TOWARDS A MAXIMAL COMPLETION OF A PERIOD MAP

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Abstract. The motivation behind this work is to construct a "Hodge theoretically maximal" completion of a period map. This is done up to finite data (we work with the Stein factorization of the period map). The image of the extension is a Moishezon variety that compactifies a finite cover of the image of the period map.

1. Introduction

The motivation behind this work is to construct completions of period mappings, and to apply those completions to study moduli. Here we are interested in a "Hodge theoretically maximal" completion. A "minimal" completion is introduced in [GGLR20]. That work raised a number of questions about the *global asymptotic structure* of a period mapping. We distinguish this from both *global properties of the period mapping* and the *local asymptotic structure*. The first concerns properties of a variation of Hodge structures over a quasi-projective base. (For example, one may assume without loss of generality that the period map is proper.) This is a classical and much studied subject beginning with [Gri70], and with recent developments including [BKT18, BBT18, BBKT, BBT20]. The second concerns local properties of degenerations of period mappings beginning with the nilpotent and SL(2) orbit theorems [Sch73, CKS86], and with significant applications including the Iitaka conjecture [Vie83a, Vie83b, Kol87] and the arithmeticity of Hodge loci [CDK95]. The orbit theorems describe the period mapping over a local coordinate chart at infinity. The period map will not (in general) be proper when restricted to this local coordinate chart; very roughly, what is meant by the "global asymptotic structure" is that: (i) we consider certain extensions of the period map across infinity, and (ii) properties over larger neighborhoods at infinity where the extensions (not only the period map) are proper. (The two extensions we are most concerned with are (2.24) below.) The purpose of this paper is to define these extensions, and investigate their "global properties at infinity." (Other approaches to the global study of extended variations of Hodge structure over a complete base include [Moc09, Sai17] and the references in those works.) The structural results obtained here are used to construct a finite cover of the desired maximal completion (Theorem 1.3).

We consider triples $(\overline{B}, Z; \Phi)$ consisting of a smooth projective variety $\overline{B}$ and a reduced normal crossing divisor Z whose complement

$$B \;=\; \overline{B}\backslash Z$$

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has a variation of (pure) polarized Hodge structure

$$\begin{array}{c} \mathcal{F}^p \subset \mathcal{V} = \tilde{B} \times_{\pi_1(B)} V \\ \downarrow \\ B \end{array} \tag{1.1a}$$

inducing a period map

$$\Phi : B \to \Gamma \backslash D\,. \tag{1.1b}$$

Here D is a period domain parameterizing pure, weight n, Q–polarized Hodge structures on the vector space V, and $\pi_1(B) \twoheadrightarrow \Gamma \subset \mathrm{Aut}(V, Q)$ is the monodromy representation.

Without loss of generality, $\Phi : B \to \Gamma\backslash D$ is proper [Gri70, §9]. Let

$$\wp = \Phi(B)$$

denote the image, and let

$$\begin{tikzcd} B \arrow[r, "\hat{\Phi}"'] \arrow[rr, bend left, "\Phi"] & \hat{\wp} \arrow[r] & \wp \end{tikzcd} \tag{1.2}$$

be the Stein factorization of the period map (1.1b); the fibres of $\hat{\Phi}$ are connected, the fibres of $\hat{\wp} \to \wp$ are finite, and $\hat{\wp}$ is a normal complex analytic space.

Theorem 1.3. *Assume that Γ is neat. The complex analytic variety $\hat{\wp}$ is Zariski open in a Moishezon variety $\hat{\wp}^{\mathsf{T}}$, and the map $\hat{\Phi} : B \to \hat{\wp}$ extends to a morphism $\hat{\Phi}^{\mathsf{T}} : \overline{B} \to \hat{\wp}^{\mathsf{T}}$ of algebraic spaces.*

Outline of proof. The set

$$\Gamma(\hat{\Phi}) = \{(b_1, b_2) \in B \times B \mid \hat{\Phi}(b_1) = \hat{\Phi}(b_2)\}$$

defines an equivalence relation on B with the property that $\hat{\Phi} : B \to \hat{\wp}$ is the quotient map. It follows from [CDK95] that there exists a projective subvariety $\hat{X} \subset \overline{B} \times \overline{B}$ with the property that

$$\Gamma(\hat{\Phi}) = \hat{X} \cap (B \times B)\,.$$

Suppose that $\hat{X}$ defines a proper, holomorphic equivalence relation on $\overline{B}$. Then [Gra83, §3, Theorem 2] asserts that the quotient $\hat{\wp}^{\mathsf{T}}$ is a compact, complex analytic variety, and the quotient map

$$\hat{\Phi}^{\mathsf{T}} : \overline{B} \to \hat{\wp}^{\mathsf{T}}$$

is a proper holomorphic completion of $\hat{\Phi}$. Since $\overline{B}$ is projective (and therefore Moishezon) it follows that $\hat{\wp}^{\mathsf{T}}$ is Moishezon [AT82, §5, Corollary 11]. As Moishezon spaces are algebraic, Serre's GAGA implies $\hat{\Phi}^{\mathsf{T}}$ is a morphism, [Art70, §7].

So the essential problem is to show that $\hat{X}$ is defines a proper, holomorphic equivalence relation. For this, it suffices to show that every point $b \in \overline{B}$ admits a neighborhood $\overline{\mathcal{O}}^1 \subset \overline{B}$ with the properties:

(i) The restriction $\Phi|_{\mathcal{O}^1}$, with $\mathcal{O}^1 = B \cap \overline{\mathcal{O}}^1$, is proper (Corollary 2.32).

(ii) There exists a proper holomorphic map $\hat{f} : \overline{\mathcal{O}}^1 \to \hat{\mathcal{O}}^1$ whose fibres coincide over $\mathcal{O}^1$ with those of $\hat{\Phi}|_{\mathcal{O}^1}$.

The period map over $\mathcal{O}^1$ can be represented by a period matrix. It is a consequence of the infinitesimal period relation (and the properness of $\Phi|_{\mathcal{O}^1}$) that the full period matrix is determined (up to constants of integration) by a subset of the matrix coefficients, that we call the *horizontal coefficients.* The horizontal coefficients $(\varepsilon_\mu, \varepsilon_\nu)$ are of two types: the ε_μ are well-defined; while the ε_ν are multivalued, but $\tau_\nu = \exp 2\pi\mathbf{i}\varepsilon_\nu$ is well-defined. The map

$$f = (\varepsilon_\mu, \tau_\nu) : \overline{\mathcal{O}}^1 \;\to\; \mathbb{C}^m \tag{1.4}$$

is proper, and $\hat{f}$ is the Stein factorization (§5.2). The theorem then follows from Proposition 5.13. □

Remark 1.5. What one would really like to show is that $\Gamma(\Phi)$ is a proper holomorphic equivalence relation. Then the argument above would yield a proper morphism $\Phi^{\mathsf{T}} : \overline{B} \to \overline{\wp}^{\mathsf{T}}$ completing the period map Φ itself (rather than a finite cover), and factoring through $\hat{\Phi}^{\mathsf{T}}$.

Remark 1.6. It follows from the infinitesimal period relation that the the map (1.4) encodes the full period matrix up to constants of integration. This is the sense in which $\overline{\wp}^{\mathsf{T}}$ is a *maximal Hodge theoretic compactification* of $\wp = \Phi(B)$. (And there is a natural surjection $\overline{\wp}^{\mathsf{T}} \twoheadrightarrow \overline{\wp}^0$ onto the minimal Hodge theoretic compactification studied in [GGLR20].) It also indicates that points of $\overline{\wp}^{\mathsf{T}}$ parameterize equivalence classes of limiting mixed Hodge structures, and so gives $\overline{\wp}^{\mathsf{T}}$ the interpretation of a relative analog of the Kato–Usui construction [KU09]; that is, $\hat{\wp}^{\mathsf{T}}$ is intuitively (a finite cover of) the sort of object one would expect to obtain if one had a Kato–Usui horizontal completion $(\Gamma\backslash D)^\Sigma$ of $\Gamma\backslash D$, and took the closure of $\wp$ in this space. For more on the relationship to toroidal constructions see §1.4.

Remark 1.7. The fact that the period map $\Phi|_{\mathcal{O}^1}$ can be represented by a period matrix is closely related to the fact that the lift of $\Phi|_{\mathcal{O}^1}$ takes value in a Schubert cell of the compact dual (§1.3). In the classical case that D is Hermitian this is immediate: there exist many Schubert cells containing the period domain; these give the so-called "bounded" and "unbounded" realizations of D, and play a key role in the construction of the Satake–Baily–Borel compactification of an arithmetic quotient [BB66]. In contrast, non-Hermitian period domains generally contain compact subvarieties, and this means that D will not be contained in any Schubert cell (an affine space). One of the main technical results of the paper is the existence of neighborhoods $\overline{\mathcal{O}}^1$ with the properties that $\Phi|_{\mathcal{O}^1}$ is proper and the lift takes value in a Schubert cell (§1.3).

