SEMINAR ON DEGENERATION OF ALGEBRAIC VARIETIES*

Conducted by Phillip A. Griffiths

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## Seminar on Degeneration of Algebraic Varieties

Lecture 1: Some background and generalities -- Phillip A. Griffiths

(1) We want to think about algebraic families of algebraic varieties and discuss what happens when the "general" variety acquires singularities. The eventual idea is to replace the non-singular variety by something simpler such as a Picard or Albanese variety, homology or cohomology groups, or a period matrix, and find out as much as possible about the degeneration of this simpler object.

I want first to make precise the data of degeneration of an algebraic variety. The rough idea is that we take projective varieties $V_t$ given by equations

$$\left\{\begin{array}{l} f_1(x_o, \ldots, x_N; t) = 0 \\ \\ f_m(x_o, \ldots, x_N; t) = 0 \end{array}\right.$$

where the $f_\alpha(x, t)$ are homogeneous polynomials in $x_o, \ldots, x_N$ whose coefficients are holomorphic functions of $t$ in the unit $t$-disc $\Delta = \{t : |t| < 1\}$, and where we assume that $V_t$ is non-singular for $t \neq 0$. To make this more precise, we assume given a complex-analytic variety $X$ together with a holomorphic mapping $f : X \longrightarrow \Delta$ which satisfies the following: (i) there is a projective embedding $X \subset \mathbb{P}_N$; (ii) $f$ is proper and connected; (iii) if we set $V_t = f^{-1}(t)$ and $X^* = X - V_o$, then $X^*$ is smooth and $f$ has everywhere maximal rank on $X^*$ (we may say that $f$ is smooth on $X^*$).

We want to use Hironaka's resolution of singularities to arrive at what we will call a standard situation. First we use Hironaka to resolve the singularities of $X$. This is done by blowing up along subvarieties in $V_o$ and so $X^*$ is left unchanged. Next, we use Hironaka again to arrange that the divisor $V_o$ on $X$ should have normal crossings. As before this is done by blowing up in $V_o$ so that $X^*$ is left unchanged. We are now in a standard situation

$$\begin{array}{ccc} X^* & \subset & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta^* & \subset & \Delta \end{array}$$

where $X$ is a complex manifold and $f$ is given by

(1) $$x_1^{a_1} \cdots x_{n+1}^{a_{n+1}} = t$$

where $x_1, \ldots, x_{n+1}$ are local holomorphic coordinates on $X$. We should think of $X$ as a nice "filling in" or compactification of $X^*$ over $t = 0$.

(2) The general program of the seminar is as follows:

(I) Degeneration of curves and Abelian varieties.

These talks will begin next week by Alan Mayer and will in particular cover some old unpublished work of his and Mumford. This situation here is fairly well understood and makes a nice story. From these talks we will also find out what happens in general when we pass from $\{V_t\}_{t \in \Delta}$ to the family of Picard or Albanese varieties.

(II) The topology of degenerating varieties.

This will be a study of the topology of a standard situation

$$\begin{array}{ccc} X^* & \subset & X \\ \downarrow & & \downarrow \\ \Delta^* & \subset & \Delta \end{array} .$$

These talks will be started by Clemens. We are especially interested in analyzing the Picard-Lefschetz transformation $T : H_q(V) \longrightarrow H_q(V)$, which by definition is the automorphism on the homology of a general variety $V$ induced by displacing cycles around $t = 0$. In particular the monodromy theorem, which says that

$$(T^N - I)^{q+1} = 0 \qquad (N = \ell.\text{c.m. of } a_1, \ldots, a_{n+1} \text{ in } (1))$$

will be proved.

(III) Regularity of Picard-Fuchs equations.

Let

$$\begin{tikzcd} X^* \arrow[r, phantom, "\subset"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ \Delta^* \arrow[r, phantom, "\subset"] & \Delta \end{tikzcd}$$

be a standard situation and let $E \longrightarrow \Delta^*$ be the flat vector bundle whose fibre $E_t = H^q(V_t, \mathbb{C})$. Thus $E$ is induced from the trivial bundle with fibre $H^q(V, \mathbb{C})$ on the universal covering of $\Delta^*$ by the action of $\pi_1(\Delta^*) \cong \mathbb{Z}$ on $H^q(V, \mathbb{C})$ via the Picard-Lefschetz transformation $T$. The sheaf of locally constant sections of $E$ is just the Leray direct image sheaf $R^q_{f_*}(\mathbb{C})$.

Now it makes sense to speak of a holomorphically varying cohomology class $\omega(t) \in H^q(V_t, \mathbb{C})$ $(t \neq 0)$, which is by definition the same as a holomorphic section of $E \longrightarrow \Delta^*$. It also makes sense to speak of a holomorphic section of $E \longrightarrow \Delta^*$ which has a finite order pole at $t = 0$. The definition is essentially the following: A holomorphic section $\omega$ of $E \longrightarrow \Delta^*$ will have a finite order pole at $t = 0$ if we can find a "suitable" divisor $Z \subset X$ such that (i) the divisors $Z_t = Z \cdot V_t$ are well-defined, (ii) the complements $U_t = V_t - Z_t$ are affine varieties, (iii) the restrictions of $\omega(t)$ to $H^q(U_t, \mathbb{C})$ are given in the deRham sense by differential forms $\frac{\psi(t)}{t^\mu}$ where $\psi(t)$ is a rational $q$-form on $V_t$ which is holomorphic in the Zariski open set $U_t$ and which depends holomorphically on $t \in \Delta$.

For $\delta \in H_q(V, \mathbb{Z})$ and $\omega(t) \in H^q(V_t, \mathbb{C})$ a holomorphic section of $E \longrightarrow \Delta^*$ with only a pole at $t = 0$, the periods

$$\int_\delta \omega(t)$$

are multi-valued holomorphic functions on $\Delta^*$ which arise as solutions of the so-called Picard-Fuchs equations. We will prove that these D.E.'s have only regular singular points at $t = 0$ by proving the estimate

$$\left|\int_\delta \omega(t)\right| \leqq c|t|^{-M}.$$

(IV) Now all of (I) - (II) - (III) are in some sense part of the general program of studying the period matrices (to be defined later) of the smooth varieties $V_t$ as $T \longrightarrow 0$. For example, for $q = 1$ the period matrix of the holomorphic 1-forms on $V_t$ is a $g \times 2g$ matrix

$$\begin{matrix} \Omega_{11} & \cdots & \Omega_{1,2g} \\ \vdots & & \vdots \\ \Omega_{g1} & & \Omega_{g,2g} \end{matrix} \quad = \Omega(t)$$

which satisfies the Riemann bilinear relations

$$\begin{cases} \Omega Q {}^t\Omega = 0 \\ i\Omega Q {}^t\bar{\Omega} > 0 \end{cases} \qquad Q = -{}^tQ \in GL(2g, \mathbb{Q}) .$$

The lattice $\Lambda_t$ in $\mathbb{C}^g$ generated by the $2g$ columns of $\Omega(t)$ gives the Albanese variety as $A(V_t) = \mathbb{C}^q/\Lambda_t$. Thus studying $A(V_t)$ and $\Omega(t)$ are closely related. Also $\Omega(t)$ is locally holomorphic on $\Delta^*$ but it is not single-valued. Analytic continuation of $\Omega(t)$ around $t = 0$ effects the change

$$\Omega \longrightarrow \Omega T \qquad (T = \text{P. L.-transformation})$$

on $\Omega$. Thus studying $\Omega(t)$ is related to the topology of the standard situation. Finally, the regularity theorem (III) gives a first rather crude estimate on the behavior of $\Omega(t)$ as $t \longrightarrow 0$.

Now the full asymptotic study of $\Omega(t)$ has by no means been carried out. Essentially there is only the case $q = 1$, a little on surfaces, and some isolated examples known. It seems almost certain that a proper understanding of the asymptotic behavior of $\Omega(t)$ will involve (i) the reduction theory of arithmetic groups (i.e. study of fundamental domains), (ii) the geometry of certain homogeneous complex manifolds (the period matrix domains) with special attention to their boundary components, and (iii) the use of what Chern calls hyperbolic complex analysis (influence of curvature on holomorphic mappings). These things will all be discussed (according to how much is known about them) toward the end of the seminar.

Lectures 2 and 3: Compactification of the variety of moduli of curves* -- Alan L. Mayer

Let $\mathcal{M}_g$ = variety of moduli of nonsingular curves of genus $g$

$\mathcal{H}_g$ = Siegel upper half-plane of rank $g$

$\Gamma_g$ = modular group of rank $g$

$\mathcal{V}_g = \mathcal{H}_g/\Gamma_g$ = variety of moduli of principally polarized abelian varieties of dimension $g$

$\mathcal{V}_g^* = \mathcal{V}_g \cup \mathcal{V}_{g-1} \cup \dots \cup \mathcal{V}_0$ = Satake's compactification, with its natural structure or a normal projective algebraic variety

Assigning to each curve its (canonically polarized) jacobian variety we have (Torelli's theorem) $\mathcal{M}_g \subset \mathcal{V}_g$ and it is known (Baily) that $\mathcal{M}_g$ is a locally closed algebraic subvariety of $\mathcal{V}_g^*$. Let $\mathcal{M}_g^*$ be the closure of $\mathcal{M}_g$ in $\mathcal{V}_g^*$. Let $\mathcal{M}'_g$ = the set of points in $\mathcal{V}_g$ representing abelian varieties which are products of jacobians of curves of genera $< g$, with the product polarization. The structure of $\mathcal{M}_g^*$ is given by:

Theorem 1. $\mathcal{M}_g^* = \bigcup_{0\leq h\leq g} (\mathcal{M}_h \cup \mathcal{M}'_h)$.

(Previously Hoyt had shown $\mathcal{M}_g^* \cap \mathcal{V}_g \subset \mathcal{M}_g \cup \mathcal{M}'_g$.) The idea of the proof is that points in $\mathcal{M}_g$ approach a boundary point when the corresponding curves degenerate to a curve with singularities, the boundary point representing the product of the jacobian varieties of the normalized components of the singular curve. One can reduce such degeneration to the case where the singular curve is a curve with nodes, i.e. a reduced connected curve with at most ordinary double points (nodes) as singularities. To be precise, let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and $\mu_N : D \longrightarrow D$ by $z \longmapsto z^N$.

Definition. A degenerating family of curves is a connected

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* These talks are based on unpublished joint work with Mumford done in 1963-64.

proper morphism $p : V \longrightarrow D$ where $V$ is a normal analytic surface, and $V - p^{-1}(0)$ is a manifold on which $dp \neq 0$. (Thus for $z \neq 0$ $p^{-1}(z)$ is a nonsingular curve.) The family is called <u>normal</u> if $V$ is nonsingular and $p^{-1}(0)$ is a curve with nodes.

We can reduce arbitrary degeneration to normal degeneration by the following "normalization theorem" of Mumford.

<u>Proposition</u> 1. Any degenerating family $p : V \longrightarrow D$ is dominated by a normal degenerating family $\tilde{p} : \tilde{V} \longrightarrow D$ in the sense that there is a commutative diagram

$$\begin{array}{ccc} \tilde{V} & \longrightarrow & V \\ \downarrow \tilde{p} & & \downarrow p \\ D & \xrightarrow{\mu_N} & D \end{array}$$

for some integer $N > 0$ such that $\tilde{V} - \tilde{p}^{-1}(0)$ is the pull-back (fibred product) of $V - p^{-1}(0)$ via $\mu_N : D - \{0\} \longrightarrow D - \{0\}$.

The idea of the proof is as follows: by the usual resolution of singularities, we may assume $V$ is nonsingular and $p^{-1}(0) = \sum_{i=1}^{r} n_i C_i$ where the $C_i$ are nonsingular and meet transversally. Let $N = \ell.c.m.\ (n_1, \ldots, n_r)$ and let $V'$ be the normalization of the fibre product $V \times_D D$ defined by $\mu_N : D \longrightarrow D$, and let $\tilde{V}$ be its desingularized model, $\tilde{p} : \tilde{V} \longrightarrow D$ the obvious projection. One must show that $\tilde{p}^{-1}(0)$ is a curve with nodes. One needs two lemmas.

<u>Lemma</u> 1. Let $(x, y)$ be local coordinates for a nonsingular point $P$ on a surface, and $z = x^m y^n$, $(m, n) = 1$. Let $\zeta^{mn} = z$. Then the integral closure $R$ of $\mathcal{O}_P[\zeta]$ is regular and has local coordinates $u, v$ with $\zeta = uv$.

<u>Proof.</u> Let $an + bm = 1$, $u = \zeta^{bn} x^{-b} y^{a}$, $v = \zeta^{am} x^{b} y^{-a}$ so $u^m = y$, $v^n = x$ and $u, v \in R$. But $\mathcal{O}_P = \mathbb{C}\{x, y\}$ giving $R = \mathbb{C}\{u, v\}$, while clearly $\zeta = uv$.

Lemma 2. Let $(x, y)$ be local coordinates at a point $P$, $z = xy$ and $\zeta^n = z$. Then $R = \mathcal{O}_P[\zeta]$ is a normal local ring and it can be desingularized by $[\frac{n}{2}]$ quadratic transformations. In the resolved model, $\zeta = 0$ defines, locally, a curve with nodes.

Proof. $R$ is the local ring of $xy = \zeta^n$ which is singular only at $x = y = \zeta = 0$ (char. = 0!) and so normal. Blowing up we get 3 affine sets

$$U_x : x^{n-2}(\frac{\zeta}{x})^n = (\frac{y}{x}) \text{ on which } \zeta = \frac{\zeta}{x}\cdot x$$

$$U_y : y^{n-2}(\frac{\zeta}{y})^n = (\frac{y}{x}) \text{ on which } \zeta = \frac{\zeta}{y}\cdot y$$

$$U_\zeta : \zeta^{n-2} = (\frac{x}{\zeta})(\frac{y}{\zeta}) \text{ which has the only}$$

singular point, which, by induction on $n$, can be blown up giving a locus $\zeta = 0$ consisting of reduced curves normally crossing.

To complete the proof of Proposition 1, one looks at a point in $\tilde{V}$ lying over a point $P \in C_i \cap C_j$ in $V$, so $z = x^{n_i} y^{n_j}$ (the pull-backs of the coordinate on $D$) with $x, y$ local coordinates at $P$. Let $d = (n_i, n_j)$, $n_i' = n_i/d$, $n_j' = n_j/d$, $n = n_i' n_j'$ and $m = N/nd$. Let $\zeta_1 = z^{1/d}$, $\zeta_2 = \zeta_1^{1/n}$, $\zeta = \zeta_2^{1/m} = z^{1/N}$. The morphism $V' \longrightarrow V$ may be factored into $V' \longrightarrow V_2 \longrightarrow V_1 \longrightarrow V$ where $V_1$ is retained by adjoining $\zeta_1$ and normalizing, etc. Adjoining $\zeta_1$ gives $\zeta_1^d = x^{n_i} y^{n_j} = (x^{n_i'} y^{n_j'})^d$ so $V_1 \longrightarrow V$ is unramified over $P$ and the locus $z = 0$ gives $d$ curves $\zeta_1 = \eta x^{n_i'} y^{n_i'}$, $\eta^d = 1$ which are reduced and cross normally. Adjoining $\zeta_2$ and normalizing, we get a nonsingular $V_2$ and by lemma 1 $\zeta_2 = 0$ will again give reduced curves with normal crossings. Finally adjoining $\zeta = \zeta_2^{1/m}$ to obtain $V'$ and blowing up one obtains $\tilde{V}$ where by lemma 2 the divisor of $\zeta$ has the desired form.

Remark. The process can be quite complicated, e.g. let $p^{-1}(0) =$ a rational curve with a cusp with $V$ nonsingular. We have

$z = x^2 - y^3$. Let $\xi = x/y$ giving $\xi^2 = y$, $y = 0$ as new curves and $z = y^2(\xi^2 - y)$. Let $\eta = \frac{y}{\xi}$ so $\xi = 0$, $\xi = \eta$, $\eta = 0$ are the new curves and $z = \xi^3\eta^2(\xi - \eta)$. Finally let $\zeta = \frac{\xi}{\eta}$, $\zeta' = \frac{\eta}{\xi}$ so $z = \eta^6\zeta^3(\zeta - 1) = \xi^6(\zeta')^2(1 - \zeta')$. So ones has 4 curves meeting normally but with multiplicities 1, 2, 3, 6 and we must take a $6^{\text{th}}$ root of $z$ normalize and blow up and down before one obtains $\tilde{p}^{-1}(0)$ which should be a nonsingular elliptic curve with $j = 0$.

Remark. Griffiths has given an analogue of Proposition 1 for abelian varieties.

Proposition 2. If $C$ is any curve with nodes, there is a normal degenerating family $p : V \longrightarrow D$ with $p^{-1}(0) \cong C$.

Proof. Let $|x| < 1$ and $|y| < 1$ define coordinate neighborhoods for the two branches of the node $x = y = 0$. Let $V_0 = D \times D$, $V_1 = \{(P, z) \in C \times D \text{ s.t. } |x(P)| > |z| \text{ if } P \in U_x\}$, $V_2$ = analogous space for $U_y$. Attach $V_0$ to $V_1 \cup V_2$ along the locus $xy \neq 0$ by

$$(x, y) \longmapsto \begin{cases} (P, xy) \text{ in } V_1 \text{ where } x(P) = x \text{ if } x \neq 0 \\ \\ (P, xy) \text{ in } V_2 \text{ where } y(P) = y \text{ if } y \neq 0 \end{cases}$$

to obtain $V$ and let $p(x, y) = xy$ for $(x, y) \in V_0$, $p(P, z) = z$ for $(P, z) \in V_1 \cup V_2$. This is the desired family. (We have assumed a single node to simplify notation.) This scissors-and-glue approach was suggested by H. Levine.

For $C$ a curve with nodes let $J = H^1(C, \mathcal{O})/H^1(C, \mathbb{Z})$ be its (generalized) jacobian variety. $J$ is the group of (Cartier) divisors whose restriction to each component has degree 0, modulo the usual (linear) equivalence. Then:

(1) There is a canonical map of $J$ into $\tilde{J}$, the product of the jacobians of the normalized components of $C$, whose kernel is of the form $(\mathbb{C}^*)^k$.

(2) The inner product $I(\alpha, \beta) = (\alpha \cup \beta)[C]$ on $H^1(C, \mathbb{Z})$, called the polarization form, is the pull-back of the usual polarization form for $\tilde{J}$.

(3) $J$ can also be defined as $\widehat{\Gamma(C, K)}/H^1(C, \mathbb{Z})$ where $K$ is the "canonical sheaf".

More generally one may call an extension of a (principally polarized) abelian variety by $(\mathbb{C}^*)^k$ a generalized abelian variety. Such a group may be written as $E/L$ with $E$ = Lie algebra, $L$ = fundamental group and $L$ carries a "polarization form" pulled back from that on its abelian part.

Definition. A normal degenerating family of abelian varieties $(\mathcal{Q}, I)$ is an exact sequence of analytic families of Lie groups over D

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0$$

where $\mathcal{E}$ is a vector bundle, $\mathcal{L}$ a closed family of discrete subgroups (which may also be thought of as a sheaf of holomorphic sections of $\mathcal{E}$) together with a continuous alternating bilinear form $I : \mathcal{L} \times \mathcal{L} \longrightarrow \mathbb{Z}$, such that

(1) For $z \neq 0$, $\mathcal{Q}_z = \mathcal{E}_z/\mathcal{L}_z$ with the form $I_z$ is a principally polarized abelian variety

(2) For $z = 0$, $\mathcal{Q}_0 = \mathcal{E}_0/\mathcal{L}_0$ is a generalized abelian variety with the form $I_0$.

Proposition 3. Let $p : V \longrightarrow D$ be a normal degenerating family of curves. Then there is a normal degenerating family $(\mathcal{J}, I)$ of abelian varieties with $(\mathcal{J}_z, I_z) \cong$ the canonically "polarized" jacobian variety (or generalized jacobian variety) of $p^{-1}(z)$ for all $z \in D$. In principle this goes back to the work of Picard and Poincaré, and work of H. Rauch in this direction suggested developments given here. There are now generalizations of this in several directions by Néron, Griffiths, and Murre. The (rather involved) proof can be outlined:

1. Using the adjunction formula one shows that $\dim H^1(p^{-1}(z), \mathcal{O}) = \text{const } \forall z \in D$.
2. By a theorem of Grauert, it follows that $R^1 p_*(\mathcal{O}_V)$ is the sheaf of sections of a vector bundle $\mathcal{E}$.
3. One shows that each fibre $p^{-1}(z)$ is a deformation retract of a neighborhood of it, so that the stalk $\mathcal{L}_z \cong H^1(p^{-1}(z), \mathbb{Z})$ where $\mathcal{L} = R^1 p_*(\mathbb{Z})$.
4. One defines $I : \mathcal{L} \times \mathcal{L} \longrightarrow \mathbb{Z}$ by topology, and this is seen to give the polarization form on each $\mathcal{L}_z$.
5. The <u>extension lemma</u>: this asserts that any section of $\mathcal{L}$ over $D - \{0\}$ extends to a section over $D$. This is far from easy to prove and uses the classical Picard-Lefschetz theorem (see Clemens' thesis).
6. From this it is not hard to show that $\mathcal{L}$ is closed in $\mathcal{E}$ (a fact equivalent to the extension lemma) by examining the monodromy of the period-matrix, a technique also used in the next proposition.

<u>Proposition</u> 4. Let $(\mathcal{Q}, I)$ be a normal degenerating family of abelian varieties, of dimension $g$. Then we have a map $\rho : D \longrightarrow \mathcal{V}^*_g$ defined as follows: For $z \neq 0$, $\rho(z)$ = the point in $\mathcal{V}_g$ representing the principally polarized abelian variety $(\mathcal{Q}_z, I_z)$, while $\rho(0)$ = the point in $\mathcal{V}_{\tilde{g}} \subset \mathcal{V}^*_g$ $(\tilde{g} \leq g)$ representing the abelian part of $(\mathcal{Q}_0, I_0)$.

<u>Proof.</u> Choose a basis $\alpha_1 \ldots \alpha_g, \beta_1 \ldots \beta_{\tilde{g}}$ for $\mathcal{L}_0$ such that $\alpha_1 \ldots \alpha_{\tilde{g}}, \beta_1 \ldots \beta_{\tilde{g}}$ come from a canonical basis for the lattice of the abelian part of $\mathcal{Q}_0$. At any point $z \neq 0$ one can complete this to a canonical basis of $\mathcal{L}_z$ by adding $\beta_{\tilde{g}+1} \ldots \beta_g$, though not in a unique way: as we move around the origin in $D$ such a basis will change via a "monodromy matrix" $\begin{pmatrix} I & S \\ 0 & I \end{pmatrix}$, with $S = \begin{pmatrix} 0 & 0 \\ 0 & S^0 \end{pmatrix}$ a a $g \times g$ matrix and $S^0$ a positive definite $k \times k$ matrix $(\tilde{g} + k = g)$.

The $\alpha_1 \ldots \alpha_g$ form a $\mathbb{C}$-basis for $\mathcal{E}$ at each $z \in D$ and we have $\beta_i(z) = \Sigma \tau_{ij}(z)\alpha_i(z)$ where the "period matrix" $\tau(z) = (\tau_{ij}(z))$ lies in $\mathcal{H}_g$, and $\bar{\tau}(z) = \tau(z) - (\frac{1}{2\pi i} \log z)S$ is 1-valued in $D - \{0\}$. Applying Riemann's theorem on removable singularities, one sees that $\bar{\tau}(z)$ extends to a holomorphic map $D \longrightarrow \mathcal{H}_g$. Then an explicit computation shows that $\tau(z)(\text{mod } \Gamma_g)$ approaches $\lim_{z \to 0} (\bar{\tau}_{ij}(z))_{1 \leq i, j \leq \tilde{g}} (\text{mod } \Gamma_{\tilde{g}})$ using Satake's original definition of the topology on $\mathcal{V}_g^*$, and this proves Proposition 4. (This sort of argument has been greatly generalized by Borel.) One uses the fact that $\mathcal{L}$ is closed in $\mathcal{E}$ to prove $S^0 > 0$, which is equivalent with the "extension lemma" in the case of jacobians. We note that $S^0$ may be interpreted in terms of the topology of $p^{-1}(0)$, in the case of jacobians. The proof of Theorem 1 now follows:

a) $\subseteq$ Let $q \in \mathcal{M}_g^*$. Then, by Baily's results, $\exists$ a degenerating family $p : V \longrightarrow D$ such that if we let $\rho(z) = \{\text{pt. in } \mathcal{M}_g \text{ representing } p^{-1}(z)\}$, $z \neq 0$, then $q = \lim_{z \to 0} \rho(z)$. By Proposition 1 however we may assume $p : V \longrightarrow D$ normal and hence by Proposition 3 and Proposition 4 $q$ represents the abelian part of the generalized jacobian variety of $p^{-1}(0)$ and so $\in \mathcal{M}_{\tilde{g}} \cup \mathcal{M}'_{\tilde{g}}$, $\tilde{g} \leq g$.

b) $\supseteq$ Let $q \in \mathcal{M}_{\tilde{g}} \cup \mathcal{M}'_g$ represent a product of jacobians of curves $C_i$ of genus $\tilde{g}_i$ with $\Sigma \tilde{g}_i = \tilde{g} \leq g$. Joining pairs of points as necessary, one obtains a $C$ of genus $g$ whose normalized components are the $C_i$. By Proposition 2 $C = p^{-1}(0)$ for $p : V \longrightarrow D$ some normal degenerating family and by Propositions 3 and 4 one has, as before, $\rho : D \longrightarrow \mathcal{V}_g^*$ continuous with $\rho(0) = q$ and $\rho(D - \{0\}) \subset \mathcal{M}_g$, so $q \in \mathcal{M}_g^*$.

<u>Definition.</u> A curve with nodes $C$ is called <u>stable</u> if $C$ has no rational nonsingular component which meets other components at $< 3$ points.

1. To each $C$ we may assign a unique stable model. If $p_i : V_i \longrightarrow D$, $i = 1, 2$ are two normal families which

agree on $D - \{0\}$ then $p_1^{-1}(0)$ and $p_2^{-1}(0)$ have the same stable model.

2. A stable $C$ of genus $g$ has a finite number of possible topological structures, each with at most $3g - 3$ nodes and at most $2g - 2$ components $(g > 1)$.
3. For $g > 1$, $C$ is stable $\Longleftrightarrow \mathrm{Aut}(C)$ is finite.
4. For any $C$ with nodes we may define an invertible sheaf $K_C$, the canonical sheaf, e.g. by $K_C \cong K_V \otimes \mathcal{O}_C$ for $C = p^{-1}(0)$. If $g > 1$ and $C$ is stable then $K_C^3$ is very ample and defines an embedding $C \hookrightarrow \mathbb{P}^{5g-6}$.

Let $\tilde{\mathcal{M}}_g = \{$stable $C$ with nodes of genus $g\}$.

<u>Theorem</u> 2. $\tilde{\mathcal{M}}_g$ has the natural structure of a Q-variety and is also a compact, Hausdorff V-manifold.

The idea of the proof is to consider the Chow set $\mathcal{C}_g$ of all stable curves of degree $6g - 6$ in $\mathbb{P}^{5g-6}$ and take its quotient by $PGL(5g-6)$. Proposition 1 and the above remarks show that this gives a compact Hausdorff space in $1 - 1$ correspondence with the points of $\tilde{\mathcal{M}}_g$, and this can be shown to be a Q-variety in Matsusaka's sense. Using results of Schlessinger which imply that $\mathcal{C}_g$ is nonsingular, and the fact that the isotropy groups are finite, one can show that $\tilde{\mathcal{M}}_g$ is a V-manifold.

We have a surjective map $\tilde{\mathcal{M}}_g \longrightarrow \mathcal{M}_g^*$, $\tilde{\mathcal{M}}_g$ consists of bi-regular isomorphism classes of curves with simple singularities, while $\mathcal{M}_g^*$ gives us something like birational equivalence classes of the same curves. One would expect an analogous pair of compactification in other cases, e.g. abelian varieties and $K_3$ surfaces. One should note that the $*$-compactification comes from a natural quasi-projective structure which the variety of moduli carries a structure closely related to its "canonical bundle", and this would seem to be a general phenomenon. The $\sim$-compactification, on the other hand, would seem to come from adding on enough Chow points of singular varieties to give a compact space (when the quotient is taken--but excluding enough to give a Hausdorff space) to the Chow variety dominating the variety of moduli, and this again might be a general phenomenon.

The point in $\mathcal{V}_g^*$ determined by a generalized abelian variety depends only on the abelian part and ignores the nature of the extension, which depends on $(g-\tilde{g})\tilde{g}$ more parameters. However the set of generalized abelian varieties would not seem to form a nice compactification of $\mathcal{V}_g$. It seems that generalized abelian varieties have various compactifications, depending on an additional $\frac{1}{2}(g-\tilde{g})(g-\tilde{g}-1)$ parameters. In general if we have a family given by $\tau(z) = \bar{\tau}(z) + (\frac{\log z}{2\pi i})S$ $\operatorname{rg}(S) = g - \tilde{g}$ we have $\tau_{\nu\nu} \longrightarrow \infty$, $\nu \geq \tilde{g}+1$ leaving us with the following possibly finite parameters

a) $\tau_{\nu\mu}(0)$, $\nu, \mu \leq \tilde{g}$ determining the abelian part of the group

c) $\tau_{\nu\mu}(0)$, $1 \leq \nu \leq \tilde{g}$, $\tilde{g}+1 \leq \mu \leq g$ determining the $(g-\tilde{g})$ extensions by $\mathbb{C}^*$

d) $\tau_{\nu\mu}(0)$, $\tilde{g}+1 \leq \nu < \mu \leq g$ some of which may be infinite, determining the parameters of compactifications.

There is yet no general theory of such compactifications except in the case of jacobians. The generalized jacobian variety $J$ of a (stable) curve with nodes has a canonical compactification which may be described as the "linear equivalence classes" of Weil divisors, of degree 0 on each component. $\bar{J}$ is not a group but is the union of a set of quotient groups of $J$. Let $\sqsubset$ be the set of nodes of $C$ and $C_{\mathfrak{a}}$ the partial normalization of $C$ at $\mathfrak{a}$. $\forall$ subsets $\mathfrak{a} \subset \sqsubset$. $\forall_{\mathfrak{a}}$ we have $J \xrightarrow{\pi_{\mathfrak{a}}} J(C_{\mathfrak{a}}) = J_{\mathfrak{a}}$ surjective and say $\mathfrak{a} \sim \mathfrak{a}'$ if $\ker(\pi_{\mathfrak{a}}) = \ker(\pi_{\mathfrak{a}'})$. Then $\bar{J} = \bigcup J_{\mathfrak{a}}$ the union being taken over a set of representatives of the equivalence classes. For $g = 2$, $C$ irreducible rational with two nodes $J = (\mathbb{C}^*)^2$ and $\bar{J}$ is $\mathbb{P}_1 \times \mathbb{P}_1$ with the following points identified:

$(0, a)$ and $(\infty, \lambda a)$, $(a, 0)$ and $(\lambda a', \infty)$ where $\lambda$ is the cross-ratio of the 4 points on $\tilde{C}$ lying over the nodes. Here $\frac{\log \lambda}{2\pi i}$ is the finite $\lim\limits_{z \to 0} \tau_{12}(z)$ of the period matrix.

For $C = C_1 \cup C_2$, $C_1$ and $C_2$ rational curves joined at 3 points. $J$ again $= (\mathbb{C}^*)^2$ but now the compactification is seem to be $\mathbb{P}^2$ with 3 coordinate vertices blown up and then identified with the opposite side of the coordinate triangle. In this case $\lim\limits_{z \to 0} \tau_{12}(z) = \infty$.

For $g = 2$ then Torelli's theorem holds for stable curves with nodes with respect to the compactified jacobians, but this breaks down if $g \geq 3$ --e.g. joining an elliptic curve and a curve of genus 2 by one varying point, the resulting curve with nodes varies, while the (already compact) jacobian is unchanged.

The above suggests types of compactifications for generalized abelian varieties--in particular a family of compactifications of $(\mathbb{C}^*)^g$ depending on $\frac{1}{2}g(g-1)$ parameters. In this case it is easy to write down the $\frac{1}{2}(g-2)(g-3)$ "period relation" which hold for the compactified generalized jacobian variety of an irreducible rational $C$ with $g$ nodes. It also would seem that the singular locus of a compactified generalized abelian variety has $\leq \frac{1}{2}g(g+1)$ components, while for jacobians this number must be $\leq 3g-3$ thus impassing topological restrictions on jacobians.

It should be stressed that in the above we are speaking about rough empirical indications rather than a complete theory, but it suggests an approach to the difficult problem of finding a "good" compactification $\tilde{\mathcal{V}}_g$ of $\mathcal{V}_g$, a problem which has been studied by Satake, Siegel, Piatetskiĭ-Sapiro, and more extensively, Igusa.

Lectures 4 and 5 -- Thomas F. Jambois

Introduction. We will give a proof of Proposition 3 in the preceding lectures. Explicitly, we wish to construct the family of (generalized) Jacobians associated to a normal family of curves and show that the family of Jacobians fits together to form a manifold and is a normal degenerating family of Abelian varieties.

The proof breaks naturally into two parts. In the first part we analyze the topology of the family $V$ of curves and construct a basis of 1-cycles for the homology of a fibre $C_t$. The basis contains $2g$ cycles $\delta_1, \ldots, \delta_g$ and $\gamma_1, \ldots, \gamma_g$ and is canonical,

i.e.,
$$\delta_i \cdot \gamma_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

In the second part we produce a basis $\rho_1(t), \ldots, \rho_g(t)$ of holomorphic 1-forms on $C_t$ which vary holomorphically with $t$. It is important to describe the behavior of these differentials as $t$ tends to zero. This being done, we construct the family of Jacobians by using the period matrix

$$\Omega(t) = \int_{b(t)} \rho(t)$$

where $\rho(t) = \begin{pmatrix} \rho_1(t) \\ \vdots \\ \rho_g(t) \end{pmatrix}$ and $b(t) = (\delta_1(t), \ldots, \delta_g(t), \gamma_1(t), \ldots, \gamma_g(t))$.

1.1. Let $\pi : V \to \Delta$ be a proper holomorphic mapping from the connected complex analytic suface $V$ to the unit disk $\Delta$ in $\mathbb{C}$ such that:

(1) The derivative $d\pi$ vanishes only at a finite number of points $p_1, \ldots, p_d$ all of which lie on $C_0 = p^{-1}(0)$.

(2) For each $p_i$ $\exists$ coordinates $(x, y)$ on a neighborhood $U_i$ of $p_i$ such that $\pi(p) = x^2(p) - y^2(p)$ for all $p \in U$ and $x(p_i) = y(p_i) = 0$.

The two conditions conveniently express the fact that $C_0$ is a curve with nodes.

1.2. Each fibre $C_t = \pi^{-1}(t)$ is a compact Riemann surface of genus $g$, except $C_0$ which is connected, consisting of $r$ irreducible components, $C_1, \ldots, C_r$. Each $C_i$ has a normalization $\widetilde{C}_i$ which is non-singular and of genus $\widetilde{g}_i$ and $\widetilde{C} = \bigcup_{i=1}^{r} \widetilde{C}_i$ maps holomorphically via a mapping $\lambda$ onto $C = C_0$. If $p$ is a double point on $C$ then $\lambda^{-1}(p)$ contains exactly two points, $p^+$ and $p^-$. If $\lambda^{-1}(p) \subset \widetilde{C}_i$ we call $p$ a <u>node</u> on $C_i$. If $p \in C_i \cap C_j$, $i \neq j$, we call $p$ a <u>crossing.</u>

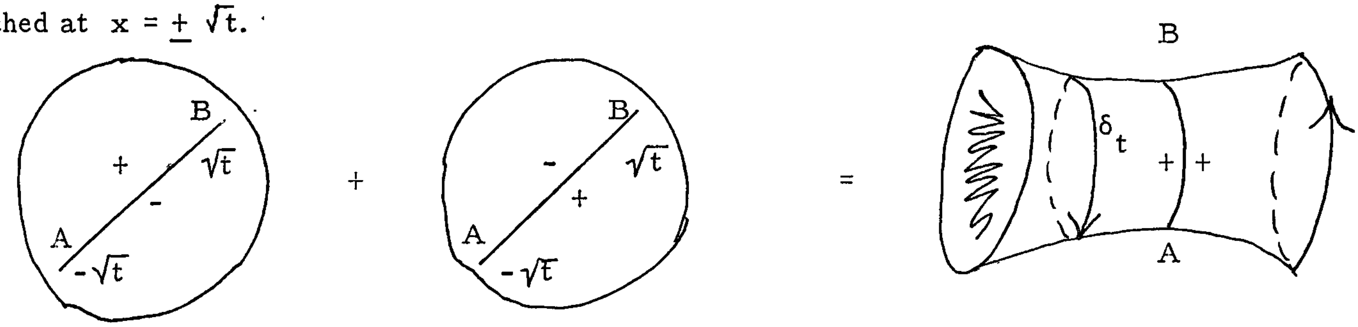
1.3. Employing coordinates $(x, y)$ as described in 1.1 near the double point $p_i$ and choosing $\epsilon > 0$ sufficiently small we define a neighborhood

$$U^i_\epsilon = \{p \in U_i : | \ x(p) \ | < \epsilon \quad | \ y(p) \ | < 2\epsilon \ \}$$

isomorphic to a polycylinder. Choosing $\delta > 0$ less than both 1 and $(\epsilon/2)^2$ insures that whenever $|x| < \epsilon$ and $|t| < \delta$ then both solutions $y$ to the equation $y^2 = x^2 - t$ satisfy $|y| < 2\epsilon$. Let $\Delta_\delta = \{t : |t| < \delta\}$ and $U^i_{\delta,\epsilon} = U^i_\epsilon \cap \pi^{-1}(\Delta_\delta)$. Given any $t \in \Delta_\delta$ we have

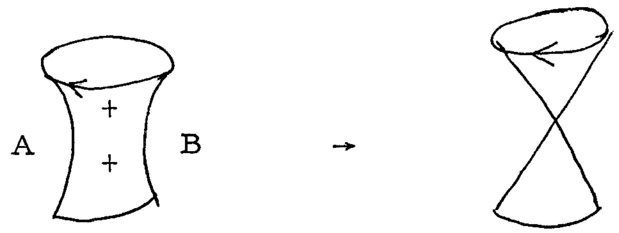
$$U^i_t = U^i_{\delta,\epsilon} \cap C_t = \{p \in U_i : |x(p)| < \epsilon \ y^2(p) = x^2(p) - t\}$$

so $U^i_t$ is isomorphic to a two sheeted branched covering of the disk $\{|x| < \epsilon\}$ branched at $x = \pm \sqrt{t}$.




Thus, $U^i_t$ is topologically a band, as pictured. A positively oriented circle in the x-plane, say $x = \frac{3}{4}\epsilon \ e^{2\pi i \theta}$ and a fixed branch of $y$ determine a loop $\delta_i(t)$ on $C_t$, called the <u>vanishing cycle</u> associated with $p_i$. The other choice of branch gives

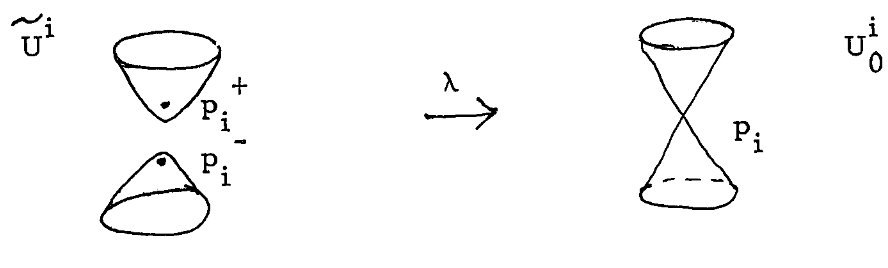
$-\delta_i(t)$. Clearly $\delta_i(t)$ generates $H_1(U_t^i, Z)$. As $t$ shrinks to zero we see that the band $U_t^i$ shrinks to a cone, which is the same as 2 discs meeting at a point.




