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SEMINAR ON CONVEX SETS

I. Introductory Material on Convex Sets in Euclidean Space,

by

P. T. Bateman

§1. Analytic geometry of n dimensions.

Euclidean space of n dimensions E_n is the vector space (over the real numbers) consisting of n-tuples of real numbers with addition, multiplication by scalars, and inner product defined as follows:

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n) ;$$

$$\alpha(a_1, \ldots, a_n) = (\alpha a_1, \ldots, \alpha a_n), \qquad \alpha \text{ real} ;$$

$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = a_1 b_1 + \ldots + a_n b_n .$$

We shall use small roman letters without subscripts to denote elements of E_n, subscripts being used to designate the coordinates. Thus $a = (a_1, \ldots, a_n)$. We shall use $|a|$ to mean $(a \cdot a)^{\frac{1}{2}}$ and refer to $|a - b|$ as the distance between a and b. Clearly $\text{abs}(a \cdot b) \leq |a|\,|b|$. If $a \neq 0$, we call $|a|^{-1} a$ the direction of a. Elements of E_n will be called either points or vectors.

By a linear manifold in E_n we mean any one of the following equivalent concepts:

(1) a non-empty subset of E_n such that if a and b belong to the set, then $\lambda a + (1 - \lambda) b$ belongs to the set for any real λ ;

(2) a set of points in E_n obtained from a subspace of E_n by adding a constant vector to each element thereof;

(3) a set of points of the form $\lambda_o a^{(o)} + \dots + \lambda_m a^{(m)}$, where $a^{(o)}, \dots, a^{(m)}$ are points of E_n, $\lambda_o, \dots, \lambda_m$ run over all real numbers such that $\lambda_o + \dots + \lambda_m = 1$, and $0 \leq m \leq n$;

(4) a set of points of the form $b^{(o)} + \mu_1 b^{(1)} + \dots + \mu_m b^{(m)}$, where $b^{(o)}, \dots, b^{(m)}$ are points of E_n, $\mu_1, \dots, \mu_m$ run over all real numbers, and $0 \leq m \leq n$.

By the <u>dimension</u> of a linear manifold we mean any one of the following, which are equivalent: the dimension of the subspace in (2) (called the parallel subspace), the minimal m in (3), or the minimal m in (4). By a <u>straight line</u> in E_n we mean a linear manifold of dimension one, i.e., either of the following, which are equivalent: a set of points of the form $b + \lambda c$, where $c \neq 0$ and λ runs over all real numbers, or a set of points of the form $\lambda a + (1-\lambda)b$, where $a \neq b$ and λ runs over all real numbers.

By a <u>plane</u> we mean either of the following, which are equivalent: a linear manifold of dimension n-1, or a subset of E_n which can be characterized as the set of points x in E_n satisfying an equation of the form $u \cdot x = u_o$, where u is a fixed vector from E_n and u_o is a fixed real number. We say that $|u|^{-1} u$ is the <u>direction-vector</u> of the plane; it depends on the equation used, although determined up to multiplication by ± 1. If $|u| = 1$, we say that the equation $u \cdot x = u_o$ is in <u>normal form</u>. With respect to the plane

$u \cdot x = u_o$

the set of points x such that $u \cdot x \geq u_o$ is called the <u>closed positive half-space</u>,

" " " " x " " $u \cdot x \leq u_o$ " " " <u>closed negative half-space</u>

the set of points x such that $u \cdot x > u_o$ is called the open positive half-space,
" " " " x " " $u \cdot x < u_o$ " " " open negative half-space.

A linear manifold can be characterized as the intersection of a set of planes having at least one point in common; the minimum number of planes is n -(the dimension).

For the facts quoted above see Steinitz, J. Reine Angew. Math. vol.143 (1913)pp.131-149.

If $a^{(o)}, \ldots, a^{(m)}$ are m+1 points ($0 \leq m \leq n$) lying in no m-1 dimensional linear manifold, then the set of points $\lambda_o a^{(o)} + \ldots + \lambda_m a^{(m)}$, where $\lambda_o, \ldots, \lambda_m$ run over all sets of non-negative real numbers with sum 1, is called the m-simplex determined by $a^{(o)}, \ldots, a^{(m)}$. The segment joining two distinct points a and b is the 2-simplex determined by a and b, i.e., the set of points of the form $\lambda a + (1-\lambda)b$, $0 \leq \lambda \leq 1$. The ray from b in the direction c is the set of points of the form $b + \lambda c$, $\lambda \geq 0$.

We recall the definitions of interior (or inner) point of a set, exterior (or outer) point, frontier (or boundary) point, cluster (or limit) point, closed set, open set, derived set of a set, closure of a set, etc. A set which is not the null set or the whole space E_n has a non-empty boundary.

A simplex in E_n has interior points if and only if it is an n-simplex. The interiors of n-simplices form a system of neighborhoods in E_n equivalent to the usual spherical neighborhoods.

§2. Basic properties of convex sets.

This section is mainly devoted to those properties of convex sets which are just as easy to prove in a general normed linear space as in a Euclidean space. In subsequent sections we shall restrict ourselves to Euclidean spaces, so that the generality of the present section is essentially a luxury.

A convex set in a normed linear space E [cf. Banach, Théorie des opérations linéaires, Warsaw, 1932, p.53] is a set of points such that if a and b belong to the set, then $\lambda a + (1-\lambda)b$ belongs to the set for $0 \leq \lambda \leq 1$. Clearly a set is convex if and only if every straight line intersects it in the null set, a point, a segment with or without either endpoint, a ray with or without the endpoint, or the whole line. The intersection of two convex sets is convex. A linear manifold is convex. An open or closed half-space (with respect to a plane) is convex. A set of points of the form $\lambda_o a^{(o)} + \dots + \lambda_p a^{(p)}$, where $p \geq 0$ and $\lambda_o, \dots, \lambda_p$ run over all sets of non-negative real numbers with sum 1, is convex.

THEOREM 1. *If a convex set K contains the points $a^{(o)}, \dots, a^{(p)}$, then it contains all points of the form $\lambda_o a^{(o)} + \dots + \lambda_p a^{(p)}$, where $\lambda_o, \dots, \lambda_p$ are non-negative numbers with sum 1.*

Proof: This theorem is true for $p = 1$ by the definition of convexity. Suppose now that the theorem has been proved for 1, 2, ..., p-1. If $\lambda_o + \dots + \lambda_{p-1} = 0$, there is nothing to prove. If $\lambda_o + \dots + \lambda_{p-1} = \lambda > 0$, then by induction

$$\frac{\lambda_o}{\lambda} a^{(o)} + \dots + \frac{\lambda_{p-1}}{\lambda} a^{(p-1)}$$

belongs to K. Since $\lambda + \lambda_p = 1$, it follows that

$$\lambda\left(\frac{\lambda_o}{\lambda}a^{(o)} + \dots \frac{\lambda_{p-1}}{\lambda}a^{(p-1)}\right) + \lambda_p a^{(p)} ,$$

or $\lambda_o a^{(o)} + \dots + \lambda_p a^{(p)}$, belongs to K, q.e.d.

<u>THEOREM 2.</u> <u>If a convex set</u> K <u>contains more than one point, then</u> $K' = \bar{K}$.

Proof: We must show that if $a \in K$, then $a \in K'$. But if b is a point of K distinct from a, all points of the segment joining a and b belong to K and thus a is a cluster point of K.

<u>THEOREM 3.</u> <u>If a convex set</u> K <u>is not the whole space</u> E, <u>then</u> $(E - K)' = \overline{E - K}$.

Proof: We must show that if $a \in E-K$, then $a \in (E-K)'$. Suppose $a \notin (E-K)'$. Then we could find a sphere with center a containing no points of E-K other than a. Joining two diametrically opposite points inside the sphere would show that $a \in K$, a contradiction.

<u>THEOREM 4.</u> <u>If a convex set</u> K <u>contains more than one point but is not the whole space,</u> Front $K = K' \cap (E - K)'$.

Proof: By Theorems 2 and 3 we have

$$\text{Front } K = \bar{K} \cap \overline{E - K} = K' \cap (E - K)'.$$

<u>THEOREM 5.</u> <u>Let</u> R(x) <u>denote the distance of a point</u> x <u>from the convex set</u> K. <u>Then the function</u> R(x) <u>has the property</u>

$$R(\{1 - \lambda\} x + \lambda y) \leq (1 - \lambda)R(x) + \lambda R(y) \ \underline{\text{for}} \ 0 \leq \lambda \leq 1.$$

Proof: There exist points x', y' of K such that

$$|x - x'| < R(x) + \delta , \ |y - y'| < R(y) + \delta ,$$

δ being any given positive number. Then $(1 - \lambda)x' + \lambda y'$ is in K and

$$|\{(1 - \lambda)x + \lambda y\} - \{(1 - \lambda)x' + \lambda y'\}| = |(1 - \lambda)(x - x') + \lambda(y - y')|$$
$$\leq (1 - \lambda)|x - x'| + \lambda|y - y'| < (1 - \lambda)R(x) + \lambda R(y) + \delta.$$

Since δ is arbitrary, our theorem is proved.

THEOREM 6. *Let* $r(x)$ *denote the distance of a point* x *from the complement* $E - K$ *of the convex set* K. *Then if* $0 \leq \lambda \leq 1$ *and* x *and* y *are in* $\bar{K}$, *we have*

$$r(\{1 - \lambda\} x + \lambda y) \geq (1 - \lambda) r(x) + r(y).$$

Proof: We may assume $0 < \lambda < 1$. If $r(x) = r(y) = 0$, there is nothing to prove. If $r(x) \neq 0$ and $r(y) \neq 0$, then $x + u$ is in K for $|u| < r(x)$ and $y + u$ is in K for $|u| < r(y)$; hence $(1 - \lambda)x + \lambda y + u$ is in K for $|u| < (1 - \lambda)r(x) + \lambda r(y)$, so that the theorem is proved in this case. If $r(y) = 0$ but $r(x) \neq 0$, then, since y is in $\bar{K}$, there is a point y' of K such that $|y - y'| < \delta$, where δ is any given positive number less than $(1 - \lambda)r(x)$. Now $x + u$ is in K for $|u| < r(x)$ and hence $(1 - \lambda)x + \lambda y' + u$ is in K for $|u| < (1 - \lambda)r(x)$. Thus

$$r(\{1 - \lambda\} x + \lambda y') \geq (1 - \lambda) r(x) ,$$

so that

$$r(\{1 - \lambda\} x + \lambda y) \geq (1 - \lambda) r(x) - \delta .$$

Since δ is arbitrary, the theorem is proved.

THEOREM 7. *If* a *is an interior point of a convex set* K *and* b *is any point of* $\bar{K}$, *then* $(1 - \lambda) a + \lambda b$ *is an interior point of* K *for* $0 \leq \lambda < 1$.

Proof: By Theorem 6

$$r(\{1 - \lambda\} a + \lambda b) \geq (1 - \lambda) r(a) > 0$$

and hence the point $(1 - \lambda) a + \lambda b$ has positive distance from $E - K$.

THEOREM 8. *If* K *is a convex set*, $a \in \text{Int } K$, *and* $b \in \text{Front } K$, *then the point* $(1 - \lambda)a + \lambda b = a + \lambda(b - a)$ *is interior to* K *for* $0 \leq \lambda < 1$ *and exterior to* K *for* $\lambda > 1$.

Proof: The first part of the conclusion is contained in Theorem 7. If $(1 - \lambda)\, a + \lambda b$ were to belong to $\bar{K}$ for some $\lambda > 1$, then by Theorem 7

$$(1 - \lambda^{-1})\, a + \lambda^{-1} \{(1 - \lambda)\, a + \lambda b\} = b$$

would be an interior point of K, a contradiction.

THEOREM 9. *If* K *is a convex set,* Int K *is also convex.*

Proof: This follows from Theorem 7.

THEOREM 10. *If* K *is a convex set,* $\bar{K}$ *is also convex.*

Proof: Suppose $a \in \bar{K}$, $b \in \bar{K}$. Then for any $\delta > 0$ there exist u, v such that $|u| < \delta$, $|v| < \delta$, $a + u \in K$, $b + v \in K$. Thus for any λ between 0 and 1 we have $(1 - \lambda)(a + u) + \lambda(b + v) \in K$ and

$$\left|\{(1 - \lambda)(a + u) + \lambda(b + v)\} - \{(1 - \lambda)\, a + \lambda b\}\right|$$

$$= |(1 - \lambda)\, u + \lambda v| < (1 - \lambda)\, \delta + \lambda\delta = \delta\,.$$

Since δ is arbitrary, $(1 - \lambda)a + \lambda b$ is in $\bar{K}$.

THEOREM 11. *If* K *is a convex set such that* Int K *is non-empty, then* Int K = Int $\bar{K}$ *and* Front K = Front $\bar{K}$ = Front (E - $\bar{K}$).

Proof: Clearly Int K $\subset$ Int $\bar{K}$. On the other hand $\bar{K}$ = Int K $\cup$ Front K and by Theorem 8 no frontier point of K can be an interior point of $\bar{K}$. Thus Int K = Int $\bar{K}$. Finally

Front $\bar{K}$ = $\bar{\bar{K}}$ - Int $\bar{K}$ = $\bar{K}$ - Int K = Front K .

THEOREM 12. *If a convex set* K *has interior points but is not the whole space, then there are points exterior to* K, *that is,* $\bar{K}$ *is not the whole space.*

Proof: Since K is not the null set or the whole space, Front K is non-empty. Thus this theorem follows from Theorem 8.

As remarked at the beginning of this section the above theorems (and their proofs) are valid for any normed linear space over the real numbers. We conclude this section with some remarks about the Euclidean case. After this section we shall consider the Euclidean case exclusively, although some of our proofs hold more generally.

The dimension of a convex set in Euclidean space E_n of n dimensions is the dimension of the smallest linear manifold containing the set. By Theorem 1 an m-dimensional convex set contains an m-simplex, but does not contain an (m+1)-simplex. Accordingly a convex set in E_n has interior points if and only if it is n-dimensional. Hence a convex set in E_n without interior points lies in some (n-1)-dimensional linear manifold. Thus in the Euclidean case the condition in Theorem 11 and 12 that Int K be non-empty may be dropped. Hence we have the following result.

THEOREM 13. *If K is any convex set in Euclidean space E_n, then* $\text{Int } K = \text{Int } \bar{K}$ *and* $\text{Front } K = \text{Front } \bar{K} = \text{Front } (E - \bar{K})$; *if in addition K is not the whole space, $\bar{K}$ is not the whole space.*

§3. Tac-planes of convex sets.

We say that the plane $u \cdot x = u_0$ is a bounding plane of a set M in E_n if either

$$\sup_{z \in M} u \cdot z < u_0 \quad \text{or} \quad \inf_{z \in M} u \cdot z > u_0 .$$

We say that the plane $u \cdot x = u_0$ is a tac-plane of a set M in E_n if either

$$\sup_{z \in M} u \cdot z = u_0 \quad \text{or} \quad \inf_{z \in M} u \cdot z = u_0 .$$

Clearly a plane is a bounding plane (or tac-plane) of M if and only if it is a bounding plane (or tac-plane) of $\bar{M}$. A set M has at most two tac-planes having a given direction.

THEOREM 14. If a set M with an interior point a has a tac-plane through each boundary point, then $\bar{M}$ and Int M are convex.

Proof: If $\bar{M}$ is the whole space, M has no tac-planes and hence no boundary points; thus Int M must also be the whole space and the theorem is trivial in this case. Hence we may assume that there exist points c not in $\bar{M}$. Then there exists a boundary point b on the segment joining a and c; clearly $b \neq a$, $b \neq c$. The tac-plane π to M through b does not contain c, for otherwise it would contain the interior point a, which is impossible. Hence the closed half-space of π containing a includes $\bar{M}$ but does not contain c. Since c is any point exterior to $\bar{M}$, we see that $\bar{M}$ is the intersection of all closed half-spaces containing M. Thus $\bar{M}$ is convex. Now Int M = Int $\bar{M}$, since a boundary point of M has a tac-plane through it and cannot therefore be an interior point of $\bar{M}$. Hence by Theorem 9 Int M is convex. This proof must be modified slightly for E_1.

THEOREM 15. If c is a point exterior to a convex set K, there is a unique point p of $\bar{K}$ closest to c.

Proof: There is at least one point of $\bar{K}$ whose distance from c is minimum, since $\bar{K}$ is closed. Suppose there were two distinct points p and q of $\bar{K}$ having the minimum distance from c. Then, since

$$\tfrac{1}{2}|p - c|^2 + \tfrac{1}{2}|q - c|^2 = |\tfrac{1}{2}(p - c) + \tfrac{1}{2}(q - c)|^2 + |\tfrac{1}{2}(p - c) - \tfrac{1}{2}(q - c)|^2$$

$$= |\tfrac{1}{2}(p + q) - c|^2 + |\tfrac{1}{2}(p - q)|^2 ,$$

the point $\frac{1}{2}(p + q)$ would have a smaller distance from c, a contradiction.

THEOREM 16. Suppose c is a point exterior to a convex set K and p is the point of $\bar{K}$ closest to c. Then the plane

$$\pi_1 : (c - p) \cdot (x - p) = 0,$$

is a tac-plane of K such that c lies in the open positive half-space thereof and K lies in the closed negative half-space. Also the plane

$$\Pi_2 : (c - p) \cdot (x - c) = 0 ,$$

is a bounding plane of K such that K lies in the open negative half-space thereof.

Proof: Clearly c lies in the open positive half-space of Π_1. To see that K lies in the closed negative half-space of Π_1 we notice that for $0 < \lambda < 1$ the point $(1 - \lambda)p + \lambda z = p + \lambda(z - p)$ is in $\bar{K}$ for $z \in K$ and thus

$$|c - p|^2 < |c - p - \lambda(z - p)|^2 = \lambda^2|z - p|^2 - 2\lambda(c - p)\cdot(z - p) + |c - p|^2$$

or

$$(c - p) \cdot (z - p) < \tfrac{1}{2}\lambda |z - p|^2 .$$

Since λ can be arbitrarily small we must have

$$(c - p) \cdot (z - p) \leq 0$$

for any z in K. Thus the statement about Π_1 is proved. To see that Π_2 is a bounding plane of K we need only observe that

$$(c - p) \cdot z \leq (c - p) \cdot p = (c - p) \cdot c - |c - p|^2$$

for any z in K.

THEOREM 17. If K is a convex set, $\bar{K}$ is the intersection of all closed half-spaces containing K.

Proof: By Theorem 16, if c is exterior to K there exists a closed half-space containing K but not c.

THEOREM 18. If b is a boundary point of a convex set K, there is at least one tac-plane of K passing through b.

