The Institute for Advanced Study

LECTURES ON QUANTUM ELECTRODYNAMICS

by

PROF. P. A. M. DIRAC

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Notes by Dr. Boris Podolsky first semester

and Dr. Nathan Rosen second semester

Dirac

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P.A.M.Dirac

The course is to be devoted principally to the quantum theory of fields. As an introduction, there is a brief presentation of the general principles of quantum mechanics which follows closely that given in the lecturer's "Quantum Mechanics" (Oxford University Press, 1930). In reporting this part of the work, the editors have decided that it is not worth while to duplicate extensively the basic presentation as given there. Instead therefore, a brief outline is given with page references together with notes on additional material or varied treatments which are not given in that work.

I. General Principles of Quantum Mechanics

1. Introduction of concept of State

The first basic concept is that of state of a dynamical system (Chapter I). One may think of the state as referring to a particular instant of time, relative to a particular Lorentz frame, or as referring to the whole development of the dynamical system throughout all time. These may be called the 3-dimensional and 4-dimensional meanings of the word respectively.

Which is preferable? Perhaps the 4-dimensional, since it is a relativistic concept, whereas the 3-dimensional is a particular section through the 4-dimensional obtained by introducing a particular Lorentz frame. But the theory has had its principal development through working with the 3-dimensional meaning in the foreground. So perhaps the 3-dimensional meaning is more fundamental than would appear if it is a mere section of the 4-dimensional state.

Here is a clash between the quantum theory and the theory of relativity.

The theory of the three-dimensional view is adequate for the non-relativistic theory. It divides naturally into (1) study of relations between states at a given instant and (2) the relation of the succession of states developing at successive instants.

Principle of superposition requires that the sum of two states shall be a state, that a meaning be ascribable to such a sum. From the principle of superposition for 4-dimensional states it follows that a linear relation between 3-dimensional states remains invariant in time.

2. Properties of three-dimensional states

The states may be pictured as vectors in an appropriate space. Only the direction of the vector has physical meaning. The vector space is complex in the sense that the components of vectors may be complex numbers. A vector represents the same state when multiplied by an arbitrary complex number.

There followed in the lectures a brief presentation of the main points contained in Chapters 2, 3, 4, and 5.

A more restricted definition of observable was given in the lectures than in the book. (Compare pp. 25-33.) There any linear operator was admitted to the status of observable. In the lectures the term observable is restricted to include the reality condition (19), p. 29; i.e. an operator must satisfy Eq. (19) in order to be an observable. Another restriction is made with regard to the expansion theorem (p. 37). The concept, observable, is now further restricted to those linear operators for which the expansion theorem is valid.

One point not treated in the book, which was treated in the lectures was that of the approximate treatment of a continuous spectrum of eigenvalues by replacing it by a discrete set. Suppose we replace the continuum of eigenvalues of ξ, between ξ' and $\xi' + d\xi'$ by a discrete number of eigenvalues in this range in such a way that the discrete number in this range is $s(\xi')d\xi'$ where $s(\xi') > 0$, for all ξ'. Then if $F(\xi')$ is any function of ξ' we shall have, approximately,

$$\sum_{\xi'} F(\xi') = \int F(\xi')\, s(\xi')\, d\xi'$$

so that the discrete case is the same as the continuous with a weighting function $s(\xi')$. From this it follows that if $(\xi'/)_D$ is the representative of a state in the discrete representation, and $(\xi'/)$ is that in the truly continuous representation, the relation is

$$(\xi'/)_D = \frac{1}{\sqrt{s(\xi')}}(\xi'/)$$

Similarly matrix components in the two schemes are related by

$$(\xi'|\alpha|\xi'')_D = \frac{1}{\sqrt{s(\xi')s(\xi'')}}(\xi'/\alpha/\xi'')$$

3. Displacement operators

The displacement of a state or an observable is a perfectly definite process physically. Thus to displace a state or observable through a distance δx in the direction of the x-axis, we should merely have to displace all the apparatus used in preparing the state, or all the apparatus used to measure the observable, through the distance δx in the direction of the x-axis, and the displaced apparatus would define the displaced state or observable. A displaced state or observable is uniquely determined by the undisplaced state or observable together with the direction and the magnitude of the displacement.

The displacement of the ψ-vector is not such a definite thing though. If we take a certain ψ-vector, it will represent a certain state and we may displace this state and get a perfectly definite new state, but this new state will not determine our displaced ψ, but only the direction of our displaced ψ. We help to fix our displaced ψ by requiring that it shall have the same length as the undisplaced, but even then it is not completely determined, but can still be multiplied by an arbitrary phase factor. We require further that the superposition relations between states remain invariant under the displacement. Thus, if before the displacement we have

$$\psi_0 = c_1\psi_1 + c_2\psi_2$$

we require that for the displaced states

$$\tilde{\psi}_0 = c_1\tilde{\psi}_1 + c_2\tilde{\psi}_2$$

This condition is satisfied only if the phase factor by which the displaced ψ's are multiplied is the same for all states.

Corresponding to a displacement we may define an operation on ψ and on an observable ξ.

$$D_x\psi = \lim_{\delta x \to 0} \frac{\tilde{\psi} - \psi}{\delta x} \quad \text{and} \quad D_x\varphi = \lim \frac{\tilde{\varphi} - \varphi}{\delta x}$$

$$D_x\xi = \lim_{\delta x \to 0} \frac{\tilde{\xi} - \xi}{\delta x}$$

If instead of $\tilde{\psi}$ we take $e^{i\gamma}\tilde{\psi}$, we get

$$D^*\psi = \lim \frac{e^{i\gamma}\tilde{\psi} - \psi}{\delta x} = \lim \frac{\tilde{\psi} - \psi}{\delta x} + \lim \frac{e^{i\gamma} - 1}{\delta x}\tilde{\psi}$$

$$= D\psi + ia\psi$$

where

$$a = \lim \frac{\gamma}{\delta x}, \text{ which must exist.}$$

Since $\tilde{\psi}$ = linear function of ψ's $= A\psi$, $D_x\psi = d_x\psi$, where d_x = linear operator acting on ψ. Suppose $\varphi_k\psi_\ell = c$, then $\tilde{\varphi}_k\tilde{\psi}_\ell = c$. Subtracting and going to the limit $(D_x\varphi_k)\psi_\ell + \varphi_k(D_x\varphi_\ell) = 0$

$(D_x\varphi_k)\psi_\ell + \varphi_k d_x\psi_\ell = 0$ for arbitrary ψ_ℓ

Hence

$$D_x \varphi_k = - \varphi_k \, d_x$$

On the other hand, by its definition

$D_x \varphi_k$ = conjugate imaginary to $D_x \Psi_k$

Hence $-\varphi_k d_x$ = conjugate imaginary to $d_x \Psi_k$

or $-d_x$ = conjugate imaginary to d_x

Therefore d_x = i $\times$ Hermitian operator.

For observables we have

$$\xi \Psi_a = \Psi_b$$

Hence $(D_x \xi)\Psi_a + \xi D_x \Psi_a = D_x \Psi_b$

or

$$(D_x \xi)\Psi_a + \xi d_x \Psi_a = d_x \Psi_b = d_x(\xi \Psi_a)$$

$$\therefore \quad D_x \xi = d_x \xi - \xi d_x$$

It is to be observed that an addition of ia Ψ to D Ψ does not alter this.

Suppose x, y, z = coordinates of the center of gravity of a system.

p_x, p_y, p_z = momenta.

$$D_x \chi = \lim \frac{\tilde{\chi} - \chi}{\delta x} = \lim \frac{(\chi - \delta x) - \chi}{\delta x} = -1$$

$$D_x y = D_x z = 0$$

$$\therefore \quad d_x \chi - \chi d_x = -1 \; ; \; etc$$

We find in this way that

$i\hbar d_x - p_x$ commutes with everything and is therefore a number.

Since d_x is already undetermined up to a constant, we can identify

$$p_x \equiv i \hbar d_x$$

If we consider two operations, D_x and D_y, we have, in general,

$$D_x D_y \psi = D_y D_x \psi + ia_z \psi$$

since the phases may change differently in the two orders of operations.

Considering rotations in a manner similar to displacements, we have

$$D_\xi \psi = \lim \frac{\tilde{\psi} - \psi}{\delta\theta} \qquad \text{undefined up to } ia\psi .$$

We also introduce d_ξ by the equation $d_\xi \psi = D_\xi \psi$; etc.

Thus we obtain

$$d_\xi d_\eta - d_\eta d_\xi = d_\zeta + ia_\zeta \; ; \text{ and two similar relations.}$$

We can get rid of ia_ζ by re-defining d_ζ etc. thus:

$$d^*_\xi = d_\xi + ia_\xi, \text{ etc.}$$

Considering combinations of rotations and displacements, we have:

$$\left[\left[d_x, d_z\right], d_\xi\right] + \left[\left[d_z, d_\xi\right], d_x\right] + \left[\left[d_\xi, d_x\right], d_z\right] = 0$$

or

$$\left[ib_y, d_\xi\right] + \left[d_y + (\quad), d_x\right] + \left[\text{number}, d_z\right] = 0$$

$$\left[ib_y, d_\xi\right] = 0$$

Therefore $\qquad \left[d_y, d_x\right] = 0$, etc.

The scheme usually assumed is the most general in <u>free space</u>. In an external field (in particular, in a magnetic field)

$$d_x d_y - d_y d_x = ib_z$$

4. Change of state in time

We have not only superposition of states in 3-dimensions, but also in 4-dimensions. Considered from 3-dimensional point of view this means that states, that are linear combinations of other states at one instant of time, remain so at all times. This requires that all states change with time according to law

$\Psi_{t_2} = A\ \Psi_{t_1}$ where $A = A(t_1, t_2)$ is a linear operator. For $t_2 - t_1 = dt$, this becomes

$$i\hbar \frac{d\Psi}{dt} = H\Psi$$

where, by analogy with the classical theory, H is called the Hamiltonian of the system, even for systems having no classical analogue. We assume that the linear operator H is an observable (Hermitian, etc.). Similarly

$$-i\hbar \frac{\partial \Phi}{\partial t} = \varphi H$$

Both are called Schrödinger's wave equations. If H is a constant,

$$\Psi_t = e^{-\frac{iHt}{\hbar}}\Psi_0$$

For a representation

$$i\hbar \frac{d}{dt}(q|) = \int (q|H|q')\,dq'\,(q'|)$$

is the practical way of statement.

Stationary Ψ is Ψ for which $\frac{d\Psi}{dt}$ is parallel to Ψ. Thus, for stationary Ψ

$$\frac{d\Psi}{dt} = \lambda\Psi$$

or

$$H\Psi = i\hbar\lambda\Psi$$

Hence, stationary Ψ is an eigen Ψ of H. Only for very special way of t entering into H can there be stationary states. The above is Schrödinger's picture.

In Heisenberg's representation the state vector is at rest but operators are considered to be functions of time. In a sense it corresponds to keeping a vector fixed with coordinates rotated. In Schrödinger's picture all operators are fixed (q, $i\hbar \frac{\partial}{\partial q}$, etc.). Thus:

	Schrödinger	Heisenberg
States as vectors	moving	fixed
Dynamical variables as linear operators	fixed	moving

A vector Ψ fixed in Heisenberg axes will appear as moving:

$$i\hbar \frac{\partial \Psi}{\partial t} = -H\Psi$$

where H is now the same function of moving operators, as before it was of fixed. This is because Heisenberg axes are considered as moving. We now have ξ fixed in any coordinate system, Ψ_a fixed in the same coordinate system

$= \Psi_b$, fixed in the same coordinate system

Then

$$i\hbar \frac{d\xi}{dt}\Psi_a + i\hbar\,\xi\frac{d\Psi_a}{dt} = i\hbar\frac{d\Psi_b}{dt}$$

or

$$i\hbar \frac{d\xi}{dt}\Psi_a - \xi H\Psi_a = -H\Psi_b = -H\xi\Psi_a$$

Hence

$$i\hbar\frac{d\xi}{dt} = \xi H - H\xi,$$

which is the Heisenberg form of the equations of motion. This unanalogous to the classical

$$\frac{d\xi}{dt} = [\xi, H]$$

In Schrödinger's representation we have no such a comparison. When ξ contains t explicitly, this becomes

$$i\hbar\frac{d\xi}{dt} = i\hbar\frac{\partial \xi}{\partial t} + (\xi H - H\xi)$$

If we put ξ = H, we obtain

$$\frac{dH}{dt} = 0 \quad \text{if} \quad \frac{\partial H}{\partial t} = 0$$

If ξ commutes with H and does not contain t explicitly, ξ is a constant of the motion. When H is made diagonal, any diagonal matrix will commute with H and will therefore be a constant of the motion.

$$(\alpha'|H|\alpha'') = H'\delta_{\alpha'\alpha''} \qquad H' = H(\alpha')$$

$$i\hbar(\alpha'|\frac{d\xi}{dt}|\alpha'') = (\alpha'|\xi H - H\xi|\alpha'') = (\alpha'|\xi|\alpha'')(H'-H'') = i\hbar\frac{d}{dt}(\alpha'|\xi|\alpha'')$$

so that $(\alpha'/\xi/\alpha'') = \text{const.}\, e^{i(H_0'-H_0'')t/\hbar}$

while in Schrödinger's picture

$$(\alpha'|\xi|\alpha'') = \text{constant}$$

Classically, when we only know that a system is in one of several possible states, distributed according to probability. Then, if ρ is the probability density in phase space = probability of <u>one</u> system being in a place (in phase space),

$$\frac{d\rho}{dt} = -[\rho, H]$$

Corresponding thing in Q.M. when $(\xi/1)$, $(\xi/2)$, etc. are possible states, let

$$P_m = \text{probability of being in m-th state}$$

$$1 \geqq P_m \geqq 0.$$

Corresponding to ρ we have

$$(\xi'|\rho|\xi'') = \sum_m (\xi'|m)\,P_m\,(m|\xi'')$$

and

$$i\hbar\frac{d\rho}{dt} = H\rho - \rho H$$

which is another way of describing equations of motion.

We may normalize ρ classically thus

$$\int \rho\, dq\, dp = 1$$

So that

$$\text{Ave. } \chi = \int \rho \chi \, dp \, dq \text{ ; etc.}$$

In Q.M.

$$\int (\xi' | \rho | \xi') \, d\xi' = 1 = \sum_m \int (\xi' | m) P_m (m | \xi') \, d\xi' = \sum_m P_m$$

$$\text{Ave. } \chi = \sum_m \int P_m (m | \xi') \, d\xi' (\xi' | \chi | \xi'') \, d\xi'' (\xi'' | m)$$

$$= \int (\xi' | \rho \chi | \xi') \, d\xi'$$

so that

$(\xi' | \rho | \xi') \, d\xi'$ = probability of ξ lying in $d\xi'$

It is interesting to consider at this point the question: In what sense, for a dynamical system of n degrees of freedom, a cell of volume in phase space is equivalent to a state. One way is to enclose the system, say a particle, in a box. Then the number of states corresponding to dp is $\frac{V \, dp}{h^3}$, where V is the volume of the box. Consider different eigen states of p. m is specified by numerical value of p. Then, the probability

$$P_{p', p'+dp'} = P_{p'} \, dp'$$

$$(\xi' | \rho | \xi'') = \int (\xi' | p') P_{p'} \, dp' (p' | \xi'')$$

In representation in which q is diagonal

$$(q' | \rho | q'') = \int (q' | p') P_{p'} \, dp' (p' | q'')$$

$$(q' | p') = \frac{1}{h^{n/2}} e^{i (q_i' p_i')/\hbar}$$

$$(q' | \rho | q'') = \frac{1}{h^n} \iint e^{i (q_k' - q_k'') p_k' / \hbar} P_{p'} \, dp'$$

$(q' | \rho | q') = \frac{1}{h^n} \int P_{p'} \, dp'$ density of particles in coordinate space

$\therefore \frac{P_{p'}}{h^n} =$ density in phase space

Change now to discrete states

$$(q' | \rho | q'') = \sum_{p'} (q' | p')_D P_{p'} (p' | q')_D$$

where $P_{p'}$ = probable number of systems in state p'.

Hence $\frac{1}{h^n}$ = density for one system in state p'

and h^n = volume of one state.

Thus, this is connected with the difference in normalization between continuous and discrete states.

Finally, when we go over to relativistic point of view, and are dealing with a single particle, time must be treated in the same way as the other variables. We should rewrite $(x\ y\ z/)_t$ $(x\ y\ z\ t/)$;

and the Schrödinger equation becomes

$$i\hbar \frac{\partial}{\partial t}(xyzt|) = H(xyzt|)$$

H must therefore be linear in $\frac{\partial}{\partial x_k}$; k = 1, 2, 3.

II. Method of Treating of Assemblies of Large Numbers of Particles

This method, although applicable primarily to a system of particles without interactions, may be extended to the case when each of the particles (or systems) interacts with an outside system, thus providing an indirect interaction. Secondly, such a system in its equations is similar to a field -- thus providing a mathematical analogy upon which the Q.M. of fields is based.

Chief applications are to the systems

a) Photons + atom = atom + field

b) Electrons + field = electrons with interactions

Unsatisfactory feature is that there is no accurate theory of interaction of electron and photon.

For each kind of particle the ψ functions are either all symmetric or all antisymmetric in the coordinates of the particles. There is no theory to tell which it must be. Experimentally we have

For symmetrical -- Einstein-Bose particles (photons)

For anti-" -- Fermi " (electrons)

The main idea of the method is to introduce a large number of similar particles and to introduce as dynamical variables numbers of particles in specified states (described by the value of q, say)

$$q', q'', q''', \ldots q^{(a)}$$

New variables are n_a = number of particles in state $q^{(a)}$.

For symmetrical case $n_a' = 0, 1, 2, \ldots \infty$

For anti- " " $n_a' = 0, 1.$

The n_a's can be treated as dynamical variables, but they are not sufficient. All n_a's commute with each other. To get a complete set we must introduce also other variables.

