the assumption (11.1a) that $E_{RS} \subset \mathcal{W}_P$ (R) provides the shortest length. Hence each point T of H_{RS} lies on E_{RS}, and, since there is one and only one point on each of the arcs of a given distance ρ, $0 \leq \rho \leq \overline{RS}$, from R, we have $H_{RS} \equiv E_{RS}$, proving the statement.

As a result of this proof we obtain the following important fact: if $Q \subset \mathcal{N}_\rho(\mathcal{P})$ and if the Weierstrass $\mathcal{E}$-function is essentially positive, each "between" point R of $\mathcal{P}$ and Q lies on the extremal arc $E_{\mathcal{P}Q}$ joining $\mathcal{P}$ to Q and lying in $\mathcal{N}_\rho(\mathcal{P})$

12. We will consider now the problem of characterizing normal coordinates, i.e. coordinates in which the extremals issuing from a point $\mathcal{P}_0$ have the equation $y_i = \xi^i s$ where ξ is a constant vector and s is equal to a constant times arc-length, by properties of the metric function $F(x, x')$. Take a normal coordinate neighborhood $\mathcal{U}(y_0)$ of the point $\mathcal{P}_0$, where the normal coordinates y are defined by $x_i = \phi_i(x_0, y)$ and where $\mathcal{U}(y_0)$ is such that if $\bar{y} \subset \mathcal{U}(y_0)$ then the whole extremal $\bar{y}t \subset \mathcal{U}(y_0)$, $0 \leq t \leq 1$. If $F(x, x')$ is the metric function in the x coordinate system,

$$F\left(\phi(x_0, y), \frac{\partial \phi}{\partial y_i} y_i'\right) = \mathcal{F}(y, y')$$

will be the metric function in the normal coordinate system. Since according to (4.1) $F(x, x')$ has derivatives of the fourth order and since $\frac{\partial \phi_i}{\partial y_j}$ exists and is continuous, and, except for $y_0 = 0$, $\frac{\partial^2 \phi_i}{\partial y_j \partial y_k}$ exists and is continuous also, the following assertions hold. $\mathcal{F}(y, y')$ is continuous; $\frac{\partial \mathcal{F}}{\partial y_i}$ is continuous likewise except at $y = y_0 = 0$; $\frac{\partial \mathcal{F}}{\partial y_i'}$ and $\frac{\partial^2 \mathcal{F}}{\partial y_i' \partial y_k'}$ are continuous except at $y' = 0$; and $\frac{\partial^2 \mathcal{F}}{\partial y_i \partial y_k'}$ is continuous except at $y' = 0$ and $y = 0$, where all of the statements of continuity are for $y \subset \mathcal{U}(y_0)$ and y' arbitrary.

If $y_i = \eta^i s$, s equal to a constant times the arc-length, is an extremal E issuing from $\mathcal{P}_0$, the Euler vector on E

$$\rho_i = \frac{\partial \mathcal{F}(y, \eta)}{\partial y_i} - \frac{d}{dt} \frac{\partial \mathcal{F}(y, \eta)}{\partial y_i'} = 0 \tag{12.1}$$

for all $s \neq 0$ and such that $y_i = \eta^i s$ is in $\mathcal{U}(y_0)$. Also, since s is equal to a constant times the arc-length, (12.2) $$\mathcal{F}(y,\eta) = k = \mathcal{F}(0,\eta)$$

for all points of E. Set

$$\eta_i(s) = \frac{\partial \mathcal{F}(y,\eta)}{\partial y_i'} .$$

$\eta_i(s)$ is continuous along E; therefore

$$\eta_i(0) = \lim_{s \to 0} \eta_i(s) = \frac{\partial \mathcal{F}(0,\eta)}{\partial y_i'} .$$

Then from (12.1) we get

$$\frac{d}{ds}\left(\eta_i(s) - \eta_i(0)\right) = \frac{\partial \mathcal{F}(y,\eta)}{\partial y_i},$$

the Euler equation in another form. Multiplying (12.2) by s we get

$$\mathcal{F}(y,y) = \mathcal{F}(0,y) ;$$

this differentiated is

$$\frac{\partial \mathcal{F}(y,y)}{\partial y_i} + \frac{\partial \mathcal{F}(y,y)}{\partial y_i'} = \frac{\partial \mathcal{F}(0,y)}{\partial y_i'}$$

for all $y \neq 0$ in the neighborhood $\mathcal{V}(y_0)$. Therefore, due to the homogeneity conditions on the terms of the above equation,

$$\frac{\partial \mathcal{F}(y,y)}{\partial y_i} + \frac{\partial \mathcal{F}(y,\eta)}{\partial y_i'} = \frac{\partial \mathcal{F}(0,\eta)}{\partial y_i'}, \quad \text{or}$$

$$\frac{\partial \mathcal{F}(y,y)}{\partial y_i} + \eta_i(s) - \eta_i(0) = 0 .$$

Multiplying by $\frac{1}{s}$,

$$\frac{\partial \mathcal{F}(y,\eta)}{\partial y_i} + \frac{\eta_i(s) - \eta_i(0)}{s} = 0, \quad s \neq 0.$$

Hence

$$\frac{d}{ds}\left(\eta_i(s) - \eta_i(0)\right) + \frac{\eta_i(s) - \eta_i(0)}{s} = 0, \quad s \neq 0.$$

Integrating this differential equation, we get

$$\eta_i(s) - \eta_i(0) = \frac{c_i}{s}, \quad s \neq 0$$

Since the limit of the left side of the above as s approaches zero is zero (we have assumed that the whole extremal, $0 \leqq s \leqq s_1$, lies in $\mathcal{V}(y_0)$) $c_i = 0$ and

$$\eta_i(s) - \eta_i(0) = 0, \qquad \text{or}$$

$$\frac{\partial \mathcal{F}(y, \eta)}{\partial y'_i} = \frac{\partial \mathcal{F}(0, \eta)}{\partial y'_i}, \qquad \text{or}$$

(12.3)
$$\frac{\partial \mathcal{F}(y, y)}{\partial y'_i} = \frac{\partial \mathcal{F}(0, y)}{\partial y'_i}.$$

(12.3) hence are necessary conditions on the function $\mathcal{F}(y, y')$ for the y coordinate system to be a normal coordinate system.

We will now show that conditions (12.3) are also sufficient. Multiplying (12.3) by y_i,

(12.4)
$$\mathcal{F}(y, y) = \mathcal{F}(0, y);$$

multiplying the above by $\frac{1}{s}$,

$$\mathcal{F}(y, \eta) = \mathcal{F}(0, \eta),$$

and hence s is proportional to the arc-length along the curve $y_i = \eta_i s$. Next it is shown that $\rho_i = 0$ along $y_i = \eta_i s$. From (12.3) and the derivative of (12.4),

$$\frac{\partial \mathcal{F}(y, y)}{\partial y_i} = 0.$$

Also

$$\frac{\partial \mathcal{F}(y, \eta)}{\partial y'_i} = \frac{\partial \mathcal{F}(0, \eta)}{\partial y'_i} = const.,$$

and hence along $y_i = \eta_i s$

$$\frac{d}{ds} \frac{\partial \mathcal{F}(y, \eta)}{\partial y'_i} = 0.$$

$$\therefore \quad \rho_i = \frac{\partial \mathcal{F}(y, \eta)}{\partial y_i} - \frac{d}{ds} \frac{\partial \mathcal{F}(y, \eta)}{\partial y'_i} = 0,$$

and the sufficiency of (12.3) is shown.

We note that

$$\frac{\partial \mathcal{F}(y, y)^k}{\partial y'_i} = \frac{\partial \mathcal{F}(0, y)^k}{\partial y'_i}, \quad k \neq 1,$$

as well as (12.3) form a set of necessary and sufficient conditions for normal coordinates. In Riemannian space

$$\mathcal{F}^2 = g_{ij}(y)\eta^i\eta^j$$

and hence the conditions become

$$2g_{ij}(y)\eta^j = 2g_{ij}(0)\eta^j, \qquad \text{or}$$

(12.5)
$$g_{ij}(y)y_j = g_{ij}(0)y_j.$$

This can be written as

$$g_{ij}(y)y_j = y_i$$

if coordinates are chosen such that $g_{ij}(0)$ are equal to δ_i^j

13. Along an extremal $y_i = \eta^i s$ we have

$$\begin{aligned} \text{arc-length} &= \int_0^s \mathcal{F}(y,\eta)\,ds \\ &= \int_0^s \mathcal{F}(0,\eta)\,ds \qquad \text{by (12.4)} \\ &= \mathcal{F}(0,\eta)\,s \\ &= \mathcal{F}(0,y) = S(y) \end{aligned}$$

if the last equality be taken as the definition of $S(y)$. (Of course this formula has been proven before.) By the considerations of section 12 we see that $\frac{\partial S}{\partial y_i}$ exists for $y \neq 0$ and

$$\frac{\partial S}{\partial y_i} = \frac{\partial \mathcal{F}(0,y)}{\partial y_i'} = \frac{\partial \mathcal{F}(y,y)}{\partial y_i'}$$
$$= \frac{\partial \mathcal{F}(y,\eta)}{\partial y_i'}.$$

Let $y_i = \eta^i s$ be an extremal E joining y_0 to y_1, and let $y_i = y_i(t)$, $0 \leqq t \leqq 1$, be any other curve C of class D^1 lying in the normal coordinate neighborhood, connecting y_0 to y_1, and passing through y_0 only for the value $t = 0$. Let $\mathcal{P}(\tau)$ be the point $y_i = y_i(\tau)$ on the curve C. Then

$$\begin{aligned} S(y_1) - S(y(\tau)) &= \int_{C\,\mathcal{P}(\tau)}^{y_1} dS \\ &= \int_{C\,\mathcal{P}(\tau)}^{y_1} \frac{\partial S}{\partial y_i}\,dy_i \\ &= \int_\tau^1 \frac{\partial \mathcal{F}(y,\eta)}{\partial y_i'}\,y_i'\,dt \end{aligned}$$

Let τ approach zero; $S(y(\tau)) = \mathcal{F}(0, y(\tau))$ approaches zero, since $y(\tau)$ approaches zero and $\mathcal{F}$ is continuous and $\mathcal{F}(0, 0) = 0$. Then, since the

$\int_0^1 \frac{\partial \mathcal{F}(y, \eta)}{\partial y_i'} y_i' \, dt$ exists,

$$S(y_1) = \int_0^1 \frac{\partial \mathcal{F}(y, \eta)}{\partial y_i'} y_i' \, dt.$$

C was any curve passing through y_0 but once, and could be E; therefore this holds for E too, in which case it also follows from the homogeneity of $\mathcal{F}(y, \eta)$.
The above formula for $S(y_1)$ has been proved for curves of class D^1 passing through the origin but once. It holds equally well for closed curves where the origin is the first and the last point on the curve, but is not any other point on the curve. (In this case $S = S(0) = 0$.) Every arc of class D^1, as can be easily shown, is a sum of a finite number of such arcs; hence the formula holds for each arc of class D^1. Let

$$\begin{aligned}
\Delta &= \text{arc-length of C} - \text{arc-length of E} \\
&= \int_C \mathcal{F}(y, y')\,dt - S(y_1) \\
&= \int_C \mathcal{F}(y, y')\,dt - \int_C \frac{\partial \mathcal{F}(y, \eta)}{\partial y_i'} y_i' \, dt \\
&= \int_C \left\{ \frac{\partial \mathcal{F}(y, y')}{\partial y_i'} - \frac{\partial \mathcal{F}(y, \eta)}{\partial y_i'} \right\} y_i' \, dt \\
&= \int_C \mathcal{E}(y, y', \eta)\,dt.
\end{aligned}$$

If $\mathcal{E} \geqq 0$, $\Delta \geqq 0$ and the extremal E is an arc not longer than any other curve connecting the two end points and lying in the normal coordinate neighborhood.

THEOREM. If in $\mathcal{N}_\rho(y_0)$, $\rho \leqq P(\mathcal{P}_0)$, the $\mathcal{E}$-function is non-negative, an extremal E from y_0 to y_1 lying in $\mathcal{N}_\rho(y_0)$ is such that $\mathcal{J}(E) \leqq \mathcal{J}(C)$, C being any other curve of class D^1 connecting y_0 to y_1. If is essentially positive, then $\mathcal{J}(E) < \mathcal{J}(C)$.

Proof. If $C \subset \mathcal{W}_\rho(y_0)$ and $\mathcal{E} \geqq 0$, then $\mathcal{J}(C) \geqq \mathcal{J}(E)$ as we have just seen. There is at least one point $\mathcal{P}$, different from y_0, on C at which $y' \neq \eta$ since $C \not\equiv E$, and at this point $\mathcal{P}$, under the assumption that the $\mathcal{E}$-function is essentially positive, $\mathcal{E} > 0$. Since we are assuming $C \subset \mathcal{W}_\rho(y_0)$, $\mathcal{P} \,\varepsilon\, \mathcal{W}_\rho(y_0)$. Now $\mathcal{E}(y, y', \eta) = \mathcal{E}(y, y', y)$, $y \neq y_0$, showing that in normal coordinates the $\mathcal{E}$-function is a continuous function of y and y' alone, with $y \neq y_0$. Hence $\mathcal{E} > 0$ in some interval on C about the point $\mathcal{P}$. Therefore $\Delta > 0$ and $\mathcal{J}(C) > \mathcal{J}(E)$.

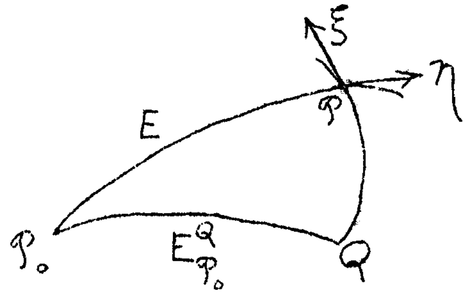
If $C \not\subset \mathcal{W}_\rho(y_0)$ there will exist a point $\mathcal{P}$ on C and on the boundary of $\mathcal{W}_\rho(y_0)$.

$$\mathcal{J}(C_{y_0}^{y_1}) \geq \mathcal{J}(C_{y_0}^{\mathcal{P}}).$$

$$\therefore \qquad \mathcal{J}(C_{y_0}^{y_1}) \geq \overline{\mathcal{P}_0\mathcal{P}} = \rho,$$

since $\mathcal{W}_\rho(\mathcal{P}_0) \equiv \mathcal{U}_\rho(\mathcal{P}_0)$. But $\mathcal{J}(E) < \rho$. Therefore $\mathcal{J}(C) > \mathcal{J}(E)$ under either of the assumptions on $\mathcal{E}$ if $C \not\subset \mathcal{W}_\rho(\mathcal{P}_0)$, and the theorem is proved.

Suppose $\mathcal{E}(x, \xi, \eta) < 0$ for some x, ξ, η in our manifold. Construct an extremal E through the point $\mathcal{P} = x$ with the tangent vector η. On E pick a point $\mathcal{P}_0$ such that $\mathcal{P} \,\varepsilon\, \mathcal{W}_{\rho(\mathcal{P}_0)}(\mathcal{P}_0)$. Then for $\eta^i = y_i$ in the normal coordinate system with origin at $\mathcal{P}_0$ $\mathcal{E}(y, \xi, y)$ is a continuous function of y and ξ, and since $\mathcal{E}(y, \xi, y) < 0$ at $\mathcal{P}$, $\mathcal{E}(y, \xi, y) < 0$ along some arc $\widehat{Q\mathcal{P}}$ of class C^1 which has ξ^i as tangent vector at $\mathcal{P}$.



Construct the extremal $E_{\mathcal{P}_0}^{Q}$ from $\mathcal{P}_0$ to Q. Take $C = E_{\mathcal{P}_0}^{Q} + \widehat{Q\mathcal{P}}$ as a comparison curve.

$$
\begin{aligned}
\Delta &= J(E) - J(C) \\
&= \int_C \mathcal{E}\, dt \\
&= \int_{E_{\mathcal{P}_0}^{Q}} \mathcal{E}\, dt + \int_{\overline{Q\mathcal{P}}} \mathcal{E}\, dt \\
&= \int_{\overline{Q\mathcal{P}}} \mathcal{E}\, dt < 0,
\end{aligned}
$$

and the extremal E is not a minimizing arc. Collecting the results we get the

THEOREM. A necessary and sufficient condition that in each $\mathcal{W}_\rho(\mathcal{P}_0)$ the extremal arcs through the origin have lengths no greater than those of any comparison curves is that the $\mathcal{E}$-function be non-negative throughout the space. If the $\mathcal{E}$-function is essentially positive the extremal arc will be shorter than any comparison curve.

14. A curve C_1 is said to be homotopic to a curve C_2 ($C_1 \approx C_2$), both connecting two given points $\mathcal{P}$ and Q, if there exists a one-parameter family of curves

$$
\mathcal{P} = \mathcal{P}(\epsilon, t), \qquad \begin{matrix} 0 \leq \epsilon \leq 1 \\ -\infty < a(\epsilon) \leq t \leq b(\epsilon) < \infty \end{matrix}
$$

joining $\mathcal{P}$ to Q such that $\mathcal{P}(\epsilon, t)$ is continuous in the domain of definition and C_1 and C_2 are members of the family. It is a simple matter to verify that the relation of homotopy is reflexive, symmetric, and transitive. Let $\{C\}$ be a homotopic class of curves joining two given points, i.e. a set of curves such that any two of them are homotopic. Due to the three properties of the homotopy relation listed above, a homotopic class is uniquely defined by any member of it. We might remark that if the space in question has properties such as to allow speaking of arcs of class D^m we can restrict our definition of homotopy to such types of curves. It can be shown that two arcs of class D^m homotopic in the general sense are homotopic in the restricted sense, and conversely.

THEOREM. There is a minimizing arc (an extremal without a corner) in each homotopic class $\{C\}$ of curves C joining $\mathcal{P}$ to Q, if the $\mathcal{E}$-function is essentially positive.

