QUANTUM LOGICS

by

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These notes constitute a summary of the essential details of the theory presented by Professor von Neumann in lectures on this subject. Only definitions, theorems and some lemmas are stated, detailed proofs being omitted. The axioms for this subject are to be found in the notes for the A. M. S. colloq uium lectures (Summer, 1937). Small Roman letters will be used for elements of L.

Geometrical Character of L: The system L is a projective (discrete) or continuous geometry, i.e., satisfies Axioms I°- VI° (Part I). Hence (Part I, Definition 6.11 theorem 6.9, also Ch. VII) there exists a unique real-valued dimension function D(a), $a \in L$ such that

$$0 \leqq D(a) \leqq 1 ,$$

$$D(0) = 0, \quad D(1) = 1 ,$$

$$D(a+b) + D(ab) = D(a) + D(b),$$

$$a < b \rightarrow \quad D(a) < D(b) ,$$

$$\lim_{i \to \infty} D(a_i) = D(\sum (a_i \; ; i = 1, 2, \ldots))$$

$$\text{if} \quad a_i \leqq a_j \qquad (i \leqq j) ,$$

$$\lim_{i \to \infty} D(a_i) = D \quad (\prod (a_i \; ; i = 1, 2, \ldots)$$

$$\text{if} \quad a_i \geqq a_j \qquad (i \leqq j) .$$

The range Δ of D is a set Δ_N (N=1,2, ...) or the set Δ_∞, where

$$\Delta_N = (0, \tfrac{1}{N}, \tfrac{2}{N}, \ldots , 1), \quad \Delta_\infty = (x; \; 0 \leqq x \leqq 1).$$

In the former cases $L = L_N$ is a projective geometry of N - 1 dimensions; in the last case $L = L_\infty$ is a continuous geometry. We assume that in case $L = L_N$, $N = 4, 5, \ldots$.

<u>The Ring</u> $\mathcal{R}$: There exists a regular ring $\mathcal{R} = (\xi, \eta, \cdots)$ unique up to isomorphisms such that L is isomorphic to the lattice $\bar{R}_{\mathcal{R}}$ of all principal right ideals in $\mathcal{R}$ (cf. Part II, Theorem 14.1). L and $\bar{R}_{\mathcal{R}}$ are identified in what follows. The lattice $L' \equiv \bar{L}_{\mathcal{R}}$ of all principal left ideals in $\mathcal{R}$ is dual to $L = \bar{R}_{\mathcal{R}}$ and hence satisfies I° - VI° also; the unique dimension function for L' is denoted by D'. There exists a unique real-valued <u>rank-function</u> $R(\xi)$, $\xi \in \mathcal{R}$ such that

$$0 \leqq R(\xi) \leqq 1,$$

$$R(0) = 0, \quad R(1) = 1,$$

$$R(\xi\eta) \leqq R(\xi), \quad R(\eta),$$

$$\epsilon^2 = \epsilon, \quad \eta^2 = \eta, \quad \epsilon\eta = \eta\epsilon = 0 \rightarrow R(\epsilon + \eta) = R(\epsilon) + R(\eta),$$

$$R(\xi + \eta) \leqq R(\xi) + R(\eta),$$

$R(\xi) = R(\eta)$ if and only if $\xi = \alpha\eta\beta$ with $\alpha, \beta \in \mathcal{R}$ such that α^{-1}, β^{-1} exist,

$R(\xi) = 1 \rightleftarrows \xi^{-1}$ exists,

$R(\xi - \eta)$ is a metric in $\mathcal{R}$; $\mathcal{R}$ is complete in this topology.

$$R(\xi) = D((\xi)_r) = 1 - D((\xi)_l{}^r)$$
$$= D'((\xi)_l) = 1 - D'((\xi)_r{}^l).$$

(cf. Part II, Lemma 17.2, Theorems 17.1, 2). The function R is invariant under any automorphism or antiautomorphism of $\mathcal{R}$ (cor. 2, Theorem 17.2). The antiautomorphism $a \rightarrow -a$ is generated by an involutory antiautomorphism $\xi \rightarrow \xi^*$ of $\mathcal{R}$, i.e.,

$$(\xi + \eta)^* = \xi^* + \eta^*, \quad (\xi\eta)^* = \eta^*\xi^*,$$

$= (\xi)_r = (\xi^*)_l{}^r, \quad \xi^{**} = \xi;$

every $\xi \varepsilon \mathcal{R}$ is of the form $\xi = \varsigma\varsigma^*\xi$ $(\varsigma \varepsilon \mathcal{R})$;

for every $\xi \varepsilon \mathcal{R}$ there is a unique $\epsilon \varepsilon \mathcal{R}$ such that