By analogy with harmonic oscillator, $H = \frac{1}{2}(p^2 + q^2)$ which has eigenvalues $\frac{1}{2}\hbar, \frac{3}{2}\hbar, \cdots, \frac{H}{\hbar} - \frac{1}{2}$ has eigenvalues 0, 1, 2, ... Thus, apart from trivial changes n_a = Hamiltonian of a harmonic oscillator. Take representation in which n_a is diagonal

$$n_a = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Introduce

$$e^{i w_a} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad e^{-i w_a} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

for each value of a. Let these commute with variables for other values of a.

$$e^{-iw_a} e^{iw_a} = 1 \;; \qquad e^{iw_a} e^{-iw_a} \neq 1$$

$$n_a e^{iw_a} = e^{iw_a}(n_a+1) \;; \quad n_a e^{-iw_a} = e^{-iw_a}(n_a-1)$$

$$f(n_a) e^{iw_a} = e^{iw_a} f(n_a+1) \;; \quad f(n_a) e^{-iw_a} = e^{-iw_a} f(n_a-1)$$

Introduce also

$$\xi_a = (n_a+1)^{1/2} e^{iw_a} = e^{-iw_a} n_a^{1/2} \qquad \text{(analogue of p + iq)}$$

$$\bar{\xi}_a = e^{iw_a}(n_a+1)^{1/2} = n_a^{1/2} e^{iw_a} \qquad \text{(analogue of p - iq)}$$

The new set of variables, ξ_a and $\bar{\xi}_a$, are sufficient to describe any symmetrical function of dynamical variables -- the only kind having a physical meaning. The problem is then to express an arbitrary symmetrical function of p's and q's in terms of ξ_a and $\bar{\xi}_a$

Let the function be $U = \sum_r U_r$ (special kind of function) where U_r depends only upon the variables of one particle. Let $U\psi_1 = \psi_2$

$$\psi_1 = (q_1' q_2' \cdots q_n' | 1) \;;$$

then

$$(q_1', q_2' \cdots q_n' | 2) = \sum_{q''} (q_1' \cdots q_n' | U | q_1'' \cdots q_n'')(q_1'' \cdots q_n'' | 1)$$

$$= \sum_r \sum_{q_r''} (q_r' | U_r | q_r'')(q_1' q_2' \cdots q_r'' \cdots q_n' | 1),$$

because of the form of U Now, since $\underline{n}$'s are functions q's only,

$$(n_1', n_2' \cdots |\;) = A\,(q_1', q_2' \cdots q_n' |)$$

Since we wish that whenever

$$\sum_{q'} |(q_1' q_2' \cdots q_n' |)|^2 = 1$$

we should have

$$\sum_{n'} |(n_1' n_2' \cdots |)|^2 = 1$$

we obtain

$$A = \sqrt{\frac{N!}{n_1'!\, n_2'!\,\ldots\ldots}}$$

Thus

$$(n_1'\ n_2'\ldots\ldots|2) = \sum_r{}' \sum_{q''} (q_r'|V|q_r'')(n_1 n_2 \cdot n_{q_r'}-1\ldots\ n_{q_r''}+1\ldots\ldots)|1)\left(\frac{n_{q_r'}-1}{n_{q_r'}+1}\right)^{1/2}$$

Let $(q_r^{(a)}|U_r|q_r^{(b)}) = U_{ab}$, since U_r is the same function for all particles, but of different variable U_{ab} is independent of r. Then

$$(n_1' n_2'\ldots|2) = \sum_a n_a U_{aa}(n_1 n_2\ldots|1) + \sum_{\substack{a,b\\ a\neq b}}{}' n_a U_{ab}(n_1 n_2\ldots\ldots n_a-1\ldots n_b+1\ldots|1)\left(\frac{n_b+1}{n_a}\right)^{1/2}$$

or

$$U = \sum_a n_a U_{aa} + \sum_{\substack{a,b\\ a\neq b}}{}' n_a^{1/2}(n_b+1)^{\frac{1}{2}} e^{iw_a} e^{-iw_b} U_{ab}$$

$$= \sum_{a,b}{}' n_a^{1/2}(n_b+1-\delta_{ab})^{1/2} U_{ab}\, e^{iw_a} e^{-iw_b}$$

$$= \sum_{ab}{}' \bar{\xi}_a U_{ab} \xi_b$$

Suppose each system satisfies equation

$$i\hbar \frac{d\Psi}{dt} = H\Psi$$

or

$$i\hbar \frac{d}{dt}(q|) = \sum_{q'}{}' (q|H_r|q')(q'|)$$

what are the Heisenberg equations of motion for ξ's.

$$H = \sum_r H_r = \sum_{ab}{}' \bar{\xi}_a H_{ab} \xi_b$$

(a, b need not be eigenstates of H. They are eigenstates of q's.) Then, since

$$\xi_a \xi_b - \xi_b \xi_a = 0$$

and

$$\xi_a \bar{\xi}_b - \bar{\xi}_b \xi_a = \delta_{ab}$$

$$i\hbar \dot{\xi}_a = \xi_a H - H\xi_a = \sum_{cb}{}' \delta_{ca} H_{cb} \xi_b = \sum_b{}' H_{ab}\xi_b$$

which is formally the same as eq. for $(q|)$, but each ξ_a is an operator. Thus, the main idea,-- Take a wave equation for a single particle, solve it, assume that the wave functions do not commute. Analogy also holds for transformations

$$\xi_A = \sum_a{}' (Q^{(A)} | q^{(a)}) \xi_a$$

Here Q's have the eigenvalues A -- this is the analogue of

$$(Q^{(A)}|) = \sum_a{}' (Q^{(A)}| q^{(a)})(q^{(a)}|)$$

Also $\bar{\xi}_a \xi_a = n_a$ (analogue to $|(q^{(a)}|)|^2$).

It is important that phases of ξ's are dynamical variables and can be observed. Thus, the idea of superquantization corresponds to giving phases a physical meaning -- which is necessary to a field theory.

III. Quantum Theory of Fields

The general method is to pass from the equation of motion (Schrödinger eq.) for a single particle

$$i\hbar \frac{\partial}{\partial t}(\alpha'|) = \sum_{\alpha''}{}' (\alpha'|H|\alpha'')(\alpha''|)$$

to the quantized equation

$$i\hbar \frac{d}{dt}\xi(\alpha') = \sum_{\alpha''}{}' (\alpha'|H|\alpha'')\xi(\alpha'') \qquad (1)$$

where $\xi(\alpha')$ are operators satisfying the commutation rules

$$\left.\begin{aligned} &\xi(\alpha')\xi(\alpha'') - \xi(\alpha'')\xi(\alpha') = 0 \\ &\bar{\xi}(\alpha')\xi(\alpha') = n(\alpha') \\ &\xi(\alpha')\bar{\xi}(\alpha'') - \bar{\xi}(\alpha'')\xi(\alpha') = \delta_{\alpha'\alpha''} \end{aligned}\right\} \qquad (2)$$

If the particle has a spin, or polarization, the Schrödinger equation is

$$i\hbar \frac{d}{dt}(x'\sigma'|) = \sum_{\sigma''}{}' \int (x'\sigma'|H|x''\sigma'')dx''(x''\sigma''|) \quad (3)$$

where x' are the coordinates, σ' the spin of the particle. The super-quantized equation is then

$$i\hbar \frac{d}{dt}\, \xi(x'\sigma') = \sum_{\sigma''} \int (x'\sigma'|H|x''\sigma'')\, dx''\, \xi(x''\sigma'') \quad (4)$$

with the commutation rules

$$\xi(x'\sigma')\,\bar{\xi}(x''\sigma'') - \bar{\xi}(x''\sigma'')\,\xi(x'\sigma') = \delta(x'-x'')\,\delta_{\sigma'\sigma''} \quad (5)$$

Equation (4), being a Heisenberg equation of motion, can be compared with the corresponding classical equations,- in this case with Maxwell's equations. It is not quite of Maxwell's form, since the eqs. div E = 0 and div H = 0 are not of this form.

We suppose that a photon has spin variables commuting with momentum, but not with position variables. Equation (3) cannot then be written. We can, however, use momentum variables:

$$i\hbar \frac{d}{dt}(p'\sigma') = \sum_{\sigma''} \int (p'\sigma'|H|p''\sigma'')\, dp''(p''\sigma''|)$$

and

$$i\hbar \frac{d}{dt}\, \xi(p'\sigma') = \sum_{\sigma''} \int (p'\sigma'|H|p''\sigma'')\, dp''\, \xi(p''\sigma'')$$

For comparison with the classical theory, we must thus resolve Maxwell's field into Fourier's components a, b, ... referring to states of light quantum (definite momentum and spin). We use the wave vector $k = p/\hbar$, with $k_a = k$ associated with a state a. Then

$$\mathcal{E} = \sum \int \mathcal{E}_a \cos[k_a \cdot x - \gamma_a]\, dk_a$$

where γ_a contains time dependence. The sum being over the two states of polarization. The direction of $\mathcal{E}_a$ is assumed to be determined by the state a. Similarly

$$\mathcal{H} = \sum \int \mathcal{H}_a \cos[k_a \cdot x - \gamma_a]\, dk_a$$

It is convenient to go over to a sum, by the old device of introducing S_a = number of discrete states per unit volume of k_a space, about k_a. Thus

$$\mathcal{E} = \sum_a \mathcal{E}_a \cos[k \cdot x - \gamma_a]\, S_a^{-1}$$

The energy can then be expressed in the form

$$\frac{1}{8\pi} \int (\mathcal{E}^2 + \mathcal{H}^2)\, dx = \sum_a (\cdots\cdots \quad)$$

The expression in parenthesis (.....) is the energy belonging to the state a, and can be put equal to $h\nu_a n_a$ The classical calculation then yields

$$|\mathcal{E}_a| = |\mathcal{H}_a| = \left(\frac{2}{\pi}\hbar\nu_a n_a S_a\right)^{1/2}$$

Thus

$$\mathcal{E} = \frac{1}{\pi}\sum_a{}' (\hbar\nu_a)^{1/2} n_a^{1/2} \cos[k_a\cdot x - \gamma_a] S_a^{-1/2}\alpha_a$$

where α_a = unit vector in the direction of $\mathcal{E}_a$. To get Hermitian operator we must replace

$$2n_a^{1/2}\cos[k_a\cdot x-\gamma_a] \text{ by } n_a^{1/2}e^{ik_a\cdot x}e^{-i\gamma_a}+e^{-ik_a\cdot x}e^{i\gamma_a}n_a^{1/2} \text{ or } e^{ik_a\cdot x}e^{-i\gamma_a}n_a^{1/2}+e^{-ik_a\cdot x}n_a^{1/2}e^{i\gamma_a}$$

where n_a and γ_a are the only non-commuting quantities. The second expression differs from the first merely in the physical meaning assigned to n_a.

Since

$$n = \begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ \cdot & 1 & & \\ \cdot & & 2 & \\ \end{pmatrix}; \quad e^{i\gamma} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdot \\ 0 & 1 & 0 & \cdot \end{pmatrix}; \quad e^{-i\gamma} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdot \\ 0 & 0 & 0 & 1 & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \end{pmatrix}$$

$$e^{i\gamma}n^{1/2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & & \\ 0 & & & \end{pmatrix}$$ with the same form for $n^{\frac{1}{2}}e^{-i\gamma}$

This is not convenient. We shall therefore use the <u>second</u> expression for $2n_a^{\frac{1}{2}}\cos[k_a\cdot x-\gamma_a]$

Now $\xi_a = e^{-i\gamma_a}n_a^{1/2}$; $\bar{\xi}_a = n_a^{1/2}e^{i\gamma_a}$

Thus

$$\mathcal{E} = \frac{1}{2\pi}\sum_a{}' (h\nu_a)^{1/2}\alpha_a\left\{\bar{\xi}_a e^{-ik_a\cdot x}+\xi_a e^{ik_a\cdot x}\right\}S_a^{-1/2} = \mathcal{E}(x)$$

$$\mathcal{H} = \frac{1}{2\pi}\sum_a{}' (h\nu_a)^{1/2}\beta_a\left\{\quad '' \qquad\qquad '' \quad\right\}S_a^{-1/2} = \mathcal{H}(x)$$

This is the transformation from the momentum variables, to coordinates. The corresponding transformation for a single photon is impossible.

Now

$$[\bar{\mathcal{E}}_\ell(x'), \mathcal{E}_m(x'')] = \frac{1}{4\pi^2}\sum_{ab}' h\nu_a^{1/2}\nu_b^{1/2}\alpha_{a\ell}\alpha_{bm}\Big\{e^{-i(k_a\cdot x')+i(k_b\cdot x'')}\delta_{ab} + e^{i(k_a\cdot x')-i(k_b\cdot x'')}\delta_{ab}\Big\}(S_aS_b)^{-1/2}$$

$$= \frac{1}{\pi}\sum_a' \nu_a\alpha_{a\ell}\alpha_{am}\sin[k_a\cdot(x'-x'')]S_a^{-1}.$$

because each term cancels with one obtained by reversing k and keeping polarization constant. Similarly $[H_\ell, H_m] = 0$. On the other hand

$$[\mathcal{E}_\ell(x'), H_m(x'')] = \frac{1}{\pi}\sum_a' \nu_a\alpha_{a\ell}\beta_{am}\sin[k_a\cdot(x'-x'')]S_a^{-1}$$

Sum for the two directions of polarization. For one $\alpha \parallel \beta$; for the other $\alpha \parallel -\beta$; $\alpha_\ell\beta_m$ leads to $\alpha_\ell\beta_m - \alpha_m\beta_\ell$ If ℓ = m this is zero. If ℓ and m are at right angles

$$\alpha_\ell\beta_m \longrightarrow (\alpha\times\beta)_n = \frac{k_n}{|k|}$$

where ℓ, m, n is the right-handed system. Then

$$[\bar{\mathcal{E}}_\ell(x'), \bar{\mathcal{H}}_m(x'')] = \frac{1}{\pi}\sum_{k_a}' \nu_a\frac{k_n}{|k|}\sin[k_a\cdot(x'-x'')]S_a^{-1} = \frac{1}{\pi}\int \nu\frac{k_n}{|k|}\sin k\cdot(x'-x'')$$

more explicitly

$$[\mathcal{E}_x(x'y'z'), \mathcal{H}_y(x''y''z'')] = -4\pi\frac{\partial}{\partial z'}\{\delta(x'-x'')\delta(y'-y'')\delta(z'-z'')\}$$

$$= -4\pi\frac{\partial}{\partial z'}\delta(r'-r'')$$

The Hamiltonian for the field is

$$H_F = \sum_a' n_a h\nu_a = h\sum_a' \nu_a\bar{\xi}_a\xi_a$$

Expressing this in terms of $\mathcal{E}$ and $\mathcal{H}$ one obtains

$$H_F = \frac{1}{8\pi}\int(\mathcal{E}^2+\mathcal{H}^2)\,dx - \sum_a{}' \frac{1}{2}h\nu_a$$

which shows that the first term alone is not a suitable expression for the quantum theory Hamiltonian. As the last term is a constant (although infinite) it does not affect equations of motion. Thus, equation of motion of $\mathcal{E}_x'$ is (prime refers to point x'y'z'),

$$\frac{d}{dt}\mathcal{E}_x' = [\mathcal{E}_x', H_F] = \int[\mathcal{E}_x', \mathcal{H}_y''^2 + \mathcal{H}_z''^2]\,dx''.$$

But

$$[\mathcal{E}_x', \mathcal{H}_y''^2] = \mathcal{H}_y''[\mathcal{E}_x', \mathcal{H}_y''] + [\qquad]$$

$$= -8\pi\,\mathcal{H}_y''\,\delta(x'-x'')\,\delta(y'-y'')\,\delta(z'-z'')$$

$$\int[\mathcal{E}_x', \mathcal{H}_y''^2] = -8\pi\frac{\partial\mathcal{H}_y'}{\partial z'}$$

Therefore

$$\frac{d}{dt}\mathcal{E}_x' = -\frac{\partial\mathcal{H}_y'}{\partial z'} + \frac{\partial\mathcal{H}_z'}{\partial y'}$$

which corresponds to correct Maxwell equation.

Now consider extension of properties of potentials. The extension is not trivial, since potentials are not uniquely determined in terms of the field quantities. The extension is necessary when interaction with charges is later considered.

We use A for the vector and A_o for the scalar potential. It is customary to assume $\frac{\partial A_0}{\partial t}$ + div A = 0, but we will neglect this at present and consider it later. We have A_ν, ν = x, y, z, 0. In terms of Fourier's components

$$A_\nu = \int A_{\nu k}\cos[\delta_k + 2\pi\nu_k t - k\cdot x]\,d\underline{k}$$

This implies $\Box A_\nu = 0$.

One needs to find the commutation relations for two different times, as well as two different places. Going over to sum

$$A_\nu = \sum_k{}' A_{\nu k}\cos[\delta_k + 2\pi\nu_k t - k\cdot x]S_k^{-1}$$

To make this into a Hermitian operator we have to split cos into exponentials and put $A_{\nu k}$ to the right of one of them, as with field quantities.

$$A_\nu = \sum_k \left\{ \bar{\eta}_{\nu k} e^{i[2\pi\nu_k t - k\cdot x]} + \eta_{\nu k} e^{-i[2\pi\nu_k t - k\cdot x]} \right\} S_k^{-1/2}$$

where η 's are analogous to ξ 's. We could write

$$A_\nu = \sum_k \left\{ \bar{\xi}_{\nu k} e^{-i k\cdot x} + \xi_{\nu k} e^{i k\cdot x} \right\} S_k^{-1/2}$$

and therefore

$$\xi_{\nu k} = \eta_{\nu k} e^{-2\pi i \nu_k t}$$

The ξ 's are dynamical variables not involving time explicitly, so

$\dot{\xi} = [\xi, H_F]$, but

$$\dot{\eta} = \frac{\partial \eta}{\partial t} + [\eta, H_F]$$

Suppose $\mathcal{E}$ and $\mathcal{H}$ given for plane wave in x-direction

$$\mathcal{E}_y = -\frac{\partial A_y}{\partial t} - \frac{\partial A_0}{\partial y}$$

gives

$$\xi_y = \frac{1}{2\pi i \nu} \cdot \frac{1}{2\pi} (h\nu)^{1/2} \xi_a$$

Hence.

$$\xi_y \bar{\xi}_y - \bar{\xi}_y \xi_y = \frac{h}{16\pi^4 \nu}$$

also

$$\xi_z \bar{\xi}_z - \bar{\xi}_z \xi_z = \frac{h}{16\pi^4 \nu}$$

This is all we get from the theory of field quantities. It tells nothing of ξ_x and ξ_0. Natural assumption is to take

$$\xi_x \bar{\xi}_x - \bar{\xi}_x \xi_x = \frac{h}{16\pi^4 \nu}$$

$$\xi_0 \bar{\xi}_0 - \bar{\xi}_0 \xi_0 = -\frac{h}{16\pi^4 \nu}$$

The minus sign in the last relation is required for relativistic invariance.

