# GEOMETRY OF COMPLEX DOMAINS

a seminar conducted by

PROFESSORS OSWALD VEBLEN and JOHN VON NEUMANN

1935-36

First term lectures by Professor Veblen
Second term lectures by Mr. J. W. Givens

Notes by Dr. A. H. Taub and Mr. J. W. Givens

The Institute for Advanced Study
Princeton, New Jersey

Reissued with corrections,
1955.

CONTENTS

# Chapter I

# SPINORS AND PROJECTIVE GEOMETRY

## THE MINKOWSKI SPACE REPRESENTED BY HERMITIAN MATRICES

1. If a fixed origin is specified in the Minkowski space of the special theory of Relativity it becomes a real vector (centered affine) space, $R_4$, of four dimensions which contains a special invariant locus, the light cone, with vertex at the origin. The points of $R_4$ may be put in a (1-1) correspondence with the two-row Hermitian matrices. To do this we recall that an Hermitian matrix $\|\Psi_{\dot{A}B}\|$ is defined by the conditions

$$\overline{\Psi}_{\dot{A}B} = \Psi_{\dot{B}A} \qquad (\dot{A}, B = 1, 2) \tag{1.1}$$

where the bar denotes the complex conjugate, and hence $\Psi_{11}$ and $\Psi_{22}$ are real while $\Psi_{12}$ and $\Psi_{21}$ are complex conjugates of one another. Every Hermitian matrix of order two is therefore expressible in terms of four real parameters $(x^1, x^2, x^3, x^4)$ by means of the equations

$$\begin{Vmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{Vmatrix} = \frac{1}{\sqrt{2}} \begin{Vmatrix} x^4 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^4 - x^3 \end{Vmatrix} . \tag{1.2}$$

This correspondence is (1-1), the inverse equations being

$$\begin{aligned} x^1 &= \frac{1}{\sqrt{2}} (\Psi_{12} + \Psi_{21}), \quad & x^2 &= \frac{1}{\sqrt{2}i} (\Psi_{12} - \Psi_{21}), \\ x^3 &= \frac{1}{\sqrt{2}} (\Psi_{11} - \Psi_{22}), \quad & x^4 &= \frac{1}{\sqrt{2}} (\Psi_{11} + \Psi_{22}). \end{aligned} \tag{1.3}$$

The light-cone is the locus of those points of $R_4$ which correspond to singular Hermitian matrices. For,

$$2|\Psi_{\dot{A}B}| \equiv 2(\Psi_{11}\Psi_{22} - \Psi_{12}\Psi_{21}) = -(x^1)^2 - (x^2)^2 - (x^3)^2 + (x^4)^2, \tag{1.4}$$

and hence a matrix $\|\Psi_{\dot{A}B}\|$ for which $|\Psi_{\dot{A}B}| = 0$ corresponds under (1.3) to a point $(X^1, X^2, X^3, X^4)$ of $R_4$ which lies on the cone

$$g_{ij}X^iX^j = -(X^1)^2 - (X^2)^2 - (X^3)^2 + (X^4)^2 = 0. \tag{1.5}$$

In the abbreviated expression for the invariant quadratic form in (1.5) we are using the summation convention which is customary in relativity theory and defining the numbers $g_{ij}$ by the matrix equation

$$\|g_{ij}\| = \begin{Vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{Vmatrix} . \tag{1.6}$$

We also use the convention that small Latin letters i, j, etc., take on the values 1, 2, 3, 4.

The rows of a singular matrix are proportional so that $\|\Psi_{\dot{A}B}\| = \|\theta_{\dot{A}}\,\varphi_B\|$. The condition (1.1) implies for a non-zero matrix that $\theta_{\dot{A}} = \pm\rho\bar{\varphi}_A$ where $\rho$ is real and positive. Writing $\psi_A = \rho^{\frac{1}{2}}\varphi_A$ we have the result that every two-row singular Hermitian matrix is of one of the forms

$$\pm\begin{Vmatrix} \bar{\psi}_1\psi_1 & \bar{\psi}_1\psi_2 \\ \bar{\psi}_2\psi_1 & \bar{\psi}_2\psi_2 \end{Vmatrix}, \tag{1.7}$$

the zero matrix being the only one which can be written in both forms.

The points of the light cone, exclusive of the vertex (0, 0, 0, 0), thus fall into two distinct classes corresponding to the two possible signs in (1.7). The points of one class, those constituting the future branch of the light-cone, are parameterized by the equations

$$X^1 = \frac{1}{\sqrt{2}}(\bar{\psi}_1\psi_2 + \bar{\psi}_2\psi_1), \qquad X^2 = \frac{1}{\sqrt{2}\,i}(\bar{\psi}_1\psi_2 - \bar{\psi}_2\psi_1), \tag{1.8}$$
$$X^3 = \frac{1}{\sqrt{2}}(\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2), \qquad X^4 = \frac{1}{\sqrt{2}}(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2),$$

the other branch being given by changing the sign of the right member in each of these equations. The points of each branch clearly form a continuous family while the two branches are connected only through the vertex (0, 0, 0, 0). Points on the future branch of the light-cone are characterized in terms of $(X^1, X^2, X^3, X^4)$ by the conditions

(1.9) $$2|\Psi_{\dot{A}B}| = g_{ij}X^iX^j = 0, \quad X^4 > 0,$$

and those on the past branch by

(1.10) $$2|\Psi_{\dot{A}B}| = g_{ij}X^iX^j = 0, \quad X^4 < 0.$$

Equations (1.8) give a correspondence,

(1.11) $$(\psi_1, \psi_2) \longrightarrow \text{point on future branch of light cone,}$$

between pairs of complex numbers, not both zero, and points on the future branch of the light-cone. This correspondence is not (1-1) for $(e^{i\theta}\psi_1, e^{i\theta}\psi_2)$, with $\theta$ real, corresponds to the same point as does $(\psi_1, \psi_2)$. The correspondence

(1.12) $$(e^{i\theta}\psi_1, e^{i\theta}\psi_2) \longleftrightarrow \text{point on future branch of light cone,}$$

between families of pairs of complex numbers and points on the future (or the past) branch of the light-cone is (1-1).

Multiplying $\psi_1$ and $\psi_2$ by the same real number r multiplies each of the coordinates $X^i$ by $r^2$ so that the pairs of complex numbers $(\rho\psi_1, \rho\psi_2)$, where $\rho$ is a complex parameter, correspond to the points of a ray (i.e. a half-line) on the future branch of the cone. Since each generator of the cone consists of two collinear rays through the vertex and is determined by either of them, there is a (1-1) correspondence,

(1.13) $$(\rho\psi_1, \rho\psi_2) \longleftrightarrow \text{line on the light-cone,}$$

between the sets of numbers $(\rho\psi_1, \rho\psi_2)$ and the lines of the light-cone.

The points of $R_4$ which do not lie on the light-cone fall into three disconnected sets, the absolute future, the absolute past, and the absolute elsewhere, which are characterized algebraically by the conditions

(1.14) $2\,|\Psi_{\dot{A}B}| = g_{ij}X^iX^j > 0, \qquad X^4 > 0,$

(1.15) $2\,|\Psi_{\dot{A}B}| = g_{ij}X^iX^j > 0, \qquad X^4 < 0,$

and

(1.16) $2\,|\Psi_{\dot{A}B}| = g_{ij}X^iX^j < 0,$

respectively.

If a point in the absolute future is joined to a point in the absolute past by a continuous curve there must be at least one point on it for which $X^4 = 0$. For such a point $g_{ij}X^iX^j \leqq 0$ and the equality sign holds only for the vertex of the cone, (0, 0, 0, 0). Hence the curve either passes through the vertex or through a point in the absolute elsewhere. A schematic representation of $R_4$ is given by the figure:

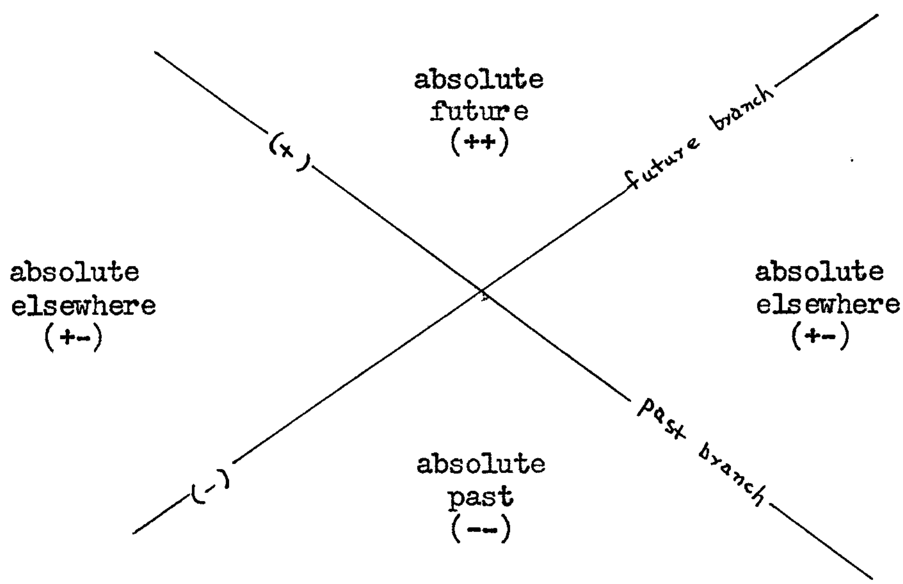



Figure 1.

The division of $R_4$ into regions is easily described in terms of the Hermitian matrices which correspond to the points of $R_4$. The condition (1.1) implies that the Hermitian form

$$\Psi_{\dot{A}B}\, \bar{Z}^A Z^B \tag{1.17}$$

is real for all values of the complex-variables $Z^1$, $Z^2$. In particular, the point (0, 0, 0, 1) of $R_4$ lies in the absolute future and corresponds under (1.2) to the Hermitian form $\frac{1}{\sqrt{2}}(\bar{Z}^1Z^1 + \bar{Z}^2Z^2)$ which is always $> 0$, if we agree to exclude (0, 0) as a possible pair of values for $(Z^1, Z^2)$. Similarly, the point (0, 0, 1, 1) lies on the future branch of the light-cone and corresponds to the form $\sqrt{2}\,\bar{Z}^1Z^1$ which is always $\geqq 0$; the point (0, 0, 1, 0) lies in the absolute elsewhere and corresponds to the form $\frac{1}{\sqrt{2}}(\bar{Z}^1Z^1 - \bar{Z}^2Z^2)$ which assumes all real values; the point (0, 0, -1, -1) corresponds to the form $-\sqrt{2}\,\bar{Z}^1Z^1$ which is always $\leqq 0$; and the point (0, 0, 0, -1) corresponds to $\frac{1}{\sqrt{2}}(-\bar{Z}^1Z^1 - \bar{Z}^2Z^2)$ which is always $< 0$.

If an Hermitian form assumes one positive value it assumes all positive values since multiplying $Z^1$ and $Z^2$ by a real number $r$ multiplies the form by the positive number $r^2$. A similar result holds for negative values and hence the range of values of an arbitrary Hermitian form is identical with the range of values of one and only one of the typical forms

$$\bar{Z}^1Z^1,\ -\bar{Z}^1Z^1,\ \bar{Z}^1Z^1 + \bar{Z}^2Z^2,\ -\bar{Z}^1Z^1 - \bar{Z}^2Z^2,\ \bar{Z}^1Z^1 - \bar{Z}^2Z^2. \tag{1.18}$$

Hence an Hermitian form in two variables falls into one of five distinct classes according as its range of values is $\geqq 0$, $\leqq 0$, $> 0$, $< 0$, or $\gtreqless 0$.

The range of values assumed by an Hermitian form is unchanged if we make the linear substitution

$$Z^A = W^B\, P_B^A \tag{1.19}$$

with $|P_A^B| \neq 0$. On making this substitution in (1.17) we find that

$$\Psi_{\dot{A}B}\, \bar{Z}^A Z^B = \Phi_{\dot{A}B}\, \bar{W}^A W^B,$$

where

(1.20) $$\Phi_{\dot{A}B} = \bar{P}_A^C\, P_B^D\, \Psi_{\dot{C}D}.$$

To determine a substitution (1.19) which will reduce (1.17) to one of the typical forms (1.18), we proceed as follows. Choose $Q_1{}^A$ such that $\Psi_{\dot{A}B}\, \bar{Q}_1{}^A\, Q_1{}^B \equiv \rho_1 \neq 0$, which is possible since we are assuming $\|\Psi_{\dot{A}B}\| \neq 0$. Then let $Q_2{}^A$ ($\neq 0$) be a solution of the single equation $\Psi_{\dot{A}B}\, \bar{Q}_1{}^A\, Q_2{}^B = 0$ (and hence not a multiple of $Q_1{}^A$) and write $\Psi_{\dot{A}B}\, \bar{Q}_2{}^A\, Q_2{}^B = \rho_2$. If $\rho_2 = 0$ the transformation (1.19) with $P_1{}^A = \rho_1{}^{-\frac{1}{2}}\, Q_1{}^A$ and $P_2{}^A = Q_2{}^A$ will reduce $\Psi_{\dot{A}B}\, \bar{Z}^A Z^B$ to $\bar{W}^1 W^1$ or $-\bar{W}^1 W^1$. If $\rho_2 \neq 0$, the transformation (1.19) with $P_1{}^A = \rho_1{}^{-\frac{1}{2}}\, Q_1{}^A$, and $P_2{}^A = \rho_2{}^{-\frac{1}{2}}\, Q_2{}^A$ will reduce $\Psi_{\dot{A}B}\, \bar{Z}^A Z^B$ to $\pm(\bar{W}^1 W^1 + \bar{W}^2 W^2)$ or $\pm(\bar{W}^1 W^1 - \bar{W}^2 W^2)$. We do not need to include both of the forms $\bar{W}^1 W^1 - \bar{W}^2 W^2$ and $-\bar{W}^1 W^1 + \bar{W}^2 W^2$ in our list of canonical forms since interchanging the variables carries one into the other.

It follows that an arbitrary Hermitian matrix $\|\Psi_{\dot{A}B}\|$ ($\neq 0$) can be transformed under (1.20) into one and only one of the matrices

(1.21) $$\left\|\begin{matrix}1 & 0\\ 0 & 0\end{matrix}\right\|,\quad \left\|\begin{matrix}-1 & 0\\ 0 & 0\end{matrix}\right\|,\quad \left\|\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\|,\quad \left\|\begin{matrix}-1 & 0\\ 0 & -1\end{matrix}\right\|,\quad \left\|\begin{matrix}1 & 0\\ 0 & -1\end{matrix}\right\|,$$

and the matrix is then said to be of signature (+), (-), (++), (--), or (+-), respectively. From (1.20)

(1.22) $$|\Phi_{\dot{A}B}| = |\bar{P}_A{}^C|\ |P_B{}^D|\ |\Psi_{\dot{C}D}|\ ,$$

and hence a matrix of signature (+-) is characterized by the condition $|\Psi_{\dot{A}B}| < 0$. Referring to (1.16) we see that the absolute elsewhere is the locus of points corresponding to matrices of signature (+-). Since (1.7) with the plus sign is of signature (+), the future branch of the light-cone is the locus of matrices of this signature and similarly the past branch of the cone is the locus of matrices of signature (-). Since the absolute future is bounded by the future branch of the cone, its points correspond to matrices of signature (++) and not of signature (--). The regions of $R_4$ are therefore characterized in a manner indicated in Figure 1.

THE COMPLEX PROJECTIVE LINE

2. An ordered pair of complex numbers $(\psi_1, \psi_2)$ may be interpreted as homogeneous coordinates of a point in a complex projective space of one dimension, i.e. a complex projective line $P_1$. Each point of $P_1$ is represented by one, and but one, family of pairs $(\rho\psi_1, \rho\psi_2)$ where $\rho$ takes on all complex values except zero. The pair of numbers (0, 0) does not correspond to any point but any other pair does determine a unique point. Equations (1.8) define a (1-1) correspondence (1.13) between points of $P_1$ and generators of the light-cone.

In a similar way the ordered sets of real numbers $(X^1, X^2, X^3, X^4)$ may be interpreted as homogeneous coordinates in a real projective space $R_3$. This amounts to recognizing that the lines through the origin of $R_4$ constitute a projective three-space. The equation (1.5) represents a real non-ruled quadric in $R_3$. Thus the lines of the light-cone in $R_4$ are points of the quadric in $R_3$. The equations (1.8) give a (1-1) correspondence between the points of this quadric and the points of the complex line $P_1$.

If we set

$$\frac{X^1}{X^4} = X, \quad \frac{X^2}{X^4} = Y, \quad \frac{X^3}{X^4} = Z, \tag{2.1}$$

we see the quadric (1.5) as the sphere

$$X^2 + Y^2 + Z^2 = 1 \tag{2.2}$$

in a Euclidean three-space $E_3$. This Euclidean space may be defined by specifying the plane $X^4 = 0$ of $R_3$ as plane at infinity and (1.5) as unit sphere. The points of the sphere (2.2) represent the lines of the light-cone in $R_4$.

Let us also set

$$\frac{\psi_1}{\psi_2} = z = x + iy \tag{2.3}$$

where x and y are real. These equations establish a (1-1) correspondence between the points of the complex line $P_1$ and the totality of complex numbers z, including ∞. The numbers x and y can be interpreted as rectangular cartesian coordinates in a Euclidean plane and we thus have a (1-1) correspondence between the points of this plane, including one point at infinity, and the points of the complex line $P_1$.

Combining (1.3) with (2.1) and (2.3) we have the formulas

$$\begin{aligned}
X &= \frac{\bar{\psi}_1\psi_2 + \bar{\psi}_2\psi_1}{\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2} = \frac{\bar{z} + z}{1 + \bar{z}\,z} = \frac{2x}{1 + x^2 + y^2}, \\
Y &= \frac{1}{i}\,\frac{\bar{\psi}_1\psi_2 - \bar{\psi}_2\psi_1}{\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2} = \frac{1}{i}\,\frac{\bar{z} - z}{1 + \bar{z}\,z} = \frac{-2y}{1 + x^2 + y^2}, \\
Z &= \frac{\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2}{\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2} = \frac{\bar{z}\,z - 1}{1 + \bar{z}\,z} = \frac{x^2 + y^2 - 1}{1 + x^2 + y^2},
\end{aligned} \tag{2.4}$$

which define a (1-1) correspondence between the sphere and the z-plane (or xy-plane). This correspondence is discussed in function theory by means of stereographic projection and indeed (2.4) are the formulas for the stereographic projection of the sphere upon an equatorial plane.

The points of $R_3$ not on the sphere may also be represented in the xy-plane. For, a point $(A^1, A^2, A^3, A^4)$ uniquely determines its polar plane

$$-A^1X^1 - A^2X^2 - A^3X^3 + A^4X^4 = 0, \tag{2.5}$$

and this plane intersects the sphere in a circle the projection of which in the xy-plane is given by the equation,

$$(2.6) \qquad (A^4 - A^3)(x^2 + y^2) - 2A^1 x + 2A^2 y + (A^4 + A^3) = 0.$$

Hence under stereographic projection circles on the sphere correspond to circles in the plane, it being understood that lines in the xy-plane are regarded as circles which pass through the point at infinity.

The locus (2.6) is a real circle, a point, or an imaginary circle (i.e. no locus at all) according as $-(A^1)^2 - (A^2)^2 - (A^3)^2 + (A^4)^2$ is $< 0$, $= 0$, or $> 0$; that is, according as the point $A^i$ is outside, on, or inside the sphere, respectively. A point outside the quadric in $R_3$ corresponds to a line lying in the absolute elsewhere of $R_4$ and passing through the vertex of the light-cone. Such a line corresponds to a pencil of indefinite Hermitian forms, that is, forms of signature (+-). Similarly, a point inside the sphere corresponds to a pencil of definite forms (of signature (++) or (--)) and a point on the sphere corresponds to a pencil of singular forms. Indeed, the point $A^i$ corresponds to the form

$$(2.7) \quad (A^4 + A^3)\bar{Z}^1 Z^1 + (A^1 + iA^2)\bar{Z}^1 Z^2 + (A^1 - iA^2)\bar{Z}^2 Z^1 + (A^4 - A^3)\bar{Z}^2 Z^2,$$

and if we put $-\dfrac{Z^2}{Z^1} = z = x + iy$, the form is a multiple of the left member of (2.6).

Two circles on the sphere are orthogonal if and only if

$$(2.8) \qquad -A^1B^1 - A^2B^2 - A^3B^3 + A^4B^4 = 0,$$

where the circles correspond to points $A^i$ and $B^i$. Stated in terms of the projective space $R_3$ this means that each of the points A and B lies on the polar plane of the other with respect to the non-ruled quadric. The condition (2.8) is, however, just the condition that the two circles in the xy-plane be orthogonal. Stereographic projection therefore carries orthogonal circles into orthogonal circles.

THE LORENTZ GROUP ISOMORPHIC TO THE QUADRIC GROUP

3. The Lorentz group is defined to be the set of all real linear transformations,

(3.1) $$Y^i = L^i{}_j X^j$$

which leave the quadratic form

(3.2) $$-(X^1)^2 - (X^2)^2 - (X^3)^2 + (X^4)^2$$

invariant and do not interchange past and future. That is,

(3.3) $$g_{ij} L^i{}_k L^j{}_\ell = g_{k\ell} \quad \text{and} \quad L^4{}_4 > 0.$$

The latter condition implies that (0, 0, 0, 1), which is a point in the absolute future, is transformed into a point in the absolute future. Taking the determinant of both members of the equation in (3.3) gives

(3.4) $$|L| = \pm 1.$$

The transformations of the full Lorentz group for which $|L| = +1$ form an invariant sub-group called the restricted, or proper, Lorentz group. The transformations for which $|L| = -1$ are called improper Lorentz transformations, but these do not form a group. An example of an improper Lorentz transformation is

(3.5) $$Y^1 = X^1, \quad Y^2 = -X^2, \quad Y^3 = X^3, \quad Y^4 = X^4.$$

The totality of improper transformations is obtained by multiplying the proper ones by a single improper one, such as (3.5).

The transformations of the restricted Lorentz group maintain the distinction between right- and left-handed systems of coordinate axes in the space-like sections of $R_4$. The transformation (3.5), however, causes a space reflection in each of the three-spaces, $X^4$ = const., and this interchange of right- and left-handed coordinate systems is typical of improper transformations. All the Lorentz transformations, both proper and improper, leave the various regions of $R_4$ invariant.

If we remove the restriction $L^4{}_4 > 0$ and only require that (3.1) leave (3.2) invariant, we call the group so defined the extended Lorentz group. This group allows the interchange of past and future as well as the change from right- to left-handed sets of coordinate axes. The extended group is generated by adjoining to the Lorentz group the transformation

(3.6) $$Y^i = -X^i$$

Beginning with the restricted Lorentz transformations, those for which

(3.7) $$|L| = +1, \text{ and } L^4{}_4 > 0,$$

the Lorentz group is obtained by adding the transformations for which

(3.8) $$|L| = -1, \text{ and } L^4{}_4 > 0.$$

The extended Lorentz group also contains the transformations satisfying

(3.9) $$|L| = +1, \text{ and } L^4{}_4 < 0,$$

and those satisfying

(3.10) $$|L| = -1, \text{ and } L^4{}_4 < 0.$$

We shall chiefly be concerned with the restricted Lorentz group and the Lorentz group, but the restricted Lorentz transformations may be combined with the transformations satisfying (3.9) or with the transformations satisfying (3.10) to form other groups.

The transformation (3.1) may also be interpreted in $R_3$ and it is then called a collineation, from its property of carrying lines into lines. Since the coordinates of points in $R_3$ are homogeneous, two matrices $\|L^i_j\|$ and $\|rL^i_j\|$, where r is real and $\neq 0$, define exactly the same collineation. For example, (3.6) is the identity transformation in $R_3$.

The quadric group in $R_3$ is defined to be the set of all collineations which leave the quadric (1.5) invariant. The homogeneous coordinates of the collineation, $\|L^i_j\|$, therefore satisfy an equation

(3.11) $$g_{ij}L^i_k L^j_\ell = \rho\, g_{k\ell}\,, \text{ with } \bar{\rho} = \rho \neq 0.$$

The exterior and interior of a non-ruled quadric cannot be interchanged by a collineation* and since points exterior to the quadric (1.5) are characterized by the condition $g_{ij}X^iX^j < 0$ we must have $\rho > 0$.

Hence a Lorentz transformation uniquely determines a collineation of the quadric group. Conversely, in every pencil of matrices, $\|rL^i_j\|$, defining a collineation of the quadric group, there is exactly one matrix satisfying (3.3). This matrix is in fact determined to within sign by the normalizing condition $|L| = \pm 1$, the plus sign being taken when $|rL^i_j| = r^4\,|L^i_j| > 0$ and the minus sign when $|rL^i_j| < 0$. The requirement that $L^4_4 > 0$ completes the normalization.

The correspondence,

(3.12) Lorentz transformation $\longleftrightarrow$ collineation of the quadric group,

thus defines a (1-1) isomorphism between the Lorentz group and the group of the quadric in $R_3$. Under this isomorphism restricted Lorentz transformations correspond to collineations with positive determinant, and conversely. The collineations of the quadric group which have positive determinant form a group which

*This is easily proved by showing that a line may contain only exterior points but cannot contain only interior points.

we call the restricted quadric group.

The one-to-one character of the isomorphism (3.12) is evident from the fact that a Lorentz transformation is completely determined by the way in which it permutes the lines through the origin of $R_4$.

THE PROJECTIVE GROUP IN $P_1$ ISOMORPHIC TO THE PROPER LORENTZ GROUP

4. The projective group on the complex projective line is the set of all linear transformations

$$\varphi_1 = a\psi_1 + b\psi_2, \qquad \varphi_2 = c\psi_1 + d\psi_2, \tag{4.1}$$

with $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, where two transformations are identical if and only if their coefficients are proportional. Thus if (4.1) is abbreviated to

$$\varphi_A = P_A{}^B\psi_B, \tag{4.2}$$

the equations $\varphi_A = \sigma P_A{}^B\psi_B$ with $\sigma$ complex and $\neq 0$, effect exactly the same permutation of the pencils of number pairs $(\rho\psi_1, \rho\psi_2)$ and hence define the same transformation. In terms of the non-homogeneous coordinate $z = \psi_1/\psi_2$, (4.1) is the general linear fractional transformation

$$w = \frac{az + b}{cz + d}, \tag{4.3}$$

where $w = \varphi_1/\varphi_2$.

Two configurations in $P_1$ (for example, two ordered sets of points) are said to be projectively equivalent if there exists a projectivity (4.2) which carries one configuration into the other. The fundamental theorem of one-dimensional projective geometry then states that there is a uniquely determined projectivity which carries three distinct points into any other three distinct points. To prove this theorem it is sufficient to show that the points (1, 0), (0, 1), (1, 1) can be carried into the distinct points $(\alpha_1, \alpha_2)$, $(\beta_1, \beta_2)$, and $(\gamma_1, \gamma_2)$, respectively, in just one way. In order that (1, 0) shall go into $(\alpha_1, \alpha_2)$ we must have in (4.1) $a = \rho\alpha_1$, $c = \rho\alpha_2$ and, for (0, 1) to go into $(\beta_1, \beta_2)$, $b = \sigma\beta_1$, $d = \sigma\beta_2$, with $\rho$ and $\sigma \neq 0$. Then $\rho$ and $\sigma$ are determined by the equations

$$\gamma_1 = \rho\alpha_1 + \sigma\beta_1,$$

$$\gamma_2 = \rho\alpha_2 + \sigma\beta_2,$$

which have a unique solution with $\rho \neq 0$ and $\sigma \neq 0$ if the points $\alpha$, $\beta$ and $\gamma$ are distinct. Introducing a factor of proportionality into the coordinates of the points alters the four components of $P_A{}^B$ by a common factor. Similarly, in n-dimensional complex projective geometry two sets of n+2 points are equivalent under a uniquely determined projectivity if no n+1 points of either set lie in the same hyper-plane.

The transformation (4.2) brings about a linear homogeneous transformation,

$$\Phi_{\dot{A}B} = \bar{P}_A{}^C P_B{}^D \Psi_{\dot{C}D} \tag{4.4}$$

of the degenerate Hermitian matrices $\|\Psi_{\dot{A}B}\| = \|\bar{\psi}_A \psi_B\|$ and if we agree that (4.4) is also induced by (4.2) on all Hermitian matrices, we have the transformation (1.20) studied in §1.

Let us write (1.3) in the abbreviated form

$$x^i = g^{i\dot{A}B}\Psi_{\dot{A}B} \quad . \tag{4.5}$$

The coefficients $g^{i\dot{A}B}$ have the values given by the four matrix equations

$$(4.6)\quad \begin{aligned} \|g^{1\dot{A}B}\| &= \frac{1}{\sqrt{2}} \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}, \qquad \|g^{2\dot{A}B}\| = \frac{1}{\sqrt{2}} \begin{Vmatrix} 0 & -i \\ i & 0 \end{Vmatrix}, \\ \|g^{3\dot{A}B}\| &= \frac{1}{\sqrt{2}} \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}, \text{ and } \|g^{4\dot{A}B}\| = \frac{1}{\sqrt{2}} \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}. \end{aligned}$$

The equations (1.2) which are inverse to (4.5) may then be written

(4.7) $$\Psi_{\dot{A}B} = g_{i\dot{A}B}\, X^i$$

and we have

(4.8) $$g^{i\dot{A}B}\, g_{j\dot{A}B} = \delta^i{}_j ,$$

where

$$||\delta^i{}_j|| = 1 ,$$

and also

(4.9) $$g_{i\dot{A}B}\, g^{i\dot{C}D} = \delta^{\dot{C}}{}_{\dot{A}}\, \delta^D{}_B ,$$

where

$$||\delta^{\dot{C}}{}_{\dot{A}}|| = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} = ||\delta^D{}_B|| .$$

The transformation (4.4) induces a transformation $X \longrightarrow Y$ of the variables $X^i$ which is represented by

(4.10) $$Y^i = g^{i\dot{A}B}\, \Phi_{\dot{A}B} = g^{i\dot{A}B}\, \bar{P}_A{}^C\, P_B{}^D\, g_{j\dot{C}D}\, X^j .$$

This is of the form (3.1) with

(4.11) $$L^i{}_j = g^{i\dot{A}B}\, \bar{P}_A{}^C\, P_B{}^D\, g_{j\dot{C}D} ,$$

and $||L^i{}_j||$ is real since (4.4) carries Hermitian matrices into Hermitian matrices.

As a transformation of the four variables $\Psi_{11}, \Psi_{12}, \Psi_{21}, \Psi_{22}$ (4.4) has a matrix

(4.12) $$\gamma = \begin{Vmatrix} a\bar{a} & b\bar{a} & a\bar{b} & b\bar{b} \\ c\bar{a} & d\bar{a} & c\bar{b} & d\bar{b} \\ a\bar{c} & b\bar{c} & a\bar{d} & b\bar{d} \\ c\bar{c} & d\bar{c} & c\bar{d} & d\bar{d} \end{Vmatrix} = \begin{Vmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{Vmatrix} \begin{Vmatrix} \bar{a} & 0 & \bar{b} & 0 \\ 0 & \bar{a} & 0 & \bar{b} \\ \bar{c} & 0 & \bar{d} & 0 \\ 0 & \bar{c} & 0 & \bar{d} \end{Vmatrix} .$$

Hence (4.11) is equivalent to

(4.13) $$||L^i{}_j|| = k\, \gamma\, k^{-1} ,$$

where k denotes the four-row matrix of (4.5) regarded as a linear transformation of $(\Psi_{11}, \Psi_{12}, \Psi_{21}, \Psi_{22})$ into $(x^1, x^2, x^3, x^4)$. Taking the determinant of both members of (4.13) gives

$$(4.14) \qquad |L^i{}_j| = |\gamma| = |P_A{}^B|^2 \, |\bar{P}_A{}^B|^2,$$

and therefore $|L^i{}_j|$ is positive. Since (4.4) changes singular matrices into singular matrices the collineation (3.1) with $L^i{}_j$ given by (4.11) is an element of the quadric group. Hence every projectivity of $P_1$ corresponds under (4.11) to a uniquely determined collineation of the restricted quadric group in $R_3$.

This correspondence is a (1-1) isomorphism between the projective group in $P_1$ and the restricted quadric group in $R_3$. We shall prove this by a simple geometric argument which makes use of the fact that a collineation of the restricted quadric group is completely determined by the fate of three points on the quadric.

The points on the quadric correspond to the points of the complex line $P_1$. We have already constructed the uniquely determined projectivity of $P_1$ which carries three distinct points into any other three distinct points. This, combined with equations (4.10), gives a collineation of the restricted quadric group which carries any three points of the quadric into any other three points. Moreover, there is just one collineation of the restricted quadric group which does this. For, if $T_1$ and $T_2$ are transformations of the restricted quadric group which have the same effect on three points, $T_1^{-1}T_2 = T$ is an element of the group which leaves three points on the quadric invariant. We shall show that in this case T is the identity.

The plane of the three points is invariant, and hence the conic in which this plane intersects the quadric is invariant. The intersection of the tangents to the conic at two of the points is a fourth point, not collinear with any two of the other three, which is also invariant. The collineation T therefore induces in this invariant plane a collineation which leaves four points invariant, no three of the points being collinear. By the fundamental theorem of two-dimensional projective geometry, this collineation in the plane is the identity and hence T leaves each point of the plane invariant.

The polar point of the plane with respect to the quadric is an invariant point not in the plane. If we draw a line through this point which intersects the quadric in two real points, $P_1$ and $P_2$, the line intersects the plane in an invariant point and is therefore itself invariant. Hence T either leaves $P_1$ and $P_2$ separately invariant, or it interchanges them. In either case the fundamental theorem for three-dimensional projective geometry assures us that there is just one collineation having the given effect. We have therefore found that there are just two collineations of the quadric group which leave three points A, B, C on the quadric invariant. One of these two transformations must have negative determinant. For, consider an arbitrary collineation with negative determinant; it carries the points A, B, C into three points which may be called A', B', C'; and a collineation with positive determinant can be found which carries A', B', C' back into A, B, C. The product of these two collineations has negative determinant and leaves the points A, B, C invariant. Hence, of the two collineations which leave three points invariant, one is the identity and the other is a collineation whose determinant is negative.

We have now completed our proof that there is a (1-1) isomorphism between the restricted quadric group in $R_3$ and the projective group in $P_1$. In the last section we established the existence of a (1-1) isomorphism between the restricted Lorentz group and the restricted quadric group. Hence, combining these two isomorphisms, we get the (1-1) correspondence

(4.15) projectivity of $P_1 \longleftrightarrow$ restricted Lorentz transformation.

To express this isomorphism as a correspondence between matrices, we normalize the matrix of a projectivity to within sign by the condition

$$|P_A{}^B| = +1. \tag{4.16}$$

The matrix $\|L^i{}_j\|$ defined by (4.11) is consequently normalized so that

$$|L^i{}_j| = +1, \tag{4.17}$$

as follows from (4.14). Moreover, we saw in §1 that (4.4), or the equivalent transformation (4.10), does not interchange the absolute future and the absolute past. Hence $\|L^i{}_j\|$ is a proper Lorentz matrix and not the negative of one.

This result means that if we are given a restricted Lorentz matrix, $\|L^i{}_j\|$, there are just two unimodular matrices, $\|P_A{}^B\|$ and $-\|P_A{}^B\|$, which satisfy (4.11). Conversely each of the two unimodular matrices $\|P_A{}^B\|$ and $-\|P_A{}^B\|$ determine the same restricted Lorentz matrix by means of (4.11). Since the product of two Lorentz matrices corresponds to the products of the corresponding unimodular matrices, we have the (2-1) isomorphism,

(4.18) unimodular matrix $\|P_A{}^B\| \longrightarrow$ restricted Lorentz matrix $\|L^i{}_j\|$,

between the group of unimodular matrices of order two (with complex elements) and the group of restricted Lorentz matrices (with real elements).

THE ANTIPROJECTIVE GROUP IN $P_1$ ISOMORPHIC TO THE LORENTZ GROUP

5. An antiprojectivity in $P_1$ is a transformation of $P_1$ defined by equations of the form

$$\varphi_A = P_A{}^{\dot{B}} \overline{\psi}_B, \tag{5.1}$$

with $|P_A{}^{\dot{B}}| \neq 0$, and two sets of equations of this form define the same antiprojectivity if and only if their coefficients are proportional. In terms of the non-homogeneous coordinate $z = \frac{\psi_1}{\psi_2}$, (5.1) is

$$(5.2) \qquad w = \frac{a\bar{z} + b}{c\bar{z} + d}$$

where $\|P_A{}^{\dot{B}}\| = \begin{Vmatrix} a & b \\ c & d \end{Vmatrix}$. Clearly, the set of all antiprojectivities results from combining all the projectivities with the special antiprojectivity

$$(5.3) \qquad \varphi_A = \bar{\psi}_A$$

The antiprojective group is the set of all projectivities and antiprojectivities. It contains as an invariant sub-group the projective group. The antiprojectivities alone do not form a group since the product of two antiprojectivities is a projectivity. Thus the product of (5.1) and the antiprojectivity $\theta_A = Q_A{}^{\dot{B}}\bar{\varphi}_B$ is the projectivity $\theta_A = (Q_A{}^{\dot{B}}\bar{P}_B{}^{\dot{C}})\psi_C$. The antiprojective group is obtained by adjoining any one antiprojectivity, say (5.3), to the projective group.

The transformation (5.3) carries the singular Hermitian matrix

$$\|\Psi_{\dot{A}B}\| = \begin{Vmatrix} \bar{\psi}_1\psi_1 & \bar{\psi}_1\psi_2 \\ \bar{\psi}_2\psi_1 & \bar{\psi}_2\psi_2 \end{Vmatrix} \quad \text{into} \quad \|\Phi_{\dot{A}B}\| = \begin{Vmatrix} \psi_1\bar{\psi}_1 & \psi_1\bar{\psi}_2 \\ \psi_2\bar{\psi}_1 & \psi_2\bar{\psi}_2 \end{Vmatrix};$$

that is,

$$(5.4) \qquad \Phi_{11} = \Psi_{11}, \quad \Phi_{12} = \Psi_{21}, \quad \Phi_{21} = \Psi_{12}, \quad \text{and} \quad \Phi_{22} = \Psi_{22},$$

and we assume that (5.4) is induced on all Hermitian matrices by (5.3). Referring to (1.2), we see that (5.4) is represented in the coordinates of $R_4$ by the transformation (3.5). Multiplying all the projectivities of $P_1$ by (5.3) gives the set of all antiprojectivities and multiplying the restricted Lorentz transformations by the corresponding transformation (3.5) gives the set of all improper Lorentz transformations, as we saw in §3.

In the last section we established a (1-1) isomorphism between the projective group in $P_1$ and the restricted Lorentz group in $R_4$. This has now been extended to include the (1-1) correspondence,

(5.5) antiprojectivity of $P_1 \longleftrightarrow$ improper Lorentz transformation,

and hence there is a (1-1) isomorphism between the antiprojective group in $P_1$ and the Lorentz group in $R_4$.

The equations defining (5.5) are got by following (5.4) with the transformation (4.4). They are

$$\Phi_{\dot{A}B} = \bar{P}_A{}^{\dot{C}} P_B{}^{\dot{D}} \Psi_{\dot{D}C}, \tag{5.6}$$

and hence in the coordinates of $R_4$ they are of the form (3.1) with

$$L^i{}_j = g^{i\dot{A}B} \bar{P}_A{}^{\dot{C}} P_B{}^{\dot{D}} g_{j\dot{D}C}. \tag{5.7}$$

Since the determinant of (3.5) is -1,

$$|L^i{}_j| = - |P_A{}^{\dot{B}}|^2 \; |\bar{P}_A{}^{\dot{B}}|^2. \tag{5.8}$$

Normalizing the coefficients of the antiprojectivities to within sign by the condition

$$|P_A{}^{\dot{B}}| = +1, \tag{5.9}$$

equations (5.7) determine a (2-1) correspondence,

(5.10) unimodular matrix $\|P_A{}^{\dot{B}}\| \longrightarrow$ improper Lorentz matrix $\|L^i{}_j\|$.

The correspondences (4.18) and (5.10) may be combined to give the theorem:

For every proper Lorentz matrix $\|L^i{}_j\|$ there are exactly two unimodular matrices, $\|P_A{}^B\|$ and $-\|P_A{}^B\|$, which satisfy (4.11), and for every improper Lorentz matrix $\|L^i{}_j\|$ there are exactly two unimodular matrices, $\|P_A{}^{\dot{B}}\|$ and $-\|P_A{}^{\dot{B}}\|$, which satisfy (5.7). Conversely, if $\|P_A{}^B\|$ is a unimodular matrix it determines a proper Lorentz transformation by means of (4.11) and if $\|P_A{}^{\dot B}\|$ is a unimodular matrix it determines an improper Lorentz transformation by means of (5.7). Moreover, the product of two Lorentz transformations corresponds to the product of the corresponding normalized transformations in $P_1$.

COORDINATE TRANSFORMATIONS AND TENSOR CALCULUS

6. Up to this point we have employed only a single coordinate system in each of the spaces considered. The linear transformation (3.1) of $R_4$ was regarded as a permutation of the points of $R_4$. The coordinate transformation

(6.1) $$X^{i*} = A^i_j X^j,$$

with $|A^i_j| \neq 0$, is of the same form as (3.1) but is to be regarded as a renaming of the points of $R_4$, rather than as a permutation of the points. Defining a coordinate system* in $R_4$ as a (1-1) correspondence between the points of $R_4$ and sets of four real numbers, the cartesian coordinate systems are those derived from the special one we have been using in the sections above, by the group of transformations (6.1) with $\|A^i_j\|$ real and constant.

Under (6.1) the quadratic form $g_{ij}X^iX^j$ becomes $g_{ij}{}^*X^{i*}X^{j*}$ where $g_{ij}{}^*$ is related to $g_{ij}$ by the usual tensor law

(6.2) $$g_{ij}{}^* = a^k_i\, a^\ell_j\, g_{k\ell} \;,$$

with $a^j_i$ defined by

(6.3) $$a^i_j\, A^j_k = \delta^i_k.$$

* For a general discussion of coordinate systems see the Cambridge Tract "Foundations of differential geometry", by Veblen and Whitehead.

In what follows $g_{ij}X^iX^j$ is a general quadratic form which can be reduced to (3.2) by means of a real transformation (6.1) but which is not necessarily identical with (3.2).

Coordinate systems in which the quadratic form is given by (3.2) are called Galilean coordinate systems. Since a transformation of coordinates between two Galilean coordinate systems satisfies (6.2) with $g_{ij}{}^* = g_{ij}$, the extended Lorentz group could have been defined as the set of all those transformations which carry one Galilean coordinate system into another one of the same kind.

The covariant tensor $g_{ij}$ and the contravariant tensor $g^{ij}$ defined by

$$g^{ij}g_{jk} = \delta^i_k, \tag{6.4}$$

enable us to raise and lower tensor indices. Thus if $X^i$ is a contravariant vector transforming by the law (6.1) and we put

$$g_{ij}X^j = X_i \tag{6.5}$$

then

$$X_i{}^* = a^j_i\, X_j \tag{6.6}$$

and this is the law of transformation of a covariant vector. Moreover,

$$X^j = g^{ji}X_i. \tag{6.7}$$

If we take the determinant of both members of (6.2), we find that

$$g^* = a^2 g \tag{6.8}$$

where $g = |g_{ij}|$ and $a = |a^j_i|$. This is the law of transformation of a relative scalar* of weight two. In general we consider relative tensors having the trans-

---

* Some writers use the term "density". We prefer to reserve density for the case to which this term has been applied in physics. Thus a relative scalar of weight one is a density.

formation law

(6.9) $$X^{ij\ldots}{}_{\ell m\ldots}^{*} = a^{w} A^{i}_{p} A^{j}_{q} \ldots a^{s}_{\ell} a^{t}_{m} \ldots X^{pq\ldots}{}_{st\ldots}$$

and the weight of such a tensor is said to be w. The right member of (6.9) is a linear and homogeneous polynomial in the components of the tensor and is a homogeneous expression in the elements of the matrix $\| A^{i}_{j} \|$.

THE ALTERNATING NUMERICAL TENSORS

7. The rule (6.5) is the familiar way of associating with a contravariant tensor a covariant one. A less familiar way of doing this employs a covariant tensor $\epsilon_{ijk\ell}$ of weight -1. In any one coordinate system we define $\epsilon_{ijk\ell}$ to be +1 if (i, j, k, $\ell$) is an even permutation of (1, 2, 3, 4), -1 if it is an odd permutation, and zero otherwise. Then the formula for the expansion of $|a^{j}_{i}|$, which is

(7.1) $$a\,\epsilon_{ijk\ell} = \epsilon_{pqrs}\, a^{p}_{i}\, a^{q}_{j}\, a^{r}_{k}\, a^{s}_{\ell}$$

insures that the tensor shall have exactly the same components in any other coordinate system. The weight of the tensor $\epsilon_{ijk\ell}$ is the exponent of <u>a</u> when (7.1) is written in the form

$$\epsilon_{ijk\ell} = a^{-1}\epsilon_{pqrs}\, a^{p}_{i}\, a^{q}_{j}\, a^{r}_{k}\, a^{s}_{\ell}\,.$$

A contravariant vector $X^{i}$ determines a covariant tensor

(7.2) $$\breve{X}_{ijk} = \epsilon_{ijk\ell} X^{\ell}.$$

This amounts merely to a renumbering of the components $X^{1}, X^{2}, \ldots, X^{4}$ as indicated in the formulas

(7.3) $$\breve{X}_{123} = -\breve{X}_{213} = \ldots = \breve{X}_{231} = X^{4},\ \ \breve{X}_{124} = -\breve{X}_{214} = \ldots = -X^{3}, \text{ etc.}$$

While the components of $\breve{X}_{ijk}$ are equal to plus or minus the components of $X^{i}$, it is nevertheless true that $X^i$ is a contravariant tensor and $\breve{X}_{ijk}$ is a covariant one. If $X^i$ is of weight w, $\breve{X}_{ijk}$ is of weight w-1.

For the corresponding operation of raising indices we use the relative tensor $\epsilon^{ijk\ell}$ of weight +1 defined by the relations

$$\epsilon^{ijk\ell} \begin{cases} = +1 \text{ if } (i,j,k,\ell) \text{ is an even permutation of } (1,2,3,4), \\ = -1 \text{ if } (i,j,k,\ell) \text{ is an odd permutation of } (1,2,3,4), \\ = 0 \text{ if two of the indices are equal.} \end{cases} \tag{7.4}$$

The expansion of $|A^i_j|$ can be written in the form

$$\epsilon^{ijk\ell} = a\,\epsilon^{pqrs} A^i_p A^j_q A^k_r A^\ell_s ,$$

since $|A^i_j| = |a^i_j|^{-1} = a^{-1}$, and this identity proves that if $\epsilon^{ijk\ell}$ is a contravariant tensor of weight +1 then its components have the same values in all coordinate systems.

A covariant index is converted into three contravariant ones by the rule

$$\breve{Y}^{jk\ell} = Y_i\,\epsilon^{ijk\ell} , \tag{7.5}$$

and if $Y_i$ is of weight w, $\breve{Y}^{jk\ell}$ is of weight 1+w. We may also sum off two of the indices of $\epsilon^{ijk\ell}$ against two covariant indices as in the equations

$$\breve{Y}^{k\ell} = \frac{1}{2}\,Y_{ij}\,\epsilon^{ijk\ell}, \tag{7.6}$$

which are an abbreviated form for

$$\breve{Y}^{12} = \frac{1}{2}\,(Y_{34} - Y_{43}),\quad \breve{Y}^{13} = \frac{1}{2}\,(Y_{42} - Y_{24}),\ \text{etc.} \tag{7.7}$$

Since the right members of these equations involve only the combinations $(Y_{ij} - Y_{ji})$, the equations will not have a unique inverse unless the components $Y_{ij}$ are restricted in some way. The simplest restriction is

$$Y_{ij} = -\,Y_{ji}, \tag{7.8}$$

and such a tensor is said to be skew-symmetric. We shall apply (7.6) only to skew-symmetric tensors and for them the equations inverse to (7.6) are

$$Y_{ij} = \frac{1}{2} \epsilon_{ijk\ell} \overset{\smile}{Y}^{k\ell}, \tag{7.9}$$

since

$$\frac{1}{2} \epsilon_{ijk\ell} \epsilon^{pqk\ell} = \delta^{pq}_{ij} \tag{7.10}$$

where

$$\delta^{pq}_{ij} = \delta^p_i \delta^q_j - \delta^p_j \delta^q_i. \tag{7.11}$$

The equations inverse to (7.2) may be written in a form similar to (7.5) since

$$\frac{1}{3!} \epsilon^{ijkp} \epsilon_{ijk\ell} = \delta^p_\ell, \tag{7.12}$$

and hence

$$X^p = \frac{1}{3!} \overset{\smile}{X}_{ijk} \epsilon^{ijkp}. \tag{7.13}$$

For a more extended treatment of the tensor calculus, the reader is referred to: "Applications of the absolute differential calculus", by A. J. McConnell; "Riemannian geometry", by L. P. Eisenhart; "The differential invariants of generalized spaces", by T. Y. Thomas; the Cambridge Tract, "Invariants of quadratic differential forms", by O. Veblen; or to any of the numerous books on the subject. The first chapter of the Cambridge Tract, "Invariants of quadratic differential forms", contains general formulas of which (7.10) and (7.11) are special cases.

DUAL COORDINATES IN $R_3$

8. The linear transformation (6.1) may be interpreted as a coordinate transformation in $R_3$ instead of in $R_4$. In doing this we must remember

that (6.1) and

(8.1) $$X^{i*} = \rho A^i_j X^j, \qquad \text{with } \rho \neq 0,$$

effect exactly the same permutation of the sets of homogeneous coordinates $(rX^1, rX^2, rX^3, rX^4)$ and therefore define the same coordinate transformation in $R_3$. The formalism in $R_3$ is identical with that in $R_4$ but many of the relations have a simpler geometric interpretation in $R_3$ than in $R_4$.

Thus in general coordinates the plane polar to $A^i$ with respect to the quadric is

(8.2) $$A^j g_{ji} X^i = 0, \quad \text{or} \quad A_i X^i = 0,$$

and so lowering an index by the rule (6.5) corresponds to the geometric operation of taking the polar plane with respect to the fundamental quadric. The inverse of this operation, raising an index by means of $g^{ij}$, corresponds to taking the polar point of a plane with respect to the quadric.

The homogeneous coordinates of a point P of $R_3$ are uniquely determined as solutions of the set of equations

(8.3) $$\breve{P}_{ijk} X^k = 0,$$

where $\breve{P}_{ijk}$ is defined in terms of $P^i$ by means of (7.2). This is evident if we use (7.3) to write out (8.3) in the form

(8.4) $$P^4X^3 - P^3X^4 = 0, \quad P^2X^4 - P^4X^2 = 0, \text{ etc.}$$

Hence a point has covariant coordinates $\breve{P}_{ijk}$ as well as contravariant ones $P^i$. These covariant coordinates, like the contravariant ones, are homogeneous.

The existence of covariant coordinates corresponds to the possibility of defining a point as the intersection of the planes containing it. Thus if $\alpha_i$, $\beta_i$ and $\gamma_i$ are three independent planes which contain $P^i$, then the tensor

$$\Pi_{ijk} = \frac{1}{3!}\delta^{pqr}_{ijk}\alpha_p\beta_q\gamma_r$$

(8.5)

$$\equiv \frac{1}{6}(\alpha_i\beta_j\gamma_k + \beta_i\gamma_j\alpha_k + \gamma_i\alpha_j\beta_k - \gamma_i\beta_j\alpha_k - \beta_i\alpha_j\gamma_k - \alpha_i\gamma_j\beta_k$$

is not zero and satisfies the equation

(8.6) $$\Pi_{ijk}P^k = 0.$$

In a coordinate system in which P has the coordinates (1, 0, 0, 0) this equation, together with the skew-symmetry of $\Pi_{ijk}$, implies that the only non-vanishing components of $\Pi_{ijk}$ are $\Pi_{234}$ and the ones obtained by a permutation of these indices. The same result holds for $\breve{P}_{ijk}$ and hence

(8.7) $$\Pi_{ijk} = \rho\,\breve{P}_{ijk}\,.$$

In equation (8.7) we must remember that if $P^i$, $\alpha_i$, $\beta_i$, and $\gamma_i$ are of weight zero, then $\breve{P}_{ijk}$ is of weight -1 and $\Pi_{ijk}$ is of weight zero so that $\rho$ must transform as a relative scalar of weight +1. We could, for example, put $\rho = \sigma(-g)^{\frac{1}{2}}$ where $\sigma$ is a scalar of weight zero.

Two points, P and Q, of $R_3$ determine the line joining them, and this line has the contravariant Plücker coordinates*

---

* A discussion of these coordinates is given in Veblen and Young's "Projective geometry", Vol. I, p. 327.

---

(8.8) $$p^{ij} = P^iQ^j - Q^iP^j$$

Replacing $P^i$ by $P^i + \lambda Q^i$ in this equation does not change the value of $p^{ij}$ and since a similar result holds for $Q^i$ the homogeneous coordinates $p^{ij}$ do not depend upon which points on the line are chosen to define it. In particular, we may take P to be the point in which the line intersects the plane $\xi$ and then $p^{ij}\xi_j = (Q^j\xi_j)P^i$. Hence $p^{ij}\xi_j$ is the point in which the line $p^{ij}$

intersects the plane $\xi_j$ and the condition that the plane contain the line is

(8.9) $$p^{ij}\xi_j = 0.$$

Similarly, if $\alpha_i$ and $\beta_i$ are planes which contain the line PQ, covariant Plücker coordinates of the line are

(8.10) $$q_{ij} = \alpha_i \beta_j - \beta_i \alpha_j$$

Then $q_{ij}$ is proportional to the tensor $\breve{p}_{ij}$ defined by

(8.11) $$\breve{p}_{ij} = \frac{1}{2}\,\epsilon_{ijk\ell}\,p^{k\ell}$$

as we readily prove by taking the coordinates of P and Q to be (1, 0, 0, 0) and (0, 1, 0, 0) and of $\alpha$ and $\beta$ to be (0, 0, 1, 0) and (0, 0, 0, 1). The tensor $\breve{p}_{ij}$ is said to be the dual of $p^{ij}$ and $\breve{q}^{ij}$ (cf. (7.6)) is the dual of $q_{ij}$. The term "dual" is appropriate since the principle of duality in a projective (or vector) space implies the existence of the two sorts of coordinates. If we apply the rule (7.6) to $\breve{p}_{ij}$ we recover $p^{ij}$ and hence the dual of $\breve{p}_{ij}$ is $p^{ij}$.

The coordinates of a line, $p^{ij}$, satisfy the equations

(8.12) $$p^{ij}\,\breve{p}_{jk} = 0,$$

since the left member is just

$$\frac{1}{2}\,\epsilon_{jk\ell m}\,p^{ij}\,p^{\ell m}$$

and for fixed i and k = 1, 2, 3, 4 this is the expression for the minors of the matrix

$$\left\|\begin{matrix} -P^iQ^1 + Q^iP^1 & -P^iQ^2 + Q^iP^2 & -P^iQ^3 + Q^iP^3 & -P^iQ^4 + Q^iP^4 \\ P^1 & P^2 & P^3 & P^4 \\ Q^1 & Q^2 & Q^3 & Q^4 \end{matrix}\right\| .$$

We shall now prove the converse proposition, namely, that if an arbitrary skew-symmetric tensor not equal to zero satisfies (8.12), its components are the coordinates of a line.

From (8.12) it follows that the sum of the nullities of $\| p^{ij} \|$ and $\| \breve{p}_{jk} \|$ is equal to or greater than the range of the index j and hence rank $\| p^{ij} \|$ + rank $\| \breve{p}_{jk} \| \leqq 4$. If rank $\| p^{ij} \| = 1$, the equations $p^{ij} \xi_j = 0$ have three independent solutions, and after a suitable transformation of coordinates we may take these solutions to be the unit vectors $\delta^1_j$, $\delta^2_j$ and $\delta^3_j$. Then $p^{i1} = p^{i2} = p^{i3} = 0$, and since $p^{44} = 0$ on account of the skew-symmetry, we have $p^{ij} = 0$, contrary to assumption. Hence rank $\| p^{ij} \| \geqq 2$. A similar result holds for $\breve{p}_{ij}$ and consequently rank $\| p^{ij} \|$ = rank $\| \breve{p}_{ij} \| = 2$. The equations $p^{ij} \xi_j = 0$ therefore have just two independent solutions and if we choose the coordinate system so that they are $\delta^3_j$ and $\delta^4_j$, $p^{ij} = \rho(\delta^i_1 \delta^j_2 - \delta^i_2 \delta^j_1)$ and hence represents a line. In the same coordinate system $\breve{p}_{ij} = \sigma(\delta^3_i \delta^4_j - \delta^4_i \delta^3_j)$ and so corresponds to the same line as does $p^{ij}$.

In this argument we have not made use of the relationship (8.11) between $p^{ij}$ and $\breve{p}_{ij}$ and we have therefore proved that if $p^{ij}$ and $\breve{q}_{ij}$ are skew-symmetric tensors different from zero and

$$p^{ij} \breve{q}_{jk} = 0, \tag{8.13}$$

then $p^{ij}$ and $\breve{q}_{ij}$ are coordinates of the same line. Indeed, by an entirely similar argument it is possible to prove that if $p^{i_1 i_2 \cdots i_a}$ and $\breve{q}_{i_1 i_2 \cdots i_b}$ are non-vanishing skew-symmetric tensors which satisfy the equation

$$p^{i_1 i_2 \cdots i_a} \breve{q}_{i_a j_1 \cdots j_{b-1}} = 0, \text{ with } a+b = k, \ (i_1, i_2, \ldots = 1, 2, \ldots, k) \tag{8.14}$$

then p and $\breve{q}$ are coordinates of the same linear space of (a-1)-dimensions in a projective space of (k-1)-dimensions.

Equations (8.13) are equivalent to

$$q^{ij}\breve{p}_{jk} = 0 \tag{8.15}$$

as follows readily from the identity

$$X^{ij}\breve{Y}_{jk} + Y^{ij}\breve{X}_{jk} \equiv -\frac{1}{2}\,(X^{pq}\breve{Y}_{pq})\,\delta^{i}_{k} \tag{8.16}$$

which holds for arbitrary skew-symmetric tensors $X^{ij}$ $(= -X^{ji})$ and $Y^{ij}$ $(= -Y^{ji})$. To verify this identity we use (8.11) and its inverse, which is of the form (7.6), to get

$$\begin{aligned} X^{ij}\breve{Y}_{jk} &\equiv \frac{1}{2}\,\epsilon^{pqij}\breve{X}_{pq}\cdot\frac{1}{2}\,\epsilon_{jkrs}Y^{rs} \\ &\equiv -\frac{1}{4}\,\delta^{pqi}_{krs}\,\breve{X}_{pq}Y^{rs} \\ &\equiv -\frac{1}{4}\,(\delta^{pq}_{kr}\delta^{i}_{s} + \delta^{pq}_{sk}\,\delta^{i}_{r} + \delta^{pq}_{rs}\,\delta^{i}_{k})\breve{X}_{pq}Y^{rs} \\ &\equiv -\,Y^{ij}\breve{X}_{jk} - \frac{1}{2}\,(\breve{X}_{pq}Y^{pq})\,\delta^{i}_{k}. \end{aligned} \tag{8.17}$$

Putting $X^{ij} = Y^{ij} = p^{ij}$ in (8.16) gives

$$p^{ij}\breve{p}_{jk} \equiv -\frac{1}{4}\,(p^{ab}\breve{p}_{ab})\,\delta^{i}_{k}, \tag{8.18}$$

and hence (8.12) is equivalent to

$$p^{ij}\breve{p}_{ij} \equiv \frac{1}{2}\,\epsilon_{ijk\ell}\,p^{ij}p^{k\ell} = 4\,(p^{12}p^{34} + p^{13}p^{42} + p^{14}p^{23}) = 0\,. \tag{8.19}$$

This equation is therefore the necessary and sufficient condition that $p^{ij}$ be the coordinates of a line. The condition for the lines with coordinates $p^{ij}$ and $q^{ij}$ to intersect can be shown to be

$$p^{ij}\breve{q}_{ij} = 0. \tag{8.20}$$

To interpret the covariant tensor $p_{ij}$ geometrically, we observe that multiplying (8.8) by $g_{ki}\,g_{\ell j}$ and summing gives

(8.21) $$p_{k\ell} = P_k Q_\ell - Q_k P_\ell$$

and hence $p_{k\ell}$ are the covariant coordinates of the line polar to $p^{ij}$ with respect to the quadric. The line $p^{ij}$ intersects its polar if

(8.22) $$p^{ij}p_{ij} = 0$$

and so this is the condition for $p^{ij}$ to be tangent to the quadric. The line will lie on the quadric if it coincides with its polar and the condition for this is (cf. (8.13))

(8.23) $$p^{ij}p_{jk} = 0.$$

This condition is not satisfied by the coordinates of any real line.