Proof. Let $d = \text{g.l.b. } \mathcal{J}(C)$, $C \in \{C\}$, and C of class D^1. Then there exists a sequence C_n such that $C_n \in \{C\}$ and

$$d_n = \mathcal{J}(C_n) \to d \quad \text{as } n \to \infty.$$

Take constants $D_1 > D > d$.

$$C_n \subset \mathcal{U}_D(\mathcal{P})$$

for all n sufficiently large. Let

$$\underline{P} = \text{g.l.b. } P(R),$$

where $R \in \overline{\mathcal{U}}_{D_1}(\mathcal{P})$ and $P(R)$ is the l.u.b. of ρ such that $\mathcal{W}_\rho(R)$ is describable by normal coordinates. Then $\underline{P} > 0$ since $\overline{\mathcal{U}}_{D_1}(\mathcal{P})$ is compact. Divide C_n into N subarcs of equal length by points P_{ni}, $i = 0, 1, \ldots, N$; $\mathcal{P}_0 = \mathcal{P}$, $\mathcal{P}_N = Q$. Then there exists a k such that

$$\mathcal{J}(\widehat{\mathcal{P}_{ni}\mathcal{P}_{ni+1}}) = \frac{d_n}{N} < k < \underline{P}$$

for sufficiently large N. As a consequence

$$\overline{\mathcal{P}_{ni}\mathcal{P}} < k < \underline{P}$$

for any point $\mathcal{P}$ on $C' = \widehat{\mathcal{P}_{ni}\mathcal{P}_{ni+1}}$ and indeed

$$\overline{\mathcal{P}_{ni}\mathcal{P}_{ni+1}} < k < \underline{P}.$$

Since $\mathcal{W}_{\underline{P}}(\mathcal{P}_{ni}) \equiv \mathcal{U}_{\underline{P}}(\mathcal{P}_{ni})$,

$$C' = \widehat{\mathcal{P}_{ni}\mathcal{P}_{ni+1}} \subset \mathcal{W}_{\underline{P}}(\mathcal{P}_{ni}).$$

Hence a unique extremal arc of length $\overline{\mathcal{P}_{ni}\mathcal{P}_{ni+1}}$

$$E' = E_{\mathcal{P}_{ni}\mathcal{P}_{ni+1}}$$

exists connecting $\mathcal{P}_{ni}$ to $\mathcal{P}_{ni+1}$, and $E' \subset \mathcal{W}_{\underline{P}}(\mathcal{P}_{ni})$. We have

$$\mathcal{J}(E') = \overline{\mathcal{P}_{ni}\mathcal{P}_{ni+1}} \le \mathcal{J}(C'). \tag{14.1}$$

Now

$$E' \approx C' \; (= \widehat{\mathcal{P}_{ni}\mathcal{P}_{ni+1}}).$$

For the proof of this we use normal coordinates in $\mathcal{W}_p(\mathcal{P}_{ni})$; then C' has as equations

$$y_i = y_i(t), \qquad a \leq t \leq b.$$

Consider the family

$$y_i(\epsilon, t) = \begin{cases} (t-a)\,\dfrac{y_i(\epsilon)}{\epsilon - a}, & a \leq t \leq \epsilon, \\ y_i(t), & \epsilon \leq t \leq b. \end{cases}$$

$y_i(\epsilon, t)$ is obviously continuous, and $\epsilon = a$ gives C' while $\epsilon = b$ gives E'; hence $E' \approx C'$. By introducing

$$C_n^* = \sum_{i=0}^{N} E_{\mathcal{P}_{ni}\mathcal{P}_{ni+1}}, \quad E_{\mathcal{P}_{ni}\mathcal{P}_{ni+1}} \equiv E',$$

we have

(14.2)
$$C_n^* \approx C_n$$

since

$$C_n = \sum_{i=0}^{N} C_{\mathcal{P}_{ni}\mathcal{P}_{ni+1}}, \quad C_{\mathcal{P}_{ni}\mathcal{P}_{ni+1}} \equiv C'.$$

Because of (14.1)

$$\mathcal{J}(C_n^*) \leq \mathcal{J}(C_n)$$

Since by (14.2) $C_n^* \in \{C\}$,

$$d \leq \mathcal{J}(C_n^*).$$

Hence, since $\mathcal{J}(C_n) \to d$ as $n \to \infty$, we have

$$\mathcal{J}(C_n^*) \to d \text{ as } n \to \infty.$$

We now show that a minimizing arc C^* exists. Take a subsequence of C_n^* such that

$$\mathcal{P}_{ni} \to \mathcal{P}_i, \qquad i = 0, 1, \ldots, N,$$

which can be done since $\mathcal{P}_{ni} \in \mathcal{U}_D(\mathcal{P})$ and $\mathcal{U}_D(\mathcal{P})$ is compact. Then

$$\mathcal{P}_i \in \overline{\mathcal{U}}_D(\mathcal{P}) \subset \mathcal{U}_{D_1}(\mathcal{P}),$$

and also since

$$\overline{\mathcal{P}_{ni}\mathcal{P}_{ni+1}} < k < \underline{P}$$

we have in the limit

$$\overline{\mathcal{P}_i\mathcal{P}_{i+1}} \leq k < \underline{P}.$$

Because of this last, there exists a unique extremal $E_{\mathcal{P}_i\mathcal{P}_{i+1}}$ joining $\mathcal{P}_i$ to $\mathcal{P}_{i+1}$ such that

$$\begin{aligned}\mathcal{J}(E_{\mathcal{P}_i\mathcal{P}_{i+1}}) &= \overline{\mathcal{P}_i\mathcal{P}_{i+1}} \\ &= \lim_{n\to\infty} \overline{\mathcal{P}_{ni}\mathcal{P}_{ni+1}} \\ &= \lim_{n\to\infty} \mathcal{J}(E_{\mathcal{P}_{ni}\mathcal{P}_{ni+1}}).\end{aligned}$$

Let $C^* = \sum_{i=0}^{N} E_{\mathcal{P}_i\mathcal{P}_{i+1}}$. Then

$$\mathcal{J}(C^*) = \lim_{n\to\infty} \mathcal{J}(C^*_n) = d,$$

and C^* is a minimizing arc relative to $\{C\}$.

Next it is shown that $C^* \in \{C\}$. This is accomplished by showing that $C^* \approx C^*_n$ for some fixed n sufficiently large. Take neighborhoods $\mathcal{U}(\mathcal{P}_i) \subset \mathcal{U}_{D_1}(\mathcal{P})$, $i = 0, 1, \ldots, N$, such that if $\mathcal{P}_i' \in \mathcal{U}(\mathcal{P}_i)$ and $\mathcal{P}_{i+1}' \in \mathcal{U}(\mathcal{P}_{i+1})$ then $\overline{\mathcal{P}_i'\mathcal{P}_{i+1}'} < \underline{P}$. Then $\mathcal{P}_{ni} \in \mathcal{U}(\mathcal{P}_i)$ for n sufficiently large and for $i = 0, 1, \ldots, N$. Connect $\mathcal{P}_i$ to $\mathcal{P}_{ni}$ by a continuous curve $\mathcal{P}_i(\epsilon)$, $0 \leq \epsilon \leq 1$, lying in $\mathcal{U}(\mathcal{P}_i)$. Then there exists a unique extremal $E_{\mathcal{P}_i(\epsilon)\mathcal{P}_{i+1}(\epsilon)}$ connecting $\mathcal{P}_i(\epsilon)$ to $\mathcal{P}_{i+1}(\epsilon)$

The totality of all these extremals $E_{\mathcal{P}_i(\epsilon)\mathcal{P}_{i+1}(\epsilon)}$ as ϵ runs from 0 to 1 will be a continuous family of extremals deforming $E_{\mathcal{P}_i\mathcal{P}_{i+1}}$ into $E_{\mathcal{P}_{ni}\mathcal{P}_{ni+1}}$. If R is a point of $E_{\mathcal{P}_i(\epsilon)\mathcal{P}_{i+1}(\epsilon)}$ we shall assign to it ϵ as its first parameter. The second parameter α may be given by

$$\alpha = \frac{\overline{\mathcal{P}_i(\epsilon)R}}{\overline{\mathcal{P}_i(\epsilon)\mathcal{P}_{i+1}(\epsilon)}} + i \tag{14.3}$$

so that on $E_{\mathcal{P}_i(\epsilon)\mathcal{P}_{i+1}(\epsilon)}$ α runs from i to i+1. R, for any point of an extremal of the family, is thus given in the form $R(\epsilon,\alpha)$, $0 \leqq \epsilon \leqq 1$, $0 \leqq \alpha \leqq N$. We show that $R(\epsilon,\alpha)$ is continuous in ϵ and α. Take sequences $\epsilon_n \to \epsilon$ and $\alpha_n \to \alpha$, ϵ_n and ϵ between 0 and 1, α_n and α between i and i+1. (The latter is no real restriction for if $i < \alpha < i+1$ then almost all α_n lie in this interval; if for instance $\alpha = i+1$, then we have to consider for α_n both intervals $i < \alpha' \leqq i+1$ and $i+1 \leqq \alpha' < i+2$, and for both intervals the proof runs along the lines given.) We show that for $\epsilon_n \to \epsilon$, $\alpha_n \to \alpha$

$$R(\epsilon_n,\alpha_n) \to R(\epsilon,\alpha) \quad \text{as} \quad n \to \infty.$$

By assumption $\mathcal{P}_i(\epsilon_n) \to \mathcal{P}_i(\epsilon)$ and $\mathcal{P}_{i+1}(\epsilon_n) \to \mathcal{P}_{i+1}(\epsilon)$. The point $R_n = R(\epsilon_n,\alpha_n)$ has the α-parameter

$$\alpha_n = \frac{\overline{\mathcal{P}_i(\epsilon_n) R_n}}{\overline{\mathcal{P}_i(\epsilon_n)\mathcal{P}_{i+1}(\epsilon_n)}} + i. \tag{14.4}$$

Since R_n is a point on $E_{\mathcal{P}_i(\epsilon)\mathcal{P}_{i+1}(\epsilon)}$,

$$\overline{\mathcal{P}_i(\epsilon_n) R_n} + \overline{R_n \mathcal{P}_{i+1}(\epsilon_n)} = \overline{\mathcal{P}_i(\epsilon_n)\mathcal{P}_{i+1}(\epsilon_n)} \tag{14.5}$$

due to the minimizing property of the extremal arc when the $\mathcal{E}$-function is non-negative. Because of the fact

$$\begin{aligned} \overline{\mathcal{P}_i R_n} &\leq \overline{\mathcal{P}_i \mathcal{P}_i(\epsilon)} + \overline{\mathcal{P}_i(\epsilon) R_n} \\ &\leq \overline{\mathcal{P}_i \mathcal{P}_i(\epsilon)} + \underline{P} \end{aligned}$$

it follows that $\overline{\mathcal{P}_i R_n}$ is bounded, and by assumption (3.1) there will be some convergent subsequence of R_n, again denoted by R_n, such that

$$R_n \to R \text{ as } n \to \infty.$$

We must now show that $R = R(\epsilon,\alpha)$. From (14.4) and (14.5)

$$\alpha = \frac{\overline{\mathcal{P}_i(\epsilon) R}}{\overline{\mathcal{P}_i(\epsilon)\mathcal{P}_{i+1}(\epsilon)}} + i, \tag{14.4'}$$

$$(14.5') \qquad \overline{\mathcal{P}_i(\epsilon)R} + \overline{R\mathcal{P}_{i+1}(\epsilon)} = \overline{\mathcal{P}_i(\epsilon)\mathcal{P}_{i+1}(\epsilon)}.$$

According to (14.5'), R is a "between" point of $\mathcal{P}_i(\epsilon)$ and $\mathcal{P}_{i+1}(\epsilon)$, and, since $E_{\mathcal{P}_i(\epsilon)\mathcal{P}_{i+1}(\epsilon)}$ is the unique Hilbert arc connecting $\mathcal{P}_i(\epsilon)$ to $\mathcal{P}_{i+1}(\epsilon)$, each "between" point lies on $E_{\mathcal{P}_i(\epsilon)\mathcal{P}_{i+1}(\epsilon)}$; therefore R lies on $E_{\mathcal{P}_i(\epsilon)\mathcal{P}_{i+1}(\epsilon)}$ and has ϵ as its first parameter. According to (14.4') R has α as second parameter,

$$\therefore \qquad R = R(\epsilon, \alpha),$$

and $C^* \approx C^*_n$ for sufficiently large n. This shows that $C^* \epsilon \{C\}$ and completes the proof of the theorem stated at the start of this section.

The following remarks can be made concerning the work of this section:

1st. If the $\mathcal{E}$-function is essentially positive, the minimizing arc C^* has no corners. For if it had one we could construct in $\{C\}$ a one-parameter family of curves $C(\epsilon)$, $-1 \leqq \epsilon \leqq 1$, such that $C^* = C(0)$ and $\mathcal{J}'(0) \neq 0$, contradicting the fact that C^* is minimizing in $\{C\}$.

2nd. The above proof shows the existence of a minimizing arc for the class of all curves of class D' joining $\mathcal{P}$ to Q, and in that case it is the easiest proof.

3rd. In quite an analogous way it can be shown that if $\mathcal{P}, Q \epsilon \mathcal{W}_{D_1}(\mathcal{P}_o)$ and $\overline{\mathcal{P}Q} \leq \underline{P}$ where $\underline{P}$ is the $\underline{P}$ belonging to $\mathcal{W}_{D_1}(\mathcal{P}_o)$, then there exists a unique extremal arc joining $\mathcal{P}$ to Q, and if R lies on this arc, R is a continuous function of $\mathcal{P}$, Q and the parameter $\alpha = \dfrac{\overline{\mathcal{P}R}}{\overline{\mathcal{P}Q}}$.

Chapter III. The Notion of the Field and the Weierstrass $\mathcal{E}$ -Function

15. The work of this section was first proved, in part, by J. H. C. Whitehead in a paper The Weierstrass $\mathcal{E}$ -Function in Differential Metric Geometry, appearing in the December 1933 number of the Quarterly Journal of Mathematics. The proof here presented for the theorem below was obtained independently by Dr. Comenetz.

THEOREM. If assumptions (2.1) and (2.2) hold and $\left|\dfrac{\partial^2 F^{\backslash}(x,x')}{\partial x'_i \, \partial x'_k}\right| \neq 0$, $F^{\backslash}(x, x') = F(x, x')^2$, the Weierstrass $\mathcal{E}$ -function is essentially positive and $\dfrac{\partial^2 F^{\backslash}(x,x')}{\partial x'_i \, \partial x'_k}$ is positive definite.

Fix x and ξ, and consider $\mathcal{E}(x, \xi, \eta) = \mathcal{E}(\eta)$ as a function of η alone. Note the important fact that $\mathcal{E}(\eta)$ is positive homogeneous of degree zero in η, i.e. in the η space $\mathcal{E}(\eta)$ is constant along a straight line issuing from the origin. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the l.u.b. and the g.l.b. respectively of $\mathcal{E}(\eta)$. Then two sequences η_n and η'_n can be picked such that $\mathcal{E}(\eta_n) \to \mathcal{E}_1$ and $\mathcal{E}(\eta'_n) \to \mathcal{E}_2$. Because of the homogeneity property of $\mathcal{E}(\eta)$ these sequences can be so picked that, say, $F(x, \eta_n) = 1$ and $F(x, \eta'_n) = 1$, that is both sequences lie on a closed compact set. They therefore both have accumulation points, and subsequences can be picked, again denoted by η_n and η'_n, such that

$$\eta_n \to \eta_1 \;, \; \eta'_n \to \eta_2 .$$

Then

$$\mathcal{E}(\eta_1) = \mathcal{E}_1, \; \mathcal{E}(\eta_2) = \mathcal{E}_2$$

and the existence of at least one maximum and one minimum of $\mathcal{E}(\eta)$ is shown. Now

(15.1)
$$\mathcal{E}(x, \xi, \eta) = F(x, \xi) - \frac{\partial F(x,\eta)}{\partial x'_i} \xi^i$$

and $\dfrac{\partial \mathcal{E}}{\partial \eta^i} = 0$ at η_1 and η_2.

(15.2) $$\frac{\partial \mathcal{E}}{\partial \eta^i} = \frac{\partial^2 F(x,\eta)}{\partial x'_i \, \partial x'_k} \xi^i = 0.$$

Since by assumption $\left| \frac{\partial^2 F}{\partial x'_i \partial x'_k} \right| \neq 0$, $\left\| \frac{\partial^2 F}{\partial x'_i \partial x'_k} \right\|$ is of rank n-1, as was proved on pages 32 and 33. Hence there is only one solution, up to a factor, of (15.2). This is $\xi^i = \alpha \eta^i$. Since the factor is unimportant, the only two cases are $\xi^i = \pm \eta^i$ Since these are the only two solutions and since at least one maximum and one minimum exist, one of them must give the maximum and the other the minimum. Putting $\eta = +\xi$ in (15.1), we get $\mathcal{E} = 0$. Putting $\eta = -\xi$ in (15.1) we get

(15.3) $$\begin{cases} \mathcal{E}(x, \xi, -\xi) = F(x, \xi) + \dfrac{\partial F(x, -\xi)}{\partial x'_i}(-\xi^i) \\ \qquad\qquad\quad\; = F(x, \xi) + F(x, -\xi), \end{cases}$$

and according to our assumption (2.2)

$$\mathcal{E}(x, \xi - \xi) > 0$$

Thus $\eta = +\xi$ gives the minimum and $\eta = -\xi$ gives the maximum, and thus $\underline{\mathcal{E}(x, \xi, \eta) \geq 0}$ Since $\mathcal{E} = 0$ is a minimum, $\mathcal{E} = 0$ only if $\eta = \xi$; hence $\mathcal{E}(x, \xi, \eta)$ is essentially positive. Note that if the assumption (2.2) is dropped, the sign of $\mathcal{E}(x, \xi, \eta)$ is independent of η, as is shown by (15.3).

Since $\eta = \xi$ gives a minimum, we see that

$$\frac{\partial^2 \mathcal{E}(\eta)}{\partial \eta^i \partial \eta^k} \zeta^i \zeta^k \geq 0$$

for $\eta = \xi$ and any ζ. Now

$$\frac{\partial^2 \mathcal{E}(\xi)}{\partial \eta^i \partial \eta^k} = -\frac{\partial^3 F(x, \xi)}{\partial x'_l \partial x'_i \partial x'_k} \xi^l$$
$$= \frac{\partial^2 F(x, \xi)}{\partial x'_i \, \partial x'_k},$$

Therefore

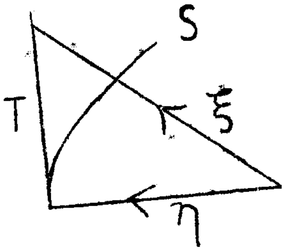
$$\frac{\partial^2 F(x, \xi)}{\partial x'_i \partial x'_k} \zeta^i \zeta^k \geq 0$$

for all ξ, ζ. Now

$$\frac{1}{2}\frac{\partial^2 F^2(x,\xi)}{\partial x'_i\,\partial x'_k}\zeta^i\zeta^k = \left(\frac{\partial F}{\partial x'_i}\zeta^i\right)^2 + F\,\frac{\partial^2 F(x,\xi)}{\partial x'_i\,\partial x'_k}\zeta^i\zeta^k \geq 0$$

and since $\left\|\frac{\partial^2 F^2}{\partial x'_i\,\partial x'_k}\right\|$ is of rank n, it is positive definite.