With plus sign the commutation rules are like for (p + iq) of ordinary harmonic oscillator. The ξ_0 is like the (p + iq) of a harmonic oscillator of negative mass.

Contribution to H_F of a wave in x-direction is

$$H_F = 16\pi^4\nu^2(\bar{\xi}_y\xi_y + \bar{\xi}_z\xi_z)$$

but this does not give the correct equation of motion of the ξ's. Instead, we take

$$H_F = 16\pi^4\nu^2[\bar{\xi}_x\xi_x + \bar{\xi}_y\xi_y + \bar{\xi}_z\xi_z - \bar{\xi}_0\xi_0]$$

The ξ's with different suffixes are assumed to commute. This leads to

$$[A_\mu, A_\nu] = 0$$

for the same time. Also

$$[A_\mu'(t'), A_\nu''(t'')] = 0 \qquad \text{for } \mu \neq \nu$$

On the other hand

$$[A_\mu'(t'), A_\mu''(t'')] = \pm\sum_k{}' \frac{1}{4\pi^3\nu}\sin[k\cdot(x'-x'')-2\pi\nu_k(t'-t'')]\,S_k^{-1}$$

+ for μ = x, y, z, minus for μ = 0.

$$= \pm\int\frac{1}{4\pi^3\nu}\sin[k\cdot(x'-x'')-2\pi\nu_k(t'-t'')]\,dk$$

$$= \pm\frac{1}{|x'-x''|}\left\{\delta(|x'-x''|+(t'-t''))-\delta\{|x'-x''|-(t'-t'')\}\right\}$$

since $\int e^{iax} = 2\pi\delta(a)$ Hence, for t' = t" the result is

$$[A_\nu', A_\nu''] = 0$$

For t" > t' the above result is

$$= \pm\ 2\delta[(x'-x'')^2-(t'-t'')^2];$$

and for t" < t'

$$\mp\ 2\delta[(x'-x'')^2-(t'-t'')^2].$$

we used the fact that for a > 0

$$\delta(x^2-a^2) = \frac{1}{2a}\{\delta(x-a)+\delta(x+a)\}$$

We also want

$$\left[A_\nu', \frac{d}{dt}A_\nu''\right] = \pm \int \cos[k\cdot(x'-x'')]\,dk = \pm 4\pi\delta(x'-x'')$$

Also

$$\left[\frac{\partial A_\nu'}{\partial t}, \frac{\partial A_\nu''}{\partial t}\right] = \pm \frac{1}{\pi}\int \nu_k \sin[k\cdot(x'-x'')]\,dk = 0$$

Put $H_F = \oint\left(A_\nu, \frac{\partial A_\nu}{\partial t}\right)$; then

$$i\frac{dA_\nu}{dt} = [A_\nu, H_F] = \frac{\partial A_\nu}{\partial t}$$

$$\frac{d}{dt}\frac{\partial A_\nu}{\partial t} = \left[\frac{\partial A_\nu}{\partial t}, H_F\right] = \nabla^2 A_\nu$$

The above commutation rules for the A's imply

$$\eta_{\nu k}\bar{\eta}_{\nu k} - \bar{\eta}_{\nu k}\eta_{\nu k} = \pm \frac{h}{16\pi^4\nu_k}\delta_{\mu\nu} \qquad \begin{array}{ll} + \text{ for} & = 1, 2, 3 \\ - \text{ for} & = 0. \end{array}$$

So far the four components of A_μ are treated as four independent scalars.

To make this fit with Maxwell's theory we have to add

$$\operatorname{div} A + \frac{\partial A_0}{\partial t} = 0 \qquad (*)$$

This, however, is inconsistent with the commutation rules already established for A's. We had

$$\left[A_\nu', \frac{\partial A_\nu''}{\partial t}\right] = \pm 4\pi\delta(x'-x'')$$

But (*) would imply

$$\left[A_0', \operatorname{div} A'' + \frac{\partial A_0''}{\partial t}\right] = 0$$

Now, $\left[A_0', \operatorname{div} A''\right] = 0$, so we get

$$\left[A_0', \operatorname{div} A'' + \frac{\partial A_0''}{\partial t}\right] = \left[A_0', \frac{\partial A_0''}{\partial t}\right] = -4\pi\delta(x'-x'') \neq 0$$

which is inconsistent with (*).

Instead of (*) we assume, with Fermi, that

$$\left(\operatorname{div} A + \frac{\partial A_0}{\partial t}\right)\psi = 0$$

is a condition on ψ. We shall call this the supplementary condition. It means that the space of all possible ψ's is larger than the space needed to represent actual states. All linear operators must then leave this space invariant. Supplementary conditions occur in other places. Thus, in the theory of many similar particles there is a supplementary condition of antisymmetry, or symmetry.

Supplementary conditions must be not too stringent. If we have two such conditions $U\psi = 0$ and $V\psi = 0$, they must be consistent.

$$\therefore [V, U]\psi = 0$$

$$[[V,U],U]\psi = 0 \; ; \quad [[V,U],V]\psi = 0 \; ; \text{ etc.}$$

All these must be consistent. If after a certain number of these constructs no new conditions arise, we may take it that our conditions are not ~~too~~ stringent. Example of too stringent conditions: $p\psi = 0$ and $q\psi = 0$; hence $(pq - qp)\psi = 0 \;\therefore \hbar\psi = 0 \;\therefore \psi = 0$. Since $\left(i\hbar\frac{\partial}{\partial t} - H\right)\psi = 0$ the condition $U\psi = 0$ implies $\left[i\hbar\frac{\partial}{\partial t} - H, U\right]\psi = 0$ etc.

In Heisenberg's representation ψ is fixed, but div A + $\dot{A}_0$ is a function of time. Hence, making Fourier resolution of div A + $\dot{A}_0$ and applying each component to ψ and equating to zero, we will get an equivalent condition. Take the special case of one component, along x-axis. The supplementary condition becomes:

$$(\eta_x - \eta_0)\psi = 0 \quad \text{and} \quad (\bar{\eta}_x - \bar{\eta}_0)\psi = 0$$

To see if these are consistent we take

$$[(\eta_x - \eta_0), (\bar{\eta}_x - \bar{\eta}_0)]\psi = 0$$

or

$$\{[\eta_x, \bar{\eta}_x] + [\eta_0, \bar{\eta}_0]\}\psi = 0$$

This is satisfied identically, because of the minus sign in the commutation rule for η_0 and $\bar{\eta}_0$. These conditions cut out two degrees of freedom, so that it turns out that ψ may depend in an arbitrary way only on the two transverse components of η_k.

In further development of the theory those quantum-mechanical equations that have for classical analogue equations requiring the use of $\operatorname{div} A + \dot{A}_0 = 0$, will appear only as supplementary conditions.

Let $H = \operatorname{curl} A$; $\mathcal{E} = -\frac{\partial A}{\partial t} - \operatorname{grad} A_0$.

These give $\operatorname{div} H = 0$ and $\frac{\partial H}{\partial t} = -\operatorname{curl} \mathcal{E}$ without the use of Eq. (*). On the other hand, $\operatorname{div} \mathcal{E} = 0$ and $\frac{\partial \mathcal{E}}{\partial t} - \operatorname{curl} H = 0$ can be derived only with the help of (*). Thus, we will have

$$(\operatorname{div} \mathcal{E})\psi = 0 \quad , \quad \text{and} \quad \left(\frac{\partial \mathcal{E}}{\partial t} - \operatorname{curl} H\right) = 0.$$

We will regard as observables only operators leaving the space of all ψ's satisfying the supplementary condition invariant. For this it is sufficient that the observable commutes with $\operatorname{div} A + \dot{A}_0$. For, suppose $B(\operatorname{div} A + \dot{A}_0) = (\operatorname{div} A + \dot{A}_0)B$; then, if $(\operatorname{div} A + \dot{A}_0)\psi = 0$,

$$(\operatorname{div} A + \dot{A}_0) B\psi = B(\operatorname{div} A + \dot{A}_0)\psi = 0$$

Taking again a component wave along x-axis, H and $\mathcal{E}$ will contain only $\eta_y, \eta_z, \bar{\eta}_y, \bar{\eta}_z, \eta_x - \eta_0, \bar{\eta}_x - \bar{\eta}_0$ and these all commute with $\eta_x - \eta_0$ and $\bar{\eta}_x - \bar{\eta}_0$. Only those quantities are observable which are gauge invariant, and these all will leave the space of ψ's satisfying the supplementary condition invariant.

For the Hamiltonian the component we considered contributes

$$16\pi^4 \nu^2(\bar{\xi}_y \xi_y + \bar{\xi}_z \xi_z)$$

We generalize this by adding

$$+16\pi^4 \nu^2(\bar{\xi}_x \xi_x - \bar{\xi}_0 \xi_0) + \gamma.$$

γ = constant which is required to make η_k vary properly with the time. If we express the new Hamiltonian in terms of the potentials, we obtain

$$H_F = \frac{1}{8\pi}\int \sum_{\nu} \pm \left\{ (\nabla A_\nu)^2 - \left(\frac{\partial A_\nu}{\partial t}\right)^2 \right\} dx - \frac{1}{2}\sum_a h\nu_a$$

which is different from the old Hamiltonian

$$H_F = \frac{1}{8\pi}\int (\mathcal{E}^2 + \mathcal{H}^2)\, dx$$

The new H_F is an observable. Further, γ can be chosen in such a way that $H_{F\,new}\Psi = H_{F\,old}\Psi$ Thus, let $\gamma = \bar{\xi}_0 \xi_0 - \xi_0 \bar{\xi}_0$ = const. Then

$$H_{F\,new} - H_{F\,old} = \bar{\xi}_x \xi_x - \xi_0 \bar{\xi}_0$$

But

$$\begin{aligned}\bar{\xi}_x \xi_x - \xi_0 \bar{\xi}_0 &= \frac{1}{2}\left(\bar{\xi}_x \xi_x + \bar{\xi}_x \xi_x - \xi_0 \bar{\xi}_0 - \xi_0 \bar{\xi}_0 \right) \\ &= \frac{1}{2}\left(\bar{\xi}_x \xi_x + \xi_x \bar{\xi}_x - \bar{\xi}_0 \xi_0 - \xi_0 \bar{\xi}_0 \right) \\ &= \frac{1}{2}\left\{ (\bar{\xi}_x + \bar{\xi}_0)(\xi_x - \xi_0) + (\xi_x + \xi_0)(\bar{\xi}_0 - \bar{\xi}_0) \right\}\end{aligned}$$

Thus:

$$\begin{aligned}(H_{F\,new} - H_{F\,old})\Psi &= (\bar{\xi}_x \xi_x - \xi_0 \bar{\xi}_0)\Psi \\ &= \frac{1}{2}(\bar{\xi}_x' + \xi_0)(\xi_x - \xi_0)\Psi + \frac{1}{2}(\xi_x + \xi_0)(\bar{\xi}_0 - \bar{\xi}_0)\Psi \\ &= 0\end{aligned}$$

for all Ψ satisfying the supplementary conditions.

Now

$$[A_0', H_F] = \frac{1}{8\pi}\int \left[A_0', \left(\frac{\partial A_0'}{\partial t}\right)^2 \right] dx = \frac{\partial A_0'}{\partial t}$$

and similarly

$$\left[\frac{\partial A_0'}{\partial t}, H_F \right] = \nabla^2 A_0'$$

When particles are present, we assume that the commutation rules, <u>for any one instant</u> of time, are independent of the presence of particles. Hence, by the theory of relativity, they are the same for any two instants outside of light cone. Further, the equivalence of waves and particles should be preserved.

Here it is appropriate to review the method of double quantization. We assume that we have a number of similar particles (photons) acted upon by a perturber (one or more charges). Let H_p = Hamiltonian for the perturber having observables β For one photon

$$i\hbar \frac{\partial \psi}{\partial t} = (H_p + U)\psi \qquad (1)$$

where

U = energy of particle (photon) + interaction (if any).

$$i\hbar \frac{d}{dt}(q^\alpha, \beta'/) = \sum_{\beta''}(\beta'/H_p/\beta'')(q^\alpha, \beta''/) + \sum_{b,\beta''}(q^\alpha\beta'/U/q^b\beta'')(q^b, \beta''/)$$

For many particles

$$i\hbar \frac{d\Psi}{dt} = H\Psi \; ; \qquad H = H_p + \sum_k U_k$$

and we can write

$$H = H_p + \sum_{a,b} n_a^{1/2} e^{iw_a} U_{ab} (n_b + 1)^{1/2} e^{-iw_b}$$

Here, however, U_{ab} is no longer just a number. It is an operator with respect to the variables of the perturber, having matrix elements

$$(\beta'/U_{ab}/\beta'') = (q^a\beta'/U/q^b\beta'')$$

Also

$$H = H_p + \sum_{a,b} \bar{\xi}_a U_{ab} \xi_b$$

ξ's commute with H_p. Thus, we would get an equation of motion for ξ's different in form from the equation (1) for a single particle. We can perform, however, a transformation that removes this difficulty.

Let $\psi^* = e^{iH_pt/\hbar}\psi$

then (1) becomes

$$\frac{d}{dt}\psi^* = -H_p e^{iH_pt/\hbar}\psi + e^{iH_pt/\hbar}\frac{d\psi}{dt} = e^{iH_pt/\hbar} U\psi$$

$$= e^{iH_pt/\hbar} U e^{-iH_pt/\hbar}\psi^* = U^*\psi^* \text{ where} \qquad (2)$$

$$U^* = e^{iH_pt/\hbar} U e^{-iH_pt/\hbar}$$

H_p is constant operator, because we are dealing with Schrödinger's representation. Using the same transformation on ξ_s 's, one can obtain

$$\iota \hbar \frac{d\xi_a^*}{dt} = \sum_b U_{ab}^* \xi_b^*$$

which is analogous to (2).

Modification is required when number of photons is not fixed. This is done by assuming a zero state for light quanta, in which they are not observed. Infinite number of light quanta may be assumed to exist in this state. Now

$$H = H_p + \sum_{ab} \bar{\xi}_a U_{ab} \xi_b$$

$$= H_p + \sum_{ab}{}' \bar{\xi}_a U_{ab} \xi_b + \sum{}' \bar{\xi}_0 U_{0b} \xi_b + \sum_a{}' \bar{\xi}_a U_{a0} \xi_0 + U_{00} \bar{\xi}_0 \xi_0$$

The last term has no physical significance, for it does not lead to anything observable. We make $U_{00} \equiv 0$. We assume $\bar{\xi}_0 U_{0b} \equiv U_b$ and $U_{a0} \xi_0 = \bar{U}_a$ to be finite. Then

$$H = H_p + \sum_{ab}{}' \bar{\xi}_a U_{ab} \xi_b + \sum_a{}' \bar{\xi}_a \bar{U}_a + \sum_a{}' \xi_a U_a$$

The last two terms will lead to emission and absorption of radiation. The next problem is to determine the form of U_{ab}, U_a and $\bar{U}_a$

While the previous theory seems to be on a fairly sure footing, the following is not certain, and seems to be essentially wrong. No satisfactory theory exists. The same difficulties that arise in the following unsatisfactory theory are also to be found in the classical theory. Both cases will be considered.

For simplicity we take the case of one dimension. Here, the field will be described by $V(x,t)$ such that

$$\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} = 0$$

From classical point of view, the action principle is of the type

$$\delta \int L \, dt = 0$$

for fixed end conditions. $\mathcal{L}$ = Lagrangian density, L = $\int \mathcal{L}\, dx$ Then

$$\delta \int \mathcal{L}\, dv = 0$$

where $dv = dx\, dt$

Here $\mathcal{L} = \frac{1}{2}\left\{ \left(\frac{1}{c}\frac{\partial \bar{V}}{\partial t}\right)^2 - \left(\frac{\partial \bar{V}}{\partial x}\right)^2 \right\}$

which can be verified as follows:

$$\delta \mathcal{L} = \frac{1}{c^2}\frac{\partial \bar{V}}{\partial t}\,\delta\frac{\partial \bar{V}}{\partial t} - \frac{\partial \bar{V}}{\partial x}\,\delta\frac{\partial \bar{V}}{\partial t}$$

$$\delta \iint \mathcal{L}\, dx\, dt = \iint \left(-\frac{1}{c^2}\frac{\partial^2 \bar{V}}{\partial t^2} + \frac{\partial^2 \bar{V}}{\partial x^2}\right)\delta \bar{V}\, dx\, dt$$

+ a surface integral which vanishes for fixed end conditions. This gives

$$-1/c^2\, \frac{\partial^2 \bar{V}}{\partial t^2} + \frac{\partial^2 \bar{V}}{\partial x^2} = 0$$

as required.