Remark 1.8. Theorem 1.3 does not assert that $\hat{\wp}^{\mathsf{T}}$ is an algebraic *variety.* We conjecture that $\hat{\wp}^{\mathsf{T}}$ does indeed admit an ample line bundle.

Remark 1.9. For the purposes of studying moduli and their compactifications, the hypothesis in Theorem 1.3 that Γ is neat should be removed; and indeed we expect that it can be dropped. Here it is a convenience allowing us avoid technical thickets that might otherwise obscure the main ideas of the paper. For more on where and how it is used see Remark 3.9.

We do not assume that Γ is arithmetic – the group may be thin.

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compact, complex analytic variety $\hat{\wp}^{\mathsf{T}}$ constructed in Theorem 1.3 is Moishezon (and by essentially the same argument as that establishing [GSTW20, Corollary 1.3]).

1.1. **Motivation and context.** Write

$$Z \;=\; Z_1 \cup Z_2 \cup \cdots \cup Z_\nu \,,$$

with smooth irreducible components Z_i. We denote by

$$Z_I \;=\; \bigcap_{i\in I} Z_i$$

the closed strata, and $Z_I^* \subset Z_I$ the Zariski open smooth locus. As we approach a point $b \in Z_I^*$ the period map Φ degenerates to a limiting mixed Hodge structure (W,F) that is polarized by nilpotent operators in the local monodromy cone σ_I. The Hodge filtration $F \in \check{D}$ will vary along Z_I^*, and is well-defined only up to the action of $\exp(\mathbb{C}\sigma_I)$ on the compact dual $\check{D}$. This induces a map

$$\Phi_I : Z_I^* \;\to\; (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I \,, \tag{1.10a}$$

cf. B.2. Nonetheless, because $N(W_a) \subset W_{a-2}$ for all $N \in \sigma_I$, the induced Hodge filtration $F^p(\mathrm{Gr}^W_a)$ on the graded quotient $\mathrm{Gr}^W_a = W_a/W_{a-1}$ is well-defined. In this way we obtain a period map

$$\Phi_I^0 : Z_I^* \;\to\; \Gamma_I\backslash D_I^0 \tag{1.10b}$$

factoring through Φ_I.

Remark 1.11. Recalling the notation of Remark 1.5, the restriction $\Phi^{\mathsf{T}}\big|_{Z_I^*}$ factors through Φ_I, and the fibres of $\Phi_I(Z_I^*) \twoheadrightarrow \Phi^{\mathsf{T}}(Z_I^*)$ are finite.

A recent attempt [GGLR20] to generalize the Satake–Bailey–Borel compactification to arbitrary period maps raised two questions.

Question 1.12. What is the global geometry of a fibre A_I^* of (1.10b)?

The maps (1.10b) can be patched together to define a proper extension $\Phi^0 : \overline{B} \to \overline{\wp}^0$ of $\Phi : B \to \wp$, cf. §§2.3-2.4. The fibre A_I^* is quasi-projective and Zariski open in a Φ^0–fibre. Let A^0 be a connected component of the fibre.

Question 1.13. Does A^0 admit a neighborhood $\overline{\mathcal{O}}^0 \subset \overline{B}$ with the following properties?

(i) The restriction of Φ^0 to $\overline{\mathcal{O}}^0$ is proper.

(ii) The holomorphic functions on $\overline{\mathcal{O}}$ separate the fibres of $\Phi^0\big|_{\overline{\mathcal{O}}^0}$.

As will be discussed in §1.3, the condition (ii) is closely related to:

(iii) Can $\Phi\big|_{B\cap\overline{\mathcal{O}}^0}$ be represented by a period matrix?

Remark 1.14. Questions 1.12 and 1.13 concern the *global study of period mappings at infinity.*

1.2. **Global geometry of A^*.** Returning to Question 1.12, the variation of limiting mixed Hodge structures over Z_I^* defines a map $\Phi^1 : A_I^* \to J_I$, with J_I an abelian variety. This map encodes the level one extension data in the variation of limiting mixed Hodge structure along A_I^*. It extends to the Zariski closure,

$$\Phi^1 : A_I \ \to \ J_I \,. \tag{1.15}$$

The abelian variety admits a family $\{\mathcal{L}_M\}$ of ample "theta" line bundles. Let A_I^0 be a connected component of A_I. Then, given any one of these bundles, there exist integers $\kappa_i = \kappa_i(M)$ so that

$$(\Phi^1\big|_{A_I^0})^*(\mathcal{L}_M) \ = \ \sum \kappa_i [Z_i]|_{A_I^0} \ = \ \sum \kappa_i \, \mathcal{N}_{Z_i/\overline{B}}\Big|_{A_I^0} \ ; \tag{1.16}$$

see (4.5) for the precise statement. This expression relates the geometry *along* A to the geometry *normal* to $Z \subset \overline{B}$. Moreover, *this is the central geometric information that arises when considering the variation of limiting mixed Hodge structure along* A_I^* (Proposition 5.1).

Example 1.17. Consider a weight $n = 1$ variation of Hodge structure with Hodge numbers $\mathbf{h} = (2,2)$. Suppose that $\dim B = 2$, and fix local coordinates $(t,w) \in \Delta^2 = \overline{\mathcal{U}}$ on $\overline{B}$ centered at a point $b \in Z$ so that $Z = \{t = 0\}$ locally. (Here $\Delta \subset \mathbb{C}$ is the unit disc.) Suppose the local nilpotent logarithm of monodromy about $t = 0$ has rank one. (This is the mildest possible non-trivial degeneration. Imagine a 2-parameter family of smooth genus two curves acquiring a node.) Then the restriction of Φ to $\Delta^* \times \Delta = \mathcal{U}$ may be represented by the period matrix

$$\widetilde{\Phi}(t,w) \ = \ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha(t,w) & \lambda(t,w) \\ \hat\nu(t,w) & \alpha(t,w) \end{bmatrix} , \tag{1.18}$$

with $\alpha(t,w)$, $\lambda(t,w)$, $\nu(t,w) = \hat\nu(t,w) - \log(t)/2\pi\mathbf{i}$ holomorphic functions on Δ^2.

We can choose the neighborhood $\overline{\mathcal{O}}$ so that the monodromy over $\mathcal{O}$ takes the form

$$\gamma \ = \ \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathsf{a} & 1 & 0 & 0 \\ \mathsf{b} & 0 & 1 & 0 \\ \mathsf{c} & \mathsf{b} & -\mathsf{a} & 1 \end{bmatrix} ,$$

with $\mathsf{a}, \mathsf{b}, \mathsf{c} \in \mathbb{Z}$. Then the period matrix $\widetilde{\Phi}(t,w)$ transforms as

$$\gamma \cdot \widetilde{\Phi}(t,w) \ = \ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha(t,w) + \mathsf{b} - \mathsf{a}\lambda(t,w) & \lambda(t,w) \\ \hat\nu(t,w) + \mathsf{c} - \mathsf{ab} + 2\mathsf{a}\alpha(t,w) + \mathsf{a}^2\lambda(t,w) & \alpha(t,w) + \mathsf{b} - \mathsf{a}\lambda(t,w) \end{bmatrix}$$

Under this action, $\nu(t,w)$ transforms as

$$\nu(t,w) \ \mapsto \ \nu(t,w) + \mathsf{c} - \mathsf{ab} + 2\mathsf{a}\alpha(t,w) + \mathsf{a}^2\lambda(t,w) \,,$$

so that

$$\tau(t,w) \ = \exp(2\pi\mathbf{i}\,\hat\nu(t,w)) \ = \ t\exp(2\pi\mathbf{i}\,\nu(t,w))$$

transforms as

$$\begin{aligned}\tau(t,w) &\mapsto t\exp 2\pi\mathbf{i}\left(\varepsilon(t,w)+\mathsf{a}^2\lambda(t,w)-2\mathsf{a}\alpha(t,w)\right)\\ &= \tau(t,w)\exp 2\pi\mathbf{i}\left(\mathsf{a}^2\lambda(t,w)-2\mathsf{a}\alpha(t,w)\right).\end{aligned}$$

This is the functional equation for the classical theta function. We may normalize our choice of coordinates (t,w) so that $\nu(t,w)=0$. Then, this computation implies that $t\cdot\vartheta$, with ϑ a section of the dual to the theta line bundle, is globally well-defined along A^0_I.

