Problems of the Theory of Dispersion Relations

by Bogoliubov, Medvedev and Polivanov

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PROBLEMS OF THE THEORY OF DISPERSION RELATIONS

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1. Introduction

The quantum field theory has recently produced a new trend which we believe will have a big future. This trend is connected with the so-called dispersion relations, that is, relations between the Hermitian part of the scattering amplitude and a certain integral over energy of its anti-Hermitian part. To a certain extent, such relations arise independently of the concrete details of the theory under consideration; to obtain them the essential thing is the requirement of microscopic causality, which is usually formulated (in connection with the conditions of relativistic invariance) in the form of a demand that the commutators of field quantities become zero at space-like points. It is precisely this general character of the dispersion relations on the one hand, and the fact that they correlate magnitudes that may be measured directly (which is nontrivial for the quantum field theory) on the other, that cuase great interest in such investigations not only on the part of theoreticians but also among the experimenters.

Despite the fact that the literature on dispersion relations runs into several dozens of papers and that dispersion relations for a whole series of concrete physical processes have been written and compared with experiment, as yet no method has been suggested for obtaining these relations that would satisfy even the usual requirements of rigor in physical work. The very fact of numerous different ways of obtaining them, suggested at times by the same authors, indicates that there is something lacking in their justification. In addition, the question of the physical assumptions that are really necessary to obtain the dispersion relations is as yet an open one; it is a question of the degree to which they are connected with the present-day scheme of quantum field theory, or to what extent the theory may be generalized, the relations remaining valid.

The present investigation is devoted to these two problems. In section two we shall attempt to formulate the basic principles, which, in our opinion, should be taken from conventional theory in order to make possible the derivation of dispersion relations. Otherwise, the construction of the theory may be arbitrary; for example, we shall not need to fix the type of Lagrangian (or in general write it out explicitly), nor shall we make use of the Hamiltonian method.

We shall mainly deal with the variational derivatives of the scattering matrix with respect to fields of real particles, the so-called radiation operators. Section three will be devoted to establishing certain general relations between such operators. The study of radiation operators is closely connected with the study of Green's functions for real particles. Therefore, Sections four and five will deal with a new proof of the well-known spectral representations of Källen-Lehmann. This proof will be based on a study of the properties of analyticity of the vacum matrix elements of the radiation operators, and has the advantage that no divergent expressions will appear anywhere, even in the intermediate stages.

Finally, Sections six and seven are devoted to the derivation of the dispersion relations themselves, and Section eight to a detailed consideration of them in a number of concrete cases.

We may note that in themselves the dispersion relations are in no way new to physics, and various types were known even before the creation of quantum field theory. As early as 1926-27, Kronig and Kramers obtained in classical electrodynamics the dispersion relations between the real and imaginary parts of the coefficient of refraction

$$\mathrm{Re}[n(\omega) - n(0)] = P \int_0^{\infty} \frac{2\omega^2 \, \mathrm{Im}\, n(\omega')}{\bar{n}\, \omega'^2 - \omega^2)} \, d\,\omega'$$

which expresses the fact that signals cannot propagate at a velocity greater than the velocity of light. At the present time, various forms of dispersion relations are widely used in radio engineering.

The principal mathematical device for obtaining dispersion relations is the well-known Cauchy theorem. Since, however, in the quantum field theory we have to deal in a number of cases with generalized functions, we must be cautious in applying this theorem.

Let us assume a certain function, $f(E)$, analytic in the upper half-plane $\mathrm{Im}\, E > 0$, with the properties: (a) for any positive δ, a constant $A(\delta)$ may be found such that

$$|f(E)| \leqslant \frac{A(\delta)}{E}$$

when $\mathrm{Im}\, E > \delta$; (b) when $\mathrm{Im}\, E \to 0$, the function $f(E)$ tends, in the improper sense, to a function integrable in the class $C(q, \tau)$ (q is a certain positive integer). And the words "$f(E)$ is a function integrable in the class $C(q, \tau)$" means that $f(E)$ is defined on the real axis as the kernel of a linear functional

$$\int_{-\infty}^{\infty} f(E')\, h(E')\, dE' \qquad (1.1)$$

in the linear space of functions h(E) that satisfy the conditions

$$\left| E^s h^{(p)} (E) \right| < \text{const.} \qquad (1.2)$$

for $-\infty < E < +\infty$; $S = 0, 1, \ldots \tau$; $p = 0, 1, \ldots, q$ and the limit in the improper sense means that there is an ordinary limit transition for the respective functionals.[x/]

We shall call an analytic function that satisfies the formulated conditions regular in the upper half-plane.

Let us now construct a closed contour formed by the line $(i\delta - R, i\delta + R)$ and a semi-circle with radius R lying in the upper half-plane. Since in virtue of property (a) the integral of $f(E')/(E'-E)$ over this semi-circle will as $R \to \infty$ tend to zero, it follows from Cauchy's theorem that

$$f(E) = \frac{1}{2\pi i} \int_{i\delta-\infty}^{i\delta+\infty} \frac{f(E')}{E'-E} \, dE' \ ; \ \text{Im } E > \delta > 0.$$

Taking into consideration property (b), we can shift the line of integration to the real axis, by letting $\delta \to 0$, and write

$$f(E) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(E')}{E'-E} \, dE' \ ; \ \text{Im } E > 0. \qquad (1.3)$$

Let us now also transfer to the real axis the point of observation $E(\text{Im } E \to 0)$ and note that for real E and E' we have a symbolic identity:

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x/ For details on the definition of integrable functions and improper limits see the work of N. N. Bogoliubov and D. V. Shirkov, Uspekhi Fiz. Nauk, 55, 149 (1955), Art. 2, p. 164.

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{1}{E'-E-i\varepsilon} = P\left(\frac{1}{E'-E}\right) + i\pi\,\delta\,(E'-E). \qquad (1.4)$$

We then obtain

$$f(E) = \frac{1}{i\pi} P \int_{-\infty}^{\infty} \frac{f(E')}{E'-E}\, dE' \; ; \; -\infty < E < +\infty \qquad (1.5)$$

If we isolate from this formula the real part, we will arrive at a typical "dispersion relation"

$$\mathrm{Re}\, f(E) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{Im}\, f(E')}{E'-E}\, dE'. \qquad (1.6)$$

However, in the majority of cases it is not possible to make use of the dispersion relation directly in the form (1.6), because in many cases of application the conditions (a), (b) prove too stringent: the real physical functions that occur in the dispersion relations may not only fail to diminish at infinity, they may even increase; however, not faster than a certain polynomial.

We shall show that it is not difficult to extend the above argument to the case of functions f(E), which are analytic in the upper half-plane and for which conditions less stringent that (a) and (b) are assumed.

(a') there is an integer $m > 0$ such that for any $\delta > 0$ constants $A_j(\delta)$ may be found such that

$$|f(E)| \leq A_0(\delta)\,|E|^m + \ldots + A_m(\delta) \text{ for } \mathrm{Im}\,(E) > \delta;$$

(b') when $\mathrm{Im}\, E \to 0$ the function f(E) tends, in the improper sense, to a function integrable on a certain class $C(q, \tau)$. We shall say that such analytic functions have at infinity a pole of the n-th order, where n is the largest of the numbers $(m+1)$ and $(\tau-2)$, or that at infinity they do not have an essential singularity.

In order to reduce this case to the preceding one, let us consider in addition to f(E) the function

$$g(E) = \frac{f(E)}{(E-E_o+i\varepsilon)^{n+1}} . \qquad (1.7)$$

It is clear that if f(E) is analytic in the upper half-plane and has at infinity a pole not higher than of the n-th order, then the function g(E) will be regular in the upper half-plane for any real E_o and positive ε. Therefore, for g(E) it is possible to make use of the relation (1.5) and write

$$f(E) = \frac{(E-E_o+i\varepsilon)^{n+1}}{i\pi} \; P\int_{-\infty}^{\infty} \frac{f(E')\,dE'}{(E'-E_o)(E'-E_o+i\varepsilon)^{n+1}} ; \qquad (1.8)$$

$$-\infty < E, E_o < +\infty .$$

With the aid of a symbolic identity analogous to (1.4)

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{1}{(E-E_o+i\varepsilon)^{n+1}} = P\left\{\frac{1}{(E'-E_o)^{n+1}}\right\} - \frac{i\pi(-1)^n}{n!}\,\delta^{(n)}(E'-E_o) \qquad (1.9)$$

we again will be able to separate in the integral (1.8) the δ-like parts and the principal values and obtain

$$f(E) = \frac{(E-Eo)^{n+1}}{i\pi} \; P\int_{-\infty}^{\infty} \frac{f(E')\,dE'}{(E'-Eo)(E'-Eo)^{n+1}} +$$

$$+ f(Eo) + \ldots + \frac{f^{(n)}(Eo)}{n!}\,(E-Eo)^n ; \qquad (1.10)$$

$$-\infty < E, Eo < +\infty$$

Thus, in the case of the functions under consideration, which have at infinity a pole not higher than the n-th order, relations of the type (1.5) may again be written. These relations, however, will not, 1) hold only up to a polynomial of degree n, and 2) have a more complicated kernel that ensures convergence of the integral (a simple integral of the type entering into (1.5) would be divergent for a function increasing at infinity).

The relation (1.10) may be also given a slightly more convenient form.

For this purpose let us select any real c_j, E_j that satisfy the conditions

$$\sum_{(j)} C_j E_j^q = 0 \quad \text{for } q = 0, 1, \ldots, n, \qquad (1.11)$$

and let us determine operation Σ as applied to any function f(E), as

$$\Sigma f(E) = \sum_{(j)} C_j f(E_j). \qquad (1.12)$$

It is clear that in virtue of (1.11) Σ gives zero when applied to any polynomial in E of degree not greater than n. Let us now note that the difference

$$\left\{ \frac{(E-Eo)^{n+1}}{(E'-E)(E'-Eo)^{n+1}} - \frac{1}{E'-E} \right\} \qquad (1.13)$$

is, with respect to E, a polynomial of the n-th degree (when both terms are combined the denominator (E'-E) cancels). Therefore

$$\Sigma \left\{ \ldots \; |1.13| \; \ldots \right\} = \sum_{(j)} C_j \left\{ \frac{(Ej-Eo)^{n+1}}{(E'-Ej)(E'-Eo)^{n+1}} - \frac{1}{E'-Ej} \right\} = 0.$$

Now applying operation Σ to both sides of (1.10) we immediately obtain

$$\sum_{(j)} C_j f(E_j) = \frac{1}{i\pi} P \int_{-\infty}^{\infty} f(E') \sum_{(j)} \frac{Cj}{E'-E_j} dE', \qquad (1.14)$$

since all the polynomials disappear.

Thus, it may be said that the "simple" relation (1.5) is retained also with respect to functions f(E) which increase polynomially at infinity, the only thing required being the application to it of the operation Σ which excludes polynomials of degree n.

It might appear that for the polynomially increasing functions f(E') the right-hand side (1.14) does not have any meaning due to the divergence of the integrals (as was noted with respect to (1.5)); however, this is not so since from an analysis of the derivation of this relation it may be seen that although the integral of each individual term in the sum does diverge, still the integral of the linear combination with c_j and E_j which enters into (1.14) must be convergent, the c_j and E_j satisfying (1.11).

Finally, taking from both sides of (1.14) the real part, we obtain the corresponding dispersion relation:

$$\sum_{(j)} C_j \operatorname{Re} f(E_j) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \operatorname{Im} f(E') \sum_{(j)} \frac{C_j}{E'-E_j} dE'. \qquad (1.15)$$

In order that one may utilize mathematical dispersion relations in the study of any process of the collision of particles, it is necessary to be sure first that the relevant scattering amplitude as a function of the energy may be properly continued to the upper half-plane. In order at once to explain the connection between the property of analytic continuity of the scattering amplitude onto the upper half-plane and the condition of causality, let us consider a purely illustrative single-dimension example.

Let us assume that the scattering amplitude f(E) is defined as

$$f(E) = \int_{-\infty}^{\infty} F(\tau)\, e^{iE\tau}\, d\tau. \qquad (1.16)$$

It will be obvious from Sections (2) and (3) that the causality condition leads precisely to relations of this type with

$$F(\tau) = 0 \qquad \text{for } \tau < 0.$$

Now passing to the upper half-plane

$$E = x + iy; \; y > 0$$

we note that the factor $e^{-y\tau}$ plays the role of a cut-off factor ensuring the convergence of the integral (1.16), since at $\tau < 0$, where $e^{-y\tau}$ increases, the function $F(\tau)$ is equal to zero.

It may be shown that even if $F(\tau)$ is a singular function, the only requirement being that it remain integrable in the sense of our definition (1.1,2), the integral (1.16) will still converge and define the function without essential singularities at infinity.

The situation is different if $F(\tau)$ is zero only for $\tau < -a$, where a is a certain "elementary length." Then by replacing in (1.16)

$$\tau \rightarrow \tau - a$$

we see that

$$f(E) = e^{-iaE} f_1(E)$$

where now there is no essential singularity at infinity only for the factor $f_1(E)$; but in the case of the factor e^{-iaE} and consequently the function $f(E)$, such an essential singularity arises. Therefore, in this case in order to obtain the function to which the dispersion relations apply, it is necessary to multiply $f(E)$ by $e^{i\alpha E}$ with $\alpha \geq a$.

Naturally, the situation will actually be considerably more complex, for the simple reason that integration in the Eqs. replacing (1.16) will take place with respect to a larger number of variables. However, as we shall see further on, despite the necessity of a considerable technical improvement of the argument given here, its basis remains intact.

As has already been mentioned, many works have been devoted to problems of the analytic continuation of the scattering amplitude. First of all, mention should be made of the fundamental work of Heisenberg which laid out a program for the direct study of the scattering matrix, which transforms the asymptotic part of the incident wave into the asymptotic part of the outgoing wave, as well as the related investigations of Ning Hu, van Kampen, M. G. Krein. In the latter an investigation was made of the process of elastic collision of two particles from the viewpoint of ordinary quantum mechanics, which reduces to the problem of the scattering of one particle by a fixed force center. For the f(E), a study was made here of the component of the scattering amplitude which corresponds to a partial wave with a definite angular momentum, chiefly the amplitude of the S-wave.

The theorems concerning the possibility of an analytic continuation of the amplitude of S - scattering, $f_s(E)$, onto the upper half-plane for the case when interaction practically disappears at distances greater than the radius of a certain "sphere of action," represent an important result obtained in this direction. However, it turns out that, as is obvious from the illustrative example given above, at infinity $f_s(E)$ may have an essential singularity which is eliminated only by multiplying by the "cut-off factor" e^{iaE}. Therefore only the following function is regular in the upper half-plane

$$f_s\, e^{iaE},$$

to which the dispersion relation (1.6) will be applied.

A dispersion relation of this type was recently applied by Hebel, Karplus, and Ruderman to elastic scattering of a pi-meson on nucleons. Utilizing the available experimental data on s-scattering, these authors arrived at the interesting result that the radius of meson-nucleon interaction must be more than 0.1 of the Compton wave length for the meson.

It should, however, be pointed out that these works proceed from the scheme of conventional quantum mechanics, which does not take account of the peculiarties of the field theory for example the possibilities of creation and destruction of particles.

The dispersion relations for the scattering of bosons in the quantum field theory were the subject of investigations of another kind represented by the works of Gell-Mann, Goldberger, Thirring, Karplus, Ruderman, Miyazawa, Oehme, and others.

Here, for f(E) a study is made of the forward scattering amplitude in the laboratory system; an investigation is made of the problem of its analytic continuation into the upper half-plane, and convincing indications are given that its singularity at infinity will not be stronger than a pole of first order.

A consideration of the forward scattering amplitude is especially convenient due to the fact that according to the so-called "optical theorem" its imaginary part is proportional to the total cross section, that is to a value which again may be determined experimentally. The optical theorem is a consequence of the unitarity of the scattering matrix and may easily be proven in the most general form.

Indeed, let us agree to designate by indices α and β the total set of quantum numbers of a complete system of states. Then the condition of the unitarity of the scattering matrix is written as follows:

$$\sum_{\beta} S_{\alpha\beta} \quad S^{*}_{\alpha\beta} = 1.$$

Let us assume

$$S_{\alpha\beta} = \delta_{\alpha\beta} + iT_{\alpha\beta} ,$$

then $T_{\alpha\beta}$ will be proportional to the scattering amplitude for the process $\alpha \longrightarrow \beta$. Substituting $T_{\alpha\beta}$ in the condition of unitarity, we find

$$i(T^*_{\alpha\alpha} - T_{\alpha\alpha}) = \sum_{(\beta)} |T_{\alpha\beta}|^2 .$$

In this relation, the left-hand side is obviously the imaginary part of the amplitude of elastic scattering at zero angle, and the right-hand side is proportional to the total cross section for all possible processes.

In the normalization used in the theory of the collision of particles, the total cross section $\sigma_\tau(E)$ is related to Im f(E) by

$$\text{Im } f(E) = 4\pi K \sigma(E) \qquad (1.17)$$

where K is the wave number corresponding to the energy E. The real part of f(E) is then found from the relation

$$[\text{Re } f(E)]^2 + [4\pi K\sigma(E)]^2 = |f(E)|^2 = \left(\frac{d\sigma}{d\Omega}\right)_{\theta=0} . \qquad (1.18)$$

The first derivation of dispersion relations in the formalism of the quantum field theory was suggested by Gell-Mann, Goldberger, and Thirring, who made use of Cauchy's theorem, first establishing the requisite analytic properties of the forward scattering amplitude. However, this proof, at any rate for particles with a rest mass different from zero, is not free from objections, the gravity of which was acknowledged by the authors themselves. Karplus and Ruderman established a dispersion relation which may be used for the scattering of neutral mesons on nucleons; however, their deduction is based on the analyticity of the scattering amplitude as a preliminary assumption. This assumption would be very simply obtained as a consequence of a series of theorems concerning the properties Green's functions formulated by Nambu.

However, no convincing proof has yet been proposed.

Finally, Goldberger recently attempted to abandon in general the problem of the analytic continuation of the scattering amplitude into the complex plane, considering the dispersion relations simply as certain identities, which follow in purely algebraic form from the definition of the dispersive and absorptive parts (that correspond to our division of (1.4) into the principal value and the δ-function) of the scattering amplitude by means of sums over intermediate states. However, it is easy to see that the definitions they used are not correct for $E < \mu$, since in this case the respective integrals diverge.

Let us add several remarks with respect to the physical meaning of the quantities that enter into dispersion relations of the type (1.6) or (1.16)). The amplitude for elastic forward scattering is found in the left-hand sides, and the total cross section is found under the integral on the right. Both of these quantities are observable only for real particles, i.e. for energies that are positive and greater than μ. At the same time, the integration on the right-hand sides is extended over all values of energy from $-\infty$ to $+\infty$. For this reason, in order to make practical use of the dispersion relations, we must rid them of integration over negative energies and the "non-observable" region $0 < E < m$.

Integration with respect to negative energies may be eliminated by using the requirement of invariance with respect to charge conjugation (or, for uncharged fields, - the reality), which leads to a relation between the scattering amplitude for negative energy and the conjugate amplitude for positive energy. Negative energies may always be eliminated through the use of this technique, however, it leads to "mixing" of cross sections for anti-particles (of opposite charge) into the dispersion relations, which, for example, in the case of the scattering of nucleons, is inconvenient, since the cross sections for antinucleons are as yet experimentally unknown.

More complex is the case of the "non-observable" region $E < m$ where, as we shall see later, δ-functions $\delta(E-E_p)$, with E_p, corresponding to possible intermediate bound states, arise under the integral. If such bound states (that are possible in the problem) have a discrete spectrum, then such integrals are easily calculated in explicit form. If, however, the spectrum of intermediate states proves continuous in any part, the situation becomes less favorable.

Section 2. Basic Physical Assumptions

As we have already stated, dispersion relations are customarily deduced from the conventional scheme of quantum field theory, to be concrete, for example, from pseudoscalar meson theory, at times even making use of the arguments of perturbation theory. However, the idea that the dispersion relations are in essence not connected with the conventional formalism and ought to be obtained only from certain basic premises (which, of course, are valid in ordinary theory), seems very plausible. Because of the fundamental importance of the problem of the applicability of dispersion relations and the possibility of their generalization, we want to formulate explicitly the physical principles which are really necessary for their deduction. This we consider all the more important since we hope that a detailed study of the radiation operators introduced below and the establishing of relations between them might form the basis of a new approach to the construction of the quantum field theory as a whole.

The conventional present day scheme of quantum field theory is based essentially on three basic assumptions: the Hamiltonian formalism, the application of perturbation theory, and the concept of adiabatic switching on and off of the interaction. The Hamiltonian formalism automatically leads to the fulfillment of the strict requirement of causality (since the nonlocal variants of the theory, which might violate this requirement, do not satisfy the condition of solvability of the equations). However, recently weighty arguments have appeared indicating that an internally consistent theory may not in general be squeezed into the narrow limits of the Hamiltonian method. Perturbation theory permits a practical execution of calculations; but apparently the corresponding series diverge even in the case of weak coupling, to say nothing of nuclear interactions to which in general, it is not applicable. The merit of the concept of adiabaticity is the apparent simplicity of the relations between actual and free fields. Since, however, in the far past and in the far future, the self-action which always exists physically is seitched off together with the interaction between particles, this concept leads to the necessity of distinguishing between fictitious and real free particles, and consequently, in the final analysis, to the whole renormalization ideology.

The general scheme, proposed in 1943 by Heisenberg for the construction of a transition matrix, rejected completely the Hamiltonian formalism, made essential use of the adiabaticity concept and had nothing to say about perturbation theory. Due to its extremely general character, this statement of the problem hardly led to any concrete results, and it should be viewed rather as a program for the construction of the theory, than as a finished scheme. We should like to point out that while formulating the basic conditions which the theory must comply with, Heisenberg did not consider the causality requirements which (at least in the form of the condition of macroscopic causality) the theory must satisfy.

A theory of the scattering matrix, worked out recently by one of the authors (N. N. B.) and Shirkov, was built by proceeding from the Heisenberg principles, which, however, were severely narrowed by the assumption of expansion in powers of the coupling constant, by accepting the concept of adiabaticity and, what is most important, by the requirement of causality, which was formulated in the form of a strict condition of microscopic causality or locality. It turned out that these assumptions, though greatly limiting the theory, led to a scheme which in essence is equivalent to the ordinary Hamiltonian method and differs from it only in the possibility of expounding the material with greater mathematical clarity.

Attempts have been made recently to continue developing the initial Heisenberg program. The first steps have been directed toward improving the basic definitions, which are especially significant in the light of the necessity to introduce bound states into the theory. In this respect, of importance is the work of Haag in which a number of problems connected with the mathematical formulation of the requirements of relativistic invariance and the procedure of second quantization have been elaborated.

From the point of view of conventional field theory, the method of introducing bound states into the Heisenberg scheme is not obvious: if we do not resort again to the adiabatic switching off of interaction, the particles that form the bound state are all the time close to each other, whereas in the Heisenberg statement of the problem, all the particles in the initial state must be spatially separated. A way out will be found if we consider each concrete bound state as a particle of a new type and if instead of speaking about the formation of a bound state we will speak of the annihilation of the initial elementary particles and the creation of a new "complex" particle. Naturally, in such an approach there arises a very complicated problem, that of describing the interaction between this large number of complex particles newly introduced. Here we shall not attempt to deal with this problem.

Now the question immediately arises as to how to describe the initial states of spatially separated particles. We may recall that in conventional theory, when the possibility of bound states is neglected, the total Hamiltonian H may be divided into the "kinetic energy" part H_o, and the interaction part V; and the initial states, no matter how many free particles there were in them, are eigenfunctions of H_o. However, in such a division both the self-action (due to which the particles in the initial state prove to be not real, but fictitious free particles) and that part of the interaction, thanks to which the complex particles exist, are thrown out of H_o. H_o will not have such particles. Now we want to construct an operator H_o, the eigenfunctions of which are the initial states such that both of these unpleasant facts may be avoided. This may be achieved with the aid of the following construction.

Let us consider a system described by the total Hamiltonian (we make the construction by proceeding from the correspondence principle with the conventional theory). Let us designate by R_1 the space of all single-particle eigenstates, i.e. states, in which there is only one real elementary particle. If the Hamiltonian under consideration permits the existence of bound states, it will then have also eigenstates, in which there is one bound complex of 2, 3 .... real elementary particles. The spaces spanned by such states we shall designate by R_2, R_3.... respectively. We may note that the base states by which the spaces R_1, R_2, ... are spanned may be characterized as single-particle ones. We have in view here two peculiarities of such states: 1) from the point of view of their observation they have a certain degree of localization (compare the clever determination through a series of gedanken-experiments by Haag); 2) they are stable.

The space, the vectors of which may be considered as functions describing the initial (or final) states we need, which states correspond to any number of mutually non-interacting (due to spatial separation) particles, will in this case be obtained obviously, as a direct product of all spaces R_1, R_2 ..., and each of these factors may enter into this product an arbitrary number of times in accordance with the fact that in the initial state there may be an arbitrary number of particles of each type;

$$R = R_o \times R_1 \times R_1 \times \ldots R_2 \times R_2 \times \ldots R_k \times \ldots \qquad (2.1)$$

A "free" Hamiltonian, the eigenfunctions of which are functions that describe the initial state of the mutually non-interacting particles, may be constructed as follows; let us introduce the operators for projecting the Hamiltonian H onto the spaces R_o, R_1, ..., R_k, ... - operators P_o, P_1, ..., P_k, ..., and determine the "free" Hamiltonian H_o as a direct sum

$$H_o = P_o H + P_1 H + P_1 H + \ldots + P_2 H + P_2 H + \ldots \qquad (2.2)$$

Then the total Hamiltonian H may be written as

$$H = H_o + V = H_o + (H - H_o) \qquad (2.3)$$

The interaction Hamiltonian V now describes precisely only the mutual actions of the particles, but not the self-action, which, due to the method of construction, is entirely contained in the Hamiltonian H_o. To be more precise, V describes only that part of the interaction which is responsible for the processes of scattering and cration of particles inasmuch as the interaction that holds the elementary particles in the complexes ("complex particles") has also been already pushed into H_o. For this reason, when considering transitions to the initial or final state, $\tau \longrightarrow \mp \infty$, we may deal with the interaction V without any special care, for example, we may simply utilize the adiabatic switching off, because now this will not lead either to the disappearance of self-action or to the disintegration of the bound states.

The references, made in the preceding construction, to the Hamiltonian and to conventional theory were, of course, of a purely illustrative character and were aimed at making the explanation as clear as possible and also at connecting it with that which is generally accepted. This construction should be considered only as an example of how, proceding from conventional theory, it might be possible to verify several of the basic physical assumptions the formulation of which we shall now undertake. We should like to point out that although on the one hand all these assumptions are complied with in conventional theory, we do not believe that they fully exhaust its content. Let us leave this extremely interesting question open. In the same way we shall not attempt to solve the more general problem of whether our assumptions form, to any extent, a non-contradictory, independent and complete system of axioms; these assumptions should be viewed not as an attempt to create such a system in the sense that mathematicians attribute to this concept, but simply as a collection of suppositions which we require for the derivation of dispersion relations.

It is convenient to divide all of our assumptions into two groups general properties which in our opinion are obligatory for an extremely broad class of possible theories, and the special properties, that are connected with the requirement that we impose concerning microscopic causality. From our point of view, this latter group of requirements is necessary in order to obtain dispersion relations of the usual kind.

1. General Properties.

1. In accordance with the above, we accept the Heisenberg statement of the problem: we shall consider that the asymptotic states of the system represent totalities of a certain number of elementary and complex particles infinitely distant from each other. The interaction between these particles is equal to zero, and for this reason such quantities as energy, momentum, etc. are additive. Such states are described by the amplitudes $| \rangle$, which are elements of a linear space, which one may imagine to be constructed by the method described above.

2. We shall consider that we have a certain group G of transformations L, which includes as a subgroup the Lorentz group Li (G may include also other transformations, for example, isotopic or gauge transformations, etc.). Under the action of G, the state amplitudes transform under a certain unitary representation of G with elements U_L. x/

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x/ In analogy with the argument of Haag, it should be noted that for one-particle states, U_L form irreducible representations of G. Further, from the construction performed above it should be possible to establish that for any asymptotic state (insofar as it is always represented by a vector in the direct product space), the infinitesimal operator U_L is a direct sum of the infinitesimal operators U_L' that correspond to irreducible representations. Hence, in particular, there would then follow the statement made above concerning the additivity of the integrals of motion.

3. If in the state $|p\rangle$ the vector of four-momentum p has a definite value, then

$$U_{L_a}|p\rangle = e^{-ipa}|p\rangle, \qquad (2.4)$$

where L_a is the translation $x \rightarrow x + a$. There exists a state $|0\rangle$, for which

$$U_{L_a}|0\rangle = |0\rangle. \qquad (2.5)$$

$|0\rangle$ the <u>vacuum</u> state.

Similar properties may be formulated also for other subgroups of G, for example, for representations that correspond to the angular momentum.

4. There exists a system of eigenstate amplitudes of four-momentum, corresponding to non-negative values of energy, which is complete, so that

$$\langle\alpha|AB|\beta\rangle = \langle\alpha|A|0\rangle\langle 0|B|\beta\rangle + \frac{1}{(2\pi)^3}\sum_n \int d\vec{k}\,\langle\alpha|A|n\vec{k}\rangle\langle n\vec{k}|B|\beta\rangle \qquad (2.6)$$

Here n signifies the set of all the remaining quantum numbers which together with k fully characterize the state.