$\epsilon^2 = \epsilon^* = \epsilon, \quad (\xi)_r = (\epsilon)_r.$

The center $\mathcal{Z} = (\xi;\ \xi\eta = \eta\xi$ for every $\eta \varepsilon \mathcal{R})$ of $\mathcal{R}$ is a commutative field.

Definitions:

ξ, η commute in case $\xi\eta = \eta\xi$.

ξ, η anticommute in case $\xi\eta = -\eta\xi$.

$\epsilon \varepsilon \mathcal{R}$ is idempotent in case $\epsilon^2 = \epsilon$,

$\xi \varepsilon \mathcal{R}$ is hermitian in case $\xi^* = \xi$.

$\xi \varepsilon \mathcal{R}$ is antihermitian in case $\xi^* = -\xi$.

$\xi \varepsilon \mathcal{R}$ is unitary in case ξ^{-1} exists and $\xi^{-1} = \xi^*$;

if δ is idempotent, then $\mathcal{R}(\delta) \equiv (\xi;\ \xi\delta = \delta\xi = \xi)$;

$\xi \varepsilon \mathcal{R}$ is partially unitary in case ξ has the mutually equivalent properties $\xi^*\xi\xi^* = \xi^*$, $\xi\xi^*\xi = \xi$, $\xi\xi^*$ idempotent, $\xi^*\xi$ idempotent;

for ξ partially unitary, $(\xi^*)_r = (\xi^*\xi)_r$ is the initial ideal of ξ, and $(\xi)_r = (\xi\xi^*)_r$ is the final ideal of ξ.

Theorem: $P(1, a) = D(a)$ $\quad (a \varepsilon L)$.

Theorem: $D(a) = D(b)$ if and only if there exists a partially unitary $\rho \varepsilon \mathcal{R}$ with initial ideal a and final b; $D(a) = D(b)$ if and only if there exists a unitary $\rho \varepsilon \mathcal{R}$ such that the ring correspondence $\xi \to \rho\xi\rho^{-1}$ carries a into b.

Definiteness: An element $\alpha \varepsilon \mathcal{R}$ is definite in case it is of the form $\alpha = \xi^*\xi$. $\alpha = 0$ if and only if $\alpha, -\alpha$ are both definite.

$$\alpha, \beta \text{ definite} \rightarrow \alpha + \beta \text{ definite}, (\alpha + \beta)_r = (\alpha)_r + (\beta)_r.$$

$$\text{if } \alpha + \beta = 0 \text{ then } \alpha = \beta = 0.$$

$$\alpha \text{ definite}, \alpha \varepsilon \mathcal{R}(\delta) \rightarrow \alpha = \xi^* \xi \text{ with } \xi \varepsilon \mathcal{R}(\delta).$$

An application of these results is that the characteristic ρ_0 of $\mathfrak{Z}$ is zero. Hence the rational numbers may be isomorphically imbedded into $\mathfrak{Z}$. Denote the set of rational numbers by $P = (\kappa, \lambda, \cdots) \; (\subset \mathfrak{Z})$. Define for $\alpha, \beta \varepsilon \mathcal{R}$, $\alpha \leqq \beta$ to mean $\beta - \alpha$ definite. This order relation coincides for rational numbers with the usual order relation for rational numbers. If ϵ, η are hermitian idempotents, then $\epsilon \leqq \eta$ is equivalent to $\mathcal{R}(\epsilon) \subset \mathcal{R}(\eta)$. The relation $<$ partially orders $\mathcal{R}$.