1.3. **Period matrices and Schubert cells.** The fact that period maps are locally liftable implies that they can always be locally represented by period matrices. Schmid's nilpotent orbit theorem implies that this property also holds at infinity: points $b\in Z$ admit local coordinates $\overline{\mathcal{U}}\subset\overline{B}$ so that the restriction of Φ to $\mathcal{U}=B\cap\overline{\mathcal{U}}$ can be represented by a period matrix. The expression (1.18) is an example of one such representation. The caveat is that the entries/coefficients of the period matrix may not be multi-valued: they may involve logarithms (as in Example 1.17).

Period matrix representations are closely related to Schubert cells (§B.4). The compact dual $\check{D}\supset D$ can be covered by Zariski open Schubert cells. Each such cell is biholomorphic to $\mathbb{C}^m$, with $m=\dim D$. (These are local coordinate charts on $\check{D}$.) In each of the cases discussed above, the local lift of the period map takes value in a Schubert cell $\mathcal{S}$. And the entries/coefficients of the period matrix are the pullbacks of the coordinates $\mathcal{S}\to\mathbb{C}^m$. In general, to say that the period map Φ can be represented by a period matrix over an open set $\mathcal{O}\subset B$ is equivalent to the following two conditions:

(i) The lift $\tilde{\Phi}:\tilde{\mathcal{O}}\to D$ of Φ to the universal cover of $\mathcal{O}$ takes value in a Schubert cell $\mathcal{S}\subset\check{D}$.

(ii) In general the monodromy $\Gamma_{\mathcal{O}}\subset\Gamma$ of the variation over $\mathcal{O}$ acts on $\check{D}$. This action must preserve the Schubert cell $\mathcal{S}$ (or at least $D\cap\mathcal{S}$).

Under these conditions, the pullback of the coordinates on $\mathcal{S}\to\mathbb{C}^m$ yields the period matrix representation of $\Phi|_{\mathcal{O}}$. If $\Gamma_{\mathcal{O}}$ is nontrivial, then the entries of the period matrix may be multi-valued (cf. the logarithm in Example 1.17). Nonetheless, we may think of this as giving us a (possibly multi-valued) coordinate representation of $\Phi|_{\mathcal{O}}$.

Given a connected component A^0 of a Φ^0–fibre, one of the main technical results of this paper is the existence of a neighborhood $\overline{\mathcal{O}}^0\subset\overline{B}$ of A^0 with the properties: (i) the restriction of Φ^0 to $\overline{\mathcal{O}}^0$ is proper, and (ii) the restriction of Φ to $\mathcal{O}^0=B\cap\overline{\mathcal{O}}^0$ *admits a period matrix representation*, cf. §3.1.

The monodromy Γ_{A^0} over $\overline{\mathcal{O}}^0$ is too complicated to extract the map (1.4) from this matrix representation. However, the map (1.15) is the restriction of a proper extension $\Phi^1:\overline{B}\to\overline{\wp}^1$ of Φ through which Φ^0 factors, cf. §§2.3–2.4. And a connected component A^1 of a Φ^1–fibre admits a neighborhood $\overline{\mathcal{O}}^1\subset\overline{\mathcal{O}}^0$ with the properties: the restriction of Φ^1 to $\overline{\mathcal{O}}^1$ is proper, and (ii) the monodromy Γ_{A^1} over $\overline{\mathcal{O}}^1$ is very simple (almost as simple as Schmid's local monodromy at infinity, cf. Proposition 5.1). The map (1.4) is constructed from this second period matrix representation in §5.2. In Example 1.17, the local expression for f is $(\alpha,\lambda,e^{\hat{\nu}}):\Delta^2\to\mathbb{C}^3$.

1.4. **Relationship to toroidal constructions.** It follows from Remark 1.6 that $f=(\varepsilon_\mu,\tau_\nu)$ encodes the full variation of mixed Hodge structure (1.10a) over $Z_I^*\cap\overline{\mathcal{O}}^1$ (up to constants of integration). In particular, $\overline{\wp}^{\mathsf{T}}$ parameterizes equivalence classes of limiting mixed Hodge structures. In the classical case that D is Hermitian symmetric and Γ is arithmetic the toroidal compactifications $\overline{\Gamma\backslash D}^{\mathsf{T}}$ of [AMRT75] are also known to parameterize equivalence classes of limiting mixed Hodge structures [CCK80]. This naturally raises the question: what is the relationship between $\overline{\wp}^{\mathsf{T}}$ and the closure of $\wp$ in $\overline{\Gamma\backslash D}^{\mathsf{T}}$? This is an interesting question, but not one we will undertake to address here, beyond the following remarks.

While it seems reasonable to expect a relationship between the two constructions, the precise nature of that relationship is not obvious because the approaches are quite different: $\hat{\wp}^{\mathsf{T}}$ is realized by the compact quotient of a proper holomorphic equivalence relation, while the toroidal construction involves attaching "boundary components" to $\Gamma\backslash D$ by a subtle "glueing" process involving fans (which do not arise in the construction of $\hat{\wp}^{\mathsf{T}}$. Instead our "boundary structure" comes from (1.10) and the period matrix representation of §1.3). Nonetheless there are some suggestive similarities (which motivate our use of superscript $^{\mathsf{T}}$).

The first is that the functions τ_ν of (1.4) seem to play a role analogous to that of the monomials defining the toric structure in [Mum75]. The second is that the glueing procedure in [Mum75] involves a factorization

$$D \;\to\; \Gamma_0\backslash D \;\to\; \Gamma_1\backslash D \;\to\; \Gamma\backslash D$$

of $D \to \Gamma\backslash D$. Our "fibre monodromies" $\Gamma_{A^1} \subset \Gamma_{A^0}$ in §1.3 are related to Mumford's groups $\Gamma_0 \subset \Gamma_1$ by

$$\Gamma_{A^1} \;\subset\; \Gamma_0 \;\subset\; \Gamma_{A^0} \;\subset\; \Gamma_1\,.$$

Remark 1.19. The existence of an extension $\overline{\Phi} : \overline{B} \to \overline{\Gamma\backslash D}^{\mathsf{T}}$ is a subtle question, and one need not exist [FS86, CMGHL17]. So it is striking that we have the extension $\hat{\Phi}^{\mathsf{T}}$ in Theorem 1.3.

1.5. **Organization of the paper.** Section 2 develops the constructions and preliminary results that will be used to study the period map in a neighborhood of the fibre A^*_I. In §3 we show that A^0 admits a neighborhood $\overline{\mathcal{O}}^0 \subset \overline{B}$ with the property that Φ is proper and can be represented by a period matrix over $B \cap \overline{\mathcal{O}}^0$. We show that the extended line bundles $\det(\mathcal{F}^p_e) \to \overline{B}$ are trivial over $\overline{\mathcal{O}}^0$ when Γ is neat (Theorem 3.12). And we construct explicit sections s_M of line bundles over $\overline{\mathcal{O}}^1$ that will be used to establish (1.16).

In §4 we study the level one extension data map (1.15) along A^0, and the "theta" line bundles $\mathcal{L}_M$ over the Jacobian variety J_I. In §5 we show that, modulo the nilpotent orbit, the higher level extension data is locally constant on fibres of (1.15). This yields a period matrix representation over punctured neighborhood $B \cap \overline{\mathcal{O}}^1$ of A^1 from which we can extract the functions (1.4).

We need to set notation and review the local behavior of period maps at infinity. Because this material is classical, we streamline the presentation by placing this this review (which also includes the proofs of a few technical lemmas) in Appendices B and C; we refer to this material as convenient.

Contents

2. Period maps at infinity

2.1. **A tower of maps: extension data.** As a first step towards answering Question 1.12, we note that what varies along the fibre $A_I^* \subset Z_I^*$ of Φ_I^0 is the extension data of the mixed Hodge structure (W, F). This invites the study of the geometry of the extension data associated to the collection of limiting mixed Hodge structures with fixed associated graded. This is done by realizing (1.10) as the extremal maps in a tower

$$\begin{tikzcd}
Z_I^* \arrow[r, "\Phi_I"] \arrow[dr, bend right=15, "\Phi_I^a"] \arrow[ddr, bend right=30, "\Phi_I^2"] \arrow[dddr, bend right=40, "\Phi_I^1"] \arrow[ddddr, bend right=50, "\Phi_I^0"] & (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I \arrow[d, two heads] \\
& (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I^a \arrow[d, two heads] \\
& (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I^2 \arrow[d, two heads] \\
& \Gamma_I\backslash D_I^1 \arrow[d, two heads] \\
& \Gamma_I\backslash D_I^0
\end{tikzcd} \tag{2.1}$$

that is defined as follows.