THE SPINOR CALCULUS IN $P_1$

9. In the complex projective line the hyperplanes are themselves points and therefore a point is equally well represented by its contravariant coordinates $(\psi^1, \psi^2)$ and by the coefficients of its equation

(9.1) $$\psi_1 X^1 + \psi_2 X^2 = 0.$$

Indeed, we have anticipated this result by writing the coordinates of a point with the indices in covariant position. In order that (9.1) shall be the equation of the point $(\psi^1, \psi^2)$ it is necessary and sufficient that

(9.2) $$\frac{\psi^1}{\psi^2} = -\frac{\psi_2}{\psi_1}.$$

This relationship can be expressed in terms of a rule, analogous to (7.2) and (7.13), for lowering and raising indices by means of matrices $\|\epsilon_{AB}\|$ and $\|\epsilon^{AB}\|$ defined by the equations*

---

* The left and right indices will always refer to the rows and columns, respectively.

(9.3) $$\|\epsilon_{AB}\| = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} = \|\epsilon^{AB}\| .$$

The covariant components of a point are related to the contravariant ones by the formula

(9.4) $$\psi_A = \epsilon_{AB}\psi^B.$$

In $R_4$ we had two methods of converting contravariant indices into covariant ones. In $P_1$, however, we have no fundamental quadratic form and so the rule (9.4) is our only way of lowering indices. For this reason we write $\psi_A$ for the left member of (9,4) instead of $\breve{\psi}_A$ as we should do by strict analogy with (7.2).

The equations inverse to (9.4) are

(9.5) $$\psi^A = \epsilon^{BA}\psi_B$$

in which it is important to notice that we sum on the first index of $\epsilon^{BA}$. The proof that (9.5) is inverse to (9.4) is contained in the equations,

(9.6) $$\psi^A = \epsilon^{BA}\psi_B = \epsilon^{BA}\,\epsilon_{BC}\psi^C = \delta^A_C\psi^C.$$

It follows at once that

(9.7) $$\phi^A\psi_A = -\,\psi^A\varphi_A \quad \text{and hence} \quad \psi^A\psi_A \equiv 0.$$

Moreover,

(9.8) $$\epsilon_{AC}\,\delta^C_D\,\epsilon^{DB} = -\,\delta^B_A,$$

where

(9.9) $$\|\delta^A_B\| = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}.$$

TRANSFORMATIONS OF COORDINATES IN $P_1$

10. A transformation of coordinates in $P_1$ is given by equations of the form

(10.1) $$X_A^* = t_A^B X_B$$

where $|t_A^B| \neq 0$. The equations

$$X_A^* = \rho t_A^B X_B$$

with $\rho$ a complex number $\neq 0$, determine exactly the same permutation of the homogeneous coordinates $(\sigma X_1, \sigma X_2)$ and therefore define the same transformation of coordinates.

In every pencil of non-singular matrices $\| \rho t_A^B \|$ there are just two unimodular ones which are

(10.2) $$s_A^B = t_A^B t^{-\frac{1}{2}}$$

where $t = |t_A^B|$ and the two values of $\| S_A^B \|$ arise from the ambiguous sign of $t^{-\frac{1}{2}}$. For the purposes of projective geometry we could therefore restrict ourselves to the transformations of coordinates

(10.3) $$\Psi_A^* = s_A^B \Psi_B$$

with unimodular matrices. We however find it advisable to make use of the general transformation (10.1) which is a composite of the coordinate transformation (10.3) and the transformation

(10.4) $$\Psi_A^* = t^{1/2} \Psi_A,$$

which multiplies the coordinates of each point by a factor but does not change the coordinate system.

We may achieve the effect of restricting ourselves to the unimodular transformations (10.3) by assigning a suitable weight to $\psi_A$. Thus if $\psi_A$ is of weight $-\frac{1}{2}$ its transformation law is

(10.5) $$\psi_A{}^* = t_A^B \psi_B t^{-\frac{1}{2}} = s_A^B \psi_B.$$

Similarly, a geometric being $\psi^A$ of weight $+\frac{1}{2}$ will have the transformation law

(10.6) $$\psi^{A*} = T_B^A \psi^B t^{1/2} = S_B^A \psi^B$$

where

(10.7) $$T_B^A t_C^B = \delta_C^A \quad \text{and} \quad S_B^A s_C^B = \delta_C^A.$$

The formula for the expansion of a two-rowed determinant is analogous to (7.1), and therefore the transformation laws of $\epsilon^{AB}$ and $\epsilon_{AB}$ are

(10.8) $$t_A^C t_B^D \epsilon_{CD} t^{-1} = \epsilon_{AB} = s_A^C s_B^D \epsilon_{CD}$$

and

(10.9) $$T_C^A T_D^B \epsilon^{CD} t = \epsilon^{AB} = S_C^A S_D^B \epsilon^{CD}$$

from which it follows that $\epsilon^{AB}$ is of weight +1 and $\epsilon_{AB}$ is of weight -1. Hence the weights $+\frac{1}{2}$ for $\psi^A$ and $-\frac{1}{2}$ for $\psi_A$ are consistent with the rules (9.4) and (9.5).

The geometric being which has the components $(\psi_1, \psi_2)$ is closely analogous to a covariant vector in the sense that we have used this term, but the coefficients of the transformation (10.1), or (10.3), are complex instead of real. This allows the more general transformation law

(10.10) $$\psi_A{}^* = t_A^B \psi_B t^w \bar{t}^d$$

for a geometric being with components $\psi_A$. In (10.10) we call w and d the weight and antiweight, respectively. When w = d the transformation may be written as $\psi_A{}^* = t_A^B \psi_B \, |t|^{2w}$ where $|t| = (t\,\bar{t})^{\frac{1}{2}}$ is the absolute value of the determinant, $t$. In this case $\psi_A$ is said to be of absolute weight $2W$.

A geometric being with the transformation law (10.10) is a particular instance of what we shall in the next chapter define to be a spinor. Whatever the values of $w$ and $d$ in (10.10), $\psi_A$ still represents a point, but we shall usually take $w = -\frac{1}{2}$, $d = 0$.

A less trivial instance of a spinor is the geometric being defining an antiprojectivity. Thus if an antiprojectivity of $P_1$ has the equations

(10.11) $$\bar{\varphi}_A = P_{\dot{A}}{}^B \psi_B$$

in one coordinate system, then in a coordinate system related to the first by (10.1), it will have the equations

(10.12) $$\bar{\varphi}_A{}^* = P_{\dot{A}}{}^{B*} \psi_B{}^*$$

where

(10.13) $$P_{\dot{A}}{}^{B*} = \bar{t}_A^C \, P_{\dot{C}}{}^D \, T_D^B \, t^{c-w} \, \bar{t}^{v-d}$$

and $v$ and $c$ are the weights of $\varphi_A$ and $w$ and $d$ of $\psi_A$. We shall usually take $v = w = -\frac{1}{2}$ and $c = d = 0$, so that (10.13) may be written

(10.14) $$P_{\dot{A}}{}^{B*} = \bar{s}_A^C \, P_{\dot{C}}{}^D \, S_D^B \; ;$$

with these weights the normalization $|P_{\dot{A}}{}^B| = 1$ is invariant. The placing of the dot over the index C in (10.13) or (10.14) serves as a reminder that a bar is to be placed over the corresponding $t_A^C$ or $s_A^C$ in the law of transformation.

Equations (10.13) are of a more general type than is encountered in the ordinary tensor calculus in that they involve not only the components of $\|t_B^A\|$ but also the complex conjugates of these components. This possibility does not arise in tensor calculus, which refers to real coordinates and real transformations. We shall consider spinors having a transformation law of the type

$$(10.15)\qquad X^{AB\ldots\dot{C}\dot{D}\ldots}{}_{EF\ldots\dot{G}\dot{H}\ldots}{}^{*} = X^{PQ\ldots\dot{R}\dot{S}\ldots}{}_{TU\ldots\dot{V}\dot{W}\ldots} T^{A}_{P}\, T^{B}_{Q}\ldots \bar{T}^{C}_{R}\, \bar{T}^{D}_{S}\ldots t^{T}_{E}\, t^{U}_{F}\ldots \bar{t}^{V}_{G}\, \bar{t}^{W}_{H}\ldots t^{w}\bar{t}^{d}.$$

It is always possible to choose the weights w and d so that the transformation can be written in terms of the unimodular matrices $\| s^{A}_{B} \|$ and $\| S^{A}_{B} \|$ and we shall usually do this.

The raising and lowering of dotted indices is accomplished with the aid of spinors $\epsilon^{\dot{A}\dot{B}}$ and $\epsilon_{\dot{A}\dot{B}}$ by the rules

$$(10.16)\qquad \psi^{\dot{A}} = \epsilon^{\dot{B}\dot{A}}\psi_{\dot{B}} \quad \text{and} \quad \psi_{\dot{A}} = \epsilon_{\dot{A}\dot{B}}\psi^{\dot{B}}$$

where

$$(10.17)\qquad \|\epsilon^{\dot{A}\dot{B}}\| = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} = \|\epsilon_{\dot{A}\dot{B}}\| .$$

These rules are consistent with the antiweights $+\frac{1}{2}$ for contravariant dotted indices and $-\frac{1}{2}$ for covariant dotted ones since $\epsilon^{\dot{A}\dot{B}}$ and $\epsilon_{\dot{A}\dot{B}}$ have the transformation laws

$$(10.18)\qquad \bar{t}\, \bar{T}^{A}_{C}\, \bar{T}^{B}_{D}\, \epsilon^{\dot{C}\dot{D}} = \epsilon^{\dot{A}\dot{B}} = \epsilon^{\dot{C}\dot{D}}\, \bar{S}^{A}_{C}\, \bar{S}^{B}_{D}.$$

and

$$(10.19)\qquad \bar{t}^{-1}\, \bar{t}^{C}_{A}\, \bar{t}^{D}_{B}\, \epsilon_{\dot{C}\dot{D}} = \epsilon_{\dot{A}\dot{B}} = \epsilon_{\dot{C}\dot{D}}\, \bar{s}^{C}_{A}\, \bar{s}^{D}_{B},$$

and hence have the antiweights +1 and -1, respectively.

We also need to consider spinors having some indices which refer to a coordinate system in $R_4$ and some which refer to a coordinate system in $P_1$. For example, the coefficients of the fundamental equations (4.7) transform under (6.1) by the rule

(10.20) $$g_{i\dot{A}B}{}^{*} = a_i^j\, g_{j\dot{A}B},$$

and under (10.1) by the rule

(10.21)
$$\begin{aligned} g_{i\dot{A}B}{}^{*} &= g_{i\dot{C}D}\, \bar{t}_A^C\, t_B^D\, t^{-\frac{1}{2}}\, \bar{t}^{-\frac{1}{2}} \\ &= g_{i\dot{C}D}\, \bar{s}_A^C\, s_B^D \end{aligned}$$

INVOLUTIONS IN $P_1$

11. Using our rule for raising and lowering indices, the equations of a projectivity in $P_1$ may be written in the four equivalent forms

(11.1) $\varphi^A = P^A{}_B \psi^B$, $\varphi_A = P_{AB}\psi^B$, $\varphi_A = -P_A{}^B\psi_B$, and $\varphi^A = -P^{AB}\psi_B$,

where if $\| P^A{}_B \| = \left\| \begin{matrix} a & b \\ c & d \end{matrix} \right\|$, then

(11.2) $\| P_{AB} \| = \left\| \begin{matrix} c & d \\ -a & -b \end{matrix} \right\|$, $\| P_A{}^B \| = \left\| \begin{matrix} -d & c \\ b & -a \end{matrix} \right\|$, and $\| P^{AB} \| = \left\| \begin{matrix} -b & a \\ -d & c \end{matrix} \right\|$,

respectively. It is to be observed that the order of the indices distinguishes $P^A{}_B$ from $P_B{}^A$. Under the transformation of coordinates (10.1) the equations of the projectivity become $\varphi^{A*} = P^A{}_B{}^{*}\psi^{B*}$ where

(11.3) $$P^A{}_B{}^{*} = T_C^A\, P^C{}_D\, t_B^D\, t^{q-p},$$

if p and q are the weights of $\psi^A$ and $\varphi^A$, respectively, and their anti-weights are zero. Taking $q = p = \frac{1}{2}$, $P^A{}_B$ and $P_A{}^B$ are of weight zero, $P_{AB}$ is of weight -1, and $P^{AB}$ is of weight +1. The normalization $|P^A{}_B| = +1$ is invariant if the weight of $P^A{}_B$ is zero.

The matrix $\| P^A{}_B \|$ will define an involution if $P^A{}_B\, P^B{}_C\, \psi^C = \rho\psi^A$ and $\| P^A{}_B \|$ is not a multiple of $\left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\|$. That is,

(11.4) $$\left\| \begin{matrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{matrix} \right\| = \left\| \begin{matrix} \rho & 0 \\ 0 & \rho \end{matrix} \right\|,$$

or a + d = 0. This is expressed in convenient forms by the equivalent equations

(11.5) $$P^A{}_A = 0, \quad \text{or} \quad P_{AB} = P_{BA}.$$

The point $\psi^A$ will be invariant under the projectivity (11.1) if and only if $\varphi_A = \rho\psi_A$. This is equivalent to $\psi^A \varphi_A = 0$, or

(11.6) $$P_{AB}\psi^A\psi^B \equiv P_{11}(\psi^1)^2 + (P_{12} + P_{21})\psi^1\psi^2 + P_{22}(\psi^2)^2 = 0.$$

All the coefficients of this quadratic equation vanish only when $P_{AB}$ is a multiple of $\epsilon_{AB}$ and in this case the projectivity is the identity. Otherwise a projectivity has just two invariant points which are distinct when the symmetric matrix $\| P_{AB} + P_{BA} \|$ is non-singular and coincident when it is singular.

We can always express the coefficients of a projectivity in the form

(11.7) $$P_{AB} = Q_{AB} + \frac{1}{2} P^C{}_C \, \epsilon_{AB}$$

Since $P_{AB} - P_{BA} = P^C{}_C \, \epsilon_{AB}$, we have

(11.8) $$Q_{AB} = \frac{1}{2}(P_{AB} + P_{BA}) = Q_{BA}$$

and so with a projectivity P there is associated the uniquely determined involution Q. The double points of P are the same as those of Q.

The invariant points of an involution completely determine it. For, the roots of (11.6) determine its coefficients to within a common factor and the additional condition $P_{12} = P_{21}$ gives $\|P_{AB}\|$ to within a factor. Indeed, the projectivity defined by

(11.9) $$Q_{AB} = \alpha_A \beta_B + \alpha_B \beta_A$$

is the involution which leaves $\alpha$ and $\beta$ invariant. Since

(11.10) $$\beta_A \alpha_B - \alpha_A \beta_B = (\beta^C \alpha_C)\, \epsilon_{AB}$$

the involution is also given by

(11.11) $$Q_{AB} = 2\alpha_A\beta_B + (\beta^C\alpha_C)\epsilon_{AB}$$

or by

(11.12) $$Q_{AB} = 2\beta_A\alpha_B - (\beta^C\alpha_C)\epsilon_{AB}.$$

A singular projectivity ($\neq 0$) is of the form $\| \rho_A\sigma_B \|$ and if this is to be an involution (that is to say, symmetric) we must have $\sigma_B = \lambda\rho_B$. Putting $\alpha_A = (\frac{\lambda}{2})^{\frac{1}{2}}\rho_A$ the general singular involution is

(11.13) $$Q_{AB} = 2\alpha_A\alpha_B,$$

and this is just what we get by putting $\beta_A = \alpha_A$ in (11.9). This singular involution carries every point, except $\alpha$, into $\alpha$.

Two points, $X^A$ and $Y^A$, determine the homogeneous scalar $X^AY_A$ the vanishing of which implies the coincidence of the points. If the scalar does not vanish, its value is changed when the coordinates $X^A$ are multiplied by a factor. Four points, however, determine the absolute scalar

(11.14) $$\lambda = \frac{(\varphi^A\alpha_A)(\psi^B\beta_B)}{(\varphi^A\beta_A)(\psi^B\alpha_B)}$$

which is called the cross-ratio of the four points, $(\varphi\,\psi|\alpha\beta)$. The value of $\lambda$ is invariant under transformations of coordinates. Moreover, since the right member of (11.14) is homogeneous of degree zero in the coordinates of the points, the value of $\lambda$ depends only on the points and not on the coordinates chosen to represent the points. Under a projectivity a set of four points goes into a new set which has the same cross-ratio as the old, but under an anti-projectivity the cross-ratio is changed into its complex conjugate.

If $\varphi_A = Q_{AB}\psi^B$ where $Q_{AB}$ is given by (11.9), we have

$$(11.15)\qquad \frac{(\varphi^A\alpha_A)(\psi^B\beta_B)}{(\varphi^A\beta_A)(\psi^B\alpha_B)} = -1.$$

When this relation holds, the points $\varphi$ and $\psi$ are said to be harmonic conjugates with respect to $\alpha$ and $\beta$. Since (11.15) determines $\varphi^A$ as a function of $\psi^A$ to within a factor, the involution with invariant points $\alpha$ and $\beta$ may be defined as the transformation which carries any point into its harmonic conjugate with respect to the pair of points, $\alpha$ and $\beta$.

A projectivity which interchanges two points is an involution. For, by a suitable choice of coordinate system we may take the covariant coordinates of the points to be (1, 0) and (0, 1) and then $\|P_A{}^B\|$ will interchange them only if $P_1{}^1 = P_2{}^2 = 0$. This implies the invariant condition $P_A{}^A = 0$, which characterizes an involution. Indeed, the projectivity

$$(11.16)\qquad Q_{AB} = \lambda\alpha_A\alpha_B + \mu\beta_A\beta_B,$$

with $\lambda$ and $\mu$ arbitrary complex numbers $\neq 0$, is an involution which interchanges $\alpha$ and $\beta$. The most general projectivity with this property is of this form. For a projectivity is determined by the fate of three points and if $\lambda$ and $\mu$ are solutions of the equations

$$(11.17)\qquad \lambda\alpha_A(\alpha_B\xi^B) + \mu\beta_A(\beta_B\xi^B) = \eta_A,$$

Q will carry $\alpha$, $\beta$, and $\xi$ into $\beta$, $\alpha$, and $\eta$, respectively, where $\xi$ and $\eta$ are arbitrary points distinct from both $\alpha$ and $\beta$.

It is an important theorem that every projectivity in $P_1$ is the product of two involutions. We prove this theorem by considering several cases. The identity is the square of an involution and we have seen that any other projectivity has just two double points, which may coincide. Hence it is sufficient to consider projectivities with two distinct double points, non-singular projectivities with one double point, and singular projectivities.

If the distinct invariant points of the projectivity P are a and b, we let $\alpha$ and $\beta$ be a pair of points harmonic conjugate with respect to them and define $Q_1$ to be the involution with double points $\alpha$ and $\beta$. Then, denoting the projectivity which results from following P by $Q_1$ by $Q_1P$, we have that $Q_1P$ interchanges a and b and hence is an involution, $Q_2$. Since $Q_1^2 = 1$, $P = Q_1^2P = Q_1Q_2$ and P is the product of two involutions.

If P is non-singular with the single invariant point a, we take $\alpha$ to be a point distinct from a and call $\beta = P\alpha$ the transform of $\alpha$ under P. Let b be the harmonic conjugate of a with respect to $\alpha$ and $\beta$, $Q_1$ the involution with double points a and b, and $Q_2$ the involution with double points a and $\beta$. Then $Q_2Q_1a = a$, $Q_2Q_1\alpha = Q_2\beta = \beta$. Moreover $Q_2Q_1$ cannot leave invariant any point $\gamma \neq a$ for $Q_2Q_1\gamma = \gamma$ would imply $Q_1\gamma = Q_2\gamma$ and $Q_1$ and $Q_2$ would both interchange $\gamma$ and $\delta = Q_1\gamma$. This and the invariance of a under both $Q_1$ and $Q_2$ would imply $Q_1 = Q_2$, which is false. Hence, P and $Q_2Q_1$ each have the single invariant point a and each carry $\alpha$ into $\beta$. Reference to a canonical coordinate system now easily gives $P = Q_2Q_1$.

A singular projectivity is given by a matrix $\|\alpha_A\beta_B\|$ and if $\alpha$ and $\beta$ are distinct points this is the product of the singular involutions $\|\alpha_A\alpha_B\|$ and $\|\beta^B\beta_C\|$. When the projectivity is a singular involution, $\|\alpha_A\alpha_B\|$, it is the product of $\|\alpha_A\alpha_B\|$ and $\|\alpha^B\gamma_C + \gamma^B\alpha_C\|$, where $\gamma$ is distinct from $\alpha$.

ANTIINVOLUTIONS IN $P_1$

12. An antiprojectivity $\psi \longrightarrow \varphi$ may be written in the four equivalent forms

$$(12.1)\quad \bar{\varphi}^A = P^{\dot{A}}{}_B\psi^B, \quad \bar{\varphi}_A = P_{\dot{A}B}\psi^B, \quad \bar{\varphi}_A = -P_{\dot{A}}{}^B\psi_B, \text{ and } \bar{\varphi}^A = -P^{\dot{A}B}\psi_B,$$

the components of the four matrices being related as in (11.2). If we take both $\phi^A$ and $\psi^A$ to be of weight $\frac{1}{2}$ and antiweight zero, we must take $P^{\dot{A}}{}_B$ to be of weight $-\frac{1}{2}$ and antiweight $+\frac{1}{2}$, $P_{\dot{A}}{}^B$ to be of weight $+\frac{1}{2}$ and antiweight $-\frac{1}{2}$, $P_{\dot{A}B}$ to be of absolute weight -1, and $P^{\dot{A}B}$ to be of absolute weight +1. With these weights the determinant of each of the four matrices is invariant and a normalization such as $|P^{\dot{A}}{}_B| = 1$ is preserved under coordinate transformations.

The invariant points of the antiprojectivity (12.1) are given by

(12.2) $$P_{\dot{A}B}\bar{\psi}^A\psi^B = 0.$$

If we put

(12.3) $$H_{\dot{A}B} = \frac{1}{2}(P_{\dot{A}B} + \bar{P}_{\dot{B}A}) \text{ and } i\,K_{\dot{A}B} = \frac{1}{2}(P_{\dot{A}B} - \bar{P}_{\dot{B}A}),$$

then $\|H_{\dot{A}B}\|$ and $\|K_{\dot{A}B}\|$ are Hermitian matrices and

(12.4) $$P_{\dot{A}B} = H_{\dot{A}B} + i\,K_{\dot{A}B}.$$

Equating the real and imaginary parts of the left member of (12.2) to zero gives

(12.5) $$H_{\dot{A}B}\bar{\psi}^A\psi^B = 0 \quad\text{and}\quad K_{\dot{A}B}\bar{\psi}^A\psi^B = 0.$$

Representing the points of $P_1$ by points of the xy-plane as in §2 by the equation (cf. (2.3))

(12.6) $$-\frac{\psi^2}{\psi^1} \equiv \frac{\psi_1}{\psi_2} = z = x + iy$$

(12.5) are the equations of two circles (real, degenerate, or imaginary) in the xy-plane. (In the special cases in which $H_{\dot{A}B} = 0$ or $K_{\dot{A}B} = 0$, one of the equations is satisfied identically and there is only one circle.)

If the two circles do not coincide, they may intersect in two points, be tangent, or fail to intersect, and the antiprojectivity will then have two, one or no invariant points, respectively. From (12.4) we see that

(12.7) $$(\lambda + i\mu)P_{\dot{A}B} = (\lambda H_{\dot{A}B} - \mu K_{\dot{A}B}) + i\,(\lambda K_{\dot{A}B} + \mu H_{\dot{A}B}),$$

and hence the homogeneous components, $\|\rho P_{\dot{A}B}\|$, of an antiprojectivity do not determine a unique pair of circles (12.5) but only the pencil of which they are members.

If equations (12.5) define one or a pair of coincident circles (real or imaginary), and only in this case, the antiprojectivity will be an antiinvolution. For, if $\|P^{\dot{A}}{}_{B}\| = \begin{Vmatrix} a & b \\ c & d \end{Vmatrix}$, the antiinvolutions are characterized by the matrix equation

(12.8) $$\begin{Vmatrix} a & b \\ c & d \end{Vmatrix} \begin{Vmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{Vmatrix} = \rho \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$$

and multiplying both members by $\begin{Vmatrix} -d & b \\ c & -a \end{Vmatrix}$ = transpose $\|P_{\dot{A}}{}^{B}\|$ gives

$$-(ad - bc) \begin{Vmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{Vmatrix} = \rho \begin{Vmatrix} -d & b \\ c & -a \end{Vmatrix}$$

using the fact that $(\widetilde{P_{\dot{B}}{}^{D}})(P^{\dot{B}}{}_{A}) = -\left|P^{\dot{E}}{}_{F}\right| (\delta_{A}{}^{D})$, $\sim$ = transpose. Thus $(\bar{P}^{\dot{D}}{}_{C}) = \dfrac{-\rho}{\text{Det}\,(P^{\dot{E}}{}_{F})} (\widetilde{P_{\dot{C}}{}^{D}})$. Lowering the index D we find that a non-singular antiinvolution therefore satisfies the equation

(12.9) $$\bar{P}_{\dot{A}B} = \sigma P_{\dot{B}A}$$

where $\sigma = -\dfrac{\rho}{ad - bc}$, and it follows from (12.3) that the two circles (12.5) coincide. Taking the determinant of both members of (12.9) we see that $\sigma\bar{\sigma} = 1$, so that

$$\bar{\sigma}^{\frac{1}{2}}\ \bar{P}_{\dot{A}B} = \sigma^{1/2}\, P_{\dot{B}A}$$

and hence $\|\sigma^{\frac{1}{2}} P_{\dot{A}B}\|$ is Hermitian. The matrix defining an antiinvolution is therefore proportional to an Hermitian matrix and conversely every Hermitian matrix defines an antiinvolution.

A non-singular antiinvolution is of one of two kinds according as the Hermitian matrices defining it are indefinite or definite. The discussion of §1 proves that a suitable choice of coordinate system will allow us to take the equation of the invariant circle to be $\bar{\Psi}^1\psi^1 - \bar{\Psi}^2\psi^2 = 0$ or $\bar{\Psi}^1\psi^1 + \bar{\Psi}^2\psi^2 = 0$ according as the antiinvolution is of the first or second kind, respectively. In terms of the non-homogeneous coordinate z, these circles are $z\bar{z} = 1$ and $z\bar{z} = -1$, and the corresponding antiprojectivities are $w = \frac{1}{\bar{z}}$ and $w = -\frac{1}{\bar{z}}$.

A singular antiinvolution ($\neq 0$) is of the form $\bar{\alpha}^A\beta_B$ and (12.8) now implies $\bar{\alpha}^A\beta_B\,(\alpha^B\bar{\beta}_C) = 0$, or $\beta_B = \rho\alpha_B$. Hence the matrices defining the singular antiinvolution are proportional to $\|\bar{\alpha}_A\,\alpha_B\|$ and the antiinvolution carries every point, except $\alpha$, into $\alpha$.

The antiinvolutions which leave two points, say $\alpha$ and $\beta$, invariant, correspond to the circles through $\alpha$ and $\beta$. These circles are linearly dependent upon any two among them so that

$$\text{(12.10)} \qquad P_{AB} = \lambda(\bar{\alpha}_A\beta_B + \bar{\beta}_A\alpha_B) + i\,\mu(\bar{\alpha}_A\beta_B - \bar{\beta}_A\alpha_B),$$

is, for a suitable choice of the real numbers $\lambda$ and $\mu$, any antiinvolution leaving $\alpha$ and $\beta$ invariant.

The involution with invariant points $\alpha$ and $\beta$ is the product of the antiinvolutions $(\bar{\alpha}_A\beta_B + \bar{\beta}_A\alpha_B)$ and $i(\bar{\alpha}_A\beta_B - \bar{\beta}_A\alpha_B)$, for

$$\text{(12.11)} \qquad i(\alpha_A\bar{\beta}_B + \beta_A\bar{\alpha}_B)(\bar{\alpha}^B\beta_C - \bar{\beta}^B\alpha_C) = i(\bar{\alpha}^B\bar{\beta}_B)(\alpha_A\beta_C + \beta_A\alpha_C).$$

Moreover, the singular involution $\alpha_A\alpha_B$ is the product of $\bar{\alpha}_A\beta_B + \bar{\beta}_A\alpha_B$ and $\bar{\alpha}_A\alpha_B$. Hence every involution is the product of two antiinvolutions. We saw in the preceding section that every projectivity was the product of two involutions and so every projectivity is the product of four antiinvolutions. Remembering that all the antiprojectivities are obtained by multiplying the projectivities by a single antiinvolution, we have the result that the antiinvolutions generate the entire antiprojective group.

POINT-PLANE REFLECTIONS IN $R_3$

13. The antiinvolution

(13.1) $$\bar{\varphi}_A = P_{\dot{A}B}\psi^B, \quad \text{with} \quad \bar{P}_{\dot{A}B} = P_{\dot{B}A},$$

induces in $R_3$ the involution

(13.2) $$Y^i = P^i{}_j X^j,$$

where (cf. (5.7))

(13.3) $$P^i{}_j = g^{i\dot{A}B} P_{\dot{C}B} P_{\dot{A}D} g_j^{\dot{C}D}.$$

Since $P_{\dot{C}B} P_{\dot{A}D} - P_{\dot{A}B} P_{\dot{C}D}$ is skew-symmetric both in the indices $(\dot{A}\dot{C})$ and in the indices (BD), we have

(13.4) $$P_{\dot{C}B} P_{\dot{A}D} = P_{\dot{A}B} P_{\dot{C}D} + \rho\epsilon_{\dot{A}\dot{C}}\,\epsilon_{BD},$$

and multiplying by $\epsilon^{\dot{A}\dot{C}}\epsilon^{BD}$ and summing gives $\rho = -\frac{1}{2}P^{\dot{E}F}P_{\dot{E}F}$. Hence, substituting from (13.4) in (13.3), we get

(13.5) $$P^i{}_j = P^i P_j - \frac{1}{2}(P^k P_k)\delta^i_j,$$

where $P^i = g^{i\dot{A}B} P_{\dot{A}B}$ is the point of $R_3$ corresponding to $P_{\dot{A}B}$ under (4.5).

The involution (13.2) leaves $P^i$ and each point of its polar plane, $P_i$, invariant. For, from (13.5), $P^i{}_j P^j = \frac{1}{2}(P^k P_k)P^i$ and, if $X^i P_i = 0$, $P^i{}_j X^j = -\frac{1}{2}(P^k P_k)X^i$. An involution of this sort is called a point-plane reflection. To find the transform of an arbitrary point X, under (13.2) we observe that the line determined by P and X intersects the plane $P_i$ in an invariant point, say Q, and hence (13.2) sets up an involution on this line with the double points P and Q. The transform of X is then its harmonic conjugate with respect to P and Q.

Two points on the quadric are interchanged by the involution if and only if they are collinear with the center, P, of the point-plane reflection. Hence there is a real pencil of antiinvolutions which interchange two points $\alpha_A$ and $\beta_A$ and the elements of this pencil are

$$P_{\dot{A}B} = \lambda \bar{\alpha}_{\dot{A}} \alpha_B + \mu \bar{\beta}_{\dot{A}} \beta_B, \tag{13.6}$$

where $\lambda$ and $\mu$ are real parameters, neither of which is zero.

LINE REFLECTIONS IN $R_3$

14. In equations (12.11) we expressed an arbitrary involution as the product of two antiinvolutions. Moreover, since $(\bar{\alpha}^{\dot{A}}\beta^B + \bar{\beta}^{\dot{A}}\alpha^B)(\bar{\alpha}_{\dot{A}}\beta_B - \bar{\beta}_{\dot{A}}\alpha_B) = 0$, the antiinvolutions corresponded to points of $R_3$ which were conjugate with respect to the quadric. Hence an involution in $P_1$ corresponds to the product of two point-plane reflections in $R_3$, the point and plane of one being incident with the plane and point of the other, respectively.

Let us denote the two point-plane reflections by $P_1$ and $P_2$, their centers by C and D, and their planes by c and d, respectively. Since the point-plane reflections leave the quadric invariant, c is the polar plane of C and d is the polar plane of D. The intersection of c and d is a line, cd, the points of which are invariant under both $P_1$ and $P_2$ and hence under their product, $P_1P_2 = Q$. Moreover, $P_1$ and $P_2$ both induce the same involution on the invariant line CD and therefore Q leaves each point of CD invariant. An involution of $R_3$ which leaves each of two skew lines pointwise invariant is called a line reflection.

To find the transform of a point X, not on CD or cd, we take the intersection of the plane determined by X and CD with the plane determined by X and cd. This is a line which intersects CD and cd in points E and F, respectively. Since E and F are invariant, the line EF is invariant and the line reflection induces on it the involution with double points E and F. Hence, the transform of X is the harmonic conjugate of X with respect to E and F. The line reflection is therefore completely determined by the two lines which it leaves pointwise invariant.

Picturing $R_3$ as a Euclidean space, and the quadric as a sphere in it, the line-reflection leaves invariant each plane of the pencils on cd and CD and hence leaves invariant two pencils of circles on the sphere. By our initial construction c and d cut the sphere in circles which intersect in the points $A^i$ and $B^i$ corresponding to the invariant points $\alpha_A$ and $\beta_A$ of the involution in $P_1$. The planes on cd therefore cut the sphere in the circles through A and B and the planes on CD cut the sphere in the pencil of circles orthogonal to the circles through A and B.

We can express the components, $Q^i{}_j$, of the line reflection in terms of the coordinates, $q^{ij}$, of cd by the formula

$$Q^i{}_k = q^{ij}q_{jk} + \frac{1}{4}\, q^{pq}q_{pq}\,\delta^i_k. \tag{14.1}$$

Indeed, if we choose a coordinate system in which the invariant points $\alpha_A$ and $\beta_A$ have coordinates (1, 0) and (0, 1), the corresponding points $A^i$ and $B^i$ in $R_3$ are, from (1.8), $\frac{1}{\sqrt{2}}$ (0, 0, 1, 1), and $\frac{1}{\sqrt{2}}$ (0, 0, -1, 1). The coordinates of cd and CD are therefore

$$(14.2)\qquad \|q^{ij}\| = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{Vmatrix}, \text{ and } \|q_{ij}\| = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{Vmatrix}.$$

Substituting in (14.1) we get

$$(14.3)\qquad \|Q^i{}_k\| = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} - \frac{1}{2}\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} = \frac{1}{2}\begin{Vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix},$$

and so the collineation $Y^i = 2Q^i{}_j X^j$ is

$$(14.4)\qquad Y^1 = -X^1, \quad Y^2 = -X^2, \quad Y^3 = X^3, \quad Y^4 = X^4,$$

which is clearly the line reflection with invariant lines $X^1 = X^2 = 0$ and $X^3 = X^4 = 0$.

FACTORIZATION OF THE FUNDAMENTAL QUADRATIC FORM

15. We began this chapter by observing that the Hermitian matrices of order two constitute a linear space of four real dimensions. If we combine this result, as expressed in (4.7), with the theorem (proved in §12) that an Hermitian matrix defines an antiinvolution, we see that $(\bar{g}_{i\dot{A}B}X^i)(g_j{}^{\dot{B}C}X^j)$ is a multiple of $\delta^A_C$ for all values of the variables $X^i$. Hence

$$(15.1)\qquad (\bar{g}_{i\dot{A}B}X^i)(g_j{}^{\dot{B}C}X^j) = \rho_{ij}X^iX^j\delta^C_A.$$

To evaluate $\rho_{ij}X^iX^j$ we set A equal to C and sum, getting

$$(15.2)\qquad g_{ij}X^iX^j = 2\rho_{ij}X^iX^j,$$

since $\bar{g}_{i\dot{A}B} = g_{i\dot{B}A}$ and

$$(15.3)\qquad g_{i\dot{B}A}g_j{}^{\dot{B}A} = g_{ij},$$

on account of (4.8). Equations (15.1) are then

(15.4) $$2(\bar{g}_{i\dot{A}B}X^i)g_j{}^{\dot{B}C}X^j) = g_{ij}X^iX^j\delta_A^C.$$

Equating coefficients in (15.4) gives the important equations

(15.5) $$\bar{g}_{i\dot{A}B}g_j{}^{\dot{B}C} + \bar{g}_{j\dot{A}B}g_i{}^{\dot{B}C} = g_{ij}\delta_A^C.$$

Equations (15.4) may be interpreted as a factorization of the quadratic form $g_{ij}X^iX^j$ into the product of two linear forms, $\sqrt{2}\,\bar{g}_{i\dot{A}B}\,X^i$ and $\sqrt{2}\,g_{j\dot{B}C}\,X^j$, with matrix coefficients. We shall be able to write $g_{ij}\,X^iX^j$ as the square of a single linear form if we combine $\|\bar{g}_{i\dot{A}B}\|$ and $\|g_i{}^{\dot{A}B}\|$ into the four-rowed matrices

(15.6) $$\gamma_i = \sqrt{2}\begin{Vmatrix} 0 & 0 & \|\bar{g}_{i\dot{A}B}\| & \\ 0 & 0 & & \\ \|g_i{}^{\dot{A}B}\| & & 0 & 0 \\ & & 0 & 0 \end{Vmatrix}$$

and observe that (15.5) and its conjugate implies

(15.7) $$\gamma_i\gamma_j + \gamma_j\gamma_i = 2g_{ij}\,1.$$

Then (15.4) may be written

(15.8) $$(\gamma_iX^i)^2 = g_{ij}X^iX^j\,1$$

# Chapter II

## UNDERLYING AND TANGENT SPACES

1. The space underlying the theory of relativity is the Minkowski space which we will call $X_4$. It is a four-dimensional space in which the distance between two space-time points (events) $x^i$ and $y^i$ is given by

$$s^2 = g_{ij}(x^i - y^i)(x^j - y^j). \tag{1.1}$$

The preferred coordinate systems of $X_4$ are those in which the distance formula (1.1) becomes

$$s^2 = -(x^1 - y^1)^2 - (x^2 - y^2)^2 - (x^3 - y^3)^2 + (x^4 - y^4)^2. \tag{1.2}$$

Cartesian coordinates are obtained from preferred ones by transformations of the form

$$x^{i*} = A^i_j x^j + a^i \tag{1.3}$$

where $A^i_j$ and $a^i$ are constants and $A = |A^i_j| \neq 0$.

If the quantities $A^i_j$ are the coefficients of an extended Lorentz transformation, the transformation (1.3) carries one preferred coordinate system into another.

Fixing an arbitrary point $y^i$ in $X_4$ changes it into the space $R_4$ considered in Chapter I for if we make the transformation to the Cartesian coordinates

$$X^i = x^i - y^i \qquad Y^i = y^i - y^i = 0 \tag{1.4}$$

equation (1.2) becomes

$$s^2 = -(X^1)^2 - (X^2)^2 - (X^3)^2 + (X^4)^2. \tag{1.5}$$

The transformations which leave the right member of (1.5) invariant in form and the point $Y^i = 0$ invariant are just the transformations of the extended Lorentz group. Thus the four-space characterized by the quadratic form (1.5) and its preferred coordinate systems is $R_4$.

Allowable coordinate systems in $X_4$ are obtained from preferred coordinate systems by transformations of the type

(1.6) $$x^{i*} = f^i(x)$$

where $f^i(x)$ are analytic functions of $x^1 x^2 x^3$ and $x^4$ such that

(1.7) $$\left| \frac{\partial f^i}{\partial x^j} \right| \neq 0.$$

Since (1.2) defines the distance between two arbitrary points of $X_4$, we may employ this formula in the usual way to define the length of a curve.

Thus the length of a segment of a curve is the integral of ds taken along the segment, where

(1.8) $$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2.$$

In allowable coordinates this equation becomes

(1.9) $$ds^2 = g_{ij}(x)dx^i dx^j.$$

The quantities $dx^1$ $dx^2$ $dx^3$ and $dx^4$ may be considered as coordinates in a space $T_4(x)$, the tangent space. Thus at every point of $X_4$ we have an associated tangent space $T_4(x)$. The transformation (1.6) of $X_4$ induces in $T_4(x)$ the linear homogeneous transformation with constant coefficients

(1.10) $$dx^{i*} = \frac{\partial x^{i*}}{\partial x^j} dx^j$$

since the quantities $\frac{\partial x^{i*}}{\partial x^j}$ are independent of $dx^j$.

The point whose coordinates have the value (0, 0, 0, 0) in one coordinate system in $T_4(x)$ has these coordinates in all coordinate systems of $T_4(x)$. We identity this point with the point $x^1 \ldots x^4$ of the underlying space and call it the point of contact of $T_4(x)$ and $X_4$. Because of the special role of the point (0, 0, 0, 0), $T_4(x)$ is a centered affine (or vector) space in which the length of the vector $dx^i$ is given by equation (1.9). The points of $T_4(x)$ which satisfy the equation

$$ds^2 = g_{ij}\, dx^i\, dx^j = 0$$

are said to be on the light cone. In a preferred coordinate system in $X_4$ this equation becomes

$$- (dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2 = 0. \tag{1.11}$$

Thus each $T_4(x)$ is a replica of the space $R_4$ studied in Chapter I.

The geometry* of any space may be characterized by the class of pre-

---

* Veblen and Whitehead, Foundations of differential geometry.

---

ferred coordinate systems in it and the pseudo group of transformations which transforms one into another. In the case of the tangent space $T_4(x)$, the preferred coordinate systems are the Galilean ones and the group is the extended Lorentz group.

From the relations between the transformations in $X_4$ and $T_4(x)$ we see that a general transformation in $X_4$ induces the satellite Cartesian transformation (1.10) in $T_4(x)$. The only coordinate systems we will use in $T_4(x)$ are the Cartesian ones.

It is possible to consider the $dx^i$ as the homogeneous coordinates of the lines through the origin in $T_4(x)$. As remarked in §2, Chapter I, these lines constitute a three-space, which will be denoted by $T_3(x)$. Since $dx^i$ and $\rho\, dx^i$ correspond to the same point in $T_3(x)$ and since they also represent the same direction in $T_4(x)$, it is evident that $T_3(x)$ is the space of directions of vectors in $T_4(x)$. Thus at every point x of $X_4$ there is associated a real projective three-dimensional space with a real non-ruled quadric, the space of directions $T_3(x)$.

SPIN AND GAUGE SPACES

2. From the correspondence, discussed in Chapter I, between $R_4$ and the complex projective line $P_1$, we see that to every $T_4(x)$ there is associated a $P_1(x)$. That is, with every point x in the underlying space there is associated a complex projective line $P_1(x)$.

The points of each $P_1(x)$ are denoted by the homogeneous coordinates $(\psi_1, \psi_2)$. A transformation of coordinates

(2.1) $$\psi_A{}^* = t_A^B \psi_B \qquad \psi^{A*} = T_B^A \psi^B$$

where the $t_B^A$ are arbitrary complex numbers such that

(2.2) $$T_B^A t_C^B = \delta_C^A \quad \text{and} \quad t = \left| t_B^A \right| \neq 0$$

changes a given homogeneous coordinate system in $P_1(x)$ to another such system. Two pairs of equations like (2.1) represent the same transformation in $P_1(x)$ if their coefficients are proportional. All transformations of the type (2.1) with proportional coefficients determine the same unimodular transformation:

(2.3) $$\psi_A{}^* = s_A^B \psi_B \qquad \psi^{A*} = S_B^A \psi^B$$

where

(2.4) $$s_B^A = t_B^A t^{-\frac{1}{2}} \quad \text{and hence} \quad s = \left| s_B^A \right| = 1.$$

These homogeneous coordinate systems are the preferred coordinate systems in $P_1(x)$. Any two of them are connected by the transformations (2.1) or (2.3).

The transformations (2.1) will be called general spin transformations.

The space $P_1(x)$ differs from the tangent space in that there is no singular point in it which may be regarded as the point of contact with the underlying space. Also the transformations in $P_1(x)$ are completely independent of the transformations in $X_4$ and of the satellite transformations in $T_4(x)$.

Transformations of the type (2.1) are generated by a unimodular transformation of the type (2.3) followed by the transformation

(2.4) $$\psi_A{}^* = t\psi_A \, .$$

If we write

(2.5) $$t = re^{i\theta}$$

where r and $\theta$ are real, we see that the transformation (2.4) may be written as the product of the following two,

(2.6) $$\psi_A{}^* = e^{i\theta}\psi_A$$

and

(2.7) $$\psi_A{}^* = r\psi_A \, .$$

The transformations (2.6) and (2.7) will be shown to be closely related to transformations which we will call gauge transformations of the first and second kind respectively. Since $P_1(x)$ is a projective space $(\psi_1, \psi_2)$ and $(t\psi_1, t\psi_2)$ represent the same point in $P_1(x)$; hence the transformations (2.4), (2.6) and (2.7) cannot be pictured in $P_1(x)$.

However, the totality of transformations (2.6) is a group simply isomorphic with the group of rotations about the origin in a real Euclidean plane and is multiply isomorphic with the group of translations of a real Euclidean line into itself. Hence, if we associate with each $P_1(x)$ a real Euclidean line, we can picture the transformations (2.6) as translations along this line. We shall call this associated space the gauge space of the first kind $G_1(x)$ and describe it by the variable $x^o$. The transformations in $G_1(x)$ are of the form

$$x^{o*} = x^o + \theta, \tag{2.8}$$

Similarly the transformations (2.7) may be pictured as translations along another real Euclidean line which we will call the gauge space of the second kind $G_2(x)$. We shall describe it by the variable $y^o$. The transformations in $G_2(x)$ are of the form

$$y^{o*} = y^o + \log r. \tag{2.9}$$

Hence we see that the general spin transformation of the type (2.1) may be pictured as a transformation in $P_1(x)$ followed by translations in $G_1(x)$ and $G_2(x)$.

Just as in tensor analysis where the geometry of all the tangent spaces is studied simultaneously by considering vectors which are functions of the coordinates of the underlying space, we propose to study the geometry of all the spin spaces $P_1(x)$ simultaneously by considering the $\psi_A$ as analytic complex scalar functions of $x^1$ $x^2$ $x^3$ $x^4$ and the transformation coefficients (2.1) and (2.3) as arbitrary analytic complex functions of $x^1 \dots x^4$ subject to the conditions (2.2) and (2.4). Similarly all the gauge spaces $G_1(x)$ and $G_2(x)$ are studied simultaneously by considering r and $\theta$ in equations (2.8) to (2.9) as arbitrary real functions of $x^1 \dots x^4$.

In the following we shall restrict ourselves to spinors of the form

$$\psi_A = e^{Ix^o} e^{Jy^o} f_A(x^1 \dots x^4) \tag{2.10}$$

where I is a pure imaginary number and J is real. Also $x^o$ and $y^o$ are arbitrary real numbers; they are the coordinates in the spaces $G_1(x)$ and $G_2(x)$ respectively. If we fix a point x in the underlying space, $\psi_1$ and $\psi_2$ are homogeneous coordinates of a definite point in the associated spin space $P_1(x)$. That is, equations (2.10) are another means of writing the infinite set $(\rho\psi_1, \rho\psi_2)$ ($\rho$ arbitrary) of homogeneous coordinates of a point in $P_1(x)$, which is specified as soon as a point in $P_1(x)$ is specified. If $x^o$ is given a definite value, that is, if a point in the first gauge space $G_1(x)$ is specified, then the argument of $\rho$ is fixed. Similarly, if $y^o$ is given a definite value, that is, a point in $G_2(x)$ is specified, the modulus of $\rho$ is fixed. Hence assigning definite values to $x^o$ and $y^o$ selects a particular pair of homogeneous coordinates of a point in $P_1(x)$.

DEFINITION OF SPINORS

3. A frame of reference is specified when we give an arbitrary coordinate system in $X_4$ and a preferred one in each of the spaces $T_4(x)$, $P_1(x)$, $G_1(x)$ and $G_2(x)$. There is a change of frame of reference whenever the coordinate system in $X_4$ or any of the associated spaces is changed.

A geometric or physical being will for our present purposes be an entity which has a unique set of components in each frame of reference. The components in two frames of reference are related by a transformation law, namely a formula which gives the components in one frame in terms of the components of the other.

Tensors are a special class of geometric beings. Their transformation law is such that only the change in reference frame produced by the transformation

$$x^{i*} = x^{i*}(x) \tag{3.1}$$

of $X_4$, produces a change in the components of the tensor. Under the transformation (3.1), the components of the tensor in the new frame of reference are linear homogeneous functions of the components in the old frame, and the coefficients are homogeneous functions in $\frac{\partial x^{i*}}{\partial x^j}$. The degree of homogeneity of these functions is determined by the number of covariant and contravariant indices and the weight of the tensor.

For example, the transformation law of a contravariant vector is

$$V^{i*}(x) = V^j(x)\,\frac{\partial x^{*i}}{\partial x^j} \tag{3.2}$$

where the right member is a homogeneous polynomial of the first degree in $\frac{\partial x^{i*}}{\partial x^j}$. For a mixed tensor of second order and weight w we have

$$T^{i*}_{j} = T^k_{\ell}\,\frac{\partial x^{\ell}}{\partial x^{j*}}\,\frac{\partial x^{i*}}{\partial x^k}\left|\frac{\partial x^{*m}}{\partial x^n}\right|^{w}. \tag{3.3}$$

The right member is a linear homogeneous functions of $T^k_{\ell}$ and the coefficients are homogeneous functions of $\frac{\partial x^{*i}}{\partial x^k}$ since $\frac{\partial x^{\ell}}{\partial x^{*j}}$ and $\left|\frac{\partial x^{m*}}{\partial x^n}\right|^{w}$ are homogeneous functions in $\frac{\partial x^{i*}}{\partial x^k}$ of degree minus one and 4w respectively.

A spinor is a geometric being which has a transformation law of the following type. It is a tensor under coordinate transformations, a scalar under gauge transformations of either kind, but under spin transformations of the type (2.1), the new components are linear homogeneous functions of the old components and the coefficients are homogeneous functions in the $t^B_A$ and $\bar{t}^B_A$. For example

$$\psi_A{}^{*} = t^B_A \psi_B t^w \tag{3.4}$$

is the transformation law of a simple covariant spinor of weight w. The transformation law of a contravariant spinor of weight w under the same transformation is

$$\psi^{A*} = T^A_B \psi^B t^w \tag{3.5}$$

where

$$T^A_B t^B_C = \delta^A_C \,.$$

The quantities $\varepsilon^{AB}$ and $\varepsilon_{AB}$ which we have used to raise and lower spin indices are spinors of weight +1 and -1 respectively. Their transformation laws are

(3.6) $$\varepsilon^{AB*} = \varepsilon^{AB} = \varepsilon^{CD} \, T^A_C \, T^D_B \, t$$

(3.7) $$\varepsilon_{AB}{}^{*} = \varepsilon_{AB} = \varepsilon_{CD} \, t^C_A \, t^D_B \, t^{-1} \,.$$

A spin density may be defined as a spinor which transforms as a scalar or weight 1 under spin transformations. If we require that this density $\rho$ have the value 1 in a particular spin coordinate system, then in any other spin coordinate system it will have the value

(3.8) $$\rho = t$$

where t is the determinant of the spin transformation which carries the first coordinate system into the second. From equations (3.6), (3.7) and (3.8) we see that $\rho\varepsilon_{AB}$ and $\frac{1}{\rho}\ \varepsilon^{AB}$ will then have the transformation laws

(3.9) $$\rho^* \varepsilon_{AB}{}^{*} = \rho\varepsilon_{CD} \, t^C_A \, t^D_B$$

(3.10) $$\frac{1}{\rho^*} \varepsilon^{AB*} = \frac{1}{\rho} \ \varepsilon^{CD} \, T^A_C \, T^B_D .$$

That is, $\frac{1}{\rho}\ \varepsilon^{AB}$ and $\rho\varepsilon_{AB}$ are weightless contravariant and covariant spinors of second order respectively. They have been used by Van der Waerden and Infeld* to raise and lower indices in place of $\varepsilon^{AB}$ and $\varepsilon_{AB}$.

---

* Infeld and Van der Waerden, "Die Wellengleichung des Electrons in der allgemeinen Relativitätstheorie", Sitzungsberichte der Preussische Akademie der Wissenschaften, 9 (1933), 330.

---

From equations (3.4) and (3.5) we see that the transformation law for spinors of the type $\varphi_{\dot A} = \overline{\psi}_{\dot A}$ is

(3.11) $$\varphi_{\dot{A}}^{\ *} = \bar{t}_{\dot{A}}^{\dot{B}} \varphi_{\dot{B}} \bar{t}^{a}.$$

We have called the member a the anti-weight of the spinor $\varphi_{\dot{A}}$.

Since the geometric transformation determined by the spinor $P_{\dot{A}B}$ is invariant, the equations

(3.12) $$\bar{\varphi}_{\dot{A}} = P_{\dot{A}B} \psi^{B}$$

in a new coordinate system are

(3.13) $$\bar{\varphi}_{\dot{A}}^{\ *} = P_{\dot{A}B}^{\ *} \psi^{B*}.$$

That is, the spinor $P_{\dot{A}B}$ has the transformation law

(3.14) $$P_{\dot{A}B}^{\ *} = P_{\dot{C}D} \bar{t}_{\dot{A}}^{\dot{C}} t_{B}^{D} t^{w} \bar{t}^{a}$$

where the numbers w and a are the weight and anti-weight of the spinor $P_{\dot{A}B}$. These are so chosen so that the weight of both sides of equation (3.12) are equal.

The above are examples of spinors which transform as scalars under coordinate transformations. An example of a spinor which transforms as a vector under a coordinate transformation is one which may be denoted by $\psi_{Ai}$. Under spin transformations it has the transformation law

(3.15) $$\psi_{Ai}^{*} = \psi_{Bi} t_{A}^{B} t^{w} \bar{t}^{a}$$

and under coordinate transformations it has the transformation law

(3.16) $$\psi_{Ai}^{*} = \psi_{Aj} \frac{\partial x^{j}}{\partial x^{i*}}.$$

Another example of this type of spinor is $g^{i\dot{A}B}$. Under a spin transformation it has the transformation law

(3.17) $$g^{i\dot{A}B*} = g^{i\dot{C}D} \bar{T}_{\dot{C}}^{\dot{A}} T_{D}^{B} \bar{t}^{a} t^{w}$$

and under a coordinate transformation it has the transformation law

(3.18) $$g^{i\dot{A}B*} = g^{j\dot{A}B} \frac{\partial x^{i*}}{\partial x^j} .$$

The weights w and a of $g^{i\dot{A}B}$ will be chosen so that the quantities

(3.19) $$J^i = g^{i\dot{A}B} P_{\dot{A}B}$$

will be components of contravariant vectors and thus be independent of spin transformations. That is, if the weight of $P_{\dot{A}B}$ is w and a then the weight of $g^{i\dot{A}B}$ must be -w and -a.

GAUGE TRANSFORMATIONS

4. The components of spinors are arbitrary complex functions of $x^1, \ldots, x^4$. They may also be functions of the gauge variables $x^\circ$ and $y^\circ$. However, these variables will only be allowed to enter as factors in the form $e^{Ix^\circ}$ and $e^{Jy^\circ}$. Thus, for simple spinors we have

(4.1) $$\psi_A = e^{Ix^\circ} e^{Jy^\circ} f_A(x)$$

where I is a pure imaginary number (usually $\pm \sqrt{-1}$ or 0) and J is a real number (usually 1 or 0), and $f_A(x)$ is an arbitrary complex function of $(x^1, \ldots, x^4)$. The numbers I and J will be referred to as the indices of the first and second kind respectively.

If we now make the gauge transformation of the first kind

(4.2) $$x^{\circ *} = x^\circ - \theta(x)$$

since spinors are scalars with respect to gauge transformations of either kind, we have

(4.3) $$\psi_A^*(x^{\circ *}) = \psi_A(x^\circ) = e^{Ix^\circ} f_A(x) = e^{I(x^{\circ *}+\theta(x))} f_A(x) = e^{Ix^{\circ *}} f_A^*(x)$$

where

(4.4) $$f_A{}^*(x) = e^{I\theta(x)} f_A(x).$$

Similarly under the gauge transformation of the second kind

(4.5) $$y^{o*} = y^o - \log \rho(x)$$

we see that

(4.6) $$\psi_A{}^*(y^{o*}) = \psi_A(y^o) = e^{Jy^o} f_A(x) = e^{Jy^{o*}} f_A^*(x)$$

where

(4.7) $$f_A{}^*(x) = \rho^J_{(x)}\, f_A(x).$$

Hence under spin transformations of the first kind we see that the geometric being whose components are $f_A(x)$, which we will call the basis of the spinor $\psi_A$, undergoes then transformation of the type (2.6). Under gauge transformations of the second kind the basis undergoes the transformation of the type (2.7).

The spinors $\varepsilon_{AB}$, $\varepsilon_{\dot{A}\dot{B}}$, $\varepsilon^{AB}$ and $\varepsilon^{\dot{A}\dot{B}}$ will be defined to be of indices $I = J = 0$. Hence they will be unaffected by gauge transformations.

This implies that the spinors $g^{i\dot{A}B}$ and $g^j_{\dot{A}B}$ will have the same indices (i.e. depend in the same manner on the gauge variables). Hence the quantities

(4.8) $$g^{ij} = g^{i\dot{A}B}\, g^j_{\dot{A}B}$$

which are the components of the metric tensor of $X_4$ (see Chapter I, section 15) will have indices 2I and 2J, where I and J are the indices of $g_{i\dot{A}B}$. If the $g^{ij}$ are to be unaffected by gauge transformations, then we must have $I = J = 0$. Since the length of a vector in $X_4$ must have an absolute meaning, we impose the condition that the tensor $g_{ij}$ is independent of gauge transformations; that is, the spinor $g^{iAB}$ has indices $I = J = 0$.

SPINORS OF WEIGHT $-\frac{1}{2}$

5. If $\psi_A$ is a spinor of weight w and anti-weight a and indices I = 0 and J = 0 so that the components are functions of $x^1, \ldots, x^4$ alone, then in each coordinate system in each $P_1(x)$ the components assume only one set of values $(\psi_1, \psi_2)$ and this gives only one of the infinite set of homogeneous coordinates $(\rho\psi_1, \rho\psi_2)$ of a point in each $P_1(x)$. In case $w \neq -\frac{1}{2}$ and $a \neq 0$, when the multiplication spin transformation

$$(5.1) \qquad \psi_A^{\;*} = \sigma\psi_A$$

is applied, the coordinate system in each $P_1(x)$ remains unchanged but the components are multiplied by $\sigma$. Thus the transformation (5.1) converts them into another pair of numbers $(\psi_1^{\;*}, \psi_2^{\;*})$ which constitute another selection of a particular pair out of the set of pairs $(\rho\psi_1, \rho\psi_2)$.

In case $w = -\frac{1}{2}$ and a = 0, the spinors $\psi_A$ have the transformation law

$$(5.2) \qquad \psi_A^{\;*} = \psi_B\, t_A^B\, t^{-\frac{1}{2}} = \psi_B\, s_A^B\,.$$

That is, the simple spinors of weight $-\frac{1}{2}$ are the substratum of the unimodular group in $P_1(x)$. Since the only multiplicative spin transformations in this group are $\pm 1$, the general transformation (5.1) cannot be applied to spinors of weight $-\frac{1}{2}$. Thus in this case the components of $\psi_A$ are not, strictly speaking, homogeneous coordinates in $P_1(x)$, but give a particular choice of one pair of numbers $(\psi_1, \psi_2)$ from the infinite set $(\rho\psi_1, \rho\psi_2)$. Thus, specifying a point in each $P_1(x)$ specifies the spinor $\psi_A$ completely.

However, simple spinors of weight $-\frac{1}{2}$ and indices I = i and J = 1 are homogeneous coordinates in each $P_1(x)$. This may be seen from the fact that in each frame of reference the components $\psi_A$ are of the form

(5.3) $$\psi_A = e^{y^o} e^{ix^o} f_A(x^1 \ldots x^4)$$

where $x^o$ and $y^o$ are arbitrary real numbers. If we apply the two gauge transformations (4.2) and (4.5), (5.3) is changed into

(5.4) $$\psi_A^* = e^{y^{o*}} e^{ix^{o*}} f_A^*(x) = \psi_A$$

where

(5.5) $$f_A^* = \rho e^{i\theta} f_A(x).$$

The basis of $\psi_A^*$ is a multiple of the basis of $\psi_A$. However $\psi_A^* = \psi_A$ since spinors are scalars under gauge transformations, hence in each $P_1(x)$, $(\psi_1^*, \psi_2^*)$ are the same homogeneous coordinates of the same point as $(\psi_1, \psi_2)$ in the same coordinate system.

In case of a spinor of this type, specifying a point in each $P_1(x)$ determines the components in the form (4.10) with $x^o$ and $y^o$ variable. If in addition a point is specified in each gauge space, the numerical value of the components $\psi_A$ is fully determined.

The spinors of weight $-\frac{1}{2}$ and indices I = 0, J = 1 evaluated at a point x of $X_4$, determine a singly infinite subset out of the doubly infinite set of homogeneous coordinates of a point in $P_1(x)$, since the components are of the form

(5.6) $$\psi_A = e^{y^o} f_A(x)$$

where $y^o$ is arbitrary and real. This subset is the set $(k\psi_1, k\psi_2)$ with k real. If we apply the gauge transformation (4.5), (5.6) is changed into

(5.7) $$\psi_A^* = e^{y^{o*}} f_A^*(x)$$

where

$$f_A{}^* = \rho f_A. \tag{5.8}$$

The basis of $\psi_A{}^*$ is a real multiple of the basis of $\psi_A$. Again, since $\psi_A{}^* = \psi_A$, in each $P_1(x)$, $(\psi_1{}^*, \psi_2{}^*)$ are the same homogeneous coordinates, selected from the subset of all homogeneous coordinates, of the same point as $(\psi_1, \psi_2)$ in the same coordinate system.