Let S be the surface $F(x, \eta) = \text{const}$. Now



$$\frac{\partial F(x,\eta)}{\partial x'_i}(\eta^i - \xi^i) = 0, \quad \text{or}$$

$$F(x,\eta) = \frac{\partial F(x,\eta)}{\partial x'_i}\xi^i$$

is the equation of the tangent plane T to S in η.

$$\mathcal{E}(x,\xi,\eta) = F(x,\xi) - \frac{\partial F(x,\eta)}{\partial x'_i}\xi^i$$
$$= F(x,\xi) - F(x,\eta)$$

Thus if $\mathcal{E} \geqq 0$, $F(x, \xi) \geqq F(x, \eta)$ and each point on the tangent plane T lies outside of S, showing the convexity of S with regard to straight lines. Conversely, if S is convex, $F(x, \xi) \geqq F(x, \eta)$ and $\mathcal{E} \geqq 0$. Thus the condition $\mathcal{E} \geqq 0$ is a necessary and sufficient condition for the convexity of the surface $F(x, \eta) = \text{const}$.

We now see that assumptions (11.1) and (11.1a) are equivalent with no need to differentiate between the two cases, and indeed that these assumptions need not be made since the work of this section shows that the previous assumptions were sufficient to show that the Weierstrass $\mathcal{E}$-function is essentially positive.

16. In a $\mathcal{W}_\rho(P_0)$ with P_0 omitted, the extremals issuing from P_0 define a vector field by means of the tangents η^i to the extremals at each point. For this vector field it has been shown that the covariant vector

$$\eta_i = \frac{\partial F(x,\eta)}{\partial y'_i}$$

is a gradient, i.e. $\eta_i = \frac{\partial S}{\partial x_i}$ where S is the arc-length on the extremal measured from P_0

We now consider a generalization of this field. Take a given open, simply-connected region of our space which may be covered by one or more coordinate neighborhoods. Let a field of directions ξ^i (contravariant vectors up to a positive factor) exist such that the covariant vector

$$\xi_i = \frac{\partial F(x,\xi)}{\partial x_i'}$$

is a gradient, or, which is equivalent, that $\int_{P_0}^{P} \xi_i \, dx_i$ depends only on the end points P_0, P of the integration path. We have previously shown that

$$\Pi_{ik} = \frac{\partial \xi_i}{\partial x_k} - \frac{\partial \xi_k}{\partial x_i} = 0$$

is a necessary and sufficient condition for the integral $\int_{P_0}^{P} \xi_i \, dx_i$ to be independent of path. In our case this becomes

$$\begin{aligned} \Pi_{ik}(\xi) = {} & \frac{\partial^2 F(x,\xi)}{\partial x_i' \partial x_k} + \frac{\partial^2 F(x,\xi)}{\partial x_i' \partial x_j'} \frac{\partial \xi^j}{\partial x_k} \\ & - \frac{\partial^2 F(x,\xi)}{\partial x_i \partial x_k'} - \frac{\partial^2 F(x,\xi)}{\partial x_k' \partial x_j'} \frac{\partial \xi^j}{\partial x_i} = 0 \end{aligned} \tag{16.1}$$

Since $\Pi_{ik}(\alpha\xi) = \Pi_{ik}(\xi)$, $\alpha > 0$, any solution $\xi^i(x_1, \ldots, x_n)$ of (16.1) is given only up to a positive factor as one would expect from the nature of the problem. Let $\xi^i(x_1, \ldots, x_n)$ be such a solution. Then the curves tangent to the vectors of the field are given as solutions of

$$\frac{dx_i}{dt} = \xi^i(x_1, \cdots, x_n). \tag{16.2}$$

Multiplying (16.1) by ξ^k we get

$$\begin{aligned} \Pi_{ik}\xi^k = {} & \frac{\partial^2 F(x,\xi)}{\partial x_i' \partial x_k} \xi^k + \frac{\partial^2 F(x,\xi)}{\partial x_i' \partial x_j'} \frac{\partial \xi^j}{\partial x_k} \xi^k \\ & - \frac{\partial F(x,\xi)}{\partial x_i} = 0, \end{aligned}$$

and since the derivative of (16.2) is

$$\frac{d^2 x_i}{dt^2} = \frac{\partial \xi^i}{\partial x_k} \xi^k .$$

this becomes

$$-\Pi_{ik}\xi^k = \frac{\partial F(x,x')}{\partial x_i} - \frac{\partial^2 F(x,x')}{\partial x_i' \partial x_k} x_k' - \frac{\partial^2 F(x,x')}{\partial x_i' \partial x_k'} x_k''$$

(16.3)

$$= \rho_i = 0 .$$

THEOREM. <u>The curves tangent to the vectors ξ^i of a given vector field are extremals if the ξ^i is chosen so that the corresponding covariant vector ξ_i is a gradient.</u>

We insert at this point the following remark. Let $\xi_i = \frac{\partial F(x,\xi)}{\partial x_i'} = \frac{\partial \varphi(x)}{\partial x_i}$ be such a field. Let C_{PQ} be a curve tangent to the contravariant vectors of the field, and compare $J(C_{PQ})$ with $J(\overline{C}_{PQ})$, $\overline{C}_{PQ}$ being any other curve in the field connecting P to Q. Then

$$\begin{aligned}
\Delta J &= \int_{\overline{C}} F(x,x')\,dt - \int_C F(x,x')\,dt \\
&= \int_{\overline{C}} F(x,x')\,dt - \int_C \xi_i x_i'\,dt \\
&= \int_{\overline{C}} F(x,x')\,dt - \int_{\overline{C}} \xi_i x_i'\,dt \\
&= \int_{\overline{C}} \left[\frac{\partial F(x,x')}{\partial x_i'} - \frac{\partial F(x,\xi)}{\partial x_i'} \right] x_i'\,dt \\
&= \int_{\overline{C}} \mathcal{E}(x,x',\xi)\,dt
\end{aligned}$$

Thus under our assumptions $\Delta J \geqq 0$ and C_{PQ} is a minimizing arc. This, too, shows that the curves tangent to the contravariant vectors are extremals. Indeed it shows more, namely that <u>any extremal arc of the field (tangent to the ξ^i) is relatively minimizing with regard to the neighborhood defined by the region of the field.</u> This shows the importance of fields for the variation problem.

In general the converse of the theorem stated above does not hold, as will be shown, but an additional condition is required. In case the space is 2-dimensional the converse does hold because (16.3) has $\Pi_{ik} = 0$ as a consequence. We see that $\Pi_{ik} = -\Pi_{ki}$ and from (16.3)

$$\Pi_{12}\xi^2 = 0, \quad \Pi_{21}\xi^1 = 0$$

Then because not both ξ^1 and ξ^2 are zero by assumption

$$\Pi_{12} = -\Pi_{21} = 0.$$

Thus the

THEOREM. In a 2-dimensional space R_2 the tangents to any field of extremals form a vector field ξ^i with ξ^i such that the corresponding ξ_i is a gradient.

If n > 2 and ξ_i is a gradient, i.e. $\xi_i = \frac{\partial \varphi(x)}{\partial x_i}$, then

$$\frac{\partial \varphi(x)}{\partial x_i} d x_i = 0$$

and the vector ξ_i is normal to the family of hyper-surfaces $\varphi(x)$ = const. But if the covariant vector ξ_i is given by means of a field of extremals, then in general for n > 2 there exists no family of hyper-surfaces $\varphi(x)$ = const. with ξ_i as a normal, showing that for n > 2 the converse is in general not true. The additional condition necessary is that the extremals of the field be normal to a family of hyper-surfaces $\varphi(x_1, \cdots, x_n)$ = const. We will show that the existence of one hyper-surface to which the extremals are normal will insure the existence of a family, thus sharpening the additional condition necessary.

Proof. Let there be given a field of extremals and a hyper-surface $\varphi^0(x_1, \cdots, x_n) = 0$ which each extremal cuts in a distinct point. Let $\varphi^0(x) = 0$ be written as $x_i = x_i^0(y_1, \ldots, y_{n-1})$. Let $\xi^i(y_1, \ldots, y_n)$ be the tangent vector to the extremal at $x^0(y)$ on the hyper-surface. Then this extremal is given by

$$x_i = \varphi_i(x^0(y), \xi(y), t), \quad -\infty < t < \infty.$$

Consider the two extremals E_{P_0P} and $E_{\bar{P}_0\bar{P}}$ passing through the points P_0 and $\bar{P}_0$ on $\varphi^0 = 0$, and set

$$\Delta J = J(E_{\bar{P}_0\bar{P}}) - J(E_{P_0P})$$
$$= \int_0^{t_1} F(x, x')\,dt - \int_0^{t_0} F(x, x')\,dt$$

where the first integral is along the arc $E_{\bar{P}_0\bar{P}}$, the second along E_{P_0P}. To calculate ΔJ let us join the points P_0 and $\bar{P}_0$ by an arc C_0 lying in $\varphi^0 = 0$.

$$C_0 : \; y_\alpha = y_\alpha(\epsilon), \quad 0 \leq \epsilon \leq 1.$$

The hyper-surface, the field of extremals and the curve C_0 may be chosen so that

$$x_i^0(y), \quad \xi_0^i(y), \quad y_\alpha(\epsilon)$$

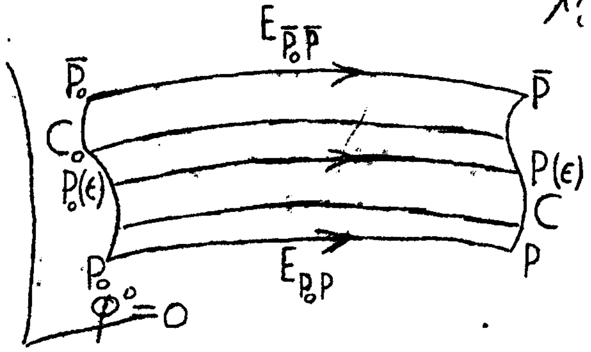
are of class C^1. The totality of extremals of the field passing through each point of the curve C_0 forms a two dimensional manifold F_2 denoted by

$$x_i = \varphi_i(x_0(y(\epsilon)), \xi_0(y(\epsilon)), t) \quad \text{or}$$

$$x_i = x_i(\epsilon, t) \quad \begin{cases} 0 \leq t \leq \alpha(\epsilon) \\ 0 \leq \epsilon \leq 1. \end{cases}$$

which contain all derivatives necessary to make the following calculations permissible. Join P to $\bar{P}$ by a curve C of class C^1 and lying in F_2.

$$C : \; t = \sigma(\epsilon), \; \sigma(0) = t_0, \; \sigma(1) = t_1, \text{ or}$$
$$x_i = x_i(\epsilon, \sigma(\epsilon)).$$



On each extremal arc $E_{P_0(\epsilon)P(\epsilon)}$

$$J(\epsilon) = \int_0^{\sigma(\epsilon)} F(x(\epsilon,t), x'(\epsilon,t))\,dt.$$

Then
$$\Delta J = J(1) - J(0)$$
$$= \int_0^1 J'(\epsilon)\,d\epsilon.$$

Now

$$J'(\epsilon) = \Big[F(x(\epsilon,t),x'(\epsilon,t))\sigma'(\epsilon)\Big]^{P(\epsilon)} + \int_0^{\sigma(\epsilon)}\left[\frac{\partial F}{\partial x_i}\frac{\partial x_i}{\partial \epsilon} + \frac{\partial F}{\partial x_i'}\frac{\partial x_i'}{\partial \epsilon}\right]dt\,.$$

Integrating this by parts we get

$$J'(\epsilon) = \Big[F(x,x')\sigma'\Big]^{P(\epsilon)} + \int_0^{\sigma(\epsilon)}\frac{\partial F}{\partial x_i}\frac{\partial x_i}{\partial \epsilon}\,dt - \int_0^{\sigma(\epsilon)}\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x_i'}\right)\frac{\partial x_i}{\partial \epsilon}\,dt + \left[\frac{\partial F}{\partial x_i'}\frac{\partial x_i}{\partial \epsilon}\right]_{P_0(\epsilon)}^{P(\epsilon)}$$

(16.4)* $$J'(\epsilon) = \Big[F\sigma'\Big]^{P(\epsilon)} + \left[\frac{\partial F}{\partial x_i'}\frac{\partial x_i}{\partial \epsilon}\right]_{P_0(\epsilon)}^{P(\epsilon)} + \int_0^{\sigma(\epsilon)}\rho_i\frac{\partial x_i}{\partial \epsilon}\,dt.$$

---

* (16.4) is invariant under coordinate transformations; therefore there is no assumption as to the number of coordinate systems necessary to cover F_2.

Since C is $x_i = x_i(\sigma(\epsilon), \epsilon)$ and C_0 is $x_i = x_i(0, \epsilon)$,

$$\left[\frac{dx_i}{d\epsilon}\right]^{P(\epsilon)} = \left[\frac{\partial x_i}{\partial \epsilon}\right]^{P(\epsilon)} + \left[\frac{\partial x_i}{\partial t}\right]^{P(\epsilon)}\sigma'$$ and

$$\left[\frac{dx_i}{d\epsilon}\right]^{P_0(\epsilon)} = \left[\frac{\partial x_i}{\partial \epsilon}\right]^{P_0(\epsilon)}.$$

Then (16.4) becomes $\left(\frac{\partial F}{\partial x_i'}x_i' = F\right)$

(16.4') $$J'(\epsilon) = \Big[F\Big]^{P(\epsilon)}\sigma' + \left[\frac{\partial F}{\partial x_i'}\frac{dx_i}{d\epsilon}\right]^{P(\epsilon)} - \Big[F\Big]^{P(\epsilon)}\sigma' - \left[\frac{\partial F}{\partial x_i'}\frac{dx_i}{d\epsilon}\right]^{P_0(\epsilon)}$$ or

(16.5) $$J'(\epsilon) = \left[\xi_i\frac{dx_i}{d\epsilon}\right]^{P(\epsilon)} - \left[\xi_i\frac{dx_i}{d\epsilon}\right]^{P_0(\epsilon)}.$$

Thus

(16.6) $$\Delta J = \int_0^1 J'(\epsilon)\,d\epsilon = \int_C \xi_i\,dx_i - \int_{C_0}\xi_i\,dx_i\,.$$

If C_o and C are closed, then $\overline{P}_o = P_o$, $\overline{P} = P$ and $\Delta J = 0$, i.e.

$$\int_C \xi_i \, dx_i = \int_{C_o} \xi_i \, dx_i$$

Now let the field of extremals be such that there exists a hyper-surface $\varphi^o(x) = 0$ to which the extremals are normal. Then $\xi_i \, dx_i = 0$ on $\varphi^o = 0$ and

(16.7) $$\int_C \xi_i \, dx_i = 0$$

Note that C was any closed curve in the field of extremals that possessed the following property: C can be split into a finite number of arcs each of which has not more than one intersection point with any one given extremal of the field.

Thus we have proved that $\varphi(x) = \int_{C_{P_oP}} \xi_i \, dx_i$ depends only on the coordinates of the end points if the integral is taken along curves possessing the above property, and by this means a $\varphi(x)$ for a fixed P_o may be defined. Now take a coordinate neighborhood at the point P so oriented that the n x-axes have the above mentioned property in this neighborhood. Then if P has the coordinates

$$\varphi(x_1, \cdots, x_{i-1}, x_i + \Delta, x_{i+1}, \cdots, x_n) - \varphi(x) = \int_{P(x)}^{P(x_1, \cdots, x_i + \Delta, \cdots, x_n)} \frac{\partial F(x, \xi)}{\partial x_i'} \, dx_i$$

where the integral is taken along the x_i axis from $x_i(P)$ to $x_i + \Delta$. By the theorem of the mean

$$\varphi(x_1, \cdots, x_{i-1}, x_i + \Delta, x_{i+1}, \cdots, x_n) - \varphi(x) = \Delta \, \frac{\partial F(\tilde{x}\, \tilde{\xi})}{\partial x_i'}$$

Hence the limit

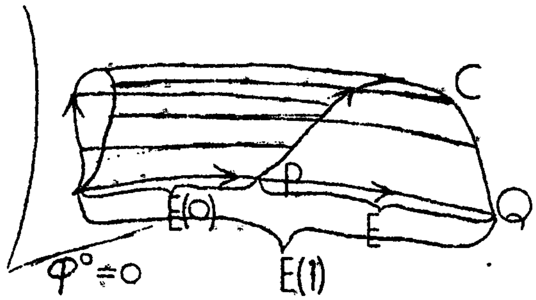
(16.8) $$\xi_i = \frac{\partial F(x, \xi)}{\partial x_i'} = \frac{\partial \varphi(x)}{\partial x_i}$$

and the covariant vector ξ_i formed from the contravariant tangent vector ξ^i of the field of extremals normal to $\varphi^o = 0$ is a gradient. This completes the proof.

We include here two remarks.

I. We have shown that under the assumption that the field of extremals is normal to a hypersurface,

$$\Delta \mathcal{J} = \mathcal{J}(E(1)) - \mathcal{J}(E(0)) = \mathcal{J}(E)$$



$$= \int_0^1 \mathcal{J}'(\epsilon)\, d\epsilon$$

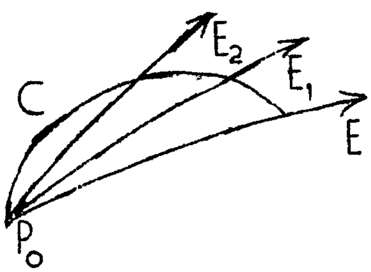
$$= \int_C \frac{\partial F(x,\xi)}{\partial x'_i}\, dx_i$$

Hence

$$(16.9)\quad \begin{cases} \mathcal{J}(C) - \mathcal{J}(E) = \int_0^1 \left[\dfrac{\partial F(x,x')}{\partial x'_i} - \dfrac{\partial F(x,\xi)}{\partial x'_i} \right] x'_i\, dt \\ \qquad\qquad\qquad = \int_0^1 \mathcal{E}(x,x',\xi)\, dt \end{cases}$$

(16.9) is correct as long as $x_i(t,\epsilon)$ has $\frac{\partial x_i}{\partial t}, \frac{\partial x_i}{\partial \epsilon}, \frac{\partial^2 x_i}{\partial t^2}, \frac{\partial^2 x_i}{\partial t\, \partial \epsilon}$, the derivatives needed to use (16.5). However, since $x_i(t,\epsilon) = \varphi_i(x_0(\epsilon), \xi_0(\epsilon), t)$ the existence of these derivatives depends only on the choice of $x_0(\epsilon)$ and $\xi_0(\epsilon)$. Thus in (16.9) there is no other restriction on C. This means that as long as an arc C can be imbedded in such a one-parameter family of extremals, then $\Delta \mathcal{J} \geq 0$. This does not contradict the fact, which will later be shown, that if Q is beyond the conjugate point of P on E, an arc C_{PQ} can be constructed such that $\Delta \mathcal{J} < 0$, for this arc obviously cannot be so imbedded.