If κ is rational and $\alpha \varepsilon \mathcal{R}$, define $\|\alpha\| \leqq \kappa$ to mean $\kappa^2 - \alpha^* \alpha$ definite, i.e., $\alpha^* \alpha \leqq \kappa^2$. The relation $\|\alpha\| \leqq \kappa$ has the properties

$$\|\alpha\| \leqq 0 \rightarrow \alpha = 0,$$

$$\kappa, \rho \varepsilon \mathcal{R}, \rho \neq 0 \rightarrow (\|\alpha\| \leqq \kappa \rightleftarrows \|\rho\alpha\| \leqq |\rho|\kappa).$$

$$\|\alpha\| \leqq \kappa, \|\beta\| \leqq \lambda \rightarrow \|\alpha + \beta\| \leqq \kappa + \lambda, \ \|\alpha\beta\| \leqq \kappa\lambda,$$

$$\|\alpha\| \leqq \kappa \rightleftarrows \|\alpha^*\| \leqq \kappa,$$

$$0 \leqq \kappa \leqq \lambda \quad \|\alpha\| \leqq \kappa \rightarrow \|\alpha\| \leqq \lambda,$$

$$\rho \varepsilon P \quad \|\rho\| \leqq \kappa \rightarrow |\rho| \leqq \kappa.$$

An element $\alpha \varepsilon \mathcal{R}$ is <u>bounded</u> in case there exists $\kappa \varepsilon P$ such that $\|\alpha\| \leqq \kappa$. Every rational κ is bounded; the set B of all bounded elements $\alpha \varepsilon \mathcal{R}$ is a <u>hermitian subring</u> of $\mathcal{R}$, i.e.,

$$\alpha, \beta \varepsilon B, \rho \varepsilon P \rightarrow \alpha^*, \alpha\beta, \alpha + \beta, \rho\alpha \varepsilon B.$$

If α is bounded, then α is of the form $\alpha = \sum_{i=1}^{n} \omega_i$ (ω_i unitary), and conversely. Moreover $\alpha \varepsilon B \rightleftarrows \kappa \leqq \alpha \leqq \lambda$ with $\kappa, \lambda \varepsilon P$; $\alpha \varepsilon B \rightleftarrows \alpha^*\alpha \varepsilon B$.

<u>Polynomials and Rational Functions:</u> A rational polynomial p(X),

X real, is a function of the form

$$p(X) = \alpha_0 X^n + \alpha_1 X^{n-1} + \dots + \alpha_n, \qquad \alpha_j \in P.$$

The set of all such polynomials is denoted by $\mathcal{P}$. For every such polynomial $p(X)$ there is defined a corresponding function

$$p(\alpha) = \alpha_0 \alpha^n + \alpha_1 \alpha^{n-1} + \dots + \alpha_n \qquad (\alpha \in \mathcal{R}).$$

The correspondence $p(X) \to p(\alpha)$ is possibly one-to-many, but it preserves the operations $+$, $\cdot$. A fundamental lemma is the following: If $\kappa, \lambda \in P$ and $p \in \mathcal{P}$ such that $\kappa \leqq X \leqq \lambda \to p(X) > 0$, then $p(\alpha)$ is definite if $\alpha \in \mathcal{R}$, $\kappa \leqq \alpha \leqq \lambda$.

A rational <u>quotient function</u> $\varphi(X)$, X real, is a function of the

$$\varphi(X) = \frac{p(X)}{q(X)} \qquad (p, q \in \mathcal{P},\ q \not\equiv 0).$$

The set of all such functions os denoted by Q. If $\varphi \in Q$ and if $\varphi(X)$ is finite for $\kappa \leqq X \leqq \lambda$, then $\varphi(X) = \dfrac{p(X)}{q(X)}$ with $p, q \in \mathcal{P}$, $q(X) \neq 0$ for $\kappa \leqq X \leqq \lambda$; under these conditions $q(\alpha)^{-1}$ exists for $\kappa \leqq \alpha \leqq \lambda$ and $\varphi(\alpha) = p(\alpha)\, q(\alpha)^{-1}$ depends only on $\varphi(X)$ and not on its representing polynomials $p(X)$ and $q(X)$.

Lemma: Let $\kappa \leqq \alpha \leqq \lambda$ (with $\kappa, \lambda \in P$) and $\varphi(x) \in Q$ such that $\varphi(X)$ is finite and >0 for $\kappa \leqq X \leqq \lambda$. Then $\varphi(\alpha)$ is definite.