5. The subject of the theory is the study of the probabilities of transitions between such asymptotic states. We shall assume that to each transition between states $|\alpha\rangle$ and $|\beta\rangle$ corresponds a definite probability, which is expressed in the usual manner by the elements, $S_{\alpha\beta}$, of a certain unitary matrix

$$\sum_\beta S_{\alpha\beta}\,(S_{\gamma\beta})^* = \delta_{\alpha\gamma} \qquad (2.7)$$

the elements of which may be regarded as the mean values of a certain operator S:

$$S_{\alpha\beta} = \langle\alpha| S |\beta\rangle \tag{2.8}$$

6. Since we consider single-particle states as the states of of real particles, our single-particle states and the vacuum will be stable, i.e.

$$S|\alpha\rangle = |\alpha\rangle \tag{2.9}$$

if $|\alpha\rangle$ is the state of the vacuum, of one elementary particle or one composite particle.

Before passing on to a description of special local properties, we shall add the following remarks.

It is obvious first of all that our asymptotic states that correspond to the presence of a definite number n of particles of definite types α_i with definite momenta $\vec{p}_i$ may be obtained by introducing in the usual way creation operators $A^{(+)}_{\alpha_i}(\vec{p}_i)$ and destruction operators $A^{(-)}_{\alpha_i}(\vec{p}_i)$ of particles of type α_i with momentum $\vec{p}_i$, and letting them act on the vacuum state:

$$\begin{aligned} &|\alpha_1 p_1; \ldots; \alpha_n p_n\rangle = \\ &= A^{(+)}_{\alpha_1}(\vec{p}_1) \ldots, A^{(+)}_{n}(\vec{p}_n) |0\rangle \end{aligned} \tag{2.10}$$

and from the circumstance that our separated particles do not interact, it will follow that the operators $A^{(+)}$, $A^{(-)}$ satisfy the usual commutation relations

$$\begin{aligned} &[A^{(-)}_{\alpha}(\vec{p}), A^{(+)}_{\alpha'}(\vec{p}^{1})] = \delta_{\alpha\alpha'} \cdot \delta(\vec{p}-\vec{p}'), \\ &[A^{(-)}_{\alpha}(\vec{p}), A^{(-)}_{\alpha'}(\vec{p}')] = 0 \end{aligned} \tag{2.11}$$

From property (2) it will then follow that in the transformation L of G the operators $A^{(\pm)}(\bar{p})$ will be transformed into

$$A^{(\pm)}(\vec{p}) \rightarrow A^{(\pm)}_{L\alpha}(L\vec{p}) = U_L \; A^{(\pm)}_{\alpha}(\vec{p}) \; U_L^{+} \qquad (2.12)$$

In order to be able to formulate the causality condition (and in general discuss the local properties of the theory), we will obviously have to learn somehow to distinguish the individual points of space-time. For this purpose we shall construct from the creation and destruction operators of elementary particles the usual space-localized combinations:

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{K}}{\sqrt{2K_o}} \left\{ e^{iKx} A^{(+)}(\vec{K}) + e^{-iKx} A^{(-)}(\vec{K}) \right\}$$
$$K^o = \sqrt{\vec{K}^2 + m^2} \quad . \qquad (2.13)$$

where, in order not to make the explanation cumbersome, we wrote out the formula relating to the real scalar field. It is precisely at this point that we meet for the first time the difference between elementary and compound particles. Although for the latter it would be possible to introduce formally a definition of the type (2.13), the physical meaning of such combinations would be rather cumbersome.

Further development in conventional theory may be conducted in approximately the following manner. The scattering matrix S may always be thought of as being a series of creation and destruction operators:

$$S = \sum_{,m=o}^{\infty} \int d\vec{K}_1 \ldots d\vec{K}_\ell \; d\vec{K}_1^1 \ldots . d\vec{K}_m^1 \; f^{\ell,m}(\vec{K}; \vec{K}^1).$$
$$A^{(-)}(\vec{K}_1) \ldots . A^{(-)}(\vec{K}_\ell) \; A^{(+)}(\vec{K}_1^1) \ldots . A^{(+)}(\vec{K}_m^{\,\prime}) \qquad (2.14)$$

(for the sake of simplicity, we shall suppose temporarily that we are working with particles of one type). Such an expansion might be rewritten with the aid of (2.13) as an expansion in terms of the normal products of the fields $\varphi(x)$:

$$S = \sum_{n=o}^{\infty} \int dK_1 \ldots dK_n \, f^n(x_1 \ldots x_n) : \varphi(x_1) \ldots \varphi(x_n) : \qquad (2.15)$$

then define the formation of the variational derivative of the S matrix with respect to the field $\varphi(x)$, $\frac{\delta S}{\delta \varphi(x)}$, by the following operation: The sum is taken of all the expressions obtained from each term (2.15) by cancellation of one factor $\varphi(x_i)$ it being replaced by $\delta(x-x_i)$.

Then the matrix elements of the scattering matrix

$$S_{\omega\omega'} = \langle \bar{p}' \alpha_1' ; \ldots , \bar{p}_\tau' \alpha_\tau' \mid S \mid \bar{p}_1 \alpha_1 , \ldots , \bar{p}_s \alpha_s \rangle \qquad (2.16)$$

might be reduced to the vacuum expectation values of the "radiation operators"

$$H(x_1, \ldots x_n) = \frac{\delta^n}{\delta \varphi_{\alpha_1}(x_1) \ldots \delta \varphi_{\alpha_n}(x_n)} S \qquad (2.17)$$

Indeed, in virtue of (2.10) the matrix element (2.16) may be rewritten as

$$\langle 0 | A_{\alpha 1}^{(-)'} (\bar{p}_1{}^1) \ldots A_{\alpha_\tau}^{(-)'} (\bar{p}_\tau{}') \, S \, A_{\alpha_1}^{(+)} (\bar{p}_1) \ldots A_{\alpha_s}^{(+)} (\bar{p}_s) | 0 \rangle \qquad (2.18)$$

Again, limiting ourselves, for the sake of simplicity, to the case of real, scalar fields, we may note that from (2.11), (2.13) we obtain

$$[A_\rho^{(+)} (\vec{p}), \varphi_{\rho'}(x)] = \frac{\delta_{\rho\rho'}}{(2\pi)^{3/2}} \frac{e^{-ipx}}{\sqrt{2p^o}}$$

$$[A_\rho^{(-)} (\bar{p}), \varphi_{\rho'}(x)] = \frac{\delta_{\rho\rho'}}{(2\pi)^{3/2}} \cdot \frac{e^{ipx}}{\sqrt{2po}} \qquad (2.19)$$

$$\text{where } p^o = \sqrt{\vec{p}^{\,2} + m^2},$$

whence follows immediately for the commutation relations of the creation and destruction operators with the scattering matrix:

$$[A_\rho^{(+)}(\vec{p}), S] = \frac{1}{(2\pi)^{3/2}} \int dx \frac{\delta S}{\delta \varphi_\rho(x)} \frac{e^{-ipx}}{\sqrt{2p^o}}$$

$$[A_\rho^{(-)}(\vec{p})\quad S] = \frac{1}{(2\pi)^{3/2}} \int dx \frac{\delta S}{\delta \varphi_\rho(x)} \cdot \frac{e^{ipx}}{2po}\,. \qquad (2.20)$$

where again $p^o = + \sqrt{\vec{p}^{\,2} + m^2}$

If we now transfer into (2.18) all the creation operators to the left-hand side and the destruction operators to the right, where acting on the vacuum function they give zero, we shall immediately find (we assume that all the momenta $\bar{p}_1{}'$, ..., $\bar{p}_\tau{}'$ and $\bar{p}_1$, ..., $\bar{p}_s$ are different, otherwise there would arise more terms of the same type, but smaller degree) that (2.16) may be written in the form of an integral

$$S_{\omega\omega'} = \frac{(-1)^s}{(2\pi)^{\frac{3(\tau+s)}{2}}} \int dx_1{}' \ldots dx_\tau{}' \ldots dx_s \frac{e^{i(\Sigma x_i{}' p_i{}' - \Sigma p_i x_i)}}{2^{\frac{\tau+s}{2}} \sqrt{p_1{}'^o \ldots p_\tau{}'^o p_1^o \ldots p_s^o}}$$

$$\cdot \left\langle 0 \right| \frac{\delta^{(\tau+s)} S}{\delta \varphi_{\rho_1'}(x_1{}')\ldots\delta \varphi_{\rho_\tau'}(x_\tau{}')\delta \varphi_{\rho_1}(x_1)\ldots\delta \varphi_{\rho_s}(x_s)} \left| 0 \right\rangle;$$

$$p_i{}^o = + \sqrt{\vec{p}_i{}^2 + m_i{}^2} \qquad (2.21)$$

Under the integral in (2.21) are precisely those vacuum expectation values of the radiation operators of (2.17) that have just been determined. Indeed, in virtue of the condition of stability of the vacuum in 1.6 we find:

$$\left\langle 0 \right| \frac{\delta^{(\tau+s)} S}{\delta \varphi_{\rho_1'}(x_1{}')\ldots\delta \varphi_{\rho_s}(x_s)} \left| 0 \right\rangle = \left\langle 0 \right| \frac{\delta^{(\tau+s)} S}{\delta \varphi_{\rho_1'}(x_1{}')\ldots\delta \varphi_{\rho_s}(x_s)} S^+ \left| 0 \right\rangle \qquad (2.22)$$

We may, however, note that this deduction was made a little too hastily and that we did not follow the actual meaning of the operations with sufficient care. First of all (2.13) <u>cannot</u>, generally speaking, be solved with respect to $A^{(\pm)}$ since this expression defines not an arbitrary $\varphi(x)$ but only a $\varphi(x)$ that satisfies the equation

$$(\Box - m^2)\varphi(x) = 0 \qquad (2.23)$$

For this reason, from (2.14) we cannot obtain (2.15), the functional determined for a broader class of operator functions $\varphi(x)$, which do not necessarily satisfy the equation (2.23). Further, we might seem to have chosen arbitrarily the rule of variational differentiation with respect to $\varphi(x)$. Finally, the commutation relations (2.19) are obtained from (2.11, 13), again only for such $\varphi(x)$ as satisfy the equation (2.23). We, however, used them for an arbitrary $\varphi(x)$.

In essence, the meaning of the transformation was that we actually extended the definition of the S matrix, removing in (2.15) the restriction of (2.23) and considering the scattering matrix as a functional of arbitrary but commuting (or, for fermion fields, anticommuting) $\varphi(x)$.[x/] And all that we shall need for further investigation is the commutation relations (2.19) of these functions with the creation and destruction operators that make it possible to establish (2.20) and the rule for reducing any matrix elements of the scattering matrix to vacuum expectation values of radiation operators. For this reason, we shall not refer any more to the analogy of conventional theory, but simply require the fulfillment of the following

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x/ It should be emphasized that such an extension does not carry us out of the framework of conventional theory. Indeed, in the usual theory "fields" $\varphi_p(x)$ play a double role: firstly, the operator S itself is considered to be a functional of these fields, and secondly, the creation and destruction operators $A^{(\pm)}$ that correspond to the fields serve in the calculation of the matrix elements of this operator. And in the first case, the fields always stand in chronological or normal products and therefore commute (anticommute) with each other. In addition, in taking variations no restrictions are imposed that are connected with the requirement that the fields satisfy some equations.

2. Local Properties

1. Elementary particles are characterized by boson and fermion fields $\varphi(x)$ with the ordinary transformation properties of free fields. The operator S possesses variational derivatives of any order with respect to these fields.[xx] The radiation operators (2.17) and their products with independent arguments are integrable, that is all the matrix elements

$$\langle \omega | H(x_1, \ldots, x_n) \ldots H(z_1, \ldots z_n) | \omega' \rangle$$

are integrable functions that belong to one of the classes C(q, r), (See (1.1), (1.2)).

2. The causality condition is satisfied in the form

$$\frac{\delta}{\delta\varphi(x)} \left(\frac{\delta S}{\delta\varphi(y)} S^{+}\right) = 0 \quad \text{for} \quad x \lesssim y. \qquad (2.24)$$

This causality condition is absolutely analogous to that used by Bogoliubov and Shirkov, to which we refer the reader for an explanation of its physical meaning.

3. The matrix elements of the scattering matrix may be transformed into the vacuum expectations value of radiation operators by using the formal relations

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x/ (continued)
Actually, this is equivalent to the assumption that the S matrix is considered to be a functional of arbitrary classical functions, $\varphi_\rho(x)$, which precisely commute (anticommute) and which have only transformation properties of quantized fields. On the contrary, when calculating matrix elements it is essential that $A^{(\pm)}$ be operators with the properties of (2.19).

xx/ Here, the variational derivatives have all of their usual properties. Their transformation character is determined by the transformation character of the fields $\varphi(x)$. The derivatives of the S matrix with respect to boson fields commute, whereas with respect to the fermion fields they anticommute among themselves.

$$[A_{\rho}^{(+)}(\vec{p}),\ \varphi_{\rho'}(x)] = \frac{\delta_{\rho'\rho}}{(2\pi)^{3/2}}\ \frac{e^{-ipx}}{\sqrt{2p^{o}}}$$

$$[A_{\rho}^{(-)}(\vec{p}),\ \varphi_{\rho'}(x)] = \frac{\delta_{\rho\rho'}}{(2\pi)^{3/2}}\ \frac{e^{ipx}}{\sqrt{2p^{o}}} \qquad (2.25)$$

$$p^{o} = \sqrt{\vec{p}^{\,2} + m^{2}}$$

and analogous for fermions x/

$$[b_{+s''}^{(-)}(\bar{p}''),\ \overline{\Psi}_{s'}(x')]_{+} = \frac{\overline{U_{s'}^{+s''}(\bar{p}'')}}{(2\pi)^{3/2}}\ e^{ip''x'};\ p''^{o} = \sqrt{\vec{p}''^{2} + \mu^{2}};$$

$$[b_{+s''}^{(-)}(\bar{p}''),\ \Psi_{\lambda}(x)\ ]_{+} = [b_{+s}^{(+)}(\bar{p}),\ \overline{\Psi}_{\lambda'}(x')]_{+} = 0; \qquad (2.26)$$

$$[\Psi_{\lambda}(x),\ b_{+s}^{(+)}(\bar{p})\ ]_{+} = \frac{\overline{U_{\lambda}^{+s}(\bar{p})}}{(2\pi)^{3/2}}\ e^{-ipx}\ ;\ \ p^{o} = \sqrt{\vec{p}^{\,2} + \mu^{2}}$$

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x/ Given below for reference are the main equations for a spinor field, in the conventional theory of a free field, in our notations. For the field operator $\Psi_{\lambda}(x)$ we write the expansion

$$\Psi_{\lambda}(x) = \frac{1}{(2\pi)^{3/2}} \int d\vec{k} \Big\{ e^{ikx}\, u_{\lambda}^{(-)\alpha s}(\bar{k})\, b_{-\alpha s}^{(+)}(\bar{k}) +$$

$$+ e^{-ikx}\, U_{\lambda}^{(+)\alpha s}(\bar{k})\, b_{+\alpha s}^{(-)}(\bar{k}) \Big\} \qquad (2.27)$$

$$k^{o} = +\sqrt{\vec{k}^{2} + \mu^{2}},$$

where $\pm$, α and 's' are quantum numbers that define the particle-antiparticle, the spin and isotopic state; $b^{(\pm)}$ ... are the creation and destruction operators of the fermion in the same state; u_{λ}.... are the spinor amplitudes. Then for the Dirac-conjugate operator $\overline{\Psi}_{\lambda}(x)^{\lambda}$ we obtain the expansion:

$$\overline{\Psi}_{\lambda}(x) = \frac{1}{(2\pi)^{3/2}} \int d\bar{k} \Big\{ e^{ikx}\, \overline{U_{\lambda}^{+\alpha s}}(\vec{k})\, b_{+\ s}^{(+)}(\vec{k}) +$$

$$+ e^{-ikx}\, \overline{U_{\lambda}^{(-)\alpha s}}(\bar{k})\, b_{-\alpha s}^{(-)}(\bar{k}) \Big\}; \qquad (2.28)$$

$$k^{o} = +\sqrt{\bar{k}^{2} + \mu^{2}}.$$

The creation and destruction operators satisfy the anti-commutation relations:

$$[b^{(-)}_{\pm\alpha s}(\bar{k}),\ b^{(+)}_{\pm'\alpha's'}(\bar{k}')]_{+} = \delta_{(\pm)(\pm)'}\,\delta_{\alpha\alpha'}\,\delta_{ss'}\,\delta(\vec{k}-\vec{k}') \qquad (2.29)$$

the remaining anticommutators being equal to zero. There are the relations of Hermitian conjugation:

$$(b^{(-)}_{+\alpha s}(\bar{k}))^{*} = b^{(+)}_{+\alpha s}(\bar{k});\ (b^{(+)}_{-\alpha s}(\bar{k}))^{*} = b^{(-)}_{-\alpha s}\ (\bar{k}) \qquad (2.30)$$

From the fact that ψ_λ satisfies the Dirac equation

$$(i\ \gamma^{i}\frac{\partial}{\partial x^{i}} - \mu)\psi\,(x) = 0 \qquad (2.31)$$

it follows that the amplitudes $U^{\pm\alpha s}$ satisfy equations in the momentum representation

$$(\gamma k-\mu)U^{+\alpha s}(\bar{k}) = 0;\ \overline{U^{+\alpha s}}\ (\bar{k})\ (\gamma k-\mu) = 0;\ k^{o} = +\sqrt{\bar{k}^{2}+\mu^{2}} \qquad (2.32)$$

and

$$(\gamma k-u)U^{-\alpha s}(-\bar{k}) = 0;\ \overline{U^{-\alpha s}}\ (-\bar{k})\ (\gamma k-\mu) = 0;\ k^{o} = -\sqrt{\bar{k}^{2}+\mu^{2}}$$

In addition to the expansions (2.27), (2.28) we shall find it convenient to make use of the Fourier transformations

$$\psi_\lambda(x) = \frac{1}{(2\pi)^4}\int e^{-ipx}\,\psi_\lambda(p)\ dp, \qquad (2.33)$$

and

$$\overline{\psi}_\lambda(x) = \frac{1}{(2\pi)^4}\int e^{ipx}\,\overline{\psi}_\lambda(p)\ dp, \qquad (2.34)$$

in which it is not assumed that $\psi_\lambda(x)$ and $\overline{\psi}_\lambda(x)$ satisfy any equations and, correspondingly, all four components of the momentum are consideres as independent. For functions that satisfy (2.31) the operators $\psi_\lambda(p)$ and $\psi_\lambda(p)$ are related with the operators $b(\pm)$ are related with the operators $b(\pm)$ $(\bar{p})$ by the following relations

$$\psi_\lambda(p) = (2\pi)(2\pi)^{3/2}\ \delta(p^{o} - \sqrt{\bar{p}^{2}+\mu^{2}})U^{+s}_{\lambda}(\bar{p})b^{(-)}_{+s}(\bar{p})\ + \qquad (2.35)$$

$$+ (2\pi)(2\pi)^{3/2}\ \delta(p^o + \sqrt{\bar{p}^2 + \mu^2})\ U_\lambda^{-s}(-\bar{p}) b_{-s}^{(+)}(-\bar{p}),$$

and

$$\overline{\Psi}_\lambda(p) = 2\pi(2\pi)^{3/2}\ \delta(p^o - \sqrt{\bar{p}^2 + \mu^2})\ U_\lambda^{+s}(\bar{p}) b_{+s}^{(+)}(\bar{p}) + \qquad (2.36)$$

$$+ 2\pi(2\pi)^{3/2}\ \delta(p^o + \sqrt{\bar{p}^2 + \mu^2})\ \overline{U_\lambda^{-s}}(-\bar{p}) b_{-s}^{(-)}(-\bar{p}).$$

From the expansions (2.27) and (2.28) and the anticommutators (2.29) we at once obtain the anticommutators (2.26) postulated in the text. Let us now add the equations for the commutators of the destruction and creation operators with the transition matrix

$$[b_{+s''}^{(-)}(\bar{p}''), S]_- = \int dx\ [b_{+s''}^{(-)}(\bar{p}''), \overline{\Psi}_\lambda(x)]_+ \frac{\delta S}{\delta \overline{\Psi}_\lambda(x)};$$

$$[S, b_{+s}^{(+)}(\bar{p})]_- = \int dx\ [\Psi_\lambda(x), b_{+s}^{(+)}(\bar{p})]_+ \frac{\delta S}{\delta \Psi_\lambda(x)} \qquad (2.37)$$

It should be noted that, as distinguished from the boson fields, when the equations (2.20) were equally applicable both to the S matrix itself and to any one of its variational derivatives, (2.37) are applicable only to the S matrix itself; to take the succeeding variations with respect to the fermion field, the equations have to be derived anew due to the anti-commutativity of the variations.

It should be noted in concluding that for the calculation of matrix elements of the S matrix we ought to have some analogous rules also for the transformation of matrix elements into radiation operators for states including complex particles. This is a very interesting and important problem, but it could form the subject of an independent investigation and so we shall not deal with it here. Fortunately, we shall not have to solve this problem in order to derive the more interesting dispersion relations.

Section 3. The relations between Radiation Operators

In the preceding section we introduced the concepts of radiation operators of different orders. Let us now consider them in more detail. First of all, it will be convenient to make this consideration more concrete by restricting ourselves to the case of some definite fields. Having in view the fact (as we have already had occasion to point out) that the greatest interest now is the meson theory, we shall select the problem of the interaction of nucleon and meson fields. With the purpose of keeping our reasoning as definite as possible we shall restrict ourselves to an account only of a charge-symmetric interaction and we shall not take into account the presence of electromagnetic and weak interactions with light particles.

As usual, we shall describe the nucleon field by the spinor

$$\psi(x) = \binom{\psi_p(x)}{\psi_n(x)}. \tag{3.1}$$

The meson field we shall consider as a field with three real pseudoscalar components $\varphi_\rho(x)$ that form a vector in isotopic space. And, as usual, the linear combinations

$$\varphi_+ = \frac{\varphi_1 + i\varphi_2}{2}; \quad \varphi_- = \frac{\varphi_1 - i\varphi_2}{2}; \quad \varphi_0 = \varphi_3 \tag{3.2}$$

will correspond to the particles $\pi^{(+)}$, $\pi^{(-)}$ and $\pi^{(o)}$. The selection of a real representation (although as a matter of principle this is not obligatory) will simplify slightly the further argument.

We shall consider that the group G includes (in addition to the Lorentz group) rotations in isotopic space and gauge-transformations of the first kind of the fermion fields

$$\psi(x) \rightarrow e^{i\alpha}\psi(x); \quad \alpha = \text{const.} \tag{3.3}$$

The operators of the first order are the simplest radiation operators. Due to the fact that real $\varphi_\rho(x)$ were selected, there will be three such operators (in place of four otherwise):

$$j_\rho(x) = i\frac{\delta S}{\delta\varphi_\rho(x)}S^+;\ \bar{\cap}(x) = i\frac{\delta S}{\delta\psi(x)}S^+;\ \cap(x) = -i\frac{\delta S}{\delta\bar{\psi}(x)}S^+ \quad (3.4)$$

We shall call these three operators currents, the first one a <u>bose</u> current, and the two latter, <u>fermi</u> currents.x/

It is easy to see that due to the unitarity of S and the reality of $\varphi_\rho(x)$, the bose current $j_\rho(x)$ is Hermitian. Indeed, if we take in (3.4, 1) the Hermitian conjugate, we obtain

$$j^+_\rho(x) = -i\,S\frac{\delta S^+}{\delta\varphi_\rho(x)}.$$

but $SS^+ = 1$ and therefore

$$\frac{\delta S}{\delta\varphi_\rho(x)}S^+ = -S\frac{\delta S^+}{\delta\varphi_\rho(x)}$$

For this reason $j^+_\rho(x)$ coincides with $j_\rho(x)$

$$j^+_\rho(x) = j_\rho(x). \quad (3.5)$$

The question of the Hermitian properties of fermion currents will be treated in the next section.

It is essential to note that the vacuum matrix elements of the current operators are identically equal to zero:

$$\langle 0|j_\rho(x)|0\rangle = 0;\ \langle 0|\bar{\cap}(x)|0\rangle = 0;\ \langle 0|\cap(x)|0\rangle = 0. \quad (3.6)$$

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x/ The term "current" is based on the analogy of perturbation theory, where the first of these operators in the first approximation is proportional simply to $\bar{\psi}\gamma_5\tau_p\psi$ - , the current of non-interacting fermi particles. However, the words "bose" and "fermi" are used here in a slightly unusual sense; they indicate the field in which <u>variation</u> was <u>conducted</u>.

Indeed, with the aid of "assumption" 1.6 and utilizing the transformation 2.3 (Section 2) such matrix elements reduce to matrix elements of the type

$$\langle 0 | A^{(\pm)} | 0 \rangle$$

which are obviously equal to zero in virtue of the definition of the vacuum.x/ ($A^{(\pm)}$ are creation/destruction operators of bosons or fermions).

In our case, there may be six second order radiation operators. The vacuum expectation values of four of them are equal to zero on the same grounds:

$$\left\langle 0 \left| \frac{\delta^2 S}{\delta \varphi_\rho (x) \delta \psi(y)} \quad S^+ \right| 0 \right\rangle = 0; \left\langle 0 \left| \frac{\delta^2 S}{\delta \varphi_\rho (x) \delta \bar{\psi}(y)} S^+ \right| 0 \right\rangle = 0$$

$$\left\langle 0 \left| \frac{\delta^2 S}{\delta \psi(x) \delta \psi(y)} \quad S^+ \right| 0 \right\rangle = 0; \left\langle 0 \left| \frac{\delta^2 S}{\delta \quad (x) \delta \quad (y)} \quad S^+ \right| 0 \right\rangle = 0 \qquad (3.7)$$

Indeed, our argument may be applied directly to the mixed boson-fermion derivatives, since after their transformation into a sum of matrix elements of the products of creation and destruction operators, each term has one fermion and one boson operator, which act in different spaces, and for this reason such matrix elements may be written in the form of direct products of matrix elements of single operators. As concerns the double fermion derivatives, after the transition to creation and destruction operators the sum may contain either terms with two creation/destruction operators,

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x/ It may be noted here that already from considerations of translation invariance it might be possible to conclude at once that the matrix elements (3.6) should be constant. In our concrete case, we might even go further and conclude that the first of these matrix elements was equal to zero due to the invariance with respect to space reflections, and the other two due to invariance with respect to gauge transformations (3.3). However, in the case, for example, of the derivative with respect to the neutral scalar field, considderations of covariance would not be sufficient to conclude that the matrix elements (3.6) are equal to zero.

obviously equal to zero, or terms with two different operators. But in the latter case, these will necessarily be, for the matrix elements of (3.7), two operators, one of which belongs to the particle, and the other to the anti-particle, in other words also acting essentially in different spaces, and again our argument is true.

Thus, if we were interested only in vacuum expectation values from radiation operators, we would have to examine only two operators

$$\frac{\delta^2 S}{\delta\,\varphi_{\rho'}(x)\delta\,\varphi_{\rho}(y)}\; S^+ \tag{3.8}$$

and

$$\frac{\delta^2 S}{\delta\bar{\psi}(x)\delta\psi(y)}\; S^+ . \tag{3.9}$$

As we shall see later, it will also be sufficient for us (in order to deduce the dispersion relations of meson-nucleon scattering) to examine only the radiation operator (3.8), or, to be more precise, its matrix element between two single-nucleon states. For this reason we shall now discuss in more detail this latter operator. It should be kept in mind that a similar investigation for other radiation operators of the second order would be nearly the same.

In order to make clear the idea of the computations we plan, let it be recalled that, as we already know (see below, the end of Section 4) the vacuum expectation value of a radiation operator (3.8) coincides essentially with the Green's function for bosons, i.e. with the vacuum expectation value of the T-product of two operators of the "real" boson field. On the other hand, a consideration, in the free field theory, not only of the T-product but also of other various products is very useful. It leads to the introduction of other singular functions in addition to the causal function. We should like to do the same thing here, i.e. introduce, in addition to the radiation operator (3.8), other operators connected with it as the various singular functions of the free field are connected with the causal function. The introduction of such operators will prove extremely useful for further investigation.

For this purpose let us now calculate the variational derivative of the current operator introduced above (3.4, 1). We obtain:

$$-i\,\frac{\delta j_{\rho'}(x)}{\delta \varphi_{\rho}(y)} = \frac{\delta^2 S}{\delta \varphi_{\rho'}(x)\delta \varphi_{\rho}(y)}\, S^+ + \frac{\delta S}{\delta \varphi_{\rho'}(x)} \cdot \frac{\delta S^+}{\delta \varphi_{\rho}(y)}, \qquad (3.10)$$

where one of the terms at the right coincides with the radiation operator (3.8). The second term at the right may be expressed as a product of two current operators, after which we arrive at the relation

$$\frac{\delta^2 S}{\delta \varphi_{\rho'}(x)\delta \varphi_{\rho}(y)}\, S^+ = -j_{\rho'}(x)\, j_{\rho}(y) - i\,\frac{\delta j_{\rho'}(x)}{\delta \varphi_{\rho}(y)}. \qquad (3.11)$$

The left-hand side of (3.11) is symmetric with respect to the permutation of $\varphi_{\rho'}(x)$ and $\varphi_{\rho}(y)$. Therefore, if we make this permutation we will obtain another expression for this same operator (3.8):

$$\frac{\delta^2 S}{\delta \varphi_{\rho'}(x)\delta \varphi_{\rho}(y)}\, S^+ = -j_{\rho}(y) j_{\rho'}(x) - i\,\frac{\delta j_{\rho}(y)}{\delta \varphi_{\rho'}(x)}. \qquad (3.12)$$

Certain individual terms in the right-hand sides of these expressions are among those new operators which we intend to introduce.

It is not difficult to grasp the meaning of the operators introduced. Indeed, the first terms in the right-hand sides (3.11, 12) are simply products of currents, that is, operators which should become analogous to the singular functions D_+ and D_- of the free field theory. Below, when we examine their matrix elements we shall see in what sense they actually contain only frequencies of one sign.