Carathéodory's Proof: (Cf. C. Carathéodory, Rend.Circ.Mat.Palermo vol.32(1911)pp.195-201 or L. L. Dines, Amer.Math.Monthly vol.46(1938)pp. 199-209). By Theorem 13 there exists a sequence of points $\{b^{(k)}\}$ exterior to K tending to b. By Theorem 16 there is a bounding plane of K through $b^{(k)}$. Thus we can assert the existence of a plane $u^{(k)}\cdot x = u_o^{(k)}$ such that

$$u^{(k)}\cdot b^{(k)} = u_o^{(k)}, \quad u^{(k)}\cdot z < u_o^{(k)} \text{ for } z \in K, \quad |u^{(k)}| = 1 .$$

Since the unit-sphere in Euclidean space E_n is compact, we may assume that the point $u^{(k)}$ tends to a limit u as k goes to infinity. Since

$$\text{abs } u_o^{(k)} = \text{abs } u^{(k)}\cdot b^{(k)} \leq |b^{(k)}| ,$$

abs $u_o^{(k)}$ is bounded and so we may assume that the number $u_o^{(k)}$ tends to a limit u_o as k goes to infinity. Clearly

$$u \cdot b = u_o, \qquad u \cdot z \leq u_o \text{ for } z \in K , \quad |u| = 1 ,$$

and so $u \cdot x = u_o$ is the equation (in normal form) of a tac-plane to K passing through b.

McShane's Proof: (Cf. T. Botts, Amer.Math. Monthly vol.69(1942) pp. 532-533. Botts also gives another nice proof). Let S be the set of points x such that $|x - b| = 1$. If R(x) is the distance of a point x from K, then $R(x) \leq |x - b|$ and hence $R(x) \leq 1$ for $x \in S$. Since R(x) is continuous and S is compact, there is a point $c \in S$ such that

$$R(x) \leq R(c) \qquad \text{for all } x \in S.$$

We claim that $R(c) = 1$. In fact if δ is a given positive number less than 1, there exists (by Theorem 13) a point d exterior to K such that $|d - b| < \delta$. Suppose Π : $u \cdot x = u_o$ is a bounding plane of K through d such that $|u| = 1$ and $\bar{K}$ lies in the negative open half-space of Π. Thus

$$u \cdot d = u_o, \quad |u| = 1, \quad \sup_{z \in \bar{K}} (u \cdot z) < u_o .$$

Since in particular $u \cdot b < u_o$, we have

$$0 < u_o - u \cdot b = u \cdot (d - b) \leq |d - b| < \delta ,$$

Hence for any z in K we have

$$|u + b - z| \geq u \cdot (u + b - z) > 1 + u \cdot b - u_o > 1 - \delta .$$

Thus $u + b$ is a point of S such that $R(u + b) > 1 - \delta$. Hence $R(c) > 1 - \delta$ and, since δ is arbitrary, $R(c) = 1$ as claimed. Now $|c - b| = 1$ and hence b must be the unique point of $\bar{K}$ closest to the exterior point c. By Theorem 16 the plane through b perpendicular to $c - b$ is a tac-plane to K.

§4. Convex hulls and convex closures.

The convex hull H(M) of a point set M is the smallest convex set containing M, that is, the intersection of all convex sets containing M. The convex closure C(M) of a point set M is the smallest closed convex set containing M, that is, the intersection of all closed convex sets containing M.

THEOREM 19. $H(\bar{M}) \subset \overline{H(\bar{M})} = \overline{H(M)} = C(M)$.

Proof: By Theorem 10 $\overline{H(M)}$ is convex as well as closed.

We shall prove later that $H(\bar{M}) = C(M)$ if M is a bounded set in E_n. If M is unbounded, $H(\bar{M})$ can be properly contained in C(M); an example of this is the set of points (x_1, x_2) in E_2 such that $x_2 = e^{-x_1^2}$.

THEOREM 20. M, H(M), and C(M) have exactly the same tac-planes and bounding planes and are contained in exactly the same closed half-spaces.

Proof: For each non-zero vector u the set of points x such that

$$\inf_{z \in M} u \cdot z \leq u \cdot x \leq \sup_{z \in M} u \cdot z$$

is a closed convex set containing M and thus contains H(M) and C(M). Hence

$$\inf_{z \in M} u \cdot z = \inf_{z \in H(M)} u \cdot z = \inf_{z \in C(M)} u \cdot z$$

and

$$\sup_{z \in M} u \cdot z = \sup_{z \in H(M)} u \cdot z = \sup_{z \in C(M)} u \cdot z$$

In view of the definitions of tac-planes, bounding planes, and closed half-spaces, this proves the theorem.

THEOREM 21. C(M) is the intersection of all closed half-spaces containing M.

Proof: By Theorem 17 C(M) is the intersection of all closed half-spaces containing C(M). But a closed half-space contains C(M) if and only if it contains M.

THEOREM 22. C(M) is the set of points through which pass no bounding planes of M.

Proof: If c is not in C(M), by Theorem 16 there is a plane through c which is a bounding plane of C(M) and therefore of M. If c is in C(M), no bounding plane of M can contain c by Theorem 20.

THEOREM 23. H(M) is the set of all points expressible in the form

$$\lambda_o a^{(o)} + \dots + \lambda_p a^{(p)}, \quad \lambda_o \geq 0, \dots, \lambda_p \geq 0, \quad \lambda_o + \dots + \lambda_p = 1 ,$$

where p is any non-negative integer and $a^{(o)}, \dots, a^{(p)}$ are any points of M.

Proof: Let K be the set of points expressible as above. Clearly $M \subset K$ and K is convex. By Theorem 1 $K \subset H(M)$. Hence K = H(M).

THEOREM 24. Suppose

$$c = \lambda_o a^{(o)} + \dots + \lambda_p a^{(p)}, \quad \lambda_o > 0, \dots, \lambda_p > 0 , \lambda_o + \dots + \lambda_p = 1,$$

and suppose r is the dimension of the smallest linear manifold containing $a^{(o)}, \ldots, a^{(p)}$. Then we can find $r+1$ or fewer of the points $a^{(o)}, \ldots, a^{(p)}$ in terms of which c can be expressed linearly with positive coefficients whose sum is unity.

Proof: If $p = r$ there is nothing to prove, so assume $p > r$. Then the p vectors $a^{(1)} - a^{(o)}, \ldots, a^{(p)} - a^{(o)}$ lie in a subspace of E_n of dimension r and thus satisfy a relation of the form

$$\mu_1(a^{(1)} - a^{(o)}) + \ldots + \mu_p(a^{(p)} - a^{(o)}) = 0 ,$$

where not all the μ's are zero. If we put $\mu_o = -\mu_1 - \ldots - \mu_p$ we have

$$\mu_o a^{(o)} + \ldots + \mu_p a^{(p)} = 0, \quad \mu_o + \ldots + \mu_p = 0, \text{ not all } \mu\text{'s zero.}$$

Let ν be that one of $\mu_o/\lambda_o, \ldots, \mu_p/\lambda_p$ which has largest absolute value. (If several have the maximum absolute value, any one may be taken). Since $\nu \neq 0$ we may write

$$c = (\lambda_o - \mu_o/\nu)\, a^{(o)} + \ldots + (\lambda_p - \mu_p/\nu)\, a^{(p)} ,$$

where the coefficients are non-negative and have sum 1 and at least one of them is zero. Since this process may be repeated, our theorem is proved.

THEOREM 25. Suppose r is the dimension of the smallest linear manifold containing a set M in E_n. Then $H(M)$ is the set of all points expressible in the form

$$\lambda_o a^{(o)} + \ldots + \lambda_r a^{(r)}, \quad \lambda_o \geq 0, \ldots, \lambda_r \geq 0, \quad \lambda_o + \ldots + \lambda_r = 1 ,$$

where $a^{(o)}, \ldots, a^{(r)}$ are any points of M.

Proof: This follows from Theorems 23 and 24.

THEOREM 26. If M is a bounded set in E_n, $H(\bar{M}) = C(M)$.

Proof: By Theorem 19 we need only prove that $H(\bar{M})$ is compact and therefore closed. Suppose $\{c^{(i)}\}$ is a sequence of points of $H(\bar{M})$. By

Theorem 25 each $c^{(i)}$ can be expressed in the form

$c^{(i)} = \lambda_{oi}\, a^{(oi)} + \ldots + \lambda_{ni}\, a^{(ni)}, \lambda_{oi} \geq 0, \ldots, \lambda_{ni} \geq 0,\ \lambda_{oi} + \ldots + \lambda_{ni} = 1$

where $a^{(oi)}, \ldots, a^{(ni)}$ are points of $\bar{M}$. Since $\bar{M}$ is compact, we may (by taking subsequences) assume that there exist points $a^{(o)}, \ldots, a^{(n)}$ of $\bar{M}$ such that $a^{(oi)}$ converges to $a^{(o)}$, $a^{(1i)}$ converges to $a^{(1)}$, etc., as i goes to infinity. Since the sequences $\{\lambda_{oi}\}, \ldots, \{\lambda_{ni}\}$ are bounded, we may likewise assume that there exist non-negative real numbers $\lambda_o, \ldots, \lambda_n$ such that λ_{oi} converges to λ_o, λ_{1i} converges to λ_1, etc. Clearly $\lambda_o + \ldots + \lambda_n = 1$ and $c^{(i)}$ converges to

$$c = \lambda_o\, a^{(o)} + \ldots + \lambda_n\, a^{(n)},$$

which is in $H(\bar{M})$, since $a^{(o)}, \ldots, a^{(n)}$ are in $\bar{M}$.

<u>THEOREM 27.</u> <u>If</u> π <u>is a tac-plane of</u> M , <u>then</u> $\pi \cap H(M) = H(\pi \cap M)$.

Proof: Let the equation of π be $u \cdot x = u_o$ and suppose $\sup_{z \in M} u \cdot z = u_o$ If c is a point of $\pi \cap H(M)$ it can be expressed in the form

$$c = \lambda_o\, a^{(o)} + \ldots + \lambda_r\, a^{(r)},\quad \lambda_o > 0, \ldots, \lambda_r > 0,\quad \lambda_o + \ldots + \lambda_r = 1,$$

for some r, $0 \leq r \leq n$, where $a^{(o)}, \ldots, a^{(r)}$ are in M. If $u \cdot a^{(i)} < u_o$ for some i, then

$$u \cdot c = \lambda_o\, u \cdot a^{(o)} + \ldots + \lambda_r u \cdot a^{(r)} < \lambda_o\, u_o + \ldots + \lambda_r\, u_o = u_o$$

and c would not be in π. Hence $u \cdot a^{(i)} = u_o$ for $i = 0, \ldots, r$; that is, each $a^{(i)}$ is in $\pi \cap M$, and so c is in $H(\pi \cap M)$.

<u>THEOREM 28.</u> <u>If</u> M <u>is a bounded set in</u> E_n <u>and</u> π <u>is a tac-plane of</u> M, <u>then</u> $\pi \cap C(M) = C(\pi \cap \bar{M})$.

Proof: By Theorem 26 and 27

$$\pi \cap C(M) = \pi \cap H(\bar{M}) = H(\pi \cap \bar{M}) = H(\overline{\pi \cap \bar{M}}) = C(\pi \cap \bar{M}).$$

§5. Guage functions.

In this section we shall be concerned with convex sets in E_n having inner points.

If a is an inner point of a convex set K, the <u>guage function</u> (or distance function or Minkowski functional) $F(x)$ of K with respect to a is the greatest lower bound (inf) of all positive λ such that $a + \lambda^{-1}(x-a)$ is in K. Clearly $F(x) = 0$ if $x = a$ or if $x \neq a$ and there is no boundary point on the ray from a in the direction $x - a$. If $x \neq a$ and there is a boundary point b on the ray from a in the direction $x - a$, then by Theorem 8 $F(x) = |x - a| / |b - a|$. Also in this latter case $F(x)$ is the least upper bound (sup) of all positive λ such that $a + \lambda^{-1}(x - a)$ is not in K. Thus $F(x) < 1$ if and only if $x \in \text{Int } K$, $F(x) = 1$ if and only if $x \in \text{Front } K$, and $F(x) > 1$ if and only if $x \in E - \bar{K}$.

Without loss of generality we take $a = 0$ throughout the rest of this section. Thus we tacitly assume that our convex sets have the origin as an inner point.

<u>THEOREM 29.</u> <u>The guage function</u> $F(x)$ <u>of a convex set</u> K <u>with respect to the origin has the following three properties:</u>

(a) $F(x) \geq 0$ <u>for each</u> x <u>in</u> E_n;

(b) $F(\mu x) = \mu F(x)$ <u>for</u> $\mu \geq 0$;

(c) $F(x+y) \leq F(x) + F(y)$.

Proof: Properties (a) and (b) are obvious from the definition. If $F(x) = F(y) = 0$, then $2\delta^{-1}x$ and $2\delta^{-1}y$ are in K for any positive δ, so that $\delta^{-1}(x+y)$ is in K for any positive δ and $F(x+y) = 0$. If $F(x) \neq 0$ but $F(y) = 0$, $\{F(x)\}^{-1}x$ is a boundary point of K and $\{\delta F(x)\}^{-1}y$ is an inner point of K for any positive δ; thus

$$F\left(\frac{1}{1+\delta}\ \{F(x)\}^{-1}x + \frac{\delta}{1+\delta}\ \{\delta F(x)\}^{-1}y\right) < 1 ,$$

$F(x+y) < (1+\delta)F(x)$, and, since δ is arbitrary,

$$F(x+y) \leq F(x) = F(x) + F(y).$$

If $F(x) \neq 0$ and $F(y) \neq 0$, then $\{F(x)\}^{-1}x$ and $\{F(y)\}^{-1}y$ are in $\bar{K}$, so that

$$F\left(\frac{F(x)}{F(x)+F(y)}\ \{F(x)\}^{-1}x + \frac{F(y)}{F(x)+F(y)}\ \{F(y)\}^{-1}y\right) \leq 1$$

and

$$F(x+y) \leq F(x) + F(y) .$$

THEOREM 30. If $F(x)$ is any real-valued function on E_n having properties (b) and (c) of Theorem 29, then there exists a non-negative number ρ such that abs $F(x) \leq \rho\ |x|$; moreover $F(x)$ is continuous.

Proof: Suppose $e^{(1)}, \ldots, e^{(n)}$ are the unit vectors in E_n. Then

$$x = \sum_{i=1}^{n} x_i\ e^{(i)} = \sum_{i=1}^{n} (\text{abs } x_i)\ (\pm\ e^{(i)}),$$

where the sign before $e^{(i)}$ is chosen as that of the coordinate x_i. Let σ be the maximum of the $2n$ numbers $F(\pm\ e^{(i)})$, $i = 1, \ldots, n$. Then

$$F(x) \leq \sum_{i=1}^{n} F(\ \{\text{abs } x_i\}\ \{\pm\ e^{(i)}\}\ ) = \sum_{i=1}^{n} (\text{abs } x_i)\ F(\pm\ e^{(i)})$$

$$\leq \sigma \sum_{i=1}^{n} \text{abs } x_i \leq \sigma\ n^{\frac{1}{2}}\left(\sum_{i=1}^{n} x_i^{\ 2}\right)^{\frac{1}{2}} = \sigma\ n^{\frac{1}{2}}\ |x| .$$

Since $0 = F(0) \leq F(x) + F(-x)$, we have also

$$F(x) \geq -\ F(-x) \geq -\ \sigma\ n^{\frac{1}{2}}\ |-x| = -\ \sigma\ n^{\frac{1}{2}}\ |x| .$$

Thus the inequality of our theorem is satisfied with $\rho = \sigma\ n^{\frac{1}{2}}$. The continuity of $F(x)$ at $x = a$ follows from the inequalities

$$-\rho|x| \leq -F(-x) \leq F(a+x) - F(a) \leq F(x) \leq \rho|x| .$$

THEOREM 31. *If $F(x)$ is any real-valued function on E_n having properties (a), (b), and (c) of Theorem 29, then the set $\{x \mid F(x) < 1\}$ is an open convex set including the origin and the set $\{x \mid F(x) \leq 1\}$ is a closed convex set having the origin as inner point; moreover $F(x)$ is the guage function of both these convex sets.*

Proof: By Theorem 30 $F(x)$ is continuous and thus the topological statements follow, since $F(0) = 0$. To show convexity we merely observe that if $F(x) < 1$ and $F(y) < 1$, then for $0 < \lambda < 1$ we have

$$F(\lambda x + \{1-\lambda\}y) \leq F(\lambda x) + F(\{1-\lambda\}y) = \lambda F(x) + (1-\lambda) F(y) < 1;$$

similarly if $F(x) \leq 1$ and $F(y) \leq 1$, then $F(\lambda x + \{1-\lambda\} y) \leq 1$ for $0 < \lambda < 1$. Condition (b) shows that $F(x)$ is actually the guage function of $\{x \mid F(x) < 1\}$ and $\{x \mid F(x) \leq 1\}$ (and that the second of these sets is the closure of the first).

THEOREM 32. *If $F(x)$ is the guage function of the convex set K with respect to the origin, then K is contained in the sphere $|x| \leq \rho$ if and only if $F(x)$ has the property $F(x) \geq \rho^{-1}|x|$.*

Proof: Suppose $F(x) \geq \rho^{-1}|x|$; then if x is in K, $F(x) \leq 1$ and so $|x| \leq \rho$. Suppose K is contained in $|x| \leq \rho$; then if $|x| = \rho$, $F(x) \geq 1$; hence by homogeneity $F(x) \geq \rho^{-1}|x|$ for any x.

THEOREM 33. *If $F(x)$ is the guage function of the convex set K with respect to the origin, then K is symmetrical with respect to the origin if and only if $F(x)$ has the property $F(-x) = F(x)$.*

Proof: Clear.

THEOREM 34. If $F(x)$ is the guage function of the convex set K with respect to the origin and if K is symmetrical with respect to the origin, then the points of E_n for which $F(x) = 0$ form a linear subspace.

Proof: If $F(x) = 0$ and $F(y) = 0$, then for any real λ we have

$$F(\lambda x + \{1-\lambda\}y) \leq F(\lambda x) + F(\{1-\lambda\}y) = (\text{abs}\,\lambda)F(x) + (\text{abs}\{1-\lambda\})\,F(y) = 0.$$

THEOREM 35. The boundary of a bounded convex set K with inner points is homeomorphic to the surface of the unit sphere in E_n.

Proof: Suppose the origin is an inner point of K and let $F(x)$ be the guage function of K with respect to the origin. Then $x \rightarrow \{F(x)\}^{-1}x$ is a one-to-one continuous mapping of the unit sphere onto the boundary of K and hence gives a homeomorphism. Actually the inverse mapping from the boundary to K to the surface of the unit sphere is simply $x \longrightarrow |x|^{-1}x$.

§6. Tac-functions.

In this section we shall be concerned with convex sets in E_n which are bounded.

The tac-function $G(u)$ of a bounded convex set K is a real-valued function on E_n defined as follows:

$$G(u) = \sup_{z \in K} u \cdot z .$$

Clearly K and $\bar{K}$ have the same tac-function.

THEOREM 36. If K is a bounded convex set and $G(u)$ is the tac-function of K, then for $u \neq 0$ the plane $u \cdot x = G(u)$ is the unique tac-plane with direction number $|u|^{-1}u$ and positive open half-space free of points of K.

Proof: This follows from the definition of tac-plane and tac-function.