Finally, we choose $\delta$ and $\epsilon$ small enough to form $U^i_{\delta,\epsilon}$ for each $p_i$ and to insure $U^i_{\delta,\epsilon} \cap U^j_{\delta,\epsilon} = \phi$ if $i \neq j$. Then let $U = \bigcup_{i=1}^{d} U^i_{\delta,\epsilon}$. The cycles $\delta_i(t)$ obviously map to zero under the inclusion homomorphism $i_*: H_1(C_t, \mathbb{Z}) \to H_1(V, \mathbb{Z})$. If $\bar{\epsilon}$ is slightly smaller than $\epsilon$, we have $\overline{U}^i_{\delta,\bar{\epsilon}} \subset U^i_{\delta,\epsilon}$, while $U^i_{\delta,\bar{\epsilon}}$ enjoys the same properties as $U^i_{\delta,\epsilon}$.

1.4. The map $(x, \pm\sqrt{x^2 - t}) \xrightarrow{r} (x, \pm x)$ defines a strong deformation retraction $r$ of $U$ onto $U_0 = U \cap C_0$ which maps $\partial U_t$ diffeomorphically onto $\partial U_0$. The map $r$ may be extended to a strong deformation retraction of $V$ onto $C_0$ such that $r_t = r / C_t$ maps $C_t - U_t$ homeomorphically onto $C_0 - U_0$. If $\tilde{U} = \lambda^{-1}(U_0)$ then $\lambda : \tilde{C} - \tilde{U} \to C_0 - U_0$ is a homeomorphism.

1.5. The inverse image under the normalization mapping $\lambda$ of the cone $U_0$ is a union of 2d disjoint 2-cells, two cells arising from each double point.



The maps $\lambda_* : H_*(\widetilde{C} - \widetilde{U}) \to H_*(C_0 - U_0)$ and $(r_t)_* : H_*(C_t - U_t) \to H_*(C_0 - U_0)$ are isomorphisms, and by excision the relative homology group $H_*(\widetilde{C}, \widetilde{C} - \widetilde{U})$ is isomorphic to the reduced homology of the union of 2d disjoint 2-spheres. In particular, $H_1(\widetilde{C}, \widetilde{C} - \widetilde{U})$ is trivial, so the following sequence is exact:

$$0 \to H_2(\widetilde{C}) \to H_2(\widetilde{C}, \widetilde{C} - \widetilde{U}) \to H_1(\widetilde{C} - \widetilde{U}) \to H_1(\widetilde{C}) \to 0.$$

Accordingly, any basis of $H_1(\widetilde{C})$ may be represented by loops supported by $\widetilde{C} - \widetilde{U}$, and the rank of $H_1(\widetilde{C} - \widetilde{U})$ is $2\widetilde{g} + 2d - r$ where $\widetilde{g} = \sum_{i=1}^{r} \widetilde{g}_i$.

1.6. Let $\overline{B}$ denote the closure of $B$ for any set $B$. By excision $H_*(C_t, \overline{U}_t)$ is isomorphic to $H_*(C_t - U_t, \partial U_t)$ which in turn is mapped isomorphically onto $H_*(C_0 - U_0, \partial U_0)$ by $(r_t)_*$. Accordingly, $(r_t)_* : H_*(C_t, \overline{U}_t) \to H_*(C_0, \overline{U}_0)$ is an isomorphism. The following diagram is exact along rows and commutative:

$$\begin{array}{ccccccccccc}
0 \to & H_2(C_t) & \to & H_2(C_t, \overline{U}_t) & \to & H_1(\overline{U}_t) & \to & H_1(C_t) & \to & H_1(C_t, \overline{U}_t) & \to & H_0(\overline{U}_t) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & H_2(C_0) & \to & H_2(C_0, \overline{U}_0) & \to & 0 & \to & H_1(C_0) & \to & H_1(C_0, \overline{U}_0) & \to & H_0(\overline{U}_0)
\end{array}$$

It follows immediately that $(r_t)_* : H_1(C_t) \to H_1(C_0)$ is surjective with kernel $\mathcal{V}$ equal to the image of $H_1(\overline{U}_t)$ in $H_1(C_t)$. Hence $\mathcal{V}$ is generated by the vanishing cycles. Since $H_2(C_t, \overline{U}_t)$ is isomorphic to $H_2(\widetilde{C}, \overline{\widetilde{U}})$ and since the latter group has rank $r$ we conclude that $\mathcal{V}$ has rank $d - r + 1$.

1.7. We wish to show that a basis for $\mathcal{V}$ can be chosen from the vanishing cycles $\{\delta_i\}$ and that this basis may be supplemented by $g - k$ additional cycles to form the first half of a canonical homology basis for $C_t$.

<u>Lemma.</u> Let $S$ be a compact oriented surface and let $a_1, \ldots, a_k$ be disjoint

regular simple closed curves on S with tubular neighborhoods $U_1, \ldots, U_k$ such that $\overline{U}_i$ is again a tubular neighborhood and $\overline{U}_i \cap \overline{U}_j = \phi$ if $i \neq j$. Then the following are equivalent:

a) The cycles $\alpha_1, \ldots, \alpha_k$ are independent.

b) The set $S - \bigcup_{i=1}^{k} U_i$ is connected.

c) The set $S - \bigcup_{i=1}^{k}$ support $\alpha_i$ is connected.

d) $\exists$ cycles $\beta_1, \ldots, \beta_k$ such that $\beta_i \cdot \beta_j = 0$ and $\alpha_i \cdot \beta_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

Proof: Let $U = \bigcup_{i=1}^{k} U_i$. Then $S - U$ is a deformation retract of $S - \bigcup_{i=1}^{k}$ support $\alpha_i$ and $S - U$ is a manifold with boundary. Accordingly, $H_2(S - U, \partial U)$ is freely generated by the components of $S - U$ and by excision this group is isomorphic to $H_2(S, \overline{U})$. The equivalence of (a) and (b) now follows from exactness in the following sequence:

$$0 \to H_2(S) \to H_2(S, \overline{U}) \to H_1(\overline{U}) \to H_1(S).$$

Of course, (b) is trivially equivalent to (c) by the first remark of this paragraph. Let $F_i$ be a fixed fibre in the bundle $\overline{U}_i$ over support $\alpha_i$ and let $q_i^+$ and $q_i^-$ be the endpoints of $F_i$, i.e., $\{q_i^+, q_i^-\} = F_i \cap \partial U_i$. The points $q_i^+$ and $q_i^-$ may be joined by a path in $S - U$. Putting this path together with the fibre $F_i$ we get a loop $\beta_i'$ and by changing direction if necessary we may suppose $\alpha_i \cdot \beta_i' = 1$. On the other hand $\alpha_j \cdot \beta_i' = 0$ when $j \neq i$ because then $\alpha_j$ and $\beta_i$ have disjoint supports. Finally, we let $\beta_i = \beta_i' + \sum_{\ell=i}^{k} (\beta_\ell' \cdot \beta_i')\, \alpha_\ell$ Then (since $\alpha_i \cdot \alpha_j = 0$) $\beta_i \cdot \alpha_j = \beta_i' \cdot \alpha_j$ and $\beta_i \cdot \beta_j = \beta_i' \cdot \beta_j' + \sum_{\ell=i}^{k} (\beta_\ell' \cdot \beta_i')\alpha_\ell \cdot \beta_j' + \sum_{m=j}^{k} (\beta_m' \cdot \beta_j')\, \beta_i' \cdot \alpha_m$. If $i < j$ then the right hand sum is zero and $\beta_i \cdot \beta_j = \beta_i' \cdot \beta_j' + \beta_j' \cdot \beta_i' = 0$. If $i > j$ the left hand sum is zero and $\beta_i \cdot \beta_j = \beta_i' \cdot \beta_j' - \beta_i' \cdot \beta_j' = 0$. In all cases then $\beta_i \cdot \beta_j = 0$ and the lemma is proved.

In the following $k = d - r + 1$.

<u>Corollary 1</u>: Let $\delta_1(t), \ldots, \delta_k(t)$ be an independent set of vanishing cycles. Then $\delta(t) = (\delta_1(t), \ldots, \delta_k(t))$ is a basis <u>(over</u> Z) of $\mathcal{V}$.

<u>Proof</u>: Certainly $\delta(t)$ is a basis over the rationals. Let $\alpha \in \mathcal{V}$. Then $\alpha = \Sigma a_i \delta_i(t)$. Hence, $\alpha \cdot \beta_i = a_i$ must be an integer.

<u>Corollary 2</u>: Let $\delta_1^{(t)}, \ldots, \delta_k(t)$ be independent. Then $\exists\, \tilde{g} = g-k$ one-cycles $\delta_1'(t), \ldots, \delta_{\tilde{g}}'(t)$, and cycles $\gamma_1'(t), \ldots, \gamma_{\tilde{g}}'(t)$, $\gamma_1(t), \ldots, \gamma_k(t)$ such that $b = (\delta', \delta, \gamma', \gamma)$ is a canonical homology basis for $C_t$.

<u>Proof</u>: Let $(\tilde{\delta}, \tilde{\gamma})$ be a canonical basis for $\tilde{C}$ represented by simple closed regular curves supported on $\tilde{C} - \tilde{U}$ such that $\tilde{\delta}_1, \ldots, \tilde{\delta}_{\tilde{g}}$ have disjoint tubular neighborhoods. Let $(\delta'(t), \gamma'(t)) = (r_t)_*^{-1}(\lambda_* \tilde{\delta}, \lambda_* \tilde{\gamma})$, and determine the remaining cycles $\gamma_1(t), \ldots, \gamma_k(t)$ required for the basis $(\delta'(t), \delta(t), \gamma'(t), \gamma(t))$ by the technique of the lemma. (We leave it to the reader to verify that $\delta_1'(t), \ldots, \delta_{\tilde{g}}'(t)$, $\delta_1(t), \ldots, \delta_k(t)$ are independent.)

<u>Corollary 3</u>: The cycles $\delta_1(t), \ldots, \delta_k(t)$ are independent if and only if $C_0 - \{p_1, \ldots, p_k\}$ is connected.

<u>Proof</u>: $C_0 - \bigcup_{i=1}^{k} U_0^i$ is a deformation retract of $C_0 - \{p_1, \ldots, p_k\}$ and $C_0 - \bigcup_{i=1}^{k} U_0^i = r_t(C_t - \bigcup_{i=1}^{k} U_t^i)$.

1.8. Let $\Delta' = \Delta - \{0\}$ be the punctured disc and $V' = \pi^{-1}(\Delta')$. Then $V'$ is a $C^\infty$ locally trivial fibre bundle over $\Delta'$ and the fundamental group $\pi_1(\Delta', t_0)$ acts as a group of automorphisms on the homology of the fibre $H_1(C_{t_0})$. In particular the positively oriented generator of $\pi_1(\Delta', t_0)$ determines the Picard-Lefschetz transformation $PL : H_1(C_{t_0}) \to H_1(C_{t_0})$. Since the family $C_t - U_t$ is trivial (for $t \in \Delta_\delta$) we conclude that $PL(\delta'(t_0)) = \delta'(t_0)$ and $PL(\gamma'(t_0)) = \gamma'(t_0)$ and by direct examination that $PL(\delta(t_0)) = \delta(t_0)$ (if $t_0 \in \Delta_\delta$). Having constructed the cycles $(\delta', \delta, \gamma', \gamma)(t_0)$ we agree to transport them to every other fibre $C_t$, $t \neq ($ by a differentiable isotopy associated with the local triviality of $V'$. This leads to a multiple-valued assignment of the cycles $\gamma(t)$ transverse to the vanishing cycles $\delta(t)$. In particular, transporting $\gamma(t_0)$ once around the origin in the counter-clockwise direction leads to $PL(\gamma(t_0))$. The transformation PL is well-understood. Indeed, for any cycle $\alpha \in H_1(C_{t_0})$ we have:

$$PL(\alpha) = \alpha + \sum_{i=1}^{d} (\delta_i(t_0)\cdot \alpha)\delta_i(t_0) \qquad \text{(cf. Clemens' lecture below)}$$

By Section 1.7 there is a $k \times d$ integral matrix of rank $k$ such that $(\delta_1(t_0), \ldots, \delta_d(t_0)) = \delta A$. Letting $\overline{\delta}(t_0) = (\delta_1(t_0), \ldots, \delta_d(t_0))$ we may express the above equation as follows:

$$\alpha + \overline{\delta}(t_0)({}^t\overline{\delta}(t_0)\cdot \alpha) = \alpha + \delta A\,{}^tA({}^t\delta\cdot \alpha)$$

$$PL(\alpha) = \alpha + \overline{\delta}(t_0)({}^t\overline{\delta}(t_0)\cdot \alpha) = \alpha + \delta A\,{}^tA({}^t\delta\cdot \alpha)$$

For the vector $\gamma(t_0)$ we have $PL(\gamma(t_0)) = \gamma(t_0) + \delta A\,{}^tA$. The matrix $M = A\,{}^tA$ is positive definite and symmetric.

2.1. We wish to construct a line bundle $L$ over $V$ whose restriction to each non-singular fibre $C_t$ is the canonical bundle $K_t$ and such that $\dim H^0(C, \mathcal{O}(L/C)) = g$. Then for example, by Grauert's theorem $\mathfrak{A} = \pi_*\mathcal{O}(L)$ is a free analytic sheaf of rank $g$. In choosing $g$ non-vanishing sections $\rho_1, \ldots, \rho_g$ of $\mathfrak{A}$ we are in fact choosing a basis $\rho_1(t), \ldots, \rho_g(t)$ of $H^0(C_t, \mathcal{O}(K_t))$ for each $t$ which varies holomorphically with $t$.

2.2. It suffices to take $L = \mathcal{K}$ = canonical bundle of $V$. For if $T^*$ is the holomorphic cotangent bundle of $V$, we define:

$$\alpha : \mathcal{O} \to \mathcal{O}(T^*) \text{ by } \alpha(f) = f d\pi$$

$$\beta : \mathcal{O}(T^*) \to \mathcal{O}(\mathcal{K}) \text{ by } \beta(\omega) = d\pi \wedge \omega$$

Over each open set not containing a double point the sequence

$$0 \to \mathcal{O} \xrightarrow{\alpha} \mathcal{O}(T^*) \xrightarrow{\beta} \mathcal{O}(\mathcal{K}) \to 0$$

is exact and so also is the restriction of the sequence to $C_t$ when $t \neq 0$. Thus $\mathcal{O}(\mathcal{K}|C_t)$ is isomorphic to $\mathcal{O}(K_t)$ when $t \neq 0$.

2.3. Now let $D = p_1^+ + p_1^- + \ldots + p_d^+ + p_d^-$ be the divisor of points on $\widetilde{C}$ mapping into double points. Given any vector bundle $E$ over $C$ and section $s$ of $E$ let $\widetilde{E}$ denote the inverse image of $E$ under $\lambda$ and let $\widetilde{s} = s \circ \lambda$.

<u>Lemma:</u> Let $\widetilde{K}$ denote the canonical bundle of $\widetilde{C}$. Then $\widetilde{\mathcal{K}}$ is equivalent to $\widetilde{K} \otimes L_D$, i.e., $\mathcal{O}(\widetilde{\mathcal{K}})$ is the sheaf of germs of meromorphic 1-forms on $\widetilde{C}$ with poles at $D$. If $\rho \in H^0(\widetilde{C}, \mathcal{O}(\widetilde{K} \otimes L_D))$ then $\rho = \widetilde{\omega}$ for some $\omega \in H^0(C, \mathcal{O}(\mathcal{K}|C))$ if and only if $\operatorname{res}_{p_i^+} \rho = -\operatorname{res}_{p_i^-} \rho$ for all $i = 1, \ldots, d$.

<u>Proof:</u> The section $d\widetilde{\pi}$ determines a sheaf homomorphism $\widetilde{\beta} : \mathcal{O}(\widetilde{T}^*) \to \mathcal{O}(\widetilde{\mathcal{K}})$, which, however, is not surjective because $d\widetilde{\pi}$ vanishes at points of $D$. Let $s$ be a section of $L_D$ whose zero locus is $D$ and define $\mu : \mathcal{O}(\widetilde{T}^*) \to \mathcal{O}(\widetilde{\mathcal{K}} \otimes L_D^{-1})$ by $\mu(\omega) = \widetilde{\beta}(\omega) \otimes s^{-1}$. Of course, $\mu$ is surjective at every point not in $D$.

If $p^+$ is in $D$ there are a neighborhood $U$ of $p = \lambda(p^+)$ and coordinates $(x, y)$ on $U$ as described in 1.3. Thus $x^2 - y^2 = \pi$ and

$d\pi = 2\,(xdx - ydy)$. Let $A_+ = \{q \in U : x(q) = y(q)\}$ and $A_- = \{q \in U : x(q) = -y(q)\}$. Then (we may suppose) $\tilde{A}_+ = \lambda^{-1}(A_+)$ is a neighborhood of $p^+$ on which $\tilde{x} = x\circ\lambda$ is a coordinate. Furthermore, $d\tilde{\pi} = 2\tilde{x}(d\tilde{x} - d\tilde{y})$ and $s = \tilde{x}\cdot e$ where $e$ is a nowhere vanishing section of $L_D$ on $\tilde{A}_+$. Then $\widetilde{dx\wedge dy} \otimes e^{-1}$ is a nowhere vanishing section of $\tilde{\mathcal{K}}\otimes L_D^{-1}$ over $\tilde{A}_+$ and $\mu(\frac{1}{2}\, d\tilde{x}) = [\tilde{\beta}(\frac{1}{2}\, d\tilde{x})/\tilde{x}] \otimes e^{-1} = (dx\wedge dy)\circ\lambda \otimes e^{-1}$ which proves $\mu$ is surjective at $p^+$. $\mu(\frac{1}{2}\, d\tilde{x}) = -(dx\wedge dy)\circ\lambda\otimes e^{-1}$ on $\tilde{A}_-$.

On the other hand, there is a natural homomorphism $\lambda^* : \mathcal{O}(\tilde{T}^*) \to \mathcal{O}(\tilde{K})$ whereby $\lambda^* d\tilde{\pi} = 0$. Thus $\lambda^* d\tilde{x} = \lambda^* d\tilde{y}$ (or $\lambda^*[(dx)\circ\pi] = d(x\circ\lambda) = d(y\circ\lambda)$) on $\tilde{A}_+$ and $\lambda^* d\tilde{x}$ is a nowhere vanishing section of $\tilde{K}$ on $\tilde{A}_+$, so $\lambda^*$ is surjective. But $\lambda^*(a\, d\tilde{x} + b\, d\tilde{y}) = 0 \iff \mu(a\, d\tilde{x} + b\, d\tilde{y}) = 0$ and it follows that the sheaves $\mathcal{O}(\tilde{\mathcal{K}}\otimes L_D^{-1})$ and $\mathcal{O}(\tilde{K})$ are isomorphic and as a result $\tilde{K}\otimes L_D$ is equivalent to $\tilde{\mathcal{K}}$.

A section $\omega$ of $\tilde{K}\otimes L_D$ becomes a meromorphic differential upon dividing by $s$ and it is the residue of $\omega/s$ of which we speak in the statement of the lemma. Tracing through the above description of the isomorphism between $\mathcal{O}(\tilde{\mathcal{K}})$ and $\mathcal{O}(\tilde{K}\otimes L_D)$ we see that a section $\rho$ of $\tilde{\mathcal{K}}$ over $\tilde{A}_+ \cup \tilde{A}_-$ represented by $a(dx\wedge dy)\circ\lambda$ corresponds to the differential form $\frac{1}{2}\, a\, \frac{dx}{x}$ on $\tilde{A}_+$ and the form $\frac{-1}{2}\, a\, \frac{dx}{x}$ on $\tilde{A}_-$. Thus $\mathrm{res}_{p^+}\rho = \frac{1}{2}\, a(p^+)$ and $\mathrm{res}_{p^-}\rho = -\frac{1}{2}\, a(p^-)$. If $\rho = \tilde{\omega}$ then $\omega = \alpha\, dx\wedge dy$ on $A_+ \cup A_-$ where $\alpha$ is a holomorphic function and $a = \alpha\circ\lambda$ which implies $a(p^+) = a(p^-)$ and $\mathrm{res}_{p^+}\rho = -\,\mathrm{res}_{p^-}\cdot\rho$. Conversely, if the latter condition is satisfied, there is a holomorphic function on $A_+ \cup A_-$ which lifts to $a$ and therefore $\rho = \tilde{\omega}$ for some $\omega \in H^0(C, \mathcal{O}(\mathcal{K}\,|C))$.

<u>Theorem</u>: $\dim H^0(C, \mathcal{O}(\mathcal{K}\,|\,C)) = g$.

Proof: We have a natural injective map $\phi : H^0(C, \mathcal{O}(\mathcal{K}|C)) \to H^0(\widetilde{C}, \mathcal{O}(\widetilde{\mathcal{K}}))$ given by $\phi(\omega) = \widetilde{\omega}$. Let $\mathrm{res}_i \rho = \mathrm{res}_{p_i^+} \rho + \mathrm{res}_{p_i^-} \rho$ for any $\rho \in H^0(\widetilde{C}, \mathcal{O}(\widetilde{\mathcal{K}}))$ and $\mathrm{res}\, \rho = (\mathrm{res}_1 \rho, \ldots, \mathrm{res}_d \rho)$.

Define $\sigma : \mathbb{C}^d \to \mathbb{C}$ by $\sigma(c_1, \ldots, c_d) = \sum_{i=1}^{d} c_i$. The sequence

$$0 \to H^0(C, \mathcal{O}(\mathcal{K}|C)) \xrightarrow{\phi} H^0(\widetilde{C}, \mathcal{O}(\widetilde{\mathcal{K}})) \xrightarrow{res} \mathbb{C}^d \xrightarrow{\sigma} \mathbb{C} \to 0$$

is exact as a result of the lemma, except possibly at $\mathbb{C}^d$. In fact it is exact at $\mathbb{C}^d$ as well. Indeed, it is sufficient to show that for each $i \neq 1$ there exists $\rho \in H^0(\widetilde{C}, \mathcal{O}(\widetilde{\mathcal{K}}))$ such that $\mathrm{res}_1 \rho = 1$ and $\mathrm{res}_i \rho = -1$. Here we may assume $p_1$ to be a crossing. For if there are no crossings, $\widetilde{C}$ is connected and the result is classical. If there is a crossing we renumber $p_1, \ldots, p_d$ so that $p_1$ is the crossing, say, between $C_1$ and $C_2$. There are two cases to consider.

Case I. If $p_j \neq p_1$ and $p_j$ is a crossing, say, between $C_j$ and $C_j'$, choose a shortest possible chain $C_{i_1}, \ldots, C_{i_k}$ such that $C_{i_1}$ is either $C_1$ or $C_2$ and $C_{i_k}$ is either $C_j$ or $C_j'$ and $C_{i_\ell} \cap C_{i_{\ell+1}} \neq \phi$. We suppose $C_1 = C_{i_1}$ and $C_j = C_{i_k}$. Then neither $C_2$ nor $C_j'$ appear in the chain. Let $C_{i_0} = C_1$. Each intersection $C_{i_\ell} \cap C_{i_{\ell+1}}$ contains a double point $p_{i_\ell}$ and we let $p_{i_\ell}^- = \lambda^{-1}(p_{i_\ell}) \cap C_{i_\ell}$ and $p_{i_\ell}^+ = \lambda^{-1}(p_{i_\ell}) \cap C_{i_{\ell+1}}$. On $\widetilde{C}_{i_\ell}$ choose a meromorphic differential $\rho_{i_\ell}$ with a pole at $p_{i_{\ell-1}}^+$ and residue $+1$ there and a pole at $p_{i_\ell}^-$ with residue $-1$ there. This is to be done for $\ell = 2, \ldots, k-1$. Let $\rho = \rho_{i_\ell}$ on $\widetilde{C}_{i_\ell}$ $\ell = 2, \ldots, k-1$ and let $\rho = 0$ elsewhere. Then $\mathrm{res}_{p_1} \rho = 1 = -\mathrm{res}_{p_j} \cdot \rho$ and $\mathrm{res}_{p_k} \rho = 0$ when $k \neq 1$ and $k \neq j$.

<u>Case II.</u> The point $p_j$ is a node on $C_j$. Again choose a minimal chain $C_{i_1}, \ldots, C_{i_k}$ connecting either $C_1$ or $C_2$ to $C_j$ where again we assume $C_1 = C_{i_1}$ and $C_j = C_{i_k}$. Of course $C_{i_j} \neq C_2$ so we may let $C_{i_0} = C_2$. Now let $\rho_{i_\ell}$ be a meromorphic differential form on $\widetilde{C}_{i_\ell}$ having a pole at $p_{i_{\ell-1}}^{+}$ with residue 1 and a pole at $p_{i_\ell}^{-}$ with residue -1 for each $\ell = 1, \ldots, k\text{-}1$. On $\widetilde{C}_j = \widetilde{C}_{i_k}$ let $\rho_{i_k}$ have a pole at $p_{i_{k-1}}^{+}$ with residue +1 and a pole at either $p_j^{+}$ or $p_j^{-}$ with residue -1. Let $\rho = \rho_{i_\ell}$ on $\widetilde{C}_{i_\ell}$ $\ell = 1, \ldots, k$ and let $\rho = 0$ elsewhere. Then, $\mathrm{res}_1 \rho = 1$ and $\mathrm{res}_j \rho = -1$ as required.

By the Riemann-Roch theorem applied to each $\widetilde{C}_i$ we have $\dim H^0(\widetilde{C}, \mathcal{O}(\widetilde{\mathcal{H}})) = \tilde{g} - r + 2d$ and by the exact sequence $\dim H^0(C, \mathcal{O}(\mathcal{H}|C)) = \tilde{g} - r + 2$

$-d +$

$= g = \tilde{g} + k$

2.4. Any holomorphic section of $\widetilde{K}$ trivially satisfies the residue condition of the lemma and hence is the pull back of a section of $\mathcal{H}$ over C. Thus $\exists\, \rho_1', \ldots, \rho_{\tilde{g}}' \in H^0(C, \mathcal{O}(C|\mathcal{H}))$ such that the $\tilde{\rho}_1', \ldots, \tilde{\rho}_g'$ form a normalized basis of holomorphic differentials on $\widetilde{C}$ relative to the canonical homology basis $(\tilde{\delta}, \tilde{\gamma})$ selected in Corollary 2, §1.7. If $\rho' = \begin{pmatrix} \rho_1' \\ \vdots \\ \rho_{\tilde{g}}' \end{pmatrix}$ then $\int_{\tilde{\delta}} \tilde{\rho}' = I_{\tilde{g}}$. In addition to $\rho'$ there are k independent sections $\rho_1, \ldots \rho_k$ of $\mathcal{K}$ over C and we may suppose that these are chosen so that if $\rho = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_k \end{pmatrix}$ then $\int_{\tilde{\delta}} \tilde{\rho} = 0$.

Given k double points $p_1, \ldots, p_k$ we call the set of normalized sections $\{\rho_1, \ldots, \rho_k\} \subset H^0(C, \mathcal{O}(\mathcal{H}|C)$ <u>associated with $p_1, \ldots, p_k$</u> if

$$\operatorname{res}_{p_i^+} \rho_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Lemma: Let $k = d - r + 1$. Then $\exists$ k double points $p_1, \ldots, p_k$ which have an associated set of differentials $\rho_1, \ldots, \rho_k$.

Proof: Let $\mathcal{N} = \{\omega : \omega$ is a normalized section of $\mathcal{K} \mid C\}$. Then $\mathcal{N}$ is k-dimensional. If $\omega \in \mathcal{N}$ and $\omega \neq 0$ then $\tilde{\omega}$ must have a pole which we suppose to be $p_1^+$. Otherwise $\tilde{\omega}$ would be holomorphic with zero periods (relative to $\tilde{\delta}$) and hence $\tilde{\omega} = 0$ by standard Riemann surface theory. Thus $\operatorname{res}_{p_1^+}$ is a non-trivial linear functional on $\mathcal{N}$ and $\dim \operatorname{kernel} \operatorname{res}_{p_1^+} = k-1$. Let $\operatorname{res}_{p_1^+} \omega_1 = 1$ and choose $\omega_2 \in \operatorname{kernel} \operatorname{res}_{p_1^+}$. Adjusting $\omega_1$ by a multiple of $\omega_2$ and continuing we arrive at a set $\omega_1, \ldots, \omega_k$ associated to $p_1, \ldots, p_k$.

Of course, if $p_1, \ldots, p_k$ are double points with an associated set of 1-forms $\omega_1, \ldots \omega_k$ then $C - \{p_1, \ldots, p_k\}$ is connected for otherwise there would be a component $\mathcal{C}$ of $C - \{p_1, \ldots, p_k\}$ and a point $p_i$ such that if $\mathcal{C}'$ is the closure of $\tilde{\mathcal{C}}$ then $p_i^+ \in \mathcal{C}'$ but $p_i^- \notin \mathcal{C}'$. Then $\sum_{p \in \mathcal{C}'} \operatorname{res}_p \tilde{\omega}_i = 1$ which is impossible (if $p_\ell^+ \in \mathcal{C}'$ and $\ell = 1, \ldots, k$ then $p_\ell^- \in \mathcal{C}'$ because $\mathcal{C}$ is a connected component).

2.5. Let $\rho = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_k \end{pmatrix}$ where $\{\rho_1, \ldots \rho_k\}$ is associated with $\{p_1, \ldots, p_k\}$. Then the cycles $\delta_1(t), \ldots \delta_k(t)$ are independent on $C_t$. As suggested in 1.7 we choose $(\delta'(t), \gamma'(t)) = (\delta_1'(t), \ldots, \delta_{\tilde{g}}'(t), \gamma_1'(t), \ldots, \gamma_{\tilde{g}}'(t))$ so that $(r_t)_*(\delta'(t), \gamma'(t)) = \lambda_*(\tilde{\delta}, \tilde{\gamma})$ and cycles $\gamma_1(t), \ldots, \gamma_k(t)$ such that $b(t) = (\delta'(t), \delta(t), \gamma'(t), \gamma(t))$ is a canonical homology basis for $\mathcal{C}_t$. Finally choose sections $\omega_1, \ldots, \omega_g$ in $H^0(\Delta; \pi_* \mathcal{O}(\mathcal{K}))$ so that the vector $\omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix}$ restricts to the vector $\begin{pmatrix} \rho' \\ \rho \end{pmatrix}$ at $t = 0$. Let

$\rho' = \begin{pmatrix}\omega_1\\ \vdots \\ \omega_g\end{pmatrix}$ $\rho = \begin{pmatrix}\omega_1\\ \vdots \\ \omega_k\end{pmatrix}$ where now $\rho'$ and $\rho$ are defined on all of $\Delta$. Hereafter we denote by $\rho_t$ the section of $H'(C_t, K_t)$ gotten by restricting $\rho$ to $C_t$. Then the functions $\int_{b(t)} \rho_t'$, $\int_{\delta'(t)} \rho_t$, $\int_{\gamma'(t)} \rho_t$, and $\int_{\delta(t)} \rho_t$ are all holomorphic in $\Delta$, and $\int_{\delta'(0)} \rho_0' = I_{\tilde{g}}$, $\int_{\delta'(0)} \rho_0 = 0$, $\int_{\delta(0)} \rho_0' = 0$, and $\int_{\delta(0)} \rho_0 = I_k$. The last equation merely expresses the fact that the matrix of residues of the differentials $\tilde{\rho}_0$ at the points $p_1^+, \ldots, p_k^+$ on $\tilde{C}$ is the identity. The matrix $B_1 = \int_{\gamma'(0)} \rho_0'$ is symmetric with positive definite imaginary part because the matrix of periods $\int_{(\delta', \gamma')(0)} \rho_0' = \int_{(\tilde{\delta}, \tilde{\gamma})} \tilde{\rho}_0'$ satisfies the Riemann bilinear relations.

The <u>full period matrix</u> $\Omega(t) = \int_{b(t)} \omega_t$ is a multiple valued holomorphic function on $\Delta'$ because $\int_{\gamma(t)} \rho_t$ is multiple-valued on $\Delta'$. We let $\Omega(t) = (A(t), B(t))$ where A and B are $g \times g$ matrices. The above remarks contain the fact that $A(t) = \int_{(\delta', \delta)(t)} \omega_t$ is holomorphic on the entire disk $\Delta$. Furthermore, from the Riemann relations on $C_t$ we know that $A(t)$ is invertible on $\Delta'$. However, $A(0) = I_g$ so A is invertible for all t. We choose a new set of differentials equal to $A^{-1}\omega$ and denote the new vector by the old letter $\omega$. Since $A(0) = I$ the vector $\omega_0$ was left unchanged. Now the full period matrix

$$\Omega(t) = \int_{b(t)} \omega_t = (I, B(t)).$$

It is trivial to check that $\int_{\gamma'(t)} \omega_t$ is still holomorphic. Explicity, $B(t) = \begin{pmatrix} B_1(t) & {}^tB_2(t) \\ B_2(t) & B_3(t)\end{pmatrix}$ and $B_1$ and $B_2$ are holomorphic functions while $B_3(t) = \int_{\gamma(t)} \rho_t$ is multiple valued.

When continued once around the origin in the positive direction $B_3(t)$ returns to

$$\int_{PL(\gamma(t))} \rho_t = \int_{\gamma(t)+\delta M} \rho_t = \int_{\gamma(t)} \rho_t + M.$$

Thus the matrix function $B_3(t) - \frac{1}{2\pi i} \log t\, M.$ is a single valued function $B'(t)$ on $\Delta'$.

Lemma: (Mayer) $B'$ extends to a holomorphic function on $\Delta$.

Proof: Regardless of the branch chosen for $B_3(t) = (\tau_{ij}(t))$ the matrix $H_3(t) = \mathrm{Im}(B_3(t))$ is positive definite and symmetric because $H(t) = \mathrm{Im}B(t)$ is positive definite and symmetric. If $\tau_{jj}$ is a diagonal element of $B_3(t)$ then $e^{2\pi i \tau_{jj}}$ is bounded on $\Delta'$ and single valued. So $e^{2\pi i \tau_{jj}}$ extends to a holomorphic function $h(t)$ on $\Delta$. Then $h(t) = t^m g(t)$ where $g(0) \neq 0$ so $\tau_{jj}(t) - \frac{m}{2\pi i} \log t$ is holomorphic. The integer $m$ being uniquely determined by the condition that $t_{jj} - \frac{m}{2\pi i} \log t$ be single-valued, we have $m = m_{jj}$ where $M = (m_{ij})$, and we have shown the diagonal elements of $B'$ are holomorphic on $\Delta$.

However, the same argument applied to the function $-2\tau_{ij} + \tau_{ii} + \tau_{jj}$ (which has positive imaginary part) shows that $\tau_{ij} - \frac{m}{2\pi i} ij \log t$ is holomorphic, which proves the lemma.

Summary: In the normalized period matrix $\Omega = (I, B)$ all the entries of $B$ are holomorphic in $\Delta$ save possibly the lower right hand $k \times k$ matrix $B_3$. Then $B_3 = B' + \frac{M}{2\pi i} \log t$ where $B'$ is holomorphic and $M$ is positive definite and symmetric. The matrix $H = \mathrm{Im}B$ is single valued on $\Delta'$.

Lemma: $H^{-1}$ extends continuously to the origin and $H^{-1}(0) = \begin{pmatrix} H_1^{-1}(0) & 0 \\ 0 & 0 \end{pmatrix}$

Proof: Let $T = (H_3 - H_2 H_1^{-1}\, {}^tH_2)^{-1}$ (since $H$ is positive definite $T$ is defined).

Then: $$H^{-1} = \begin{pmatrix} H_1^{-1} + H_1^{-1}\, {}^tH_2\, T\, H_2\, H_1^{-1} & -H_1^{-1}\, {}^tH_2\, T \\ -T\, H_2\, H_1^{-1} & T \end{pmatrix}$$

However, $H_3 = \mathrm{Im}(B_3) = \mathrm{Im}B' - \frac{M}{2\pi} \log |t|$; so $\lim_{t\to 0} T = 0$ and the lemma follows.

2.6. Let $B(0) = \begin{pmatrix} B_1(0) & 0 \\ B_2(0) & 0 \end{pmatrix}$ and $\Omega(0) = (I, B(0))$ and define an equivalence relation "$\sim$" on $\tilde{J} = \Delta \times \mathbb{C}^2$ as follows:

$$(t,x) \sim (t', x') \iff t = t' \text{ and}$$

$$x - x' = \Omega(t)\begin{pmatrix} n \\ m \end{pmatrix} = n + B(t)m \text{ where}$$

$$n, m \in \mathbb{Z}^g \text{ are integral column vectors.}$$

The above relation is well-defined (i.e., does not depend on the choice of branch chosen for $\Omega(t)$). Let $J$ be the quotient space of $\tilde{J}$ under $\sim$ and let $pr: \tilde{J} \to J$ be the projection.

Theorem: a) The relation $\sim$ is closed.

b) The map $pr$ is a local homeomorphism

c) If $U_1$ and $U_2$ are open sets mapped homeomorphically onto $U$ by $pr_1 = pr \mid U_1$ and $pr_2 = pr \mid U_2$ respectively then $pr_1^{-1} \circ pr_2$ is an analytic isomorphism.

d) $J$ is a complex manifold with a complex structure such that $pr$ is an analytic map.

Proof: We may agree to multiply $\Omega(0)$ only by vectors $\begin{pmatrix} n \\ m \end{pmatrix}$ such that the last $k$ components of $m$ are zero without altering the equivalence relation. Then if $x = \Omega(t)\begin{pmatrix} n \\ m \end{pmatrix} = n + B(t)m$, we have $m = H^{-1}(t)\mathrm{Im}(x)$ even when $t = 0$.

a) Let $(t_i, x_i)$ be a sequence of periods in $\tilde{J}$ tending to the limit $(t, x)$. For each $i$ we have $x_i = n_i + B(t_i)m_i$ and if $t \neq 0$ we agree to choose the branches $B(t_i)$ so they approach $B(t)$ in the limit. The sequence of integers $m_i = H^{-1}(t_i)\mathrm{Im}(x_i)$ being convergent is eventually constant so $m_i = m$ if $i$ is sufficiently large. Then $n_i = x_i - B(t_i)m_i$. If $t \neq 0$ the right hand side is convergent so $n_i$ is eventually constant. If $t = 0$ then $m = \lim_{t \to 0} H^{-1}(t_i)\mathrm{Im}(x_i)$ is zero in the last $k$ components so $\lim_{t_i \to 0} B(t_i)m_i = \lim_{t_i \to 0} B(t_i)m$ exists and $n$ is again eventually constant. In either case $x = n + B(t)m$ for some pair of integral vectors $n, m$ and $(t, x)$ is a period. This proves part a).

(b) Given $(t, x) \in \widetilde{J}$ choose neighborhoods $U_1$ and $U_2$ of $t$ and $x$ respectively such that $t' \in U_1$ and $x', x'' \in U_2 \Rightarrow ||x' - x''|| < \frac{1}{2}$ and $||A(t')\mathrm{Im}(x' - x'')|| < \frac{1}{2}$. Then pr is injective on $U = U_1 \times U_2$. For, given $(t', x')$ and $(t', x'')$ in $U$ and $\mathrm{pr}(t', x') = \mathrm{pr}(t', x'')$ then $x' - x'' = n + B(t')m \Rightarrow ||m|| = ||A(t')\mathrm{Im}(x' - x'')|| < \frac{1}{2}$ $\Rightarrow m = 0 \Rightarrow ||n|| = ||x' - x''|| < \frac{1}{2} \Rightarrow n = 0$. Furthermore, pr is an open map because:

$\mathrm{pr}^{-1}(\mathrm{pr}(U)) = \widetilde{U} = \{(t, x + n + B(t)m) : (t, x) \in U, n, m \in \mathbb{Z}^g\}$

If $(t', x') \in \widetilde{U}$ then either $t' \neq 0$ in which case we choose an analytic branch $B(t)$ for $t$ near $t'$ or $t' = 0$ in which case we choose $m$ to be zero in its last $k$ components. In either case if $x' = n + B(t')m + x$ then $x' - n - B(t')m$ is continuous and it follows that $\widetilde{U}$ is open. Thus pr is a local homeomorphism proving (b).