Let us introduce the Lagrangian of a particle by $L_{part} = \frac{1}{2} m\dot{X}^2 - eV(X$

Assume for the total L the sum of the two

$$\delta \left\{ \int L_p\, dt + \int \mathcal{L}\, dx\, dt \right\} = 0$$

$$\delta L_p = m\dot{X}\,\delta\dot{X} - e\,\delta[V(X)]$$

$$\delta[V(X)] = \delta V_{at\ x=X} + \left(\frac{\partial V}{\partial x}\right)_{x=X} \delta X$$

The variation principle then gives

$$\int \left\{ -m\ddot{X}\,\delta X - e\left[\delta V(X) + \left(\frac{\partial V}{\partial x}\right)_X \delta X\right]\right\} dt$$
$$+ \iint \left[-\frac{1}{c^2}\frac{\partial^2 \bar{V}}{\partial t^2} + \frac{\partial^2 \bar{V}}{\partial x^2}\right]\delta \bar{V}\, dx\, dt = 0$$

$$\therefore\ m\ddot{X} + e\left(\frac{\partial \bar{V}}{\partial x}\right)_X = 0 \qquad \text{(Eq. of motion (1))}$$

and

$$\iint \left\{-\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} + \frac{\partial^2 \bar{V}}{\partial x^2} - e\,\delta(x-X)\right\}\delta V\, dx\, dt = 0$$

or

$$-\frac{1}{c^2}\frac{\partial^2 \bar{V}}{\partial t^2} + \frac{\partial^2 \bar{V}}{\partial x^2} = e\,\delta(x-X) \qquad \text{(Eq. of motion (2)).}$$

To find a solution in static case, $\frac{\partial \bar{V}}{\partial t} = 0$, and the particle fixed at $X = 0$.

$$\frac{\partial V}{\partial x} = -\frac{e}{2} \quad \text{for } x < 0$$
$$\frac{\partial V}{\partial x} = +\frac{e}{2} \quad \text{for } x > 0$$
$$\therefore \; V = \frac{1}{2} e |x|$$

This gives attraction between particles of like sign. Only admitting oscillations of negative mass would change this. In the 3-dimensional case it is the oscillations of terms corresponding to ϕ that gives this. Equation $m\ddot{X}$ = etc. gives undetermined $\left(\frac{\partial V}{\partial x}\right)_X$ (force) which is unsatisfactory.

In quantum mechanics it is better to use $H \equiv \sum_r p_r \dot{q}_r - \mathcal{L} =$ $= \sum_r \frac{\partial L}{\partial \dot{q}_r}\dot{q}_r - L$ In our case the Hamiltonian density $\mathcal{H} = \frac{1}{c^2}\left(\frac{\partial V}{\partial t}\right)^2 - \mathcal{L} =$ $= \frac{1}{2}\left\{\frac{1}{c^2}\left(\frac{\partial V}{\partial t}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right\}$ For the particle energy = $m/2\, \dot{X}^2 + e\bar{V}$

Total energy = $H = \sum\left(\frac{m\dot{X}^2}{2} + eV\right) + \frac{1}{2}\int\left\{\frac{1}{c^2}\left(\frac{\partial V}{\partial t}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right\}dx$

and $\quad i\hbar\frac{\partial \Psi}{\partial t} = H\Psi$

We will solve this equation for the case when just two particles are present.

We must express V in terms of Fourier's components.

$$V = \int_{-\infty}^{\infty}\left\{a_\nu e^{i\nu(t+\frac{x}{c})} + b_\nu e^{i\nu(t-x/c)}\right\}d\nu$$

This leads to

$$H_F = \int_0^\infty \nu^2(a_\nu a_{-\nu} + b_\nu b_{-\nu})\,d\nu$$

The commutation relations are then

$$[a_\nu, a_{\nu'}] = \frac{ic}{\nu}\delta(\nu+\nu')$$
$$[b_\nu, b_{\nu'}] = \frac{ic}{\nu}\delta(\nu+\nu')$$
$$[a_\nu, b_{\nu'}] = 0$$

When two particles are present

$$\left\{i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} + \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} - e_1 V(x_1,t) - e_2 V(x_2,t)\right\}\Psi = 0$$

where H_F was eliminated by the transformation

$$\psi^* = e^{iH_F t/\hbar}\,\psi$$

and subsequently dropping the star.

Assume that

$$\psi = \psi_0 + \psi_1 + \psi_2 + \cdots\cdots$$

in powers of e. Then

$$\left\{ i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} + \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} \right\}\psi_0 = 0$$

$$\left\{ \quad " \quad\quad " \quad\quad " \quad \right\}\psi_1 = \left\{ e_1 V(x_1 t) + e_2 V(x_2 t) \right\}\psi_0$$

$$\left\{ \quad " \quad\quad " \quad\quad " \quad \right\}\psi_2 = \left\{ \quad " \quad\quad " \quad \right\}\psi_1$$

As ψ_0 corresponding to two particles moving with definite momenta we take

$$\psi_0 = e^{\frac{i}{\hbar}(p_1 x_1 + p_2 x_2) - iWt/\hbar}\cdot\delta_{n0}$$

δ_{n0} specifies that there are no quanta, $\delta_{n0} = 0$ unless n = 0).

Then

$$\psi_1 = e_1\int_{-\infty}^{\infty}\left\{ \frac{a_\nu e^{i\nu(t+\frac{x}{c})}}{W - \hbar\nu - \frac{(p_1 + \frac{\hbar\nu}{c})^2}{2m_1} - \frac{p_2^2}{2m_2}} + \frac{b_\nu\cdots\cdots}{\quad} \right\} d\nu\,\psi_0$$

$$+ e_2\int_{-\infty}^{\infty}\left\{ \quad " \quad " \quad\cdot\quad\cdot \quad \right\}$$

ψ_2 is quadratic in a_ν and b_ν There will be terms corresponding to 2 light quanta or no light quanta. The latter have the factors $a_\nu a_{-\nu}$ or $b_\nu b_{-\nu}$ and $a_{-\nu} a_\nu$, or $b_{-\nu} b_\nu$ For $\nu > 0$, a_ν is like $p + iq$; $a_{-\nu} \sim p - iq$.

$$a_\nu a_{-\nu} \sim (p+iq)(p-iq) = E$$

$$a_{-\nu} a_\nu \sim (p-iq)(p+iq) = E + \text{one quantum}$$

Since, for n = 0, E = 0,

$$a_\nu a_{-\nu}\,\psi = 0$$

$$a_{-\nu} a_\nu\,\psi = \psi \qquad \text{with one quant.}$$

Thus we only need to consider terms with $a_{-\nu}a_{\nu}$ and $b_{-\nu}b_{\nu}$ Then we will have terms in e_1^2, e_1e_2, e_2e_1, and e_2^2. The term with e_1e_2 is if $\nu << c$

$$-\frac{e_1e_2}{\hbar}\int_0^{\infty}\frac{d\nu}{\nu^2}\cos\left[\frac{\nu(x_1-x_2)}{c}\right]\psi_0$$

$$=\frac{e_1e_2}{\hbar}\int_0^{\infty}\frac{d\nu}{\nu^2}\left\{1-\cos\left[\frac{\nu}{c}(x_1-x_2)\right]\right\}\psi_0-\frac{e_1e_2}{\hbar}\int\frac{d\nu}{\nu^2}\psi_0$$

$$=\left\{2\pi e_1e_2\,|x_1-x_2|+K\right\}\psi_0$$

K is infinite, but contains no x.

All together

$$\psi_2=\left\{4\pi e_1e_2\,|x_1-x_2|+K'\right\}\psi_0$$

Such result would be obtained if we were to solve for the 1st order correction the equation

$$\left\{i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2}+\frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2}-4\pi e_1e_2|x_1-x_2|+K'\right\}\psi=0$$

Chapter I. Theory When Charges Present

We now consider the theory when charged particles are present in the case of a four-dimensional space, instead of the two-dimensional space of the previous example.

We assume that the field is described by the dynamical variables in the same way as if no charges were present and that at one instant of time the same commutation rules hold as when no particles are present. This is analogous to the situation in quantum mechanics where it is assumed that the same commutation rules hold for the dynamical variables of a system whether it is interacting with another system or not. It is natural to take the Hamiltonian H of the form

$$H = H_F + \sum_r{}' H_r \tag{1}$$

where H_F is the Hamiltonian of the field and H_r is that of the r'th particle in interaction with the field. H_F is of the same form in terms of the variables A_μ, $\frac{\partial A_\nu}{\partial t}$, as when no charged particles are present. H_r is of the type occurring in the relativistic wave equation for the electron, this being the only satisfactory relativistic equation that we now have. (Protons are not to be treated at present since the equation which they satisfy is not yet known.)

$$H_r = e_r A_{or} - (\alpha_r, p_r - e_r A_r) - \alpha_{mr} m_r \tag{2}$$

where the subscript r indicates a function of the coordinates of the r'th particle.

It is readily verified that the Hamiltonian H leads to the correct equations of motion for the particles:

$$\begin{aligned} \dot{x}_r &= -\alpha_r \\ \dot{p}_r &= -e_r \frac{\partial A_{or}}{\partial x_r} - e_r \frac{\partial}{\partial x_r}(\alpha_r, A_r) \end{aligned} \tag{3}$$

The equations of motion for the field are

$$\frac{dA_\nu}{dt} = [A_\nu, H] = [A_\nu, H_F] = \frac{\partial A_\nu}{\partial t} \tag{4}$$

since

$$[A_\nu(x'), A_\nu(x'')] = 0$$

Furthermore

$$\frac{d}{dt}\left(\frac{\partial A_\nu}{\partial t}\right) = \left[\frac{\partial A_\nu}{\partial t}, H\right] = \left[\frac{\partial A_\nu}{\partial t}, H_F\right] + \sum_r \left[\frac{\partial A_\nu}{\partial t}, H_r\right]$$
$$= \nabla^2 A_\nu + \sum_r' \left[\frac{\partial A_\nu}{\partial t}, e_r A_{0r} + e_r(\alpha_r, A_r)\right]$$

and making use of the commutation relations

$$\left[\frac{\partial A_0(x')}{\partial t}, A_0(x'')\right] = 4\pi\,\delta(x'-x'')$$
$$\left[\frac{\partial A_1(x')}{\partial t}, A_1(x'')\right] = -4\pi\,\delta(x'-x'')$$

one finds

$$\frac{d}{dt}\left(\frac{\partial A_0}{\partial t}\right) = \nabla^2 A_0 + 4\pi \sum_r' e_r\,\delta(x-x_r)$$
$$\frac{d}{dt}\left(\frac{\partial A_1}{\partial t}\right) = \nabla^2 A_1 - 4\pi \sum_r' e_r \alpha_{1,r}\,\delta(x-x_r) \tag{5}$$

These equations are of the same form as in the classical theory in view of equation (3).

We cannot derive from these equations the relation

$$\text{div } E = 4\pi\rho$$

since for that it is necessary to have

$$\frac{\partial A_\nu}{\partial x_\nu} = 0$$

which we do not have as yet. It will be considered later.

In order to obtain equations of relativistic form it seems necessary to go to the Schrödinger picture. We then have a wave function for the system satisfying the Schrödinger equation

$$i\hbar \frac{d\psi}{dt} = H\psi = (H_F + \sum_r' H_r)\psi.$$

This is still not relativistic since one coordinate, the time t, is preferred and is common to the whole system. To remedy this we can introduce many t's, one for each particle. First, however, we must remove H_F from the Hamiltonian by the contact transformation

$$\beta^* = e^{iH_F t/\hbar}\,\beta\, e^{-iH_F t/\hbar}$$
$$\psi^* = e^{iH_F t/\hbar}\,\psi \tag{6}$$

where β is any dynamical variable. We then have

$$\begin{aligned} i\hbar \frac{d\psi^*}{dt} &= -H_F\psi^* + e^{iH_F t/\hbar}(H_F + \sum_r' H_r)\psi, \\ &= e^{iH_F t/\hbar}(\sum_r' H_r)e^{-iH_F t/\hbar}\,\psi^* \\ &= \sum_r' H_r^*\,\psi^*. \end{aligned} \tag{7}$$

Now we can introduce the many t's. We replace this single equation by the set of equations

$$i\hbar \frac{\partial \Psi}{\partial t_r} = H_r^*(t_r)\Psi \tag{8}$$

and take them as the fundamental equations for the system. If we add the equations together and put all the t's equal to t, since

$$\frac{\partial}{\partial t}\left[\Psi_{t_1=t_2=\cdots=t}\right] = \left[\sum_r' \frac{\partial}{\partial t_r}\Psi\right]_{t_1=t_2=\cdots=t},$$

we see that Ψ, for all the t's equal to t, satisfies the same equation (7) as ψ^*.

To see the relativistic invariance we note that, since $\dot{H}_F$ commutes with P_r and α_r:

$$H_r^* = e_r A_{0r}^* - (\alpha_r, p_r - e_r A_r^*) - \alpha_{mr} m_r$$

where

$$A_{\mu r}^* = e^{iH_F t/\hbar} A_{\mu r} e^{-iH_F t/\hbar}$$

We have already found in the discussion of the field in the absence of charges that we can write

$$A_\mu = \sum_k' \left\{ \bar{\mathcal{S}}_{\mu k} e^{-i(k,x)} + \mathcal{S}_{\mu k} e^{i(k,x)} \right\} S_k^{-1/2}$$

where the $\mathcal{S}$'s are dynamical variables (the last factor being used because we write a sum instead of an integral). Or we can write

$$A_\mu = \sum_k' \left\{ \bar{\eta}_{\mu k} e^{i[2\pi\nu_k t - (k,x)]} + \eta_{\mu k} e^{-i[2\pi\nu_k t - (k,x)]} \right\} S_k^{-1/2}$$

where

$$\mathcal{S}_{\mu k} = e^{-2\pi i \nu_k t} \eta_{\mu k}$$

In going from the Heisenberg to the Schrödinger picture the $\mathcal{S}_{\mu k}$ must become constant operators satisfying, however, the same commutation rules as before. We have then

$$A_\mu^* = \sum_k \left\{ \bar{\mathcal{S}}_{\mu k}^* e^{-i(k,x)} + \mathcal{S}_{\mu k}^* e^{i(k,x)} \right\} S_k^{-1/2}$$

where

$$\mathcal{S}_{\mu k}^* = e^{iH_F t/\hbar} \mathcal{S}_{\mu k} e^{-iH_F t/\hbar}$$

involves the time explicitly. By direct differentiation

$$i\hbar \frac{d\mathcal{S}_{\mu k}^*}{dt} = -H_F e^{iH_F t/\hbar} \mathcal{S}_{\mu k} e^{-iH_F t/\hbar} + e^{iH_F t/\hbar} \mathcal{S}_{\mu k} e^{-iH_F t/\hbar} H_F \tag{9}$$

$$= -H_F \mathcal{S}_{\mu k}^* + \mathcal{S}_{\mu k}^* H_F$$

or

$$\frac{d\mathcal{S}_{\mu k}^*}{dt} = [\mathcal{S}_{\mu k}^*, H_F]$$

Since we have (from previous lectures)

$$H_F = 16\pi^4 \nu_k^2 \bar{\xi}_{\nu k} \xi_{\nu k} \text{ + terms commuting with } \xi_{\nu k}$$

then

$$[\xi_{\nu k}, H_F] = -2\pi i \nu_k \xi_{\nu k}$$

and hence

$$[\xi^*_{\nu k}, H_F] = -2\pi i \nu_k \xi^*_{\nu k} \tag{10}$$

From (9) and (10) we see that

$$\xi^*_{\nu k} = e^{-2\pi i \nu_k t} \xi_{\nu k}$$

that is, the dependence of $\xi^*_{\nu k}$ on t in the Schrödinger picture is the same as that of $\xi_{\nu k}$ in the Heisenberg picture in the absence of charges. We then get

$$A^*_{\nu r} = \sum_k \left\{ \bar{\xi}_{\nu k} e^{i[2\pi\nu_k t - (k, x_r)]} + \xi_{\nu k} e^{-i[2\pi\nu_k t - (k, x_r)]} \right\} \xi_k^{-1/2}$$

and we see that $A^*_{\nu r}$ and hence H^*_r are of relativistic form in x_r and t.

To go over to many t's is justifiable if the two forms of equations correspond, that is, if (1) the Ψ with all t's put equal to t satisfies the same equation (7) as $\psi^*(t)$, and (2) every solution $\psi^*(t)$ can be generalized to a Ψ. We have already verified that (1) holds and shall therefore investigate (2). We have the set of equations (8) and try to find a Ψ which is equal to ψ^* when all the t's are put equal to t and which satisfies (8) also for the t's not equal. This is possible only if

$$\frac{\partial}{\partial t_r}\left(\frac{\partial \Psi}{\partial t_s}\right) = \frac{\partial}{\partial t_s}\left(\frac{\partial \Psi}{\partial t_r}\right)$$

for all values of r and s. But (if r and s are different)

$$i\hbar\frac{\partial}{\partial t_r} i\hbar\frac{\partial}{\partial t_s}\Psi = i\hbar\frac{\partial}{\partial t_r} H^*_s(t_s)\Psi = H^*_s i\hbar\frac{\partial}{\partial t_r}\Psi = H^*_s H^*_r \Psi$$

since $H_s(t_s)$ does not contain t_r. Hence the integrability condition is that

H_r^* and H_s^* commute. The only possibility of their not commuting lies in the field variables. It is necessary to consider the expressions $\left[A^*_{\mu r}(t_r), A^*_{\mu s}(t_s)\right]$ These are the same as in the Heisenberg picture in the absence of particles since the A's are expressed in terms of the η's in the same way as the A^*'s in terms of the ζ's. Hence we have that $\left[A^*_{\mu r}(t_r), A^*_{\mu s}(t_s)\right] = 0$ except when $(t_r - t_s)^2 - (x_r - x_s)^2 = 0$. Consequently there does not exist a general solution of the equations. We must restrict $(t_1 t_2 \ldots x_1 x_2 \ldots)$ to the domain for which $(t_r - t_s)^2 < (x_r - x_s)^2$. In this domain we can get a solution corresponding to any ψ^* in the case of the t's equal.

This restriction has physical justification. Suppose we have a wave function $\Psi(x_1, y_1, z_1, t \ldots s_1, s_2 \ldots n_1, n_2, n_3 \ldots)$ where $s_1, s_2 \ldots$ are the spin variables, and where for convenience the field is described by the number of quanta in the various energy states, $n_1, n_2, \ldots$. By generalizing the usual interpretation of $\bar{\psi}\psi$ as a probability, it is natural to take $\bar{\Phi}\cdot\Psi$ (where Φ is the conjugate imaginary to Ψ and the $\cdot$ denotes multiplication with summation over the spin variables) as the probability of the r'th particle being within a unit volume Δx_r about the point x_r at the time t_r, etc., and the field being in the specified state. However, it is not to be expected that it is possible to measure the positions of all the particles unless the conditions $(x_r - x_s)^2 > (t_r - t_s)^2$ are fulfilled; otherwise the measurement of one particle disturbs the observation of the others. If these inequalities are satisfied the disturbance cannot reach the other particles fast enough. The interpretation of $\bar{\Phi}\cdot\Psi$ as the probability of the various particles being at specified places at specified times with the field in a given state, is analogous to the problem of scattering in quantum mechanics where one interprets $|\psi(x, y, z, t, J)|^2$ as the probability of the scattered particle being at (x, y, z) at the time t and the scatterer being in a state of quantum number J.