THEOREM 37. If K is a bounded convex set and $G(u)$ is its tac-function, $\bar{K}$ is the set of all points x such that $u \cdot x \leq G(u)$ for every u; in other words $\bar{K}$ is the intersection of the closed half-spaces $u \cdot x \leq G(u)$.

Proof: Cf. Theorems 16 and 17.

THEOREM 38. The tac-function $G(u)$ of a bounded convex set K has the property

$$G(u) + G(-u) \geq 0 ,$$

with equality if and only if $u = 0$ or if $u \neq 0$ and K lies in the plane $u \cdot x = G(u)$.

Proof: In fact

$$G(u) = \sup_{z \in K} u \cdot z \geq \inf_{z \in K} u \cdot z = - \sup_{z \in K} (-u \cdot z) = - G(-u).$$

THEOREM 39. If $|u| = 1$ the tac-function $G(u)$ of a bounded convex set K is the distance from the origin to the tac-plane $u \cdot x = G(u)$, the distance being reckoned as positive or negative according as the origin is in the negative or positive open half-space of $u \cdot x = G(u)$. (K is in the closed negative half-space of $u \cdot x = G(u)$).

Proof: In fact the vector $G(u)\, u$ is the shortest vector from the origin to the plane $u \cdot x = G(u)$.

THEOREM 40. If $G(u)$ is the tac-function of a bounded convex set K and $H(u)$ is the tac-function of a bounded convex set L, then $G(u) \leq H(u)$ for every u if and only if $K \subset L$.

Proof: This follows from Theorem 37 and the definition of tac-function.

THEOREM 41. If $G(u)$ is the tac-function of a bounded convex set K, then the tac-function of $K + a$ is $G(u) + a \cdot u$.

Proof: In fact

$$\sup_{z \in K+a} u \cdot z = \sup_{y \in K} u \cdot (y+a) = \sup_{y \in K} u \cdot y + u \cdot a .$$

Some examples of tac-functions are as follows. The tac-function of a point a is $a \cdot u$, that of the unit sphere is $|u|$, that of the segment from a to b is $\max(a \cdot u, b \cdot u)$, that of the segment from $-a$ to a is $|a \cdot u|$, that of the cube $\max_{k=1,\ldots,n} |x_k| \leq 1$ is $\sum_{k=1}^{n} |u_k|$, that of the octahedron $\sum_{k=1}^{n} |x_k| \leq 1$ is $\max_{k=1,\ldots,n} |u_k|$.

THEOREM 42. The tac-function $G(u)$ of a bounded convex set K has the properties

(b) $G(\mu u) = \mu G(u)$ for $\mu \geq 0$;

(c) $G(u+v) \leq G(u) + G(v)$.

Proof: Property (b) is obvious from the definition of tac-function. As for (c) we have

$$\sup_{z \in K} (u+v) \cdot z \leq \sup_{z \in K} u \cdot z + \sup_{z \in K} v \cdot z .$$

We wish to show that any real-valued function on E_n with properties (b) and (c) is the tac-function of some bounded convex set. For this purpose we need a theorem on the tac-planes of a convex cone. A convex cone with vertex a is a convex set C containing a and points other than a such that if $b \neq a$ and $b \in C$, then

$$(1 - \lambda) a + \lambda b = a + \lambda(b - a)$$

is in C for any positive λ.

THEOREM 43. Any tac-plane of a convex cone passes through the vertex of the cone.

Proof: Suppose $u \cdot x = u_0$ is a tac-plane of the convex C with vertex a and suppose $\sup_{z \in C} u \cdot z = u_0$. Then $u \cdot a \leq u_0$. Thus if a does not lie on $u \cdot x = u_0$ we must have $u \cdot a = u_0 - \delta$, $\delta > 0$. Since $\sup_{z \in C} u \cdot z = u_0$, there is a point b of C such that $u \cdot b = u_0 - \theta\delta$, where $0 \leq \theta < 1$. Now for any positive λ the point $(1-\lambda)a + \lambda b$ is in C. But the quantity

$$u \cdot \{(1-\lambda)a + \lambda b\} - u_0 = (1-\lambda)(u_0 - \delta) + \lambda(u_0 - \theta\delta) - u_0 = \{(1-\theta)\lambda - 1\}\delta$$

is positive for large λ, a contradiction. Hence $u \cdot a = u_0$.

THEOREM 44. If $G(u)$ is any real-valued function on E_n having properties (b) and (c) of Theorem 42, then the set $K = \{x \mid u \cdot x \leq G(u) \text{ for every } u\}$ is a closed bounded convex set of which $G(u)$ is the tac-function. (By Theorem 37 K is the only such set).

Proof: (Cf. Rademacher, Sitzungsberichte der Berliner Mathematischen Gesellschaft 20(1920) 14-19). That K is closed and convex follows from the fact that it is the intersection of closed half-spaces. To prove boundedness we note that by Theorem 30 $G(u)$ is continuous and so has a finite maximum σ on the set $|u| = 1$. Hence for every u with $|u| = 1$ the set K is contained in the half-space $u \cdot x \leq \sigma$. In particular if $x \neq 0$ and $x \in K$, then $|x|^{-1}x \cdot x \leq \sigma$ or $|x| \leq \sigma$. Thus K is bounded.

To show that $G(u)$ is the tac-function of K we need only show that for each fixed $\bar{u}$, there exists an $\bar{x}$ such that

$$\bar{u} \cdot \bar{x} = G(\bar{u}), \quad u \cdot \bar{x} \leq G(u) \quad \text{for any } u .$$

(In particular, this will show that K is non-empty). To this end consider the set W in E_{n+1} consisting of those points $(u, u_{n+1}) = (u_1, \ldots, u_n, u_{n+1})$ such that $u_{n+1} \geq G(u)$. The set W is a convex cone with vertex at the origin; for if $u_{n+1} \geq G(u)$, then $\lambda u_{n+1} \geq \lambda G(u)$ for any non-negative λ, and if

$u_{n+1} \geq G(u)$ and $v_{n+1} \geq G(v)$, then $\lambda u_{n+1} + (1-\lambda) v_{n+1} \geq \lambda G(u) + (1-\lambda) G(v) =$ $= G(\lambda u) + G(\{1-\lambda\}v) \geq G(\lambda u + \{1-\lambda\}v)$ for $0 \leq \lambda \leq 1$. If we put $\bar{u}_{n+1} = G(\bar{u})$, the point $(\bar{u}, \bar{u}_{n+1})$ is a boundary point of W. There exists a tac-plane π to W through $(\bar{u}, \bar{u}_{n+1})$. By Theorem 43, π passes through the origin and hence has the form

$$a \cdot u + a_{n+1} u_{n+1} = 0 ,$$

where a is an n-dimensional vector such that (a, a_{n+1}) is a non-zero $(n+1)$-dimensional vector. We may assume that W is in the closed positive half-space of π . Thus we have

$$a \cdot \bar{u} + a_{n+1} \bar{u}_{n+1} = 0, \quad a \cdot u + a_{n+1} u_{n+1} \geq 0 \text{ for } (u, u_{n+1}) \in W$$

For any given u the point (u, u_{n+1}) is in W for large u_{n+1}, and hence $a_{n+1} \geq 0$. If a_{n+1} were zero, $a \cdot u$ would be non-negative for all u, so that in particular $a \cdot (-a) \geq 0$ and $a = 0$, a contradicion to the fact that (a, a_{n+1}) is a non-zero vector. Hence we have $a_{n+1} > 0$. If we put $\bar{x} = -a_{n+1}^{-1} a$, we have

$$\bar{x} \cdot \bar{u} = \bar{u}_{n+1}, \quad \bar{x} \cdot u \leq u_{n+1} \text{ for } (u, u_{n+1}) \in W .$$

In other words

$$\bar{x} \cdot \bar{u} = G(\bar{u}), \quad \bar{x} \cdot u \leq u_{n+1} \text{ if } u_{n+1} \geq G(u) \text{ (for any u)}.$$

Thus

$$\bar{x} \cdot \bar{u} = G(\bar{u}), \quad \bar{x} \cdot u \leq G(u) \text{ for any u}.$$

Accordingly Theorem 44 is proved.

THEOREM 45. *If* $G(u)$ *is the tac-function of a bounded convex set* K, *then* K *contains the sphere* $|x| < \rho$ *if and only if* $G(u)$ *has the property* $G(u) \geq \rho\, |u|$.

Proof: If the sphere $|x| < \rho$ is contained in K, then $\frac{\rho}{|u|}$ u is in $\bar{K}$ for every non-zero u and hence

$$G(u) = \sup_{x \in \bar{K}} (u \cdot x) \geq u \cdot (\frac{\rho}{|u|} u) = \rho\, |u| .$$

If G(u) has the property $G(u) \geq \rho\, |u|$, then for each u the half-space $u \cdot x \leq G(u)$ contains the sphere $|x| \leq \rho$, for if $|x| \leq \rho$, then $u \cdot x \leq |u||x| \leq \rho\, |u| \leq G(u)$. Since $\bar{K} = \{x \mid u \cdot x \leq G(u) \text{ for every } u\}$, it follows that $\bar{K}$ contains the sphere $|x| \leq \rho$ and hence K contains the sphere $|x| < \rho$ (by Theorem 13).

§7. Polar reciprocal convex bodies.

Throughout this section a guage function will be understood to mean a guage function with respect to the origin.

A convex set in E_n has both a guage function and a tac-function only if it is bounded and has the origin as inner point. In this section we consider convex sets with these properties and the additional property of being closed (so that no two distinct sets will have the same guage function or tac-function). A convex set which is bounded, closed, and has the origin as inner point we call a convex body.

THEOREM 46. If F(x) is the guage function of a convex set K having the origin as inner point, then K is bounded if and only if $F(x) > 0$ for $x \neq 0$.

Proof: If $F(x) > 0$ for $x \neq 0$, then $F(x) \geq \rho^{-1}x$, where ρ^{-1} is the minimum of the continuous function F(x) on the set $|x| = 1$. The result then follows from Theorem 32.

THEOREM 47. If $G(u)$ is the tac-function of a bounded convex set K, then K has the origin as inner point if and only if $G(u) > 0$ for $u \neq 0$.

Proof: Cf. Theorem 45 and the proof of Theorem 46.

THEOREM 48. Both the guage function and the tac-function of a convex body have the following three properties:

(a^*) $H(x) > 0$ for $x \neq 0$;

(b) $H(\mu x) = \mu H(x)$ for $\mu \geq 0$;

(c) $H(x+y) \leq H(x) + H(y)$.

Moreover any real valued function with these three properties is both the guage function of a unique convex body and the tac-function of a unique convex body.

Proof: Cf. Theorems 31, 44, 46, 47.

THEOREM 49. If a convex body K has guage function $F(x)$ and tac-function $G(u)$, then the convex body L with guage-function G has tac-function F.

Proof: Since

$$G(u) = \sup_{x \in K} u \cdot x = \sup_{x \neq 0} \frac{u \cdot x}{F(x)} ,$$

we have

$$u \cdot x \leq G(u)\, F(x) \quad \text{for any } u \text{ and } x,$$

where for each non-zero u equality holds for at least one non-zero x. Moreover we claim that for each non-zero x equality holds for at least one non-zero u. For by homogeneity we may take $F(x) = 1$, so that x is a boundary point of K. Then there exists at least one tac-plane of K passing through x, that is, there is at least one non-zero u such that $u \cdot x = G(u)$. Thus our assertion is proved. It follows that

$$F(x) = \sup_{u \neq 0} \frac{u \cdot x}{G(u)} = \sup_{u \in L} u \cdot x ,$$

and hence $F(x)$ is the tac-function of L.

When two convex bodies are so related that the guage function of one is the tac-function of the other, we say that they are polar reciprocal convex bodies. Polar reciprocity can also be defined more generally for closed convex sets including the origin, without making use of guage functions and tac-functions.

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SEMINAR ON CONVEX SETS

II, Polar Reciprocity

by

Hans Rådström

§1. We will denote the inner product of two vectors x and y by xy. To any point a in the space E_n there corresponds a closed point set, which we will call a closed halfspace, and which is characterized by the requirement that the inner product of any one of its points with a is less than or equal to 1. If, in particular, a is the origin, then the corresponding half-space is the whole space. We observe that a closed half-space in our present terminology always contains the origin.

Definition: Let S be any point set in the space. We define its polar reciprocal set S^*, to be the set of all points a such that their corresponding half-spaces contain S. Formally $S^* = \{a \mid x \in S \Rightarrow ax \leq 1\}$.

Examples:

1. S is the set of one point. Then S^* is the closed half-space determined by this point. In particular $\{0\}^*$ = the whole space.

2. Let S be a linear subspace. Then S^* is the orthogonal linear subspace.

3. Let C(r) denote a closed solid sphere with radius r and with centre at the origin. Then $C(r)^* = C(\frac{1}{r})$.

§2. Theorem: S^* is closed, convex and contains the origin.

Proof: The proof is more or less trivial. We prove only the convexity. Let a_1 and a_2 be elements of S^*. Therefore, for all $x \in S$ we have $a_1 x \leq 1$. Let $0 \leq \alpha \leq 1$. We get : For all $x \in S$: $[\alpha a_1 + (1-\alpha)a_2]x \leq 1$ which proves that S^* is convex.

§3. Theorem: $A \subset B \Rightarrow A^* \supset B^*$.

Proof: Let $a \in B^*$. This means that for all $x \in B$ we have $ax \leq 1$ so that in particular $ax \leq 1$ holds for all $x \in A$. Therefore $a \in A^*$.

§4. Theorem: 1. S bounded $\Rightarrow S^*$ has the origin as inner point.

2. S has the origin as inner point $\Rightarrow S^*$ bounded.

Proof: 1. Let S be contained in $C(r)$. Then (Theorem §3) S^* contains $C(r)^* = C(\frac{1}{r})$.

Proof: 2. The other way around.

§5. Theorem: S^{**} = convex closure of $(\{0\} \cup S)$.

Proof: S^{**} is the set of all points x for which it is true that all a whith the property $y \in S \Rightarrow ay \leq 1$ also have the property $ax \leq 1$, or in other words: S^{**} is the set of all points x which are contained in all closed half-spaces which contain S. Thus S^{**} is the intersection of the closed half-spaces which contain S, and the theorem now follows from a result on convex closure which has been proved earlier in this seminar (p.13).

§6. Theorem: $(\bigcup_\alpha S_\alpha)^* = \bigcap_\alpha S_\alpha^*$

Proof: First, assume $a \in (\bigcup S_\alpha)^*$.

Thus: For all $x \in \bigcup S_\alpha$ we have $ax \leq 1$,

or: For all $x \in S_\alpha$ we have $ax \leq 1$.

Therefore: $a \in S_\alpha^*$. But this holds for any α, so that $a \in \bigcap S_\alpha^*$.

Secondly assume $a \in \bigcap S_\alpha^*$. It is easy to verify that the previous argument can be done in the opposite direction.

§7. From now on we are going to restrict ourselves to considering only closed, convex sets containing the origin. Denote the class of such sets by $\mathcal{K}$. According to theorem §2 the polar reciprocal of any set is a set in $\mathcal{K}$. We are going to show conversely, that any set in $\mathcal{K}$ is the polar reciprocal of some set and more specifically of a unique set in $\mathcal{K}$. This follows at once from the following theorem

§8. Theorem: If $K \in \mathcal{K}$ then $K^{**} = K$.

Proof: Direct consequence of theorem §5 and the definition of $\mathcal{K}$

§9. Theorem: If $K_\alpha \in \mathcal{K}$ then $(\bigcap K_\alpha)^* =$ convex closure of $\bigcup K_\alpha^*$.

Proof: Theorems §8 and §6 give

$$\bigcap K_\alpha = \bigcap K_\alpha^{**} = (\bigcup K_\alpha^*)^*$$

Thus $(\bigcap K_\alpha)^* = (\bigcup K_\alpha^*)^{**}$ which according to Theorem §5 is = convex closure of $(\{0\} \cup \bigcup K_\alpha^*)$ = convex closure of $\bigcup K_\alpha^*$.

§10. For convex cones there is some confusion of terminology in the literature. If K is a convex cone with vertex at the origin then in our notation $K^* = \{a \mid x \in K \Rightarrow ax \leq 1\}$, which is equivalent to saying $K^* = \{a \mid x \in K \Rightarrow ax \leq 0\}$, because of the fact that K is a cone. This is the definition adopted by Bonnesen-Fenchel. Some authors have instead $K^* = \{a \mid x \in K \Rightarrow ax \geq 0\}$ which is the cone obtained from ours by a reflection in the origin.

III. The Coefficient Problem for Analytic Functions with Positive Real Part.

§1. We are going to consider a special problem of the following general type: Let S be a set of functions analytic in the open unit circle. Assume also

(a) S is convex, that is, $f_1 \in S$, $f_2 \in S$, $0 \leq \alpha \leq 1 \Longrightarrow \alpha f_1 + (1-\alpha)f_2 \in S$.

(b) S is compact. Here is meant compact in the topology in which convergence is defined as uniform convergence on every compact subset of the open unit circle. (This is equivalent to saying: S is normal and contains all its limit functions, and the infinite constant is not a limit-function).

§2. Examples: (a) S is the set of functions with modulus bounded by 1 or: $f \in S \Longleftrightarrow |f(x)| \leq 1$ for all x with $|x| < 1$.

(b) S is the set of functions which take the value $\frac{1}{2}$ at the origin and which have real part ≥ 0. In this example the requirement $f(0) = \frac{1}{2}$ serves the purpose of preventing the infinite constant from being a limit function. The peculiar choice of $\frac{1}{2}$ instead of 1 will be explained later. The coefficient problem for this class of functions is the problem we are going to consider. But first some general considerations.

§3. Each function of S can be written as a power series $\sum_{\nu=0}^{\infty} c_\nu x^\nu$ convergent for $|x| < 1$. Consider in a 2n + 2 dimensional Euclidean space E_{2n+2} the set K_{2n+2} of all points with the property that their coordinates $a_0, b_0, a_1, b_1, \dots, a_n, b_n$ correspond to a power series $\sum_{\nu=0}^{\infty} c_\nu x^\nu \in S$ with $c_\nu = a_\nu - i b_\nu$ for $0 \leq \nu \leq n$.

<u>Proposition:</u> K_{2n+2} is closed and convex.

Proof: Let $x_\nu \in K_{2n+2}$ and $x_\nu \longrightarrow x$, $\nu = 1\ 2\ 3 \ldots\ldots$. Choose corresponding functions $f_\nu \in S$ so that the coefficient sequence of f_ν begins with something corresponding to the coordinates of x_ν. S being compact, we can choose a convergent subsequence of the sequence $\{f_\nu\}$. Obviously the limit function corresponds to the point x which therefore has to be in K_{2n+2}. - The convexity part is easy.