II. In the case that the hyper-surface $\varphi^0 = 0$ degenerates to a point, we have



$$\mathcal{J}(E_1) - \mathcal{J}(E_2) = \int_{C_{(E_1,E_2)}} \frac{\partial F(x,\xi)}{\partial x'_i}\, dx_i$$

and if $E_1 = E$, and $E_2 = 0$,

$$\mathcal{J}(E) = \int_C \frac{\partial F(x,\xi)}{\partial x^i}\, dx^i .$$

Thus

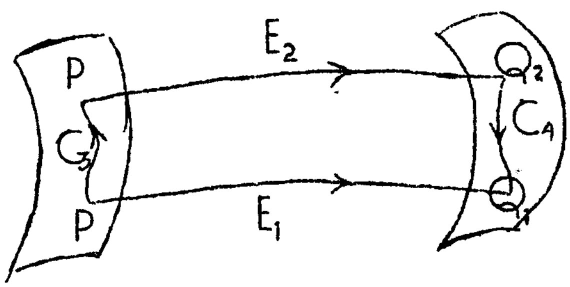
(16.10) $$\Delta \mathcal{J} = \mathcal{J}(C) - \mathcal{J}(E) = \int_C \mathcal{E}(x, x', \xi)\, dt .$$

(16.10) is quite analogous to (16.9) and holds whenever a one-parameter family of extremals $x_i = x_i(t, \epsilon)$ with the above stated derivatives exists passing through a point and such that C is imbedded in it.

At a point P_o take a normal coordinate neighborhood $\mathcal{W}_\rho(P_o)$. If $y_i = y_i(\epsilon)$ are the equations of a curve C passing through P_o, $y_i = y_i(\epsilon)t$ is a one-parameter family of extremals passing through P_o and such that the curve C is imbedded in it. Thus we have another proof for (16.10) in the case of a $\mathcal{W}_\rho(P_o)$.

We now consider two immediate consequences of this work.

THEOREM. Given a field of extremals with tangent vector ξ^i such that the corresponding covariant vector ξ_i is a gradient, and two hyper-surfaces normal to the extremals of the field. Any two extremal arcs intercepted by the two hyper-surfaces have equal length.



Let the two extremal arcs be E_1 and E_2, and let C_3 and C_4 be curves lying one in each of the two hyper-surfaces and connecting the points of intersection of E_1 and E_2 with the hyper-surfaces (see diagram). Consider E_1 and $C_3 + E_2 + C_4$, both joining P_1 to Q_1. Set $\mathcal{J}(E_1) = \mathcal{J}_1$, $\mathcal{J}(E_2) = \mathcal{J}_2$, $\mathcal{J}(C_3) = \mathcal{J}_3$, $\mathcal{J}(C_4) = \mathcal{J}_4$. Then

$$\mathcal{J}(C_3 + E_2 + C_4) - \mathcal{J}(E_1) = \mathcal{J}_3 + \mathcal{J}_2 + \mathcal{J}_4 - \mathcal{J}_1$$

$$= \int_{C_3+E_2+C_4} \left[\frac{\partial F(x,x')}{\partial x'_i} - \frac{\partial F(x,\xi)}{\partial x'_i} \right] x'_i \, dt.$$

Now

$$\int_{C_3} \left[\frac{\partial F(x,x')}{\partial x'_i} - \frac{\partial F(x,\xi)}{\partial x'_i} \right] x'_i \, dt = \int_{C_3} \frac{\partial F(x,x')}{\partial x'_i} x'_i \, dt = J_3,$$

$$\int_{C_4} \left[\frac{\partial F(x,x')}{\partial x'_i} - \frac{\partial F(x,\xi)}{\partial x'_i} \right] x'_i \, dt = \int_{C_4} \frac{\partial F(x,x')}{\partial x'_i} x'_i \, dt = J_4,$$

$$\int_{E_2} \left[\frac{\partial F(x,x')}{\partial x'_i} - \frac{\partial F(x,\xi)}{\partial x'_i} \right] x'_i \, dt = 0.$$

Therefore

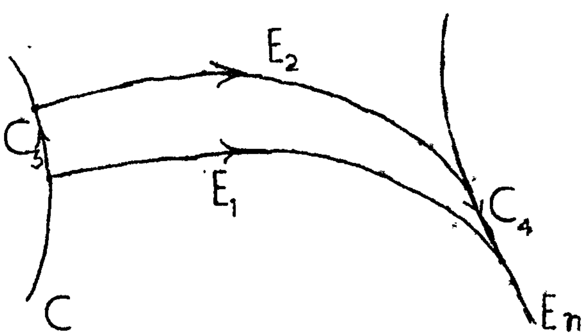
$$J_3 + J_2 + J_4 - J_1 = J_3 + J_4, \text{ or}$$

$$J_2 = J_1$$

completing the proof.

THE ENVELOPE THEOREM FOR R_2. <u>Let an extremal exist in R_2 with C a normal curve and E_n an envelope. Then</u>

$$J(E_2) + J(C_4) = J(E_1).$$



For $J(C_3) + J(E_2) + J(C_4) - J(E_1)$

$$= \int_{C_3+E_2+C_4} \mathcal{E}(x, x', \xi) \, dt$$

$$= J(C_3)$$

since $J(C_4) = 0$ ($\mathcal{E}$ function vanishes along the envelope).

$$\therefore \quad J(E_2) + J(C_4) = J(E_1).$$

17. We now consider the fields of extremals obtained from the Jacobi-Hamilton normal form. Take the $F^2 = \mathcal{F}$ problem and assume $\left| \dfrac{\partial^2 \mathcal{F}}{\partial x'_i \partial x'_k} \right| \neq 0$.

As before, we obtain the Jacobi-Hamilton normal form

(17.1)
$$\begin{cases} \dfrac{d\nu_i}{dt} = \dfrac{\partial H(\chi,\nu)}{\partial \chi_i} \\ \dfrac{d\chi_i}{dt} = -\dfrac{\partial H(\chi,\nu)}{\partial \nu_i}, \end{cases}$$

where $\nu_i = \dfrac{\partial F^{\wedge}(\chi,\chi')}{\partial \chi_i'}$ and $H(x,\nu) = F^{\wedge}(\chi,\mathcal{G}) - \nu_i \mathcal{G}^i = -F^{\wedge}(\chi,\mathcal{G})$.
$\mathcal{G}^i(\chi,\nu)$ was obtained by solving $\nu_i = \dfrac{\partial F^{\wedge}(\chi,\chi')}{\partial \chi_i'}$ to get $x_i' = \mathcal{G}^i(x,\nu)$.
Consider a field of extremals (solutions of (17.1) defined over a given domain and having the additional property that

(17.2)
$$\nu_i = \frac{\partial V}{\partial \chi_i},$$

that is ν is a gradient. V will be called the field function. Then

(17.3)
$$H\left(\chi_1,\cdots,\chi_n,\frac{\partial V}{\partial \chi_1},\cdots,\frac{\partial V}{\partial \chi_n}\right) = \text{const.}$$

are partial differential equations for the field function. As proof, differentiate the left side of (17.3), and by using (17.1) and (17.2)

$$\frac{d}{d\chi_i}H = \frac{\partial H}{\partial \chi_i} + \frac{\partial H}{\partial \nu_K}\frac{\partial \nu_K}{\partial \chi_i} = \frac{d\nu_i}{dt} - \frac{d\chi_K}{dt}\frac{\partial \nu_i}{\partial \chi_K} = 0.$$

Conversely, if we have given a solution $V(x_1 \,..\, x_n)$ of (17.3) for some constant, then

(17.4)
$$\frac{d\chi_i}{dt} = -\frac{\partial H\left(\chi,\frac{\partial V}{\partial \chi}\right)}{\partial \nu_i}$$

defines a field of curves $x_i(t)$. Along each such curve $V(x)$ and $\nu_i(\chi) = \dfrac{\partial V}{\partial \chi_i}$ are functions of t and $\dfrac{d\nu_i}{dt} = \dfrac{\partial \nu_i}{\partial \chi_K}\dfrac{d\chi_K}{dt}$. But from (17.3) we get

$$0 = \frac{\partial H}{\partial \chi_i} + \frac{\partial H}{\partial \nu_K}\frac{\partial \nu_K}{\partial \chi_i} = \frac{\partial H}{\partial \chi_i} - \frac{d\chi_K}{dt}\frac{\partial \nu_i}{\partial \chi_K}$$
$$= \frac{\partial H}{\partial \chi_i} - \frac{d\nu_i}{dt}$$

(17.4')
$$\frac{\partial H}{\partial \chi_i} = \frac{d\nu_i}{dt}.$$

$x_i(t)$ and $\gamma_i(t)$ hence are solutions of (17.1) and the curves are extremals. Thus the

THEOREM. <u>The field function V of a field of extremals with $\gamma_i = \frac{\partial V}{\partial x_i}$ is a solution of (17.3) and any solution of (17.3) gives a field function for some field of extremals with γ_i a gradient.</u>

We can get the same result by considering the problem of finding a contravariant field $\xi^i(x_1 \,..\, x_n)$ such that $\frac{\partial \gamma_i}{\partial x_k} = \frac{\partial \gamma_k}{\partial x_i}$, γ_i being the corresponding covariant vector. As in the preceding section we get

$$\Pi_{ik} = \frac{\partial \gamma_i}{\partial x_k} - \frac{\partial \gamma_k}{\partial x_i} = 0$$

and

$$\Pi_{ik}\xi^k = \rho_i - \frac{d}{dx_i}\overline{F}(x,\xi) = 0.$$

Hence if we have a field of extremals, $\rho_i = 0$ and therefore

$$\overline{F}\ (x,\xi) = \text{const. or}$$

$$H\ (x,\gamma) = \text{const.}$$

since $\overline{F}(x,\xi) = \overline{F}(x,\xi(x,\gamma)) = -H(x,\gamma)$. Conversely, if $H(x,\gamma) = \text{const.}$, $\overline{F}(x,\xi) = \text{const.}$ and $\rho_i = 0$, giving a field of extremals.

In this section and the one preceding we have given two different definitions of extremal fields. In the first case we started with such vector fields $\xi^i(x_1 \,..\, x_n)$ that $\mu_i = \frac{\partial F(x,\xi)}{\partial x_i'}$ is a gradient, and showed that each curve tangent to the vector field was an extremal. In the second case we started with an extremal field with $\gamma_i = \frac{\partial \overline{F}}{\partial x_i'}$ $(\overline{F} = F^2)$ being a gradient, and showed that $H(x,\gamma) = -\overline{F}(x,\xi)$ was constant, and hence also $F(x,\xi)$. Now

$$\gamma_i = 2F\,\mu_i \tag{17.5}$$

and if γ_i is a gradient μ_i is also one since F is a constant throughout the space. <u>Thus each extremal field in the second sense is an extremal field in the first sense.</u>

Start now from the first point of view with $\mu_i = \dfrac{\partial F(x,\xi)}{\partial x'_i}$ a gradient. Now since ξ^i is only fixed up to a positive factor we so chose it that $F(x, \xi) =$ const. throughout the space. Then, using (17.5) again, ν_i is seen to be a gradient. *Hence each extremal field in the first sense, if so parameterized that ks is parameter (s arc-length, k fixed throughout the space) is an extremal field in the second sense.*

We consider now the special case of the Riemannian n-space, where $\mathcal{F} = g_{ik}x'_i x'_k$ and $\nu_i = \dfrac{\partial \mathcal{F}}{\partial x'_i} = 2g_{ik}x'_k$. This last has the solution $x'_k = \mathfrak{g}^k(x, \nu) = \frac{1}{2}g^{kp}\nu_p$. Hence

$$H(x,\nu) = -\mathcal{F}(x, \mathfrak{g})$$

$$= -\frac{g_{ik}}{4} g^{ip} g^{kq} \nu_p \nu_q$$

$$= -\frac{1}{4} g^{pq} \nu_p \nu_q$$

We thus see that

$$g^{pq}\frac{\partial V}{\partial x_p}\frac{\partial V}{\partial x_q} = \text{const.}$$

defines the field function V for extremal fields defined by

$$x'_i = -\frac{\partial H}{\partial \nu_i} = \frac{1}{2} g^{ik}\frac{\partial V}{\partial x_k}.$$

(The hyper-surfaces V = const. are such that the extremals of the field are normal to them.)

Chapter IV. Minimum Points and Conjugate Points

18. Consider $E_{P_0P(t)}$ the extremal arc issuing from P_0. We know that there exists some sphere neighborhood in which $E_{P_0P(t)}$ is minimizing. We also know that if $E_{P_0P(t_1)}$ is minimizing $E_{P_0P(\tau)}$, $0 \leqq \tau \leqq t_1$ is too, and if $E_{P_0P(t_2)}$ is not minimizing neither is $E_{P_0P(\tau)}$, $t_2 \leqq \tau$ Thus a class division of all points $P(t)$, $0 \leqq t < \infty$, $P(0) = P_0$, is defined where a point $P(t)$ belongs to class $\mathfrak{A}$ if it and hence all preceding points $P(t')$, $t' < t$, are such that

$E_{P_oP(t)}$ are minimizing, and all other points comprise $\mathcal{L}$. This class division defines a point $P_1 = P(t_1)$ which is either the last point of $\mathfrak{A}$ or the first point of $\mathcal{L}$. P_1 is called the minimum point on E with regard to P_o*.

---

* We note that the work of this section could be carried through considering minimum point to the left on E with regard to P_o, i.e. a point $\tilde{P}_1$ such that if P is to the right of $\tilde{P}_1$ E_{PP_o} is a minimizing arc.

THEOREM. $P_1 \in \mathfrak{A}$

Proof. $P_1 = P(t_1)$ is the minimum point on E with regard to P_o; hence there exists a sequence $t_n \to t_1$, $t_n < t_1$, such that each $E_{P_oP_n}$ $P_n = P(t_n)$, is minimizing.

$$\mathcal{J}(E_{P_oP_n}) = \overline{P_oP_n}.$$

But

$$\mathcal{J}(E_{P_oP_n}) \to \mathcal{J}(E_{P_oP_1}) \quad \text{and} \quad \overline{P_oP_n} \to \overline{P_oP_1} \quad \text{as } n \to \infty$$

Therefore

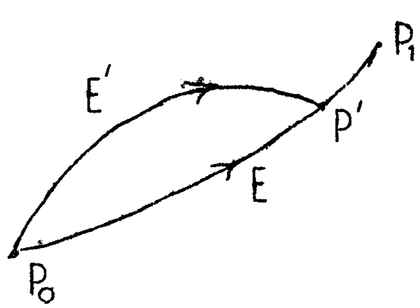
$$\mathcal{J}(E_{P_oP_1}) = \overline{P_oP_1},$$

and $E_{P_oP_1}$ is minimizing.

Obviously $\mathfrak{A}$ contains an interval, namely $\mathcal{N}_\rho(P_o) \cap E_{P_oP(t)}$, $\rho \leq P(P_o)$

THEOREM. If E_{P_oP} is minimizing and P' is on E_{P_oP} between P_o and $\hat{P}$, then no other minimizing arc exists joining P_o to P'.

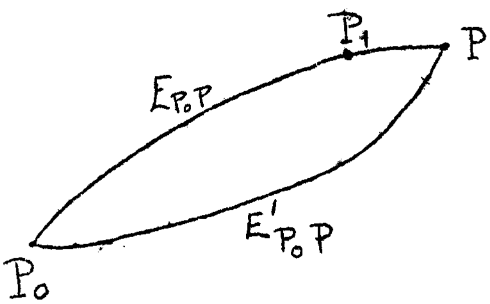
Proof. Let $\hat{E}'$ be another minimizing arc connecting P_o to P' other than E. Then $E' + E_{P'P_1}$ would be minimizing. There exists a corner in this arc at P', for otherwise E and E' would be identical. But, since $\mathcal{E}$ is essentially positive, there can be no such corner in a minimizing arc. Therefore $E' \equiv E$ and E was unique.



The conjugate point to P_o on $E_{P_oP(t)}$ is defined as the first point on $E_{P_oP(t)}$ other than P_o at which $\Delta = \left| \dfrac{\partial \varphi_i(x, y)}{\partial \xi_k} \right|$ vanishes, when $y_i = \xi^i t$ is the extremal $E_{P_oP(t)}$.

THEOREM. If P_1 is not the conjugate point to P_o on $E_{P_oP(t)}$, then there are at least two minimizing arcs joining P_o to P_1.

Proof. Take a point P beyond P_1 on $E_{P_oP(t)}$. Since by definition of P_1, E_{P_oP} is not minimizing and since according to the theorem of section 14 there exists an unbroken minimizing extremal arc $E'_{P_oP_1}$, $E'_{P_oP} \not\equiv E_{P_oP}$ and



$$J(E'_{P_oP}) = \overline{P_oP}.$$

Take a sequence $t_n \to t_1$, $t_n > t_1$, and the corresponding sequence $P(t_n) = P_n \to P_1$, P_n being beyond P_1. As we have just seen, there exist minimizing extremal arcs $E_{nP_oP_n} \not\equiv E_{P_oP_n}$;

$$E_{nP_oP_n}: \quad x_i = \varphi_i(x_o, y_n t)$$

so parameterized that

$$P_n: \quad x_{ni} = \varphi_i(x_o, y_n)$$

Now $F(x_o, y_n) = \overline{P_oP_n}$ and $\overline{P_oP_n} < \overline{P_oP_1} + \epsilon = M$ for some $\epsilon > 0$ and for all n larger than a sufficiently large N. Therefore

$$B \max y_n \leq F(x_o, y_n) < M$$

and the y_n are bounded. We can then pick a convergent subsequence, again denoted by y_n, such that $y_n \to y'$. Consider now

$$E': \quad x_i = \varphi_i(x_o, y' t).$$

$E_n \to E'$ in the sense that $\varphi_i(x_o, y_n t_n) \to \varphi_i(x_o, y' \tau)$ if $t_n \to \tau$. In particular for $t_n = 1$, $x_{ni} = \varphi_i(x_o, y_n) \to \varphi_i(x_o, y')$ and the point P_1 is on E', that is $E'_{P_oP_1}$ is an extremal connecting P_o to P_1. $E'_{P_oP_1}$ is also minimizing, for

$$E_{nP_oP_n} \to E'_{P_oP_1} \quad \text{and}$$

$$\mathcal{J}(E_{nP_oP_n}) = \int_0^{t_n} F(\varphi(x_o, y_n t), \varphi'(x_o, y_n t))\, dt$$

$$\to \int_0^1 F(\varphi(x_o, y' t), \varphi'(x_o, y' t))\, dt$$

$$= \mathcal{J}(E'_{P_oP_1}) .$$

But also $\mathcal{J}(E_{nP_oP_n}) = \overline{P_oP_n}$ and $\overline{P_oP_n} \to \overline{P_oP_1}$.

$$\therefore \quad \mathcal{J}(E'_{P_oP_1}) = \overline{P_oP_1}$$

To complete the proof now we show that if $E'_{P_oP_1} \equiv E_{P_oP_1}$ then P_1 is a conjugate point. Assume $E'_{P_oP_1} \equiv E_{P_oP_1}$, that is that $y'_i = y_i t_1$. Now P_n is given on E and E_n by the parameters t_n and 1 respectively. Then

$$0 = \varphi_i(x_o, y_n) - \varphi_i(x_o, y t_n) \tag{18.1}$$

Consider

$$\Phi_i(\tau) = \varphi_i(x_o, y t_n + \tau(y_n - y t_n))$$

According to (18.1)

$$\Phi_i(1) - \Phi_i(0) = 0$$

or according to the law of the mean

$$\Phi'_i(\theta_i) = \frac{\partial \varphi_i(x_o, y t_n + \theta_i(y_n - y t_n))}{\partial y_p}\left[y_{np} - y_p t_n\right] = 0, \quad 0 < \theta_i < 1. \tag{18.2}$$

Since $E_n \not\equiv E$, $y_{nP} \neq y_p t_n$. Therefore

$$\left| \frac{\partial \varphi_i(x_o, y t_n + \theta_i(y_n - y t_n))}{\partial y_p} \right| = 0, \quad 0 < \theta_i < 1.$$

Now $yt_n \to yt_1$ and $y_n \to y' = yt_1$ as $n \to \infty$ and we have

$$\left| \frac{\partial \varphi_i(x_o, y t)}{\partial y_p} \right| = 0. \tag{18.3}$$

We will later show that since $E'_{P_oP_1}$ is a minimizing arc this determinant does not vanish on $E_{P_oP_1}$ between P_o and P_1. Hence P_1 is the conjugate point of P_o on E, and the theorem is established.