The spinors of weight $-\frac{1}{2}$ and indices I = i, J = 0 evaluated at a point x of $X_4$ form another singly infinite subset of the doubly infinite set of homogeneous coordinates of a point in $P_1(x)$, since the components $\psi_A$ are of the form

$$\psi_A = e^{ix^o} f_A(x) \tag{5.9}$$

where $x^o$ is an arbitrary real number. This subset is the set $(e^{i\theta}\psi_1, e^{i\theta}\psi_2)$ where $\theta$ is real. If we apply the gauge transformation (4.2), (5.9) is changed into

$$\psi_A{}^* = e^{ix^{o*}} f_A{}^*(x) \tag{5.10}$$

where

$$f_A{}^*(x) = e^{i\theta} f_A(x). \tag{5.11}$$

Thus, the basis of $\psi_A{}^*$ is the basis of $\psi_A$ multiplied by a factor of absolute value one. Since $\psi_A{}^* = \psi_A$ in each $P_1(x)$, $(\psi_1{}^*, \psi_2{}^*)$ are the same homogeneous coordinates, selected from the subset of all homogeneous coordinates, of the same point as $(\psi_1, \psi_2)$ in the same coordinate system.

The space underlying the Pauli quantum theory of the electron is the infinitely many dimensional space $P_\infty$ formed by taking the direct sum of all the associated spin spaces $P_1(x)$. The physical states of any quantum mechanical system are represented by points in this space. The coordinates of the points in this space are normalized so that an infinite dimensional hermitian form is always equal to one. Two points are to be identified if the functions which represent them in a given coordinate system have a ratio of absolute value one. This is, the space $P_\infty$ is a projective space where only a singly infinite subset of the doubly infinite set of homogeneous coordinates is used. It is the same subset which in the case of $P_1(x)$ we have described by means of spinors of weight $-\frac{1}{2}$ and indices I = i and J = 0. In the following we shall restrict ourselves to spinors of this type.

SPINORS OF OTHER WEIGHTS

6. The same subset of the infinite set of homogeneous coordinates may be described by the spinors of weight -1/4 anti-weight -1/4, and indices I = J = 0. This may be seen from the transformation law for these spinors. It is

(6.1) $$\psi_A^* = \psi_B\, t_A^B\, t^{-\frac{1}{4}}\, \bar{t}^{-\frac{1}{4}} = \psi_B\, s_A^B$$

where

(6.2) $$s_A^B = t_A^B\, |t|^{-\frac{1}{2}}$$

and the bars denote the absolute value. That is, spinors of weight -1/4 and anti-weight -1/4 are the substratum of the linear homogeneous group in $P_1(x)$ with determinant of absolute value one, since

(6.3) $$s = t/\bar{t} = e^{i\varphi}$$

where $\varphi$ is arbitrary.

The only multiplicative spin transformations of this group are of the form (5.1) where $\sigma = e^{i\theta}$ that is

(6.4) $$\psi_A^* = e^{i\theta}\psi_A.$$

Hence the numbers $(\psi_1^*, \psi_2^*)$ are another selection of homogeneous coordinates of a point in each $P_1(x)$ out of the subset $(e^{i\varphi}\psi_1, e^{i\varphi}\psi_2)$ of the infinite set of homogeneous coordinates.

The spinors of weight -1/4, anti-weight -1/4, and indices I = 0, J = 1, are homogeneous coordinates from the latter set since in each frame of reference they are of the form (5.6). Thus if we have the homogeneous coordinate $(\psi_1, \psi_2)$ and wish to obtain a new pair $(\rho\psi_1, \rho\psi_2)$, we need only make the multiplicative spin transformation (6.4) where $\theta = \sqrt{\rho/\bar{\rho}}$ and choose a new $y'^{o}$ so that $y^{o'} - y^{o} = \log\sqrt{\rho\bar{\rho}}$. This implies specifying a new point in $G_2(x)$ and is not a gauge transformation.

Similarly the subsets $(k\psi_1, k\psi_2)$, k real, of the set of homogeneous coordinates may be described by the spinors of weight -1/4, anti-weight +1/4, and indices I = J = 0. Their transformation law is

(6.5) $$\psi_A^{\,*} = \psi_B\, t_A^B\, t^{-\frac{1}{4}}\, \bar{t}^{\frac{1}{4}} = s_A^B \psi_B$$

where now

(6.6) $$s_A^B = t_A^B\, t^{-\frac{1}{4}}\, \bar{t}^{\frac{1}{4}} .$$

We note that

(6.7) $$s = t^{\frac{1}{2}}\, \bar{t}^{\frac{1}{2}} = |t|^{\frac{1}{2}} > 0 \quad \text{and real.}$$

That is, these spinors are the substratum of the linear homogeneous group in $P_1(x)$ with real and positive determinants. The only multiplicative spin transformations are of the form (4.8) where $\sigma = r$ and r is real, that is

(6.8) $$\psi_A^{\,*} = r\psi_A .$$

Hence the numbers $(\psi_1^{\,*}, \psi_2^{\,*})$ are another selection of homogeneous coordinates of a point in each $P_1(x)$ out of the subset $(k\psi_1, k\psi_2)$ k real, of the infinite set of homogeneous coordinates $(\rho\psi_1, \rho\psi_2)$.

The spinors of weight -1/4, anti-weight +1/4, and indices I = i and J = 1 are homogeneous coordinates from the latter set since in each frame of reference they are of the form (5.10). Thus if we have the homogeneous coordinates $(\psi_1, \psi_2)$ and wish to obtain a new pair $(\rho\psi_1, \rho\psi_2)$ we need only make the multiplicative spin transformation (6.8) where $r = \sqrt{\rho\bar{\rho}}$ and choose a new $x^{o'}$ so that $e^{i(x^{o'}-x^{o})} = \sqrt{\rho/\bar{\rho}}$. This implies specifying a new point in $G_1(x)$ and is not a gauge transformation.

SPINORS OF INDICES I ≠ 0 and J = 0

7. The equations derived in Chapter I to express the correspondence between points in $P_1(x)$ and vectors on the light cone at the point x of $X_4$ are

$$x^i = g^{i\dot{A}B}\bar{\psi}_A\psi_B. \tag{7.1}$$

If $\psi_A$ is a spinor of indices I ≠ 0 and J = 0, that is if it is of the form

$$\psi_A(x^o, x) = e^{Ix^o}f_A(x) \tag{7.2}$$

and if $\psi_A(x^o, x)$ is a solution of equation (7.1), then $\psi_A(x^o + k, x)$, where k is an arbitrary real number, is also a solution of equations (7.1).

Thus the correspondence is (1-1) between null vectors in $X_4$ and spinors of the type (7.2). These spinors characterize the single infinite subset of the doubly infinite set of homogeneous coordinates of a point in $P_1(x)$, namely the set $(e^{Ix^o}\psi_1, e^{Ix^o}\psi_2)$ where $x^o$ is an arbitrary real number. For, if in a definite frame of reference, we specify a point x in $X_4$ and a point $x^o$ in $G_1(x)$, then a numerical value of $\psi_A$ is obtained. If we change the value of $x^o$ then the numerical value of $\psi_A$ is altered but it is still in the same subset of homogeneous coordinates of a point in $P_1(x)$.

DIFFERENTIATION OF SPINORS

8. What we propose to do next is analogous to the treatment of differentiation in the special relativity theory by limiting the discussion to Cartesian coordinate systems in $X_4$. The partial derivatives of the components of any tensor with respect to these coordinates behave like the components of a tensor with one additional covariant index, under a Cartesian transformation of coordinates.

For example

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} \tag{8.1}$$

are the components of a tensor of the sort indicated by the arrangement of indices on the left-hand side of the equation. The equation (8.1) holds in all Cartesian coordinate systems in $X_4$ but not in general ones. The right-hand member satisfies the tensor law of transformation under transformations from one Cartesian coordinate system to another, since the coefficients of such a transformation are constants. But the tensor whose components are given in the preferred coordinate system by equation (8.1) is given in an arbitray coordinate system by a formula involving the Christoffel symbols of the $g_{ij}$'s.

In a like manner we may write

$$\psi_{A,i} = \frac{\partial \psi_A}{\partial x^i} . \tag{8.2}$$

The right-hand member will obviously transform as a covariant vector under arbitrary allowable coordinate transformations since $\psi_A$ is a scalar function of $x^1$ to $x^4$. It will also transform as a spinor under constant spin transformations. Thus a spinor which is defined by equations (8.2) in a particular coordinate system and spin frame is given in any other coordinate system and any spin frame obtainable from the first by a constant spin transformation by the transformation laws (3.15) and (3.16).

In order to obtain the spinor defined by equations (8.2) in more general spin coordinate systems, it will be necessary to discuss the covariant differentiation of spinors. This will be done in Chapter III.

If $\psi_A$ has indices I = J = 0, then it is unaffected by gauge transformations. Hence under a gauge transformation we have

$$\frac{\partial \psi_A^{*}}{\partial x^i} = \frac{\partial \psi_A}{\partial x^i} \quad \text{or} \quad \psi_{A,i}^{*} = \psi_{A,i}. \tag{8.3}$$

In case $\psi_A$ has indices $I \neq 0$ and $J = 0$, then under the gauge transformation of the first kind

$$x^{o*} = x^o + \theta(x) \tag{8.4}$$

the equation

$$\psi_A = e^{Ix^o} f_A(x) \tag{8.5}$$

becomes

$$\psi_A^{*} = e^{Ix^{o*}} f_A^{*}(x) \tag{8.6}$$

where

$$f_A^{*} = e^{-I\theta(x)} f_A(x). \tag{8.7}$$

From equation (8.5) we have

$$\frac{\partial \psi_A}{\partial x^i} = e^{Ix^o} \frac{\partial f_A}{\partial x^i}. \tag{8.8}$$

From equation (8.6) we have

$$\frac{\partial \psi_A^{\ *}}{\partial x^i} = e^{Ix^{o*}} \frac{\partial f_A^{\ *}}{\partial x^i} = e^{Ix^{o*}} \left( \frac{\partial}{\partial x^i} e^{-I\theta(x)} f_A(x) \right)$$

$$= e^{Ix^{o*}} e^{-I\theta(x)} \left( \frac{\partial f_A(x)}{\partial x^i} - I \frac{\partial \theta}{\partial x^i} f_A(x) \right)$$

$$= e^{Ix^{o}} \left( \frac{\partial f_A}{\partial x^i} - I \frac{\partial \theta}{\partial x^i} f_A(x) \right).$$

In virtue of equations (8.8) and (8.6) this becomes

(8.9)
$$\frac{\partial \psi_A^*}{\partial x^i} = \frac{\partial \psi_A}{\partial x^i} - I \frac{\partial \theta}{\partial x^i} \psi_A$$

or

(8.10)
$$\frac{\partial \psi_A}{\partial x^i} = \frac{\partial \psi_A^*}{\partial x^i} + I \frac{\partial \theta}{\partial x^i} \psi_A^{\ *}.$$

Since the vector potential of the electromagnetic theory, $A_j$, may be changed by the addition of the gradient of a scalar without affecting the electromagnetic field corresponding to it, we see that it has in addition to the usual vector transformation law the transformation law

(8.11)
$$A_j^{\ *} = A_j - \frac{\partial \lambda}{\partial x^j} .$$

Hence if $\varphi_j$ is the vector $A_j$, and if $\lambda$ in equation (8.11) is identified with $\theta(x)$ in the gauge transformation (8.4), it is evident that

(8.12)
$$\frac{\partial \psi_A}{\partial x^j} - I \varphi_j \psi_A = \frac{\partial \psi_A^{\ *}}{\partial x^j} - I \varphi_j^{\ *} \psi_A^{\ *}.$$

Hence, if $\psi_A$ is a spinor of indices $I \neq 0$ and $J = 0$, the geometric being

(8.13)
$$\frac{\partial \psi_A}{\partial x^j} - I \varphi_j \psi_A$$

transforms as a vector under arbitrary coordinate transformation as a spinor under constant spin transformations, and as a scalar under gauge transformations of the first kind. It is unaffected by gauge transformations of the second kind.

We shall denote the differential operator $\frac{h}{2\pi i}\ (\frac{\partial}{\partial x^j} - I\varphi_j)$ where h is Planck's constant, by $p_j$. Thus the quantity

$$p_j\psi_A = \frac{h}{2\pi i}\ (\frac{\partial\psi_A}{\partial x^j} - I\varphi_j\psi_A)$$

is a geometric being with the same transformation laws as those of the geometric being defined by the expression (8.13).

INVARIANT DIFFERENTIAL EQUATIONS

9. The differential equations

(9.1) $$p_j\psi_A = \frac{h}{2\pi i}\left(\frac{\partial\psi_A}{\partial x^j} - I\varphi_j\psi_A\right) = 0$$

where I and $\varphi_j$ are different from zero, have no solutions other than $\psi_A = 0$ unless $\varphi_j$ is the gradient of a scalar (that is, unless the electromagnetic field is absent).

However, the equations

(9.2) $$\frac{dx^j}{ds}\, p_j\psi_A = \frac{h}{2\pi i}\left(\frac{\partial\psi_A}{\partial x^j} - I\varphi_j\psi_A\right)\frac{dx^j}{ds} = 0$$

have solutions for arbitrary values of $\varphi_j$. Equations (9.2) may be considered as the equations for a displacement of spinors along a curve in $X_4$ whose tangent vector is $\frac{dx^j}{ds}$. Then equations (9.1) are the equations for the displacement which is independent of the curve in $X_4$.

Although the equations (9.1) have no solutions other than $\psi_A = 0$ for arbitrary $\varphi_j$, the simplest linear combinations of these equations, namely,

(9.3) $$g^{j\dot{A}B}p_j\psi_B = g^{j\dot{A}B}\,\frac{h}{2\pi i}\left(\frac{\partial\psi_B}{\partial x^j} - I\varphi_j\psi_B\right) = 0$$

do have other solutions. By taking the complex conjugate of equation (9.3), we have

$$(9.4) \qquad g^{j\dot{B}A} \frac{h}{2\pi i} \left( \frac{\partial \overline{\psi}_B}{\partial x^j} + I \varphi_j \overline{\psi}_B \right) = 0$$

since $\overline{g}^{j\dot{A}B} = g^{j\dot{B}A}$ and since I is a pure imaginary number. Multiplying equation (9.3) by $\overline{\psi}_A$ and equation (9.4) by $\psi_A$ and adding, we obtain

$$(9.5) \qquad \overline{\psi}_A g^{j\dot{A}B} p_j \psi_B + p_j \overline{\psi}_A g^{j\dot{A}B} \psi_B = \frac{h}{2\pi i} \frac{\partial}{\partial x^j} (\overline{\psi}_A g^{j\dot{A}B} \psi_B) = 0$$

or

$$(9.6) \qquad \frac{\partial J^j}{\partial x^j} = 0$$

where

$$(9.7) \qquad J^j = \overline{\psi}_A g^{j\dot{A}B} \psi_B.$$

From the fundamental correspondence between $T_4(x)$ and $P_1(x)$ discussed in Chapter I, we know that the vector $J^i$ defined by equations (9.7) is the vector on the light cone in $T_4(x)$ corresponding to the point whose coordinates are $(\psi_1, \psi_2)$ in $P_1(x)$. Since $J^i$ satisfies the equation (9.6) which is a continuity equation, it may be interpreted as the current vector of a particle whose wave function satisfies equation (9.3). That is, $J^4_{(x)}$ is the probability at time t that the particle will be found in the volume element bounded by the points x, y, z and x + dx, y + dy, z + dz, and $J^1$, $J^2$, and $J^3$ are the probabilities that the particle will cross the yz, xz, and xy planes at (x, y, z, t) in unit time respectively.

However, since $J^i$ is on the light cone, this particle travels with the velocity of light. The only particles treated in the physical theories with this property are the photons and neutrinos. Since photons, or the light fields which they represent, are described fully by the vector and tensor representations of the Lorentz group (that is, have integer angular momenta), we see that the spinor satisfying equation (9.3) probably does not correspond to a photon. It may correspond to a neutrino. If in equation (9.3) we set I = 0, it becomes the equation for the neutrino used by Fermi.*

---

* E. Fermi, Versuch einer Theories der $\beta$-strahlen, Zeitschr. f. Phys. 88 (1934), 161.

---

Since the spinor $g^{i\dot{A}B}$ transforms as a contravariant vector under coordinate transformations, it is evident that equations (9.3) and also (9.4) are invariant under arbitrary coordinate and gauge transformations, and constant spin transformations. Equations (9.3) are also invariant in form under Lorentz transformations, as we shall now show.

The spinors $g^{j\dot{A}B}$ have the numerical values given in Chapter I in a frame of reference consisting of a particular galilean coordinate system in $X_4$ and a particular coordinate system in each spin space $P_1(x)$. They have the weight $+\frac{1}{2}$ and anti-weight $+\frac{1}{2}$. The values of the $g^{j\dot{A}B}$ in any other frame of reference are obtained from these by means of the transformation law embodied in equations (3.17) and (3.18). In particular a constant unimodular spin transformation changes the numerical values of the $g^{i\dot{A}B}$ as follows

$$g^{j\dot{A}B*} = g^{j\dot{C}D}\, \bar{S}^{\dot{A}}_{\dot{C}}\, S^{B}_{D}. \tag{9.8}$$

From the fundamental isomorphism between $X_4$ and $P_1(x)$ we know that to each constant unimodular spin transformation there corresponds a unique proper Lorentz transformation with constant coefficients, namely

$$L^{i}{}_{j} = g^{i\dot{A}B}\, \bar{S}^{\dot{C}}_{\dot{A}}\, S^{D}_{B}\, g_{j\dot{C}D}. \tag{9.9}$$

This transformation carries one galilean coordinate system in $X_4$ into another one. If we define $\ell^{i}{}_{j}$ by the equations

$$(9.10)\qquad \ell^{i}_{j}\, L^{j}_{k} = \delta^{i}_{k}$$

then

$$(9.11)\qquad \ell^{i}_{j} = g^{i\dot{A}B}\, \bar{s}^{C}_{A}\, s^{D}_{B}\, g_{j\dot{C}D}$$

where

$$(9.12)\qquad s^{A}_{B}\, s^{B}_{C} = \delta^{A}_{C}.$$

Hence, if we simultaneously perform the transformation (9.11) in $X_4$ and the transformation

$$(9.13)\qquad \psi^{A*} = s^{A}_{B}\psi^{B}$$

in $P_1(x)$, we have from equations (3.17) and (3.18)

$$(9.14)\qquad g^{i\dot{A}B*} = g^{j\dot{C}D}\, \bar{s}^{A}_{C}\, s^{B}_{D}\, \ell^{i}_{j} = g^{j\dot{C}D}\, \bar{s}^{A}_{C}\, s^{B}_{D}\, g^{i\dot{E}F}\, \bar{s}^{G}_{E}\, s^{H}_{F}\, g_{j\dot{G}H}$$

$$= g^{i\dot{A}B}$$

since

$$g^{j\dot{C}D}\, g_{j\dot{G}H} = \delta^{\dot{C}}_{\dot{G}}\delta^{D}_{H}.$$

That is, the spinors $g^{i\dot{A}B}$ are numerically invariant if we make a constant spin transformation and then perform the proper Lorentz transformation corresponding to the inverse of this transformation.

In virtue of equations (9.14), equations (9.3) may be written:

$$(9.15)\qquad g^{k\dot{C}D}\, \bar{s}^{A}_{C}\, s^{B}_{D}\, \ell^{j}_{k} \left(\frac{\partial\psi_B}{\partial x^j} - I\varphi_j\psi_B\right) = 0$$

or

$$(9.16)\qquad g^{k\dot{C}D}\left(\frac{\partial\psi_D^{\;*}}{\partial x^{k*}} - I\varphi_k^{\;*}\psi_D^{\;*}\right) = 0$$

where

(9.17) $$x^i = \ell^i_j x^{j*}, \quad \varphi_j^* = \ell^k_j \varphi_k, \quad \psi_D^* = S^B_D \psi_B.$$

This is, equations (9.3) are invariant in form if we perform a proper Lorentz transformation, given by the first of equations (9.17) and simultaneously perform the spin transformation corresponding to the inverse of this transformation.

DIRAC EQUATIONS

10. In this section we shall consider a set of differential equations which are slightly more complicated than (9.1) and involve two spinors:

(10.1) $$g^{j\dot{A}B}\left(\frac{\partial \psi_B}{\partial x^j} - I\varphi_j \psi_B\right) = a\bar{\chi}^A$$

and

(10.2) $$g^{j\dot{A}B}\left(\frac{\partial \chi_B}{\partial x^j} + I\varphi_j \chi_B\right) = a\bar{\varphi}^A.$$

In these equations a is a constant, $\psi_A$ is a spinor of weight $-\frac{1}{2}$, index of first kind I, and index of second kind zero, and $\chi_A$ is a spinor of weight $-\frac{1}{2}$, index of first kind -I and index of second kind zero.

From the previous sections it is evident that equations (10.1) and (10.2) are invariant under arbitrary coordinate and gauge transformations and constant spin transformations. They are also invariant in form under proper Lorentz transformations. This will be proved by the argument used in section 9. From equations (9.14) we see that equation (10.1) may be written as

(10.3) $$g^{k\dot{C}D}\, \bar{S}^A_C\, S^B_D\, \ell^i_k \left(\frac{\partial \psi_B}{\partial x^j} - I\varphi_j \psi_B\right) = a\bar{\chi}^A$$

or

(10.4) $$g^{k\dot{C}D}\left(\frac{\partial \psi_D^*}{\partial x^{k*}} - I\varphi_k^* \psi_D^*\right) = a\bar{\chi}^{C*}$$

where

(10.5) $\chi^{i} = \ell^{i}{}_{j}\chi^{j*}$, $\varphi_{j}{}^{*} = \ell^{k}{}_{j}\varphi_{k}$, $\psi_{D}{}^{*} = s^{B}_{D}\varphi_{B}$, and $\chi^{A*} = s^{A}_{C}\chi^{C}$.

Similarly equations (10.2) become

(10.6) $$g^{k\dot{C}D}\left(\frac{\partial \chi_{D}^{*}}{\partial x^{k*}} + I\varphi_{k}^{*}\chi_{D}^{*}\right) = a\bar{\psi}^{C*}.$$

By a similar argument it can be shown that the equations (10.1) and (10.2) are invariant in form under improper Lorentz transformations provided that

(10.7) $$\chi_{D}^{*} = \bar{s}^{B}_{D}\bar{\psi}_{B} \qquad \psi^{A*} = \bar{s}^{\dot{A}}_{C}\bar{\chi}^{C}$$

where $s^{A}_{B}$ is the anti-projectivity corresponding to the inverse of the improper Lorentz transformation.

Equations (10.1) and (10.2) are equivalent to the Dirac equations for an electron provided $\varphi_{j}$ is the electromagnetic potential vector and

(10.8) $$I = \frac{2\pi ei}{hc} \qquad \text{and} \qquad a = \frac{\sqrt{2}\,\pi mci}{h}$$

where h is Planck's constant, c is the velocity of light, and e and m are the charge and mass of the electron respectively. This we shall prove in section 12.

DIRAC EQUATIONS (Continued)

11. Dirac's derivation of the differential equation for the wave function of a free electron employed four-rowed matrices and hence the wave functions were really four-component spinors. His argument is essentially the following one. The equation for the wave function of the free electron must be linear, invariant under Lorentz and gauge transformations, and contain only first-order time derivatives. Because of the Lorentz invariance the equation may only contain first order spatial derivatives. In addition, the solutions of this equation must be solutions of the relativistic analogue to the Schrödinger equation, namely the equation

(11.1) $$-\frac{h^2}{4\pi^2}\left(-\frac{\partial^2}{\partial x^2}\Psi-\frac{\partial^2}{\partial y^2}\Psi-\frac{\partial^2}{\partial z^2}\Psi+\frac{\partial^2}{c^2\partial t^2}\Psi\right)=m^2c^2\Psi.$$

In terms of the tensor notation, equation (11.1) may be written as

(11.2) $$-\frac{h^2}{4\pi^2}g^{ij}\frac{\partial^2}{\partial x^i\partial x^j}\Psi=m^2c^2\Psi.$$

The conditions enumerated above can be satisfied if the second-order differential operator occurring on the left of equation (11.2) can be written as the square of a linear operator. The differential operator

(11.3) $$\gamma^j\frac{h}{2\pi i}\frac{\partial}{\partial x^j}$$

has this property provided the quantities $\gamma^j$ are a set of four constant matrices satisfying the conditions

(11.4) $$\frac{1}{2}(\gamma^k\gamma^j+\gamma^j\gamma^i)=g^{ij}\cdot 1.$$

The existence of a set of four four-rowed square matrices satisfying equations (11.4) was shown in section 15, Chapter I. Therefore the equations

(11.5) $$\gamma^i\frac{h}{2\pi i}\frac{\partial}{\partial x^j}\Phi=mc\,\Phi$$

satisfy the requirements given above. In equation (11.5) $\Phi$ is the matrix of one column

(11.6) $$\|\Phi\| = \left\|\begin{matrix}\Phi^1\\ \Phi^2\\ \Phi^3\\ \Phi^4\end{matrix}\right\|$$

and the $\gamma^i$ are the matrices given in section 15, Chapter I.

The wave equation for an electron in a field of force whose vector potential is $\varphi_j$ is

(11.7) $$\gamma^j p_j\Phi=mc\,\Phi$$

where $p_j$ is the differential operator defined in section 9, and I has the value given by equation (10.8).

Equations (11.5) or (11.7) may be transformed directly to the form given by Dirac by making a substitution of the form

$$\Phi = T\,\Psi \tag{11.8}$$

where T is a properly chosen matrix, and multiplying equation (11.7) by $T^{-1}\rho$ where $\rho$ is another matrix. Then we have

$$\alpha^i p_i \Psi - mc\,\alpha_m \Psi = 0 \tag{11.9}$$

where

$$\alpha^i = T^{-1}\rho\gamma^i T \qquad\qquad \alpha_m = T^{-1}\rho T. \tag{11.10}$$

If $\rho$ and T are the matrices

$$\rho = \left\| \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix} \right\| \qquad T = \left\| \begin{matrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{matrix} \right\|$$

the matrices $\alpha_i$ and $\alpha_m$ are those given by Dirac*.

---

* P.A.M.Dirac, "Quantum mechanics", Oxford (1930), p. 243. Observe that the Dirac $\alpha$-matrices are the covariant set $\alpha_i$, where

$$\alpha_1 = -\alpha^1,\ \alpha_2 = -\alpha^2,\ \alpha_3 = -\alpha^3 \qquad \text{and} \qquad \alpha_4 = \alpha^4.$$

---

DIRAC EQUATIONS (Continued)

12. We prove the equivalence between the Dirac equations and equations (10.1) and (10.2) by showing that the latter may be written in the form of equations (11.7). We first write equation (10.2) in the form

$$\bar{g}^j_{AB}\left(\frac{\partial \bar{\chi}^B}{\partial x^j} - \frac{2\pi ei}{hc}\,\varphi_j\,\bar{\chi}^B\right) = \frac{\sqrt{2}\,\pi mci}{h}\,\psi_A \tag{12.1}$$

where we have raised and lowered indices by means of $\varepsilon^{AB}$ and $\varepsilon_{AB}$ and have taken the complex conjugates of equations (10.2).

If we now let $\Phi$ be the matrix of one column

$$(12.2) \qquad \Phi = \begin{Vmatrix} \psi_1 \\ \psi_2 \\ \bar{\chi}^1 \\ \bar{\chi}^2 \end{Vmatrix}$$

Equations (10.1) and (12.1) may be combined into the single matrix equation

$$(12.3) \qquad g^j p_j \Phi = mc \rho \Phi$$

where

$$(12.4) \qquad g^j = \sqrt{2} \begin{Vmatrix} \|g^{j\dot{A}B}\| & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \|\bar{g}^j_{\dot{A}B}\| \end{Vmatrix} \quad \text{and } \rho = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}$$

Multiplying equation (12.3) by $\rho$ we obtain

$$(12.5) \qquad \gamma^j p_j \Phi = mc \Phi$$

where

$$(12.6) \qquad \gamma^j = \rho g^j = \sqrt{2} \begin{Vmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \|\bar{g}^j_{\dot{A}B}\| \\ \|g^{j\dot{A}B}\| & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{Vmatrix}$$

The matrices $\gamma^j$ are those given in section 15, Chapter I, and satisfy the relation

$$(12.7) \qquad \frac{1}{2}(\gamma^i \gamma^j + \gamma^j \gamma^i) = g^{ij} 1.$$

Since equations (10.1) and (10.2) are equivalent to the Dirac equation, we see that the invariance properties of the latter may be treated by decomposing the four Dirac equations into two sets of two equations and treating the invariance properties of these equations by means of two-component spinors. In so doing we restrict the spin transformations of the four-component spinors to those involving a certain pairing of the components. The Dirac equations, however, have invariance properties under the larger group of linear homogeneous transformations which underlies the four-component theory. We shall return to this question in a later chapter.

CURRENT VECTOR

13. The vector

$$J^i = g^{i\dot{A}B} (\overline{\psi}_A \psi_B + \overline{\chi}_A \chi_B) \tag{13.1}$$

satisfies the equation

$$\frac{\partial J^i}{\partial x^i} = 0 \tag{13.2}$$

as a consequence of equations (10.1) and (10.2). This may be proved by first taking the complex conjugates of these equations. Then we have

$$g^{j\dot{B}A} \left( \frac{\partial \overline{\psi}_B}{\partial x^j} + I \varphi_j \overline{\psi}_B \right) = - a \chi^A \tag{13.3}$$

$$g^{j\dot{B}A} \left( \frac{\partial \overline{\chi}_B}{\partial x^j} - I \varphi_j \overline{\chi}_B \right) = - a \psi^A \tag{13.4}$$

since $\overline{g}^{j\dot{A}B} = g^{j\dot{B}A}$ and since a is a pure imaginary number.

If we now multiply equations (10.1) and (10.2) by $\overline{\psi}_A$ and $\overline{\chi}_A$ respectively, and then multiply equations (13.3) and (13.4) by $\psi_A$ and $\chi_A$ respectively and add, we have

$$g^{j\dot{A}B} \frac{\partial}{\partial x^j} \left( \overline{\psi}_A \psi_B + \overline{\chi}_A \chi_B \right) = \frac{\partial J^j}{\partial x^j} = 0.$$

The vector $J^i$ is a time-like vector. This may be seen from the fundamental correspondence between vectors in $X_4$ and hermitian forms in $P_1(x)$. In Chapter I it was shown that the vectors corresponding to positive definite hermitian matrices are timelike. Since

$$P_{\dot{A}B} = \bar{\psi}_{\dot{A}}\psi_B + \bar{\chi}_{\dot{A}}\chi_B \tag{13.5}$$

is a positive definite hermitian matrix, the vector $J^i$ is timelike. This fact may also be seen by computing the length of the vector $J^i$, we have

$$\begin{aligned} J^i J_i &= g_{ij} g^{i\dot{A}B} g^{j\dot{C}D} (\bar{\psi}_{\dot{A}}\psi_B + \bar{\chi}_{\dot{A}}\chi_B)(\bar{\psi}_{\dot{C}}\psi_D + \bar{\chi}_{\dot{C}}\chi_D) \\ &= 2\,(\chi_A\psi^A)(\bar{\chi}_{\dot{C}}\bar{\psi}^{\dot{C}}) = 2\,|\chi_A\psi^A| \end{aligned} \tag{13.6}$$

since $g^{i\dot{A}B} g_i^{\dot{C}D} = \varepsilon^{\dot{A}\dot{C}}\,\varepsilon^{BD}$, and since $\chi^A\chi_A = \psi^A\psi_A = 0$. The right-hand member of equation (13.6) is always positive if $\chi_A$ is not proportional to $\psi_A$ and hence $J^i$ is a timelike vector.

Since $\psi_A$ and $\chi_A$ are both spinors with indices of first kind different from zero and indices of second kind equal to zero, $J^i$ is independent of the gauge variables.

Since $J^i$ is a timelike vector, whose divergence vanishes, it may be interpreted as the current vector for the electron which is described by the two spinors $\psi_A$ and $\chi_A$ which satisfy equations (10.1) and (10.2) respectively.

# Chapter III

## COVARIANT DIFFERENTIATION

1. A galilean frame of reference is given by specifying the gauge coordinate systems, a galilean coordinate system in $X_4$, and a spin coördinate system in $P_1$ such that the spinors $g^i{}_{\dot{A}B}$ take on the numerical values given in Chapter I, section 4. We have shown, in section 9, Chapter II, that the spinors $g^i{}_{\dot{A}B}$ are numerically invariant if we make a spin transformation with constant coefficients and then perform the proper Lorentz transformation corresponding to the inverse of this transformation. Since the latter transformation carries galilean coordinate systems in $X_4$ into galilean coordinate systems, we see that the group of transformations which carries galilean frames of reference into galilean frames is the restricted Lorentz group.

A cartesian frame of reference is obtained from a galilean one by making a cartesian transformation of coordinates in $X_4$ or a transformation with constant coefficients in $P_1$, or both. That is, in a cartesian frame of reference the spinors $g^i{}_{\dot{A}B}$ may have any constant values subject to the quadratic identity between them (i.e. independent of the coordinates of $X_4$).

The covariant derivative of a spinor is a spinor with one additional covariant tensor index, which reduces to the ordinary derivative of the spinor in a cartesian frame of reference. In the case of the simple spinor $\psi_A$, in a cartesian frame of reference we have

$$\psi_{A,i} = \frac{\partial \psi_A}{\partial x^i} \quad . \tag{1.1}$$

In section 8, Chapter II we have shown that $\psi_{A,i}$ transforms as a vector under arbitrary coordinate transformations and as a spinor under constant spin transformations.

We shall now use the definition of the covariant derivative of a spinor to compute $\psi_{A,i}$ after a general spin transformation. If this transformation is

(1.2) $$\psi_A{}^* = \rho_A^B \psi_B$$

where $\rho_A^B$ are arbitrary complex functions of $x^1, \ldots, x^4$ such that $\rho$, the determinant of $\rho_A^B$ is different from zero, then since $\psi_{A,i}$ is a spinor, we have

(1.3) $$\psi_{A,i}{}^* = s_A^B \psi_{B,i}$$

where

(1.4) $$s_A^B = \rho_A^B \; \rho^w \; \bar{\rho}^a$$

and w and a are the weight and anti-weight of $\psi_A$ respectively.

Since

(1.5) $$\frac{\partial \psi_A{}^*}{\partial x^j} = \frac{\partial \psi_B}{\partial x^j} s_A^B + \psi_B \frac{\partial s_A^B}{\partial x^j} = \frac{\partial \psi_B}{\partial x^j} s_A^B + S_B^C \psi_C{}^* \frac{\partial s_A^B}{\partial x^j}$$

where $S_B^A \, s_C^B = \delta_C^A$

we have

(1.6) $$\frac{\partial \psi_B}{\partial x^j} s_A^B = s_A^B \psi_{B,j} = \frac{\partial \psi_A{}^*}{\partial x^j} - S_B^C \frac{\partial s_A^B}{\partial x^j} \psi_C{}^*$$

Hence if we define

(1.7) $$\Lambda_{Aj}^{C\,*} = S_B^C \frac{\partial s_A^B}{\partial x^j}$$

we have

(1.8) $$\psi_{A,j}{}^* = \frac{\partial \psi_A{}^*}{\partial x^j} - \Lambda_{Aj}^{C\,*} \psi_C{}^*$$

Equation (1.8) is the formula for the covariant derivative of a spinor of weight w and anti-weight a in any frame of reference. If $\psi_A$ are the components of a spinor in a general coordinate system, equation (1.8) may be written without the asterisks, thus

(1.9) $$\psi_{A,j} = \frac{\partial \psi_A}{\partial x^j} - \Lambda_{Aj}^C \psi_C$$

where $\Lambda_{Aj}^C$ is defined by equations (1.7) and (1.4).

The transformation law for the quantities $\Lambda_{Cj}^A$ may be computed from equation (1.9). Under a general transformation of spin coordinates equation (1.9) becomes

(1.10) $$\frac{\partial \psi_A^*}{\partial x^i} - \Lambda_{Ai}^{C\,*} \psi_C^* = \psi_{A,i}^* = \psi_{B,i}\, s_A^B \;.$$

Since $\psi_A^* = \psi_B\, s_A^B$, equation (1.10) may be written

$$s_A^B \frac{\partial \psi_B}{\partial x^i} + \frac{\partial s_A^B}{\partial x^i} \psi_B - \Lambda_{Ai}^{C\,*} s_C^B \psi_B = s_A^B \left( \frac{\partial \psi_B}{\partial x^i} - \Lambda_{Bi}^{C} \psi_C \right) \;.$$

Hence

$$\left[ \Lambda_{Ai}^{B\,*} s_B^C - \left( s_A^D \Lambda_{Di}^{C} + \frac{\partial s_A^C}{\partial x^i} \right) \right] \psi_C = 0 \;.$$

Since this is to hold for arbitrary $\psi_C$, we must have

(1.11) $$\Lambda_{Ai}^{B\,*} = s_C^B \left( \Lambda_{Di}^{C}\, s_A^D + \frac{\partial s_A^C}{\partial x^i} \right) \;.$$

Since $s_B^A$ are scalar functions of $x^1, \ldots, x^4$, it is evident that $\Lambda^A_{\ Bj}$ transforms as a covariant vector with respect to coordinate transformations.

If we set B = A and sum, equations (1.11) become

(1.12) $$\Lambda^B_{\ Bi}{}^* = \Lambda^B_{\ Bi} + \frac{\partial \log s}{\partial x^i}$$

where s is the determinant of the transformation $s_B^A$.

From equations (1.4) we see that equations (1.7) may be written as

(1.13) $$\Lambda^C_{\ Aj} = P_B^C\, \rho^{-w} \bar{\rho}^{-a} \frac{\partial}{\partial x^j} \left( \rho_A^B \rho^w \bar{\rho}^a \right)$$

$$= \Gamma_{Aj}^C + w\, \Gamma_{Bj}^B\, \delta_A^C + a\, \bar{\Gamma}_{Bj}^B\, \delta_A^C$$

where

(1.14) $$\Gamma_{Aj}^C = P_B^C \frac{\partial \rho_A^B}{\partial x^j}$$

and hence

(1.15) $$\Gamma_{Cj}^C = P_B^C \frac{\partial \rho_C^B}{\partial x^j} = \frac{\partial \log \rho}{\partial x^j}$$

by the definition of the derivative of a determinant. The geometric being $\rho$ is a scalar under coordinate transformations and a scalar density under spin transformations.

The geometric being with the components $\Gamma^{A}_{Bi}$ will be called a spin connection. It transforms as a covariant vector under a coordinate transformation. Its transformation law under spin transformations will be obtained below. Instead of equation (1.10) we may write

(1.16) $$\psi_{A,j} = \frac{\partial \psi_A}{\partial x^j} - \Gamma^{C}_{Aj}\psi_C - w\,\Gamma^{C}_{Cj}\,\psi_A - a\,\bar{\Gamma}^{C}_{Cj}\,\psi_A$$

as an alternative expression for the covariant derivative of a spinor of weight w and anti-weight a.

THE TRANSFORMATION LAW OF $\Gamma^{A}_{Bj}$

2. The transformation law for the spin connection may be obtained from equations (1.11) and (1.13). If we set w = a = 0 in equations (1.13), we have

(2.1) $$\Lambda^{C}_{Aj} = \Gamma^{C}_{Aj}$$

and

(2.2) $$s^{B}_{A} = t^{B}_{A}$$

where $t^{B}_{A}$ is a general spin transformation. Hence in virtue of equations (1.13) we have

(2.3) $$\Gamma^{B}_{Aj}{}^{*} = T^{B}_{C}\left(\Gamma^{C}_{Dj}\,t^{D}_{A} + \frac{\partial t^{C}_{A}}{\partial x^j}\right)$$

and

(2.4) $$\Gamma^{C}_{Cj}{}^{*} = \Gamma^{C}_{Cj} + \frac{\partial \log t}{\partial x^j} .$$

If we now compare equations (2.3) and (1.14) we see that they are the same if in equation (2.3) we set $\Gamma^{C}_{Dj} = 0$ and $t^{C}_{A} = \rho^{C}_{A}$. That is in the cartesian coordinate system the components of the spin connection are zero and their law of transformation under spin coordinate transformations then determines them in any spin coordinate system ( $\rho_A^{\ C}$ is the transformation from the cartesian coordinate system to the general one). Thus the $\Gamma^A_{\ Bj}$ are determined in a general spin coordinate system as soon as we know the transformation which carries a cartesian frame of reference into a general one.

From equation (1.15) we see that just giving the value of the determinant of the transformation from a cartesian to a general spin coordinate system determines the trace of the spin connection. That is the spin scalar of weight one, $\rho$ , determines the trace of the spin connection. Equation (1.15) may be written as

(2.5) $$\frac{\partial \rho}{\partial x^i} - \rho \Gamma^C_{\ Ci} = 0 \ .$$

But this is just the statement that the covariant derivative of $\rho$ is zero. By specifying the absolute value of $\rho$ or the argument of $\rho$ we specify the real or imaginary part of the trace of the spin connection. This may be seen as follows: From equation (1.15) we have

(2.6) $$\bar{\Gamma}^C_{\ Cj} = \frac{\partial \log \bar{\rho}}{\partial x^j}$$

hence

(2.7) $$\frac{1}{2}( \Gamma^C_{\ Cj} + \bar{\Gamma}^C_{\ Cj}) = \frac{1}{2} \frac{\partial \log \rho\bar{\rho}}{\partial x^j} = \frac{\partial \log \sqrt{\rho\bar{\rho}}}{\partial x^j} \ .$$

But the left member of this equation is just the real part of the trace of the spin connection and the right member involves only the absolute value of $\rho$ . Hence giving the absolute value of $\rho$ determines the real part of the trace of the spin connection. Equation (2.7) may be written as

(2.8) $$\frac{\partial (\rho\bar{\rho})}{\partial x^j} - \rho\bar{\rho} \ (\Gamma^C_{\ Cj} + \bar{\Gamma}^C_{\ Cj}) = (\rho\bar{\rho})_{,j} = 0 \ .$$

That is, the covariant derivative of the real scalar density $\rho\bar{\rho}$ is zero.

Similarly, we have

(2.9) $$\left(\Gamma^{C}_{Cj} - \bar{\Gamma}^{C}_{Cj}\right) = \frac{\partial}{\partial x^{j}} \log \rho/\bar{\rho} \quad .$$

Hence we see that giving the argument of $\rho$ determines the imaginary part of the trace of the spin connection. Equation (2.9) may be written as

(2.10) $$\frac{\partial}{\partial x^{j}} ( \rho/\bar{\rho} ) - \rho/\bar{\rho} \ ( \Gamma^{C}_{Cj} - \bar{\Gamma}^{C}_{Cj} ) = ( \rho/\bar{\rho} )_{,j} = 0 \quad .$$

That is, the covariant derivative of the pure imaginary scalar $\rho/\bar{\rho}$ is zero.

EXAMPLES OF COVARIANT DIFFERENTIATION

3. The formula for the covariant derivative of a contravariant spinor may be obtained in the same manner as that of a covariant spinor. In the cartesian frame of reference we have

(3.1) $$\psi^{A}_{,i} = \frac{\partial \psi^{A}}{\partial x^{i}} \quad .$$

If we make the transformation (1.2) in the spin space, then

(3.2) $$\psi^{A*} = P^{A}_{B} \psi^{B} \rho^{-w} \bar{\rho}^{-a} = S^{A}_{B} \psi^{B}$$

where -w and -a are the weight and anti-weight of $\psi^{A}$ respectively and where

(3.3) $$P^{A}_{B} \rho^{B}_{C} = \delta^{A}_{C} \ .$$

Since

$$\frac{\partial \psi^{A*}}{\partial x^{j}} = S^{A}_{B} \frac{\partial \psi^{B}}{\partial x^{j}} + \psi^{B} \frac{\partial S^{A}_{B}}{\partial x^{j}} = S^{A}_{B} \frac{\partial \psi^{B}}{\partial x^{j}} + s^{B}_{C} \psi^{C*} \frac{\partial S^{A}_{B}}{\partial x^{j}}$$

we have

(3.4) $$S^{A}_{B} \frac{\partial \psi^{B}}{\partial x^{j}} = \frac{\partial \psi^{A*}}{\partial x^{j}} - s^{B}_{C} \frac{\partial S^{A}_{B}}{\partial x^{j}} \psi^{C*} \quad .$$

But

(3.5) $$s^{B}_{C} \frac{\partial S^{A}_{B}}{\partial x^{j}} = - S^{A}_{B} \frac{\partial s^{B}_{C}}{\partial x^{j}} = - \Lambda^{A}_{Cj} \quad .$$

Therefore we have in any coordinate system

(3.6) $$\psi^{A}_{,i} = \frac{\partial \psi^A}{\partial x^i} + \Lambda^{A}_{Ci}\psi^C .$$

An alternative expression for equation (3.6) is

(3.7) $$\psi^{A\,*}_{,i} = \frac{\partial \psi^A}{\partial x^i} + \Gamma^{A}_{Ci}\,\psi^C + w\,\Gamma^{C}_{Ci}\psi^A + a\,\overline{\Gamma}^{C}_{Ci}\psi^A$$

for the covariant derivative of a contravariant spinor of weight -w and anti-weight -a.

The formula for the covariant derivative of a spinor with a dotted index may be obtained by considering the simple spinor

(3.8) $$\varphi_{\dot{A}} = \overline{\psi}_A .$$

In a cartesian frame of reference we have

(3.9) $$\varphi_{\dot{A},i} = (\overline{\psi}_A)_{,i} = \frac{\partial \overline{\psi}_A}{\partial x^i} = \overline{\left(\frac{\partial \psi_A}{\partial x^i}\right)} = (\overline{\psi_{A,i}}) .$$

Therefore, in general frame of reference we have

(3.10) $$\varphi_{\dot{A},i} = \frac{\partial \varphi_{\dot{A}}}{\partial x^i} - \varphi_{\dot{C}}\,\overline{\Gamma}^{C}_{Ai} - w\,\varphi_{\dot{A}}\,\Gamma^{C}_{Ci} - a\,\varphi_{\dot{A}}\,\overline{\Gamma}^{C}_{Ci}$$

for the covariant derivative of a spinor of weight w and anti-weight a with a single dotted index.

Equations (3.9) express the fact that the process of taking the complex conjugate of a spinor is commutative with the process of taking its covariant derivative. Equation (3.10) can also be derived from the definition of the covariant derivative in the same manner as equation (1.16) was. It holds for any spinor with a dotted index.

The formula for the covariant derivative of a product of simple spinors is the same as the formula for the ordinary derivative of a product. For in the galilean frame of reference we have

(3.11) $$(\psi_A \chi_B)_{,i} = \frac{\partial(\psi_A \chi_B)}{\partial x^i} = \psi_A \frac{\partial \chi_B}{\partial x^i} + \frac{\partial \psi_A}{\partial x^i} \chi_B = \psi_{A,i} \chi_B + \psi_A \chi_{B,i} .$$

Hence in any frame of reference we have

(3.12) $$(\psi_A \chi_B)_{,i} = \psi_{A,i} \chi_B + \psi_A \chi_{B,i} .$$

By the same argument it can be shown that the covariant derivative of the product of any types of spinors obey the same law as the ordinary derivative of the product.

From equation (3.12) we see that the formula for the covariant derivative of the spinor,

(3.13) $$\Phi_{AB} = \psi_A \chi_B$$

is

(3.14) $$\Phi_{AB,j} = \frac{\partial \Phi_{AB}}{\partial x^j} - \Phi_{AC} \Gamma^C_{Bj} - \Phi_{CB} \Gamma^C_{Aj} - w \Gamma^C_{Cj} \Phi_{AB} - a \Gamma^C_{Cj} \Phi_{AB}$$

where w and a are the weights and anti-weights of $\Phi_{AB}$ respectively. Equation (3.14) is obtained from equation (3.12) where the values of $\chi_{a,i}$ and $\psi_{A,i}$ are given by equations of the type (1.16). Equation (3.14) is also valid for any spinor with two covariant spin indices, as can be proved by a consideration of the transformation law of $\Phi_{AB}$.

In particular we have for the spinor $\varepsilon_{AB}$ defined in Chapter I, the relation

(3.15) $$\varepsilon_{AB,i} = \frac{\partial \varepsilon_{AB}}{\partial x^i} - \varepsilon_{CB} \Gamma^C_{Ai} - \varepsilon_{AC} \Gamma^C_{Bi} + \varepsilon_{AB} \Gamma^C_{Ci}$$

(3.16) $$\varepsilon_{AB,i} = - \varepsilon_{CB} K^C_{Ai} - \varepsilon_{AC} K^C_{Bi} = \varepsilon_{BC} K^C_{Ai} - \varepsilon_{AC} K^C_{Bi}$$

where

(3.17) $$K^A_{Bi} = \Gamma^A_{Bi} - \frac{1}{2} \Gamma^C_{Ci} \delta^A_C .$$

That is, $K^A_{Bi}$ is the traceless part of $\Gamma^A_{Bi}$. Since the trace of $K^A_{Bi}$ is zero we have

(3.18) $$\varepsilon_{AC} K^C_{Bi} \equiv K_{ABi} = K_{BAi} .$$

Hence

(3.19) $$\varepsilon_{AB,i} = 0 .$$

This means that the process of lowering spin indices is commutative with the process of covariant differentiation. We shall now verify that

(3.20) $$\varphi_{A,i} = (\mathcal{E}_{AC}\psi^{C})_{,i} = \mathcal{E}_{AC}\psi^{C}_{,i} \quad .$$

Since the relation

(3.21) $$\varphi_{A} = \mathcal{E}_{AC}\psi^{C}$$

holds in all coordinate systems if the weight of $\psi^{C}$ is one greater than the weight of $\varphi_{C}$, we need only verify equation (3.20) in the cartesian frame of reference. Since the ordinary derivative of $\mathcal{E}_{AC}$ is zero, equation (3.20) holds in this frame and therefore in any frame of reference.

Equation (3.12) together with equations (3.7) and (1.16) enables us to show that the covariant derivative of the spinor

(3.22) $$\Phi_{A}{}^{B} = \psi_{A}\chi^{B}$$

is

(3.23) $$\Phi_{A}{}^{B}{}_{,i} = \frac{\partial\Phi_{A}{}^{B}}{\partial x^{i}} - \Phi_{C}{}^{B}\Gamma^{C}_{Aj} + \Phi_{A}{}^{C}\Gamma^{B}_{Cj} - w\,\Phi_{A}{}^{B}\Gamma^{C}_{Cj} - a\,\Phi_{A}{}^{B}\bar{\Gamma}^{C}_{Cj}$$

where w and a are the weight and anti-weight of the spinor $\Phi_{A}^{B}$ respectively. Equation (3.23) is also valid for any spinor with one covariant and one contravariant spin index as may be proved by a consideration of the transformation law of $\Phi_{A}{}^{B}$.

In particular it holds for the spinor $\delta^{A}_{B}$, the Kronecker delta.

Thus

(3.24) $$\delta^{A}_{B,i} = \frac{\partial\delta^{A}_{B}}{\partial x^{i}} - \delta^{A}_{C}\Gamma^{C}_{Bi} + \delta^{C}_{B}\Gamma^{A}_{Ci} = 0 \ .$$

Since $\delta^{A}_{B}$ is a constant spinor with both weights zero.

From the relation

(3.25) $$\mathcal{E}_{AB}\mathcal{E}^{AC} = \delta^{C}_{B}$$

and the rule for differentiating a product we see that

(3.26) $$\mathcal{E}_{AB}\mathcal{E}^{AC}{}_{,j} = 0$$

since $\mathcal{E}_{AB,j} = 0$ and $\delta^{C}_{B,j} = 0$. Hence we have

(3.27) $$\mathcal{E}^{AC}{}_{,j} = 0 \ .$$

That is, the process of raising indices is commutative with the process of covariant differentiation. It is readily verified that

$$\varphi^{A}{}_{,i} = (\varepsilon^{CA}\psi_{C})_{,i} = \varepsilon^{CA}\psi_{C,i} \quad (3.28)$$

The formula for the covariant derivative of a spinor with any number of spin indices may be obtained in a similar manner from the definition of the covariant derivative and its transformation law.

Then we have

$$\begin{aligned}\Phi^{A\ldots\dot{B}\ldots}{}_{D\ldots\dot{E}\ldots,i} = {} & \frac{\partial\Phi^{A\ldots\dot{B}\ldots}{}_{D\ldots\dot{E}\ldots}}{\partial x^{i}} + \Phi^{C\ldots\dot{B}\ldots}{}_{D\ldots\dot{E}\ldots}\Gamma^{A}_{Ci} \ldots \\ & + \Phi^{A\ldots\dot{C}\ldots}{}_{D\ldots\dot{E}\ldots}\overline{\Gamma}^{B}_{Ci} - \Phi^{A\ldots\dot{B}\ldots}{}_{C\ldots\dot{E}\ldots}\Gamma^{C}_{Di} \\ & - \Phi^{A\ldots\dot{B}\ldots}{}_{D\ldots\dot{C}\ldots}\overline{\Gamma}^{C}_{Ei} - w\,\Phi^{A\ldots\dot{B}\ldots}{}_{D\ldots\dot{E}\ldots}\Gamma^{C}_{Ci} \\ & - a\,\Phi^{A\ldots\dot{B}\ldots}{}_{D\ldots\dot{E}\ldots}\overline{\Gamma}^{C}_{Ci} \,. \end{aligned} \quad (3.29)$$

COVARIANT DIFFERENTIATION OF SPINORS WITH TENSOR INDICES

4. Before taking up the discussion of covariant differentiation of spinors with tensor indices, we shall briefly outline the theory of covariant differentiation of tensors. The covariant derivative of a tensor with respect to the $g_{ij}$ is another tensor with one additional covariant index which reduces to the ordinary derivative in the cartesian coordinate systems in $X_4$. Thus in a cartesian coordinate system we have for the vector $V_i$

$$V_{i,j} = \frac{\partial V_i}{\partial x^j} \,. \quad (4.1)$$

If we now make the transformation to the general coordinate $x^i$ where

$$x^{j_*} = x^{j_*}(x) \quad (4.2)$$

the vector $V_i$ undergoes the transformation

(4.3) $$V_i^* = V_j t_i^j$$

where

(4.4) $$t_i^j = \frac{\partial x^j}{\partial x^{i*}} \quad .$$

By an argument* analogous to that given in section 2 we see that

---

* See O. Veblen, "Invariants of quadratic differential forms," Cambridge Tract No. 24, (1927).

---

(4.5) $$V_{i,j}{}^* = \frac{\partial V_i{}^*}{\partial x^{j*}} - V_k{}^* \Gamma_{ij}^{k\,*}$$

where

(4.6) $$\Gamma_{ij}^{k\,*} = T_\ell^k \frac{\partial t_i^\ell}{\partial x^{*j}}$$

and

(4.7) $$T_\ell^k t_i^\ell = \delta_i^k \quad .$$

We note that

(4.8) $$\Gamma_{ij}^{k\,*} = \Gamma_{ji}^{k\,*}$$

as a consequence of equation (4.4).

Hence in a general coordinate system we have

(4.6) $$V_{i,j} = \frac{\partial V_i}{\partial x^j} - V_k \Gamma_{ij}^k \quad .$$

The transformation law for the geometric being with components $\Gamma_{ij}^k$ may be obtained in the same manner as that for $\Lambda_{Ci}^A$ was.

The relation between the quantities $\Gamma_{kj}^i$ and the Christoffel symbols of the space $X_4$ may be obtained from the definition of cartesian coordinate systems in $X_4$. In such a coordinate system the metric tensor $g_{ij}$ is a constant. Hence we have in the cartesian coordinate systems

(4.7) $$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} = 0 \quad .$$

Since $g_{ij,k}$ is a tensor and since it vanishes in one coordinate system, it must vanish in all coordinate systems. Therefore we have

$$\frac{\partial g_{ij}}{\partial x^k} - g_{i\ell}\,\Gamma^{\ell}_{jk} - g_{j\ell}\,\Gamma^{\ell}_{ik} = 0 \quad . \tag{4.8}$$

Equations (4.8) may be considered as the equations relating the quantities $\Gamma^{\ell}_{jk}$ and the derivatives of the tensor $g_{ij}$. Since $\Gamma^{\ell}_{jk} = \Gamma^{\ell}_{kj}$, the only solutions of these equations are

$$\Gamma^{\ell}_{jk} = \left\{ {\ell \atop jk} \right\} = g^{\ell m}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right) \quad . \tag{4.9}$$

Thus the $\Gamma^{i}_{jk}$'s are the Christoffel symbols formed from the $g_{ij}$.

Consider the spinor $g^{i\dot{A}B}$. In a cartesian frame of reference it is a constant; hence we have

$$g^{i\dot{A}B}{}_{,k} = \frac{\partial g^{i\dot{A}B}}{\partial x^k} = 0 \quad . \tag{4.10}$$

If we now change the frame of reference by going over to a general coordinar system in $X_4$, equation (4.10) becomes

$$g^{i\dot{A}B}{}_{,k} = \frac{\partial g^{i\dot{A}B}}{\partial x^k} + g^{j\dot{A}B}\left\{ {i \atop jk} \right\} \quad . \tag{4.11}$$

If we go over to another frame of reference by changing to a general spin coordinate system, we have

$$g^{i\dot{A}B}{}_{,k} = \frac{\partial g^{i\dot{A}B}}{\partial x^k} + g^{j\dot{A}B}\left\{ {i \atop jk} \right\} + g^{i\dot{C}B}\,\bar{\Gamma}^{\dot{A}}_{\dot{C}k} + g^{i\dot{A}C}\,\Gamma^{B}_{Ck} - \frac{1}{2}g^{i\dot{A}B}\,\Gamma^{C}_{Ck} - \frac{1}{2}g^{i\dot{A}B}\,\bar{\Gamma}^{\dot{C}}_{\dot{C}k} \tag{4.12}$$

since $g^{i\dot{A}B}$ is a spinor with weight $+\frac{1}{2}$ and anti-weight $+\frac{1}{2}$.

Since $g^{i\dot{A}B}{}_{,k}$ is a spinor which is zero in the cartesian frame of reference it is zero in any frame of reference. We will use this fact in the next section to obtain relations between the $\Gamma^{A}_{Bk}$ and $\left\{ {i \atop jk} \right\}$.

## RELATIONS BETWEEN $\Gamma^{A}_{Bi}$ and $\left\{ {i \atop jk} \right\}$.

5. Equation (4.12) may be written as

$$g^{i}_{\dot{A}B,k} = \frac{\partial g^{i}_{\dot{A}B}}{\partial x^k} + g^{j}_{\dot{A}B}\left\{ {i \atop kj} \right\} - g^{i}_{\dot{C}B}\,\bar{K}^{\dot{C}}_{\dot{A}k} - g^{i}_{\dot{A}C}\,K^{C}_{Bk} = 0 \tag{5.1}$$

where

$$(5.2) \qquad K^{A}_{Bj} = \Gamma^{A}_{Bj} - \frac{1}{2}\delta^{A}_{B}\Gamma^{C}_{Cj} \quad .$$

The equations (5.1) may be solved for the $K^{A}_{Bj}$ in terms of the $\left\{ {i \atop kj} \right\}$ or vice versa. If we multiply equations (5.1) by $g_i^{\dot{A}E}$ and sum, we have

$$(5.3) \qquad 2K^{E}_{Bk} = g^{j}_{\dot{A}B} \left\{ {i \atop kj} \right\} g_i^{\dot{A}E} + g_i^{\dot{A}E} \frac{\partial g^{i}_{\dot{A}B}}{\partial x^k}$$

since $g_i^{\dot{A}B} g^{i}_{\dot{C}D} = \delta^{\dot{A}}_{\dot{C}} \delta^{B}_{D}$ and since $K^{A}_{A} = 0$. Equations (2.6) express the traceless part of the spin connection in terms of the Christoffel symbols formed from the $g_{ij}$ of the underlying space $X_4$.

If we multiply equations (5.1) by $g_m^{\dot{A}B}$ and sum, we have

$$(5.4) \qquad \left\{ {i \atop mj} \right\} = g_m^{\dot{A}B} \left( g^{i}_{\dot{A}C} K^{C}_{Bj} + g^{i}_{\dot{C}B} \bar{K}^{\dot{C}}_{\dot{A}j} - \frac{\partial g^{i}_{\dot{A}B}}{\partial x^j} \right)$$

since $g_m^{\dot{A}B} g^{i}_{\dot{A}B} = \delta^{i}_{m}$ . The Christoffel symbols of $X_4$ are thus expressed in terms of the traceless part of the spin connection. The former must be real since $X_4$ is a real space. This condition is satisfied since the right-hand side of equation (5.4) is real as a consequence of the fact that $g^{i}_{\dot{A}B}$ is hermitian.

Since equation (5.1) only involves the traceless part of $\Gamma^{A}_{Bi}$, $\Gamma^{C}_{Cj}$ may be completely arbitrary and equations (5.1) are unaffected.

6. In the preceding we have determined the $K^{A}_{Bi}$ in a general spin coordinate system in terms of the unimodular transformation from a cartesian spin coordinate system to the general one. In this section we shall consider the problem of reversing this procedure; that is, we shall assume that we are given the quantities $K^{A}_{Bi}$ in a general spin coordinate system and find the transformation which carries it into a cartesian one.