§4. If S is known then the convex sets K_{2n+2} can be constructed (at least theoretically) for every value of $n = 0\ 1\ 2 \ldots\ldots$. Conversely, if the sets K_{2n+2} are known for <u>an infinite number</u> of values of n, then the set S is determined, or in other words, two different sets S_1 and S_2 can not give the same sets K_{2n+2} for an infinite number of values of n. This means that the knowledge of an infinite number of sets K solves the coefficient problem for the set S. In order to prove this we assume that $f(x) = \sum_{\nu=0}^{\infty} (a_\nu - i\, b_\nu)x^\nu$ is an element of S_1 and that S_1 and S_2 give the same sets K_{2n+2} for $n = n_1, n_2, n_3, \ldots$. We shall prove that $f \in S_2$. Let $f_{n_1}, f_{n_2}, \ldots$ be functions $\in S_2$ with their coefficient sequences starting with (respectively):

$$a_0 - i\, b_0\ ,\ a_1 - i\, b_1\ ,\ \ldots\ ,\ a_{n_1} - i\, b_{n_1}$$

$$a_0 - i\, b_0\ ,\ a_1 - i\, b_1\ ,\ \ldots\ ,\ a_{n_2} - i\, b_{n_2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

S_2 being compact, the sequence $\{f_{n_\nu}\}$ has a convergent subsequence tending to a function $g \in S_2$. But the convergence is uniform so the coefficients of the functions also converge to the coefficients of the limit function which gives $g = f$ and $f \in S_2$.

§5. We shall now determine the sets K_{2n+2} for the set S of Example 2(b). In this case $a_0 = \frac{1}{2}$, $b_0 = 0$, so it is enough to consider points in E_{2n} with coordinates $(a_1, b_1, a_2, b_2, \dots, a_n, b_n)$. Denote the set of such points L_{2n}. These are the sets which solve the coefficient problem. The determination of these sets is due to Carathéodory (Rend. Circ. Mat. Palermo vol.32(1911)pp.193 - 217), who proves the following theorem.

Theorem: L_{2n} is equal to the convex hull of the curve which is represented by the equations

$$\begin{cases} x_\nu = \cos \nu\theta \\ y_\nu = \sin \nu\theta \end{cases}$$

where $x_1, y_1, x_2, y_2, \dots, x_n, y_n$ are the coordinates of a point in E_{2n} and θ a real parameter.

Proof: Denote the hull of the curve by H_{2n}. We first prove $H_{2n} \subset L_{2n}$. Let $p \in H_{2n}$. Then there exist λ_μ with $\lambda_\mu \geq 0$ and $\sum \lambda_\mu = 1$ and points p_μ on the curve so that $p = \sum \lambda_\mu p_\mu$. (We can always assume that μ runs from 0 to n, but this is not essential for the proof). Let θ_μ be values of the parameter corresponding to the points p_μ.

Now, observe that the function $\chi_\mu(x) = \frac{1}{2} + \sum_{\nu=1}^{\infty} e^{-i\nu\theta_\mu} x^\nu$ has the coefficients $\frac{1}{2}$, $\cos\theta_\mu - i\sin\theta_\mu$, $\cos 2\theta_\mu - i \sin 2\theta_\mu$, ... and therefore corresponds to the point p_μ. Further, $\chi_\mu(0) = \frac{1}{2}$ and $R\,e\,\chi_\mu(x)$ is not negative, for

$$\chi_\mu(x) = -\frac{1}{2} + \frac{1}{1 - e^{-i\theta_\mu} x},$$

and the function $z = \frac{1}{1-x}$ maps the unit circle conformally on the half-plane $R\,e\,z > \frac{1}{2}$. Therefore, the function $f(x) = \sum \lambda_\mu \chi_\mu(x)$ satisfies

$$f(0) = \tfrac{1}{2}, \quad \mathrm{Re}\,(f(x)) \geq 0 \text{ and thus } f \in S.$$

On the other hand f has a coefficient sequence which corresponds to p. Hence $p \in L_{2n}$.

The second part of the proof is to prove that $L_{2n} \subset H_{2n}$. Assume $p = (a_1, b_1, \ldots, a_n, b_n) \in L_{2n}$. Therefore, there is a function $f \in S$ whose coefficients start with $\frac{1}{2}$, $a_1 - i\,b_1$, $a_2 - i\,b_2$, ..., $a_n - i\,b_n$. Put $x = r \cdot e^{i\theta}$ and $\mathrm{Re}\,(f(x)) = U(r;\theta) = \frac{1}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos\nu\theta + b_\nu \sin\nu\theta) r^\nu$. We have $U(0;\theta) = \frac{1}{2}$ and $U(r;\theta) \geq 0$. Now Euler's formulas for the Fourier coefficients give:

$$\frac{1}{\pi}\int_0^{2\pi} U(r;\theta)\,d\theta = 1,$$

$$\frac{1}{\pi}\int_0^{2\pi} U(r;\theta)\cos\nu\theta\,d\theta = a_\nu r^\nu \qquad (\nu = 1,2,\ldots),$$

$$\frac{1}{\pi}\int_0^{2\pi} U(r;\theta)\sin\nu\theta\,d\theta = b_\nu r^\nu \qquad (\nu = 1,2,\ldots).$$

As $\frac{U}{\pi}$ is a positive function of θ, we can consider it as defining a positive mass distribution over the curve $(\cos\theta, \sin\theta, \ldots, \cos n\theta, \sin n\theta)$. The first of the above equations tells us that the total mass is 1 and therefore the point $(a_1 r, b_1 r, \ldots, a_n r^n, b_n r^n)$ has to belong to the convex hull of the curve, that is to H_{2n}. As H_{2n} is closed (the convex hull of a closed bounded set) we can let r tend to 1 and get the result $p \in H_{2n}$, which proves the theorem.

§6. We shall now consider the corresponding coefficient problem for polynomials. Let $f(x)$ be a polynomial of degree n and suppose $f(x) \in S$ that is $f(x) = \frac{1}{2} + \sum_{\nu=1}^{n} c_\nu x^\nu$ where $\mathrm{Re}\, f(x) \geq 0$. Denote by M_{2n} the set of points in E_{2n} such that the corresponding polynomial is $\in S$. In the notation from the previous lecture we have :

<u>Theorem:</u> $M_{2n} = -\frac{1}{2} L^*_{2n}$

Proof: Let $q(\theta)$ be the point $(\cos \theta, \sin \theta, \ldots, \cos n\theta, \sin n\theta)$ Let $p = (a_1, b_1, \ldots, a_n, b_n)$ be a point in E_{2n}. The necessary and sufficient condition that p be in M_{2n} is, by definition $R\,e(\frac{1}{2} + \sum_{\nu=1}^{n}(a_\nu - i\, b_\nu)x^\nu) \geq 0$ or, if we put $x = r \cdot e^{i\theta}$: $\frac{1}{2} + \sum_{\nu=1}^{n}(a \cos \nu\theta + b_\nu \sin \nu\theta)\, r^\nu \geq 0$ for $r < 1$. But this is equivalent to saying that this expression is ≥ 0 for $r = 1$. Assume namely that it is ≥ 0 for $r < 1$. Letting r tend to 1 we get ≥ 0 for $r = 1$. On the other hand, suppose it is ≥ 0 for $r = 1$; then the minimum principle for harmonic functions tells us that it is ≥ 0 for $r = 1$. We see thus that p is in M_{2n} if and only if

$$\frac{1}{2} + \sum_{\nu=1}^{n} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta) \geq 0 ,$$

which can be written : $\frac{1}{2} + p \cdot q(\theta) \geq 0$, or $p \in M_{2n} \Longleftrightarrow (-2p) \cdot q(\theta) \leq 1$ for all θ, that is $-2M_{2n} = \{q(\theta)\}^* =$ (convex closure of $\{q(\theta)\})^* = L^*_{2n}$, which proves the theorem.

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IV. CONNECTEDNESS AND CONVEX HULLS

BY

O. Hanner

Let M be an arbitrary set in an n-dim Euclidean space, E^n.

Definition. A point $p \in E^n$ has the k-point property (k - p.p.) with respect to M if there exist k (or fewer) points $q_1, q_2, \ldots, q_k \in M$ such that $p \in H(q_1, \ldots, q_k)$ (= the convex hull of the finite set $\{q_1, \ldots, q_k\}$).

If p has the k - p.p. with respect to M, then $p \in H(M)$.

Definition. If every point $p \in H(M)$ has the k - p.p. with respect to M, M is said to have the k - p.p.

Examples: Take n = 2. If M consists of the three vertices of a triangle, M has the 3 - p.p. but not the 2 - p.p. If M is the whole boundary of the triangle, M has the 2 - p.p. but not the 1 - p.p. (A set has the 1 - p.p. if and only if it is convex).

Now there is a well-known theorem (cf. p.14 of these notes) which in this terminology may be stated:

Theorem 1. Every set $M \subset E^n$ has the (n+1) - p.p.

The number n+1 in this theorem is best possible, for we can take M to be the n+1 vertices of an n-simplex. However, if we know something about the connectedness of the set M, it may be possible to lower the number n+1 to n. To show this will be the subject of this lecture. We will consider two cases. Firstly we take connectedness in the topological sense. In this case no further restrictions are necessary on the set M. (Theorem 2.) Secondly, we introduce a new, weaker notion of connectedness, but then we must take M compact (Theorem 3.)

Theorem 2. If $M \subset E^n$ has at most n components (in particular if M is connected), then M has the n - p.p.

Example: If M is the union of the coordinate axes, M has the n - p.p. but not the (n-1)- p.p. Hence the number n in this theorem is best possible.

Proof of Theorem 2. If n = 1 the theorem holds, since on a line every connected set is convex.

Suppose $n \geq 2$. Let $p \in H(M)$ be a point which does not have the n - p.p. with respect to M. By Theorem 1 we have

$$p = a_1 q_1 + \dots + a_{n+1} q_{n+1}, \quad a_i \geq 0, \quad \sum a_i = 1 .$$

Here $\{q_1, \dots, q_{n+1}\} \subset M$ can not lie in any (n-1)-plane (use Theorem 1 with n-1 instead of n) and $a_i > 0$. Change the coordinate system so that p is the origin. Then

$$(1) \qquad a_1 q_1 + \dots + a_{n+1} q_{n+1} = 0 , \quad a_i > 0, \quad \sum a_i = 1 ,$$

and there is no other linear relation between $q_1 \dots q_{n+1}$.

Now for $(i = 1, \dots, n+1)$ define the sets

$$A_i = \{ r \mid r = b_1 q_1 + \dots + b_{i-1} q_{i-1} + b_{i+1} q_{i+1} + \dots + b_{n+1} q_{n+1};\ b_k \leq 0 \}.$$

We have

(i) $\cup A_i = E^n$

For $q_{n+1} - q_i (i=1,\dots,n)$ are n independent vectors, so for any $r \in E^n$, we have

$$r = c_1 q_1 + \dots + c_{n+1} q_{n+1} .$$

Using formula (1) we easily prove (i).

(ii) $\text{Int } A_i \cap \text{Int } A_j = 0 \quad (i \neq j)$

Take a point $r \in A_1 \cap A_2$.

$$r = b_2 q_2 + b_3 q_3 + \dots = c_1 q_1 + c_3 q_3 + \dots \qquad b_k \leq 0 , \; c_k \leq 0 .$$

We must have $b_2 = 0$ and $c_1 = 0$, for otherwise there would be another linear relation among $q_1, \dots, q_{n+1}$. Hence $r \in \text{Bdry } A_1 \cap \text{Bdry } A_2$.

(iii) $(\text{Bdry } A_i) \cap M = 0$

For if $q_o \in (\text{Bdry } A_i) \cap M$ we would have

$$q_o = b_1 q_1 + \dots + b_{i-1} q_{i-1} + b_{i+1} q_{i+1} + \dots + b_{\ell-1} q_{\ell-1} + b_{\ell+1} q_{\ell+1} + \dots ,$$

$$b_k \leq 0 .$$

But then

$$p = 0 = q_o - b_1 q_1 \dots ;$$

which would mean that p had the n - p.p.

(iv) $A_i \cap M \neq 0$

For the point q_i belongs to this set. (From formula (1)).

Geometrically the sets A_i can be described as follows. $\{-q_1 \dots -q_{n+1}\}$ are the vertices of an n-simplex with p as an interior point. A_i is the (infinite, convex) cone from p through the $(n-1)$-face opposite to $-q_i$.

Now we have from (i) and (iii)

$$M = (\text{Int } A_1 \cap M) \cup \dots \cup (\text{Int } A_{n+1} \cap M).$$

(ii) and (iv) show that this is a mutual separation of M into n+1 non-void sets open in M. This contradicts that M does not have more than n components.

Definition. A set M is called convexly connected if there is no $(n-1)$-plane π such that $\pi \cap M = 0$ and M contains points in both half-spaces determined by π.

Examples. Take $n = 2$. 1) M consists of a number of concentric circles. 2) M consists of a hyperbola and one of its asymptotes.

The union of convexly connected sets having a point in common is convexly connected. Hence we can make the following definition:

Definition. Let M be any set in E^n, $p \in M$. The union of all convexly connected subsets of M containing p is a convexly connected set M_p, which we call a convexly connected component.

If $q \in M_p$ then $M_p = M_q$. Hence there is a unique decomposition of M into convexly connected components.

Example. $n = 2$. Let (r, φ) be polar coordinates. Take

$$A_i = \{(r, \varphi) \quad 0 < r \leq 1, \ \frac{2\pi}{3} i \leq \varphi < \frac{\pi}{3} + \frac{2\pi}{3} i\} \quad (i = 0, 1, 2),$$

$$M = A_0 \cup A_1 \cup A_2 .$$

This set M is convexly connected (but not connected). Hence there is only one convexly connected component. The origin does not have the 2 - p.p. This example shows, that the following theorem may be false for non-compact sets.

Theorem 3. If $M \subset E^n$ is compact and has at most n convexly connected components (in particular if M is convexly connected) then M has the n - p.p.

Theorem 4. Suppose $M \subset E^n$ is compact, $p \in H(M)$, p does not have the n - p.p. Then for every set $q_1, \ldots, q_{n+1} \in M$ such that $p \in H(q_1, \ldots, q_{n+1})$ there is an $(n-1)$-plane through p not intersecting M separating q_{n+1} from $q_1, \ldots, q_n$.

Theorem 5. Suppose $M \subset E^n$ is compact, $p \in H(M)$, p does not have the n - p.p. A is a closed subset of M, $p \in H(A)$. Then there is an (n-1)-plane π through p such that

$$\pi \cap M = 0 \quad \text{and} \quad \pi \cap H(A) = 0 .$$

Proofs: We prove Theorem 5 and show Theorem 5 $\Rightarrow$ Theorem 4 $\Rightarrow$ Theorem 3.

(i) Theorem 4 implies Theorem 3. For suppose Theorem 3 false. Then there is a point $p \in H(M)$ which does not have the n - p.p. Then by Theorem 1 there are $q_1, \dots, q_{n+1} \in M$ such that $p \in H(q_1, \dots, q_{n+1})$. Theorem 4 (used n+1 times) shows that $q_1, \dots, q_{n+1}$ belong to different convexly connected components. Hence there would be at least n+1 such components which is a contradiction.

(ii) Theorem 5 implies Theorem 4. For let $q_1, \dots, q_{n+1}$ be given. Then take $A = \{q_1, \dots, q_n\}$. Theorem 5 then gives us a plane π. This separates q_{n+1} and A, since $p \in H(q_1, \dots, q_{n+1})$.

(iii) Proof of Theorem 5. By making use of the compactness of M we can construct $M_1 \supset M$, so that M_1 is a finite union of solid spheres, and so that p does not have the n - p.p. with respect to M_1. We suppose this already done, so that M is the union of a finite number of solid spheres. We can also suppose that A contains interior points.

Now consider the set of (n-1)-planes:

$S = \{\pi \mid p \in \pi, \pi$ is bounding plane or tac-plane of $A\}$.

This is a compact set in its natural topology(the topology of the corresponding normalized direction vectors). Define on it the non-negative valued function $V(\pi)$ = the volume of the part of M which lies in the half-space determined by π not containing A. There is a plane π_1 such that

$$V(\pi_1) = \text{g.l.b. } V(\pi).$$

We claim that this plane π_1 or a plane near it satisfies the conditions of Theorem 5. There will be two cases.

Firstly $\pi_1 \cap M = 0$. Then π_1 can not be a tac-plane to $H(A)$ since A is a closed subset of M. Hence π_1 is a bounding plane, and $\pi_1 \cap H(A) = 0$.

Secondly $\pi_1 \cap M = N \neq 0$. Then $p \notin H(N)$, since N is a subset of M (use Theorem 1 with $n-1$ instead of n). Hence there is in π_1 an $(n-2)$-dimensional bounding plane B of N passing through p and dividing π_1 into the two $(n-1)$-dimensional closed halfspaces $\pi_{11} \supset N$ and π_{12}. We will show that we can turn π_1 around B a sufficiently small angle so that the plane we get does not intersect M.

This is possible with π_{12} in both directions since $\pi_{12} \cap M = 0$ and M is a compact set.

This is possible with π_{11} in the direction into the side of π_1 not containing A. For, since M is a finite union of solid spheres, there would otherwise be a plane π with $V(\pi) < V(\pi_1)$.

The plane we have gotten in this way must still belong to S (A lies completely in one of its half-spaces). For the same reason as before it must be a bounding plane. This completes the proof of Theorem 5 and hence of Theorem 3.

Historical Note. Theorem 2 was first proved for compact sets by Fenchel [Math. Ann. vol.101(1929)pp.238-252]. He used the method with covering of the set with solid spheres which we here have used to prove Theorem 5. For non-compact sets Theorem 2 was proved (in essentially the same way as here) by L. N. H. Bunt, Bijdrage zot de theorie der convexe puntverzamelingen; Amsterdam, 1934.

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V. Compactness Theorems

By

A. M. Macbeath

1. Theorem of Blaschke.

This theorem shows that the set of all bounded closed convex sets in E^n is locally compact, if topologized by a natural metric, to be described. It is best understood as a special case of a similar result for general bounded sets in E^n.

The metric is defined as follows. Let A be a bounded closed set in E^n. Let $A_{(\rho)}$ be the union of all closed spheres of radius $\rho > 0$ with centre in A. The distance d(A,B) between two such sets A,B is defined by

$$d(A,B) = \inf \rho \; [A_{(\rho)} \supset B, \; B_{(\rho)} \supset A].$$

It follows easily from the definitions of distance and closed set that

1) $d(A,B) = 0 \Longleftrightarrow A = B$.

2) $d(A,B) = d(B,A)$.

3) $d(A,B) \leq d(A,C) + d(B,C)$.

A sequence $\{A_\nu\}$ of such sets is said to converge to the bounded closed set T ($\lim A_\nu = T$) if $\lim d(A_\nu, T) = 0$.

Selection Theorem.

Let W be a fixed closed cube of E^n and let there be an infinite set of closed subsets A of W. Then there is a sequence of sets A_ν contained in this set such that

$$\lim A_\nu = T .$$

We prove first (a) (completeness): Let A be a Cauchy sequence; i.e., for $\varepsilon > 0$ there exists an N such that, for all $\nu, \mu > N$, $d(A_\nu, A_\mu) < \varepsilon$. Then there is a non-empty set T such that $\lim A_\nu = T$.