In the course of this proof use was made of the following

Lemma. If an arc $P(t)$, $0 \leqq t \leqq 1$ is covered by a finite number N of neighborhoods $A_o, \ldots, A_{N-1}$, $P(0) \in A_o$, $P(1) \in A_{N-1}$, such that the subarc defined by $a_i \leqq t \leqq a_{i+1}$ lies in A_i, $(i = 0, \ldots, ;\ N-1;\ a_o = 0, a_N = 1)$ and if $P(t)$ is the $\epsilon = 0$ member of a continuous one-parameter family $P(t, \epsilon)$ of arcs, $0 \leqq \epsilon \leqq 1$, $0 \leqq t \leqq 1$, then for sufficiently small ϵ each ϵ arc has the property that its subarc defined by $a_i \leqq t \leqq a_{i+1}$ lies in A_i.

Proof. If this were not so there would exist a sequence ϵ_n such that for some i which is a function of n and for some t_n the point $P(t_n, \epsilon_n)$, $a_i \leqq t_n \leqq a_{i+1}$ would lie in the closed complement $C(A_i)$ of A_i. But there are only a finite number of i. Hence for a subsequence $\epsilon_{n'}$ of ϵ_n a $t_{n'}$, $a_i \leqq t_{n'} \leqq a_{i+1}$ exists for each $\epsilon_{n'}$ such that $P(t_{n'}, \epsilon_{n'}) \in C(A_i)$ for at least one i. Take a convergent subsequence $t_{n''}$ of $t_{n'}$; $t_{n''}$ t. For this subsequence $P_{n''} = P(t_{n''}, \epsilon_{n''}) \rightarrow P(t, 0)$, $a_i \leqq t \leqq a_{i+1}$. This, however, is a contradiction since $P_{n''}$ lies in closed point set $C(A_i)$ while $\lim P_{n''} = P(t, 0)$ lies in A_i.

This lemma was used in writing equation (18.2). For by this lemma the extremal arcs E_n from a certain n on, and the extremal arc E are describable by the same chain $A_o, \ldots, A_{N-1}$ of coordinate neighborhoods, and hence $\varphi_i(x_o, y'_n)$ and $\varphi_i(x_o, y'_n)$ are the same set of functions.

$x_i = \varphi_i(x_o, \xi t)$, $0 \leqq t < \infty$. The equations for an extremal are constructed by using a chain of coordinate neighborhoods covering this extremal arc. These equations depend on this chain in that another extremal arc having the same end points and the same orientation but going along another chain of coordinate neighborhoods may be described in the coordinate neighborhood of the end-point, which may be the same for both chains, by a different set of functions. If

$$E : x_i = \varphi_i(x_0, \xi t)$$

is an extremal, we have only proved by use of the existence theorems that for a neighborhood of $\xi_1, \ldots, \xi_n, t$, (x_0 fixed) this system of functions φ_i exists and is of class C^2. (Of course also this remark shows the use of (18.2) to be permissible.) A second extremal E' passing through $\mathfrak{U}(x)$, a neighborhood of x_i, will have in general initial values ξ' not in the above-mentioned neighborhood of $\xi_1, \ldots, \xi_n, t$. However, in our case E_n from a certain n on will stay in that neighborhood. If E and E' are describable by the same chain of neighborhoods, then the very construction of the sets of functions φ_i show both sets to be the same n functions.

19. The relative minimum point can be defined in the same manner as the minimum point P_1 (more properly called the absolute minimum point). An extremal arc $\dot{E}_{P_oP'}$ is said to be relatively minimizing if its length is less than or equal to the length of any arc connecting P_o to P' and lying in some definite neighborhood $\mathfrak{U}(E_{P_oP'})$ of the extremal arc. Also as before, if $E_{P_oP'}$ is relatively minimizing and P'' is a point on it between P_o and P', then $E_{P_oP''}$ is relatively minimizing. If we describe an E_{P_oP} by $P = P(t)$, $0 \leq t < \infty$, we have by the above remarks a class division $(\mathfrak{A}, \mathfrak{L})$ in the t interval, where $t \in \mathfrak{A}$ if $E_{P_oP(t)}$ is relatively minimizing, and $t \in \mathfrak{L}$ if not. This class division defines a parameter t_2, and, if $t_2 < \infty$, a corresponding point $P_2 = P(t_2)$ which will be called the relative minimum point of P_o on E_{P_oP}. We cannot say in general whether t_2 belongs to class $\mathfrak{A}$ or $\mathfrak{L}$; examples of both cases exist. Due to the existence of $\mathfrak{U}_\rho(P_o)$ we see that $P_2 \neq P_o$. Also it is immediately obvious that the absolute minimum point P_1 either comes before or coincides with the relative minimum point P_2.

We note that the notions of "left" minimum points, relative or absolute, can be defined and the theorems proved concerning the "right" minimum points can be proved for the "left" ones. (See the footnote on page 86.)

Let $x_i = \varphi_i(x_0, \xi t)$, $0 \leq t < \infty$, be a set of extremals issuing from the point $P_0(x_0)$. Along one extremal E_{P_0P} given by ξ we define

$$\Delta(t) = \left|\frac{\partial \varphi_i}{\partial \xi^k}\right|.$$

Since $\dfrac{\partial \varphi_i}{\partial \xi^k} = \dfrac{\partial \varphi_i}{\partial y_k} t$, where $y_k = \xi^k t$, we can write this as

$$\Delta(t) = t^n \left|\frac{\partial \varphi_i}{\partial y_k}\right|$$

Hence $\Delta(0) = 0$, and, since $\left|\dfrac{\partial \varphi_i}{\partial y_k}\right| = 1$ at the point P_0 and is a continuous function, there will be some t-interval, $0 < t < \epsilon$, in which $\Delta(t) \neq 0$. Let $\{t\}_\Delta$ be the set of t values, $t \neq 0$, for which $\Delta(t)$ vanishes. Since $\Delta(t)$ is continuous, $\{t\}_\Delta$ is a closed point set which does not have $t = 0$ as an accumulation point. (In fact it can be shown that $\{t\}_\Delta$ has no accumulation points.) Thus τ, the greatest lower bound of $\{t\}_\Delta$, belongs to $\{t\}_\Delta$, and is the first parameter other than zero at which $\Delta(t) = 0$. $P(\tau)$ is called the conjugate point to P_0 on E_{P_0P}. This definition of conjugate point is independent of the coordinate system. Making a transformation in $\mathcal{U}(P_0)$ and $\mathcal{U}(P(\tau))$,

$$\bar{x}_i = \bar{x}_i(x_1, \cdots, x_n),$$

$$\bar{\xi}^i = \frac{\partial \bar{x}_i}{\partial x_k} \xi^k,$$

we get

$$\frac{\partial \bar{x}_i}{\partial \bar{\xi}^k} = \left[\frac{\partial \bar{x}_i}{\partial x_j}\right]_{\mathcal{U}(P(\tau))} \frac{\partial x_j}{\partial \xi^h} \left[\frac{\partial \xi^h}{\partial \bar{\xi}^k}\right]_{\mathcal{U}(P_0)} \tag{19.1}$$

$$= \left[\frac{\partial \bar{x}_i}{\partial x_j}\right]_{\mathcal{U}(P(\tau))} \frac{\partial x_j}{\partial \xi^h} \left[\frac{\partial x_h}{\partial \bar{x}_k}\right]_{\mathcal{U}(P_0)}$$

and thus

$$\bar{\Delta} = \Delta \left|\frac{\partial \bar{x}_i}{\partial x_j}\right|_{\mathcal{U}(P(\tau))} \left|\frac{\partial x_i}{\partial \bar{x}_j}\right|_{\mathcal{U}(P_0)}. \tag{19.2}$$

Since the two Jacobians in the above formula do not vanish, we see that our statement is correct. (19.1) can be interpreted as showing that $\frac{\partial x_i}{\partial \xi_k}$ is a mixed tensor which is a contravariant vector at the point P and a covariant vector at the point P_o.

We consider now the question of double points on the extremal E_{P_oP}. A point P(t) is called a double point of the extremal if there exists a t', $0 \leq t' < t$, such that

$$(\alpha) \qquad P(t) = P(t').$$

THEOREM. <u>If there are any double points on an extremal, there is a first double point.</u>

Proof. From the set $0 \leq t < \infty$ we take the set $\{t\}_\alpha$ of all t for which the property (α) holds. Let $\sigma = \text{g.l.b.}\{t\}_\alpha$. By the definition of σ there exists a sequence $t_n \to \sigma$, $t_n \geq \sigma$, such that

$$P(t_n) = P(t_n'), \quad 0 \leq t_n' < t_n .$$

Take a subsequence of t_n such that the corresponding t_n' converges to a σ'. Then

$$P(\sigma) = P(\sigma'), \quad 0 \leq \sigma' \leq \sigma.$$

If $\sigma' < \sigma$, $P(\sigma)$ is a double point, and, since $\sigma = \text{g.l.b.}\{t\}_\alpha$, it is the <u>first</u> double point. If $\sigma' = \sigma$ then in each neighborhood of σ there would be two different parameters t_n and t_n' whose maps on the extremal E_{P_oP} are identical. This, however, is in contradiction to the fact that to each point of an arc of class C^1 there exists a neighborhood on the curve and a neighborhood on the parameter interval which are in one-to-one correspondence. (This proof actually shows more, namely that the set of all double points on an arc of class C^1 is closed.)

20. THEOREM. <u>t_2 is the smaller of τ and σ; that is, the relative minimum point coincides with the first of the two points, the conjugate point and the first double point.</u>

The rather long proof of this theorem will be presented in two cases:

<u>First case.</u> Let $\sigma \geq \tau$. We will show that $t_2 = \tau$. The proof of this case is in two parts:

We will show that $t_2 \geq \tau$. Let $\bar{E}_{P_0 P(t')} = E'$, $0 < t' < \tau$. Since $\sigma \geq \tau$, there are no double points on E'; to each t, $0 \leq t \leq t'$, corresponds one and only one point $P(t)$ on E'. Also on E'

$$\tilde{\Delta}(t) = \left| \frac{\partial \varphi_i}{\partial y_k} \right| \neq 0$$

We now have the following situation: there exists a continuous mapping of $\{y\}$, the n-dimensional number space, onto our point space γ; under this mapping our extremal arc E' corresponds to the closed compact "straight line"

$$e' : \ \xi^i t, \ 0 \leq t \leq t',$$

in the y space; the correspondence between e' and E' is one to one; and, since $\left| \frac{\partial \varphi_i}{\partial y_k} \right| \neq 0$ on E', there exists for each y on e' a neighborhood $\mathcal{U}(y)$ which is in one-to-one correspondence with its map. Thus all of the assumptions of the Bolza Theorem (page 50) are satisfied, and we can conclude that there exists a neighborhood $\mathcal{U}(e')$ of e' in one-to-one correspondence with its map $\mathcal{U}(E')$.

Lemma. <u>There exists a neighborhood $\mathcal{U}_1(e') \subset \mathcal{U}(e')$ such that $\left| \frac{\partial \varphi_i}{\partial y_k} \right| \neq 0$ for any y in $\mathcal{U}_1(e')$.</u>

If this were not true, take a sequence of neighborhoods $\mathcal{U}_{\rho_n}(e')$, $\rho_n \to 0$, $\rho_1 > \rho_2 > \rho_3 > \ldots$, where $\mathcal{U}_{\rho_n}(e')$ is defined by means of any metric in the y space. Obviously for all n greater than some N, $\mathcal{U}_{\rho_n}(e') \subset \mathcal{U}(e')$, since $\mathcal{U}(e')$ is an open point set. Now if the lemma were not true there would be a sequence of points $y_n \in \mathcal{U}_{\rho_n}(e')$, such that $y_n \to \tilde{y}$, $\tilde{y} \in e'$, and

$$\left| \frac{\partial \varphi_i(x_0, y_n)}{\partial y_k} \right| = 0$$

Then $\left| \frac{\partial \varphi_i(x_0, \tilde{y})}{\partial y_k} \right| = 0$, $\tilde{y} \in e'$, contradicting our assumptions. Thus the lemma is true and the existence of $\mathcal{V}(e') \leftrightarrow \mathcal{V}(E')$, such that $\left|\frac{\partial \phi_i}{\partial y_k}\right| \neq 0$ in $\mathcal{V}(e')$, is shown.

The theorem of Dini shows that the inverse function $y = f(P)$ will be of class C^1, and of class C^2 for $P \neq P_o$. Thus we have a neighborhood of E' describable by the normal coordinates of the point P_o.

We now try to find a neighborhood in $\mathcal{V}(e')$ with the property that if it contained a point y it contained the entire "straight line" yt, $0 \leq t \leq 1$, that is to say, a neighborhood containing an extremal field. If e' is $y = y_1 t$, $0 \leq t \leq 1$, consider extremals $\tilde{y}t$ such that the euclidean angle $\measuredangle \tilde{y}y_1 < \alpha$ and the range of t is the interval from 0 to $1 + \alpha$; $\tilde{e} : y = \tilde{y}t$, $0 \leq t \leq 1 + \alpha$. For sufficiently small α, we state that all $\tilde{e}$ will lie in $\mathcal{V}(e')$. If not, then for some sequence $\alpha_n \to 0$ we could find two sequences y_n and t_n such that

$$\measuredangle\, y_n y_1 < \alpha_n \,, \qquad 0 \leq t_n \leq 1+\alpha,$$

but $y_n t_n \in C(\mathcal{V}(e'))$, the complement of $\mathcal{V}(e')$. Take convergent subsequences of y_n and t_n, to be again denoted by y_n and t_n, such that $y_n \to \bar{y}$, $t_n \to \bar{t}$. Then

$$\measuredangle\, \bar{y}y_1 = 0, \; 0 \leq \bar{t} \leq 1,$$

together with $\bar{y}\bar{t} \in C(\mathcal{V}(e'))$. But then $\bar{y} = y_1$ and $\bar{y}\bar{t}$ lies on e', giving rise to a contradiction. Thus for α less than a certain ϵ all extremal arcs yt of length less than $1 + \alpha$ and lying in the cone given by the point P_o and the euclidean angle $\measuredangle \bar{y}y < \alpha$ will lie in $\mathcal{V}(e')$. Take the open point set consisting of all these extremal arcs and add to it a neighborhood $\mathcal{W}_\rho(y_o) \subset \mathcal{V}(e')$; the result, $\tilde{\mathcal{V}}(e')$, will be a neighborhood of the desired property. We can use our previous results for neighborhoods describable by normal coordinates and obtain the result that the extremal arc e' is minimizing relative to all admissable arcs in $\tilde{\mathcal{V}}(e')$. But e' can extend as close as desired to the conjugate point; hence the relative minimum point can not come before the conjugate point, that is $t_2 \geq \gamma$.

Before proving $t_2 \leq \gamma$ in the first case, to obtain $t_2 = \gamma$ we shall consider the second case.

Second case. Let $\gamma \geq \sigma$ We will show that $t_2 = \sigma$ Take a t' such that $0 \leq t' < \sigma$ On $E_{P_0P(t')}$ there is no double point, and

$$\tilde{\Delta} = \left| \frac{\partial \varphi_i}{\partial y_k} \right| \neq 0.$$

Hence by the results just obtained in Case I, $E_{P_0P(t')}$ is minimizing relative to some neighborhood of it, and $t_2 \geq \sigma$ But if $t' \geq \sigma$, the extremal arc $E_{P_0P(t')}$ contains a double point, and hence can't be minimizing relative to any neighborhood. Thus $t_2 = \sigma$, and Case II is proved.

In order to prove $t_2 \leq \gamma$ if $\sigma \geq \gamma$, the second part of the First Case of the theorem, it is necessary to have a knowledge of the second variation and its properties. Therefore, keeping in mind that the proof of the theorem is as yet incomplete, we turn to the consideration of the second variation.

21. Let $x = x(t)$, $0 \leq t \leq 1$ be the equations of an extremal E, and embed it in a one-parameter family of curves, $x_i = x_i(t, \epsilon)$, $0 \leq t \leq 1$, $-1 \leq \epsilon \leq 1$, which pass through the end points of E, and of which E is the $\epsilon = 0$ member. Consider then the two integrals

$$\mathcal{I}(\epsilon) = \int_0^1 F(x, x')\,dt,$$

$$\mathcal{J}(\epsilon) = \int_0^1 \mathcal{F}(x, x')\,dt,$$

where $\mathcal{F} = F^2$. Let us assume the family $x_i(t, \epsilon)$ to be continuous and possess the following continuous derivatives: $\frac{\partial x_i}{\partial t}$, $\frac{\partial x_i}{\partial \epsilon}$, $\frac{\partial^2 x_i}{\partial \epsilon \partial t}$, $\frac{\partial^2 x_i}{\partial \epsilon^2}$, $\frac{\partial^3 x_i}{\partial \epsilon^2 \partial t}$. We can now differentiate the above integrals, getting

$$\mathcal{J}'(\epsilon) = 2\int_0^1 F \frac{\partial F}{\partial \epsilon}\,dt$$

and

$$\frac{1}{2}\mathcal{J}''(\epsilon) = \int_0^1 \left(\frac{\partial F}{\partial \epsilon}\right)^2 dt + \int_0^1 F\frac{\partial^2 F}{\partial \epsilon^2}\, dt.$$

$\mathcal{J}''_0 = \mathcal{J}''(0)$ and $J''_0 = J''(0)$ are called the second variations of E (for the F^2 and F problems respectively) with respect to the family of comparison curves $x_i(t, \epsilon)$. Parameterizing E, the $\epsilon = 0$ member, with a constant times arc-length with the same limits, 0, 1, we get, since F = k along E,

(21.1) $$\mathcal{J}'_0 = 2k\,J'_0$$

(21.2) $$\frac{1}{2}\mathcal{J}''_0 = \int_0^1 \left(\frac{\partial F}{\partial \epsilon}\right)^2 dt + k\,J''_0, \quad k > 0.$$

(21.1) gives us the known result that an extremal of the F problem if parameterized with a constant times arc-length is an extremal of the $\mathcal{F} = F^2$ problem.