Lemma: Let $\kappa \leqq \alpha \leqq \lambda$ (with $\kappa, \lambda \in P$) and $\varphi(X) \in Q$

(a) If $\mu, \nu \in P$ such that $\mu < \varphi(X) < \nu$ whenever $\kappa \leqq X \leqq \lambda$, then $\mu \leqq \varphi(\alpha) \leqq \nu$.

(b) If $\nu \in P$ such that $|\varphi(X)| < \nu$ whenever $\kappa \leqq X \leqq \lambda$, then $\|\varphi(\alpha)\| \leqq \nu$.

Traces: If δ is a hermitian idempotent, then a real-valued function $t(\alpha)$, $\alpha \varepsilon \mathcal{R}(\delta)\cdot B$ is called a trace in $\mathcal{R}(\delta)$ in case

a) $t(\delta) = R(\delta)$

b) $t(\alpha + \beta) = t(\alpha) + t(\beta)$

c) $t(\alpha^*) = t(\alpha)$

d) $\alpha > 0 \rightarrow t(\alpha) > 0$

e) $\beta \varepsilon \mathcal{R}(\delta)$, β unitary $\rightarrow t(\beta\alpha\beta^*) = t(\alpha)$.

Two essential properties of a trace are $t(\epsilon) = R(\epsilon)$ for $\epsilon \leqq \delta$, ϵ a hermitian idempotent, and $t(\alpha\beta) = t(\beta\alpha)$. The first of these is equivalent to a) if b) - e) are assumed, and the second is equivalent to e) if b) - d) are assumed. If δ, ϵ are hermitian idempotents such that $\delta \leqq \epsilon$, then a trace t in $\mathcal{R}(\epsilon)$ is an extension of a trace t' in $\mathcal{R}(\delta)$ in case $t'(\alpha) = t(\alpha)$ for $\alpha \varepsilon \mathcal{R}(\delta)$. If either t or t' is given, the other is uniquely determined, provided $\epsilon, \delta \neq 0$.

Absolute Values: For every trace t in $\mathcal{R}$ we define

$$|\alpha| \equiv \sqrt{t(\alpha^*\alpha)} = \sqrt{t(\alpha\alpha^*)} \geq 0$$

for every $\alpha \varepsilon B$. For $\rho \varepsilon P$, $|\rho|$ is the usual absolute value of ρ regardless of the choice of t. The function $|\alpha|$ (for a fixed trace) has the properties

$$|0| = 0; \quad \alpha \neq 0 \rightarrow |\alpha| > 0;$$

$$\rho \varepsilon P \rightarrow |\rho\alpha| = |\rho||\alpha| \qquad (\alpha \varepsilon \mathcal{R});$$

$$|\alpha + \beta| \leqq |\alpha| + |\beta|.$$

Hence $|\alpha - \beta|$ is a linear metric for $\mathcal{R}$ (with coefficients from P). This metric is the trace metric for t. If $\|\alpha\| \leqq K$, then $|\alpha\beta|$,

$|\beta\alpha| \geqq \kappa|\beta|$; an important special case is obtained for $\beta = 1$, i. e., $\|\alpha\| \leqq \kappa \rightarrow |\alpha| \leqq \kappa$.

Lemmas:

1) For every $\alpha \in \mathcal{R}$ $(1+\alpha^*\alpha)^{-1}$, $(1+\alpha\alpha^*)^{-1}$ exist and are >0, $\leqq 1$; moreover

$$\alpha^*\alpha(1+\alpha^*\alpha)^{-1} = \alpha^*(1+\alpha\alpha^*)^{-1}\alpha = (1+\alpha^*\alpha)^{-1}\alpha^*\alpha,$$

$$\alpha\alpha^*(1+\alpha\alpha^*)^{-1} = \alpha(1+\alpha^*\alpha)^{-1}\alpha^* = (1+\alpha\alpha^*)^{-1}\alpha\alpha^*,$$

and these quantities are all $\geqq 0$, < 1. Finally,

$$\alpha(1+\alpha^*\alpha)^{-1} = (1+\alpha\alpha^*)^{-1}\alpha,$$

$$\alpha^*(1+\alpha\alpha^*)^{-1} = (1+\alpha^*\alpha)^{-1}\alpha^*,$$

and these quantities satisfy $\|\xi\| \leqq \frac{1}{2}$.