We first note that our previous definition of a cartesian frame of reference, namely, one in which the tensor $g_{ij}$ and the spinor $g^{i\dot{A}B}$ have constant values*, is equivalent to the following: A cartesian frame of reference is one

* $g^{i\dot{A}B}$ = constants implies $g_{ij}$ = constants of course.

in which both $\left\{ {i \atop jk} \right\}$ and $K^A_{Bi}$ vanish. It is well known that the necessary and sufficient condition for the tensor $g_{ij}$ to be constant in a given coordinate system is that the Christoffel symbols vanish in that coordinate system. From equation (5.3) it follows that if $\left\{ {i \atop jk} \right\} = 0$ and if the $g_{i\dot{A}B}$ are constants, then $K^A_{Bi} = 0$. Conversely if $\left\{ {i \atop jk} \right\} = 0$ and $K^A_{Bi} = 0$, then the $g^{i\dot{A}B}$ are constant. This proves the equivalence of the two definitions of cartesian frames of reference.

In terms of this definition of a cartesian spin coordinate system and the transformation law of the $K^A_{Bj}$ (equation (1.11) where $S^A_B$ are coefficients of unimodular transformations) the problem may be stated as follows: To find the transformation coefficients $s^A_B$ such that

$$K^{B\,*}_{Aj} = S^B_C\left(K^C_{Dj}\, s^D_A + \frac{\partial s^C_A}{\partial x^j}\right) = 0 \tag{6.1}$$

subject to the condition that $|s^A_B| = 1/|S^A_B| \neq 0$. In virtue of this condition equations (6.1) may be written as

$$\frac{\partial s^C_A}{\partial x^j} = -\,K^C_{Dj}\, s^D_A \quad . \tag{6.2}$$

The integrability conditions for these equations are

$$\frac{\partial^2 s^C_A}{\partial x^j \partial x^k} - \frac{\partial^2 s^C_A}{\partial x^k \partial x^j} = -\,B^C_{Ejk}\, s^E_A = 0 \tag{6.3}$$

where

$$B^C_{Ejk} = \left(\frac{\partial K^C_{Ej}}{\partial x^k} - \frac{\partial K^C_{Ek}}{\partial x^j} + K^C_{Dk} K^D_{Ej} - K^C_{Dj} K^D_{Ek}\right) \quad . \tag{6.4}$$

The geometric being with components $B^C_{Ejk}$ is a spinor as may readily be seen from the analogue of the Ricci identity

$$\psi^A_{\ ,ij} - \psi^A_{\ ,ji} = \psi^C B^A_{Cij} \tag{6.5}$$

where the $\psi^A$ is an arbitrary spinor of weight $+\frac{1}{2}$.

Hence the necessary and sufficient conditions which must be satisfied in order to solve equations (6.1) are that the spinor $B^A_{Cij}$ must be zero. Since this spinor is zero in a cartesian frame of reference, it is zero in every frame of reference. Hence we can solve equations (6.1) for the $s_B^A$.

## 7. EXTENSION TO THE GENERAL THEORY OF RELATIVITY

The general theory of relativity differs from the special theory in that the underlying space is a certain general four-dimensional Riemannian manifold instead of a flat space. That is, there exists no coordinate system in which the tensor $g_{ij}$ is constant throughout the space. If we replace the underlying flat space of Chapter II by a more general Riemannian space, everything but the theory of covariant differentiation of spinors is unaffected. We must find a new definition for the covariant derivative of a spinor for the old one is based on the notion of a cartesian frame of reference and in general relativity we cannot introduce such frames.

However, at any point of the underlying space we can introduce a coordinate system in a neighborhood of the point in which the first derivatives of the tensor $g_{ij}$ vanish at this point, i.e. the Christoffel symbols vanish there. Such coordinates are called the geodesic coordinates.* We shall briefly review

---

* Eisenhart, Riemannian Geometry, Princeton University Press, 1926, p. 53.

---

some of the facts about geodesic coordinates.

If the coordinates of an arbitrary point $P_o$ are $x_o^i$ in a coordinate system in which the Christoffel symbols have the values $\left\{ {i \atop jk} \right\}_o$ at the point $x_o^i$, then the series

$$(7.1) \qquad x^i = x_o^i + x'^i + \tfrac{1}{2} c^i_{jk} x'^j x'^k + \frac{1}{3!} c^i_{jk\ell} x'^j x'^k x'^\ell + \ldots;$$

where the c's are arbitrary constants symmetric in the subscripts so chosen so that the series converges in a neighborhood of $x_o^i$, defines a transformation of coordinates. The constants $c^i_{jk}$ may be chosen so that the Christoffel symbols vanish in the primed coordinate system.

From equations (7.1) we have that in the primed coordinate system the point $P_o$ has coordinates $x'^i = 0$. Hence at the point $P_o$ we have

$$(7.2) \qquad \left(\frac{\partial x^i}{\partial x'^j}\right)_o = \delta^i_j \; ; \left(\frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\right)_o = c^i_{\ell m}\, \delta^\ell_j \delta^m_k = c^i_{\ell m} \left(\frac{\partial x^\ell}{\partial x'^j}\right)_o \left(\frac{\partial x^m}{\partial x'^k}\right)_o \; .$$

From the transformation law of the Christoffel symbols we have

$$(7.3) \qquad \left\{{i \atop jk}\right\}'_o = \left(\frac{\partial x'^i}{\partial x^p}\right)_o \left(\left\{{p \atop \ell m}\right\}_o \left(\frac{\partial x^\ell}{\partial x'^j}\right)_o \left(\frac{\partial x^m}{\partial x'^k}\right)_o + \left(\frac{\partial^2 x^p}{\partial x'^j \partial x'^k}\right)_o\right) \; .$$

In virtue of equations (7.2) this may be written as

$$\left\{{i \atop jk}\right\}'_o = \left(\left\{{i \atop jk}\right\}_o + c^i_{jk}\right) \; .$$

Hence a necessary and sufficient condition that $\left\{{i \atop jk}\right\}'_o = 0$ is that $c^i_{jk} = -\left\{{i \atop jk}\right\}_o$. Therefore the equations

$$(7.4) \qquad x^i = x^i_o + x'^i - \frac{1}{2}\left\{{i \atop jk}\right\}_o x'^j x'^k + \frac{1}{3!} c^i_{\ell jk}\, x'^\ell x'^j x'^k + \ldots$$

where the c's are arbitrary constants symmetric in the subscripts define a transformation to a coordinate system in which the first derivatives of the $g_{ij}$ at $P_o$ vanish.

From equations (7.4) we see that to every point in the underlying space there corresponds a family of geodesic coordinates. If we perform a linear transformation on the $x'^i$, such as

$$(7.5) \qquad x''^j = a^j_i x'^i$$

we again get a geodesic coordinate system. In particular we may choose the $a^i_j$ so that the tensor $g_{ij}$ has the values

(7.6)
$$g_{ij} = \begin{Vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

The group of linear homogeneous transformations which carry geodesic coordinate systems of this type into another one of the same type is the Lorentz group.

Another type of transformation which carries a geodesic coordinate system at $P_o$ into another at $P_o$ is the following

(7.7)
$$x''^i = x'^i + F^i(x'^i)$$

where $F^i$ is an arbitrary function of the third order in the $x'^i$, that is,

(7.8)
$$\left(\frac{\partial F^i}{\partial x'^j}\right)_o = \left(\frac{\partial^2 F^i}{\partial x'^j \partial x'^k}\right)_o = 0 \quad .$$

This is evident from the transformation law of the Christoffel symbols. The general transformation which carries geodesic coordinate systems into systems of the same type is a combination of the type (7.7) and (7.5).

When the coordinates $x^i$ are subjected to an arbitrary analytic transformation, the geodesic coordinates at $P_o$ determined by (7.4) undergo a linear transformation with constant coefficients up to terms of third order.

## 8. GEODESIC SPIN COORDINATES

If at a point $P_o$ of the underlying space we introduce a geodesic coordinate system, we may then pick a spin coordinate system in which the first derivatives of the spinor $g_{i\dot{A}B}$ vanish at $P_o$. We recall that the spinor $g_{i\dot{A}B}$ satisfies the relations

(8.1)
$$\bar{g}_{i\dot{A}B}\, g_j^{\;\dot{B}C} + \bar{g}_{j\dot{A}B}\, g_i^{\;\dot{B}C} = g_{ij}\, \delta_A^C$$

where the $g_{ij}$ are the components of the metric tensor of the underlying space.

From these equations it is evident that if the derivatives of the spinor $g_{i\dot{A}B}$ vanished throughout the space, then the derivatives of the $g_{ij}$ would vanish, which is impossible. Hence we must expect that the spin coordinate system can at most be so chosen that the first derivatives of the spinor $g_{i\dot{A}B}$ vanish at a point.

The differential equations which determine the unimodular spin transformation coefficients are

(8.2) $$\frac{\partial g_{i\dot{A}B}{}^{*}}{\partial x^j} = \frac{\partial}{\partial x^j} \left( g_{i\dot{C}D} \bar{s}^C_A s^D_B \right) = 0 \; .$$

That is,

(8.3) $$\frac{\partial g_{i\dot{C}D}}{\partial x^j} \bar{s}^C_A s^D_B + g_{i\dot{C}D} \frac{\partial \bar{s}^C_A}{\partial x^j} s^D_B + g_{i\dot{C}D} \bar{s}^C_A \frac{\partial s^D_B}{\partial x^j} = 0 \; .$$

Equations (8.3) may be written as

(8.4) $$\frac{\partial g_{i\dot{F}E}}{\partial x^j} + g_{i\dot{C}E} \frac{\partial \bar{s}^C_A}{\partial x^j} \bar{S}^A_F + g_{i\dot{F}D} \frac{\partial s^D_B}{\partial x^j} S^B_E = 0$$

where $S^A_B$ is such that

(8.5) $$S^A_B \, s^B_C = \delta^A_C \; .$$

Multiplying by $g^{i\dot{F}N}$ and summing, we have

(8.6) $$g^{i\dot{F}N} \frac{\partial g_{i\dot{F}E}}{\partial x^j} + \delta^N_E \frac{\partial \bar{s}^C_A}{\partial x^j} \bar{S}^A_C + 2 \frac{\partial s^N_B}{\partial x^j} S^B_E = 0 \; .$$

Since

(8.7) $$\frac{\partial \bar{s}^C_A}{\partial x^j} \bar{S}^A_C = \frac{\partial \log |\bar{s}|}{\partial x^j} = 0$$

in virtue of the fact that $|s| = 1$, equations (8.6) may be written as

(8.8) $$\frac{\partial s^N_B}{\partial x^j} = -\frac{1}{2} \left( g^{i\dot{F}N} \frac{\partial g_{i\dot{F}E}}{\partial x^j} \right) s^E_B \; .$$

If we set

(8.9) $$\frac{1}{2} g^{i\dot{F}N} \frac{\partial g_{i\dot{F}E}}{\partial x^j} = K^N_{Ej}$$

these equations (8.8) become

(8.10) $$\frac{\partial s^N_B}{\partial x^j} = - K^N_{Ej} s^E_B \quad .$$

The integrability conditions for these equations have been considered in the previous section. In section 10 we shall show that they are satisfied if and only if the underlying space is a flat space. Hence in the general case there exists no spin coordinate system in which the derivatives of the spinor $g^i_{\dot{A}B}$ vanish.

Nevertheless equations (8.10) may be used to define a spin coordinate system in which the first derivatives of the spinor $g^i_{\dot{A}B}$ vanish at the origin of the geodesic coordinate system. If we start with an arbitrary spin coordinate system and define the $K^A_{Bj}$ by (8.9), then a set of transformation coefficients $s^A_B$ are given by the power series

(8.11) $$s^A_B = s^A_{oB} - K^A_{Ej} s^E_{oB} x^j + C^A_{Ejk} s^E_{oB} x^j x^k + \ldots$$

where $s^A_{oB}$ are constants and the coefficients of the second and higher order terms are arbitrary. Spin coordinate systems in which the derivatives of the spinor $g^i_{\dot{A}B}$ vanish at the origin of geodesic coordinates will be called geodesic spin coordinate systems.

Just as in the case of the geodesic coordinate systems of the underlying space, at every point $P_o$ there exists a family of geodesic spin coordinate systems. The members of a family of geodesic spin coordinate systems are permuted among themselves by transformations of the type

(8.12) $$t^A_B = \ell^A_E \left( \delta^E_C + F^E_C(x) \right)$$

where $\ell^A_E$ is a constant spin transformation and $F^E_C$ is of second or higher order in the x's, that is

(8.13) $$F_o = \left(\frac{\partial F_C^E}{\partial x^j}\right)_o = 0 \quad .$$

The transformation (8.12) induces the linear homogeneous transformation of the components of a spinor $\psi_{EF}$ (e.g.) at the point $P_o$

(8.14) $$(\psi_{\mathring{A}B})_o = \ell_A^E \, \ell_B^F \, (\psi_{EF})_o \quad .$$

We can find a geodesic spin coordinate system in which the spinor $g^{i\mathring{A}B}$ has the values given in §4, Chapter I, at the point $P_o$, namely

(8.15) $$\left\| g^{1\mathring{A}B} \right\| = \frac{1}{\sqrt{2}} \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\| \qquad \left\| g^{2\mathring{A}B} \right\| = \frac{1}{\sqrt{2}} \left\| \begin{matrix} 0 & -i \\ i & 0 \end{matrix} \right\|$$
$$\left\| g^{3\mathring{A}B} \right\| = \frac{1}{\sqrt{2}} \left\| \begin{matrix} 0 & -i \\ i & 0 \end{matrix} \right\| \qquad \left\| g^{4\mathring{A}B} \right\| = \frac{1}{\sqrt{2}} \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\|$$

A frame of reference defined by a geodesic coordinate system of the underlying space and a geodesic spin coordinate system will be called a geodesic frame of reference. A geodesic frame of reference in which the $g_{ij}$ have the form (7.6) and the $g^{i\mathring{A}B}$ the form (8.15) will be called a normal geodesic frame of reference

A transformation of the type of the combination of (7.5) and (7.7) in the underlying space and one of the type (8.12) in spin space induces the following transformation on the components of the spinor $g^i_{\mathring{A}B}$ at the point $P_o$:

(8.16) $$(g^i_{\mathring{A}B})^*_o = a^i_j \, \bar{\ell}^C_A \, \ell^D_B \, (g^j{}_{\mathring{C}D})_o \quad .$$

If we now require that

(8.17) $$(g^i_{\mathring{A}B})^*_o = (g^i_{\mathring{A}B})_o$$

equation (8.16) defines an isomorphism between a linear group in the tangent space at $P_o$ and a group in the spin space.

## 9. DEFINITION OF THE COVARIANT DERIVATIVE OF A SPINOR

The covariant derivative of a spinor is a spinor with one additional tensor index which reduces to the ordinary derivative at the origin of a geodesic frame of reference when evaluated in that frame. Thus

$$(\psi_{A,i})_o = \left(\frac{\partial\psi_A}{\partial x^i}\right)_o \quad . \tag{9.1}$$

By the argument of §8, Chapter II, it is evident that $\psi_{A,i}$ transforms as a vector under arbitrary coordinate transformations and as a spinor under arbitrary spin transformations.

If $\psi_A$ undergoes the unimodular spin transformation

$$\psi_A^* = s_A^B\,\psi_B \tag{9.2}$$

then by definition

$$(\psi_{A,i})_o^* = (s_A^B\,\psi_{B,i})_o \quad . \tag{9.3}$$

Since

$$\left(\frac{\partial\psi_A^*}{\partial x^i}\right)_o = \left(\frac{\partial\psi_B}{\partial x^j}\,s_A^B + \psi_B\,\frac{\partial s_A^B}{\partial x^j}\right)_o = \left(\frac{\partial\psi_B}{\partial x^j}\,s_A^B + S_B^C\,\psi_C^*\,\frac{\partial s_A^B}{\partial x^j}\right)_o \tag{9.4}$$

then

$$\begin{aligned}(s_A^B)_o\,(\psi_{B,i})_o = \left(\frac{\partial\psi_B}{\partial x^j}\right)_o (s_A^B)_o &= \left(\frac{\partial\psi_A^*}{\partial x^j} - \frac{\partial s_A^B}{\partial x^j}\,S_B^C\,\psi_C^*\right)_o \\ &= \left(\frac{\partial\psi_A^*}{\partial x^j} + s_A^B\,\frac{\partial S_B^C}{\partial x^j}\,\psi_C^*\right)_o \quad .\end{aligned} \tag{9.5}$$

If $s_B^A$ are the transformation coefficients from a geodesic spin coordinate system to an arbitrary one in which the $K_{Bj}^A$ defined by equations (8.9) have the values $(K_{Bj}^A)_o$ at $P_o$, then $S_B^C$ are the transformation coefficients of the inverse transformation and equation (8.10) applies to them. Thus

(9.6) $$\left(\frac{\partial s_B^C}{\partial x^j}\right)_o = -K_{Ejo}^C \; s_{Bo}^E \quad .$$

Equations (9.5) then become

(9.7) $$(s_A^B)_o \, (\psi_{B,i})_o = \left(\frac{\partial \psi_A^*}{\partial x^j} - K_{Aj}^C \, \psi_C^*\right)_o = (\psi_{A,j}^*)_o \quad .$$

Since $P_o$ was an arbitrary point, we have that at any point of the underlying space in an arbitrary spin-coordinate system

(9.8) $$\psi_{A,j} = \frac{\partial \psi_A}{\partial x^j} - K_{Aj}^C \, \psi_C \quad .$$

The covariant derivative of the spinor $g_{\dot{A}B}^i$ is zero; for its value at any point $P_o$ in the underlying space in a geodesic frame of reference is

(9.9) $$(g_{\dot{A}B,j}^i)_o = \left(\frac{\partial g_{\dot{A}B}^i}{\partial x^j}\right)_o$$

and by the definition of a geodesic frame of reference the right member vanishes. By the argument of section 4 of this chapter, we have that in a general frame of reference and at any point x

(9.10) $$g_{\dot{A}B,j}^i = \frac{\partial g_{\dot{A}B}^i}{\partial x^j} + g_{\dot{A}B}^k \left\{ {i \atop kj} \right\} - g_{\dot{C}B}^i \, \overline{K}_{Aj}^C - g_{\dot{A}C}^i \, K_{Bj}^C = 0 \quad .$$

These equations may be solved for the $K_{Bi}^A$ in terms of the $\left\{ {i \atop jk} \right\}$ and the derivatives of the $g_{\dot{A}B}^i$ and we again obtain equation (5.3). Equations (5.4) may also be obtained from (9.10). It is readily seen that equations (5.3) reduce to equations (8.10) at $P_o$ in a frame of reference in which the coordinate system of the underlying space is a geodesic one at $P_o$.

With the definition of the covariant derivative of a spinor given in this section all the results of the preceding sections of this chapter and of Chapter II may be taken over to general relativity.

## 10. RELATIONS BETWEEN THE CURVATURE TENSOR AND THE CURVATURE SPINOR

In Riemannian geometry it is well known that the vanishing of the Riemann Christoffel tensor $R^{i}_{jk\ell}$ where*

---

* In the notation of Eisenhart, op. cit.

---

$$(10.1)\qquad R^{\ell}_{ijk} = \frac{\partial}{\partial x^{j}}\begin{Bmatrix}\ell\\ ik\end{Bmatrix} - \frac{\partial}{\partial x^{k}}\begin{Bmatrix}\ell\\ ij\end{Bmatrix} + \begin{Bmatrix}m\\ ik\end{Bmatrix}\begin{Bmatrix}\ell\\ mj\end{Bmatrix} - \begin{Bmatrix}m\\ ij\end{Bmatrix}\begin{Bmatrix}\ell\\ mk\end{Bmatrix} .$$

is the necessary and sufficient condition that there exist a coordinate system in which the tensor $g_{ij}$ has constant values. From the results of the section 6 we see that the necessary and sufficient condition that there exist a coordinate system in which the $K^{A}_{Bi}$ vanish is that the spinor $B^{A}_{Bij}$ vanish. If both $R_{ijk\ell}$ and $B^{A}_{Bij}$ vanish, then from equations (5.3) it follows that there exists a cartesian frame of reference.

However, from relations between the $g_{ij}$ and the $g^{i\dot{A}B}$ it is to be expected that these two conditions are not independent. In fact, we will show that the vanishing of $R_{ijk\ell}$ implies the vanishing of $B^{A}_{Bij}$ and conversely. We will then have the result: The necessary and sufficient condition for the existence of a cartesian frame of reference in that $R_{ijk\ell} = 0$ (or $B^{A}_{Bij} = 0$).

The relations between the tensor $R_{ijk\ell}$ and the spinor $B^{A}_{Bj\ell}$ may be obtained from the following generalization of the Ricci identity:

$$(10.2)\qquad g^{i\dot{A}B}_{\ ,k\ell} - g^{i\dot{A}B}_{\ ,\ell k} = -\, g^{m\dot{A}B} R^{i}_{mk\ell} - g^{i\dot{A}E} B^{B}_{Ek\ell} - g^{i\dot{E}B} \bar{B}^{A}_{Ek\ell} .$$

Since the covariant derivative of the spinor $g^{i\dot{A}B}$ vanishes, we have

$$(10.3)\qquad g^{m\dot{A}B} R^{i}_{mk\ell} = -\, g^{i\dot{A}E} B^{B}_{Ek\ell} - g^{i\dot{E}B} \bar{B}^{A}_{Ek\ell} .$$

If we multiply equation (10.3) by $g_{n\dot{A}B}$ and sum, we have

$$R^{i}_{nk\ell} = - g_{n\dot{A}B} \left( g^{i\dot{A}E} B^{B}_{Ek\ell} + g^{i\dot{E}B} \bar{B}^{A}_{Ek\ell} \right) \tag{10.4}$$

since $g_{n\dot{A}B} \, g^{m\dot{A}B} = \delta^{m}_{n}$ .

From equations (15.5) of Chapter I we have

$$g_{n\dot{A}B} \, g^{i\dot{A}E} = \bar{g}_{n\dot{B}A} \, g^{i\dot{A}E} = s^{i\ E}_{nB} + \frac{1}{2} \delta^{i}_{n} \delta^{E}_{B}$$

where

$$s^{i\ E}_{nB} = \frac{1}{2}\left( \bar{g}_{n\dot{B}A} \, g^{i\dot{A}E} - \bar{g}^{i}_{\dot{B}A} \, g_{n}^{\ \dot{A}E} \right) \ .$$

Similarly

$$g_{n\dot{A}B} \, g^{i\dot{E}B} = s^{i\ \dot{E}}_{n\dot{A}} + \frac{1}{2} \delta^{i}_{n} \delta^{\dot{E}}_{\dot{A}} = \bar{s}^{i\ E}_{nA} + \frac{1}{2} \delta^{i}_{n} \delta^{\dot{E}}_{\dot{A}} \ .$$

Hence equation (10.4) may be written as

$$R^{i}_{nk\ell} = - s^{i\ E}_{nB} B^{B}_{Ek\ell} - \bar{s}^{i\ E}_{nA} \bar{B}^{A}_{Ek\ell} \tag{10.5}$$

in virtue of the fact that $B^{A}_{Ak\ell} = 0$.

Thus we have

$$-R_{ink\ell} = s_{inB}^{\ \ \ \ E} B^{B}_{Ek\ell} + \bar{s}_{inB}^{\ \ \ \ E} \bar{B}^{B}_{Ek\ell} \ . \tag{10.6}$$

That is, $R_{ink\ell}$ is automatically real and has the proper symmetry properties.

Equations (10.3) may be solved for the $B^{A}_{Ek\ell}$ in terms of the $R^{i}_{mk\ell}$ as follows. Multiply equation (10.3) by $g_{i\dot{M}N}$ and sum on i. Then

$$\begin{aligned} g_{i\dot{M}N} \, g^{m\dot{A}B} R^{i}_{mk\ell} &= - g_{i\dot{M}N} \left( g^{i\dot{A}E} B^{B}_{Ek\ell} + g^{i\dot{E}B} \bar{B}^{A}_{Ek\ell} \right) \\ &= - \left( \delta^{\dot{A}}_{\dot{M}} B^{B}_{Nk\ell} + \delta^{B}_{N} \bar{B}^{A}_{Mk\ell} \right) \end{aligned} \tag{10.7}$$

since

$$g_{i\dot{M}N}\, g^{i\dot{A}B} = \delta^{\dot{A}}_{\dot{M}}\, \delta^{B}_{N} \quad .$$

Setting A = M and summing, we have

$$g_{i\dot{A}N}\, g^{m\dot{A}B}\, R^{i}_{mk\ell} = -\, 2\, B^{B}_{Nk\ell} \tag{10.8}$$

since $B^{A}_{Ak\ell} = 0$. Equation (10.8) may be written as

$$B^{B}_{Nk\ell} = -\,\frac{1}{2}\, s_{i}{}^{m}{}_{N}{}^{B}\, R^{i}{}_{mk\ell} \,. \tag{10.9}$$

From equations (10.6) and (10.9) it is evident that the vanishing of $B^{A}_{Bk\ell}$ implies the vanishing of $R^{i}_{jk\ell}$ and conversely.

## 11. DIRAC EQUATIONS

The Dirac equations in general relativity must satisfy the following requirements. They must be invariant under arbitrary coordinate, gauge and spin transformations, be first order in the time derivative of the wave function, and must reduce to the Dirac equations of special relativity in case the underlying space is a flat space.

The simplest equations which satisfy these conditions are obtained from the Dirac equations in special relativity (Chapter II, equations (10.1) and (10.2)) by replacing ordinary differentiation by covariant differentiation. Thus we have for the two-component form of the Dirac equations in general relativity:

$$g^{j\dot{A}B}\left(\psi_{B,j} - \frac{2\pi i}{h}\,\frac{e}{C}\,\varphi_j\,\psi_B\right) = \sqrt{2}\,\frac{\pi m C}{h}\, i\, \bar{\chi}^{A} \tag{11.1}$$

and

$$g^{j\dot{A}B}\left(\chi_{B,j} + \frac{2\pi i}{h}\,\frac{e}{C}\,\varphi_j\,\chi_B\right) = \sqrt{2}\,\frac{\pi m C}{h}\, i\, \bar{\psi}^{A} \quad . \tag{11.2}$$

These equations may be written in the four-component form just as equations (10.1) and (10.2) of Chapter II. The equations thus obtained are included in the class of Dirac equations discussed in the paper "The Dirac Equation in Projective Relativity".*

---

* The Dirac equation in projective relativity, A. H. Taub, O. Veblen, and J. von Neumann, Proc. Nat. Acad. Sci., vol. 20 (1934), pp. 383-388.

References to other papers on the Dirac equation in general relativity and covariant differentiation of spinors are:

H. Weyl, Gravitation and the electron, Proc.Nat.Acad.Sci. vol. 15 (1929), 323
Electron and gravitation, Zeit. Phys. vol. 56 (1929), 330

B. L. Van der Waerden, Spinoranalyse, Göttinger Nachrichten, vol. 100 (1929)

W. Infeld and B. L. Van der Waerden, Die Wellungleichung des Electrons in der allgemeinen Relativitätstheorie, Berlin Berich., vol. 9 (1933), 380

V. Fock, Geometrisierung der Diracschen Théorie des Electrons, Zeitsch. Phys. vol. 57 (1929), 261

J. A. Schouten, Dirac equations in general relativity, 1. Four dimensional theory, 2. Five dimensional theory, Jour. Math. and Phys. vol. 10 (1931), 239

E. Schrödinger, Diracsches Electron im Schwerefeld 1, Sitzungsberichte der Preussischen Akademie der Wissenschaften, Phys.-Math. Klasse, 11 (1932), 105-128

W. Pauli and J. Solomon, La théorie unitaire d'Einstein et Mayer et les équations de Dirac I and II, Journ. de Phys. et la Radium, vol. 3, series 7, (1932), 452, 582

W. Pauli, Die Diracschen Gleichungen für die Materiewellen, Ann. der Phys., vol. 18 (1933), 337

O. Veblen and A. H. Taub, Projective differentiation of spinors, Proc. Nat. Acad. Sci., vol. 20 (1934), 85-92

# Chapter IV

# PROJECTIVE GEOMETRY OF (k-1)-DIMENSIONS

## 1. DEFINITION OF PROJECTIVE SPACES

The structure of a projective (k-1)-space $P_{k-1}$, is defined by a certain class of preferred coordinate systems which are related by linear homogeneous transformations of coordinates. In any one of these coordinate systems any ordered set of k numbers $(X^1, X^2, \ldots, X^k)$, not all zero, corresponds to one and only one point. Conversely, each point of the space corresponds to at least one ordered set of numbers $(X^1, X^2, \ldots, X^k)$ and also to all sets $(\rho X^1, \rho X^2, \ldots, \rho X^k)$ where $\rho$ is any number different from zero, but not to any set $(Y^1, Y^2, \ldots, Y^k)$ unless $Y^A = \rho X^A$.

The space $P_{k-1}$ is a real projective space if the numbers $X^A$, $Y^A$ and $\rho$ in the above statements are real. If they are complex the space $P_{k-1}$ is a complex projective space. Topologically a complex $P_{k-1}$ is a (2k-2)-dimensional manifold. The quantities $X^A$ could also be taken to be the elements of an arbitrary field and then $P_{k-1}$ would be a projective space in the broad sense*.

*Cf. Vol. 1 of Veblen and Young, Projective Geometry.

In a complex $P_{k-1}$, if the coordinates of a point are $X^A$ in one preferred coordinate system and $X^{A*}$ in another, then

$$X^{A*} = T^A_B X^B \tag{1.1}$$

where $T^A_B$ are complex numbers subject only to the condition that the determinant of the matrix

$$T = \begin{Vmatrix} T^1_1 & T^1_2 & \cdots & T^1_k \\ T^2_1 & T^2_2 & \cdots & T^2_k \\ \\ T^k_1 & T^k_2 & \cdots & T^k_k \end{Vmatrix} \qquad (1.2)$$

is not zero. These equations effect a transformation of coordinate $X \rightarrow X^*$ which is a renaming of the points of $P_{k-1}$. The inverse transformation is

$$X^A = t^A_B X^{B*} \qquad (1.3)$$

where

$$t^A_B T^B_C = \delta^A_C = \begin{cases} 0 \text{ if } A \neq C \\ 1 \text{ if } A = C \end{cases} \qquad (1.4)$$

That is, the matrix $\|t^A_B\|$ is the inverse of $\|T^A_B\|$. We shall denote the determinant of this matrix by t.

Since the $X^A$ are homogeneous coordinates in $P_{k-1}$, the transformation (1.1) and the transformation

$$X^{A*} = U^A_B X^B \qquad (1.5)$$

effect the same renaming of the points of $P_{k-1}$ if their coefficients are proportional and therefore determine the same transformation of homogeneous coordinates. Furthermore all transformations of the type (1.1) with proportional coefficients determine the same unimodular transformations

$$X^{A*} = S^A_B X^B \tag{1.6}$$

where

$$S^A_B = T^A_B t^{1/k} \quad . \tag{1.7}$$

There are k of these transformations corresponding to the k values of $t^{1/k}$. Thus the group of all coordinate transformations is $(1 \rightarrow k)$ isomorphic with the unimodular matrix group.

The inverse of the transformation (1.6) is

$$X^A = s^A_B X^{B*} \tag{1.8}$$

where

$$s^A_B = t^A_B t^{-1/k} \text{ and hence } s = \left| s^A_B \right| = 1. \tag{1.9}$$

In this chapter the discussion of coordinate transformations will be made in terms of unimodular matrices.

## 2. HYPERPLANES

The points on the hyperplane whose homogeneous coordinates are $\psi_A$ satisfy the equation

$$\psi_A X^A = 0 \quad . \tag{2.1}$$

Under the transformation (1.6) the coordinates of the points on this hyperplane are transformed into coordinates satisfying

(2.2) $$\psi_A{}^{*} x^{A*} = 0 \quad ,$$

where

(2.3) $$\psi_A{}^{*} = s_A^B \psi_B \quad .$$

That is, under a transformation of coordinates the homogeneous coordinates $\psi_A$ of a hyperplane transform contragrediently to the homogeneous coordinates of points.

The homogeneous coordinates of the hyperplane passing through the k-1 independent points $x_1^A, x_2^A, \ldots, x_{k-1}^A$, are the (k-1)-rowed determinants of the matrix

(2.4) $$\left\| \begin{matrix} x_1^1 & x_2^1 \ldots & x_{k-1}^1 \\ x_1^2 & x_2^2 \ldots & x_{k-1}^2 \\ \\ x_1^k & x_2^k \ldots & x_{k-1}^k \end{matrix} \right\| \quad .$$

An explicit formula for them is

(2.5) $$\psi_A = \varepsilon_{ABC\ldots D} x_1^B x_2^C \ldots x_{k-1}^D$$

where $\varepsilon_{ABC\ldots D}$ is +1 if (A, B, C, ..., D) is an even permutation of

(1, 2, 3, ..., k), -1 if it is an odd permutation, and zero otherwise. From the formula for the expansion of a determinant we have

(2.6) $$s\,\varepsilon_{ABC\ldots D} = \varepsilon_{EFG\ldots H}s^E_A s^F_B s^G_C \ldots s^H_D \quad .$$

Hence the formula (2.5) is unaffected by coordinate transformations.

From the symmetry properties of $\varepsilon_{AB\ldots D}$, it is evident that

(2.7) $$\psi_A X^A_i = 0 \qquad\qquad i = 1, 2, \ldots, k-1 .$$

Thus the points on the hyperplane are those linearly dependent on $X^A_1, \ldots, X^A_{k-1}$.

Similarly the homogeneous coordinates of the point determined by the k-1 independent planes $\psi_{1A}, \psi_{2A}, \ldots \psi_{k-1A}$ are

(2.8) $$X^A = \varepsilon^{ABC\ldots D}\psi_{1B}\psi_{2C} \ldots \psi_{k-1D}$$

where $\varepsilon^{ABC\ldots D}$ is +1 or -1 according as (A, B, C, ..., D) is an even or an odd permutation of (1, 2, ..., k) and zero otherwise. Again from the formula for the expansion of a determinant, we have in a new coordinate system

(2.9) $$\varepsilon^{AB\ \ldots D*} = \varepsilon^{AB\ \ldots D} = s\varepsilon^{EF\ldots G}s^A_E s^B_F \ldots s^D_G \quad .$$

Therefore equation (2.8) is unaffected by coordinate transformations.

3. COORDINATES OF LINEAR SUBSPACES

The homogeneous coordinates of a point $Y^A$ in $P_{k-1}$ satisfy the set of equations

(3.1) $$Y_{ABC\ldots D}X^D = 0$$

where $Y_{ABC\ldots D}$ has k-1 indices and is defined by the equations

(3.2) $$Y_{ABC\ldots D} = \varepsilon_{ABC\ldots DE}Y^E \quad .$$

That is, the quantities $Y_{AB\ldots D}$ are just a renumbering of the coordinates $Y^E$ with appropriate plus and minus signs. If we write $Y_{AB\ldots D}$ as a matrix with the different combinations of the first k-2 indices labeling the rows, and the last index labeling the columns, the matrix will be of rank k-1. Thus the homogeneous coordinates $X^D$ are determined uniquely by equations (3.1).

The quantities $Y_{AB\ldots D}$ are called the covariant coordinates of the point $Y^A$.

Similarly the quantities

(3.2) $$\psi^{AB\ldots D} = \varepsilon^{ABC\ldots DE}\psi_E$$

where $\psi^{AB\ldots D}$ has k-1 indices. These are the contravariant coordinates of the hyperplane $\psi_E$. Also the homogeneous coordinates of the hyperplane are uniquely determined as solutions of the equations

(3.3) $$\psi^{AB\ldots D}\theta_D = 0 \quad .$$

The existence of the covariant coordinates of points enables us to express some aspects of the principle of duality analytically. Thus the covariant coordinates of a point may be expressed in terms of the coordinates of k-1 hyperplanes on the point as follows:

$$X_{AB\ldots C} = \delta^{EF\ldots G}_{AB\ldots C}\, \psi_{1E}\psi_{2F}\ldots\psi_{k-1G}$$

where* $\delta^{EF\ldots G}_{AB\ldots C}$ is the generalized Kronecker delta and has k-1 subscripts and

---

*For general formulas involving the $\varepsilon$'s and $\delta$'s see Veblen, Invariants of Quadratic Differential Forms, Cambridge Tract, p. 9.

---

k-1 superscripts; the value of the symbol is zero unless both sets of numbers are the same, then it has the value +1 or -1 according as an even or an odd permutation is required to arrange the superscripts in the same order as the subscripts. This follows from (2.8) and (3.2) and the identity

$$\varepsilon_{A\ldots BE}\,\varepsilon^{C\ldots DE} = \delta^{C\ldots D}_{A\ldots B} \quad .$$

The homogeneous covariant coordinates of the m-dimensional linear subspace spanned by the m+1 independent points $X_1^A$, $X_2^A$, ..., $X_{m+1}^A$, are the (m+1)-rowed determinants of the matrix

$$\left\|\begin{matrix} X_1^1 & X_2^1 \ldots & X_{m+1}^1 \\ X_1^2 & X_2^2 \ldots & X_{m+1}^2 \\ \\ X_1^k & X_2^k \ldots & X_{m+1}^k \end{matrix}\right\| \quad . \tag{3.5}$$

An explicit formula for them is

$$X_{AB\ldots C} = \varepsilon_{AB\ldots CDE\ldots F}\, X_1^D X_2^E \ldots X_{m+1}^F \tag{3.6}$$

where $X_{AB\ldots C}$ has k-m-1 indices and is antisymmetric in all of them. From the antisymmetry properties of the $\varepsilon_{AB\ldots F}$ it is evident that

$$X_{AB\ldots C}X^{C} = 0 \tag{3.7}$$

is satisfied by $X_i$ ($i = 1, \ldots, m+1$). Hence also the points linearly dependent on the $X_i^A$ satisfy the equations (3.7). The m-dimensional subspace is determined by the equations (3.7).

The contravariant coordinates of the m-dimensional subspace are

$$X^{AB\ldots C} = \frac{1}{(k-m-1)!}\,\varepsilon^{AB\ldots CDE\ldots F}X_{DE\ldots F} \quad . \tag{3.8}$$

They have m+1 indices and are antisymmetric in all of them.

The necessary and sufficient condition that an arbitrary quantity $X^{AB\ldots C}$ with m+1 antisymmetric indices be the coordinates of a linear subspace is

$$X^{AB\ldots C}X_{CD\ldots E} = 0 \tag{3.9}$$

where $X_{CD\ldots E}$ is obtained from $X^{AB\ldots C}$ by means of the equations

$$X_{CD\ldots E} = \frac{1}{(m+1)!}\,\varepsilon_{CD\ldots EFG\ldots H}X^{FG\ldots H} \tag{3.10}$$

and is the covariant coordinate of the subspace.

This theorem is a generalization of the theorem proved in § 8, Chapter I, and is proved in the same manner.

A line is a simple example of a linear subspace. It is a one-dimensional space determined by two points. The homogeneous contravariant coordinates of the line determined by the two points $X_1^A$ and $X_2^A$ are

$$X^{AB} = (X_1^A X_2^B - X_1^B X_2^A) \quad . \tag{3.11}$$

From (3.11) it is evident that if we replace $X_1^A$ by $X_1^A + \mu X_2^A$, $X^{AB}$ remains unaltered. Since a similar result holds for $X_2$, we see that any two points on the line determine the same homogeneous coordinates of that line. The covariant coordinates of a line are obtained from the quantities $X^{AB}$ by means of the $\mathcal{E}_{AB\ldots C}$.

The linear subspace determined by the intersection of the two hyperplanes $\psi_{1A}$ and $\psi_{2A}$ has the covariant coordinates

$$\psi_{AB} = (\psi_{1A}\psi_{2B} - \psi_{1B}\psi_{2A}) \quad . \tag{3.12}$$

The contravariant coordinates of this subspace are obtained by means of the $\mathcal{E}^{AB\ldots F}$.

In $P_3$ the intersection of two planes is again a line. If the two planes $\psi_{1A}$ and $\psi_{2A}$ contain the line $X_1^A X_2^A$, the $\psi_{AB}$ are proportional to the covariant coordinates of $X^{AB}$. That is

$$X_{AB} = \tfrac{1}{2}\mathcal{E}_{ABCD}X^{CD} = \rho\,\psi_{AB} \quad . \tag{3.13}$$

## 4. ANTIPROJECTIVE GROUP

A geometric transformation is a permutation of the elements (points, hyperplanes, lines, etc.) of $P_{k-1}$. It should be sharply distinguished from a coordinate transformation which simply renames them. Geometric transformations $x^A \to y^A$, of the types

$$(4.1) \qquad y^A = P^A_{\ B} x^B,$$

and

$$(4.2) \qquad \bar{y}^A = \dot{P}^A_{\ B} x^B,$$

where the bar denotes the complex conjugate, are called collineations and anticollineations respectively. Both transformations carry points into points, and linear subspaces into linear subspaces of the same kind, but in the latter transformation the complex conjugates of the coordinates appear and the transformation is said to be anti-linear.

Under the collineation (4.1) the points on a hyperplane

$$(4.3) \qquad \xi_A x^A = 0$$

go into the points on the hyperplane

$$(4.4) \qquad \eta_A y^A = 0$$

where

$$(4.5) \qquad \eta_A = P_A^{\ B} \xi_B$$

and

$$(4.6) \qquad P_A^{\ C} P^A_{\ B} = \delta^C_B = P_B^{\ A} P^C_{\ A} \quad .$$

Here we must assume that the determinant of the matrix $\| P^A_{\ B} \|$ is not zero. That is, the collineation (4.1) is non-singular.

Equation (4.5) represents the transformation induced on the hyperplanes by the transformation (4.1).

Similarly the transformation induced on the hyperplanes by the anticollineation (4.2) is

(4.7) $$\bar{\eta}_A = P_A^{\cdot B} \xi_B$$

where

(4.8) $$P_{\dot{A}}^{\cdot B} P^{\dot{A}}_{\ C} = \delta^B_C = \delta^{\dot{B}}_{\dot{C}} = P^{\dot{B}}_{\ A} P_C^{\cdot A} \quad .$$

Again we assume that the determinant of the matrix $\| P^{\dot{A}}_{\ B} \|$ is not zero.

The product of the collineation (4.1) and the collineation

(4.9) $$z^A = Q^A_{\ B} y^B$$

is the collineation

(4.10) $$z^A = Q^A_{\ B} P^B_{\ C} x^C \quad .$$

Hence the set of all non-singular collineations form a group, the collineation group in $P_{k-1}$.

The product of the anticollineation (4.2) and the anticollineation

(4.11) $$\bar{z}^A = Q^{\dot{A}}_{\ B} y^B$$

is the transformation

(4.12) $$z^A = \bar{Q}^{\dot{A}}_{\ B} P^{\dot{B}}_{\ C} x^C$$

which is a collineation. Similarly the product of the collineation (4.9) and the anticollineation (4.2) is the transformation

(4.13) $$\bar{z}^A = \bar{Q}^A_{\ B} P^{\dot{B}}_{\ C} x^C$$

which is an anticollineation. The product of (4.2) and (4.9) is also an anticollineation. Hence the set of all non-singular collineations and anti-collineations form a group. This is called the anticollineation group in $P_{k-1}$. The anticollineations alone, however, do not form a group.

The transformations

$$\xi_A = P_{AB}X^B \tag{4.14}$$

and

$$\bar{\xi}_A = P_{\dot{A}B}X^B \tag{4.15}$$

are called correlations and anticorrelations respectively. Both transformations carry points into hyperplanes, but the latter transformation is anti-linear. The transformations are non-singular if the determinants of the matrices $\| P_{AB} \|$ and $\| P_{\dot{A}B} \|$ are different from zero.

The transformations induced on the hyperplanes by the non-singular correlation (4.14) and the non-singular anticorrelation (4.15) are

$$X^A = P^{AB}\xi_B \tag{4.16}$$

and

$$\bar{X}^A = P^{\dot{A}B}\xi_B \tag{4.17}$$

respectively, where

$$P^{AB}P_{AC} = P^{BA}P_{CA} = \delta^B_C \tag{4.18}$$

and

$$P^{\dot{A}B}P_{\dot{A}C} = \delta^B_C = \delta^{\dot{B}}_{\dot{C}} = P^{\dot{B}A}P_{\dot{C}A} \quad . \tag{4.19}$$

Correlations and anticorrelations may be described as transformations which carry hyperplanes into points as well as transformations which carry points into hyperplanes.

In a $P_1$ the hyperplanes are again points. Hence in this case correlations and anticorrelations carry points into points and are indistinguishable from collineations and anticollineations respectively.

In $P_3$ the points on a line are carried into planes on a line by transformations of these types. Hence in a three-dimensional space they are transformations which permute the lines of this space.

The product of the two correlations, $X^A \to \xi_A$ given by (4.14) and $\xi_A \to Y^A$ given by

(4.20) $$Y^A = Q^{AB}\xi_B$$

is the transformation

(4.21) $$Y^A = Q^{AB}P_{BC}X^C$$

which carries points into points and hence is a collineation. Also the product of the correlation (4.20) and the collineation (4.1) is the transformation

(4.22) $$\xi_A = Q_{AB}P^B{}_C X^C$$

which carries points into hyperplanes and hence is a correlation. We see, then, that although the correlations alone do not form a group, the non-singular collineations together with the non-singular correlations do form a group. This group is called the projective group.

The product of the correlation (4.20) and the anticorrelation (4.15) is the transformation

(4.23) $$\bar{X}^A = P^{AB}Q_{BC}Y^C$$

which is anti-linear and carries points into points and is therefore an anti-collineation.

Similar considerations give us the multiplication table*

---

*Cartan, Géométrie Projective Complexe, p. 97.

---

| | collineation | correlation | anticollineation | anticorrelation |
|---|---|---|---|---|
| collineation | collineation | correlation | anticollineation | anticorrelation |
| correlation | correlation | collineation | anticorrelation | anticollineation |
| anticollineation | anticollineation | anticorrelation | collineation | correlation |
| anticorrelation | anticorrelation | anticollineation | correlation | collineation |

From this table it is evident that all non-singular correlations, collineations, anticorrelations, and anticollineations form a group. This group is called the antiprojective group. It may be obtained from the collineation group by adjoining two transformations of different types, for example, a correlation and an anticollineation.

However, if we adjoin to the collineation group one transformation of a different type, we obtain subgroups of the antiprojective group. We have already seen how the projective group and the anticollineation group can be obtained in this manner. By adjoining an anticorrelation to the collineation group we obtain another subgroup of the antiprojective group.

The transformations of the antiprojective group have been defined above by stating what they do to points. Each such transformation, however, brings about a permutation of all the linear subspaces of $P_{k-1}$. The formulas which describe the effect of the transformations on the linear subspaces of various dimensionality are obtainable by generalizing the derivation of equations (4.5), (4.7), (4.16), and (4.17).

## 5. MATRIX NOTATION

The matrix notation may be used to write equations (1.1) as

$$X^* = TX \tag{5.1}$$

where T is the matrix (1.2) and X and X* are both matrices of one column of the form

$$\left\| \begin{matrix} X^1 \\ X^2 \\ X^3 \\ \cdot \\ \cdot \\ \cdot \\ X^k \end{matrix} \right\| \quad \text{and} \quad \left\| \begin{matrix} X^{1*} \\ X^{2*} \\ X^{3*} \\ \cdot \\ \cdot \\ \cdot \\ X^{k*} \end{matrix} \right\| \tag{5.2}$$

respectively. That is, we consider $X^A$ as a two-index quantity $X^{A1}$, the second index taking on just one value.

Similarly equations (1.6) may be written as

$$X^* = SX \tag{5.3}$$

where S is a unimodular matrix obtained from the matrix (1.2) by the formula (1.7).

In the following we shall write $\psi_A$ also as a matrix of one column. The transformation induced on $\psi_A$ by the transformation (5.3) may be written as

$$\psi^* = s'\psi \qquad (5.4)$$

where s is the inverse matrix to that used in (5.3) and the prime denotes the transposed matrix.

The transpose of the matrix X is denoted by X' and is a matrix of one row. Thus we may write for $X^A\psi_A$ the matrix $X'\psi$ which has only one row and one column and is therefore equal to its transpose $\psi'X$.

The matrices S and s of a coordinate transformation have indices which refer to different coordinate systems. We take account of this fact by writing the contravariant index of the transformation matrix directly above the covariant one, thus $S^A_B$ and $s^A_B$. For a matrix of this type the contravariant index is assumed to come first (i.e., be leftmost). The inverse of a matrix is indicated by the exponent -1 so that $s = S^{-1}$. With these coventions equation (5.4) may be rewritten as

$$\psi^* = S'^{-1}\psi = S^{\vee}\psi$$

where $S^{\vee} = S'^{-1}$.

The equations of geometric transformations may be written in matrix notation. Thus equation (4.1) may be written as

$$Y = \| P^A{}_B \| X \qquad (5.5)$$

where X and Y are matrices of one column and $\| P^A{}_B \|$ is the matrix representing the collineation. In the index notation for a geometric transformation, and indeed for any geometric being, we never place one index directly above another. Thus it always has a meaning to say that one index precedes another even if one is a covariant and the other is a contravariant index. We shall use the convention that the first index refers to the rows of the matrix and the second refers to the columns.

With these conventions equations (4.2), (4.14) and (4.15) may be written as

(5.6) $$\bar{\dot{Y}} = \| P^{\dot{A}}{}_{B} \| \, \dot{X}$$

(5.7) $$\xi = \| P_{AB} \| \, X$$

(5.8) $$\bar{\xi} = \| P_{\dot{A}B} \| \, X \quad .$$

The transformations induced by the non-singular collineation (5.5) and the non-singular anticollineation (5.6) on the hyperplanes may be written as

(5.9) $$\eta = \| P^{A}{}_{B} \|^{\cup} \, \xi$$

and

(5.10) $$\bar{\eta} = \| P^{\dot{A}}{}_{B} \|^{\cup} \, \xi$$

where we denote the transposed inverse of any matrix Q by $Q^{\cup}$, and we call the latter the matrix contragredient to Q.

Similarly the transformations induced on the hyperplanes by the non-singular correlations (5.7) and non-singular anticorrelations (5.8) are

(5.11) $$X = \| P_{AB} \|^{\cup} \, \xi$$

and

(5.12) $$\bar{X} = \| P_{\dot{A}B} \|^{\cup} \, \xi \quad .$$

If we make the transformation of coordinates (5.3) equations (5.5) may be written as

(5.13) $$S^{-1}Y* = \| P^A{}_B \| S^{-1}X*$$

or

(5.14) $$Y* = \| P^A{}_B \|* X*$$

where

(5.15) $$\| P^A{}_B \| * = S \| P^A{}_B \| S^{-1} \quad .$$

Thus in a new coordinate system, the components of the matrix $\| P^A{}_B \|$ which determine the collineation, are linear homogeneous functions of the components in the old coordinate system. The coefficients are homogeneous functions of the elements of the transformation matrix. Geometric beings* with this type

*A "geometric being" is an entity described by components in every coordinate system. The components in two coordinate systems are related by a transformation law, namely a formula which gives the components in one coordinate system in terms of the components in the other.

of transformation law are special types of spinors. Equation (5.15) is a special case of a transformation law which satisfies the two homogeneity requirements given above, and hence $P^A{}_B$ are the components of a spinor.

Similarly, we see that in a new coordinate system, equations (5.6) may be written as

(5.16) $$\bar{Y}* = \| P^{\dot{A}}{}_B \|* X*$$

where

(5.17) $$\| P^{\dot{A}}{}_B \|* = \bar{S} \| P^{\dot{A}}{}_B \| S^{-1} \quad .$$

Also equations (5.7) and (5.8) may be written as

(5.18) $$\xi * = \| P_{AB} \|* X*$$

and

(5.19) $$\bar{\xi} * = \| P_{\dot{A}B} \|* X*$$

where

$$\| P_{AB} \| * = S^{\vee} \| P_{AB} \| S^{-1} \tag{5.20}$$

and

$$\| P_{\dot{A}B} \| * = \bar{S}^{\vee} \| P_{\dot{A}B} \| S^{-1} \quad . \tag{5.21}$$

Equations (5.17), (5.20) and (5.21) are all special cases of transformation laws for spinors. From these equations, together with equations (5.15), we see that if in one coordinate system a single matrix, for example the unit matrix, represents a collineation, a correlation, an anticollineation and an anticorrelation, in another coordinate system, these different transformations will not in general be represented by the same matrix. We have used the convention that placing a dot over one of the indices of a spinor implies that the complex conjugates of the elements of the transformation matrix enter into the law of transformation of the spinor.

## 6. COMMUTING TRANSFORMATIONS

In this section we will discuss the transforms of geometric transformations when a different geometric transformation is made. This differs from the work of the previous section where we obtained the transformation laws of the components of a geometric transformation under a transformation of coordinates.

Let point Y be the transform of the point X under the non-singular collineation P, that is,

$$Y = PX \quad . \tag{6.1}$$

If we now perform the non-singular collineation Q, the points X and Y are transformed into the points Z and W where

(6.2) $Z = QX$ and $W = QY$ .

Equations (6.1) then become

(6.3) $$Q^{-1}W = PQ^{-1}Z$$

or

(6.4) $$W = RZ$$

where

(6.5) $$R = QPQ^{-1} \quad .$$

That is, the collineation P goes into the collineation $R = QPQ^{-1}$ under the collineation Q.

Two geometric transformations are said to be geometrically commutative if each transforms the other into itself. It is geometrically obvious that if a geometric transformation A transforms a transformation B into itself, then B will transform A into itself. Thus if $R = \rho P$ where $\rho$ is an arbitrary number, the two collineations P and Q are geometrically commutative. In this case equations (6.5) may be written as

(6.6) $$\rho P = QPQ^{-1}$$

or

(6.7) $$QP = \rho PQ \quad .$$

From equation (6.7) we see that geometrical commutativity of two collineations does not distinguish between commutativity and anticommutativity of the matrices representing the collineations. A geometric distinction between these two cases will be given in the next chapter.

If instead of performing a collineation, we perform the anti-collineation Q:

(6.8) $\bar{Z} = QX$ and $\bar{W} = QY$ .

Equations (6.1) then become

(6.9) $$Q^{-1}\bar{W} = PQ^{-1}\bar{Z}$$

or

(6.10) $$W = RZ$$

where

(6.11) $$\bar{R} = QPQ^{-1} .$$

Thus the collineation P and the anticollineation Q are geometrically commutative if

(6.12) $$QP = \rho \bar{P}Q .$$

If the transformation Q is a correlation, then the points X and Y go into the hyperplanes $\xi$ and $\eta$ . That is,

(6.14) $\xi = QX$ and $\eta = QY$ .

Equation (6.1) then becomes

(6.15) $$Q^{-1}\eta = PQ^{-1}\xi$$

or

(6.16) $$\eta = R\xi$$

where

(6.17) $$R = QPQ^{-1} .$$

Thus the transform of the collineation given by equations (6.1) by the correlation Q is a transformation which carries hyperplanes into hyperplanes and hence is a collineation. If the two transformations are commutative, the collineation R must be the same as the collineation P. However, since R is a transformation that transforms hyperplanes, it must be the contragredient transformation to P. Thus

$$(6.18) \qquad \rho P^{\cup} = QPQ^{-1} \qquad \text{or} \qquad QP = \rho P^{\cup} Q$$

is the condition that the correlation Q be commutative with the collineation R.

Similar considerations give us the result that the collineation

$$(6.19) \qquad \bar{R} = QPQ^{-1}$$

is the transform of the collineation P under the anticorrelation Q. It should be remembered that R acts on hyperplanes and hence we must have

$$(6.20) \qquad \rho \bar{P}^{\cup} = QPQ^{-1} \qquad \text{or} \qquad QP = \rho \bar{P}^{\cup} Q$$

as the condition that the anticorrelation Q be commutative with the collineation P.

It is readily verified that the transform of an anticollineation Q by a collineation P is the anticollineation

$$(6.21) \qquad R = \bar{P}QP^{-1} \quad .$$

Setting $R = \rho Q$ we again obtain equation (6.1) for the condition that these two transformations be geometrically commutative.

Also, the transform of the correlation Q by the collineation P is the transformation

$$R = P^{\vee} Q P^{-1} \tag{6.22}$$

which is again a correlation. The transform of an anticorrelation Q by a collineation P is the transformation

$$R = \bar{P}^{\vee} Q P^{-1} \tag{6.23}$$

which is again an anticorrelation.

From equation (6.22) we see that the conditions that two collineations P and R be commutative with the correlation Q may be written as

$$\rho Q = P^{\vee} Q P^{-1} \quad \text{and} \quad \rho Q = R^{\vee} Q R^{-1} \ . \tag{6.24}$$

From these equations it is evident that

$$(PR)^{\vee} Q (PR)^{-1} = \rho Q \ . \tag{6.25}$$

Hence the set of collineations commutative with a correlation form a group. Similar arguments show that the set of all geometric transformations commutative with a given one form a group.

## 7. INVOLUTIONS

A collineation P which carries the point Y into the point X = PY, which is not the identity collineation and such that its square is the identity collineation, is said to be of period two. It is also called an involution.

It satisfies

$$P^2 = \rho 1 \tag{7.1}$$

where $\rho$ is arbitrary. The collineation is a non-singular involution if $\rho \neq 0$ and a singular one if $\rho = 0$.

Since $X^A$ and $Y^A$ are homogeneous coordinates of points P and $\sigma$ P where $\sigma$ is an arbitrary scalar, determine the same geometric transformation. Hence we may normalize the non-singular involutions so that equation (7.1) becomes

$$P^2 = 1 \quad . \tag{7.2}$$

For a given non-singular involution there are just two normalized matrices satisfying this condition. One of them is the negative of the other. This normalization is invariant under transformations of coordinates.

Thus the matrix of a non-singular involution satisfies the equation $X^2 - 1 = 0$. The elementary divisors of the matrix are therefore simple. Hence there exists a coordinate system in $P_{k-1}$ in which the matrix of an involution has the form

$$\left\| P^A{}_B \right\| = \left\| \begin{matrix} 1_p & 0 \\ 0 & -1_q \end{matrix} \right\| \tag{7.3}$$

where $1_p$ and $1_q$ are the unit p- and q-dimensional matrices respectively and the two 0's represent rectangular blocks of zeros. The numbers p and q are such that $p + q = k$ and must both be different from zero; for if either were zero the collineation P would be the identity.

The involutions which have the further property that

$$\text{trace } P = 0 \tag{7.4}$$

that is, p = q in equation (7.3), we shall call axial reflections. Axial reflections can only exist in case k is even.

In $P_1$ all involutions are axial reflections. If P is the matrix

$$\left\| \begin{matrix} a & b \\ c & d \end{matrix} \right\|$$

then $P^2$ is the matrix

$$\left\| \begin{matrix} a^2 + bc & ab + bd \\ ac + cd & d^2 + bc \end{matrix} \right\| .$$

If the latter is to be a multiple of the unit matrix we must have d = -a. But this means that the trace of P is zero.

An aribtrary involution in $P_1$ may be reduced to the form

$$\left\| \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right\| . \tag{7.5}$$

The invariant points of this involution are the points (1, 0) and (0, 1). These points characterize the involution. The transform of any point X is the point which is harmonically conjugate to X with respect to these two points.

In $P_3$ the axial reflections are called line reflections. They may be reduced to the form

$$(7.6) \qquad P = \left\| \begin{matrix} +1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} \right\| .$$

The involution (7.6) is characterized by the fact that it leaves the two non-intersecting lines $X^1 = X^2 = 0$ and $X^3 = X^4 = 0$ pointwise invariant. These lines are referred to as the axes. The transform of a point Y not on either of the axes is found by constructing the uniquely determined line through Y which meets both axes. Then the transform of Y is its harmonic conjugate with respect to the two points of intersection. This is because the involution in $P_3$ induces an involution on this line. The invariant lines of a line reflection are the two axes and all the lines which meet them both.

In $P_{k-1}$ (k even) the axial reflections are characterized by their invariant axes which are non-intersecting subspaces of dimensions $\frac{k}{2}-1$. These are left pointwise invariant by the involution. The transform of a point Y not on either of the axes is again found by constructing the uniquely determined line which meets both axes. Then the transform of Y is the point on this line which is harmoniously conjugate to Y with respect to the two points of intersection.

In $P_3$ the only other types of involutions are those which may be reduced to the form

$$P = \left\| \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} \right\| \quad . \tag{7.7}$$

This involution leaves the point (0, 0, 0, 1) and every point in the plane $X^4 = 0$ invariant. It is called a point-plane reflection with center (0, 0, 0, 1). It is characterized completely by its invariant points for the transform of a point X is found by joining it to the center C by a line XC, and taking the harmonic conjugate of X with respect to C and the point of intersection of the line XC and the invariant plane of the involution. The invariant lines of the involution are those which pass through C and meet the invariant plane or lie in the invariant plane of the involution.

If the matrix of the involution in $P_{k-1}$ is reduced to the form (7.3), the pointwise invariant spaces of the involution are

$$\psi^{p+1} = \psi^{p+2} = \ldots = \psi^{k} = 0 \quad \text{and} \quad \psi^{1} = \psi^{2} = \ldots \psi^{p} = 0 \quad . \tag{7.8}$$

We shall designate them by $[P]^+$ and $[P]^-$, respectively. Points on $[P]^+$ and $[P]^-$ are characterized algebraically by the conditions

$$P\psi = +\psi \tag{7.9}$$

and

$$P\psi = -\psi \tag{7.10}$$

respectively, where we have used the matrix notation. If we define $[-P]^+$ as the cones of points satisfying $-P\psi = \psi$ we have $[-P]^+ = [P]^-$. Similarly $[-P]^- = [P]^+$. Hence an involution in $P_{k-1}$ determines a pair of pointwise invariant spaces but does not impose an order on them while giving a particular matrix P in the form (7.3) determines a positive and negative space.