(21.2) gives us the inequality,

(21.3) $$\tfrac{1}{2}\mathcal{J}''_0 \geq k\,J''_0, \quad k > 0,$$

between the second variations of the two problems, F and F^2.

We make the following note:

$$\begin{aligned}\frac{\partial F}{\partial \epsilon} &= \frac{\partial F}{\partial x_i}\frac{\partial x_i}{\partial \epsilon} + \frac{\partial F}{\partial x'_i}\frac{\partial x'_i}{\partial \epsilon} \\ &= \left(\rho_i + \frac{d}{dt}\frac{\partial F}{\partial x'_i}\right)\frac{\partial x_i}{\partial \epsilon} + \frac{\partial F}{\partial x'_i}\frac{\partial x'_i}{\partial \epsilon} \\ &= \rho_i\frac{\partial x_i}{\partial \epsilon} + \frac{d}{dt}\left(\frac{\partial F}{\partial x'_i}\frac{\partial x_i}{\partial \epsilon}\right).\end{aligned}$$

Hence

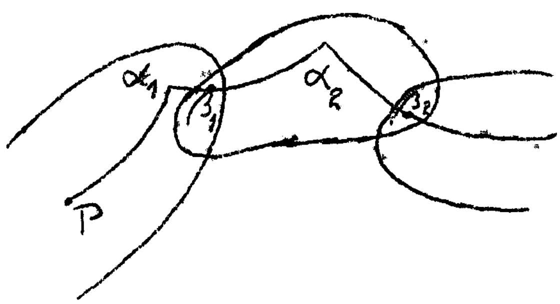
$$\left(\frac{\partial F}{\partial \epsilon}\right)_{\epsilon=0} = \left[\frac{d}{dt}\left(\frac{\partial F}{\partial x'_i}\frac{\partial x_i}{\partial \epsilon}\right)\right]_{\epsilon=0}.$$

If the family be so chosen that $\frac{\partial F}{\partial x'_i}\frac{\partial x_i}{\partial \epsilon} =$ const. on the $\epsilon = 0$ member, then (21.2) becomes $\frac{1}{2}\mathcal{J}''_0 = k\,J''_0$, $k > 0$.

Because of the vanishing of the determinant $\left|\frac{\partial^2 F}{\partial x'_i \partial x'_k}\right|$ the use of the second variation $\mathcal{J}''_0$ of the $\mathcal{F}$ problem rather than J''_0, that of the F problem, will be necessary. However, the two second variations are connected by

(21.2) and (21.3).

22. In deriving the expressions for the second variation, an accurate description of the nature of the family $x_i(t, \epsilon)$, $0 \leq t \leq 1$, $-1 \leq \epsilon \leq 1$, is necessary. We take $x_i(t, \epsilon)$ of class D^2, where we mean by that that the t-interval can be divided into a finite set of sub-intervals in each of which any member of the family is of class C^2. If α_i, $i = 0, 1, \ldots, N-1$, be the corner points of a member of the family, we require that the member be continuous at α_i, and that α_i be independent of ϵ. Besides the α-points, other points β_j $j = 1, \ldots, M$, are of importance in this work. A β-point is one lying in the intersection of two coordinate neighborhoods, which is the end-point of an arc described by one coordinate system. We shall take care that no β-point is also a α-point, which of course can be done. We also assume in each sub-interval of the curve in which it is of class C^2 the existence and continuity of all derivatives necessary to differentiate under the integral sign and to integrate by parts in what follows. The existence and continuity of the first and second partial derivatives of x and x' and the first partial derivative of x", all with respect to ϵ will suffice.



We note that since $x_i(0, \epsilon)$ and $x_i(1, \epsilon)$ are independent of ϵ, and since $x_i(\alpha^-, \epsilon) = x_i(\alpha^+, \epsilon)$ for all ϵ, $\frac{\partial x_i}{\partial \epsilon}$ and $\frac{\partial^2 x_i}{\partial \epsilon^2}$ vanish at $t = 0$ and 1, and are continuous at the corner points.

We derive a formula for the second variation of an extremal arc E_{P_0P} for the F' problem.

hence

$$J'(\epsilon) = \int_0^1 \left(\frac{\partial F}{\partial x_i}\frac{\partial x_i}{\partial \epsilon} + \frac{\partial F}{\partial x'_i}\frac{\partial x'_i}{\partial \epsilon}\right) dt.$$

Split the path up into intervals containing α and β points as end-points only; we can then integrate by parts and get

$$J'(\epsilon) = \int_0^1 \left(\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial x'_i}\right)\frac{\partial x_i}{\partial \epsilon}\, dt$$

$$+ \left[\frac{\partial F}{\partial x'_i}\frac{\partial x_i}{\partial \epsilon}\right]_{\alpha +}^{\alpha -} + \left[\frac{\partial F}{\partial x'_i}\frac{\partial x_i}{\partial \epsilon}\right]_{\beta +}^{\beta -},$$

where the bracket terms are summed for all α and β points. However, since $\frac{\partial F}{\partial x'_i}\frac{\partial x_i}{\partial \epsilon}$ is an invariant $\left[\frac{\partial F}{\partial x'_i}\frac{\partial x_i}{\partial \epsilon}\right]_{\beta +}^{\beta -} = 0$, and the above equation can be written as

(22.1) $$J'(\epsilon) = \int_0^1 \rho_i \frac{\partial x_i}{\partial \epsilon}\, dt - \left[\frac{\partial F}{\partial x'_i}\frac{\partial x_i}{\partial \epsilon}\right]_{\alpha -}^{\alpha +}.$$

Hence

(22.2) $$J''(0) = \int_0^1 \frac{\partial \rho_i}{\partial \epsilon}\frac{\partial x_i}{\partial \epsilon}\, dt - \left[\frac{\partial}{\partial \epsilon}\left(\frac{\partial F}{\partial x'_i}\right)\frac{\partial x_i}{\partial \epsilon} + \frac{\partial F}{\partial x'_i}\frac{\partial^2 x_i}{\partial \epsilon^2}\right]_{\alpha -}^{\alpha +},$$

where ϵ is set equal to zero throughout the right-hand side. This equation, in spite of its appearance, is an <u>invariant equation</u>; since $\rho_i = 0$ on E, the $\epsilon = 0$ member of the family, $\left(\frac{\partial \rho_i}{\partial \epsilon}\right)_{\epsilon = 0}$ is a covariant vector, while each term in the brackets is an invariant as we shall show. Since $\frac{\partial x_i}{\partial \epsilon}$ and $\frac{\partial F}{\partial x'_i}$ (E is an unbroken extremal) are continuous at the α points, the bracket term can be written as

$$\left[\frac{\partial}{\partial \epsilon}\left(\frac{\partial F}{\partial x'_i}\right)\right]_{\alpha -}^{\alpha +}\frac{\partial x_i(\alpha, 0)}{\partial \epsilon} + \frac{\partial F}{\partial x'_i}\left[\frac{\partial^2 x_i}{\partial \epsilon^2}\right]_{\alpha -}^{\alpha +}.$$

Let $\bar{x}_i = \bar{x}_i(x_1, \ldots, x_n)$. Then

$$\frac{\partial \bar{x}_i}{\partial \epsilon} = \frac{\partial \bar{x}_i}{\partial x_k}\frac{\partial x_k}{\partial \epsilon},$$

$$\frac{\partial F}{\partial \bar{x}'_i} = \frac{\partial x_k}{\partial \bar{x}_i}\frac{\partial F}{\partial x'_k},$$

and

$$\frac{\partial^2 \bar{x}_i}{\partial \epsilon^2} = \frac{\partial \bar{x}_i}{\partial x_k}\frac{\partial^2 x_k}{\partial \epsilon^2} + \frac{\partial^2 \bar{x}_i}{\partial x_p \partial x_q}\frac{\partial x_p}{\partial \epsilon}\frac{\partial x_q}{\partial \epsilon},$$

$$\frac{\partial}{\partial \epsilon}\left(\frac{\partial F}{\partial \bar{x}'_i}\right) = \frac{\partial x_k}{\partial \bar{x}_i}\frac{\partial}{\partial \epsilon}\left(\frac{\partial F}{\partial x'_k}\right) + \frac{\partial^2 x_k}{\partial \bar{x}_i \partial \bar{x}_l}\frac{\partial F}{\partial x'_k}\frac{\partial \bar{x}_l}{\partial \epsilon}.$$

Since $\left[\frac{\partial x_p}{\partial \epsilon}\right]_{\alpha-}^{\alpha+} = 0$, $\left[\frac{\partial F}{\partial x'_k}\right]_{\alpha-}^{\alpha+} = 0$, we see that $\left[\frac{\partial^2 x_i}{\partial \epsilon^2}\right]_{\alpha-}^{\alpha+}$ is a contravariant vector and $\left[\frac{\partial}{\partial \epsilon}\left(\frac{\partial F}{\partial x'_i}\right)\right]_{\alpha-}^{\alpha+}$ is a covariant vector, and hence that each term in the bracket is an invariant. In fact, the second term in the bracket drops out and we get

$$J''(0) = \int_0^1 \frac{\partial \rho_i}{\partial \epsilon}\frac{\partial x_i}{\partial \epsilon}\,dt - \left[\frac{\partial}{\partial \epsilon}\left(\frac{\partial F}{\partial x'_i}\right)\right]_{\alpha-}^{\alpha+}\frac{\partial x_i(\alpha, 0)}{\partial \epsilon}. \tag{22.3}$$

<u>Notice that this expression for the second variation is independent of $\frac{\partial^2 x_i}{\partial \epsilon^2}$</u>

Another expression for $J''(0)$ will be deduced in the following manner. Differentiate the equation for the Euler vector ρ_i, and get

$$\frac{\partial \rho_i}{\partial \epsilon} = \frac{\partial^2 F}{\partial x_i \partial x_k}\frac{\partial x_k}{\partial \epsilon} + \frac{\partial^2 F}{\partial x_i \partial x'_k}\frac{\partial x'_k}{\partial \epsilon} - \frac{d}{dt}\left(\frac{\partial^2 F}{\partial x'_i \partial x_k}\frac{\partial x_k}{\partial \epsilon} + \frac{\partial^2 F}{\partial x'_i \partial x'_k}\frac{\partial x'_k}{\partial \epsilon}\right).$$

We set

$$P_{ik}(x, x') = \frac{\partial^2 F}{\partial x_i \partial x_k},\quad Q_{ik}(x, x') = \frac{\partial^2 F}{\partial x_i \partial x'_k} \neq Q_{ki},\quad R_{ik}(x, x') = \frac{\partial^2 F}{\partial x'_i \partial x'_k}$$

(only $R_{ik}(x, x')$ is a tensor), and set up the quadratic form

(22.4) $2\Omega(\eta,\dot\eta) = P_{ik}\eta^i\eta^k + 2Q_{ik}\eta^i\dot\eta^k + R_{ik}\dot\eta^i\dot\eta^k$

in the 2n variables $\eta^i(t)$, $\dot\eta^i(t)$, $i = 1, \ldots, n$. Then

$$\frac{\partial f_i}{\partial \epsilon} = P_{ik}\eta^k + Q_{ik}\dot\eta^k - \frac{d}{dt}(Q_{ki}\eta^k + R_{ik}\dot\eta^k)$$

$$= \frac{\partial\Omega(\eta,\dot\eta)}{\partial\eta^i} - \frac{d}{dt}\frac{\partial\Omega(\eta,\dot\eta)}{\partial\dot\eta^i} = \psi_i(\eta)$$

where $\eta^i = \frac{\partial x_i}{\partial\epsilon}$, $\dot\eta^i = \frac{\partial x_i'}{\partial\epsilon}$, and where that equation is taken as the definition of $\psi_i(\eta)$. Thus $\psi_i(\eta)$, a function of the arbitrary contravariant vector $\eta^k(t)$ is a covariant vector along E. This is evident since $x_i(t) + \epsilon\,\eta^i(t)$ defines, on each segment of E which is covered by one coordinate system, a family for which $\eta^i = \frac{\partial x_i}{\partial\epsilon}$. Then $\psi_i(\eta) = \left(\frac{\partial f_i}{\partial\epsilon}\right)_{\epsilon=0}$ is a covariant vector. (22.3) can now be written as

$$\text{(22.4)} \qquad \mathcal{J}''(0) = \int_0^1 \psi_i(\eta)\eta^i dt - \left[\frac{\partial}{\partial\epsilon}\left(\frac{\partial F}{\partial x_i'}\right)\right]_{\alpha-}^{\alpha+}\eta^i(\alpha).$$

Since

$$\frac{\partial\Omega}{\partial\dot\eta^i} = Q_{ki}\eta^k + R_{ki}\dot\eta^k = \frac{\partial}{\partial\epsilon}\left(\frac{\partial F}{\partial x_i'}\right)$$

we get

$$\text{(22.4')} \qquad \mathcal{J}''(0) = \int_0^1 \psi_i(\eta)\eta^i dt - \left[\frac{\partial\Omega}{\partial\dot\eta^i}\right]_{\alpha-}^{\alpha+}\eta^i(\alpha).$$

Another expression for $\mathcal{J}''(0)$ can be obtained by integrating both sides of the equation

$$\text{(22.5)} \qquad \frac{\partial^2 F}{\partial\epsilon^2} = 2\Omega\left(\frac{\partial x}{\partial\epsilon}, \frac{\partial x'}{\partial\epsilon}\right) + \frac{\partial F}{\partial x_i}\frac{\partial^2 x_i}{\partial\epsilon^2} + \frac{\partial F}{\partial x_i'}\frac{\partial^2 x_i'}{\partial\epsilon^2}$$

$$= 2\Omega + f_i\frac{\partial^2 x_i}{\partial\epsilon^2} + \frac{d}{dt}\left(\frac{\partial F}{\partial x_i'}\frac{\partial^2 x_i}{\partial\epsilon^2}\right)$$

This gives

$$\mathcal{J}''(0) = \int_0^1 \frac{\partial^2 F^1}{\partial \epsilon^2}\, dt \tag{22.6}$$
$$= 2\int_0^1 \Omega\left(\frac{\partial x}{\partial \epsilon}, \frac{\partial x'}{\partial \epsilon}\right) dt + \left(\frac{\partial F^1}{\partial x_i'} \frac{\partial^2 x_i}{\partial \epsilon^2}\right)_{\beta+}^{\beta-}$$

The middle term of (22.5) drops out in the integration since $\rho_i = 0$ on E. The second term in (22.6) is not usually given, but is nécessary if one is working in more than one coordinate system. This equation, (22.6), involves $\frac{\partial^2 x_i}{\partial \epsilon^2}$ but this expression can be removed as follows:

$$\left(\frac{\partial F^1}{\partial x_i'} \frac{\partial^2 x_i}{\partial \epsilon^2}\right)_{\beta+}^{\beta-} = \frac{\partial}{\partial \epsilon}\left(\frac{\partial F^1}{\partial x_i'} \frac{\partial x_i'}{\partial \epsilon}\right)_{\beta+}^{\beta-} - \left[\frac{\partial}{\partial \epsilon}\left(\frac{\partial F^1}{\partial x_i'}\right)\frac{\partial x_i}{\partial \epsilon}\right]_{\beta+}^{\beta-};$$

but

$$\left(\frac{\partial F^1}{\partial x_i'} \frac{\partial x_i'}{\partial \epsilon}\right)_{\beta+}^{\beta-} = 0$$

and, as before,

$$\frac{\partial}{\partial \epsilon}\left(\frac{\partial F^1}{\partial x_i'}\right) = \frac{\partial \Omega}{\partial \dot{\eta}^i};$$

hence we can write (22.6) as

$$\mathcal{J}''(0) = 2\int_0^1 \Omega\left(\frac{\partial x}{\partial \epsilon}, \frac{\partial x'}{\partial \epsilon}\right) dt - \left[\frac{\partial \Omega}{\partial \dot{\eta}^i} \frac{\partial x_i}{\partial \epsilon}\right]_{\beta+}^{\beta-} \tag{22.7}$$

an equation which could also be obtained from (22.4).

23. The equations for $\eta^i(t)$,

$$\psi_i(\eta) = P_{ik}(x,x')\eta^k + Q_{ik}(x,x')\eta'^k \tag{23.1}$$
$$-\frac{d}{dt}\left(Q_{ki}(x,x')\eta^k + R_{ki}(x,x')\eta'^k\right) = 0$$

are called the Jacobi differential equations of the variation problem F^1. They are linear and of the second order in $\eta(t)$, and as we have seen, are invariant under coordinate transformation if x(t) is an extremal arc. Since $|R_{ik}(x\ \ x')| \neq 0$ for the F^1 problem, they can be solved for η'' and put in the normal form

(23.2) $$\eta''^{i} = A_{iK}(t)\eta^{K} + B_{iK}(t)\eta'^{K}$$

This reduction to normal form is impossible in the F-problem; for this reason we must use the $\overline{F}$ -problem in this work on the second variation.

We now endeavor in the following work to characterize conjugate points by means of solutions of the Jacobi differential equations. From the existence theorem for the set of equations (23.2) we know that any solution $\eta^{i}(t)$ of $\psi_i(\eta) = 0$ is uniquely determined by arbitrarily given initial values $\eta^{i}(t_o)$. $\frac{d}{dt}\eta^{i}(t_o)$ at a given point t_o. From this we deduce that if any 2n solutions η^{i}_{α} $\alpha = 1, \dots, 2n$, of (23.2) are known such that

(23.3) $$D(t) = \left|\eta^{i}_{\alpha}, \frac{d}{dt}\eta^{i}_{\alpha}\right| \neq 0$$

for t_o, then these 2n solutions form a basis for all solutions, i.e. any solution of $\psi_i(\eta) = 0$ is linearly expressible with constant coefficient in terms of the 2n η^{i}_{α}. Since $\psi_i(\eta) = 0$ are linear, any such linear combination of the η^{i}_{α} form a solution, and since we have assumed $D(t_o) = 0$, the equations

$$\eta^{i}(t_o) = A_{\alpha}\eta^{i}_{\alpha}$$

$$\frac{d}{dt}\eta^{i}(t_o) = A_{\alpha}\frac{d}{dt}\eta^{i}_{\alpha}$$

with given $\eta^{i}(t_o)$ $\frac{d}{dt}\eta^{i}(t_o)$ are uniquely solvable for A_{α}. Hence the solution $A_{\alpha}\eta^{i}_{\alpha}$ is the unique solution with given initial values.