2) If $\beta \geqq \kappa > 0$ or $\beta \leqq -\kappa < 0$ with $\kappa \in P$ then there exists β^{-1}, $\|\beta^{-1}\| \leqq \frac{1}{\kappa}$, and $0 < \beta^{-1} \leqq \frac{1}{\kappa}$ or $-\frac{1}{\kappa} \leqq \beta^{-1} < 0$ in the respective cases.

3) If $p(X) \in \mathcal{P}$ and $\kappa, \lambda \in P$, then there exists a real $\gamma_0 = \gamma_0(p, \kappa, \lambda)$ such that

$$|p(\alpha) - p(\ )| \leqq \gamma_0|\alpha - \beta|$$

for every α, β with $\kappa \leqq \alpha \leqq \lambda$, $\kappa \leqq \beta \leqq \lambda$, and for every trace t in $\mathcal{R}$.

4) Let a hermitian idempotent δ and a sequence $(\alpha_i;\ i=1,2, \ldots)$ be given such that $\alpha_i \in \mathcal{R}(\delta)$, $\|\alpha_i\| \leqq \frac{1}{3}$ and $\lim_{i,j\to\infty} |\alpha_i - \alpha_j| = 0$ uniformly for all traces t. Then there exists a sequence $(\beta_i;\ i=1,2, \ldots)$ such that $\beta_i \in \mathcal{R}(\delta)$ and

$$\lim_{i\to\infty} |\alpha_i - 2\beta_i(1+\beta_i^*\beta_i)^{-2}(1-\beta_i^*\beta_i)| = 0$$

$$\lim_{i,j\to\infty} |(1+\beta_i^*\beta_i)^{-1}(1-\beta_i^*\beta_i) - (1+\beta_j^*\beta_j)^{-1}(1-\beta_j^*\beta_j) = 0,$$

$$\lim_{i,j\to\infty} |\beta_i(1+\beta_i^*\beta_i)^{-1} - \beta_j(1+\beta_j^*\beta_j)^{-1}| = 0,$$

each limit being uniformly for all traces t in $\mathcal{R}$.

Theorem: If $0 \neq a \leq b$, then $P(a, b) = \frac{D(b)}{D(a)}$.

The functions $\Pi_h(a, b)$: Define for $a \neq 0, 1$

$$\Pi_1(a, B) = D(a)P(a, b)$$

$$\Pi_2(a, b) = D(b) - D(-a)P(-a, b).$$

Then for $h = 1, 2$

$\Pi_h(a, b+c) = \Pi_h(a, b) + \Pi_h(a, c)$ for b, c orthogonal.

$\Pi_h(a, b) = D(b)$ $\qquad (b \leq a)$.

$\Pi_h(a, b) = 0$ $\qquad$ for a, b orthogonal.

$\Pi_h(a, b) > 0$ $\qquad$ for a, b not orthogonal.

Π_h is invariant under any $(<, -)$-automorphism of L.

Lemmas:

1) $\xi, \eta \in \mathcal{R}(\delta)$ (δ a hermitian idempotent) implies that $\eta^* \eta = \xi^* \xi$ is equivalent to the existence of ω such that $\omega^* \omega = \omega \omega^* = \delta$, $\omega \xi = \eta$.

2) $\delta, \epsilon', \epsilon''$ hermitian idempotents such that $\delta \epsilon' \delta = \delta \epsilon'' \delta$ implies the existence of a unitary element ω such that $\omega \delta = \delta$, $\omega \epsilon' \delta = \epsilon'' \delta$.

3) $\delta, \epsilon', \epsilon''$ hermitian idempotents such that $\delta \epsilon' \delta = \delta \epsilon'' \delta$ implies $\Pi_h((\delta)_r, (\epsilon')_r) = \Pi_h((\delta)_r, (\epsilon'')_r)$ $(h = 1, 2)$.

4) If $\gamma \leq \delta$, γ, δ hermitian idempotents, and if $R(\delta) + 2R(\gamma) \leq 1$, then ξ_1, ξ_2, ξ_3 definite, $\xi_1 + \xi_2 + \xi_3 = \gamma$ implies the existence of three hermitian idempotents $\epsilon_1, \epsilon_2, \epsilon_3$ such that

(a) $\epsilon_i \epsilon_j = 0$ $(i \neq j)$, whence $\sum_{i=1}^{3} \epsilon_i$ hermitian idempotent,

(b) $\gamma \leq \sum_{i=1}^{3} \epsilon_i$, whence $\sum_{i=1}^{3} \epsilon_i - \gamma$ hermitian idempotent,

(c) $\sum_{i=1}^{3} \epsilon_i - \gamma \leq 1 - \delta$

(d) $\xi_i = \delta \epsilon_i \delta$ $\qquad (i = 1, \ldots, 3)$.