2.1.1. *Mixed Hodge structures.* Given a MHS (W, F_0), define Hodge numbers $f_\ell^p := \dim F_0^p(\mathrm{Gr}_\ell^W)$, and set

$$D_W \;=\; \{F \in \check{D} \mid (W, F) \text{ is a MHS, } \dim F^p(\mathrm{Gr}_\ell^W) = f_\ell^p\}\,.$$

Set

$$G \;=\; \mathrm{Aut}(V, Q)\,,$$

and let $P_W \subset G$ be the $\mathbb{Q}$–algebraic group stabilizing the weight filtration. (See §B.1.1 for further discussion of group notation.) Given any $g \in P_W$, there is an induced action on the quotients $W_\ell/W_{\ell-a}$. The normal subgroups

$$P_W^{-a} \;=\; \{g \in P_W \mid g \text{ acts trivially on } W_\ell/W_{\ell-a}\ \forall\ \ell\}$$

define a filtration $P_W = P_W^0 \supset P_W^{-1} \supset \cdots$. The group

$$G_W \;=\; (P_{W,\mathbb{R}}/P_{W,\mathbb{R}}^{-1}) \ltimes P_{W,\mathbb{C}}^{-1} \tag{2.2}$$

acts transitively D_W, [KP16]. (See §B.1.1 for a review of the group notation.) Let

$$D_W^a \;=\; P_{W,\mathbb{C}}^{-a-1}\backslash D_W$$

be the quotient. This yields a tower of fibre bundles

$$D_W \;\twoheadrightarrow\; D_W^a \;\twoheadrightarrow\; D_W^0\,.$$

Set

$$\Gamma_W \;=\; \Gamma \cap P_{W,\mathbb{Q}}^{-1}\,. \tag{2.3}$$

Definition 2.4 (Extension data of MHS). If $\delta_W = \delta_{W,F} \subset D_W$ is the fibre of the surjection $D_W \to D_W^0$, then $\Gamma_W\backslash\delta_{W,F}$ is the *extension data of the MHS* (W,F). The image $\delta_W^a = \delta_{W,F}^a$ of δ_W under the projection $D_W \to D_W^a$ is also a fibre of $D_W^a \twoheadrightarrow D_W^0$, and we say that $\Gamma_W\backslash\delta_{W,F}^a$ is the *extension data of level* $\le a$.

Remark 2.5. Our treatment of extension data will focus on its Lie theoretic properties as a locally homogeneous space. For a geometric perspective see [Car87].

2.1.2. *Limiting mixed Hodge structures.* Now suppose that the MHS (W,F_0) is polarized by a nilpotent cone

$$\sigma_I \;=\; \mathrm{span}_{\mathbb{R}_{>0}}\{N_i \mid i\in I\} \;\subset\; \mathrm{End}(V_\mathbb{R},Q)$$

of commuting logarithms of monodromy. (Here $\exp(N_i)$ is a local monodromy operator about Z_i^*.) Define

$$D_I \;=\; \{F\in D_W \mid (W,F) \text{ is polarized by } \sigma_I\}\,.$$

Then $W=W(\sigma_I)$ implies that the $\mathbb{Q}$–algebraic group $C_I\subset \mathrm{Aut}(V,Q)$ centralizing the cone σ_I is a subgroup of P_W. Note that this centralizer also admits a filtration $C_I = C_I^0 \supset C_I^{-1} \supset \cdots$ by normal subgroups

$$C_I^{-a} \;=\; C_I \cap P_W^{-a}\,.$$

The group

$$G_I \;=\; (C_{I,\mathbb{R}}/C_{I,\mathbb{R}}^{-1}) \ltimes C_{I,\mathbb{C}}^{-1}$$

acts transitively D_I, [KP16]. Let

$$\Gamma_I \;=\; \Gamma \cap C_{I,\mathbb{Q}}\,.$$

2.1.3. *Definition of the tower.* The variation of limiting mixed Hodge structures along Z_I^* in §1.1 induces the map Φ_I of (2.1), cf. §B.2.4. The maps Φ_I^a are defined by passing to the quotient spaces $D_I^a = C_{I,\mathbb{C}}^{-a-1}\backslash D_I$. Define

$$\wp_I^a \;=\; \Phi_I^a(Z_I^*)\,.$$

We have natural surjections $\wp_I^{a+1} \twoheadrightarrow \wp_I^a$. Proposition 5.1(c) implies

Theorem 2.6. *The maps $\wp_I^{a+1} \twoheadrightarrow \wp_I^a$ are finite to one for all $a \ge 2$.*

Remark 2.7. Theorem 2.6 asserts that the level two extension data map Φ^2_I determines the full extension data map Φ_I up to constants of integration. Additionally the level 2 extension data is discrete. (The data not given by constants of integration is given by sections of line bundles with fixed divisor, Proposition 5.1(c) and Remark 5.2.) So it is then not surprising that we will see that the answer to Question 1.12 is to be found in studying the restriction of Φ^1 to A^*_I. This restriction takes value in some $\Gamma_I\backslash\delta^1_{I,F}$. The spaces $\Gamma_I\backslash\delta^1_{I,F}$ and $\Gamma\backslash\delta^1_{W,F}$ of level one extension data carry rich geometric structure. As observed by Carlson, these spaces are tori, and $\Gamma_I\backslash\delta^1_{I,F}$ is an abelian subvariety when $F^p(\mathrm{Gr}^W_{-1})$ defines a level one Hodge structure [Car87]. To this we add Theorem 4.3, and the corollary (4.5) that encodes the central geometric information that arises when considering the VLMHS along A^*.

We say that the quotient D^a_I has automorphism group $G^a_I = G_I/C^{-a-1}_{I,\mathbb{C}}$ to indicate that G_I acts on D^a_I, with the normal subgroup C^{-a-1}_I acting trivially. The base space D^0_I is a Mumford–Tate domain with Mumford–Tate group G^0_I. Again we have a tower of fibre bundles

$$D_I \twoheadrightarrow D^a_I \twoheadrightarrow D^0_I\,.$$

Definition 2.8 (Extension data of LMHS). If $\delta_I = \delta_{I,F} = \delta_{W,F}\cap D_I$ is the fibre of the surjection $D_I \to D^0_I$, then $\Gamma_I\backslash\delta_{I,F}$ is the *(polarized) extension data of the limiting mixed Hodge structure* (W,F). The image $\delta^a_I = \delta^a_{W,I}$ of δ_I under the projection $D_I\to D^a_I$ is also a fibre of $D^a_I \twoheadrightarrow D^0_I$, and we say that $\Gamma_I\backslash\delta^a_{I,F}$ is the *(polarized) extension data of level* $\le a$.

2.2. **Reduced limit period map.** The purpose of this section is to describe an important relationship between the period map $\Phi^0_I : Z^*_I \to \Gamma_I\backslash D^0_I$ and the topological boundary ∂D of the period domain in the compact dual $\check{D}$. In general, the limit Hodge filtration F associated with a point $b\in Z^*_I$ (as in §B.2.4) will not lie in the boundary. However, there is a "naïve", or *reduced limit* $F_\infty(b)$, that does lie in ∂D (§2.2.1). Each of these limits takes value in a $G_\mathbb{R}$–orbit $\mathcal{O}_I\subset\partial D$, and there is an induced map

$$\Phi^\infty_I : Z^*_I \to \Gamma_I\backslash\mathcal{O}_I\,. \tag{2.9}$$

Let

$$\wp^\infty_I = \Phi^\infty_I(Z^*_I) \subset \Gamma_I\backslash\mathcal{O}_I$$

denote the image.

Proposition 2.10. *The period map Φ^0_I factors through the reduced limit period map Φ^∞_I. Moreover, the map Φ^∞_I is locally constant on Φ^0_I–fibres. In particular, the map $\pi_I : \wp^\infty_I \to \wp_I$ is finite.*

Remark 2.11. The proposition (proved in §§2.2.2–2.2.4) has the important Corollary 2.16. The later imposes an additional constraint on the monodromy over a neighborhood $\mathcal{O}^0$ of a A^0_I (Lemma 3.3). This constraint makes it possible for us to show that $\Phi|_{\mathcal{O}^0}$ admits a period matrix representation (Corollary 3.6).