The determinant D(t) is such that if it is zero for some t it is zero for all t. To show this, differentiate D(t). In differentiating the columns involving the η^{i}_{α} you get determinants equal to zero; in differentiating the columns involving $\frac{d}{dt}\eta^{i}_{\alpha}$ you substitute from (23.2) and simplify, getting determinants of the form $B_{ii}D$. Hence

$$D' = \sum B_{ii}D$$

and

$$D = C e^{\int \sum_i B_{ii}\, dt}$$

thus D has no zeros unless C = 0, in which case D ≡ 0. Now using the fact that

$\rho_i \equiv 0$ in x_0, ξ, t, for $x_i = \varphi_i(x_0, \xi t)$, we get

(23.4) $$\left[\frac{\partial F(x,x')}{\partial x_i} - \frac{d}{dt}\frac{\partial F(x,x')}{\partial x'_i}\right]_{x=\varphi(x_0,\xi t)} \equiv 0 .$$

Therefore, since $\psi_i(\eta) = \frac{\partial \rho_i}{\partial \epsilon}$,

$$\eta^i = \frac{\partial \varphi_i}{\partial x_{0j}}, \quad \eta^i = \frac{\partial \varphi_i}{\partial \xi^j}, \quad \eta^i = \frac{\partial \varphi}{\partial t}, \quad j = 1, \cdots, n,$$

are solutions of $\psi_i(\eta) = 0$. For example, differenciate (23.4) with respect to ξ^j and get

$$0 \equiv \frac{\partial^2 F(\varphi,\varphi')}{\partial x_i\, \partial x_k}\frac{\partial \varphi_k}{\partial \xi^j} + \frac{\partial^2 F(\varphi,\varphi')}{\partial x_i\, \partial x'_k}\frac{\partial \varphi'_k}{\partial \xi^j} - \frac{d}{dt}\left[\frac{\partial^2 F(\varphi,\varphi')}{\partial x'_i\, \partial x_k}\frac{\partial \varphi_k}{\partial \xi^j} + \frac{\partial^2 F(\varphi,\varphi')}{\partial x'_i\, \partial x'_k}\frac{\partial \varphi'_k}{\partial \xi^j}\right], \text{ or}$$

$$\psi_i\left(\frac{\partial \varphi}{\partial \xi^j}\right) = 0.$$

(Note that all derivatives entering into this calculation exist.)

In a similar manner it can be shown that $\frac{\partial \varphi_i}{\partial x_{0j}}$ and $\frac{\partial \varphi_i}{\partial t}$ are also solutions.

Since $\varphi_i(x_0, 0) = x_0 i$, $\frac{d}{dt}\varphi_i(x_0, 0) = \xi^i$, we see that

(23.5) $$\begin{cases} \left.\dfrac{\partial \varphi_i}{\partial x_{0j}}\right|_{t=0} = \delta^i_j, & \left.\dfrac{d}{dt}\dfrac{\partial \varphi_i}{\partial x_{0j}}\right|_{t=0} = 0, \\ \left.\dfrac{\partial \varphi_i}{\partial \xi^j}\right|_{t=0} = 0, & \left.\dfrac{d}{dt}\dfrac{\partial \varphi_i}{\partial \xi^j}\right|_{t=0} = \delta^i_j . \end{cases}$$

Therefore for the 2n solutions $\frac{\partial \varphi}{\partial x_{0j}}, \frac{\partial \varphi}{\partial \xi^j}$ at $t_0 = 0$

$$D(0) = \left|\begin{array}{cc:cc} 1 & 0 & & \\ 0 & 1 & \multicolumn{2}{c}{O} \\ \hdashline & & 1 & 0 \\ \multicolumn{2}{c:}{O} & 0 & 1 \end{array}\right| = 1,$$

and the 2n solutions $\frac{\partial \phi}{\partial x_{0j}}, \frac{\partial \phi}{\partial \xi^j}$ of $\psi_i(\eta) = 0$ are linearly independent, and form a basis for all solutions. Thus for any solution $\eta^i(t)$ of $\psi_i(\eta) = 0$ we have

$$\eta^i(t) = \alpha^j \frac{\partial \phi_i}{\partial x_{0j}} + \beta^j \frac{\partial \phi_i}{\partial \xi^j}$$

where α^j and β^j are constants and contravariant vectors at the point t = 0. Because of (23.5) we have

$$\eta^i(0) = \alpha^i$$

and hence $\alpha^i = 0$ for any solution of $\psi_i(\eta) = 0$ vanishing at t = 0; therefore

$$\eta^i(t) = \beta^j \frac{\partial \phi_i}{\partial \xi^j}, \qquad \beta^j \text{ const.,} \tag{23.6}$$

is the general solution of the Jacobi differential equations which vanishes at t = 0.

τ, the parameter of the conjugate point $P(\tau)$ of P_0 on E, was defined as the first parameter different from zero for which $\left|\frac{\partial \phi_i}{\partial \xi^j}\right| = 0$. Hence the equations

$$\beta^j \left(\frac{\partial \phi_i}{\partial \xi^j}\right)_{t=\tau} = 0$$

are solvable by a non-trivial set $\beta^1, \ldots, \beta^n$. The corresponding solution $\eta^i(t)$ of the Jacobi equations (see (23.6)) therefore vanishes at τ. Also, since τ was the first parameter different from zero at which $\left|\frac{\partial \phi_i}{\partial \xi^j}\right| = 0$, there is no solution of $\psi_i(\eta) = 0$ which vanishes at t = 0 and at $t' < \tau$. Thus the

THEOREM: τ, the parameter of the conjugate point $P(\tau)$ of P_0 on E, is the first parameter after t = 0 in the given orientation for which a solution $\eta(t)$ of the Jacobi differential equations which vanishes at t = 0 will vanish again.

This will be called the second characteristic property of the conjugate point, the first being the vanishing of $\left|\frac{\partial \varphi_i}{\partial \xi^j}\right|$.

It was important to know in this work that ~~we~~ had, in (23.6), the <u>general</u> solution of $\psi_i(\eta) = 0$ which vanished at t = 0; to obtain this we had to work with the $\overline{F}$-problem rather than the F-problem.

In the same way that we defined $P(\tau)$, the conjugate point on the right of P_0 on E, as the first point other than t = 0 at which $\left|\frac{\partial \varphi_i}{\partial \xi^j}\right| = 0$, we can define $P(\overline{\tau})$, the conjugate point on the left of P_0 on E, as the last point other than t = 0 at which $\left|\frac{\partial \psi_i}{\partial \xi^j}\right| = 0$, where ψ_i, as in (8.2)", is the expression for an extremal E coming into P_0. The proof we have just completed can be easily modified for this case, and we have the second characteristic property for left conjugate points. Thus for $P(\overline{\tau})$, there exists a solution of the Jacobi equations vanishing at $P(\overline{\tau})$ and at P_0, and there is no solution vanishing at P_0 and $\overline{P}$, $P(\overline{\tau}) \leq \overline{P} < P_0$. However the existence of a solution of $\psi_i(\eta) = 0$ which vanishes at $P(\overline{\tau})$ and at $\overline{P}$, $P(\overline{\tau}) < \overline{P} < P_0$, is still a logical possibility; that is, the right conjugate point of the left conjugate point of a point P_0 need not be P_0. Later, by using a third characteristic property of the conjugate point which we shall prove, we shall exclude this possibility, and have that the $\begin{smallmatrix}\text{left}\\\text{right}\end{smallmatrix}$ conjugate point of the $\begin{smallmatrix}\text{right}\\\text{left}\end{smallmatrix}$ conjugate point of a point P is P.

Using the definition of $\psi_i(\eta)$ we obtain

$$\begin{aligned}
\xi^i\psi_i(\eta) - \eta^i\psi_i(\xi) &= \left[\frac{\partial\Omega(\eta,\dot\eta)}{\partial\eta^i} - \frac{d}{dt}\frac{\partial\Omega(\eta,\dot\eta)}{\partial\dot\eta^i}\right]\xi^i \\
&\quad - \left[\frac{\partial\Omega(\xi,\dot\xi)}{\partial\xi^i} - \frac{d}{dt}\frac{\partial\Omega(\xi,\dot\xi)}{\partial\dot\xi^i}\right]\eta^i \\
&= \left[\frac{\partial\Omega(\eta,\dot\eta)}{\partial\eta^i}\xi^i + \frac{\partial\Omega(\eta,\dot\eta)}{\partial\dot\eta^i}\dot\xi^i\right] - \frac{d}{dt}\left(\frac{\partial\Omega(\eta,\dot\eta)}{\partial\dot\eta^i}\xi^i\right) \\
&\quad - \left[\frac{\partial\Omega(\xi,\dot\xi)}{\partial\xi^i}\eta^i + \frac{\partial\Omega(\xi,\dot\xi)}{\partial\dot\xi^i}\dot\eta^i\right] + \frac{d}{dt}\left(\frac{\partial\Omega(\xi,\dot\xi)}{\partial\dot\xi^i}\eta^i\right).
\end{aligned}$$

Set $$\Omega(\eta,\dot\eta;\xi,\dot\xi) = \frac{\partial\Omega(\eta,\dot\eta)}{\partial\eta^i}\xi^i + \frac{\partial\Omega(\eta,\dot\eta)}{\partial\dot\eta^i}\dot\xi^i .$$

Then
$$\Omega(\eta,\dot\eta;\xi,\dot\xi) = P_{ik}\eta^k\xi^i + Q_{ik}(\dot\eta^k\xi^i + \dot\xi^k\eta^i) + R_{ik}\dot\xi^i\dot\eta^k$$
$$= \Omega(\xi,\dot\xi;\eta,\dot\eta).$$

Hence we obtain

$$\xi^i\psi_i(\eta) - \eta^i\psi_i(\xi) = \frac{d}{dt}\left[\frac{\partial\Omega(\xi,\dot\xi)}{\partial\dot\xi^i}\eta^i - \frac{\partial\Omega(\eta,\dot\eta)}{\partial\dot\eta^i}\xi^i\right]. \tag{23.7}$$

24. We return now to the unfinished theorem of section 20. The fact that remained to be proved was that if the conjugate point were not beyond the first double point $(\sigma \geq \tau)$ then $t_2 \leq \tau$, where t_2 was the parameter of the relative minimum point. We have already shown that $t_2 \geq \tau$ in this case, so this would show that $t_2 = \tau$ and complete the proof of the theorem.

We first show that <u>if T is a parameter greater than τ, a comparison family</u>

$$x_i(t,\epsilon), \qquad \begin{matrix} -\eta \leq \epsilon \leq \eta, \\ 0 \leq t \leq T, \end{matrix}$$

<u>can be constructed for which $J''(0) < 0$.</u>

The proof follows that of Schwartz given in Bolza, Variationsrechnung, p. 84. Let $z^i(t)$ be a non-trivial solution of $\psi_i(\eta) = 0$ such that $z^i(0) = z^i(\tau) = 0$. Let $y^i(t)$ be any other contravariant vector of class C^2 defined for $0 \leq t \leq T$, and such that $y^i(0) = y^i(T) = 0$. Along the extremal $E_{P_oP(T)}$ we define

$$\begin{aligned} \eta^i(t) &= z^i(t) + ky^i(t), \quad 0 \leq t \leq \tau, \\ &= ky^i(t), \quad \tau \leq t \leq T, \end{aligned} \tag{24.1}$$

where k is a constant to be determined later. $\eta^i(t)$ is of class D^2 with τ the only point of discontinuity of $\frac{d}{dt}\eta^i(t)$; at this point we have

Hence $\left.\dfrac{d\eta^i}{dt}\right|_{\tau+} = k\dot{y}^i \quad , \left.\dfrac{d\eta^i}{dt}\right|_{\tau-} = \dot{z}^i + k\dot{y}^i .$

$$\left.\frac{d\eta^i}{dt}\right|_{\tau-} - \left.\frac{d\eta^i}{dt}\right|_{\tau+} = \dot{z}^i .$$

From (22.4') we get

$$\mathcal{J}''(0) = \int_0^\tau \psi_i(z+ky)(z^i+ky^i)\,dt + \int_\tau^T \psi_i(ky)\,ky^i\,dt - \left[\frac{\partial\Omega}{\partial\dot{\eta}^i}\right]_{\tau-}^{\tau+}\eta^i(\tau).$$

Because of the linearity of $\psi_i(\eta)$ we have

$$\psi_i(z+ky) = \psi_i(z) + k\,\psi_i(y).$$

Therefore, since $\psi_i(z) = 0$ and $\eta^i(\tau) = ky^i(\tau)$

(24.2) $$\mathcal{J}''(0) = k\int_0^\tau \psi_i(y)z^i\,dt + k^2\int_0^T \psi_i(y)y^i\,dt - k\left[\frac{\partial\Omega}{\partial\dot{\eta}^i}\right]_{\tau-}^{\tau+} y^i(\tau).$$

(23.7) yields

$$\psi_i(y)z^i = \psi_i(z)y^i + \frac{d}{dt}\left[\frac{\partial\Omega}{\partial\dot{z}^i}y^i - \frac{\partial\Omega}{\partial\dot{y}^i}z^i\right].$$

Integration of this from 0 to τ^- gives

$$\int_0^{\tau-}\psi_i(y)z^i\,dt = \left[\frac{\partial\Omega}{\partial\dot{z}^i}y^i - \frac{\partial\Omega}{\partial\dot{y}^i}z^i\right]_0^{\tau-}$$
$$= \left[\frac{\partial\Omega}{\partial\dot{z}^i}\right]^{\tau-} y^i(\tau)$$

since $z^i(0) = z^i(\tau) = y^i(0) = 0$. (24.2) then becomes

$$\mathcal{J}''(0) = k\left\{\left[\frac{\partial\Omega}{\partial\dot{z}^i}\right]^{\tau-} + \left[\frac{\partial\Omega}{\partial\dot{\eta}^i}\right]_{\tau+}^{\tau-}\right\}y^i(\tau) + k^2\int_0^T \psi_i(y)y^i\,dt.$$

Since $\dfrac{\partial\Omega}{\partial\dot{\eta}^i} = Q_{ki}\eta^k + R_{ki}\dot{\eta}^k$, $\eta^i(\tau^+) = \eta^i(\tau^-) = k\,y^i(\tau)$,

$\dot{\eta}^i(\tau^+) = k\,\dot{y}^i(\tau)$, and $\dot{\eta}^i(\tau^-) = k\,\dot{y}^i(\tau) + \dot{z}^i(\tau)$,

we have
$$\left\{\left[\frac{\partial\Omega}{\partial\dot{z}^i}\right]^{\tau-}+\left[\frac{\partial\Omega}{\partial\dot{\eta}^i}\right]^{\tau-}_{\tau+}\right\}=$$
$$[R_{ki}\,\dot{z}^k]+[Q_{ki}ky^k+R_{ki}(k\dot{y}^k+\dot{z}^k)]$$
$$-[Q_{ki}ky^k+R_{ki}k\dot{y}^k]$$
$$=2R_{ki}\,\dot{z}^k.$$

Therefore

(24.3) $$\mathcal{J}''(0)=2kR_{ki}\,\dot{z}^k y^i\Big|_{\tau}+k^2\int_0^T\psi_i(y)\,y^i\,dt.$$

(24.3) is an invariant equation since $\dot{z}^k(\tau)$ is a contravariant vector. Since $z^i(\tau)=0$, $\dot{z}^i(\tau)\neq 0$; otherwise $z^i(t)\equiv 0$ and is a trivial solution of $\psi_i(\eta)=0$. Therefore, because $|R_{ik}|=0$, $R_{ik}\dot{z}^i(\tau)\neq 0$. We now see that we are able to choose the vector $y^i(\tau)$ so that $R_{ik}\dot{z}^k y^i\Big|_{\tau}\neq 0$. Hence if k is chosen with the correct sign the first term of (24.3) is negative, and if k is chosen sufficiently small, the second term, containing as it does a k^2 factor, cannot influence the sign of $\mathcal{J}''(0)$. Therefore if $T>\tau$, a comparison family can be picked for which $\mathcal{J}''(0)<0$. According to (21.3) we see immediately that for the same family $\mathcal{J}''(0)$, the second variation for the F-problem, is less than zero. But such an extremal arc with negative second variation can't be relatively minimizing. If it were possible, the members of $x_i(t,\epsilon)$, for sufficiently small ϵ, would lie in the $\mathcal{U}(E_{P_oP(T)})$ for which $E_{P_oP(T)}$ is supposed to be minimizing and in this case $\mathcal{J}(\epsilon)-\mathcal{J}(0)\geq 0$, and $\mathcal{J}''(0)\geq 0$. Hence the relative minimum point cannot lie after the conjugate point, i.e. $t_2\leq\tau$, and the proof of the unfinished theorem of section 20 is completed.

We have assumed in the reasoning that it is always possible to find a family $x_i(t,\epsilon)$ corresponding to a given $x_i(t)$ and $\eta^i(t)=\dfrac{\partial x_i(t,0)}{\partial\epsilon}$.

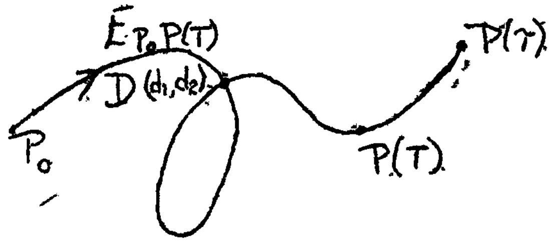
In the case of one coordinate system only,

$$x_i(t,\epsilon) = x_i(t) + \eta^i(t)\epsilon$$

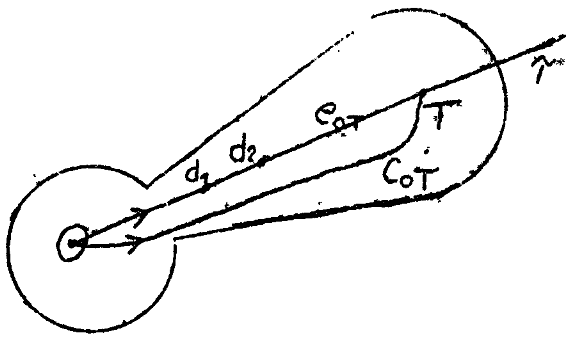
is such a family.

We also notice that this theorem of section 20 can be stated for left conjugate, first double and relative minimum points.