5) If $\gamma \leq \delta$, γ, δ hermitian idempotents, and if $R(\delta) + 2R(\gamma) \leq 1$, then $\alpha \varepsilon \mathcal{R}(\delta)$, $0 \leq \alpha \leq 1$ if and only if α is of the form $\alpha = \delta \epsilon \delta$ with ϵ a hermitian idempotent, $\epsilon \leq 1 - \delta + \gamma$. The value of $\Pi_h((\delta)_r, (\epsilon)_r)$ depends only on $\delta \epsilon \delta$ and h.

6) If $\gamma \leq \delta$, γ, δ hermitian idempotents, if $R(\delta) + 2R(\gamma) \leq 1$, and if h = 1, 2, then there exists one and only one real-valued function $\tau(\alpha)$ defined for $0 \leq \alpha \leq 1$ such that $\tau(\delta \epsilon \delta) = \Pi_h((\delta)_r, (\epsilon)_r)$ for every hermitian idempotent $\epsilon \varepsilon \mathcal{R}(1 - \delta + \gamma)$. (The function $\tau(\alpha)$ depends on δ, γ and h = 1, 2.)

7) The function $\tau(\alpha)$ has the properties

(a) $\tau(\gamma) = R(\gamma)$,

(b) $0 \leq \alpha, \beta$, $\alpha + \beta \leq 1 \rightarrow \tau(\alpha + \beta) = \tau(\alpha) + \tau(\beta)$,

(c) $\tau(\alpha) \begin{cases} = 0 \text{ for } \alpha = 0 \\ > 0 \text{ for } \alpha \neq 0, \end{cases}$

(d) $\omega^* \omega = \omega \omega^* = \gamma \rightarrow \tau(\omega \alpha \omega^*) = \tau(\alpha)$.

8) If $\gamma \leq \delta$, γ, δ hermitian idempotents and if $R(\gamma) - 2R(\gamma) \leq 1$, then every hermitian bounded $\xi \varepsilon \mathcal{R}(\gamma)$ may be represented in the form $\xi = \rho \alpha + \sigma \gamma$, with $\rho, \sigma \varepsilon$ P, $\alpha \varepsilon \mathcal{R}(\gamma)$, $0 \leq \alpha \leq 1$. The same ρ, σ can be used for two elements ξ_1, ξ_2, and ρ may be chosen either positive or negative. The value of $\rho \tau(\alpha) + \sigma \tau(\gamma)$ depends only on ξ and not on its representation. This defines $\tau(\xi)$, for $\xi \varepsilon \mathcal{R}(\gamma)$, ξ hermitian, bounded. If $\eta \varepsilon \mathcal{R}(\gamma)$, η bounded, then define $t^\circ(\eta) = \tau(\frac{\xi + \xi^*}{2})$. The value of $t^\circ(\eta)$ depends only on η. The function $t^\circ(\eta)$ is a trace in $\mathcal{R}(\gamma)$. It depends on γ, δ and h = 1, 2.

Since there exist hermitian idempotents γ, δ with $0 \neq \gamma \leq \delta$, $R(\delta) + 2R(\gamma) \leq 1$, the existence of a trace $t^\circ(\eta)$ in $\mathcal{R}(\gamma)$ is established; hence a unique trace $\bar{t}(\alpha)$ in $\mathcal{R}$ exists such that

$\bar{t}(\delta\epsilon) = \bar{t}(\delta\epsilon\delta) = \pi_h((\delta)_{r^\circ}(\epsilon)_r)$ for $\epsilon \leqq 1 - \delta + \gamma$

This trace, $t_{\theta, h}(\alpha)$ depends on $R(\delta) = \theta$ and h only.