If we are to interpret $\Phi \cdot \Psi$ as the probability mentioned, we must verify that it leads to a conservation law for the particles. Now, in the elementary case of a single particle in an unquantized field, the conservation law is derived as follows:

The wave equations for a particle of charge e can be written

$$\left[W - eA_0 + (\alpha, p - eA) + \alpha_m m\right]\Psi = 0$$
$$\phi\left[W - eA_0 + (\alpha, p - eA) + \alpha_m m\right] = 0$$

where

$p = -i\hbar$ grad, $W = i\hbar\frac{\partial}{\partial t}$ when operating to the right,

$p = i\hbar$ grad, $W = -i\hbar\frac{\partial}{\partial t}$ " " " " left.

Defining

$$\alpha_0 = 1,$$

we can write the equations

$$[\alpha_\nu(p_\nu - eA_\nu) + \alpha_m m]\Psi = 0$$
$$\phi[\alpha_\nu(p_\nu - eA_\nu) + \alpha_m m] = 0$$

We multiply the first from the left by $\phi\cdot$, and the second from the right by $\cdot\Psi$ ($\cdot$ denoting also summation over the components) and subtract. Since

$$\phi\cdot\alpha_m\Psi = \phi\alpha_m\cdot\Psi$$
$$\phi\cdot\alpha_\nu A_\nu\Psi = \phi\alpha_\nu A_\nu\cdot\Psi$$

we are left with

$$\phi\alpha_\nu p_\nu\cdot\Psi - \phi\cdot\alpha_\nu p_\nu\Psi = 0$$

and from the definition of p_ν this gives

$$\frac{\partial}{\partial x_\nu}(\phi\alpha_\nu\Psi) = 0$$

This is the required conservation equation.

We consider how the corresponding argument can be carried out in the present case. We see that it is applicable to each electron:

$$\Phi\cdot[\alpha_\nu(p_\nu - eA_\nu(x_r,t_r) + \alpha_m m]\Psi = 0$$
$$\Phi\,[\alpha_\nu(p_\nu - eA_\nu(x_r,t_r) + \alpha_m m]\cdot\Psi = 0$$

But A_ν is now an operator. It is necessary that the terms involving it cancel. This will happen if we sum over all the variables n of the field (since then the terms involving A_ν will differ from the symbolic products only in that the latter involve, in addition, integration over x_r, but since $\alpha_\nu\, A_\nu$ commutes with x_r the terms will cancel without this integration). We therefore define $\Phi\cdot$ to involve summation over the n's; then we have again

$$\frac{\partial}{\partial x_\nu}(\Phi\cdot\alpha_\nu\Psi) = 0$$

We can give a physical interpretation to $\Phi\cdot\Psi$ when not summed over n; but we cannot expect conservation since the fields are changing. When summed, $\Phi\cdot\Psi$ is the total probability, i.e. for all fields.

One might ask for a probability including also the probability of the field-variables having specified values at certain points (and time). One would need a representation in which the field quantities are diagonal. This is not convenient here because one wants a maximum set of commuting variables, and since in this case there are variables for every point of space their number is infinity of the order of the number of points on a line; and this is too large.

Now, for a variable f having an eigenvalue f'

$$|\psi(f')|^2 = \sum_f{}' \phi(f)\,\delta(f-f')\,\psi(f)$$

This is convenient here. The probability of a field quantity having a specified value and the electrons being at specified points is therefore given by $\Phi\cdot\delta(f - f')\Psi$ (summed over spin and the n's of the field) where f is a field quantity at some point (x, y, z, t) satisfying the inequality

$$(x - x_r)^2 > (t - t_r)^2 \tag{11}$$

for all values of r. One can show that this expression satisfies a conservation law. Thus, proceeding as before, one finds

$$\Phi\, \delta(f-f') \cdot [\alpha_\nu (p_\nu - e A_\nu) + \alpha_m m]\, \Psi = 0$$

$$\Phi\, [\alpha_\nu (p_\nu - e A_\nu) + \alpha_m m] \cdot \delta(f-f')\, \Psi = 0$$

For a conservation theorem to hold, one must have

$$\Phi\, \delta(f-f') \cdot \alpha_\nu A_\nu \Psi = \Phi\, \alpha_\nu A_\nu \cdot \delta(f-f')\, \Psi$$

This is fulfilled in the present case, however, since the field variables commute with one another in virtue of the inequality (11).

In the case of several field quantities, the corresponding expression for the probability is of the form $\Phi \cdot \delta(f - f')\, \delta(g - g')\, \Psi$. The conservation theorem holds in this case provided all the points considered satisfy the inequality (11).

We consider next the question of the supplementary conditions on the wave function. In the classical theory the potentials were made to satisfy the condition

$$\frac{\partial A_0}{\partial t} + \operatorname{div} A = 0.$$

At present we have too general a theory and must impose a restriction analogous to this. We say that only those wave functions are allowed which satisfy a supplementary condition. For this condition we try taking

$$\left(\frac{\partial A_0^*}{\partial t} + \operatorname{div} A^* \right) \Psi = 0. \tag{12}$$

However, this is not satisfactory for it is not consistent with the set of equations which Ψ satisfies, namely

$$\left(i\hbar \frac{\partial}{\partial t_r} - H_r^* \right) \Psi = 0 \tag{13}$$

where by the consistency of two operator equations $A\Psi = 0$, $B\Psi = 0$, we mean that as a consequence of these equations or without imposing further conditions on Ψ, the equation $[A, B]\Psi = 0$ is satisfied. In the present case, if we consider a typical part of the expression in (13), $\left(i\hbar\frac{\partial}{\partial t_r} - e_r A^*_{or}\right)\Psi$,

$$\left[\frac{\partial A^*_0}{\partial t} + \operatorname{div} A^*, i\hbar\frac{\partial}{\partial t_r} - e_r A^*_{or}\right] = -e_r\left[\frac{\partial A^*_0}{\partial t}, A^*_0(x_r, t_r)\right]$$

$$= -e_r\frac{\partial}{\partial t}\left[A^*_0, A^*_0(x_r, t_r)\right]$$

$$= e_r\frac{\partial}{\partial t}\, D(x - x_r, t - t_r)]$$

where (since the A^*'s have the same commutation rules as the previous A's)

$$\begin{aligned} D(x - x_r, t - t_r) &= 2\,\delta\{(x - x_r)^2 - (t - t_r)^2\}, & t < t_r, \\ &= 0, & t = t_r, \\ &= -2\,\delta\{(x - x_r)^2 - (t - t_r)^2\}, & t > t_r, \end{aligned} \tag{14}$$

so that

$$\left[\frac{\partial A_0^*}{\partial t} + \operatorname{div} A^* - e_r D(x - x_r, t - t_r), i\hbar\frac{\partial}{\partial t_r} - e_r A^*_{or}\right] = 0.$$

One can readily verify that the first member in the brackets commutes also with the other terms in (13). Hence we take as the supplementary condition:

$$\left\{\frac{\partial A^*_0}{\partial t} + \operatorname{div} A^* - \sum_r e_r D(x - x_r, t - t_r)\right\}\Psi = 0 \tag{15}$$

This equation really represents a large number of conditions on Ψ, one for each point of space time. All these conditions for different values of x and t are however consistent with one another. This had been shown in earlier lectures for the case in which no particles are present. The additional term in the present case involving D commutes with all the operators in the supplementary conditions. Hence the consistency continues to hold in the presence of particles.

There is a final supplementary condition to be considered. If several electrons are present Ψ must be antisymmetric in their coordinates, times and spin variables.

If we differentiate (15) with respect to t and then put all the t's equal to t since

$$\frac{\partial^2 A_0}{\partial t^2} + \frac{\partial}{\partial t}\operatorname{div} A = \nabla^2 A_0 + \frac{\partial}{\partial t}\operatorname{div} A = -\operatorname{div}\mathcal{E}$$

$$\left[\frac{\partial}{\partial t} D(x-x_1, t-t_1)\right]_{t_1=t} = -4\pi\,\delta(x-x_1)$$

we get

$$\left[\operatorname{div}\mathcal{E} - 4\pi\sum_r e_r\,\delta(x-x_r)\right]\Psi = 0 \qquad (16)$$

corresponding to one of the equations of the Maxwell theory.

To get the other Maxwell equations, one first puts all the t's equal to t in the supplementary condition (15). It then becomes, in virtue of (14),

$$\left(\frac{\partial A_0^*}{\partial t} + \operatorname{div} A^*\right)\psi^* = 0$$

If the A^*'s are replaced by their expressions (6) in term of the A's and one notes that H_F is independent of x and t and in the Schrödinger picture the A's are independent of t, one gets

$$\left[\frac{i}{h}(H_F A_0 - A_0 H_F) + \operatorname{div} A\right]\Psi = 0,$$

which, by (4), becomes

$$\left(\frac{\partial A_0}{\partial t} + \operatorname{div} A\right)\Psi = 0. \qquad (17)$$

If we now go over to the Heisenberg picture we have (5) for the equations of motion of the A's; and these together with (17) lead to all the Maxwell equations. However, this procedure destroys the relativistic invariance (in spite of the fact that the classical equations <u>are</u> relativistically invariant) since in quantum mechanics t is treated differently from x, y, z. For example, in the equation of motion

$$\frac{\partial^2 A_0}{\partial t^2} - \nabla^2 A_0 - 4\pi\sum_r e_r\,\delta(x-x_r) = 0$$

the x_r are operators whereas the t is not.

We consider one more transformation to simplify the equations and make them more convenient. This consists in eliminating the longitudinal components of the field from the equations. Thus for a Fourier component corresponding to a wave moving in the direction of the x-axis and having components ζ_x, ζ_y, ζ_z, ζ_o, the ζ_x and ζ_o are the longitudinal components and can be eliminated by making use of the supplementary conditions. This destroys the relativistically invariant form of the equations. The main idea of the transformation is as follows: We take a representation in which the ζ's are diagonal so that our wave function is $\Psi(\zeta_\ell, \zeta_m)$ where the ζ_ℓ, represent the longitudinal components and ζ_m the remaining components. The supplementary conditions then tell us that

$$\Psi(\zeta_\ell, \zeta_m) = f(\zeta_\ell)\, X(\zeta_m)$$

where f is completely determined and X is arbitrary. Introducing this expression into the equations for Ψ, we get

$$\left[i\hbar \frac{\partial}{\partial t_n} - H_n^* \right] f(\zeta_\ell)\chi(\zeta_m) = 0$$

If we now bring f to the left of the operator, obtaining additional terms in the latter, and then cancel f, we obtain a set of equations for X:

$$\left[i\hbar \frac{\partial}{\partial t_n} - H_n^* - \cdots \right] \chi(\zeta_m) = 0$$

This was first done by Fermi for the case of all the times equal. It can be done more generally for all t's. By different choices of ζ_ℓ different forms may be obtained which are equivalent although not obviously so.

Consider one Fourier component of the field moving along the x-axis, so that ζ_o and ζ_x are the longitudinal components. Let

$$\zeta_o - \zeta_x = \lambda + i\mu$$
$$\bar{\zeta}_o - \bar{\zeta}_x = \lambda - i\mu$$

where λ and μ are both real or Hermitian operators. One can readily verify that λ and μ commute with each other. For this Fourier component the supplementary condition gives two equations

$$\left[S_0 - S_x - \sum_r' \frac{e_r s^{-1/2}}{16\pi^4\nu^2} e^{2\pi i\nu(t_r - x_r)} \right] \Psi = 0$$

and

$$\left[\bar{S}_0 - \bar{S}_x - \sum_r' \frac{e_r s^{-1/2}}{16\pi^4\nu^2} e^{-2\pi i\nu(t_r - x_r)} \right] \Psi = 0$$

where $s^{-\frac{1}{2}}$ is introduced because the Fourier components are considered here as discrete instead of continuous. If we add and subtract these equations we get

$$(\lambda - a)\Psi = 0 \qquad a = \sum_r' \frac{e_r}{16\pi^4\nu^2} \cos 2\pi\nu(t_r - x_r)$$

$$(\mu - b)\Psi = 0 \qquad b = \sum_r' \frac{e_r}{16\pi^4\nu^2} \sin 2\pi\nu(t_r - x_r)$$

Let us take as representation one in which λ and μ are diagonal.

Since

$$[S_0 + S_x, \bar{S}_0 - \bar{S}_x] = \frac{i}{4\pi^2\nu}$$

$$[S_0 + S_x, S_0 - S_x] = 0$$

it follows that

$$S_0 + S_x = -\frac{\hbar}{8\pi^3\nu}\left(\frac{\partial}{\partial\lambda} + i\frac{\partial}{\partial\mu}\right)$$

$$\bar{S}_0 + \bar{S}_x = -\frac{\hbar}{8\pi^3\nu}\left(\frac{\partial}{\partial\lambda} - i\frac{\partial}{\partial\mu}\right)$$

We now have to solve the supplementary condition. The solution is obviously

$$\Psi = \delta(\lambda - a)\delta(\mu - b)\chi(S_m, x_r)$$

This solution is simple because of the choice of λ and μ and of the representation. The general solution might involve λ and μ in χ; but, since

$$\delta(\lambda - a)f(\lambda) = \delta(\lambda - a)f(a)$$

we can replace λ and μ by a and b. Hence in χ the longitudinal components have disappeared.

By carrying out this procedure for every Fourier component, we can now write equation (8) for Ψ in the form

$$\left(i\hbar\frac{\partial}{\partial t_r} - H_r^*\right)\prod \delta(\lambda - a)\delta(\mu - b)\chi = 0$$

where $\prod$ indicates the product due to all Fourier components. Since $(i\hbar\frac{\partial}{\partial t_r} - H_r^*)$ commutes with the supplementary condition and hence with its Fourier components it commutes with $\delta(\lambda - a)$. The equations become

$$\prod \delta(\lambda - a)\,\delta(\mu - b)\left(i\hbar\frac{\partial}{\partial t_r} - H_r^*\right)\chi = 0$$

The Fourier component of H_r^* is

$$e_r\left\{\left[\bar{S}_0 + (\alpha_r, \bar{S})\right]e^{2\pi i\nu(t_r - x_r)} + \left[S_0 + (\alpha_r, S)\right]e^{-2\pi i\nu(t_r - x_r)}\right\}S^{-\frac{1}{2}},$$

and the part due to the longitudinal components is

$$e_r\left\{\left[\bar{S}_0 + \alpha_{rx}\bar{S}_x\right]e^{2\pi i\nu(t_r - x_r)} + \left[S_0 + \alpha_{rx}S_x\right]e^{-2\pi i\nu(t_r - x_r)}\right\}S^{-\frac{1}{2}}$$

If we note that

$$S_0\chi = \left[\tfrac{1}{2}(S_0 + S_x) + \tfrac{1}{2}(S_0 - S_x)\right]\chi$$
$$= \tfrac{1}{2}\left[S_0 - S_x\right]\chi$$

since χ does not contain λ or μ, then the part of the Fourier component of H_r^* due to the longitudinal components of the field can be written

$$e_r(1 - \alpha_{rx})\left[\lambda\cos 2\pi\nu(t_r - x_r) + \mu\sin 2\pi\nu(t_r - x_r)\right]S^{-1/2}$$

We can substitute a for λ and b for μ because of the factor $\delta(\lambda - a)\ (\mu - b)$. We then get

$$e_r(1 - \alpha_{rx})\sum_s{}' \frac{e_r}{16\pi^4\nu^2}\cos 2\pi\nu\left[(t_r - t_s) - (x_r - x_s)\right]S^{-1}$$

Integrating over the Fourier components, we finally have

$$e_r\sum_s{}' \frac{e_s}{2|x_r - x_s|} - e_r\sum_s{}' \frac{e_s(t_r - t_s)}{|x_r - x_s|^3}(x_r - x_s, \alpha_r)$$

We therefore get for the equations

$$\left\{i\hbar\frac{\partial}{\partial t_r} + \left(\alpha_r, p_r - \tfrac{e}{c}A(x_r, t_r)\right) - \alpha_{m r}m_r - \sum_s{}' \frac{e_r e_s}{2|x_r - x_s|} + \sum_s{}' \frac{e_r e_s}{|x_r - x_s|^3}(x_r - x_s, \alpha_r)\right\}\chi = 0 \qquad (18)$$

Putting the t's equal and summing over r,

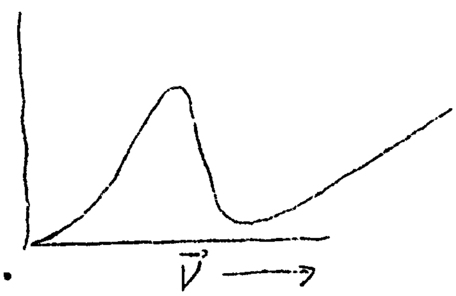
$$\left\{i\hbar\frac{\partial}{\partial t} + \quad\cdots\quad - \sum_{r\neq s}{}' \frac{e_r e_s}{|x_r - x_s|} - \sum_r{}' \frac{e_r^2}{2|x_r - x_r|}\right\}\chi = 0$$

The first new term is the Coulomb interaction. The last term is infinite and corresponds to the self-energy of the electron. This is the first appearance of infinity in the equations of the present theory and shows that the theory is incorrect. Actually this is not serious since the infinity involved is independent of the dynamical variables. One can remove it by taking

$$\chi_{new} = \chi_{old}\, e^{i \text{ const } t}$$

However, even when the infinite constant is dropped other infinities come in due to relativity. This corresponds to the old non-relativistic theory of radiation in which one dropped the scalar potential and the longitudinal component of the electromagnetic field and used the Coulomb interaction between particles. This theory has the same Hamiltonian as in the present case and is a consequence.