Finally, if $\mathrm{pr}_1 : U_1 \to U$ and $\mathrm{pr}_2 : U_2 \to U$ are homeomorphisms let $(t, x) \in U_1$ and choose a branch of $B(t)$ at $t$ if $t \neq 0$. Then $\mathrm{pr}_2^{-1} \circ \mathrm{pr}_1(t, x) = (t, x + n + B(t)m)$. If $t \neq 0$ then $B(t)$ is analytic and $\mathrm{pr}_2^{-1} \circ \mathrm{pr}_1$ is continuous so $m, n$ are constant near $t$ so $\mathrm{pr}_2^{-1} \circ \mathrm{pr}_1$ is analytic near $(t, x)$. If $t = 0$ then we can choose $m$ with zero as its last components so $n + B(t)m$ is continuous and both $m$ and $n$ must be constant on a neighborhood of $(t, x)$. Again the composition $\mathrm{pr}_2^{-1} \circ \mathrm{pr}_1$ is analytic and this proves (c). Statement (d) follows trivially from (a), (b), and (c).

We have thus constructed explicitly a family $J$ of Jacobian varieties over the disk such that $J_t = \pi^{-1}(t)$ is the ordinary Jacobian variety of $C_t$ when $t \neq 0$ and $J_0$ is the generalized Jacobian of $C_0$.

There is clearly a projection of $J_0$ onto $J(\widetilde{C})$ with kernel $(\mathbb{C}^*)^k$ so $J_0$ is an extension of $\widetilde{J}$ by a complex Lie group.

3.0. The above results could also be obtained using the sheaf $\pi_* H^1(V,\mathcal{O})$ (i.e. the sheaf on $\Delta$ whose sections over $U$ are just the group $H^1(\pi^{-1}(U),\mathcal{O})$). It is necessary to show that $\dim H^1(C,\mathcal{O}) = g$ in order to apply Grauert's theorem. This may be accomplished by referring to the exact sequence:

$$0 \to \mathcal{O} \to \lambda_* \widetilde{\mathcal{O}} \to Q \to 0$$

over $C$ where $\widetilde{\mathcal{O}}$ is the structure sheaf on $\widetilde{C}$ and $Q$ is the quotient sheaf. The sheaf $Q$ is easily seen to be a "skyscraper" sheaf supported by the double points $p_1, \ldots, p_d$. Hence the sequence:

$$0 \to H^0(C,\mathcal{O}) \to H^0(C, \pi_*\widetilde{\mathcal{O}}) \to \mathbb{C}^d \to H^1(C,\mathcal{O}) \to H^1(C, \pi_*\widetilde{\mathcal{O}}) \to 0$$

is exact. The desired result is obtained upon noticing that $H^*(C, \pi_*\widetilde{\mathcal{O}})$ is isomorphic to $H^*(\widetilde{C},\widetilde{\mathcal{O}})$, a fact which follows from the spectral sequence of the fibration $\lambda : \widetilde{C} \to C$.

Thus the sheaf $\pi_* H^1(V,\mathcal{O})$ is isomorphic to the sheaf of germs of a complex vector bundle $E$ of rank $g$. Each fibre $E_t$ is canonically isomorphic to $H^1(C_t,\mathcal{O})$ and in each fibre is contained a lattice $L_t = H^1(C_t, \mathbb{Z})$. Furthermore, the union $\bigcup_{t\in\Delta} L_t$ is a closed complex submanifold of $E$ and it follows that the family of groups $E_t/L_t$ is a complex manifold. The group $H^1(V,\mathcal{O})/H^1(V,\mathbb{Z})$ is just the group of global sections of the family $J = E/L$.

Lectures 6-10 -- H. Clemens.

Let $W$ be a non-singular (open) complex manifold and

$$p : W \to D$$

a proper analytic morphism of $W$ onto the unit disc. Put $D' \doteq D - \{0\}$ and suppose that $p$ is of maximal rank on $p^{-1}(D')$. Suppose further that:

$$p^{-1}(0) = A(1) \cup \ldots \cup A(h) \quad \ldots$$

where the $A(j)$ are compact non-singular submanifolds of $W$ meeting transversely. As $z$ approaches $0$, each component $A(j)$ of $p^{-1}(0)$ is acquired with a certain multiplicity, which we shall call $m(j)$. Let

$$A(J) = \cap \{A(j) : j \in J\}.$$

Finally set $S_z = p^{-1}(z)$ for $z \in D$. For $z \in D'$, $S_z$ is a non-singular compact complex manifold.

If $W$ is any (possibly open) complex manifold and $Y$ is a compact submanifold of $W$, let $V$ be an open set in $W$.

Definition 1: $\mu : V \to V \cap Y$ will be called a regular normal bundle over $V \cap Y$ if:

i) $\mu$ is a $C^\infty$-projection everywhere of maximal rank;

ii) for each $P \in V \cap Y$, there exists a neighborhood $U$ of $P$ in $Y$ and a $C^\infty$-map:

$$w : \mu^{-1}(U) \times U \to \mathbb{C}^r$$

such that for $P' \in U$:

a) $w(\ ,P')$ is analytic and of maximal rank;

b) $\mu^{-1}(P') = \{Q : w(Q, P') = 0\}$.

Now let M be a closed subset of Y. A mapping $\mu' : N \to M$ will be called a <u>regular</u> <u>normal</u> <u>bundle</u> <u>over</u> <u>M</u> if there exists an open neighborhood V of M in W and a regular normal bundle

$$\mu : V \to V \cap Y$$

such that $N = \mu^{-1}(M)$ and $\mu' = \mu|_N$.

<u>Lemma</u> 2: Let Y and W be as above. Let

$$\mu' : N \to M$$

be a regular normal bundle over the closed set M in Y. Then $\mu'$ extends to a regular normal bundle over all of Y.

<u>Proof</u>: Let $U_0$ be an open neighborhood of M in Y such that $\mu'$ extends to a regular normal bundle $\mu_0$ over $U_0$. Let $\{U\}$ be a finite cover of Y such that for each $U \in \{U\}$:

either i) $U \subseteq U_0$ and there exists $w_U : \mu_0^{-1}(U) \times U \to \mathbb{C}^r$ which gives $\mu_0$ as in Definition 1;

or ii) $U \subseteq Y - M$ and there exists a $w_U$ which gives a regular normal bundle over U as in Definition 1.

Let $\rho_U$ be a $C^\infty$-partition of unity subordinate to $\{U\}$. For $P \in U \cap U'$, put

$$\Delta(U, U'; P) = \left(\frac{\partial(w_U(\ ,P)|_U)}{\partial(w_{U'}(\ ,P)|_{U'})}\right)_P$$

the Jacobian matrix which is non-degenerate by Definition 1, ii).

Let $V_U$ be a sufficiently small neighborhood of U in W and define:

$$y_U : V_U \times U \to \mathbb{C}^r$$

by $y_U(Q,P) =$

$$\Sigma\{\rho_{U'}(P)\,\Delta(U,U';P)\,w_{U'}(Q,P) : U' \subseteq \{U\}\}.$$

Now $y_U(Q,P)$ defines a regular normal bundle over $U$ since the Jacobian

$$\left(\frac{\partial y_U(\ ,P)|_U}{\partial w_U(\ ,P)|_U}\right)_P = \text{identity matrix.}$$

Also $y_U(\ ,P) = \Delta(U,U';P)\,y_{U'}(\ ,P)$ over $P \in U \cap U'$ so that $y_U$ and $y_{U'}$ determine the same regular normal bundle over $U \cap U'$. This then gives the lemma.

We shall use Lemma 2 to construct inductively a special system of regular normal bundles around the various $A(J)$.

Now a regular normal bundle

$$\mu : V \to Y$$

gives at each point $Q \in V$ a linear transformation of complex cotangent spaces:

$$I^* : T^*(V,Q) \to T^*(\mu^{-1}(\mu(Q)),Q).$$

The kernel of $I^*$ is precisely the image of $T^*(Y, \mu(Q))$ under the map $\mu^*$. Hence we have an exact sequence of <u>complex</u> vector spaces:

$$0 \to T^*(Y,P) \xrightarrow{\mu^*} T^*(V,Q) \xrightarrow{I^*} T^*(\mu^{-1}(P),Q) \to 0$$

where $\mu(Q) = P$.

Put $N(\mu) = T^*(V,\ )\,/\,\mu^*\,T^*(Y,\ )$. Then $N(\mu)$ is a $C^\infty$-complex vector bundle on $V$ and $N(\mu)|_{\mu^{-1}(P)}$ is a complex analytic vector bundle which is canonically identified with $T^*(\mu^{-1}(P),\ )$. Thus we can define

$$d : \Lambda^s N(\mu) \to \Lambda^{s+1} N(\mu)$$

and can speak of "closed" and "exact" sections of $\Lambda^s N(\mu)$.

<u>Definition</u> 3: Let $\omega$ be a section of $\Lambda^s N(\mu)$ over a set $U \subseteq V$. Then $\omega$ will be called <u>regular</u> if for any $P$ with $U \cap \mu^{-1}(P) \neq \phi$:

i) $\omega|_{U \cap \mu^{-1}(P)}$ is holomorphic;

ii) there exists a neighborhood $U_P$ of $P$ in $Y$ and a closed $C^\infty$-section $\omega_0$ of $\Lambda^s T^*(V, \ )$ on $\mu^{-1}(U_P) \cap U$ such that:

$$I^*(\omega_0) = \omega|_{\mu^{-1}(U_P) \cap U}.$$

(Thus a regular section of $\Lambda N(\mu)$ is always closed.)

We now proceed to apply the above to the case $Y = A(J)$, $J \subseteq \{1, \dots, h\}$

<u>Lemma</u> 4: Shrinking $D$ around $0$ (and hence $W$ around $S_0$) as necessary, there exists a system of regular normal bundles:

$$\mu(J) : V(J) \to A(J)$$

of $A(J)$ in $W$ and a collection of regular sections:

$$\omega(j, J) \text{ of } N(J) \text{ over } (V(J) - A(J))$$

where $N(J) = N(\mu(J))$ and $j \in J$ such that:

i) $W = \bigcup \{V(j) : j \in J\}$;

ii) $V(J) \cap V(K) = V(J \cup K)$;

iii) $\omega(j, J)$ has logarithmic pole along $A(j)$;

iv) $(x_j = e^{2\pi i \int \omega(j, J)})_{j \in J}$ gives holomorphic coordinates on $\mu(J)^{-1}(P)$ for $P \in A(J)$ such that, for $K \subseteq J$, $\mu(K)$ is given on $\mu(J)^{-1}(P)$ by setting

$$x_j = 0 \qquad j \in K$$

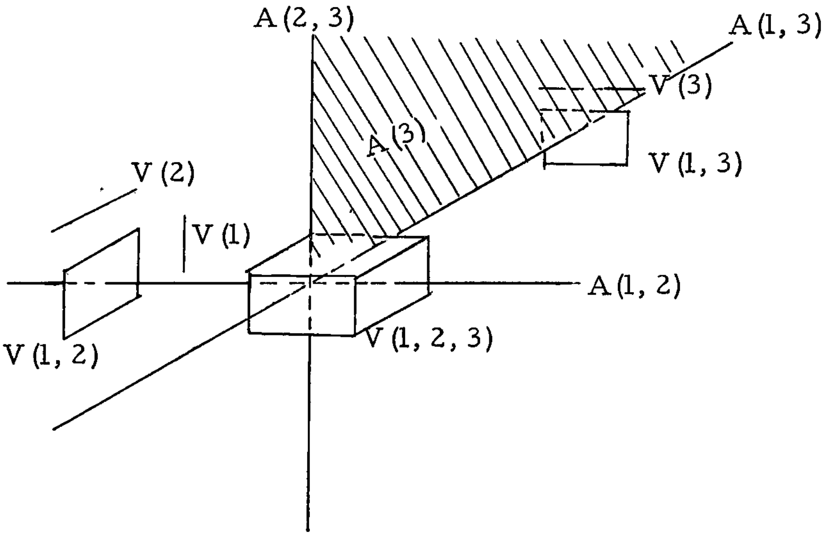
and leaving the remaining $x_j$ constant;

v) for $K \subseteq J$, $\omega(j, K)$ and $\omega(j, J)$ correspond under the induced map

$$N(J) \to N(K);$$

vi) $\Sigma\{m(j)\,\omega(j,J) : j \in J\} \equiv (1/2\pi i)\, d \log z.$

Picture:




Proof: We proceed by induction. Assume we have a system of regular normal bundles:

$$\mu(J) : V(J) \to A(J)$$

on each $J$ with $|J| > s$ and regular sections $\omega(j, J)$ over

$$V(J) - A(j)$$

such that ii) through vi) are satisfied. It will suffice to construct $\mu(J)$ and $\omega(j, J)$ for some fixed $J$ with $|J| = s$. Let $V = \bigcup\{V(K) : J \subsetneq K\}$. Put $x_j = e^{2\pi i \int \omega(j,K)}$ and then by the induction assumption one can define a regular normal bundle:

$$\mu(J) : V \to A(J) \cap V$$

by putting $x_j = 0$ for $j \in J$ and keeping the remaining $x_j$ constant on $\mu(K)^{-1}(P)$. By v), $\mu(J)$ is well-defined. By Lemma 2, $\mu(J)$ can be extended to a regular normal bundle:

$$\mu(J) : V(J) \to A(J).$$

(It may be necessary to shrink $V(K)$ with $K \supsetneq J$ slightly at this point.)

Next let $M$ be a compact neighborhood of $\bigcup\{A(K) : J \subsetneq K\}$ in $A(J) \cap V$. Shrinking $V(J)$ around $A(J)$ if necessary, one can find a finite cover $\{U\}$ of $A(J) - M$ and analytic coordinates:

$$x_{j,U}, \; j \in J$$

such that $p : W \to D$ is given on $\mu(J)^{-1}(U)$ by

$$\pi\{x_{j,U}^{m(j)} : j \in J\} = z.$$

Complete $\{U\}$ to a cover of $A(J)$ by adding the set

$$U_0 = A(J) \cap V.$$

Let $\rho_U$ be a $C^\infty$-partition of unity subordinate to this cover and define $\omega(j, J) = \rho_{U_0}\,\omega(j) + (1/2\pi i)\Sigma\{\rho_U \, d \log x_{j,U} : U \in \{U\}\}$ where $\omega(j)$ is the section of $N(J)$ induced by $\omega(j, K)$ for $K \supsetneq J$ by the maps

$$N(K) \to N(J).$$

Shrinking $V(J)$ and $V(K)$ for $K \supseteq J$ as necessary, one achieves the induction step and hence the lemma.

Note that the $x_j = e^{2\pi i \int \omega(j, J)}$ give holomorphic local coordinates on $\mu(J)^{-1}(P)$ such that

$$z = c\pi\{x_j^{m(j)} : j \in J\}.$$

Also we can assume, by further shrinking $D$ if necessary, that $\{x_j\}$ gives an isomorphism

$$\mu(J)^{-1}(P) \to \pi\{|x_j| < \text{constant } R_j : j \in J\},$$

that is, $\mu(J)^{-1}(P)$ with coordinates $x_j$ is a polydisc. Having done this we will write:

$$W(q) = \bigcup \{V(J) : |J| \geq q\}$$

$$V'(J) = V(J) - W(|J| + 1) \; ; \; A'(J) = A(J) - W(|J| + 1)$$

$$\mu'(J) : V'(J) \to A'(J).$$

and
$$N'(J) \to V'(J).$$

Thus there is a well-defined function

$$\tau(j) : V(j) \to [0,1]$$

given by $\tau(j)'(Q) = |x_j(Q)| / R_j$. $\tau(j)$ is well-defined since $x_j$ is determined on $\mu(J)^{-1}(P)$ up to multiplicative constant. We can extend $\tau(j)$ to all of $W$ by putting $\tau(j) = 1$ on $W - V(j)$. We can assume that the $V(J)$ have been so defined that $\tau(j)$ is $C^\infty$ on $W - (A(j) \cup \partial V(j))$ and continuous on all $W$.

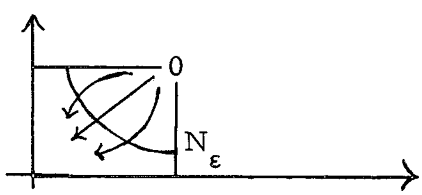
Let $\nabla_h = [0,1]^h - \{(1, \ldots, 1)\}$. Then we have a map given by the $\tau(j)$:

$$\tau : W \to \nabla_h.$$

We shall now construct an appropriate retraction and partition of unity on $\nabla_h$ which will "lift back" to give a "Retraction Theorem" and a "Picard-Lefschetz Theorem" on

$$p : W \to D.$$

In $\nabla_2$, let $N_\varepsilon = \{(r_1, r_2) : r_1^{m(j)} r_2^{m(k)} = \varepsilon\}$.




Consider the vector field on $(0,1)^2$ given by

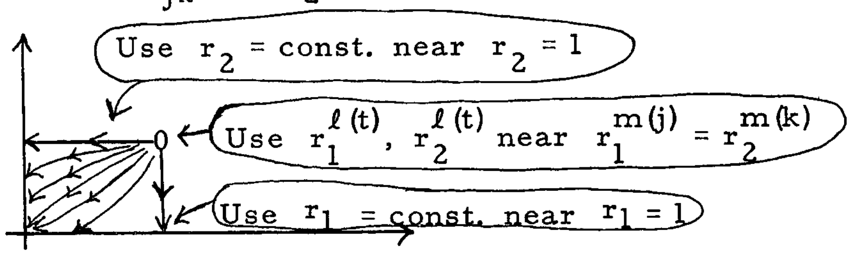
$$\{(r_1^{\ell(t)}, r_2^{\ell(t)}) : 0 < t < 1\}$$

for $(r_1, r_2) \in N_\varepsilon$, $\ell(t) = \log_\varepsilon t$. Use a partition of unity to average this vector field with the horizontal and vertical vector fields given by:

$$\{(r_1^{\ell(t)}, r_2) : 0 < t < 1\}, \quad r_1^{m(j)} = \varepsilon$$

$$\{(r_1, r_2^{\ell(t)}) : 0 < t < 1\}, \quad r_2^{m(k)} = \varepsilon$$

in order to achieve a $C^\infty$-vector field $\sigma_{jk}$ on $\nabla_2$:




such that:

i) $\sigma_{jk}$ is transverse to $N_{\varepsilon'}$ for all $\varepsilon'$;

ii) for the integral curves $(r_1(t), r_2(t))$ of $\sigma_{jk}$:

$$(r_1(t), r_2(t)) \in N_{\varepsilon^{\ell(t)}} = N_t.$$

Proceeding inductively average on $\nabla_3$:

$$\{(r_1^{\ell(t)}, r_2^{\ell(t)}, r_3^{\ell(t)}) : 0 \le t \le 1\}$$

near $r_1^{m(j)} = r_2^{m(k)} = r_3^{m(\ell)}$,

$$\{\sigma_{jk}(r_1, r_2), \ r_3 = \text{constant}\}$$

near $r_3 = 1$,

$$\{\sigma_{k\ell}(r_2, r_3), \ r_1 = \text{constant}\}$$

near $r_1 = 1$,

$$\{\sigma_{j\ell}(r_1, r_3), \ r_2 = \text{constant}\}$$

near $r_2 = 1$

to get $\sigma_{jk\ell}$ with analogous properties to i) and ii) above.

Proceeding in this manner we inductively construct a $C^{\infty}$-vector field

$$\sigma = \sigma_{123\ldots h} \quad \text{on } \nabla_h$$

with the following list of properties:

Property 5a: $\sigma$ is normal to $N_\varepsilon = \{(r_1, \ldots, r_h) : \pi r_j^{m(j)} = \varepsilon\}$ for all $\varepsilon \in (0,1)$.

Denote $C(J) = \{(r_1, \ldots, r_h) : r_j^{m(j)} \leq r_k^{m(k)}$ for all $j \in J$, $1 \leq k \leq h\}$;

$Z(J) = \{(r_1, \ldots, r_h) : r_j = 0$ for all $j \in J\}$.

Property 5b: The integral curves of $\sigma$ give a strong deformation retraction:

$$R : \nabla_h \times [0,1] \to \nabla_h$$

such that:

i) for $(r_j) \in N_{\varepsilon_1}$, $0 \leq \varepsilon_2 \leq \varepsilon_1 < 1$ :

$$R((r_j), \varepsilon_2/\varepsilon_1) \in N_{\varepsilon_2} \;;$$

ii) $R(\ ,0)$ maps a neighborhood of $C(J)$ in $(0,1)^h$ onto $Z(J)$ for any $J \subsetneq \{1, \ldots, h\}$.

By the last property the sets

$$[\text{interior } (R(\ ,0)^{-1} Z(j))]$$

cover $N_\varepsilon$ for any $\varepsilon \neq 0$. Let $\rho_j$ be a $C^{\infty}$-partition of unity subordinate to this cover. Then the $\rho_j$ can be extended uniquely to a partition of unity $\rho_j$ on $\nabla_h$ such that:

Property 6: Each $\rho_j$ is constant along each integral curve of $\sigma$.

On $\mu(J)^{-1}(P)$ we have local coordinates $x_j : e^{2\pi i \int \omega(j, J)}$. We can now define

$$\psi : [0,1] \times \mathbb{R} \times V'(J) \to V'(J)$$

by $$\psi(r, \theta; (P, x_j)) = (P, R(\tau, r)_j \, e^{2\pi i \rho_j(\tau)\theta / m(j)} x_j).$$

By construction $\psi$ is independent of the choice of multiplicative constant in the definition of $x_j$. Also the $\psi$'s fit together to give a map (which is $C^\infty$ by Definition 3 and Lemma 4):

$$\Psi : [0,1] \times \mathbb{R} \times W \to W.$$

<u>Theorem</u> 7: The map $\Psi$ has the following properties:

i) $\Psi(r, \theta, \ )|_{\dot{S}_0}$ = identity map;

ii) $\Psi(r_2, \theta_2, \ ) \circ \Psi(r_1, \theta_1, \ ) = \Psi(r_1 r_2, \theta_1 + \theta_2, \ )$;

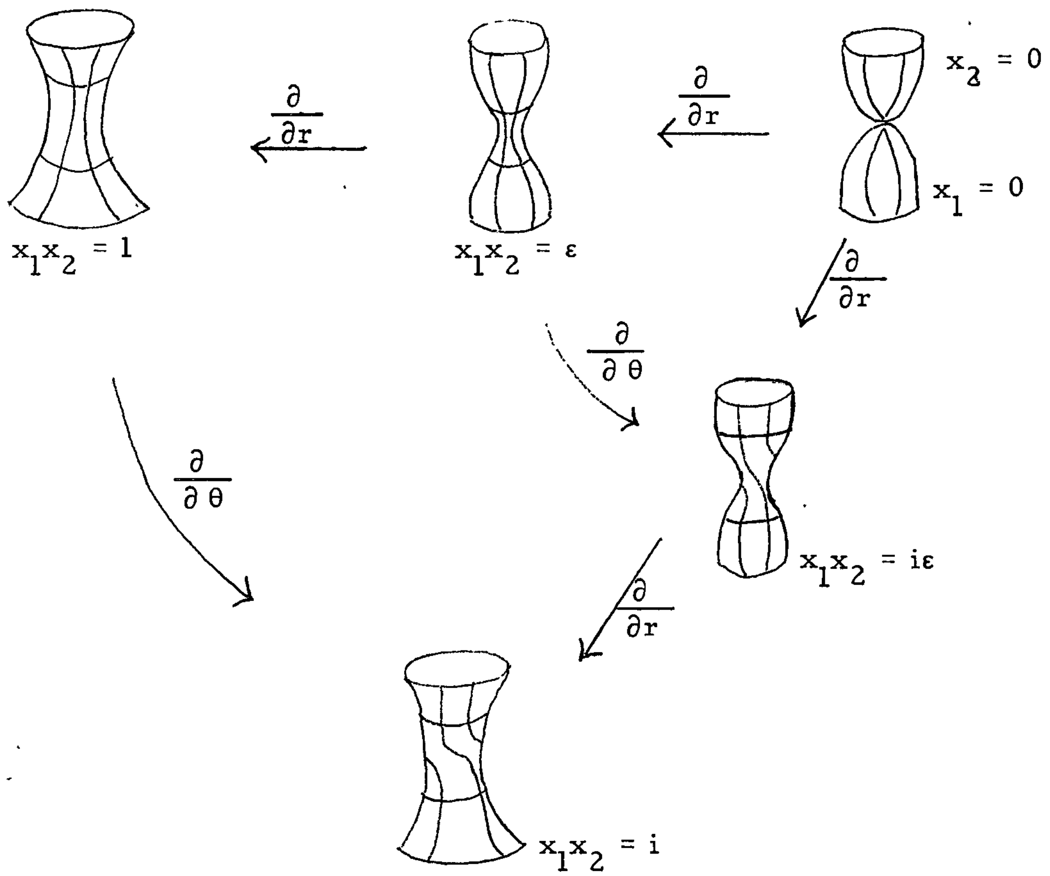
iii) $\mu'(J) \circ \Psi = \mu'(J)$ for $J \subseteq \{1, \ldots, h\}$;

iv) $p \circ \Psi(r, \theta, \ ) = r\, e^{2\pi i \theta} p$.

<u>Proof:</u> i) and ii) follow from Property 5b, i), Property 6 and the definition of $\Psi$. iii) is immediate and iv) follows from Property 5b, i) and the fact that $p$ is given on $\mu(J)^{-1}(P)$ by:

$$c\pi\{x_j^{m(j)} : j \in J\}.$$

Picture: Case of $\mu'(J)^{-1}(P)$ for $J = \{1, 2\}$ and $m(1) = m(2) = 1$:




where the tops of the various tubes correspond by fixing the value of $x_1$ and the bottoms by fixing the value of $x_2$.

Theorem 7 gives a retraction of $W$ onto $S_0$ which covers the standard retraction of $D$ onto $0$. This retraction also "commutes" with the family of diffeomorphisms:

$$\Psi(1, \theta, \quad ) : S_z \to S_{z'}$$

induced by the fundamental group of $D'$. We now wish to partially characterize the isomorphisms:

$$T^m = \Psi(1, m, \quad )^* : H^*(S_z; \mathbb{C}) \to H^*(S_z; \mathbb{C})$$

for $z \in D'$, $m = \text{l.c.m.}\ \{m(j) : j = 1, \ldots, h\}$.

First we can sum the bundles:

$$0 \to T^*(A(J), P) \to T^*(V(J), Q) \to N(J)_Q \to 0$$

with their complex conjugates to get an exact sequence:

$$0 \to L(A(J), P) \xrightarrow{\mu^*} L(V(J), Q) \xrightarrow{I^*} M(J)_Q \to 0$$

where $L$ denotes the complex vector space gotten by tensoring the real cotangent space with $\mathbb{C}$. Then $\Psi(1, \theta, \ )$ induces:

$$B^\theta : M'(J) \to M'(J).$$

But $\{\omega(j, J), \overline{\omega}(j, J)\}_{j \in J}$ gives a frame for $M'(J)$ over $(V'(J) - S_0)$ so that $B^\theta$ is given there by the formulas:

$$\text{(8)} \qquad \begin{aligned} B^\theta(\omega(j, J)) &= \omega(j, J) + (\theta / m(j))\, d\rho_j ; \\ B^\theta(\overline{\omega}(j, J)) &= \overline{\omega}(j, J) + (\theta / m(j))\, d\rho_j ; \\ B^\theta(d\rho_j) &= d\rho_j . \end{aligned}$$

(These formulas follow directly from the definitions of $\rho_j$ and $\Psi$.) Put:

$$\sigma(j, J) = (1/2)(\omega(j, J) + \overline{\omega}(j, J)).$$

Now from Lemma 4 vi) we have:

$$\text{(9)} \qquad \Sigma\{m(j)\, k_z^* \sigma(j, J) : j \in J\} = 0$$

where $k_z : S_z \to W$ is the inclusion map. Let $J_0 = J - \{\text{one fixed element of } J\}$. Let $U \subseteq A'(J)$ and let $\Omega(K, J, U)$ be the algebra of closed sections $\alpha$ of $\Lambda L(V'(J) \cap S_z, \ )$ over $U$ such that:

$$I^*(\alpha) = k_z^* \sigma(K, J)$$

where $K \subseteq J_0$ and $\sigma(K, J) = \Lambda\{\sigma(j, J) : j \in K\}$. Let $B(J, U) =$ algebra of sections of

$$\{\Lambda L(A'(J), P) \otimes H^0(\mu'(J)^{-1}(P) \cap S_z ; \mathbb{C})\}_{P \in U} .$$

Then any element $\alpha \in B(J,U)$ can be considered as a form on $\mu'(J)^{-1}(U) \cap S_z$ by pulling $\alpha|_P$ back to $\mu'(J)^{-1}(P) \cap S_z$ and then evaluating the coefficient at the appropriate element of $H_0(\mu'(J)^{-1}(P) \cap S_z\, ; \mathbb{C})$. We shall denote this form on $\mu'(J)^{-1}(U)$ again by $\alpha$ and the set of such forms again by $B(J,U)$.

<u>Definition</u> 10: A differential form $\phi$ on $S_z$ will be said to be in <u>normal form</u> if for each $J \subseteq \{1, \ldots, h\}$ and for each $P \in A'(J)$, there is an open neighborhood $U$ of $P$ in $A'(J)$ such that

$$\phi|_{S_z \cap \mu'(J)^{-1}(U)}$$

is given by:

$$\Sigma\{\alpha_K \wedge \pi_K : K \subseteq J_0\}$$

where $\alpha_K \in B(J,U)$ and $\pi_K \in \Omega(K,J,U)$.

We shall now utilize a conjecture which can be avoided (see Trans. A.M.S., Vol. 136, pp. 101-103) but which is quite reasonable and serves to illuminate the computations which follow:

<u>Conjecture</u> 11: Every deRham cohomology class on $S_z$ has a representative in normal form.

Now if $\phi$ is in normal form, it follows immediately from the formulas (8) that:

$$(T^m - \text{identity})^q(\phi|_{\mu'(J)^{-1}(U) \cap S_z}) = \begin{cases} 0 & \text{if } q \geq |J| \\ q!\,(m^q/\pi\{m(j) : j \in J_0\})\,(\alpha_{J_0} \wedge \rho_{J_0}) & \text{if } q = |J| - 1, \end{cases} \tag{12}$$

where $\rho_{J_0} = \Lambda\{d\rho_j : j \in J_0\}$. Then since the $\rho_{J_0}|_{\mu'(J)^{-1}(U)\cap S_z}$ are restrictions of closed forms $\rho_{J_0}$ on all of $S_z$: If $\phi$ is closed then the $a_{J_0} \in B(J,U)$ must be closed and must piece together to give a deRham class in $H^*(A'(J)\,;\, H^0(\mu'(J)^{-1}(P)\cap S_z\,;\,\mathbb{C}))$. If we let $\{\gamma_k\}$ be a basis for this last cohomology group and $\{\gamma_k'\}$ be the dual basis in $H^*(A'(J), \partial A'(J); H_0(\text{fibre}))$, then

$$\int_{A'(J)} \gamma_k \wedge \gamma_\ell' = \delta_{k\ell}.$$

For $P \in A'(J)$, put:

$$T(P) = \{Q \in (S_z \cap \mu'(J)^{-1}(P)) : \tau(j)(Q)^{m(j)} = \tau(k)(Q)^{m(k)} \ \forall j,\ k \in J\}.$$

Then $T(P)$ has $m(J)$ components $T_1(P), \ldots, T_{m(J)}(P)$ corresponding to the $m(J)$ components of $(\mu'(J)^{-1}(P)\cap S_z)$, where:

$$m(J) = \text{g.c.d. } \{m(j) : j \in J\}.$$

Also:

$$\begin{aligned} \int_{T_k(P)} \pi_{J_0} &= (1/m(J)) \int_{T(P)} \pi_{J_0} \\ &= m(j_0)/m(J) \int_{\tau(j)(Q)} = \text{const. for } j \in J_0 \pi_{J_0} \\ &= \pm\, m(j_o)/m(J) \end{aligned} \tag{13}$$

where $\qquad \{j_0\} = J - J_0.$

If we put $T(J) = \cup\{T(P) : P \in A'(J)\}$, then $\overline{V'(J)}\cap S_z$ is a fibre space over $T(J)$ with fibre a $(|J| - 1)$-dimensional cell and $T(J)$ is itself a fibre space over an $m(J)$-sheeted covering space $C(J)$ of $A'(J)$ with fibre a $(|J| - 1)$-dimensional torus. Also:

$$H^*(C(J)\,;\,\mathbb{C}) \approx H^*(A'(J)\,;\,H^0(\mu'(J)^{-1}(P)\cap S_z\,;\,\mathbb{C})).$$

Thus in homology we have Gysin maps for $q = |J| - 1$:

$$G : H_P(A'(J); H_0(\text{fibre})) \to H_{p+q}(S_z)$$

$$G : H_P(A'(J), \partial A'(J); H^0(\text{fibre})) \to H_{p+q}(S_z, W(q+2))$$

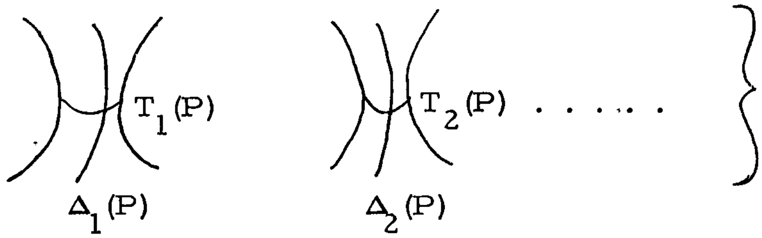
gotten by putting $G(\gamma)$ = locus of $T(P)$ as $P$ traces out $\gamma$. Denote by $\Gamma_P(J)$ a basis for $H_P(A'(J), \partial A'(J); H^0(\text{fibre}))$. For $\gamma \in \Gamma_p(J)$ let $\gamma'$ denote the dual element in $H_{P'}(A'(J); H_0(\text{fibre}))$.

For each $T_k(P)$, put $\Delta_k(P)$ = some component of

$\{Q \in (\mu'(J)^{-1}(P) \cap S_z) : \text{Arg } x_j(Q) = \text{Arg } x_k(Q),\ \forall j,\ k \in J\}$

which intersects $T_k(P)$. $(x_j = e^{2\pi i \int \omega(j, J)}.)$

Picture:




$\mu'(J)^{-1}(P) \cap S_z$ where $m(J) > 1$.

Then
$$\int_{\Delta_k(P)} \rho_{J_0} = \pm \text{ volume } (\{r_1, \ldots, r_{|J_0|}) : r_k \geq 0 \text{ and } \Sigma r_k \leq 1\}.$$
$$= \pm (1/q!) \tag{14}$$

where $q = |J| - 1$.

From this it follows that for the closed form $\alpha_{J_0}$ on $A'(J)$ appearing in formula (12):

$$\alpha_{J_0} = \Sigma_k (\int_{A'(J)} \alpha_{J_0} \wedge \gamma_k') \gamma_k$$
$$= \pm (m(J)\, q!/m(j_0))\, \Sigma_k (\int_{S_z} \alpha_{J_0} \wedge \pi_{J_0} \wedge \gamma_k' \wedge \rho_{J_0})\, \gamma_k$$
$$= \pm (m(J)\, q!\, / \, m(j_0)\, \Sigma_k (\int_{S_z} \phi \wedge \gamma_k' \wedge \rho_{J_0})\, \gamma_k$$

where $q = |J| - 1$.

Hence by (12):

$$(T^m - \text{identity})^q(\phi) = \tag{15}$$

$$\pm (q!)^2 m^q \Sigma_{|J| = q+1} \Sigma_k (m(J)/\pi_J m(j))(\int_{S_z} \phi \wedge \gamma_k' \wedge \rho_{J_0})\, \gamma_k \wedge \rho_{J_0}.$$

on $S_z - W(q+2)$.

It is perhaps more enlightening to use Poincaré duality and give (15) as an intersection formula in homology. (14) gives that $q!\, \rho_{J_0}$ is dual to $T_P$. Then the Poincaré dual of (15) is:

<u>Theorem</u> 16: The mapping

$(T_m - \text{identity})^q : H_{p+q}(S_z; \mathbb{C}) \to H_{p+q}(S_z, W(q+2); \mathbb{C})$ is given by

$(T_m - \text{identity})^q(a) =$

$(-1)^r m^q \Sigma\{(m(J)/\pi_J m(j))(a \cdot G(\gamma'))G(\gamma) : |J| = q+1,\ \gamma \in \Gamma_p(J)\}$

(where $r = (2p+q)(q-1)/2$ by direct calculation).

<u>Corollary</u> 17 (A. Landman): If $n = \dim_{\mathbb{C}} S_z$, $T = \Psi(1,1,\ )_* : H_*(S_z; \mathbb{C}) \to H_*(S_z; \mathbb{C})$ satisfies the polynomial equation:

$$(\lambda^m - 1)^{n+1} = 0.$$

Let $\eta = (T_m - \text{identity}) : H_*(S_z) \to H_*(S_z)$.

<u>Definition</u> 18: $a \in H_*(S_z)$ is q-<u>invariant</u> if:

$$\eta^q(a) = 0.$$

<u>Lemma</u> 19: For $a \in H_q(S_z)$ or $a \in H_{2n-q}(S_z)$, $a$ is $(q+1)$-invariant.

<u>Proof</u>: For $\phi \in H^q(S_z)$, locally $\phi = \Sigma a_k \wedge \pi_K$ where $|K| \leq q$. Hence just as in

formula (12) $(T^m - \text{identity})^{q+1}\phi = 0$. Now use Poincaré duality and Lefschetz duality.

Lemma 20: Let $\alpha$ be (q+1)-invariant. Then $\alpha$ is q-invariant if and only if

$$\alpha \cdot \beta = 0$$

for all $\beta \in \eta^q(H_*(S_z))$.

Proof: This is a purely formal result. Using that $T_j(\alpha) \cdot T_k(\alpha) = T_{j+\ell}(\alpha) \cdot T_{k+\ell}(\alpha)$ for all $j, k$, and $\ell$, one calculates that:

$$\eta^q(\alpha) \cdot \beta + (-1)^{q+1} \; \alpha \cdot \eta^q(\beta) = q\, \eta^{q+1}(\alpha) \cdot \beta.$$

The lemma then follows.

The above lemma gives a weak characterization of q-invariance. A stronger characterization would follow if we had a more explicit topological characterization of $\eta^q(H_*(S_z))$.

Definition 21: $\alpha \in H_*(S_z)$ is q-vanishing if :

i) $\alpha$ lies in $W(q+1)$;

ii) $\alpha$ is in the kernel of the homology map induced by the retraction:

$$S_z \to S_0.$$

(See Theorem 7 i) and ii).)

Theorem 22: If $\alpha \cdot \beta = 0$ for every q-vanishing cycle $\beta$, then $\alpha$ is q-invariant.

Proof: By Theorem 7 we have a commutative diagram:

$$\begin{tikzcd}[column sep=small] H_*(S_z) \arrow[rr, "T"] \arrow[dr] & & H_*(S_z) \arrow[dl] \\ & H_*(S_0) & \end{tikzcd} \qquad (23)$$

Now $\eta^q(H_*(S_z))$ lies in $W(q+1)$ by Theorem 16. Also $\eta^q(H_*(S_z))$ lies in the kernel of the retraction map by diagram (23). The lemma now follows from Lemma 20.

The converse to Theorem 22 has been conjectured by Griffiths for $q = 1$. This seems to be a difficult problem. At present one has the converse only in the case $q = n$:

Theorem 24: If $\alpha \in H_n(S_z)$ is n-invariant, then $\alpha \cdot \beta = 0$ for every n-vanishing cycle $\beta$.

Proof: $\eta^n(\alpha) = (-1)^{n(n-1)/2} m^n \Sigma\{(m(J)/\pi_J m(j))(\alpha\cdot G(\gamma))\, G(\gamma) : |J| = n+1,\ \gamma \in \Gamma_0(J)\}$.
If $\eta^n(\alpha) = 0$ then $\alpha\cdot\eta^n(\alpha) = 0$ hence $(\alpha\cdot G(\gamma))^2 = 0$ for each $\gamma$. But the $G(\gamma)$ generate the n-vanishing cycles of dimension $n$ since each component of $W(n+1)\cap S_z$ has the homotopy type of an n-torus.