Theorem: For every $\theta \in \Delta$, $0 < \theta \leqq 1$, there exists a trace $\bar{t}_\theta(\alpha)$ in R such that

$$\left.\begin{array}{l}\delta, \epsilon \text{ hermitian idempotents} \\ R(\delta) = \theta\end{array}\right\} \rightarrow P((\delta)_{r^\circ}(\epsilon)_r) = \frac{\bar{t}_\theta(\delta\epsilon)}{R(\delta)}.$$

For each trace t_θ there is defined an absolute value $|\alpha|_\theta = \sqrt{t_\theta(\alpha\alpha^*)}$.

The completeness axiom (IX) yields the following completeness property of R:

Theorem: There exists $\theta \in \Delta$, $0 < \theta \leq 1$, with the following property: For every sequence (ξ_i; $i = 1, 2, \ldots$) such that

(a) there exists $K \in P$ such that $\| \xi_i \| \leq K$ $(i = 1, 2, \ldots)$,

(b) $\lim\limits_{i,j \to \infty} | \xi_i - \xi_j | = 0$ uniformly for all traces $t(\xi)$,

there exists a bounded $\xi \in R$ such that $\lim\limits_{i \to \infty} | \xi_i - \xi |_\theta = 0$.

Rational polynomial and rational quotient functions of $\alpha \in \mathfrak{R}$ have already been defined. Continuous functions may be defined in the following way. (The element $\theta \in \Delta$ appearing in subsequent statements is given by the preceding theorem; it need not be unique, but its existence is assured.) Let $\varphi(X)$ be a real valued function of the real variable X, defined and continuous for $K \leq X \leq \lambda$ $(K, \lambda \in P)$. Let $\alpha \in \mathfrak{R}$ be hermitian and bounded, $K \leq \alpha \leq \lambda$. Then there exists (Weierstrass' approximation theorem) a sequence ($p_i(X)$; $i = 1, 2, \ldots$) of rational polynomials such that

$$\lim_{i \to \infty} p_i(X) = p(X) \text{ uniformly in } K \leq X \leq \lambda.$$

For every such sequence p_i there exists a hermitian, bounded $\beta \in \mathfrak{R}$ such that

$$\lim_{i \to \infty} | p_i(\alpha) - \beta | = 0.$$

The element β is uniquely determined by $\varphi(X)$ and by α and is thus independent of the choice of the sequence p_i. The unique β is denoted by $\varphi(\alpha)$. When $\varphi(X)$ is a rational polynomial or rational quotient function, this definition of $\varphi(\alpha)$ agrees with that already given.

Theorem: Let $K \leq \alpha \leq \lambda$ $(K, \lambda \in P,\ \alpha \in \mathfrak{R})$. Suppose that to every real-valued function $\varphi(X)$ of the real variable X, defined and continuous for $K \leqq X \leqq \lambda$, there corresponds a hermitian element $\varphi'(\alpha) \varepsilon \mathfrak{H}$ such that the correspondence $\varphi(X) \to \varphi'(\alpha)$, though possibly many-to-one, preserves addition and multiplication, and that 1 and X correspond respectively to 1 and α. Then this isomorphism is uniquely determined and $\varphi'(\alpha) = \varphi(\alpha)$. It possesses the properties

(a) $\varphi(X) \geqq 0$ for $K \leqq X \leqq \lambda$ implies $\varphi(\alpha)$ definite,

(b) $\rho \leqq \varphi(X) \leqq \sigma$ (with $\rho, \sigma \varepsilon$ P) for $K \leqq X \leqq \lambda$ implies $\rho \leqq \varphi(\alpha) \leqq \sigma$,

(c) $|\varphi(X)| \leqq \sigma$ (with $\sigma \varepsilon$ P) for $K \leqq X \leqq \lambda$ implies $\|\varphi(\alpha)\| \leqq \sigma$,

(d) $\varphi(\alpha)$ is bounded for each α ($K \leqq \alpha \leqq \lambda$),

(e) if α, β commute (where $\beta \varepsilon \mathfrak{H}$), then $\varphi(\alpha), \beta$ commute.

Theorem: The set R of all real numbers may be mapped isomorphically on hermitian elements of $\mathfrak{Z}$ in one and only one way. This mapping leaves rational numbers fixed and carries each real number into a bounded element of $\mathfrak{Z}$. It is an isomorphism with respect to <.