Proof: Put $S_k = \bigcup_{\nu=k}^{\infty} A_\nu$; $T_k = \bar{S}_k$. Then $T_k \supset T_{k+1}$ so $T = \bigcap_{i=1}^{\infty} T_i$ is a non-empty closed set. We assert

$$(\alpha) \qquad \lim T_k = T ;$$

for otherwise there is an $\varepsilon > 0$ such that the set $D_k = T_k - (\operatorname{int} T_{(\varepsilon)}) \cap T_k$ is not empty for every k. Now D_k is closed and $D_k \supset D_{k+1}$. Hence $D = D_1 \cap D_2 \cap D_3 \cap \ldots$ is closed and non-empty. Now $D_k \cap (\operatorname{int} T_{(\varepsilon)}) = \emptyset$, so $D \cap T = \emptyset$; but $D_k \subset T_k \Longrightarrow D \subset T$, a contradiction.

Now it follows from (α) that, given $\varepsilon > 0$, there is an M such that $T_{(\varepsilon)} \supset T_m$ for all $m \geq M$. Since $T_m \supset A_m$, the relation

$$(\beta) \qquad T_{(\varepsilon)} \supset A_m$$

holds for all $m \geq M$. For this same ε, there exists by hypothesis an M' such that $(A_m)_{(\varepsilon)} \supset A_\nu$ for $\nu \geq m \geq M'$. Hence $(A_m)_{(\varepsilon)} \supset T_m$, or

$$(\gamma) \qquad (A_m)_{(\varepsilon)} \supset T \quad \text{for all } m > M'.$$

(β) and (γ) show that $d(T, A_m) \leq \varepsilon$ for all $m \geq \max(M, M')$, so $\lim A_m = T$.

Now we prove assertion (b) (compactness): There exists a sequence contained in each infinite system of closed sets of W which converges to a closed set T.

Proof: Without loss of generality take W to be the unit cube. By ith subdivision, W is split into 2^{in} closed little cubes of side length 2^{-i}. Consider the set of all aggregates of these little cubes. The number of possible aggregates is finite: $2^{2^{in}}$. Now associate with each A of our system the aggregate of little cubes which have points in common with A. At least one such aggregate occurs infinitely often.

Let $\{A_{1\nu}\}$ be an (infinite) sequence of sets of our system, each element associated with the same aggregate in the first subdivision. Then let $\{A_{2\nu}\}$ be an (infinite) subsequence of $\{A_{1\nu}\}$, each element associated with the same aggregate in the second subdivision, etc. Thus $\{A_{i+1,\nu}\}$ is an (infinite) subsequence of $\{A_{i\nu}\}$.

Then $d(A_{i\nu}, A_{i\mu}) \leq \sqrt{n}/2^i$, and, since $\{A_{j\nu}\}$ is a subsequence of $\{A_{i\nu}\}$ if $j > i$, we have $d(A_{jj}, A_{ii}) \leq \sqrt{n}/2^i$ for $j > i$. Thus the sequence $\{A_{\nu\nu}\}$ is a Cauchy sequence and (b) follows from (a).

The theorem of Blaschke is the same as the selection theorem proved above with the words "closed set" replaced by "convex closed set" everywhere. It suffices to show, therefore, that a limit of convex closed sets is convex. Let $\{B_\nu\}$ be a sequence of convex closed sets and let $\lim B_\nu = B$. Suppose that p, q are points of B and let $\varepsilon_\nu = d(B, B_\nu)$. Then

$$p, q \in (B_\nu)_{(\varepsilon_\nu)} \Rightarrow \lambda p + (1-\lambda)q \in (B_\nu)_{(\varepsilon_\nu)} \Rightarrow \lambda p + (1-\lambda)q \in (B)_{(2\varepsilon_\nu)}$$

Since $\varepsilon_\nu \longrightarrow 0$ and B is closed $\lambda p + (1-\lambda)q \in B$ and so B is convex.

2. Compactness of affine classes.

2.1. Linear combinations of convex bodies.

In this paragraph the term "convex body" will denote a bounded closed convex set with inner points in E^n.

If K_1, K_2 are two convex bodies, and λ, μ are real non-negative numbers, we denote by $\lambda K_1 + \mu K_2$ the set of points $\lambda x + \mu y$, where $x \varepsilon K$, $y \varepsilon K$. Clearly $\lambda K_1 + \mu K_2$ is a convex body.

Let H_1, H_2 be the tac-functions of K_1, K_2 respectively. $H_1(u)$, $H_2(u)$ are the maxima of $u.x$ for x in K_1, K_2 respectively. Hence the maximum of $u.(\lambda x + \mu y)$ is $\lambda H_1(u) + \mu H_2(u)$; i.e., the tac-function of $\lambda K_1 + \mu K_2$ is $\lambda H_1 + \mu H_2$.

It follows from Theorem 40 (Chapter I of these notes) that

$$K_1 + \mu K' \supset K_2 + \mu K' \Longrightarrow H_1 + \mu H' \geq H_2 + \mu H' \Longrightarrow H_1 \geq H_2 \Longrightarrow K_1 \supset K_2 ,$$

and that

$$K_1 \supset K_1', \quad K_2 \supset K_2' \Longrightarrow \lambda K_1 + \mu K_2 \supset \lambda K_1' + \mu K_2'$$

Finally we note that the metric defined in §1 of this chapter can, for convex bodies, be written in the form

$$d(K_1, K_2) = \inf \rho[K_1 + \rho S \supset K_2, \; K_2 + \rho S \supset K_1] ,$$

where S is the solid unit sphere with centre O.

2.2 The space of affine classes.

Consider the group G of affine transformations σ of E^n, given by

$$(\sigma x)_i = \sum_{j=1}^{n} \sigma_{ij} x_j + \sigma_{io} ,$$

where $\det(\sigma_{ij})$ $(i,j=1,\dots,n)$ is different from zero.

Each σ induces in a natural manner a transformation (which we shall also call σ) of the space C of all convex bodies in E^n. The relation is

$$\sigma x \in \sigma K \Longleftrightarrow x \in K$$

Now introduce the equivalence relation

$$K_1 \sim K_2 \Longleftrightarrow K_1 = \sigma K_2 \quad \text{for some } \sigma \in G .$$

This divides the elements of C into equivalence classes which we shall call <u>affine classes</u>. The affine classes then form a topological space C^* by the usual identification; i.e., a set of classes is open in C^* if and only if the union of classes of the set is open in C. The main result of this section is

<u>Theorem.</u> C^* <u>is compact and metrizable.</u>

2.3. The invariant ρ .

The proof of the assertion about C^* depends on the introduction of a real-valued function $\rho(K_1,K_2)$ which is affine invariant in both variables separately, i.e. it is a function of pairs of elements of C^*.

Because of the continuity properties of the boundary, each convex body K has a well-defined volume (or content) V(K) with the properties:

1) If $K_1 \supset K_2$, then $V(K_1) \geq V(K_2)$, with equality only if $K_1 = K_2$. The last assertion is proved as follows: Suppose $K_1 \supset K_2$, but $K_1 \neq K_2$. Let $p \in K_1 - K_2$, $q \in \text{int}K_1 \subset \text{int}K_1$. Since K_2 is closed, the segment pq intersects the boundary of K_2 at $r \neq p$. Then $r \in \text{int}K_1$(Chapter I, Th. 7) so there is a sphere with centre r in K_1. Since K_2 has a tac-plane at r, half of the sphere is in $K_1 - K_2$, so $V(K_1) > V(K_2)$.

2) $V(\sigma K) = |\det \sigma| \cdot V(K)$.

3) V is a continuous function on $\mathcal{C}$.

Proof: Suppose that $K \subset K' + \varepsilon S$, $K' \subset K + \varepsilon S$. Choose an inner point of K as origin O. Certainly $K \supset \lambda S$ for some $\lambda > 0$, so (cf. 2.1)

$$K' \subset K + \frac{\varepsilon}{\lambda} K, \qquad K \subset K' + \frac{\varepsilon}{\lambda} K$$
$$\Longrightarrow K' \subset K + \frac{\varepsilon}{\lambda} K, \qquad (1 - \frac{\varepsilon}{\lambda})K + \frac{\varepsilon}{\lambda} K \subset K' + \frac{\varepsilon}{\lambda} K$$
$$\Longrightarrow (1 + \frac{\varepsilon}{\lambda}) K \supset K' \supset (1 - \frac{\varepsilon}{\lambda}) K$$
$$\Longrightarrow (1 + \frac{\varepsilon}{\lambda})^n V(K) \geq V(K') \geq (1 - \frac{\varepsilon}{\lambda})^n V(K),$$

and the continuity is proved.

Now define

$$\rho(K_1, K_2) = \inf \frac{V(\sigma K_1)}{V(K_2)},$$

where σ runs over all those elements of G such that $\sigma K_1 \supset K_2$.

Lemma 1. The bound $\rho(K_1, K_2)$ is actually attained, i.e. there is a $\sigma \in G$ such that $\sigma K_1 \supset K_2$, and $V(\sigma K_1) = \rho V(K_2)$.

Proof: There is a sequence of bodies $\{\sigma_\nu K_1\}$ with the properties

$$\sigma_i K_1 \supset K_2, \lim_{\nu \to \infty} V(\sigma_\nu K_1) = \rho V(K_2)$$

It suffices to show that the set Σ of $\sigma \in G$ such that

$$\sigma K_1 \supset K_2 \qquad |\det \sigma| \leq A$$

is compact, for each positive A. For, if we write $A = \rho V(K_2)/V(K_1) + \varepsilon$, all but a finite number of elements of the sequence σ_ν belong to Σ. Hence we may assume that $\lim \sigma_\nu = \tau \in G$, so $\lim(\sigma_\nu K_1) = \tau K_1$, so $\tau K_1 \supset K_2$, $V(\tau K_1) = \rho V(K_2)$.

The space G is homeomorphic to an open set of E^{n^2+n}, namely the whole space with the algebraic manifold $|\det \sigma| = 0$ removed. It suffices to show that Σ is bounded and closed in E^{n^2+n}.

Let S_1 be a sphere containing K_1, S_2 a sphere contained in K_2. Then if $\sigma \varepsilon \Sigma$,

$$\sigma S_1 \supset S_2$$

but $S_1 = \tau_1 S$, $S_2 = \tau_2 S$, where τ_1, τ_2 are affine transformations and S is the unit sphere with centre at the origin. Hence

$$\tau_2^{-1} \sigma \tau_1 S \supset S , \qquad |\det \sigma| \leq A \tag{*}$$

Let Σ_1 denote the subset of G whose elements satisfy (*). Since Σ is a closed subset of Σ_1, it is enough to show that Σ_1 is compact.

Now let $\mathcal{O}$ denote the orthogonal group, which leaves S fixed and is a compact subgroup of G. Any $n \times n$ matrix is the product of an orthogonal matrix and a positive symmetric matrix (Chevalley, Lie Groups, p. 14). The symmetric matrix, by the usual reduction of quadratic forms, can be written in the shape $o\ d\ o'$, where d is a diagonal matrix and o is orthogonal. Applying the same transformation to the element $\tau_2^{-1}\sigma\tau_1$ of G, we have

$$\tau_2^{-1} \sigma \tau_1 = o_1 \delta^{-1} o_2 \qquad\qquad o_1, o_2 \varepsilon \mathcal{O}$$

where the homogeneous part of δ is a diagonal matrix, i.e. $\delta_{ij} = 0$ if $j \neq 0, i$. Hence

$$\Sigma_1 = \tau_2 \mathcal{O} \triangle^{-1} \mathcal{O} \tau_1^{-1} \tag{**}$$

where $\triangle$ is the set of all δ such that

$$S \supset \delta S, \qquad |\det \delta| = |\Pi \delta_{ii}| \geq A'.$$

Δ is bounded and closed, for, applying $S \supset \delta S$ to the centre of S and the unit vectors, we have

$$\sum \delta_{io}^2 \leq 1 \qquad (\delta_{ii} + \delta_{io})^2 \leq 1 \qquad\qquad |\det \delta| \geq A' .$$

Since Θ, Δ are compact and the operations of inversion and left and right translation are homeomorphisms, it follows from (**) that Σ_1 is compact. Thus Lemma 1 is proved.

From Lemma 1 and property 1) of the volume, it follows that $\rho(K_1, K_2) \geq 1$, with equality only if $K_1 \sim K_2$.

If $\tau_1, \tau_2 \in G$,

$$\rho(\tau_1 K_1, \tau_2 K_2) = \inf \frac{V(\sigma \tau_1 K_1)}{V(\tau_2 K_2)} \qquad \text{for } \sigma \tau_1 K_1 \supset \tau_2 K_2$$

but

$$\sigma \tau_1 K_1 \supset \tau_2 K_2 \Longleftrightarrow \tau_2^{-1} \sigma \tau_1 K_1 \supset K_2$$

and

$$\frac{V(\sigma \tau_1 K_1)}{V(\tau_2 K_2)} = \frac{V(\tau_2^{-1} \sigma \tau_1 K_1)}{V(K_2)}$$

by property 2) of $V(K)$. Since $\tau_2^{-1} \sigma \tau_1$ runs over G if σ does,

$$\rho(\tau_1 K_1, \tau_2 K_2) = \rho(K_1, K_2) .$$

Thus ρ is really a function on $C^* \times C^*$. Now suppose that we have three bodies K_1, K_2, K_3. For suitable σ, σ' we have

$$\rho(K_1, K_2) = \frac{V(\sigma K_1)}{V(K_2)} , \qquad \rho(K_2, K_3) = \frac{V(\sigma' K_2)}{V(K_3)}$$

where $\quad \sigma K_1 \supset K_2 , \quad \sigma' K_2 \supset K_3 \quad$ so $\quad \sigma' \sigma K_1 \supset \sigma' K_2 \supset K_3$.

Hence

$$\rho(K_1,K_3) \leq \frac{V(\sigma'\sigma K_1)}{V(K_3)} = \frac{V(\sigma'\sigma K_1)}{V(\sigma' K_2)} \cdot \frac{V(\sigma' K_2)}{V(K_3)} = \rho(K_1 K_2)\rho(K_2,K_3).$$

Lemma 2. $\rho(K_1,K_2)$ is a continuous function on $C \times C$.

Proof: We have to show that $\rho(K_1',K_2') \longrightarrow \rho(K_1,K_2)$ as $K_1' \longrightarrow K_1$, $K_2' \longrightarrow K$. The inequalities

$$\rho(K_1',K_2') \leq \rho(K_1',K_1)\rho(K_1,K_2)\rho(K_2,K_2')$$
$$\rho(K_1,K_2) \leq \rho(K_1,K_1')\rho(K_1',K_2')\rho(K_2',K_2)$$

show that it is enough to prove that if $K' \longrightarrow K$, then

$$\rho(K,K') \longrightarrow 1 \quad \text{and} \quad \rho(K',K) \longrightarrow 1 .$$

Suppose that $d(K,K') = \varepsilon$. Then, as in the proof of continuity of the volume, we have

$$(1 + \frac{\varepsilon}{\lambda})\, K \supset K' \supset (1 - \frac{\varepsilon}{\lambda})\, K$$

Since the expansion by $(1 \pm \frac{\varepsilon}{\lambda})$: 1 can be regarded as a transformation of G,

$$\rho(K,K') \leq (1 + \frac{\varepsilon}{\lambda})^n \ ; \qquad \rho(K',K) \leq (1 - \frac{\varepsilon}{\lambda})^{-n}$$

and Lemma 2 is proved.

Lemma 3. $\rho(K_1,K_2)$ is bounded.

Proof: Let T be the simplex of largest content contained in K_1, vertices $a_0,a_1,\ldots,a_n$. Let π_0 be the plane through a_0 parallel to the face opposite a_0. Then π_0 is a tac-plane of K_1, for otherwise there would be a point a_0' of K_1 on the other side of π_0 from a_1, and the simplex $a_0'a_1\ldots a_n$ would have larger volume than T. Similarly there are planes

$\pi_1, \pi_2, \ldots, \pi_n$ through $a_1, a_2, \ldots, a_n$, and these, with π_0 bound a simplex T' such that $T' \supset K$ and $V(T')/V(T) = n^n$.

Hence

$$\rho(T,K_1)\,\rho(K_1,T) \leq n^n .$$

So

$$\rho(K_1,T) \leq n^n , \qquad \rho(T,K_2) \leq n^n$$
$$\rho(K_1,K_2) \leq n^{2n}$$

This proves Lemma 3.

The compactness of C^* now follows easily. Consider the subset B of C consisting of those convex bodies K such that

$$K \subset T , \quad V(K) \geq n^{-n}V(T).$$

The mapping of B into C^* where each element is mapped on the class to which it belongs is continuous, and, by the proof of Lemma 3, onto. But B is compact, by Blaschke's theorem and the continuity of V, so C^* is compact.

Now consider the function

$$\triangle(K,K') = \log \rho(K,K') + \log \rho(K',K).$$

This is a metric for the set of affine classes and characterizes them as a metric space C'. Consider the mapping

$$\varphi : C^* \longrightarrow C'$$

where each affine class, considered as an element of C^*, is mapped on the same class, considered as an element of C'. An open set in C' is the union of spheroids about all its points, and thus corresponds, since ρ is a continuous function on C^*, to a union of open sets in C^*. Thus φ is continuous.

φ is a 1-1 mapping of a compact space onto a metric space; therefore φ is a homeomorphism. Thus C^* and C' give rise to the same topology of the set of affine classes and this proves our assertion.

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SEMINAR ON CONVEX SETS

VI. The Packing of Convex Bodies in Euclidean Space.

by

C. A. Rogers

§1. Introduction. Let E_n be the real Euclidean n-dimensional space of points $X = (x_1, \ldots, x_n)$. By a convex body we mean a set of E_n which is non-empty, bounded, open and convex. Let K be such a convex body and suppose the origin O is in K. Take a large cube S of side s and volume $V(S) = s^n$. Take points $X_1, \ldots, X_\ell$ in S such that

$$K + X_\lambda \subset S, \qquad \text{for } \lambda = 1, \ldots, \ell$$

$$(K + X_\lambda) \cap (K + X_\mu) \text{ is empty, for } 1 \leq \lambda < \mu \leq \ell.$$

We call this arrangement of sets $K + X_1, \ldots, K + X_\ell$ a packing of ℓ sets K into S. The density of the packing is defined to be

$$\frac{\ell \cdot V(K)}{V(S)} .$$

We write

$$\rho(K,S) = \max \frac{\ell V(K)}{V(S)} ,$$

the maximum being taken over all packings of K into S, and we write

$$\rho(K) = \overline{\lim_{s \to \infty}} \rho(K,S) .$$

We call $\rho(K)$ the density of the closest packing of K. For alternative definitions of $\rho(K)$ see Hlawka, Monatsh. für Math., 53(1949), 81-131.

We have the trivial result $\ell V(K)/V(S) \leq 1$, so that $\rho(K) \leq 1$ for any convex body K. Note that when K is a cube we have $\rho(K) = 1$.

In section 2 we find a simple lower bound for $\rho(K)$. In section 3 we discuss the case of the packing of convex domains in the plane. In section 4 we use Blichfeldt's method to obtain his bound for $\rho(K)$ when K is an n dimensional sphere. In section 5 we show how Blichfeldt's method may be used to obtain a bound for the minimum of the product of n linear forms.