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# SEMINAR ON DEGENERATION OF ALGEBRAIC VARIETIES

Lectures by N. M. Katz

Institute for Advanced Study, Princeton, New Jersey

Second Term, 1969-1970

# SEMINAR ON DEGENERATION OF ALGEBRAIC VARIETIES

## Lectures 11 - 15: Nilpotent connections and the Monodromy Theorem Applications of a Result of Turrittin-- N. M. Katz.

### Introduction.

0.0 Let $\bar{S}/\underset{=}{C}$ be a projective non-singular connected curve, and $S = \bar{S} - \{\mathcal{Y}_1, \ldots, \mathcal{Y}_r\}$ a Zariski-open subset of S. Suppose that

0.0.0 $$\pi : X \to S$$

is a proper and smooth morphism. From the $\mathcal{C}^\infty$ viewpoint, $\pi$ is a locally trivial fibre space, so that, for $s \in S$ variable and $i \geq 0$ a fixed integer, the $\underset{=}{C}$-vector spaces "complex cohomology of the fibre"

0.0.1 $$H^i(X_s, \underset{=}{C})$$

form a local system on $S^{anal}$.

This local system may be constructed in a purely algebraic manner, by using the algebraic de Rham cohomology sheaves $H^i_{DR}(X/S)$. For each $i \geq 0$, $H^i_{DR}(X/S)$ is a locally free coherent algebraic sheaf on S, whose "fibre" at each point $s \in S$ is the $\underset{=}{C}$-vector space $H^i(X_s, \underset{=}{C})$, and has an integrable connection $\nabla$, the "Gauss-Manin connection". From this data, the local system of $H^i(X_s, \underset{=}{C})$ may be recovered as the sheaf of germs of horizontal sections of the associated coherent analytic sheaf on $S^{anal}$

0.0.2 $$H^i_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_S^{anal}$$

0.1 Now in down-to-earth terms, $\underset{=}{H}^i_{DR}(X/S)$ is an algebraic differential equation on S (classically called the Picard-Fuchs equations), and the local system of $H^i(X_s, \underline{\underline{C}})$ is the local system of germs of solutions of that equation.

0.2 The Griffiths-Landman-Grothendieck "Local Monodromy Theorem" asserts that if we restrict the local system of the $H^i(X_s, \underline{\underline{C}})$ to a small punctured disc $D^*$ around one of the "missing" points $\mathcal{Y} \in \overline{S} - S$, then picking a base point $s_0 \in D^*$, the automorphism T of $H^i(X_{s_0}, \underline{\underline{C}})$ induced by the canonical generator of $\pi_1(D^*, s_0)$ (the generator being "turning once around $\mathcal{Y}$ counterclockwise") has a very special Jordan decomposition:

0.2.0 $$T = D \cdot U = U \cdot D$$

where

0.2.1 D is semisimple of finite order (i.e., its eigenvalues are roots of unity)

and

0.2.2 $\underline{\underline{U}}$ is unipotent, and $(1 - U)^{i+1} = 0$ (i.e., the local monodromy has exponent of nilpotence $\leq i+1$)

0.3 We can interpret the 'Local Monodromy Theorem" as a statement about the local monodromy of the Picard-Fuchs equations around the singular point

$\mathcal{Y}$. Griffiths, by estimating the rate of growth of the periods as we approach the singular point $\mathcal{Y}$, was able to prove that the Picard-Fuchs equations have a "regular singular point" (in the sense of Fuchs) at $\mathcal{Y}$.

Given that the Picard-Fuchs equations have a regular singular point at $\mathcal{Y}$, the statement that the eigenvalues of its local monodromy are roots of unity is precisely the statement that the exponents of the Picard-Fuchs equation at $\mathcal{Y}$ are rational numbers. (In fact, Brieskorn [2] has recently given a marvelous proof of the rationality of the exponents via Hilbert's 7th Problem.)

0.4 The purpose of this paper is to give an arithmetic proof that the Picard-Fuchs equations have only regular singular points, rational exponents, and exponent of nilpotence $i+1$ (for $H^i$).

0.5 The method is first to "thicken"

0.5.0
$$X \xrightarrow{\pi} S \to \mathrm{Spec}\ (\underline{\underline{C}})$$

to a family

0.5.1
$$\underline{\underline{X}} \xrightarrow{\underline{\underline{\pi}}} \underline{\underline{S}} \to \mathrm{Spec}\ (R)$$

where $R$ is a subring of $\underline{\underline{C}}$, finitely generated over $\underline{\underline{Z}}$, $\underline{\underline{S}}/\mathrm{Spec}\ (R)$ is a smooth connected curve which "gives back" $S/\underline{\underline{C}}$ after extension of scalars $R \hookrightarrow \underline{\underline{C}}$, and $\underline{\underline{\pi}} : \underline{\underline{X}} \to \underline{\underline{S}}$ is a proper and smooth morphism which "gives back" $\pi : X \to S$ after the base change $S \to \mathbb{S}$.

For instance, the Legendre family of elliptic curves, given in homogeneous coordinates by

0.5.2 $$Y^2Z - X(X - Z)(X - \lambda Z) \text{ in } \operatorname{Spec}\left(\mathbb{C}\left[\lambda, \frac{1}{\lambda(1-\lambda)}\right]\right) \times \mathbb{P}^2$$

is (projective and) smooth over $\operatorname{Spec}\left(\mathbb{C}[\lambda, \frac{1}{\lambda(1-\lambda)}]\right) = \mathbb{A}^1 - \{0,1\}$. A natural thickening is just to keep the equation 0.5.2, but replace $\mathbb{C}[\lambda, \frac{1}{\lambda(1-\lambda)}]$ by $\mathbb{Z}[\lambda, \frac{1}{2(\lambda)(1-\lambda)}]$, and replace $\mathbb{C}$ by $\mathbb{Z}[1/2]$.

The thickening completed, we look at $H^i_{DR}(\mathbb{X}/\mathbb{S})$; replacing $\mathbb{S}$ by a Zariski open-subset, we can suppose

0.5.3 $\mathbb{S}$ is affine, say $\mathbb{S} = \operatorname{Spec}(\mathcal{S})$, and is étale over $\mathbb{A}^1_T$ (i.e., $\mathcal{S}$ is étale over $R[\lambda]$).

0.5.4 $M = H^i_{DR}(\mathbb{X}/\mathbb{S})$ is a free $\mathcal{S}$-module of finite rank.

The data of the Gauss-Manin connection is that of an $R$-linear mapping

0.5.5 $$\nabla\left(\frac{d}{d\lambda}\right) : M \to M$$

which satisfies, for $f \in \mathcal{S}$, $m \in M$

0.5.6 $$\nabla\left(\frac{d}{d\lambda}\right)(fm) = \frac{df}{d\lambda} \cdot m + f \cdot \nabla\left(\frac{d}{d\lambda}\right)(m)$$

The next step is to prove that this connection is globally nilpotent on $\underset{=}{\mathcal{S}}$ of exponent $i+1$, which by definition means that for every prime number $p$, the R-linear operation

0.5.7 $$(\nabla(\tfrac{d}{d\lambda}))^{p(i+1)} : M \to M$$

induces the zero mapping of $M/pM$.

To prove this, we use the fact that, $M = H^i_{DR}(\underset{=}{X}/\underset{=}{S})$ being free, we have

0.5.8 $$M/pM \simeq H^i_{DR}(\underset{=}{X}\otimes\underset{=}{F}_p / \underset{=}{S}\otimes\underset{=}{F}_p)$$

(the right hand side being an $\underset{=}{\mathcal{S}}/p\underset{=}{\mathcal{S}}$ module). The problem is then to prove the nilpotence of the Gauss-Manin connection in characteristic $p$; this is done in Section 5.

0.5.9 The final step is to deduce, from the global nilpotence of exponent $i+1$, that the Picard-Fuchs equations have only regular singular points, and rational exponents, and that the exponent of nilpotence of the local monodromy is $\leq i+1$.

This deduction (13.0) is made possible by the fantastic Theorem 11.10 of Turrittin, which allows us to really see what keeps a singular point of a differential equation from being a regular singular point.

0.6 The first sections (1 -4) review the formalism of connections. They represent joint work with Oda, and nearly all of the results are either contained in or implicit in [31], which unfortunately was not cast in sufficient generality for the present applications.

Sections 5 - 6 take up nilpotent connections in characteristic $p > 0$. The notion of a nilpotent connection is due to Berthelot (cf. [1]). We would like to call attention to the beautiful formula 5.3.0 of Deligne. The main result (5.10) is that, in characteristic $p$, the Gauss-Manin connection on $H^i_{DR}(X/S)$ is nilpotent of exponent $\leq i+1$ (or $\leq 2n-i+1$, if $i > n = \dim(X/S)$).

Section 7 is entirely due to Deligne. He had the idea of using the Cartier operation to lower the exponent of nilpotence of $H^i_{DR}(X/S)$ from $i+1$ to the number of pairs $(p,q)$ of integers with $h^{p,q}(X/S) = H^q(X/S, \Omega^p_{X/S}) \neq 0$, and $p + q = i$, thus relating the exponent of nilpotence to the Hodge structure.

Section 8 is a review of standard base-changing theorems, and Section 9 precises the notion of global nilpotence. Section 10 combines the results of Sections 7, 8 and 9 to show that $H^i_{DR}(X/S)$ is globally nilpotent of exponent i+1, or (by Deligne), the number of non-zero terms in the Hodge decomposition of $H^i(X_s, \underset{=}{C})$, s any $\underset{=}{C}$-valued point of S.

Section 11 reviews the classical theory of regular singular points, and proves Turrittin's theorem. I am grateful to E. Brieskorn for having made me aware of the paper of D. Lutz [24], from which I learned of the existence of Turrittin's Theorem.

Section 12 recalls the classical theory of the local monodromy around a regular singular point. It is a pleasure to be able to refer to the elegant paper [25] of Manin for the main result (12.0).

In Section 13 we establish that global nilpotence of a differential equation implies that all of its singular points are regular singular points, with rational exponents (13. 0). This theorem was originally conjectured by Grothendieck (and proved by him for a rank-one equation on $\underline{\underline{P}}^1$!). Needless to say, that conjecture was the starting point of the work presented here.

In Section 14, we "tie everything together", and give the final statement of the Local Monodromy Theorem (14.1), with Deligne's improvement on the exponent of nilpotence in terms of the Hodge structure. We also give Deligne's extension of the theorem 14.3 for non-proper smooth families, proved via the systematic use of Hironaka's resolution of singularities and Deligne's technique of systematically working with differentials having only logarithmic singularities along the divisor at $\infty$.

It is a pleasure to acknowledge the overwhelming influence of Grothendieck and Deligne on this work.

1.0 Let $T$ be a scheme, $f : S \to T$ a smooth $T$-scheme, and $\mathcal{E}$ a quasi-coherent sheaf of $\mathcal{O}_S$-modules. A $T$-connection on $\mathcal{E}$ is a homomorphism $\nabla$ of abelian sheaves

1.0.0
$$\nabla : \mathcal{E} \to \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}$$

such that

1.0.1 $$\nabla(ge) = g\nabla(e) + dg\otimes e$$

where g and e are sections of $\mathcal{O}_S$ and $\mathcal{E}$ respectively over an open subset of S, and dg denotes the image of g under the canonical exterior differentiation $d : \mathcal{O}_{\underline{\underline{S}}} \to \Omega^1_{S/T}$. The kernel of $\nabla$, noted $\mathcal{E}^{\nabla}$, is the sheaf of germs of <u>horizontal sections</u> of $(\mathcal{E},\nabla)$.

A T-connection $\nabla$ may be extended to a homomorphism of abelian sheaves

$$\nabla_i : \Omega^i_{S/T}\otimes_{\mathcal{O}_S}\mathcal{E} \to \Omega^{i+1}_{S/T}\otimes_{\mathcal{O}_S}\mathcal{E}$$

by

1.0.2 $$\nabla_i(\omega\otimes e) = d\omega\otimes e + (-1)^i\omega\wedge\nabla(e)$$

where $\omega$ and e are sections of $\Omega^i_{S/T}$ and $\mathcal{E}$ respectively over an open subset of S, and where $\omega\wedge\nabla(e)$ denotes the image of $\omega\otimes\nabla(e)$ under the canonical map

$$\Omega^i_{S/T}\otimes_{\mathcal{O}_S}(\Omega^1_{S/T}\otimes_{\mathcal{O}_S}\mathcal{E}) \to \Omega^{i+1}_{S/T}\otimes_{\mathcal{O}_S}\mathcal{E}$$

which sends $\omega\otimes\tau\otimes e$ to $(\omega\wedge\tau)\otimes e$.

The curvature $K = K(\mathcal{E},\nabla)$ of the T-connection $\nabla$ is the $\mathcal{O}_S$-linear map

$$K = \nabla_1\circ\nabla : \mathcal{E} \to \mathcal{E}\otimes_{\mathcal{O}_S}\Omega^2_{S/T}$$

One easily verifies that

$$\nabla_{i+1} \circ \nabla_i (\omega \otimes e) = \omega \wedge K(e)$$

where $\omega$ and $e$ are sections of $\Omega^i_{S/T}$ and $\mathcal{E}$ over an open subset of $S$.

The T-connection $\nabla$ is called <u>integrable</u> if $K = 0$. An integrable T-connection $\nabla$ on $\mathcal{E}$ thus gives rise to a complex (the de Rham complex of $(\mathcal{E}, \nabla)$)

1.0.3 $$0 \to \mathcal{E} \xrightarrow{\nabla} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{\nabla_1} \Omega^2_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{\nabla_2} \cdots$$

which we denote simply by $\Omega^{\cdot}_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}$ when the integrable T-connection $\nabla$ is understood.

Let <u>Der</u> $(S/T)$ denote the sheaf of germs of T-derivations of $\mathcal{O}_S$ into itself. We note that <u>Der</u> $(S/T)$ is naturally a sheaf of $f^{-1}(\mathcal{O}_T)$-Lie algebras, while, as $\mathcal{O}_S$-module, it is isomorphic to $\mathrm{Hom}_{\mathcal{O}_S}(\Omega^1_{S/T}, \mathcal{O}_S)$.

Let $\underline{\mathrm{End}}_T(\mathcal{E})$ denote the sheaf of germs of $f^{-1}(\mathcal{O}_T)$-linear endomorphisms of $\mathcal{E}$. We note that $\underline{\mathrm{End}}_T(\mathcal{E})$ is naturally a sheaf of $f^{-1}(\mathcal{O}_T)$ Lie algebras.

Now fix a T-connection $\nabla$ on $\mathcal{E}$; $\nabla$ gives rise to an $\mathcal{O}_S$-linear mapping

$$\nabla : \underline{\mathrm{Der}}(S/T) \to \underline{\mathrm{End}}_T(\mathcal{E})$$

send $D$ to $\nabla(D)$, where $\nabla(D)$ is the composite

$$\mathcal{E} \xrightarrow{\nabla} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{D \otimes 1} \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{E} \simeq \mathcal{E}$$

We have

1.0.4 $$\nabla(D)(fe) = D(f)e + f\nabla(D)(e)$$

whenever $D, f$ and $e$ are sections of $\underline{\mathrm{Der}}(S/T), \mathcal{O}_S$ and $\mathcal{E}$ respectively over an open subset of $S$. Conversely, because $S/T$ is smooth, any $\mathcal{O}_S$-linear mapping

$$\underline{\mathrm{Der}}(S/T) \to \underline{\mathrm{End}}_T(\mathcal{E})$$

satisfying 1.0.4 arises from a unique T-connection $\nabla$.

The T-connection $\nabla$ is integrable precisely when the mapping $\underline{\mathrm{Der}}(S/T) \to \mathrm{End}_T(\mathcal{E})$ is also a Lie-algebra homomorphism. This is seen by using the well-known fact that for $D_1$ and $D_2$ sections of $\underline{\mathrm{Der}}(S/T)$ over an open subset of $S$, we have

1.0.5 $$[\nabla(D_1), \nabla(D_2)] - \nabla([D_1, D_2]) = (D_1 \wedge D_2)(K)$$

where the right-hand side is the composite mapping

$$\mathcal{E} \xrightarrow{K} \Omega^2_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{D_1 \wedge D_2} \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{E} \simeq \mathcal{E}$$

1.1 Let $(\mathcal{E}, \nabla)$ and $(\mathcal{F}, \nabla')$ be quasi-coherent $\mathcal{O}_S$-modules with

T-connections. An $\mathcal{O}_S$-linear mapping

$$\phi : \mathcal{E} \to \mathcal{F}$$

is called <u>horizontal if</u>

1.1.0 $$\phi(\nabla(D)(e)) = \nabla'(D)(\phi(e))$$

whenever $D$ and $e$ are sections of $\underline{\mathrm{Der}}(S/T)$ and $\mathcal{E}$ respectively over an open subset of $S$.

We denote by $MC(S/T)$ the abelian category whose objects are pairs $(\mathcal{E}, \nabla)$ as above, and whose morphisms are the horizontal ones (MC = modules with connection). The category $MC(S/T)$ has an internal <u>Hom</u> and a tensor product, constructed as follows;

$$\underline{\mathrm{Hom}}_{\mathcal{O}_S}((\mathcal{E},\nabla), (\mathcal{F},\nabla')) = (\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{E},\mathcal{F}), \nabla'')$$

$\nabla''$ defined by the formula

1.1.1 $$(\nabla''(D)(\phi))(e) = \nabla'(D)(\phi(e)) - \phi(\nabla(D)(e))$$

where $D$, $\phi$, and $e$ are sections of $\underline{\mathrm{Der}}(S/T)$, $\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{E},\mathcal{F})$ and $\mathcal{E}$ respectively over an open subset of $S$.

$$(\mathcal{E}, \nabla) \otimes (\mathcal{F}, \nabla') = (\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{F}, \nabla'''),$$

$\nabla'''$ defined by the formula

1.1.2 $$\nabla''' (D)(e \otimes f) = \nabla(D)(e) \otimes f + e \otimes \nabla' (D)(f)$$

where $D$, $e$, and $f$ are sections of $\underline{\mathrm{Der}}(S/T)$, $\mathcal{E}$, and $\mathcal{F}$ respectively over an open subset of $S$.

We denote by $\mathrm{MIC}(S/T)$ the full (abelian) subcategory of $\mathrm{MC}(S/T)$ consisting of sheaves of quasi-coherent $\mathcal{O}_S$-modules with <u>integrable</u> connections. This subcategory is <u>stable</u> under the internal <u>Hom</u> and tensor product of $\mathrm{MC}(S/T)$.

We remark that the categories $\mathrm{MC}(S/T)$ and $\mathrm{MIC}(S/T)$ have an evident functionality in the smooth morphism $f : S \to T$. Explicitly, if $f' : S' \to T'$ is a smooth morphism, and

1.1.3 $$\begin{array}{ccc} S' & \xrightarrow{g} & S \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{h} & T \end{array}$$

is a commutative diagram, there is an "inverse image" functor

1.1.4 $$(g, h)^* \cdot \mathrm{MC}(S/T) \to \mathrm{MC}(S'/T')$$

(which maps $\mathrm{MIC}(S/T)$ to $\mathrm{MIC}(S'/T')$), as follows. Let $(\mathcal{E}, \nabla)$ be an object of

MC(S/T). Taking the usual inverse image by $(g, h)$ of the mapping

1.1.5 $$\nabla : \mathcal{E} \to \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}$$

gives a mapping

1.1.6 $$g^*(\mathcal{E}) \to (g, h)^* \Omega^1_{S/T} \otimes_{\mathcal{O}_{S'}} g^*(\mathcal{E})$$

The canonical mapping

1.1.7 $$(g, h)^* \Omega^1_{S/T} \to \Omega^1_{S'/T'} ,$$

tensorized by $g^*(\mathcal{E})$, gives a map

1.1.8 $$(g, h)^* \Omega^1_{S/T} \otimes_{\mathcal{O}_{S'}} g^*(\mathcal{E}) \to \Omega^1_{S'/T'} \otimes_{\mathcal{O}_{S'}} g^*(\mathcal{E}).$$

The composition of 1.1.6 and 1.1.8 is thus a mapping $(g, h)^*(\nabla)$

1.1.9 $$(g, h)^*(\nabla) : g^*(\mathcal{E}) \to \Omega^1_{S'/T'} \otimes_{\mathcal{O}_{S'}} g^*(\mathcal{E})$$

which is easily seen to be a T'-connection on $g^*(\mathcal{E})$. The inverse image $(g, h)^*(\mathcal{E}, \nabla)$ is, by definition, $(g^*(\mathcal{E}), (g, h)^*(\nabla))$.

One checks immediately that the curvature element

$$K(g^*(\mathcal{E}),\ (g,h)^*(\nabla)) \in \mathrm{Hom}_{S'}(g^*(\mathcal{E}),\ \Omega^2_{S'/T'} \otimes_{\mathcal{O}_{S'}} g^*(\mathcal{E}))$$

is the <u>inverse image</u> of $K(\mathcal{E},\nabla) \in \mathrm{Hom}_S(\mathcal{E},\ \Omega^2_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E})$.

1.2 We remark that the category MIC$(S/T)$ has enough injectives, being (tautologically) equivalent to the category of quasicoherent modules over an appropriate sheaf of enveloping algebras (the sheaf P-D Diff. of Berthelot [ 1 ], or, equivalently, the enveloping algebra of Kostant, Rosenberg, and Hochschild [19 ]).

2.0 We define the de Rham cohomology sheaves on $T$ of an object $(\mathcal{E},\nabla)$ in MIC$(S/T)$ by

2.0.1 $$H^q_{DR}(S/T,\ \mathcal{E},\nabla) = \underline{\underline{R}}^q f_*(\Omega^{\cdot}_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E})$$

where $\Omega^{\cdot}_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}$ is the de Rham complex of $(\mathcal{E},\nabla)$, cf. 1.3, and $\underline{\underline{R}}^q f_*$ are the hyper-derived functors of $\underline{\underline{R}}^0 f_*$. In particular, $H^0_{DR}(S/T,\ (\mathcal{E},\nabla)) = f_*(\mathcal{E}^{\nabla})$. As is proved in [17] and also in [19] , the functors $H^q_{DR}(S/T,\ ?)$ are the right derived functors of the left exact functor

2.0.2 $$H^0_{DR}(S/T,\ ?) : \mathrm{MIC}(S/T) \to \mathrm{MIC}(T/T) = (\text{quasicoherent sheaves on}$$

3.0 Suppose now $\pi : X \to S$ is a smooth morphism. The natural forgetful functor

$$\text{3.0.1} \qquad \begin{aligned} \mathrm{MIC}(X/T) &\to \mathrm{MIC}(X/S) \\ (\mathcal{E},\nabla) &\rightsquigarrow (\mathcal{E},\nabla \mid \underline{\mathrm{Der}}(X/S)) \end{aligned}$$

allows us to define the de Rham complex of $(\mathcal{E},\nabla \mid \mathrm{Der}(X/S))$, which we will denote simply by $\Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$. Further abusing notation, we write

3.0.2 $$H^q_{DR}(X/S, (\mathcal{E},\nabla)) = \underline{\underline{R}}^q \pi_* (\Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$$

Exactly as in [31], we may construct a canonical T-connection $\nabla$ on the quasi-coherent $\mathcal{O}_S$-module $H^q_{DR}(X/S, (\mathcal{E},\nabla))$, the "Gauss-Manin connection", so that the functors $H^q_{DR}(X/S, ?)$ may be interpreted as an exact connected sequence of cohomological functors

$$\mathrm{MIC}(X/T) \to \mathrm{MIC}(S/T)$$

3.1 Remark. There is no difficulty in checking that these functors are none other than the right derived functors of

$$H^0_{DR}(X/S, ?) : \mathrm{MIC}(X/T) \to \mathrm{MIC}(S/T)$$

where the T-connection on $H^0_{DR}(X/S, (\mathcal{E},\nabla \mid \underline{\mathrm{Der}}(X/S))) = \pi_*(\mathcal{E}^{\nabla \mid \underline{\mathrm{Der}}(X/S)})$ is defined by using the exactness of the sequence of sheaves on $X$

3.1.0 $$0 \to \underline{\mathrm{Der}}(X/S) \to \underline{\mathrm{Der}}(X/T) \to \pi^{*}\underline{\mathrm{Der}}(S/T) \to 0$$

3.2 For computational purposes, however, we recall the construction given in [31], of the entire de Rham complex $\Omega^{\cdot}_{S/T} \otimes_{\mathcal{O}_S} H^q_{DR}(X/S, (\mathcal{E}, \nabla))$. Consider the canonical filtration of $\Omega^{\cdot}_{X/T}$ by locally free subsheaves

3.2.0 $$\Omega^{\cdot}_{X/T} = F^0(\Omega^{\cdot}_{X/T}) \supset F^1(\Omega^{\cdot}_{X/T}) \supset \ldots$$

given by

3.2.1 $$F^i(\Omega^{\cdot}_{X/T}) = \text{image of } \pi^{*}(\Omega^i_{S/T}) \otimes_{\mathcal{O}_X} \Omega^{\cdot - i}_{X/T} \to \Omega^{\cdot}_{X/T}$$

By smoothness, the associated graded objects $gr^i = F^i/F^{i+1}$ are given the (locally free) sheaves

3.2.2 $$gr^i(\Omega^{\cdot}_{X/T}) = \pi^{*}(\Omega^i_{S/T}) \otimes_{\mathcal{O}_X} \Omega^{\cdot - i}_{X/S}\, .$$

We filter the de Rham complex $\Omega^{\cdot}_{X/T} \otimes_{\mathcal{O}_X} \mathcal{E}$ by the subcomplexes

3.2.3 $$F^i(\Omega^{\cdot}_{X/T} \otimes_{\mathcal{O}_X} \mathcal{E}) = F^i(\Omega^{\cdot}_{X/T}) \otimes_{\mathcal{O}_X} \mathcal{E}\ ;$$

the associated graded objects are the $f^{-1}(\mathcal{O}_S)$ - linear complexes

3.2.4 $$gr^i(\Omega^{\cdot}_{X/T} \otimes_{\mathcal{O}_X} \mathcal{E}) \simeq \pi^{*}(\Omega^i_{S/T}) \otimes_{\mathcal{O}_X} (\Omega^{\cdot - i}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$$

(the differential in this complex is $1\otimes$(the differential of $\Omega^{\cdot - i}_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E}$)) .

Consider the functor $\underline{\underline{R}}^0\pi_*$ from the category of complexes of abelian sheaves on X to the category of abelian sheaves on S. Applying the spectral sequence of a finitely filtered object, we obtain a spectral sequence abutting to (the associated graded object with respect to the filtration of) $\underline{\underline{R}}^{\cdot}\pi_*(\Omega^{\cdot}_{X/T}\otimes_{\mathcal{O}_X}\mathcal{E})$, while

3.2.5
$$\begin{aligned} E_1^{p,q} = \underline{\underline{R}}^{p+q}\pi_*(\mathrm{gr}^p) &= \underline{\underline{R}}^{p+q}\pi_*(\pi^*(\Omega^p_{S/T})\otimes_{\mathcal{O}_X}(\Omega^{\cdot - p}_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E})) \\ &= \Omega^p_{S/T}\otimes_{\mathcal{O}_S}\underline{\underline{R}}^q\pi_*(\Omega^{\cdot}_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E}) \\ &= \Omega^p_{S/T}\otimes_{\mathcal{O}_S}H^q_{DR}(X/S,(\mathcal{E},\nabla)) \end{aligned}$$

The de Rham complex of $H^q_{DR}(X/S,(\mathcal{E},\nabla))$ is then the complex $(E_1^{\cdot,q}, d_1^{\cdot,q})$, the q'th <u>row</u> of $E_1$ terms of the above spectral sequence.

3.3 <u>Remark.</u> The zealous reader who wishes to construct the "Leray spectral sequence" of de Rham cohomology for $X\xrightarrow{\pi}S\xrightarrow{f}T$

3.3.0
$$E_2^{p,q} = H^p_{DR}(S/T,(H^q_{DR}(X/S,(\mathcal{E},\nabla)),\nabla)) \Longrightarrow H^{p+q}_{DR}(X/T,(\mathcal{E},\nabla))$$

without availing himself of the previous remark (whose truth reduces the question to one of the usual composite functor spectral sequence) may employ the following trick, due to Deligne.

Let $\mathcal{B}$ and $\mathcal{C}$ be abelian categories, $\mathcal{B}$ having enough injectives, and let

$N : \mathcal{B} \to \mathcal{C}$ be a left exact additive functor. Let $K^{\cdot}$ be a complex $(K^i = 0$ for $i<0)$ over $\mathcal{B}$. By a C-E resolution with respect to $N$ of $K^{\cdot}$ we mean an augmented first quadrant bicomplex

$$K^{\cdot} \to M^{\cdot\cdot}$$

such that, for each $i \geq 0$, the complex $M^{\cdot,i}$ is a resolution of $K^i$ by N-acyclic objects, and such that for each $p \geq 0$, the complex

$$H^p(M^{0,\cdot}) \to H^p(M^{1,\cdot}) \to H^p(M^{2,\cdot}) \to \ldots$$

is a resolution of $H^p(K^{\cdot})$ by N-acyclic objects.

If $K^{\cdot}$ is a finitely filtered complex over $\mathcal{B}$

$$K^{\cdot} = F^0(K^{\cdot}) \supset F^1(K^{\cdot}) \supset \ldots ,$$

then by a filtered C-E resolution of $K^{\cdot}$ with respect to $N$ we mean an augmented first quadrant finitely filtered bicomplex

$$M^{\cdot\cdot} = F^0(M^{\cdot\cdot}) \supset F^1(M^{\cdot\cdot}) \supset \ldots$$

such that, for $i \geq 0$,

$$F^i(K^{\cdot}) \to F^i(M^{\cdot\cdot})$$

and

$$\mathrm{gr}^i(K^\cdot) \to \mathrm{gr}^i(M^{\cdot\cdot})$$

are C-E resolutions with respect to $N$ of $F^i(K^\cdot)$ and $\mathrm{gr}^i(K^\cdot)$ respectively.

Proposition 3.3.1. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be three abelian categories, $\mathcal{A}$ and $\mathcal{B}$ with enough injectives, and let

$$L : \mathcal{A} \to \mathcal{B} \quad , \quad N : \mathcal{B} \to \mathcal{C}$$

be left exact additive functors, such that $L$(an injective) is T-acyclic. Suppose further that every finitely filtered complex over $\mathcal{B}$ admits a filtered C-E resolution with respect to $N$.

Let $A \supset F^1(A) \supset F^2(A) \supset \ldots$ be a finitely filtered object of $\mathcal{A}$. The spectral sequence of a finitely filtered object for the functor $L$ gives a spectral sequence

$$E_1^{p,q}(A) = R^{p+q}(L)(\mathrm{gr}^p A) \Longrightarrow R^{p+q}(L)(A).$$

For each $q$, we denote by $E_1^{\cdot,q}(A)$ the complex

$$(E_1^{\cdot,q}(A),\ d_1^{\cdot,q}).$$

Then there is a spectral sequence

3.3.2 $$E_2^{p,q} = \underline{\underline{R}}^p(N)(E_1^{\cdot,q}(A)) \Longrightarrow R^{p+q}(NL)(A).$$

3.3.3 Remark. If $\mathcal{B}$ is the category of abelian sheaves on a topological space $S$, $\mathcal{C}$ the category of abelian sheaves on a topological space $T$, and $N$ the functor $f_*$, where $f : S \to T$ is a continuous map, then taking "the canonical flasque resolution" componentwise functorially provides every finitely filtered complex over $\mathcal{B}$ with a finitely filtered C-E resolution with respect to $N$.

To apply the proposition, we take

$$\begin{cases} \mathcal{A} = \text{complexes of abelian sheaves on } X \\ \mathcal{B} = \text{abelian sheaves on } S \\ \mathcal{C} = \text{abelian sheaves on } T \end{cases}$$

$$\begin{cases} L = \underline{\underline{R}}^0 \pi_* \\ N = f_* \\ A = \Omega^{\cdot}_{X/T} \otimes_{\mathcal{O}_X} \mathcal{E} \text{ with the filtration 3.2.3.} \end{cases}$$

Outline of proof: Take a finitely filtered injective resolution $I^{\cdot}$ of $A$, so that, for each $i \geq 0$, $F^i I^{\cdot}$ and $gr^i(I^{\cdot})$ are injective resolutions of $F^i(A)$ and $gr^i(A)$ respectively. Put $K^{\cdot} = L(I^{\cdot})$, $F^i(K^{\cdot}) = L(F^i(I^{\cdot}))$. Let $M^{\cdot\cdot}$ be a filtered C-E resolution with respect to $N$ of $K^{\cdot\cdot}$, and define a new filtration $\widetilde{F}$ on $M^{\cdot\cdot}$ by defining

$$\widetilde{F}^i(M^{p,q}) = F^{i-p}(M^{p,q})$$

Now let $P^{\cdot\cdot} = N(M^{\cdot\cdot})$, filtered by $\widetilde{F}^i(P^{p,q}) = N(\widetilde{F}^i M^{p,q}) = N(F^{i-p}M^{p,q})$. The desired spectral sequence is that of the "totalized" complex of $P^{\cdot\cdot}$, with the filtration $\widetilde{F}$.

3.4 We now recall from [31] the explicit calculation of the Gauss-Manin connection. The question being local on $S$, we will suppose that $S$ is affine.

Choose a finite covering of $X$ by affine open sets $\{\mathcal{U}_\alpha\}$ such that each $\mathcal{U}_\alpha$ is étale over $\underline{\underline{A}}^n_S$, so that, on $\mathcal{U}_\alpha$, the sheaf $\Omega^1_{X/S}$ is a free $\mathcal{O}_X$-module, with base $\{dx_1^\alpha, \ldots, dx_n^\alpha\}$.

For any object $(\mathcal{E}, \nabla)$ of MIC(X/T), the S-modules $\underline{\underline{R}}^i\pi_*(\Omega^\cdot_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$ may be calculated as the total homology of the bicomplex of $\mathcal{O}_S$-modules

$$C^{p,q}(\mathcal{E}) \overset{\text{def'n}}{=} C^p(\{\mathcal{U}_\alpha\}, \Omega^q_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$$

of alternating Cech cochains on the nerve of the covering $\{\mathcal{U}_\alpha\}$. We will now describe a T-connection (in general not integrable) on the totalized complex associated to the bicomplex $C^p\{\mathcal{U}_\alpha\}, \Omega^q_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$, which upon passage to homology yields the Gauss-Manin connection.

Let $D$ be any T-derivation of the coordinate ring of $S$. For each index $\alpha$, let $D_\alpha \in \mathrm{Der}_T(\mathcal{O}_{\mathcal{U}_\alpha}, \mathcal{O}_{\mathcal{U}_\alpha})$ be the unique extension of $D$ which kills $dx_1^\alpha, \ldots, dx_n^\alpha$. $D_\alpha$ induces a T-linear endomorphism of sheaves (a "Lie derivative")

3.4.0 $$D_\alpha : \Omega^q_{\mathcal{U}_\alpha/S} \to \Omega^q_{\mathcal{U}_\alpha/S}$$

by

3.4.1 $$D_\alpha(h\,dx^\alpha_{i_1}\wedge\ldots dx^\alpha_{i_q}) = D_\alpha(h)\,dx^\alpha_{i_1}\wedge\ldots\wedge dx^\alpha_{i_q}$$

where $h$ is a section of $\mathcal{O}_X$ over an open subset of $\mathcal{U}_\alpha$. Similarly, $D_\alpha$ induces a T-linear endomorphism

3.4.2 $$D_\alpha : \Omega^q_{\mathcal{U}_\alpha/S}\otimes_{\mathcal{O}_{\mathcal{U}_\alpha}}\mathcal{E} \longrightarrow \Omega^q_{\mathcal{U}_\alpha/S}\otimes_{\mathcal{O}_{\mathcal{U}_\alpha}}\mathcal{E}$$

by

3.4.3 $$D_\alpha(\omega\otimes e) = D_\alpha(\omega)\otimes e + \omega\otimes\nabla(D_\alpha)(e)$$

where $\omega$ and $e$ are sections of $\Omega^q_{X/S}$ and $\mathcal{E}$ respectively over an open subset of $\mathcal{U}_\alpha$.

Choose a total ordering on the indexing set of the covering $\{\mathcal{U}_\alpha\}$. We define a T-linear endomorphism $\widetilde{D}$ of bidegree $(0,0)$ of the bigraded $\mathcal{O}_S$-module $C^{p,q}(\mathcal{E}) = D^p(\{\mathcal{U}_\alpha\}, \Omega^q_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E})$ by setting

3.4.4. $$\widetilde{D}\mid \Gamma(\mathcal{U}_{\alpha_0}\cap\ldots\cap\mathcal{U}_{\alpha_p}, \Omega^q_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E}) = D_{\alpha_0}$$

if $\alpha_0 < \alpha_1 < \ldots < \alpha_p$.

For each pair $\alpha$, $\beta$ of indices, we define an $\mathcal{O}_X$-linear mapping of sheaves (the interior product with $D_\alpha - D_\beta$)

3.4.5 $$\lambda(D)_{\alpha,\beta} : \Omega^q_{X/S} \mid \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \Omega^{q-1}_{X/S} \mid \mathcal{U}_\alpha \cap \mathcal{U}_\beta$$

by

3.4.6 $$\lambda(D)_{\alpha,\beta}(h\,dx_1 \wedge \ldots \wedge dx_q) = h\Sigma(-1)^i (D_\alpha - D_\beta)(x_i)dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_q$$

where $h, x_1, \ldots, x_q$ are sections of $\mathcal{O}_X$ over an open subset of $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$. (We put $\lambda(D)_{\alpha,\beta} = 0$ on $\mathcal{O}_X$). Similarly, we define an $\mathcal{O}_X$-linear mapping

3.4.7 $$\lambda(D)_{\alpha,\beta} : \Omega^q_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \mid \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \Omega^{q-1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \mid \mathcal{U}_\alpha \cap \mathcal{U}_\beta$$

by

3.4.8 $$\lambda(D)_{\alpha,\beta}(\omega \otimes e) = \lambda(D)_{\alpha,\beta}(\omega) \otimes e$$

We define an $\mathcal{O}_X$-linear endomorphism $\lambda(D)$ of bidegree $(1,-1)$ of the bigraded $\mathcal{O}_S$-module $C^{\cdot\cdot}(\mathcal{E})$

3.4.9 $$\lambda(D) : C^p(\{\mathcal{U}_\alpha\}, \Omega^q_{X/S} \otimes \mathcal{E}) \to C^{p+1}(\{\mathcal{U}_\alpha\}, \Omega^{q-1}_{X/S} \otimes \mathcal{E})$$

by

3.4.10 $$(\lambda(D)(\sigma))_{\alpha_0,\ldots,\alpha_{p+1}} = (-1)^q \lambda(D)_{\alpha_0,\alpha_1}(\sigma_{\alpha_1,\ldots,\alpha_{p+1}}) \quad \text{if } \alpha_0 < \ldots < \alpha_{p+1}$$

where $\sigma$ is the alternating p-cochain whose value on $\mathcal{U}_{\alpha_0} \cap \ldots \cap \mathcal{U}_{\alpha_p}$, $\alpha_0 < \ldots, \alpha_p$, is $\sigma_{\alpha_0,\ldots,\alpha_p}$.

Notice that $\{D_\alpha - D_\beta\}$ is a 1-cocycle on the nerve of the covering $\{\mathcal{U}_\alpha\}$ with values in Der (X/S), whose cohomology class in $H^1(X, \underline{\mathrm{Der}}(X/S))$ is the value at $D \in \mathrm{Der}_T(\mathcal{O}_S, \mathcal{O}_S)$ of the Kodaira-Spencer map

3.4.11 $$\rho_{X/S} : \mathrm{Der}_T(\mathcal{O}_S, \mathcal{O}_S) \to H^1(X, \underline{\mathrm{Der}}(X/S)).$$

The cochain map $\lambda(D)$ is just the cup-product with the representative cocycle $\{D_\alpha - D_\beta\}$.

"The" Gauss-Manin connection on the bicomplex $C^{\cdot\cdot}(\mathcal{E})$ is given by

3.4.12 $$\nabla : D \in \mathrm{Der}_T(\mathcal{O}_S, \mathcal{O}_S) \rightsquigarrow \nabla(D) = \tilde{D} + \lambda(D).$$

This explicit formula has a number of immediate consequences, which we will now record.

Theorem 3.5.