The infinite part of the constant arises from the high frequencies (in the Fourier integrals) and it is therefore hoped that the theory is correct for low frequencies. In problems of absorption or emission the infinity shows itself in that the curves obtained are of the type shown in the diagram. One generally cuts off such curves (beyond the maximum) and takes integrals over the finite region. For this to be justified the maximum must occur for not too large a frequency, i.e. the wave length must be long compared to the classical electron radius.



One may ask whether quantum electrodynamics is of any value. If one takes the theory with the usual unquantized field as perturbation one gets results without the infinite part. It thus appears that this theory is better than the quantum electrodynamics theory. However, this theory would not deal correctly with the problem of several photons in the same state. Hence it does not give correct results for processes involving radiation falling on an atom, in particular for spontaneous emission. Processes with stimulated transitions are treated correctly. Quantum electrodynamics does treat spontaneous emission properly.

Another application of quantum electrodynamics is to give the interaction between two electrons. By means of it one can derive:

(1) Breit's formula, for velocities not too great,

(2) Moller's formula, for all velocities but to the order of e^2.

Beyond these quantum electrodynamics cannot go. Thus quantum electrodynamics has never as yet given any result not previously obtained otherwise.

It is interesting to investigate the connection between quantum electrodynamics and the old radiation theory -- how it is that the two agree for elementary problems and not for advanced problems. Suppose we have an atomic system with Coulomb forces and we consider the emission, absorption and scattering of radiation. In the elementary theory we have the system absorbing a quantum and emitting a different quantum in a different direction. Suppose the incident field consists of two beans which we denote schematically by

$$a\, e^{i\nu(t-x)} + \bar{a}\, e^{-i\nu(t-x)},$$

$$b\, e^{i\nu(t-y)} + \bar{b}\, e^{-i\nu(t-y)},$$

where x and y refer to directions arbitrarily related. Solving by the perturbation method we get as wave function for the system

$$\psi = \psi_0 + a\psi_a + \bar{a}\psi_{\bar{a}} + b\psi_b + \bar{b}\psi_{\bar{b}}$$
$$+ a\bar{a}\,\psi_{a\bar{a}} + \dots\dots + a\bar{b}\,\psi_{a\bar{b}} + \dots$$

For the present the important terms are those increasing with time. These correspond to conservation of energy and momentum. Consider ψ_{ab}. This corresponds to the energy and momentum of the system being increased by one quantum of the first beam and decreased by one quantum of the second beam. The transition probability for this is given by

$$|a\bar{b} \cdot \text{coeff.}|^2 \propto I_a I_b$$

where I_a is the intensity of the first beam. We thus have a transition probability proportional to I_a and I_b. If we are interested in the case of no incident radiation in the second beam, we find that the transition probability is zero according to this elementary calculation. On the other hand if we use Einstein's laws of radiation we find that the transition probability is proportional to

$$I_a\,(I_b + \frac{h\nu^3}{c^2}).$$

Hence to get spontaneous radiation we should replace I_b by $\frac{h\nu^3}{c^2}$ This can be applied to all frequencies and gives a definite answer. In this way the Klein-Nishina formula is calculated (although probably incorrect for high frequencies).

Now let us suppose that the above amplitudes are operators and that they satisfy the following commutation relations:

$$a\,\bar{a} - \bar{a}\,a = -1,$$
$$b\,\bar{b} - \bar{b}\,b = -1,$$
$$a \text{ and } \bar{a} \text{ commute with } b \text{ and } \bar{b}.$$

We can solve the equations in the same way as before, but must be careful about the order of a and $\bar{a}$, etc. Let us take a representation in which $a\,\bar{a}$ is diagonal and equal to n_a and in which $b\,\bar{b}$ is diagonal and equal to n_b where n_a and n_b have the eigenbalues 0, 1, 2, ... Then all matrix elements of a and $\bar{b}$ vanish except

$$(n_a|\,a\,|n_a - 1) \sim n_a^{\frac{1}{2}},$$
$$(n_b - 1|\,\bar{b}\,|n_b) \sim n_b^{\frac{1}{2}},$$

or

$$(n_b|\,\bar{b}\,|n_b + 1) \sim (n_b + 1)^{\frac{1}{2}}.$$

If in the expression $a\,\bar{b}\cdot\text{coeff.}^2$ we introduce the matrix element of a for n_a changing from n_a to $n_a - 1$ and that of $\bar{b}$ for n_b changing from n_b to $n_b + 1$, we get

$$|\,a\,\bar{b}\cdot\text{coeff.}|^2 \sim n_a(n_b + 1),$$

which means

$$\left| a\,\bar{b} \cdot \text{coeff.} \right|^2 \sim I_a\left(I_b + \frac{h\nu^3}{c^2}\right).$$

Hence we obtain numerical agreement with the non-quantum theory provided the latter makes use of Einstein's law.

This procedure, however, is not quite quantum electrodynamics. In the latter we have perturbation not only due to the two fields above, but also due to a field with all directions, frequencies and amplitudes at a time. Hence in ψ there will be extra terms not present before. Thus there will be higher order terms corresponding to the same state as before, e.g. $a\,\bar{b}\,c\,\bar{c}\,\psi_{a\bar{b}c\bar{c}}$, where c is the amplitude of an arbitrary plane monochromatic wave. All such terms ought to be included and these will make a difference. For although in our representation

$$(o\,|\,c\,\bar{c}\,|\,o) = o,$$

we have

$$(o\,|\,\bar{c}\,c\,|\,o) \neq o.$$

The terms will be small in many problems because of their higher order (in powers of e which is in the coefficient, i.e. $\frac{e^2}{hc} \sim 0$). This is valid only if the frequency is not too high. This shows why the two theories are in agreement for low perturbation orders. Actually the approximation is not justified however, because the process of calculating successive orders is here divergent.

Chapter II. Work of Wentzel

We now consider briefly some recent work of Wentzel. It is interesting but one finds it difficult to give ~~to~~ it a physical meaning. In the Schrödinger picture we had many times but we had to go to one time in order to go over to the Heisenberg picture. The role of the Heisenberg picture is to give equations comparable to the classical equations. This the Schrödinger picture does not do. The present work gives a Heisenberg picture with many times. Hence it allows one to go over to the classical theory with many times.

Suppose an arbitrary atomic system with Hamiltonian H. Then for any dynamical variable ξ,

$$\frac{d\xi}{dt} = [\xi, H]$$

Let us arbitrarily split up H into two parts not involving the time explicitly and not necessarily commuting with each other,

$$H = H_1 + H_2$$

and let us introduce two times t_1 and t_2 for the two corresponding parts of the system. Assume that

$$\frac{d\xi}{dt_1} = [\xi, H_1]$$

$$\frac{d\xi}{dt_2} = [\xi, H_2]$$

These must be consistent, or

$$\frac{\partial}{\partial t_2}\frac{\partial \xi}{\partial t_1} = \frac{\partial}{\partial t_1}\frac{\partial \xi}{\partial t_2}.$$

But

$$\frac{\partial}{\partial t_2}\frac{\partial \xi}{\partial t_1} = \frac{\partial}{\partial t_2}[\xi, H_1] = [[\xi, H_1], H_2]$$

$$\frac{\partial}{\partial t_1}\frac{\partial \xi}{\partial t_2} = [[\xi, H_2], H_1]$$

and, because of an identity existing for Poisson brackets, the difference between the two expressions is

$$\left[\xi, \left[H_1, H_2 \right] \right]$$

In general this does not vanish. Hence the procedure is not satisfactory.

We modify the procedure by introducing

$$\overset{*}{H_2} = e^{iH_1(t_1-t_2)/h} H_2 e^{-iH_1(t_1-t_2)/h} \tag{19}$$

and we assume

$$\begin{aligned} \frac{d\xi}{dt_1} &= [\xi, H_1] \\ \frac{d\xi}{dt_2} &= [\xi, H_2^*] \end{aligned} \tag{20}$$

We now get

$$\frac{\partial}{\partial t_1}\frac{\partial \xi}{\partial t_2} = \frac{\partial}{\partial t_1}[\xi, H_2^*] = \left[\frac{\partial \xi}{\partial t_1}, H_2^*\right] + \left[\xi, \frac{\partial H_2^*}{\partial t_1}\right]$$

$$\frac{\partial}{\partial t_2}\frac{\partial \xi}{\partial t_1} = \frac{\partial}{\partial t_2}[\xi, H_1] = \left[\frac{\partial \xi}{\partial t_2}, H_1\right]$$

so that for consistency we must have

$$0 = -[\xi, [H_2^*, H_1]] + \left[\xi, \frac{\partial H_2^*}{\partial t_1}\right]$$

By (19) we find, however, that

$$\frac{\partial H_2^*}{\partial t_1} = [H_2^*, H_1]$$

so that the equations (20) are consistent.

We now apply this to the Hamiltonian we have used before,

$$H = H_F + \sum_r{}' H_r$$

and set up the equations of motion

$$\begin{aligned} \frac{d\xi}{dt} &= [\xi, H_F] \\ \frac{d\xi}{dt_r} &= [\xi, H_r^*] \end{aligned} \tag{21}$$

where

$$\overset{*}{H_r} = e^{iH_F(t-t_r)/h} H_r e^{-iH_F(t-t_r)/h} \tag{22}$$

The first equation of motion is consistent with the others by the previous proof.

For the remaining equations to be consistent with one another, since

$$\frac{\partial}{\partial t_r}\frac{\partial}{\partial t_s}\xi = \frac{d}{dt_r}[\xi, H_s^*] = [[\xi, H_r^*], H_s^*]$$

(H_s^* does not depend on t_r) we must have

$$[\xi, [H_r, H_s]] = 0$$

As

$$H_r^* = e_r A_0^*(x_r) + (\alpha_r, -p_r + e_r A(x_r)) - m_r m_r$$

the only source of non-commutation is that the potentials may not commute, so that the commutator vanishes everywhere except on the light cone,

$$[H_r^*, H_s^*] = -e_r e_s(1 - (\alpha_r, \alpha_s))\, D\,(x_r - x_s, t_r - t_s).$$

Consequently, if ξ is a function only of the field variables and of the coordin-

ates of the electrons but not of the spins or momenta of the electrons, the equa-

tions are consistent. These are generalizations of the Heisenberg equations of

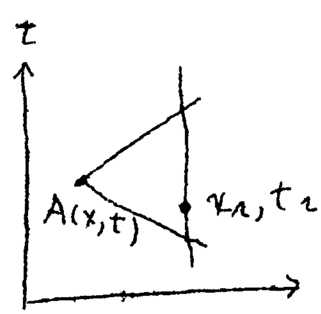
motion. The ψ on which the operators act must be made to agree with that of

the latter when we put all the t's equal. For if we put all the t's equal to T

$$\frac{d\xi}{dT} = \left(\frac{d\xi}{dt} + \sum_r \frac{d\xi}{dt_r}\right)_T = [\xi, H]$$

Note that the restriction $t_r - t_s < x_r - x_s$ is not required here and is meaningless in fact since the x's are operators.

Wentzel takes ξ as function only of field variables, e.g. $\xi = A(x,t)$. Then $\frac{d\xi}{dt_r} = [\xi, H_r] = 0$ except for $(x-x_r)^2 - (t-t_r)^2 = 0$. At these points ξ has sudden discontinuities. Thus $A(x, t)$ for an electron at (x_r, t_r) changes discontinuously when t_r is varied until it crosses the light-cone. Wentzel finds that the change in $A(x, t)$ is given by the classical formula for the retarded potential, but now with operators. Similarly, if t_r crosses the other part of the light-cone one gets advanced potentials.



Consider the classical solutions of Maxwell's equation:

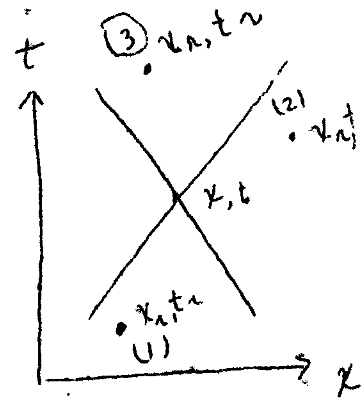
$$\Box A = \text{charge, current.}$$

The solutions are of two types:

A = incoming waves + retarded potential solution

or alternatively

A = outgoing waves + advanced potential solution.



Hence in the present case we shall have for the various positions of (x_r, t_r) with respect to the light-cone through x, t shown in the diagram,

(1) A is incoming waves,

(2) " " " " + retarded potential solution,

(3) A is outgoing waves.

In (1) and (3) the retarded and advanced potentials are canceled by the discontinuous changes on the light-cone.

Wentzel tries to overcome the infinite energy of the electron by saying that the Lorentz force is to be derived from the mean of the incoming and outgoing potentials instead of in the usual way. His calculations appear to be not entirely correct however.

A criticism that can be raised against the theory is that the equations for two t's are not really consistent since the vector potential A depends on the spin variables of the electrons

$$\frac{dA_x}{dt_r} = [A_x, H_r^*] = \alpha_{rx} e_r D(x - x_r, t - t_r).$$

For one electron there is no difficulty; for more than one electron there is. It is to be noted also that the light-cone is defined by <u>operators</u>; hence its meaning is not clear.

Chapter III. Quantization of Electron-waves

In applying the theory to electrons or protons in cases where there are many particles of the same kind, it is necessary to introduce the further assumption that the wave function is antisymmetric in all the particles of the same nature. It is then possible to introduce a new kind of procedure which is formally similar to the second quantization that has been discussed before. Mathematically it is equivalent to the ordinary treatment with an antisymmetric wave function. It can be applied to any number of particles (even infinity) and hence it will turn out later to be useful in the treatment of positrons.

Suppose we have a system of n similar particles (e.g. electrons) with wave function

$$(q_1\ q_2\ q_3\ \dots\ q_n|)$$

where q_k represents the set of dynamical variables for the k'th particle and the wave function is antisymmetric in the q's. We now pass to a new representation where the number of particles in each of the various states is diagonal. Let all the q's be denoted by q, and suppose that q has the eigenvalues

$$q^{(1)}\ q^{(2)}\ \dots\ q^{(k)}\ \dots$$

and introduce

$$n_1\ n_2\ \dots\ n_k\ \dots$$

where n_k is the number of variables having the eigenvalue $q^{(k)}$. Here n_k can be only 0 or 1 because of the antisymmetry of the wave function. In the new representation the wave function is $(n_1\ n_2\ \dots\ )$ where in a practical case the number of n's is infinite even when the number of q's is finite. (The advantage of the method shows up when the number of q's is also infinite.) The transformation here is not a general contact transformation, but is an extended point transformation since the n's are functions of the q's (and not of the p's).

Hence apart from the normalizing factor we can take

$$(q_1, q_2, \ldots, q_n|) = \pm(n_1, n_2, \ldots, |)$$

and the normalization factor here is not necessary (in the previous second quantization it was $\sqrt{\frac{n!}{n_1! n_2! \cdots\cdots}}$). The $\pm$ sign however is needed because if the n's are given we do not know which q's have the given values and the sign is affected by the order of the q's. To fix the sign we choose arbitrarily an order for the set of eigenvalues, which we call the standard order, and take the + or - sign according to whether the actual order of the q's is an even or odd permutation of the standard order (with gaps omitted).

Consider the dynamical variable

$$U = \sum_r U_r$$

where U_r is a function of q_r only. All dynamical variables must be symmetrical between all the particles to be physical observables. This U is the simplest such variable. Let us write

$$U_{ab} = (q_r^{(a)} | U_r | q_r^{(b)}).$$

Suppose we have the relation

$$\Psi_2 = U \Psi_1$$

In the q-representation this can be written

$$(q_1\, q_2 \cdots q_n | 2) = \sum_r \sum_{q_r'} (q_r | U_r | q_r')(q_1\, q_2 \cdots q_r' \text{ for } q_r \cdots q_n | 1).$$

For convenience we separate out the diagonal elements on the right-hand side:

$$\left\{\sum_r (q_r | U_r | q_r)\right\}(q_1\, q_2 \cdots q_n | 1) + \sum_r \sum_{q_r' \neq q_r} (q_r | U | q_r')(q_1 q_2 \cdots q_r' \text{ for } q_r \cdots q_n | 1)$$

We write it now in the n-representation

$$(n_1\, n_2 \cdots | 2) = \left\{\sum_a n_a U_{aa}\right\}(n_1 n_2 \cdots | 1) + \sum_{a \neq b} \pm U_{ab}(n_1 n_2 \cdots n_a - 1 \cdots n_b + 1 \cdots | 1),$$

where we have taken $q_r = q^{(a)}$, $q_r' = q^{(b)}$ and $(n_1\, n_2 \cdots | 1) = 0$ unless the n's are 0 or 1. The + sign will occur whenever the two functions

$$(q_1\, q_2 \cdots q_n|) = \pm(n_1\, n_2 \cdots |)$$

and

$$(q_1\, q_2 \ldots q_r' \text{ for } q_r \ldots |) = \pm\, (n_1\, n_2 \ldots n_{q_r} - 1 \ldots n_{q_r'} + 1 \ldots |)$$

have the same sign. This will be the case if the number of q's between q_r and q_r' is even. This is

$$\sum_{(q^{(c)} \text{ between } q_r \text{ and } q_r' \text{ (not inclusive))}} n_c$$

We now introduce variables conjugate to the n's. Since the n's have the eigenvalues 0 and 1, they no longer correspond to the harmonic oscillator but more nearly to the spin variables, having the representation

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{23}$$

which satisfy the relations

$$\begin{aligned} \sigma_x^2 = \sigma_y^2 = \sigma_z^2 &= 1 \\ \sigma_x \sigma_y = -\sigma_y \sigma_x &= i\, \sigma_z \quad etc. \end{aligned} \tag{24}$$

We can take

$$\sigma_{za} = 1 - 2n_a \tag{25}$$

and also the corresponding σ_{xa}, σ_{ya} and the variables for different <u>a</u> commute. In this case σ_{xa} and σ_{ya} play the part of angle variables. We have, in the present representation,

$$\begin{aligned} \tfrac{1}{2}(\sigma_x - i\,\sigma_y) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \tfrac{1}{2}(\sigma_x + i\,\sigma_y) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{26}$$

Since

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix}$$

it follows that when $\frac{1}{2}(\sigma_{xa} - i\,\sigma_{ya})$ operates on $(n_a|)$ one obtains $(n_a - 1|)$. Similarly $\frac{1}{2}(\sigma_{xa} + i\,\sigma_{ya})$ operating on $(n_a|)$ results in $(n_a + 1|)$.