The pointwise invariant spaces of an involution determine it completely. To prove this we let $S^+$ and $S^-$ be the two non-intersecting pointwise invariant subspaces of dimensions r-1 and k-r-1 respectively, and construct the unique involution in $P_{k-1}$ with these spaces as pointwise invariant spaces. If $\psi$ is any point not on $S^+$ or $S^-$ there is a unique line through it and intersecting $S^+$ and $S^-$. The involution then leaves points on $S^+$ and $S^-$ invariant and transforms any other point $\psi$ into its harmonic conjugate with respect to the two points in which the line through $\psi$ meets $S^+$ and $S^-$.

The invariant lines of an involution are all the lines which meet both $S^+$ and $S^-$ or lie entirely in either one of them. Indeed if $P_{r-1}$ is any space of r-1 dimensions which is invariant under the involution, then it is either pointwise invariant and lies in $S^+$ or $S^-$, or it intersects them in spaces of dimensions $r_1-1$ and $r_2-1$ with $r_1 + r_2 = r$. The proof of this depends on the fact that if a space is invariant, the involution induces in it either the identity or an involution which necessarily has two pointwise invariant subspaces.

## 8. ANTI-INVOLUTIONS

An anticollineation

$$\bar{X} = PY \tag{8.1}$$

will be of period two and will be called an anti-involution if and only if

$$P\bar{P} = \rho 1 \tag{8.2}$$

where $\rho$ is a number different from zero. If we take the trace of both sides of equation (8.2) we see that $k\rho$ = trace $(P\bar{P})$ = trace $(\bar{P}P) = k\bar{\rho}$ . Hence $\rho$ must be real.

From the transformation law of P (cf. equation (5.17)) we see that the normalization condition, $|P| = 1$, where $|P|$ is the determinant of P, is invariant under coordinate transformations. Under this normalization we have $\rho^k = 1$. Hence for odd k, $\rho = +1$, and for even k we have $\rho = \pm 1$. Thus for even k we have the two cases

(8.3) $$P\bar{P} = 1$$

and

(8.4) $$P\bar{P} = -1 \quad .$$

Anti-involutions which satisfy (8.3) are said to be of the first kind and those which satisfy (8.4) are of the second kind.

Any two anti-involutions of the same kind are equivalent under a transformation of coordinates. The proof of this statement is contained in an unpublished theorem of Dr. N. Jacobson which states that two non-singular anticollineations are projectively equivalent if the matrices of their squares are equal.*

---

*The proof follows:
Let the two non-singular anticollineations be P and Q. Then the hypothesis is

$$P\bar{P} = Q\bar{Q} .$$

Let us now subject Q to the transformation of coordinates with matrix T where

$$T = e^{i\theta}1 + e^{-i\theta}P^{-1}Q$$

and $\theta$ is real and such that $|T| \neq 0$. This gives

$$Q^* = \bar{T}QT^{-1} = P$$

since

$$T^{-1} = (e^{i\theta}P + e^{-i\theta}Q)^{-1}P \quad .$$

The unit matrix satisfies (8.3) and hence in a suitable coordinate system we have for any anti-involution of the first kind

$$(8.5)\qquad P = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix} \quad \text{or } \bar{X}^A = Y^A \quad .$$

The invariant points of this transformation are all points having a set of real coordinates in this coordinate system and hence are a real projective (k-1)-space.

The totality of points in a complex projective (k-1)-space which in a given preferred coordinate system have real coordinates, is called a (k-1)-chain, Thus any anti-involution of the first kind has the points of a (k-1)-chain as invariant points, and transforms each point into its conjugate imaginary with respect to the (k-1)-chain. Conversely, each (k-1)-chain determines a unique anti-involution of the first kind.

A set of k+1 points, say $X_1, X_2, \ldots, X_{k+1}$, no k of which are linearly dependent, may be chosen as the basic points, (1, 0, 0, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, 0, ..., 0, 1) and (1,1, ..., 1) of a projective coordinate system in $P_{k-1}$. Hence this set of points determines a unique chain the points of which are given by

$$r^1 X_1^A + r^2 X_2^A + \ldots + r^k X_k^A$$

where $r^1, r^2, \ldots, r^k$ are real parameters and the coordinates chosen to represent the k points $X_1, \ldots, X_k$ are such that

$$X_1^A + X_2^A + \dots + X_k^A = X_{k+1}^A \quad .$$

In a suitable coordinate system anti-involutions of the second kind, which exist only in case k is even, may be reduced to the form

$$(8.6) \qquad P = \left\| \begin{matrix} 0 & 1 & 0 & 0 & 0 & 0 \,.. & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \,.. & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \,.. & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \,.. & 0 & 0 \\ . & . & . & . & . & . \,.. & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 \,.. & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \,.. & -1 & 0 \end{matrix} \right\| .$$

If a point X were invariant under an anti-involution of the second kind, we would have $\rho \overline{X} = PX$. On repeating the transformation we would have $P\overline{P}\overline{X} = \overline{\rho} \rho \overline{X}$. Since $P\overline{P} = -1$, and $\rho \overline{\rho} > 0$, we must have $X = 0$, contrary to the assumption that X represents a point. Hence an anti-involution of the second kind has no invariant points. By the principle of duality it has no invariant hyperplanes. However, through each point in space, there passes a single invariant line, the line joining the point to its transform. The set of invariant lines of an anti-involution of the second kind is called an anti-congruence. Dual to it there is, of course, a set of invariant $P_{k-3}$'s.

## 9. POLARITIES

A non-singular correlation C which carries the point X into the hyperplane $\xi = CX$ will be of period two if

$$(9.1) \qquad \overset{\cup}{C} C = \rho 1$$

as one sees on looking at (4.21). Multiplying this equation by $C'$ and remembering that $C^{\nu} = C'^{-1}$ this condition becomes

(9.2) $$C = \rho C' \quad .$$

Since the operation of taking the transpose of a matrix is of period two, we must have $\rho^2 = 1$. That is, $\rho = \pm 1$. In case $\rho = +1$, the matrix representing the correlation is symmetric. In this case we have

(9.3) $$C_{AB} = C_{BA}$$

and the correlation is a polarity with respect to the non-degenerate quadric

(9.4) $$C_{AB}X^A X^B = 0 \quad .$$

A point and its hyperplane transform are incident if and only if the point lies on the quadric.

In a suitable coordinate system the quadratic form on the left-hand side of (9.4) becomes a sum of squares and the matrix C reduces to the unit matrix.

In case $\rho = -1$, the matrix C is anti-symmetric and will be non-singular only if k is even. Correlations of this type are called null-polarities. Since $C_{AB}X^A X^B$ vanishes identically, every point lies on the polar hyperplane. Non-singular null polarities exist only in spaces of odd dimensionality. In a suitable coordinate system they are of the form

$$\left\| \begin{matrix} 0 & 1_{k/2} \\ -1_{k/2} & 0 \end{matrix} \right\|$$

where $1_{k/2}$ is the unit matrix in k/2 dimensions and we have normalized the correlation to have determinant $\pm 1$.

## 10. ANTIPOLARITIES

A non-singular anticorrelation C which carries the point X into the hyperplane $\xi$ where $\bar{\xi}$ = CX will be of period two and will be called an antipolarity if

$$\rho \bar{X}^D = C^{\dot{D}B}\bar{C}_{\dot{B}A}\bar{X}^A$$

that is if (cf. §3)

(10.1) $$\check{C}\bar{C} = \rho 1 \ .$$

Multiplying by C' this condition becomes

(10.2) $$\bar{C} = \rho C' \qquad \text{or} \qquad \bar{C}_{\dot{A}B} = \rho C_{\dot{B}A} \ .$$

From the second of equations (10.2) we see that

$$C_{\dot{A}B} = \bar{\rho}\,\bar{C}_{\dot{B}A} = \bar{\rho}\rho\, C_{\dot{A}B}$$

hence $\rho\bar{\rho} = 1$. Replacing the homogeneous coefficients $C_{\dot{A}B}$ of the antipolarity by $H_{\dot{A}B} = \rho^{\frac{1}{2}}C_{\dot{A}B}$ gives

(10.3) $$\bar{H} = H' \ .$$

A matrix satisfying this condition is said to be hermitian. Hence the necessary and sufficient condition for an anticorrelation to be of period two is that the matrix of its coefficients be proportional to an hermitian one. The normalization made to obtain equation (10.3) is invariant under coordinate transformations as may be verified from equation (5.21).

The locus of points which lie on their antipolar planes is the antiquadric*

---

*These loci are called hyperquadrics by Segre and Cartan, but it seems less confusing to let a hyperquadric be the analogue of a hyperplane and let an antiquadric be the locus defined by an antipolarity.

---

(10.4)
$$C_{\dot{A}B}\bar{X}^{A}X^{B} = 0 \quad .$$

The left number of this equation is real for arbitrary complex values of the coordinates $X^A$, hence if we regard $P_{k-1}$ as a real (not projective) space of 2k-2 dimensions, the points on an antiquadric constitute a manifold of 2k-3 real dimensions, provided it exists at all. Whereas the surface of a quadric in this space has 2k-4 real dimensions.

In a suitable coordinate system an antipolarity may be reduced to the form

(10.5)
$$C = \left\| \begin{matrix} \pm 1 & 0 & 0 & 0 & 0 & . \\ 0 & \pm 1 & 0 & 0 & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & \pm 1 \end{matrix} \right\| \quad .$$

The signature of the hermitian form (10.4) is determined by the number of plus and minus signs in the matrix (10.5) and determines the geometric properties of the antiquadric. However, if we multiply C by -1 we change its signature but the antiquadric is unchanged. Hence in $P_{k-1}$ there are just $\left[\frac{k+2}{2}\right]$ distinct types of antiquadrics.

# Chapter V

# LINEAR FAMILIES OF REFLECTIONS

## 1. STATEMENT OF THE PROBLEM

In this chapter we shall inquire into the question of the existence of linear families of involutoric collineations in a complex projective space $P_{k-1}$. The matrices of such a family are given by the formula

$$(1.1) \qquad \| X^A{}_B \| = X^\alpha \| \gamma_\alpha{}^A{}_B \| \equiv X^\alpha \gamma_\alpha \qquad \begin{pmatrix} A, B = 1, 2, \ldots, k \\ \alpha = 0, 1, \ldots, m \end{pmatrix}$$

in which $X^0, X^1, \ldots, X^m$ are arbitrary complex numbers and $\gamma_\alpha$ are m+1 linearly independent matrices of involutoric collineations. We shall use the word "involution" as an abbreviation for "involutoric collineation".

That the matrix X in (1.1) represents a collineation of period two implies

$$(1.2) \qquad (X^\alpha \gamma_\alpha)^2 = \tfrac{1}{2} X^\alpha X^\beta (\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha) = \rho 1$$

Taking the trace of the members of this equation, we find

$$(1.3) \qquad \rho = \gamma_{\alpha\beta} X^\alpha X^\beta ,$$

where $\gamma_{\alpha\beta}$ are numbers defined by

$$(1.4) \qquad \tfrac{1}{2}(\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha) = \gamma_{\alpha\beta} 1 .$$

Thus every set of numbers $(X^0, X^1, \ldots, X^m)$ determines a matrix of an involution and the sets which satisfy

(1.5) $$\gamma_{\alpha\beta} X^{\alpha} X^{\beta} = 0$$

correspond to degenerate involutions. Two different sets of numbers, $X^{\alpha}$ and $Y^{\alpha}$, determine different matrices, X and Y.

The equation

(1.6) $$(X^{\alpha} \gamma_{\alpha})^2 = \gamma_{\alpha\beta} X^{\alpha} X^{\beta} 1$$

can be regarded as a factorization of the quadratic form $\gamma_{\alpha\beta} X^{\alpha} X^{\beta}$ into two equal linear factors with matrix coefficients. Equally well, it may be regarded as the statement that the matrix

$$\frac{1}{\sqrt{\gamma_{\alpha\beta} X^{\alpha} X^{\beta}}} \quad (X^{\alpha} \gamma_{\alpha})$$

is a square root of the identity matrix.

## 2. THE CENTERED EUCLIDEAN SPACE $E_{m+1}$

The family of matrices X given by (1.1) is in one-to-one correspondence with a complex vector space which we will denote by $E_{m+1}$. The linear independence of the $\gamma_{\alpha}$'s implies that $E_{m+1}$ is an (m+1)-dimensional space. The parameters $X^{\alpha}$ may be interpreted as the components of a vector in $E_{m+1}$ in a definite coordinate system.

A linear homogeneous transformation of the form

(2.1) $$Y^{\alpha} = A^{\alpha}{}_{\beta} X^{\beta}$$

where the determinant $\left| A^{\alpha}{}_{.\beta} \right| \neq 0$, leaves the family of matrices X of the form (1.1) unaltered. It may be interpreted as a renaming of the members of the family, that is, as a coordinate transformation in $E_{m+1}$.

In the following we shall assume that the matrices $\gamma_\alpha$ are such that the quantities $\gamma_{\alpha\beta}$ defined by equation (1.4) have a determinant different from zero.* In this case the quadratic form (1.3) may be put

---

*The necessary and sufficient condition that the determinant of $\gamma_{\alpha\beta}$ be different from zero is that there does not exist a vector $X^\alpha$ different from zero such that

$$\gamma_{\alpha\beta} X^\beta = 0$$

From equation (1.4) we see that this implies that there exists no vector $X^\alpha$ other than $X^\alpha = 0$ such that

$$\gamma_\alpha (\gamma_\beta \ X^\beta) + (\gamma_\beta \ X^\beta) \gamma_\alpha = 0 .$$

That is, the requirement that the determinant of $\| \gamma_{\alpha\beta} \|$ be different from zero is equivalent to the requirement that the matrices $\gamma_\alpha$ be such that no matrix of the family (except the zero matrix) should anticommute with all of them.

---

into the form

(2.2) $$\rho = (X^o)^2 + (X^1)^2 + \dots + (X^m)^2$$

by a linear transformation of the type (2.1). The coordinate systems in $E_{m+1}$ in which the fundamental quadratic form reduces to (2.2) will be called Cartesian coordinate systems. The vector space $E_{m+1}$ is thus a centered Euclidean space of m+1 dimensions in which there are preferred Cartesian coordinate systems. This means that we have assigned to the point (0,0, ..., 0) a special role, namely that its coordinates are invariant under transformations of coordinates. This point corresponds to the matrix all of whose elements are zero.

When Cartesian coordinates are used in $E_{m+1}$ the relations (1.4) reduce to

$$\tfrac{1}{2}(\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha) = \begin{cases} 1 \text{ if } \alpha = \beta \\ 0 \text{ if } \alpha \neq \beta \end{cases} . \tag{2.3}$$

Hence the $\gamma_\alpha$, if existent, are a set of m+1 anticommuting k-row matrices each of which is a square root of the identity matrix.

In the space $E_{m+1}$ we shall use the familiar notations of tensor analysis. In particular we shall raise and lower indices by means of the fundamental tensor $\gamma_{\alpha\beta}$ and the tensor $\gamma^{\alpha\beta}$ defined by

$$\gamma^{\alpha\beta} \gamma_{\beta\gamma} = \delta^\alpha_\gamma \qquad \text{where } \| \delta^\alpha_\gamma \| = 1 \tag{2.4}$$

This gives a definite meaning to $\gamma^{\cdot\alpha}$ and (1.4) may be written in the form

$$\tfrac{1}{2}(\gamma^\alpha \gamma_\beta + \gamma_\beta \gamma^\alpha) = \delta^\alpha_\beta \, 1 . \tag{2.5}$$

Of course in Cartesian coordinates $\gamma^{\cdot\alpha}$ is the same matrix as $\gamma_\alpha$.

It is clear that constant matrices $\gamma_\alpha$ which satisfy (1.6) must also satisfy the differential relation

$$\gamma^{\alpha} \frac{\partial}{\partial X^\alpha} \left( \gamma_\beta \frac{\partial \psi}{\partial X^\beta} \right) = \gamma^{\alpha\beta} \frac{\partial^2 \psi}{\partial X^\alpha \partial X^\beta} , \tag{2.6}$$

which on putting in indices becomes

$$\gamma^{\alpha\,A}{}_{B}\frac{\partial}{\partial X^{\alpha}}\left(\gamma^{\beta\,B}{}_{C}\frac{\partial\psi^{C}}{\partial X^{\beta}}\right)=\gamma^{\alpha\beta}\;\frac{\partial^{2}\psi^{A}}{\partial X^{\alpha}\,\partial X^{\beta}}$$

Thus we have what may be regarded as a factorization of the quadratic differential operator $\gamma^{\alpha\beta}\dfrac{\partial^2}{\partial X^{\alpha}\,\partial X^{\beta}}$ which is expressed as the square of a linear differential operator $\gamma^{\alpha}\dfrac{\partial}{\partial X^{\alpha}}$.

This observation, made by P. A. M. Dirac, was the starting point in his theory of the spinning electron which has given the theory of spinors its vogue.

## 3. TRANSFORMATIONS OF LINEAR FAMILIES OF INVOLUTIONS

A non-singular collineation $\psi \longrightarrow \varphi$

$$\varphi^{A} = P^{A}{}_{B}\,\psi^{B} \tag{3.1}$$

in $P_{k-1}$ brings about a transformation $X \to Y$ of the collineations given by (1.1) according to the formula

$$Y = PXP^{-1} = X^{\alpha}\eta_{\alpha} \tag{3.2}$$

where

$$\eta_{\alpha} = P\gamma_{\alpha}P^{-1} \tag{3.3}$$

In case the family as a whole is transformed into itself we have also

(3.4) $$Y = Y^{\alpha} \gamma_{\alpha} \quad .$$

Comparing (3.4) and (3.2) we have

(3.5) $$Y^{\alpha} \gamma_{\alpha} = (P \gamma_{\alpha} P^{-1}) X^{\alpha}$$

To solve this equation for $Y^{\alpha}$ we make use of the relation

(3.6) $$\text{Trace } (\gamma^{\alpha} \gamma_{\beta}) = k \, \delta^{\alpha}_{\beta} \quad ,$$

which is obtained from (2.5) by taking the trace of both members. Multiplying (3.5) by $\gamma^{\beta}$ and taking the trace, gives

(3.7) $$Y^{\beta} = L^{\beta}_{\alpha} \; X^{\alpha}$$

where

(3.8) $$L^{\beta}_{\alpha} = \frac{1}{k} \text{ Trace } (\gamma^{\beta} P \gamma_{\alpha} P^{-1}) \; .$$

Hence in case (3.1) transforms the family (1.1) into itself, it induces the non-singular transformation (3.7) in the Euclidean space $E_{m+1}$ .

Since the degenerate transformations of the family (1.1) are carried into themselves, the points on the cone (1.5) in $E_{m+1}$ are permuted among themselves. Indeed we may prove that the quadratic form (1.3) is left invariant by (3.7). To do this we first substitute from (3.7) for $Y^{\alpha}$ in (3.5) and equate coefficients of the arbitrary variables $X^{\alpha}$ to get

$$\eta_{\alpha} = \eta_{\beta} L^{\beta}_{\alpha} . \tag{3.9}$$

But

$$\tfrac{1}{2}(\eta_{\alpha} \bar{\eta}_{\beta} + \eta_{\beta} \eta_{\alpha}) = \gamma_{\alpha\beta} 1 \tag{3.10}$$

is a consequence of (3.3) and (1.4) and substitution from (3.9) in this equation now gives

$$\gamma_{\lambda\mu} L^{\lambda}_{\alpha} L^{\mu}_{\beta} = \gamma_{\alpha\beta} . \tag{3.11}$$

Multiplying through by $X^{\alpha} X^{\beta}$ and summing, we obtain

$$\gamma_{\alpha\beta} Y^{\alpha} Y^{\beta} = \gamma_{\alpha\beta} X^{\alpha} X^{\beta} . \tag{3.12}$$

Equation (3.12) is equivalent to the statement that (3.7) is a rotation of the Euclidean space $E_{m+1}$ about the origin. Hence the group of collineations in $P_{k-1}$ which leave the linear family (1.1) invariant is isomorphic (perhaps multiply) with a sub-group H of the group of rotations about the center of $E_{m+1}$.

We shall find that the group H thus defined coincides with the whole orthogonal group when $m + 1 = 2\nu$ and $k = 2^{\nu}$. It is the proper orthogonal group when $m + 1 = 2\nu + 1$ and $k = 2^{\nu}$. We shall also prove that the isomorphism (3.8) between the collineations which leave the family $X^{\alpha}\gamma_{\alpha}$ invariant and the group H is (1-1) in these cases. This implies that the equation (3.8) is satisfied by a unique collineation $\| \rho P \|$ when $\| L^{\beta}_{\alpha} \|$ defines an arbitrary transformation of H.

## 4. EXISTENCE OF LINEAR FAMILIES OF INVOLUTIONS

In § 2 we saw that the existence of a linear family $X^{\alpha}\gamma_{\alpha}$ satisfying (1.2) implied the existence of a set of matrices $\gamma_{\alpha}$ ($\alpha = 0, 1, \ldots, m$) satisfying (2.3). The converse is also true, for a set of matrices satisfying

$$(4.1) \qquad (\gamma_{\alpha})^2 = 1, \qquad \text{and} \qquad \gamma_{\alpha}\gamma_{\beta} = -\gamma_{\beta}\gamma_{\alpha} \ (\alpha \neq \beta)$$

serve to define a linear family $X^{\alpha}\gamma_{\alpha}$ which satisfies (1.2). The existence of linear families of involutions in $P_{k-1}$ is then reduced to the problem of discovering sets of matrices of order k satisfying (4.1). We call an <u>ordered</u> set of linearly independent matrices satisfying (4.1) a

$\gamma$ -set and say that it is even (or odd) if the number of matrices is even (or odd). Two $\gamma$ -sets $\gamma_\alpha$ and $\eta_\alpha$ are said to be similar if (3.3) is satisfied for a non-singular matrix P.

The existence of a $\gamma$ -set in the case k = 2 has been known since quaternions were represented by two-row matrices. Such a set is

$$(4.2) \qquad \gamma_1 = \left\| \begin{matrix} 0 & i \\ -i & 0 \end{matrix} \right\|, \quad \gamma_2 = \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\|, \quad \gamma_0 = -i\,\gamma_1\gamma_2 = \left\| \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right\|$$

A $\gamma$ -set containing more than one matrix can exist only if the dimensionality, k-1, of the space is odd (see § 10) so that in an even-dimensional space a $\gamma$ -set consists of a single non-singular involution. For an odd-dimensional space the problem reduces essentially to the case $k = 2^\nu$ as we shall see later. When $\nu = 1$ we have the $\gamma$ -set exhibited in (4.2) and when $\nu = 2$ (k = 4) a $\gamma$ -set is

$$(4.3) \qquad \gamma_0^{(2)} = \left\| \begin{array}{cc|cc} 1 & 0 & & \\ 0 & -1 & \multicolumn{2}{c}{0} \\ \hline & & -1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1 \end{array} \right\|, \quad \gamma_1^{(2)} = \left\| \begin{array}{cc|cc} 0 & i & & \\ -i & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & -i \\ \multicolumn{2}{c|}{0} & i & 0 \end{array} \right\|, \quad \gamma_2^{(2)} = \left\| \begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & -1 \\ \multicolumn{2}{c|}{0} & -1 & 0 \end{array} \right\|$$

$$\gamma_3^{(2)} = \left\| \begin{array}{cc|cc} & & i & 0 \\ \multicolumn{2}{c|}{0} & 0 & i \\ \hline -i & 0 & & \\ 0 & -i & \multicolumn{2}{c}{0} \end{array} \right\| \quad \text{and} \quad \gamma_4^{(2)} = \left\| \begin{array}{cc|cc} & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1 \\ \hline 1 & 0 & & \\ 0 & 1 & \multicolumn{2}{c}{0} \end{array} \right\|$$

It is easy to construct $\gamma$-sets of $2\nu + 1$ matrices of $k = 2^{\nu}$ rows and columns from $\gamma$-sets of $2(\nu - 1) + 1$ matrices of $k_1 = 2^{\nu - 1}$ rows and columns. Formulas for doing this are

$$\gamma_p^{(\nu)} = \left\| \begin{array}{c|c} \gamma_p^{(\nu-1)} & O \\ \hline O & -\gamma_p^{(\nu-1)} \end{array} \right\| , \; p = 0, 1, \ldots, 2(\nu - 1)$$

(4.4)

$$\gamma_{2\nu-1}^{(\nu)} = \left\| \begin{array}{c|c} O & i1_{k_1} \\ \hline -i1_{k_1} & O \end{array} \right\| , \qquad \gamma_{2\nu}^{(\nu)} = \left\| \begin{array}{c|c} O & 1_{k_1} \\ \hline 1_{k_1} & O \end{array} \right\|$$

where $\gamma_p^{(\nu-1)}$ $(p = 0, 1, \ldots, 2\nu - 2)$ is a $\gamma$-set of matrices of order $k_1 = 2^{\nu - 1}$ and $1_{k_1}$ is the unit matrix of order $k_1$. Thus a set of matrices for $\nu = 3$ is

(4.5)

$$\gamma_0^{(3)} = \left\| \begin{array}{cc|cc|cc|cc} 1 & 0 & & & & & & \\ 0 & -1 & & O & & & & \\ \hline & & -1 & 0 & & & O & \\ & O & 0 & 1 & & & & \\ \hline & & & & -1 & 0 & & \\ & & & & 0 & 1 & & O \\ \hline & O & & & & & 1 & 0 \\ & & & & & O & 0 & -1 \end{array} \right\| , \quad \gamma_1^{(3)} = \left\| \begin{array}{cc|cc|cc|cc} 0 & i & & & & & & \\ -i & 0 & & O & & & & \\ \hline & & 0 & -i & & & O & \\ & O & i & 0 & & & & \\ \hline & & & & 0 & -i & & \\ & & & & i & 0 & & O \\ \hline & O & & & & & 0 & i \\ & & & & & O & -i & 0 \end{array} \right\| ,$$

$$\gamma_2^{(3)} = \left\| \begin{array}{cc|cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right\| , \quad \gamma_3^{(3)} = \left\| \begin{array}{cc|cc|cc|cc} 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ \hline -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ \hline 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \end{array} \right\|$$

$$\gamma_4^{(3)} = \left\| \begin{array}{cc|cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{array} \right\| , \quad \gamma_5^{(3)} = \left\| \begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ \hline -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \end{array} \right\|$$

and $$\gamma_6^{(3)} = \left\| \begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right\|$$

Using (4.4) we readily show by induction on $\nu$ that the equations

$$\gamma_0^{(\nu)} = (-i)^{\nu} \gamma_1^{(\nu)} \gamma_2^{(\nu)} \cdots \gamma_{2\nu-1}^{(\nu)} \gamma_{2\nu}^{(\nu)} , \tag{4.6}$$

and

$$\text{Trace } (\gamma_0^{(\nu)} \gamma_1^{(\nu)} \cdots \gamma_{2\nu}^{(\nu)}) = (+i)^{\nu} \text{ Trace } 1 = (+i)^{\nu}\, 2^{\nu} \tag{4.7}$$

hold for the $\gamma$ -sets defined in this explicit manner.

## 5. EQUIVALENCE OF $\gamma$ -SETS

The remainder of this chapter will be devoted to a discussion of the geometric properties of the linear family of involutions determined by a $\gamma$ -set, and to the equivalence of $\gamma$-sets under collineation. The main results are contained in the following theorems.

Theorem (5.1). If $k = 2^{\nu} \ell$ ($\ell$ odd), the maximum number of matrices of order k in a $\gamma$ -set is $2\nu + 1$ and this maximum is attained.

Theorem (5.2). Two even $\gamma$ -sets are similar if and only if their matrices are of the same order and equal in number.

Theorem (5.3). Two odd $\gamma$ -sets, $\eta_\alpha$ and $\mathcal{J}_\alpha$ ($\alpha = 0, 1, \ldots, 2p$) of matrices of order k are similar if and only if

$$\text{Trace } (\eta_0 \eta_1 \cdots \eta_{2p}) = \text{Trace } (\mathcal{J}_0 \mathcal{J}_1 \cdots \mathcal{J}_{2p}) . \tag{5.1}$$

Theorem (5.4). If $k = 2^\nu$, an arbitrary $\gamma$-set $\eta_i, \eta_0$ ($i = 1, \ldots, 2\nu$) is either similar to $\gamma_i^{(\nu)}, \gamma_0^{(\nu)}$ or is similar to $\gamma_i^{(\nu)}, -\gamma_0^{(\nu)}$ according as Trace $(\eta_0, \eta_1, \ldots, \eta_{2\nu})$ is $(+i)^\nu 2^\nu$ or $-(+i)^\nu 2^\nu$.

The proof of these theorems is quite simple when one assumes sufficient knowledge of (k-1)-dimensional projective geometry. Not wishing to presuppose this knowledge, we shall first (§6) derive some algebraic properties of $\gamma$-sets, then (§7) deduce a few consequences of Theorems (5.1) to (5.4), and in §§ 9 to 12 derive some general theorems on axial reflections, and finally (§13) give the proofs of Theorems (5.1) to (5.4).

## 6. ALGEBRAIC PROPERTIES OF $\gamma$-SETS

Some important properties of $\gamma$-sets may be established without using the theorems of §5 or the numerically given matrices of § 4. In this section we take $\gamma_\alpha$ to be any $\gamma$-set of $2\nu + 1$ matrices of order $2^\nu$, where $\nu \geq 1$.

The trace of each of the matrices $\gamma_\alpha$ is zero because $\gamma_\beta \gamma_\alpha \gamma_\beta^{-1} = -\gamma_\alpha$, where $\alpha \neq \beta$ and $\beta$ is not summed. Indeed, the product of $p < 2\nu + 1$ different $\gamma$-matrices, $\gamma_{\alpha_1} \gamma_{\alpha_2} \cdots \gamma_{\alpha_p}$ has a zero trace because

$$(6.1) \qquad \gamma_\beta\ (\gamma_{\alpha_1} \gamma_{\alpha_2} \cdots \gamma_{\alpha_p})\, \gamma_\beta^{-1} = -(\gamma_{\alpha_1} \gamma_{\alpha_2} \cdots \gamma_{\alpha_p})$$

if we take $\beta$ different from all the indices $\alpha_1, \alpha_2, \ldots, \alpha_p$ when p is odd, and equal to some one of them when p is even. Hence the product $\gamma_{\alpha_0}\gamma_{\alpha_1}\cdots\gamma_{\alpha_{2\nu}}$ of $2\nu+1$ factors, some of which may coincide, has zero trace unless all the indices are different, so that in any case

(6.2) $$\text{Trace}\,(\gamma_{\alpha_0}\gamma_{\alpha_1}\cdots\gamma_{\alpha_{2\nu}}) = \epsilon_{\alpha_0\alpha_1\cdots\alpha_{2\nu}}(\text{Trace}(\gamma_0\gamma_1\cdots\gamma_{2\nu})),$$

where $\epsilon_{\alpha_0\alpha_1\cdots\alpha_{2\nu}}$ is the alternating quantity defined in Chapter IV. Using (4.7) we get for the canonical matrices $\gamma_\alpha^{(\nu)}$

(6.3) $$\text{Trace}\ \gamma_{\alpha_0}^{(\nu)}\gamma_{\alpha_1}^{(\nu)}\cdots\gamma_{\alpha_{2\nu}}^{(\nu)} = \epsilon_{\alpha_0\alpha_1\cdots\alpha_{2\nu}}(+i)^{\nu}2^{\nu}.$$

The matrices

(6.4) $$1,\gamma_\alpha,\gamma_\alpha\gamma_\beta\ (\alpha<\beta),\gamma_\alpha\gamma_\beta\gamma_\gamma\ (\alpha<\beta<\gamma),\ldots,\gamma_{\alpha_1}\gamma_{\alpha_2}\cdots\gamma_{\alpha_\nu}\ (\alpha_1<\alpha_2<\cdots<\alpha_\nu)$$

are a set of $1 + (2\nu+1) + \frac{(2\nu+1)(2\nu)}{2} + \ldots + \frac{(2\nu+1)!}{\nu!\ (\nu+1)!} = \frac{1}{2}(1+1)^{2\nu+1}$

$$= 2^{2\nu}$$

matrices which are linearly independent. For, if we had

(6.5) $$A = a1 + a^{\alpha}\gamma_\alpha + a^{\alpha\beta}\gamma_\alpha\gamma_\beta + \ldots + a^{\alpha_1\alpha_2\cdots\alpha_\nu}\gamma_{\alpha_1}\gamma_{\alpha_2}\cdots\gamma_{\alpha_\nu} = 0$$

where $a^{\alpha\beta}$, $a^{\alpha\beta\gamma}$, ... etc., were alternating in their indices, taking the trace of $A\gamma_\lambda \gamma_\mu \cdots \gamma_\sigma$ would give $a^{\lambda\mu\cdots\sigma} = 0$. Hence there can exist no non-trivial linear relation between the matrices of (6.4). Since there are $2^{2\nu}$ of them, they form a basis for all matrices of order $2^\nu$, that is to say, any $2^\nu$-rowed matrix B is expressible in the form

$$(6.6) \qquad B = b1 + b^\alpha \gamma_\alpha + b^{\alpha\beta}\gamma_\alpha \gamma_\beta + \cdots + b^{\alpha_1\cdots\alpha_\nu} \gamma_{\alpha_1} \cdots \gamma_{\alpha_\nu}.$$

An exactly similar argument proves that the $2^{2\nu}$ matrices

$$(6.7) \qquad 1,\ \gamma_i,\ \gamma_i\gamma_j\ (i<j),\ \gamma_i\gamma_j\gamma_k\ (i<j<k),\ \ldots,\ \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_{2\nu}}$$

$$(i, j, \ldots = 1, 2, \ldots, 2\nu) \qquad\qquad (i_1 < i_2 < \cdots < i_{2\nu})$$

which are obtained by taking products of the matrices $\gamma_1, \gamma_2, \ldots, \gamma_{2\nu}$ also form a basis for all matrices of order $2^\nu$. Any matrix which commutes with all the $\gamma_i (i = 1, \ldots, 2\nu)$ will therefore commute with all matrices and must therefore be a multiple of the identity matrix. In particular, if $\sigma$ is a matrix which anticommutes with the $\gamma_i$, the matrix $\gamma_0\sigma$ will commute with $\gamma_i$ and hence

$$(6.8) \qquad \gamma_0\sigma = \rho 1 \qquad \text{or} \qquad \sigma = \rho\gamma_0.$$

It follows that a $\gamma$-set of $2\nu$ matrices of order $2^\nu$ can be extended to a $\gamma$-set of $2\nu + 1$ matrices in just two ways, by adding to the set the matrix $\gamma_0 = (-i)^\nu \gamma_1\gamma_2 \cdots \gamma_{2\nu}$ or its negative. The linear independence of the resulting set of $2\nu + 1$ matrices is obvious since the matrices are proportional to $2\nu + 1$ elements of the basis (6.7).

## 7. REPRESENTATION OF ROTATION GROUP IN $E_{m+1}$ BY COLLINEATIONS IN $P_{k-1}$

Using the theorems of § 5 on $\gamma$-sets which we have stated (but not yet proved), it is now possible to establish the theorems:

Theorem (7.1). The group $H^{+}_{2\nu+1}$ of all proper rotations about the origin in $E_{2\nu+1}$ is (1-1) isomorphic with the group G of collineations in $P_{2\nu-1}$ which transform the linear family of involutions defined by the matrices

$$X^0 \gamma_0 + X^1 \gamma_1 + \dots + X^{2\nu} \gamma_{2\nu}$$

into itself.

Theorem (7.2). The group $H_{2\nu}$ of all rotations, proper and improper, about the origin in $E_{2\nu}$ is (1-1) isomorphic with the group $G_0$ of collineations in $P_{2\nu-1}$ which transform the linear family of involutions defined by the matrices

$$X^1 \gamma_1 + X^2 \gamma_2 + \dots + X^{2\nu} \gamma_{2\nu}$$

into itself.

We take the points of $E_{2\nu+1}$ to be the matrices of the linear family $X = X^{\alpha}\gamma_{\alpha}$ ($\alpha = 0, 1, \dots, 2\nu$) of order $2^{\nu}$ and use Cartesian coordinates so that the matrices $\gamma_{\alpha}$ form a $\gamma$-set. In § 3 we showed that a collineation in $P_{2\nu-1}$ which left the linear family $X^{\alpha}\gamma_{\alpha}$ invariant induced in $E_{2\nu+1}$ an orthogonal transformation $Y^{\alpha} = L^{\alpha}_{\beta} X^{\beta}$. This was shown by means of the equations (3.3) and (3.9) which combine into

(7.1) $$P \gamma_\alpha P^{-1} = \gamma_\beta L^\beta_\alpha .$$

We shall use these equations to establish the isomorphism of Theorem (7.1).

If P is a matrix defining a collineation of the group G, we already know (cf. (3.8)) that equation (7.1) uniquely determines an orthogonal matrix $\| L^\beta_\alpha \|$. But from (7.1)

(7.2) $$P(\gamma_0\gamma_1 \cdots \gamma_{2\nu})P^{-1} = (\gamma_\alpha \gamma_\beta \cdots \gamma_\lambda) L^\alpha{}_0 L^\beta{}_1 \cdots L^\lambda{}_{2\nu} ,$$

and taking the trace gives

(7.3) $$\text{Trace } (\gamma_0\gamma_1 \cdots \gamma_{2\nu}) = \text{Trace } (\gamma_0\gamma_1 \cdots \gamma_{2\nu})\epsilon_{\alpha\beta \cdots \lambda} L^\alpha{}_0 L^\beta{}_1 \cdots L^\lambda{}_{2\nu}$$

or

(7.4) $$1 = | L^\beta_\alpha | .$$

Consequently a collineation of G always induces in $E_{2\nu+1}$ a proper rotation.

Conversely, if $\| L^\beta_\alpha \|$ is a proper orthogonal matrix, the matrices $\eta_\alpha = \gamma_\beta L^\beta_\alpha$ again form a $\gamma$-set which, by the computation of the last paragraph, satisfy the relation

(7.5) $$\text{Trace } (\eta_0\eta_1 \cdots \eta_{2\nu}) = | L^\alpha{}_\beta | \text{ Trace } (\gamma_0\gamma_1 \cdots \gamma_{2\nu}) = \text{Trace } (\gamma_0\gamma_1 \cdots \gamma_{2\nu})$$

Theorem (5.3) can now be applied to give us the existence of a matrix P such that $P\gamma_\alpha P^{-1} = \eta_\alpha$ and this matrix will then correspond to $\| L^\alpha{}_\beta \|$ under the isomorphism of (7.1).

The proof of Theorem (7.1) will be complete if we can show that the transformation $Y^\alpha = L^\alpha{}_\beta X^\beta$ cannot be induced in $E_{2\nu+1}$ by two different collineations of $P_{k-1}$. That is, we must prove that (7.1) and $Q \gamma_\alpha Q^{-1} = \gamma_\beta L^\beta_\alpha$ imply $P = \rho Q$ for some number $\rho$. But these equations give

$$P \gamma_\alpha P^{-1} = Q \gamma_\alpha Q^{-1} \qquad \text{or} \qquad (Q^{-1}P)\gamma_\alpha = \gamma_\alpha (Q^{-1}P) \ .$$

Since $Q^{-1}P$ commutes with $\gamma_\alpha$ it commutes with all the matrices of (6.4). This can be the case only if $Q^{-1}P = \rho 1$, or $P = \rho Q$.

Theorem (7.2) follows as a corollary of Theorem (7.1) by observing that the full rotation group $H_{2\nu}$ in $E_{2\nu}$ is essentially the sub-group of $H^+_{2\nu+1}$ which leaves the hyperplane $X^o = 0$ invariant. Indeed, if $Y^i = L^i_j X^j$ is a proper rotation in $E_{2\nu}$

(7.6) $$Y^i = L^i_j X^j \ , \qquad Y^o = X^o$$

is a proper rotation in $E_{2\nu+1}$ and if $Y^i = L^i_j X^j$ is an improper rotation in $E_{2\nu}$

(7.7) $$Y^i = L^i_j X^j \ , \qquad Y^o = -X^o \ ,$$

is a proper rotation in $E_{2\nu+1}$. Conversely, any proper rotation in $E_{2\nu+1}$ which leaves the hyperplane $X^o = 0$ invariant, is either of the form (7.6) or (7.7), where in (7.6) $|L^i_j| = +1$ and in (7.7) $|L^i_j| = -1$.

The collineations of G which induce in $E_{2\nu+1}$ rotations of the special forms (7.6) and (7.7) are those which satisfy

(7.8) $$P \gamma_0 P^{-1} = \gamma_0 \qquad \text{and} \qquad P \gamma_0 P^{-1} = -\gamma_0$$

respectively. The collineations of G which satisfy either one or the other of these conditions constitute a group isomorphic to $H_{2\nu}$. This is, however, the group $G_0$ of all collineations which leave the linear family $X^i \gamma_i$ $(i = 1, 2, \ldots, 2\nu)$ invariant. To prove this we suppose that P transforms $E_{2\nu}$ into itself, so that $P\gamma_i P^{-1} \equiv \eta_i = \gamma_j L^j{}_i$. Then both $\gamma_0$ and $P\gamma_0 P^{-1}$ anticommute with all the $\eta_i$ and hence (cf. (6.8)) $P\gamma_0 P^{-1} = \pm \gamma_0$.

## 8. THE CASE k = 2.

The geometry of $\gamma$-sets as interpreted in $P_{k-1}$ makes considerable use of the geometry of involutions in the complex projective line $P_1$. As a preliminary to the geometry in the general case, we therefore discuss the case k = 2.

A matrix X of order two will define an involution if and only if its trace is zero (cf. § 11, Chapter I), and so the matrices of involutions in $P_1$ constitute the linear family

$$(8.1) \qquad X = \begin{Vmatrix} X^0 & iX^1+X^2 \\ -iX^1+X^2 & -X^0 \end{Vmatrix} \equiv X^\alpha \gamma_\alpha \qquad (\alpha = 0, 1, 2),$$

where the $\gamma_\alpha$ are given by (4.2). The numbers $X^\alpha$ $(\alpha = 0, 1, 2)$ are then coordinates of points in the centered complex Euclidean space $E_3$ and the plane in $E_3$ given by $X^0 = 0$ is our space $E_2$.

Using the notation described in § 7, Chapter IV, the invariant points of $\gamma^1$ and $\gamma^2$ are

$$(8.2) \quad [\gamma_1]^+ = (1, -i),\ [\gamma_1]^- = (1, i),\ [\gamma_2]^+ = (1, 1), \text{ and } [\gamma_2]^- = (1, -1).$$

In terms of a non-homogeneous coordinate $(z = \frac{\psi_1}{\psi_2})$ they are

$$(8.3) \qquad [\gamma_1]^+ = i, \; [\gamma_1]^- = -i, \qquad [\gamma_2]^+ = 1, \qquad [\gamma_2]^- = -1 \; .$$

These are two pairs of points which are harmonic conjugates with respect to one another. Moreover, this is a sufficient as well as a necessary condition in order that two involutions shall anticommute.

A harmonic set of points is determined by one pair of points and one point of the other pair. Hence in order to find a collineation carrying the $\gamma$-set $(\gamma_1, \gamma_2)$ into another $\gamma$-set $(\eta_1, \eta_2)$ it is only necessary to find a collineation carrying $[\gamma_1]^+$, $[\gamma_1]^-$ and $[\gamma_2]^+$ into $[\eta_1]^+$, $[\eta_1]^-$ and $[\eta_2]^+$, respectively. The existence and uniqueness of such a collineation is the fundamental theorem of projective geometry for the line. In terms of the non-homogeneous coordinate $z = \frac{\psi_1}{\psi_2}$ the collineations of $P_1$ are just the linear fractional transformations (cf. §4, Chapter I)

$$(8.4) \qquad w = \frac{az + b}{cz + d}$$

where a, b, c, and d are complex numbers and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$. It is evident that the ratios of a, b, c and d may be chosen so as to transform any three distinct points into any other three distinct points.

The invariant points of an involution anticommutative at once with $\gamma_1$ and $\gamma_2$ are harmonic conjugates with respect to both the pairs i, -i and 1, -1. The points are therefore uniquely determined to be 0 and $\infty$. Corresponding to the two possible orders in which these points may be given, we have the matrices $\begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}$ and $\begin{Vmatrix} -1 & 0 \\ 0 & +1 \end{Vmatrix}$ for $\gamma_0$. Choosing the first of these, we have $\gamma_0 = -i\gamma_1\gamma_2$. For $\nu = 1$ we have now proved the theorems of § 5.

Since by (8.1) the linear family $X^\alpha \gamma_\alpha$ includes all involutions in $P_1$, an arbitrary collineation must leave the space $E_3$ invariant and hence the group G of Theorem (7.1) is the entire collineation group in $P_1$. Theorem (7.1) states in this case that the proper orthogonal transformations on three variables are (1-1) isomorphic with the non-singular collineations in $P_1$. The parameterization of orthogonal matrices given by (3.8) becomes, for this special case,

$$(8.5)\quad \begin{Vmatrix} L^o{}_o & L^o{}_1 & L^o{}_2 \\ L^1{}_o & L^1{}_1 & L^1{}_2 \\ L^2{}_o & L^2{}_1 & L^2{}_2 \end{Vmatrix} = \frac{1}{2(ad-bc)} \begin{Vmatrix} 2(ad+bc) & -2i(ac+bd) & 2(bd-ac) \\ 2i(ab+cd) & a^2+b^2+c^2+d^2 & i(-a^2+b^2-c^2+d^2) \\ 2(cd-ab) & i(a^2+b^2-c^2-d^2) & (a^2-b^2-c^2+d^2) \end{Vmatrix}$$

where $P = \begin{Vmatrix} a & b \\ c & d \end{Vmatrix}$. Every proper orthogonal matrix of order three is uniquely expressible in this form for suitable values of the homogeneous parameters a, b, c, and d, and conversely the matrix is proper orthogonal provided only $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

An arbitrary matrix of order two is expressible in the form (cf. 6.5))

$$(8.6)\quad P = p1 + p^\alpha \gamma_\alpha = \begin{Vmatrix} p+p^0 & ip^1+p^2 \\ -ip^1+p^2 & p-p^0 \end{Vmatrix} .$$

This matrix will commute with $\gamma_0$ if and only if $p^1 = p^2 = 0$, that is if

$$(8.7)\quad P = \begin{Vmatrix} p+p^0 & 0 \\ 0 & p-p_0 \end{Vmatrix} .$$

We may put this matrix in the form

$$P = \rho \begin{Vmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{+i\frac{\theta}{2}} \end{Vmatrix} \tag{8.8}$$

if we take $\rho = \pm\sqrt{(p)^2-(p^0)^2}$ and $\theta = +i \log \frac{p+p^0}{p-p^0}$. Of course $\theta$ is in general complex.

The group $G_0^+$ of collineations which leave the linear family $X^1\gamma_1 + X^2\gamma_2$ invariant and commute with $\gamma_0$ therefore consists of collineations described by matrices of the form (8.8). Substituting from (8.8) in (8.5) gives

$$\begin{Vmatrix} L^o{}_o & L^o{}_1 & L^o{}_2 \\ L^1{}_o & L^1{}_1 & L^1{}_2 \\ L^2{}_o & L^2{}_1 & L^2{}_2 \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & +\sin\theta & \cos\theta \end{Vmatrix} \tag{8.9}$$

so that the equations $Y^i = L^i{}_j X^j (i, j = 1, 2)$ give the usual formulas for a rotation about the origin in the Euclidean plane.

If in a similar fashion we investigate the collineations which leave the family $X^1\gamma_1 + X^2\gamma_2$ invariant and which anticommute with $\gamma_0$ we find that their matrices are of the form

$$P = \rho \begin{Vmatrix} 0 & e^{+i\frac{\theta}{2}} \\ e^{-i\frac{\theta}{2}} & 0 \end{Vmatrix} \tag{8.10}$$

Substitution in (8.5) then gives for the corresponding orthogonal matrix

$$(8.11)\qquad \begin{Vmatrix} L^0{}_0 & L^0{}_1 & L^0{}_2 \\ L^1{}_0 & L^1{}_1 & L^1{}_2 \\ L^2{}_0 & L^2{}_1 & L^2{}_2 \end{Vmatrix} = \begin{Vmatrix} -1 & 0 & 0 \\ 0 & -\cos\theta & \sin\theta \\ 0 & \sin\theta & +\cos\theta \end{Vmatrix}$$

Since $\gamma_1$ and $\gamma_2$ anticommute with $\gamma_0$, so does every involution of the pencil $\lambda\gamma_1 + \mu\gamma_2$. Hence the invariant points of $\lambda\gamma_1 + \mu\gamma_2$ are harmonic conjugates with respect to the invariant points of $-i\gamma_1\gamma_2 = \gamma_0$. But if $\psi^A$ are coordinates of any point in $P_1$, not $[\gamma_0]^+$ or $[\gamma_0]^-$ we can choose $\lambda$ and $\mu$ so that

$$(8.12)\qquad (\lambda\gamma_1{}^A{}_B + \mu\gamma_2{}^A{}_B)\psi^B = \psi^A$$

and hence by a suitable choice of $\lambda$ and $\mu$ the involution $\lambda\gamma_1 + \mu\gamma_2$ will have as double points any given pair of distinct points harmonically conjugate with respect to $[\gamma_0]^+$ and $[\gamma_0]^-$.

The singular elements of the pencil $\lambda\gamma_1 + \mu\gamma_2$ are $\gamma_1 + i\gamma_2$ and $\gamma_1 - i\gamma_2$. The first of these carries every point, except $[\gamma_0]^+$ into $[\gamma_0]^+$ as we see from the equation

$$(8.13)\qquad \gamma_1 + i\gamma_2 = \begin{Vmatrix} 0 & 2i \\ 0 & 0 \end{Vmatrix}$$

Similarly, $\gamma_1 - i\gamma_2$ carries every point, except $[\gamma_0]^-$, into $[\gamma_0]^-$.

## 9. COMMUTATIVE AND ANTICOMMUTATIVE INVOLUTIONS

If A and B are matrices defining two non-singular involutions in

$P_{k-1}$, they satisfy the equations

(9.1) $$A^2 = a1, \text{ and } B^2 = b1$$

where, since k is the order of the matrices,

(9.2) $$a = \frac{1}{k} \text{ Trace } A^2 \quad \text{and} \quad b = \frac{1}{k} \text{ Trace } B^2 .$$

Regarding the involutions as permutations of the points of $P_{k-1}$, the condition that their product be the same in either order is

(9.3) $$A^{B} = \rho^{BA}$$

where $\rho$ is any number $\neq 0$. To determine $\rho$, we multiply the equation on the left and on the right by A, getting $a(BA) = \rho a(AB)$. Hence $BA = \rho AB = \rho^2 BA$ and $\rho = \pm 1$. We distinguish between the case in which $\rho = \pm 1$ and the case in which $\rho = -1$ by saying that the involutions are commutative or anticommutative, respectively. In either case the involutions are commutative in the sense of transformations, but there is a geometric distinction between commutative and anticommutative involutions.

If the matrices A and B ($A^2 = B^2 = 1$) of two involutions commute, then $AB = BA$ so that if $\psi$ is a point of $[A]^+$, we have

$$A(B\psi) = B\psi$$

and hence $B\psi$ is also a point of $[A]^+$. Hence $[A]^+$ is invariant under B and must therefore cross $[B]^+$ and $[B]^-$ or be contained in one of them.

Similarly, $[A]^-$ crosses $[B]^+$ and $[B]^-$ or is contained in one of them. Choosing the vertices of the reference k-point to be points in the four intersections of $[A]^+$ and $[A]^-$ with $[B]^+$ and $[B]^-$, the matrices A and B are of the forms

$$(9.4) \qquad A = \left\| \begin{array}{cc} 1_r & 0 \\ 0 & -1_s \end{array} \right\| \qquad \text{and} \qquad B = \left\| \begin{array}{cc|cc} 1_{r_1} & 0 & & \\ 0 & -1_{r_2} & & 0 \\ \hline & & 1_{s_1} & 0 \\ & 0 & 0 & -1_{s_2} \end{array} \right\|$$

where $r_1 + r_2 = r$ and $s_1 + s_2 = s = k-r$. Continuing the specialization of the coordinate system, it is possible simultaneously to take any number of commutative involutions in diagonal form. This is a result well known from the theory of matrix algebra.

In a three-dimensional projective space this result means that any set of commuting involutions leaves a tetrahedron invariant and consists at most of (1) the point-plane reflections in the four vertices and their opposite faces, and (2) the three line reflections in the pairs of opposite edges. In the general case a set of commuting involutions is a subset of the set of all reflections in pairs of opposite linear subspaces of a k-point in a (k-1)-dimensional space.

## 10. EQUIVALENCE OF PAIRS OF ANTICOMMUTING INVOLUTIONS

If A and B are matrices of anticommuting involutions

$$(10.1) \qquad AB = -BA$$

and if $A\psi = +\psi$ then $A(B\psi) = -(B\psi)$, and conversely. Hence B exchanges the two pointwise invariant subspaces $[A]^+$ and $[A]^-$. Similarly, A exchanges $[B]^+$ and $[B]^-$. This is possible only if the dimensions of $[A]^+$ and $[A]^-$ as well as of $[B]^+$ and $[B]^-$ are equal. Of course no two of these four axes can intersect. From (9.4) we see that the difference between the dimensions of $[A]^+$ and $[A]^-$, $(r-1)-(k-r-1)$, is equal to trace $PAP^{-1}$ = trace A, so that if A and B satisfy (10.1) we must have*

*These equations also follow by taking the trace of both members of the equations $BAB^{-1} = -A$ and $ABA^{-1} = -B$. It was this method that we used to prove Trace $\gamma_\alpha = 0$ in § 6, but we now see the geometric interpretation of this result.

(10.2) $$\text{Trace } A = \text{Trace } B = 0.$$

If the matrices A and B are of order k, they define involutions in a complex projective space of (k-1)-dimensions, $P_{k-1}$. The pointwise invariant spaces of A and B must be of equal dimensionality and so are $(\frac{k}{2}-1)$-spaces. This is possible only if k is even and consequently pairs of anticommuting involutions exist only in projective spaces of an odd number of dimensions. We shall speak of $[A]^+$ and $[A]^-$ as the axes of the involution determined by A and as a natural extension of this usage we shall call any linear $(\frac{k}{2}-1)$-dimensional sub-space of $P_{k-1}$ an axis. The "axes" of a space are then the self-dual linear sub-spaces of the space.

The two ordered pairs of axes $[A]^+$, $[A]^-$, and $[B]^+$, $[B]^-$ completely determine (in a given coordinate system) the two matrices A and B in view of the normalization $A^2 = B^2 = 1$. However, because A and B anticommute it is only necessary to give three of these spaces, the fourth, say $[B]^-$, being determined as the transform of $[B]^+$ by the involution with axes $[A]^+$ and $[A]^-$.

Putting $k = 2k_1$, $[B]^+$ is a $(k_1-1)$-space and so is determined as the join of any $k_1$ of its points which do not lie in the same $(k_1-2)$-space. Through each of the points of such a set of $k_1$ points there goes a unique line intersecting both $[A]^+$ and $[A]^-$. These $k_1$ lines intersect $[A]^+$ and $[A]^-$ in two sets of $k_1$ points which are $2k_1 = k$ linearly independent points. We are now in a position to observe that any pair of anticommuting matrices, say A and B, are equivalent under collineation to any other pair, say $A_1$ and $B_1$. For, since any k linearly independent points may be carried into any other similar set, we may transform $[A]^+$ and $[A]^-$ into the axes $[A_1]^+$ and $[A_1]^-$ and indeed may carry any set of $k_1$ independent lines intersecting $[A]^+$, $[A]^-$ and $[B]^+$ into any similar set intersecting $[A_1]^+$, $[A_1]^-$ and $[B_1]^+$. Then a suitable collineation on each of these $k_1$ lines will carry $[B]^+$ into $[B]^-$ and yet leave $[A]^+ = [A_1]^+$ and $[A]^- = [A_1]^-$ invariant.

Thus we have the theorem:

<u>Theorem (10.1)</u>. If A, B, $A_1$ and $B_1$ are matrices of order $k = 2k_1$ satisfying the conditions

$$A^2 = B^2 = 1, \qquad AB = -BA,$$

and

$$A_1^2 = B_1^2 = 1, \qquad A_1B_1 = -B_1A_1$$

then there exists a non-singular collineation represented by the matrix P such that

$$PAP^{-1} = A_1 \text{ and } PBP^{-1} = B_1 \ .$$

## 11. THE REGULI DETERMINED BY TWO ANTICOMMUTING INVOLUTIONS

A line which intersects any three of the four axes of A and B will also intersect the fourth. For example, if the line intersects $[A]^+$, $[A]^-$, and $[B]^+$, it is invariant under A and the transform under A of its intersection with $[B]^+$ is a point of $[B]^-$ which also lies on the line. By analogy with the usual definition of a regulus in 3-space, we shall call the system of lines which intersect three non-intersecting axes a regulus of lines or, briefly, a line-regulus.

The lines of a line regulus may also be defined as the lines invariant under each of two anticommuting involutions. This suggests the desirability of considering the set of all linear sub-spaces of $P_{k-1}$ which are invariant under A and B. This set contains no points since we have already seen that the axes of A cannot intersect the axes of B. Hence no two lines of the line-regulus can intersect.

No space of an even number of dimensions can be invariant under both A and B for if there were such a space, A and B would induce in it two anticommuting involutions which we know cannot exist. Furthermore, any space pointwise invariant under one of the involutions A or B, is not invariant under the other.

Any linear space S, of p-1 dimensions, which is invariant under A and B is also invariant under the product $K = -iAB$. Moreover $K^2 = 1$ and K anticommutes with A and with B. Consequently S intersects $[K]^+$ and $[K]^-$ in the axes of an involution in S. Hence the intersection of S and $[K]^+$ is a space, $S^+$, of $\frac{p}{2} - 1$ dimensions. Since $S^+$ is transformed into $S^-$, the intersection of S and $[K]^-$, by the matrix A (or B), S is determined by $S^+$. Moreover if $S^+$ is an arbitrary subspace of $[K]^+$ then its join with its transform under A (or B) is invariant under both A and B.

The join of two invariant spaces S and T will again be an invariant space and will intersect $[K]^+$ in the join of $S^+$ and $T^+$. A similar remark holds for the intersection of two invariant spaces and therefore the linear spaces invariant under both A and B are in (1-1) correspondence with the linear subspaces of $[K]^+$. This correspondence is such that a point of $[K]^+$ corresponds to the line of the line-regulus which intersects it; a line of $[K]^+$ corresponds to a three-space containing the lines of the line regulus which intersect it; and so on.

In 3-dimensional projective space the regulus conjugate to a line-regulus is again a line-regulus, but in $P_{k-1}$ $(k > 4)$ this is not the case. Indeed, in $P_{k-1}$ the conjugate regulus by definition consists of the set of axes each of which intersects all the lines of the line-regulus. We may call this conjugate regulus an axis-regulus.

The lines of a line regulus could have been defined as the system of lines invariant under both A and B. The axes of the conjugate regulus may be defined as the axes of the pencil of involutions.

$$(11.1) \qquad I = \lambda A + \mu B \qquad (\lambda \text{ and } \mu \text{ not both zero}).$$

The axes of I belong to the axis-regulus, for if $\psi$ is a point on any line of the line regulus, then $I\psi = \lambda A\psi + \mu B\psi$ is a point of the same line, which is therefore invariant under I. To show that every axis of the regulus is an axis of I for suitable values of $\lambda$ and $\mu$ may be accomplished by observing that the problem reduces to the consideration of any one (necessarily typical) line of the line regulus and referring to the discussion of § 8.

The singular elements of the pencil, namely,

$$K_1 = A + iB, \quad \text{and} \quad K_2 = A - iB , \tag{11.2}$$

transform every point not on $[K]^+$ or $[K]^-$, respectively, into a point of an axis of K. Thus if $K_1\psi = \psi_1 \neq 0$ we have

$$K\psi_1 = (KA + iKB)\psi = (iB + A)\psi = \psi_1$$

and consequently $\psi_1$ is a point of $[K]^+$, and it is indeed any such point. Similarly $K_2$ carries an arbitrary point, not on $[K]^-$, into a point of $[K]^-$. We may regard $[K]^+$ and $[K]^-$ as the coincident axes of $K_1$ and $K_2$, respectively.

12. The geometric discussion of the two preceding paragraphs may easily be paralleled by an algebraic one. Thus the matrix $K = -iAB$ may be taken in the form

$$K = \begin{Vmatrix} {}^1k_1 & 0 \\ 0 & -{}^1k_1 \end{Vmatrix} \tag{12.1}$$

by performing a collineation which carries $[K]^+$ and $[K]^-$ into the axes

$$\psi^{k_1+1} = \psi^{k_1+2} = \dots = \psi^k = 0 \quad \text{and} \quad \psi^1 = \psi^2 = \dots = \psi^{k_1} = 0 ,$$

respectively, where $k_1 = k/2$.

Since B anticommutes with K and has its square equal to one, a simple calculation shows it to be of the form

$$B = \begin{Vmatrix} 0 & B_1 \\ B_1^{-1} & 0 \end{Vmatrix} \tag{12.2}$$

The collineation with matrix $P = \begin{Vmatrix} B_1^{-1} & 0 \\ 0 & 1 \end{Vmatrix}$ will therefore transform B into

$$PBP^{-1} = \begin{Vmatrix} 0 & 1_{k_1} \\ 1_{k_1} & 0 \end{Vmatrix} \tag{12.3}$$

and leave K invariant. After this collineation is performed the matrix A = iKB will be

$$\begin{Vmatrix} 0 & i1_{k_1} \\ -i1_{k_1} & 0 \end{Vmatrix} \tag{12.4}$$

We therefore have

Theorem (12.1). If A and B are matrices of anticommuting involutions and $A^2 = B^2 = 1$, then there exists a non-singular matrix P such that

$$PAP^{-1} = \begin{Vmatrix} 0 & i1_{k_1} \\ -i1_{k_1} & 0 \end{Vmatrix} \quad \text{and} \quad PBP^{-1} = \begin{Vmatrix} 0 & 1_{k_1} \\ 1_{k_1} & 0 \end{Vmatrix} \tag{12.5}$$

Hence the normalized matrices of two anticommuting involutions may simultaneously be taken in the given canonical forms.