3.5.1 $\nabla(D)$ is compatible with the "Zariski" filtration $F_{zar}$ of $C^{\cdot\cdot}(\mathcal{E})$, $F^i_{zar} = \sum_{p \geq i} C^{p,q}(\mathcal{E})$, hence acts on the associated spectral sequence

3.5.1.0 $$E_1^{p,q} = C^p(\{\mathcal{U}_\alpha\}, \mathcal{H}^q_{DR}(X/S, (\mathcal{E},\nabla)) \Longrightarrow H^{p+q}_{DR}(X/S, (\mathcal{E},\nabla))$$

where $\mathcal{H}^q_{DR}(X/S, (\mathcal{E},\nabla))$ denotes the presheaf on X with values in MIC(S/T)

$$\mathcal{U} \rightsquigarrow H^q_{DR}(\mathcal{U}/S, (\mathcal{E},\nabla) \mid \mathcal{U})$$

3.5.2 $\nabla(D)$ is <u>not</u> compatible with the "Hodge" filtration $F_{\text{Hodge}}$ of $C^{\cdot\cdot}(\mathcal{E})$, $F^i_{\text{Hodge}} = \sum_{q \geq i} C^{p,q}(\mathcal{E})$, and does <u>not</u> act on the associated spectral sequence.

3.5.2.0 $$E_1^{p,q} = H^q(X, \Omega^p_{X/S} \otimes \mathcal{E}) \Longrightarrow H^{p+q}_{DR}(X/S, (\mathcal{E},\nabla)).$$

However, $\nabla(D)$ <u>does</u> respect the Hodge filtration on $H_{DR}(X/S, (\mathcal{E},\nabla))$ to a shift of <u>one</u>, i.e.,

$$\nabla(D) F^i_{\text{Hodge}} \subset F^{i-1}_{\text{Hodge}}$$

(<u>"Griffith's transversality theorem"</u>) and so induces, by passage to quotients an S-linear mapping

3.5.2.1 $$\nabla(D) : \mathrm{gr}^p_{\text{Hodge}} H^{p+q}_{DR}(X/S, (\mathcal{E},\nabla)) \to \mathrm{gr}^{p-1} H^{p+q}_{DR}(X/S, (\mathcal{E},\nabla))$$

In particular, <u>if</u> the spectral sequence 3.5.2.0 degenerates ($E_1 = E_\infty$), (this is the case for example, if X is <u>proper</u> and smooth over S, S is of characteristic zero, and $\mathcal{E} = \mathcal{O}_X$ with the standard connection (cf. 8.7)) this induced mapping 3.5.2.1

$$\nabla(D) : H^q(X, \Omega^p_{X/S} \otimes \mathcal{E}) \to H^{q+1}(X, \Omega^{p-1}_{X/S} \otimes \mathcal{E})$$

is none other than the cup-product with the Kodaira-Spencer class $\rho_{X/S}(D) \in H^1(X, \underline{\mathrm{Der}}(X/S))$.

3.5.3 If $X$ is itself étale over $\underline{\underline{A}}^n_S$, with $\Omega^1_{X/S}$ free with base $\{dx_1, \ldots, dx_n\}$, then

$$H^q_{DR}(X/S, (\mathcal{E}, \nabla)) = H^q(\Gamma(X, \Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})),$$

and, for $D \in \mathrm{Der}_T(\mathcal{O}_S, \mathcal{O}_S)$, the action of $\nabla(D)$ on $H^q_{DR}(X/S, (\mathcal{E}, \nabla))$ is that induced from the T-endomorphism $\tilde{D}$ of $\Gamma(X, \Omega^{\cdot}_{X/S} \otimes \mathcal{E})$:

$$\tilde{D}(\omega \otimes e) = D_0(\omega) \otimes e + \nabla(D_0)(e)$$

where $D_0 \in \mathrm{Der}_T(\mathcal{O}_X, \mathcal{O}_X)$ is the unique extension of $D$ which kills $dx_1, \ldots, dx_n$, and where $\omega$ and $e$ are sections of $\Omega^{\cdot}_{X/S}$ and $\mathcal{E}$ respectively over $X$.

## 4. Connections having logarithmic singularities

4.0 Let $\pi : X \to S$ be a smooth morphism, and let $i : Y \hookrightarrow X$ be the inclusion of a divisor with normal crossings relative to $S$, $j : X - Y \hookrightarrow X$ the inclusion of its complement. "Normal crossings" means that $X$ may be covered by affine open sets $\mathcal{U}$ such that

4.0.1 $\mathcal{U}$ is étale over $\underline{\underline{A}}^n_S$, via "coordinates" $x_1, \ldots, x_n$

4.0.2 $Y \mid \mathcal{U}$ is defined by an equation $x_1 \ldots x_\nu = 0$ (i.e., $Y$ is the inverse image of the union of the first $\nu \leq n$ of the coordinate hyperplanes in $\underline{\underline{A}}^n_S$.

4.1 We define a locally free $\mathcal{O}_X$-module, $\Omega^1_{X/S}(\log Y)$, by giving, as base over an open set $\mathcal{U}$ as above, the elements $\frac{dx_1}{x_1}, \ldots, \frac{dx_\nu}{x_\nu}$, $dx_{\nu+1}, \ldots, dx_n$. We define $\Omega^i_{X/S}(\log Y) = \Lambda^i_{\mathcal{O}_X}(\Omega^1_{X/S}(\log Y))$. Viewing $\Omega^\cdot_{X/S}(\log Y)$ as a subsheaf of $j_*(\Omega^\cdot_{X-Y/S})$, we see that the usual exterior differentiation in $j_*(\Omega^\cdot_{S-Y/S})$ preserves $\Omega^\cdot_{X/S}(\log Y)$, which is thus (given the structure of) a complex ("the de Rham complex of $X/S$ with logarithmic singularities along $Y$").

Now let $M$ be a quasicoherent $\mathcal{O}_X$-module. An S-connection on $M$, with logarithmic singularities along $Y$, is a homomorphism of abelian sheaves

4.1.0 $$\nabla : M \to \Omega^1_{X/S}(\log Y) \otimes_{\mathcal{O}_X} M$$

such that

4.1.1 $$\nabla(gm) = g\nabla(m) + dg \otimes m$$

where $g$ and $m$ are sections of $\mathcal{O}_X$ and $M$ respectively over an open subset of $X$. We denote by $M^\nabla$ the kernel of $\nabla$; $M^\nabla$ is the sheaf of germs of horizontal sections.

4.2 Just as for "ordinary" connections, we say that $\nabla$ is <u>integrable</u>

if the canonical extensions 1.0.2 of $\nabla$ to maps

4.2.0 $$\nabla_i : \Omega^i_{X/S}(\log Y)\otimes_{\mathcal{O}_X} M \to \Omega^{i+1}_{X/S}(\log Y)\otimes_{\mathcal{O}_X} M$$

make $\Omega^{\cdot}_{X/S}(\log Y)\otimes_{\mathcal{O}_X} M$ into a <u>complex</u> ("the de Rham complex of $(M,\nabla)$ with logarithmic singularities along $Y$").

Let $\underline{\mathrm{Der}}_Y(X/S)$ be the sheaf on $X$ defined by

4.2.1 $$\underline{\mathrm{Der}}_Y(X/S) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}(\log Y), \mathcal{O}_X).$$

Over an open $\mathcal{U}$ as above, $\underline{\mathrm{Der}}_Y(X/S)$ is $\mathcal{O}_X$-free on $x_1\frac{\partial}{\partial x_1}, \ldots, x_\nu\frac{\partial}{\partial x_\nu}, \frac{\partial}{\partial x_{\nu+1}}, \ldots, \frac{\partial}{\partial x_n}$.

$\underline{\mathrm{Der}}_Y(X/S)$ is a sheaf of $f^{-1}(\mathcal{O}_S)$-Lie algebras, and an <u>integrable S-connection</u> in $M$ with logarithmic singularities along $Y$ is nothing other than an $\mathcal{O}_X$-linear mapping

4.2.2 $$\nabla : \underline{\mathrm{Der}}_Y(X/S) \to \mathrm{End}_{\mathcal{O}_S}(M)$$

which is compatible with brackets, and such that

4.2.3 $$\nabla(D)(gm) = D(g)m + g\nabla(D)(m)$$

where $D$, $g$ and $m$ are sections of $\underline{\mathrm{Der}}_Y(X/S)$, $\mathcal{O}_X$ and $M$ respectively over an open subset of X.

4.3 We denote by $\mathrm{MIC}(X/S(\log Y))$ the abelian category of pairs $(M,\nabla)$, $M$ a quasicoherent $\mathcal{O}_X$-module and $\nabla$ an integrable S-connection on $M$ with logarithmic singularities along $Y$. (The morphisms are the horizontal ones). Just as before (cf. 1.2), $\mathrm{MIC}(X/S(\log Y))$ has enough injectives, and has an internal $\underline{\mathrm{Hom}}$ and a tensor product.

4.4 The de Rham cohomology sheaves on $S$ of an object $(M,\nabla)$ in $\mathrm{MIC}(X/S(\log Y))$ are defined by

4.4.0
$$H^q_{DR}(X/S(\log Y),(M,\nabla)) = \underline{\underline{R}}^q\pi_*(\Omega^{\cdot}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} M).$$

Thus

4.4.1
$$H^0_{DR}(X/S(\log Y),\ (M,\nabla)) = \pi_*(M^{\nabla})$$

and the arguments of [17] or [19] show that the $H^q_{DR}$ are the right derived functors of $H^0_{DR}$.

4.5 Suppose now that $f : S \to T$ is a smooth morphism. Then $f\pi : X \to T$ is a smooth morphism, and $i : Y \hookrightarrow X$ is a divisor with normal crossings relative to $T$. As in 3.0 there is a natural forgetful functor

$$\text{4.5.0} \qquad \begin{aligned} \mathrm{MIC}(X/T(\log Y)) &\to \mathrm{MIC}(X/S(\log Y)) \\ (\mathcal{E},\nabla) &\mapsto (\mathcal{E},\nabla \mid \underline{\mathrm{Der}}_Y(X/S)) \end{aligned}$$

so that, just as in 3.0, we may define an exact connected sequence of cohomological functors

$$\text{4.5.1} \qquad H^q_{DR}(X/S(\log Y), \ ?\ ) : \mathrm{MIC}(X/T(\log Y)) \to \mathrm{MIC}(S/T)$$

by putting, for $(\mathcal{E},\nabla)$ an object of $\mathrm{MIC}(X/T(\log Y))$

$$\text{4.5.2} \qquad H^q_{DR}(X/S(\log Y), (\mathcal{E},\nabla)) = \underline{\underline{R}}^q \pi_*(\Omega^{\cdot}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{E}).$$

4.6 The Gauss-Mamin connection is constructed as before, using the canonical filtration of $\Omega^{\cdot}_{X/T}(\log Y)$ by the subcomplexes

$$\text{4.6.0} \qquad F^i(\Omega^{\cdot}_{X/T}(\log Y)) = \text{image } \pi^*(\Omega^i_{S/T}) \otimes_{\mathcal{O}_X} \Omega^{\cdot - i}_{X/T}(\log Y)$$

whose associated graded complexes are

$$\text{4.6.1} \qquad \mathrm{gr}^i(\Omega^{\cdot}_{X/T}(\log Y)) = \pi^*(\Omega^i_{S/T}) \otimes_{\mathcal{O}_X} \Omega^{\cdot - i}_{X/S}(\log Y)$$

We then filter the de Rham complex of $(\mathcal{E},\nabla)$ with logarithmic singularities along $Y$ by the subcomplexes

4.6.2 $$F^i(\Omega^{\cdot}_{X/T}(\log Y)\otimes_{\mathcal{O}_X}\mathcal{E}) = F^i(\Omega^{\cdot}_{X/T}(\log Y))\otimes_{\mathcal{O}_X}\mathcal{E}$$

whose associated graded objects are given by

4.6.3 $$gr^i(\Omega^{\cdot}_{X/T}(\log Y)\otimes_{\mathcal{O}_X}\mathcal{E}) = \pi^*(\Omega^i_{S/T})\otimes_{\mathcal{O}_X}(\Omega^{\cdot-i}_{X/S}(\log Y)\otimes_{\mathcal{O}_X}\mathcal{E})$$

Then the de Rham complex $\Omega^{\cdot}_{S/T}\otimes_{\mathcal{O}_S} H^q_{DR}(X/S(\log Y), (\mathcal{E},\nabla))$ of the Gauss-Manin connection on $H^q_{DR}(X/S(\log Y), (\mathcal{E},\nabla))$ is the complex $(E_1^{\cdot,q}, d_1^{\cdot,q})$ of $E_1$ terms of the spectral sequence of the filtered (as above) object $\Omega^{\cdot}_{X/T}(\log Y)\otimes_{\mathcal{O}_X}\mathcal{E}$ and the functor $\underline{\underline{R}}^0\pi_*$.

When $S$ is affine, the Gauss-Mamin connection can be "lifted" to a connection on the Cech bicomplex

4.6.4 $$C^p(\{\mathcal{U}_\alpha\}, \Omega^q_{X/S}(\log Y)\otimes_{\mathcal{O}_X}\mathcal{E})$$

by exactly the same formulas as before, provided that:

4.6.5 we use a covering of $X$ by $\mathcal{U}$'s as in 4.0.1, and use coordinates $x_1,\ldots,x_n$ on $\mathcal{U}$ so that $Y$ is defined by $x_1\ldots x_\nu = 0$

4.6.6 we lift $D \in \mathrm{Der}(S/T)$ to the derivation of $\mathcal{O}_X$ which extends it and kills $dx_1,\ldots,dx_n$ (so that, in particular, the lifting is <u>tangent</u> to $Y$).

The Gauss-Manin connection acts on the spectral sequence over an affine S associated to a covering of X by affine open sets $\mathcal{U}_i$ verifying 4.0.1 and 4.0.2. (by using the Zariski filtration (3.5.1) of the Cech bicomplex)

4.6.7 $$E_1^{p,q} = C^p(\{\mathcal{U}_i\}, \mathcal{H}^q_{DR}(X/S(\log Y), (\mathcal{E},\nabla)) \Longrightarrow H^{p+q}_{DR}(X/S(\log Y), (\mathcal{E},\nabla))$$

where $\mathcal{H}^q_{DR}(X/S(\log Y), (\mathcal{E},\nabla))$ is the presheaf on X with values in MIC(S/T) given by

4.6.8 $$\mathcal{U} \rightsquigarrow H^q_{DR}(\mathcal{U}/S(\log Y), (\mathcal{E},\nabla)\,|\,\mathcal{U})$$

## 5. <u>Connections in Characteristic p(>0)</u>

5.0 In this section, we suppose the base scheme T to be of characteristic $p>0$, i.e., that $p\mathcal{O}_T = 0$. As before, let $f : S \to T$ be a smooth T-scheme. Recall the Leibniz rule

5.0.1 $$D^n(gh) = \Sigma_{i=0}^n \binom{n}{i} D^i(g) D^{n-i}(h)$$

where D, g, and h are sections of <u>Der</u> (S/T), $\mathcal{O}_S$ and $\mathcal{O}_S$ respectively over an open subset of S. Putting $n = p$, we find (being in characteristic p) that

5.0.2 $$D^p(gh) = D^p(g) \cdot h + gD^p(h)$$

i.e., that the p-th iterate of a derivation is a derivation, so that <u>Der</u> (S/T) is a sheaf of restricted p-Lie algebras.

Let $(\mathcal{E},\nabla)$ be an object of MIC(S/T). Since $\underline{\mathrm{End}}_T(\mathcal{E})$ is also a sheaf of restricted p-Lie algebras (taking the p'th iterate of a T-endormorphism), it is natural to ask whether or not the homomorphism

$$\nabla : \underline{\mathrm{Der}}(S/T) \to \mathrm{End}_T(\mathcal{E})$$

is compatible with the p-structures, i.e., whether or not it is the case that

$$\nabla(D^p) = (\nabla(D))^p$$

whenever D is a section of $\underline{\mathrm{Der}}(S/T)$ over an open subset of S.

With this question in mind, we define the "p-curvature" $\psi$ of the connection $\nabla$ as a mapping of sheaves

5.0.3 $$\psi : \underline{\mathrm{Der}}(S/T) \to \underline{\mathrm{End}}_T(\mathcal{E})$$

by setting

5.0.4 $$\psi(D) = (\nabla(D))^p - \nabla(D^p)$$

We remark that $\psi$ "is" actually a mapping

5.0.5 $$\psi : \underline{\mathrm{Der}}(S/T) \to \underline{\mathrm{End}}_S(\mathcal{E})$$

(i.e., that $\psi(D)$ is S-linear). To see this, we use the Leibniz rule

5.0.6 $$(\nabla(D))^m(ge) = \Sigma_{i=0}^{m}\binom{m}{i}D^i(g)(\nabla(D))^{m-i}(e)$$

where D, g, and e are sections of $\underline{\mathrm{Der}}(S/T)$, $\mathcal{O}_S$ and $\mathcal{E}$ over an open subset of S. Putting $m = p$, we get

5.0.7 $$(\nabla(D))^p(ge) = D^p(g)e + g(\nabla(D))^p e$$

Since we have also the "connection-rule"

5.0.8 $$\nabla(D^p)(ge) = D^p(g)e + g\nabla(D^p)(e),$$

subtracting 5.0.8 from 5.0.7 gives the desired formula

5.0.9 $$\psi(D)(ge) = g\psi(D)(e)$$

We recall that having "p-curvature zero" means having enough horizontal sections. More precisely

Theorem 5.1 (Cartier) Let $f: S \to T$ be a smooth T-scheme of characteristic p.

5.1.0 Let $F_{abs}: T \to T$ be the absolute Frobenius (i.e., the p'th power mapping on $\mathcal{O}_T$), and

5.1.1 $\qquad S^{(p)} = \underset{=}{S} \times_{T^{F_{abs}}} T$ , the fibre product of $F_{abs} : T \to T$ and $f : S \to T$.

Let $F : S \to S^{(p)}$ be the relative frobenius (i.e., elevation of vertical coordinates to the p'th power).

There is an equivalence of categories between the category of quasi-coherent sheaves on $S^{(p)}$ and the full subcategory of MIC(S/T) consisting of objects $(\mathcal{E},\nabla)$ whose p-curvature is zero. This equivalence may be given explicitly as follows:

Let $\mathcal{F}$ be a quasicoherent sheaf on $S^{(p)}$. Then there is a unique T-connection $\nabla_{can}$, integrable and of p-curvature zero, on $F^*(\mathcal{F})$, such that

$$\mathcal{F} \xrightarrow{\sim} (F^*(\mathcal{F}))^{\nabla_{can}}.$$

The desired functor is $\mathcal{F} \rightsquigarrow (F^*(\mathcal{F}), \nabla_{can})$.

Given an object $(\mathcal{E},\nabla)$ of MIC(S/T) of p-curvature zero, we form $\mathcal{E}^{\nabla}$, which is in a natural way a quasicoherent sheaf on $S^{(p)}$. The desired inverse functor is

$$(\mathcal{E},\nabla) \rightsquigarrow \mathcal{E}^{\nabla}$$

Proof: The only point requiring proof is that, if an object $(\mathcal{E},\nabla)$ of MIC(S/T) has p-curvature zero, then the canonical mapping of $\mathcal{O}_S$-modules

5.11.0 $$F^*(\mathcal{E}^\nabla) \to \mathcal{E}$$

is an isomorphism. The question being local on S, we may suppose S is affine, and étale over $\underline{\underline{A}}^r_T$, with $\Omega^1_{S/T}$ free on $\{ds_1, \ldots, ds_r\}$. Consider the $F^{-1}(\mathcal{O}_{S^{(p)}})$-linear endomorphism P of $\mathcal{E}$, given by

5.1.2 $$P = \Sigma_w \prod_{i=1}^r \left(\frac{(-s_i)^{w_i}}{(w_i)!}\right) \prod_{i=1}^r \nabla\left(\frac{\partial}{\partial s_i}\right)^{w_i}$$

the sum taken over all r-tuples $(w_i, \ldots, w_r)$ of integers satisfying $0 \leq w_i \leq p-1$.

One immediately verifies that:

5.1.3 $$P(\mathcal{E}) \subset \mathcal{E}^\nabla$$

5.1.4 $$P\,|\,\mathcal{E}^\nabla = \mathrm{id}$$

5.1.5 $P^2 = P$ is a projection onto $\mathcal{E}^\nabla$

5.1.6 $$\bigcap_w \text{Kernel of } P \cdot \prod_{i=1}^r \nabla\left(\frac{\partial}{\partial s_i}\right)^{w_i} = \{0\},$$

the intersection extended to all r-tuples of integers $(w_1, \ldots, w_r)$ with $0 \leq w_i \leq p-1$

It follows that the mapping inverse to 5.1.1.0 is given explicitly by

5.1.7 $$\text{Taylor} : \mathcal{E} \to F^*(\mathcal{E}^\nabla)$$

$$\text{Taylor}(e) = \Sigma_w \prod_{i=1}^r \frac{(s_i)^{w_i}}{(w_i)!} P \cdot \prod_{i=1}^r \left(\nabla\left(\frac{\partial}{\partial s_i}\right)\right)^{w_i}(e).$$

We now develop the basic properties of the p-curvature.

Proposition 5.2.

5.2.0 The mapping $\psi : \underline{\mathrm{Der}}(S/T) \to \underline{\mathrm{End}}_S(\mathcal{E})$ is p-linear, i.e., it is additive, and $\psi(gD) = g^p\psi(D)$ whenever $g$ and $D$ are sections of $\mathcal{O}_S$ and $\mathrm{Der}(S/T)$ over an open subset of $S$.

5.2.1 If $D$ is a section of $\underline{\mathrm{Der}}(S/T)$ over an open subset $\mathcal{U}$ of $S$, the three T-endomorphisms of $\mathcal{E}|\mathcal{U}$

$$\nabla(D),\ \nabla(D^p),\ \psi(D)$$

mutually commute.

5.2.2 If $D$ and $D'$ are any two sections of $\underline{\mathrm{Der}}(S/T)$ over an open subset of $S$, then $\psi(D)$ and $\psi(D')$ commute.

5.2.3 $\psi$ Takes values in the sheaf of germs of horizontal S-endomorphisms of $\mathcal{E}$.

Proof: To prove that $\psi$ is additive, we use the Jacobson formula. If $a$ and $b$ are elements of an associate ring $R$ of characteristic $p$, then

5.2.4 $$(a+b)^p = a^p + b^p + \Sigma_{i=1}^{p-1} s_i(a,b)$$

where the $s_i(a,b)$ are the <u>universal</u> <u>Lie</u> <u>polynomials</u> obtained by passing to $R[t]$, $t$ an indeterminate, and writing

5.2.5 $$(\mathrm{ad}(ta+b))^{p-1}(a) = \Sigma_{i=1}^{p-1} i s_i(a,b) t^{i-1}.$$

Now let $D, D' \in \mathrm{Der}(S/T)$. Then by 5.2.4

5.2.6 $$(D+D')^p = D^p + (D')^p + \Sigma_{i=1}^{p-1} s_i(D, D')$$

and, as the connection $\nabla$ is integrable and the $s_i$ are Lie polynomials, we have

5.2.7 $$\nabla((D+D')^p) = \nabla(D^p) + \nabla((D')^p) + \Sigma_{i=1}^{p-1} s_i(\nabla(D), \nabla(D'))$$

Again applying 5.2.4, we have

5.2.8 $$(\nabla(D) + \nabla(D'))^p = (\nabla(D))^p + (\nabla(D'))^p + \Sigma_{i=1}^{p-1} s_i(\nabla(D), \nabla(D'))$$

Subtracting, we find that $\psi(D+D') = \psi(D) + \psi(D')$.

We next prove that $\psi$ is p-linear. For this we use Deligne's identity for $(gD)^p$.

<u>Proposition 5.3</u> (Deligne) Let $A$ be an associative ring of characteristic $p$, $g$ and $D$ two elements of $A$. For each integer $n \geq 0$, put

$g^{(n)} = (\mathrm{ad}(D))^n(g) = \underbrace{[D, [D, \ldots [Dg]]}_{n \text{ times}} \ldots ]$. Suppose that the elements $g^{(n)}$, $n \geq 0$ mutually commute. Then

5.3.0 $$(gD)^p = g^p D^p + g(g^{p-1})^{(p-1)} D$$

<u>Proof</u>: Reducing to the "universal" case, we may suppose that $g$ is invertible; let $h = g^{-1}$. By induction, it is easily seen that for each positive integer $n$, we have

5.3.1 (*n) $$(gD)^n = (h^{-1}D)^n = h^{-2n} \Sigma A_{\underline{m}} \prod h^{(m_i)} D^{n-\Sigma m_i}$$

the sum being over all n-tuples of integers $\underline{m} = (m_1, \ldots, m_n)$ having $0 \leq m_1 \leq m_2 \leq \ldots \leq m_n$ and $\Sigma m_i < n$, with the $A_{\underline{m}} \in \mathbb{F}_p$.

Consider now the special case of the ring of additive endomorphisms of the field $\mathbb{F}_p(X_0, \ldots, X_{p-2}, T)$, and let $D = \frac{d}{dT}$, $h = \Sigma_{i=0}^{p-2} X_i T^i / i!$ . Then $D^p = 0$, and because $h = D(k)$ with $k = \Sigma_{i=0}^{p-2} X_i \frac{T^{i+1}}{(i+1)!}$ we have $h^{(p-1)} = D^{p-1}(h) = 0$ and $(h^{-1}D)^p = 0$ (since $h^{-1}D = \frac{d}{dk}$). On the other hand, $D, D^2, \ldots, D^{p-1}$ are linearly independent over $K$. Putting $n = p$ in 5.3.1, we thus find that for each integer $j$, $0 < j < p$,

5.3.2 $$0 = \Sigma A_{\underline{m}} \prod_{i=1}^{n} h^{(m_i)}$$

the sum being over those $\underline{m} = (m_1, \ldots, m_p)$, $0 \leq m_1 \leq \ldots \leq m_p$ with $\Sigma m_i = j$.

As $g^{(0)}, \ldots, g^{(p-2)}$ are algebraically independent over $\mathbb{F}_p$ (their values at $T = 0$ being $X_0, \ldots, X_{p-2}$), we have

5.3.3 $$A_{\underline{m}} = 0 \text{ if } \underline{m} = (m_1, \ldots, m_p),\ 0 \leq m_1 \leq \ldots \leq m_p$$
$$\Sigma m_i < p \text{ and each } m_i < p-1.$$

The only possibly non-zero $A_{\underline{m}}$ in $(*p)$ are thus $A_{(0,\ldots,0)}$ and $A_{(0,\ldots,0,p-1)}$. Returning to $(*n)$, it is immediately verified by induction on $n$ that $A(0, \ldots, 0) = 1$ and $A(0, \ldots, 0, n-1) = 1$, so that 5.3.1 with $n = p$ becomes the desired formula:

5.3.4 $$\begin{aligned}(gD)^p = (h^{-1}D)^p &= h^{-2p}\{h^p D^p + h^{p-1}h^{(p-1)}D\} \\ &= g^p D^p + g(g^{p-1})^{(p-1)}D\end{aligned}$$

5.4 We now return to the proof of 5.2. Applying 5.3.0 to $g$ and $D$ in $\mathrm{End}_T(\mathcal{O}_S)$, we have

5.4.0 $$\begin{aligned}(gD)^p &= g^p D^p + g(\mathrm{ad}(D))^{p-1}(g^{p-1}) \cdot D \\ &= g^p D^p + gD^{p-1}(g^{p-1}) \cdot D\end{aligned}$$

whence

5.4.1 $$\nabla((gD)^p) = g^p \nabla(D^p) + gD^{p-1}(g^{p-1})D\ ;$$

Applying 5.3.0 to $g$ and $\nabla(D)$ in $\mathrm{End}_T(\mathcal{E})$, we have

5.4.2 $$(\nabla(gD))^p = (g\nabla(D))^p = g^p(\nabla(D))^p + g(\mathrm{ad}(\nabla(D))^{p-1}(g^{p-1})\nabla(D)$$
$$= g^p\nabla(D)^p + gD^{p-1}(g^{p-1})\nabla(D)$$

Subtracting 5.4.1 from 5.4.2 gives the desired p-linearity.

To prove 5.2.1, we remark that $D$ and $D^p$ commute, thus, $\nabla$ being integrable, so do $\nabla(D)$ and $\nabla(D^p)$, whence $\psi(D) = (\nabla(D))^p - \nabla(D^p)$ commutes with $\nabla(D)$ and $\nabla(D^p)$.

We now prove 5.2.2 and 5.2.3. The question being local on $S$, we may suppose that $S$ is affine, and is étale over $\underline{\underline{A}}^r_T$, so that $\Omega^1_{S/T}$ is free, with base $\{ds_1, \ldots, ds_r\}$. We denote by $\{\frac{\partial}{\partial s_1}, \ldots, \frac{\partial}{\partial s_r}\}$ the dual base of $\mathrm{Der}(S/T)$. Let

5.4.3 $$D = \Sigma a_i\frac{\partial}{\partial s_i}, \quad D' = \Sigma b_i\frac{\partial}{\partial s_i};$$

we must prove that

5.4.4 $$[\psi(D), \psi(D')] = 0 = [\psi(D), \nabla(D')]$$

But $$\psi(D) = \Sigma a_i^p\psi(\frac{\partial}{\partial s_i}) = \Sigma a_i^p(\nabla(\frac{\partial}{\partial s_i}))^p,$$

$$\psi(D') = \Sigma b_i^p(\nabla(\frac{\partial}{\partial s_i}))^p, \text{ and } \nabla(D') = \Sigma b_i\nabla(\frac{\partial}{\partial s_i}),$$

so that $[\psi(D), \psi(D')] = \Sigma a_i^p b_j^p[(\nabla(\frac{\partial}{\partial s_i}))^p, (\nabla(\frac{\partial}{\partial s_j}))^p] = 0$, and $[\psi(D), \nabla(D')] =$

$$\Sigma a_i^p b_j[(\nabla(\frac{\partial}{\partial s_i}))^p, \nabla(\frac{\partial}{\partial s_j})] = 0.$$

<u>Corollary 5.5</u>: Let $f: S \to T$ be a smooth T-scheme of characteristic p,

$(\mathcal{E},\nabla)$ an object of MIC(S/T), and $n \geq 1$ an integer. The following conditions are equivalent.

5.5.0 There exists a filtration of $(\mathcal{E},\nabla)$ of length $\leq n$ (i.e., $F^0$ = all, $F^n = \{0\}$) whose associated graded object has p-curvature zero.

5.5.1 Wherever $D_1, \ldots, D_n$ are sections of $\underline{\mathrm{Der}}(S/T)$ over an open subset of S, $\psi(D_1)\psi(D_2) \ldots \psi(D_n) = 0$.

5.5.2 There exists a covering of S by affine open subsets $\mathcal{U}$, and on each $\mathcal{U}$ "coordinates" $u_1, \ldots, u_r$ (i.e., sections of $\mathcal{O}_S$ over $\mathcal{U}$ such that $\Omega^1_{\mathcal{U}/T}$ is free on $du_1, \ldots, du_r$) such that for every r-tuple $(w_1, \ldots, w_r)$ of integers with $\Sigma w_i = n$,

$$(\nabla(\frac{\partial}{\partial u_1}))^{pw_1} \cdots (\nabla(\frac{\partial}{\partial u_r}))^{pw_r} = 0.$$

Proof: 5.5.0 <==> 5.5.1 is clear.

5.5.1 ==> 5.5.2 because $\psi(\frac{\partial}{\partial u_i}) = (\nabla(\frac{\partial}{\partial u_i}))^p$.

5.5.2 ==> 5.5.1 by the p-linearity of $\psi$; for, covering any open set by its intersection with the covering of (3), we are immediately reduced to the case in which $D_1, \ldots, D_n \in \mathrm{Der}(\mathcal{U}/T)$. We expand each $D_i$ using the given coordinates on $\mathcal{U}$

$$D_i = \sum a_{ij} \frac{\partial}{\partial u_j}$$

whence

$$\psi(D_i) = \Sigma\, a_{ij}^p\, \psi(\frac{\partial}{\partial u_j}) = \Sigma\, a_{ij}\, (\nabla(\frac{\partial}{\partial u_j}))^p,$$

and the assertion is clear.

<u>Definition 5.6.</u> We say that $(\mathcal{E},\nabla)$ is <u>nilpotent of exponent</u> $\leq n$ when one of the equivalent conditions of 5.5 is verified. We say that $(\mathcal{E},\nabla)$ is nilpotent if there exists a positive integer $n$ such that it is nilpotent of exponent $\leq n$. We denote by Nilp $(S/T)$ the full subcategory of MIC$(S/T)$ of objects $(\mathcal{E},\nabla)$ which are nilpotent, by $\mathrm{Nilp}^n(S/T)$ those which are nilpotent of exponent $\leq n$. $\mathrm{Nilp}^1$ consists of those of p-curvature zero. We record for future reference:

<u>Proposition 5.7</u>

5.7.0 Nilp$(S/T)$ is an exact abelian subcategory of MIC$(S/T)$.

5.7.1 Each $\mathrm{Nilp}^n(S/T)$ is stable under the operations of taking sub-objects and quotient objects.

5.7.2 Nilp$(S/T)$ is stable under the operations of internal hom and internal tensor product, and if $A$ and $B$ are objects of $\mathrm{Nilp}^n(S/T)$ and $\mathrm{Nilp}^m(S/T)$ respectively, then $A \otimes B$ and $\underline{\mathrm{Hom}}(A,B)$ are in $\mathrm{Nilp}^{n+m-1}(S/T)$.

<u>Proposition 5.8.</u> $(\mathcal{E},\nabla)$ is nilpotent if and only if for any section $D$ of

Der(S/T) (over an open $\mathcal{U}$ subset of S) which, as a T-endomorphism of $\mathcal{O}_{\mathcal{U}}$, is nilpotent, the corresponding T-endomorphism $\nabla(D)$ of $\mathcal{E}|\mathcal{U}$ is nilpotent.

Proof ($\Longrightarrow$) If $D^p = 0$ in Der($\mathcal{U}$/T), $\psi(D) = (\nabla(D))^p$ is nilpotent by assumption. By induction on the integer $\nu$ such that $D^{p^\nu} = 0$ in Der($\mathcal{U}$/T), we may suppose already proven the nilpotence of $\nabla(D^p)$ (since $(D^p)^{p^{\nu-1}} = 0$). But $(\nabla(D))^p = \psi(D) + \nabla(D^p)$, a sum of commuting (5.2.1) nilpotents.

($\Longleftarrow$) take a finite covering of S by affine open sets $\mathcal{U}$ which are étale over $\underline{\underline{A}}^r_T$. On each $\mathcal{U}$, choose "coordinates" $u_1, \ldots, u_r$ (i.e., sections of $\mathcal{O}_{\mathcal{U}}$ which define an étale morphism $\mathcal{U} \to \underline{\underline{A}}^r_T$). Then each $(\frac{\partial}{\partial u_i})^p = 0$ in Der($\mathcal{U}$/T). Let $n_{\mathcal{U}}$ be an integer such that, for each i, $(\nabla(\frac{\partial}{\partial u_i}))^{p^{n_u}} = 0$ in $\mathrm{End}_T(\mathcal{E}/\mathcal{U})$ ; and take $n = \mathrm{Sup}_{\{u\}} n_u$. Then $(\mathcal{E}, \nabla)$ is nilpotent of exponent $\leq n^2$.

Theorem 5.9. Let $f : S \to T$ and $f' : S' \to S'$ be smooth morphisms, and

5.9.0
$$\begin{array}{ccc} S' & \xrightarrow{g} & S \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{h} & T \end{array}$$

a commutative diagram. Suppose T is of characteristic p. Then under the inverse image functor

5.9.1 $$(g,h)^* : \mathrm{MIC}(S/T) \to \mathrm{MIC}(S'/T')$$

we have, for every integer $n \geq 1$

5.9.2 $$(g,h)^{*}(\mathrm{Nilp}^{n}(S/T)) \subset \mathrm{Nilp}^{n}(S'/T')$$

Proof: The proof is by induction on $n$, the exponent of nilpotence. Suppose first the theorem proven for $\nu = 1, \ldots, n-1$, and take an object $(\mathcal{E},\nabla)$ in $\mathrm{Nilp}^{n}(S/T)$. By definition there is an exact sequence in $\mathrm{MIC}(S/T)$

$$0 \to (\mathcal{E}',\nabla') \to (\mathcal{E},\nabla) \to (\mathcal{E}'',\nabla'') \to 0$$

with $(\mathcal{E}',\nabla') \in \mathrm{Nilp}^{1}(S/T)$, $(\mathcal{E}'',\nabla'') \in \mathrm{Nilp}^{n-1}(S/T)$. Pulling back gives an exact sequence in $\underline{\underline{\mathrm{MIC}}}(S'/T')$

$$(g,h)^{*}(\mathcal{E}',\nabla') \to (g,h)^{*}(\mathcal{E},\nabla) \to (g,h)^{*}(\mathcal{E}'',\nabla'') \to 0$$

and by hypothesis, $(g,h)^{*}(\mathcal{E}',\nabla') \in \mathrm{Nilp}^{1}(S'/T')$, and $(g,h)^{*}(\mathcal{E}'',\nabla'') \in \mathrm{Nilp}^{n-1}(S'/T')$, whence, by definition, $(g,h)^{*}(\mathcal{E},\nabla) \in \mathrm{Nilp}^{n}(S'/T')$.

Thus it suffices to prove that $\mathrm{Nilp}^{1}$ is stable under inverse image. To do this, we make use of the fibre product to reduce to checking two cases.

Case 1 $S = S \times_{T} T'$, $g = \mathrm{pr}_1$, $f' = \mathrm{pr}_2$. The question is local on $S$, which we suppose is affine, and étale over $\underline{\underline{A}}^{r}_{T}$, with $\Omega^{1}_{S/T}$ free on $ds_1, \ldots, ds_r$. Then $S'$ is étale over $\underline{\underline{A}}^{r}_{T'}$, with $\Omega^{1}_{S/T}$ free on $ds'_1, \ldots, ds'_r$, where $s'_i = g^{*}(s_i)$. By the p-linearity of $\psi$, it suffices to check that $\psi(\frac{\partial}{\partial s'_i}) = (\nabla(\frac{\partial}{\partial s'_i}))^p = 0$ in $\mathrm{End}_{S'}(g^*(\mathcal{E}))$. But $\nabla(\frac{\partial}{\partial s'_i}) \in \mathrm{End}_{T'}(g^*(\mathcal{E}))$ is the T'-linear endomorphism of $g^*(\mathcal{E}) \simeq \mathcal{E} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'}$ deduced from the T-linear endomorphism $\nabla(\frac{\partial}{\partial s_i})$ of $\mathcal{E}$ by extension of scalars $\mathcal{O}_T \to \mathcal{O}_{T'}$.

<u>Case 2.</u> $T' = T$, $h = \mathrm{id}$. We have the commutative diagram of T-schemes (cf. 5.1.1)

$$\begin{tikzcd} S' \arrow[r, "g"] \arrow[d, "F'"] & S \arrow[d, "F"] \\ S'^{(p)} \arrow[r, "g^{(p)}"] & S^{(p)} \end{tikzcd}$$

By Cartier's theorem 5.1, any object $(\mathcal{E}, \nabla) \in \mathrm{MIC}(S/T)$ with p-curvature zero is isomorphic to $(F^*(\mathcal{F}), \nabla_{can})$, where $\mathcal{F}$ is a quasicoherent $S^{(p)}$-module (namely $\mathcal{E}^{\nabla}$). Clearly we have

$$(g, \mathrm{id})^*(F^*(\mathcal{F}), \nabla_{can}) = (F'^*(g^{(p)*}(\mathcal{F})), \nabla_{can}),$$

an object of p-curvature zero.

We now prove the stability of nilpotence under higher direct images.

<u>Theorem 5.10.</u> Let $\pi : X \to S$ and $f : S \to T$ be smooth morphisms, with $T$ a scheme of characteristic $p$. Let $n$ be the relative dimension of $X/S$, supposed

constant. Suppose S is affine, and consider the spectral sequence 3.5.1.0 associated to a covering $\{\mathcal{U}_\alpha\}$ of X by open subsets étale over $\underline{\underline{A}}^n_S$ and an object $(\mathcal{E},\nabla) \in$ $\mathrm{Nilp}^\nu(X/T)$.

5.10.0 $$E_1^{p,q} = C^p(\{\mathcal{U}_\alpha\}, \mathcal{H}^q_{DR}(X/S, (\mathcal{E},\nabla)) \Longrightarrow H^{p+q}_{DR}(X/S, (\mathcal{E},\nabla)),$$

on which Der(S/T) acts through the Gauss-Manin connection.