The real numbers are identified with those elements of $\mathfrak{Z}$ to which they correspond. Thus $P \subset R \subset \mathfrak{Z} \subset \mathfrak{X}$. Moreover, if $\rho \varepsilon R$, $K \varepsilon P$, then $\|\rho\| \leqq K$ is equivalent to $|\rho| \leqq K$; also, if $\rho > 0$ and $\alpha < \beta$, then $\rho\alpha < \rho\beta$. Finally, for every $\rho \varepsilon$ R and every trace $t(\alpha)$ we have $t(\rho\alpha) = \rho t(\alpha)$ and $|\rho\alpha| = |\rho||\alpha|$ (where $|\rho|$ means the numerical absolute value of ρ).

Lemmas for the spectral theorem:

(1) If $\rho \varepsilon$ R and if $\varphi_\rho(X) \equiv \rho$, then for every hermitian bounded $\alpha \varepsilon \mathfrak{X}$ we have $\varphi_\rho(\alpha) = \rho$.

(2) Let $\alpha \varepsilon \mathcal{R}$ be hermitian, bounded. Then there exists a system of hermitian idempotents $(\epsilon_\rho(\alpha);\ \rho \varepsilon R)$ such that $\epsilon_\rho(\alpha)(\rho - \alpha)$ definite, $(1 - \epsilon_\rho(\alpha))(\alpha - \rho)$ definite, $\epsilon_\rho(\alpha) \leqq \epsilon_\rho(\sigma)$ for $\rho \leqq \sigma$ and $\epsilon_\rho(\alpha), \beta$ commute when α, β commute.

(3) If $K, \lambda \ \varepsilon\ R$ and $K \leqq \alpha \leqq \lambda$, α hermitian, bounded, then

(a) $\rho > \lambda$ implies $\epsilon_\rho(\alpha) = 1$.

(b) $\rho < K$ implies $\epsilon_\rho(\alpha) = 0$.

(c) $\rho \leqq \sigma$ implies $(\epsilon_\sigma(\alpha) - \epsilon_\rho(\alpha))(\sigma - \alpha)$ definite and $(\epsilon_\sigma(\alpha) - \epsilon_\rho(\alpha))(\alpha - \rho)$ definite.

(4) Under the hypotheses of (3),

$$\sum_{i=1}^{\ell} \rho_{i-1}(\epsilon_{\rho_i}(\alpha) - \epsilon_{\rho_{i-1}}(\alpha)) \leqq \alpha \leqq \sum_{i=1}^{\ell} \rho_i(\epsilon_{\rho_i}(\alpha) - \epsilon_{\rho_{i-1}}(\alpha))$$

for every choice of real numbers $\rho_0, \rho_1, \ldots, \rho_\ell$ such that $K > \rho_0 \leqq \rho_1 \leqq \ldots \leqq \rho_{\ell-1} \leqq \rho_\ell > \lambda$.

Spectral theorem: Let $\alpha \varepsilon \mathcal{R}$ $K, \lambda \ \varepsilon\ R$, $K \leqq \alpha \leqq \lambda$, α bounded, hermitian, and let $(\epsilon_\rho(\alpha);\ \rho \varepsilon R)$ be the hermitian idempotents of (2). Let $\rho_0, \rho_1, \ldots, \rho_\ell, X_1, \ldots, X_\ell, \delta$ be real numbers such that

$$K > \rho_0 \leqq X_1 \leqq \rho_1 \leqq \ldots \leqq X_{\ell-1} \leqq \rho_{\ell-1} \leqq X_\ell \leqq \rho_\ell > \lambda$$

$$\delta \varepsilon R,\ \delta > 0,\ \rho_i - \rho_{i-1} \leqq \delta \quad (i = 1, \ldots, \ell).$$

Then

$$\| \alpha - \sum_{i=1}^{\ell} X_i(\epsilon_{\rho_i}(\alpha) - \epsilon_{\rho_{i-1}}(\alpha)) \| \leqq \delta.$$

Theorem: There exists one and only one trace $t(\alpha)$ in $\mathcal{R}$.

This theorem enables us to define $|\alpha|$ in a unique fashion as $\sqrt{t(\alpha^*\alpha)} \geqq 0$; thus we have a unique trace metric $|\alpha - \beta|$ in $\mathcal{R}$. Every set $\|\alpha\| \leqq K$ (K fixed and in P) is complete in the trace metric.