In order to determine the meaning of the second terms in (3.11, 12), let us express them by the S matrix and turn to the condition of causality (2.2, Section 2). We will then see that

$$-i\,\frac{\delta j_{\rho'}(x)}{\delta \varphi_{\rho}(y)} = \frac{\delta}{\delta \varphi_{\rho}(y)}\left(\frac{\delta S}{\delta \varphi_{\rho'}(x)}\,S^{+}\right) = 0 \quad \text{if } y \lesssim x \qquad (3.13)$$

$$-i\,\frac{\delta j_{\rho}(y)}{\delta \varphi_{\rho'}(x)} = \frac{\delta}{\delta \varphi_{\rho'}(x)}\left(\frac{\delta S}{\delta \varphi_{\rho}(y)}\,S^{+}\right) = 0 \text{ if } x \lesssim y \qquad (3.14)$$

that is, operators

$$-i\,\frac{\delta j_{\rho'}(x)}{\delta \varphi_{\rho}(y)} \quad \text{and} \quad -i\,\frac{\delta j_{\rho}(y)}{\delta \varphi_{\rho'}(x)}$$

behave just like advanced and retarded Green's functions.

If we now take account of the relations (3.13), (3.14), then from (1.11, 12) we will obtain

$$\frac{\delta^2 S}{\delta \varphi_{\rho'}(x)\delta \varphi_{\rho}(y)}\,S^{+} = \begin{cases} -j_{\rho'}(x)j_{\rho}(y), & \text{when } x \gtrsim y, \\ -j_{\rho}(y)j_{\rho'}(x), & \text{when } y \gtrsim x, \end{cases} \qquad (3.15')$$

that is

$$\frac{\delta^2 S}{\delta \varphi_{\rho'}(x)\delta \varphi_{\rho}(y)}\,S^{+} = -T\,(j_{\rho'}(x)j_{\rho}(y)\,), \qquad (3.15)$$

as we expected from the very beginning.[x/]

Finally, if in (3.11, 12) we carry out anti-symmetrization and symmetrization, we will arrive at combinations that are similar to the singular functions D and $D^{(1)}$:

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x/ It should be pointed out that equations (3.15'), which are actually a definition of the T-product in (3.15), determine it only for $x \neq y$. For $x = y$, the value of (3.15) remains indefinite, which fact will give rise later on to the possibility of adding certain arbitrary polynomials to the Fourier transforms.

$$-i\,\frac{\delta j_{\rho'}(x)}{\delta \varphi_{\rho}(y)} + i\,\frac{\delta j_{\rho}(y)}{\delta \varphi_{\rho'}(x)} = j_{\rho'}(x)\, j_{\rho}(y) - j_{\rho}(y) j_{\rho'}(x), \qquad (3.16)$$

and

$$j_{\rho'}(x) j_{\rho}(y) + j_{\rho}(y) j_{\rho'}(x) = -2\,\frac{\delta^2 S}{\delta \varphi_{\rho'}(x)\delta \varphi_{\rho}(y)} - i\,\frac{\delta j_{\rho}(y)}{\delta \varphi_{\rho'}(x)} - i\,\frac{\delta j_{\rho'}(x)}{\delta \varphi_{\rho}(y)}. \qquad (3.17)$$

If we take account of (3.13), 14), it follows from (3.16), in particular, that

$$[j_{\rho'}(x),\, j_{\rho}(y)] = 0 \quad \text{for} \quad x \sim y, \text{ i.e. } (x-y)^2 < 0 \qquad (3.18)$$

i.e. that for the radiation operator which is similar to the D-function is retained its most important property that of being zero outside the light cone. We should like to emphasize here that this peculiarity is due entirely to the causality requirement that we imposed (2.2 from Section 2). What is more, a number of authors used precisely this requirement as a causality condition in the deduction of dispersion relations.

Later on we shall also need the relation

$$-i\,\frac{\delta j_{\rho'}(x)}{\delta \varphi_{\rho}(y)} = \left(i\,\frac{\delta j_{\rho'}(x)}{\delta \varphi_{\rho}(y)}\right)^{+}, \qquad (3.19)$$

the validity of which becomes obvious if we recall that both $j_{\rho'}(x)$ and $\varphi_{\rho}(y)$ are Hermitian.

Let us now proceed to establish the effects which, for the matrix elements of the radiation operators with respect to any states, follow from the requirement of translation invariance. It is well-known that for the matrix elements of operators of the second order with respect to vacuum this requirement leads to a situation where the matrix elements may depend only on the difference x-y. As for matrix elements with respect to any states, the situation is slightly more complex, although in essence it remains the same.

Obviously it is sufficient to restrict oneself to the consideration of matrix elements with respect to states $|p, s\rangle$ having a definite total momentum p (the remaining quantum numbers we shall designate by S) which, according to assumption 1.4 in Section 2, span the total system. Thus, let us consider for example the matrix element between states $|p, s\rangle$ and $|p', s'\rangle$ of the radiation operator $\frac{\delta^2 S}{\delta \varphi_{\rho'}(x) \delta \varphi_{\rho}(y)} S^+$:

$$\langle p's' | \frac{\delta^2 S}{\delta \varphi_{\rho'}(x) \delta \varphi_{\rho}(y)} S^+ | ps \rangle$$

In virtue of the translation invariance

$$\langle p's' | \frac{\delta^2 S}{\delta \varphi_{\rho'}(x) \delta \varphi_{\rho}(y)} S^+ | ps \rangle =$$

$$= \langle p's' | e^{i\hat{p}a} \frac{\delta^2 S}{\delta \varphi_{\rho'}(x-a) \delta \varphi_{\rho}(y-a)} S^+ e^{-i\hat{p}a} | ps \rangle =$$

$$= e^{i(p'-p)a} \langle p's' | \frac{\delta^2 S}{\delta \varphi_{\rho'}(x-a) \delta \varphi_{\rho}(y-a)} S^+ | ps \rangle$$

where $\hat{p}$ is the operator of the total four-momentum, the eigenstates of which, according to assumption, are the states $|ps\rangle$ and $|p's'\rangle$. Selecting now 'a' equal to $\frac{x+y}{2}$ we see that the operator under the sign of the matrix element proves to be dependent only on the difference (x-y). Therefore, we may write:

$$\langle p's' | \frac{\delta^2 S}{\delta \varphi_{\rho'}(x) \delta \varphi_{\rho}(y)} S^+ | ps \rangle = i\, e^{i \frac{p'-p}{2}(x+y)} F^{c}_{\alpha\omega}(x-y) \qquad (3.20)$$

where the factor i is added from considerations of correspondence with the definition of the singular functions for a free field. For the same reason we insert the sign (C) in $F^{(c)}$.

The matrix elements for other radiation operators of the second order may be represented in exactly the same way:

$$\left\langle p's' \left| \frac{\delta j_\rho(y)}{\delta \varphi_{\rho'}(x)} \right| ps \right\rangle = -e^{i\frac{p'-p}{2}(x+y)} \cdot F^{ret}_{\alpha\omega}(x-y), \qquad (3.21)$$

$$\left\langle p's' \left| \frac{\delta j_{\rho'}(x)}{\delta \varphi_{\rho}(y)} \right| ps \right\rangle = -e^{i\frac{p'-p}{2}(x+y)} \cdot F^{adv}_{\alpha\omega}(x-y), \qquad (3.22)$$

$$\left\langle p's' \left| j_{\rho'}(x) j_\rho(y) - j_\rho(y) j_{\rho'}(x) \right| ps \right\rangle = -i\, e^{i\frac{p'-p}{2}(x+y)} \cdot F_{\alpha\omega}(x-y), \qquad (3.23)$$

$$\left\langle p's' \left| j_{\rho'}(x) j_\rho(y) + j_\rho(y) j_{\rho'}(x) \right| ps \right\rangle = e^{i\frac{p'-p}{2}(x+y)} \cdot F^{(1)}_{\alpha\omega}(x-y) \qquad (3.24)$$

$$\frac{1}{2}\left\langle p's' \left| \frac{\delta j_\rho(y)}{\delta \varphi_{\rho'}(x)} + \frac{\delta j_{\rho'}(x)}{\delta \varphi_{\rho}(y)} \right| ps \right\rangle = -e^{i\frac{p'-p}{2}(x+y)} \cdot \bar{F}_{\alpha\omega}(x-y). \qquad (3.25)$$

In equations (3.20-25), for the sake of brevity the set of indices ρ, p, s is designated by α, and ρ', p', s' by ω.

For the current products one may, of course, write expressions that take account of the translation invariance:

$$\left\langle p's' \left| j_{\rho'}(x) j_\rho(y) \right| ps \right\rangle = -i\, e^{i\frac{p'-p}{2}(x+y)} \cdot F^{(-)}_{\alpha\omega}(x-y), \qquad (3.26)$$

$$\left\langle p's' \left| j_\rho(y) j_{\rho'}(x) \right| ps \right\rangle = i\, e^{i\frac{p'-p}{2}(x+y)} \cdot F^{(+)}_{\alpha\omega}(x-y), \qquad (3.27)$$

However, we may go further still and determine the structure of the function $F^{(-)}_{\alpha\omega}$ in more detail. Precisely because of assumption 1.2 of Section 2 concerning the completeness of the system of functions with definite momenta, we may expand the matrix element on the lefthand side (3.26) into a product of matrix elements of the current:

$$\langle p's' | \, j_{\rho'}(x) \, j_{\rho}(y) \, | ps \rangle =$$

$$= \frac{1}{(2\pi)^3} \int d\bar{k} \sum_n \langle p's' | \, j_{\rho'}(x) \, | \bar{k} \; n \rangle \langle \bar{k} n \, | \, j_{\rho} \, (y) | \; ps \rangle . \tag{3.28}$$

We can now make use of the demand for translation invariance for each of the current operators in (3.28), and the right-hand side will be as follows:

$$\frac{1}{(2\pi)^3} \sum_n \int d\bar{k} \langle p's' | \; j_{\rho'}(0) | \bar{k} \; n \rangle \langle \bar{k} n \, | \, j_{\rho}(0) \, | \; ps \rangle \; e^{-i K_n (x-y) + ip'x - ipy} .$$

Comparing this expression with (3.26), we obtain an explicit form of the function $F^{(-)}_{\alpha\omega}$, now expressed only by the matrix elements of the currents at the origin of coordinates:

$$F^{(-)}_{\alpha\omega} (x) = \frac{i}{(2\pi)^3} \sum_n \int d\bar{k} \langle p's' | \; j_{\rho'}(0) \; | \bar{k} \; n \rangle \langle \bar{k} \, n \, | \, j_{\rho}(0) \, | \, ps \rangle \, . \tag{3.30}$$

$$\exp \left\{ -i \left(\sqrt{M_n^2 + \bar{k}^2} - \frac{po + po'}{2} \right) x^o + i \left(\bar{k} - \frac{\bar{p} + \bar{p}'}{2} \right) \bar{x} \right\}$$

where $M_n^2 = K_n^2$, that is, in the state $\bar{k}$, n: $K^o = \sqrt{M_n^2 + \bar{k}^2}$.

The importance of Eq. (3.30) consists in the fact that, since all radiation operators are related to each other, it makes it possible in principle to express all the radiation operators of the second order by operators of the first order. Still, as we shall demonstrate in the next section with vacuum matrix elements, one must be careful in virtue of the singular behaviour of radiation operators when the arguments coincide.

From definitions (3.20-27) and the relations obtained earlier between the radiation operators there follows a large number of relations between the functions $F^{(c)} \ldots . F^{(+)}$. First of all, by permutation of x and y in (3.22) and (3.27),

$$F^{adv}_{\alpha\omega}(x) = P_{\rho\rho'} \cdot F^{ret}_{\alpha\omega}(-x) \quad (3.31)$$

and

$$F^{(+)}_{\alpha\omega}(x) = -P_{\rho\rho'} \, F^{(-)}_{\alpha\omega}(-x) \quad (3.32)$$

where $P_{\rho\rho'}$ denotes the operator permuting ρ and ρ'. Now from (3.11) and (3.12) we obtain

$$F^{(c)}_{\alpha\omega}(x) = F^{(-)}_{\alpha\omega}(x) + F^{adv}_{\alpha\omega}(x) = \quad = F^{(-)}_{\alpha\omega}(x) + P_{\rho\rho'} \cdot F^{ret}_{\alpha\omega}(-x) \quad (3.33.1)$$

$$F^{(c)}_{\alpha\omega}(x) = -F^{(+)}_{\alpha\omega}(x) + F^{ret}_{\alpha\omega}(x) = P_{\rho\rho'} \, F^{(-)}_{\alpha\omega}(-x) + F^{ret}_{\alpha\omega}(x). \quad (3.33.2)$$

And now applying also (3.16), we find that

$$F_{\alpha\omega}(x) = -F^{adv}_{\alpha\omega}(x) + F^{ret}_{\alpha\omega}(x) = F^{(-)}_{\alpha\omega}(x) + F^{(+)}_{\alpha\omega}(x) \quad (3.34)$$

or

$$F_{\alpha\omega}(x) = F^{(-)}_{\omega}(x) - P_{\rho\rho'} \, F^{(-)}_{\alpha\omega}(-x) = F^{ret}_{\alpha\omega}(x) - P_{\rho\rho'} \, F^{ret}_{\alpha\omega}(-x) \quad (3.34a)$$

Comparing (3.25), (3.21) and (3.22) we see that

$$\bar{F}_{\alpha\omega}(x) = \frac{F^{ret}_{\alpha\omega}(x) + F^{adv}_{\alpha\omega}(x)}{2} = \frac{F^{ret}_{\alpha\omega}(x) + P_{\rho\rho'} \, F^{ret}_{\alpha\omega}(-x)}{2} \quad (3.35)$$

Finally, comparing (3.24) and (3.26, 27) and taking account of (3.17), we note that

$$F^{(1)}_{\alpha\omega}(x) = i\, F^{(-)}_{\alpha\omega}(x) - i\, F^{(+)}_{\alpha\omega}(x) = i\left(F^{(-)}_{\alpha\omega}(x) + P_{\rho\rho'} \, F^{(-)}_{\alpha\omega}(-x)\right) \quad (3.26.1)$$

and

$$F^{(1)}_{\alpha\omega}(x) = 2i\, F^{c}_{\alpha\omega}(x) - i\, F^{ret}_{\alpha\omega}(x) - i\, F^{adv}_{\alpha\omega}(x) = 2i(F^{c}_{\alpha\omega}(x) - \bar{F}_{\alpha\omega}(x)) \quad (3.36.2)$$

We now write out the relations for Hermitian conjugation. In (3.30) we perform complex conjugation and find that

$$F^{(-)}_{\alpha\omega}(x) = -\overset{*}{F}{}^{(-)}_{\omega\alpha}(-x) \qquad (3.37)$$

Now, from (3.19) it is easy to obtain

$$F^{ret}_{\alpha\omega}(x) = P_{\rho\rho'}\overset{*}{F}{}^{ret}_{\omega\alpha}(x) \qquad (3.38)$$

With the aid of these equations and expressions (3.34a, 35) for the functions $F_{\alpha\omega}$ and $\overline{F}_{\alpha\omega}$ in terms of $F^{ret}_{\alpha\omega}$, it is easy to obtain also the rules for conjugation of the functions $F_{\alpha\omega}$ and $\overline{F}_{\alpha\omega}$;

$$\overset{*}{F}_{\omega\alpha}(x) = P_{\rho\rho'}F_{\alpha\omega}(x); \; \overset{*}{\overline{F}}_{\omega\alpha}(x) = P_{\rho\rho'}\overline{F}_{\alpha\omega}(x). \qquad (3.39)$$

However, in addition to complex conjugation and permutation of the indices α and ω, the full Hermitian conjugation should include also the substitution of x by -x (the permutation of x and y). In order to determine the rules for such conjugation, let us find the properties of symmetry of the functions $F_{\alpha\omega}$ and $\overline{F}_{\alpha\omega}$, which are also of interest in themselves. These properties are quite obvious from (3.34a, 35)

$$F_{\alpha\omega}(-x) = -P_{\rho\rho'}F_{\alpha\omega}(x); \; \overline{F}_{\alpha\omega}(-x) = P_{\rho\rho'}\overline{F}_{\alpha\omega}(x), \qquad (3.40)$$

and differ from the properties of symmetry of the respective free singular functions only in the appearance of the operator $P_{\rho\rho'}$, which in the case of a free field reduces to a unit operator.

With the aid of (3.40) we may write straightforwardly

$$F_{\alpha\omega}(x)^{+} = \overset{*}{F}_{\omega\alpha}(-x) = -F_{\alpha\omega}(x); \; \overline{F}_{\alpha\omega}(x)^{+} = \overset{*}{\overline{F}}_{\omega\alpha}(-x) = \overline{F}_{\alpha\omega}(x) \qquad (3.41)$$

that is, the matrix $F_{\alpha\omega}(x)$ is anti-Hermitian, and the matrix

$\overline{F}_{\alpha\omega}(x)$ is Hermitian. Noting now that

$$F^{ret}_{\alpha\omega}(x) = \overline{F}_{\alpha\omega}(x) + \frac{1}{2} F_{\alpha\omega}(x)$$
$$\overset{*}{F}{}^{ret}_{\omega\alpha}(-x) = \overline{F}_{\alpha\omega}(x) - \frac{1}{2} F_{\alpha\omega}(x) \qquad (3.42)$$

we see that $\overline{F}_{\alpha\omega}(x)$ is the Hermitian part of the retarded matrix $F^{ret}_{\alpha\omega}(x)$, and $\frac{1}{2i}F_{\alpha\omega}(x)$ its anti-Hermitian part.

Returning now to the symmetry properties (3.40), we see that from the Hermitian and anti-Hermitian parts of the matrix $F^{ret}_{\alpha\omega}(x)$ we may form two combinations that are <u>even</u> with respect to the reflection $x \longrightarrow -x$:

$$(1 - P_{\rho\rho'})\, F_{\alpha\omega}(-x) = (1 - P_{\rho\rho'})\, F_{\alpha\omega}(x) \qquad (3.43)$$

and

$$(1 + P_{\rho\rho'})\, \overline{F}_{\alpha\omega}(-x) = (1 + P_{\rho\rho'})\, \overline{F}_{\alpha\omega}(+x), \qquad (3.44)$$

and two <u>odd</u> combinations:

$$(1 + P_{\rho\rho'})\, F_{\alpha\omega}(-x) = -(1 + P_{\rho\rho'})\, F_{\alpha\omega}(x) \qquad (3.45)$$

and

$$(1 - P_{\rho\rho'})\, \overline{F}_{\alpha\omega}(-x) = -(1 + P_{\rho\rho'})\, \overline{F}_{\alpha\omega}(x) \qquad (3.46)$$

These symmetry properties, rewritten in momentum space, will make it possible for us later, in the derivation of the dispersion relations, to escape integrations over negative values of energy.

Let us return once more, in concluding, to a discussion of relations (3.15). As we have already pointed out, they express a very curious situation that arises in the theory: on the ond hand, (3.15') expresses the radiation operator $\frac{\delta^2 S}{\delta \varphi_{\rho'}(x')\delta \varphi_{\rho}(y)} S^+$ of second order by the product of two currents, that is, by radiation operators of first order. On the other hand, however, such a reduction of operators of second order to those of first order cannot be carried out in full: equations (3.15') say nothing about the significance of the radiation operator of second order when the points x and y coincide (more precisely, of course, about rules of integration in the neighborhood of x = y). One might say that the radiation operator of the second order $\frac{\delta^2 S}{\delta \varphi_{p'}(x)\delta \varphi_{p}(y)} S^+$ reduces to the radiation operators of the first order, accurate to an arbitrary quasi-local operator (for a definition of quasi-local operators see). When passing on to the momentum representation, this quasi-local operator will be expressed in the form of an arbitrary polynomial added to the Fourier transform (compare the discussion in Section 4 following (4.38) and in Section 6 following (6.15) below). This result is extremely close to that obtained by one of the authors (N.N.B.) and Shirkov in the construction of a theory of the S matrix on the basis of an expansion in terms of the small coupling constant. The essential differences consist in the fact that first of all these authors, having at their disposal a Lagrangian, could determine with its aid the degree of the arbitrary polynomial, and, secondly, that the constants entering in the arbitrary polynomials at various places were finally combined into one generalized Lagrangian. In our method of constructing the theory, the degree of the polynomial has to be introduced into the theory without the use of the Lagrangian as a new requirement, the basis being, of course, correspondence with experiment.

The situation that arises for the radiation operator of the second order is not an exception, but a rule valid also for all the radiation operators of higher orders. Indeed, by a consistent application of the causality condition (2.24) it may be shown that any radiation operator of the $(l + m + n)$ th order reduces to a chronological product of currents:

$$\frac{\delta^{\ell+m+n} S}{\delta\overline{\psi}(x_1)\ldots\delta\overline{\psi}(x_\ell)\delta\psi(y_1)\ldots\delta\psi(y_m)\delta\varphi(z_1)\ldots\delta\varphi(z_n)} S^{+} = \tag{3.47}$$

$$= i^{\ell+3m+3n}\, T(\eta(x_1)\ldots\eta(x_\ell)\overline{\eta}(y_1)\ldots\overline{\eta}(y_m)j(z_1)\ldots j(z_n)).$$

Hence, it will, of course, immediately follow that any matrix elements of such an operator will for distinct values of the arguments $x_1 \ldots z_n$ be expressed in terms of matrix elements of the currents by means of sums analogous to (3.30). However, in the case of any coincidence of any points $x_1, \ldots, z_n$ there will arise an arbitrariness connected with the not entirely definite quality of the T-product, which arbitrariness may be expressed by adding the product of an arbitrary quasi-local operator of the coincident points and the currents at the remaining points. A more detailed elaboration of these ideas would lead us beyond the limits of the derivation of dispersion relations and, in our opinion, might serve as a basis for a new approach to the construction of the quantum field theory.

Section 4. Vacuum Expectation Values of Boson Radiation Operators of the Second Order.

In this section we shall investigate in more detail the vacuum expectation values of the radiation operators (3.8) considered in Section 3 and the operators connected with them. It is clear that in the special case of vacuum expectation values all general relations between matrix elements (these relations were derived in Section 3) will be valid. The vacuum expectation values of radiation operators will be determined by Eqs. (3.20-27), on the right-hand sides of which the factor $e^{-i\frac{p-p'}{2}(x+y)}$ will now vanish, and the indices α and ω will simply be converted into ρ and ρ'. For example,

$$\left\langle 0 \left| \frac{\delta^2 S}{\delta \varphi_{\rho'}(x) \delta \varphi_{\rho}(y)} S^+ \right| 0 \right\rangle = i F^{(c)}_{\rho\rho'}(x-y).$$

Further, due to isotopic invariance, the dependence upon isotopic indices will now become diagonal and we shall write

$$F^{(?)}_{\rho\rho'}(x) = \delta_{\rho\rho'} f^{(?)}(x) \qquad (4.1)$$

where (?) denotes one of the signs (c), ...; (+).

Let us begin by considering the representation (3.30) for $F^{(-)}_{\alpha\omega}$ using the matrix elements of the currents. It will now be written as follows:

$$\delta_{\rho\rho'} f^{(-)}(x) = \frac{i}{(2\pi)^3} \sum_n \int d\bar{k} \langle 0 | j_{\rho'}(0) | \bar{k}, n \rangle \langle \bar{k}, n | j_{\rho}(0) | 0 \rangle e^{-i k_n^0 x^0 + i \bar{k}\bar{x}} \qquad (4.2)$$

We shall show that in the sum (4.2) the first terms are absent. Indeed, the term with n = 0 (vacuum) is zero in virtue of (3.6). We shall assume that the states with one, two, three, etc. mesons are, in the sum (4.2), the lowest energy states (that is, we shall assume that there do not exist any bound complexes of mesons and nucleons with a mass less than 3m, i.e. three meson masses). Then the term in (4.2) with n = (1 meson) will also be zero. Indeed, according to the definition of current $\langle a'| j_{\rho'}(x)| a\rangle \sim \langle a'| \frac{\delta S}{\delta \varphi_{\rho'}(x)} S^+ | a\rangle$. Further, $\frac{\delta S}{\delta \varphi_{\rho'}(x)} S^+ \sim [\varphi, S]S^+$ and therefore $\langle a'| j_{\rho'}(x)| a\rangle \sim \langle a'| [\varphi_{\rho'}(x), S]\, S^+ |a\rangle$. Opening up the commutator here, we find that

$$\langle a'| j_{\rho'}(x)| a\rangle \sim \langle a'| \varphi_{\rho'}(x) | a\rangle - \langle a'| S\, \varphi_{\rho'}(x) | a\rangle$$

If we now assume that the states $|a'\rangle$, $|a\rangle$ are vacuum states $|0\rangle$ or one-particle states $|1\rangle$, then in virtue of the stability conditions $S^+ |a\rangle = |a\rangle$, $\langle a| S = \langle a|$ the two terms of the commutator cancel. Thus, it is proved that matrix elements of the type $\langle 0| j_{\rho'} | 1\rangle$ and $\langle 1| j_\rho | 0\rangle$ are zero. Finally, in virtue of the pseudoscalarity of mesons, the matrix elements of the current between the vacuum and two-meson states will be equal to zero too.

Thus, the sum in (4.2) begins only with three-meson states, that is, the least value of K_n^o is 3m.

Let us now rewrite the sum (4.2) in the form of a four-dimensional Fourier integral:

$$\delta_{\rho\rho'}\, f^{(-)}(x) = \frac{i}{(2\pi)^3} \int d\bar{k} \sum_{n \geq 3} \langle 0| j_{\rho'}(0)| n,\bar{k}\rangle \langle n,\bar{k}| j_\rho(0) |0\rangle e^{-ikx}$$

$$\delta(k^o - \sqrt{M_n^2 + \bar{k}^2}) \qquad (4.3)$$

Introducing now the Fourier transforms for all functions by the definition

$$f^{(?)}(x) = \frac{1}{(2\pi)^4} \int dk\, e^{-ikx}\, g^{(?)}(k), \qquad (4.4)$$

we see that (in virtue of the proportionality of the left-hand side of (4.3) to $\delta_{\rho\rho'}$, the right-hand side must be zero for $\rho \neq \rho'$ and independent of ρ for $\rho = \rho'$)

$$g^{(-)}(k) = 2\pi i \sum_{n \geq 3} \left| \langle 0 | j_\rho (o) | n\bar{k} \rangle \right|^2 \delta(k^o - \sqrt{\mu_n^2 + k^2}). \qquad (4.5)$$

However, from the pseudoscalarity of $\varphi_\rho(x)$ it follows that the function $f^{(-)}(x)$, and consequently $g^{(-)}(k)$, must be invariant with respect to Lorentz transformations, excluding time reflection. For this reason, $g^{(-)}(k)$ can depend only on k^2 and the sign of k^o, i.e. upon $\theta(k^o)$. But from (4.5) it may be seen that it contains only positive frequencies. Therefore we may write

$$g^{(-)}(k) = 2\pi i \, \theta(k^o) \, I(k^2) \qquad (4.6)$$

where the function I depends only on k^2. Comparing this expression with (4.5), we note that (4.5) may be rewritten as

$$g^{(-)}(k) = 2\pi i \sum_{n \geq 3} \left| \langle 0 | j_\rho (0) | n\bar{k} \rangle \right|^2 2\sqrt{\mu_n^2 + k^2} \; \delta(k^2 - \mu_n^2) \; \theta(k^o). \qquad (4.5)$$

Thus, we represented $g^{(-)}(k)$, which in virtue of (4.6) should be a product of an invariant function and $\theta(k^o)$, in the form of a product of $\theta(k^o)$ and a function which obviously does not depend on the choice of time direction. Thus, we may state that

$$I(k^2) = \sum_{n \geq 3} \left| \langle 0 | j_\rho (o) | n\bar{k} \rangle \right|^2 2\sqrt{\mu_n^2 + k^2} \; \delta(k^2 - \mu_n^2). \qquad (4.7)$$

The two basic properties of the function $I(k^2)$ follow immediately from (4.7):

$$\begin{aligned} &1. \quad I(k^2) = 0 \qquad \text{for } k^2 \leq (3m)^2, \\ &2. \quad I(k^2) \geqq 0. \end{aligned} \qquad (4.8)$$

We may note further that in virtue of (4.8.1), (4.6) may be rewritten as

$$g^{(-)}(k) = 2\pi i\ \theta(k^0) \int_{(3m)^2}^{\infty} \delta(k^2-m^2)I(m^2)dm^2 = \int_{(3m)^2}^{\infty} g^{o(-)}(k,m^2)I(m^2)dm^2. \quad (4.9)$$

This so-called "spectral" representation for a function which is simply related (see below) to $g^{(-)}(k)$, was first obtained by Kallen and Lehmann. They also established the properties (4.8).

Thus,

$$\left\langle 0 \middle| j_{\rho'}(x) j_{\rho}(y) \middle| 0 \right\rangle = \frac{\delta_{\rho\rho'}}{(2\pi)^3} \int dk\ e^{-ik(x-y)}\theta(k^0)I(k^2). \quad (4.10)$$

Substituting here $x \leftrightarrow y$, we obtain

$$\left\langle 0 \middle| j_{\rho}(y) j_{\rho'}(x) \middle| 0 \right\rangle = \frac{\delta_{\rho\rho'}}{(2\pi)^3} \int dk\ e^{-ik(x-y)}\ \theta(-k^0)I(k^2). \quad (4.11)$$

This justifies the signs (-) and (+) introduced earlier; the negative-frequency function really contains only negative frequencies, and the positive-frequency function, only positive frequencies. It should be stressed that this is shown only for vacuum matrix elements; generally speaking, this property may not hold for matrix elements between arbitrary states.