2.2.1. *Definition.* Fix a local lift $\widetilde{\Phi}(t,w)$, and let (W,F,σ) be the associated limiting mixed Hodge structure (§B.2.4). The *reduced limit period*

$$F_\infty(w) = \lim_{y\to\infty}\widetilde{\Phi}(z,w) = \lim_{y\to\infty}\exp(\mathbf{i}yN)\xi(0,w)\cdot F \in \overline{D}$$

is independent of our choice of $N \in \sigma$, [GGK13, KP14, GGR17]. (The limit is understood to be taken with x bounded.) The two filtrations F and $F_\infty(0)$ are related by the Deligne splitting (§B.3)

$$F^p \;=\; \bigoplus_{a\geq p} V^{a,b}_{W,F} \quad\text{and}\quad F^p_\infty(0) \;=\; \bigoplus_{b\leq n-p} V^{a,b}_{W,F}\,.$$

In particular, the Lie algebra $\mathfrak{f}_\infty$ of the stabilizer $\mathrm{Stab}_{G_\mathbb{C}}(F_\infty(0))$ is

$$\mathfrak{f}_\infty \;=\; \bigoplus_{q\leq 0} \mathfrak{g}^{p,q}_{W,F}\,.$$

Recalling that the map $\xi(0,w)$ takes value in $C_{I,\mathbb{C}}$ (§B.2.4), we see that

$$F_\infty(w) \;=\; \xi(0,w)\cdot F_\infty(0)\,. \tag{2.12}$$

In particular, the map $F_\infty : \{0\}\times\Delta^r \to \check D$ is holomorphic, and takes value in the $C_{I,\mathbb{C}}$–orbit of $F_\infty(0)$. What is less obvious is that: (i) The holomorphic $F_\infty(0,w)$ takes value in the real orbit

$$\mathcal{O}_I \;=\; C_{I,\mathbb{R}}\cdot F_\infty(0) \;\subset\; \check D\,.$$

(ii) The real orbit $\mathcal{O}_I$ is open in the (complex) orbit $C_{I,\mathbb{C}}\cdot F_\infty(0)$, and so is a complex submanifold of $\check D$.

The reduced limit F_∞ is independent of the local coordinates (t,w) expressing $\widetilde\Phi$. So the reduced period limit induces a well-defined holomorphic map (2.9).

2.2.2. *Proof: period map factors through reduced limit.* Observe that there is a natural identification

$$D^0_I \;\simeq\; C^{-1}_{I,\mathbb{R}}\backslash\mathcal{O}_I\,.$$

This identification induces

$$\pi_I : \Gamma_I\backslash\mathcal{O}_I \;\to\; \Gamma_I\backslash D^0_I\,. \tag{2.13}$$

We have

$$\Phi^0_I \;=\; \pi_I\circ\Phi^\infty_I\,. \tag{2.14}$$

In particular, $\pi_I : \wp^\infty_I \twoheadrightarrow \wp_I$.

Remark 2.15. When D is Hermitian the map (2.13) is an isomorphism and $\Phi^0_I = \Phi^\infty_I$.

Let $C^{-1}_{I,\infty,\mathbb{C}}$ denote the stabilizer in $C^{-1}_{I,\mathbb{C}}$ of the filtration $F_\infty(0)\in\check D$.

Corollary 2.16. *The map $\Phi^0_{A^0,I} : A^0\cap Z_I \to \delta_I$ takes value in*

$$C^{-1}_{\sigma,\infty,\mathbb{C}}\cdot F \;\subset\; C^{-1}_{I,\mathbb{C}}\cdot F \;=\; \delta_I\,.$$

2.2.3. *Proof of finiteness: formulation of the argument.* It is enough to show that $F_\infty(w)$ is constant along the Φ^0–fibres in $\{0\} \times \Delta^r$. This is a consequence of the infinitesimal period relation. The essential point is that the map

$$w \ \mapsto \ \xi(0,w)\cdot F \quad \text{is horizontal.} \tag{2.17}$$

Recall that $\xi(t,w)$ takes value in $\exp(\mathfrak{f}^\perp)$, and $\xi(0,w)$ takes value in $\exp(\mathfrak{c}_{I,\mathbb{C}})$, cf. §B.3 and §B.2.4. We have

$$\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}} \ = \ \bigoplus_{\substack{p<0\\ p+q\le 0}} \mathfrak{c}^{p,q}_{I,F}\,.$$

Note that

$$\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}} \cap \mathfrak{f}_\infty \ = \ \bigoplus_{\substack{p<0\\ q\le 0}} \mathfrak{c}^{p,q}_{I,F}\,,$$

and consider the decomposition

$$\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}} \ = \ \mathfrak{d} \ \oplus \ \mathfrak{e} \ \oplus \ (\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}} \cap \mathfrak{f}_\infty)$$

defined by

$$\mathfrak{d} \ = \ \bigoplus_{\substack{p<0\\ p+q=0}} \mathfrak{c}^{p,q}_{I,F} \quad \text{and} \quad \mathfrak{e} \ = \ \bigoplus_{\substack{p<0<q\\ p+q<0}} \mathfrak{c}^{p,q}_{I,F}\,.$$

Each of these three summands is a Lie subalgebra of $\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}}$.

Since $\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}}$ is nilpotent, the function $\xi(0,w)$ may be uniquely decomposed as

$$\xi(0,w) \ = \ e(w)f(w)s(w)$$

with $f(w) \in \exp(\mathfrak{d})$, $e(w) \in \exp(\mathfrak{e})$ and $s(w) \in \exp(\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}} \cap \mathfrak{f}_\infty)$. Since $\xi(0,w) = e(w)\,f(w)s(w)f(w)^{-1}\,f(w)$, and both $e(w)$ and $f(w)s(w)f(w)^{-1}$ take value in the unipotent radical $C^{-1}_{I,\mathbb{C}}$, we may

$$\text{identify } \Phi^0_I(0,w) \text{ with } f(w).$$

Furthermore, since $\mathfrak{f}_\infty$ is the stabilizer of $F_\infty(0)$ in $\mathfrak{f}$, (2.12) implies we may

$$\text{identify } F_\infty(w) \text{ with } e(w)f(w).$$

So to prove the lemma, it suffices to show that

$$e(w) \text{ is locally constant along } f\text{–fibres.}$$

So we assume

$$\mathrm{d}f = 0\,, \tag{2.18a}$$

and will show that $\mathrm{d}e = 0$; equivalently,

$$e^{-1}\mathrm{d}e \ = \ 0\,. \tag{2.18b}$$

2.2.4. *Proof of finiteness: horizontality.* Horizontality is the condition

$$(\xi^{-1}\mathrm{d}\xi)^{p,q} \;=\; 0\,, \quad \forall\, p \le -2\,, \tag{2.19}$$

with $(\xi^{-1}\mathrm{d}\xi)^{p,q}$ the component of the $\mathfrak{f}^\perp$–valued $\xi^{-1}\mathrm{d}\xi$ taking value in $\mathfrak{g}^{p,q}_{W,F}$, cf. §B.4 and §B.6. At $(0,w)$ we have

$$\begin{array}{rcl} \xi^{-1}\mathrm{d}\xi & = & (efs)^{-1}\mathrm{d}(efs) \\ & = & \mathrm{Ad}_{fs}^{-1}(e^{-1}\mathrm{d}e) \;+\; \mathrm{Ad}_s^{-1}(f^{-1}\mathrm{d}f) \;+\; s^{-1}\mathrm{d}s \\ & \overset{(2.18a)}{=} & \mathrm{Ad}_{fs}^{-1}(e^{-1}\mathrm{d}e) \;+\; s^{-1}\mathrm{d}s\,. \end{array} \tag{2.20}$$

Note that $e^{-1}\mathrm{d}e$ and $s^{-1}\mathrm{d}s$ take value in $\mathfrak{e}$ and $\mathfrak{f}_\infty$, respectively. Furthermore, (B.6d) and $fs \in \exp(\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}})$ imply that

$$e^{-1}\mathrm{d}e \;=\; 0 \quad \text{if and only if} \quad \Big(\mathrm{Ad}_{fs}^{-1}(e^{-1}\mathrm{d}e)\Big)^{p,q} \;=\; 0$$

for all $q > 0$ and $p + q < 0$. At the same time (B.6d), (2.19) and (2.20) imply that

$$0 \;=\; (\xi^{-1}\mathrm{d}\xi)^{p,q} \;=\; \Big(\mathrm{Ad}_{fs}^{-1}(e^{-1}\mathrm{d}e)\Big)^{p,q}$$

for all $q > 0$ and $p + q < 0$. The desired (2.18b) now follows, completing the proof of Proposition 2.10.

2.3. **Extension to proper maps.** Along each strata Z_I^* there is a well-defined Γ–congruence class $[W^I]$ of weight filtrations. Let

$$Z_W \;=\; \bigcup_{[W^I]=[W]} Z_I^*$$

be the union of those strata with the same "weight class." The intersection $Z_I \cap Z_W$ is the *weight-closure* of Z_I^*. There is a subset $I_W \supset I$ with the property that

$$Z_I \,\cap\, Z_W \;=\; \bigcup_{I \subset J \subset I_W} Z_J^*$$

(Corollary C.10). The maps Φ_I^0 and Φ_I^1 in the tower (2.1) extend to the weight-closure (Lemma C.14); in particular, we have a commutative diagram

$$\begin{tikzcd} Z_I^* \arrow[r, hook] & Z_I \cap Z_W \arrow[r, "\Phi_I^1"'] \arrow[rr, bend left, "\Phi_I^0"] & \Gamma_I\backslash D_I^1 \arrow[r, two heads] & \Gamma_I\backslash D_I^0\,. \end{tikzcd}$$

These two extensions to $Z_I \cap Z_W$ are the restrictions of well-defined proper maps on Z_W that are defined as follows.