5.10.1 Each term $E_r^{p,q} \in \mathrm{Nilp}^\nu(S/T)$.

5.10.2 For each integer $i \geq 0$ we put

$$\tau(i) = \text{the number of integers } p \text{ with } E_\infty^{p,i-p} \neq 0$$

Then

$$H^i_{DR}(X/S, (\mathcal{E},\nabla)) \in \mathrm{Nilp}^{\nu\cdot\tau(i)}(S/T).$$

5.10.3 $$\tau(i) \leq i+1, \text{ and } \tau(i) \leq 2n - i + 1$$

<u>Proof:</u> To prove 5.10.1, it suffices to show each $E_1^{p,q} \in \mathrm{Nilp}^\nu(S/T)$. But

5.10.4 $$E_1^{p,q} = \prod_{i_0<\cdots<i_p} H^q_{DR}(\mathcal{U}_{i_0} \cap \cdots \cap \mathcal{U}_{i_p}/S, (\mathcal{E},\nabla) \mid \mathcal{U}_{i_0} \cap \cdots \cap \mathcal{U}_{i_p})$$

so that we must prove that if $(\mathcal{E},\nabla) \in \mathrm{Nilp}^{\nu}(S/T)$ and $X$ is étale over $\underline{\underline{A}}^n_S$, then $H^i_{DR}(X/S, (\mathcal{E},\nabla)) \in \mathrm{Nilp}^{\nu}(S/T)$.

Let us remark first that, if $X$ is étale over $\underline{\underline{A}}^n_S$, the Gauss-Manin connection $\psi$ on $H_{DR}(X/S, (\mathcal{E},\nabla)) = \underline{\underline{R}}\pi_*(\Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$ is deduced from an integrable T-connection on the complex $\Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$ (cf. 3.5.3) which may be described explicitly as follows. Let $\Omega^1_{X/S}$ be free on $dx_1, \ldots, dx_n$, and for each $D \in \mathrm{Der}(S/T)$ denote by $D_0 \in \mathrm{Der}(X/T)$ the unique extension of $D$ which kills $dx_1, \ldots, dx_n$. Then the Gauss-Manin connection is deduced from the integrable connection

$$\psi : \mathrm{Der}(S/T) \to \mathrm{End}_T(\Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$$

given by

$$\psi(D)(dx_{i_1} \wedge \ldots dx_{i_r} \otimes e) = dx_{i_1} \wedge \ldots \wedge dx_{i_r} \otimes \nabla(D_0)(e)$$

Clearly we have

$$(\psi(D))^p (dx_{i_1} \wedge \ldots \wedge dx_{i_r} \otimes e) = dx_{i_1} \wedge \ldots \wedge dx_{i_r} \otimes (\nabla(D_0))^p(e),$$

thus the hypothesis $(\mathcal{E},\nabla) \in \mathrm{Nilp}^{\nu}(X/T)$ implies that, for any $D^{(1)}, \ldots, D^{(\nu)} \in \mathrm{Der}(S/T)$, the endomorphism $\psi(D_0^{(1)}) \ldots \psi(D_0^{(\nu)})$ of $\Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$ is zero, and hence it is zero on $H_{DR}(X/S, (\mathcal{E},\nabla))$, which concludes the proof of 5.10.1.

To prove 5.10.2, notice that

5.10.5 $$E_\infty^{p,q} = gr^p_{zar} H^{p+q}_{DR}(X/S, (\mathcal{E},\nabla)), \text{ so that}$$

$H^i_{DR}(X/S, (\mathcal{E},\nabla))$ has a horizontal filtration with $\tau(i)$ non-zero quotients, each quotient in $\mathrm{Nilp}^\nu(S/T)$.

To prove 5.10.3, we observe first that $E_1^{p,q} = 0$, unless $p \geq 0$ and $0 \leq q \leq n = \text{rel. dim } (X/S)$.

To conclude the proof, it suffices to show that $E_2^{p,q} = 0$ (and hence $E_\infty^{p,q} = 0$) if $p > n$. (The problem is that, while $\pi : X \to S$ has cohomological dimension $\leq n$ for sheaves, our $E_2^{p,q}$ terms are, a priori, only the Cech cohomology of certain presheaves. But being in characteristic $p$ will allow us to circumvent these difficulties by using an idea of Deligne.

Let $S \xrightarrow{F_{abs}} S$ be the absolute Frobenius, $X^{(p)}$ the fibre product of $\pi : X \to S$ and $F_{abs} : S \to S$, and $F : X \to X^{(p)}$ the relative Frobenius (cf. 5.1.0).

The complex $\Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$ is linear over $\pi^{-1}(\mathcal{O}_S)$ and over $(\mathcal{O}_X)^p$; in other words, $F_*(\Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$ is an $\mathcal{O}_{X^{(p)}}$ - linear complex of quasicoherent $\mathcal{O}_{X^{(p)}}$ - modules. Thus the cohomology presheaves of this complex, $\mathcal{H}^i(F_*(\Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}))$, are sheaves of quasicoherent $\mathcal{O}_{X^{(p)}}$ - modules. As we have an isomorphism of presheaves on $X^{(p)}$

5.10.6 $$F_* \mathcal{H}^i_{DR}(X/S, (\mathcal{E},\nabla)) \xrightarrow{\sim} \mathcal{H}^i(F_*(\Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}))$$

it follows that the presheaves $\mathcal{H}^i_{DR}(X/S, (\mathcal{E},\nabla))$ are in fact sheaves. Furthermore, we have

5.10.7 $$E_2^{p,q} = \check{H}^p(X, \{\mathcal{U}_\alpha\}, \mathcal{H}^q_{DR}(X/S, (\mathcal{E},\nabla)) = \check{H}^p(X^{(p)}, \{F(\mathcal{U}_\alpha)\}, F_*\mathcal{H}^q_{DR}(X/S, (\mathcal{E},\nabla)))$$
$$= H^p(X^{(p)}, F_*\mathcal{H}^q_{DR}(X/S, (\mathcal{E},\nabla)))$$

the last equality because $F_*\mathcal{H}^q_{DR}(X/S, (\mathcal{E},\nabla))$ is quasicoherent on $X^{(p)}$, and $\{F(\mathcal{U}_\alpha)\}$ is a covering of $X^{(p)}$ by affine open sets. As $X^{(p)}/S$ is of cohomological dimension $\leq n$, we have $E_2^{p,q} = 0$ if $p > n$, which concludes the proof of 5.10.3.

5.10.8 Remark: The interpretation 5.10.7 of the $E_2^{p,q}$ term of the spectral sequence 5.10.0 shows that the zariski filtration it defines on the $H^i_{DR}(X/S, (\mathcal{E},\nabla))$ is independent of the choice of covering of $X$ by affine open sets étale over $\underline{\underline{A}}^n_S$. Indeed, it shows that the entire spectral sequence, from $E_2$ on, is independent of that choice. We do not know if this is true when $S$ is no longer of characteristic $p$.

6. Connections in characteristic $p > 0$ having logarithmic singularities

6.0 Let $\pi : X \longrightarrow S$ be a smooth morphism, $i : Y \hookrightarrow X$ the inclusion of a divisor with normal crossings relative to S, and $f : S \longrightarrow T$ a smooth morphism, with T of characteristic p.

We define $\mathrm{Nil}_p^{\nu}(X/T(\log Y))$ to be the full subcategory of $\mathrm{MIC}(X/T(\log Y))$ consisting of objects admitting a filtration which has $\leq \nu$ non-zero quotients, each of p-curvature zero. In this context, the p-curvature of an object $(\mathcal{E}, \nabla)$ in $\mathrm{MIC}(X/T(\log Y))$ is the p-linear mapping

6.0.1
$$\psi : \underline{\mathrm{Der}}_Y(X/T) \longrightarrow \underline{\mathrm{End}}_T(\mathcal{E})$$
$$\psi(D) = (\nabla(D))^p - \nabla(D^p) \ .$$

The proof of 5.10 carries over mutatis mutandi to give

Theorem 6.1 = 5.10 bis. Assumptions as above, suppose that S is affine, and let $n = \mathrm{rel.dim}(X/S)$. Let $(\mathcal{E}, \nabla)$ be an object of $\mathrm{MIC}(X/T(\log Y))$. Consider the spectral sequence 4.6.7,

6.1.0 $$E_1^{p,q} = C^p(\{\mathcal{U}_\alpha\}, \mathcal{H}^q_{DR}(X/S(\log Y), (\mathcal{E}, \nabla)) \Longrightarrow H^{p+q}_{DR}(X/S(\log Y), (\mathcal{E}, \nabla))$$

which by 5.10.7, has

6.1.1 $$E_2^{p,q} = H^p(X^{(p)}, F_*\mathcal{H}^q_{DR}(X/S(\log Y), (\mathcal{E}, \nabla))$$

and, from $E_2$ on, is independent of choice of covering. Der(S/T) acts on this spectral sequence through the Gauss-Manin connection. Suppose $(\mathcal{E}, \nabla) \in \mathrm{Nil}_p^{\nu}(X/T(\log Y))$. Then

6.1.2 Each term $E_r^{p,q} \in \mathrm{Nil}_p^{\nu}(S/T)$.

6.1.3 For each integer $i \geq 0$, put

$$\tau(i) = \text{the number of integers } p \text{ with } E_\infty^{p,i-p} \neq 0.$$

Then $$H^i_{DR}(X/S(\log Y), (\mathcal{E}, \nabla)) \in \mathrm{Nil}_p^{\nu\cdot\tau(i)}(S/T).$$

6.1.4 $$\tau(i) \leq i+1 \text{ and } \tau(i) \leq 2n-i+1 \ .$$

## 7. Ordinary de Rham cohomology in characteristic p

7.0 Let $\pi : X \longrightarrow S$ be a smooth morphism, $i : Y \hookrightarrow X$ the inclusion of a divisor with normal crossings relative to $S$. The structural sheaf $\mathcal{O}_X$, with the integrable $S$-connection "exterior differentiation"

7.0.1
$$d_{X/S} : \mathcal{O}_X \longrightarrow \Omega^1_{X/S}$$

defines an object in $MIC(X/S)$.

We denote the de Rham cohomology sheaves on $S$ of this object simply $H_{DR}(X/S)$, i.e. by definition

7.0.2
$$H^q_{DR}(X/S) = \underline{\underline{R}}^q \pi_*(\Omega^{\bullet}_{X/S}) .$$

Similarly, by composing 7.0.2 with the canonical inclusion $\Omega^1_{X/S} \hookrightarrow \Omega^1_{X/S}(\log Y)$, we obtain an object in $MIC(X/S(\log Y))$ whose de Rham cohomology sheaves on $S$ are denoted $H_{DR}(X/S(\log Y))$, i.e., by definition

7.0.3
$$H^q_{DR}(X/S(\log Y)) = \underline{\underline{R}}^q \pi_*(\Omega^{\bullet}_{X/S}(\log Y)) .$$

For any smooth morphism $f : S \longrightarrow T$, the objects of $MIC(X/S)$ and $MIC(X/S(\log Y))$ defined by $(\mathcal{O}_X, d_{X/S})$ come via 3.0.1 and 4.5.0 from the objects $MIC(X/T)$ and $MIC(X/T(\log Y))$ defined by $(\mathcal{O}_X, d_{X/T})$. Thus the sheaves $H^q_{DR}(X/S)$ and $H^q_{DR}(X/S(\log Y))$ are provided with a canonical integrable $T$-connection whenever $f : S \longrightarrow T$ is a smooth morphism.

7.1. Suppose now that $S$ is of characteristic $p$. As before (5.1.0) we denote by $X^{(p)}$ the scheme which makes the following diagram cartesian

7.1.0
$$\begin{tikzcd} X^{(p)} \arrow[r, "W"] \arrow[d, "\pi^{(p)}"] & X \arrow[d, "\pi"] \\ S \arrow[r, "F_{abs}"] & S \end{tikzcd}$$

(i.e., $X^{(p)}$ is the fibre product of $\pi : X \longrightarrow S$ and $F_{abs} : S \longrightarrow S$) and we denote by $F : X \longrightarrow X^{(p)}$ the relative Frobenius. The diagram

7.1.1
$$\begin{tikzcd} X^{(p)} \arrow[r, "W"] \arrow[d, "\pi^{(p)}"] & X \arrow[r, "F"] \arrow[d, "\pi"] & X^{(p)} \arrow[dl, "\pi^{(p)}"] \\ S \arrow[r, "F_{abs}"] & S & \end{tikzcd}$$

is commutative, and $W \circ F$ is the absolute Frobenius endomorphism of $X$, $F \circ W$ the absolute Frobenius of $X^{(p)}$. We denote by $Y^{(p)}$ the fibre product of $Y \overset{i}{\hookrightarrow} X$ and $X^{(p)} \xrightarrow{W} X$; $Y^{(p)}$ is a divisor in $X^{(p)}$ with normal crossings relative (via $\pi^{(p)}$) to $S$. The spectral sequences of ordinary de Rham cohomology of $X/S$ may be written (writing $\mathcal{H}^q$ for cohomology sheaf)

7.1.2 $$E_2^{p,q} = R^p\pi_*^{(p)}(\mathcal{H}^q(F_*(\Omega^\bullet_{X/S})) \Longrightarrow \underline{\underline{R}}^{p+q}\pi_*(\Omega^\bullet_{X/S}) = H^{p+q}_{DR}(X/S)$$

and that for de Rham cohomology of $X/S(\log Y)$ may be similarly written

7.1.3 $$E_2^{p,q}(\log Y) = R^p\pi_*^{(p)}(\mathcal{H}^q(F_*(\Omega^\bullet_{X/S}(\log Y))) \Longrightarrow \underline{\underline{R}}^{p+q}\pi_*(\Omega^\bullet_{X/S}(\log Y))H^{p+q}_{DR}(X/S(\log Y))$$

The $E_2$ terms have a remarkably simple interpretation due to Deligne, via the Cartier operation.

<u>Theorem</u> 7.2 (Cartier). There is a unique isomorphism of $\mathcal{O}_{X^{(p)}}$-modules

7.2.0 $$C^{-1} : \Omega^i_{X^{(p)}/S} \xrightarrow{\sim} \mathcal{H}^i(F_*\Omega^\bullet_{X/S})$$

which verifies

7.2.1 $$C^{-1}(1) = 1$$

7 2.2 $$C^{-1}(\omega \wedge \tau) = C^{-1}(\omega) \wedge C^{-1}(\tau)$$

7.2.3 $$C^{-1}(d(W^{-1}(f)) = \text{the class of } f^{p-1}df \text{ in } \mathcal{H}^1(F_*\Omega^\bullet_{X/S}) .$$

Furthermore, $C^{-1}$ induces an isomorphism of $\mathcal{O}_{X^{(p)}}$ modules (by restricting $C^{-1}$ to $\Omega^i_{X^{(p)}/S}(\log Y^{(p)}) \subset \Omega^i_{X^{(p)}-Y^{(p)}/S}$

7.2.4 $$C^{-1} : \Omega^i_{X^{(p)}/S}(\log Y^{(p)}) \longrightarrow \mathcal{H}^i(F_*\Omega^{\bullet}_{X/S}(\log Y)) \ .$$

Proof. First, we construct $C^{-1}$, following a method of Deligne. We need only construct $C^{-1}$ for $i = 1$, for then the asserted multiplicativity (2) will determine it uniquely for $i \geq 1$, and for $i = 0$ the condition (1) and $\mathcal{O}_{X^{(p)}}$-linearity suffice.

An $\mathcal{O}_{X^{(p)}}$-linear mapping

7.2.5 $$C^{-1} : \Omega^1_{X^{(p)}/S} \longrightarrow \mathcal{H}^1(F_*\Omega^{\bullet}_{X/S})$$

is nothing other than a $(\pi^{(p)})^{-1}(\mathcal{O}_S)$-linear derivation of $\mathcal{O}_{X^{(p)}}$ into $\mathcal{H}^1(F_*\Omega^{\bullet}_{X/S})$. Making explicit use of the definition of $X^{(p)}$ as a fibre product, we have

7.2.6 $$\mathcal{O}_{X^{(p)}} = \mathcal{O}_X \otimes_{\pi^{-1}(\mathcal{O}_S)} \pi^{-1}(\mathcal{O}_S) \quad \text{(where } \pi^{-1}(\mathcal{O}_S) \text{ is a module over itself by } F_{abs})$$

so that such a derivation is a mapping of sheaves

7.2.7 $$\delta : \mathcal{O}_X \times \pi^{-1}(\mathcal{O}_S) \longrightarrow \mathcal{H}^1(\Omega^{\bullet}_{X/S})$$
$$(f, s) \longrightarrow \delta(f, s)$$

which is biadditive and verifies

7.2.8 $$\begin{cases} \delta(fs, s') = \delta(f, s^p s') \\ \delta(gf, s) = g^p\delta(f, s) + f^p\delta(g, f) \\ \delta(f, 1) = \text{the class of } f^{p-1}df \ . \end{cases}$$

We define

7.2.9 $$\delta(f, s) = \text{the class of } sf^{p-1}df$$

The properties 7.2.8 are obvious; as for biadditivity, we calculate

7.2.10
$$\begin{aligned}&\delta(f+g,\ s) - \delta(f,\ s) - \delta(g,\ s)\\ &= s((f+g)^{p-1}(df+dg) - f^{p-1}df - g^{p-1}dg)\\ &= d\left(s\cdot\left(\frac{(f+g)^p - f^p - g^p}{p}\right)\right)\end{aligned}$$

Having defined $C^{-1}$, we must now prove it is an isomorphism. The question being local on X, we may suppose X is étale over $\underline{\underline{A}}^n_S$ via coordinates $x_1, \ldots, x_n$, such that the divisor Y is defined by the equation $x_1 \ldots x_\nu = 0$. Then $F_*(\Omega^\bullet_{X/S})$ is the $\mathcal{O}_{X^{(p)}}$-linear complex

7.2.11
$$\mathcal{O}_{X^{(p)}} \otimes_{\underline{\underline{F}}_p} K^\bullet(n)$$

where for every integer $n \geq 1$, $K^\bullet(n)$ is the complex of $\underline{\underline{F}}_p$-vector spaces with basis the differential forms

$$x_1^{w_1} \ldots x_n^{w_n}\, dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_j}$$
$$\begin{cases} 0 \leq w_i \leq p-1 & \text{for } i = 1, \ldots, n\\ 1 \leq \alpha_1 < \ldots < \alpha_j \leq n\end{cases}$$

and differential the usual exterior derivative in n variables. Thus

7.2.12
$$H^i(F_*\Omega^\bullet_{X/S}) \simeq \mathcal{O}_{X^{(p)}} \otimes_{\underline{\underline{F}}_p} H^i(K^\bullet(n)) .$$

Similarly, $F_*(\Omega^\bullet_{X/S}(\log Y))$ is the $\mathcal{O}_{X^{(p)}}$-linear complex

7.2.13
$$\mathcal{O}_{X^{(p)}} \otimes_{\underline{\underline{F}}_p} L^\bullet(n,\ \nu)$$

where for each pair of integers $1 \leq \nu \leq n$, $L^\bullet(n,\ \nu)$ is the complex of $\underline{\underline{F}}_p$-vector spaces with basis the differential forms

$$x_1^{w_1} \cdots x_n^{w_n} \omega_{\alpha_1} \wedge \cdots \wedge \omega_{\alpha_j}$$
$$\begin{cases} 1 \le w_i \le p-1 \text{ for } i = 1, \ldots, n \\ 1 \le \alpha_1 < \cdots < \alpha_j \le n \\ \omega_i = \begin{cases} dx_i/x_i & i = 1, \ldots, \nu \\ dx_i & i = \nu+1, \ldots, n \end{cases} \end{cases}$$

and differential the usual exterior derivative in $n$ variables. We have

7.2.14 $$H^i(F_*(\Omega^{\bullet}_{X/S}(\log Y)) \simeq \mathcal{O}_{X^{(p)}} \otimes_{\mathbb{F}_p} H^i(L^{\bullet}(n, \nu)) .$$

What must be proved, then, is that

a) $H^0(L^{\bullet}(n, \nu)) = H^0(K^{\bullet}(n)) = \mathbb{F}_p$

b) $H^1(K^{\bullet}(n))$ has as base the classes $x_i^{p-1}dx_i$, $i = 1, \ldots, n$

and $H^1(L^{\bullet}(n, \nu))$ has as base the classes $\begin{cases} \omega_i & i = 1, \ldots, \nu \\ x_i^{p-1}\omega_i & i = \nu+1, \ldots, n \end{cases}$

c) $H^i(K^{\bullet}(n)) = \Lambda^i H^1(K(n))$ and $H^i(L^{\bullet}(n, \nu)) = \Lambda^i H^1(L^{\bullet}(n, \nu))$.

To see this, we observe that the complexes $K^{\bullet}(n)$ and $L^{\bullet}(n, \nu)$ are easily expressed as tensor products of "1-variable" complexes. Namely

$$K^{\bullet}(n) = \underbrace{K^{\bullet}(1) \otimes_{\mathbb{F}_p} K^{\bullet}(1) \otimes_{\mathbb{F}_p} \otimes \cdots \otimes_{\mathbb{F}_p} K^{\bullet}(1)}_{n \text{ times}}$$

$$L^{\bullet}(n, \nu) = \underbrace{L^{\bullet}(1, 1) \otimes_{\mathbb{F}_p} \cdots \otimes_{\mathbb{F}_p} L^{\bullet}(1, 1)}_{\nu \text{ times}} \otimes_{\mathbb{F}_p} \underbrace{K^{\bullet}(1) \otimes \cdots \otimes_{\mathbb{F}_p} K^{\bullet}(1)}_{n-\nu \text{ times}}$$

By Kunneth, it suffices to show

$$H^i(K^{\bullet}(1)) = \begin{cases} \mathbb{F}_p \cdot 1 & i = 0 \\ \mathbb{F}_p \text{ (class of } x^{p-1}dx), & i = 1 \\ 0 & i \ge 2 \end{cases}$$

and

$$H^i(L^\cdot(1,\ 1)) = \begin{cases} \underline{\underline{F}}_p \cdot 1 & i = 0 \\ \underline{\underline{F}}_p \text{ class of } \omega = dx/x & i = 1 \\ 0 & i \geq 2 \end{cases}$$

which is clear.

This concludes the proof of theorem 7.2.

7.3 Thus the spectral sequences 7.1.2 and 7.1.3 may be written

7.3.0 $$E_2^{p,q} = R^p\pi_*^{(p)}(\Omega^q_{X^{(p)}/S}) \Longrightarrow \underline{\underline{R}}^{p+q}\pi_*(\Omega^\cdot_{X/S}) = H^{p+q}_{DR}(X/S)$$

and

7.3.1 $$E_2^{p,q}(\log Y) = R^p\pi_*^{(p)}(\Omega^q_{X^{(p)}/S}(\log Y^{(p)})) \Longrightarrow \underline{\underline{R}}^{p+q}\pi_*(\Omega^\cdot_{X/S}(\log Y)) = H^{p+q}_{DR}(X/S(\log Y))$$

Suppose now that <u>either</u> of the two following conditions is true.

7.3.2 For each pair of integers $p,\ q \geq 0$, the formation of the sheaves $R^p\pi_*(\Omega^q_{X/S})$ and $R^p\pi_*(\Omega^q_{X/S}(\log Y))$ commutes with arbitrary base change.

7.3.3 The morphism "absolute Frobenius" $F_{abs} : S \longrightarrow S$ is <u>flat</u>; this is the case, for example, if $S$ is smooth over a field, or locally admits a "p-base" (cf. 4).

Since the diagram (7.1.0)

7.3.4 $$\begin{array}{ccc} X^{(p)} & \xrightarrow{W} & X \\ \Big\downarrow{\scriptstyle \pi^{(p)}} & & \Big\downarrow{\scriptstyle \pi} \\ S & \xrightarrow{F_{abs}} & S \end{array}$$

is <u>cartesian</u>, and

7.3.5 $$\begin{cases} \Omega^q_{X^{(p)}/S} = W^*(\Omega^q_{X/S}) \\ \Omega^q_{X^{(p)}/S}(\log Y^{(p)}) = W^*(\Omega^q_{X/S}(\log Y)) \end{cases}$$

either of the assumptions (1) or (2) implies that, for all $p, q \geq 0$, we have isomorphisms

7.3.6 $$E_2^{p,q} = R^p\pi_*^{(p)}(\Omega^q_{X^{(p)}/S}) = F^*_{abs}(R^p\pi_*(\Omega^q_{X/S}))$$

7.3.7 $$E_2^{p,q}(\log Y) = R^p\pi_*^{(p)}(\Omega^q_{X^{(p)}/S}(\log Y^{(p)}) = F^*_{abs}(R^p\pi_*(\Omega^q_{X/S}(\log Y))) \ .$$

Remark. When (1) or (2) holds, the above isomorphisms provide (via Cartier's theorem 5.1) an a priori construction of the Gauss-Manin connection on the $E_2$ terms of 7.12 and 7.13.

We summarize our findings.

Theorem 7.4. Let $\pi : X \longrightarrow S$ be a smooth morphism, $i : Y \hookrightarrow X$ the inclusion of a divisor with normal crossings. Suppose that $S$ is a scheme of characteristic $p$, and that either

7.4.0 for each pair of integers $p, q \geq 0$, the formation of the sheaves $R^p\pi_*(\Omega^q_{X/S})$ and $R^p\pi_*(\Omega^q_{X/S}(\log Y))$ commutes with arbitrary base change;

7.4.1 the morphism "absolute Frobenius" $F_{abs} : S \longrightarrow S$ is flat.

Then the spectral sequences 7.1.2 and 7.1.3 may be rewritten

7.4.2 $$E_2^{p,q} = F^*_{abs}(R^p\pi_*(\Omega^q_{X/S})) \Longrightarrow H^{p+q}_{DR}(X/S)$$

7.4.3 $$E_2^{p,q}(\log Y) = F^*_{abs}(R^q\pi_*(\Omega^q_{X/S}(\log Y))) \Longrightarrow H^{p+q}_{DR}(X/S(\log Y)) \ .$$

For any smooth morphism $f : S \longrightarrow T$, these spectral sequences are endowed with a canonical integrable $T$-connection, that of Gauss-Manin, which has p-curvature zero on the terms $E_r^{p,q}$, $r \geq 2$.

Corollary 7.5 (Deligne). For each integer $i \geq 0$, let $h(i)$ (respectively $h_Y(i)$) be the number of integers $p$ for which $E_2^{p,i-p}$ (resp. $E_2^{p,i-p}(\log Y)$) is non-zero. (Clearly $h(i)$ and $h_Y(i)$ are $\leq \sup(i+1, 2\dim(X/S)+1-i)$.) Then for any smooth morphism $f : S \longrightarrow T$, we have

7.5.0 $$H^i_{DR}(X/S) \in \mathrm{Nilp}^{h(i)}(S/T)$$

7.5.1 $$H^i_{DR}(X/S(\log Y)) \in \mathrm{Nilp}^{h_Y(i)}(S/T) .$$

## 8. Base-changing the de Rham and Hodge cohomology

We first recall a rather crude version of the "base-changing" theorems (cf. EGA [14], Mumford [29], and Deligne [6]).

Theorem 8.0. Let $S$ be a noetherian scheme, and $\pi : X \longrightarrow S$ a proper morphism. Let $K^{\cdot}$ be a complex of abelian sheaves on $S$, such that

8.0.0 $K^i = 0$ if $i < 0$ and for $i >> 0$

8.0.1 each $K^i$ is a coherent $\mathcal{O}_X$-module, flat over $S$

8.0.2 the differential of the complex $K^{\cdot}$ is $\pi^{-1}(\mathcal{O}_S)$-linear.

Then the following conditions are equivalent:

8.0.3 For every integer $n \geq 0$, the coherent $\mathcal{O}_S$-modules $\underline{\underline{R}}^n\pi_*(K^{\cdot})$ are locally free.

8.0.4 For every morphism $g : S' \longrightarrow S$, we form the (cartesian) diagram

8.0.4.0
$$\begin{tikzcd} S' \times_S X \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & X \arrow[d, "\pi"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

The canonical morphism of base-change,

8.0.4.1 $$g^*\underline{\underline{R}}^n\pi_*(K^{\cdot}) \longrightarrow \underline{\underline{R}}^n pr_{1_*}(pr_2^*(K^{\cdot}))$$

is an isomorphism for every integer $n \geq 0$.

8.0.5 Same as 8.0.4 for every morphism $g : S' \longrightarrow S$ which is the inclusion of a point of $S$.

Furthermore, there is a non-empty open subset $\mathcal{U}$ of $S$ such that, for the morphism $\pi_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U}$ and the complex $K^{\cdot}/\mathcal{U}$, each of these equivalent conditions is satisfied.

When these conditions are satisfied, we say that the formation of the $\underline{\underline{R}}^n\pi_*(K^{\bullet})$ commutes with base change.

Corollary 8.1. Let $\pi : X \longrightarrow S$ be a proper and smooth morphism, and suppose $S$ is noetherian. There is a non-empty open subset $\mathcal{U} \subset S$, such that each of the coherent sheaves on $S$

8.1.0 $$R^q\pi_*(\Omega^p_{X/S}),\ p,\ q \geq 0$$

8.1.1 $$H^n_{DR}(X/S) = \underline{\underline{R}}^n\pi_*(\Omega^{\bullet}_{X/S}),\ n \geq 0$$

is locally free over $\mathcal{U}$.

8.2 Let us define the Hodge cohomology of $X/S$ by

8.2.0 $$H^n_{Hodge}(X/S) = \prod_{p+q=n} R^q\pi_*(\Omega^p_{X/S})$$

it is bigraded;

8.2.1 $$H^{p,q}_{Hodge}(X/S) = R^q\pi_*(\Omega^p_{X/S}) .$$

Corollary 8.3. Let $\pi : X \longrightarrow S$ be a proper and smooth morphism, and suppose $S$ is noetherian. Suppose that each of the coherent $\mathcal{O}_S$-modules

8.3.0 $$H^{p,q}_{Hodge}(X/S),\ p,\ q \geq 0$$

8.3.1 $$H^n_{DR}(X/S),\ n \geq 0$$

is locally free.

Then for any change of base $g : S' \longrightarrow S$, the canonical morphisms of sheaves on $S'$

8.3.1 $$g^*H^{p,q}_{Hodge}(X/S) \longrightarrow H^{p,q}_{Hodge}(S' \times_S X/S')$$

8.3.2 $$g^*H^n_{DR}(X/S) \longrightarrow H^n_{DR}(S' \times_S X/S')$$

are isomorphisms for all values of $p$, $q$ and $n$. In particular, the Hodge and de Rham cohomology sheaves of $S' \times_S X/S'$ are locally free sheaves on $S'$.

Corollary 8.4. Under the assumptions of 8.1, let

$$i : Y \hookrightarrow X$$

be the inclusion of a divisor with normal crossings relative to $S$. Then there is a non-empty open subset $\mathcal{U} \subset S$ such that each of the coherent sheaves on $S$

8.4.0 $$R^q\pi_*(\Omega^p_{X/S}(\log Y)) \quad p, q \geq 0$$

8.4.1 $$(H^n_{DR}(X/S(\log Y)) = \underline{\underline{R}}^n\pi_*(\Omega^{\bullet}_{X/S}(\log Y))$$

is locally free over $\mathcal{U}$.

8.5 Let us define the Hodge cohomology of $X/S(\log Y)$ by

8.5.0 $$H^n_{Hodge}(X/S(\log Y)) = \prod_{p+q=n} R^q\pi_*(\Omega^p_{X/S}(\log Y))$$

it is again bigraded

8.5.1 $$H^{p,q}_{Hodge}(X/S(\log Y)) = R^q\pi_*(\Omega^p_{X/S}(\log Y)) .$$

Corollary 8.6. Assumptions as in 8.4, suppose that each of the coherent sheaves on $S$

8.6.0 $$H^{p,q}_{Hodge}(X/S(\log Y)) \quad p, q \geq 0$$

8.6.1 $$H^n_{DR}(X/S(\log Y)) \quad n \geq 0$$

is locally free.

Then for any change of base $g : S' \longrightarrow S$, denoting by $Y'$ the fibre product of $i : Y \hookrightarrow X$ and $pr_2 : S' \times_S X \longrightarrow X$ (cf. 8.0.4.0), which is a divisor in $S' \times_S X$ with normal crossings relative to $S'$, the canonical morphisms of sheaves on $S'$

8.6.2 $$g^*H^{p,q}_{Hodge}(X/S(\log Y)) \longrightarrow H^{p,q}_{Hodge}(S' \times_S X/S'(\log Y'))$$

8.6.3 $$g^*H^n_{DR}(X/S(\log Y)) \longrightarrow H^n_{DR}(S' \times_S X/S'(\log Y'))$$

are isomorphisms for all values of p, q and n. In particular, the Hodge and de Rham cohomology sheaves of $S' \times_S X/S'(\log Y')$ are locally free sheaves on $S'$.

8.7 It is proven in [6] that, if S is of characteristic zero, then the open subset of Corollary 8.1 is all of S, and the spectral sequence of sheaves on S

8.7.1 $$E_1^{p,q} = H^{p,q}_{Hodge}(X/S) \Longrightarrow H^{p+q}_{DR}(X/S)$$

degenerates at $E_1$.

Similar arguments, using Deligne's theory of "mixed Hodge structures," (unpublished) allow one to prove that, if S is of characteristic zero, then the open subset of Corollary 8.4 is all of S, and that the spectral sequence of sheaves on S

8.7.2 $$E_1^{p,q} = H^{p,q}_{Hodge}(X/S(\log Y)) \Longrightarrow H^{p+q}_{DR}(X/S(\log Y))$$

degenerates at $E_1$.

There is an elementary proof of the fact that the $H_{DR}(X/S)$ are locally free on S, if X/S is proper and smooth, and S is smooth over a field k at characteristic zero. It is based only on the fact that the $H_{DR}(X/S)$ are coherent sheaves on S, with an integrable k-connection (that of Gauss-Manin!).

Proposition 8.8. Let S be smooth over a field k of characteristic zero, and let $(M, \nabla)$ be an object of MIC(S/k), such that M is coherent. Then M is a locally free sheaf on S.

Proof. The question being local on S, it suffices to prove that, for every closed point $s \in S$, the module $M_s$ over $\mathcal{O}_{S,s}$ is free. As $M_s$ is finitely generated over $\mathcal{O}_{S,s}$ by hypothesis, it suffices to prove that $\hat{M}_s = M_s \otimes_{\mathcal{O}_{S,s}} \hat{\mathcal{O}}_{S,s}$, the completion of $M_s$ for the topology defined by powers of the maximal ideal of $\mathcal{O}_{S,s}$, is free over $\hat{\mathcal{O}}_{S,s}$. Thus it suffices to prove an analogue of Cartier's theorem 5.1.

Proposition 8.9. Let $K$ be a field of characteristic zero, $K[[t_1, \ldots, t_n]]$ the ring of formal power series over $K$ in $n$ variables. Let $M$ be a finitely generated module over $K[[t_1, \ldots, t_n]]$, given with an integrable connection $\nabla$ (for the continuous $K$-derivations of $K[[t_1, \ldots, t_n]]$ to itself). Then $M^{\nabla}$, the $K$-space of horizontal elements of $M$, is finite-dimensional over $K$, and the pair $(M, \nabla)$ is isomorphic to the pair $(M^{\nabla} \otimes_K K[[t_1, \ldots, t_n]], 1 \otimes d)$ where $d$ denotes the "identical" connection on $K[[t_1, \ldots, t_n]]$.

Proof. We begin by constructing an additive endomorphism of $M$. For $i = 1, \ldots, n$, let

8.9.0 $$D_i = \nabla\left(\frac{\partial}{\partial t_i}\right)$$

and for each integer $j \geq 0$, let

8.9.1 $$D_i^{(j)} = \frac{1}{j!}\left(\nabla\left(\frac{\partial}{\partial t_i}\right)\right)^j ; D_i^{(0)} = 1 .$$

For an $n$-tuple $J = (j_1, \ldots, j_n)$ of non-negative integers, we put

8.9.2 $$D^{(J)} = \prod_{i=1}^{n} D_i^{(j_i)}$$

8.9.3 $$t^J = \prod_{i=1}^{n} (t_i)^{j_i}$$

8.8.4 $$(-1)^J = \prod_{i=1}^{n} (-1)^{j_i} .$$

We then define a $K$-linear endomorphism $P$ of $M$

8.9.5 $$P : M \longrightarrow M$$
$$P = \sum_J (-1)^J t^J D^{(J)}$$

One successively verifies

8.9.6 $\qquad P(fm) = f(o)P(m)$ for $f \in K[[t_1, \ldots, t_n]]$ and $m \in M$ ,

by Leibniz's rule, so that $\mathrm{Kernel}(P) \supset (t_1, \ldots, t_n)M$

8.9.7 $\qquad P(m) \equiv m$ modulo $(t_1, \ldots, t_n)M$, so that

$$\mathrm{Kernel}(P) = (t_1, \ldots, t_n)\cdot M$$

8.9.8 $\qquad P|M^{\nabla} = \mathrm{id}.$

8.9.8 $\qquad P(M) \subset M^{\nabla}$ (by a "telescoping"), so

$P^2 = P$ is a projection onto $M^{\nabla}$

8.9.10 $\qquad P$ induces an isomorphism of K-vector spaces.

$$P : M/(t_1, \ldots, t_n)M \xrightarrow{\sim} M^{\nabla} \quad .$$

This shows that $M^{\nabla}$ is finite dimensional, and that (by Nakayama), the canonical mapping

8.8.11 $$M^{\nabla} \otimes_K K[[t_1, \ldots, t_n]] \longrightarrow M$$

is <u>surjective</u>. To show it is an isomorphism, we must show that if $m_1, \ldots, m_r$ are K-linearly independent elements of $M^{\nabla}$, then there can be no non-trivial relation

8.9.12 $$\Sigma f_i m_i = 0 \text{ in } M \ .$$

Supposing the contrary, assume $f_1$ is $\neq 0$. Then for some $J = (j_1, \ldots, j_n)$, we have

8.9.13 $$\left( \prod_{\nu=1}^{n} \frac{1}{j_\nu !} \left( \frac{\partial}{\partial t_\nu} \right)^{j_\nu} (f_1) \right) (0) \neq 0 \ .$$

Since the $m_i$ are horizontal, applying $D^{(J)}$ to 8.9.12 gives

8.9.14 $$O = D^{(J)}(\Sigma f_i m_i) = \sum_i \prod_{\nu=1}^{n} \frac{1}{j_\nu!} \left(\frac{\partial}{\partial t_\nu}\right)^{j_\nu}(f_i)\cdot m_i$$

a relation of the form

8.9.15 $$\Sigma g_i m_i = 0, \ g_1(0) \neq 0 \ .$$

Applying P to 7.11.14 gives a relation

$$\Sigma\, g_i(0) m_i = 0$$

which is impossible. Q.E.D.

Remark 8.9.16. Heuristically, $P(m)(t_1, \ldots, t_n) = m\,(t_1 - t_1, \ldots, t_n - t_n)$ expanded in Taylor series. In fact, the proof of 7.11 is just a concrete spelling-out of the formal descent theory (with a section, no less) as indicated in Grothendieck's "Crystals and de Rham Cohomology" in "Dix Exposés."

Remark 8.10 (an afterthought). Of course when $S = \mathrm{Spec}(\underline{\underline{C}})$, $X/\underline{\underline{C}}$ proper and smooth, $Y \hookrightarrow X$ a divisor with normal crossings, we have isomorphisms

8.10.1 $$\begin{aligned} &H^i_{DR}(X/\underline{\underline{C}}) \simeq H^i(X^{anal}, \underline{\underline{C}}) \\ &H^i_{DR}(X/\underline{\underline{C}}(\log Y)) \simeq H^i_{DR}(X-Y/\underline{\underline{C}}) \simeq H^i(X^{anal} - Y^{anal}, \underline{\underline{C}}) \end{aligned}$$

For S any scheme of characteristic zero, if $\pi : X \longrightarrow S$ is proper and smooth, $i : Y \hookrightarrow X$ the inclusion of a divisor with normal crossings relative to S, $j : X - Y \hookrightarrow X$ the inclusion of its complement, then the canonical morphism of complexes of sheaves on X

8.10.2 $$\Omega^{\bullet}_{X/S}(\log Y) \longrightarrow j_* \Omega^{\bullet}_{X-Y/S}$$

is a quasi-isomorphism (i.e., an isomorphism on cohomology sheaves) (cf. Atiyah-Hodge [ 0 ]), from which it follows that the maps deduced by applying the $\underline{\underline{R}}^i \pi_*$ to 7.12.2

8.10.3 $$H^i_{DR}(X/S(\log Y)) \longrightarrow H^i_{DR}(X-Y/S)$$

are <u>isomorphisms</u> of sheaves on $S$.