Recalling now the relations (3.33) and passing in them to the Fourier transforms with the aid of (4.4), and inserting the expression (4.6) for $g^{(-)}(k)$ and the expression for $g^{(+)}(k)$

$$g^{(+)}(k) = -2\pi i\ \theta(-k^0)\ I(k^2) \quad (4.12)$$

that follows from (11), we obtain:

$$\begin{aligned} g^c(k) &= 2\pi i\ \theta(k^0)\ I(k^2) + g^{adv}(k), \\ g^c(k) &= 2\pi i\ \theta(-k^0)I(k^2) + g^{ret}(k). \end{aligned} \quad (4.13)$$

One very important consequence follows from these equations. Due to the property (4.8.1) of the spectral function $I(k^2)$ just established, we see that at small momenta $k^2 < (3\mu)^2$ the Fourier transforms of all three functions g^c, g^{adv} and g^{ret} coincide:

$$g^c(k) = g^{adv}(k) = g^{ret}(k) \quad \text{when} \quad k^2 < (3\mu)^2. \tag{4.14}$$

This circumstance will serve as a basis for establishing the analytic properties of the functions $g^{(-)}(k)$, $g^{adv}(k)$ and $g^{ret}(k)$, which we shall now investigate.

Let us consider in detail the Fourier transform

$$g^{ret}(k) = \int f^{ret}(x)\, e^{ikx}\, dx$$

in which, in virtue of the causality condition

$$f^{ret}(x) = 0 \quad \text{for} \quad x \lesssim 0. \tag{4.15}$$

We shall show that this Fourier transform may be continued into the region of complex k by substituting

$$k \rightarrow k = p + i\Gamma, \; p = \mathrm{Re}\, k; \; \Gamma = \mathrm{Im}\, k$$

if the four-vector Γ satisfies the condition

$$\Gamma > 0 \tag{4.16}$$

and p is arbitrary. We then have

$$g^{ret}(k) = \int f^{ret}(x)\, e^{ipx}\, e^{-\Gamma x}\, dx = g^{ret}(p+i\Gamma).$$

It is clear that in this integral the exponent $e^{-\Gamma x}$ will be a cut-off factor ensuring its convergence. Indeed, in virtue of (4.16) we will always be able to select a frame of reference in which $\vec{\Gamma} = 0$;

therefore, the exponent will take the form

$$e^{-\Gamma^{o} x^{o}}$$

But according to (4.15) the integration extends only over the inside of the upper half of the light-cone, where $x^{o} \geqslant 0$ and $\vec{x}^{2} \leqq x_{o}^{2}$.

Thus, the function $h(x) = e^{ipx} e^{-\Gamma x}$ will belong to a certain class $C(q, r)$, in which

$$h_{mn} = \text{Sup} \; |x|^{m} \left| \frac{\partial^{n} h(x)}{\partial x_{\alpha_{i}} \ldots \partial x_{\alpha_{n}}} \right| \leqslant \text{const.}$$

for any $m = 0, 1, \ldots\ldots, 2$; $n = 0, 1, \ldots\ldots, q$.

On the other hand, according to condition (2.1) the function $f^{ret}(x)$ must be integrable and therefore the integral

$$\int f^{ret}(x) \, h(x) dx = \int f^{ret}(x) \, e^{ipx} \, e^{-\Gamma x} \, dx \qquad (4.17)$$

may be viewed as a linear functional in the space of functions $h(x)$. For this reason, both the integral (4.17) itself and its derivatives with respect to k will converge:

$$i^{s} \int f^{ret}(x) \, x_{\alpha_1} \ldots x_{\alpha_s} \, e^{ikx} \, dx < \infty, \quad s = 0, 1 \ldots .$$

Thus, $g^{ret}(k)$ will be an analytic function of k in the region (4.16).

Let us note further that the integral (4.17), being a linear functional in $c(q, r)$, must ipso facto be limited in absolute value by a linear combination of values h_{mn}. Since the derivatives of e^{iky} with respect to x are proportional to powers of k, we see that the function $g^{ret}(k)$ increases at infinity not faster than a certain polynomial in k (here we deal, of course, with the region k in which inequalities (4.16) are not relaxed).

The Fourier transform $g^{ret}(k)$ for real k may now be <u>defined</u> as an improper limit of the integral (4.17), thus

$$\lim_{\substack{\Gamma \to 0 \\ \Gamma > 0}} g^{ret}(p + i\Gamma) = g^{ret}(p) \tag{4.18}$$

In the same way the Fourier transform

$$g^{adv}(x) = \int f^{adv}(x)\, e^{ikx}\, dx \tag{4.19}$$

may be continued into the complex plane with the condition that

$$\Gamma < 0 \tag{4.20}$$

and after this the integral (4.19) may be defined as an improper limit

$$\lim_{\substack{\Gamma \to 0 \\ \Gamma < 0}} g^{adv}(p + i\Gamma) = g^{adv}(p). \tag{4.21}$$

Thus, we have introduced two functions $g^{ret}_{(k)}$ and $g^{adv}_{(k)}$ and have proved their analyticity in the regions (4.16) and (4.20) respectively.

It is easy to see that a relation between these functions follows from the parity relation (3.31) deduced earlier. Between

$$g^{ret}(k) = g^{ret}(p + i\Gamma) = \int_{\Gamma > 0} f^{ret}(x)\, e^{ipx}\, e^{-\Gamma x} dx \tag{4.22}$$

and

$$g^{adv}(k) = g^{adv}(p + i\Gamma) = \int_{\Gamma < 0} f^{adv}(x)\, e^{ipx}\, e^{-\Gamma x}\, dx$$

there exists the relation

$$g^{adv}(-p - i\Gamma) = g^{ret}(p + i\Gamma). \quad \Gamma > 0 \tag{4.22}$$

For further argument let us fix the frame of reference so that $\vec{\Gamma} = 0$. Since Γ is time-like, this is always possible and does not restrict generality.

Let us first investigate the function g^{ret}; the function g^{adv} can always be obtained from it with the aid of (4.28). From considerations of relativistic invariance $f^{ret}(x)$ can depend only on x^2 and sign x^o. But then from (4.22) it may be seen that the values of the integral (4.22), for any two values of k connected by a Lorentz transformation L^+, which does not include time reflection, will simply coincide. But any two complex vectors k, for which the integral (4.22) is defined, are necessarily related by a transformation L^+, since only such transformations retain the condition $\Gamma^o > 0$. Consequently, the left-hand side of (4.22) is (for all k that satisfy the condition $\Gamma^2 > 0$; $\Gamma^o > 0$) <u>a function only of</u> k^2. Thus $g^{ret}\ (p + i\Gamma)$ is a certain analytic function $G(k^2)$ only of k^2:

$$g^{ret}\ (p + i\ \ ) = G(k^2), \tag{4.23}$$

defined only for k satisfying

$$(\mathrm{Im}\ K)^2 > 0;\ (\mathrm{Im}\ K)^o > 0.$$

In order to find the region of analyticity of this function on the complex k^2 plane, let us note that in virtue of the proved analyticity of $f^{ret}\ (p + i\Gamma)$ in the upper k^o half-plane, the function $G(k^2)$ will obviously be analytic at a certain point

$$\begin{gathered} k^2 = \zeta = \xi + i\eta \\ \xi = p^2 - \Gamma^{o2};\ \eta = 2\,p^o\Gamma^o \end{gathered} \tag{4.24}$$

since one may find at least one vector $k = \{p^0 + i\Gamma^0, \vec{p}\}$, that satisfies (4.24), the fourth complex component of which would lie strictly in the upper half-plane. From the equations (4.24) connecting $\zeta = \xi + i\eta$ and k, it may be seen immediately that this can always be done for any points of the complex plane $\zeta = \xi + i\eta$, with the exception of the real positive semi-axis:

$$\eta = \text{Im } (k^2) = 0; \quad \xi = \text{Re } (k^2) \geq 0. \qquad (4.25)$$

Thus, the function $G(k^2)$ is analytic in the complex k^2 plane everywhere with the exception of the positive semi-axis. But the function $G(k^2)$ is a function of one scalar variable, and it does "not know" what vector raised to the second power gave rise to this argument. For this reason, the reservation made after (4.23) is now no longer needed: $G(k^2)$ will be an analytic function for any complex vector k, the square of which is not a real positive number. Finally, in virtue of the remark after equation (4.17), at infinity $G(k^2)$ may increase not faster than a polynomial.

Let us now define the two (maybe generalized) functions $G_+(p^2)$ and $G_-(p^2)$ as improper limits:

$$G_+(p^2) = \lim_{\text{Im } K \to 0} G(k^2), \ \text{Im } (k^2) > 0, \ \text{Re } (k^2) > 0, \qquad (4.26.1)$$

and

$$G_-(p^2) = \lim_{\text{Im } K^2 \to 0} G(k^2), \ \text{Im } (k^2) < 0, \ \text{Re } (k^2) > 0. \qquad (4.26.2)$$

If we now compare with the aid of Eqs. (4.24) the limit transition to the real axis in the function $f^{ret}_{(k)}$ and in $G(k^2)$, given the condition $Re(k^2) > 0$, we shall see that

$$g^{ret}_{p^2>0} = \begin{cases} G_+(p^2), & p^o > 0, \\ G_-(p^2), & p^o < 0. \end{cases} \qquad (4.27)$$

The property of symmetry (4.23) now gives us straightforwardly

$$g^{adv}_{p^2>0} = \begin{cases} G_-(p^2), & p^o > 0, \\ G_+(p^2), & p^o < 0. \end{cases} \qquad (4.28)$$

Thus we obtain expressions for the generalized functions $f^{ret}_{(p)}$ and $f^{adv}_{(p)}$ in the form of improper limits of a certain single analytic function $G(k^2)$. Returning again to (4.24) we find that these limit relations may be written also in a more simple and symmetric way:

$$g^{ret}(p) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} G(p^2 + i\varepsilon p^o), \qquad (4.29.1)$$

and

$$g^{adv}(p) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} G(p^2 - i\varepsilon p^o). \qquad (4.29.2)$$

Now, noting that in virtue of (4.13)

$$g^c(p) = \begin{cases} g^{adv}(p) & p^o < 0, \\ g^{ret}(p) & p^o > 0, \end{cases}$$

we see that the function $g^c(p)$ may be written in the form of an improper limit

$$g^c(p) = G_+(p^2) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} G(p^2 + i\varepsilon). \qquad (4.29.3)$$

It may be noted that in Eqs. (4.29) we dropped the condition of the vector p being time-like on the grounds that for $p^2 < 0$ the function

$G(p^2)$ is regular and consequently the limit is independent of the direction of approach.

Finally, subtracting (4.13) from each other, we find the difference of the limit values on the cut

$$G_+(p^2) - G_-(p^2) = 2\pi i\ I(p^2). \qquad (4.30)$$

From the property (4.8.1) of the spectral function $I(p^2)$ established earlier, it may now be seen that not the whole positive semi-axis will represent the cut on the complex plane k^2 for $G(k^2)$, but only the part

$$\mathrm{Im}\ (k^2) = \theta,\ \mathrm{Re}\ (k^2) \geqslant (3m)^2. \qquad (4.31)$$

The properties of the analytic function $G(\zeta)$ that we have established, and also its property of not increasing at infinity faster than a certain polynomial in ζ, permit us (using the limit equations (4.29) which we have just deduced) to construct for the functions $g^c(k)$, $g^{ret}(k)$ and $g^{adv}(k)$ spectral representations of the same type as those obtained above for $g^{(-)}(k)$. For this purpose we shall use Cauchy's theorem, as discussed in detail in Section 1.

We shall assume that the function $G(\zeta)$ increases at infinity not faster than ζ^n. Then according to Section 1 Cauchy's theorem may be applied (if we disregard the integration over the large circle) to the function

$$h(\zeta) = \frac{G(\zeta)}{(\zeta - m^2)^{n+1}}, \qquad (4.32)$$

which will have a pole at $\zeta = m^2$, in addition to a cut from $(3m)^2$ to ∞. Therefore, we shall select the integration contour in the following manner: beginning from the origin, it will proceed slightly above the real axis to $+\infty$, then it will include the large circle and return to the origin slightly below the real axis. Due to the properties of the function $h(\zeta)$, the integral over such a contour will reduce to the difference of the integrals over the upper and lower sides of the cut, and the small contour G_m^- around the point $\zeta = m^2$, which it passes in the negative sense. Thus, we may write:

$$G(\zeta) = \frac{(\zeta - m^2)^{n+1}}{2\pi i} \oint_{C_m^-} \frac{d\zeta'}{(\zeta' - m^2)^{n+1}} \frac{(\zeta')}{\zeta' - \zeta} + \frac{(\zeta - m^2)}{2\pi i} \int_{(3m)^2}^{\infty} \frac{G_+(\zeta') - G_-(\zeta')}{(\zeta' - \zeta)(\zeta' - m^2)^{n+1}} d\zeta' \tag{4.33}$$

In place of the difference of the integrals over the upper and lower side of the cut, we wrote (in accordance with the definitions in (4.26)) the difference $G_{(+)}(\zeta) - G_-(\zeta)$. The integral over C_m^- gives

$$\frac{1}{2\pi i} (\zeta - m^2)^{n+1} \oint_{C_m^-} \frac{d\zeta' \, G(\zeta')}{(\zeta' - m^2)^{n+1}(\zeta' - \zeta)} = \sum_{j=o}^{n} \frac{(\zeta - m^2)^j}{j!} G^{(j)}(m^2),$$

and therefore

$$G(k^2) = (k^2-m^2)^{n+1} \int_{(3m)^2}^{\infty} \frac{I(\zeta)\, d\zeta}{(\zeta - m^2)^{n+1}(\zeta - k^2)} + \sum_{j=o}^{n} G^{(j)}(m^2) \frac{(k^2-m^2)^j}{j!} \tag{4.33}$$

This equation may be considered as a spectral representation for the function $G(k^2)$, in which we find the same spectral function $I(k^2)$.

If we now pass to the corresponding limits, we shall obtain spectral representations also for the functions $g^c(p^2)$, $g^{ret}(p^2)$ and $g^{adv}(p^2)$ which are of immediate interest to us:

$$g^c(p) = G_+(p^2) = (p^2-m^2)^{n+1} \int_{(3m)^2}^{\infty} \frac{I(\zeta)\, d\zeta}{(\zeta - m^2)^{n+1}(\zeta - p^2 - i\varepsilon)} + \sum_{j=o}^{n} G^{(j)} (m^2) \frac{(p^2-m^2)^j}{j!} \tag{4.34}$$

$$g^{\left(\substack{ret\\adv}\right)}(p) = (p^2-m^2)^{n+1} \int_{(3m)^2}^{\infty} \frac{I(\zeta)\, d\zeta}{(\zeta - m^2)^{n+1}(\zeta - p^2 \mp i\varepsilon p^o)} +$$

$$+ \sum_{j=o}^{n} G^{(j)}(m^2) \frac{(p^2-m^2)^j}{j!} . \qquad (4.35)$$

Let us now establish certain properties of the unknown coefficients $G^{(j)}(m^2)$..., that enter into (4.33-35). Let us first of all show that the coefficient $G^{(o)}$ is zero. For this purpose let us consider the matrix element of the S matrix between two one-meson states $|\vec{p}\,'\,\rho'\rangle$ and $|\vec{p}\,\rho\rangle$. According to (2.10) it will be equal to

$$\langle p'\,\rho'|\,S\,|p\rho\rangle = \langle 0|\, b_{\rho'}^{(-)}(\vec{p})\, S\, b_{\rho}^{(+)}(p)\,|\,0\rangle .$$

If we put the creation amplitudes to the left and the destruction amplitudes to the right with the aid of commutation relations (2.25) and (2.11) and make use of the definition of the function $g^c(p)$, we will obtain for this matrix element

$$\langle p'\,\rho'|\,S|\,p\rho\rangle = \langle 0|\, b_{\rho'}^{(-)}(\vec{p}\,')\, b_{\rho}^{(+)}(\vec{p})\,|\,0\rangle + \frac{in\delta_{\rho\rho'}}{p^o}\,\delta(p'-p)g^c(p').$$

On the other hand, in virtue of the stability conditions of one-particle states (1.6 from Section 2), this matrix element is equal only to the first term on the right-hand side (-). Thus,

$$g^c(p) = 0 \qquad \text{when} \qquad p^2 = m^2.$$

It follows that

$$G^{(o)} = 0. \qquad (4.36)$$

We shall now show that all the constants $G^{(1)}$ ... $G^{(n)}$ must be real. This assertion is proved immediately, if we note that from the conjugation relations

$$f^{ret}(x) = f^{* \, ret}(x) \qquad (4.38)$$

there follows immediately

$$g^{ret}(k) = g^{* \, ret}(-k), \qquad (4.37)$$

and the integral (4.35) also has this property. Thus, the Fourier transforms of all three "Green like matrix elements" (delayed, advanced and causal) have spectral representations of the type:

$$g^{c}(p^2) = (p^2-m^2)^{n+1} \int_{(3m)^2}^{\infty} \frac{I(\zeta)\, d\zeta}{(\zeta - m^2)^{n+1}(\zeta - p^2 - i\varepsilon)} + \sum_{m=1}^{\infty} C_m (m^2-p^2)^m$$

$$(4.38)$$

$$g^{\left(\frac{ret}{adv}\right)}(p^2) = (p^2-m^2)^{n+1} \int_{(3m)^2}^{\infty} \frac{I(\zeta)\, d\zeta}{(\zeta - m^2)^{n+1}(\zeta - p^2 \mp i\varepsilon p^0)} + \sum_{m=1}^{\infty} C_m (m^2-p^2)^m$$

and the C_m are real.

With the aid of the relations found in Section 3 between the radiation operators possessing the spectral representations (4.38), one may obtain immediately the spectral representations for the vacuum matrix elements of all the remaining radiation operators also. We may note that in the formation of linear combinations that correspond to all the "non-Green-like" matrix elements ($\mathcal{D}^{(+)}$, $\mathcal{D}^{(-)}$ $\mathcal{D}$ and $\bar{\mathcal{D}}$), the unknown polynomials that enter into (4.38) vanish, and under the integral there arises $\delta(\zeta - p^2)$, due to which the factors $(p^2 - m^2)^n$ within and without the integrals cancel, and we then obtain representations of exactly the same type as the representation (4.9) for $g^{(-)}$ found above. A representation of the type (4.38) with substitution of $P\frac{1}{\zeta - p^2}$ for $\frac{1}{\zeta - p^2 - i\varepsilon}$ will be found for the Green-like function D^1.

Thus, we find that with respect to the spectral representations of vacuum matrix elements, the radiation operators of the second order are divided into two groups. Simple spectral representations of the type (4.9) are obtained for matrix elements of "non-Green-like" operators, whereas for "Green-like" operators complex representations of the type (4.38) are obtained. These latter were obtained in a rather cumbersome ay, by investigating the analytic behaviour of the corresponding functions.

We might make a different attempt to pass directly from the spectral representation (4.9) for $g^{(-)}(k)$ and from a similar one for $g^{(+)}(k)$ to the spectral representations for "Green-like" functions, with the aid (for example for $g^c(k)$) of equations of the type (3.15). Direct calculation would then lead us to a spectral representation of the "simple" type

$$g^c(k) = \int_{(3m)^2}^{\infty} \frac{dm^2}{m^2-k^2-i\varepsilon} I(m^2) \qquad (4.39)$$

for g^c, and to the same representations (differing only in the method of circumventing the pole at $m^2 = k^2$) for the other "Green-like" functions. This was the method used in the ork of Lehmann.

However, in actuality these simple representations uld, generally speaking (if we did not impose a stringent restriction on the degree of increase of the spectral function at infinity), be devoid of any meaning insofar as the integral over m^2 diverges. Indeed, whereas for "non-Green-like" functions the kernels of the spectral representations necessarily contain a δ-function and the integration in equations of the type (4.9) actually takes place only in the neighborhood of one point, and the behaviour of $I(m^2)$ at infinity is inessential to the convergence of the integral; in the spectral representations of the type (4.39) for "Green-like" functions the integration extends over the entire interval $(-\infty, +\infty)$, which leads to divergence in the case of insufficiently rapid diminishing of $I(m^2)$ at ∞.

The reason for this difference in behaviour is already clear from the equations (3.15'). Indeed, these equations define the function $F^c_{\alpha\omega}(x-y)$ only for $x > y$ or $x < y$, whereas its value at $x = y$ remains indefinite. And in virtue of the well-known singularity of all F functions on the light-cone, this value at an individual point is essential for the construction of Fourier transforms. In other words, for a full definition of a T-product it is not sufficient to define it only for $x > y$ and $x < y$; we must also give the rules of its integration in the neighborhood of zero. Otherwise, the meaning of expressions of the type $T(j_{\rho'}(x)j_{\rho}(y))$ remains not entirely defined, which manifests itself in the origin of meaningless expressions diverging at large momenta.

The arbitrariness that arises during the integration of a T-product near zero is most simply expressed by adding to its definition in coordinate space a certain number of derivatives of $\delta(x-y)$ with indefinite coefficients (see, for example,), which will add to the right-hand side of (4.39) a certain polynomial in k^2:

$$g^c(k) = \int_{(3m)^2} \frac{I(z)\,dz}{z-k^2-i} + p(k^2). \qquad (4.39')$$

It is precisely the coefficients (which may be divergent) of this polynomial that are to compensate for the divergences in the integral. In practice this compensation is most simply performed by making use of the well known subtraction procedure.

Indeed, we would then obtain

$$\frac{1}{z'-k^2} = \frac{1}{z'-m^2-(k^2-m^2)} =$$

$$= \frac{1}{z'-m^2}\left\{1 + \frac{k^2-m^2}{z'-m^2} + \dots + \left(\frac{k^2-m^2}{z'-m^2}\right)^n\right\} + \frac{(k^2-m^2)^{n+1}}{(z'-k^2)(z'-m^2)^{n+1}},$$

and therefore

$$g^c(k) = i\ (k^2-m^2)^{n+1} \int_{(3m)^2}^{\infty} \frac{I(z)\,dz}{(z-m^2)^{n+1}(z-k^2-i\varepsilon)} + \sum_{0 \leqq m \leqq n} (k^2-m^2)^m \int_{(3m)^2}^{\infty} \frac{I(z)\,dz}{(z-m^2)^{m+1}} + p(k^2). \tag{4.40}$$

If we select n sufficiently large we might make the first integral on the right-hand side (4.40) convergent; and the divergent terms expanded in powers of (k^2-m^2) might be compensated by the polynomial $p(k^2)$, of which there would then remain only a finite polynomial, the same as in our "complex" spectral representations of the type (4.38).

Thus, also along this line we would finally arrive at the same relations (4.38), the derivation of which, however, would be less convincing due to the necessity of having to do "along the way" with divergent expressions. It is precisely in the possibility of escaping this difficulty entirely that we see the chief advantage of the method of argument we have chosen.

We shall show how the well known result of Lehmann-Kallen which refers to the spectral representation of the ordinary Green's function is obtained from our spectral representations for the variational derivatives of the scattering matrix.

The Green's function G(x, y) is usually defined as

$$G_{\rho\rho'}(x,y) = \delta_{\rho\rho'} G(x-y) = i \left\langle 0 \right| T(\varphi_{\rho'}(x) S \varphi_{\rho}(y) \left| 0 \right\rangle . \tag{4.41}$$

Using Wick's theorem for transformation of the T-product on the right-hand side, we obtain

$$\delta_{\rho\rho'} G(x-y) = i \, \overline{\varphi_{\rho'}(x) \, \varphi_{\rho}}(y) \langle 0 | S \; | 0 \rangle +$$

$$+ i \, \sum_{\rho'' \rho'''} \int dx' \, dy' \, \overline{\varphi_{\rho'}(x) \, \varphi_{\rho''}}(x') \langle 0 \left| \frac{\delta^2 S}{\delta \varphi_{\rho''}(x') \delta \varphi_{\rho'''}(y')} \right| 0 \rangle$$

$$\overline{\varphi_{\rho'''}(y') \, \varphi_{\rho}}(y), \tag{4.42}$$

where $\overline{\varphi_{\rho'}(x) \, \varphi_{\rho}}(y) = -i\delta \mathcal{D}^c(x-y)$

the usual chronological pairing for non-interacting operators, and

$$\mathcal{D}^c(x) = \frac{1}{(2\pi)^4} \int e^{-ikx} \mathcal{D}^c(k) dk, \quad \mathcal{D}^c(k) = \frac{1}{m^2-k^2-i\varepsilon} .$$

Passing in (4.42) to the Fourier transforms, we find

$$G(k) = \frac{1}{m^2-k^2-i\varepsilon} + \frac{1}{(m^2-k^2-i\varepsilon)^2} \; f^c(k) \tag{4.43}$$

whence, on the basis of (4.38),

$$G(k) = (1 + c_1)(m^2-k^2-i\varepsilon)^{-1} + \Sigma \, C_m (m^2-k^2)^{m-2} +$$

$$+ (m^2-p^2)^{n+1} \int_{(3m)^2}^{\infty} \frac{I(z) \, dz}{(m^2-z)^{n+1}(z-k^2-i\varepsilon)} . \tag{4.44}$$

The Kallen-Lehmann representation will now be obtained if we make another additional assumption to the effect that the "degree of increase" is equal to unity. Then

$$G(k) = (1 + C_1)(m^2+k^2-i\varepsilon)^{-1} + \int_{(3m)^2}^{\infty} \frac{\mathcal{J}(z) dz}{z-k^2 i} \; ; \; \mathcal{J}(z) = \frac{I(z)}{(z-m^2)^2} , \tag{4.45}$$

and the factor $(1 + C_1)$ may be eliminated by a finite renormalization:

$$G(p) \rightarrow (1 + C_1) \, G(p). \tag{4.46}$$

Section 5. The Vacuum Expectation Values of Fermion Radiation Operators of the Second Order[x/]

In Section 3 we found that of all the radiation operators not higher than the second order, only the operators of type (3.8) and (3.9) have vacuum expectation values different from zero. The first one of them was considered in the preceding section; we shall now investigate the vacuum expectation values

$$\left\langle 0 \right| \frac{\delta^2 S}{\delta \bar{\psi}(x) \delta \psi(y)} \cdot \overset{+}{S} \left| 0 \right\rangle = i\, \theta^{(c)}(x-y) \qquad (5.1)$$

In performing the differentiations with respect to the fermion fields, one must take into account their anti-commutativity. This will firstly lead to the left and right derivatives differing as to sign if an even function of the fermion operators is differentiated. For the sake of definiteness we shall deal henceforth only with the left-hand derivatives. Further, the anti-commutativity of the fields will lead to anti-commutativity of the derivatives in the case of multiple differentiation, for example,

$$\frac{\delta^2 A}{\delta \psi_1 \delta \psi_2} = - \frac{\delta^2 A}{\delta \psi_2 \delta \psi_1} . \qquad (5.2)$$

The formula for differentiation of a product will also change. If left-hand derivatives are used, it will take the form

$$\frac{\delta (AB)}{\delta \psi} = \frac{\delta A}{\delta \psi} B + (-1)^{\eta_A} A \frac{\delta B}{\delta \psi} , \qquad (5.3)$$

where η_A is the number of Fermi operators that enter into A multiplicatively. Finally, we may note that in performing Hermitian conjugation our left-hand derivatives will pass over to the right-hand side, and in order to

x/ The reader who is interested only in the derivation of dispersion relations may omit this section.

return them to our standard order we will need an additional change of sign, if an expression that is even in the spinors is differentiated.

As in the boson case, we shall establish a relation between (5.1) and the vacuum expectation value of a product of the currents. It is therefore useful first to determine the rules of conjugation for the two currents $\Lambda(x)$ and $\bar{\Lambda}(x)$ introduced above (/3.4/). In virtue of the unitarity of the transition matrix, the expression for $\Lambda(x)$ may be written in two forms

$$\Lambda(x) = -i \frac{\delta S}{\delta \bar{\psi}(x)} S^{+} = i S \frac{\delta S^{+}}{\delta \bar{\psi}(x)} . \qquad (5.4)$$

Performing the Dirac conjugation we find:

$$\bar{\Lambda}(x) = \Lambda^{+}(x)\beta = i S(\frac{\delta S}{\delta \bar{\psi}(x)})^{+}\beta = -i S(\frac{S^{+}\delta}{\delta \bar{\psi}(x)})\beta = -i S(\frac{\delta}{\delta \bar{\psi}(x)})^{+}\beta S^{+}$$

In order to determine the meaning of $\bar{\Lambda}(x)$ let us consider the local variation

$$\delta_x S = \delta \bar{\psi}(x) (\frac{\delta S}{\delta \bar{\psi}(x)}) + \delta \psi(x) (\frac{\delta S}{\delta \psi(x)})$$

Performing now the Hermitian conjugation, we find

$$\delta_x S^{+} = (\frac{\delta S}{\delta \bar{\psi}(x)})^{+} (\delta \bar{\psi}^{+}(x)) + (\frac{\delta S}{\delta \psi(x)})^{+} \delta(\psi(x))^{+} =$$

$$= (\frac{\delta S}{\delta \bar{\psi}(x)})^{+} \beta\, \delta\psi(x) + (\frac{\delta S}{\delta \psi(x)})^{+} \delta\bar{\psi}(x)\beta =$$

$$= \delta\psi(x) (\frac{\delta}{\delta \bar{\psi}(x)})^{+}\beta S^{+} + \delta\bar{\psi}(x) (\frac{\delta}{\delta \psi(x)})^{+}\beta S^{+}$$

But, on the other hand,

$$\delta_x S^{+} = \delta\bar{\psi}(x) \frac{\delta S^{+}}{\delta \bar{\psi}(x)} + \delta\psi(x) \frac{\delta S^{+}}{\delta \psi(x)}$$

Therefore

$$\left(\frac{\delta}{\delta\bar{\psi}(x)}\right)^{+}\beta = \frac{\delta}{\delta\psi(x)}; \quad \left(\frac{\delta}{\delta\psi(x)}\right)^{+}\beta = \frac{\delta}{\delta\bar{\psi}(x)} \tag{5.5}$$

Thus, for the expression Dirac-conjugate to (5.4) we obtain

$$\bar{\Lambda}(x) = i\,\frac{\delta S}{\delta\psi(x)}\,S^{+} = -i\,S\,\frac{\delta S^{+}}{\delta\psi(x)}\,. \tag{5.6}$$

which justifies the designations $\Lambda(x)$ and $\bar{\Lambda}(x)$.