The inclusions $D_I \hookrightarrow D_W$ and $\Gamma_I \subset \Gamma_W$ induce

$$\Gamma_I\backslash D_I^a \;\to\; \Gamma_W\backslash D_W^a\,.$$

The maps Φ^0_W and Φ^1_W defined by the diagram

$$
\begin{tikzcd}
Z_I^* \arrow[r, hookrightarrow] \arrow[d, "\Phi^1_I"] \arrow[dd, bend right=60, "\Phi^0_I"'] & Z_W \arrow[d, "\Phi^1_W"'] \arrow[dd, bend left=60, "\Phi^0_W"] \\
\Gamma_I\backslash D^1_I \arrow[r] \arrow[d, twoheadrightarrow] & \Gamma_W\backslash D^1_W \arrow[d, twoheadrightarrow] \\
\Gamma_I\backslash D^0_I \arrow[r] & \Gamma_W\backslash D^0_W
\end{tikzcd}
\tag{2.21}
$$

are proper and analytic (Lemma C.1). The proper mapping theorem implies that the images

$$\wp^0_W \;=\; \Phi^0_W(Z_W) \quad \text{and} \quad \wp^1_W \;=\; \Phi^1_W(Z_W)$$

are complex analytic spaces.

Remark 2.22. The fibres A of Φ^0_W are compact analytic subvarieties of $\overline{B}$. And given $Z_I^* \subset Z_W$, the intersection $A \cap Z_I^*$ is the fibre A^* of Question 1.12.

Remark 2.23. In general, the maps Φ^a_I do not extend when $a \geq 3$. (The case $a = 2$ is subtle, cf. §C.4.1).

2.4. **Two topological completions.** Set

$$\overline{\wp}^0 \;=\; \bigcup_W \wp^0_W \quad \text{and} \quad \overline{\wp}^1 \;=\; \bigcup_W \wp^1_W$$

(with the finite unions taken over a single representative $W \in [W]$). Define maps

$$
\begin{tikzcd}
\overline{B} \arrow[r, "\Phi^1"'] \arrow[rr, bend left, "\Phi^0"] & \overline{\wp}^1 \arrow[r, twoheadrightarrow] & \overline{\wp}^0
\end{tikzcd}
\tag{2.24}
$$

by specifying $\Phi^0\big|_{Z_W} = \Phi^0_W$ and $\Phi^1\big|_{Z_W} = \Phi^1_W$.

Let $\alpha = 1, 2$. Fix a Riemannian metric on on $\overline{B}$. Since the fibres of Φ^α are compact, there is an induced metric on $\overline{\wp}^\alpha$. Endow $\overline{\wp}^\alpha$ with the metric topology.

Proposition 2.25. *The topology on $\overline{\wp}^\alpha$ is Hausdorff. The induced subspace topology on $\wp^\alpha_W = \Phi^\alpha_W(Z_W) \subset \overline{\wp}^\alpha$ coincides with the natural topology on $\wp_W$ as a complex analytic space. The map $\Phi^\alpha : \overline{B} \to \overline{\wp}^\alpha$ is continuous and proper.*

Remark 2.26. The completion $\Phi^0 : \overline{B} \to \overline{\wp}^0$ was introduced in [GGLR20]. It encodes the variations of limiting mixed Hodge structures *modulo extension data* along the strata. This is the sense in which $\overline{\wp}^0$ is a *minimal Hodge theoretic compactification*. In the classical case that D is Hermitian and Γ is arithmetic, $\overline{\wp}^0$ is the closure of $\wp$ in the Satake-Baily-Borel compactification of $\Gamma\backslash D$.

Proof. It is clear that that the induced subspace topology coincides with the natural topology on $\wp^\alpha_W$. The topology on $\overline{\wp}^\alpha$ is Hausdorff if an only if the map Φ^α is continuous. In this case, the map is necessarily proper. So if suffices to establish the continuity of Φ^α.

Suppose that $b_i \in \overline{B}$ is a sequence of points converging to $b_\infty \in \overline{B}$. Let A_i and A_∞ be the fibres of Φ^α through b_i and b_∞, respectively. Now let $b_i' \in A_i$. Since $\overline{B}$ is compact,

$\{b'_i\}$ contains a convergent subsequence; abusing notation, let $\{b'_i\}$ denote that convergent subsequence with limit b'_∞. The essential point is to prove that

$$b'_\infty \;=\; \lim_{i\to\infty} b'_i \;\in\; A_\infty\,. \tag{2.27}$$

Informally this says

$$\lim_{i\to\infty} A_i \;\subset\; A_\infty\,.$$

The local analog of this assertion is Lemma C.4. The "globalization" will follow from a certain finiteness result for Siegel domains.

First assume that both sequences $\{b_i\}$ and $\{b'_i\}$ are contained in B. Fix two coordinate charts $\overline{\mathcal{U}}$ and $\overline{\mathcal{U}}'$ centered at b_∞ and b'_∞ respectively, and local lifts $\widetilde{\Phi}(t,w)$ and $\widetilde{\Phi}'(t,w)$. Without loss of generality, $b_i \in \mathcal{U}$ and $b'_i \in \mathcal{U}'$. Since $b_i, b'_i \in A_i$, there exists $\gamma_i \in \Gamma$ so that $\widetilde{\Phi}'(b'_i) = \gamma_i \cdot \widetilde{\Phi}(b_i)$. Shrinking $\overline{\mathcal{U}}$ if necessary, there exists a finite union $\mathfrak{D} \subset D$ of Siegel sets so that $\widetilde{\Phi}(\widetilde{\mathcal{U}}) \subset \mathfrak{D}$. (In the case of one-variable degenerations this is a corollary of Schmid's SL(2) orbit theorem [Sch73, (5.26)]. In the general case, this is [BKT18, Theorem 1.5], and is key to the Bakker–Klingler–Tsimerman result that period maps are $\mathbb{R}_{\mathrm{an,exp}}$–definable.) Likewise, we have a finite union $\mathfrak{D}' \subset D$ of Siegel sets so that $\widetilde{\Phi}'(\widetilde{\mathcal{U}}') \subset \mathfrak{D}'$. It follows that there are only finitely many distinct γ_i. Restricting to a subsequence with all $\gamma_i = \gamma$ equal, we have $\widetilde{\Phi}'(b'_i) = \gamma \cdot \widetilde{\Phi}(b_i)$. Since we may replace the local lift $\widetilde{\Phi}'$ with $\gamma^{-1}\widetilde{\Phi}'$, this forces b_∞ and b'_∞ to lie in the same Φ^α–fibre. This establishes the desired (2.27) in the case that $\{b_i\}$ and $\{b'_i\}$ are contained in B.

For the general case, we may assume without loss of generality that $\{b_i\} \subset Z_I^*$ and $\{b'_i\} \subset Z_{I'}^*$ with $W^I = W^{I'}$. We leave it an exercise for the reader to verify that Lemma C.4 allows us to modify the argument above to treat the general case. □

Remark 2.28. Recently Bakker–Brunebarbe–Tsimerman have applied the o-minimal structures of model theory to prove the long standing conjecture that the image $\wp = \Phi(B)$ of the period map is quasi-projective [BBT18]. In particular, they show that the (*augmented*) *Hodge line bundle*

$$\Lambda \;=\; \det(\mathcal{F}^n) \otimes \det(\mathcal{F}^{n-1}) \otimes \cdots \otimes \det(\mathcal{F}^{\lceil (n+1)/2 \rceil})$$

is free over B, and that $\wp = \mathrm{Proj}\left(\oplus_d H^0(B, d\Lambda)\right)$. Note however that this result does not suffice to establish the existence of a *completion* of the period map: if $\overline{\wp}^{\mathrm{BBT}}$ is the projective closure, it does *not* a priori follow that there is an extension $\overline{B} \to \overline{\wp}^{\mathrm{BBT}}$ of Φ. (And one wants extensions in order to apply Hodge theory to study moduli spaces and their compactifications [GGR21b].) What is missing is to show that the extended Hodge line bundle is free over $\overline{B}$. This is conjectured to be the case in [GGLR20], and proven for $\dim \wp = 1, 2$.

Remark 2.29. In contrast, the topological space $\overline{\wp}^1$ will not admit a compatible complex analytic structure: the fibre dimensions of $\overline{\wp}^1 \to \overline{\wp}^0$ may *drop* on proper subvarieties. For example, if the variation of limiting mixed Hodge structures is Hodge–Tate over Z_I^*, then it is Hodge–Tate over Z_I and the fibres of $\overline{\wp}^1 \to \overline{\wp}^0$ over $\Phi^0(Z_I)$ are *finite*.