§2. A lower bound for $\rho(K)$. Let K be a convex body containing the origin O. Suppose that the sets $K + X_1, \ldots, K + X_\ell$ form a packing of K into a large cube S with side s, and that ℓ is as large as is possible, so that

$$\rho(K,S) = \frac{\ell V(K)}{V(S)} .$$

Let $DK = K + (-K)$ be the difference set of K, that is the set of all points of the form $X - Y$ where X and Y are points of K. Consider the sets

$$DK + X_1, \ldots, DK + X_\ell .$$

These sets will overlap in general. We prove that, when s is sufficiently large, they cover most of the cube S. Let d be the diameter of K. Let S' be the cube of side $s - 2d$ centrally placed inside S. (Here we suppose that $s > 2d$). Let X_0 be any point of S'. Consider the sets $K + X_0, \ldots, K + X_\ell$. As X_0 is in S' it is clear that $K + X_0$ is in S. So the sets $K + X_0, \ldots, K + X_\ell$ are all contained in S. By our choice of ℓ it follows that two of these sets have a point in common. So for some integer λ with $1 \leq \lambda \leq \ell$ and for some point Y we have

$$Y \text{ is in both } K + X_0 \text{ and } K + X_\lambda \ .$$

So both $Y - X_0$ and $Y - X_\lambda$ are in K. Hence the point

$$X_0 - X_\lambda = (Y - X_\lambda) - (Y - X_0)$$

is in DK and the point

$$X_0 = (X_0 - X_\lambda) + X_\lambda$$

is in $DK + X_\lambda$. Thus every point of S' belongs to at least one of the sets $DK + X_1, \ldots, DK + X_\ell$. Comparing volumes we have

$$(s - 2d)^n \leq \ell \ V(DK).$$

So

$$\rho(K,S) = \frac{\ell \cdot V(K)}{V(S)} \geq \frac{V(K)}{V(DK)} \left(\frac{s-2d}{s} \right)^n .$$

Taking the upper limit as s tends to infinity we obtain

$$\rho(K) \geq \frac{V(K)}{V(DK)} \ .$$

In the particular case, when K has $\mathcal{O}$ as centre, we have DK = 2K so that

$$\rho(K) \geq (\tfrac{1}{2})^n .$$

We remark that this lower bound for $\rho(K)$ can be improved by application of a theorem stated by Minkowski and proved by Hlawka, Math. Zeitschr., 49(1944), 285-312; or by application of refined forms of Hlawka's theorem due to Mahler and to Davenport and Rogers, Duke, Math. J., 13(1946), 611-621 and 14(1947), 367-375. But these improvements only show that $\rho(K) \geq c_n V(K)/V(DK)$, where the sequence $c_1, c_2, \ldots$ is bounded.

§3. We consider the problem of <u>the closest packing of plane convex domains</u>. (I am preparing a detailed account of the work described in this section for publication.) Before we can state the main result we need to introduce and discuss the concept of a lattice packing. Let K be a convex domain (i.e. a plane, non-empty, bounded open and convex set) containing the origin $\mathcal{O}$. Let Λ be any lattice with the property that, if X and Y are any two distinct points of Λ, then the sets K + X and K + Y have no point in common. If S is a square of side s with $\mathcal{O}$ as centre, the system of sets $K+X_1,\ldots,K+X_\ell$ is said to be a lattice packing of K into S with lattice Λ, when $X_1,\ldots,X_\ell$ are just those points X of Λ for which K+X is contained in S. It is clear that such a system $K+X_1,\ldots,K+X_\ell$ forms a packing of K into S with density

$$\frac{\ell V(K)}{V(S)},$$

in the sense we introduced in §1. The density of the closest lattice packing of K into S is defined to be

$$\rho^*(K,S) = \overline{\mathrm{bd}}\ \frac{\ell V(K)}{V(S)},$$

the upper bound being taken over all lattice packings of K into S. The density of the closest lattice packing of K is defined to be

$$\rho^*(K) = \overline{\lim_{s\to+\infty}}\ \rho^*(K,S).$$

Comparing this definition with that of $\rho(K)$ we see that

$$\rho^*(K) \leq \rho(K) \leq 1 .$$

Now $\rho^*(K)$ is not too difficult to evaluate. Let D(K) be the lower bound of the areas $D(\Lambda)$ of the fundamental parallelograms of the lattices Λ, such that, if X and Y are distinct points of Λ, then $K+X$ and $K+Y$ have no point in common. Then this lower bound is attained and there is a lattice Λ_0 of this type with $D(\Lambda_0) = D(K)$. Now consider any lattice packing $K+X_1, \ldots, K+X_\ell$ with lattice Λ into a large square S with sides s. To each of the points $X_1, \ldots, X_\ell$ we assign a neighbouring fundamental parallelogram of Λ, so that these parallelograms fit together without overlapping. The parallelograms together are contained in a square of side $s + 2d$ and contain a square of side $s - 2d$, where d is the diameter of the parallelogram. Thus

$$(s-2d)^2 \leq \ell\, D(\Lambda) \leq (s+2d)^2 . \qquad (1)$$

Using the lattice Λ_0 we get

$$\rho^*(K,S) = \overline{\mathrm{bd}}\, \frac{\ell V(K)}{V(S)} \geq \frac{V(K)}{D(\Lambda_0)} \left(1 - \frac{2d_0}{s}\right)^2$$
$$= \frac{V(K)}{D(K)} \left(1 - \frac{2d_0}{s}\right)^2 .$$

Letting $s \longrightarrow \infty$ we obtain

$$\rho^*(K) \geq \frac{V(K)}{D(K)} .$$

But applying (1) to a lattice $\Lambda(s)$ for which ℓ has its maximum value we get

$$\rho^*(K,S) = \frac{\ell^*(K,S)V(K)}{V(S)} \leq \frac{V(K)}{D(\Lambda(s))} \left(1 + \frac{2d(s)}{s}\right)^2$$
$$\leq \frac{V(K)}{D(K)} \left(1 + \frac{2d(s)}{s}\right)^2 .$$

Now it is clear that, if we choose a suitable fundamental parallelogram for $\Lambda(s)$, we have $d(s) = o(s)$ as $s \longrightarrow \infty$; otherwise ℓ could not possibly have its maximum value for the lattice $\Lambda(s)$. Thus we have

$$\rho^*(K) \leq \frac{V(K)}{D(K)} .$$

Combining these results we have

$$\frac{V(K)}{D(K)} = \rho^*(K) \leq \rho(K) \leq 1 \ . \tag{2}$$

For a more detailed discussion of these and of alternative definitions of $\rho(K)$ and $\rho^*(K)$ we refer the reader to a recent paper by Hlawka, Monatsh. für Math., 53(1949), 81-131 (but see also my review in Math. Rev., 11(1950),12). We remark that $D(K)$ is in fact the 'critical determinant' of the set $DK = K+(-K)$; it is a relatively easy quantity to find, when K is known explicitly.

Until recently $\rho(K)$ was only known for a few sets K. When K is a parallelogram or a convex symmetrical hexagon there is a lattice packing of K, which covers the whole space, except for a set of measure zero. In this case we have the trivial result

$$\rho^*(K) = \rho(K) = 1 \ .$$

We also know $\rho(K)$ when K is a circle, or ellipse. In fact Thue proved in 1892 that in this case

$$\rho^*(K) = \rho(K) = \frac{\pi}{\sqrt{12}} \ . \tag{3}$$

[A. Thue, Forhandlingerne ved de Skandinavske Naturforskeres, 14(Copenhagen, 1892), 352-353 (in Norwegian); for a more complete and accessible proof see A. Thue, Skrifter Videnskabo-Selskabet i Christiania, Math.-Nat.Klasse, 1910, No.1.] Proofs of this result have also been given by Féjes Tóth (Math. Zeit., 46(1940), 83-85) and by Segre and Mahler (Amer.Math.Monthly, 51(1944),261-270). A couple of years ago I was able to prove that, for any convex domain K,

$$\rho^*(K) = \rho(K). \tag{4}$$

In this seminar I gave a detailed proof of this result. But the proof is very complicated and was only intelligible because it was accompanied by a large number of diagrams; it is not suitable for reproduction in these notes. So I merely make a few remarks about the proof.

The proof is inductive and it seems to be necessary to prove a result which is rather more precise than the result (4), but which applies only to strictly convex domains. A domain K is said to be strictly convex if it is such that, for every pair of distinct points A, B on the boundary of K, every inner point C of the line segment AB is in K. We prove

Theorem 1. Let K *be an open bounded strictly convex set. Let* $X_0, X_1, \ldots, X_n = X_0, X_{n+1}, \ldots, X_{n+m}$ *be points such that:*

(1) *the polygon* $X_0X_1\ldots X_n$ *is a Jordan polygon bounding a domain* π *of area* $V(\pi)$;

(2) *the sets* $K + X_{r-1}$ *and* $K + X_r$ *have a common boundary point, if* $1 \leq r \leq n$;

(3) *the points* $X_{n+1},\ldots,X_{n+m}$ *lie in the interior or on the boundary of* π; *and*

(4) *the sets* $K + X_r$ *and* $K + X_s$ *have no points in common, if* $1 \leq r < s \leq n+m$.

Then

$$(m + \tfrac{1}{2}n-1)\, D(K) \leq V(\pi). \tag{5}$$

Here one should think of the sets $K+X_1,\ldots,K+X_n$ as forming a rigid frame and of the sets $K+X_{n+1},\ldots,K+X_{n+m}$ as forming a packing of m sets K into the region bounded by the frame.

We first remark that it is easy to deduce the result (4) for any strictly convex domain K, once this theorem has been proved. Consider a large square S of side s. Let $K+Y_1,\dots,K+Y_\ell$ be a packing of the largest possible number of sets K into S. Let d be the diameter of K, let d_1 and d_2 be the lengths of the longest chords of K parallel to the x_1 and x_2 axes respectively. Choose integers n_1, n_2 such that

$$(n_i - 1)d_i < s + 2d \leq n_i d_i \, , \qquad i = 1,2.$$

Then it is clear that we can choose points $X_0, X_1, \dots, X_{2n_1+2n_2} = X_0$ such that the sets $K+X_0,\dots,K+X_{2n_1+2n_2}$ satisfy the conditions (1), (2), (4) with $n = 2n_1 + 2n_2$, where π is the rectangle $X_0 X_{n_1} X_{n_1+n_2} X_{2n_1+n_2} X_{2n_1+2n_2} = X_0$ with sides $n_1 d_1$, $n_2 d_2$ centrally placed round the square S. Taking $X_{n+1} = Y_1$, $\dots$ $X_{n+m} = Y_\ell$, $m = \ell$ we see that the conditions of Theorem 1 are satisfied. Hence

$$(\ell + n_1 + n_2 - 1)\, D(K) \leq V(\pi) = n_1 d_1 n_2 d_2$$

and

$$\rho(K,S) = \frac{\ell V(K)}{V(S)} \leq \frac{V(K)}{D(K)} \, \frac{n_1 d_1 n_2 d_2}{s^2}$$

$$\leq \frac{V(K)}{D(K)} \left(1 + \frac{2d + d_1}{s}\right)\left(1 + \frac{2d + d_2}{s}\right).$$

Thus letting $s \to \infty$ we have

$$\rho(K) \leq \frac{V(K)}{D(K)} \, .$$

Combining this with (2) we obtain

$$\frac{V(K)}{D(K)} = \rho^*(K) = \rho(K) \leq 1 \, . \qquad (6)$$

Once this has been established when K is strictly convex it can be extended to all convex K by simple continuity arguments.

We now consider Theorem 1. We first remark that for any points X, Y the sets K+X and K+Y have a common point (or a common boundary point) if and only if the sets $\frac{1}{2}$ DK+X and $\frac{1}{2}$ DK+Y have a common point (or a common boundary point). It follows from this remark that the truth of Theorem 1 for the domain $\frac{1}{2}$ DK implies the truth of the theorem for K. Thus we have only to prove the theorem in the case when K has σ as centre. For the rest of this section we confine our attention to this case.

It is clear that in proving Theorem 1 we may suppose that m has its maximum possible value for the given $X_0, X_1, \dots, X_n$. So we may insert the extra condition that

(5) <u>it is not possible to find points</u> $Z_0, Z_1, \dots, Z_m$ <u>in or on the boundary of</u> π <u>such that no two of the sets</u> $K+X_1, \dots, K+X_n$, $K+Z_0, \dots, K+Z_m$ <u>have a common point</u>.

It is also clear that we may insert the extra condition that

(6) <u>it is not possible to find points</u> $Z_1, Z_2, \dots, Z_m$ <u>in or on the boundary of</u> π <u>such that no two of the sets</u> $K+X_1, \dots, K+X_n$, $K+Z_1, \dots, K+Z_m$ <u>have a common point and such that the sum of the second coordinates of</u> $Z_1, \dots, Z_m$ <u>is less than the sum of the second coordinates of</u> $X_{n+1}, \dots, X_{n+m}$.

After these extra conditions have been inserted Theorem 1 may be proved by induction on m. Suppose that $m \geq 1$ and that the modified form of Theorem 1 is true for all smaller values of m. Then it follows from the condition (6) that it is possible to choose integers r, s with $0 \leq r < s \leq n$ and a sequence of points $Z_1, \ldots, Z_\ell$ from the points $X_{n+1}, \ldots, X_{n+m}$ such that $K+X_r$ touches $K+Z_1$, $K+Z_1$ touches $K+Z_2, \ldots$, and $K+Z_\ell$ touches $K+X_s$. (We say that two sets touch if they have a common boundary point without having a common point.) We may arrange that $Z_1, \ldots, Z_\ell$ are distinct. Then the broken line $X_r Z_1 Z_2 \ldots Z_\ell X_s$ divides the polygon π into two subpolygons π_1 and π_2. The sets $K+X_{n+1}, \ldots, K+X_{n+m}$ other than the sets $K+Z_1, \ldots, K+Z_\ell$ will be distributed between the polygons π_1 and π_2. It is easy to see that we may apply the modified form of the theorem to the polygons π_1 and π_2 and that if we combine the inequalities so obtained we get the inequality (5). Thus the theorem may be proved by induction provided we can prove the modified form of the theorem in the case when $m = 0$.

In fact it is convenient to prove a slightly more general result by an inductive method. We prove the following lemma.

Lemma 1. Let K *be an open strictly convex set with* O *as centre. Let* $0 \leq \sigma < 1$, *and let* Z, $X_1, \ldots, X_n$ *be points, with* $n \geq 3$, *such that:*

(1) *the polygon* $X_1 X_2 \ldots X_n X_1$ *is a Jordan polygon bounding a domain* π *of area* $V(\pi)$;

(2) *the sets* $K+X_r$ *and* $K+X_{r+1}$ *touch for* $r = 1, \ldots, n-1$ *and* $\sigma K+Z$ *touches* $K+X_1$ *and* $K+X_n$;

(3) *no two of the sets* $\sigma K+Z$, $K+X_1, \ldots, K+X_n$ *have a point in common;*

(4) *there is no point* Y *in or on the boundary of* π *such that no two of the sets* $K+Y$, $K+X_1, \ldots, K+X_n$ *have a point in common.*

Then

$$V(\pi) \geq (\tfrac{1}{2} n - 1)\, D(K). \tag{7}$$

If we take $\sigma = 0$ in this lemma we obtain the modified form of Theorem 1 in the case $m = 0$.

It is rather awkward to prove this lemma rigourously; but the idea of the proof is quite simple. The proof is by induction on n. We vary Z and σ moving Z between the points X_1 and X_n towards π or into π, taking care to ensure that $\sigma K+Z$ continues to touch $K+X_1$ and $K+X_n$, and continuing the movement until $\sigma K+Z$ comes into contact with one of the sets $K+X_2,\dots,K+X_{n-1}$, say the set $K+X_r$. It follows from the condition (4) that we will have $\sigma < 1$ when $\sigma K+Z$ reaches its final position. It is possible to show that the triangle $X_1X_rX_n$ will be contained in π. Thus π will split up into the triangle $X_1X_rX_n$ and two polygons π_1 and π_2 with vertices $X_1X_2\dots X_rX_1$ and $X_rX_{r+1}\dots X_nX_r$ (these polygons will sometimes degenerate to a line taken twice). We are able to apply the lemma with a smaller value of n to the polygons π_1 and π_2. We apply Lemma 2 stated below to the triangle $X_1X_rX_n$. Adding the inequalities obtained in this way we arrive at the inequality (7). In the initial case, when $n = 3$, we necessarily have $r = 2$, and the inequality (7) follows from Lemma 2 without use of any inductive hypothesis. Thus we can prove Lemma 1 by induction provided we can prove Lemma 2.

Lemma 2. *Let* K *be an open strictly convex set with* O *as centre. Let* $0 \leq \sigma < 1$, *and let* Z, X_1, X_2, X_3 *be points such that:*

(1) $\sigma K+Z$ *touches* $K+X_r$ *for* $r=1,2,3$;

(2) *no two of the sets* $K+X_r$, $r=1,2,3$ *have a point in common.*

Then, if $V(T)$ *is the area of the triangle* T *with vertices* X_1, X_2, X_3, *we have*

$$V(T) \geq \tfrac{1}{2} D(K). \qquad (8)$$

It is easy to reduce the problem of proving Lemma 2 to that of proving the lemma in the special case when two of the sets $K+X_1$, $K+X_2$, $K+X_3$ touch, say the sets $K+X_1$ and $K+X_2$. In this case we write

$$X_{rs} = r(X_2 - X_1) + s(X_3 - X_1)$$

for $r,s = 0, \pm 1, \pm 2, \dots$. We prove that no two of the sets

$$K + X_{rs} \qquad r,s = 0, \pm 1, \pm 2, \dots,$$

have a common point. Since the points X_{rs}, $r,s = 0, \pm 1, \pm 2$, form a lattice with fundamental parallelogram $X_{00}X_{10}X_{11}X_{01}$ of area $2V(T)$, it follows from the definition of $D(K)$ that $2V(T) \geq D(K)$. This implies (8), and enables us to complete the proof of the theorem.

§4. In this section we consider the problem of the closest packing of spheres in n-dimensional space. Although the closest lattice-packings of n-dimensional spheres are known for $n \leq 8$, the closest unrestricted packing is unknown for $n \geq 3$. Every physicist knows that the method used for stacking cannon balls (or oranges) is one of the closest packings of 3-dimensional spheres, but it has never been proved that there is not a packing which is closer than this well known packing.

In this section we use Blichfeldt's method to prove his result [H.F.Blichfeldt, Trans.Amer.Meth.Soc., 15(1914), 227-235, Theorem 2; see also H. F. Blichfeldt, Math. Annalen, 101(1929),605-608] giving an upper bound for $\rho(K)$ when K is an n-dimensional sphere.

Let S be a large cube of side s and let K be the sphere with centre $\mathcal{O}$ and radius 1. Let $K+X_1,\dots,K+X_\ell$ be a packing of ℓ spheres K into S. Blichfeldt's method is to replace each sphere by a somewhat larger spherical distribution of matter, to show that the total density at every point of space is not too large. This gives an upper bound for the amount of matter in a cube slightly larger than S and leads to an upper bound for the number of spheres in S.