If we determine what conditions the equations

$$PAP^{-1} = A \qquad \text{and} \qquad PBP^{-1} = B$$

impose on P when A and B are in their canonical forms (12.5), we find that

$$P = \begin{Vmatrix} P_1 & 0 \\ 0 & P_1 \end{Vmatrix} \tag{12.6}$$

where $P_1$ is an arbitrary non-singular matrix of order $k_1 = k/2$. A collineation which leaves both A and B invariant therefore determines an arbitrary collineation on $[K]^+$ and the same collineation on $[K]^-$. Since it leaves $[A]^+$, $[A]^-$, $[B]^+$ and $[B]^-$ invariant, it leaves invariant every axis of the axis-regulus containing them and effects the same collineation in each of these axes. The collineation in $P_{k-1}$ therefore determines and is determined by the collineation which it induces in the $P_{k/2-1}$ consisting of the linear sub-spaces invariant under both A and B.

A matrix $\Gamma$ will anticommute with A and B if and only if the product $\Gamma K$ commutes with both A and B. For,

$$(\Gamma K)A = A(\Gamma K) \quad \text{implies} \quad \Gamma A = -A\Gamma \ , \tag{12.7}$$

and conversely. A similar statement of course holds for B. Hence $\Gamma$ must be of the form

$$\Gamma = (\Gamma K)K = \begin{Vmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_1 \end{Vmatrix} \begin{Vmatrix} 1_{k_1} & 0 \\ 0 & -1_{k_1} \end{Vmatrix} = \begin{Vmatrix} \Gamma_1 & 0 \\ 0 & -\Gamma_1 \end{Vmatrix} \tag{12.8}$$

Performing a collineation of the form (12.6) gives

$$P\Gamma P^{-1} = \begin{Vmatrix} P_1\Gamma_1 P_1^{-1} & 0 \\ 0 & -P_1\Gamma_1 P_1^{-1} \end{Vmatrix} \tag{12.9}$$

and so has the effect of an arbitrary collineation, with matrix $P_1$, on the matrix $\Gamma_1$.

The matrices anticommuting with A and B are of the form (12.8) and $\Gamma^2 = 1$ is equivalent to $(\Gamma_1)^2 = 1$. Two matrices, $\Gamma$ and $\Lambda$ of the form (12.8) with $\Gamma^2 = \Lambda^2 = 1$ will be similar under collineations of the form (12.6) if and only if (cf. § 7, Chapter IV)

$$\text{Trace } \Gamma_1 = \text{Trace } \Lambda_1 , \tag{12.10}$$

that is to say, if the corresponding axes of $\Gamma$ and $\Lambda$ intersect $[K]^+$ in spaces of the same number of dimensions. This condition is equivalent to

$$\text{Trace } K\Gamma = \text{Trace } K\Lambda \tag{12.11}$$

on account of the fact that $\Gamma$ and $\Lambda$ anticommute with both A and B and are therefore of the form (12.8).

## 13. PROOF OF THEOREMS (5.1) to (5.4)

With the geometrical discussion of the preceding sections in mind, we are in a position to prove the theorems of § 5.

Theorem (5.1) has been proved for $k = 2^{\nu}\ell$ ($\ell$ odd) when $\nu = 0$, for we have seen that two anticommuting involutions cannot exist when $P_{k-1}$ is of even dimensionality. Of course a single matrix of order k satisfying $\gamma^2 = 1$ can always be found; it is 1, -1 or the matrix of an involution. We may therefore prove the theorem by induction on $\nu$, taking $\ell$ fixed. But any $\gamma$-set of N (> 2) matrices of order k gives rise to a $\gamma$-set of N-2 matrices of order $k_1 = k/2$ by taking two matrices of the first set in the canonical forms (12.5) so that the remaining matrices are of the form (12.8). Then the maximum number of matrices of order $k = 2^{\nu}\ell$ in a $\gamma$-set is two greater than the maximum number in a set of matrices of order $k_1 = 2^{\nu-1}\ell$. But for every odd $\ell$, this latter maximum is $2(\nu-1) + 1$ by the induction hypothesis. Hence there are $2(\nu-1) + 1 + 2 = 2\nu + 1$ matrices in a maximal $\gamma$-set of order $k = 2^{\nu}\ell$, which proves the theorem.

Theorem (5.2) states the similarity of any two even $\gamma$-sets containing the same number of matrices and this was proved in the lowest case (for two matrices) in Theorem (12.1). We may therefore use induction on the number of matrices, the induction going by steps of two matrices and the assumption being that the theorem is true for 2p matrices of any order. But if $\eta_i$ $(i = 1, 2, \ldots, 2p+2)$ is a $\gamma$-set of 2p+2 matrices it is similar, by the results of § 12, to the set

$$(13.1)\quad \begin{Vmatrix} \eta_{1j} & 0 \\ 0 & -\eta_{1j} \end{Vmatrix} \quad (j = 1, 2, \ldots, 2p), \quad \begin{Vmatrix} 0 & i1_{k_1} \\ -i1_{k_1} & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 1_{k_1} \\ 1_{k_1} & 0 \end{Vmatrix} .$$

By the hypothesis of the induction, the set $\eta_{1j}$ is similar to any other set of the same number of matrices. Since any $\gamma$-set of 2p+2 matrices is similar to the set given by equations (13.1), any two $\gamma$-sets of 2p+2 matrices are similar.

Theorem (5.3) was proved in § 7, Chapter (IV) for a $\gamma$-set consisting of one matrix. We again use induction on the number of matrices in the set, supposing the theorem true for $\gamma$-sets of 2p-1 matrices of any order and proving it for 2p+1 matrices. As before, we take two matrices, say

$\eta_{2p-1}$ and $\eta_{2p}$ of the $\gamma$-set $\eta_\alpha$ ($\alpha = 0, 1, \ldots, 2p$) to be in the canonical form of (12.5), and then the remaining ones will be of the form

$$(13.2) \qquad \begin{Vmatrix} \eta_{1\beta} & 0 \\ 0 & -\eta_{1\beta} \end{Vmatrix} \qquad (\beta = 0, 1, \ldots, 2p-2).$$

Similarly, we may reduce the matrices of another $\gamma$-set, say $\zeta_\alpha$, to be of the same form. Using the matrices (12.5) for $\eta_{2p-1}$ and $\eta_{2p}$ it is clear that the equation

$$(13.3) \qquad \text{Trace } (\eta_0 \eta_1 \eta_2 \cdots \eta_{2p}) = 2i \text{ Trace } (\eta_{10} \eta_{11} \eta_{12} \cdots \eta_{1(2p-2)})$$

holds for the set $\eta_\alpha$ and a similar equation for $\zeta_\alpha$. By the hypothesis of Theorem (5.3),

$$(13.4) \qquad \text{Trace } (\eta_0 \eta_1 \cdots \eta_{2p}) = \text{Trace } (\zeta_0 \zeta_1 \cdots \zeta_{2p}) \quad .$$

This, however, implies the hypothesis of the theorem for the $\gamma$-sets $\eta_{1\beta}$ and $\zeta_{1\beta}$ ($\beta = 0, 1, \ldots, 2p-2$) which are therefore similar by the hypothesis of the induction.

To prove Theorem (5.4) we make use of Theorem (5.2) to establish the existence of a matrix P such that

$$(13.5) \qquad P \eta_i P^{-1} = \gamma_i^{(\nu)} \quad (i = 1, 2, \ldots 2\nu)$$

Then $P \eta_0 P^{-1}$ and $\gamma_0^{(\nu)}$ both anticommute with all the matrices $\gamma_i^{(\nu)}$. Hence

$$(13.6) \qquad P \eta_0 P^{-1} = \pm \gamma_0 ,$$

where the sign agrees with that in the equation

(13.7) $\quad \text{Trace}\,(\eta_0\eta_1\cdots\eta_{2\nu}) = \pm\,\text{Trace}\,(\gamma_0^{(\nu)}\gamma_1^{(\nu)}\cdots\gamma_{2\nu}^{(\nu)}) = \pm(i)^{\nu}2^{\nu}.$

# Chapter VI

# THE EXTENSION TO CORRELATIONS

## 1. THE DUAL MAPPING OF $E_{2\nu+1}$ ONTO ITSELF

In the last chapter we regarded the matrices $X = X^o\gamma_o + X^1\gamma_1 + \dots + X^{2\nu}\gamma_{2\nu}$ of the linear family $X^\alpha\gamma_\alpha$ as defining point $\longrightarrow$ point transformations of a complex projective space $P_{2\nu-1}$ by means of the equation

(1.1) $$||\varphi^A|| = X||\psi^B|| , \quad \text{or} \quad \varphi^A = X^A{}_B\psi^B \ .$$

If the matrix X is non-singular, the collineation (1.1) may be described equally well as a hyperplane $\longrightarrow$ hyperplane transformation by the equations

(1.2) $$||\varphi_A|| = \rho\check{X}||\psi_B|| ,$$

where

$$\check{X} = ||X^A{}_B||'^{-1} \ .$$

The operation of taking the transposed inverse is one which maps the set of all non-singular matrices upon itself. It is well known that this mapping does not extend in a (1-1) and continuous way to the singular matrices. However, by a suitable choice of the factor $\rho$, the mapping

(1.3) $$X \longrightarrow \rho\check{X}$$

of the non-singular matrices of the $(2\nu+1)$-parameter set $X^\alpha\gamma_\alpha$ may be extended to include the singular matrices of the set. Indeed, since $X^2 = 2^{-\nu}(\text{Trace } X^2)1$,

(1.4) $$\check{X} = 2^\nu(\text{Trace } X^2)^{-1}X' ,$$

and consequently the mapping (1.3) may be written in the form $X \longrightarrow \rho 2^\nu(\text{Trace } X^2)X'$. Choosing $\rho$ to be $\sigma 2^{-\nu}$ Trace $X^2$, where $\sigma$ is independent of X, the mapping becomes linear and is extended to the singular matrices of the family in the obvious way.

We shall find it convenient to take $\rho = (-1)^{\nu+1}2^{-\nu}$ Trace $X^2$ and we accordingly associate with a point $X^\alpha$ of $E_{2\nu+1}$ the hyperplane $\longrightarrow$ hyperplane transformation

(1.5) $$||\varphi_A|| = (-1)^{\nu+1}\, X'\,||\psi_B||$$

as well as the point $\longrightarrow$ point transformation (1.1). When X is a singular matrix of the linear family $X^\alpha\gamma_\alpha$, (1.1) transforms the points of a hyperplane into points of a linear space of at most $2^{\nu-1}-1$ dimensions and so the point $\longrightarrow$ point transformation does not induce a hyperplane $\longrightarrow$ hyperplane transformation in the same way that a non-singular transformation does. However, both (1.1) and (1.5) determine a permutation of the axes (i.e., the subspaces of $2^{\nu-1}-1$ dimensions) of $P_{2^\nu-1}$ and by choosing a coordinate system in which (cf. (11.2) and (12.5) of Chapter V)

(1.6) $$X = 2i \begin{Vmatrix} 0 & 1_{2^{\nu-1}} \\ 0 & 0 \end{Vmatrix}$$

it is easily shown that these two permutations are identical. It is in this sense that we shall speak of (1.1) and (1.5) as defining the same (singular) involution.

## 2. REPRESENTATION OF IMPROPER ORTHOGONAL TRANSFORMATIONS

A correlation $\psi^B \longrightarrow \varphi^*_A$ defined by the equations

(2.1) $$\varphi^*_A = Q_{AB}\psi^B$$

transforms the point $\longrightarrow$ point collineation (1.1) into the hyperplane $\longrightarrow$ hyperplane collineation

(2.2) $$\varphi^*_A = Q_{AB}X^B{}_C Q^{DC}\psi^*_D\,, \qquad \text{or} \qquad ||\varphi^*_A|| = (QXQ^{-1})||\psi^*_B||\,,$$

where $Q = ||Q_{AB}||$ and $||Q^{DC}|| = Q^\cup$.

If the correlation is to transform every involution of the linear family into an involution belonging to the family, the matrix $Q(X^\alpha\gamma_\alpha)Q^{-1}$ must define a collineation of the set (1.5) with matrix $(-1)^{\nu+1}(Y^\alpha\gamma_\alpha)'$ so that

(2.3) $$(Q\gamma_\beta Q^{-1})X^\beta = (-1)^{\nu+1}\gamma'_\alpha Y^\alpha$$

for all values of $X^\beta$. Hence the correlation (2.1) induces in $E_{2\nu+1}$ the linear transformation

(2.4) $$Y^\alpha = L^\alpha{}_\beta \; X^\beta$$

by means of the equations

(2.5) $$Q\gamma_\beta Q^{-1} = (-1)^{\nu+1} \gamma'_\alpha L^\alpha{}_\beta \quad .$$

Solving these last equations for $L^\alpha{}_\beta$ we obtain

(2.6) $$L^\alpha{}_\beta = (-1)^{\nu+1} 2^{-\nu} \cdot \text{Trace} \; (\gamma^{\alpha'} Q \gamma_\beta Q^{-1}) \quad .$$

Instead of using the point ⟶ hyperplane equations (2.1) of a correlation, we might have employed hyperplane ⟶ point equations

(2.7) $$\varphi^{A*} = R^{AB} \psi_B$$

where $||R^{AB}|| = ||R_{AB}||'^{-1}$. The family of transformations (1.5) would then be carried into the family (1.1) by the equations

(2.8) $$\check{R}[(-1)^{\nu+1} X']R' = Y \qquad \text{or}$$
$$[(-1)^{\nu+1} \check{R} \gamma'_\alpha R']X^\alpha = \gamma_\beta Y^\beta \quad .$$

Consequently, $Y^\alpha = M^\alpha{}_\beta X^\beta$ and $(-1)^{\nu+1} \check{R} \gamma'_\alpha R' = \gamma_\beta M^\beta{}_\alpha$. When $Q = R$ these equations are equivalent to (2.4) and (2.5).

Combining the correlations (2.1) and (2.7) we get the collineation $||\varphi^A|| = P||\psi^B||$, where $P = \check{R}Q$. The corresponding transformation in $E_{2\nu+1}$ is

(2.9) $$Y^\alpha = (M^\alpha{}_\beta \, L^\beta{}_\gamma) X^\gamma$$

where

(2.10) $$P\gamma_\gamma P^{-1} = \gamma_\alpha (L^\alpha{}_\beta \, M^\beta{}_\gamma) \; .$$

Comparing these equations with equations (7.1) of Chapter V, we see that $||L^\alpha{}_\beta M^\beta{}_\gamma||$ is the proper orthogonal matrix corresponding to the collineation $||\varphi^A|| = P||\psi^B||$ under the isomorphism of Theorem (7.1), Chapter V.

The improper orthogonal matrices of order $2\nu + 1$ are all obtained if we multiply the proper orthogonal matrices by a single improper one. Similarly, all correlations which transform $E_{2\nu+1}$ into itself are obtained by multiplying the collineations leaving $E_{2\nu+1}$ invariant by a single correlation with this property. We shall therefore extend the isomorphism of Theorem (7.1), Chapter V, to a representation of improper orthogonal matrices by means of correlations in the spin space if we show that a single correlation corresponds under (2.5) to an improper orthogonal matrix. In a coordinate system in which the matrices $\gamma_\alpha$ take the forms given in (4.4), Chapter V, we have

$$(2.11) \qquad \gamma'_\alpha = (-1)^\alpha \gamma_\alpha \qquad (\alpha \text{ not summed!}) ,$$

and consequently in this coordinate system the correlation described by the unit matrix corresponds under (2.5) to the linear transformation

$$(2.12) \quad Y^o = (-1)^{\nu+1}X^o, \; Y^1 = (-1)^{\nu}X^1, \; Y^2 = (-1)^{\nu+1}X^2, \; \ldots, \; Y^{2\nu} = (-1)^{\nu+1}X^{2\nu}.$$

The right members of these equations contain $(\nu+1)$ minus signs if $\nu$ is even and $\nu$ minus signs if $\nu$ is odd. Hence the orthogonal transformation induced in $E_{2\nu+1}$ by the "unit" correlation is always improper. This result of course depends upon our having chosen the numerical factor in (2.5) in the way that we did.

Our results on the spinor representation of the orthogonal group are summarized in the theorem

<u>Theorem</u> <u>(2.1)</u>. The group $H_{2\nu+1}$ of all rotations about the origin in $E_{2\nu+1}$ is (1-1) isomorphic with the group $\mathcal{G}$ of all the collineations and correlations of the spin space $P_{2\nu-1}$ which transform the linear family of involutions defined by the matrices $X^\alpha\gamma_\alpha$ into itself. In this isomorphism the collineation with matrix $P = ||P^A{}_B|| = ||P_A{}^B||^{-1}$ corresponds to the proper orthogonal matrix determined by the equations

$$(2.13) \qquad P\gamma_\alpha P^{-1} = \gamma_\beta L^\beta{}_\alpha, \qquad \text{or} \qquad P^A{}_B\gamma_\beta{}^B{}_C P_D{}^C = \gamma_\alpha{}^A{}_D L^\alpha{}_\beta \;,$$

$$|L^\alpha{}_\beta| = -1,$$

and the correlation with matrix $R = ||R_{AB}|| = ||R^{AB}||'^{-1}$ corresponds to the improper orthogonal matrix determined by the equations

$$(2.4) \qquad R\gamma_\alpha R^{-1} = (-1)^{\nu+1}\gamma_\beta' L^\beta{}_\alpha, \quad \text{or} \quad R_{AB}\gamma_\beta{}^B{}_C R^{DC} = (-1)^{\nu+1}\gamma_\alpha{}^D{}_A L^\alpha{}_\beta, \quad |L^\alpha{}_\beta| = -1 .$$

We shall refer to the isomorphism of this theorem as the isomorphism (or representation) $\Delta_{2\nu+1}$. If only the correspondence between proper rotations and collineations in the spin space is in question, we shall speak of the isomorphism $\Delta^+_{2\nu+1}$.

## 3. THE INVARIANT POLARITY

The improper rotation

$$(3.1) \qquad Y^\alpha = -X^\alpha$$

is of period two and is commutative with all rotations. Hence its image in the spin space under $\Delta_{2\nu+1}$ is a correlation

$$(3.2) \qquad ||\varphi_A|| = C||\psi^B||, \quad \text{or} \quad \varphi_A = C_{AB}\psi^B ,$$

which is of period two and commutes with all the collineations and correlations of the group $\mathcal{G}$.

In order for (3.2) to be of period two we must have $C' = eC$, where $e = \pm 1$. To determine e we observe that (3.1) is the product of (2.12) and a proper rotation. In the coordinate system of (4.4), Chapter V, this proper rotation is the image under $\Delta_{2\nu+1}$ of the collineation with matrix $\gamma_1\gamma_3\gamma_5 \cdots \gamma_{2\nu-1}$. Hence in this coordinate system

$$(3.3) \qquad C = \rho\gamma_1\gamma_3\gamma_5 \cdots \gamma_{2\nu-1}, \quad \text{or} \quad C_{AB} = \rho\delta_{AC}\gamma_1{}^C{}_D\gamma_3{}^D{}_E \cdots \gamma_{2\nu-1}{}^M{}_B .$$

Using (2.11) we find

(3.4) $$C' = eC, \quad \text{where } e = (-1)^{\frac{\nu(\nu+1)}{2}} = \begin{cases} +1 \text{ when } \nu \equiv 3 \text{ or } 0 \pmod 4, \\ -1 \text{ when } \nu \equiv 1 \text{ or } 2 \pmod 4. \end{cases}$$

Since (3.1) and (3.2) correspond to one another under $\Delta_{2\nu+1}$, equations (2.14) give $C\gamma_\alpha C^{-1} = (-1)^\nu \gamma'_\alpha$, or

(3.5) $$(C\gamma_\alpha)' = f(C\gamma_\alpha) \quad \text{where} \quad f = (-1)^\nu e.$$

Multiplying by $X^\alpha$ and summing, we observe that the product of the invariant polarity and an arbitrary involution of the linear family is of period two. This property of the correlation defined by C is sufficient to determine C to within a factor. For, if $(DX)' = g(DX)$ for all matrices $X = X^\alpha\gamma_\alpha$ and D is assumed non-singular, the scalar g cannot depend on X and we have $D\gamma_\alpha \check{D} = g^{-1}\gamma'_\alpha$. Consequently $(\check{D}D)\gamma_\alpha = \gamma_\alpha(\check{D}D)^{-1}$ (since $g^2 = 1$) so that $\check{D}D$ is a multiple of the identity matrix and $D' = \pm D$. Using this we get $(\check{D}C)\gamma_\alpha(\check{D}C)^{-1} = \pm\gamma_\alpha$. Here the minus sign is excluded, for it would imply $(\check{D}C)\nu_\alpha = -\nu_\alpha(\check{D}C)$, $\alpha = 0, 1, \ldots, 2\nu$, and the argument at the end of Chapter V, §6 shows this to be impossible for $\check{D}C \neq 0$. This then implies that C is a multiple of $D'$ and hence a multiple of D.

We have now proved the theorem.

<u>Theorem (3.1)</u>. There exists exactly one correlation which transforms every involution of the family $X^\alpha\gamma_\alpha$ into itself. This correlation is the image of $Y^\alpha = -X^\alpha$ under $\Delta_{2\nu+1}$ and its matrix. C is determined to within a factor by the equations $(C\gamma_\alpha)' = f(C\gamma_\alpha)$, which have a solution only if $f = (-1)^{\frac{\nu+\nu(\nu+1)}{2}}$. The type of polarity described by C and by $C(X^\alpha\gamma_\alpha)$ is given in the table:

| | C defines a polarity with respect to a | $C(X^\alpha\gamma_\alpha)$ defines a polarity with respect to a | The axes of $(X^\alpha\gamma_\alpha)$ are |
|---|---|---|---|
| $\nu \equiv 0 \pmod 4$ | quadric | quadric | exchanged |
| $\nu \equiv 1 \pmod 4$ | linear complex | quadric | separately invariant |
| $\nu \equiv 2 \pmod 4$ | linear complex | linear complex | exchanged |
| $\nu \equiv 3 \pmod 4$ | quadric | linear complex | separately invariant |

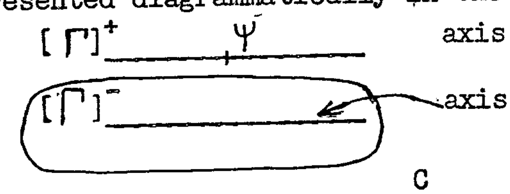
The last column of this table will be explained in the next section. All four possible combinations of signs for e and f occur and the corresponding geometrical figure in the spin space is accordingly dependent upon the value of $\nu$.

4. GEOMETRICAL PROPERTIES OF THE INVARIANT POLARITY

The geometrical relationship between the invariant polarity and the axial reflections of the family $X^\alpha \gamma_\alpha$ will be made clear by a discussion of the possible relationships which may exist between a polarity C and a single axial reflection, $\Gamma$, with the pointwise-invariant spaces $[\Gamma]^+$ and $[\Gamma]^-$.

If C is a polarity in a quadric and the axes of $\Gamma$ are interchanged by C, the situation is represented diagrammatically in the figure:

(4.1)

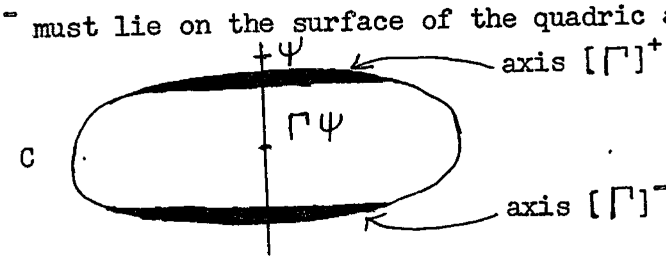



The product $C\Gamma$ is a polarity in a quadric since if $\psi$ is a point on $[\Gamma]^+$, then $(C\Gamma)\psi = C\psi$ is a hyperplane which does not contain $\psi$. Hence

(4.2) $$C' = +C, \quad (C\Gamma)' = +C\Gamma \quad .$$

If C is a polarity in a quadric and the axes are separately invariant, then $[\Gamma]^+$ and $[\Gamma]^-$ must lie on the surface of the quadric and we have the diagram:

(4.3)

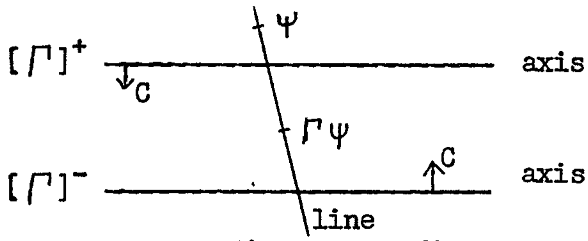



If $\psi$ is any point in the space its transform under $C\Gamma$ is a hyperplane containing it and hence $C\Gamma$ is a null-polarity. That is,

(4.4) $$C' = +C \quad (C\Gamma)' = -(C\Gamma) \quad .$$

Similarly if C defines a null-polarity which interchanges the axes of $\Gamma$, we have the diagram:

(4.5)

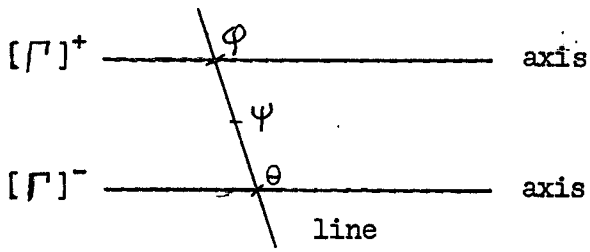



The transform of an arbitrary point, $\psi$, under $C\Gamma$ contains the point and hence

(4.6) $$C' = -C, \qquad (C\Gamma)' = -(C\Gamma) \quad .$$

Finally, if C defines a null-polarity which leaves $[\Gamma]^+$ and $[\Gamma]^-$ separately invariant, we can choose a point $\varphi$ in $[\Gamma]^+$ and then take $\theta$ to be a point of $[\Gamma]^-$ not contained in the hyperplane $C\Gamma\varphi$.

(4.7)




It follows that if $\psi$ is any point on the line joining $\varphi$ and $\theta$ and different from them, then $C\Gamma\psi$ is a hyperplane not containing $\psi$. Hence $C\Gamma$ defines a polarity in a quadric and

(4.8) $$C' = -C, \qquad (C\Gamma)' = +(C\Gamma) \quad .$$

A commutative axial reflection and polarity are necessarily related to one another as in (4.1), (4.3), (4.5), or (4.7) and comparing (4.2), (4.4), (4.6) and (4.8) with the first two columns of the table in Theorem (3.1) gives the last column of that table. When $\nu$ is even, the axes of every involution $X^{\alpha}\gamma_{\alpha}$ are exchanged, and when $\nu$ is odd they are all separately invariant. In each case the invariant polarity may be either with respect to a quadric or with respect to a linear complex.

The correlation defined by the matrix C may be constructed for a given value of $\nu$ from the correlation in the immediately preceding case $(\nu-1)$ just as in the case of $\gamma$-sets. In the lowest case, $\nu = 1$, each collineation determines a unique correlation which has the same effect on the points of $P_1$ but which interchanges the empty set and the whole line. The matrix $||\mathcal{E}_{AB}||$ serves as C in this case and transforms the coordinates $\psi^A$ of a point into coordinates $\psi_A$ of the same point regarded as a hyperplane.

The polarity induces on the axis regulus containing the axes of $\gamma_{2\nu-1}$ and $\gamma_{2\nu}$ a collineation which either leaves four axes separately invariant or exchanges them by pairs. In either event $[i\gamma_{2\nu-1}\gamma_{2\nu}]^+ \equiv S^+$ and $[i\gamma_{2\nu-1}\gamma_{2\nu}]^+ \equiv S^-$ are separately invariant. To construct $C^{(\nu)}$ we first define $C^{(\nu-1)}$ on the lines of the line regulus. The transform of a line of the regulus is then a $P_{2\nu-3}$ which intersects every axis of the axis regulus in a $P_{2\nu-1-2}$.

When $\nu$ is odd, $C^{(\nu)}$ leaves every axis of the axis regulus invariant, and when $\nu$ is even it induces on the axis regulus an involution with double elements $S^+$ and $S^-$. If $\varphi$ is any point on a line, $\ell$, of the line regulus, it is determined as the intersection of this line with an axis, A. We therefore define the hyperplane $C^{(\nu)}\varphi$ to be the join of $C^{(\nu)}\ell$ and $C^{(\nu)}A$. This extends the definition of $C^{(\nu)}$ to all the lines which contain two points lying on lines of the line regulus. Every point of $P_{2\nu-1}$ is either the intersection of two such lines or lies on a line of the line regulus, and in either event its transform under $C^{(\nu)}$ has been defined.

## 5. THE (1-2) MATRIX REPRESENTATION OF $H_{2\nu+1}$

Since the improper rotation $Y^\alpha = -X^\alpha$ commutes with all rotations in $E_{2\nu+1}$, the correlation $\varphi_A = C_{AB}\psi^B$ commutes with all the collineations and correlations of the group, $\mathcal{G}$, isomorphic to the rotation group $H_{2\nu+1}$. Hence if P is the matrix of a collineation belonging to $\mathcal{G}$, $CP = \rho \check{P}C$, or $P'CP = \rho C$, and the matrix of the collineation may be normalized to within sign by the equation

$$P'CP = C \quad (5.1)$$

Similarly the matrix of a correlation may be normalized to within sign by the equation

(5.2) $$Q'\check{C}Q = C.$$

It may be observed that the normalization (5.1) is unchanged if the matrix C is multiplied by a scalar factor while the normalization (5.2) is relative to a particular choice of the matrix C. One may readily verify that the normalized pairs of matrices form a group provided we multiply correlations by collineations in the appropriate way. Hence we have the first part of the theorem.

Theorem (5.1). If the collineations and correlations of $\vartheta$ are described by matrices normalized by (5.1) and (5.2) respectively, then the collineation isomorphism $\Delta_{2\nu+1}$ becomes a (1-2) matrix representation of the rotation group on $2\nu+1$ variables by means of matrices of order $2^\nu$. This representation cannot be sharpened to a (1-1) representation.

If it were possible to select one out of every pair of matrices $\pm P$ satisfying (5.1) in such a way that the resulting matrices formed a group, then it would be necessary to include matrices corresponding to the proper rotations

(5.3) $$Y^0 = X^0,\ Y^1 = -X^1,\ Y^2 = -X^2,\ \ldots,\ Y^{2\nu} = -X^{2\nu} \quad \text{and}$$

(5.4) $$Y^0 = -X^0,\ Y^1 = +X^1,\ Y^2 = -X^2,\ \ldots,\ Y^{2\nu} = -X^{2\nu} \quad .$$

In the coordinate system of (4.4), Chapter V, these rotations correspond to matrices which are multiples of $\gamma_0$ and $\gamma_1$ and these matrices anticommute while (5.3) and (5.4) commute. Hence the assumed selection of matrices would fail to determine a single matrix corresponding to the product of (5.3) and (5.4). It follows that the matrix representation of $H_{2\nu+1}$ is essentially double valued and cannot be sharpened to a (1-1) representation.

## 6. LINEAR FAMILIES OF CORRELATIONS

If the matrices $\gamma_\alpha$ form a $\gamma$-set, then the matrices

(6.1) $$||C_{oAB}|| = ||C_{AC}\gamma_o{}^C{}_B||,\ \ldots\ ||C_{2\nu AB}|| = ||C_{AC}\gamma_{2\nu}{}^C{}_B|| \quad \text{and} \quad ||C_{2\nu+1\ AB}|| = i||C_{AB}||$$

define $2\nu+2$ correlations which form a linear family. That is to say, if we define

(6.2) $$||C_\sigma{}^{AB}|| = ||C_{\sigma AB}||'^{-1} \qquad (\sigma = 0, 1, \ldots, 2\nu+1)$$

then

(6.3) $$(X^\sigma C_{\sigma AC})(X^\tau C_\tau{}^{BC}) = \Sigma (X^\sigma)^2 \delta_A^B$$

and consequently the equations

(6.4) $$\varphi_A = (X^\sigma C_{\sigma AB})\psi^B \qquad \text{and} \qquad \varphi^A = (X^\sigma C_\sigma{}^{AB})\psi_B$$

define the same correlation.

In general, we shall say that two sets of p+1 matrices of order k, $C_\sigma$ and $D_\sigma (\sigma = 0, 1, \ldots, p)$, define a linear family of correlations if $||\varphi_A|| = (X^\sigma C_\sigma)||\psi^B||$ and $||\varphi^A|| = (X^\sigma D_\sigma)||\psi_B||$ define the same correlation, that is, if

(6.5) $$(X^\sigma C_\sigma)(X^\tau D'_\tau) = \rho 1 \quad .$$

Taking the trace of this equation, we have $\rho = \gamma_{\sigma\tau} X^\sigma X^\tau$ where $\gamma_{\sigma\tau} = \frac{1}{2k}$ Trace $(C_\sigma D'_\tau + C_\tau D'_\sigma)$. We shall consider only those linear families of correlations for which $||\gamma_{\sigma\tau}||$ is non-singular.

The problem of determining all linear families of correlations is on account of (6.5) identical with the problem of factoring a quadratic form into two linear factors which may not be identical. If we choose a coordinate system in which $\rho = \Sigma(X^\sigma)^2$ equating coefficients in (6.5) gives $D'_\sigma = C_\sigma^{-1}$ and consequently the matrices $C_\sigma$ satisfy the equation

(6.6) $$(X^\sigma C_\sigma)(X^\tau C_\tau^{-1}) = \sum_{\sigma=0}^{p} (X^\sigma)^2 1 \quad .$$

Conversely, any such set of matrices defines a linear family of correlations. We shall say that an ordered set of matrices $C_0, C_1, C_2, \ldots, C_p$, satisfying (6.6) is a C-set. A $\gamma$-set is also a C-set but we regard the matrices of a C-set as having two covariant indices while the matrices of a $\gamma$-set have one contravariant and one covariant index.

Multiplying equations (6.6) on the left by $iC_p^{-1}$ and on the right by $-iC_p$, the resulting equations are equivalent to

(6.7) $$(X^{\alpha}S_{\alpha})^2 = \Sigma(X^{\alpha})^2\, 1 \qquad \alpha = 0, 1, \ldots, p-1,$$

where

(6.8) $$S = iC_p^{-1}C_{\alpha} \quad .$$

Hence we have the theorem

<u>Theorem</u> <u>(6.1)</u>. If $S_{\alpha}(\alpha = 0, 1, \ldots, p-1)$ is a $\gamma$-set of p matrices and $C_p$ is an arbitrary non-singular matrix, then

(6.9) $$C_{\alpha} = -iC_pS_{\alpha},\ C_p = C_p$$

is a C-set of p+1 matrices. Conversely, if $C_{\sigma}$ is a C-set of p+1 matrices, then $S_{\alpha} = iC_p^{-1}C_{\alpha}$ is a $\gamma$-set of p matrices.

This theorem enables us to use the properties of $\gamma$-sets to prove a number of theorems on C-sets. We omit the proofs where they can easily be supplied.

<u>Theorem</u> <u>(6.2)</u>. The maximum number of matrices in a C-set of matrices of order $2^{\nu}\ell$ ($\ell$ odd) is $2\nu+2$ and this maximum is attained.

<u>Theorem</u> <u>(6.3)</u>. If $C_{\sigma}$ is a C-set, so is $AC_{\sigma}B$, where A and B are any non-singular matrices.

<u>Theorem</u> <u>(6.4)</u>. If $C_{\sigma}$ is a C-set, so are $C_{\sigma}'$, $C_{\sigma}^{\cup}$, and $C_{\sigma}^{-1}$.

<u>Theorem</u> <u>(6.5)</u>. If $C_{\sigma}$ is a C-set of $2\nu+2$ matrices of order $2^{\nu}$ and $A_1C_{\sigma}B_1 = A_2C_{\sigma}B_2$, then

(6.10) $$A_2 = \rho\, A_1 \quad \text{and} \quad B_2 = \frac{1}{\rho}\, B_1 .$$

<u>Theorem</u> <u>(6.6)</u>. If $C_{\sigma}$ and $D_{\sigma}$ are C-sets of $2\nu+2$ matrices of order $2^{\nu}$, there exist two non-singular matrices A and B such that either

(6.11) $$AC_{\sigma}B = D_{\sigma},$$

or

(6.12) $$AC_{\alpha}B = -D_{\alpha}(\alpha = 0, 1, \ldots, 2\nu), \quad \text{and} \quad AC_{2\nu+1}B = D_{2\nu+1},$$

according as the trace of $(C_0C_1^{-1}C_2 \ldots C_{2\nu}^{-1}C_{2\nu+1})$ is equal to the trace of

$(D_o D_1^{-1} \ldots D_{2\nu+1})$ or its negative, respectively. If (6.11) holds, there does not exist a pair of matrices satisfying (6.12) and if (6.12) holds there does not exist a pair of matrices satisfying (6.11).

This theorem implies that every C-set in $P_{2^\nu-1}$ is obtained from (6.1) by a transformation of the type (6.11) or by such a transformation combined with a change in the sign of one matrix. When $\nu \equiv 0 \pmod 4$ all the matrices (6.1) are symmetric, and when $\nu \equiv 2 \pmod 4$ they are all skew-symmetric. Hence when $\nu$ is even there exists a maximal linear family of correlations all of which are of period two. Such a set of transformations will be called a linear family of polarities. Using the fact that the equations $(C\gamma_\alpha)' = \pm(C\gamma_\alpha)$ determine C to within a factor, it is not difficult to prove the theorem

Theorem (6.7). A $(2\nu+2)$-parameter linear family of polarities exists in $P_{2^\nu-1}$ if and only if $\nu$ is even.

## 7. THE REPRESENTATION OF $H^+_{2\nu+2}$ BY COLLINEATIONS IN $P_{2^\nu-1}$

Theorem (7.1). If $C_\sigma$ is a C-set and $||L^\sigma{}_\tau||$ is an orthogonal matrix, then $C_\sigma L^\sigma{}_\tau$ is also a C-set and if $|L^\sigma{}_\tau| = +1$, there exists a pair of matrices A and B such that

(7.1) $$AC_\tau B = C_\sigma L^\sigma{}_\tau \quad .$$

These equations may be regarded as establishing a correspondence

(7.2) $$||L^\sigma{}_\tau|| \longleftrightarrow (\rho A, \frac{1}{\rho} B)$$

between proper orthogonal matrices and pairs of matrices A and B which, by Theorem (6.5) are determined to within the scalar factors $\rho$ and $\frac{1}{\rho}$, respectively. It is evident that if $||L_1{}^\sigma{}_\tau||$ corresponds to $(\rho A_1, \frac{1}{\rho} B_1)$, then $||L^\sigma{}_\tau L_1{}^\tau{}_\mu||$ corresponds to $(\rho AA_1, \frac{1}{\rho} B_1 B)$ and hence the correspondence

(7.3) $$||L^\sigma{}_\tau|| \longrightarrow \rho A$$

is a representation of the orthogonal group $H^+_{2\nu+2}$ by collineations in $P_{2^\nu-1}$.

Except for the order of multiplication the correspondence $||L^\sigma_\tau|| \longrightarrow \frac{1}{\rho} B$ is a representation of $H^+_{2\nu+2}$. Consequently equations (7.1) also determine a representation

(7.4) $$||L^\sigma_\tau|| \longrightarrow \rho P$$

where $P = B^{-1}$.

Theorem (7.2). If $C_\sigma$ is a C-set of $2\nu+2 > 4$ matrices of order $2^\nu$ and

(7.5) $$C_\sigma B = C_\tau L^\tau{}_\sigma$$

for some matrix $||L_\sigma||$, then $B = b1$ and $L^\tau_\sigma = b\delta^\tau_\sigma$. Similarly, if $AC_\sigma = C_\tau L^\tau{}_\sigma$, then $A = a1$ and $L^\tau_\sigma = a\delta^\tau_\sigma$.

A C-set of $2\nu+2 = 4$ matrices of order 2 is a basis for all matrices of order 2 and hence if B is any matrix there will exist coefficients $L^\tau_\sigma$ satisfying (7.5). The theorem may be proved by multiplying (7.5) on the left by $iC^{-1}_{2\nu+1}$ and using the linear independence of the matrices 1, $S_\alpha = iC^{-1}_{2\nu+1}C_\alpha$, and $S_\alpha S_\beta (\alpha < \beta$ and $\alpha, \beta = 0, 1, \ldots, 2\nu)$. This linear independence was proved in §6, Chapter V, and of course depends upon the hypothesis that $2\nu+2 > 4$, or $\nu > 1$.

Theorem (7.3). When $\nu > 1$, the correspondences (7.3) and (7.4) established by equations (7.1) are (2-1) representations of the proper orthogonal group $H^+_{2\nu+2}$ by collineations in $P_{2^\nu-1}$. When $\nu = 1$ the representations are $(1 - \infty^3)$.

We have already observed that a proper orthogonal matrix determines A and $P = B^{-1}$ to within a common factor. To prove the converse we assume that $AC_\sigma B = C_\tau L^\tau{}_\sigma$ and $AC_\sigma B_1 = C_\tau L_1{}^\tau{}_\sigma$. Then $C_\sigma(BB_1^{-1}) = C_\tau M^\tau{}_\sigma$, where $L^\sigma{}_\mu = L_1{}^\sigma{}_\tau M^\tau{}_\mu$, and by Theorem (7.2), $BB_1^{-1} = b1$ and $M^\tau_\sigma = b\,\delta^\tau_\sigma$ so that $L^\sigma{}_\tau = bL_1{}^\sigma{}_\tau$ and b may equal +1 or -1 since the order is even (i.e., $-1_{2\nu+2}$ is proper orthogonal). When $\nu = 1$, a given matrix A will occur in (7.1) associated with $\infty^4$ matrices B, thus with $\infty^3$ collineations.

If $\nu$ is even, the matrices (6.1) are either all symmetric or all skew-symmetric. Hence, if $C_\sigma$ is this set of matrices, taking the transpose of (7.1) gives

(7.6) $$AC_\sigma B = B' C_\sigma A'$$

and by Theorem (6.5), $A = \rho B'$. $(= \rho \breve{P})$ By making a suitable relative normalization of A and B so that $A = B'$, we obtain the theorem

<u>Theorem</u> <u>(7.4)</u>. If $\nu$ is even, there exists a C-set of $2\nu+2$ matrices of order $2^\nu$ such that

(7.7) $$C_\sigma' = (-1)^{\frac{1}{2}\nu(\nu+1)} C_\sigma \quad .$$

For such a set the equations

(7.8) $$\breve{P} C_\sigma P^{-1} = C_\tau L^\tau{}_\sigma$$

establish a (2-1) matrix isomorphism

(7.9) $$\pm P \longleftrightarrow ||L^\tau{}_\sigma||$$

between normalized matrices of collineations in $P_{2\nu-1}$ which leave the linear family $X^\sigma C_\sigma$ invariant and the proper orthogonal matrices of order $2\nu+2$).

For $\nu = 2$ the matrices $C_\sigma$ of this theorem are a set of six skew-symmetric matrices which form a basis for all four-rowed skew-symmetric matrices. Such a set of matrices has been used as a starting point for a discussion of the Pluecker-Klein correspondence* and the geometry underlying the representation

---

* "Geometry of Four-Component Spinors", by O. Veblen, Proc. Nat. Acad. Sci., 19 (1933), pp. 503-517.

---

(7.9) may be regarded as a generalization of the Pluecker-Klein correspondence.

<u>Theorem</u> <u>(7.5)</u>. The isomorphism of Theorem (7.4) may be extended to include the representation of improper orthogonal matrices by means of correlations by the equations

(7.10) $$Q^{-1} C_\sigma \breve{Q} = \breve{C}_\tau L^\tau{}_\sigma$$

or by the equivalent equations

(7.11) $$Q' \breve{C}_\sigma Q = C_\tau L^\tau{}_\sigma \qquad (\text{i.e., } C_\sigma^{CD} Q_{CA} Q_{DB} = C_{\tau AB} L^\tau{}_\sigma) \quad .$$

# Chapter VII

## TENSOR COORDINATES OF LINEAR SPACES

### INTRODUCTION

1. Homogeneous coordinates of linear subspaces of a projective space were defined by Grassman and have been studied by Severi,[1] Antonelli,[2] and others.

---

[1] Annali di Matematica, 24 (3), 1915, pp. 89-120.

[2] Annali della R. Scuola normale Sup. di Pisa, Bd. III, 1883, pp. 71-77.

---

These studies have not, however, employed the notation and methods of the tensor calculus. In this chapter we define contravariant and covariant tensor coordinates of the linear subspaces of a projective space and derive ab initio the quadratic relations which they satisfy.

These coordinates enable us to give elegant algebraic expression to the geometric operation of perspection and section, where the center of perspection is a linear space of any number of dimensions. Two expressions are found for the coordinate tensor of the join and intersection of two spaces and in verifying the equivalence of these two expressions we are led to a useful identity.

In §5 we introduce a non-singular quadratic form into the geometry and employ the matrix of this form to lower and raise tensor indices in the familiar fashion. We then derive some of the properties of the linear spaces which lie on a quadric. In particular, we find the algebraic characterization of the two families of rulings on a quadric in a complex projective space of $2\nu-1$ dimensions and obtain the intersection properties of these rulings. Certain exceptional features of the quadric in 3-space are noted which imply special properties of the orthogonal group on four variables.

In §8, Chapter I, we gave a detailed discussion of the linear subspaces of a 3-dimensional space. That section may be read as an introduction to the present chapter, but we do not need to make use of the special results there obtained.

## 2. DEFINITION OF THE COORDINATE TENSORS

An (r-1)-dimensional linear subspace, V, of the real or complex projective space $P_{n-1}$ is determined by any r linearly independent points which it contains. If $A_1^i, A_2^i, A_3^i, \ldots, A_r^i$ $(i = 1, 2, \ldots, n)$ are coordinates of r such points, then the tensor

$$(2.1)\quad \begin{aligned} V^{i_1 i_2 \cdots i_r} &= \frac{1}{r!}\,\delta^{i_1 i_2 \cdots i_r}_{s_1 s_2 \cdots s_r} A_1^{s_1} A_2^{s_2} \cdots A_r^{s_r} \\ &\equiv A_1^{[i_1} A_2^{i_2} \cdots A_r^{i_r]} \end{aligned}$$

where $\delta^{i_1 i_2 \cdots i_r}_{s_1 s_2 \cdots s_r}$ is the generalized Kronecker delta,[3] is said to be a contravariant coordinate tensor of V. Thus the coordinate tensor of the line determined by points $A^i$ and $B^i$ is $V^{ij} = \frac{1}{2}(A^iB^j - B^iA^j)$, and the components of the coordinate tensor of an (r-1)-dimensional space are proportional to the minors of the matrix of r rows and n columns formed by the coordinates of r points of the space.

In computations involving many indices we replace a set of indices such as $i_1 i_2 \ldots i_r$ by a single underlined index $\underline{i}$. The number of indices in a set $\underline{i}$ is denoted by $|\underline{i}|$, or $|i|$, in words, the length of $\underline{i}$. Thus (2.1) becomes in this

---

[3] For the definition and properties of these symbols see Chapter I of the Cambridge Tract No. 24, "Invariants of Quadratic Differential Forms", by O. Veblen.

notation

$$(2.2) \qquad V^{\underline{i}} = \frac{1}{|i|!} \delta^{\underline{i}}_{\underline{s}} A_1^{s_1} A_2^{s_2} \cdots A_{|s|}^{s_{|s|}} ,$$

where $|s| = |i| = r$. We shall also use symbols such as $\underline{k}_1$ to denote sets of ordinary indices, the underline not being extended under the subscript for typographical reasons.

If we define $\epsilon_{\underline{i}}$, $|i| = n$, to be +1 if $\underline{i}$ is an even permutation of (1, 2, ... n), -1 if it is an odd permutation, and zero otherwise, and define $\epsilon_{\underline{i}}$ to be a covariant tensor of weight -1, then the components of $\epsilon_{\underline{i}}$ have the same values in all coordinate systems. This numerically invariant tensor is always present in our geometry and we employ it to define a covariant tensor

$$(2.3) \qquad V_{\underline{j}}^{\cup} = \frac{\rho(|i|)}{|i|!} \epsilon_{\underline{j}\,\underline{i}} V^{\underline{i}}$$

associated with $V^{\underline{i}}$. In this definition $\rho$ is assumed to be a scalar (which may depend on $|i|$) of weight +1 so that $V_{\underline{j}}^{\cup}$ and $V^{\underline{i}}$ have the same weight, which we shall take to be zero. For many purposes in projective geometry it would be sufficient to take $\rho = 1$ in an arbitrarily chosen coordinate system. When we introduce the matrix of a quadratic form we will determine the value of $\rho(|i|)$ so as to give a convenient calculus.

Observing that equations (2.3) amount to no more than a renumbering of the components of $V^{\underline{i}}$, we may invert the equations to get

$$(2.4) \qquad V^{\underline{i}} = \frac{1}{\rho(|i|)\ |j|!} V_j^{\cup} \epsilon^{\underline{j}\,\underline{i}} ,$$

where $\epsilon^{\underline{i}}$ is the numerically invariant contravariant tensor of weight +1 defined by the equations $\epsilon^{\underline{i}} = \epsilon_{\underline{i}}$. In (2.3) and (2.4) we have given rules for lowering and raising sets of skew-symmetric indices, the hook $(\cup)$ being added or removed by the operation. We also use (2.4) to define $W^{\underline{i}}_{\cap}$ in terms of $W_{\underline{j}}$ and then (2.3) gives $W_{\underline{j}}$ in terms of $W^{\underline{i}}_{\cap}$. So far we have no way of defining a tensor $V_{\underline{i}}$ associated with $V^{\underline{i}}$.

The operations of raising and lowering sets of skew-symmetric indices give algebraic expression to the law of duality as applied to linear spaces. Thus if we choose coordinates so that $A^i_a = \delta^i_a$ $(a = 1, 2, \ldots, r)$ in (2.2), $V^{\cup}_{\underline{j}}$ will vanish unless $\underline{j}$ is a permutation of ((r+1), (r+2), ..., n) and consequently $V^{\cup}_{\underline{j}}$ is a multiple of $\delta^{r+1}_{[j_1}\delta^{r+2}_{j_2}\ldots\delta^{n}_{j_{n-r}]}$. Hence a set of r independent points in V determines $V^{\underline{i}}$ to within a factor and a set of n-r linearly independent planes containing V determine $V^{\cup}_{\underline{j}}$ to within a factor. Also, the coordinates $X^i$ of a point in V satisfy the linear equations

(2.5) $$V^{\cup}_{\underline{k}\,s}X^s = 0\ ,$$

and the coordinates of a hyperplane containing V satisfy the equations

(2.6) $$V^{\underline{k}\,s}X_s = 0\ .$$

Moreover, writing (2.5) in the form $\epsilon_{\underline{i}\,\underline{k}\,s}A_1^{k_1}A_2^{k_2}\ldots A_r^{k_r}X^s = 0$, we see that every solution of (2.5) is linearly dependent on $A^i_1$, $A^i_2$, ..., and $A^i_r$ and so determines a point of V. A similar statement holds for (2.6). This completes the proof of the theorem.

Theorem (2.1). A linear space determines its coordinate tensors to within a factor and is uniquely determined by them.

## 3. THE QUADRATIC IDENTITIES

The coordinate tensors of a linear space are necessarily skew-symmetric and non-vanishing. Not all such tensors, however, are expressible in the form

(2.1) and consequently a tensor $X^{\underline{i}} = X^{[\underline{i}]} \neq 0$ must satisfy additional conditions in order to be a coordinate tensor. One form of these conditions is given in the theorem.

<u>Theorem (3.1)</u>. A necessary and sufficient condition that $V^{\underline{i}} = V^{[\underline{i}]} \neq 0$ shall be the coordinate tensor of a linear space is that

$$\operatorname{rank} \| V^{\underline{k}\, s} \| = |\underline{k}\, s| \quad ,$$

where the set of indices $\underline{k}$ numbers the rows of the matrix and the single index s numbers the columns. (Thus $\| V^{\underline{k}\, s} \|$ is rectangular with $n^{|k|}$ rows and n columns.)

We first prove the Lemma:

<u>Lemma 1</u>. If $V^{\underline{i}} = V^{[\underline{i}]} \neq 0$, then

$$\operatorname{rank} \| V^{\underline{k}\, s} \| \geqq |\underline{k}\, s| \ .$$

If rank $\| V^{\underline{k}\, s} \|$ were less than $|\underline{k}\, s| = r$, equations (2.6) would have more than n-r independent solutions. By a suitable choice of coordinate system a linearly independent set of these solutions could be taken to be $\delta^1_i, \delta^2_i, \ldots, \delta^t_i$, where $t > n-r$. Then $V^{\underline{k}\, 1} = V^{\underline{k}\, 2} = V^{\underline{k}\, 3} = \ldots = V^{\underline{k}\, t} = 0$ and since r indices $\underline{i}$ cannot be chosen different from one another and from all the numbers (1, 2, 3, ..., t), we would conclude from the skew-symmetry of $V^{\underline{i}}$ that $V^{\underline{i}} = 0$, contrary to hypothesis.

The necessity of Theorem (3.1) now follows from Lemma 1 and the observation that equations (2.6) have at least n-r independent solutions so that rank $\| V^{\underline{k}\, s} \| \leqq r$. To prove the sufficiency of the condition we need only choose coordinates so that a set of n-r linearly independent solutions of (2.6) are $\delta^{r+1}_i, \delta^{r+2}_i, \ldots,$ and $\delta^n_i$; then $V^{\underline{i}}$ will be a multiple of $\delta^{[i_1}_1 \delta^{i_2}_2 \cdots \delta^{i_r]}_r$.

<u>Theorem (3.2)</u>. If $X^{\underline{i}\, s} Y_{\underline{j}\, s} = 0$, where $X^{\underline{x}} = X^{[\underline{x}]} \neq 0$, $Y_{\underline{y}} = Y_{[\underline{y}]} \neq 0$,

and the indices range from one to n, then

(1) $|x| + |y| \leqq n$, and

(2) if $|x| + |y| = n$, then $Y_{\underline{y}} = \lambda X^{\cup}_{\underline{y}}$ and $X^{\underline{x}}$ and $Y_{\underline{y}}$ are coordinate tensors of the same linear space.

For each choice of the indices $\underline{j}$ the vector $Y_{\underline{j}\,s}$ belongs to the right null-space of the matrix $\| X^{\underline{i}\,s} \|$ and consequently

$$(3.1) \qquad \text{rank } \| Y_{\underline{j}\,s} \| \leqq n - \text{rank } \| X^{\underline{i}\,s} \| .$$

But by Lemma 1,

$$(3.2) \qquad |x| \leqq \text{rank } \| X^{\underline{i}\,s} \| \text{ and } |y| \leqq \text{rank } \| Y_{\underline{j}\,s} \| .$$

Combining these inequalities we have the first part of the theorem. Moreover, if $|x| + |y| = n$, the equality signs must hold in (3.2) so that by Theorem (3.1) $X^{\underline{x}}$ is a coordinate tensor. Choosing coordinates so that $X^{\underline{x}} = \lambda\, \delta^{[x_1}_{1}\, \delta^{x_2}_{2} \cdots \delta^{x_{|x|}]}_{|x|}$, the hypothesis of the theorem implies that $Y_{\underline{j}\,1} = Y_{\underline{j}\,2} = \cdots = Y_{\underline{j}\,|x|} = 0$ and consequently $Y_{\underline{y}} = \mu\, \delta^{|x|+1}_{[y_1}\, \delta^{|x|+2}_{y_2} \cdots \delta^{n}_{y_{|y|}]}$ is a multiple of $X^{\cup}_{\underline{y}}$.

<u>Theorem (3.3)</u>. A necessary and sufficient condition that $V^{\underline{i}} = V^{[\underline{i}]} \neq 0$ shall be a coordinate tensor is that

$$(3.3) \qquad V^{\underline{k}\,s}\, V^{\smile}_{\underline{j}\,s} = 0 .$$

The sufficiency of the condition follows from Theorem (3.2) by putting $X^{\underline{i}} = V^{\underline{i}}$ and $Y_{\underline{j}} = V^{\cup}_{\underline{j}}$. The necessity follows from the observation that when $V^{\underline{i}}$ is expressible in the form (2.2), then for each set of values of the indices $\underline{k}$ and $\underline{j}$, $V^{\underline{k}\,s}\, V^{\smile}_{\underline{j}\,s}$ is a determinant in which two rows are equal.

Equations (3.3) obviously imply the vanishing of $V^{\underline{k}\,\underline{s}}\, V^{\smile}_{\underline{j}\,\underline{s}}$ if $\underline{s}$ contains more than one index. In Theorem (3.4) we shall prove that $V^{\underline{k}\,r\,s}\, V_{\underline{j}\,r\,s} = 0$

implies the apparently stronger condition (3.3). This will prove the theorem.

Theorem (3.4). A necessary and sufficient condition that $V^{\underline{i}} = V^{[\underline{i}]} \neq 0$ shall be a coördinate tensor is that

$$V^{\underline{k}\ r\ s}\ V_{\underline{j}\ r\ s}^{\smile} = 0\ . \tag{3.4}$$

The quadratic relations satisfied by the components of a coordinate tensor are usually given in the non-tensorial form[4]

---

[4] Bertini, "Einführung in die Projektive Geometrie Mehrdimensionaler Räume", page 43, formula (14).

---

$$V^{\underline{m}\ a\ b}\ V^{\underline{m}\ c\ d} + V^{\underline{m}\ a\ c}\ V^{\underline{m}\ d\ b} + V^{\underline{m}\ a\ d}\ V^{\underline{m}\ b\ c} = 0\ , \tag{3.5}$$

where the set of indices $\underline{m}$ occurs twice but without being summed. These equations are obtained by multiplying (3.3) by $\epsilon^{\underline{j}\ \underline{m}\ b\ c\ d}$, summing, and putting $\underline{m}$ a for $\underline{k}$. We shall here employ only the tensor form of the quadratic relations, (3.3) or (3.4).

## 4. JOINS AND INTERSECTIONS

We call the linear space of smallest dimensionality which contains two linear spaces, X and Y, the join of X and Y and denote it by X + Y. Similarly, we call the space of greatest dimensionality contained in both X and Y the intersection (meet) of X and Y and denote it by XY. We write 0 and 1 for the null-set and the entire space, respectively.

Theorem (4.1). If X, Y and J = X + Y are linear spaces with the respective coordinate tensors $X^{\underline{x}}$, $Y^{\underline{y}}$, and $J^{\underline{j}}$, and XY = 0, then $J_{\underline{i}}^{\cup}$ is proportional to $Y_{\underline{i}\ \underline{x}}^{\smile} X^{\underline{x}}$, and to $X_{\underline{i}\ \underline{y}}^{\smile} Y^{\underline{y}}$. If XY $\neq$ 0, $Y_{\underline{i}\ \underline{x}}^{\smile} X^{\underline{x}} = X_{\underline{i}\ \underline{y}}^{\smile} Y^{\underline{y}} = 0$.

If X is the join of the $|x|$ points $A_1, A_2, \ldots, A_{|x|}$ and Y is the join of the $|y|$ points $B_1, B_2, \ldots, B_{|y|}$, then $J = A_1 + A_2 + \ldots + A_{|x|} + B_1 + B_2 + \ldots + B_{|y|}$ and $XY = 0$ implies that these $|x| + |y|$ points are linearly independent. Consequently,

$$J^{\cup}_{\underline{i}} = \epsilon_{\underline{i}\,\underline{x}\,\underline{y}} A_1^{x_1} A_2^{x_2} \ldots A_{|x|}^{x_{|x|}} B_1^{y_1} B_2^{y_2} \ldots B_{|y|}^{y_{|y|}}$$

is different from zero and is by definition a covariant coordinate tensor of J. Using the definition of $X^{\underline{x}}$ and $Y^{\underline{y}}$, $J^{\cup}_{\underline{i}} = \epsilon_{\underline{i}\,\underline{x}\,\underline{y}} X^{\underline{x}} Y^{\underline{y}} = (-1)^{|x|\,|y|} \epsilon_{\underline{i}\,\underline{y}\,\underline{x}} Y^{\underline{y}} X^{\underline{x}}$ so that $J^{\cup}_{\underline{i}}$ is proportional to $Y^{\cup}_{\underline{i}\,\underline{x}} X^{\underline{x}}$ and to $X^{\cup}_{\underline{i}\,\underline{y}} Y^{\underline{y}}$. If $XY \neq 0$, the points $A_1, A_2, \ldots, A_{|x|}, B_1, B_2, \ldots, B_{|y|}$ are linearly dependent so that $J^{\cup}_{\underline{i}} = 0$.

A similar proof gives the dual theorem.

<u>Theorem (4.2)</u>. If $X + Y = 1$ and $XY = I$, then $I^{\underline{i}}$ is proportional to $X^{\underline{i}\,\underline{r}} Y^{\cup}_{\underline{r}}$ and to $Y^{\underline{i}\,\underline{s}} X^{\cup}_{\underline{s}}$. If $X + Y \neq 1$, then $X^{\underline{i}\,\underline{r}} Y^{\cup}_{\underline{r}} = Y^{\underline{i}\,\underline{s}} X^{\cup}_{\underline{s}} = 0$.