2.5. **A "Stein factorization" of Φ^α.** Let

$$\begin{tikzcd} Z_W \arrow[r, "\hat\Phi^\alpha_W"'] \arrow[rr, bend left, "\Phi^\alpha_W"] & \hat\wp^\alpha_W \arrow[r] & \wp^\alpha_W \end{tikzcd} \tag{2.30}$$

be the Stein factorization of Φ^α_W; the fibres of $\hat\Phi^\alpha_W$ are connected, the fibres of $\hat\wp^\alpha_W \to \wp^\alpha_W$ are finite, and $\hat\wp^\alpha_W$ is normal. Set

$$\hat\wp^\alpha \;=\; \bigcup \hat\wp^\alpha_I \,,$$

and define maps

$$\begin{tikzcd} \overline{B} \arrow[r, "\hat\Phi^\alpha"'] \arrow[rr, bend left, "\Phi^\alpha"] & \hat\wp^\alpha \arrow[r] & \overline{\wp}^\alpha \end{tikzcd} \tag{2.31}$$

by specifying that the restriction of (2.31) to Z_W coincides with (2.30).

Corollary 2.32. *Let $\hat A \subset \overline{B}$ be a fibre of $\hat\Phi^\alpha$.* (Equivalently, $\hat A$ is a connected component of a Φ^α–fibre.) *Fix a neighborhood $\hat{\mathcal{O}} \subset \hat\wp^\alpha$ of $\hat\Phi^\alpha(\hat A) \in \hat\wp$. Then $\overline{\mathcal{O}} = (\hat\Phi)^{-1}(\hat{\mathcal{O}}) \subset \overline{B}$ is a neighborhood of $\hat A$ with the property that $\Phi|_{B\cap\overline{\mathcal{O}}}$ is proper.*

3. Neighborhood of a compact fibre

3.1. **Monodromy about the fibre.** Now take the case $\Phi^\alpha = \Phi^0$. Let A^0 be the fibre $\hat A$ of Corollary 2.32. The restriction $\mathcal{V}|_{\mathcal{O}^0} = \tilde{\mathcal{O}}^0 \times_{\pi_1(\mathcal{O}^0)} V$ of the VHS over B to $\mathcal{O}^0 = \overline{\mathcal{O}}^0 \cap B$ induces a period map

$$\Phi_{A^0} : \mathcal{O}^0 \;\to\; \Gamma_{A^0}\backslash D \tag{3.1}$$

with monodromy $\Gamma_{A^0} \subset \Gamma$.

Let (W, F, σ_I) be any LMHS arising along A^0 (as in §B.2.4). Let

$$I(A^0) \;=\; \{i \mid A^0 \cap Z^*_i \neq \emptyset\}\,.$$

By definition of Φ^0, $W = W^I$ is independent of I. Then Corollary C.10 implies that

$$I \;\subset\; I(A^0) \;\subset\; I_W$$

and $W = W^{I(A^0)}$. We have $C_{I(A^0)} \subset P_W$, and $G_{I(A^0)} = \mathrm{Aut}(D^0_{I(A^0)}) = C_{I(A^0)}/C^{-1}_{I(A^0)}$ (§2.1).

Lemma 3.2. *We may choose the neighborhood $\overline{\mathcal{O}}^0$ of Corollary 2.32 so that*

$$\Gamma_{A^0} \;\subset\; G_{I(A^0),\mathbb{Q}} \ltimes P^{-1}_{W,\mathbb{Q}}\,,$$

and the image of Γ_{A^0} under the quotient $G_{I(A^0)} \ltimes P^{-1}_W \twoheadrightarrow G_{I(A^0)}$ is finite and stabilizes $F(\mathrm{Gr}^W)$.

Proof. The weight filtration W is independent of our choice of LMHS (W, F, σ_I) along A^0. So we may choose the neighborhood $\overline{\mathcal{O}}^0$ so that $\Gamma_{A^0} \subset P_{W,\mathbb{Q}}$. Likewise, the Hodge structure $F(\mathrm{Gr}^W) \in D^0_W$ is independent of the choice of LMHS. So we may further assume that Γ_{A^0} fixes $F(\mathrm{Gr}^W)$; equivalently, the discrete quotient $\Gamma_{A^0}/(\Gamma_{A^0} \cap P^{-1}_W)$ stabilizes $F(\mathrm{Gr}^W)$.

Given $N \in \sigma_{I(A^0)}$, Lemma C.24 asserts that $Q_{n+k} = Q(\cdot, N^k \cdot)$ polarizes the Hodge structure $F(\mathrm{Prim}^N_{n+k}) \subset F(\mathrm{Gr}^W_{n+k})$. So we may also choose the neighborhood $\overline{\mathcal{O}}^0$ so that Prim^N_{k+k} and Q_{n+k} are invariant under Γ_{A^0}. This implies $\Gamma_{A^0}/(\Gamma_{A^0} \cap P^{-1}_W) \subset G_{I(A^0)}$. And since Γ_{A^0} stabilizes the Hodge filtration $F(\mathrm{Gr}^W_{n+k})$, this forces the discrete $\Gamma_{A^0}/(\Gamma_{A^0} \cap P^{-1}_W)$ to be finite. □

Lemma 3.2 can be further strengthened. Without loss of generality $I = \{1, \ldots, k\}$. Let $\mathrm{Stab}_{G_\mathbb{C}}(F_\infty)$ denote the stabilizer in $G_\mathbb{C}$ of the reduced period limit filtration $F_\infty \in \check{D}$ defined by

$$F_\infty \;=\; \lim_{y\to\infty} \exp(\mathbf{i}yN) \cdot F\,.$$

This filtration is independent of the choice of $N \in \sigma_{I_W}$, and is related to the Deligne splitting (§B.3) by

$$F^q_\infty \;=\; \bigoplus_{b \le n-q} V^{a,b}_{W,F}\,.$$

Lemma 3.3. *We may choose the neighborhood $\overline{\mathcal{O}}^0$ so that $\Gamma_{A^0} \subset P_{W,\mathbb{Q}} \cap \mathrm{Stab}_{G_\mathbb{C}}(F_\infty)$.*

Proof. The IPR forces a very close relationship between Φ^0 and the reduced period limit map (Proposition 2.10): the reduced period limit is locally constant on Φ^0–fibres. On strata $A^0 \cap Z^*_I \cap \overline{\mathcal{U}}$ this implies Corollary 2.16. Over $A^0 \cap \overline{\mathcal{U}}$ this implies that the map $\widetilde{\Phi}_W$ of (C.22) takes value in $\exp(\mathbb{C}\sigma_{I_W})\backslash(\mathrm{Stab}_{G_\mathbb{C}}(F_\infty) \cap P^{-1}_{W,\mathbb{C}}) \cdot F \subset \exp(\mathbb{C}\sigma_{I_W})\backslash\delta_W$. (We have $\exp(\mathbb{C}\sigma_{I_W}) \subset \mathrm{Stab}_{G_\mathbb{C}}(F_\infty) \cap P^{-1}_{W,\mathbb{C}}$.) □

Lemma 3.3 has some strong consequences for Φ_{A^0}. Consider the Schubert cell (§B.4)

$$\mathcal{S} \;=\; \exp(\mathfrak{f}^\perp) \cdot F \;=\; \left\{ \tilde{F} \in \check{D} \mid \dim(\tilde{F}^a \cap \overline{F^b_\infty}) = \dim(F^a \cap \overline{F^b_\infty}),\ \forall\, a, b \right\}. \tag{3.4}$$

Lemma 3.5. *The action of Γ_{A^0} on $\check{D}$ preserves the cell $\mathcal{S} \subset \check{D}$.*

Corollary 3.6. *Every local lift of Φ_{A^0} over a chart $\overline{\mathcal{U}}$ centered at a point $b \in A^0$ takes value in $\mathcal{S}$. In particular, the lift of Φ_{A^0} to the universal cover $\tilde{\mathcal{O}}^0 \to \mathcal{O}^0$ takes value in the Schubert cell:*

$$\begin{array}{ccc} \tilde{\mathcal{O}}^0 & \xrightarrow{\widetilde{\Phi}_{A^0}} & \mathcal{S} \cap D \\ \big\downarrow & & \big\downarrow \\ \mathcal{O}^0 & \xrightarrow{\Phi_{A^0}} & \Gamma_{A^0}\backslash D\,. \end{array}$$

First proof of Lemma 3.5. Since Γ_{A^0} is both real and stabilizes F_∞, it follows that Γ_{A^0} stabilizes $\overline{F_\infty}$; that is,

$$\Gamma_{A^0} \;\subset\; \mathrm{Stab}_{G_\mathbb{C}}(F_\infty, \overline{F}_\infty) \;=\; \mathrm{Stab}_{G_\mathbb{C}}(F_\infty) \cap \mathrm{Stab}_{G_\mathbb{C}}(\overline{F}_\infty)\,. \tag{3.7}$$

Since $\mathcal{S}$ is by definition those filtrations $\tilde{F} \in \check{D}$ intersecting $\overline{F_\infty}$ generically, it follows that $\mathcal{S}$ is preserved by Γ_{A^0}. □

It is instructive to consider a second proof.