## 9. <u>Nilpotence over a global base</u>

9.0 Let $R$ be an integral domain which is finitely generated (as a ring) over $\mathbb{Z}$, and whose field of fractions has characteristic zero. Let $T = \operatorname{Spec}(R)$; we call $T$ a "<u>global affine variety.</u>" Let $f : S \longrightarrow T$ be a smooth morphism.

Let $p$ be prime number which is <u>not</u> invertible on $S$. This excludes a finite set of primes. Put

9.0.1 $$T \otimes \mathbb{F}_p = \operatorname{Spec}(R/pR) = \operatorname{Spec}(R \otimes_{\mathbb{Z}} \mathbb{F}_p)$$

and

9.0.2 $$S \otimes \mathbb{F}_p = S \times_{\mathbb{Z}} \mathbb{F}_p \ .$$

We have the diagram (in which all squares are cartesian)

9.0.3 $$\begin{tikzcd} S \otimes \mathbb{F}_p \arrow[r] \arrow[d] & S \arrow[d] \\ T \otimes \mathbb{F}_p \arrow[r] \arrow[d] & T \arrow[d] \\ \operatorname{Spec}(\mathbb{F}_p) \arrow[r] & \operatorname{Spec}(\mathbb{Z}) \end{tikzcd}$$

9.1 Let $(M, \nabla)$ be an object of $\mathrm{MIC}(S/T)$, with $M$ locally free of finite rank on $S$. Taking its inverse image (cf. 1.1.3) in $\mathrm{MIC}(S\otimes\mathbb{F}_p/T\otimes\mathbb{F}_p)$, which we denote $(M\otimes\mathbb{F}_p, \nabla\otimes\mathbb{F}_p)$, we may ask whether or not it is nilpotent, and, if nilpotent, then nilpotent of what exponent?

We will say that $(M, \nabla)$ is <u>globally nilpotent on</u> $S/T$ if, for every prime $p$ which is not invertible on $S$, we have

9.1.0 $$(M\otimes\mathbb{F}_p, \nabla\otimes\mathbb{F}_p) \in \mathrm{Nilp}(S\otimes\mathbb{F}_p/T\otimes\mathbb{F}_p) .$$

Let $\nu$ be an integer, $\nu \geq 1$. We will say that $(M, \nabla)$ is <u>globally nilpotent of exponent</u> $\nu$ on $S/T$, if, for every prime $p$ which is not invertible on $S$, we have

9.1.1 $$(M\otimes\mathbb{F}_p, \nabla\otimes\mathbb{F}_p) \in \mathrm{Nilp}^{\nu}(S\otimes\mathbb{F}_p/T\otimes\mathbb{F}_p) .$$

Clearly we have

<u>Proposition</u> 9.2. Let $f : S \longrightarrow T$ and $f' : S' \longrightarrow T'$ be smooth morphisms, with $T$ and $T'$ global affine varieties (cf. 9.0), and suppose given a commutative diagram of morphisms

9.2.0 $$\begin{tikzcd} S' \arrow[r, "g"] \arrow[d, "f'"] & S \arrow[d, "f"] \\ T' \arrow[r, "h"] & T \end{tikzcd}$$

Let $(M, \nabla)$ be an object of $\mathrm{MIC}(S/T)$, with $M$ locally free of finite rank on $S$. Then

9.2.1 If $(M, \nabla)$ is globally nilpotent on $S/T$, then its inverse image $(g, h)^*(M, \nabla)$ is globablly nilpotent on $S'/T'$.

9.2.2 If $(M, \nabla)$ is globally nilpotent of exponent $\nu$ on $S/T$, then its inverse image $(g, h)^*(M, \nabla)$ is globablly nilpotent of exponent $\nu$ on $S'/T'$.

We also have the self evident

<u>Proposition</u> 9.3. Let $T$ be a global affine variety, $f : S \longrightarrow T$ a smooth morphism, and $g : S' \longrightarrow S$ a proper étale morphism. Let $(M, \nabla)$ be an object of $\mathrm{MIC}(S'/T)$, with $M$ locally free of finite rank on $S'$. Then

9.3.0 $(M, \nabla)$ is globally nilpotent on $S'/T$ if and only if $(g_*M, \nabla)$ is globally nilpotent on $S/T$.

9.3.1 $(M, \nabla)$ is globally nilpotent of exponent $\nu$ on $S'/T$ if and only if

$(g_*M, \nabla)$ is globally nilpotent of exponent $\nu$ on $S/T$.

10. Global nilpotence of de Rham cohomology

Putting together sections 7, 8, and 9, we find

Theorem 10.0. Let $T$ be a global affine variety (cf. 9.0), $f : S \longrightarrow T$ a smooth morphism, with $S$ connected, and $\pi : X \longrightarrow S$ a proper and smooth morphism.

Suppose that each of the coherent sheaves on $S$ (cf. 8.2)

$$H^{p,q}_{\mathrm{Hodge}}(X/S) \qquad p, q \geq 0$$
$$H^n_{\mathrm{DR}}(X/S) \qquad n \geq 0$$

is locally free on $S$ (a hypothesis which is always verified over a non-empty open subset of $S$, cf. 8.4).

For each integer $i \geq 0$, let $h(i)$ be the number of integers $a$ such that $H^{a,i-a}_{\mathrm{Hodge}}(X/S)$ is non-zero. (Thus $h(i)$ is the number of non-zero groups $H^{i-a}(X_s, \Omega^a_{X_s/\underline{\underline{C}}})$ which occur in the Hodge decomposition of the i'th cohomology group the fibre $X_s$ of $\pi$ over any $\underline{\underline{C}}$-valued point of $S$.)

Then for each integer $i \geq 0$, the locally free sheaf $H^i_{\mathrm{DR}}(X/S)$, with the Gauss-Manin connection, is globally nilpotent of exponent $h(i)$ on $S/T$.

Theorem 10.0 (log Y). Let $T$ be a global affine variety, $f : S \longrightarrow T$ a smooth morphism, with $S$ connected, $\pi : X \longrightarrow S$ a proper and smooth morphism, and $i : Y \hookrightarrow X$ the inclusion of a divisor with normal crossings relative to $S$. Suppose that each of the coherent sheaves (cf. 8.5)

$$H^{p,q}_{\mathrm{Hodge}}(X/S(\log Y)) \qquad p, q \geq 0$$
$$H^n_{\mathrm{DR}}(X/S(\log Y)) \qquad n \geq 0$$

is locally free on $S$ (a hypothesis always verified on a non-empty open subset of $S$).

For each integer $i \geq 0$, let $h_Y(i)$ be the number of integers $a$ with $H^{a,i-a}_{\mathrm{Hodge}}(X/S(\log Y))$ non-zero. Then for each integer $i \geq 0$, the locally free sheaf $H^i_{DR}(X/S(\log Y))$, with the Gauss-Manin connection, is globally nilpotent of exponent $h_Y(i)$ on $S/T$.

## 11. Classical theory of regular singular points

11.0 Let $k$ be a field of characteristic zero, and $K$ a field of functions in one variable over $k$, i.e., $K$ is the function field of a projective, smooth, absolutely irreducible curve over $k$.

Let $W$ be a finite-dimensional vector space over $K$. A $k$-connection $\nabla$ on $W$ is an additive mapping

11.0.1
$$\nabla : W \longrightarrow \Omega^1_{K/k} \otimes_K W$$

which satisfies

11.0.2
$$\nabla(fw) = df \otimes w + f\nabla(w)$$

for $f \in K$, $w \in W$. Equivalently, $\nabla$ "is" a $K$-linear mapping

11.0.3
$$\nabla : \mathrm{Der}(K/k) \longrightarrow \mathrm{End}_k(W)$$

such that

11.0.4
$$(\nabla(D))(fw) = D(f)w + f(\nabla(D))(w)$$

for $D \in \mathrm{Der}(K/k)$, $f \in K$, and $w \in W$.

The connection is necessarily integrable, i.e., compatible with brackets, since $\Omega^2_{K/k} = 0$.

If $(W, \nabla)$ and $(W', \nabla')$ are two such objects, a horizontal morphism $\varphi$ from $(W, \nabla)$ to $(W', \nabla')$ is a $K$-linear mapping of $W$ to $W'$ which is compatible with the connections, i.e.,

11.0.5
$$\varphi(\nabla(D)(w)) = \nabla'(D')(\varphi(w)) \ .$$

The objects $(W, \nabla)$ as above, with morphisms the horizontal ones, form an abelian category $MC(K/k)$. (NB - this notation is slightly misleading, since, unlike what was required in the geometrical case of a smooth morphism $S \longrightarrow k$, we are requiring that $W$ be finite dimensional over $K$ (i.e., <u>coherent</u>), rather than just quasicoherent.) Just as in 1.1, $MC(K/k)$ has an internal <u>Hom</u> and a tensor product.

11.1 Let $\mathcal{Y}$ be a place of $K/k$ (i.e., a closed point of the projective and smooth curve over $k$ whose function field is $K$), $\mathcal{O}_{\mathcal{Y}}$ its local ring, $m_{\mathcal{Y}}$ its maximal ideal, $\mathrm{ord}_{\mathcal{Y}} : K \longrightarrow \underline{\underline{Z}} \cup \{\infty\}$ associated valuation "order of zero at $\mathcal{Y}$." Thus

11.1.0 $$\mathcal{O}_{\mathcal{Y}} = \{f \in K \mid \mathrm{ord}_{\mathcal{Y}}(f) \geq 0\}$$

11.1.1 $$m_{\mathcal{Y}} = \{f \in K \mid \mathrm{ord}_{\mathcal{Y}}(f) \geq 1\} \ .$$

Let $\mathrm{Der}_{\mathcal{Y}}(K/k)$ denote the $\mathcal{O}_{\mathcal{Y}}$-submodule of $\mathrm{Der}(K/k)$

11.1.2 $$\mathrm{Der}_{\mathcal{Y}}(K/k) = \{D \in \mathrm{Der}(K/k) \mid D(m_{\mathcal{Y}}) \subset m_{\mathcal{Y}}\} \ .$$

In terms of a uniformizing parameter $h$ at $\mathcal{Y}$, $\mathrm{Der}_{\mathcal{Y}}(K/k)$ is the free $\mathcal{O}_{\mathcal{Y}}$ module with basis $h\frac{d}{dh}$. <u>In fact, for any function</u> $y \in K^*$, <u>which is not a unit at</u> $\mathcal{Y}$, $y\frac{d}{dy}$ <u>is an</u> $\mathcal{O}_{\mathcal{Y}}$<u>-basis for</u> $\mathrm{Der}_{\mathcal{Y}}(K/k)$.

11.2 Let $(W, \nabla)$ be an object of $MC(K/k)$. We say that $(W, \nabla)$ has a <u>regular singular point at</u> $\mathcal{Y}$ if there exists an $\mathcal{O}_{\mathcal{Y}}$-lattice $W_{\mathcal{Y}}$ of $W$ (i.e., a subgroup of $W$ which is a free $\mathcal{O}_{\mathcal{Y}}$-module of rank $= \dim_K(W)$) such that

11.2.0 $$\mathrm{Der}_{\mathcal{Y}}(K/k)(W_{\mathcal{Y}}) \subset W_{\mathcal{Y}} \ .$$

In more concrete terms, we ask if there is a base $\underline{\underline{e}} = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$ of $W$ as $K$-space, such that

11.2.1 $$\nabla(h\frac{d}{dh})\underline{\underline{e}} = B\underline{\underline{e}} \quad \text{with} \quad B \in M_n(\mathcal{O}_{\mathcal{Y}})$$

for some (and hence for any) uniformizing parameter $h$ at $\mathcal{Y}$.

<u>Remark</u> 11.2.2. If $\mathcal{Y}$ is a regular singular point of $(W, \nabla)$, there is no

unicity in the lattice $W_{\mathscr{Y}}$ which "works" in 11.2.0. We will return to this question later (cf. especially 12.0 and 12.5).

Proposition 11.3. Suppose

11.3.0 $$0 \longrightarrow (V, \nabla') \longrightarrow (W, \nabla) \longrightarrow (U, \nabla'') \longrightarrow 0$$

is an exact sequence in $MC(K/k)$. Then $(W, \nabla)$ has a regular singular point at $\mathscr{Y}$ if and only if both $(V, \nabla')$ and $(U, \nabla'')$ have a regular singular point at $\mathscr{Y}$.

Proof. Suppose first that $(V, \nabla')$ and $(U, \nabla'')$ have regular singular points at $\mathscr{Y}$. This means we can choose a $K$-base of $W$ of the form

11.3.1 $$\begin{pmatrix} \underline{\underline{e}} \\ \underline{\underline{f}} \end{pmatrix} = \begin{pmatrix} e_1 \\ \vdots \\ e_{n1} \\ f_1 \\ \vdots \\ f_{n2} \end{pmatrix}$$

of $W$ so that $\underline{\underline{e}}$ is a base of $V$, and $\underline{\underline{f}}$ projects to a base of $U$, in terms of which the connection is expressed

11.3.2 $$\nabla(h\frac{d}{dh}) \begin{pmatrix} \underline{\underline{e}} \\ \underline{\underline{f}} \end{pmatrix} = \begin{pmatrix} A & O \\ B & C \end{pmatrix} \begin{pmatrix} \underline{\underline{e}} \\ \underline{\underline{f}} \end{pmatrix}$$

with $A \in M_{n_1}(\mathcal{O}_{\mathscr{Y}})$, $C \in M_{n_2}(\mathcal{O}_{\mathscr{Y}})$. The problem is that $B$ may not be holomorphic at $\mathscr{Y}$. But for any integer $\nu$, we readily compute

11.3.3 $$\nabla(h\frac{d}{dh}) \begin{pmatrix} \underline{\underline{e}} \\ h^{\nu}\underline{\underline{f}} \end{pmatrix} = \begin{pmatrix} A & O \\ h^{\nu}B & C+\nu \end{pmatrix} \begin{pmatrix} \underline{\underline{e}} \\ h^{\nu}\underline{\underline{f}} \end{pmatrix},$$

and for $\nu >> 0$, we have $h^{\nu}B$ holomorphic at $\mathscr{Y}$.

Conversely, suppose that $(W, \nabla)$ has a regular singular point at $\mathscr{Y}$, and let $W_{\mathscr{Y}}$ be an $\mathcal{O}_{\mathscr{Y}}$-lattice in $W$ which "works" for 11.2.0. Then $V \cap W_{\mathscr{Y}}$ is an $\mathcal{O}_{\mathscr{Y}}$-lattice in $V$ (elementary divisors) which works for 11.2.0. Similarly, $U \cap$ (image of $W_{\mathscr{Y}}$ in $U$) is an $\mathcal{O}_{\mathscr{Y}}$-lattice in $U$ which works for 11.2.0.

Remark 11.3.4. The full abelian subcategory of $MC(K/k)$ consisting of objects with a regular singular point at $\mathscr{Y}$ is stable under the internal Hom and tensor product of $MC(K/k)$.

11.4. Let us say that $(W, \nabla)$ is cyclic if there is a vector $w \in W$, such that for some (and hence for any) non-zero derivation $D \in Der(K/k)$, the vectors $w$, $\nabla(D)w$, $(\nabla(D))^2 w$, $(\nabla(D))^3 w$, ... span $W$ over $K$. (We should remark that for $w \in W$, the $K$-span of the vectors $w$, $\nabla(D((w)$, $(\nabla(D))^2(w)$, ... is independent of the choice of non-zero $D \in Der(K/k)$, and is thus a $Der(K/k)$ stable subspace of $W$.)

Corollary 11.5. Let $(W, \nabla)$ be an object of $MC(K/k)$. Then $(W, \nabla)$ has a regular singular point at $\mathscr{Y}$ if and only if every cyclic subobject of $(W, \nabla)$ has a regular singular point at $\mathscr{Y}$.

Proof. "Only if" by 11.3, "if" because $(W, \nabla)$ is a quotient of a direct sum of finitely many of its cyclic subobjects (so apply 11.3 again).

11.6 Let $(W, \nabla)$ be an object of $MC(K/k)$, and $W_{\mathscr{Y}}$ an $\mathcal{O}_{\mathscr{Y}}$-lattice in $W$. We say that $(W, \nabla)$ satisfies "Jurkat's Estimate" (J) at $\mathscr{Y}$ for the lattice $W_{\mathscr{Y}}$ if there is an integer $\mu$, such that, for every integer $j \geq 1$ and every $j$-tuple $D_1, \ldots, D_j \in Der_{\mathscr{Y}}(K/k)$, we have (denoting by $h$ a uniformizing parameter at $\mathscr{Y}$)

11.6.0 $$\nabla(D_1)\cdot \nabla(D_2) \ldots \nabla(D_j)(W_{\mathscr{Y}}) \subset h^{\mu}(W_{\mathscr{Y}}) .$$

Let us reformulate this condition. Let $D_o$ be an $\mathcal{O}_{\mathscr{Y}}$ base of $Der_{\mathscr{Y}}(K/k)$. One quickly checks by induction that for any $D_1, \ldots, D_j \in Der_{\mathscr{Y}}(K/k)$, one has

$$\nabla(D_1)\cdot\nabla(D_2)\cdot\ldots\cdot\nabla(D_j) = \sum_{\nu=0}^{j} a_\nu(\nabla(D_0))^\nu$$

with $a_0, \ldots, a_j \in \mathcal{O}_{\mathcal{Y}}$. Thus 11.6.0 holds for all $j \geq 1$ if and only if, for some $\mathcal{O}_{\mathcal{Y}}$-base $D_0$ of $\mathrm{Der}_{\mathcal{Y}}(K/k)$, one has

11.6.0 bis $\quad (\nabla(D_0))^j(W_{\mathcal{Y}}) \subset h^\mu(W_{\mathcal{Y}})$ for all $j \geq 1$.

In terms of an $\mathcal{O}_{\mathcal{Y}}$-base $\underline{\underline{e}}$ of $W_{\mathcal{Y}}$, the condition 11.6.0 bis may be expressed as follows. For each $j \geq 1$, define a matrix $B_j \in M_n(K)$ by

11.6.1 $\quad (\nabla(D_0))^j\underline{\underline{e}} = B_j\underline{\underline{e}}$.

Then 11.6.0 bis is equivalent to

11.6.2 $\quad \mathrm{ord}_{\mathcal{Y}}(B_j) \geq \mu$ for all $j \geq 1$.

In applications we will speak of a K-base $\underline{\underline{e}}$ of W as satisfying (J) at $\mathcal{Y}$, rather than of the lattice given by its $\mathcal{O}_{\mathcal{Y}}$-span. Also, we will usually take as $\mathcal{O}_{\mathcal{Y}}$-base of $\mathrm{Der}_{\mathcal{Y}}(K/k)$ a derivation $h\frac{d}{dh}$, h a uniformizing parameter at $\mathcal{Y}$, although we may occasionally use $y\frac{d}{dy}$ as base, for a non-zero y which is a <u>non-unit</u> at $\mathcal{Y}$.

<u>Proposition</u> 11.6.3. If $(W, \nabla)$ satisfies (J) at $\mathcal{Y}$ for one base, it satisfies it for every base.

<u>Proof.</u> Let $\underline{\underline{e}}$ be a base of W and $\mu$ an integer such that, for all $j \geq 1$

11.6.3.0 $\quad (\nabla(h\frac{d}{dh}))^j\underline{\underline{e}} = B_j\underline{\underline{e}},\ \mathrm{ord}_{\mathcal{Y}}(B_j) \geq \mu$.

Let $\underline{\underline{f}}$ be another base of W, so that

11.6.3.1 $\quad \underline{\underline{f}} = A\underline{\underline{e}},\ \underline{\underline{e}} = A^{-1}\underline{\underline{f}},\ A \in GL_n(K)$.

We define the sequence of matrices $C_j$ by

11.6.3.2 $\quad (\nabla(h\frac{d}{dh}))^j\underline{\underline{f}} = C_j\underline{\underline{f}}$.

We easily calculate the $C_j$ in terms of the $B_j$, by using Leibniz's rule.

11.6.3.3 $$(\nabla(h\frac{d}{dh}))^j \underline{\underline{f}} = (\nabla(h\frac{d}{dh}))^j A\cdot \underline{\underline{e}})$$
$$= \Sigma_{i=o}^{j}\binom{j}{i}[(h\frac{d}{dh})^{j-i}(A)]\cdot B_i \underline{\underline{e}}$$

whence

11.6.3.4 $$C_j = \Sigma_{i=o}^{j}\binom{j}{i}[(h\frac{d}{dh})^{j-i}(A)]\cdot B_i A^{-1} \quad .$$

Since for any element $f \in K$ we have

11.6.3.5 $$\mathrm{ord}_{\mathscr{Y}}(h\frac{df}{dh}) \geq \mathrm{ord}_{\mathscr{Y}}(f) \quad ,$$

11.6.3.4 gives immediately

11.6.3.6 $$\mathrm{ord}_{\mathscr{Y}}(C_j) \geq \min_{i=o}^{j}(\mathrm{ord}_{\mathscr{Y}}(A) + \mathrm{ord}_{\mathscr{Y}}(B_i) + \mathrm{ord}_{\mathscr{Y}}(A^{-1}))$$

i.e.,

11.6.3.7 $$\mathrm{ord}_{\mathscr{Y}}(C_j) \geq \mu + \mathrm{ord}_{\mathscr{Y}}(A) + \mathrm{ord}_{\mathscr{Y}}(A^{-1}) \quad .$$

<u>Proposition</u> 11.7. If $(W, \nabla)$ has a regular singular point at $\mathscr{Y}$, then it satisfies (J) at $\mathscr{Y}$.

<u>Proof</u>. Indeed in a suitable base $\underline{\underline{e}}$, we have

11.7.0 $$\nabla(h\frac{d}{dh})\underline{\underline{e}} = B\underline{\underline{e}},\ B \in M_n(\mathcal{O}_{\mathscr{Y}}) \quad .$$

As the $B_j$ are formed successively according to the rule

11.7.1 $$B_{j+1} = h\frac{d}{dh}(B_j) + B_j B$$

we see that each $B_j$ is holomorphic at $\mathscr{Y}$, i.e., $\mathrm{ord}_{\mathscr{Y}}(B_j) \geq 0$.

11.8 Let $a$ be a positive integer. In the extension field $K(h^{1/a})$ of $K$, there is a unique place $\mathscr{Y}^{1/a}$, which extends $\mathscr{Y}$, and $h^{1/a}$ is a uniformizing parameter there.

Proposition 11.8.1. Let $(W, \nabla)$ be an object of $MC(K/k)$. Then $(W, \nabla)$ satisfies (J) at $\mathscr{Y}$ if and only if its inverse image in $MC(K(h^{1/a})/k)$ satisfies (J) at $\mathscr{Y}^{1/a}$.

Proof. Calculate the matrices $B_j$ of 11.6.1, using a K-base $\underline{\underline{e}}$ of W, and $h\frac{d}{dh}$ as $\mathcal{O}_{\mathscr{Y}}$-base, for both $(W, \nabla)$ and its inverse image. They are the same matrices.

Theorem 11.9 (Fuchs [ 8 ], Turrittin [ 34 ], Lutz [ 24 ]). Let $(W, \nabla)$ be a cyclic object of $MC(K/k)$, $w \in W$ a cyclic vector, $\mathscr{Y}$ a place of $K/k$, h a uniformizing parameter at $\mathscr{Y}$, $n = \dim_K(W)$. The following conditions are equivalent.

11.9.1 $(W, \nabla)$ does not have a regular singular point at $\mathscr{Y}$.

11.9.2 In terms of the base

$$\underline{\underline{e}} = \begin{pmatrix} w \\ \nabla(h\frac{d}{dh})(w) \\ \vdots \\ (\nabla(h\frac{d}{dh}))^{n-1}(w) \end{pmatrix}$$

of W, the connection is expressed

$$\nabla(h\frac{d}{dh})\underline{\underline{e}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \bigcirc \\ & \bigcirc & & \ddots & 1 \\ & & \cdots & 0 & 1 \\ f_0 & f_1 & & \cdots & f_{n-1} \end{pmatrix} \underline{\underline{e}} \qquad 11.9.2.0$$

and, for some value of i, we have $\operatorname{ord}_{\mathscr{Y}}(f_i) < 0$.

11.9.3 For every multiple a of n!, the inverse image of $(W, \nabla)$ in $MC(K(h^{1/a})/k)$ admits a base $\underline{\underline{f}}$ in terms of which the connection is expressed (putting $t = h^{1/a}$)

11.9.3.0 $$\nabla(t\frac{d}{dt})\underline{\underline{f}} = B\underline{\underline{f}}$$

such that, for an integer $\nu \geq 1$, we have

11.9.3.1 $$B = t^{-\nu}B_{-\nu}, \quad B_{-\nu} \in M_n(\mathcal{O}_{\mathcal{Y}^{1/a}}) ,$$

and the image of $B_{-\nu}$ in $M_n(k(\mathcal{Y}))$ (i.e., the value of $B_{-\nu}$ at $\mathcal{Y}^{1/a}$) is <u>not</u> nilpotent.

11.9.4 For every multiple $a$ of $n!$, the inverse image of $(W, \nabla)$ in $MC(K(h^{1/a})/k)$ does <u>not</u> satisfy (J) at $\mathcal{Y}^{1/a}$ (using $h^{1/a}$ as parameter).

11.9.5 $(W, \nabla)$ does not satisfy (J) at $\mathcal{Y}$.

<u>Proof</u> (11.9.1 $\Longrightarrow$ 11.9.2) by definition of a regular singular point

(11.9.2 $\Longrightarrow$ 11.9.3). After the base change $K \longrightarrow K(t)$, $t^a = h$, we have, in terms of the given base $\underline{\underline{e}}$,

11.9.6 $$\frac{1}{a}\nabla(t\frac{d}{dt})\underline{\underline{e}} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ f_0 & & \cdots & & f_{n-1} \end{pmatrix}\underline{\underline{e}} = C\underline{\underline{e}} .$$

By assumption, we have $\mathrm{ord}_{\mathcal{Y}^{1/a}}(f_i) < 0$ for at least one value of $i$, while for every value of $j$, the integer $\mathrm{ord}_{\mathcal{Y}^{1/a}}(f_j)$ is divisible by $n!$. Consider the strictly positive integer $\nu$ defined by

11.9.7 $$\nu = \max_{j=0}^{n-1} (-\,\mathrm{ord}_{\mathcal{Y}^{1/a}}(f_j)/n-j) .$$

Consider the basis $\underline{\underline{f}}$ of $W \otimes_K K(t)$, $t = h^{1/a}$, given by

11.9.8 $$\underline{\underline{f}} = \begin{pmatrix} 1 & & & & \\ & t^{\nu} & & & \\ & & t^{2\nu} & & \\ & & & \ddots & \\ & & & & t^{(n-1)\nu} \end{pmatrix}\underline{\underline{e}} = A\underline{\underline{e}} .$$

We readily calculate (cf. 11.9.6)

11.9.9 $$\nabla(t\frac{d}{dt})\underline{\underline{f}} = \nabla(t\frac{d}{dt})(A\underline{\underline{e}}) = [(t\frac{d}{dt})(A)]\cdot \underline{\underline{e}} + A\cdot \nabla(t\frac{d}{dt})(\underline{\underline{e}})$$
$$= ([t\frac{d}{dt}(A)]A^{-1} + ACA^{-1})\underline{\underline{f}} = B\underline{\underline{f}}$$

and an immediate computation then yields

11.9.10 $$\frac{1}{a}\nabla(t\frac{d}{dt})\underline{\underline{f}} = B\underline{\underline{f}}$$

with

$$B = \begin{pmatrix} 0 & & & & \\ & \nu & & & \\ & & 2\nu & & \\ & & & \ddots & \\ & & & & (n-1)\nu \end{pmatrix} + \begin{pmatrix} 0 & t^{-\nu} & & & & \vdots \\ \vdots & 0 & \ddots & & & \vdots \\ \vdots & \vdots & & t^{-\nu} & & \vdots \\ \vdots & \vdots & & & \ddots & 0 \\ 0 & 0 & & & & t^{-\nu} \\ t^{(n-1)\nu}f_0 & t^{(n-2)\nu}f_1 & \cdots & t^{(n-i-1)\nu}f_i & \cdots & f_{n-1} \end{pmatrix}$$

By definition of $\nu$, (11.9.7), we have

11.9.11 $$\operatorname{ord}_{\mathcal{Y}^{1/a}}(t^{(n-j-1)\nu}f_j) = (n-j-1)\nu + \operatorname{ord}_{\mathcal{Y}^{1/a}}(f_j) \geq -\nu$$

and, for at least one value of i, we have

$$\operatorname{ord}_{\mathcal{Y}^{1/a}}(t^{(n-i-1)\nu}f_i) = -\nu \quad .$$

Thus we may write $B = t^{-\nu}B_{-\nu}$, $B_{-\nu} \in M_n(\mathcal{O}_{\mathcal{Y}^{1/a}})$. In fact, putting

11.9.12 $$g_j = t^{n-1-j}f_j, \; j = 0, \ldots, n-1$$

so that

11.9.13 $$\begin{cases} g_j \in \mathcal{O}_{\mathcal{Y}^{1/a}} & \text{for all } j \\ g_i \notin m_{\mathcal{Y}^{1/a}} & \text{for some } i \end{cases}$$

we have

$$\text{11.9.14} \qquad B_{-\nu} \equiv \begin{pmatrix} 0 & 1 & \cdots & & 0 \\ & 0 & \ddots & 0 & \vdots \\ 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ g_0 & g_1 & \cdots & & g_{n-1} \end{pmatrix} \text{ modulo } \mathfrak{y}^{1/a}$$

which is <u>not</u> nilpotent modulo $\mathfrak{y}^{1/a}$. Indeed, we have

$$\text{11.9.15} \qquad \det(TI_n - B_{-\nu}) \equiv T^n - \Sigma_{i=0}^{n-1} g_i T^i, \text{ modulo } \mathfrak{y}^{1/a}$$

so that $B_{-\nu}$ is nilpotent modulo $\mathfrak{y}^{1/a}$ if and only if each $g_j \in m_{\mathfrak{y}^{1/a}}$, which (11.9.13) is not the case. This concludes the proof that 11.9.2 $\Longrightarrow$ 11.9.3.

(11.9.3 $\Longrightarrow$ 11.9.4) We use the base $\underline{\underline{f}}$ to test the estimate. Writing

$$\text{11.9.16} \qquad (\nabla(t\frac{d}{dt}))^j \underline{\underline{f}} = B_j \underline{\underline{f}}$$

we have

$$\text{11.9.17} \qquad B_{j+1} = (t\frac{d}{dt})B_j + B_j B$$

and one checks immediately (despite the confusing notation) that

$$\text{11.9.18} \qquad \begin{aligned} &B_j = t^{-\nu j} B_{-\nu j} \quad \text{with } B_{-\nu j} \in M_n(\mathcal{O}_{\mathfrak{y}^{1/a}}) \\ &\text{and } B_{-\nu j} \equiv (B_{-\nu})^j \text{ modulo } \mathfrak{y}^{1/a} \end{aligned}$$

so that

$$\text{11.9.19} \qquad \operatorname{ord}_{\mathfrak{y}^{1/a}}(B_j) = -\nu j$$

so that (J) is not satisfied. To conclude the proof of 11.9, we note that 11.9.4 $\Longrightarrow$ 11.9.5 by 11.8.1, and 11.9.5 $\Longrightarrow$ 11.9.1 by 11.7.

<u>Corollary</u> 11.9.20 (Manin [25 ]). Let $(W, \nabla)$ be an object of $MC(K/k)$, $\mathfrak{y}$ a place of $K/k$. Then $(W, \nabla)$ has a regular singular point at $\mathfrak{y}$

if and only if, for every $w \in W$, the smallest $\mathcal{O}_{\mathscr{Y}}$-module stable under $\mathrm{Der}_{\mathscr{Y}}(K/k)$ (cf. 11.1.2) and containing $w$ (if $h$ is a uniformizing parameter at $\mathscr{Y}$, this is the $\mathcal{O}_{\mathscr{Y}}$-span of $w$, $\nabla(h\frac{d}{dh})(w)$, ..., $(\nabla(h\frac{d}{dh}))^i(w)$, ...) is of finite type over $\mathcal{O}_{\mathscr{Y}}$.

Proof. If $(W, \nabla)$ has a regular singular point at $\mathscr{Y}$, then for any element $w \in W$, the K-span of the elements $(\nabla(h\frac{d}{dh}))^i(w)$, $i \geq 0$, "is" a cyclic object of $MC(K/k)$ having a regular singular point at $\mathscr{Y}$. Letting $n_1$ be the K-dimension of this span, we see by 11.9 that the $\mathcal{O}_{\mathscr{Y}}$-span of the elements $(\nabla(h\frac{d}{dh}))^i(w)$, $i \geq 0$, is free of rank $n_1$ over $\mathcal{O}_{\mathscr{Y}}$. (In fact, the elements $(\nabla(h\frac{d}{dh}))^i(w)$, $i = 0, \ldots, n_1-1$ form an $\mathcal{O}_{\mathscr{Y}}$-base.).

Conversely, suppose that for every $w \in W$, the $\mathcal{O}_{\mathscr{Y}}$ span of the elements $(\nabla(h\frac{d}{dh}))^i(w)$, $i \geq 0$, is of finite type. This means that every $w \in W$ is annihilated by monic polynomial in $\nabla(h\frac{d}{dh})$ whose coefficients are in $\mathcal{O}_{\mathscr{Y}}$, hence that the K-span of the elements $(\nabla(h\frac{d}{dh}))^i(w)$, $i \geq 0$, is a quotient in $MC(K/k)$ of an object with a regular singular point at $\mathscr{Y}$. We conclude, by 11.5, that $(W, \nabla)$ has a regular singular point at $\mathscr{Y}$.

Theorem 11.10 (Turrittin). Let $(W, \nabla)$ be an object of $MC(K/k)$, $\mathscr{Y}$ a place of $K/k$, $h$ a uniformizing parameter at $\mathscr{Y}$, $n = \dim_K(W)$. The following conditions are equivalent.

11.10.1 $(W, \nabla)$ does not have a regular singular point at $\mathscr{Y}$.

(11.10.2) For every integer a multiple of $n!$, there exists a base $\underline{\underline{f}}$ of $W \otimes_K K(h^{1/a})$ in terms of which the connection is expressed (putting $t = h^{1/a}$)

$$(11.10.2.0)\quad \begin{cases} \nabla(t\frac{d}{dt})\underline{\underline{f}} = B\underline{\underline{f}} \\ B = t^{-\nu}B_{-\nu},\ \nu \text{ an integer} \geq 1, \text{ and} \\ B_{-\nu} \in M_n(\mathcal{O}_{\mathscr{Y}^{1/a}}) \text{ has } \underline{\text{non-nilpotent}} \text{ image in } M_n(k(\mathscr{Y})) . \end{cases}$$

11.10.3 The estimate (J) is not satisfied at $\mathscr{Y}$.

<u>Proof.</u> The implications 11.10.2 $\Longrightarrow$ 11.10.3 and 11.10.3 $\Longrightarrow$ 11.10.1 are obvious, using 11.9.16-19 and 11.8.1 for the first, and 11.7 for the second. We now turn to the serious part of the proof.

(11.10.1) $\Longrightarrow$ (11.10.2) We proceed by induction. (If $n = 1$, we are in the cyclic case.) If $(W, \nabla)$ has no proper non-zero subobjects, it is necessarily cyclic. If $(W, \nabla)$ has a non-trivial subobject $(V, \nabla')$, we have a short exact sequence in $MC(K/k)$

$$0 \longrightarrow (V, \nabla') \longrightarrow (W, \nabla) \longrightarrow (U, \nabla'') \longrightarrow 0$$

with $n_1 = \dim_K(V) < n$, $n_2 = \dim_K(U) < n$.

By 11.3.0, either $(V, \nabla')$ <u>or</u> $(U, \nabla'')$ does not have a regular singular point at $\mathcal{Y}$. So by induction, there exists a basis of $W \otimes_K K(h^{1/a})$ of the form $\binom{\underline{\underline{e}}}{\underline{\underline{f}}}$, where $\underline{\underline{e}}$ is a basis of $V \otimes_K K(h^{1/a})$, and where $\underline{\underline{f}}$ projects to a basis of $U \otimes_K K(h^{1/a})$, in terms of which the connection is expressed (putting $t = h^{1/a}$)

11.10.4 $$\nabla\left(t\frac{d}{dt}\right)\binom{\underline{\underline{e}}}{\underline{\underline{f}}} = \begin{pmatrix} A & O \\ B & C \end{pmatrix} \binom{\underline{\underline{e}}}{\underline{\underline{f}}}$$

such that

11.10.5 $$A = t^{-\nu} A_{-\nu} \quad \text{for an integer } \nu \geq 0$$
$$A_{-\nu} \in M_{n_1}(\mathcal{O}_{\mathcal{Y}^{1/a}})$$

if $\nu > 0$, $A_{-\nu}$ has <u>non-nilpotent</u> image in $M_{n_1}(k(\mathcal{Y}))$ and

11.10.6 $$C = t^{-\tau} C_{-\tau} \quad \text{for an integer } \tau \geq 0$$
$$C_{-\tau} \in M_{n_2}(\mathcal{O}_{\mathcal{Y}^{1/a}})$$

if $\tau > 0$, then $C_{-\tau}$ has <u>non-nilpotent</u> image in $M_{n_2}(k(\mathcal{Y}))$ and finally

11.10.7 $$\nu + \tau > 0 \ .$$

Replacing the basis $(\underline{\underline{e}}_{\underline{\underline{f}}})$ by the basis $(t^{\bar{\bar{N}}_{\underline{\underline{e}}_{\underline{\underline{f}}}}})$ for N large, we obtain the new connection matrix

$$\text{11.10.8} \qquad \nabla(t\frac{d}{dt})(t^{\bar{\bar{N}}}\underline{\underline{e}}_{\underline{\underline{f}}}) = \begin{pmatrix} A & O \\ t^N B & C+NI \end{pmatrix} (t^{\bar{\bar{N}}}\underline{\underline{e}}_{\underline{\underline{f}}})$$

with A, B, C as before (but now $t^N B$ is holomorphic at $\mathfrak{y}^{1/a}$). Clearly this connection matrix 11.10.8 has a pole at $\mathfrak{y}^{1/a}$ of order $\sup(\nu, \tau)$, and

$$t^{\sup(\nu,\tau)} \begin{pmatrix} A & O \\ t^N B & C+NI \end{pmatrix}$$

has <u>non-nilpotent</u> image in $M_n(k(\mathfrak{y}))$. This concludes the proof of Turrittin's theorem.

<u>Proposition</u> 11.11. Let

$$F \hookrightarrow K \hookrightarrow L$$

be a tower of function fields in one variable over a field k of characteristic zero, with $\deg(K/F) < \infty$ and $\deg(L/K) < \infty$. Let $\mathfrak{p}$ be a place of L/k, $\mathfrak{y}'$ the induced place of K/k, and $\mathfrak{y}''$ the induced place of F/k. Let $\mathfrak{y}'_1, \ldots, \mathfrak{y}'_r$ be <u>all</u> the places of K/k which lie over the place $\mathfrak{y}''$ of F/k.

Let $(W, \nabla)$ be an object of MC(K/k). Then:

11.11.1 $(W, \nabla)$ has a regular singular point at $\mathfrak{y}'$ if and only if the inverse image $(W \otimes_K L, \nabla_L)$ of $(W, \nabla)$ in MC(L/k) has a regular singular point at $\mathfrak{p}$.

11.11.2 The "direct image" (W as F-space, $\nabla|\mathrm{Der}(F/k)$) of $(W, \nabla)$ in MC(F/k) has a regular singular point at $\mathfrak{y}''$, if and only if $(W, \nabla)$ has a regular singular point at each place $\mathfrak{y}'_i$ of K/k which lies over $\mathfrak{y}''$.