25. We have just shown that for an extremal arc $E_{P_oP(T)}$, $T > \tau$, there is always a comparison family such that $\mathcal{J}''(0)$, and consequently $J''(0)$ is negative. We will not prove that for no point before the conjugate point does there exist a family $x(t, \epsilon)$, $0 \leq t \leq T < \tau$, whose second variation $J''(0)$ is negative. According to (21.3) we see that $\mathcal{J}''(0)$ is likewise not negative.



x Space

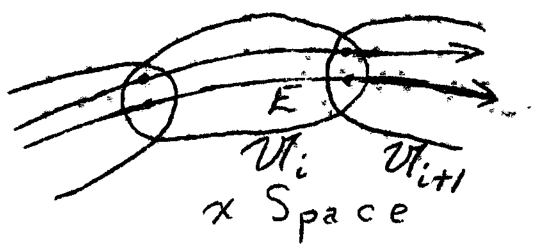


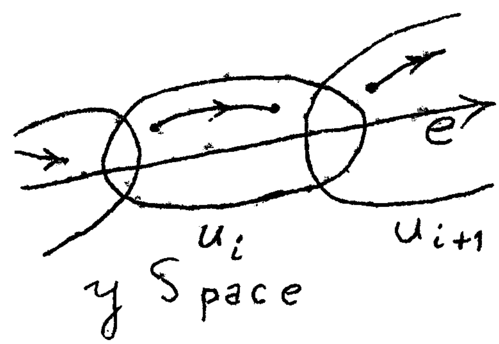
y Space

We note that we might have double points on the extremal arc $E_{P_oP(T)}$. In the y space $\Delta(t) = \left|\frac{\partial \phi_i}{\partial y_k}\right| \neq 0$ on e_{oT}, the straight line $y_i = \xi^i t$, $0 \leq t \leq T$, corresponding to $E_{P_oP(T)}$, since $T < \tau$. There exists, therefore, some neighborhood $\mathcal{U}(e_{oT})$ of e_{oT} in which $\left|\frac{\partial \phi_i}{\partial y_k}\right|$, being continuous, will not vanish. Let us choose this neighborhood $\mathcal{U}(e_{oT})$ so that if y is a point of it the entire straight line yt, $0 \leq t \leq 1$, will also lie in it, as can easily be done. The map $\mathcal{U}(E_{P_oP(T)})$ of $\mathcal{U}(e_{oT})$ will be an open point set containing $E_{P_oP(T)}$ with the property that there exists to each point P of it an extremal which lies in $\mathcal{U}(E_{P_oP(T)})$ joining P_o to P in $\mathcal{U}(E_{P_oP(T)})$.

Any arc C_{oT}: $y_i(\epsilon)$, $0 \leq \epsilon \leq 1$, of class D^1 which connects $y = 0$ to $y(T)$ and which lies in $\mathcal{U}(e_{oT})$ can be embedded in a continuous family of straight lines $y_i = y_i(\epsilon)t$, $0 \leq \epsilon \leq 1$, $0 \leq t \leq 1$, having all the properties necessary to obtain the expression for $\Delta \mathcal{J} = \mathcal{J}(C_{oT}) - \mathcal{J}(e_{oT})$ in terms of the $\mathcal{E}$-function, and hence obtain $\Delta \mathcal{J} \geq 0$. Therefore the second variation, $\mathcal{J}''(0)$, is not negative for any family $x_i(t, \epsilon)$ <u>whose members, for sufficiently small ϵ, are maps of such curve C_{oT}.</u> It is still necessary to show that this holds for <u>any</u> family $x_i(t, \epsilon)$ whose $\epsilon = 0$ member is the extremal arc $E_{P_oP(T)}$.

Take a finite set of neighborhoods $\mathcal{U}_1, \ldots, \mathcal{U}_N$ in $\mathcal{U}(E_{P_oP(T)})$ that cover the extremal $E_{P_oP(T)}$, where $\mathcal{U}_i$ contains the sub-arc of $E_{P_oP(T)}$ defined by $\alpha_i \leq t \leq \alpha_{i+1}$. The $\mathcal{U}_i$ can be taken such that N corresponding neighborhoods $u_1, \ldots, u_N$ lying in $\mathcal{U}(e_{oT})$ exist covering e_{oT}, each u_i homeomorphic to $\mathcal{U}_i$. To do this, split $E_{P_oP(T)}$ into a finite number of sub-arcs each homeomorphic to the corresponding t-interval and hence to the corresponding sub-arc of e_{oT}. Then by the Bolza theorem each such sub-arc of e_{oT} has a neighborhood u homeomorphic to its map $\mathcal{U}$. For sufficiently small ϵ the members of the given family $x_i(t, \epsilon)$ will all lie in the sum of the N neighborhoods $\mathcal{U}_i$ (the sub-arc $\alpha_i \leq t \leq \alpha_{i+1}$ in $\mathcal{U}_i$), and will be split by these into N sub-arcs to each of which will correspond a sub-arc in u_i in the y space. (See the Lemma, p. 89.)





The sub-arcs for such an ϵ member lying in $u_1, \ldots, u_N$, which are thus obtained, need not be connected; $x_k(\alpha_{i+1}, \epsilon)$, as a point in $\mathcal{V}_i$ and as a point in $\mathcal{V}_{i+1}$ might have two distinct maps lying in u_i and u_{i+1}. These two points corresponding to $x_k(\alpha_{i+1}, \epsilon)$ approach a common limit point $y_k = \xi^k \alpha_{i+1}$ lying in $u_i \cap u_{i+1}$ as $\epsilon \to 0$. Hence, for sufficiently small ϵ, the two corresponding points in u_i and u_{i+1} will both lie in the intersection $u_i \cap u_{i+1}$, in which case they will coincide. Since the number of neighborhoods u_i is finite, the maps in the y space of the members of the family $x_i(t, \epsilon)$, for sufficiently small ϵ, will be connected arcs of class D^1 lying in the sum of $u_1, \ldots, u_N$ and connecting $y = 0$ to $y(T)$. But we have already shown that for such a family, $\mathcal{J}''(0) \geq 0$. Thus for no point before the conjugate point does there exist a family whose second variation $\mathcal{J}''(0)$, and consequently also $\mathcal{J}''(0)$, is negative. We also see that for the arc $E_{P_0 P(\tau)}$, $P(\tau)$ the conjugate point, there exists no family such that $\mathcal{J}'' < 0$. For if such a family $x_i(t, \epsilon)$ were to exist, let

$$\eta^i(t) = \frac{\partial x_i(t,0)}{\partial \epsilon}, \quad \eta^i(0) = \eta^i(\tau) = 0.$$

Let t' be a parameter value between that of the last β-point and τ, and let $\eta^i_\alpha(t)$, $\alpha > 1$, be defined in the following way:

$$\begin{cases} \eta^i_\alpha(t) = \eta^i(t), & 0 \leq t \leq t' \\ = \eta^i(t' + \alpha(t - t')), & t' \leq t \leq t' + \frac{\tau - t'}{\alpha} \\ = 0, & t' + \frac{\tau - t'}{\alpha} \leq t \leq \tau. \end{cases}$$

Obviously $\eta^i_\alpha(t) \to \eta^i(t)$ as $\alpha \to 1$. Also, as we see from the continuity of (22.6), $(\mathcal{J}''(0))_{\eta_\alpha} \to (\mathcal{J}''(0))_\eta$ as $\alpha \to 1$. But since $(\mathcal{J}''(0))_{\eta_\alpha} \geq 0$, we see that $(\mathcal{J}''(0))_\eta$ cannot be negative.

THEOREM A: <u>The $\frac{\text{right}}{\text{left}}$ conjugate point of the $\frac{\text{left}}{\text{right}}$ conjugate point of the point P_0 on E is P_0.</u>

We prove the theorem for the upper of the two choices indicated in its statement; the proof for the other case will then be obvious. Let $P(\tau)$ be the right conjugate point of P_o on E. Then there will exist a solution $\eta(t)$ of the Jacobi differential equations vanishing at $t = 0$ and at $t = \tau$. Because of the existence of this solution $P' = P(t')$, the left conjugate point of $P(\tau)$ on E, cannot lie before P_o. If $P' \neq P_o$, then there would exist a second variation $\gamma''(0) < 0$ for $E_{P_oP(\tau)}$. But we have just seen that this is impossible; hence $P' = P_o$ and the theorem is proved. According to Theorem A we see that conjugate points can be associated in pairs.

THEOREM B: <u>Two pairs of conjugate points, P_o, $P(\tau)$ and P_o', $P'(\tau')$ always separate, i.e. they fall in the order $P_o < P_o' < P(\tau) < P'(\tau')$.</u>

Let P_o' be such that $P_o < P_o' < P(\tau)$, $P_o' \neq P_o$. $P'(\tau')$, the right conjugate point of P_o', cannot lie before $P(\tau)$ since on $E_{P_o'P(\tau)}$, and hence on $E_{P_oP(\tau)}$, $\gamma''(0)$ would be negative for some family. Likewise $P'(\tau') \neq P(\tau)$, for in that case P_o' and not P_o would be the left conjugate point of $P(\tau)$, contradicting Theorem A. Hence $P'(\tau')$ must lie after $P(\tau)$, and the proof is completed.

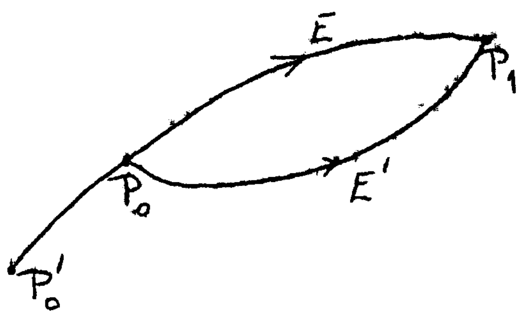
THEOREM C. <u>$P(\tau(t))$, the right (or left) conjugate point of $P(t)$, moves continuously with $P(t)$.</u>

Let $P(\tau(t))$ be the right conjugate point of $P(t)$ on E. Let $\overline{P(\tau_1)P(\tau_2)}$ be an interval containing $P(\tau(t))$. Let $P(t_1)$, $P(t_2)$ be the left conjugate points of $P(\tau_1)$, $P(\tau_2)$ respectively. Then $\overline{P(t_1)P(t_2)}$ contains $P(t)$, and, considering $\tau(t)$ as a mapping, map $\overline{P(t_1)P(t_2)} = \overline{P(\tau_1)P(\tau_2)}$, proving the theorem.

Three analogous theorems involving (absolute) minimum points will now be given.

THEOREM A_1. *The $\frac{\text{right}}{\text{left}}$ minimum point of the $\frac{\text{left}}{\text{right}}$ minimum point of the point P_o on E is P_o.*

Let P_1 be the right minimum point of P_o on E, and P_o' be the left minimum point of P_1 on E. P_o' cannot lie between P_o and P_1 since $E_{P_oP_1}$ is minimizing. If P_o' were to lie to the left of P_o (see the diagram), then P_1 could not be the right *conjugate* point of P_o on E, since according to Theorem B the right conjugate point of P_o' would have to lie between P_o and P_1, and in this case $E_{P_o'P_1}$ would not even be relatively minimizing. Therefore if P_o' were to lie before P_o there would have to be a second minimizing arc $E'_{P_oP_1}$ connecting P_o to P_1. But then $E_{P_o'P_o} + E'_{P_oP_1}$ would be a minimizing arc *with a corner* connecting P_o' to P_1, which of course is impossible. Therefore $P_o' = P_o$ and the theorem is proved.



THEOREM B_1. *Two pairs of minimum points P_o, P_1 and P_o', P_1' always separate.*

Let P_o' lie between P_o and P_1 and not coincide with P_o. P_1', the right minimum point of P_o' on E, cannot lie before P_1 since $E_{P_o'P_1}$ is minimizing; it cannot coincide with P_1, for in that case P_o' and not P_o would be the left minimum point of P_1, contradicting Theorem A_1. Hence P_1' must lie after P_1, and the proof is completed.

THEOREM C_1. *P_1, the right (or left) minimum point of P(t) moves continuously with P(t).*

The proof parallels that of Theorem C.

The three similar theorems for *relative* minimum points are not valid.

25. Consider an extremal arc $E_{P_oP_1}$, where P_1 is the (absolute) minimum point of P_o on E. If $E_{P_oP_1}$ is defined by $x_i = \varphi_i(x_o, \xi t)$, the closed segment $y_i = \xi^i t$, $0 \leq t \leq t_1(\xi)$, corresponds to it in the y-space. Let $\mathfrak{M}$ be the point-set in the y-space consisting of all such closed segments for all ξ, and let $\mathfrak{M}_1$ be the point-set in the y-space defined by $y_i = \xi^i t_1(\xi)$ for all ξ. $\mathfrak{M}_1$ is such that its map in $\mathfrak{X}$, the x-space, is the locus of minimum points with regard to P_o. We show the following properties concerning $\mathfrak{M}$ and $\mathfrak{M}_1$.

1) The map of $\mathfrak{M}$ under $P = P(y)$ ($x_i = \varphi_i(x_o, \xi t)$) is the entire space $\mathfrak{X}$. For each point P of $\mathfrak{X}$ there exists a minimizing extremal connecting P_o to it; hence there exists a point y in $\mathfrak{M}$ such that $P = P(y)$.

2) $\mathfrak{M}$ is closed. A point y of $\mathfrak{M}$ is characterized by the equation $F(x_o, y) = \overline{P_oP(y)}$; since both sides of this equation are continuous in y we see that a limit point of points in $\mathfrak{M}$ is in $\mathfrak{M}$.

3) If $y \in \mathfrak{M} - \mathfrak{M}_1$, there exists a neighborhood $\mathfrak{U}(y)$ in the y space which is homeomorphic to its map. Since the conjugate point to P_o on E does not lie before the minimum point to P_o on E, $\Delta(y) = \left|\frac{\partial \varphi_i}{\partial y_k}\right| \neq 0$ if $y \in \mathfrak{M} - \mathfrak{M}_1$; hence the existence of a $\mathfrak{U}(y)$ homeomorphic to its map.

4) $\mathfrak{M} - \mathfrak{M}_1$ is homeomorphic to its map. To any y corresponds only one point P of $\mathfrak{X}$. Since y lies before the minimum point, only one minimizing extremal arc $E_{P_oP(y)}$ exists, and hence to a point P of the map of $\mathfrak{M} - \mathfrak{M}_1$ corresponds only one y of $\mathfrak{M} - \mathfrak{M}_1$.

5) If $y, y' \in \mathfrak{M}$, and if $P(y) = P(y')$ then either $y = y'$ or $y, y' \in \mathfrak{M}_1$. The argument is the same as in 4).

6) $\mathfrak{M} - \mathfrak{M}_1$ is open. Consider $y_i = \xi^i t$, $0 \leq t < t_1(\xi)$, a point of $\mathfrak{M} - \mathfrak{M}_1$. According to property 3), there exists a $\mathfrak{U}(y)$ which is homeomorphic to its map, and hence is such that $\Delta(\bar{y}) \neq 0$ if $\bar{y} \in \mathfrak{U}(y)$. Using the Euclidean

metric in the y space, take a series of sphere neighborhoods $\mathcal{U}_{\sigma_n}(y)$, $\sigma_n \to 0$; for sufficiently large n $\mathcal{U}_{\sigma_n}(y) \subset \mathcal{U}(y)$. If 6) were false, then there would exist in each $\mathcal{U}_{\sigma_n}$ a $y'_{\sigma_n} \in C(\mathfrak{M} - \mathfrak{M}_1)$, the complement of $\mathfrak{M} - \mathfrak{M}_1$. y'_{σ_n} is either in $\mathfrak{M}_1$ or $C(\mathfrak{M})$; in either case, since $\Delta(y'_{\sigma_n}) \neq 0$, there would be a <u>minimizing</u> extremal $e_{oy''_{\sigma_n}}$ different from $e_{oy'_{\sigma_n}}$ and such that $y''_{\sigma} \in \mathfrak{M}$ and $P(y'_{\sigma_n}) = P(y''_{\sigma_n})$. Therefore, according to 3), $y''_{\sigma_n} \notin \mathcal{U}_{\sigma_n}(y)$. Since $e_{oy''_{\sigma_n}}$ is minimizing,

$$F(x_o, y''_{\sigma_n}) \leq F(x_o, y'_{\sigma_n}).$$

Since $F(x_o, y'_{\sigma_n}) \to F(x_o, y)$ as $n \to \infty$, we have that $F(x_o, y''_{\sigma_n}) < M$; using the lemma of p. 4, we see that y''_{σ_n} is bounded, and a convergent subsequence, again denoted by y''_{σ_n}, can be picked: $y''_{\sigma_n} \to y''$. Since $y''_{\sigma_n} \notin \mathcal{U}_{\sigma_n}(y)$, $y'' \notin \mathcal{U}_{\sigma_n}(y)$, and hence $y'' \neq y$. According to 2), $y'' \in \mathfrak{M}$. But now we have

$$P(y) = P(y''),$$

$$y \neq y'',$$

$$y \in \mathfrak{M} - \mathfrak{M}_1, \; y'' \in \mathfrak{M},$$

in contradiction to 5).

7) <u>$\mathfrak{M}_1$ is closed.</u> This follows immediately out of 2) and 6).

8) <u>The condition that $\mathfrak{M}$ is bounded in the y-space is equivalent to the condition that γ is compact.</u>

a) Assume γ is compact. Then, $\overline{P_oP}$ being a continuous function of P in γ, $\overline{P_oP} < M$ for all P; accordingly $F(x_o, y) < M$ for $y \in \mathfrak{M}$. Then by the lemma on p. 4, y_i is bounded.

b) Assume $\mathfrak{M}$ is bounded. Then $|y_i| < k$ if $y \in \mathfrak{M}$. Then $F(x_o, y) < M$ for $y \in \mathfrak{M}$, and consequently $\overline{P_oP} < M$ for all P. Assumption (3.1) then shows the space γ to be compact.

9) $\underline{\mathfrak{M}_1 \text{ is the boundary of } \mathfrak{M} - \mathfrak{M}_1}$. Since $\mathfrak{M} - \mathfrak{M}_1$ is open, its boundary consists of all of its accumulation points not in itself. But each accumulation point of $\mathfrak{M} - \mathfrak{M}_1$ is in $\mathfrak{M}$, and hence either in $\mathfrak{M}_1$ or in $\mathfrak{M} - \mathfrak{M}_1$. On the other hand, every point of $\mathfrak{M}_1$ is an accumulation point of points of $\mathfrak{M} - \mathfrak{M}_1$ (for instance of points on the extremal to this point).

10) <u>$t_1(\xi)$, the parameter of the minimum point of P_0 on the extremal $x_i = \varphi_i(x_0, \xi t)$, is a continuous single-valued function of ξ which is homogeneous of degree -1 in ξ.</u> Let $\xi^i_m \to \xi^i$, and $y_{mi} = \xi^i_m t_1(\xi_m)$ be the corresponding minimum points. All $y_{mi} \in \mathfrak{M}_1$, and hence, since $\mathfrak{M}_1$ is closed, $y_{mi} \to y_i$, the minimum point on the extremal given by ξ^i.

$$y_{mi} \to y_i = \xi^i t_1(\xi).$$

Since $\xi^i \neq 0$, some component, say ξ^1, does not vanish; then $\xi^1_m \neq 0$ for sufficiently large m. We then have

$$t_1(\xi_m) = \frac{y_{m1}}{\xi^1_m} \to \frac{\xi^1 t_1(\xi)}{\xi^1} = t_1(\xi)$$

We notice also that the minimum points of P_0 on γ are continuous functions of ξ.