If we now vary the expression $\left(\frac{\delta S}{\delta\psi(y)}\right)S^{+}$ with respect to $\bar{\psi}(x)$:

$$\frac{\delta^2 S}{\delta\bar{\psi}(x)\delta\psi(y)}\,S^{+} = \frac{\delta}{\delta\bar{\psi}(x)}\left(\frac{\delta S}{\delta\psi(y)}\,S^{+}\right) + \frac{\delta S}{\delta\psi(y)}\,\frac{\delta S^{+}}{\delta\bar{\psi}(x)}$$

and using the definitions of the currents (5.4, 6), we find that

$$\frac{\delta^2 S}{\delta\bar{\psi}(x)\delta\psi(y)}\,S^{+} = -i\,\frac{\delta\bar{\Lambda}(y)}{\delta\bar{\psi}(x)} - \bar{\Lambda}(y)\,\Lambda(x). \tag{5.7}$$

In the same way we obtain

$$\frac{\delta^2 S}{\delta\bar{\psi}(x)\delta\psi(y)}\,S^{+} = -\frac{\delta^2 S}{\delta\psi(y)\delta\bar{\psi}(x)}\,S^{+} = -i\,\frac{\delta\Lambda(x)}{\delta\psi(y)} + \Lambda(x)\,\bar{\Lambda}(y). \tag{5.8}$$

From (5.7), (5.8) there immediately follows

$$\Lambda(x)\bar{\Lambda}(y) + \bar{\Lambda}(y)\Lambda(x) = i\left(\frac{\delta\Lambda(x)}{\delta\psi(y)} - \frac{\delta\bar{\Lambda}(y)}{\delta\bar{\psi}(x)}\right), \tag{5.9}$$

- an analogy of the former commutation relation (3.16), where now the commutator is replaced by the anti-commutator. On the other hand, making use of the causality condition, we obtain

$$\frac{\delta^2 S}{\delta\bar{\psi}(x)\delta\psi(y)}\,S^{+} = \begin{cases} \Lambda(x)\bar{\Lambda}(y) & x \gtrsim y \\ -\bar{\Lambda}(y)\Lambda(x) & x \lesssim y \end{cases} \tag{5.10}$$

$$= T(\Lambda(x)\bar{\Lambda}(y)).$$

In complete analogy with the boson case, we determine vacuum matrix elements

$$\left\langle 0 \,\middle|\, T(\Lambda_{\alpha}(x)\,\bar{\Lambda}_{\beta}(y) \,\middle|\, 0\right\rangle = i\,\theta^{(c)}_{\alpha\beta}\,(x;y) \tag{5.11}$$

$$\langle 0 \,|\, \Lambda_\alpha(x)\, \overline{\Lambda}_\beta(y) \,|\, 0 \rangle = i\, \theta^{(-)}_{\alpha\beta} \;(x, y), \tag{5.12}$$

$$\langle 0 \,|\, \overline{\Lambda}_\beta(y)\, \Lambda_\alpha(x) \,|\, 0 \rangle = i\, \theta^{(+)}_{\alpha\beta} \;(x, y), \tag{5.13}$$

$$\langle 0 |\; \Lambda_\alpha(x)\, \overline{\Lambda}_\beta(y) + \overline{\Lambda}_\beta(y)\, \Lambda_\alpha(x) \,|\, 0 \rangle = i\, \theta_{\alpha\beta} \;(x,y) = $$
$$= i(\theta^{(-)}_{\alpha\beta} \;(x, y) + \theta^{(+)}_{\alpha\beta} \;(x, y)), \tag{5.14}$$

$$-i \langle 0 | \frac{\delta\, \overline{\Lambda}_\beta(y)}{\delta\, \overline{\psi}_\alpha(x)} | 0 \rangle = \langle 0 | \theta\,(x - y)[\, \Lambda_\alpha(x),\; \overline{\Lambda}_\beta\,(y)]_+ \,|\, 0 \rangle = $$
$$= i\, \theta^{(ret)}_{\alpha\beta} \;(x, y), \tag{5.15}$$

$$-i \langle 0 | \frac{\delta\, \Lambda_\alpha(x)}{\delta\, \psi_\beta(y)} | 0 \rangle = - \langle 0 | \theta\,(y - x)\; [\, \Lambda_\alpha(x), \overline{\Lambda}_\beta(y)]_+ \,|\, 0 \rangle = $$
$$= i\, \theta^{(adv)} \;(x, y). \tag{5.16}$$

We shall need the following relations between matrix elements of radiation operators that follow obviously from these definitions:

$$\theta^{(c)} \;(x, y) = \theta^{ret} \;(x, y) - \theta^{(+)} \;(x, y),$$
$$\theta^{(c)} \;(x, y) = 0^{adv} \;(x, y) + \theta^{(-)} \;(x, y), \tag{5.17}$$

$$\theta^{ret} \;(-x) = \theta^{adv} \;(x). \tag{5.18}$$

Of course, the retarded and advanced matrix elements have the property:

$$\theta^{ret}(x, y) = 0 \;\text{ for }\; x \lesssim y;\; \theta^{adv}(x, y) = 0 \;\text{ for }\; x \gtrsim y. \tag{5.19}$$

It is clear that due to translation and isotopic invariance, all the functions $\theta^{(?)}$ must have the form

$$\theta^{(?)}(x, y) = \delta_{st}\, \theta^{(?)} \;(x - y) \tag{5.20}$$

where s, t are isotopic (proton-neutron) indices, and $0^{(?)}(x-y)$ are ordinary spinor matrices. Further, in virtue of the invariance with respect to Lorentz transformations it is clear that $\theta^{(?)}(x-y)$ must in turn have the structure

$$\theta^{(?)}(x) = i \hat{\partial} \theta_1^{(?)}(x) + \theta_2^{(?)}(x) \qquad (5.21)$$

where $\theta_1^{(?)}$ and $\theta_2^{(?)}$ are scalar functions. Substituting in (5.17) the expression for $\theta^{(?)}(x, y)$ obtained from (5.20, 21) in terms of $\theta_1^{(?)}$, $\theta_2^{(?)}$, we immediately obtain relations between the scalar functions $\theta_1^{(?)}$ and $\theta_2^{(?)}$:

$$\theta_{1;2}^{(c)} = \theta_{1;2}^{(ret)}(x) - \theta_{1;2}^{(+)}(x),$$
$$\theta_{1;2}^{(c)} = \theta_{1;2}^{(adv)}(x) + \theta_{1;2}^{(-)}(x). \qquad (5.17')$$

Let us now consider the function $\theta^{(-)}(x)$. Using, as always, the completeness condition and translation invariance, we may write for it:

$$\theta_{\alpha\beta}^{(-)}(x-y) = \frac{-i}{(2\pi)^3} \sum_n \int d\bar{k} \langle 0 | \Lambda_\alpha(o) | n, \bar{k} \rangle \langle n, \bar{k} | \bar{\Lambda}_\beta(o) | 0 \rangle e^{-ik(x-y)} \qquad (5.22)$$

In virtue of (3.6), the vacuum term with n = 0 will be absent in the sum. In virtue of arguments precisely the same as those used in the boson case (following Eq. /4.2/) the sum will not include terms that correspond to intermediate states with one nucleon or an arbitrary number of mesons. Thus, the sum in (5.22) must begin with the term having n = 2, but not a two-meson term. Thus, a minimum mass of the intermediate state will be obtained if the state has one nucleon and one meson, that is

$$k^2 \geqslant (\mu + m)^2. \qquad (5.23)$$

Writing the expansion (5.22) in the form of a four-dimensional Fourier integral and defining the Fourier transform $\sigma^{(?)}(k)$ of the function $\theta^{(?)}(x)$ in our standard way, we see that

$$\sigma^{(-)}(k) = -2\pi i \sum_n \langle 0 | \eta_\alpha(x) | n, \bar{k} \rangle \langle n, \bar{k} | \bar{\eta}_\beta(o) | 0 \rangle \, \delta\left(k^o - \sqrt{\mu_n^2 + k^2}\right) \tag{5.24}$$

It is clear that the Fourier transform $\sigma^{(?)}(k)$ will have a matrix structure that is quite analogous to the matrix structure of $\theta^{(?)}(x)$;

$$\sigma^{(?)}(k) = (\gamma k)\, \sigma^{(?)}(k) + \sigma_2^{(?)}(k), \tag{5.25}$$

where $\sigma_1^{(?)}(k)$ and $\sigma_2^{(?)}(k)$ are scalar functions, which are precisely the Fourier transforms of the scalar functions $\theta_1^{(?)}(x)$ and $\theta_2^{(?)}(x)$.

Repeating the argument in Section 4 we see that the functions $\sigma_1^{(-)}$ and $\sigma_2^{(-)}$ are of the form

$$\sigma_{1;2}^{(-)}(k) = -2\pi i\, \theta(k^o).\, \rho_{1;2}(k^2), \tag{5.26}$$

where the spectral functions $\rho_1(k^2)$ and $\rho_2(k^2)$, depending only on k^2, are determined by the condition

$$\theta(k^o) \left\{ (\gamma k)\, \rho_1(k^2) + \rho_2(k^2) \right\} = \tag{5.27}$$

$$= \sum_n \langle 0 | \eta_\alpha(o) | n\, \bar{k} \rangle \langle n\, \bar{k} | \bar{\eta}_\beta(o) | 0 \rangle\, \theta(k^o)\, 2\sqrt{\bar{k}^2 + \mu_n^2}.\, \delta(k^2 - \mu_n^2).$$

In order to separate, on the right-hand side, the terms that refer to ρ_1 and ρ_2, let us multiply (5.27) by γ^o; then taking the Spur and replacing Dirac conjugation by Hermitian conjugation, we shall have

$$\theta(k^o) k^o \rho_1(k^2) = \frac{\theta(k^o)}{2}\, k^o \sum_{n,\alpha} \left| \langle 0 | \eta_\alpha(o) | n\, \bar{k} \rangle \right|^2 \delta(k^2 - \mu_n^2). \tag{5.28}$$

The factors $\theta(k^o)k^o$ cancel, leaving

$$\rho_1(k^2) = \frac{1}{2} \sum_{n,\alpha} \left| \langle 0 | \Lambda_\alpha(o) | n, \bar{k} \rangle \right|^2 \delta(k^2-\mu_n^2), \quad (5.29)$$

whence it may be seen that

1. $\rho_1(k^2) \geqslant 0$

2. $\rho_1(k^2) = 0$ for $k^2 \leq (\mu + m)^2$, (from /5.23/). (5.30)

In order to find an expression for $\rho_2(k^2)$, we take the Spur of (5.27) directly,

$$\rho_2(k^2) = \frac{\theta(k^o)k^o}{2} \sum_{n,\alpha,\beta} \langle o | \Lambda_\alpha(o) | n \bar{k} \rangle \langle n k | \Lambda^*_\beta(o) | 0 \rangle \gamma^o_{\beta\alpha} \delta(k^2-\mu_n^2).$$

Using the representation $\gamma^o = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ for the Dirac matrices, this gives

$$\theta(k^o)\rho_2(k^o) = \frac{\theta(k^o)k^o}{2} \sum_{n;\alpha=1,2} \left| \langle 0 | \Lambda_\alpha(o) | n \bar{k} \rangle \right|^2 \delta(k^2-\mu_n^2) - \quad (5.31)$$
$$- \frac{\theta(k^o)k^o}{2} \sum_{n;\alpha=3,4} \left| \langle 0 | \Lambda_\alpha(o) | n \bar{k} \rangle \right|^2 \delta(k^2-\mu_n^2)$$

By adding and subtracting (5.28) and (5.31) we will arrive at two more inequalities

$$\theta(k^o)[k^o \rho_1(k^2) + \rho_2(k^2)] \geq 0; \quad \theta(k^o)[k^o \rho_1(k^2) - \rho_2(k^2)] \geq 0.$$

If we now note that

$$k^o = \sqrt{\vec{k}^2 + m^2} = \sqrt{\vec{k}^2 + k^2} \geq + \sqrt{k^2} = |k|$$

we will see that the condition, necessary and sufficient for these inequalities to hold obtained (the inequalities must hold in any frame of reference), is

$$|k| \rho_1(k^2) + \rho_2(k^2) \geq 0; \quad |k| \rho_1(k^2) - \rho_2(k^2) \geq 0.$$

Thus the function $\rho_2(k^2)$ must satisfy the conditions:

$$1. \quad -|k|\rho_1(k^2) \leq \rho_2(k^2) \leq |k|\rho_1(k^2), \qquad (5.32)$$

$$2. \quad \rho_2(k^2) = 0 \text{ for } k^2 < (\mu + m)^2, \text{ (from/5.23).}$$

Thus, we have found a spectral representation for the functions $\sigma_1^{(-)}$ and $\sigma_2^{(-)}$:

$$\sigma_{1;2}^{(-)}(k) = -2\pi i \int_{(\mu+m)^2}^{\infty} \theta(k^o)\delta(k^2-\mu^2)\rho_{1;2}(\mu^2)\, d\mu^2, \qquad (5.33)$$

which is entirely analogous to the spectral representation (4.9) for the function $g^{(-)}(k)$.

In order to pass on to the construction of spectral representations for other functions $\sigma^{(?)}(k)$, we shall first have to establish several more relations between the functions of various upper indices, which follow from invariance with respect to charge conjugation. The condition of such invariance may be written in the form

$$\langle 0 | \bar{\Omega}_\beta(y)\, \Omega_\alpha(x) | 0 \rangle = \langle 0 | \bar{\Omega}'_\beta(y)\, \Omega'_\alpha(x) | 0 \rangle \qquad (5.34)$$

where the charge-conjugate operators $\bar{\Omega}'$, Ω' are connected with $\bar{\Omega}, \Omega$ by the well-known relations

$$\Omega'(x) = c\,\bar{\Omega}(x) = \bar{\Omega}(x)\, c^t; \quad \bar{\Omega}'(x) = c^{-1}\Omega(x), \qquad (5.35)$$

and the following conditions are fulfilled by the matrix C:

$$\overset{t}{C}\, C^{-1} = -1; \; C^t = -C; \; C^{-1}\gamma^k C = -\overset{t}{\gamma}{}^k. \qquad (5.36)$$

Applying condition (5.34) to the product on the left-hand side of (5.13) and utilizing (5.35) and (5.36), we shall (after rather simple matrix transformations) arrive at the relation:

$$\theta^{(+)}(x) = -(C^{-1}\theta^{(-)}(-x)\,C)^T \quad (5.37)$$

which establishes the connection between negative - and positive - frequency functions. The obvious linear relations between different type functions will make it possible for us to deduce immediately from (5.37) another series of relations, from among which we shall write only the following:

$$\theta^{ret}(x) = (C^{-1}\ \theta^{adv}(-x)\ C)^T, \quad (5.38)$$

which we shall need further on.

If with the aid of (5.20, 21) we write $\theta^{(?)}(x, y)$ using $\theta^{(?)}_{1;2}(x-y)$ and utilize (5.36) in order to get rid of the matrix C and then separate the parts containing and not containing the matrixes (which may be done by evaluating the traces), we will obtain from (5.37) and (5.38) the corresponding "parity relations" for the scalar functions $\theta^{(?)}_{1;2}$:

$$\theta^{(+)}_{1;2}(x) = -\theta^{(-)}_{1;2}(-x);\ \theta^{adv}_{1;2}(x) = \theta^{ret}_{1;2}(-x).$$

Taking Fourier transforms we have

$$\sigma_{1;2}^{(+)}(k) = -\sigma_{1;2}^{(-)}(-k), \quad (5.39)$$

$$\sigma_{1;2}^{adv}(k) = +\sigma_{1;2}^{ret}(-k). \quad (5.40)$$

The relation (5.39) makes it possible to write immediately the spectral representations for the functions $\sigma_{1;2}^{(+)}(k)$:

$$_{1;2}^{(+)}(k) = 2\pi i\ \theta(-k^o)\,\rho_{1;2}(k^2) =$$

$$= 2\pi i \int_{(\mu+m)^2}^{\infty} \theta(-k^o)\ \delta(k^2-\mu^2)\,\rho_{1;2}(\mu^2)\ d\,\mu^2. \quad (5.41)$$

Substituting this spectral representation as well as (5.26) in the relations obtained from (5.17') by taking Fourier transforms, we obtain:

$$\sigma_{1;2}^{(c)}(k) = -2\pi i\,\theta(k^0)\,\rho_{1;2}(k^2) + \sigma_{1;2}^{adv}(k),$$
$$\sigma_{1;2}^{(c)}(k) = -2\pi i\,\theta(-k^0)\,\rho_{1;2}(k^2) + \sigma_{1;2}^{ret}(k). \qquad (5.42)$$

Thus, we find that (as in the boson case) at small momenta $k^2 < (\mu+m)^2$ the Fourier transforms of all three "Green-like" functions coincide:

$$\sigma^{(c)}(k) = \sigma^{ret}(k) = \sigma^{adv}(k), \text{ if } k^2 < (\mu+m)^2. \qquad (5.43)$$

Since we have established the spectral representations (5.26), (5.41), the properties of the spectral functions (5.30.2) and (5.32.2), equations (5.42) and (5.40), we may now repeat word for word the conclusion of the preceding section. For this reason, let us copy out at once the "complex" spectral representations for the Green-like functions $\sigma_{1;2}^{(c)}$, $\sigma_{1;2}^{ret}$ and $\sigma_{1;2}^{adv}$:

$$\sigma_{1;2}^{(c)}(p) = -(p^2-\mu^2)^{n+1}\int_{(\mu+m)^2}^{\infty}\frac{\rho_{1;2}(\zeta)\,d\zeta}{(\zeta-\mu^2)^{n+1}(\zeta-p^2-i\varepsilon)} + \sum_{j=0}^{n}\mathfrak{a}_j^{(1;2)}(p^2-\mu^2)^j, \qquad (5.44)$$

and

$$\sigma_{1;2}^{\substack{ret\\adv}}(p) = -(p^2-\mu^2)^{n+1}\int_{(\mu+m)^2}^{\infty}\frac{\rho_{1;2}(\zeta)\,d\zeta}{(\zeta-\mu^2)^{n+1}(\zeta-p^2\mp i\varepsilon p^0)}\ \sum_{j=0}^{n}\mathfrak{a}_j^{(1;2)}(p^2-\mu^2)^j \qquad (5.45)$$

In practice, it is more convenient to have the spectral representations written in a slightly different form: in place of $\rho_{1;2}$ let us introduce two functions that are non-negative in virtue of (5.30.32):

$$\mathcal{J}_1(\nu) = \frac{\rho_1(\nu^2) - \frac{\rho_2(\nu^2)}{\nu}}{2} \quad ; \quad \mathcal{J}_2(\nu) = \frac{\rho_1(\nu^2) + \frac{\rho_2(\nu^2)}{\nu}}{2} \tag{5.46}$$

where $\nu = +\sqrt{k^2} = |k|$. Then

$$(k\gamma)\rho_1(\nu^2) + \rho_2(\nu^2) = (k\gamma - \nu)\mathcal{J}_1(\nu) + (k\gamma + \nu)\mathcal{J}_2(\nu),$$

(here ν is considered a variable independent of k), and if we construct combinations

$$\Sigma^{(?)}(k) = (k\gamma)\,\sigma_1^{(?)}(k) + \sigma_2^{(?)}(k),$$

which are, as it is easy to see, Fourier transforms of $\Theta^{(?)}(x)$ (the index (?) denotes (C), (adv) or (ret)), then for the full Green-like function $\Sigma^{(?)}(k)$ we will obtain the spectral representation

$$\Sigma^{(?)}(k) = -(k^2-\mu^2)^{n+1} \int_{(\mu+m)^2}^{\infty} \frac{(\hat{k}-\nu)\mathcal{J}_1(\nu) + (\hat{k}+\nu)\mathcal{J}_2(\nu)}{(\nu^2-\mu^2)^{n+1}\,(\nu^2-k^2)}\, d\nu^2 +$$
$$+ (\hat{k}\,\mathcal{U}_0^{(1)} + \mathcal{U}_0^{(2)}) + (\hat{k}\,\mathcal{U}_1^{(1)} + \mathcal{U}_1^{(2)})\,(k^2-\mu^2) + \ldots \tag{5.47}$$

If we wish to establish the spectral representations for the individual functions Σ^c, Σ^{ret} and Σ^{adv}, we need only select the proper rule for circumventing the pole $k^2 = \mu^2$.

In order to reduce the representation (5.47) to a still more transparent form, we may note that the differences

$$\frac{(k^2-\mu^2)^{n+1}}{(\nu^2-\mu^2)^{n+1}}\ \frac{1}{\nu-\hat{k}} - \frac{(\hat{k}-\mu)^{2n+2}}{(\nu-\mu)^{2n+2}}\ \frac{1}{\nu-\hat{k}}$$

and

$$\frac{(k^2-\mu^2)^{n+1}}{(\nu^2-\mu^2)^{n+1}}\ \frac{1}{\nu+\hat{k}} - \frac{(\hat{k}-\mu)^{2n+2}}{(\nu+\mu)^{2n+2}}\ \frac{1}{\nu+\hat{k}}$$

are, with respect to $\hat{k}$, polynomials of degree (2n + 1). Indeed, by reducing both terms of such a difference to a common denominator and extracting the linear factor from the difference of (n+1)' th powers in the numerator, we find that it necessarily contains a factor $(\nu-\hat{k})$ (or $(\nu+\hat{k})$), which will cancel the only factor of the denominator which depends on $\hat{k}$. Therefore, if we form such differences under the integral in (5.47), then the powers of $\hat{k}$ in each term of the polynomial may be placed outside the integral, and the integration with respect to ν will give a pure number, that is, integration of such a difference will simply lead to a polynomial of degree (2n + 1) in $\hat{k}$.

This enables us to transform (5.47) to:

$$\Sigma(k) = -(\hat{k}-\mu)^{2n+2} \int \left\{ \frac{I_1(\nu)}{\nu+\hat{k}} + \frac{I_2(\nu)}{\nu-\hat{k}} \right\} d\nu + \\ + \mathcal{B}_o + \mathcal{B}_1(\hat{k}-\mu) + \dots + \mathcal{B}_{2n+1}(\hat{k}-\mu)^{2n+1}, \tag{5.48}$$

where the following new spectral functions were introduced

$$I_1(\nu) = \frac{2\,J_1(\nu).\nu}{(\nu+\mu)^{2n+2}} \quad \text{and} \quad I_2(\nu) = \frac{2\,J_2(\nu).\nu}{(\nu-\mu)^{2n+2}} \quad . \tag{5.49}$$

As in the boson case, it may be proved that

$$\mathcal{B}_o = 0.$$

For this purpose it is sufficient to consider the matrix element of S between two one-nucleon states, just as in Section 4 we considered the matrix element of S between two one-meson states. Finally, the relation (5.18) leads to the conclusion that

$$\text{all } \mathcal{B}_m \text{ are real.} \tag{5.50}$$

The representation of Kallen-Lehmann for the fermion Green's function might be obtained again by means of considerations entirely analogous to those used in the preceding section, with the additional assumption that the degree of increase n = 0. Here we again come face to face with the interesting fact that in our system of conditions (Section 2) we have to give the "degree of increase" of the spectral function instead of the form of the Lagrangian.

Section 6. The Construction of Dispersion Relations.

This section is devoted to the derivation of concrete dispersion relations for a definite process: the scattering of mesons by nucleons. To simplify the argument, we shall first obtain them by a simple method, which in essence is entirely analogous to that used by a number of authors. Notwithstanding its simplicity (or rather due to it) this method has certain defects: at several points one is forced to carry out mathematically incorrect operations. These places will be noted below, and in Section 7 we shall give the rigorous derivation of dispersion relations, which should be free from such incorrect steps. It should be emphasized that in deriving the dispersion relations we shall not refer at any point to conventional theory, but will proceed only from our basic principles formulated in Sec. 2.

Thus, we shall consider the problem of the scattering of π-mesons described by the real field operators $\varphi_\rho(x)$ (ρ is the isotopic index; charge symmetric theory is considered) on nucleons described by the spinor field $\psi(x) = \begin{pmatrix}\psi_P(x)\\ \psi_N(x)\end{pmatrix}$. We shall assume that before scattering the nucleon is in a certain definite state, characterized by momentum p and spin and isotopic quantum numbers which as a whole are designated by s; the values of the same quantities after collision will be denoted by a prime: p', s'. Similarly, the momentum and isotopic index of the π-meson before the collision will be designated by q and ρ, and after the collision by q' and ρ'.

It will be convenient to select from among all the quantum numbers the momentum q (q') of the meson, designating the totality of the remaining numbers for the initial (final) state by one letter

$$\alpha = (p, s, \rho) \qquad (w = (p', s', \rho')) .$$

We will assume that $q = q'$.

Then the transition matrix element will be written as follows, using normalization conventional for scattering theory:

$$
\begin{aligned}
S(\alpha, q; w, q') &= (2\pi)^3 \left\langle \vec{p}'s', \vec{q}'\rho' \right| S \left| \vec{p}s, \vec{q}\rho \right\rangle = \\
&= (2\pi)^3 \left\langle \vec{p}'s' \right| a_{\rho'}^{(-)}(\vec{q}) S a_{\rho}^{(+)}(\vec{q}) \left| \vec{p}\, s \right\rangle ; \\
p^0 &= \sqrt{\vec{p}^2 + M^2}, \ldots, q^0 = \sqrt{\vec{q}^2 + m^2}, \ldots
\end{aligned}
\tag{6.1}
$$

where we made use of (2.10). With the aid of assumption 2.3 from Sec. 2, performing a transformation of the type (2.20-22), we reduce this matrix element to the form:

$$
\begin{aligned}
S(\alpha, q; w, q') &= \int dx dy \frac{e^{i(q'x-qy)}}{\sqrt{2q_0'q_0}} \left\langle p's' \right| \frac{\delta^2 S}{\delta\varphi_{\rho'}(x)\,\delta\varphi_{\rho}(y)} S^+ \left| ps \right\rangle \\
p^0 &= \sqrt{\vec{p}^2 + M^2}, \ldots, q^0 = \sqrt{\vec{q}^2 + m^2}, \ldots
\end{aligned}
\tag{6.2}
$$

Now let us return to the consideration of the functions $F_{\alpha w}^{(c)}(x) - F_{\alpha w}^{(+)}(x)$ introduced in Sec. 3; now, however, we shall assume that the states $|ps\rangle$ and $|p's'\rangle$, between which the matrix elements are taken, are precisely the initial and final states of the nucleon. Let us introduce (in the same way as in Sec. 4 (Eq./4.4/)) the Fourier transforms of these functions:

$$
F_{\alpha w}^{(?)}(x) = \frac{1}{(2\pi)^4} \int dk\, e^{-ikx}\, T_{\alpha w}^{(?)}(k) . \tag{6.3}
$$

Now substituting in (6.2) the expression (3.20) for the matrix element of the causal radiation operator, and passing in it with the aid of (6.3) to the Fourier transform, we obtain for $S(\alpha, q; w, q')$

$$S(\alpha,q;\ w,q') = \frac{-i(2\pi)^4}{\sqrt{4q_o q_o'}}\ \delta(q + p - q' - p')T^c_{\alpha w}(\frac{q+q'}{2})$$
$$p^o = \sqrt{\vec{p}^2 + M^2},\ \dots,\ q^o = \sqrt{\vec{q}^2 + m^2}\ . \qquad (6.4)$$

For further argument it will be convenient to introduce an auxiliary function, the "retarded" matrix element

$$H(\alpha,q;\ w,q') = \int dxdy\ \frac{e^{i(q'x-qy)}}{\sqrt{4q^o q^{o'}}}\ \left\langle p's'\right|\ i\ \frac{\delta\ \dot{j}_\rho(y)}{\delta\ \varphi_{\rho'}(x)}\left|\ ps\right\rangle$$
$$= \frac{-i(2\pi)^4}{\sqrt{4q^o q^{o'}}}\ \delta\ (q + p - q' - p')T^{ret}_{\alpha w}\ (\frac{q+q'}{2})\ . \qquad (6.5)$$

It may be noted now that in virtue of (3.33.2)

$$T^c_{\alpha w}(\frac{q+q'}{2}) - T^{ret}_{\alpha w}\ (\frac{q+q'}{2}) = \mathcal{P}_{\rho\rho'}\ T^{(-)}_{\alpha w}(-\frac{q+q'}{2}) =$$
$$= \int e^{-i\frac{q+q'}{2}x}\ \mathcal{P}_{\rho\rho'}\ F^{(-)}_{\alpha w}(x)dx\ . \qquad (6.6)$$

Substituting in (6.6) the expression (3.30) for $F^{(-)}_{\alpha w}(x)$ and carrying out the integration, we obtain for the difference (6.6)

$$T^c_{\alpha w}(\frac{q+q'}{2}) - T^{ret}_{\alpha w}(\frac{q+q'}{2}) = 2\pi i\ \sum_n \delta\left(\sqrt{\vec{K}^2 + M_n^2} + \frac{q^o + q^{o'}}{2} - \frac{p^o + p^{o'}}{2}\right)$$
$$\left\langle p's'\right|\ j_\rho\ (0)\left|\ \vec{K}n\right\rangle\left\langle\ \vec{K}n\right|\ j_{\rho'}\ (0)\left|\ \vec{p}s\right\rangle\Big|_{\vec{K} = \frac{\vec{p} + \vec{p}' - \vec{q} - \vec{q}'}{2}} \qquad (6.7)$$

Therefore, if we form the difference S - H, in which the δ-function $\delta(q+p-q'-p')$ expresses the conservation of four-momentum, then the argument of the δ-function that enters in (6.7) will be equal to

$$\sqrt{M_n^2 + (\vec{p} - \vec{q})^2} + q^{o'} - p^o\ ,$$

which, if we take account of the expressions for $q^{o'}$ and p^o in terms of momenta, will equal

$$\sqrt{M_n^2 + (\vec{p} - \vec{q}')^2} + \sqrt{m^2 + \vec{q}'^2} - \sqrt{M^2 + \vec{p}^2} \; ,$$
$$M_n \geq M + m \; . \tag{6.8}$$

We will now assume that the system of a nucleon and meson does not have bound states with a mass less than the mass of the nucleon. Then the latter expression is essentially positive, and consequently the argument (6.8) of the δ-function in (6.7) can nowhere be zero.