More explicitly we introduce mass - distributions $M_1,\dots,M_\ell$ the density of the distribution M_λ at the point X being given by

$$\rho\,(X) = \begin{cases} 2 - |X - X_\lambda|^2 \;, & \text{if } |X - X_\lambda| \leq \sqrt{2}, \\ 0 \;, & \text{if } |X - X_\lambda| \geq \sqrt{2}, \end{cases}$$

for $\lambda = 1,\dots,\ell$, where $|X - X_\lambda|$ denotes the distance between the points X and X_λ. We take an arbitrary point X and show that the total density

$$\rho(\overline{X}) = \sum_{\lambda=1}^{\ell} \rho_\lambda(\overline{X})$$

at $\overline{X}$ is not too large. Let $Y_1,\dots,Y_m$ be those of the points $X_1,\dots,X_\ell$ for which

$$|Y_\mu - \overline{X}| = |X_\lambda - \overline{X}| < \sqrt{2}\,.$$

Then

$$\rho(\overline{X}) = \sum_{\lambda=1}^{\ell} \rho_\lambda(\overline{X}) = \sum_{\mu=1}^{m} \left\{2 - |\,\overline{X} - Y_\mu|^2\right\}.$$

We want to show that $|\overline{X} - Y_\mu|^2$ is fairly large on the average. We have to use the fact that the spheres $K+Y_1,\dots,K+Y_m$ do not overlap, that is the inequalities:

$$|Y_\mu - Y_\nu| \geq 2\,, \quad \text{if } \mu \neq \nu\,,\ \mu,\nu = 1,\dots,m.$$

Write $Z_\mu = Y_\mu - \overline{X} = (z_1^{(\mu)}, \ldots, z_n^{(\mu)})$ for $\mu = 1, \ldots, m$. Then

$$|Z_\mu - Z_\nu| \geq 2 ,$$

or

$$(z_1^{(\mu)} - z_1^{(\nu)})^2 + \ldots + (z_n^{(\mu)} - z_n^{(\nu)})^2 \geq 4 ,$$

for $\mu \neq \nu$, $\mu, \nu = 1, \ldots, m$. Summing these inequalities over all pairs μ, ν, we have

$$\sum_{1 \leq \mu < \nu \leq m} \sum_{r=1}^{n} (z_r^{(\mu)} - z_r^{(\nu)})^2 \geq 2m(m-1).$$

Re-arranging the left hand side this yields

$$\sum_{r=1}^{n} \left\{ (m-1) \sum_{\mu=1}^{m} (z_r^{(\mu)})^2 - 2 \sum_{1 \leq \mu < \nu \leq m} z_r^{(\mu)} z_r^{(\nu)} \right\} \geq 2m(m-1)$$

or

$$\sum_{r=1}^{n} \left\{ m \sum_{\mu=1}^{m} (z_r^{(\mu)})^2 - (\sum_{\mu=1}^{m} z_r^{(\mu)})^2 \right\} \geq 2m(m-1).$$

Thus

$$m \sum_{r=1}^{n} \sum_{\mu=1}^{m} (z_r^{(\mu)})^2 \geq 2m(m-1) .$$

Dividing by m we obtain

$$\sum_{\mu=1}^{m} |Z_\mu|^2 = \sum_{\mu=1}^{m} \sum_{r=1}^{n} (z_r^{(\mu)})^2 \geq 2m - 2$$

so that

$$\sum_{\mu=1}^{m} \left\{ 2 - |Z_\mu|^2 \right\} \leq 2 .$$

By our choice of the mass-distributions (which was in fact suggested by this inequality) we deduce that

$$\rho(\overline{X}) = \sum_{\mu=1}^{m} \{2 - |\overline{X} - Y_\mu|^2\}$$

$$= \sum_{\mu=1}^{m} \{2 - |Z_\mu|^2\} \leq 2.$$

The inequality $\rho(\overline{X}) \leq 2$ is valid for all points $\overline{X}$. If $\overline{X}$ is not in the cube S' of side $s+2\sqrt{2}$ centered round S, it is clear that $\rho(\overline{X}) = 0$. So integrating over the whole space we have

$$\int \rho(X)dX \leq \int_{S'} \rho(X)dX \leq \int_{S'} 2dX \leq 2(s + 2\sqrt{2})^n .$$

But we also have

$$\int \rho(X)dX = \int \sum_{\lambda=1}^{\ell} \rho_\lambda(X)dX = \ell M,$$

where M is the total mass of one mass distribution. So

$$\ell \leq \frac{2}{M}(s + 2\sqrt{2})^n.$$

Now let J_n be the volume of the unit sphere K in n-dimensional space. Splitting the mass-distribution into spherical shells, we see that

$$M = \int_0^{\sqrt{2}} (2 - r^2)\, d\, J_n r^n$$

$$= n J_n \int_0^{\sqrt{2}} \{2r^{n-1} - r^{n+1}\} dr$$

$$= n J_n \left[\frac{2r^n}{n} - \frac{r^{n+2}}{n+2} \right]_0^{\sqrt{2}}$$

$$= \frac{4J_n\, 2^{n/2}}{n+2} .$$

So the density of the closest packing of K into S satisfies

$$\rho(K,S) = \frac{\ell J_n}{s^n} \leq \frac{n+2}{2^{(n+2)/2}} \left(1 + \frac{2\sqrt{2}}{s}\right)^n .$$

Letting s tend to infinity we obtain the result that

$$\rho(K) \leq \frac{n+2}{2^{(n+2)/2}} . \tag{9}$$

This should be compared with the result

$$\rho(K) \geq \frac{1}{2^n} \tag{10}$$

obtained in §2. The inequality (9) has been improved by Blichfeldt and by Rankin, [H.F.Blichfeldt, Math.Annalen,101(1929), 605-608, R.A.Rankin, Annals of Math., 48(1947),1062-1081.] but the improvements are very small when n is large. The result (10) has also been improved slightly when K is a sphere. [H. Davenport and C.A.Rogers, Duke Math.J.,14(1947), 367-375, Theorem 2.]

§5. In this section we obtain an upper bound of <u>the minimum of the product of n real linear forms</u> by application of Blichfeldt's method. Let

$$x_i = a_{i1}u_1 + \dots + a_{in}u_n , \qquad i = 1, \dots, n \tag{11}$$

be n real linear forms with determinant 1. We want to prove that there are integers $u_1, \dots, u_n$ not all zero such that the product $x_1 \dots x_n$ is fairly small numerically.

Let Λ be the lattice of all points X with coordinates given by (11) where $u_1, \dots, u_n$ take all possible integral values. We take a large cube

S of side s. Let $X_1, \dots, X_\ell$ be the points of Λ which lie in this cube. Then, as Λ has determinant 1, we have

$$\ell = s^n(1 + o(1)) , \qquad \text{as } s \longrightarrow \infty .$$

We choose an integer $m \geq 2$ and a number ε with $0 < \varepsilon < 1$. Let K be the region

$$\sum_{i=1}^{n} |x_i| < d \quad ,$$

where

$$d = \tfrac{1}{2}\{(m - 1 + \varepsilon)(n!)\}^{1/n}.$$

Consider the bodies $K+X_1, \dots, K+X_\ell$. These bodies are contained in a cube of volume $(s + 2d)^n$; their total volume is

$$\ell . 2^n d^n/n! = s^n\{1 + o(1)\}(m - 1 + \varepsilon).$$

So, provided s is sufficiently large, the total volume of the bodies will be greater than (m - 1) times the volume of the cube containing the bodies. So there is some point which is in at least m of the bodies. Suppose then that X is a point in the bodies $K+Y_\mu$, $\mu = 1, \dots, m$ where $Y_1, \dots, Y_m$ are selected from the points $X_1, \dots, X_\ell$. Then we have

$$\sum_{i=1}^{n} |y_i^{(\mu)} - x_i| < d \quad , \qquad \mu = 1, \dots, m.$$

Write $Z_\mu = Y_\mu - X$, so that

$$\sum_{i=1}^{n} |z_i^{(\mu)}| < d \, , \qquad \mu = 1, \dots, m. \tag{12}$$

Now, if $\mu \neq \nu$, the point $Z_\mu - Z_\nu = Y_\mu - Y_\nu$ is a lattice point other than O. We want to show that

$$\prod_{i=1}^{n} |z_i^{(\mu)} - z_i^{(\nu)}|$$

is small for some integers μ, ν. To do this we show that the product of all such products is small.

Let K_m be the least positive number such that

$$\prod_{1 \le \mu < \nu \le m} |\zeta_\mu - \zeta_\nu| \le \left\{ \frac{K_m}{m} \sum_{\mu=1}^{m} |\zeta_\mu| \right\}^{\frac{1}{2} m(m-1)}, \qquad (13)$$

for all real numbers $\zeta_1, \ldots, \zeta_m$. It does not seem to be easy to obtain a good estimate for K_m when m is large. But I was able to prove [C.A.Rogers, Acta Math., 82(1950), 185-208, Lemma 5.] that

$$K_m \le \frac{\pi}{2\sqrt{e}} \left(\frac{e^3 \pi^2 m(m-1)^2}{16} \right)^{\frac{1}{(2m-2)}}. \qquad (14)$$

Using the inequality (13), the inequality of the arithmetic and geometric means, and the inequality (12), we obtain

$$\prod_{1 \le \mu < \nu \le m} \prod_{i=1}^{n} |z_i^{(\mu)} - z_i^{(\nu)}| \le \left[\left\{ \frac{K_m}{m} \right\}^n \left\{ \prod_{i=1}^{n} \sum_{\mu=1}^{m} |z_i^{(\mu)}| \right\} \right]^{\frac{1}{2} m(m-1)}$$

$$\le \left[\left\{ \frac{K_m}{m} \right\}^n \left\{ \frac{1}{n} \sum_{i=1}^{n} \sum_{\mu=1}^{m} |z_i^{(\mu)}| \right\}^n \right]^{\frac{1}{2} m(m-1)}$$

$$< \left[\frac{K_m}{m} \cdot \frac{md}{n} \right]^{\frac{1}{2} nm(m-1)}$$

$$= \left[\left(\frac{K_m}{2n} \right)^n (m - 1 + \varepsilon)(n!) \right]^{\frac{1}{2} m(m-1)}.$$

So for some μ, ν with $\mu \ne \nu$ we have

$$\prod_{i=1}^{n} |z_i^{(\mu)} - z_i^{(\nu)}| < (m - 1 + \varepsilon)(n!) \left(\frac{K_m}{2n} \right)^n.$$

Thus for each integer $m \ge 2$ and for each ε with $0 < \varepsilon < 1$ we can first find a lattice point X other than $\mathcal{O}$ satisfying

$$|x_1 \cdots x_n| < (m - 1 + \varepsilon)(n!)\left(\frac{K_m}{2n}\right)^n . \tag{15}$$

If we take m = [n log n] and use the inequality (14) we see that the right hand side of (15) is less than or equal to

$$\left(\frac{\pi}{4e\sqrt{e}} + o(1)\right)^n \tag{16}$$

as $n \longrightarrow \infty$. It is conceivable that one can always find a lattice point X other than $\mathcal{O}$ satisfying

$$|x_1 \cdots x_n| \leq M_n$$

where

$$M_n = \{o(1)\}^n$$

as $n \longrightarrow \infty$, but the result given by (15) and (16) is the best result known [See C.A.Rogers, Acta Math., loc.cit., and references given there.] at the moment for large n .

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VII. Convex Sets in Linear Spaces: Two Applications of Zorn's Lemma.

Talk by B.J. Pettis, notes by V. L. Klee.

§1. Kakutani's theorem on decomposition into convex sets.

By a linear system we shall mean a module over the real number field (or what Banach calls "espace linéaire".) The following result, due to S. Kakutani [1], seems to provide the simplest approach to the separation and support theorems on convex sets:

(1.1) Suppose that A and B are disjoint convex sets in a linear system L. Then there are complementary convex sets C and D such that $C \supset A$ and $D \supset B$.

Proof: Let Z be the family of all pairs of convex sets (X,Y) such that $X \supset A$, $Y \supset B$, and $X \cap Y = \Lambda$. Write $(X,Y) < (X',Y')$ if $X \subset X'$ and $Y \subset Y'$. Z is partially ordered by "<" and each totally ordered subfamily has an upper bound in Z. From Zorn's lemma it follows that there is in Z a maximal element (C,D). Consider an arbitrary point $p \in L$. Writing xyz to indicate that $y \in [x,z]$ (the line-segment including its endpoints) note that we cannot have simultaneously pc_1d_1 and pd_2c_2 for c_i's in C and d_i's in D, for this would imply that $[c_1,c_2]$ intersects $[d_1,d_2]$, contradicting the fact that $C \cap D$ is empty. Suppose, then, that pc_1d_1 cannot subsist and let D' be the convex hull of $D \cup \{p\}$. Then $(C,D') \in Z$, so from the maximality of (C,D) it follows that $D = D'$. Hence $C \cup D = E$ and the proof is complete.

§2. The separation theorem.

A linear proper subset of a linear system will be called a subspace and each translate of a subspace is a variety. For a subset X of a linear system, lin X will denote the union of X with the set of all points y such that $[x,y) \subset X$ for some x. $X^{o} \equiv E - \mathrm{lin}(E-X)$. As a first lemma for proving the separation theorem, we have

(2.1) *If C and D are complementary convex subsets of the linear system L and $M = \mathrm{lin}\, C \cap \mathrm{lin}\, D$, then either M is a maximal variety or $M = L$.*

Proof: X convex implies lin X convex, so M must be convex. Now let x and y be distinct points of M and suppose $y \in (x,z)$. If $z \notin M$ then $z \in S^{o}$ where S is either C or D. But then it follows easily that $y \in S^{o}$ which is impossible. Thus each line determined by two points of M is contained entirely in M and M must be a variety. We may suppose without loss of generality that $\phi \in M$. Now let $p \in L-M$; say $p \in C^{o}$. Then $-p \in D^{o}$. But then $[x,-p]$ intersects M if $x \in C$ and $[x,p]$ intersects M if $x \in D$. Hence $M + Rp = L$, so either M is a maximal variety or $M = L$.

Following Hille, by topological linear space we mean a linear system with an associated topology for which $x + ty$ is continuous separately in x, t, and y. To each maximal variety M corresponds a linear functional $f \neq 0$ and a constant c such that $M = [f;c] \equiv \{x \mid f(x) = c\}$. M is called a hyperplane if f is continuous. If $X \subset [f;\leqq c]$ or $X \subset [f;\geqq c]$, we say that X is bounded by M.

(2.2) *In a topological linear space, a maximal variety is a hyperplane if and only if it bounds some convex body* (i.e., a convex set having interior points.)

Proof: If $[f;c]$ is a hyperplane then f is continuous and hence $[f;>c]$ is a convex body bounded by $[f;c]$. Suppose conversely that $f \neq 0$ and $[f;0]$ bounds a convex body K. Since K cannot intersect both $[f;>0]$ and $[f;<0]$ we may assume that $K \subset [f;\geqq 0]$. Let $p \in [f;1]$. Then $K + p$ is a convex body contained in $X = [f;>0]$ and it follows easily that X is open. Now for each open interval $(a,b) \subset R$ we have $f^{-1}(a,b) = [X + ap] \cap [-X + bp]$, which is open. Hence f is continuous and the proof is complete.

If $[f;c]$ is a maximal variety in the linear system L and A and B are subsets of L, we say that $[f;c]$ <u>separates</u> A from B provided either $A \subset [f;\geqq c]$ and $B \subset [f;\leqq c]$ or $A \subset [f;\leqq c]$ and $B \subset [f;\geqq c]$.

(2.3) (Separation Theorem) <u>Suppose that E is a topological linear space, A is a convex body in E, and B is a convex subset of E which does not intersect the interior of A. Then A and B can be separated by a hyperplane.</u>

Proof: By (1.1) there are complementary convex sets $C \supset \text{Int } A$ and $D \supset B$. Let $M = \text{lin } C \cap \text{lin } D$. The continuity axiom for topological linear spaces implies that A° is non-empty and hence $M \neq E$. Thus from (2.1) it follows that M is a maximal variety, and clearly M separates A and B. That M is a hyperplane follows from (2.2).

The separation theorem has a number of interesting consequences, some of which will be developed in VIII. At present, however, we need the following result for use in proving a theorem on extreme points of convex sets.

(2.4) <u>Suppose that E is a topological linear space with the property that for each $x \neq \emptyset$ there is a continuous linear functional f on E such that $f(x) \neq 0$. Then if K is a compact convex subset of E and $p \in E-K$, there is a continuous linear functional g on E such that $g(p) > \sup_{y \in K} g(y)$.</u>

Proof: From the hypothesis on the existence of continuous linear functionals, it follows that for each $q \in K$ there is a half-space $[f;<c]$ which contains q but not p. Since K is compact it is covered by a finite number of such half-spaces, and the intersection of the corresponding complementary open half-spaces (those of the form $[f;>c]$)is an open convex set which contains p and misses K. Then the desired conclusion follows almost at once from (2.3).

§3. Kelly's theorem on extreme points.

If K is a convex subset of a linear system, then by an extreme point of K is meant a point $y \in K$ such that whenever $x, z \in K$, $y \notin (x,z)$. A well-known theorem due to M. Krein and D. Milman [2] asserts that if E is a Banach space and K is a bounded regularly convex subset of the adjoint space E^*, then K is the closed convex hull of its set of extreme points. In a paper which has not yet been published, J.L.Kelley has generalized the theorem as follows:

(3.1) Let E be a topological linear space with the property that *for each $x \neq \emptyset$ there is a continuous linear functional f for which $f(x) \neq 0$. Let K be a compact convex subset of E. Then K is the closed convex hull of its set of extreme points.*

Proof: Let Z be the class of all closed subsets X of K which have the property that if a line segment s of K has a non-endpoint in X, then $s \subset X$. Z is non-empty, since $K \in Z$. Z is partially ordered by set inclusion and the intersection of each totally ordered subclass of Z is in Z. Hence by Zorn's lemma Z has a minimal element M. Now suppose M contains two different points, x and y. There is a continuous linear functional f on E such that $f(x) \neq f(y)$. Let M' be the subset of M on which f attains its maximum value. Since M is compact, M' is non-empty. Now if s is a line segment in K which has a non-endpoint in M', then by the definition of Z, $s \subset M$. But then from the definition of M' it follows that $s \subset M'$. M' cannot contain both x and y, so M' is a proper subset of M and $M' \in Z$, contradicting the minimality of M. Thus M contains a single point, which must be an extreme point of K. We have shown that K must have at least one extreme point.

Now let K' be the closed convex hull of the set of all extreme points of K. We know that K' is non-empty and wish to show that $K' = K$. The assumption that $K' \neq K$ implies (by use of (2.4)) that there is a continuous linear functional f on E such that if Y is the set points at which f attains its maximum on K, then $Y \cap K' = \Lambda$. But the result of the above paragraph implies that Y has at least one extreme point and clearly this must be an extreme point of K also, which is a contradiction. Thus the proof is complete.