We can now write our equation

$$\psi_2 = \sum_a n_a \psi_1 + \sum_{a \neq b} \pm U_{ab} \tfrac{1}{2}(\sigma_{xa} - i\sigma_{ya}) \cdot \tfrac{1}{2}(\sigma_{xb} + i\sigma_{yb}) \psi_1$$

whence

$$U = \sum_a n_a U_{aa} + \sum_{a \neq b} \pm U_{ab} \tfrac{1}{2}(\sigma_{xa} - i\sigma_{ya}) \tfrac{1}{2}(\sigma_{xb} + i\sigma_{yb})$$

To eliminate the $\pm$ sign we introduce new operators

$$\xi_b = \sigma_{z1} \sigma_{z2} \cdots \sigma_{z\,b-1} \cdot \tfrac{1}{2}(\sigma_{xb} + i\sigma_{yb})$$
$$\bar{\xi}_a = \tfrac{1}{2}(\sigma_{xa} - i\sigma_{ya}) \sigma_{z1} \sigma_{z2} \cdots \sigma_{z,a-1} \tag{27}$$

where the numbering is as in the standard sequence and all the factors commute.

From the properties of the σ's we see that

$$\bar{\xi}_a \xi_b = \tfrac{1}{2}(\sigma_{xa} - i\sigma_{ya} \begin{Bmatrix} \sigma_{za}\, \sigma_{z,a+1} \cdots \sigma_{z,b-1} \\ \sigma_{z,b}\, \sigma_{z,b+1} \cdots \sigma_{z,a-1} \end{Bmatrix} \tfrac{1}{2}(\sigma_{xb} + i\sigma_{yb})$$

where the upper line in the braces is for the case a before b, the lower line for a after b. Since

$$(\sigma_{xa} - i\sigma_{ya})\sigma_{za} = -i\sigma_{ya} + \sigma_{xa}$$

$$\sigma_{zb}(\sigma_{xb} + i\sigma_{yb}) = i\sigma_{yb} + \sigma_{xb}$$

we can omit the factors $\sigma_{z,a}$ and $\sigma_{z,b}$ above. The expression in the braces can be written

$$\begin{Bmatrix} (1 - 2n_{a+1})(1 - 2n_{a+2}) \cdots (1 - 2n_{b-1}) \\ (1 - 2n_{b+1})(1 - 2n_{b+2}) \cdots (1 - 2n_{a-1}) \end{Bmatrix}$$

For $n = 0$ each factor is 1; for $n = 1$ it is -1. Hence in each line the number of (-1)'s is equal to the number of n's which do not vanish, i.e. the number of q's lying between $q^{(a)}$ and $q^{(b)}$. Hence $\bar{\xi}_a \xi_b$ has the properties we require and we can now write

$$U = \sum_a n_a U_{aa} + \sum_{a \neq b} \bar{\xi}_a \xi_b U_{ab} \tag{27a}$$

It is easily verified that

$$\zeta_a \zeta_b = -\zeta_b \zeta_a, \qquad a \neq b,$$
$$\zeta_a^2 = 0$$

so that

$$\zeta_a \zeta_b + \zeta_b \zeta_a = 0 \tag{28}$$

and similarly

$$\bar{\zeta}_a \bar{\zeta}_b + \bar{\zeta}_b \bar{\zeta}_a = 0 \tag{29}$$

also

$$\bar{\zeta}_a \zeta_b + \zeta_b \bar{\zeta}_a = 0 \tag{30}$$

whereas

$$\bar{\zeta}_a \zeta_a = \frac{1}{4}(\sigma_{xa} - i\sigma_{ya})(\sigma_{xa} + i\sigma_{ya}) \tag{31}$$
$$= \frac{1}{2}(1 - \sigma_{za}) = n_a$$

and

$$\zeta_a \bar{\zeta}_a = \frac{1}{2}(1 + \sigma_{za}) = 1 - n_a \tag{32}$$

so that finally

$$\zeta_a \bar{\zeta}_a + \bar{\zeta}_a \zeta_a = 1 \tag{33}$$

We can combine equations (30) and (33) into

$$\zeta_a \bar{\zeta}_b + \bar{\zeta}_b \zeta_a = \delta_{ab}$$

If by equation (31) we substitute $\bar{\zeta}_a \zeta_a$ for n_a in equation (27a) we get

$$U = \sum_{a,b} \bar{\zeta}_a U_{ab} \zeta_b$$

which is of the same form as in the Einstein-Bose case previously discussed.

If we suppose that U is the Hamiltonian of the system, so that there is no interaction between the particles,

$$\begin{aligned} i\hbar\dot{\xi}_a &= \xi_a U - U\xi_a \\ &= \xi_a \sum_{b,c}' \bar{\xi}_c U_{cb}\xi_b - \sum_{b,c}' \bar{\xi}_c U_{cb}\xi_b\xi_a \\ &= \xi_a \sum_{b,c}' \bar{\xi}_c U_{cb}\xi_b + \sum_{b,c}' \bar{\xi}_c \xi_a U_{cb}\xi_b \\ &= \sum_{b,c}' \delta_{ac} U_{cb}\xi_b \\ &= \sum_b' U_{ab}\xi_b \end{aligned} \tag{34}$$

We see that as in the Einstein-Bose case we obtain for ξ_a an equation which is similar to the Schrödinger equation for one particle

$$i\hbar \frac{d}{dt}(q^{(a)}|) = \sum_b' U_{ab}(q^{(b)}|)$$

This is the process of second quantization.

We now extend the theory to the case of a system consisting of two parts, the inside or perturbed system and the outside or perturbing system. We take as Hamiltonian

$$H = H_p + \sum_r' U_r + \sum_r' V_r$$

where H_p is the Hamiltonian for the perturber, U_r is the energy of each particle of the inside system in the absence of the perturber, and V_r is the energy of each particle due to the perturbation. Denoting the dynamical variables of the inside system by q's and those of the outside system by α's

$$H = H_p + \sum_{a,b}' \bar{\xi}_a U_{ab}\xi_b + \sum_{a,b}' \bar{\xi}_a V_{ab}\xi_b \tag{35}$$

where

$$(\alpha^{(g)}|V_{ab}|\alpha^{(h)}) = (q_r^{(a)}, \alpha^{(g)}|V_r|q_r^{(b)}\alpha^{(h)})$$

We consider the outside system as composed of similar particles so that

$$\begin{aligned} H_p &= \sum_k' T_k \\ &= \sum_{g,h}' \bar{\eta}_g T_{gh}\eta_h \end{aligned}$$

where T_k is the energy of one particle and the η 's play the same role for the outside particles as the ξ 's do for the inside ones.

We assume that there is an interaction coupling between each particle of one kind and each of the other kind. Hence

$$V_r = \sum_k' V_{rk}$$

and because of the symmetry of the operators, $(q_r^{(a)}, \alpha_k^{(g)} | V_{rk} | q_r^{(b)} \alpha_k^{(h)})$ is independent of r and k and will be denoted by $V_{ag,bh}$. Hence V_{ab} is an operator on the α 's and we can write

$$V_{ab} = \sum_{g,h} \bar{\eta}_g V_{ag,bh} \eta_h$$

Finally, since the ξ 's and η 's commute,

$$H = \sum_{a,b} \bar{\xi}_a U_{ab} \xi_b + \sum_{g,h}' \bar{\eta}_g T_{gh} \eta_h + \sum_{a,b,g,h}' \bar{\xi}_a \bar{\eta}_g V_{ag,bh} \xi_a \eta_b \tag{36}$$

We can now calculate the equations of motion:

$$\begin{aligned} i\hbar \dot{\xi}_a &= \xi_a H - H\xi_a = \sum_b U_{ab}\xi_b + \sum_{b,g,h}' \bar{\eta}_g V_{ag,bh} \eta_h \xi_b \\ i\hbar \dot{\eta}_g &= \eta_g H - H\eta_g = \sum_h' V_{gh}\eta_h + \sum_{a,b,h}' \bar{\xi}_a V_{ag,bh} \xi_b \eta_h \end{aligned} \tag{37}$$

These are of the same form as the Hartree equations of <u>one</u> particle of one system interacting with <u>one</u> particle of the other system. In this case we have a wave function $\Psi(q, \alpha)$ satisfying a Schrödinger equation. Hartree's procedure is to assume

$$\Psi(q, \alpha) = f(q)\, F(\alpha)$$

and to determine what are the best possible functions f and F. It turns out that f must satisfy the equation

$$i\hbar \frac{d}{dt} f(q^a) = \sum_b' U_{ab} f(q^b) + \sum_{b,g,h} \bar{F}(\alpha^g) V_{ag,bh} F(\alpha^h) f(q^b)$$

where q^a is a particular value of q. The essential difference between this case and ours is that here one has ordinary functions whereas in our case we are dealing with operators.

We see then that in the case of one set of similar particles we obtain the equations for the system by quantizing the Schrödinger equation for one particle; in the case of two sets of similar particles we get the equations by quantizing the Hartree equations for two particles.

We next consider the problem of electrons interacting with radiation. It turns out that the treatment to be presented involving second quantization is mathematically equivalent to the previous one with many times. Continuing the notation of the preceding example, we have

$$H = \sum_{g,h}' \bar{\eta}_g T_{gh} \eta_h + \sum_{a,b} \bar{\xi}_a U_{ab} \xi_b + \sum_{a,b,g,h}' \bar{\xi}_a \bar{\eta}_g V_{ag,bh} \xi_b \eta_h$$

It is necessary to decide what to take for the basic states of the systems. We take for η the states of the photons, one state for each momentum and polarization; for ξ we take one state for each position x and spin k. We have then for the commutation rules (those for photons having been derived some time before)

$$\bar{\eta}_a \eta_b - \eta_b \bar{\eta}_a = \delta_{ab}$$

$$\bar{\xi}_{xk} \xi_{x'k'} + \xi_{x'k'} \bar{\xi}_{xk} = \delta_{kk'} \delta(x - x')$$

In the second commutation relation the passage to continuous variables was made by replacing the delta by the delta-function.

Since the number of photons is not conserved, the Hamiltonian must be altered accordingly. There will be additional terms which are linear (instead of bilinear) in the η's or $\bar{\eta}$'s, e.g.

$$\sum_{a,b,g}' \bar{\xi}_a \bar{\eta}_g V_{ag,b} \xi_b + \sum_{a,b,h}' \bar{\xi}_a V_{a,bh} \xi_b \eta_h$$

the first term corresponding to processes in which one light quantum is absorbed and the second to a light quantum emitted.

We now substitute for the matrix elements the elementary expressions for the interactions. Neglecting the spin for the present, we can write (since the electron charge is -e)

$$H = H_F - \int dx\, \bar{\xi}_x \cdot \left[e A_0(x) + \left(\alpha, -i\hbar\frac{\partial}{\partial x} + eA(x)\right) + \alpha_m m \right] \xi_x$$

and it will be recalled that the A's are linear functions of the η 's and $\bar{\eta}$'s. Taking spin into account, we have for the corresponding expression

$$H = H_F - \sum_{kk'} \int dx\, \bar{\xi}_{xk} \left[e A_0(x) \delta_{kk'} + \left(\alpha_{kk'}, -i\hbar\frac{\partial}{\partial x} + eA(x)\right) + \alpha_{m\,kk'} m \right] \xi_{xk'} \tag{38}$$

In this case there are no terms in the interaction between particles and radiation which are bilinear in η 's and $\bar{\eta}$'s.

The equations of motion for the ξ 's are

$$\begin{aligned} i\hbar \frac{d}{dt} \xi_{xk} &= (\xi_{xk} H - H \xi_{xk}) \\ &= -\sum_{k'} \left[e A_0(x) \delta_{kk'} + \left(\alpha_{kk'}, -i\hbar\frac{\partial}{\partial x} + eA(x)\right) + \alpha_{m\,kk'} m \right] \xi_{xk'} \end{aligned} \tag{39}$$

with corresponding equations for the $\bar{\xi}$'s. To get the equations of motion for the variables of the field we must use the commutation rules for the A's previously found. We get

$$\begin{aligned} \frac{dA}{dt} &= [A, H] = [A, H_F] = \frac{dA}{dt} \\ \frac{d^2 A_0}{dt^2} &= \left[\frac{dA_0}{dt}, H\right] = \nabla^2 A_0 - 4\pi e \sum_k \bar{\xi}_k \xi_k \\ \frac{d^2 A}{dt^2} &= \nabla^2 A + 4\pi e \sum_{kk'} \bar{\xi}_{xk} \alpha_{kk'} \xi_{xk'} \end{aligned} \tag{40}$$

We also need the supplementary condition

$$\left(\frac{dA_0}{dt} + \operatorname{div} A \right) \Psi = 0 \tag{41}$$

where Ψ is a constant vector in the Heisenberg picture. There is one such condition for each point of space-time.

Now it is necessary to show (1) that the commutation rules that are being used are invariant under a Lorentz transformation, and (2) that the supplementary conditions are consistent, both with one another for various points and also with the other equations.

For (1) we consider an infinitesimal rotation through an angle ε in the plane of x_1 and t. We have then for a variable $\beta(x)$

$$\beta^*(x) = \beta(x) + \varepsilon x_1 \frac{d\beta(x)}{dt}$$

The commutation rules for the A's are the same as for the case of a vacuum since the new terms in the equations of motion commute with the A's. The proper commutation rules for $[\xi^*, \xi^{*\prime}]$ follow since the $\xi^{*\prime}$'s are linear functions of the ξ's. Hence we must consider

$$\bar{\xi}^*_{x'k'}\xi^*_{x''k''} + \xi^*_{x''k''}\bar{\xi}^*_{x'k'} = \left(\bar{\xi}_{x'k'} + \varepsilon x_1' \frac{d\bar{\xi}_{x'k'}}{dt}\right)\left(\;\cdots\;\right) + \cdots$$

$$= \delta(x'-x'')\delta_{k'k''} + \varepsilon x_1'\left(\frac{d\bar{\xi}_{x'k'}}{dt}\xi_{x''k''} + \xi_{x''k''}\frac{d\bar{\xi}_{x'k'}}{dt}\right)$$

$$+ \varepsilon x_1''\left(\frac{d\xi_{x''k''}}{dt}\bar{\xi}_{x'k'} + \bar{\xi}_{x'k'}\frac{d\xi_{x''k''}}{dt}\right).$$

Using the expression for $\frac{d\xi}{dt}$ in (39) this becomes

$$\delta(x'-x'')\delta_{k'k''} - \frac{\varepsilon x_1''}{i\hbar}\sum_{k'''}\left[eA_0(x'')\delta_{k''k'''} + \left(\alpha_{k'',k'''}, -i\hbar\frac{\partial}{\partial x''} + eA(x'')\right) + \alpha_{m\,k''k'''}m\right]\delta_{k'k'''}\delta(x'-x'')$$

$$+ \frac{\varepsilon x_1'}{i\hbar}\sum_{k'''}\left[eA_0(x')\delta_{k'''k'} + \left(\alpha_{k'''k'}, i\hbar\frac{\partial}{\partial x'} + eA(x')\right) + \alpha_{m\,k'''k'}m\right]\delta_{k''k'''}\delta(x'-x'')$$

$$= \delta(x'-x'')\delta_{k'k''} + \varepsilon x_1''\left(\alpha_{k''k'}, \frac{\partial}{\partial x''}\delta(x'-x'')\right) + \varepsilon x_1'\left(\alpha_{k''k'}, \frac{\partial}{\partial x'}\delta(x'-x'')\right).$$

By use of the relation

$$(x'-x'')\delta'(x'-x'') = -\delta(x'-x'')$$

this can be reduced to

$$\bar{\xi}^*_{x'k'}\xi^*_{x''k''} + \xi^*_{x''k''}\bar{\xi}^*_{x'k'} = \left(\delta_{k'k''} - \varepsilon\alpha_{k''k'}\right)\delta(x'-x'') \tag{42}$$

We must now take into account the spin transformation. The procedure up to this point would have sufficed if the ξ's were scalars. As they are spinors they undergo the further transformation:

$$\xi^{+} = e^{\frac{1}{2}\varepsilon\alpha_1}\,\xi^{*}$$
$$\bar{\xi}^{+} = \bar{\xi}^{*}\,e^{\frac{1}{2}\varepsilon\alpha_1}$$

where ξ^{+} is the final function in the transformed coordinate system. Since ε is an infinitesimal this can be written

$$\xi^{+}_{k''} = \xi^{*}_{k''} + \sum_{k'''}{}' \tfrac{1}{2}\varepsilon\alpha_{1\,k''k'''}\,\xi^{*}_{k'''} \tag{43}$$
$$\bar{\xi}^{+}_{k''} = \bar{\xi}^{*}_{k''} - \sum_{k'''} \tfrac{1}{2}\varepsilon\alpha_{1\,k'''k''}\,\bar{\xi}^{*}_{k'''}$$

Combining (42) and (43) leads to

$$\bar{\xi}^{+}_{x'k'}\,\xi^{+}_{x''k''} + \xi^{+}_{x''k''}\,\bar{\xi}^{+}_{x'k'} = \delta_{k'k''}\,\delta(x'-x'')$$

which shows that the commutation rules are invariant under a Lorentz transformation.