Thus we proved that if the matrix elements S and H are taken for real particles, for which the momentum fixes the magnitude and positive sign of the energy, and the conservation of four-momentum is fulfilled, then

$$T^c_{\alpha w}(K) = T^{ret}_{\alpha w}(K) \; , \tag{6.9}$$

that is, the matrix element $S(\alpha,q;\, w,q')$ may be replaced by the matrix element $H(\alpha,q;\, w,q')$.

Let us consider in more detail the function $T^{ret}_{\alpha w}(K)$. First of all, dividing it into Hermitian and anti-Hermitian parts

$$T^{ret}_{\alpha w}(K) = D_{\alpha w}(K) + iA_{\alpha w}(K) \; , \tag{6.10}$$

and comparing it with (3.42), we see that

$$D_{\alpha w}(K) = \bar{T}_{\alpha w}(K); \; A_{\alpha w}(K) = \frac{1}{2i} T_{\alpha w}(K) \; . \tag{6.11}$$

Therefore, taking Fourier transforms, of the symmetry relations (3.43-46) for the functions $\bar{F}(x)$ and $F(x)$, we obtain at once symmetry relations for the

Hermitian and anti-Hermitian parts of the function $T^{ret}_{\alpha w}(K)$,

$$(1 + P_{\rho\rho'})D_{\alpha w}(\tfrac{q+q'}{2}) = (1 + P_{\rho\rho'})D_{\alpha w}(-\tfrac{q+q'}{2}) \;, \tag{6.12.1}$$

$$(1 - P_{\rho\rho'})D_{\alpha w}(\tfrac{q+q'}{2}) = -(1 - P_{\rho\rho'})D_{\alpha w}(-\tfrac{q+q'}{2}) \;, \tag{6.12.2}$$

$$(1 + P_{\rho\rho'})A_{\alpha w}(\tfrac{q+q'}{2}) = -(1 + P_{\rho\rho'})A_{\alpha w}(-\tfrac{q+q'}{2}) \;, \tag{6.12.3}$$

$$(1 - P_{\rho\rho'})A_{\alpha w}(\tfrac{q+q'}{2}) = (1 - P_{\rho\rho'})A_{\alpha w}(-\tfrac{q+q'}{2}) \;. \tag{6.12.4}$$

Let us now determine how $D_{\alpha w}$ and $A_{\alpha w}$ may be found using only the function $F^{(-)}(x)$, the only one for which we have an explicit expression (3.30). For the anti-Hermitian part, the relation (3.34a) gives us at once

$$A_{\alpha w}(\tfrac{q+q'}{2}) = \frac{1}{2i}\,T_{\alpha w}(\tfrac{q+q'}{2}) = \frac{1}{2i}\int e^{i\frac{q+q'}{2}x}\,(F^{(-)}_{\alpha w}(x) - P_{\rho\rho'}F^{(-)}_{\alpha w}(-x))dx \;. \tag{6.13}$$

Comparing now equations (3.33) and (3.34), we see that due to the causality condition in the form (3.13-14) $\bar{F}(x) = \frac{1}{2}F(x)$ when $x > 0$, and $\bar{F}(x) = -\frac{1}{2}F(x)$ when $x < 0$. These relations may be combined as

$$\bar{F}_{\alpha w}(x) = \frac{\epsilon(x)}{2}F(x) \;, \tag{6.14}$$

whence we at once obtain

$$D_{\alpha w}(\tfrac{q+q'}{2}) = \bar{T}_{\alpha w}(\tfrac{q+q'}{2}) = \frac{1}{2}\int e^{i\frac{q+q'}{2}x}\,\epsilon(x)\,(F^{(-)}_{\alpha w}(x) - P_{\rho\rho'}F^{(-)}_{\alpha w}(-x))dx \;. \tag{6.15}$$

We may point out here that at this point we meet with the first inaccuracy of the usual derivation. Indeed, from (3.33-34) we are able to get only a relation between the functions D and $F^{(-)}$ for $x > 0$, $x < 0$, but not at $x = 0$. But the function $F^{(-)}$ is strongly singular at zero, therefore its multiplication by the discontinuous function $\epsilon(x)$ is not permissible until the rules for integrating such an expression near zero are worked out in full. The absence of such rules may lead to divergences. Here, we again have the same situation as in writing (3.15) in the form of a T-product. Let us recall that in conducting the analysis of vacuum matrix elements in Sec. 4 we discovered that such a situation arises each time we want to pass from "non-Green-like" functions to "Green-like" functions, and that in this case there actually does arise a certain arbitrariness, which may be expressed by adding to the Fourier transform an arbitrary polynomial. Therefore, in place of (6.15) it would be more accurate to write:

$$D_{\alpha w}\left(\frac{q+q'}{2}\right) = \frac{1}{2}\int e^{i\frac{q+q'}{2}x}\epsilon(x)\left\{F^{(-)}_{\alpha w}(x) - P_{\rho\rho'}F^{(-)}_{\alpha w}(-x)\right\}dx + \mathcal{P}^{n}_{\alpha w}\left(\frac{q+q'}{2}\right) \quad (6.15')$$

with an arbitrary polynomial of degree n, $\mathcal{P}^{n}_{\alpha w}\left(\frac{q+q'}{2}\right)$. It may be noted that the origin of this polynomial is essentially connected with the behavior of $T_{\alpha w}\left(\frac{q+q'}{2}\right)$ at large $\frac{q^o+q^{o\prime}}{2}$. Indeed, if $T_{\alpha w}\left(\frac{q+q'}{2}\right)$ decreases sufficiently rapidly when $\frac{q^o+q^{o\prime}}{2} \longrightarrow \infty$, then the Fourier transform inverse to (6.13) would define a sufficiently regular function $F_{\alpha w}(x)$, and the multiplication by $\epsilon(x^o)$ would not lead to any arbitrariness. If however $T_{\alpha w}\left(\frac{q+q'}{2}\right)$ increases at infinity, the function $F_{\alpha w}(x)$ becomes singular at zero and its multiplication by $\epsilon(x)$ is devoid of any direct meaning. We have to give meaning to such a product by means of a certain regularization procedure; and here it is that the polynomial $\mathcal{P}_{\alpha w}$ arises. It is clear that the degree of this polynomial is determined by the order of increase of $T_{\alpha w}\left(\frac{q+q'}{2}\right)$ when $\frac{q^o+q^{o\prime}}{2}$ tends to ∞.

Before proceeding further, we must go into the details of our system of coordinates. The usual center-of-mass coordinate system proves inconvenient in this case, because it leads to additional singularities in the energy dependence. For this reason we will make use of the now generally accepted system, in which the sum of the momenta of the nucleon, before and after scattering, is equal to zero,

$$\vec{p} + \vec{p}\,' = 0 . \tag{6.16}$$

This system reduces to the laboratory system when considering forward scattering. (Below we shall deal not with functions of arbitrary combinations of nucleon and meson momenta, but only with functions of momenta that satisfy the conservation laws, in accordance with the δ-function in the definitions of $S(\alpha,q,w,q')$ and $H(\alpha,q;w,q')$.)

In the selected system $\vec{p}^{\,2} = \vec{p}\,'^2$ and, in virtue of energy conservation, also $\vec{q}^{\,2} = \vec{q}\,'^2$. From momentum conservation we obtain then

$$\vec{p} = \frac{\vec{q}\,' - \vec{q}}{2} \quad \text{and} \quad (\vec{q} + \vec{q}\,')\vec{p} = 0 . \tag{6.17}$$

Therefore we may put

$$\frac{\vec{q}\,' + \vec{q}}{2} = \lambda\vec{e} \tag{6.18}$$

where $\vec{e}$ is a unit vector normal to $\vec{p}$, $e^2 = 1$, $\vec{e}\cdot\vec{p} = 0$. With a given $\vec{p}$, the vector $\vec{e}$ may be considered as fixed. For second variable we choose the scalar λ. Then

$$\vec{q} = -\vec{p} + \lambda\vec{e} ; \quad \vec{q}\,' = \vec{p} + \lambda\vec{e} ,$$
$$q^o = q^{o\,'} = \sqrt{m^2 + \vec{p}^{\,2} + \lambda^2} \; ; \quad \vec{q}\,'^2 = \vec{q}^{\,2} = \vec{p}^{\,2} + \lambda^2 . \tag{6.19}$$

In place of λ we may also use the variable

$$E = \sqrt{m^2 + \vec{p}^{\,2} + \lambda^2} \tag{6.20}$$

which is the energy of the meson simply.

In the chosen frame of reference, the expressions (6.13, 15) for $A_{\alpha w}$ and $D_{\alpha w}$ will have the form:

$$D_{\alpha w}(E, \vec{e}) = \frac{1}{2} \int e^{i(Ex^0 - \lambda \vec{e}\vec{x})} \epsilon(x^0) g(x) dx \tag{6.21}$$

and

$$A_{\alpha w}(E, \vec{e}) = \frac{1}{2i} \int e^{i(Ex^0 - \lambda \vec{e}\vec{x})} g(x) dx , \tag{6.22}$$

where we put

$$g(x) = F_{\alpha w}^{(-)}(x) - P_{\rho\rho'} F_{\alpha w}^{(-)}(-x) . \tag{6.23}$$

The function g(x) is, strictly speaking, a function not only of x, but also of the nucleon momenta $\vec{p}$ and $\vec{p}\,'$. However, since in our coordinate system $\vec{p}\,'$ depends only on $\vec{p}$, we may (by fixing $\vec{p}$) consider g(x) as a function only of x. The expression that enters in the exponents in (6.20, 21) and is considered as a function of E, will, in virtue of (6.20), have branch-points. In order to exclude them, let us introduce the operations of symmetrization with respect to $\vec{e}$ and antisymmetrization with respect to $\vec{e}$ with division by λ, and let us put for any $f(\lambda, \vec{e})$,

$$\mathcal{S}_{\vec{e}} f(\lambda, \vec{e}) = f(\lambda, \vec{e}) + f(\lambda, -\vec{e}) , \tag{6.24}$$

and

$$\mathcal{A}_{\vec{e}} f(\lambda, \vec{e}) = \frac{1}{\lambda} \; f(\lambda, \vec{e}) - f(\lambda, -\vec{e}) . \tag{6.25}$$

Combining now the expressions (6.21) and (6.22), we obtain an integral representation for T^{ret},

$$T_{\alpha w}^{ret}(E, \vec{e}) = \int e^{i(Ex^0 - \lambda \vec{e}\vec{x})} \theta(x^0) g(x) dx . \tag{6.26}$$

Applying the operations $\mathcal{S}_{\vec{e}}$ and $\mathcal{U}_{\vec{e}}$ we obtain two new integral representations,

$$\mathcal{S}_{\vec{e}}\, T^{ret}_{\alpha w}(E, \vec{e}) = 2 \int e^{iEx^o} \cos(\lambda\vec{e}\vec{x})\,\theta(x^o) g(x) dx \tag{6.27}$$

and

$$\mathcal{U}_{\vec{e}}\, T^{ret}_{\alpha w}(E, \vec{e}) = \frac{2}{i} \int e^{iEx^o} \frac{\sin(\lambda\vec{e}\vec{x})}{\lambda}\,\theta(x^o) g(x) dx\ . \tag{6.28}$$

These integral representations will serve as the basis for deriving dispersion relations. For this purpose, let us consider two identities,

$$e^{i\nu t} \cos\sqrt{\nu^2 - \mu^2} = \frac{1}{i\pi} P \int_{-\infty}^{\infty} \frac{e^{i\omega t} \cos\sqrt{\omega^2-\mu^2}}{\omega - \nu}\, d\omega \tag{6.29}$$

and

$$e^{i\nu t} \frac{\sin\sqrt{\nu^2-\mu^2}}{{}^2-\mu^2} = \frac{1}{i\pi} P \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega - \nu} \cdot \frac{\sin\sqrt{\omega^2-\mu^2}}{\sqrt{\omega^2-\mu^2}}\, d\omega, \tag{6.30}$$

the validity of which, when the conditions

$$t > |\xi| > 0 \tag{6.31}$$

are fulfilled, is easily checked by calculating the integrals on the right-hand side by residue theory. In evaluating (6.31), it is always possible to select a radius of a semicircular contour so large that the negative term in the exponent (which originates from $e^{i\omega t}$) is greater than a possible positive term arising from the sine or cosine.

On the right-hand side of (6.27) let us now substitute in place of the expression $e^{iEx^o} \cos(\lambda\vec{e}\vec{x})$ its integral representation (6.29):

$$\mathcal{S}_{\vec{e}}\, T^{ret}_{\alpha w}(E,\vec{e}) =$$

$$= \frac{2}{i\pi} \int dx\,\theta(x^o) g(x) P \int_{-\infty}^{\infty} dE' \frac{e^{iE'x^o} \cos\sqrt{E'^2 - \mu^2\vec{e}\vec{x}}}{E' - E}\ . \tag{6.32}$$

To what extent is this substitution justified? The integral representation

(6.29) is valid when the condition (6.31) holds, that is (since the modulus $|\vec{e}|$ is equal to unity) when $x^0 > |\vec{x}| > 0$. Points with negative x^0 do not occur in the integral (6.27) due to the function $\theta(x^0)$ under the integral. Again there will be no points with $|\vec{x}| > x^0 > 0$, since by the causality condition (3.13, 14) the function $g(x)$ is zero outside the light cone. Thus, the only "dangerous" points (in the sense of the use of the integral representation /6.29/ being justified) in the integral over x in (6.32) are the points of the light cone with $x^0 = |\vec{x}| > 0$. For the integral in (6.32) over E', such points do present a difficulty, since at $x^0 = |\vec{x}|$ this integral over the real axis will be divergent, and consequently there will be doubts as to the validity of using the integral representation (6.29). We may however imagine that a subtraction procedure of the type explained in Sec. 1 (below we shall actually apply this subtraction procedure) has been applied to the inside integral in (6.32). Even a rise in the power of E' in the denominator by unity will be sufficient to make the integral over E' convergent, and then, by continuity, its value at $x^0 = |\vec{x}|$ will coincide with the limit as $|\vec{x}| \longrightarrow x^0$ of the same integral calculated on the assumption $|\vec{x}| < x^0$ by closing the contour of integration in the upper half-plane. By adding (through the use of the subtraction procedure) a larger number of powers of E' in the denominator, any derivative of this integral may be made continuous. Thus, the replacement in (6.27) of the expression $e^{iEx^0}\cos(\lambda\vec{e}\vec{x})$ by its integral representation (6.29) is valid (at any rate, if we have in view the further application of subtraction procedures).

Let us now change in (6.32) the order of integration over x and E',

$$\mathcal{S}_{\vec{e}}\, T^{ret}_{\alpha w}(E, \vec{e}) =$$

$$= \frac{1}{i\pi} P \int_{-\infty}^{\infty} \frac{dE'}{E'-E}\, 2 \int dx\, \theta(x^0) g(x) e^{iE'x^0} \cos\sqrt{E'^2 - \mu^2}\, \vec{e}\vec{x} \,. \qquad (6.33)$$

This change in the order of integration is, strictly speaking, unlawful. Indeed, whereas before this procedure, in (6.32) both the inside integral over E' and the outside integral over x were convergent, now for E' lying in the unphysical region $E'^2 < \mu^2$ the inside integral over x becomes divergent. For such E' the square-root in the argument of the cosine becomes imaginary, and the trigonometric functions become hyperbolic, increasing exponentially at infinity. This is an organic defect of such methods of deriving dispersion relations ( ).
In the next section a method of derivation will be elaborated which will eliminate this "difficulty of changing the order of integration."

Equation (6.33) already contains the desired dispersion relation; noting that by (6.27) the inside integral in it is equal to $\mathcal{S}_{\vec{e}} T^{ret}_{\alpha w}(E', \vec{e})$, we at once obtain

$$\mathcal{S}_{\vec{e}} T^{ret}_{\alpha w}(E, \vec{e}) = \frac{1}{i\pi} P \int_{-\infty}^{\infty} \frac{\mathcal{S}_{\vec{e}} T^{ret}_{\alpha w}(E', \vec{e})}{E' - E}\, dE' + P^n_{\alpha w}(E) \,, \qquad (6.34)$$

where $P^n_{\alpha w}(E)$ are those polynomials which appear when substituting (6.15') for (6.15). We shall not explicitly write out these polynomials in the three following equations. We will return to them again in connection with the subtraction procedure. Taking the real part of (6.34) we find the dispersion relation

$$\mathcal{S}_{\vec{e}} D_{\alpha w}(E) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathcal{S}_{\vec{e}} A_{\alpha w}(E')}{E' - E}\, dE' \,. \qquad (6.35)$$

Using identical arguments, we obtain from (6.28), with the aid of the integral representation (6.30), the second dispersion relation for the combination antisymmetric in $\vec{e}$,

$$\mathcal{A}_{\vec{e}} D_{\alpha w}(E) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathcal{A}_{\vec{e}} A_{\alpha w}(E')}{E' - E} dE' \qquad (6.36)$$

In virtue of the remark following equation (6.32), the integrals in (6.34-6.36) may diverge, and if they are calculated by means of any limit process there will arise on the right-hand side a certain additional polynomial in E. Below, applying the subtraction procedure explicitly, we shall arrive at dispersion relations which do not require such a proviso.

Meanwhile, however, we may note that the relations (6.35-36) possess also the drawback that integration in them is extended to negative energies. To get rid of this drawback, let us make use of the symmetry (6.12) with respect to substitution of the argument E by -E. We then obtain

$$\begin{aligned}
\mathcal{S}_{\vec{e}}(1+P_{\rho\rho'})D_{\alpha w}(E,\vec{e}) &= \frac{2}{\pi} P \int_0^{\infty} \frac{\mathcal{S}_{\vec{e}}(1+P_{\rho\rho'})E' A_{\alpha w}(E',w)}{E'^2 - E^2} dE', \\
\mathcal{S}_{\vec{e}}(1-P_{\rho\rho'})D_{\alpha w}(E,\vec{e}) &= \frac{2}{\pi} P \int_0^{\infty} \frac{\mathcal{S}_{\vec{e}}(1-P_{\rho\rho'})E A_{\alpha w}(E',w)}{E'^2 - E^2} dE', \\
\mathcal{A}_{\vec{e}}(1+P_{\rho\rho'})D_{\alpha w}(E,\vec{e}) &= \frac{2}{\pi} P \int_0^{\infty} \frac{\mathcal{A}_{\vec{e}}(1+P_{\rho\rho'})E A_{\alpha w}(E',\vec{e})}{E'^2 - E^2} dE', \\
\mathcal{A}_{\vec{e}}(1-P_{\rho\rho'})D_{\alpha w}(E,\vec{e}) &= \frac{2}{\pi} P \int_0^{\infty} \frac{\mathcal{A}_{\vec{e}}(1-P_{\rho\rho'})E' A_{\alpha w}(E'\vec{e})}{E'^2 - E^2} dE'.
\end{aligned} \qquad (6.37)$$

Now let us return to the question of improving the convergence of the integrals. We shall assume that the power n of the polynomial in (6.34) is equal to unity. This is a certain additional assumption; in Sec. 4 and 5 we already met such a situation, when we had to postulate (since we do not refer to any concrete form of Lagrangian) the order of increase of the matrix elements at infinity. We may eliminate the polynomial by using the subtraction procedure developed in Sec. 1 (/1.11/-/1.15/), which in our case will lead to[x)]

$$\mathcal{S}_{\vec{e}}(1+P_{\rho\rho'})D_{\alpha W}(E) - \mathcal{S}_{\vec{e}}(1+P_{\rho\rho'})D_{\alpha W}(E_o) =$$
$$= \frac{2}{\pi}(E^2-E_o^2)P\int_o^\infty \frac{E'}{(E'^2-E^2)(E'^2-E_o^2)}\mathcal{S}_{\vec{e}}(1+P_{\rho\rho'})A_{\alpha W}(E')dE') ,$$

$$\mathcal{S}_{\vec{e}}(1-P_{\rho\rho'})D_{\alpha W}(E) - \frac{E}{E_o}\mathcal{S}_{\vec{e}}(1-P_{\rho\rho'})D_{\alpha W}(E_o) =$$
$$= 2E(E^2-E_o^2)\frac{1}{\pi}P\int_o^\infty \frac{1}{(E'^2-E^2)(E'^2-E_o^2)}\mathcal{S}_{\vec{e}}(1-P_{\rho\rho'})A_{\alpha W}(E')dE' , \tag{6.38}$$

$$\mathcal{U}_{\vec{e}}(1+P_{\rho\rho'})D_{\alpha W}(E) - \frac{E}{E_o}\mathcal{U}_{\vec{e}}(1+P_{\rho\rho'})D_{\alpha W}(E_o) =$$
$$= 2E(E^2-E_o^2)\frac{1}{\pi}P\int_o^\infty \frac{\mathcal{U}_{\vec{e}}(1+P_{\rho\rho'})A_{\alpha W}(E')}{(E'^2-E^2)(E'^2-E_o^2)}dE' ,$$

$$\mathcal{U}_{\vec{e}}(1-P_{\rho\rho'})D_{\alpha W}(E) - \mathcal{U}_{\vec{e}}(1-P_{\rho\rho'})D_{\alpha W}(E_o) =$$
$$= 2(E^2-E_o^2)\frac{1}{\pi}P\int_o^\infty \frac{\mathcal{U}_{\vec{e}}(1-P_{\rho\rho'})A_{\alpha W}(E')E'}{(E'^2-E^2)(E'^2-E_o^2)}dE' ,$$

where E_o is any arbitrary energy in the physical region, that is $E_o \geq \sqrt{m^2+\vec{p}^2}$.

The integrals in (6.38) no longer contain integrations over negative energies, but they do not exclude integration over the unphysical region $0 < E < \sqrt{m^2 + \vec{p}^2}$, where

$$\lambda^2 = E^2 - m^2 - \vec{p}^2 < 0 , \tag{6.39}$$

and consequently the momenta are complex. Our problem now will be to exclude

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[x)] If we assume that the integrals in (6.38) converge, then the assumption $n = 1$ is obligatory: a higher power polynomial would not be eliminated by the subtraction procedure and would lead to an order of increase of $D_{\alpha W}(E)$ with energy greater than linear, which would contradict experiment.

this part of the integral.

Let us return to the expression (6.13) for $A_{\alpha\nu}\left(\frac{q'+q}{2}\right)$. Substituting in it the sums (3.30) for $F^{(-)}(x)$ and carrying out the integration, we obtain:

$$A_{\alpha\nu}(E,\vec{e}) = \pi\sum_n \delta\left(E-\sqrt{M_n^2+\lambda^2}+\sqrt{\vec{p}^2+M^2}\right)\left\langle P'S'\middle| j_{\rho'}(0)\middle|\lambda\vec{e},n\right\rangle\left\langle\lambda\vec{e}n\middle| j_{\rho}(0)\middle|PS\right\rangle$$
$$-\pi\sum_n\delta\left(E-\sqrt{\vec{p}^2+M^2}+\sqrt{M_n^2+\lambda^2}\right)\left\langle P'S'\middle| j_{\rho}(0)\middle|-\lambda\vec{e},n\right\rangle\left\langle-\lambda\vec{e},n\middle| j_{\rho'}(0)\middle|PS\right\rangle . \qquad (6.39')$$

For further consideration it is convenient to isolate the term with n = 1, for which the quantum number n is simply a spin-isotopic index S" and $M_n = M$. The sum of all the remaining terms we shall denote by $B_{\alpha\nu}(E)$. Then

$$A_{\alpha\nu}(E,\vec{e}) = \pi\sum_{S''}\langle P'S'| j_{\rho'}(0)|\lambda\vec{e},S''\rangle\langle\lambda\vec{e},S''| j_{\rho}(0)|PS\rangle\,\delta\left(E-\sqrt{M_n^2+\lambda^2}+\sqrt{M^2+\vec{p}^2}\right)$$
$$-\pi\sum_{S''}\langle P'S'| j_{\rho}(0)|-\lambda\vec{e},S''\rangle\langle-\lambda\vec{e},S''| j_{\rho'}(0)|PS\rangle\,\delta\left(E+\sqrt{M_n^2+\lambda^2}-\sqrt{M^2+\vec{p}^2}\right)+B_{\alpha\nu}(E,\vec{e}) . \qquad (6.40)$$

Let us now recall that according to our assumption (6.8) the system nucleon+meson has no bound states with mass less than M + m. In addition, we shall suppose that we are in a region of not too great momentum of the scatterer, so that

$$\vec{p}^2 < \frac{Mm - m^2/2}{2} . \qquad (6.41)$$

It is easy to see that in the unphysical region the argument of the first δ-function, in virtue of (6.39), is essentially positive, and therefore the first sum in (6.40) disappears. In the second sum, the condition that the argument of the δ-function be zero leads to the relation

$$E = \frac{m^2+2\vec{p}^2}{2\sqrt{M^2+\vec{p}^2}} = E_p . \qquad (6.41')$$

Let us now pass on to the sums entering in $B_{\alpha\nu}(E,\vec{e})$. In this case, the

δ-function in the first and second sum for each n will lead to

$$E = E_{1,2} = \pm \frac{M_n^2 - M^2 - m^2 - 2\vec{p}^2}{2\sqrt{M^2 + \vec{p}^2}} .$$

But by (6.8) the numerator here is greater than or equal to $(M+m)^2 - m^2 - - M^2 - 2\vec{p}^2 = 2(Mm-\vec{p}^2)$. Therefore in the first sum, for each n,

$$E = E_1 = \frac{M_n^2 - M^2 - m^2 - 2\vec{p}^2}{2\sqrt{M^2 + \vec{p}^2}} \geq \frac{Mm - \vec{p}^2}{\sqrt{M^2 + \vec{p}^2}} > E_p ,$$

the latter in virtue of the supposition (6.41). However, in the second sum, for each n

$$E = E_2 = - \frac{M_n^2 - M^2 - m^2 - 2\vec{p}^2}{2\sqrt{M^2 + \vec{p}^2}} \leq - \frac{Mm - \vec{p}^2}{\sqrt{M^2 + \vec{p}^2}} < - E_p < 0 ,$$

and consequently the root of the δ-function will be outside the region of integration and the sum will be zero.

Thus, the investigation of the region of unphysical energies shows that it is divided into two subregions

$$\sqrt{m^2 + \vec{p}^2} > E > \frac{Mm - \vec{p}^2}{\sqrt{M^2 + \vec{p}^2}} , \text{ and } 0 < E < \frac{Mm - \vec{p}^2}{\sqrt{M^2 + \vec{p}^2}} , \qquad (6.42)$$

in the second of which

$$A_{\alpha w}(E, \vec{e}) = B_{\alpha w}(E, \vec{e})$$

whereas in the first

$$A_{\alpha w}(E, \vec{e}) =$$

$$= -\pi \sum_{S''} < P'S' \Big| j_\rho(0) \Big| -\lambda\vec{e}, S'' > < -\lambda\vec{e}, S'' \Big| j_{\rho'}(0) \Big| PS > \delta\left(E + \sqrt{M^2 + E^2 - m^2 - \vec{p}^2} - \sqrt{M^2 + \vec{p}^2}\right) .$$

Transforming the δ-function into $\delta(E-E_p)$, we find that in the first unphysical subregion

$$A_{\alpha w}(E,\vec{e}) = -\pi \cdot \frac{\sqrt{M^2+\vec{p}^2}-E_p}{\sqrt{M^2+\vec{p}^2}}\,\delta(E-E_p)\,x$$

$$x \sum_{S''} < P'S'\Big| j_\rho(0)\Big| -\lambda\vec{e},S'' >< -\lambda\vec{e},S''\Big| j_{\rho'}(0)\Big| P,S> . \qquad (6.43)$$

For a further simplification of this expression, we shall consider the matrix element of the current between two one-nucleon states

$$< P''S''\big| j_\rho(0)\big| PS > .$$

In virture of translation invariance (See /3.28/ and below) and the definition of the current (3.4) we may write

$$e^{-i(P-P'')x} < P''S''\big| j_\rho(0)\big| PS > = i < P''S'\Big| \frac{\delta S}{\delta \varphi_\rho(x)}\Big| PS > .$$

Passing now from variation with respect to (x) to variation with respect to $\varphi_\rho(q)$ (the Fourier transform for arbitrary q, connected with $\varphi_\rho(x)$ by the usual relation

$$\varphi_\rho(x) = \frac{1}{(2\pi)^4}\int e^{-iqx}\,\varphi_\rho(q)dq ,$$

we arrive at the equation

$$i < P''S''\Big| \frac{\delta S}{\varphi_\rho(q)}\Big| PS > = < P''S''\big| j_\rho(0)\big| PS > \delta(q+p-p'') . \qquad (6.44)$$

At this point we shall have to resign ourselves (temporarily until the next section) to still one more defect of the usual derivation. The fact of the matter is that the δ-function in (6.44) arises, as usual, from the integral $(2\pi)^4 \int \exp\left\{-i(q+p-p'')x\right\} dx$, which naturally has δ - like properties only for <u>real</u> components, q, p, p'' and has no meaning for complex components. And in the unphysical region under consideration we have to use this δ-function (Cf. below /6.48/) for imaginary spatial components of the vectors q and p". The matrix element to the left may be transformed with the aid of (2.3) from Sec. 2. The successive replacement of commutators of the operators $b_{+S}^{(+)}(\vec{p})$ and $b_{+S''}^{(-)}(\vec{p}'')$ with $\frac{\delta S}{\delta \varphi_\rho(q)}$ by variational differentiation (with account taken of the anticommutativity of the variational derivatives with respect to spinor fields, and the right and left variational derivatives) gives us

$$< \vec{p}''S''| \frac{\delta S}{\delta \varphi_\rho(q)} |\vec{p}S> =$$

$$= \sum_{\lambda\lambda'} \int dx' dx [b_{+S''}^{(-)}(\vec{p}''), \bar{\psi}_{\lambda'}(x')]_+ [\psi_\lambda(x), b_{+S}^{(+)}(\vec{p})]_+ < 0| \frac{\delta^3 S}{\delta \psi_\lambda(x)\, \delta\varphi_\rho(q)\, \delta\bar{\psi}_{\lambda'}(x')} |0>$$

whence, using (2.26) and the definitions (2.33), (2.34) of the Fourier transforms $\psi(x)$ and $\bar{\psi}(x)$, we obtain

$$i < \vec{p}''S''| \frac{\delta S}{\delta \varphi_\rho(q)} |\vec{p}S> =$$

$$= -2\pi i(2\pi)^4 \sum_{\lambda\lambda'} \bar{U}_{\lambda'}^{+S''}(\vec{p}'') < 0| \frac{\delta^3 S}{\delta\bar{\psi}_{\lambda'}(p'')\, \delta\varphi_\rho(q)\, \delta\psi_\lambda(p)} |0> U_\lambda^{+S}(\vec{p}) , \tag{6.45}$$

$$p_0 = \sqrt{\vec{p}^2 + M^2} ,$$

$$p_0'' = \sqrt{\vec{p}''^2 + M^2} .$$

Let us consider the matrix element of the third variational derivative that enters here. It is easy to see that the most general expression for it satisfying the demands of translation, Lorentz (including reflection) and isotopic invariance is

$$< 0| \frac{\delta^3 S}{\delta\bar{\psi}_{\lambda'}(p'')\, \delta\varphi_\rho(q)\, \delta\psi_\lambda(p)} |0> = \tag{6.46}$$

$$= \sum_{w_1,w_2=0,1} [(p''\gamma)^{w_1} \gamma_5 (p\gamma)^{w_2}]_{\lambda'\lambda} \tau^\rho_{t_1 t_2} h_{w_1 w_2}(p^2, p''^2, q^2)\, \delta(p+q-p'') ,$$

where $h_{w_1w_2}(p_3^2, p''^2, q^2)$ are arbitrary <u>scalar</u> functions depending only on three four-dimensional squares p^2, p''^2 and q^2.