(Since each bounded regularly convex set is compact in the weak* topology, (3.1) implies the Krein-Milman result.)

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VIII. Separation and Support Properties of Convex Sets

by

V. L. Klee

§1. Some consequences of the separation theorem.

In this section we will develop some consequences of the separation theorem, whose proof was presented by Professor Pettis. Let us first state this theorem as

(1.1) Suppose that E is a topological linear space (in the sense of Hille), A is a convex body in E, and B is a convex subset of E which does not intersect the interior of A. Then A and B can be separated by a hyperplane.

(1.2) to (1.5) below follow immediately from (1.1).

(1.2) Suppose that E is a topological linear space, C is a convex body in E, and V is a variety which contains no interior point of C. Then C is bounded by a hyperplane which contains V.

(1.3) In a topological linear space, a convex body is supported by a hyperplane at each of its boundary points.

(1.4) A variety in a topological linear space is contained in some hyperplane if and only if its complement contains some convex body.

(1.5) There is a non-trivial continuous linear functional on a topological linear space E if and only if some proper subset of E is a convex body.

For normed spaces, (1.1) is due to Tukey [8], (1.2) and (1.3) to Mazur [7]. (1.5) was proved by LaSalle [6] for Hausdorff linear spaces (in the sense of Hille).

A subset X of a normed linear space E will be called a <u>distance-set</u> if to each $y \in E$ corresponds at least one $q \in X$ such that $\|q - y\| = \inf_{x \in X} \|x - y\|$.

(1.6) <u>In a normed linear space E, a convex distance-set is supported by a hyperplane at each point of a set dense in its boundary.</u>

Proof: Let p be an arbitrary boundary point of the convex distance-set C, and let y_i be a sequence of points of E-C such that $y_i \longrightarrow p$. For each i let q_i be a point of C which is nearest to y_i, and let S_i be the sphere having center y_i and radius $\|y_i - q_i\|$. From (1.1) it follows that S_i and C can be separated by a hyperplane H_i which clearly must support C at q_i. And since $q_i \longrightarrow p$ the proof is complete.

In [5], (1.6) was proved by the author in a somewhat different way.

Bourgin [2] calls a set $C \subset E$ <u>regularly</u> E <u>convex</u> if for each $x \in E-C$ there is a continuous linear functional f such that $f(x) > \sup_{c \in C} f(c)$. We have the following result, proved by Bourgin for Hausdorff linear spaces.

(1.7) <u>In a topological linear space, each closed convex body is regularly</u> E <u>convex.</u>

Proof: Suppose that C is a closed convex body and $x \in E-C$. Let z be an arbitrary interior point of C. Then the line segment (x,y) must fail to intersect C for some $y \in (x,z)$. Let [f;c] be a hyperplane which separates (x,y) and C; say $(x,y) \subset [f; \geqq c]$ and $C \subset [f; \leqq c]$. We wish to show that $f(x) > c$. Suppose not. Then we have $f(x) \leqq c$, $f(y) \geqq c$, and $f(z) \leqq c$ with

$y \in (x,z)$, and hence $f(x) = f(y) = f(z) = c$. But then $z \in [f; c]$, which contradicts the fact that $z \in \text{Int } C$ and completes the proof.

For normed spaces, the following result was proved in [5] by the author:

(1.8) In a locally convex Hausdorff linear space E, each closed convex cone $\neq E$ is supported at its vertex by a hyperplane.

Proof: Let C be a convex cone of the type described and $p \in E-C$. Then there is a convex open set U such that $p \in U \subset E-C$, and by (1.1) U and C can be separated by a hyperplane H. It is easy to see that the translate of H which contains the vertex of C must actually support C.

(1.9) Suppose that A and B are disjoint convex subsets of the locally convex Hausdorff linear space E, A is weakly compact, and B is closed. Then there is a continuous linear functional f on E such that $\inf_{a \in A} f(a) > \sup_{b \in B} f(b)$.

Proof: Let E_w denote the space E in its weak topology. As Bourgin has noted [2], E_w is a locally convex Hausdorff linear space and a convex set is closed in E_w if and only if it is closed in E. Since E_w is a uniform space, A is a compact and B a closed subset of E_w, it follows from a result of Bourbaki [1;pp.111-112] that there is a convex set $V \ni \phi$ such that V is open in E_w (and hence in E) and $(A + V) \cap (B + V) = \Lambda$. Then by (1.1) we can obtain a hyperplane $[f;c]$ which separates these sets. Clearly either f or $-f$ has the desired property.

This result constitutes a strengthening of Bourgin's theorem [2; pp.643-646] that each closed convex set in E is regularly E convex, and also an extension of Tukey's separation theorem for weakly compact sets [8], which was proved by him for normed spaces only.

(1.10) <u>Suppose that K is a closed, locally weakly compact, convex subset of the locally convex Hausdorff linear space E. Then at each point of a set dense in its boundary, K is supported by a hyperplane.</u>

Proof: (For normed spaces this was proved in [5] by showing that K is a distance-set and then applying (1.6). The proof here is quite analogous.) Let p be an arbitrary boundary point of K and U an arbitrary neighborhood of p. We wish to show that K is supported at some point of $U \cap K$.

From our hypotheses it follows that there is a convex open set V such that $p \in V \subset \overline{V} \subset U$ and $\overline{V} \cap K$ is weakly compact. Let $p \in V-K$ and let W be a convex open set such that $p \in W \subset \overline{W} \subset V-K$. Let Z be the class of all closed convex sets C such that $W \subset C \subset \overline{V}$ and $C \cap K \neq \Lambda$. Then $\overline{V} \in Z$ and Z is partially ordered by set-inclusion. Now suppose Z' is a totally ordered subclass of Z and define $C' = \bigcap_{C \in Z'} C$. Then C' is closed and convex and $W \subset C' \subset \overline{V}$. And the collection of weakly closed sets $\{C \cap K \mid C \in Z'\}$ has the finite intersection property, so from the weak compactness of $\overline{V} \cap K$ it follows that $C' \cap K \neq \Lambda$. Hence $C' \in Z$.

Now we can apply Zorn's lemma to obtain a minimal element M of Z. M is a convex body since $W \subset M$. Thus if we can show that $K \cap \operatorname{Int} M = \Lambda$ we obtain by (1.1) an hyperplane which separates K and M, and thus supports K at each point of $K \cap M \subset U$, which completes the proof. So suppose that $z \in K \cap \operatorname{Int} M$. Then there are points x and y of $\operatorname{Int} M \cap K$ such that $y \in (x,z)$. We see from (1.2) that there is a hyperplane $[f;c]$ such that $f(y) = c$ and $W \subset [f; \leqq c]$. Now let $M_1 = M \cap [f; \leqq c]$. Then $M_1 \in Z$ but $x \notin M_1$, which contradicts the minimality of M and completes the proof.

To show that the compactness requirement in (1.10) cannot be entirely omitted, Dieudonné [3] gave an example in the space ℓ of two disjoint bounded closed convex sets which cannot be separated by a hyperplane. The corresponding question with respect to (1.10) is still open.

§2. Non-support points of convex sets. For a convex subset C of a linear space, let N(C) denote the set of all points of C at which C fails to be supported by a hyperplane. In a finite-dimensional normed space, N(C) is empty if and only if C is actually contained in some hyperplane. For infinite-dimensional spaces, however, this need not be the case, as we see from the examples below.

(a) In the space ℓ^p $(p \geqq 1)$ let E_1 be the set of all points which have at most a finite number of non-zero coordinates. Let X be the collection of all points of ℓ^p which have exclusively non-negative coordinates. Then $X \cap E_1$ is convex and is supported by a hyperplane at each of its points, but is not contained in any hyperplane.

(b) Let E_1 be the space. Then $X \cap E_1$ is closed in E_1 and is supported by a hyperplane at each of its points, but is not even contained in a variety of E_1.

(c) Let X be an uncountable set and let E be the space $\ell^p(X)$. Then E is a non-separable Banach space. Let K be the class of all non-negative functions in E. Then K is closed, convex, and supported at each of its points, but not contained in any variety.

The question remains open as to whether such an example of a closed convex set can exist in a complete separable normed linear space. Each of (a), (b), and (c) fulfills two of the three underlined conditions.) We shall see, however, that if C does have non-support points it must have a lot of them.

(2.1) Suppose that E is a topological linear space and C is a convex subset of E for which N(C) is not empty. Then N(C) is a dense convex subset in C

Proof: Let $x \in N(C)$, $y \in C$, and $z \in (x,y)$. Clearly each hyperplane which supports C at z must contain x, so there can be no such hyperplane and we must have $(x,y) \subset N(C)$, which clearly completes the proof.

(2.2) Suppose that in a linear space, C is a convex set which at some point $x \in C$ has a unique supporting hyperplane. Then either $C \subset H$ or N(C) is non-empty.

Proof: Suppose there is some point $z \in C-H$ and let $y \in (x,z)$. If $y \notin N(C)$ then C is supported at y by a hyperplane H'. But then H' supports C at x, so by hypothesis H' = H and hence $y \in H$. But then also $Z \in H$, which is a contradiction.

(2.3) Suppose that C is a convex subset of the separable normed linear space E and that N(C) is non-empty. Then N(C) is a residual G_δ-set in C.

Proof: For each $v \in C$ let K_v be the cone from v over C. Suppose v_i is a sequence of points of C for which $v_i \longrightarrow v \in C$ and let $y \in K_v$. For some $k > 0$ and $x \in C$ we have $y = v + k(x - v)$. But now $y_i \longrightarrow y$ where $y_i = v_i + k(x - v_i) \in K_{v_i}$, so $y \in \liminf K_{v_i}$. Thus the mapping $K_v \mid v \in C$, and hence also the mapping $\overline{K}_v \mid v \in C$, is lower-semi-continuous. Now from (1.8) it follows that $\overline{K}_v = E$ if and only if $v \in N(C)$, and since N(C) is dense in C the mapping $\overline{K}_v \mid v \in C$ is continuous precisely at points of N(C). But then the desired conclusion follows at once from a theorem of M.K. Fort [4].

§4. A generalization of the separation theorem. (Although most of the proofs of this section are valid in every topological linear space, the theorems will be stated only for a finite-dimensional Euclidean space E.) A collection of n+1 convex subsets of E will be called an n-set in E provided each n of the sets have a common interior point, although the intersection of all n+1 interiors is empty. The separation theorem says that if $\{C_o, C_1\}$ is a 1-set, then C_o and C_1 can be separated by a hyperplane. This result is generalized below by showing that if $\{C_o,\ldots,C_n\}$ is an n-set, then there is variety V of deficiency n in E such that V intersects no set Int C_i, although in each direction away from V, V has a translate which intersects some set Int C_i.

The following result will be useful in the sequel.

(3.1) Suppose that $C_o,\ldots,C_n$ are closed convex subsets of E, each n of which have a point in common, and that $V_{i=0}^{n} C_i$ is convex. Then there is a point in common to all C_i's.

Proof: We may assume without loss of generality that all the C_i's are compact. For n = 0 the theorem is trivial. Now suppose it holds for n = k-1 and consider the case n = k. If $\bigcap_{i=o}^{k} C_i = \Lambda$ then C_o and $P = \bigcap_{i=1}^{k} C_i$ are disjoint compact convex sets, so can be separated by a hyperplane H disjoint from both of them. Let $C_i' = C_i \cap H (1 \leqq i \leqq k)$. For an arbitrary integer j between 1 and k let $X = \bigcap_{1 \leqq i \leqq k,\, i \neq j} C_i$. Since each k of the C_i's have a point in common, X intersects C_o. And since furthermore $P \subset X$, X must intersect H and hence $\bigcap_{1 \leqq i \leqq k,\, i \neq j} C_i' \neq \Lambda$. But $V_{i=1}^{k} C_i'$ is convex, so it follows from the inductive hypothesis that $\bigcap_{i=1}^{k} C_i' \neq \Lambda$. Since this contradicts the fact that $P \cap H = \Lambda$, the proof is complete.

If X_o, X_1, ..., X_n are points of E, $[x_o, x_1, \ldots, x_n]$ will denote the convex hull of the set $\{x_o, x_1, \ldots, x_n\}$, $(x_o, x_1, \ldots, x_n) \equiv [x_o, x_1, \ldots, x_n] - \{x_o\}$, etc., (" - " is used for both set and vector differences, since in each case the meaning is clear from the context. "+" is used for vector sum, "U" for set union.)

The proof of our theorem is effected by means of two lemmas, the first of which is the following:

(3.2) <u>If</u> $\{C_o, \ldots, C_n\}$ <u>is an n-set in</u> E, <u>then there are convex sets</u> $K_i \supset C_i$ <u>such that</u> $\{K_o, \ldots, K_n\}$ <u>is an n-set which covers</u> E.

Proof: We will show that if x is an arbitrary point of E then there are convex sets $C_i' \supset C_i$ such that $\{C_o', \ldots, C_i'\}$ is an n-set and, in addition, $x \in \bigcup_{i=o}^{n} C_i'$. (3.2) follows from this fact by a straightforward application of Zorn's lemma.

For $o \leq j \leq n$, $D_j \equiv \bigcap_{i \neq j} \text{Int } C_i$. If from some j we cannot have $d_j \in (c_j, x)$, with $d_j \in D_j$ and $c_j \in C_j$, then we merely let C_j' be the convex hull of $C_j \cup \{x\}$, $C_i' = C_i$ for $i \neq j$, and the sets C_i' will have the desired properties. Suppose this is not the case; that is, that there are points $d_o, \ldots, d_n, c_o, \ldots, c_n$ such that for $o \leq i \leq n$, $c_i \in C_i$, and $d_i \in D_i \cap (c_i, x)$. For each j let $X_j = (c_j, d_o, \ldots, d_{j-1}, d_{j+1}, \ldots, d_n)$. Then $X_j \subset \text{Int } C_j$. But by use of Cramer's rule it can be shown that all the X_i's have a point in common and hence that $\bigcap_{i=o}^{n} \text{Int } C_i \neq \Lambda$, which is a contradiction completing the proof of (3.2).

A linear subset of E is called a <u>subspace</u>, and each translate of a subspace is a <u>variety</u>. The <u>deficiency</u> in E of a subspace (and of its translates) is the dimension of a subspace complementary to it.

(3.3) <u>Suppose that</u> $\{K_o, \ldots, K_n\}$ <u>is an n-set which covers</u> E <u>and that</u> $V = \bigcap_{i=o}^{n} \overline{K}_i$. <u>Then</u> $V = E - \bigcup_{i=o}^{n} K_i$, <u>and is a variety of deficiency</u> n <u>in</u> E.

Proof: Let $W = E - \bigcup_{i=o}^{n} \text{Int } K_i$ and (for each j) $\pi_j = \bigcap_{i \neq j} \text{Int } K_i$. From (3.1) we see that V is non-empty. We show first that $V \subset W$. For if not, there is a point p and an integer j such that $p \in V \cap \text{Int } K_j$. Let $q \in \pi_j$. Then, since for each $i \neq j$ we have $p \in \overline{K}_i$ and $q \in \text{Int } K_i$, $(p,q) \subset \pi_j$. But also $p \in \text{Int } K_j$, so (p,q) intersects Int K_j and $\bigcap_{i=o}^{n} \text{Int } K_i \neq \Lambda$, which is a contradiction.

To see that $W \subset V$, let $y \in W$ and $z \in \pi_j$ for some j. Consider an arbitrary point x such that $y \in (x,z)$. If, for any $i \neq j$, $x \in K_i$, then we have $y \in \text{Int } K_i$, which contradicts the fact that $y \in W$. Hence $x \in K_j$. Thus we have shown that $y \in \overline{K}_j$ for each j, and consequently $y \in V$. Since $W \subset V$ and $V \subset W$, $V = W$.

Obviously V is convex. To prove that it is actually a variety we must show that if $y \in V$, $z \in V$, and $y \in (x,z)$, then $x \in V$. But if $x \notin V$, then (since $V = W$)$x \in \text{Int } K_j$ for some j and hence $y \in \text{Int } K_j$, which contradicts the fact that $y \in W$. Hence V is a variety and it remains only to show that the deficiency of V in E is n.

We assume without loss of generality that V contains the origin. Let S be a subspace of E complementary to V. Each point x of E has a unique expression in the form $v_x + x^*$ where $v_x \in V$ and $x^* \in S$. From the fact that $V = W$ it follows that no translate of V other than V itself can intersect all the sets K_i. This in turn implies that $\{K_o^*, K_1^*, \ldots, K_n^*\}$ is an n-set in S. Now Helly's theorem applies to an arbitrary <u>finite</u> collection of convex sets even though they may not be compact, so we can conclude that S is at least n-dimensional.

If $p_i \in \pi_i$ for each i then the variety U determined by $\{p_o, \ldots, p_n\}$ is n-dimensional. Let x be an arbitrary point of E. For a sufficiently small positive t we have $p_i + tx \in \pi_i$ for each i. Now for each i let $K_{i_t} = K_i \cap (U + tx)$. $\{K_{o_t}, K_{1_t}, \ldots, K_{n_t}\}$ is an n-set in $U + tx$, so V must intersect $U + tx$. From this it follows that V must intersect every translate of U, and hence that the deficiency of V in E is no greater than n. This completes the proof of (3.3).

<u>Theorem</u>: <u>If</u> $\{C_o, \ldots, C_n\}$ <u>is an n-set in</u> E, <u>then there is a variety</u> V <u>of deficiency</u> n <u>in</u> E <u>such that</u>

(a) V <u>intersects no set</u> Int C_i;

(b) <u>if</u> V' <u>is any variety of deficiency</u> n-1 <u>which contains</u> V, <u>and</u> H <u>is either of the half-space into which</u> V <u>separates</u> V', <u>then</u> H <u>intersects some set</u> Int C_i.

Proof: Let the K_i's be as in (3.2) and V as in (3.3). For each j let $z_j \in D_j = \bigcap_{i \neq j} C_i$ and let S be the variety determined by $\{z_o, \ldots, z_n\}$. S is a variety which is intersected by V in a single point P, and $\sigma = [z_o, \ldots, z_1]$ is an n-simplex whose boundary (relative to S) is contained in the union of the C_i's. In fact, if F_j is the face determined by $\{z_j | i \neq j\}$, then $F_j \subset \text{Int } C_j$. Now if V' is a variety of deficiency n-1 (in E) which contains V, then V' intersects S in a line through p. Hence to prove the Theorem we need merely show that $P \in \sigma$. But if $P \notin \sigma$ then for some j there is a point $q \in F_j$ such that either $q \in (P, z_j)$ or $z_j \in (q, P)$. In the first case this implies that $p \in \pi_j$, in the second that $z_j \in \text{Int } C_j$, so in either case that $\bigcap_{i=o}^{n} \text{Int } C_i \neq \Lambda$, which is a contradiction completing the proof.

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Partial lis of errata

Page 18, Theorem 32. Should be compared with Theorem 46, page 24

Page 23, Theorem 45. Should be compared with Theorem 47, page 25

Page 16, line 26, Replace "guage" by "gauge".

Page 34, line 11, Replace "r = 1" by "r < 1".