<u>Proof</u>. We have (cf. 11.1.2

$$\text{11.11.4} \qquad \mathrm{Der}_{\mathfrak{p}}(L/k) \xleftarrow{\sim} \mathrm{Der}_{\mathfrak{y}'}(K/k) \otimes_{\mathcal{O}_{\mathfrak{y}'}} \mathcal{O}_{\mathfrak{p}}$$

and

11.11.5 $$\mathrm{Der}_{\mathcal{Y}_i'}(K/k) \xleftarrow{\sim} \mathrm{Der}_{\mathcal{Y}''}(F/k) \otimes_{\mathcal{O}_{\mathcal{Y}''}} \mathcal{O}_{\mathcal{Y}_i'}, \quad i = 1, \ldots r .$$

To prove 11.11.1, observe that if $W_{\mathcal{Y}'}$ is an $\mathcal{O}_{\mathcal{Y}'}$-lattice in $W$, stable under $\mathrm{Der}_{\mathcal{Y}'}(K/k)$, then $W_{\mathcal{Y}'} \otimes_{\mathcal{O}_{\mathcal{Y}'}} \mathcal{O}_{\mathcal{P}}$ is an $\mathcal{O}_{\mathcal{P}}$-lattice in $W \otimes_K L$, stable under $\mathrm{Der}_{\mathcal{P}}(L/k)$. To prove 11.11.2 $\Longrightarrow$ 11.11.1, observe that if $(W \otimes_K L)_{\mathcal{P}}$ is an $\mathcal{O}_{\mathcal{P}}$-lattice in $W \otimes_K L$, stable under $\mathrm{Der}_{\mathcal{P}}(L/k)$, then $W \cap (W \otimes_K L)_{\mathcal{P}}$ is an $\mathcal{O}_{\mathcal{Y}'}$-lattice in $W$ which is stable under $\mathrm{Der}_{\mathcal{Y}'}(K/k)$.

Similarly, to prove 11.11.3, note that if for $i = 1, \ldots, r$ $W_{\mathcal{Y}_i'}$ is an $\mathcal{O}_{\mathcal{Y}_i'}$-lattice in $W$, stable under $\mathrm{Der}_{\mathcal{Y}_i'}(K/k)$, then $\oplus_i W_{\mathcal{Y}_i'}$ an $\mathcal{O}_{\mathcal{Y}''}$ lattice in $W$, stable under $\mathrm{Der}_{\mathcal{Y}''}(F/k)$. To prove the converse we simply apply the criterion 11.9.20 of Manin.

Corollary 11.12. Let $K/k$ be a function field in one variable over a field $k$ of characteristic zero, $\mathcal{Y}$ a place of $K/k$, $\bar{k}$ an algebraic closure of $k$, $\bar{\mathcal{Y}}$ the induced place of $K\bar{k}/\bar{k}$, $(W, \nabla)$ an object of $MC(K/k)$, and $(W_{\bar{k}}, \nabla_{\bar{k}})$ its inverse image in $MC(K\bar{k}/\bar{k})$. Then $(W, \nabla)$ has a regular singular point at $\mathcal{Y}$ if and only if $(W_{\bar{k}}, \nabla_{\bar{k}})$ has a regular singular point at $\bar{\mathcal{Y}}$.

Proof. Use the equivalence 11.10.1 $\Longleftrightarrow$ 11.10.3, calculating with a $K$-base of $W$, and a parameter at $\mathcal{Y}$.

## 12. The Monodromy around a Regular Singular Point

We refer to the elegant paper [25] of Manin for a proof of the following theorem, which ought to be well-known.

Theorem 12.0. Let $K/k$ be a function field in one variable, with $k$ of characteristic zero. Let $\mathcal{P}$ be a place of $K/k$ which is rational, i.e. $k(\mathcal{P}) = k$. Suppose that $(W,\nabla)$ is an object of $MC(K/k)$ which has, at $\mathcal{P}$, a regular singular point. In terms of a uniformizing parameter $t$ at $\mathcal{P}$, and a basis $\underline{e}$ of an $\mathcal{O}_{\mathcal{P}}$-lattice $W_{\mathcal{P}}$ of $W$ which is stable under $\nabla(t\frac{d}{dt})$, we express the connection

12.0.1 $$\nabla(t\frac{d}{dt})\underline{e} = B\underline{e}, \qquad B \in M_n(\mathcal{O}_{\mathcal{P}}) .$$

Suppose that the matrix $B(\mathcal{P}) \in M_n(k)$ (the value of $B$ at $\mathcal{P}$, whose conjugacy class depends only on the lattice $W_{\mathcal{P}}$, not on the particular choice of a base of $W_{\mathcal{P}}$ or on the choice of the uniformizing parameter $t$) has all of its proper values in $k$. Then

12.0.2 The set of images in the additive group $k^+/\underline{\underline{Z}}$ of the proper values of $B(\mathcal{P})$ (the exponents of $(W,\nabla)$ at $\mathcal{P}$) is independent of the choice of $\nabla(t\frac{d}{dt})$-stable $\mathcal{O}_{\mathcal{P}}$-lattice $W_{\mathcal{P}}$ in $W$.

12.0.3 Fix a set-theoretic section $\varphi: k^+/\underline{\underline{Z}} \longrightarrow k^+$ of the projection mapping $k^+ \longrightarrow k^+/\underline{\underline{Z}}$. (For instance, if $k = \underline{\underline{C}}$, we might require $0 \leq \mathrm{Re}(\varphi) < 1$.) There exists a unique $\mathcal{O}_{\mathcal{P}}$-lattice $W'_{\mathcal{P}}$ of $W$, stable under $\nabla(t\frac{d}{dt})$, in terms a base $\underline{e}'$ of which the connection is expressed

12.0.3.1 $$\nabla(t\frac{d}{dt})\underline{e}' = C\underline{e}' , \quad C \in M_n(\mathcal{O}_{\mathcal{P}})$$

and such that the proper values of $C(\mathcal{Y}) \in M_n(k)$ are all <u>fixed</u> by the composition $k^+ \xrightarrow{\text{proj.}} k^+/\mathbb{Z} \xrightarrow{\varphi} k^+$. (The point is that non-equal proper values of $C(\mathcal{Y})$ do <u>not</u> differ by integers.)

12.0.4 The <u>completion</u> $\hat{W}'_{\mathcal{Y}}$ of the $\mathcal{O}_{\mathcal{Y}}$-lattice $W'_{\mathcal{Y}}$ of $W$ above admits a base $\hat{\underline{\underline{e}}}$ in terms of which the connection is simply

$$\nabla(t\frac{d}{dt})\hat{\underline{\underline{e}}} = C(\mathcal{Y})\cdot\hat{\underline{\underline{e}}} \quad .$$

<u>Remark</u> 12.1. If we require of $\varphi$ that $\varphi(\mathbb{Z}) = \{0\}$, and if $B(\mathcal{Y})$ has all its proper values in $\mathbb{Z}$, then the matrix $C(\mathcal{Y})$ is <u>nilpotent</u>.

<u>Remark</u> 12.2. In general, let $C(\mathcal{Y}) = D + N$, $[D,N] = 0$ be the Jordan decomposition of $C(\mathcal{Y})$ as a sum of a semi-simple matrix $D$ and a nilpotent matrix $N$. Then the conjugacy class of $N$ is independent of the choice of $\varphi$. (And the eigenvalues of $D$ are, modulo $\mathbb{Z}$, the <u>exponents</u> (cf. 12.0.2) at $\mathcal{Y}$.)

<u>Remark</u> 12.3. Suppose $k \subset \mathbb{C}$, and let $\mathcal{O}_{\mathcal{Y}}^{\text{anal}}$ be the local ring of germs of analytic functions at $\mathcal{Y}$. Then the base $\hat{\underline{\underline{e}}}$ of $\hat{W}'_{\mathcal{Y}}$ comes by extension of scalars $\mathcal{O}_{\mathcal{Y}}^{\text{anal}} \longrightarrow \hat{\mathcal{O}}_{\mathcal{Y}}$ from a base $\underline{\underline{e}}^{\text{anal}}$ of $W'_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{Y}}^{\text{anal}}$. In terms of this base $\underline{\underline{e}}^{\text{anal}}$, a multivalued "fundamental matrix of horizontal sections" over a small <u>punctured</u> disc around $\mathcal{Y}$ is given by

12.3.1 $$t^{-C(\mathcal{Y})} = \exp(-C(\mathcal{Y})\log t) \quad .$$

Thus, when "t turns once around $\mathcal{Y}$ counterclockwise," $\log(t)$ becomes $\log(t) + 2\pi i$, and the fundamental matrix

12.3.2 $$t^{-C(\mathcal{Y})}$$

becomes

12.3.3 $$\exp(-2\pi i C(\mathcal{Y}))t^{-C(\mathcal{Y})}$$

or, what is the same,

12.3.4 $$\exp(-2\pi i D)\exp(-2\pi i N)t^{-D-N} \quad .$$

In particular, the proper values of the monodromy substitution for "t turning once around $\mathcal{Y}$ counterclockwise" are the numbers $\exp(-2\pi i\sigma_1),\ldots,\exp(-2\pi i\sigma_n)$, where $\sigma_1,\ldots,\sigma_n$ are the <u>exponents</u> at $\mathcal{Y}$.

<u>Definition</u> 12.4. Let $K/k$ be a function field in one variable, with $k$ a field of characteristic zero. Let $\mathcal{Y}$ be a place of $K/k$ which is rational, $(W,\nabla)$ an object of $MC(K/k)$ which has a regular singular point at $\mathcal{Y}$. We say that the local monodromy at $\mathcal{Y}$ is <u>quasi-unipotent</u> if the <u>exponents</u> at $\mathcal{Y}$ are rational numbers. If the local monodromy at $\mathcal{Y}$ is quasi-unipotent, we say that its <u>exponent of nilpotence is</u> $\leq \nu$ if, in the notation of 12.2, we have $N^{\nu} = 0$.

<u>Definition</u> 12.4 bis. If $\mathcal{Y}$ is any place of $K/k$ (<u>not</u> necessarily rational), at which $(W,\nabla)$ has a regular singular point, we say that the local monodromy at $\mathcal{Y}$ is quasi-unipotent (resp., quasi-unipotent with exponent of nilpotence $\leq \nu$) if this <u>becomes</u> true after the change of base $k \longrightarrow \bar{k}$ = an algebraic closure of $k$, at the induced place $\bar{\mathcal{Y}}$ of $K\cdot\bar{k}/\bar{k}$.

12.5. <u>An example</u>. Let $k = \underline{\underline{C}}$, $K = \underline{\underline{C}}(z)$, $(W,\nabla)$ the object of $MC(K/k)$ given by

12.5.1 W, a K-space of dimension, with basis $e_1, e_2$.

In terms of this base, the connection

12.5.2 $$\nabla(z\frac{d}{dz})\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = B\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & -z \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \quad .$$

Thus $(W,\nabla)$ has a regular singular point at the place $\mathcal{Y} : z = 0$, and its <u>exponents</u> there (the proper values, mod $\underline{\underline{Z}}$, of $B(0)$) are <u>integers</u>.

Although $\exp(2\pi i B(0)) = I$, the <u>monodromy</u> of local horizontal sections in a punctured disc around zero is <u>non-trivial</u>. Indeed, a basis of these (multi-valued) horizontal sections is

12.5.3
$$\begin{cases} v_1 = z e_2 \\ v_2 = \frac{1}{2\pi i}(e_1 + z\log(z)\cdot e_2) \end{cases} .$$

After a counterclockwise turn around $z = 0$,

(12.5.4
$$\begin{cases} v_1 \longrightarrow v_1 \\ v_2 \longrightarrow v_2 + v_1 \end{cases} .$$

In terms of a section $\varphi\colon \underline{\underline{C}}/\underline{\underline{Z}} \longrightarrow \underline{\underline{C}}$ which maps $\underline{\underline{Z}}$ to $0$, the unique $\mathcal{O}_{\mathcal{L}}$-lattice of 12.0.3 is the $\mathcal{O}_{\mathcal{L}}$ span of the vectors

12.5.5
$$\begin{cases} e_1' = e_1 \\ e_2' = -z e_2 \end{cases}$$

in terms of which the connection is expressed (cf. 12.0.3).

$$\nabla(z\frac{d}{dz})\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = C\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} .$$

<u>Remark</u> 12.6. Returning to the "abstract" case, suppose that $\mathcal{L}$ is a rational place of $K/k$ at which $(W,\nabla)$ has a regular singular point, and $W_{\mathcal{L}}$ is an $\mathcal{O}_{\mathcal{L}}$-lattice of $W$ which is stable under $\nabla(t\frac{d}{dt})$. Suppose the completion $\hat{W}_{\mathcal{L}}$ of $W_{\mathcal{L}}$ admits a base $\underline{\underline{\hat{e}}}$ in terms of which the connection is expressed

12.6.1
$$\nabla(t\frac{d}{dt})\underline{\underline{\hat{e}}} = C\underline{\underline{\hat{e}}} \quad \text{with} \quad C \in M_n(k) .$$

If the proper values of $C$ all lie in $k$, then we may <u>rechoose</u> the base $\underline{\underline{\hat{e}}}$ of $\hat{W}_{\mathcal{L}}$ so that the connection is expressed

12.6.2
$$\nabla(t\frac{d}{dt})\hat{e} = C\hat{e}\ , \qquad C \in M_n(k)$$

and such that $C$ is in the form

12.6.3
$$\begin{pmatrix} C_1 & & & \bigcirc \\ & \ddots & & \\ & & \ddots & \\ \bigcirc & & & C_r \end{pmatrix}$$

where each $C_i$ is a square matrix of size $\nu_i$ whose <u>only</u> proper value is $\lambda_i$. In terms of the section $\varphi$, of $k^+ \longrightarrow k^+/\underline{\underline{Z}}$ we put $n_i = \varphi(\lambda_i) - \lambda_i$. Then we replace the lattice $W_{\mathcal{Y}}$ by the lattice $W'_{\mathcal{Y}}$ whose <u>completion</u> admits as base

12.6.4
$$\hat{\underline{\underline{e}}}' = \begin{pmatrix} t^{n_1} I_{\nu_1} & & & \bigcirc \\ & \ddots & & \\ & & \ddots & \\ \bigcirc & & & t^{n_r} I_{\nu_r} \end{pmatrix} \hat{\underline{\underline{e}}} \quad .$$

In terms of the base $\hat{\underline{\underline{e}}}'$, the connection is expressed

12.6.5
$$\nabla(t\frac{d}{dt})\hat{\underline{\underline{e}}}' = \begin{pmatrix} C_1+n_1 I_{\nu_1} & & & \bigcirc \\ & \ddots & & \\ & & \ddots & \\ \bigcirc & & & C_r+n_r I_{\nu_r} \end{pmatrix} \hat{\underline{\underline{e}}}' = C'\hat{\underline{\underline{e}}}' \quad .$$

It follows (the proper values of $C'$ being fixed by $\varphi$) that $W'_{\mathcal{Y}}$ is the unique lattice specified in 12.0.3 by the choice of $\varphi$. Noting $C$ and $C'$ have the <u>same</u> nilpotent parts in their Jordan decomposition, we have

<u>Proposition</u> 12.6.6. Suppose $(W,\nabla)$ has a regular singular point at the rational place $\mathcal{Y}$ of $K/k$, and there exists an $\mathcal{O}_{\mathcal{Y}}$-lattice $W_{\mathcal{Y}}$ whose completion admits a base $\hat{\underline{\underline{e}}}$ in terms of which the connection is expressed

$$\nabla(t\frac{d}{dt})\hat{\underline{\underline{e}}} = C\hat{\underline{\underline{e}}}\,, \quad \text{with} \quad C \in M_n(k) \quad .$$

Then the local monodromy of $(W,\nabla)$ at $\mathcal{Y}$ is quasi-unipotent of exponent of nilpotence $\leq \nu$ if and only if in the Jordan decomposition of $C$,

$$C = D + N,\ [D,N] = 0$$

with $D$ semisimple and $N$ nilpotent, the proper values of $D$ are rational numbers, and $N^{\nu} = 0$.

Proposition 12.7. Let $F \hookrightarrow K \hookrightarrow L$ be a tower of function fields in one variable over a field $k$ of characteristic zero. Let $\mathcal{P}$ be a place of $L/k$, $\mathcal{P}'$ the induced place of $K/k$, and $\mathcal{P}''$ the induced place of $F/k$. Let $\mathcal{P}'_1,\ldots,\mathcal{P}'_r$ be all the places of $K/k$ which lie over the place $\mathcal{P}''$ of $F/k$. Let $(W,\nabla)$ be an object of $MC(K/k)$ which has regular singular points at each place $\mathcal{P}'_1,\ldots,\mathcal{P}'_r$, and $\nu \geq 1$ an integer. Then:

12.7.1 The inverse image $(W\otimes_K L,\nabla_L)$ of $(W,\nabla)$ in $MC(L/k)$, which has a regular singular point at $\mathcal{P}$ by 11.12, has quasi-unipotent local monodromy at $\mathcal{P}$ of exponent of nilpotence $\leq \nu$, if and only if $(W,\nabla)$ has quasi-unipotent local monodromy at $\mathcal{P}'$, of exponent of nilpotence $\leq \nu$.

12.7.2 The direct image ($W$ as $F$-space, $\nabla/\mathrm{Der}(F/k)$) of $(W,\nabla)$ in $MC(F/k)$ has quasi-unipotent local monodromy at $\mathcal{P}''$ of exponent of nilpotence $\leq \nu$ if and only if $(W,\nabla)$ has quasi-unipotent local monodromy of exponent of nilpotence $\leq \nu$ at each place $\mathcal{P}'_i$ of $K/k$ lying over $\mathcal{P}''$.

Proof. By making the base-change $k \longrightarrow \bar{k}$ = an algebraic closure of $k$, we are immediately reduced to the case of $k$ algebraically closed. Let $t$ be a uniformizing parameter at $\mathcal{P}''$.

To prove 12.7.1, we choose an $\mathcal{O}_{\mathcal{P}'}$-lattice $W_{\mathcal{P}'}$ in $W$, stable under $\nabla(t\frac{d}{dt})$, whose completion $\hat{W}_{\mathcal{P}'}$ admits a base $\hat{\underline{\underline{e}}}$ in terms of which the connection is expressed (putting $\varepsilon(\mathcal{P}'/\mathcal{P}'')$ = the ramification index)

12.7.3 $$\varepsilon(\mathcal{P}'/\mathcal{P}'')\nabla(t\frac{d}{dt})\hat{\underline{\underline{e}}} = C\hat{\underline{\underline{e}}} \qquad C \in M_n(k)\ .$$

Consider the lattice $W_{\wp'} \otimes_{\mathcal{O}_{\wp'}} \mathcal{O}_{\mathfrak{p}}$ in $W \otimes_K L$; its completion admits the "same" base $\hat{\underline{\underline{e}}}$, and the connection is

12.7.4 $$\varepsilon(\mathfrak{p}/\wp'')\nabla(t\frac{d}{dt})\hat{\underline{\underline{e}}} = \varepsilon(\mathfrak{p}/\wp')C\hat{\underline{\underline{e}}} \ .$$

We conclude the proof of 12.7.1 by applying the criterion 12.6.6 to the matrices $C$ and $\varepsilon(\mathfrak{p}/\wp')C$.

Now let us prove 12.7.2. For each point $\wp_i'$ lying over $\wp''$, we choose a lattice $W_{\wp_i'}$ in terms of a base $\hat{\underline{\underline{e}}}_i$ of whose completion the connection is expressed (writing $\varepsilon_i = \varepsilon(\wp_i'/\wp'')$)

12.7.5 $$\varepsilon_i\nabla(t\frac{d}{dt})(\hat{\underline{\underline{e}}}_i) = C(\wp_i)\hat{\underline{\underline{e}}}_i \ , \qquad C(\wp_i) \in M_n(k) \ .$$

Consider the $\mathcal{O}_{\wp''}$-lattice $\oplus_i W_{\wp_i'}$ in $W$ considered as $F$-space. In the natural basis of its completion, consisting of the blocks of vectors

12.7.6 $$(t)^{a/\varepsilon_i}\hat{\underline{\underline{e}}}_i \ , \qquad a = 0,1,\dots,\varepsilon_i-1, \ i = 1,\dots,r$$

the connection, <u>stable</u> on the span of each block $t^{a/\varepsilon_i}\hat{\underline{\underline{e}}}_i$, is expressed on each block as

12.7.7 $$\nabla(t\frac{d}{dt})(t^{a/\varepsilon_i}\hat{\underline{\underline{e}}}_i) = \frac{1}{\varepsilon_i}\left(C(\wp_i) + a\right)\left(t^{a/\varepsilon_i}\hat{\underline{\underline{e}}}_i\right) \ .$$

Again, we conclude by using condition 12.6.6, which is satisfied by each of the matrices $C(\wp_i)$ if and only if it is satisfied by each of the matrices $\frac{1}{\varepsilon_i}\left(C(\wp_i) + a\right)$, $a = 0,1,\dots,\varepsilon_i-1$.

## 13. <u>Consequences of Turrittin's Theorem</u>

We are now in a position to apply Turrittin's theorem 11.10 to the study of globally nilpotent connections.

<u>Theorem</u> 13.0. Let $T$ be a global affine variety (cf. 9.0) and $f: S \longrightarrow T$

a smooth morphism of relative dimension one, whose generic fibre is geometrically connected. Let $(M,\nabla)$ be an object of $MIC(S/T)$, with $M$ locally free of finite rank on $S$. Let $k$ denote the function field of $T$, $K$ the function field of $S$. Thus $K$ is a field of functions in one variable over a field $k$ of characteristic zero.

13.0.1 Suppose that $(M,\nabla)$ is globally nilpotent on $S/T$ (cf. 9.1). Then the inverse image of $(M,\nabla)$ in $MC(K/k)$ has a regular singular point at <u>every</u> place $\mathfrak{p}$ of $K/k$, and has quasi-unipotent local monodromy at every place $\mathfrak{p}$ of $K/k$.

13.0.2 Suppose that $(M,\nabla)$ is globally nilpotent of exponent $\nu$ on $S/T$. Then at every place $\mathfrak{p}$ of $K/k$, the local monodromy of the inverse image of $(M,\nabla)$ in $MC(K/k)$ is quasi-unipotent of exponent $\leq \nu$.

<u>Proof</u>. Using 9.3.1 and 9.2, and 11.12.2 and 12.7.2, we are immediately reduced to the case:

13.0.3 $S$ is a principal open subset of $\underset{=}{A}^1_T$, i.e., $T = \mathrm{Spec}(R)$, and $S = \mathrm{Spec}\left(R[t]\left[\frac{1}{g(t)}\right]\right)$ with $g(t) \in R[t]$

13.0.4 we wish to check at the place of $K = k(t)$ defined by $t = 0$

13.0.5 $M$ is a free $R[t]\left[\frac{1}{g(t)}\right]$ module

13.0.6 $g(t) = t^j h(t)$, with $h(t) \in R[t]$ and $j \geq 1$ (otherwise there is no singularity at $t = 0$)
and $h(0)$ an <u>invertible</u> element of $R$ (at the expense of localizing $R$ at $h(0)$).

Suppose that $(M,\nabla)$ is globally nilpotent, but that $t = 0$ is not a regular singular point of its restriction to $MC(k(t)/k)$. Let $n$ be the rank of $M$. Let us make the base change (putting $z = t^{1/n!}$)

13.0.7 $$R[t]\left[\frac{1}{g(t)}\right] \longrightarrow R[z]\left[\frac{1}{g(z)}\right] .$$

By 9.2, the inverse image of $(M,\nabla)$ on $R[z]\left[\frac{1}{g(z)}\right]$ is still globally nilpotent, but by Turrittin's theorem 11.10, there exists a basis $\underline{\underline{m}}$ of $M$ (over an open subset of $\operatorname{Spec}\left(R[z]\left[\frac{1}{g(z)}\right]\right)$, which, by "enlarging" $g$, we may suppose to be all of $\operatorname{Spec}\left(R[z]\left[\frac{1}{g(z)}\right]\right)$, in terms of which the connection is expressed

13.0.8 $$\nabla(z \frac{d}{dz})\cdot\underline{\underline{m}} = z^{-\mu}(A + zB)\underline{\underline{m}} , \quad \mu \geq 1$$

13.0.9 $$\begin{cases} A \in M_n(R) \ \ \underline{\text{non-nilpotent}} \\ \\ B \in M_n\left(R[z]\left[\frac{1}{h(z)}\right]\right) \quad \text{(and } h(0) \text{ invertible in } R) . \end{cases}$$
13.0.10

An immediate calculation then shows that, for each integer $j \geq 1$, we have

13.0.11 $$\left(\nabla(z \frac{d}{dz})\right)^j \underline{\underline{m}} = z^{-\mu j}(A^j + zB_j)\underline{\underline{m}} \ \text{ with } \ B_j \in M_n\left(R[z]\left[\frac{1}{h(z)}\right]\right) .$$

Now let $p$ be a prime number. Recall that in $\operatorname{Der}(\underline{\underline{F}}_p[z]/\underline{\underline{F}}_p)$ we have $(z \frac{d}{dz})^p = z \frac{d}{dz}$. Thus the hypothesis of global nilpotence is that, for every prime number $p$, there is an integer $\alpha(p)$ such that

13.0.12 $$\left(\left(\nabla(z \frac{d}{dz})\right)^p - \nabla(z \frac{d}{dz})\right)^{\alpha(p)} M \subset pM$$

or, equivalently, using 5.0.9, that, for every prime number $p$

13.0.12 $$\left(z^{-\mu p}(A^p + zB_p) - z^{-\mu}(A + zB)\right)^{\alpha(p)} \in pM_n\left(R[z]\left[\frac{1}{g(z)}\right]\right) .$$

Hence looking at the most polar term, we conclude

13.0.13 $$A^{p\cdot\alpha(p)} \in pM_n(R) \quad \text{for every prime } p.$$

Now look at the characteristic polynomial of $A$, $\det(XI_n - A)$. According to 13.0.13, its <u>value</u> at every <u>closed</u> <u>point</u> of $T = \mathrm{Spec}(R)$ is $X^n$, and hence

13.0.14 $$\det(XI_n - A) = X^n$$

which implies that $A$ is <u>nilpotent</u>, a contradiction. This proves that $t = 0$ was a regular singular point of the inverse image of $(M,\nabla)$ in $MC(k(t)/k)$.

We now turn to proving quasi-unipotence of the local monodromy at $t = 0$. By definition of a regular singular point, there exists a basis $\underline{\underline{m}}$ of $M$ (over an open subset of $\mathrm{Spec}\, R[t]\left[\frac{1}{g(t)}\right]$, which by "enlarging" $g$, we may suppose to be <u>all</u> of $\mathrm{Spec}\left(R[t]\left[\frac{1}{g(t)}\right]\right)$ in terms of which the connection is expressed

13.0.15 $$\nabla(t\tfrac{d}{dt})\underline{\underline{m}} = (A + tB)\underline{\underline{m}}$$
$$\text{with } A \in M_n(R),\ B \in M_n\left(R[t]\left[\frac{1}{h(t)}\right]\right),$$
$$\text{and } h(0) \text{ invertible in } R.$$

By adjoining to the ring $R$ the proper values of $A$, and perhaps localizing the resulting ring a bit, we can assume that the Jordan decomposition is defined over $R$, i.e. that

13.0.16 $$A = D + N,\ [D,N] = 0$$
$$\begin{cases} D \in M_n(R) & \underline{\text{diagonal}} \\ N \in M_n(R) & \underline{\text{nilpotent}}\ \underline{\text{super-triangular}} \end{cases}.$$

Suppose that $(M,\nabla)$ is <u>globally</u> <u>nilpotent</u>. For each prime number $p$, we thus have

13.0.17 $$\left(\left(\nabla(t\tfrac{d}{dt})\right)^p - \nabla(z\tfrac{d}{dz})\right)^{\alpha(p)} M \subset pM \quad .$$

As before (13.0.11), an immediate calculation shows that, for each integer $j \geq 1$

13.0.18 $$\left(\nabla(t \frac{d}{dt})\right)^{j} \underline{\underline{m}} = (A^{j} + tB_{j})\underline{\underline{m}}$$
$$\text{with} \quad B_{j} \in M_{n}\left(R[t]\left[\frac{1}{h(t)}\right]\right), \quad h(0) \text{ invertible in } R.$$

Now, using 5.0.9 and looking at the <u>constant</u> term of the matricial expression of 13.0.6, we find

13.0.19 $$(A^{p} - A)^{\alpha(p)} \in pM_{n}(R) \quad \text{for every prime } p .$$

Writing $A = D + N$ (cf. 13.0.15), we have (because $[D,N] = 0$)

13.0.20 $$0 \equiv (A^{p} - A)^{\alpha(p)} \equiv (D^{p} - D + N^{p} - N)^{\alpha(p)} \quad \text{modulo} \quad pM_{n}(R)$$

and, looking at the diagonal terms, we find

13.0.21 $$(D^{p} - D)^{\alpha(p)} \in pM_{n}(R) .$$

Let $d$ be a proper value of $D$; then $d$ is quantity in an integral domain $R$ of finite type over $\underline{\underline{Z}}$, whose quotient field is of characteristic zero, such that at every closed point $\mathscr{P}$ of Spec(R), the image of $d$ in the residue field $R/\mathscr{P}$ at $\mathscr{P}$ lies in the <u>prime field</u>. As is well-known, this implies that $d \in R \cap Q$. This proves the quasi-unipotence of the local monodromy. Now we must estimate the exponent of nilpotence of the local monodromy, assuming $(M,\nabla)$ globally nilpotent of exponent $\nu$. At a closed point $\mathscr{P}$ of $R$ of residue characteristic $p$, we have ($D$ being diagonal)

13.0.22 $$D^{p} \equiv D \bmod \mathscr{P}$$

so that 13.0.20 gives (since we may take $\alpha(p) = \nu$ for all $p$)

13.0.23 $$(N^{p} - N)^{\nu} \equiv 0 \mod \mathscr{P}.$$

But $N$ is <u>nilpotent</u>; let us write

13.0.24 $$(N^p - N)^\nu = (-1)^\nu N^\nu (1 - N^{p-1})^\nu$$

and notice that $(1 - N^{p-1})^\nu$ is <u>invertible</u> in $M_n(R)$, so 13.0.23 is equivalent to

13.0.25 $$N^\nu \equiv 0 \bmod \mathscr{Y} \text{ for } \underline{\text{every}} \text{ closed point } \mathscr{Y}$$

which implies that $N^\nu = 0$ in $M_n(R)$. Q.E.D.

13.1 <u>A counter-example</u> (d'après Deligne)

Let $\pi: S \longrightarrow T$ be a smooth morphism. There is a bijective correspondence between T-connections $\nabla$ on $\mathcal{O}_S$ and global sections of $\Omega^1_{S/T}$. Namely, to a T-connection $\nabla$ on $\mathcal{O}_S$

13.1.0 $$\nabla : \mathcal{O}_S \longrightarrow \Omega^1_S$$

corresponds the global section of $\Omega^1_{S/T}$

13.1.1 $$\omega = \nabla(1) \quad .$$

Conversely, to a global section $\omega$ of $\Omega^1_{S/T}$ corresponds the T-connection $\nabla_\omega$ on $\mathcal{O}_S$, defined by

13.1.2 $$\nabla_\omega(f) = df + f\omega \quad .$$

The <u>curvature</u> $K_\omega$ of the connection $\nabla_\omega$ is

13.1.3 $$\begin{cases} K_\omega : \mathcal{O}_S \longrightarrow \Omega^1_{S/T} \\ K_\omega(f) = f \cdot d\omega \quad . \end{cases}$$

Thus $\nabla_\omega$ is <u>integrable</u> precisely when $\omega$ is <u>closed</u>.

Suppose that $T$ (and hence $S$) is a <u>reduced</u> scheme of characteristic $p$, and let $\omega$ be a <u>closed</u> global section of $\Omega^1_{S/T}$. What does it mean that the connection $\nabla_\omega$ be <u>nilpotent</u>? First, since $\mathcal{O}_S$ is free of rank one, S is reduced, and the p-curvature $\psi_\omega(D)$ of a local section of <u>Der</u>(S/T) is a nilpotent $\mathcal{O}_S$-linear endomorphism of $\mathcal{O}_S$, it means that $\nabla_\omega$ has p-<u>curvature zero</u>. By Cartier's theorem (5.1), the $\mathcal{O}_S$-span of the horizontal (for $\nabla_\omega$) sections of $\mathcal{O}_S$ is all of $\mathcal{O}_S$, and hence there exists an open covering $\mathcal{U}_i$ of S, and sections $f_i$ of $\mathcal{O}_S^*$ over $\mathcal{U}_i$ such that $f_i$ is <u>horizontal</u> for $\nabla_\omega$, i.e.

13.1.4 $$\omega = -df_i/f_i \quad \text{on } \mathcal{U}_i \quad .$$

Thus, if $T = \mathrm{Spec}(\mathbb{F}_p)$, and S is an elliptic curve E over $\mathbb{F}_p$, and $\omega$ is a (non-zero) differential of the first kind on E, then $\nabla_\omega$ is <u>nilpotent</u> if and only if the "Hasse invariant" of E is 1, i.e. if and only if

13.1.6 $$\mathrm{Card}(E(\mathbb{F}_p)) \equiv 0 \quad \text{modulo } p$$

where $E(\mathbb{F}_p)$ denote the group of <u>rational points</u> of E. By the Riemann Hypothesis for elliptic curves,

13.1.7 $$\sqrt{p} - 1 \leq \sqrt{\mathrm{Card}(E(\mathbb{F}_p))} \leq \sqrt{p} + 1 \quad .$$

Thus if $p \geq 7$, and if $E(\mathbb{F}_p)$ has a non-trivial element of order <u>two</u> (so that $\mathrm{Card}(E(\mathbb{F}_p))$ is <u>even</u>), 13.1.6 and 13.1.7 are incompatible, and so $\nabla_\omega$ is not nilpotent. Thus may we construct counter-examples to the converse of 13.0.

<u>Example</u> 13.2. Let $a,b \in \mathbb{Z}$, with $a^2 \neq 4b$. Consider the projective and smooth elliptic curve $\underline{\underline{E}}$ over $\mathrm{Spec}\left(\mathbb{Z}\left[\frac{1}{30(a^2-4b)}\right]\right)$ given in homogeneous coordinates X, Y, Z by the equation

13.2.1 $$Y^2 = X(X^2 + aXZ + bZ^2) \quad .$$

Then the connection in $\mathcal{O}_{\underline{\underline{E}}}$ given by

$$f \longrightarrow df + f\omega$$

(where $\omega = d(X/Z)\big/Y/Z$ is the differential of the first kind on $\underline{\underline{E}}$) gives a connection on the function field of $\underline{\underline{E}}_{Q}$ for which every place is a regular singular point (indeed not a singular point at all) and has quasi-unipotent monodromy (namely none at all). However, the connection, far from being globally nilpotent, induces on the structure sheaf of the fibre over every closed point of the base a non-nilpotent connection.

Remark 13.3. If we project this example to the x-axis, we get a rank-two counter-example over an open subset of $\underline{\underline{A}}^1_{\underline{\underline{Z}}}$, whose inverse image on $Q(x)$ has singular points precisely at $0, \infty$, and the roots of $x^2 + ax + b$. (These are the points over which the x-coordinate is not étale; compare with 12.7.6-7).

13.4 In the "positive" direction, Messing (unpublished) has shown that, if $a,b,c \in Q$, then the rank two module over

$$\underline{\underline{Z}}\left[x, \frac{1}{n\cdot x(x-1)}\right] \quad (n \in \underline{\underline{Z}} \text{ so chosen that } a,b,c \in \underline{\underline{Z}}\left[\frac{1}{n}\right])$$

corresponding to the hypergeometric differential equation with parameters $\{a,b,c\}$, is globally nilpotent. Of course here there are only three singular points, $x = 0, 1,$ or $\infty$.

## 14. Application to the Local Monodromy Theorem

14.0 Let $S/\underline{\underline{C}}$ be a smooth connected curve, and let $\pi: X \longrightarrow S$ be a proper and smooth morphism. Clearly there exist:

14.0.1 a subring $R$ of $\underline{\underline{C}}$ which is of finitely generated over $\underline{\underline{Z}}$,

14.0.2 a smooth connected curve $\underline{\underline{S}}/R$, which "gives back" $S/\underline{\underline{C}}$ after the base change $R \hookrightarrow \underline{\underline{C}}$, 14.0.3 a proper and smooth morphism $\underline{\underline{\pi}}: \underline{\underline{X}} \longrightarrow \underline{\underline{S}}$ which "gives back" $\pi: X \longrightarrow S$ after the base change $S \longrightarrow \underline{\underline{S}}$.

Combining 10.0 and 13.0, we find

<u>Theorem</u> 14.1 (the Local Monodromy Theorem).

Let $S/\underline{\underline{C}}$ be a smooth connected curve, $K/\underline{\underline{C}}$ its function field, $\pi : X \longrightarrow S$ a proper and smooth morphism, $X_K/K$ the generic fibre of $\pi$.

For each integer $i \geq 0$, let $h(i)$ (cf. 10.0) be the number of pairs $(p,q)$ of integers with $p + q = i$ and $h^{p,q}(X_K/K) = \dim_K H^q(X_K, \Omega^p_{X_K/K}) = \operatorname{rank}_{\mathcal{O}_S} R^q\pi_*(\Omega^p_{X/S})$ <u>non-zero</u>. Then the inverse image of $H^i_{DR}(X/S)$, with the Gauss-Manin connection, in $MC(K/\underline{\underline{C}})$ (or what is the same, the K-space $H^i_{DR}(X_K/K)$ with the Gauss-Manin connection) has regular singular points at every place of $K/\underline{\underline{C}}$ (indeed has <u>no</u> singularity at any place <u>in</u> S) and quasi-unipotent local monodromy, whose exponent of nilpotence is $\leq h(i)$.

14.2 Let $K/\underline{\underline{C}}$ be the function field of a smooth connected curve $S/C$, and let

14.2.1 $$\pi: \mathcal{U} \longrightarrow \operatorname{Spec}(K)$$

be a smooth morphism (<u>not</u> necessarily proper).

By Hironaka [18], there exists a finite extension $L/K$, a proper and smooth morphism $\rho: X \longrightarrow \operatorname{Spec}(L)$, and a divisor, $i: Y \hookrightarrow X$, with normal crossings relative to $\operatorname{Spec}(L)$, such that the morphism

14.2.2 $$\pi_L : \mathcal{U}_L = \mathcal{U} \times_{X_K} L \longrightarrow \operatorname{Spec}(L)$$

<u>is</u> the morphism

14.2.3 $$\rho | X-Y : X - Y \longrightarrow \operatorname{Spec}(L).$$

Clearly there exist

14.2.4 a subring $R$ of $\underline{\underline{C}}$, finitely generated over $\underline{\underline{Z}}$,

14.2.5 a smooth connected curve $\underline{\underline{S}}/R$, the generic point of whose fibre over the given point $\mathrm{Spec}(\underline{\underline{C}}) \longrightarrow \mathrm{Spec}(R)$ is $L$,

14.2.5 a proper and smooth morphism $\underline{\underline{\rho}}\colon \underline{\underline{X}} \longrightarrow \underline{\underline{S}}$, and a divisor $i\colon \underline{\underline{Y}} \hookrightarrow \underline{\underline{X}}$ with normal crossings relative to $\underline{\underline{S}}$, whose fibres over the given point $\mathrm{Spec}(L) \longrightarrow \underline{\underline{S}}$ are $\rho\colon X \longrightarrow S$ and $i\colon Y \hookrightarrow X$ respectively.

Applying 10.0 (log Y), 13.0, 8.10, the fact that $H^i_{DR}(X\text{-}Y/L) = H^i_{DR}(\mathcal{U} \times_K L/L) = H^i_{DR}(\mathcal{U}/K) \otimes_K L$, 11.12.1 and 12.7.1, we find

<u>Theorem</u> 14.3 (Deligne). (The "Open" Local Monodromy Theorem). Assumptions and notations as in 14.2.1-3, let $\pi\colon \mathcal{U} \longrightarrow \mathrm{Spec}(K)$ be a smooth morphism. For each integer $i \geq 0$, let $h_Y(i)$ (cf. 10.0 (log Y)) be the number of pairs $(p,q)$ of integers with $p + q = i$ and $\dim_L H^q(X, \Omega^p_{X/L}(\log Y))$ <u>non-zero</u>. Then the object of $MC(K/\underline{\underline{C}})$ given by $H^i_{DR}(\mathcal{U}/K)$ with the Gauss-Manin connection, has regular singular points at every place of $K/\underline{\underline{C}}$, and at each the local monodromy is quasi-unipotent, of exponent of nilpotence $\leq h_Y(i)$.

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