It should now be noted that the matrix element to the left is invariant if we perform in it Dirac conjugation and then transpose p and p". After performing the same transformation on the right-hand side, we arrive at the equality

$$\sum_{w_1w_2=0,1} (p\gamma)^{w_1} \gamma_5 (p''\gamma)^{w_2} \tau^{\rho} h_{w_1w_2}(p^2, p''^2, q^2) =$$

$$= \sum_{w_1w_2=0,1} (p\gamma)^{w_2} \gamma_s (p''\gamma)^{w_1} \tau^{\rho} h^{*}_{w_1w_2}(p^2, p''^2, q^2) ,$$

whence there follow the rules of complex conjugation for the scalar functions $h_{w_1w_2}$:

$$h_{w_1w_2}(p^2, p''^2, q^2) = h^{*}_{w_1w_2}(p''^2, p^2, q^2) . \qquad (6.47)$$

In our special case, the components of the momenta p and p" are not arbitrary, but are fixed (since they refer to states of real particles) by the relations

$$p^o = \sqrt{\vec{p}^2 + M^2} \; ; \; p^o = \sqrt{\vec{p}''^2 + M^2} = \sqrt{\lambda_p^2 + M^2} . \qquad (6.48)$$

And the components of the vector q are determined by the δ-function in (6.46). Using this definition, the fact that $\vec{p}'' = -\lambda_p \vec{e}$, and having in view (6.39) and (6.41), we find that

$$q^{o2} - \vec{q}^2 = q^2 = m^2 . \qquad (6.49)$$

Finally, the spinor amplitudes $\overline{U^{+S''}}(p'')$ and $U^{+S}(p)$ satisfy (in virtue of /2.32/) the Dirac equation:

$$\overline{U^{+S''}}(\vec{p}'')(\gamma p'') = \overline{U^{+S''}}(\vec{p}'')M; \ (\gamma p)U^{+S}(p) = MU^{+S}(p) \ . \qquad (6.50)$$

Therefore, if the matrix element (6.46) stands between two spinor amplitudes $\overline{U^{+S''}}(p'')$ and $\overline{U^{+S}}(p)$, and (6.48) and (6.50) are fulfilled, then it may be replaced by

$$\gamma_5 \tau_\rho \, \delta\,(p+q-p'') \sum_{w_1 w_2 = 0,1} M^{w_1} M^{w_2} h_{w_1 w_2}(M^2,M^2,q^2) \ .$$

Let us introduce the notation

$$g(q^2) = -(2\pi)^5 \sum_{w_1 w_2 = 0,1} M^{w_1} M^{w_2} h_{w_1 w_2}(M^2,M^2,q^2) \ . \qquad (6.51)$$

It is easy to see that, by the rules of complex conjugation (6.47), the function $g(q^2)$ thus introduced is real. And finally it is clear that its value $g(m^2)$ at $q^2 = m^2$ would coincide in conventional theory with what is called there the experimental mesic charge of the nucleon.

Returning now to (6.45) and collecting the results, we obtain:

$$i < \vec{P}''S''| \frac{\delta S}{\delta \varphi_\rho(q)} |\vec{P}S > = ig\overline{U^{+S''}}(\vec{P}'')\gamma_5 \tau^\rho U^{+S}(\vec{P})\,\delta\,(q+p-p'') \ ,$$

that is, if we recall (6.44),

$$< \vec{P}''S''| \ j_\rho\,(0)|\vec{P}S > = ig\overline{U^{+S''}}(\vec{P}'')\gamma_5 \tau^P U^{+S}(\vec{P}) \ ,$$

$$\text{at } p + q - p' = 0; \ p^2 = p''^2 = M^2; \ q^2 = m^2 \ . \qquad (6.52)$$

Now let us split the spin-isotopic index s into two indices: the spin index s and the isotopic index t, and let us pass to the nonrelativistic case, leaving only terms up to $(\frac{P^2}{M^2})$ inclusive, and disregarding terms of order (m^2/M^2). Then the usual summation over the spin indices will give us

$$\bar{U}^{S''+}(\vec{P}'')\,\gamma_5\,\tau^{\rho}\,U^{+S}(\vec{P}) = -\,i\,\frac{\tau^{\rho}_{t''t}}{2M}\left\{\vec{\sigma}(\vec{P}-\vec{P}'')\right\}_{S''S}\,, \tag{6.53}$$

in virtue of which

$$<P''S''|\,j_{\rho}(0)\,|\vec{P}S> = g\,\frac{\tau^{\rho}_{t''t}}{2M}\left\{\vec{\sigma}(\vec{P}-\vec{P}'')\right\}_{S''S}\,. \tag{6.54}$$

Substituting this expression for the current matrix elements entering into (6.43), and passing everywhere to the nonrelativistic limit and also performing the obvious summations with respect to isotopic and spin indices, we finally obtain for $A_{\alpha w}(E_1\vec{e})$ in the first unphysical subregion the expression:

$$A_{\alpha w}(E,\vec{e}) = \frac{\pi g^2}{4M^2}\,\delta(E-E_P)\left\{(\lambda^2-\vec{P}^2)\,\delta_{s's} + 2i\lambda(\vec{e}\vec{P}\,\vec{\sigma}_{s's})\right\}\times$$
$$\times\left\{\delta_{\rho\rho'}\,\delta_{t't} - i e_{\rho'\rho\rho''}\,\tau^{\rho''}_{t't}\right\}\,. \tag{6.55}$$

Applying to it the operations of symmetrization and antisymmetrization with respect to $\vec{e}$ [(6.24), (6.25)], and symmetrization and antisymmetrization in the charge [$(1+P_{\rho\rho'})$ and $(1-P_{\rho\rho'})$], we obtain

$$\mathcal{S}_{\vec{e}}(1+P_{\rho\rho'})A_{\alpha w}(\vec{e},E) = 4\pi\left(\frac{g}{2M}\right)^2\delta(E-E_P)\,\delta_{ss'}(\lambda^2-\vec{P}^2)\,\delta_{\rho\rho'}\,\delta_{t't} \tag{6.56}$$

and

$$\mathcal{S}_{\vec{e}}(1-P_{\rho\rho'})A_{\alpha w}(\vec{e},E) = -\,4\pi i\left(\frac{g}{2M}\right)^2\delta(E-E_P)\,\delta_{ss'}(\lambda^2-\vec{P}^2)\,e_{\rho'\rho\rho''}\,\tau^{\rho''}_{t't} \tag{6.57}$$

$$\mathcal{A}_{\vec{e}}(1+P_{\rho\rho'})A_{\alpha w}(\vec{e},E) = 8\pi i\left(\frac{g}{2M}\right)^2\delta(E-E_P)\,\delta_{ss'}(\vec{e}\vec{P}\,\vec{\sigma}'_{ss})\,\delta_{\rho'\rho}\,\delta_{t't} \tag{6.58}$$

and

$$\mathcal{A}_{\vec{e}}(1-P_{\rho\rho'})A_{\alpha w}(\vec{e},E) = 8\pi\left(\frac{g}{2M}\right)^2\delta(E-E_P)(\vec{e}\vec{P}\,\vec{\sigma}_{s's})\,e_{\rho'\rho\rho''}\,\tau^{\rho''}_{t't} \tag{6.59}$$

Substituting these expressions in the dispersion relations (6.38) and carrying out the integration, which is not at all difficult, we obtain dispersion relations in which it is no longer necessary to integrate over the first of the unphysical subregions (6.42), but its contribution is expressed by a certain term standing outside the integral. As regards the second unphysical subregion, we have not succeeded in evaluating its contribution to the integration and so it simply has to be neglected. It may be noted though that for forward scattering, when $\vec{P} = 0$, this subregion will not exist since it will collapse into a point. It likewise will not exist in the approximation of infinitely heavy nucleons.

Here we shall not write out the explicit form of the dispersion relations with the term that takes account of the contribution of the unphysical region, since we first want to pass from the scattering amplitudes considered so far that have a definite symmetry, to the amplitudes that correspond to the scattering of particles in concrete spin and charge states, which will be done in Sec. 8. There also we shall write out in final form the dispersion relations for the amplitudes that are of direct interest to the experimenter. However, before passing to this purely technical problem, we want to devote the next section to a derivation of dispersion relations that is free from the objections mentioned above.

Section 7. The Rigorous Derivation of Dispersion Relations

Let us now pass on to the rigorous derivation of dispersion relations. In the previous section we had to do with the Fourier transforms

$$T^{(?)}_{\alpha\omega}(E, e) = \int e^{i(Ex^{o} - \lambda\vec{e}\vec{x})} F^{(?)}_{\alpha\omega}(x)\, dx, \qquad (6.26)$$

taken at

$$\lambda^2 = E^2 - (m^2 + \vec{p}^{\,2}).$$

Now it will be more convenient to consider these expressions as functions of two variables E and τ, where

$$\lambda^2 = E^2 - \tau. \qquad (7.1)$$

Let us put

$$\begin{aligned} T^{ret}(E, \tau) &= \int e^{i(Ex^{o} - \sqrt{E^2-\tau}\,.\,\vec{e}\vec{x})} F^{ret}(x)\, dx, \\ T^{adv}(E, \tau) &= \int e^{i(Ex^{o} - \sqrt{E^2-\tau}\,.\,\vec{e}\vec{x})} F^{adv}(x)\, dx, \end{aligned} \qquad (7.2)$$

and

$$\begin{aligned} T(E, \tau) &= T^{ret}(E, \tau) - T^{adv}(E, \tau) = \\ &= \int e^{i(Ex^{o} - \sqrt{E^2-\tau}\,\vec{e}\cdot\vec{x})} F(x)\, dx. \end{aligned} \qquad (7.3)$$

The vector $\vec{e}$ (perpendicular to $\vec{p}$) is not explicitly written among the arguments; the indices α , ω are likewise not written out since here this will not lead to misunderstanding.

In order not to deal with the two signs of the square root we shall always consider (as in the preceding section) not the functions T themselves, but their symmetrized or anti-symmetrized forms

$$ST;\ S = S_{e},\ A_{e}$$

It may be noted that

$$ST^{ret}\ (E,\ \tau)$$

is an analytic function of E, λ, regular in the region

$$\text{Im}\ E > \left|\text{Im}\ \lambda\right|$$

and ipso facto an analytic function of E, τ, regular in the region:

$$\text{Im}\ E > \left|\text{Im}\sqrt{E^2-\tau}\right|. \qquad (7.4)$$

And quite analogously

$$ST^{adv}\ (E,\ \tau)$$

is an analytic function of E, τ, regular in the region

$$\text{Im}\ E < -\left|\text{Im}\sqrt{E^2-\tau}\right|. \qquad (7.5)$$

Let us first consider the case when τ is fixed and has a real negative value

$$\tau < -\vec{p}^{\,2}. \qquad (7.6)$$

In this case

$$\left|\text{Im}\sqrt{E^2-\tau}\right| < \left|\text{Im}\ E\right| \quad \text{if} \quad \text{Im}\ E \neq 0,$$

and therefore

$$ST^{ret}\ (E,\ \tau) \qquad (7.7)$$

is an analytic function of E in the region Im E $>$ 0, and

$$ST^{adv}\ (E,\ \tau) \qquad (7.8)$$

in the region

$$\text{Im } E < 0.$$

Let us take their difference ST (E, τ) for real E and let us repeat the argument of the preceding section, which, however, will no longer contain unjustified elements, since in the case under consideration not only E but also $\lambda = \sqrt{E^2 - \tau}$ are always real.

We find

$$ST(E,\ \tau) =$$

$$= -2\pi i \frac{\sqrt{M^2+\vec{p}^{\,2}} - E_p(\tau)}{\sqrt{M^2+p^2}}\ \delta(E - E_p(\tau)) \sum_{s''} S \langle p's' | j_\rho(0) | -\lambda e S'' \rangle \langle -\lambda e S'' | j_{\rho'}(0) | p,\ s \rangle +$$

$$+ 2\pi i \frac{\sqrt{M^2+\vec{p}^{\,2}} - E_p(\tau)}{\sqrt{M^2+p^2}}\ \delta(E + E_p(\tau)) \sum_{s''} S \langle p's' | j_{\rho'}(0) | \lambda e\, s'' \rangle \langle \lambda \vec{e} s'' | j_\rho(0) | p,\ s \rangle ,$$

$$E_p(\tau) = \frac{\sqrt{\tau + \vec{p}^{\,2}}}{2\sqrt{M^2+\vec{p}^{\,2}}}, \qquad (7.10)$$

if

$$|E| < \frac{2Mm + m^2 - p^2 - \tau}{2\sqrt{M^2+p^2}} . \qquad (7.11)$$

Then, developing the expression on the right-hand side of (7.9) we obtain:

$$ST(E,\ \tau) = S\, f(E,\ \tau) +$$

$$+ 2\pi i \frac{\sqrt{M^2+p^2} - E_p(\tau)}{\sqrt{M^2+p^2}}\ \delta(E - E_p(\tau))\, g^2(\tau - \vec{p}^{\,2}) \sum_{s''} S \Big\{ \overline{U^{+s'}}(-\vec{p})\, \gamma_5\, \tau^\rho\, U^{+s''}(-\lambda\vec{e}) \Big\} \times \qquad (7.12)$$

$$\times\ \overline{U^{+s''}}(-\lambda\vec{e})\, \gamma_5\, \tau^{\rho'}\, U^{+s}(p) \Big\} -$$

$$- 2\pi i \frac{\sqrt{M^2+p^2-E_p(\tau)}}{\sqrt{M^2+p^2}} \delta(E-E_p(\tau))\, g(\tau-\vec{p}^{\,2}) \sum_{s''} \mathcal{S} \left\{ \overline{u^{+s'}(-\vec{p})}\, \gamma_\sigma \tau_\rho \,, U^{+s''}(\lambda\vec{e}) \right\} \cdot$$

$$\cdot \left\{ \overline{u^{+s''}(\lambda\vec{e})}\, \gamma_\sigma \; \tau^\rho\, u^{+s}(\vec{p}) \right\} \qquad (7.13)$$

where

$$Sf(E, \tau) = 0 \quad \text{if} \quad |E| < \frac{2Mm+m^2-\vec{p}^{\,2}-\tau}{2\sqrt{M^2+p^2}} .$$

Already from (7.9), (7.10) we see that the functions (7.7), (7.8) are one and the same analytic function $\tilde{T}(E, \tau)$, regular in the region Im $E \neq 0$ with cuts on the real axis for

$$E < - \frac{2Mm+m^2-\vec{p}^{\,2}-\tau}{2\sqrt{M^2+p^2}} ; \qquad E > \frac{2Mm+m^2-\vec{p}^{\,2}-\tau}{2\sqrt{M^2+p^2}} ,$$

and with poles of the first order at points $E = \pm E_p(\tau)$. Approaching the real axis from the upper half-plane we obtain a retarded function, and from the lower half-plane an advanced function.

Proceeding from the definitions (7.2) we note that in the case under consideration when τ is fixed the analytic function $ST(E, \tau)$ increases at infinity (when $|\mathrm{Im}\, E| > \delta > 0$) more slowly than a certain polynomial.

Thus, $S\tilde{T}(E, \tau)$ has properties that guarantee the correctness of applying the Cauchy theorem:

$$S\tilde{T}(E, \tau) = \frac{(E-E_o)^{n+1}}{2\pi i} \int_{-\infty}^{\infty} \frac{sf\,(E', \tau)}{(E'-E_o)^{n+1}(E'-E)} dE' + \frac{g^2(\tau-\vec{p}^{\,2})\, A(\tau)}{(E-E_p(\tau))} + \qquad (7.14)$$

$$+ \frac{g^2(\tau-\vec{p}^{\,2})\, B(\tau)}{(E+E_p(\tau))} + \sum_{(0 \le r \le n)} C_r(\tau) E^r ,$$

where n is a sufficiently large integer, E_o is an arbitrary real parameter, which we take in the interval (7.11) so that the denominator $(E'-E_o)^{n+1}$ does not become zero in the actual region of integration in (7.14). Further

$$A(\tau) = -2\pi i \frac{\sqrt{M^2+\vec{p}^{\,2}}-E_p(\tau)}{\left(M^2+p^2\right)^{1/2}} \sum_{s''} S \left\{u^{+s'}(-\vec{p})\gamma_\sigma \tau_\rho u^{+s''}(-\lambda\vec{e})\right\} \left\{u^{+s''}(-\lambda\vec{e})\gamma_\sigma \tau_{\rho'} u^{+s}(\vec{p})\right\} \tag{7.15}$$

$$B(\tau) = 2\pi i \frac{\sqrt{M^2+p^2}-E_p(\tau)}{\left(M^2+p^2\right)^{1/2}} \sum_{s''} S \left\{u^{+s'}(-\vec{p})\gamma_\sigma \tau_{\rho'} u^{+s''}(+\lambda\vec{e})\right\} \left\{u^{+s''}(\lambda\vec{e}) \tau_\rho \gamma_\sigma u^{+s}(\vec{p})\right\}$$

It should be emphasized that we established the "dispersion relations" (7.14) only for negative τ which satisfy the inequality (7.6).

Now the real dispersion relations that were spoken of in the preceding section are obtained directly from (7.14) only when it proves ture also for $\tau = m^2+\vec{p}^{\,2}$.

In order to extend the region of τ for which the relation (7.14) is valid, let us make use of the methods of analytic continuation. We shall prove that the function $Sf(E, \tau)$ has the following important property of analytic representation, namely

$$Sf(E, \tau) = F_1\left\{2E\sqrt{M^2+\vec{p}^{\,2}} + \tau;\ \tau\right\} + F_2\left\{-2E\sqrt{M^2+\vec{p}^{\,2}} + \tau;\ \tau\right\} \tag{7.16}$$

if E, τ, $\sqrt{E^2-\tau}$ are all real and $\tau < \left\{(1+\rho)m^2+\vec{p}^{\,2}\right\}$. The functions $F_1(\xi,\tau)$; $F_2(\xi, \tau)$ are generalized functions of the real variable ξ and analytic functions of the complex variable τ, regular in the region

$$-\vec{p}^{\,2} + \mathrm{Re}\,\tau < (1+\rho)m^2, \quad |\mathrm{Im}\,\tau| < \rho m^2, \tag{7.17}$$

where ρ is a certain positive, sufficiently small number.

In addition:

$$\left.\begin{array}{l} F_1(\xi, \tau) = 0 \\ F_2(\xi, \tau) = 0 \end{array}\right\} \text{ for } \xi < 2Mm + m^2 - \vec{p}^{\,2}. \tag{7.18}$$

Before passing to the proof of the representation (7.16) we shall show that the correctness of the relations for the required values $\tau = m^2 + \vec{p}^{\,2}$ follow directly from it.

(7.19)

For this purpose, let us again take negative τ that satisfy the inequality (7.6). For such τ both the representation (7.16) (as we have temporarily assumed, prior to the proof) and the relation (7.14) are correct. Then substituting (7.16) in (7.14) we find

$$ST(E, \tau) = \Phi(E, \tau) + \frac{g^2(\tau - \vec{p}^{\,2})A(\tau)}{E - E_p(\tau)} + \frac{g^2(\tau - \vec{p}^{\,2})B(\tau)}{E + E_p(\tau)} + \sum_{(0 \leq r \leq n)} C_r(\tau)E^r, \tag{7.20}$$

where

$$\Phi(E, \tau) = \frac{(E-E_0)^{n+1}}{2\pi i} \int_{-\infty}^{\infty} \frac{F_1\left\{2E'\sqrt{M^2+\vec{p}^{\,2}};\ \tau\right\} dE'}{\left(E' - \frac{\tau}{2\sqrt{M^2+\vec{p}^{\,2}}} - E_0\right)^{n+1}\left(E' - \frac{\tau}{2\sqrt{M^2+\vec{p}^{\,2}}} - E\right)} +$$

$$+ \frac{(E-E_0)^{n+1}}{2\pi i} \int_{-\infty}^{\infty} \frac{F_2\left\{2E'\sqrt{M^2+\vec{p}^{\,2}};\ \tau\right\} dE'}{\left(-E' + \frac{\tau}{2(M^2+\vec{p}^{\,2})^{1/2}} - E_0\right)^{n+1}\left(\frac{\tau}{2(M^2+\vec{p}^{\,2})^{1/2}} - E' + E\right)} \tag{7.21}$$

Let us take an arbitrary E_0 in the interval

$$|E_0| < \frac{2(Mm - \rho m^2)}{2(M^2+\vec{p}^{\,2})^{1/2}} \tag{7.22}$$

Then in virtue of the stated properties of the functions F_1, F_2, the equations (7.21) define an analytic function of E, τ in the region

$$-\vec{p}^{\,2} + \mathrm{Re}\,\tau < (1+\rho)m^2; \quad |\mathrm{Im}\,\tau| < \rho\, m^2; \quad |\mathrm{Im}\,\tau| < 2\sqrt{M^2+\vec{p}^{\,2}}\;|\mathrm{Im}\,E| \; . \tag{7.23}$$

On the other hand $\widetilde{ST}(E, \tau)$ is analytic in the regions (7.4), (7.5). Thus the analytic function

$$\left\{ \widetilde{ST}(E, \tau) - \phi(E, \tau) \right\} \; (E^2 - E_p^{\,2}(\tau)) \tag{7.24}$$

is regular in the region

$$\begin{gathered} -\vec{p}^{\,2} + \mathrm{Re}\,\tau < (1+\rho)m^2; \quad |\mathrm{Im}\,\tau| < \rho\, m^2; \\ |\mathrm{Im}\,\tau| < 2\sqrt{M^2+\vec{p}^{\,2}}\;|\mathrm{Im}\,E| \; ; \quad \left|\mathrm{Im}\sqrt{E^2-\tau}\right| < |\mathrm{Im}\,E| \; . \end{gathered} \tag{7.25}$$

In accordance with (7.20) the function (7.24), for negative τ satisfying the inequality (7.6), coincides with the polynomial

$$g^2(\tau-\vec{p}^{\,2})A(\tau)(E+E_p(\tau)) + g^2(\tau-\vec{p}^{\,2})B(\tau)\;(E-E_p(\tau)) + (E^2-E_p^{\,2}(\tau)) \sum_{(0 \leq r \leq n)} Cr(\tau)E^r . \tag{7.26}$$

Therefore it must also be a polynomial in E throughout the region of regularity (7.25).

Since on the other hand $A(\tau)$, $B(\tau)$, $E_p(\tau)$ are analytic functions of τ according to the definitions (7.10), (7.15), we see that $g^2(\tau-\vec{p}^{\,2})$ and $C_r(\tau)$ must allow analytic continuation.

Let us take any $\tau = \tau^*$ from the region (7.17), not lying on the real axis, $\mathrm{Im}\,\tau^* \neq 0$, and let us construct the corresponding $E = E^*$, putting

$$E^* = E_r + i\,E_i$$

where

$$2E_rE_i = \text{Im}\,\tau^*;\ E_r < M;\ E_r^2 - E_i^2 - \text{Re}\,\tau^* > 0.$$

It is clear that such E^*, τ^* belong to the region (7.25) and therefore τ^* is contained in the region of analyticity of the functions $g^2(\tau - \vec{p}^{\,2})$, $C_r(\tau)$. Hence we conclude that the functions are analytic in the region (7.17) with a possible cut lying on the real axis. Let us show that in actuality this cut does not exist, so that these functions will be regular throughout the region (7.17).

For this purpose, let us consider real $\tau_r < (1+\rho)m^2 + \vec{p}^{\,2}$ and set

$$\tau = \tau_\pm = \tau_r \pm i\eta;\ \eta > 0;\ E = E_\pm = E_r \pm \frac{i\eta}{2E_r},\ E_r > 0$$

$$\vec{p}^{\,2} + (1+\rho)m^2 < E_r^2 < M^2.$$

For sufficiently small η, such (E, τ) obviously are included in the region (7.25). Let now η tend to zero. From (7.21) we find

$$\phi(E_+, \tau_+) \to (E_r + i\varepsilon, \tau_r);\ \phi(E_-, \tau_-) \to \phi(E_r - i\varepsilon, \tau_r), \quad (7.27)$$

and taking into account (7.16) we obtain

$$\phi(E_+, \tau_+) - \phi(E_-, \tau_-) \to Sf(E_r, \tau_r) = S\tilde{T}(E_r, \tau_r). \quad (7.28)$$

On the other hand, on the basis of (7.2),

$$S\tilde{T}(E_+, \tau_+) = \int e^{-\frac{\eta}{2E_2} x_0}\, F^{ret}(x)\, S e^{i\left(E_r x^0 - \vec{e}\,\vec{x}\sqrt{E_r^2 - \tau_r - \left(\frac{\eta}{2E_r}\right)^2}\right)} dx$$

and therefore

$$S\tilde{T}(E_+, \tau_+) \to ST^{ret}(E_r, \tau_r),$$

Quite analogously

$$\widetilde{ST}(E_-, \tau_-) \rightarrow ST^{adv}(E_r, \tau_r).$$

We therefore have

$$\lim \left\{\widetilde{ST}(E_+, \tau_+) - \phi(E_+, \tau_+)\right\} = \lim \left\{\widetilde{ST}(E_-, \tau_-) - \phi(E_-, \tau_-)\right\}$$

Since for $E_\pm$, $\tau_\pm$ the function (7.24) is equal to the polynomial (7.26), this relation will obtain for it also. Hence it follows that

$$g^2(\tau_r + i\eta - \vec{p}^{\,2}),\ C_s(\tau_r + i\eta)$$

tend to the same limits as $g^2(\tau_r - i\eta - \vec{p}^{\,2})$, $C_s(\tau_r - i\eta)$. Thus, the cuts for the functions g^2, C_s does not exist, and they are regular throughout the region (7.17). Taking note of this fact, let us return to the relation (7.20), which as we now see holds for points (E, τ) of the region (7.25).

The analytic function of the complex variables E, τ on the right-hand side of the above mentioned relation is regular in a broader region, in the region (7.23).

We may therefore extend the analytic function $\widetilde{ST}(E, \tau)$ in such a way that it will equal the right-hand side of (7.20) throughout the region (7.23).