We next consider (2) the question of the consistency of the supplementary conditions. We assume that at one instant of time but for all x's the supplementary conditions are

$$\left\{\frac{dA_0}{dt} - \operatorname{div} A\right\}\Psi = 0 \tag{44}$$

and also

$$\frac{d}{dt}\left\{\frac{dA_0}{dt} + \operatorname{div} A\right\}\Psi = 0 \tag{45}$$

which by the use of the equations of motion (40) can be put into the form

$$\left(\mathcal{E} = -\nabla A_0 - \frac{\partial A}{\partial t}\right)$$

$$\left\{\operatorname{div}\mathcal{E} + 4\pi e \sum_{k}{}' \bar{\xi}_k\,\xi_k\right\}\Psi = 0 \tag{46}$$

The last condition is consistent with the preceding one since the terms in $\bar{\xi}\xi$ commute with the A's and the case for a vacuum has already been proved. One must investigate however whether the equations (46) for different points of space are consistent with one another. The only possibility of non-commuting lies in the ξ's. But

$$\bar\xi_x \xi_x + \bar\xi_{x'}\xi_{x'} = \bar\xi_x\left(-\bar\xi_{x'}\xi_x + \delta(x-x')\right)\xi_{x'}$$
$$= -\bar\xi_x \xi_{x'}\xi_x\xi_{x'} + \bar\xi_x\xi_{x'}\delta(x-x')$$
$$= -\bar\xi_{x'}\bar\xi_x \xi_{x'}\xi_x + \xi_x\xi_x\delta(x-x')$$
$$= +\bar\xi_{x'}\left(\xi_{x'}\bar\xi_x - \delta(x-x')\right)\xi_x + \bar\xi_x\xi_{x'}\delta(x-x')$$
$$= \bar\xi_x \xi_{x'}\bar\xi_x\xi_x + \left(-\bar\xi_{x'}\xi_x + \bar\xi_x\xi_{x'}\right)\delta(x-x')$$
$$= \bar\xi_{x'}\xi_{x'}\bar\xi_x\xi_x$$

so that the conditions for different points commute.

We now show that the supplementary conditions for all time follow from the above supplementary conditions in virtue of the equations of motion. Thus

$$\frac{d^2}{dt^2}\left\{\frac{dA_0}{dt} + \mathrm{div}\, A\right\}\Psi = \nabla^2\left(\frac{dA_0}{dt} + \mathrm{div}\, A\right)\Psi$$
$$-4\pi e\left\{\frac{d}{dt}\sum_k{}' \bar\xi_k\xi_k + \mathrm{div}\left(-\sum_{kk'}{}' \bar\xi_k \alpha_{kk'}\xi_{k'}\right)\right\}\Psi$$

On the right hand side the first term vanishes because of the original supplementary condition being valid for all space, and in the second term the operator itself vanishes as a consequence of the equations of motion for the 's and asserts the conservation of charge. Similarly the higher derivatives of (44) with respect to t can be shown to vanish, and hence (44) holds for all values of t.

A gauge-invariant quantity β is defined by the fact that $\beta\Psi$ satisfies the supplementary conditions if Ψ does. This will be true if β commutes with the operators of the supplementary conditions. Thus if the supplementary condition is denoted by

$$\Delta\Psi = 0$$

then

$$\Delta(\beta\Psi) = \beta\,\Delta\Psi = 0$$

Among the gauge-invariant quantities are $\mathcal{E}$ and $\mathcal{H}$ (since the supplementary conditions differ from those for a vacuum at most by terms which commute with these) and the charge-current vector. To show that the charge-current is gauge-invariant we take the time in the supplementary conditions the same as for the charge-current vector and note that the only possibility of non-commutation is in the case of $\bar{\xi}_{xk'}\xi_{xk''}$ in the charge with $\xi_{x'l'}\xi_{x'l'}$ in the second supplementary condition. Consider

$$\begin{aligned}
\bar{\xi}_{xk'}\xi_{xk''}\xi_{x'l'}\xi_{x'l''} &= \bar{\xi}_{xk'}\left(-\bar{\xi}_{x'l'}\xi_{xk''}+\delta_{k''l'}\delta(x-x')\right)\xi_{x'l''}\\
&= -\bar{\xi}_{x'l'}\bar{\xi}_{x'l'}\xi_{x'l''}\xi_{xk''}+\bar{\xi}_{xk'}\xi_{x'l''}\delta_{k''l'}\delta(x-x')\\
&= \bar{\xi}_{x'l'}\left(\xi_{x'l''}\bar{\xi}_{xk'}-\delta_{k'l''}\delta(x-x')\right)\xi_{xk''}+\text{''}\\
&= \bar{\xi}_{x'l'}\xi_{x'l''}\bar{\xi}_{xk'}\xi_{xk''}+\left(\bar{\xi}_{xk'}\xi_{x'l''}\delta_{k''l'}-\bar{\xi}_{x'l'}\xi_{xk''}\delta_{k'l''}\right)\delta
\end{aligned}$$

If we put $l' = l''$ and sum over all values of l', the last term vanishes and we have

$$\bar{\xi}_{xk'}\xi_{xk''}\sum_{l'}\bar{\xi}_{x'l'}\xi_{x'l'} = \sum_{l'}\bar{\xi}_{x'l'}\xi_{x'l'}\bar{\xi}_{xk'}\xi_{xk''}$$

Hence the charge and current vector $\left(e\sum\bar{\xi}_{xk'}\xi_{xk'},\ -e\sum\bar{\xi}_{xk'}\alpha_{k'k''}\xi_{xk''}\right)$ is gauge-invariant.

In the elementary theory if A_μ is changed to $A_\mu + \dfrac{\partial S}{\partial x_\nu}$ where

$$\frac{\partial^2 S}{\partial t^2} - \nabla^2 S = 0$$

then ψ is changed to $e^{-ieS/\hbar}\psi$ and nothing is essentially altered thereby. We consider the corresponding change in the present theory.

In the elementary theory the expression $e^{\frac{ie}{\hbar}\int A_\nu dx^\nu}\psi$ (where the integral is taken along <u>any</u> curve from the given point to infinity) is invariant under the above transformation. In the present theory we must prove that

$$\Gamma_{xk} = e^{\frac{ie}{\hbar}\int_x^\infty A_\nu dx^\nu}\cdot\xi_{xk}$$

is invariant.

To prove this we specialize for convenience by taking the integral in Γ to lie in the plane t=const. Hence the integral does not involve A_0 and Γ commutes with the first supplementary condition. For treating the second supplementary condition we first integrate the latter over a small three-dimensional volume and change the first term to a surface integral, obtaining

$$\int (\mathcal{E}, ds) - 4\pi e \int \bar{\xi}_x \xi_x \, dv = 0$$

where we are dropping the spin indices since they are not important here. We now have

$$\int \bar{\xi}_{x'} \xi_{x'} \, dv' \xi_x = - \int \bar{\xi}_{x'} \xi_x \xi_{x'} \, dv'$$
$$= \int \xi_x \bar{\xi}_{x'} \xi_{x'} \, dv' - \int \delta(x - x') \xi_{x'} \, dv'$$
$$= \xi_x \int \bar{\xi}_{x'} \xi_{x'} \, dv' - \xi_x , \quad \text{when x lies in volume (Case I)}$$
$$= \xi_x \int \bar{\xi}_{x'} \xi_{x'} \, dv' \quad \text{when x does not lie in volume (Case II)}$$

Hence

$$\int \bar{\xi}_{x'} \xi_{x'} \, dv' \Gamma_x - \Gamma_x \int \bar{\xi}_{x'} \xi_x \, dx' = - \Gamma_x \qquad \text{(Case I)}$$
$$= 0 \qquad \text{(Case II)}$$

Now we consider the case in which the point lies in the volume and the curve along which the integral is taken cuts the bounding surface only once. Then

$$\left[\int (\mathcal{E}'', ds''), \int (A', dx') \right] = - \left[\int \left(\frac{\partial A''}{\partial t}, ds''\right), \int (A', dx') \right]$$
$$= 4\pi \int \delta(x_1' - x_1'') \delta(x_2' - x_2'') \delta(x_3' - x_3'') \cos\vartheta \, dv' ds''$$
$$= 4\pi$$

where ϑ is the angle between the vectors A" and A' and the commutation rules for $\frac{\partial A''}{\partial t}$ and A' have been used. Hence

$$\left[\int (\mathcal{E}', ds'), \Gamma_x \right] = \frac{4\pi i e}{\hbar} \Gamma_x$$

and it follows that Γ_x commutes with the operator of the second supplementary condition.

In the case that the point lies outside of the volume :

$$[(\int(\mathcal{E}'', ds''), \int(A', dx'))]$$ vanishes and Γ_x is again gauge-invariant.

If the curve crosses the bounding surfaces several times there is a contribution of $\pm \frac{4\pi i e}{\hbar} \Gamma_x$ for each crossing and the sum is the same as above.

Similarly one can show that $\bar{\xi}_x e^{-\frac{ie}{\hbar}\int A_\nu dx^\nu}$ is gauge-invariant.

Chapter IV. Theory of the Positron

In the formalism we have developed there is a mathematical symmetry between the concepts of "full" and "empty". Thus we have

$$\sigma_3 = 1 - 2\eta$$

and the theory could be expressed entirely in terms of σ_z with complete symmetry of $\pm$ values. Thus

$$\sigma_z = 1, \qquad n = 0, \qquad \text{empty},$$
$$\sigma_z = -1, \qquad n = 1, \qquad \text{full}.$$

In the theory up to the present it has been implied that most of the states are empty. Now we consider that most of the negative-energy states are full and most of the positive-energy states are empty.

The difficulty in this case is that the operators $\bar{\xi}_x \xi_x$ and $\bar{\xi}_x \alpha \xi_x$ become infinite, i.e. when they operate on Ψ they give an infinite result. We must modify the Hamiltonian H to remove these infinities. Thus we have

$$H = \sum_{g,h}' \bar{\eta}_g T_{gh} \eta_h + \sum_{a,b}' \bar{\xi}_a U_{ab} \xi_b + \sum_{a,b,g}' \bar{\xi}_a \eta_g V_{ag,b} \xi_b + \cdots$$

This involves a dissymmetry between full and empty because in each term the barred variables come before the unbarred. Suppose (although this is not strictly true) that we could distinguish the positive-energy states from those of negative energy, and let us use subscripts of the type a' for the former and a'' for the latter. Suppose we take

$$H = \sum_{g,h}{}' \bar\eta_g T_{gh} \eta_h + \sum_{a'b'}{}' \bar\xi_{a'} U_{a'b'} \xi_{b'} + \sum_{a'b''}{}' \bar\xi_{a'} U_{a'b''} \xi_{b''} + \sum_{a'',b'}{}' \bar\xi_{a''} U_{a''b'} \xi_{b'}$$
$$- \sum_{a'',b''}{}' \xi_{b''} U_{a''b''} \bar\xi_{a''} + \sum_{a',b',g}{}' \bar\xi_{a'} \bar\eta_g V_{a';g,b'} \xi_{b'} + \cdots \quad - \sum_{a'',b'',g}{}' \xi_{b''} \bar\eta_g V_{a''g,b''} \bar\xi_{a''}$$

It is to be noted that $\bar\xi_{a'}$ and $\xi_{b''}$ anti-commute; this removes the apparent dissymmetry due to the - sign.

Consider

$$U_{a'a'} \bar\xi_{a'} \xi_{a'} = U_{a'a'} \eta_{a'}$$
$$U_{a''a''} \xi_{a''} \bar\xi_{a''} = U_{a''a''} (1 - \eta_{a''})$$

We see that the occupied positive-energy state is treated symmetrically to the unoccupied negative-energy state.

The change in H (old H - new H) is

$$\sum_{a'',b''}{}' \left(\bar\xi_{a''} U_{a''b''} \xi_{b''} + \xi_{b''} U_{a''b''} \bar\xi_{a''} \right) + \sum_{a'',b'',g}{}' \left(\bar\xi_{a''} \bar\eta_g V_{a''g,b''} \xi_{b''} + \xi_{b''} \bar\eta_g V_{a''g,b''} \bar\xi_{a''} \right)$$
$$= \sum_{a''}{}' U_{a''a''} + \sum_{a''g}{}' \bar\eta_g V_{a''g,a''} + \sum{}' V_{a'';a''b} \eta_b$$

This change involves the η's but not the ξ's. Hence the equations of motion of the ξ's are unchanged but those of the η's are altered.

In the above Hamiltonian we have

$$\bar\xi_{x''} \xi_{x'} = \sum_{a,b}{}' \bar\xi_a (a|x'')(x'|b) \xi_b$$

in which we are replacing $\bar\xi_a \xi_b$ by $\bar\xi_{a'} \xi_{b'}$, $\bar\xi_{a''} \xi_{b'}$, $\bar\xi_{a'} \xi_{b''}$, $-\xi_{b''} \bar\xi_{a''}$. Hence we have here

$$\bar{\xi}_{x''}\xi_{x'} \rightarrow \sum_{a,b} \bar{\xi}_a (a|x'')(x'|b)\xi_b - \sum'_{a''b''} \bar{\xi}_{a''}(a''|x')(x'|b'')\xi_{b''} - \sum'_{a'',b''} \xi_{b''}(a''|x')(x'|b'')\bar{\xi}_{a''}$$

$$\longrightarrow \bar{\xi}_{x''}\xi_x - \sum'_{a''}(x'|a'')(a''|x')$$

Strictly speaking, it is not possible to make a hard and fast distinction between the positive and negative energy states, which is relativistically and gauge invariant. One can use this theory, however, as a hint for the accurate theory. It is convenient to proceed in two steps:

1. We replace $\bar{\xi}_{x''}\xi_{x'}$ by $\frac{1}{2}(\bar{\xi}_{x''}\xi_{x'} - \xi_{x'}\bar{\xi}_{x''})$ which is more symmetrical
2. We then see what other changes are needed. The elementary theory says to add to this

$$-\sum'_{a''}(x'|a'')(a''|x'') + \frac{1}{2}\left(\bar{\xi}_{x''}\xi_{x'} + \xi_{x'}\bar{\xi}_{x''}\right) = \frac{1}{2}\delta(x'-x'') - \sum'_{a''}(x'|a'')(a''|x'')$$

$$= \frac{1}{2}\sum'_{a}(x'|a)(a|x'') - \sum'_{a''}(x'|a'')(a''|x'')$$

$$= \frac{1}{2}\left\{\sum'_{a'}(x'|a')(a'|x'') - \sum'_{a''}(x'|a'')(a''|x'')\right\}$$

corresponding to occupied positive-energy states and unoccupied negative-energy states. Using the expressions for a free electron (solutions of the relativistic equation) one finds for this correction

$$\frac{1}{\pi^2}\frac{(\alpha, x'-x'')}{|x'-x''|^4} \quad + \text{ terms with weaker singularities for } x' = x'' \text{ (diagonal)}. \tag{47}$$

For the more correct theory we must have symmetry in the time and coordinates. We go out from

$$\frac{1}{2}\left(\bar{\xi}_{x''t''}\xi_{x't'} - \xi_{x't'}\bar{\xi}_{x''t''}\right)$$

and subtract something to remove all the singularities from the diagonal $x' = x''$, $t' = t''$. Now this expression satisfies the following equation

$$\left\{i\hbar\frac{\partial}{\partial t'} + eA_0(x',t') + (\alpha, -i\hbar\frac{\partial}{\partial x'} + eA(x't')) + \alpha_m m\right\}\frac{1}{2}\left(\bar{\xi}_{x't''}\xi_{x't'} - \xi_{x't'}\bar{\xi}_{x''}\right.$$

since $\xi_{x't'}$ satisfies this equation and the operators do not act on $\xi_{x''t''}$. Similarly it satisfies a second differential equation in x'', t''. Let

$$x = x' - x'',$$

$$t = t' - t''.$$

The expression must be of the form

$$\frac{1}{2}\left(\bar{\xi}_{x''t''}\,\xi_{x't'} - \xi_{x't'}\,\bar{\xi}_{x''t''}\right) = u_1 \frac{t+(\alpha,x)}{(t^2-x^2)^2} + \frac{u_2}{t^2-x^2} + u_3 \ln(t^2-x^2) + u_4 \quad (48)$$

in order that for t = 0 we get the previous expression (47). Here the u's represent functions which are regular in the neighborhood of the diagonal. The expression (48) is not quite unique since we can replace u_1 by $u_1 + (t-(\alpha, x))f$ and u_2 by $u_2 - f$ where f is a regular function since

$$\frac{f(t-(\alpha,x))(t+(\alpha,x))}{(t^2-x^2)^2} = \frac{f}{(t^2-x^2)}.$$

Similarly we can replace u_3 by $u_3 + (t^2 - x^2)g$ and u_4 by $u_4 - g$. There is further a numerical factor at our disposal, but we choose it so that the worst singularity has the same coefficient as in (47).

The gauge and relativistic invariance are not sufficient to specify the u's completely. We take the mathematically simplest u's. These u's are functions only of the field variables (A's and their derivatives) except u_4 which contains the electron coordinates.

Finally we can write

$$\frac{1}{2}\left(\bar{\xi}_{x''t''}\,\xi_{x't'} - \xi_{x't'}\,\bar{\xi}_{x''t''}\right) = (x't'|\,R_a\,|x''t'') + u_4$$

and this R_a is to be subtracted from the operator $\frac{1}{2}(\bar{\xi}_{x''t''}\,\xi_{x't'} - \xi_{x't'}\,\bar{\xi}_{x''t''})$.

Hence in the Hamiltonian we must replace $\bar{\xi}_{x''}\,\xi_{x'}$ by u_4. Thus we have only u_4 left which satisfies an inhomogeneous equation

$$\{\qquad\} u_4 = -\{\qquad\}(x't'|\,R_a\,|x''t'') = \text{known}.$$

This means that matter can be created and annihilated.

In the equations of motion for the field variables, the A's, the quantities $\bar{\xi}_{x'}\xi_{x'}$ are to be replaced by $(u_4)_{x'=x''}$.

In this way we remove some of the infinities. The previous infinities due to the self-energy of the electrons remain. There is also infinite self-energy for the photon (pointed out by Heisenberg) analogous to that of the electron. This is due to the fact that some singularities still remain in u_4 (although the worst ones have been removed). Just as the self-energy of the electron can be regarded as due to many nascent light quanta surrounding it, so the theory gives around each photon many nascent electrons and positrons which give it a self-energy; i.e. the Hamiltonian contains terms corresponding to such transitions as cause the creation of electrons and positrons. The procedure for calculating this is as follows. Consider the matrix element

$$(\text{one photon}\,|\,H\,|\,\text{arbitrary}).$$

Among the non-vanishing components there will be some with the right-hand side having one new photon, one electron and one positron. In a stationary state such terms will give a change in energy (by the perturbation method)

$$\sum_{\text{arbitrary states}}{}' \frac{|(\text{one photon}\,|\,H\,|\,\text{arb.})|^2}{W_{\text{one photon}} - W_{\text{arb.}}}$$

and this turns out to be infinite.