<u>Theorem (4.3)</u>. If X and Y are linear spaces with coordinate tensors $X^{\underline{x}}$ and $Y^{\underline{y}}$, respectively, then the "projection operator" $P^{\underline{i}}_{\underline{j}} = Y^{\underline{i}\,\underline{s}} X^{\cup}_{\underline{j}\,\underline{s}}$ defines a transformation

$$\varphi^{\underline{i}} = P^{\underline{i}}_{\underline{j}} \psi^{\underline{j}} \tag{4.1}$$

with the following properties:

1. If $\psi^{\underline{j}}$ is the coordinate tensor of a linear space $\psi$ such that $X\psi = 0$ and $Y + X + \psi = 1$, then $\varphi^{\underline{i}}$ ($\neq 0$) is a coordinate tensor of the linear space $Y(X + \psi)$.
2. If either or both of the conditions $X\psi = 0$ and $Y + X + \psi = 1$ fail to hold, then $\varphi^{\underline{i}} = 0$.

The theorem follows immediately by applying Theorem (4.1) to the spaces X and $\psi$ and then Theorem (4.2) to the spaces Y and $(X + \psi)$. Dualizing Theorem (4.3) gives

Theorem (4.4). $\psi_{\underline{i}}\, P^{\underline{i}}_{\underline{j}}$ is a coordinate tensor of $\psi Y + X$ unless $\psi + Y \neq 1$ or $\psi YX \neq 0$, and then it is zero.

Theorem (4.5). If $X^{\underline{x}}$ and $Y^{\underline{y}}$ are coordinate tensors of linear spaces and we form the sequence of tensors

(4.2) $$Y^{\underline{i}} X^{\nu}_{\underline{j}},\; Y^{\underline{i}_1\underline{s}_1} X_{\underline{j}_1\underline{s}_1}^{\smile},\; Y^{\underline{i}_2\underline{s}_2} X_{\underline{j}_2\underline{s}_2}^{\smile},\; \dots,\; Y^{\underline{i}_a\underline{s}_a} X^{\upsilon}_{\underline{s}_a} \text{ or } Y^{\underline{s}_a} X_{\underline{j}_a\underline{s}_a}^{\smile},$$

where $|s_r| = r$ and a is the minimum of the two numbers $n - |x|$ and $|y|$, then the last non-vanishing tensor factors into the form

(4.3) $$I^{\underline{i}_r}\, J_{\underline{j}_r}$$

and $I^{\underline{i}_r}$ and $J_{\underline{j}_r}$ are coordinate tensors of $I = XY$ and $J = X + Y$, respectively.

The spaces X and Y intersect in a space of dimensionality $\leqq |x| - 1$ and $\leqq |y| - 1$. Hence if $I^{\underline{i}}$ is a coordinate tensor of $I = XY$, we may put $|i| = |y| - r$ where $r \geqq 0$. Now applying Theorem (4.3), the expression

(4.4) $$\varphi^{\underline{i}_r} = Y^{\underline{i}_r\underline{s}_r} X_{\underline{j}_r\underline{s}_r}^{\smile}\, \psi^{\underline{j}_r}$$

is either zero or is the coordinate tensor of $\varphi = Y(X + \psi)$. In the latter case $\varphi$ includes XY and its dimension, $|i_r| - 1 = |y| - r - 1$, equals the dimension of XY. Hence $\varphi = XY$ and $\varphi^{\underline{i}_r} = \rho\, I^{\underline{i}_r}$.

It is now convenient to regard (4.4) as a vector equation $\varphi = M\psi$ where M is the rectangular matrix $\| Y^{\underline{i}_r\underline{s}_r} X_{\underline{j}_r\underline{s}_r}^{\smile} \|$. Choosing as a basis for the set of all vectors $\psi^{\underline{k}}$ the vectors $\delta^{\underline{k}}_{\underline{p}}$, where $|k| = |j_r|$ and $\underline{p}$ is any set of $|k|$ numbers between one and n, the matrix M transforms every vector of the basis either into the zero vector or into a multiple of $I^{\underline{i}_r}$. Hence M is of rank one and therefore factors into the product of two vectors. That is,

$$(4.5) \qquad Y^{\underline{i}_r \underline{s}_r} X^{\cup}_{\underline{j}_r \underline{s}_r} = X^{\underline{i}_r} J_{\underline{j}_r}$$

where $I^{\underline{i}_r}$ is a coordinate tensor of XY. Applying Theorem (4.4) to this projection operator proves that $J_{\underline{j}_r}$ is a coordinate tensor of X + Y.

Summing off r + 1 indices of $Y^{\underline{y}}$ against r + 1 indices of $X^{\cup}_{\underline{j}}$ gives zero since each component of this tensor is a sum of terms of the form $P^i Q_i$, where P is a point of I = XY and Q is a hyperplane containing J = X + Y and <u>a fortiori</u> containing P. It is obvious that if any one of the terms of (4.2) vanish, then all following terms also vanish.

If we exchange X and Y in (4.2) we get the sequence

$$(4.6) \qquad \begin{gathered} X^{\underline{i}} Y^{\cup}_{\underline{j}},\ X^{\underline{i}_1 s_1} Y^{\cup}_{\underline{j}_1 s_1},\ X^{\underline{i}_2 \underline{s}_2} Y^{\cup}_{\underline{j}_2 \underline{s}_2}, \ldots \\ \ldots,\ X^{\underline{i}_a \underline{s}_a} Y^{\cup}_{\underline{s}_a} \text{ or } X^{\underline{s}_a} Y^{\cup}_{\underline{j}_a \underline{s}_a}, \end{gathered}$$

where $|s_r| = r$ and a is the minimum of $n - |y|$ and $|x|$. It is to be observed that the number of terms in the sequences (4.2) and (4.6) are, respectively, $n - |x| + 1$ and $n - |y| + 1$ if $n \leqq |x| + |y|$, and $|y| + 1$ and $|x| + 1$ if $n \geqq |x| + |y|$. Hence the sequences are in general of unequal length. For coordinate tensors $X^{\underline{x}}$ and $Y^{\underline{y}}$, Theorem (4.5) implies a special case of the theorem

<u>Theorem (4.6)</u>. If $X^{\underline{x}}$ and $Y^{\underline{y}}$ are any two skew-symmetric tensors, then the last non-vanishing terms in (4.2) and (4.6) are proportional.

To prove this theorem without restricting $X^{\underline{x}}$ and $Y^{\underline{y}}$ to be coordinate tensors we use the definition (2.3) and its inverse (2.4) to express $Y^{\cup}_{\underline{j}}$ and $X^{\underline{i}}$ in terms of $Y^{\underline{k}}$ and $X^{\cup}_{\underline{m}}$, respectively. This gives

$$(4.7)\qquad \begin{aligned} X^{\underline{i}\,\underline{r}}\, Y^{\smile}_{\underline{j}\,\underline{r}} &= \lambda\, \delta^{\underline{i}\,\underline{m}}_{\underline{y}\,\underline{j}}\, Y^{\underline{y}}\, X^{\cup}_{m} \\ &= \lambda_1\, Y^{\underline{i}\,\underline{s}}\, X^{\smile}_{\underline{j}\,\underline{s}} + \lambda_2\, \delta^{[i_1}_{[j_1}\, Y^{i_2 \cdots i_{|i|}]\underline{s}_1}\, \overbrace{X_{j_2 \cdots j_{|j|}]\underline{s}_1}} \end{aligned}$$

+ (additional terms in which more than $|s|$ + 1 indices are summed),

with $|s_1| = |s| + 1$ where the integer $|s|$ and scalars $\lambda$, $\lambda_1$ and $\lambda_2$ (possibly zero) depend on the lengths of the various indices.* Consequently, if $Y^{\underline{i}\,\underline{s}}\, X^{\smile}_{\underline{j}\,\underline{s}}$

---

* A meaningless summand in (4.7) implies that its coefficient $\lambda_p$ is zero.

---

is the last non-vanishing term in (4.2), then $X^{\underline{i}\,\underline{r}}\, Y^{\smile}_{\underline{j}\,\underline{r}}$ is the last non-vanishing term in (4.6), and

$$(4.8)\qquad X^{\underline{i}\,\underline{r}}\, Y^{\smile}_{\underline{j}\,\underline{r}} = \lambda_1\, Y^{\underline{i}\,\underline{s}}\, X^{\smile}_{\underline{j}\,\underline{s}}\,.$$

## 5. THE QUADRATIC FORM

We now introduce into our geometry a fundamental quadratic form $\gamma_{ij}\, X^i X^j$ and use the symmetric matrix $\|\gamma_{ij}\|$ to lower single tensor indices by the familiar rule, $X_i = \gamma_{ij} X^j$. Covariant indices are raised by the rule $X_i \gamma^{ij} = X^j$, where $\gamma^{ki}\gamma_{kj} = \delta^i_j$.

If we lower each of the indices of a skew-symmetric tensor $X^{\underline{j}}$ by $\gamma_{ij}$ the resulting tensor is skew-symmetric and is denoted by $X_{\underline{j}}$. Applying the rule (2.4) to raise the indices $\underline{j}$ by means of $\dfrac{1}{\rho(|i|)|j|!}\, \epsilon^{\underline{j}\,\underline{i}}$ gives the tensor $X^{\underline{i}}_{\cap}$. Alternatively, we could first lower the indices of $X^{\underline{j}}$ by the rule (2.3) to get $X^{\cup}_{\underline{i}}$ and then raise them with $\gamma^{ij}$. The result of these two processes is the same to within a factor. Indeed, computation with (2.3) and (2.4) shows that

$$(5.1) \qquad \gamma^{\underline{s}\,\underline{i}}\, X^{\cup}_{\underline{s}} = [\rho(|i|)\,\rho(n-|i|)\,\gamma^{-1}(-1)^{|i|(n-|i|)}]X^{\underline{i}}_{\cap}$$

where $\gamma^{\underline{s}\,\underline{i}} = \gamma^{s_1 i_1}\gamma^{s_2 i_2}\cdots\gamma^{s_{|i|}\, i_{|i|}}$ and $\gamma$ is the determinant $|\gamma_{ij}|$.

The scalar factor in the right member of (5.1) will be +1 if we put

$$(5.2) \qquad \rho(|i|) = \gamma^{\frac{1}{2}}(-1)^{\frac{1}{2}|i|(n-|i|)} .$$

This choice has the disadvantage that $\rho(|i|)$ is sometimes imaginary. This cannot always be avoided without introducing additional complications into the calculus for in the special case in which $n = 2\nu$ and $|i| = \nu$, (5.1) reduces to

$$(5.3) \qquad \gamma^{\underline{s}\,\underline{i}}\, X^{\cup}_{\underline{s}} = \rho^{2}(\nu)\,\gamma^{-1}\,(-1)^{\nu}\, X^{\underline{i}}_{\cap}$$

so that $\rho(\nu) = \gamma^{1/2}(-1)^{\nu/2}$. However, if $\|\gamma_{ij}\|$ is a real matrix of signature (+-+-...+-), then $\rho(\nu)$ is real (cf. Theorem (6.1)).

We now restate our rules for lowering and raising sets of skew-symmetric indices. They are

$$X^{\cup}_{\underline{j}} = \frac{\{i\}\,\gamma^{1/2}}{|i|!}\,\epsilon_{\underline{j}\,\underline{i}}\, X^{\underline{i}} \qquad \text{and}$$

$$(5.4) \qquad X^{\underline{i}}_{\cap} = \frac{\{j\}^{-1}\gamma^{-1/2}}{|j|!}\, X_{\underline{j}}\,\epsilon^{\underline{j}\,\underline{i}} ,$$

where

$$(5.5) \qquad \{i\} = (-1)^{\frac{1}{2}|i|(n-|i|)} .$$

In (5.4) if a hook ($\cup$) is added to the X in the right member it is erased from the left member. We observe that $\{i\} = \{k\}$ if $|i| + |k| = n$. The operations of calculus are summarized in the theorem[5]

---

[5] If $\gamma_{ij} = -\gamma_{ji}$, a factor $(-1)^{|i|}$ must be added to the right member of (5.1). Theorem (5.1) will continue to hold if we put $\{i\} = (-1)^{\frac{1}{2}|i|(n-|i|+1)}$ in (5.4). In order that $|\gamma_{ij}| \neq 0$, n must be even and hence $\{i\}$ is always real. Equations (5.7) of course require modification when $\gamma_{ij} = -\gamma_{ji}$.

---

Theorem (5.1). If single tensor indices are lowered and raised by $\gamma_{ij}$ and $\gamma^{ij}$ while sets of skew-symmetric indices are lowered and raised by the rules (5.4), the resulting calculus is consistent and gives the relations indicated in the diagram

(5.6)
$$\begin{array}{ccc} X^{\underline{i}} & \overset{\epsilon}{\longleftrightarrow} & X_{\underline{j}} \\ \gamma \updownarrow & & \updownarrow \gamma \\ X_{\underline{i}} & \underset{\epsilon}{\longleftrightarrow} & X^{\underline{j}}_{\cap} \end{array}$$

The rules regulating the manner in which sets of summed indices may be raised and lowered are computed to be

(5.7)
$$X^{\underline{i}} Y_{\underline{i}} = X_{\underline{i}} Y^{\underline{i}}, \qquad X^{\underline{i}}_{\cap} Y^{\cup}_{\underline{i}} = X^{\cup}_{\underline{i}} Y^{\underline{i}}_{\cap}$$

and

(5.8)
$$\frac{1}{|i|!} X^{\underline{i}} Y_{\underline{i}} = \frac{\{i\}^2}{|k|!} X^{\cup}_{\underline{k}} Y^{\underline{k}}_{\cap} .$$

The numerical coefficients in equations (4.7) may be evaluated by computation with (5.4). The result of this computation is contained in the theorem

Theorem (5.2). If $X^{\underline{x}}$ and $Y^{\underline{y}}$ are skew-symmetric tensors, $|i|$ is $\leq |x|$ and $\leq |y|$, and $|j|$ is $\leq n - |x|$ and $\leq n - |y|$, then

$$\frac{\{r\}}{|r|!} X^{\underline{i}\,\underline{r}} Y_{\underline{j}\,\underline{r}} = (-1)^{|r||s|} \frac{\{s\}}{|s|!} Y^{\underline{i}\,\underline{s}} X_{\underline{j}\,\underline{s}} \tag{5.9}$$
$$+ \sum_{|v|=|s|+1}^{M} \rho(|v|) \cdot \delta^{[\underline{i}_1}_{[\underline{j}_1} Y^{\underline{i}_2]\underline{v}} X_{\underline{j}_2]\underline{v}_j}$$

where $\underline{i} = \underline{i}_1\underline{i}_2$, $\underline{j} = \underline{j}_1\underline{j}_2$, and M is the smaller of the two numbers $|i|$ and $|j|$.

This useful identity depends only upon the rules for lowering and raising sets of skew-symmetric indices by means of the numerical tensors and does not involve the tensor $\gamma_{ij}$. If $|\gamma_{ij}|$ is not available we suppose the factor $\gamma$ in (5.4) to be an arbitrary scalar of weight two.

Special cases of (5.9) are stated in the following theorems.

Theorem (5.3). If $X^{\underline{x}}$ and $Y^{\underline{y}}$ are skew-symmetric and $|x| + |y| = n$, then

$$\frac{1}{|r|!} X^{\underline{i}\,\underline{r}} Y_{\underline{j}\,\underline{r}} = (-1)^{|x||y|+|i|} \frac{1}{|s|!} Y^{\underline{i}\,\underline{s}} X_{\underline{j}\,\underline{s}} \tag{5.10}$$

+ (additional terms in which more than $|s|$ indices are summed).

Theorem (5.4). If $V^{\underline{v}\,r\,\underline{s}} V_{\underline{w}\,r\,\underline{s}} = 0$ and $|s|$ is odd, then $V^{\underline{v}\,m\,\underline{s}} V_{\underline{w}\,p\,\underline{s}} = 0$.

To prove this we put $X^{\underline{i}} = Y^{\underline{i}} = V^{\underline{i}}$ in (5.10) and replace $\underline{i}$ and $\underline{j}$ by $\underline{v}$ r and $\underline{w}$ p, respectively. This theorem, for $|s| = 1$, was used in the proof of Theorem (3.4).

Theorem (5.5). If $X^{\underline{x}}$ and $Y^{\underline{y}}$ are skew-symmetric and $|x| + |y| = n$, then

$$\frac{1}{|r|!} X^{i\,\underline{r}} Y_{j\,\underline{r}} + (-1)^{|x||y|} \frac{1}{|s|!} Y^{i\,\underline{s}} X_{j\,\underline{s}} = \tag{5.11}$$
$$= \frac{(X^{\underline{x}} Y_{\underline{x}})}{|x|!} \delta^i_j .$$

This is the special case of (5.10) in which all but one of the indices of $X^{\underline{x}}$ are summed against indices of $Y^{\cup}_{\underline{x}}$. The scalar factor in the right member of (5.11) is most readily obtained by taking the trace of the left member.

An application of our calculus to an elementary problem in projective geometry is contained in the theorem

Theorem (5.6). If $X^{\underline{x}}$ and $Y^{\underline{y}}$ are coordinate tensors of linear spaces, X and Y, where X + Y = 1 and XY = 0, then

$$S^i{}_j = \frac{1}{|r|!} X^{i\,\underline{r}} Y^{\smile}_{j\,\underline{r}} - (-1)^{|x||y|} \frac{1}{|s|!} Y^{i\,\underline{s}} X^{\smile}_{j\,\underline{s}} \tag{5.12}$$

defines an involutoric collineation $\varphi^i = S^i{}_j \psi^j$ with the pointwise invariant spaces X and Y.

In proving this theorem it is convenient to abbreviate (5.12) to

$$S = X\cdot Y - \lambda Y\cdot X , \tag{5.13}$$

and by a similar abbreviation to replace the identity (5.11) by

$$X\cdot Y + \lambda Y\cdot X = \mu 1 . \tag{5.14}$$

Then S can be written in the forms

$$S = 2\, X\cdot Y - \mu 1 , \tag{5.15}$$

and

$$S = \mu 1 - 2\, \lambda\, Y\cdot X . \tag{5.16}$$

The collineation $\varphi^i = S^i{}_j \psi^j$ is of period two, for

$$\begin{aligned} SS &= (2\, X\cdot Y - \mu 1)\,(\mu 1 - 2\lambda Y\cdot X) \\ &= 2\,\mu (X\cdot Y + \lambda\, Y\cdot X) - \mu^2\, 1 \\ &= \mu^2\, 1 , \end{aligned}$$

where the "product" $-4\ \lambda(X\cdot Y)(Y\cdot X)$ vanishes since it contains the tensor $Y_{\underset{j\ \underline{r}}{\smile}}\ Y^{j\ \underline{s}}$ which is zero by Theorem (3.3). The collineation is non-singular since $\mu = 0$ would imply $XY \neq 0$, contrary to the hypothesis. A point $\psi^i$ in the space Y satisfies the conditions $Y_{\underset{j\ \underline{r}}{\smile}}\ \psi^j = 0$, so that using (5.15) we have

$$S^i{}_j\ \psi^j = (2X^{i\ \underline{r}}\ Y_{\underset{j\ \underline{r}}{\smile}} - \mu\,\delta^i{}_j)\,\psi^j = -\mu\psi^i\ .$$

Hence the points of Y are invariant. Similarly, using (5.16), the points of X are invariant.

## 6. LINEAR SPACES ON A QUADRIC

The tensors $V^{\underline{i}}$ and $V_{\underset{\underline{j}}{\smile}}$ have been interpreted as coordinate tensors of the same linear space. If we write

$$V^{\underline{i}} = \sum \pm\, A^{i_1}\, B^{i_2} \ldots H^{i_{|i|}}\ ,$$

we observe that

$$V_{\underline{i}} = \sum \pm\, A_{i_1}\, B_{i_2} \ldots H_{i_{|i|}}\ .$$

The hyperplanes $A_i$, $B_i$, ..., $H_i$ are the polars, respectively, of the points $A^i$, $B^i$, ..., $H^i$ in the quadric

(6.1) $$Q:\quad \gamma_{ij}\, X^i\, X^j = 0\ .$$

Hence $V_{\underline{i}}$ is a covariant coordinate tensor of the polar of the space $V^{\underline{i}}$. Of course $V^{\underset{\frown}{j}}$ is a contravariant coordinate tensor of this same space. Since a linear space $V^{\underline{i}}$ lies on Q if and only if it is contained in its polar space, we have as a special case of Theorem (4.5),

<u>Theorem (6.1)</u>. A space with coordinate tensor $V^{\underline{v}}$ lies on Q if and only if

(6.2) $$V^{\underline{i}\,s}\,V_{\underline{j}\,s} = 0 \,.$$

Reference to the first part of Theorem (3.2) shows that (6.2) implies $2\,|v| \leq n$, so that the maximum number of linearly independent points in a space on Q is $\frac{n-1}{2}$ when n is odd and $\frac{n}{2}$ when n is even. We now restrict our considerations to the latter case so that (6.1) defines a quadric in a projective space, $P_{2\nu-1}$, of $2\nu-1$ dimensions. A linear sub-space of $P_{2\nu-1}$ of $\nu-1$ dimensions will be called an "axis". An axis then has a coordinate tensor $V^{\underline{i}}$, where $|i| = \nu$. The polar of an axis is again an axis so that $V^{\underline{i}}$ will lie on the quadric if and only if it coincides with its polar space. That is

(6.3) $$V^{\underline{i}} = \rho\, V^{\underline{i}}_{\cap} \,.$$

Lowering the indices $\underline{i}$ with $\gamma_{\underline{s}\,\underline{i}}$ and raising them again with $\epsilon^{\underline{s}\,\underline{i}}$ gives $V^{\underline{i}}_{\cap} = \rho V^{\underline{i}}$ and consequently $\rho = \pm 1$.

<u>Theorem (6.2)</u>. A necessary and sufficient condition that a coordinate tensor $V^{\underline{i}}$ $(|i| = \nu)$ shall determine an axis on the quadric (6.1) is that either

(6.4) $$F_+ : \quad V^{\underline{i}} = +V^{\underline{i}}_{\cap} \,,$$

or

(6.5) $$F_- : \quad V^{\underline{i}} = -V^{\underline{i}}_{\cap} \,.$$

The axes on Q are thus separated into the two families $F_+$ and $F_-$, characterized by (6.4) and (6.5) respectively. The association of the plus sign with one family and the minus sign with the other is a matter of arbitrary choice. Indeed it is determined by the selection of one of the two values of the factor $\gamma^{\frac{1}{2}}$ occurring in (5.4).

Theorem (6.3). If V and W are two axes on Q, then

(6.6) $$\text{dimension of } (VW) \equiv \nu - 1 \pmod 2$$

if V and W belong to the same family, and

(6.7) $$\text{dimension of } (VW) \equiv \nu \pmod 2$$

if V and W belong to different families.[6]

---

[6] A proof of this theorem based upon stereographic projection is given by Bertini, "Einführung in die Projektive Geometrie", p. 143.

---

Since V and W are axes on Q,

(6.8) $$V^{\underline{i}} = \lambda_V V^{\underline{i}}_{\cap}, \qquad W^{\underline{i}} = \lambda_W W^{\underline{i}}_{\cap}, \qquad |i| = \nu,$$

and the spaces will belong to the same family if $\lambda_V \lambda_W = +1$ and to different families if $\lambda_V \lambda_W = -1$. Let us put D(VW) = 1 + dimension of (VW). Assuming that D(VW) = $\mu$ +1, using (6.8), and applying Theorem (4.5), we have $V^{\underline{i}\,\underline{s}}_{\frown} W_{\underline{j}\,\underline{s}}^{\smile} = 0$ for $|s| \geq \nu - \mu$. Hence if we put $X^{\underline{v}} = V^{\underline{v}}$ and $Y_{\underline{v}}^{\smile} = W_{\underline{v}}$ in equations (5.10) they reduce to

(6.9) $$V^{\underline{i}\,\underline{r}} W_{\underline{j}\,\underline{r}} = -(-1)^{\nu - \mu} W^{\underline{i}\,\underline{r}}_{\frown} V_{\underline{j}\,\underline{r}}^{\smile}, \text{ where } |i| = \mu + 1.$$

On account of (6.8) these equations imply

(6.10) $$V^{\underline{i}\,\underline{r}} W_{\underline{j}\,\underline{r}} = -(-1)^{\nu - \mu} \lambda_V \lambda_W W^{\underline{i}\,\underline{r}} V_{\underline{j}\,\underline{r}}.$$

By Theorem (4.5) $W^{\underline{i}\,\underline{r}} V_{\underline{j}\,\underline{r}} = I^{\underline{i}} J_{\underline{j}}$ where $I^{\underline{i}}$ and $J_{\underline{j}}$ are coordinate tensors of VW and V + W, respectively. Since V + W is the polar of VW, $J_{\underline{j}} = \rho \gamma_{\underline{j}\,\underline{s}} I^{\underline{s}}$, and consequently $W^{\underline{i}\,\underline{r}} V^{\underline{j}}_{\underline{r}} = \rho I^{\underline{i}} I^{\underline{j}}$ is symmetric in the sets $\underline{i}$ and $\underline{j}$. Hence $W^{\underline{i}\,\underline{r}} V^{\underline{j}}_{\underline{r}} = W^{\underline{j}\,\underline{r}} V^{\underline{i}}_{\underline{r}} = V^{\underline{i}\,\underline{r}} W^{\underline{j}}_{\underline{r}}$. Substituting in (6.10) now gives

$$(1 + \lambda_V \lambda_W (-1)^{\nu - \mu}) V^{\underline{i}\,\underline{r}} W_{\underline{j}\,\underline{r}} = 0 , \tag{6.11}$$

and therefore if $\lambda_V \lambda_W = +1$ $V^{\underline{i}\,\underline{r}} W_{\underline{j}\,\underline{r}} \neq 0$ implies $\nu - \mu \equiv 1 \pmod 2$, or if $\lambda_V \lambda_W = -1$, implies $\nu - \mu \equiv 0 \pmod 2$.

Theorem (6.4). If $S_{2\nu-1}$ is complex or if it is real and the signature of Q is zero, then Q contains axes. If $S_{2\nu-1}$ is real, Q contains axes only if its signature is zero.

We omit the proof.

Theorem (6.5). If Q contains axes, then axes V and W on Q exist for which $D(VW) = \alpha$, where $\alpha$ is any integer $\geqq 0$ and $\leqq \nu$.

Under the hypothesis of the theorem we may choose coordinates in the real or complex $S_{2\nu-1}$ so that the equation of Q is

$$X^1 X^{\nu+1} + X^2 X^{\nu+2} + \dots + X^{\nu} X^{2\nu} = 0 . \tag{6.12}$$

Let us call the vertices of the coordinate $2\nu$-point $E_1, E_2, \dots, E_{\nu}$, $E'_1, E'_2, \dots$, and $E'_{\nu}$, where the coordinates of $E_a$ $(a = 1, 2, \dots, \nu)$ are $\delta^i_a$ and of $E'_a$ are $\delta^i_{a+\nu}$. Then the spaces

$$\begin{aligned} V &= E_1 + E_2 + \dots + E_{\alpha} + E_{\alpha+1} + \dots + E_{\nu} \quad \text{and} \\ W &= E_1 + E_2 + \dots + E_{\alpha} + E'_{\alpha+1} + \dots + E'_{\nu} \end{aligned} \tag{6.13}$$

lie on Q and $D(VW) = \alpha$.

Theorem (6.6). If the quadric Q in $S_{2\nu-1}$ contains the distinct axes V and W, $\nu > 2$, and $\beta$ is any integer $\geqq 0$ and $\leqq \nu$, then there exists an axis A on Q such that $D(AV) = \beta \neq D(AW)$.

The theorem would be untrue for $\nu = 2$, $\beta = 1$, and D(VW) = 0 for then V and W would be non-intersecting lines on a quadric in $S_3$ and any line on Q which intersects one of two such lines in a point also intersects the other in a point. In the other cases which occur when $\nu = 1$ or 2 the theorem is trivially satisfied.

Let us put D(VW) = $\alpha$. Then it can be shown that by a suitable choice of coordinate system we may take the equation of Q to be (6.12) while V and W are the faces of the coordinate $2\nu$-point given by (6.13). We now distinguish several cases:

Case 1. $\beta < \alpha$. Take $A = E_1 + E_2 + \dots + E_\beta + E'_{\beta+1} + \dots + E'_\nu$. Then D(AV) = $\beta$, and D(AW) = $\beta + \nu - \alpha > \beta$.

Case 2. $\beta \geqq \alpha$ and $2\beta \neq \nu + \alpha$. Take A as in Case 1. Then D(AV) = $\beta$, and D(AW) = $\alpha + \nu - \beta \neq \beta$.

Case 3. $2\beta = \nu + \alpha$. Take $A = E'_1 + E'_2 + \dots + E'_{\nu-\beta} + E'_{\nu-\beta+1} + \dots + E_\nu$. Then D(AV) = $\beta$, and if $\alpha + \beta \geqq \nu$, D(AW) = $\alpha + \beta - \nu \neq \beta$, or if $\alpha + \beta < \nu$, D(AW) = $\nu - \alpha - \beta$, which is different from $\beta$ unless $2\beta = \nu - \alpha$. In this event $\alpha = 0$, and $2\beta = \nu$.

Case 4. $2\beta = \nu > 2$, $\alpha = 0$. The 3-space $E_{\nu-1} + E_\nu + E'_{\nu-1} + E'_\nu$ intersects Q in a non-degenerate quadric which contains the non-intersecting lines $E_{\nu-1} + E_\nu$ and $E'_{\nu-1} + E'_\nu$. Let F + G another line of the regulus containing these two lines, and take F on $E_{\nu-1} + E'_\nu$ and G on $E'_{\nu-1} + E_\nu$. If we put $A = E_1 + E_2 + \dots + E_\beta + E'_{\beta+1} + \dots + E'_{\beta-2} + F + G$, then D(AV) = $\beta = \nu/2$ and D(AW) = $\beta - 2 \neq \beta$.

<u>Theorem (6.7)</u>. If two collineations of $S_{2\nu-1}$ ($\nu > 2$) leave Q invariant and effect the same permutation of the axes of $F_+$, then they are identical.

By the preceding theorem an axis of $F_-$ is unambiguously determined by the axes of $F_+$ which intersect it in a space of $\beta - 1$ dimensions, where $\beta$ is any number congruent to $\nu - 1$ (mod 2) and between 0 and $\nu - 1$. Hence the two collineations must effect the same permutation on the axes of $F_-$. Since every point on Q is the complete intersection of two suitably chosen axes on Q (of the same or different families), the two collineations transform the points of Q in the same way and are therefore identical throughout the space.

# Chapter VIII

## REPRESENTATION OF LINEAR SPACES ON A QUADRIC IN $P_{2\nu-1}$

### 1. THE PROJECTIVE SPACE $P_{2\nu-1}$

In this chapter we shall apply the correspondence of Chapters V and VI between points in $E_{2\nu+1}$ and matrices of order $2^\nu$ to study the linear spaces on a quadric in a projective space of $(2\nu-1)$ dimensions. To do this we first introduce a coordinate system in $E_{2\nu+1}$ in which the cone $\gamma_{\alpha\beta} X^\alpha X^\beta = 0$ $(\alpha, \beta = 0, 1, \ldots, 2\nu)$ has the equation

$$(1.1) \qquad (X^o)^2 + \gamma_{ij} X^i X^j = 0 . \qquad (i, j = 1, 2, \ldots, 2\nu) .$$

The form of this equation will be preserved under transformations of coordinates in $E_{2\nu+1}$ if we allow only transformations with equations

$$(1.2) \qquad X^{o*} = X^o, \; X^{i*} = T^i_j X^j, \; (i, j = 1, 2, \ldots, 2\nu)$$

where $|T^i_j| \neq 0$.

The group of transformations (1.2) leaves the hyperplane $X^o = 0$ invariant and induces in it the linear transformation $X^{i*} = T^i_j X^j$, where $\|T^i_j\|$ is an arbitrary non-singular matrix. Hence the hyperplane $X^o = 0$ is a $2\nu$-dimensional complex affine space $E_{2\nu}$ in which the coordinates of a point are $(X^1, X^2, \ldots, X^{2\nu})$. Thus the coordinate vector $(X^o, X^1, X^2, \ldots, X^{2\nu})$ of a point in $E_{2\nu+1}$ is made up of a scalar $X^o$, and the coordinate vector $(X^1, X^2, \ldots, X^{2\nu})$ of a point in $E_{2\nu}$.

Under (1.2) a covariant vector has the law of transformation

$$(1.3) \qquad Y^*_o = Y_o, \quad Y^*_i = t^j_i Y_j ,$$

where $t^i_j T^j_k = \delta^i_k$. Consequently the coordinate vector $(Y_o, Y_1, Y_2, \ldots, Y_{2\nu})$ of a hyperplane in $E_{2\nu+1}$ is composed of the scalar $Y_o$ and the vector $(Y_1, Y_2, \ldots, Y_{2\nu})$.

Indeed, since

$$(1.4) \qquad \|\gamma_{\alpha\beta}\| = \left\| \begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \cdot & & & \\ \cdot & & \|\gamma_{ij}\| & \\ \cdot & & & \\ 0 & & & \end{array} \right\| \quad \text{and} \quad \|\gamma^{\alpha\beta}\| = \left\| \begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \cdot & & & \\ \cdot & & \|\gamma^{ij}\| & \\ \cdot & & & \\ 0 & & & \end{array} \right\| ,$$

where $\gamma^{ij}\gamma_{jk} = \delta^i_k$, the covariant components $X_\alpha = \gamma_{\alpha\beta} X^\beta$ of a vector $X^\beta$ are related to the contravariant ones by the equations

$$(1.5) \qquad X_0 = X^0, \qquad X_i = \gamma_{ij} X^j .$$

The rules for raising and lowering Greek indices ($\alpha$, $\beta$, etc.) with the range $(0, 1, 2, \ldots, 2\nu)$ may therefore be replaced by the corresponding rules for raising and lowering Latin indices (i, j, etc.) with the range $(1, 2, \ldots, 2\nu)$ by means of $\gamma^{ij}$ and $\gamma_{ij}$.

If we regard the numbers $X^i$ as being homogeneous rather than non-homogeneous coordinates of a point, we shall be dealing with a complex projective space $P_{2\nu-1}$ of $2\nu-1$ dimensions. In effect, this amounts to taking the lines through the origin in $E_{2\nu}$ to be the points of $P_{2\nu-1}$. The points of $P_{2\nu-1}$ with coordinates $\rho X^i$ therefore correspond to a pencil $\rho\|X^A{}_B\|$, of matrices of order $2^\nu$ by means of the equations

$$(1.6) \qquad X = X^i \gamma_i .$$

Since $\gamma_{\alpha\beta} X^\alpha X^\beta = (X^0)^2 + \gamma_{ij} X^i X^j$, we have from (1.6), Chapter V,

$$(1.7) \qquad X^2 = (X^i\gamma_i)^2 = \gamma_{ij} X^i X^j 1 .$$

The fundamental quadric

(1.8) $$\gamma_{ij}\, X^i X^j = 0$$

in $P_{2\nu-1}$ thus appears as the locus of singular elements of the linear family of involutions with matrices $X^i \gamma_i$.

In the following sections we shall pass freely from the affine space $E_{2\nu}$ with non-homogeneous coordinates $X^i$ to the projective space $P_{2\nu-1}$ with homogeneous coordinates $\rho X^i$. Equations (1.6) will then be regarded as establishing the correspondence $X^i \longleftrightarrow \| X^A{}_B \|$ between points of $E_{2\nu}$ and matrices in $P_{2\nu-1}$ or, equally well, the correspondence $\rho X^i \longleftrightarrow \rho \| X^A{}_B \|$ between points of $P_{2\nu-1}$ and involutions in $P_{2\nu-1}$.

## 2. THE CORRESPONDENCE BETWEEN TENSOR SETS AND MATRICES

In §6 of Chapter V we proved that if the $\gamma_i$ form a $\gamma$-set, then the matrices

(2.1) $$1,\ \gamma_i,\ \gamma_i \gamma_j (i < j),\ \ldots,\ \gamma_1 \gamma_2 \gamma_3 \cdots \gamma_{2\nu}$$

are linearly independent. Since there are $2^{2\nu}$ of them every matrix of order $2^\nu$ can be expressed as a linear sum of these matrices. That is, an arbitrary matrix $X = \| X^A{}_B \|$ of order $2^\nu$ is expressible in the form

(2.2) $$X = \sum_{|i|=0}^{2\nu} \frac{1}{|i|!} X^{\underline{i}} s_{\underline{i}},$$

where

(2.3) $$s_{\underline{i}} = \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_{|i|}}.$$

It is understood that when $|i| = 0$, $s_{\underline{i}} = 1$. When $X$ is a matrix of the linear family $X^i \gamma_i$ the coefficients $X^{\underline{i}}$ in (2.2) vanish unless $|i| = 1$ so that equations

(2.2) include (1.6) as a special case.

When the $\gamma_i$ are elements of a $\gamma$-set, the quantity $s_{\underline{i}} = \| s_{\underline{i}}{}^A{}_B \|$ is skew-symmetric in the indices $\underline{i}$. With the definition (2.3) this property is not preserved under transformations of coordinates in $E_{2\nu}$. We therefore define $s_{\underline{i}}$ by the formula

$$(2.4) \qquad s_{\underline{i}} = \frac{1}{|j|!} \delta^{\underline{j}}_{\underline{i}} \; \gamma_{j_1} \gamma_{j_2} \cdots \gamma_{j_{|i|}} ,$$

which reduces to (2.3) in a cartesian coordinate system, that is, when $\gamma_{ij} X^i X^j = \sum (X^i)^2$ so that the matrices $\gamma_i$ anticommute.

If we take the coefficients $X^{\underline{i}}$ in (2.2) to be skew-symmetric, we can solve for them in terms of the matrix X. To do this we employ cartesian coordinates to prove that Trace $s^{\underline{i}} s_{\underline{j}} = 0$ if $|i| \neq |j|$ and otherwise it is a multiple of $\delta^{\underline{i}}_{\underline{j}}$. The exact relation

$$(2.5) \qquad \text{Trace } (s^{\underline{i}} s_{\underline{j}}) = (-1)^{\frac{1}{2}|i|(|i|-1)} 2^{\nu} \, \delta^{|i|}_{|j|} \delta^{\underline{i}}_{\underline{j}}$$

may be verified in a cartesian coordinate system and evidently remains valid under coordinate transformations. Multiplying (2.2) by $s^{\underline{i}}$ and taking the trace therefore gives

$$(2.6) \qquad X^{\underline{j}} = (-1)^{-\frac{1}{2}|j|(|j|-1)} 2^{-\nu} \text{ Trace } X \, s^{\underline{j}}$$

and we have the theorem

<u>Theorem (2.1)</u>. Equations (2.2) and (2.6) establish a (1-1) correspondence

$$(2.7) \qquad \| X^A{}_B \| \longleftrightarrow \{ X, X^i, X^{ij} = -X^{ji}, \ldots, X^{\underline{i}} = X^{[\underline{i}]} (|i| = 2\nu) \}$$

between matrices of order $2^{\nu}$ and sets of skew-symmetric tensors.

## 3. THE COLLINEATIONS CORRESPONDING TO LINEAR SPACES ON THE QUADRIC

The correspondence of this theorem allows us to represent a linear subspace of $P_{2\nu-1}$ as a collineation in the spin space $P_{2^{\nu}-1}$. To do this it is only necessary to take the tensor set to be $\{0, 0, \ldots, 0, A^{\underline{i}}, 0, \ldots, 0\}$ where $A^{\underline{i}}$ is a coordinate tensor of the linear subspace.

A lemma which we shall need in the proof of the next theorem is

Lemma. If A and B are square matrices, $A^2 = B^2 = 0$, and $AB = \pm BA$, then

$$\text{rank } AB \leqq \tfrac{1}{2} \text{ rank } A \quad (3.1)$$

This is obtained from the general relation*

* C.C.MacDuffee, "The Theory of Matrices," Berlin 1933, Theorem 8.3.

$$\text{rank } AB + \text{rank } BC \leqq \text{rank } B + \text{rank } ABC$$

by putting $C = A$.

Theorem (3.1). If $L^{\underline{i}}$ is a coordinate tensor of a linear space L on the quadric (1.8) and $\lambda = \frac{1}{|i|!} L^{\underline{i}} s_{\underline{i}}$, then $\lambda^2 = 0$ and rank $\lambda = 2^{\nu - |i|}$.

Let us take $L_1, L_2, \ldots, L_{|i|}$ to be $|i|$ linearly independent points on L. Then $L = L_1 + L_2 + \ldots + L_{|i|}$. It is possible to find additional points $L_{|i|+1}, \ldots, L_{\nu}$ such that $A = L_1 + \ldots + L_{\nu}$ is an axis on the quadric. Then $\gamma_{ij} L_a^i L_b^j = 0$ for $a, b = 1, 2, \ldots, \nu$, where $L_a^i$ is a coordinate vector of the point $L_a$.

Equating coefficients of the arbitrary variables $X^i$ in (1.7) gives

$$\tfrac{1}{2}(\gamma_i \gamma_j + \gamma_j \gamma_i) = \gamma_{ij} 1 . \quad (3.2)$$

Multiplying both members of $L_a^i L_b^j$ and summing gives

(3.3) $$\lambda_a \lambda_b = - \lambda_b \lambda_a \qquad (a, b = 1, \ldots, \nu)$$

where $\lambda_a = L_a^i \gamma_i$ ($= L_a^i s_i$).

In particular, $(\lambda_a)^2 = 0$, so that rank $\lambda_a \leqq \frac{1}{2}$(order of $\lambda_a$) $= 2^{\nu - 1}$. The product of any number of the matrices $\lambda_a$ also has square zero for $(\lambda_a \lambda_b \cdots \lambda_p)^2 = \pm (\lambda_a)^2 (\lambda_b)^2 \cdots (\lambda_p)^2 = 0$. Since $L^{\underline{i}} = L_1^{[i_1} L_2^{i_2} \cdots L_{|i|}^{i_{|i|}]}$, $\lambda = \lambda_1 \lambda_2 \cdots \lambda_{|i|}$ and there $\lambda^2 = 0$.

Repeated application of the lemma now gives the chain of inequalities

(3.4)
$$\begin{array}{lll}
\text{rank } \lambda_1 & \leqq 2^{\nu - 1} & \\
\text{rank } (\lambda_1 \lambda_2) & \leqq \frac{1}{2} \text{rank } \lambda_1 & \leqq 2^{\nu - 2} \\
\vdots & \vdots & \vdots \\
\text{rank } (\lambda_1 \lambda_2 \cdots \lambda_r) & \leqq \frac{1}{2} \text{rank } (\lambda_1 \lambda_2 \cdots \lambda_{r-1}) & \leqq 2^{\nu - r} \\
\vdots & \vdots & \vdots \\
\text{rank } (\lambda_1 \lambda_2 \cdots \lambda_\nu) & \leqq \frac{1}{2} \text{rank } (\lambda_1 \lambda_2 \cdots \lambda_{\nu - 1}) & \leqq 2^{\nu - \nu} = 1
\end{array}$$

However, if $A^{\underline{i}}$ is a coordinate tensor of $A = L_1 + L_2 + \cdots + L_\nu$, then $\frac{1}{|i|!} A^{\underline{i}} s_{\underline{i}} = \lambda_1 \lambda_2 \cdots \lambda_\nu \neq 0$ and consequently rank $(\lambda_1 \lambda_2 \cdots \lambda_\nu) = 1$. Hence the inequality sign cannot hold at any step and rank $(\lambda_1 \lambda_2 \cdots \lambda_{|i|}) = 2^{\nu - |i|}$.

Theorem (3.2). If $A^{\underline{i}}$ ($|i| = \nu$) is the coordinate tensor of an axis on the quadric and $\alpha = \frac{1}{|i|!} A^{\underline{i}} s_{\underline{i}}$, then

(3.5) $$\alpha \equiv \| \alpha^A{}_B \| = \| \psi^A \phi_B \|.$$

By Theorem (3.1) the rank of $\alpha$ is one and hence all the columns of the matrix are proportional to any non-vanishing column with elements $\psi^1, \psi^2, \ldots, \psi^{2\nu}$. If the factors of proportionality are called $\phi_B$, we have (3.5). Interpreting $\psi^A$ as the coordinates of a point in the spin space, we have a representation

(3.6) $$A \longrightarrow \psi$$

of the axes on the quadric in $P_{2\nu-1}$ by means of points in $P_{2\nu-1}$. In the following sections we shall study the figures in the spin space determined by linear spaces on the quadric and apply these results to the representation (3.6).

## 4. SPACES IN $P_{2\nu-1}$ DETERMINED BY SPACES ON THE QUADRIC IN $P_{2\nu-1}$

The matrix $\beta = \frac{1}{|i|!} B^{\underline{i}} s_{\underline{i}}$ corresponding to a linear space B with coordinate tensor $B^{\underline{i}}$ is, by Theorem (3.1), singular if B lies on the quadric. Hence the collineation determined by $\beta$ transforms all the points of $P_{2\nu-1}$ into the points of a subspace $R_\beta$, the "rank" space of $\beta$. A point $\psi$ with coordinates $\psi^A$ will therefore be contained in $R_\beta$ if and only if there exists a coordinate vector $\theta^B$ such that $\psi^A = \beta^A{}_B \theta^B$.

The singular points of the collineation form a linear space $N_\beta$, the "null" space of $\beta$. Thus a point $\psi$ belongs to $N_\beta$ if and only if $\beta\psi = 0$. We recall the notation of Chapter VII:

(4.1) $D(S) = 1 + \text{dimension of } (S) = \left(\begin{array}{l}\text{number of points necessary to determine the} \\ \text{linear space S.}\end{array}\right)$

It is then evident that $D(R_\beta) = \text{rank}\,\beta$ and $D(N_\beta) = \text{nullity of } \beta$, so that

(4.2) $$D(R_\beta) + D(N_\beta) = 2^\nu .$$

In the following theorems of this and the next section, we shall suppose that B is a linear space on the quadric and that $\beta$ is the matrix corresponding to B under (2.7).

<u>Theorem (4.1)</u>. $R_\beta$ is a subspace of $N_\beta$.

If $\psi$ belongs to $R_\beta$, $\psi = \beta\theta$. But $\beta^2 = 0$, so that $\beta\psi = \beta^2\theta = 0$ and hence $\psi$ also belongs to $N_\beta$. The space $R_\beta$ will be a proper subspace of $N_\beta$ if the rank of $\beta$ is less than the nullity of $\beta$ and the spaces will coincide if the rank and nullity are equal. By Theorem (3.1) the latter case will arise only when B is a point and then $R_\beta$ and $N_\beta$ are the coincident axes of the degenerate involution $B^i \gamma_i$ (cf. the last paragraph of §11, Chapter V).

Theorem (4.2). The invariant polarity interchanges $R_\beta$ and $N_\beta$.

The invariant polarity in $P_{2\nu -1}$ was discussed in detail in Chapter VI. For our present purposes it is sufficient to recall that it was determined by a matrix $C = \|C_{AB}\|$ which satisfied the equations (cf. (3.4) and (3.5), Chapter VI).

(4.3) $$C' = eC , \quad e = (-1)^{\frac{1}{2}\nu(\nu+1)} ,$$

and

(4.4) $$(C\gamma_i)' = f(C\gamma_i) , \quad f = (-1)^{\nu} e .$$

Let us put D(B) = r and take $B_1^i, B_2^i, \ldots, B_r^i$ to be coordinate vectors of r linearly independent points in B. Multiplying (4.4) through by $B_p^i$ ($p = 1, 2, \ldots, r$) and putting $\beta_p = B_p^i \gamma_i$, we get $\beta_p' C' = fC\beta_p$, or, using (4.3),

(4.5) $$C\beta_p = \pm \beta_p' C .$$

Since $\|C_{AB}\|$ is non-singular, D (polar of $R_\beta$) = $2^\nu$ - D($R_\beta$) and combining this equation with (4.2) gives D (polar of $R_\beta$) = D($N_\beta$). Hence to show that the invariant polarity transforms $R_\beta$ into $N_\beta$ it will be sufficient to prove that the polar of $R_\beta$ contains $N_\beta$. This will be the case if the polar hyperplane of an arbitrary point $\psi$ of $R_\beta$ contains every point of $N_\beta$. We shall therefore have the theorem if we can prove that $\varphi^A C_{AB} \psi^B = 0$ whenever

$\psi^B = \beta^B{}_C \theta^C$ and $\beta^D{}_A \varphi^A = 0$.

Calling $\|\psi^A\|$, $\|\varphi^A\|$ and $\|\theta^A\|$ matrices of one column, the condition is that $\varphi' C \psi = 0$ whenever $\psi = \beta\theta$ and $\beta\varphi = 0$. But since $\beta = \beta_1 \beta_2 \cdots \beta_r$, we can employ (4.5) to get

$$\varphi' C \psi = \varphi' C \beta \theta = \varphi' C \beta_1 \beta_2 \cdots \beta_r \theta = \pm (\varphi' \beta_1' \beta_2' \cdots \beta_r') C \theta .$$

The points $B_p$ are conjugate by pairs in the quadric and consequently the matrices $\beta_p$ anticommute by pairs. Hence

$$(\varphi' \beta_1' \beta_2' \cdots \beta_r') = (\beta_r \cdots \beta_2 \beta_1 \varphi)' = \pm (\beta_1 \beta_2 \cdots \beta_r \varphi)' = \pm (\beta\varphi)' = 0$$

and substituting in the last equations gives the required condition, $\varphi' C \psi = 0$.

## 5. PROPERTIES OF $R_\beta$ AND $N_\beta$

Under the restricted group (1.3) of transformations in $E_{2\nu+1}$, the matrix $\|\gamma_o{}^A{}_B\|$ behaves as a scalar. Indeed we saw in Chapter V that the equations

$$(5.1) \qquad (\gamma_o)^2 = 1 \qquad \text{and} \qquad \gamma_o \gamma_i = - \gamma_i \gamma_o$$

determine $\gamma_o$ to within sign and it is evident that these conditions are not changed if we replace $\gamma$ by $t_i^j \gamma_j$ where $|t_i^j| \neq 0$. The invariant nature of the matrix $\gamma_o$ is also put into evidence if we define

$$(5.2) \qquad \gamma_o = \frac{(-1)^{\frac{\nu}{2}} \, |\gamma_{ij}|^{-\frac{1}{2}}}{(2\nu)!} \, \epsilon^{i} \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_{2\nu}} ,$$

which we may do since the right member reduces in a cartesian coordinate system to $\pm (-1)^{\nu/2} \gamma_1 \gamma_2 \cdots \gamma_{2\nu}$ and this matrix satisfies (5.1). The advantage of (5.2) is that it determines a particular one of the two matrices satisfying (5.1) -- at least to within a choice of one of the square roots of the determinant $|\gamma_{ij}|$.

If this root is the same as that used in the definition of $X^{\cup}_{\underline{i}}$ in terms of $X^{j}$ ((5.4) Chapter VII), $\gamma_o$ satisfies the equations

$$(5.3) \qquad \gamma_o s_{\underline{i}} = (-1)^{\frac{1}{2}(\nu+|i|)} s^{\cup}_{\underline{i}}$$

and

$$(5.4) \qquad \gamma_o \left(\frac{1}{|i|!} X^{\underline{i}} s_{\underline{i}}\right) = (-1)^{\frac{1}{2}(\nu-|i|)} \left(\frac{1}{|j|!} X^{\underline{j}}_{\cap} s_{\underline{j}}\right) .$$

These equations are most easily verified in a cartesian coordinate system and their tensor form then insures their validity in all centered affine coordinate systems in $E_{2\nu}$. The corresponding formulas for multiplication on the right by $\gamma_o$ are obtained from these by using the invariant relations

$$(5.5) \qquad \gamma_o s_{\underline{i}} = (-1)^{|i|} s_{\underline{i}} \gamma_o .$$

<u>Theorem (5.1)</u>. The involution defined by $\gamma_o$ leaves $R_\beta$ and $N_\beta$ each invariant.

Since $\beta = \frac{1}{|i|!} B^{\underline{i}} s_{\underline{i}}$, where $B^{\underline{i}}$ is a coordinate tensor of the space B on the quadric, we can use (5.5) to get $\gamma_o \beta = (-1)^{|i|} \beta \gamma_o$. If $\psi$ is a point of $R_\beta$ so that $\psi^B = \beta^B{}_C \theta^C$, then $\gamma_o{}^A{}_B \psi^B = \beta^A{}_B [(-1)^{|i|} \gamma_o{}^B{}_C \theta^C]$ and hence the involution transforms $\psi$ into a point which again belongs to $R_\beta$. The invariance of $N_\beta$ follows from Theorem (4.2) by using the commutativity of the invariant polarity and the involution $\gamma_o$.

If A is an axis on the quadric in $P_{2\nu-1}$, its corresponding matrix is, by Theorem (3.2), of the form $\alpha = \| \psi^A \varphi_B \|$ where $R_\beta$ is the point with coordinates $\psi^A$ and $N_\beta$ is the hyperplane with coordinates $\varphi_B$. By Theorem (4.2) the vector $\varphi_B$ is determined as a multiple of $C_{AB}\psi^B$ and therefore the point $\psi$ determines the matrix $\alpha$ to within a factor. But the matrix $\alpha$ determines a coordinate tensor of A by the equations (2.6) and hence if $\psi$ corresponds to any axis under (3.6) it uniquely determines the axis. This proves the first part of the theorem

Theorem (5.2). The equations

$$\frac{1}{\nu!} A^{\underline{i}} s_{\underline{i}}{}^{A}{}_{B} = \psi^{A} C_{BC} \psi^{C} , \tag{5.6}$$

and their inverses

$$A^{\underline{i}} = (-1)^{\nu} C_{AC} s^{\underline{i}}{}^{C}{}_{B} \psi^{A} \psi^{B} , \tag{5.7}$$

establish a (1-1) correspondence

$$A \longleftrightarrow \psi \tag{5.8}$$

between the axes on the quadric in $P_{2\nu-1}$ and points in $P_{2\nu-1}$. Under the correspondence axes belonging to $F_+$ correspond to points of the axis $[\gamma_o]^+$ and axes belonging to $F_-$ correspond to points belonging to the axis $[\gamma_o]^-$.

To prove the last part of the theorem we apply (5.4) to the matrix $\alpha = \frac{1}{|i|!} A^{\underline{i}} s_{\underline{i}}$ to get $\gamma_o \alpha = \pm \alpha$ where the plus sign is used if $A^{\underline{i}} = +A^{\underline{i}}_{\cap}$ and the minus sign if $A^{\underline{i}} = -A^{\underline{i}}_{\cap}$. Hence $\gamma_o{}^{A}{}_{B} \psi^{B} = + \psi^{A}$ if A belongs to $F_+$, and $\gamma_o{}^{A}{}_{B} \psi^{B} = - \psi^{A}$ if A belongs to $F_-$. Theorem (5.1) could have been used to prove that $\psi$ lies in either $[\gamma_o]^+$ or $[\gamma_o]^-$, but the argument just given proves in addition that points corresponding to axes of the same family lie in the same axis of $\gamma_o$ and points corresponding to axes of different families lie in different axes of $\gamma_o$.

The invariance of $R_\beta$ under $\gamma_o$ implies that $R_\beta$ intersects $[\gamma_o]^+$ and $[\gamma_o]^-$ in spaces $R_\beta[\gamma_o]^+$ and $R_\beta[\gamma_o]^-$ such that

$$D(R_\beta[\gamma_o]^+) + D(R_\beta[\gamma_o]^-) = D(R_\beta). \tag{5.9}$$

If B is not an axis, the statement that $R_\beta[\gamma_o]^+$ and $R_\beta[\gamma_o]^-$ are spaces of the same dimensionality is contained in the theorem

Theorem (5.3). If $D(R_\beta) > 1$, then

$$D(R_\beta[\gamma_o]^+) = D(R_\beta[\gamma_o]^-) = \tfrac{1}{2}D(R_\beta) \,. \tag{5.10}$$

If B is not an axis, its polar space does not lie entirely on the quadric and hence we may choose a point in it in such a way that its corresponding collineation $\Gamma$ is non-singular. Of course $\Gamma$ commutes (or anticommutes) with the collineation $\beta$ corresponding to B. If $\psi = \beta\theta$ is any point of $R_\beta$, $\Gamma\psi = \Gamma\beta\theta = \pm\beta(\Gamma\theta)$ and hence $\Gamma$ leaves $R_\beta$ invariant. Since $\Gamma$ is an involution of the family $X^i\gamma_i$, it anticommutes with $\gamma_o$ and interchanges $[\gamma_o]^+$ and $[\gamma_o]^-$. Hence $\Gamma$ interchanges $R_\beta[\gamma_o]^+$ and $R_\beta[\gamma_o]^-$ and the spaces are of the same dimension.

An (r+1)-dimensional linear space $B_1$, on the quadric is determined as the join of an r-dimensional subspace B and a point $L_1$ in $B_1$ but not in B. The spaces corresponding to B and $L_1$ determine the spaces corresponding to $B_1$ in the way stated in the theorem

Theorem (5.4). If $B_1 = B + L_1$ is a linear space on the quadric, $L_1$ is a point, $BL_1 = 0$, and $B_1$, B and $L_1$ correspond to matrices $\beta_1, \beta$ and $\lambda_1$ respectively, then

$$R_{\beta_1} = R_\beta R_{\lambda_1} \qquad \text{and} \qquad N_{\beta_1} = N_\beta + N_{\lambda_1} \,. \tag{5.11}$$

The second of these equations is obtained from the first by taking polars with respect to the invariant polarity. Moreover, since $\beta_1 = \beta\lambda_1 = \pm\lambda_1\beta$ it is evident that $R_{\beta_1}$ is included in both $R_\beta$ and $R_\lambda$ and therefore in $R_\beta R_{\lambda_1}$. Hence it is sufficient to prove that $D(R_\beta R_{\lambda_1}) = D(R_{\beta_1})$.

Let us choose $L_2$ to be a point on the intersection of the quadric Q with the polar of B. This can always be done in such a way that $L_2$ is not conjugate to $L_1$ and then $B_2 = B + L_2$ will lie on Q and the line $L_1 + L_2$ will not lie on Q. We now take $M_1$ and $M_2$ to be two distinct points on $L_1 + L_2$ which are conjugate with respect to Q and call their corresponding matrices $\mu_1$ and $\mu_2$. Then $\mu_1 = a\lambda_1 + b\lambda_2$ and $\mu_2 = c\lambda_1 + d\lambda_2$. Since both $\lambda_1$ and $\lambda_2$ commute with $\beta$ so do $\mu_1$ and $\mu_2$, and hence the non-singular involutions with matrices $\mu_1$ and $\mu_2$ transform $R_\beta$ into itself. The same is therefore true of their product, $\mu_1\mu_2$. In §11, Chapter V, we discussed the axis regulus determined by a pencil of involutions and there proved that the axes of a product, $\mu_1\mu_2$, of two anti-commutative involutions are the rank spaces of the singular elements, $\lambda_1$ and $\lambda_2$, of the pencil. Hence $D(R_\beta R_{\lambda_1}) + D(R_\beta R_{\lambda_2}) = D(R_\beta)$. But $\mu_1$ interchanges $R_{\lambda_1}$ and $R_{\lambda_2}$ and leaves $R_\beta$ invariant so that $D(R_\beta R_{\lambda_1}) = D(R_\beta R_{\lambda_2})$. Therefore $D(R_\beta R_{\lambda_1}) = \frac{1}{2}D(R_\beta)$, which is equal to $D(R_{\beta_1})$ by Theorem (3.1).

<u>Theorem (5.5)</u>. If $L_1$ and $L_2$ are points on Q and $\lambda_1$ and $\lambda_2$ are their corresponding matrices, then either

(5.12) $$D(R_{\lambda_1} R_{\lambda_2}) = 2^{\nu-2}$$

and the line $L_1 + L_2$ lies on Q, or

(5.13) $$D(R_{\lambda_1} R_{\lambda_2}) = 0$$

and the line $L_1 + L_2$ does not lie on Q.

If $L_1 + L_2$ lies on Q we may put $B = L_2$ in Theorem (5.4) and use the fact that $D(R_{\beta_1}) = 2^{\nu - D(B_1)} = 2^{\nu-2}$ to get (5.12). The only other possibility is that $L_1 + L_2$ does not lie on Q, and in this case $R_{\lambda_1}$ and $R_{\lambda_2}$ are distinct axes of the axis regulus determined by the pencil of involutions $a\lambda_1 + b\lambda_2$. Distinct axes of a non-degenerate axis regulus do not intersect and we therefore have (5.13).

## 6. GEOMETRY OF A GENERALIZATION OF THE PLUECKER-KLEIN CORRESPONDENCE

The properties of the geometrical correspondence

$$(6.1) \qquad B \longrightarrow R_\beta$$

between linear spaces on Q and the spaces into which the corresponding matrices transform the whole of $P_{2\nu-1}$ will be made clearer if we discuss some special cases. When $\nu = 1$ the quadric in $P_1$ consists of two distinct points and these points correspond, respectively, to two distinct points in the spin space, which is again a $P_1$.