For the analytic function $\widetilde{ST}(E, \tau)$ extended in this way the usual relations with improper limits holds,

$$\widetilde{ST}(E + i\varepsilon, \tau) = ST^{ret}(E, \tau), \tag{7.28}$$

$$\widetilde{ST}(E - i\varepsilon, \tau) = ST^{adv}(E, \tau), \tag{7.29}$$

if only E, τ and $\lambda = \sqrt{E^2 - \tau}$ are all real, and

$$\tau < (1 + \rho)m^2 + \vec{p}^{\,2}.$$

Indeed, on the basis of (7.20) we see that

$$\lim_{\substack{\delta \to 0 \\ \delta > 0}} \widetilde{ST}(E + i\delta, \tau) = \lim \widetilde{ST}(E + i\delta, \tau + i\alpha E \delta),$$

$$\alpha = (1 + (\tfrac{E}{M})^2)^{-1}.$$

But given sufficiently small δ, the point

$$E_+ = E + i\delta, \ \delta_+ = \tau + i\alpha E \delta,$$

belongs to the region (7.25), in which we have the right to make use of equation (7.2) and to write

$$\widetilde{ST}(E + i\delta, \tau + i\alpha E \delta) =$$

$$= S \int \exp \left\{ -\delta x^0 + \vec{x}\, \vec{e}\, \mathrm{Im} \sqrt{E^2_+ - \tau_+} \right\} \exp i \left\{ Ex_0 - \vec{x}\, \vec{e}\, \mathrm{Re} \sqrt{E_+{}^2 - \tau_+} \right\} F^{ret}(x)dx$$

$$\left| \mathrm{Im} \sqrt{E_+{}^2 - \tau_+} \right| < \delta.$$

Therefore $\lim \widetilde{ST}(E + i\delta, \tau + i\alpha E \delta) = ST^{ret}(E, \tau)$. Analogously we verify also the property (7.29). We established the relation (7.20) throughout the region (7.23). But $\tau = m^2 + \vec{p}\,^2$, together with any E that does not lie on the real axis, belongs to this region. Therefore, (7.20) is true when

$$\mathrm{Im}\, E \neq 0, \ \tau = m^2 + \vec{p}\,^2.$$

By a reverse substitution of the variable of integration we transform (7.20) to the form (7.14), in which in place of $Sf(E', m^2 + \vec{p}\,^2)$ there will be the expression

$$F_1 \left\{ 2E' \sqrt{M^2 + \vec{p}\,^2} + m^2 + \vec{p}\,^2; \ m^2 + \vec{p}\,^2 \right\} + F_2 \left\{ -2E' \sqrt{M^2 + \vec{p}\,^2} + m^2 + \vec{p}\,^2; \ m^2 + \vec{p}\,^2 \right\}, \tag{7.30}$$

coinciding with it when $E'^2 > m^2 + \vec{p}\,^2$. The expression (7.30) is zero (just like $Sf(E', m^2 + \vec{p}\,^2)$) for $|E'| < \frac{Mm - \vec{p}\,^2}{(M^2 + p^2)^{1/2}}$. In the interval

$$\frac{Mm-\vec{p}^{\,2}}{(M^2+\vec{p}^{\,2})^{1/2}} < |E'| < \sqrt{m^2+\vec{p}^{\,2}} \qquad (7.31)$$

a direct definition of the function

$$Sf(E', m^2+\vec{p}^{\,2}) = ST(E', m^2+\vec{p}^{\,2}) \qquad (7.32)$$

by the integral (7.3) has no sense, and the expression (7.30) may be considered as its proper extension in the interval (7.31).

Thus, we obtained the relation (7.14) for the required value of τ with the extended function (7.32). In order to pass, in it, to real E, we have the equations (7.28), (7.29). Thus the validity of the dispersion relations of the preceding section is established.

In order to complete the proof, we still have to prove the representation (7.16).

Let us use equation (3.23),

$$e^{i\frac{p'-p}{2}(x+y)} F_{\alpha\omega}(x-y) = i \langle p's' | j_{\rho'}(x) j_{\rho}(y) - j_{\rho}(y) j_{\rho'}(x) | p\, s \rangle ,$$

from which we obtain

$$T_{\alpha\omega}(\frac{p'-p}{2}) + p_3)\, \delta(p'-p + p_3 + p_4) =$$

$$= \frac{i}{(2\pi)^{"}} \int \langle p's' | j_{\rho'}(x_3) j_{\rho}(x_4) - j_{\rho}(x_4)\, j_{\rho'}(x_3) | ps \rangle e^{i(p_3x_3+p_4x_4)} dx_3dx_4.$$

Let us here express the matrix elements $\langle p's' | \dots | ps \rangle$ in terms of vacuum expectation values with the aid of the principle (II, 3). We find

$$T_{\alpha\omega}(\frac{p'-p}{2} + p_3)\, \delta(p'-p + p_3 + p_4) =$$

$$= \frac{-i}{(2\pi)^7} \overline{u^{+s'}(\bar{p}')} \int e^{i(p'x_1-px_2+p_3x_3+p_4x_4)} \langle 0 | \frac{\delta}{\delta\bar{\psi}(x_1)} \frac{\delta}{\delta\psi(x_2)} \qquad (7.33)$$

$$\{ j_{\rho'}(x_3) j_{\rho}(x_4) - j_{\rho}(x_4) j_{\rho'}(x_3) \} | 0 \rangle u^{+s}(p)\, dx_1 \dots dx_4$$

Let us now introduce a convenient notation. For any translation-invariant function $F(x_1, \ldots, x_4)$ of four-vecotrs $x_1, \ldots, x_4$, the full Fourier transform is proportional to $\delta(p_1 + \ldots + p_4)$. We shall agree to denote by $\widetilde{F}(p_1, \ldots, p_4)$ the coefficient that appears here and call it the Fourier transform of F. Thus, we have

$$\int F(x_1, \ldots, x_4)\, e^{i(p_1x_1 + \ldots + p_4x_4)}\, dx_1 \ldots dx_4 = \delta(p_1 + \ldots + p_4)\widetilde{F}(p_1, \ldots p_4)(2\pi)^4. \tag{7.34}$$

Using this notation, we shall rewrite the relation (7.33) in the form:

$$T_{\alpha\omega}(q) = \frac{-i}{(2\pi)^3}\ \overline{U^{+s'}}(p')\widetilde{\mathcal{D}}(p', -p, p_3, p_4)\ U^{+s}(p), \tag{7.35}$$

where

$$q = \frac{p'-p}{2} + p_3,\quad p'-p + p_3 + p_4 = 0 \tag{7.36}$$

and

$$\mathcal{D}(x_1, x_2, x_3, x_4) =$$
$$= \left\langle 0 \right| \frac{\delta}{\delta\overline{\psi}(x_1)}\ \frac{\delta}{\delta\psi(x_2)} \left\{ j_{\rho'}(x_3) j_\rho(x_4) - j_\rho(x_4) j_{\rho'}(x_3) \right\} \left| 0 \right\rangle . \tag{7.37}$$

Evaluating the second variational derivative we find:

$$\mathcal{D}\,(x_1, \ldots x_4) = \sum_{(1 \leq a \leq 8)} \mathcal{D}^a(x_1, \ldots x_n) \tag{7.38}$$

where

$$D^{(1)}(x_1, \ldots, x_4) = \left\langle 0 \right| \frac{\delta j_{\rho'}(x_3)}{\delta\overline{\psi}(x_1)}\ \frac{\delta j_\rho(x_4)}{\delta\psi(x_2)} \left| \ 0 \right\rangle ,$$
$$D^{(2)}(x_1, \ldots, x_4) = -\left\langle 0 \right| \frac{\delta j_{\rho'}(x_3)}{\delta\psi(x_2)}\ \frac{\delta j_\rho(x_4)}{\delta\overline{\psi}(x_1)} \left| \ 0 \right\rangle , \tag{7.39}$$

$$D^{(3)}(x_1, \ldots, x_4) = -\left\langle 0 \right| \frac{\delta j_\rho(x_4)}{\delta \overline{\psi}(x_1)} \frac{\delta j_{\rho'}(x_3)}{\delta \psi(x_2)} \left| 0 \right\rangle ,$$

$$D^{(4)}(x_1, \ldots, x_4) = -\left\langle 0 \right| \frac{\delta j_\rho(x_4)}{\delta \psi(x_2)} \frac{\delta j_{\rho'}(x_3)}{\delta \overline{\psi}(x_1)} \left| 0 \right\rangle , \quad (7.39)$$

$$D^{(5)}(x_1, \ldots, x_4) = \left\langle 0 \right| \left\{ \frac{\delta}{\delta \overline{\psi}(x_1)} \frac{\delta}{\delta \psi(x_2)} j_{\rho'}(x_3) \right\} j_\rho(x_4) \left| 0 \right\rangle ,$$

$$D^{(6)}(x_1, \ldots, x_4) = \left\langle 0 \right| j_{\rho'}(x_3) \frac{\delta}{\delta \overline{\psi}(x_1)} \frac{\delta}{\delta \psi(x_2)} j_\rho(x_4) \left| 0 \right\rangle ,$$

$$D^{(7)}(x_1, \ldots, x_4) = -\left\langle 0 \right| \left\{ \frac{\delta}{\delta \overline{\psi}(x_1)} \frac{\delta}{\delta \psi(x_2)} j_\rho(x_4) \right\} j_{\rho'}(x_3) \left| 0 \right\rangle ,$$

$$(7.40)$$

$$D^{(8)}(x_1, \ldots, x_4) = -\left\langle 0 \right| j_\rho(x_4) \frac{\delta}{\delta \overline{\psi}(x_1)} \frac{\delta}{\delta \psi(x_2)} j_{\rho'}(x_3) \left| 0 \right\rangle ,$$

Let us take the expression $D^{(5)}$ and apply to it the completeness relation (2.6). We have

$$D^{(5)}(x_1, \ldots, x_4) = \left\langle 0 \right| \frac{\delta}{\delta \overline{\psi}(x_1)} \frac{\delta}{\delta \psi(x_2)} j_{\rho'}(x_3) \left| 0 \right\rangle \left\langle 0 \right| j_\rho(x_4) \left| 0 \right\rangle +$$

$$+ \left(\frac{1}{2\pi}\right)^3 \sum_n \int d\vec{k} \left\langle 0 \right| \frac{\delta}{\delta \overline{\psi}(x_1)} \frac{\delta}{\delta \psi(x_2)} j_{\rho'}(x_3) \left| n\vec{k} \right\rangle \left\langle n\vec{k} \right| j_\rho(x_4) \left| 0 \right\rangle .$$

On the other hand, as has already been pointed out in Section 4:

$$\left\langle 0 \right| j_\rho(x_4) \left| 0 \right\rangle = 0,$$

$$\left\langle n\vec{x} \right| j_\rho(x_4) \left| 0 \right\rangle = \left\langle n\vec{k} \right| j_\rho(0) \left| 0 \right\rangle e^{i\left\{ E_n(\vec{k})x^0_4 - (\vec{k}\vec{x}_4) \right\}}.$$

Further, the expression $\left\langle n\vec{k} \right| j_\rho(0) \left| 0 \right\rangle$ is equal to zero for one-meson and two-meson states, and so only states for which $E_n^2(k) - \vec{k}^2 \geqslant (3m)^2$ can contribute to the sum (7.41). Thus $D^{(5)}$, considered as a function of x_4, is represented by a superposition of exponentials $e^{i(p_4 x_4)}$ with p_4

satisfying the inequality $p_4^2 \geq (3m)^2$. Therefore:

$$\tilde{D}^{(5)}(p_1 \ldots p_4) = 0 \text{ if } p_4^2 < (3m)^2. \qquad (7.42)$$

quite analogously we find

$$\tilde{D}^{(6)}(p_1 \ldots p_4) = 0 \text{ if } p_3^2 < (3m)^2,$$

$$\tilde{D}^{(7)}(p_1 \ldots p_4) = 0 \text{ if } p_3^2 < (3m)^2, \qquad (7.43)$$

$$\tilde{D}^{(8)}(p_1 \ldots p_4) = 0 \text{ if } p_4^2 < (3m)^2$$

Now let us return to the relation (7.35) and multiply both sides of it by $\left\{(q + \frac{p+p'}{2})^2 - M^2\right\} \left\{(-q + \frac{p+p'}{2})^2 - M^2\right\}$. We obtain

$$\left\{(q + \frac{p+p'}{2})^2 - M^2\right\} \left\{(-q + \frac{p+p'}{2})^2 - M^2\right\} T_{\alpha\omega}(q) =$$

$$= \frac{-i}{(2\pi)^4} \overline{U^{+s'}(p')} \tilde{M}(p', -p, p_3, p_4) U^{+s}(p), \qquad (7.44)$$

and note that

$$\tilde{M}(p_1, p_2, p_3, p_4) = \left\{(p_1 + p_3) - M^2\right\} \left\{(p_1 + p_4)^2 - M^2\right\} \tilde{D}(p_1, p_2, p_3, p_4), \qquad (7.45)$$

$$M(x_1, \ldots x_4) = \left\{(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3})^2 + m^2\right\} \left\{(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4})^2 + M^2\right\} D(x_1, \ldots x_4).$$

Further, on the basis of (7.38), (7.42), (7.43), we may write

$$M(x_1 \ldots x_4) = \sum_{(1 \leq a \leq 4)} M^{(a)}(x_1 \ldots x_4) + N(x_1 \ldots x_4) \qquad (7.46)$$

where

$$M^{(a)}(x_1 \ldots x_4) =$$

$$= \left\{(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3})^2 + M^2\right\} \left\{(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4})^2 + M^2\right\} D^{(a)}(x_1 \ldots x_4). \qquad (7.47)$$

To determine the analytic structure of the functions $M^{(1)} \ldots M^{(4)}$, we make use of the following theorem.

Suppose a set of translation-invariant generalized functions to be given,

$$F_{ij;\nu}(x_1, \ldots x_4) \qquad \begin{array}{l} i = r,\ a \\ j = r,\ a \\ y = 1, \ldots, N \end{array}$$

transforming linearly under transformations L of the Lorentz group

$$F_{ij;\nu}(L\,x_1 \ldots L\,x_4) = \sum_{(1 \leq \nu' \leq N)} A_{\nu\,\nu'}(L)\, F_{i,j,\nu'}(x_1 \ldots x_4)$$

according to a representation A(L) of this group, which may be any of the usual tensor and spinor representations.

In addition, let the functions satisfy the conditions

$$F_{r,r,\nu}(x_1, \ldots x_4) = 0,\ x_3 \lesssim x_1,\ x_4 \lesssim x_2;\ F_{a,r,\nu}(x_1 \ldots x_4) = 0,\ x_3 \gtrsim x_1,\ x_4 \lesssim x_2; \tag{7.48}$$

$$F_{r,a,\nu}(x_1 \ldots x_4) = 0,\ x_3 \lesssim x_1,\ x_4 \gtrsim x_2;\ F_{a,a,\nu}(x_1, \ldots x_4) = 0,\ x_3 \gtrsim x_1,\ x_4 \gtrsim x_2;$$

$$\tilde{F}_{rj\nu}(p_1 \ldots p_4) - \tilde{F}_{aj\nu}(p_1 \ldots p_4) = 0;\ p_1^2 < (M + m)^2,\ p_3^2 < \sigma^2 m^2,$$

$$\tilde{F}_{ir\nu}(p_1 \ldots p_4) - \tilde{F}_{ia\nu}(p_1 \ldots p_4) = 0;\ p_2^2 < (M + m)^2;\ p_3^2 < \sigma^2 m^2, \tag{7.49}$$

where σ is a number greater than unity,

$$F_{ij\nu}(p_1 \ldots p_4) = 0,\ (p_1 + p_3)^2 < (M + m)^2. \tag{7.50}$$

Then there exists a positive number ρ_1 (depending only on σ) such that for $p_1, \ldots, p_4$ belonging to the region

$$p_1 + \dots + p_4 = 0$$

$$p_1^2 < M^2 + \rho_1 m^2, \; p_3^2 < (1 + \rho_1) m^2$$

$$p_2^2 < M^2 + \rho_1 m^2, \; \rho_4^2 < (1 + \rho_1) m^2, \; (p_1 + p_2)^2 < \rho_1 m^2,$$

a representation

$$\tilde{F}_{ij\nu}(p_1, \dots p_4) = \Sigma \, p_{i_1}^{\alpha_1} \dots p_{is}^{\alpha_s} \, \Phi_\omega \left\{ p_1^2, p_2^2, p_3^2, p_4^2, (p_1+p_2)^2, (p_1+p_3)^2 \right\} \tag{A}$$

holds, with a finite number of terms in the sum.

Here, $\Phi_\omega(Z_1, Z_2, Z_3, Z_4, Z_5, Z_6)$ are generalized functions of a real variable Z_6 and analytic functions of the complex variables $Z_1, \dots z_5$, regular in the region:

$$\text{Re } Z_1 < M^2 + \rho_1 m^2, \quad \text{Re } Z_3 < (1 + \rho_1) m^2, \quad \text{Re } Z_5 < \rho_1 m^2,$$

$$\text{Re } Z_2 < M^2 + \rho_1 m^2, \quad \text{Re } Z_4 < (1 + \rho_1) m^2, \quad |\text{Im } Z_\gamma| < \rho_1 m^2, \quad \gamma = 1, \dots 5.$$

In addition,

$$\Phi_\omega(Z_1, \dots, Z_5, Z_6) = 0 \quad \text{for} \quad Z_6 < (M + m)^2.$$

Let us first of all use this theorem for the components of the function $M^{(1)}(x_1 \dots x_4)$.

Let us put $D^{(1)}_{r,r}(x_1 \dots x_4) = D^{(1)}(x_1 \dots x_4)$,

$$D^{(1)}_{a,r}(x_1 \dots x_4) = - \left\langle 0 \left| \frac{\delta \eta(x_1)}{\delta \varphi_{\rho'}(x_3)} \cdot \frac{\delta \dot{j}_\rho(x_4)}{\delta \psi(x_2)} \right| 0 \right\rangle$$

$$D^{(1)}_{r,a}(x_1 \dots x_4) = \left\langle 0 \left| \frac{\delta j_{\rho'}(x_3)}{\delta \bar{\psi}(x_1)} \; \frac{\delta \bar{\eta}(x_2)}{\delta \varphi_\rho(x_4)} \right| 0 \right\rangle$$

$$D^{(1)}_{a,a}(x_1 \dots x_4) = - \left\langle 0 \left| \frac{\delta \eta(x_1)}{\delta \varphi_{\rho'}(x_3)} \; \frac{\delta \bar{\eta}(x_2)}{\delta \varphi_\rho(x_4)} \right| 0 \right\rangle \tag{7.51}$$

where, in accordance with (3.4)

$$\Lambda(x) = -i \frac{\delta S}{\delta \bar{\psi}(x)} S^+; \ \bar{\Lambda}(x) = i \frac{\delta S}{\delta \psi(x)} S^+.$$

Let us also put

$$M_{ij}^{(1)}(x_1, \ldots, x_4) = \left\{\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\right)^2 + M^2\right\} \left\{\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}\right)^2 + M^2\right\} D_{ij}^{(1)}(x_1, \ldots x_4).$$

The functions $D_{ij}^{(1)}$, and consequently $M_{ij}^{(1)}$ also will be translation-invariant and will transform as products of spinors.

In virtue of the causality condition, $D_{ij}^{(1)}$, $M_{ij}^{(1)}$ have the properties (7.48).

Let us now verify the conditions (7.49) with $\sigma = 3, m = m$. For this purpose, we shall proceed from the identities

$$\frac{\delta j(x_3)}{\delta \bar{\psi}(x_1)} + \frac{\delta \Lambda(x_1)}{\delta \varphi(x_3)} = i\,[\Lambda(x_1)\, j(x_3) - j(x_3) \Lambda(x_1)],$$

$$\frac{\delta j(x_4)}{\delta \psi(x_2)} - \frac{\delta \bar{\Lambda}(x_3)}{\delta \varphi(x_4)} = i\,[j(x_4)\bar{\Lambda}(x_2) - \bar{\Lambda}(x_2)\, j(x_4)],$$

which follow from (3.4).

We have, for example,

$$D_{r,r}^{(1)}(x_1, \ldots x_4) - D_{a,r}^{(1)}(x_1, \ldots x_4) =$$

$$= i \left\langle 0 \right| \Lambda(x_1)\, j(x_3) \frac{\delta j(x_4)}{\delta \psi(x_2)} \left| 0 \right\rangle - i \left\langle 0 \right| j(x_3) \Lambda(x_1) \frac{\delta j(x_4)}{\delta \psi(x_2)} \left| 0 \right\rangle$$

Applying here the completeness property to the first term on the right-hand side, we find that its Fourier transform is zero when $p_1^2 < (M + m)^2$. The Fourier transform of the second term of the right-hand side is zero for

$$p_3^2 < (3m)^2.$$

Therefore $\tilde{D}^{(1)}_{r,r}(p_1 \ldots, p_4) - \tilde{D}^{(1)}_{a,r}(p_1, \ldots p_4) = 0;\ p_1{}^2 < (M + m)^2;\ p_3{}^2 < (3m)^2;$

and $\tilde{M}^{(1)}_{r,r}(p_1 \ldots, p_4) - \tilde{M}^{(1)}_{a,r}(p_1, \ldots p_4) = 0;\ p_1{}^2 < (M + m)^2;\ p_3{}^2 < (3m)^2.$

The remaining conditions of this group are verified in the same way. Let us pass, at last, to the final condition (7.50). We have

$$\left\{ \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\right)^2 + M^2 \right\} D^{(1)}_{r,r}(x_1, \ldots, x_4) =$$

$$= \frac{1}{(2\pi)^3} \sum_n \int d\vec{k} \left\langle 0 \right| A(x_1, x_3) \left| n\vec{k} \right\rangle \left\langle n\vec{k} \right| \mathcal{B}(x_2, x_4) \left| 0 \right\rangle, \quad (7.52)$$

where

$$A(x_1, x_3) = \left\{ \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\right)^2 + M^2 \right\} C(x_1, x_3),$$

$$C(x_1, x_3) = \frac{\delta j(x_3)}{\delta \psi(x_1)};\ \mathcal{B}(x_2, x_4) = \frac{\delta j\ (x_4)}{\delta \psi(x_2)}.$$

But

$$\left\langle 0 \right| C(x_1, x_3) \left| n\vec{k} \right\rangle = \left\langle 0 \right| C(x_1 - x_3, 0) \left| n\vec{k} \right\rangle e^{-i\left\{E_n(\vec{k})x_3{}^0 - \vec{k}\vec{x}_3\right\}} \quad (7.53)$$

$$\left\langle n\vec{k} \right| \mathcal{B}(x_2, x_4) \left| 0 \right\rangle = \left\langle n\vec{k} \right| \mathcal{B}(0, x_4 - x_2) \left| 0 \right\rangle e^{i\left\{E_n(\vec{k})x_2{}^0 - \vec{k}\vec{x}_2\right\}}$$

and therefore

$$\left\langle 0 \right| A(x_1, x_3) \left| n\,\vec{k} \right\rangle =$$

$$= \left\langle 0 \right| C(x_1 - x_3, 0) \left| n\,\vec{k} \right\rangle \left\{ M^2 - E_n{}^2(\vec{k}) + \vec{k}^2 \right\} e^{-i\left\{E_n(\vec{k})x_3{}^0 - \vec{k}\vec{x}_3\right\}} \quad (7.54)$$

On the other hand the expression

$$\left\langle n\,\vec{k} \right| \frac{\delta \dot{j}(x_4)}{\delta \psi(x_2)} \left| 0 \right\rangle$$

is equal to zero for states without nucleons, in accordance with the condition of the conservation of nuclear charge. For states n with only one nucleon

$$E_n^2(\vec{k}) - \vec{k}^2 = M^2,$$

as a result of which

$$\left\langle 0 \right| A(x_1, x_3) \left| n\, \bar{k} \right\rangle = 0.$$

Thus, to the sum (7.52) only those states contribute which contain at least one nucleon and one meson; for such states

$$E_n^2(\vec{k}) - \vec{k}^2 > (M + m)^2.$$

We now notice from equations (7.52), (7.53), (7.54) that the function on the left-hand side of (7.52) is a super position of the exponentials of the type

$$\exp i \left\{ q_1(x_1 - x_3) + q_2(x_4 - x_2) + k(x_2 - x_3) \right\} \quad ; k^2 \geqslant (M + m)^2.$$

Introducing the usual four-vectors:

$$p_1 = -q_1;\ p_2 = q_2 - k;\ p_3 = q_1 + k;\ p_4 = -q_2,$$

we see that

$$(p_1 + p_3)^2 = k^2 \geqslant (M + m)^2.$$

Hence it follows that the Fourier transform on the left-hand side of (7.52), and therefore also $M_{2,2}^{(1)}(p_1, \ldots, p_4)$, is zero when $(p_1 + p_3)^2 < (M + m)^2$. The remaining conditions (7.50) are verified in the same way.

We may now make use of the representation (A) for $\widetilde{M}^{(1)}(p_1 \ldots, p_4)$. It is not difficult to show that a representation of the same type is true for $\widetilde{M}^{(4)}(p_1, \ldots, p_4)$. For $\widetilde{M}^{(2)}(p_1, \ldots, p_4)$ and $\widetilde{M}^{(3)}(p_1, \ldots, p_4)$ the representation is obtained from (A) by substituting p_4 for p_3.

Thus, taking into consideration (7.44), (7.47), we arrive at the following result.

If

$$p^2 < M^2 + \rho_1 m^2, \; p'^2 < M^2 + \rho_1 m^2, \tag{7.55}$$

$$\left(\frac{p-p'}{2} + q\right)^2 < (1 + \rho_1)m^2; \; \left(\frac{p-p'}{2} - q\right)^2 < (1 + \rho_1)m^2,$$

(ρ_1 is a number $(p-p')^2 < \rho_1 m^2$, < 8), then the following representation holds

$$\left\{\left(q + \frac{p+p'}{2}\right)^2 - M^2\right\} \left\{\left(-q + \frac{p+p'}{2}\right)^2 - M^2\right\} T_{\alpha\omega}(q) = \tag{7.56}$$

$$= \overline{U^{+s}}(\bar{p}')\,\mathcal{M}(p, p', q)\, U^{+s}(\bar{p}),$$

$$\mathcal{M}(p, p', q) =$$

$$= \sum_\nu \mathcal{P}_\nu(p, p', q)\, \phi_\nu \left\{p^2, p'^2, \left(\frac{p-p'}{2} + q\right)^2, \left(\frac{p-p'}{2} - q\right)^2, (p-p')^2, \left(\frac{p+p'}{2} + q\right)^2\right\}$$

$$+ \sum_\mu Q_\mu(p, p', q)\, \phi_\mu \left\{p^2, p'^2, \left(\frac{p-p'}{2} + q\right)^2, \left(\frac{p-p'}{2} - q\right)^2, (p-p')^2, \left(\frac{p+p'}{2} - q\right)^2\right\} \tag{7.57}$$

in which

$$\mathcal{P}_\nu(p, p', q), \; Q_\mu(p, p', q),$$

are polynomials in the components of p, p', q, and $\phi(Z_1 \ldots Z_5, Z_6)$ are generalized functions of the real variable Z_6 and analytic functions of the complex variables $z_1 \ldots z_5$, regular in the region

$$\mathrm{Re}\, Z_1 < M^2 + \rho_1 m^2, \; \mathrm{Re}\, Z_3 < (1 + \rho_1)m^2, \; \mathrm{Re}\, Z_5 < \rho_1 m^2,$$

$$\text{Re } Z_2 < M^2 + \rho_1 m^2, \ \text{Re } Z_4 < (1 + \rho_1)m^2, \quad |\text{Im } Z_\nu| < \rho_1 m^2, \ \nu = 1, \ldots 5.$$

In addition, $\phi(Z_1, \ldots Z_5, Z_6) = 0$ when $Z_6 < (M + m)^2$. We use this result for the case when

$$p^o = p'^o = \sqrt{M^2 + \vec{p}^{\,2}}, \ \vec{p} + \vec{p}\,' = 0, \ (p-p')^2 = -4\vec{p}^{\,2},$$

$$q^o = E, \ \vec{q} = \lambda\vec{e}, \ \vec{e} \cdot \vec{p} = 0,$$

taking account only of the dependence on the variables E, λ. Since the operation S cancels the odd powers of λ from (7.56), (7.57), we find

$$\left\{(\tau + \vec{p}^{\,2})^2 - 4E^2(M^2 + \vec{p}^{\,2})\right\} ST(E, \tau) = \tag{7.58}$$
$$= \phi_1\left\{M^2 + \vec{p}^{\,2} + \tau + 2E\sqrt{M^2 + \vec{p}^{\,2}}, \tau\right\} + \phi_2\left\{M^2 + \vec{p}^{\,2} + \tau - 2E\sqrt{M^2 + \vec{p}^{\,2}}, \tau\right\}$$

if $\tau - \vec{p}^{\,2} < (1 + \rho_1)m^2$, $E^2 > \tau$. Here $\phi_1(\xi, \tau)$, $\phi_2(\xi, \tau)$ are generalized functions of the real variable ξ, and analytic functions of the complex variable τ, regular in the region

$$\text{Re}(\tau) < (1 + \rho_1)m^2 + \vec{p}^{\,2}; \quad |\text{Im } \tau| < \rho_1 m^2. \tag{7.59}$$

In addition,

$$\left.\begin{aligned} \phi_1(\xi, \tau) &= 0 \\ \phi_2(\xi, \tau) &= 0 \end{aligned}\right\} \text{for} \quad \xi < (M+m)^2, \tag{7.60}$$

Let $\vec{p}^{\,2}$ be restricted by the inequality

$$\vec{p}^{\,2} < \frac{Mm - (\frac{1}{2} + \rho)m^2}{2}.$$

Then $2\tau + \vec{p}^{\,2} - t < 0$ if $t + \vec{p}^{\,2} < 2Mm + m^2$. Therefore we may define by the equations

$$F_1(t, \tau) = \frac{\Phi_1(t + M^2 + \vec{p}^{\,2}, \tau)}{2\tau + \vec{p}^{\,2} - t} ,$$

$$F_2(t, \tau) = \frac{\phi_2(t + M^2 + \vec{p}^{\,2}, \tau)}{2\tau + \vec{p}^{\,2} - t}$$

two generalized functions of t which are analytic with respect to τ and regular in the region (7.59) and satisfy

$$\left.\begin{array}{l} F_1(t, \tau) = 0 \\ F_2(t, \tau) = 0 \end{array}\right. \quad \text{for } t < 2Mm + m^2 - \vec{p}^{\,2} .$$

Since $\mathcal{S}\, T(E, \tau)$ coincides with $\mathcal{S}\, f(E, \tau)$ when

$$(\tau + \vec{p}^{\,2})^2 - 4E^2(M^2 + \vec{p}^{\,2}) \neq 0,$$

we obtain from (7.58) the final result,

$$\mathcal{S}\, f(E, \tau) = F_1(2E\sqrt{M^2 + \vec{p}^{\,2}} + \tau, \tau) + F_2(-2E\sqrt{M^2 + \vec{p}^{\,2}} + \tau, \tau),$$

which completes our proof.