When $\nu = 2$ the spaces $P_{2\nu-1}$ and $P_{2\nu-1}$ are again of the same number of dimensions but the axes on the quadric are now represented by points of two non-intersecting lines $\ell_1$ and $\ell_2$ in the spin space $P_3$. Two lines of the same regulus correspond to points of the same line in the spin space. Two lines on the quadric which intersect in a point $P_o$ correspond to two points $L_1$ and $L_2$ lying on the lines $\ell_1$ and $\ell_2$ respectively, and the point $P_o$ corresponds to the line $L_1 + L_2$ crossing $\ell_1$ and $\ell_2$.

When $\nu = 3$ the quadric is in $P_5$ and the spin space is 7-dimensional. The axes $[\gamma_o]^+$ and $[\gamma_o]^-$ are non-intersecting 3-spaces. A point on Q corresponds to a $P_3$ in $P_7$ which intersects each of the 3-spaces $[\gamma_o]^+$ and $[\gamma_o]^-$ in a line. Two points $B_1$ and $B_2$ on Q therefore determine a pair of lines in $[\gamma_o]^+$ and a pair in $[\gamma_o]^-$. These two pairs of lines may intersect in points $L_1$ and

$L_2$ and if they do the line $B_1 + B_2$ lies on Q and corresponds to the line $L_1 + L_2$ crossing $[\gamma_o]^+$ and $[\gamma_o]^-$. If the plane $A = B_1 + B_2 + B_3$ determined by the three points $B_1$, $B_2$ and $B_3$ lies on Q and belongs to $F_+$, then the points determine in $[\gamma_o]^-$ the edges of a non-degenerate triangle while in $[\gamma_o]^+$ they determine three lines through the point corresponding to A under (6.1).

For $\nu = 1, 2,$ or $3$ it can be shown that all the points of $[\gamma_o]^+$ and $[\gamma_o]^-$ occur as images of axes on Q. When $\nu > 3$ this is no longer the case, as could be proved by comparing the dimensions $2^{\nu-1}-1$ of $[\gamma_o]^+$ with the dimension* $\frac{1}{2}\nu(\nu-1)$ of the families of axes on Q.

* Bertini, "Einführung in die Projektive Geometrie Mehrdimensionaler Räume", Vienna 1924, p. 142.

Referring to the table of Theorem (3.1), Chapter VI, we observe that the invariant polarity determined by $C = \| C_{AB} \|$ interchanges $[\gamma_o]^+$ and $[\gamma_o]^-$ if $\nu$ is even, and leaves them separately invariant if $\nu$ is odd. Hence when $\nu$ if even a point $\psi$ of $[\gamma_o]^+$ is transformed into a hyperplane $C\psi$ which contains $[\gamma_o]^-$ and intersects $[\gamma_o]^+$ in a space of $2^{\nu-1}-2$ dimensions. That is, when $\nu$ is even, C induces a polarity within the space $[\gamma_o]^+$ and, similarly, within $[\gamma_o]^-$.

When $\nu$ is odd, however, a point $\psi$ of $[\gamma_o]^+$ is transformed into a hyperplane $C\psi$ which contains $[\gamma_o]^+$ and intersects $[\gamma_o]^-$ in a space of $2^{\nu-1}-2$ dimensions. The invariant polarity therefore defines a mapping of the subspaces of $[\gamma_o]^+$ into subspaces of $[\gamma_o]^-$ in which points correspond to hyperplanes (with respect to $[\gamma_o]^-$), lines correspond to spaces of $2^{\nu-1}-3$ dimensions, and so on.

If $R_\alpha$ is an axis in $P_{2\nu-1}$ corresponding to a point on Q, $R_\alpha = N_\alpha$ and Theorem (4.2) states that the polarity C leaves $R_\alpha$ invariant. However, $R_\alpha = R_\alpha[\gamma_o]^+ + R_\alpha[\gamma_o]^-$ and hence the mapping of $[\gamma_o]^+$ into $[\gamma_o]^-$ transforms $R_\alpha[\gamma_o]^+$ into $R_\alpha[\gamma_o]^-$. The space $R_\alpha$ is therefore determined by its intersection with $[\gamma_o]^+$. Similarly, a point of $[\gamma_o]^-$ is determined by the $(2^{\nu-1}-2)$-dimensional space in which its polar hyperplane intersects $[\gamma_o]^+$. These results can be combined with the theorems of §5 to give the theorem

Theorem (6.1). The points of a quadric Q in $P_{2\nu-1}$ may be made to correspond to axes in $P_{2^{\nu-1}-1}$ in such a way that the points of an (r-1)-dimensional space, $r < \nu$, on Q correspond to axes all of which contain the same $(2^{\nu-1-r}-1)$-dimensional space. Under the correspondence the points of an axis on Q either correspond to axes all of which contain the same point or to axes all of which lie in the same hyperplane.

For $\nu = 3$ this theorem gives the Pluecker-Klein correspondence between points on a quadric in $P_5$ and lines in $P_3$. In this case all the lines in $P_3$ enter into the correspondence. Lines on the quadric correspond to pencils of lines in $P_3$ and the points of a plane on the quadric correspond to all the lines through a point or to all the lines in a plane.

## 7. COLLINEATION REPRESENTATION OF $H^+_{2\nu}$ FOR $\nu > 2$

The geometrical correspondences established in this chapter give rise to representations of the proper orthogonal group on $2\nu$ variables, $H^+_{2\nu}$. In this and the next two sections we shall discuss these representations.

In Chapter V we used the equations $P\gamma_\beta P^{-1} = \gamma_\alpha L^\alpha_\beta$ to establish a (1-1) collineation isomorphism between the proper orthogonal group $H^+_{2\nu+1}$ on $2\nu+1$ variables and the group of collineations leaving the linear family $X^\alpha\gamma_\alpha$

invariant. In §7 of that chapter we observed that the full orthogonal group $H_{2\nu}$ is a subgroup of $H^{+}_{2\nu+1}$. This leads to the representation of $H_{2\nu}$ stated in the theorem

<u>Theorem (7.1)</u>. The equations

$$P\gamma_j P^{-1} = \gamma_i L^i_{\ j} \tag{7.1}$$

establish a (1-1) isomorphism

$$\| L^i_{\ j} \| \longleftrightarrow \rho P \tag{7.2}$$

between the group of orthogonal matrices of order $2\nu$ and the group of collineations of $P_{2\nu-1}$ which leave the linear family $X^i\gamma_i$ invariant. Under this isomorphism

$$H^{+}_{2\nu}: \qquad \gamma_o P = + P\gamma_o \quad \text{if} \quad |L^i_{\ j}| = +1 , \tag{7.3}$$

and

$$\gamma_o P = - P\gamma_o \quad \text{if} \quad |L^i_{\ j}| = -1 . \tag{7.4}$$

If we choose a coordinate system in the spin space so that the points $\delta^A_1$, $\delta^A_2$, ..., and $\delta^A_{2^{\nu-1}}$ lie in $[\gamma_o]^+$ and the points $\delta^A_{2^{\nu-1}+1}$, $\delta^A_{2^{\nu-1}+2}$, ..., and $\delta^A_{2^\nu}$ lie in $[\gamma_o]^-$, then

$$\gamma_o = \begin{Vmatrix} 1_{2^{\nu-1}} & 0 \\ 0 & -1_{2^{\nu-1}} \end{Vmatrix} \tag{7.5}$$

Equations (7.3) show that in this coordinate system a collineation corresponding to a matrix of $H^{+}_{2\nu}$ is defined by a matrix of the form

$$(7.6) \qquad P = \begin{Vmatrix} P_+ & 0 \\ 0 & P_- \end{Vmatrix}$$

Similarly, equations (7.4) show that improper orthogonal matrices correspond to collineations with matrices of the form

$$(7.7) \qquad P = \begin{Vmatrix} 0 & P_{12} \\ P_{21} & 0 \end{Vmatrix}$$

It is evident that collineations of the type (7.6) leave $[\gamma_o]^+$ and $[\gamma_o]^-$ separately invariant while collineations of the form (7.7) interchange them. Using the (1-1) correspondence of Theorem (5.2) between axes on the quadric and points in $[\gamma_o]^+$ and $[\gamma_o]^-$ we get the theorem

<u>Theorem (7.2)</u>. Collineations in $P_{2\gamma-1}$ defined by proper orthogonal matrices ($|L^i{}_j| = +1$) leave the two families of axes on the quadric separately invariant while collineations defined by improper orthogonal matrices interchange them.

If all the matrices of a group are of the form (7.6), elementary arguments suffice to show that the matrices $P_+$ form a group which is isomorphic (perhaps multiply) with the original group. Combining this isomorphism with (7.2) we get the isomorphism

$$(7.8) \qquad \| L^i{}_j \| \longrightarrow \rho P_+$$

between $H^+_{2\nu}$ and a collineation group in $P_{2^{\nu-1}-1}$. Moreover, two different matrices $\| L^i{}_j \|$ and $\| M^i{}_j \|$ which correspond to the same pencil $\rho P_+$, induce the same permutation of points in $[\gamma_o]^+$ and hence the transformations of $P_{2\gamma-1}$ which they define effect the same permutation of the axes of $F_+$. If $\gamma > 2$, Theorem (6.7), Chapter VII, states that the two transformations must be identical. This will be the case only if $\|L^i{}_j\|$ and $\|M^i{}_j\|$ are proportional. Since both the matrices are orthogonal, $\|M^i{}_j\| = \pm \|L^i{}_j\|$.

The matrix $\gamma_o$ anticommutes with all the matrices $\gamma_i$ so that $\gamma_o \gamma_i \gamma_o^{-1} = - \gamma_i$. Comparing this with (7.1) we see that under (7.2)

$$(7.9) \qquad \|-\delta^i_j\| \longleftrightarrow \rho\gamma_o = \rho \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}.$$

Hence if $\|L^i{}_j\|$ corresponds to $\rho P_+$ under (7.8), $\|-L^i{}_j\|$ also corresponds to the same pencil of matrices. The isomorphism inverse to (7.8) is therefore (1-2) and we have the theorem

Theorem (7.3). If $\nu > 2$, the matrix group $H^+_{2\nu}$ is (2-1) isomorphic with a collineation group in $P_{2^{\nu-1}-1}$.

We have determined the isomorphism of this theorem by equations (7.1), which are equations between matrices of order $2^\nu$. If, however, we employ cartesian coordinates in $E_{2\nu}$ and also take $\gamma_o$ in the form (7.5), the matrices $\gamma_i$ are of the form

$$(7.10) \qquad \gamma_i = \begin{Vmatrix} 0 & C_i \\ C_i^{-1} & 0 \end{Vmatrix} ,$$

and equations (7.1) reduce to

$$(7.11) \qquad P_+ C_j P_-^{-1} = C_i L^i{}_j ,$$

which are equations between matrices of order $2^{\nu-1}$. The other equations that we obtain from (7.1), $P_- C_j^{-1} P_+ = C_i^{-1} L^i{}_j$, follow from (7.11) by taking the inverse of both members.

Comparing (7.11) with equations (7.1) of Chapter VI, we see that the isomorphism which we there obtained from the study of linear families of correlations has now been obtained from the geometry of linear families of involutions. Equations (7.10) establish the connection between these two approaches to the representation.

## 8. MATRIX REPRESENTATION OF $H^{+}_{2\nu}$ for $\nu > 2$

We normalize the matrices P entering into equations (7.1) to within sign by the requirement that

(8.1) $$P'CP = C \ ,$$

as we proved we could do in §5 of Chapter VI. The collineation isomorphism (7.2) then becomes the (1-2) matrix isomorphism

(8.2) $$\| L^i{}_j \| \longleftrightarrow \pm P = \pm \begin{Vmatrix} P_+ & 0 \\ 0 & P_- \end{Vmatrix} .$$

If $\nu > 2$, we may again conclude that the matrices $\pm P_+$ form a group isomorphic (perhaps multiply) to the group of matrices $\pm P$, and so obtain the isomorphism

(8.3) $$\| L^i{}_j \| \longrightarrow \pm P_+ \ .$$

We shall refer to this correspondence between proper orthogonal matrices of order $2\nu$ and matrices of order $2^{\nu-1}$ as the representation (or isomorphism) $\Delta_+$. Similarly, there is an isomorphism $\| L^i{}_j \| \longrightarrow \pm P_-$ which we shall refer to as $\Delta_-$.

We shall now determine the nature of the correspondence inverse to (8.3). Since the isomorphism of Theorem (7.3) is (2-1), the orthogonal matrices corresponding to a given pair of spin matrices $\pm P_+$ are at most the two matrices $L = \| L^i{}_j \|$ and $-L$. Moreover, if $\pm P_+$ corresponds to both $-L$ and $+L$, then

$\pm(P_+)(P_+^{-1}) = \pm 1_{2^{\nu-1}}$ corresponds to $-L(L^{-1}) = -\|\delta^i_j\|$. Conversely, if $-\|\delta^i_j\|$ corresponds to $\pm 1_{2^{\nu-1}}$, then both L and -L correspond to the same pair of spin matrices $\pm P_+$.

By (7.9) the orthogonal matrix $-\|\delta^i_j\|$ corresponds to two matrices (of order $2^\nu$) out of the pencil $\rho\gamma_o$. To determine the values of $\rho$ for which the matrix $\rho\gamma_o$ satisfies the normalizing condition, we make use of the equations (cf. (3.4) and (3.5), Chapter VI), $(C\gamma_o)' = f(C\gamma_o)$ and $C' = eC$, where $ef = (-1)^\nu$. Hence,

$$[(-1)^{\nu/2}\gamma_o]'C[(-1)^{\nu/2}\gamma_o] = C \tag{8.4}$$

and we have for $\nu$ odd

$$\|-\delta^i_j\| \longrightarrow \pm i1 , \tag{8.5}$$

and for $\nu$ even

$$\|-\delta^i_j\| \longrightarrow \pm 1 . \tag{8.6}$$

The properties of the representation $\Delta_+$ therefore depend essentially upon whether $\nu$ is odd or even.

Theorem (8.1). If $\nu > 2$ is odd, $\Delta_+$ is a (1-2) representation in which

$$\begin{aligned} \|L^i_j\| &\longleftrightarrow \pm P_+, \text{ and} \\ -\|L^i_j\| &\longleftrightarrow \pm iP_+ . \end{aligned} \tag{8.7}$$

The group $H^+_{2\nu}$ contains with each matrix $\|L^i_j\|$ its negative and no other matrices proportional to it; the group G of matrices of order $2^{\nu-1}$ contains with any matrix $P_+$ also the matrices $-P_+$, $iP_+$ and $-iP_+$ and no other matrices proportional to $P_+$. It is not possible to sharpen $\Delta_+$ to a (1-1) representation. The group G does not contain a proper subgroup in which there occurs at least one out of each set of four matrices $\pm P_+$, $\pm iP_+$.

Proof: in order to sharpen $\Delta_+$ to a (1-1) representation it would be necessary to select a single matrix of order $2^{\nu-1}$ to correspond to each of the proper orthogonal transformations

(8.8) $$Y^1 = -X^1,\ Y^2 = -X^2,\ Y^3 = X^3,\ Y^4 = X^4,\ \ldots,\ Y^{2\nu} = X^{2\nu},$$ and

(8.9) $$Y^1 = -X^1,\ Y^2 = X^2,\ Y^3 = -X^3,\ Y^4 = X^4,\ \ldots,\ Y^{2\nu} = X^{2\nu}.$$

In a cartesian coordinate system these transformations correspond to collineations defined by $\rho\gamma_1\gamma_2$ and $\rho\gamma_1\gamma_3$. A (1-1) representation would therefore include two partial matrices, say $(a\gamma_1\gamma_2)_+$ and $(b\gamma_1\gamma_3)_+$, out of these pencils. These partial matrices anticommute while (8.8) and (8.9) commute, so that the (1-1) representation would break down for their product.

To prove the last statement of the theorem: any subgroup of G containing at least one out of every set of four proportional matrices would contain the matrices $(a\gamma_1\gamma_2)_+$ and $(b\gamma_1\gamma_3)_+$ for some choice of the scalars a and b. Hence the subgroup would contain the matrix $(a\gamma_1\gamma_2)_+^{-1}(b\gamma_1\gamma_3)_+^{-1}(a\gamma_1\gamma_2)_+(b\gamma_1\gamma_3)_+$, which is equal to -1. If the subgroup contained three matrices out of any one pencil, it would contain the matrices 1, -1, i1, and -i1 and therefore all the elements of G. The only remaining possibility is for the subgroup to contain one of the two pairs $\pm P_+$ and $\pm iP_+$ out of each set of four proportional matrices. Such a selection of matrices out of G would lead to the determination of a subgroup of $H^+_{2\nu}$ which would contain one out of every pair of orthogonal matrices

$\pm \|L^i{}_j\|$. Such a subgroup cannot exist for it would necessarily contain either

$$\begin{Vmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & & \\ & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{Vmatrix} \tag{8.10}$$

or its negative and also the square of the matrix chosen, which is in both cases -1.

This proof shows, incidentally, that $H^+_{2\nu}$ is not the direct product of the two-element group consisting of 1 and -1 and a proper subgroup of itself. We may also conclude from Theorem (8.1) that no identification of the matrices $\|L^i{}_j\|$ and $-\|L^i{}_j\|$ can lead to a (2-2) isomorphism of the sort given in the following theorem.

<u>Theorem (8.2)</u>. If $\nu > 2$ is even, $\Delta_+$ is a (2-2) representation in which

$$\pm \|L^i{}_j\| \longleftrightarrow \pm P_+ . \tag{8.11}$$

It is not possible to sharpen $\Delta_+$ to a (2-1), (1-2) or (1-1) representation.

The first part of the theorem follows immediately from (8.6) and the preceding discussion. A (2-1) representation cannot be obtained from $\Delta_+$ since the group of matrices of order $2^{\nu-1}$ contains elements which anticommute. A (1-2) representation cannot exist since it would imply that $H^+_{2\nu}$ is the direct product of the two element group and a subgroup of itself. Finally, a (1-1) representation cannot be obtained from $\Delta_+$ for the commutative transformations (8.8) and (8.9) correspond to anticommuting matrices.

In §6 we discussed the way in which the invariant polarity determines a mapping of $[\gamma_0]^-$ into itself when $\nu$ is even, and into $[\gamma_0]^+$ when $\nu$ is odd.

From (8.4) it follows that the matrix C commutes with $\gamma_0$ when $\nu$ is even and anticommutes with it when $\nu$ is odd. Hence

(8.12) $$C = \begin{Vmatrix} C_1 & 0 \\ 0 & C_2 \end{Vmatrix}, \quad C_1' = eC_1, \; C_2' = eC_2 ,$$

if $\nu$ is even and

(8.13) $$C = \begin{Vmatrix} 0 & C_1 \\ C_2 & 0 \end{Vmatrix}, \quad C_2' = eC_1 ,$$

if $\nu$ is odd. The normalizing equations (8.1) may therefore be written in the forms

(8.14) $$C_1P_+C_1^{-1} = \breve{P}_+ \quad \text{and} \quad C_2P_-C_2^{-1} = \breve{P}_-$$

for $\nu$ even, and

(8.15) $$C_2P_+C_2^{-1} = \breve{P}_- \quad \text{and} \quad C_1P_-C^{-1} = \breve{P}_+$$

for $\nu$ odd. The last equations give an explicit formula for $P_-$ in terms of $P_+$ when $\nu$ is odd.

A discussion of the representation $\Delta_+$ in the terminology of representation theory is contained in §6 of the paper "Spinors in n Dimensions" (Am. Journ. Math., LVII, 1935, pp. 425-449) by R. Brauer and H. Weyl. We have here given the geometry underlying the representation and given a more complete account of the differences between the cases in which $\nu$ is even and those in which $\nu$ is odd.

## 9. REPRESENTATIONS OF $H_2^+$ and $H_4^+$

We complete the discussion of $\Delta_+$ by treating the cases $\nu = 1$ and $\nu = 2$ in the two following theorems.

Theorem (9.1). If the quadratic form in $E_2$ is taken to be $X^1X^2$, the proper "orthogonal" matrices are of the form

$$(9.1) \qquad \| L^i{}_j \| = \begin{Vmatrix} a & 0 \\ 0 & \frac{1}{a} \end{Vmatrix}$$

and the representation $\Delta_+$ is the (1-2) correspondence

$$(9.2) \qquad \| L^i{}_j \| \longrightarrow \pm \sqrt{a} \ .$$

To prove this theorem we recall that when $\nu = 1$, $C = \| c\, \epsilon_{AB} \|$ so that (8.1) is the condition $|P| = +1$. Hence $P = \begin{Vmatrix} p & 0 \\ 0 & \frac{1}{p} \end{Vmatrix}$. Taking $X^i \gamma_i = \begin{Vmatrix} 0 & X^1 \\ X^2 & 0 \end{Vmatrix}$, we have $(X^i \gamma_i)^2 = X^1X^2 1$. The equations of the isomorphism, $P(X^i \gamma_i)P^{-1} = Y^j \gamma_j$, are then

$$(9.3) \qquad Y^1 = p^2 X^1 \text{ and } Y^2 = \frac{1}{p^2} X^2 \ .$$

Hence $\| L^i{}_j \| = \begin{Vmatrix} a & 0 \\ 0 & \frac{1}{a} \end{Vmatrix} = \begin{Vmatrix} p^2 & 0 \\ 0 & \frac{1}{p^2} \end{Vmatrix}$ and $p = \pm \sqrt{a}$.

Theorem (9.2). A proper orthogonal matrix $\| L^i{}_j \|$ of order four corresponds under (8.2) to a pair of matrices

$$(9.4) \qquad \pm P = \pm \begin{Vmatrix} A & 0 \\ 0 & B \end{Vmatrix}$$

where $|A| = |B| = 1$. Conversely, if A and B are any two unimodular matrices of order two, then the matrices (9.4) correspond to a proper orthogonal matrix under (8.2). The representation $\Delta_+$ is $(2-\infty^3)$.

When $\gamma = 2$, the matrix C is skew-symmetric and since it is of the form (8.12) we must have

$$C = \left\| \begin{array}{cc|cc} 0 & c_1 & & \\ -c_1 & 0 & & O \\ \hline & & 0 & c_2 \\ & O & -c_2 & 0 \end{array} \right\| \qquad (9.5)$$

so that the matrices (9.4) satisfy the normalizing condition if and only if $|A| = |B| = 1$.

It is a theorem of 3-dimensional projective geometry that a collineation may be constructed which simultaneously effects arbitrarily given collineations on the two reguli of lines on a quadric. Since we have represented the lines on Q by points of $[\gamma_0]^+$ and $[\gamma_0]^-$, the matrices A and B describe the collineations on the two reguli and may be taken to be any two unimodular matrices. We could also show this directly by proving that every matrix of the form (9.4) transforms the linear family of collineations into itself.

# Chapter IX

# THE LORENTZ GROUPS

## 1. DEFINITION OF THE LORENTZ GROUPS

In studying the real linear transformations which leave a non-singular quadratic form invariant, it is necessary to take account of the signature of the quadratic form. If the symmetric matrix $\| \gamma_{\alpha\beta} \|$ is real, a suitable real linear substitution on the variables $X^\alpha$ ($\alpha$ = 0, 1, ..., m) will give

$$(1.1) \qquad \gamma_{\alpha\beta} X^\alpha X^\beta = -(X^0)^2 - (X^1)^2 - \ldots - (X^t)^2 + (X^{t+1})^2 + \ldots + (X^m)^2 .$$

The difference between the number of plus signs and the number of minus signs, $(m-t) - (t+1) = m - 2t - 1$, is an invariant under real transformations of coordinates and is called the signature of the quadratic form. If m is even, the signature s can assume only the values $\pm 1, \pm 3, \ldots, \pm(m+1)$. If m is odd, the signature has one of the values $0, \pm 2, \pm 4, \ldots, \pm(m+1)$.

The Lorentz group $L_{m+1;s}$ is defined to be the group of all real linear transformations

$$(1.2) \qquad Y^\alpha = L^\alpha_{\ \beta} X^\beta$$

which leave the quadratic form (1.1) invariant. The matrix $L = \| L^\alpha_{\ \beta} \|$ is therefore subject to the two conditions

$$(1.3) \qquad L' \| \gamma_{\alpha\beta} \| L \equiv \| \gamma_{\alpha\beta} L^\alpha_{\ \lambda} L^\beta_{\ \mu} \| = \| \gamma_{\mu\nu} \|$$

and

$$(1.4) \qquad L = \bar{L} .$$

On account of its tensor character, the first of these conditions is invariant in form under an arbitrary transformation of coordinates

$$X^{\alpha *} = T^{\alpha}_{\beta} X^{\beta} \tag{1.5}$$

where the coefficients $T^{\alpha}_{\beta}$ may be complex as well as real. We are regarding (1.2) as a point $\longrightarrow$ point transformation and not as a coordinate transformation. Equation (1.4), however, retains its form only if $\|T^{\alpha}_{\beta}\|$ is real. In order to make use of the theory of the preceding chapters, which always referred to general coordinate systems in a complex affine space, we shall replace (1.4) by a condition which retains its form under complex as well as real coordinate transformations.

This is accomplished by observing that the reality of $\|L^{\alpha}_{\beta}\|$ is equivalent to the statement that (1.2) commutes with the antilinear transformation of period two with the equations $\bar{Y}^{\alpha} = X^{\alpha}$, or

$$\bar{Y}^{\alpha} = A^{\dot{\alpha}}_{\beta} \; X^{\beta} , \tag{1.6}$$

where it is only in special coordinate systems that $A^{\dot{\alpha}}_{\beta} = \delta^{\alpha}_{\beta}$. Indeed, under (1.5),

$$A^{*} = \bar{T}AT^{-1} , \tag{1.7}$$

where $A = \|A^{\dot{\alpha}}_{\cdot\beta}\|$ and $T = \|T^{\alpha}_{\beta}\|$.

Equations (1.4) are, in general coordinates,

$$AL = \bar{L}A, \text{ or } A^{\dot{\alpha}}_{\beta} \; L^{\beta}_{\gamma} = \bar{L}^{\alpha}_{\beta} \; A^{\dot{\beta}}_{\gamma} . \tag{1.8}$$

In what follows we shall use the symbol $L_{m+1;s}$ to refer to the group of transformations (1.2) which satisfy (1.3) and (1.8). The group is characterized completely by the cone and the antilinear transformation (1.6). The invariant points of (1.6) constitute a real $E_{2\nu+1}$ which intersects the cone in a real cone with the signature s.

2. DEFINITION AND SPINOR REPRESENTATION OF THE ANTIORTHOGONAL GROUP $A_{m+1}$

We have regarded the orthogonal group as the group of all the linear transformations which leave $\gamma_{\alpha\beta} X^\alpha X^\beta$ invariant. We may extend the orthogonal group to a group which we shall call the antiorthogonal group $A_{m+1}$ by adding to it all the antilinear transformations

(2.1) $$\bar{Y}^\alpha = L^{\dot{\alpha}}_{\beta} X^\beta$$

which leave $\gamma_{\alpha\beta} X^\alpha X^\beta = X^\alpha X_\alpha$ invariant in the sense that $X^\alpha X_\alpha = \bar{Y}^\alpha \bar{Y}_\alpha$, or

(2.2) $$\bar{\gamma}_{\alpha\beta} L^{\dot{\alpha}}_{\lambda} L^{\dot{\beta}}_{\mu} = \gamma_{\lambda\mu} .$$

Taking the determinant of both members of this equation, we get

(2.3) $$| L^{\dot{\alpha}}_{\beta} |^2 = \frac{|\gamma_{\mu\nu}|}{|\bar{\gamma}_{\mu\nu}|}$$

so that $|L^{\dot{\alpha}}_{\beta}|$ is of absolute value one. By taking $|\gamma_{\mu\nu}|$ to be real we will have

(2.4) $$|L^{\dot{\alpha}}_{\beta}| = \pm 1 ,$$

and this partial normalization of $\gamma_{\lambda\mu}$ and $L^{\dot{\alpha}}_{\beta}$ will be preserved if we restrict the transformations (1.5) to be unimodular.

Theorem (2.1). The antiorthogonal group $A_{2\nu+1}$ on $2\nu+1$ variables is (1-1) isomorphic with the subgroup of the antiprojective group in $P_{2\nu-1}$ which leaves the linear family $X^0\gamma_0 + X^1\gamma_1 + \dots + X^{2\nu}\gamma_{2\nu}$ of involutions invariant. The equations establishing this isomorphism are:

$$(2.5)\qquad P = \| P^A{}_B \| , \quad P \gamma_\beta P^{-1} = \gamma_\alpha L^\alpha{}_\beta , \quad | L^\alpha{}_\beta | = +1 ;$$

$$(2.6)\qquad Q = \| Q_{AB} \| , \quad Q \gamma_\beta Q^{-1} = (-1)^{\nu+1} \gamma'_\alpha L^\alpha{}_\beta , \quad | L^\alpha{}_\beta | = -1 ;$$

$$(2.7)\qquad R = \| R^{\dot{A}}{}_B \| , \quad R \gamma_\beta R^{-1} = (-1)^{\nu} \bar{\gamma}_\alpha L^{\dot{\alpha}}{}_\beta , \quad | L^{\dot{\alpha}}{}_\beta | = +1 ;$$

and

$$(2.8)\qquad S = \| S_{\dot{A}B} \| , \quad S \gamma_\beta S^{-1} = - \bar{\gamma}'_\alpha L^{\dot{\alpha}}{}_\beta , \quad | L^{\dot{\alpha}}{}_\beta | = -1 .$$

Under this isomorphism collineations and anticollineations correspond to linear and antilinear transformations, respectively, of determinant +1 and correlations and anticorrelations correspond to linear and antilinear transformations, respectively, of determinant -1.

Equations (2.5) and (2.6) have already been used to establish a collineation-correlation representation of the orthogonal group. To show that equations (2.7) and (2.8) extend this to a representation of the antiorthogonal group we should first show that when we multiply two transformations in $P_{2\nu-1}$ the corresponding transformations in $E_{2\nu+1}$ multiply in the proper way. This is an elementary computation which we omit.

The antiorthogonal group is obtained from the orthogonal group by adjunction of any one antilinear transformation which leaves the quadratic form invariant. Similarly, the group of all antiprojectivities which leave the family $X^\alpha \gamma_\alpha$ invariant is obtained from the group of all projectivities with this property by adjunction of a single anticollineation or anticorrelation which leaves the family invariant. Hence the isomorphism stated in the theorem will be established if we show that a single anticorrelation in $P_{2\nu-1}$ has as its image in $E_{2\nu+1}$ an antiorthogonal transformation. We do this by using special coordinates.

Choosing a cartesian coordinate system in $E_{2\nu+1}$ and making a suitable choice of coordinate system in $P_{2\nu-1}$, we may suppose that the matrices $\gamma_\alpha$ are given by the formulas (4.2) and (4.4) of Chapter V. We shall call this combination of coordinate systems a canonical frame of reference. In such a frame the matrices $\gamma_\alpha$ anticommute by pairs and satisfy the equations

(2.9) $$\gamma_\alpha = (-1)^\alpha \gamma'_\alpha = (-1)^\alpha \bar{\gamma}_\alpha = \bar{\gamma}'_\alpha \quad (\alpha \text{ not summed}) .$$

If we now put S = 1 in (2.8) we get $\gamma_\beta = -\gamma_\alpha L^{\dot{\alpha}}{}_\beta$ so that the antipolarity

(2.10) $$\bar{\varphi}_A = \psi^A$$

in the antiquadric $\sum_{A=1}^{2^\nu} \bar{\psi} A \psi A = 0$ corresponds under (2.8) to the antiorthogonal transformation

(2.11) $$\bar{Y}^\alpha = -X^\alpha .$$

<u>Theorem (2.2)</u>. The matrices $P = \| P^A{}_B \|$, $Q = \| Q_{AB} \|$, $R = \| R^{\dot{A}}{}_B \|$, and $S = \| S_{\dot{A}B} \|$, defining antiprojectivities which leave $X^\alpha \gamma_\alpha$ invariant, may be normalized to within sign by the equations

(2.12) $$P'CP = C, \quad Q'\breve{C}Q = C, \quad R'\bar{C}R = C, \text{ and } S'\breve{\bar{C}}S = C ,$$

respectively. With these normalizations the equations of Theorem (2.1) determine a (1-2) matrix representation of the antiorthogonal group.

The possibility of normalizing the matrices of the anticollineations and anticorrelations follows from the fact that their corresponding antiorthogonal transformations commute with $Y^\alpha = -X^\alpha$ in a canonical frame, which is the image of $\varphi_A = C_{AB}\psi^B$. The proof of the theorem is similar to the proof preceding Theorem (5.1), Chapter VI.

## 3. THE INVARIANT ANTIINVOLUTION AND ANTIPOLARITY

By regarding $L_{2\nu+1;s}$ as a subgroup of $A_{2\nu+1}$ we obtain from Theorems (2.1) and (2.2) representations of $L_{2\nu+1;s}$. We shall describe these representations only for the group $L^{+}_{2\nu+1;s}$ consisting of the Lorentz transformations with determinant plus one. The extension to the representation of Lorentz transformations with determinant minus one by means of correlations in $P_{2\nu-1}$ can easily be made.

Theorem (3.1). The Lorentz group $L^{+}_{2\nu+1;s}$ is (1-1) isomorphic with the group of collineations of $P_{2\nu-1}$ which leave invariant the linear family $X^{\alpha}\gamma_{\alpha}$ and an antiinvolution

(3.1) $$\bar{\varphi}^{A} = \alpha^{\dot{A}}{}_{B}\psi^{B} .$$

The antiinvolution of the theorem is the image under (2.7) of the antilinear transformation (1.6) in $E_{2\nu+1}$ of period two. Thus the transformation of coordinates

(3.2) $$X^{0*} = iX^{0},\ X^{1*} = iX^{1} \ldots X^{t*} = iX^{t},\ X^{t+1*} = X^{t+1},\ \ldots X^{2\nu *} = X^{2\nu}$$

carries the quadratic form (1.1) into

(3.3) $$\sum_{\alpha=0}^{2\nu}(X^{\alpha *})^{2}$$

and carries the antilinear transformation $\bar{Y}^{\alpha} = X^{\alpha}$ into

(3.4) $$\bar{Y}^{0*} = -X^{0*},\ \ldots,\ \bar{Y}^{t*} = -X^{t*},\ \bar{Y}^{(t+1)*} = X^{(t+1)*},\ \ldots,\ \bar{Y}^{(2\nu)*} = X^{(2\nu)*} .$$

In this coordinate system the matrix $\alpha = \|\alpha^{\dot{A}}{}_B\|$ of (3.1) is determined to within a factor by the equations

$$(3.5) \qquad \alpha \gamma_\beta \alpha^{-1} = (-1)^\nu \bar{\gamma}_\alpha A^{\dot{\alpha}}{}_\beta$$

where

$$(3.6) \qquad \| A^{\dot{\alpha}}{}_\beta \| = \pm \begin{Vmatrix} -1_{t+1} & 0 \\ 0 & 1_{2\nu-1} \end{Vmatrix}$$

and the sign is chosen so that $|A^{\dot{\alpha}}{}_\beta| = +1$.

Proof: by employing a canonical frame of reference (dropping the stars on $X^{\alpha *}$, $Y^{\alpha *}$), we may give an explicit formula for $\alpha$ in terms of the matrices $\gamma_\alpha$. In view of the fact that (2.10) and (2.11) correspond under (2.8), and that $Y^\alpha = -X^\alpha$ corresponds to $\varphi_A = C_{AB}\psi^B$ under (2.6), the image of $\bar{Y}^\alpha = X^\alpha$ is the antiinvolution $\|\bar{\varphi}^A\| = C\,\|\varphi^A\|$. We are here using the matrix (cf. (3.3), page 6-5)

$$(3.7) \qquad C = \gamma_1 \gamma_3 \cdots \gamma_{2\nu-1},$$

which was originally introduced as a matrix of the type $\|Q_{AB}\|$, as a matrix of the type $\|R^{\dot{A}}{}_B\|$. It should be emphasized that the numerical equality of the matrix defining the image of $Y^\alpha = -X^\alpha$ and the matrix defining the image of $\bar{Y}^\alpha = X^\alpha$ is due to our special choice of coordinate system.

Using (2.9) and equations (3.4) and (3.5) of Chapter VI, it follows that the matrix C given by (3.7) satisfies the equations

$$(3.8) \qquad \bar{C}'C = 1 \qquad C\gamma_\alpha C^{-1} = (-1)^\nu \gamma'_\alpha = (-1)^\nu \bar{\gamma}_\alpha$$

$$\text{and } [(-1)^{\frac{\nu}{2}} \gamma_\alpha]' \, C \, [(-1)^{\frac{\nu}{2}} \gamma_\alpha] = C .$$

Hence if we define

$$(3.9) \qquad \alpha = (-1)^{\frac{\nu}{2}(t+1)} C\gamma_0\gamma_1\gamma_2 \cdots \gamma_t$$

we shall have

$$(3.10) \qquad \alpha\gamma_\beta\alpha^{-1} = \begin{cases} (-1)^{\nu+t}\bar{\gamma}_\beta & \text{if } \beta \leqq t \\ -(-1)^{\nu+t}\bar{\gamma}_\beta & \text{if } \beta > t . \end{cases}$$

Comparing these equations with (3.5) we see that the matrix $\alpha$ does indeed define the antiinvolution which is the image of (3.4). Moreover, the matrix $\alpha$ satisfies the normalizing equation (cf. (2.12))

$$(3.11) \qquad \alpha'\bar{C}\alpha = C .$$

This equation and (3.5) are sufficient to determine $\alpha$ to within sign.

<u>Theorem (3.2)</u>. Under the (1-2) matrix representation of the anti-orthogonal group given by Theorem (2.2), a transformation of $L^{+}_{2\nu+1;s}$ is represented by a pair of matrices $\pm P$ which satisfy the commutation rule

$$(3.12) \qquad \alpha P = p\bar{P}\alpha,$$

where $p = \pm 1$ and $\alpha$ is either of the two normalized matrices $(\pm\alpha)$ corresponding to $\bar{Y}^{\dot{\alpha}} = A^{\dot{\alpha}}{}_{\beta} X^{\beta}$ under (2.7).

The statement that (3.12) is satisfied for some value of p is just the translation into $P_{2\nu-1}$ of the condition (1.8) which was used to define $L_{2\nu+1;s}$. To show that $p = \pm 1$, we form the expression $(\alpha P)'\bar{C}(\alpha P)$ which is equal to C on account of the equations $\alpha'\bar{C}\alpha = C$ and $P'CP = C$. But, using (3.12), the expression is also equal to $p^2(\bar{P}\alpha)'\bar{C}(\bar{P}\alpha) = p^2\alpha'(\bar{P}'\bar{C}\bar{P})\alpha =$

$= p^2 \alpha' \bar{C} \alpha = p^2 C$ and hence $p = \pm 1$.

Equations (3.12) imply that

(3.13) $$\bar{P}' H P = pH ,$$

where

(3.14) $$H = \bar{C}\alpha , \text{ or } H_{\dot{A}B} = \bar{C}_{AC} \alpha^{\dot{A}}{}_{B} .$$

From (3.7) and (3.9) we have, in a canonical frame of reference, after a little computation

(3.15) $$H = (-1)^{\frac{\nu}{2}(\nu + t + 2)} \gamma_0 \gamma_1 \cdots \gamma_t$$

and

(3.16) $$\bar{H}' = (-1)^{\frac{s^2 - 1}{8} - \frac{\nu(\nu+1)}{2}} H$$

where $s = (2\nu - t) - (t+1)$ is the signature of the quadratic form. For a quadratic form in $2\nu + 1$ variables the signature is always odd so that $\bar{H}' = \pm H$ and H is either Hermitian or skew-Hermitian, depending upon the values of $\nu$ and s.

Equations (3.13) may be interpreted geometrically as the statement that the collineations which correspond to transformations of $L^+_{2\nu+1;s}$ leave invariant the antiquadric

$$H_{\dot{A}B} \bar{\psi}^A \psi^B = 0 .$$

This is obvious, for the antipolarity

(3.17) $$\bar{\varphi}_A = H_{\dot{A}B} \psi^B$$

is the image of the antilinear transformation $\bar{Y}^{\alpha} = -A^{\dot{\alpha}}{}_{\beta}\, X^{\beta}$.

## 4. REALITY OF THE SPINOR REPRESENTATION OF $L_{2\nu+1;s}$

We have seen that the images of $\bar{Y}^{\alpha} = A^{\dot{\alpha}}{}_{\beta}\, X^{\beta}$ and $\bar{Y}^{\alpha} = -A^{\dot{\alpha}}{}_{\beta}\, X^{\beta}$ are the antiinvolution $\bar{\varphi}^{A} = \alpha^{\dot{A}}{}_{B}\, \psi^{B}$ and the antipolarity $\bar{\varphi}_{A} = H_{\dot{A}B}\psi^{B}$. In this section we study these invariant transformations more closely and apply the results to obtain information about the spinor representation of $L^{+}_{2\nu+1;s}$.

To determine whether the invariant antiinvolution is of the first or second kind, we observe that (3.11) can be written $(\bar{C}\alpha)'\alpha = eC$, where $e = (-1)^{\frac{\nu(\nu+1)}{2}}$ and we have used the relation $C' = eC$. Hence, $H'\alpha\bar{\alpha} = eC\bar{\alpha} = e\bar{H}$, and comparing this with (3.16) gives

$$\bar{\alpha}\alpha = (-1)^{\frac{s^2-1}{8}}\, 1 . \tag{4.1}$$

Hence the invariant antiinvolution is of the first kind if $s^2 \equiv 1 \pmod{16}$ and of the second kind if $s^2 \equiv 9 \pmod{16}$.

Using the fact that $s = 2\nu - 2t - 1$, these conditions on s may be put into the form given in the theorem

<u>Theorem (4.1)</u>. The invariant antiinvolution is of the first kind if $\nu \equiv t$ or $t+1 \pmod 4$, and it is of the second kind if $\nu \equiv t+2$ or $t+3 \pmod 4$.

Since a suitable choice of coordinate system will allow us to take $\alpha$ to be the unit matrix when the antiinvolution is of the first kind, we have the first part of the theorem

<u>Theorem (4.2)</u>. If $s^2 \equiv 1 \pmod{16}$, the (1-2) spinor representation of $L^{+}_{2\nu+1;s}$ is real in a suitably chosen coordinate system. If in any coordinate system in $P_{2\nu-1}$ every matrix of the spinor representation of $L^{+}_{2\nu+1;s}$ is

either real or pure imaginary, then $s^2 \equiv 1 \pmod{16}$.

Without affecting the real (or pure imaginary) character of the matrices $P = \|P^A{}_B\|$ we can choose cartesian coordinates in $E_{2\nu+1}$ so that the antilinear transformation (1.6) determining $L^+_{2\nu+1;s}$ has the form (3.4). Then the matrix $(-1)^\nu \gamma_\lambda \gamma_\mu$ is properly normalized and corresponds to the transformation of $L^+_{2\nu+1;s}$ which changes the sign of $X^\lambda$ and $X^\mu$ and leaves all the other variables $X^\alpha$ unaltered. Hence $\gamma_\lambda \gamma_\mu$ is either real or pure imaginary and so is $\gamma_0 = (-1)^{\frac{\nu}{2}} (\gamma_1\gamma_2)(\gamma_3\gamma_4) \cdots (\gamma_{2\nu-1}\gamma_{2\nu})$. From this we can conclude that every matrix $\gamma_\alpha = \gamma_0(\gamma_0\gamma_\alpha)$ is either real or pure imaginary.

The antiinvolution $\bar{\varphi}^A = \psi^A$ therefore leaves the linear family $X^\alpha \gamma_\alpha$ invariant and its image in $E_{2\nu+1}$ is a transformation

(4.2) $$\bar{Y}^0 = e_0 X^0, \quad \bar{Y}^1 = e_1 X^1, \ \ldots \ \bar{Y}^{2\nu} = e_{2\nu} X^{2\nu}, \text{ with } e_\alpha = \pm 1 .$$

Since $\bar{\varphi}^A = \psi^A$ commutes (or anticommutes) with every collineation $\varphi^A = P^A{}_B \psi^B$ of the group in $P_{2\nu-1}$, (4.2) commutes (or anticommutes) with every transformation of $L^+{}_{2\nu+1;s}$. The product of (4.2) and (3.4) is therefore an orthogonal transformation with this property. Such an orthogonal transformation is either the identity or $Y^\alpha = -X^\alpha$ and so $\bar{\varphi}^A = \psi^A$ is the invariant antiinvolution corresponding to $\bar{Y}^\alpha = A^{\dot{\alpha}}{}_\beta X^\beta$. By Theorem (4.1) this is possible if and only if $s^2 \equiv 1 \pmod{16}$.

Theorem (4.3). The invariant antipolarity is a polarity in an antiquadric of signature $(+ - + - \ldots + -)$ unless $s = \pm(2\nu+1)$ and then the signature is $(+ + \ldots +)$, or $(- - \ldots -)$. The spinor representation of the real orthogonal group is then unitary in a suitably chosen coordinate system.

Choosing the index $\alpha$ equal to zero if t + 1 is even, and equal to t + 2 if t + 1 is odd, we have, from (3.15) and (2.9), in a canonical frame of reference

$$\gamma_\alpha' \, H \, \gamma_\alpha = -H \qquad (4.3)$$

and hence the signature of H is equal to the signature of -H and is therefore (+ - + - ... + -). Our choice of $\gamma_\alpha$ fails only if t + 1 = 0 or $2\nu + 1$ and then $s = \pm(2\nu + 1)$ and H is a multiple of the unit matrix. Equations (3.13) cannot be satisfied if p = -1 for the signature of H is invariant under transformations of the form $\bar{P}'HP$. For p = +1 and H = h1, (3.13) is the condition that P be unitary.

## 5. $\gamma$-SETS IN WHICH EACH MATRIX IS REAL OR PURE IMAGINARY

The existence of $\gamma$-sets of $2\nu+1$ matrices of order $2^\nu$ in which each matrix is either real or pure imaginary may be settled in the following way.

If $\nu = t + 4p$ or $\nu = t + 4p + 1$, theorem (4.1) states that the invariant antiinvolution associated with $L^+_{2\nu+1;s}$ ($s = 2\nu-2t-1$) is of the first kind. In a suitably chosen coordinate system its matrix $\alpha$ will be the unit matrix. For $\nu = t + 4p$, equations (3.10) imply that $2\nu - t = \nu + 4p$ of the matrices $\gamma_\alpha$ are pure imaginary and $t + 1 = \nu - 4p + 1$ are real. In the other case, $\nu = t + 4p + 1$, the number of pure imaginary ones is $t + 1 = \nu - 4p$. Hence in either event the number of pure imaginary ones is congruent to $\nu$ modulus 4. Since p may have any value which does not make either t + 1 or $2\nu - t$ negative, we have the sufficiency of the condition stated in the theorem

Theorem (5.1). There exists a $\gamma$-set of $2\nu + 1$ matrices of order $2^{\nu}$ in which I of the matrices are pure imaginary and R are real $(I+R=2\nu+1)$ if and only if

$$I \equiv \nu \pmod 4 . \tag{5.1}$$

This condition is equivalent to the congruence

$$R - I \equiv 1 \pmod 8 ^{*} \tag{5.2}$$

---

* In a paper by M.H.A.Newman entitled "Note on an Algebraic Theorem of Eddington" (Journ. Lond. Math. Soc., 7, 1932, pp. 93-99) it is incorrectly stated that R-I = 1 or -7.

---

The necessity of the condition follows from the observation that a $\gamma$-set of I pure imaginary and R real matrices could be used to obtain a representation of a Lorentz group for which the invariant antiinvolution is of the first kind.

Theorem (5.2). A real maximal $\gamma$-set exists if and only if $\nu \equiv 0 \pmod 4$. A pure imaginary maximal $\gamma$-set exists if and only if $\nu \equiv 3 \pmod 4$.

This theorem is obtained by putting $I = 0$ and $I = 2\nu+1$ in (5.1).

## 6. SPATIAL AND TEMPORAL SIGNATURES OF LORENTZ MATRICES

We return for the moment to a coordinate system in which $\gamma_{\alpha\beta} X^{\alpha} X^{\beta}$ is given by (1.1) and determine the conditions which the elements of a real matrix L must satisfy in order that it define a transformation of the Lorentz group. Writing

$$L = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix} , \tag{6.1}$$

where A is a square matrix of t + 1 rows and columns, these conditions are found to be

$$\begin{aligned} A'A - C'C &= 1_{t+1} , \\ B'B - D'D &= -1_{m-t} , \text{ and} \\ A'B - C'D &= 0 . \end{aligned} \tag{6.2}$$

If we form the product of the matrices $\begin{Vmatrix} A' & -C' \\ 0 & 1 \end{Vmatrix}$ and $\begin{Vmatrix} A & B \\ C & D \end{Vmatrix}$ , we may use the first and third of equations (6.2) to get

$$\begin{Vmatrix} A' & -C' \\ 0 & 1 \end{Vmatrix} \begin{Vmatrix} A & B \\ C & D \end{Vmatrix} = \begin{Vmatrix} 1 & 0 \\ C & D \end{Vmatrix} . \tag{6.3}$$

Taking determinants gives

$$|A|\,|L| = |D| . \tag{6.4}$$

The determinant $|A|$ is not equal to zero, for then there would exist a vector $(X^0, X^1, \ldots, X^t, 0, \ldots, 0)$ such that $Y^\alpha = L^\alpha{}_\beta X^\beta$ is zero for $\alpha \leqq t$ and this is impossible since $X^\alpha X_\alpha = Y^\alpha Y_\alpha$ and $X^\alpha X_\alpha < 0$ while $Y^\alpha Y_\alpha \geqq 0$.

We shall call $X^0$, $X^1$, ..., and $X^t$ the "spatial" coordinates of $E_{2\nu+1}$ as a natural extension of the terminology of the Special Theory of Relativity. The variables $X^{t+1}$, ..., $X^{2\nu}$ will be called the "temporal" variables. We shall also speak of the spatial and temporal signatures $\sigma_-(L)$ and $\sigma_+(L)$ of the Lorentz matrix L. They are defined by the relations

$$\sigma_- = \sigma_-(L) = \begin{cases} +1 \text{ if } |A| > 0 \\ -1 \text{ if } |A| < 0 \end{cases} \tag{6.5}$$

and

$$\sigma_+ = \sigma_+(L) = \begin{cases} +1 \text{ if } |D| > 0 \\ -1 \text{ if } |D| < 0 \end{cases} \tag{6.6}$$

respectively. When $t + 1 = 0$ we put $\sigma_- = +1$ and when $t + 1 = 2\nu + 1$ we put $\sigma_+ = +1$. If we also put $\sigma = \sigma(L) = |L|$ $(=\pm 1)$, equations (6.4) imply

$$\sigma = \sigma_- \sigma_+ . \tag{6.7}$$

For transformations of $L^+_{2\nu+1;s}$ $\sigma(L) = +1$ and so $\sigma_-(L) = \sigma_+(L)$. We shall now prove that the factor p in (3.12) is equal to this common value. We have from previous theorems that

$$P\gamma_\beta P^{-1} = \gamma_\alpha L^\alpha_\beta \text{ , } P'CP = C \quad \text{and} \quad \alpha P = p\bar{P}\alpha \tag{6.8}$$

and we wish to prove $p = \sigma_-(L)$.

From the first of equations (6.8) we have

$$\begin{aligned} P(\gamma_0\gamma_1 \cdots \gamma_t)P^{-1} &= (P\gamma_0P^{-1})\cdots(P\gamma_tP^{-1}) \\ &= (\gamma_{\alpha_0}\gamma_{\alpha_1}\cdots\gamma_{\alpha_t})L^{\alpha_0}{}_0 L^{\alpha_1}{}_1 \cdots L^{\alpha_t}{}_t . \end{aligned} \tag{6.9}$$

Using a canonical frame of reference, the matrices $\gamma_\alpha$ anticommute and we get

$$P(\gamma_0\gamma_1\cdots\gamma_t)P^{-1} = |A|\,\gamma_0\gamma_1\cdots\gamma_t + \left(\begin{array}{l}\text{multiples of products of } t+1 \\ \text{matrices } \gamma_\alpha \text{ in which some} \\ \text{or all of the factors are} \\ \text{difference from } \gamma_0,\ \gamma_1,\ \ldots, \\ \text{and } \gamma_t.\end{array}\right) \tag{6.10}$$

Multiplying on the right by $\gamma_{t+1}\gamma_{t+2} \cdots \gamma_{2\nu}$ and taking the trace, all the terms after the first one in the right member vanish and we have

(6.11) $$\text{Trace}\,[P(\gamma_0\gamma_1\cdots\gamma_t)P^{-1}(\gamma_{t+1}\cdots\gamma_{2\nu})] = |A|\,\text{Trace}\,(\gamma_0\gamma_1\cdots\gamma_{2\nu}) .$$

From the second and third of equations (6.8) we find that $\bar{P}'H = \bar{P}'\bar{C}\alpha = \bar{C}P^{-1}\alpha = p\bar{C}\alpha P^{-1} = pHP^{-1}$. But by (3.15), H is a multiple of $\gamma_0\gamma_1 \cdots \gamma_t$, so that $(\gamma_0\gamma_1 \cdots \gamma_t)P^{-1} = \frac{1}{p}\bar{P}' (\gamma_0\gamma_1 \cdots \gamma_t)$. Substituting in (6.11) and using the fact that $\gamma_0\gamma_1 \cdots \gamma_{2\nu}$ is a multiple of the identity matrix, we get

(6.12) $$\text{Trace}\,(P\bar{P}') = p|A| .$$

However, Trace $P\bar{P}' = \sum_{A,B} P^A{}_B \bar{P}^A{}_B > 0$ so that p has the same sign as $|A|$. Since $p = \pm 1$, $p = \sigma_-(L)$.

If $P = \|P^A{}_B\|$ corresponds to $L = \|L^\alpha{}_\beta\|$ and $Q = \|Q^A{}_B\|$ corresponds to $M = \|M^\alpha{}_\beta\|$ so that $\alpha P = \sigma_-(L)\bar{P}\alpha$ and $\alpha Q = \sigma_-(M)\bar{Q}\alpha$, then $\alpha PQ = \sigma_-(L)\bar{P}\alpha Q = \sigma_-(L)\sigma_-(M)\overline{PQ}\alpha$. However $\alpha PQ = \sigma_-(LM)\overline{PQ}\alpha$ and consequently

(6.13) $$\sigma_-(L)\sigma_-(M) = \sigma_-(LM) .$$

This relation has been proved only for Lorentz matrices with determinant +1, but it is easily extended to all Lorentz matrices by observing that it holds if one of the matrices, say L, is -1. Since $\sigma(L)\sigma(M) = \sigma(LM)$, we may use (6.7) to conclude that

(6.14) $$\sigma_+(L)\,\sigma_+(M) = \sigma_+(LM) .$$

This multiplicative property of the signatures enables us to define subgroups of $L_{m+1;s}$ by the conditions

(6.15)

$$L^{+}_{m+1;s} : \sigma = +1 \text{ (i.e. } \sigma_- = \sigma_+ = +1, \text{ or } \sigma_- = \sigma_+ = -1)$$

$$L^{+\,+}_{m+\bar{1};s} : \sigma_- = +1 \qquad \sigma_+ = \pm 1$$

$$L^{+\,+}_{\bar{m}+1;s} : \sigma_- = \pm 1 \qquad \sigma_+ = +1 \qquad \text{and}$$

$$L^{++}_{m+1;s} : \sigma_- = +1 \qquad \sigma_+ = +1 \,.$$

These groups are the generalizations to m+1 dimensions of the groups defined in §3, Chapter I.

7. THE INVARIANT SPINORS ASSOCIATED WITH $L_{2\nu;s}$

We shall employ the results of §7, Chapter V, in order to consider the full Lorentz group $L_{2\nu;s}$ on $2\nu$ variables as a subgroup of $L^{+}_{2\nu+1;s-1}$, which is a group on $2\nu+1$ variables. In doing this we put $\gamma_{\alpha\beta}X^{\alpha}X^{\beta} = -(X^{o})^2 + \gamma_{ij}X^{i}X^{j}$ so that $\gamma_{\alpha\beta}X^{\alpha}X^{\beta}$ has signature s-1 if $\gamma_{ij}X^{i}X^{j}$ has signature s. We then have in addition to the invariant spinors $C, \alpha$ , and $H = \bar{C}\alpha$, the four additional spinors

(7.1) $\qquad \gamma_o,\ D = C\gamma_o,\ \beta = \alpha\gamma_o,$ and $K = H\gamma_o$ .

The commutation rules between normalized collineation matrices in $P_{2\nu-1}$ and the invariant spinors are given in the theorem

<u>Theorem (7.1)</u>. If $\sigma$ , $\sigma_-$ and $\sigma_+$ are the determinant, the spatial signature and the temporal signature, respectively, of the Lorentz matrix $\|L^{i}{}_{j}\|$ (i, j = 1, 2, ..., $2\nu$) corresponding to the normalized collineation matrix P, then

(7.2)

$$P\gamma_o = \sigma\gamma_o P,\ \alpha P = \sigma_+\bar{P}\alpha,\ \beta P = \sigma_-\bar{P}\beta$$

$$\bar{P}'HP = \sigma_+ H \text{ and } \bar{P}'KP = \sigma_- K \,.$$

We verify only two of these commutation rules since the others may be proved in a similar fashion. The rule $P\gamma_o = \sigma\gamma_o P$ follows from the fact that P corresponds to a Lorentz matrix of order $2\nu+1$ of the form

$$(7.3) \qquad \| L^{\alpha}_{\beta} \| = \left\| \begin{matrix} \pm 1 & 0 \; 0 \ldots 0 \\ \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} & \| L^{i}_{j} \| \end{matrix} \right\| .$$

The determinant of this matrix is +1 and consequently the sign in the upper corner must agree with the sign of the determinant $|L^i_j|$. This sign, however, is determined by the equation $P\gamma_o P^{-1} = \pm\gamma_o$. To prove that $\alpha P = \sigma_+ \bar{P}\alpha$ we combine the rule $\alpha P = \sigma_-(L^{\alpha}_{\beta})\bar{P}\alpha$ with the equations

$$\sigma_-(L^{\alpha}_{\beta}) = \sigma(L^i_j)\sigma_-(L^i_j) = \sigma_+(L^i_j) .$$

If we had written the quadratic form $\gamma_{\alpha\beta}X^{\alpha}X^{\beta}$ as $+(X^o)^2 + \gamma_{ij}X^iX^j$ instead of as $-(X^o)^2 + \gamma_{ij}X^iX^j$, the only effect would have been to interchange the roles of $\alpha$ and $\beta$ and of H and K in (7.2).

Restricting ourselves to the subgroup $L^+_{2\nu;s}$ of the Lorentz group, in a coordinate system in which $\gamma_o = \left\| \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right\|$, the matrices of the group in $P_{2\nu-1}$ have the form

$$(7.4) \qquad P = \left\| \begin{matrix} P_+ & 0 \\ 0 & P_- \end{matrix} \right\| ,$$

as we saw in §7 of Chapter VIII. By (3.15), the matrix H is a multiple of $\gamma_o\gamma_1 \cdots \gamma_t$ in a canonical frame of reference and so in such a frame it commutes with $\gamma_o$ when t is even, and anticommutes with $\gamma_o$ when t is odd. That is, (7.5) $$H = \begin{Vmatrix} H_1 & 0 \\ 0 & H_2 \end{Vmatrix}$$

for t even, and

(7.6) $$H = \begin{Vmatrix} 0 & H_1 \\ H_2 & 0 \end{Vmatrix}$$

for t odd.

Equations (3.13) now become

(7.7) $$H_1 P_+ H_1^{-1} = \sigma_+ \breve{\bar{\bar{P}}}_+ \quad \text{and} \quad H_2 P_- H_2^{-1} = \sigma_+ \breve{\bar{\bar{P}}}_-$$

for t even, and

(7.8) $$P_- = \sigma_+ H_1^{-1} \breve{\bar{\bar{P}}}_+ H_1$$

for t odd. In equations (8.15), Chapter VIII, we have already obtained an explicit expression for $P_-$ in terms of $P_+$ for odd values of $\nu$. Equations (7.8) extend this result to even values of $\nu$ provided we restrict ourselves to the transformations of $L^+_{2\nu;2(\nu-t)}$ with t odd.

If $\nu = 2$ and $\gamma_{ij}X^iX^j = -(X^1)^2 - (X^2)^2 - (X^3)^2 + (X^4)^2$, we have to do with the group of the Special Theory of Relativity. We may choose coordinates so that

(7.9) $$H = h \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix},$$

and then equations (7.8) give

(7.10) $$P = \begin{Vmatrix} P_+ & 0 \\ 0 & \breve{\bar{\bar{P}}}_+ \end{Vmatrix}$$

to be the form of a matrix corresponding to a transformation of the restricted proper Lorentz group, $L^{++}_{4;-2}$.

Since the matrices $\gamma_i$ anticommute with $\gamma_o$, we have

$$(7.11) \qquad \gamma_i = \begin{Vmatrix} 0 & \eta_i \\ e_i\eta_i^{-1} & 0 \end{Vmatrix} ,$$

where $e_1 = e_2 = e_3 = -1$ and $e_4 = +1$. The equations of the isomorphism $P\gamma_i P^{-1} = \gamma_j L^j{}_i$, may now be written in the form

$$(7.12) \qquad P_+\eta_i\bar{P}'_+ = \eta_j L^j{}_i \ .$$

Putting $\eta_i = \| \bar{g}_{i\dot{A}B} \|$ and $P_+ = \| P_A{}^C \|$, equations (7.12) are, after taking complex conjugates,

$$(7.13) \qquad \bar{P}_A{}^C P_B{}^D g_{i\dot{C}D} = g_{j\dot{A}B} L^j{}_i ,$$

which are equivalent to equations (4.4) of Chapter I.