Second proof of Lemma 3.5. The essential point is to note that the Lie algebra of $\mathrm{Stab}_{G_\mathbb{C}}(F_\infty, \overline{F}_\infty)$ is

$$\mathfrak{m} \;=\; \bigoplus_{p,q\leq 0} \mathfrak{g}^{p,q}_{W,F}\,. \tag{3.8}$$

It follows from (B.6d) that $\mathfrak{f}^\perp + \mathfrak{m}$ is a nilpotent subalgebra of $\mathfrak{g}_\mathbb{C}$, and $\mathfrak{f}^\perp$ is an ideal of $\mathfrak{f}^\perp + \mathfrak{m}$. This implies that the action of $\mathrm{Stab}_{G_\mathbb{C}}(F_\infty, \overline{F}_\infty)$ on $\check{D}$ preserves $\mathcal{S}$. □

Remark 3.9 (Assume unipotent monodromy about A^0). It will be convenient at times to assume that the action of Γ_{A^0} on Gr^W is trivial; equivalently, the monodromy group

$$\Gamma_{A^0} \;\subset\; P^{-1}_{W,\mathbb{Q}} \tag{3.10}$$

is unipotent. This is equivalent to the hypothesis that Γ_{A^0} is neat; in particular, (3.10) holds whenever Γ is neat. When (3.10) holds, the fact that $P^{-1}_{W,\mathbb{C}}$ is unipotent implies that there is a well-defined logarithm

$$\log\Gamma_{A^0} \;\subset\; \mathfrak{m}^{-1} \;=\; \bigoplus_{\substack{p,\,q\,\leq\,0\\(p,\,q)\,\neq\,(0,0)}} \mathfrak{g}^{p,q}_{W,F}\,, \tag{3.11}$$

and the map $\Gamma_{A^0} \to \log\Gamma_{A^0}$ is a bijection.

3.2. **Trivializations about the fibre.** Recall that Deligne's extension $\mathcal{F}^p_\mathrm{e} \subset \mathcal{V}_\mathrm{e} \to \overline{B}$ of the Hodge vector bundle (1.1a) is trivial over $\overline{\mathcal{U}}$ [Del97]. Together (3.11) and Corollary 3.6 make it possible to trivialize $\det(\mathcal{F}^p_\mathrm{e})$ in the neighborhood $\overline{\mathcal{O}}^0$ of the fibre.

Theorem 3.12. *If* (3.10) *holds, then the bundles* $\det(\mathcal{F}^p_\mathrm{e})$ *are trivial over* $\overline{\mathcal{O}}^0$*.*

Corollary 3.13. *If* (3.10) *holds, then extended Hodge line bundle*

$$\Lambda_\mathrm{e} \;=\; \det(\mathcal{F}^n_\mathrm{e}) \otimes \det(\mathcal{F}^{n-1}_\mathrm{e}) \otimes \cdots \otimes \det(\mathcal{F}^{\lceil (n+1)/2\rceil}_\mathrm{e})\,.$$

is trivial over $\overline{\mathcal{O}}^0$*.*

Theorem 3.14. *Assume* (3.10) *holds. Let* Z_W *be the weight strata containing* A^0*. The induced Hodge bundles* $F^p(\mathrm{Gr}^W_a)$ *on the associated graded* $\mathrm{Gr}^W_a = W_a/W_{a-1}$ *are trivial over* $\overline{\mathcal{O}}^0 \cap Z_W$*.*

The theorems are proved in §§3.2.1–3.2.4.

3.2.1. *Preliminaries.* The obvious map $\exp(\mathfrak{f}^\perp) \to \exp(\mathfrak{f}^\perp)\cdot F = \mathcal{S}$ is a biholomorphism. So Corollary 3.6 implies that there is a uniquely determined holomorphic

$$g : \tilde{\mathcal{O}}^0 \;\to\; \exp(\mathfrak{f}^\perp)$$

so that

$$\tilde\Phi(\zeta) \;=\; g(\zeta)\cdot F\,.$$

We have

$$\tilde\Phi(\zeta\cdot\gamma) \;=\; \gamma^{-1}\cdot\tilde\Phi(\zeta)\,;$$

equivalently,

$$g(\zeta\cdot\gamma)\cdot F \;=\; \gamma^{-1} g(\zeta)\cdot F\,.$$

Remark 3.15. Were it the case that $\Gamma_{A^0} \subset \exp(\mathfrak{f}^\perp)$, then we would have $g(\zeta\cdot\gamma) = \gamma^{-1}g(\zeta)$. This would imply that the function $\tilde{\mathcal{O}}^0 \to V$ sending $\zeta \mapsto g(\zeta)v$ defines a section of $\mathcal{V} \to \mathcal{O}^0$, and we would have a framing of $\mathcal{F}^p_{\mathrm{e}}$ over $\overline{\mathcal{O}}^0$.

However, while γ^{-1} preserves the Schubert cell $\mathcal{S}$, it need not be an element of $\exp(\mathfrak{f}^\perp)$. So we can not assert that $g(\zeta\cdot\gamma) = \gamma^{-1}g(\zeta)$. In order to determine $g(\zeta\cdot\gamma)$ we must first factor the monodromy.

3.2.2. *Factorization of monodromy.* In order to explicitly describe the action of $\gamma \in \Gamma_{A^0}$ on $\delta_W \subset \mathcal{S}$ we first need to factor the monodromy group. Any element $\gamma \in \mathrm{Stab}_{G_\mathbb{C}}(F_\infty, \overline{F}_\infty)$ may be uniquely factored as

$$\begin{array}{rcl} \gamma & = & \alpha\,\beta\,, \qquad \text{with} \\ \beta & \in & \mathrm{Stab}_{G_\mathbb{C}}(F_\infty, \overline{F}_\infty, F) \qquad \text{and} \\ \alpha & \in & \exp(\mathfrak{m}\cap\mathfrak{f}^\perp) \ = \ \exp(\mathfrak{f}^\perp) \cap \mathrm{Stab}_{G_\mathbb{C}}(F_\infty, \overline{F}_\infty)\,. \end{array}$$

(The proof of [ČS09, Theorem 3.1.3] applies here.) Then the action of γ on $\xi\cdot F \in \mathcal{S}$ is given by

$$\gamma\xi\cdot F \ = \ \alpha\beta\xi\cdot F \ = \ \alpha\,\beta\xi\beta^{-1}\,\beta\cdot F \ = \ \alpha\,(\beta\xi\beta^{-1})\cdot F\,. \tag{3.16}$$

Note that $\mathfrak{m}\cap\mathfrak{f}^\perp = \mathfrak{m}^{-1}\cap\mathfrak{f}^\perp$. The fact that $\mathfrak{m}^{-1}$ is nilpotent implies that the exponential map $\mathfrak{m}^{-1} \to \exp(\mathfrak{m}^{-1})$ is a biholomorphism. So there exists a unique $a \in \mathfrak{m}\cap\mathfrak{f}^\perp$ such that

$$\alpha \ = \ e^a\,.$$

Likewise β admits a unique factorization as

$$\beta \ = \ \beta_0\, e^b\,,$$

with the adjoint action of $\beta_0 \in G_\mathbb{C}$ on $\mathfrak{g}_\mathbb{C}$ preserving each $\mathfrak{g}^{p,q}_{W,F}$ and $b \in \mathfrak{m}^{-1}\cap\mathfrak{f}$ (again by [ČS09, Theorem 3.1.3]).

We have $\gamma \in P^{-1}_{W,\mathbb{C}}$ if and only if $\beta_0 = 1$ is the identity; equivalently β is unipotent. In this case there exists a unique $c \in \mathfrak{m}^{-1}$ so that $\gamma = e^c$.

3.2.3. *Proof of Theorem 3.12.* While we do not expect to have a framing of $\mathcal{F}^p_{\mathrm{e}}$ over $\overline{\mathcal{O}}^0$ (Remark 3.15), we do have a framing of $\det(\mathcal{F}^p_{\mathrm{e}})$ over $\overline{\mathcal{O}}^0$ when (3.10) holds. This is a consequence of the factorization in §3.2.2. We have

$$\gamma^{-1} \ = \ \alpha\beta\,,$$

with $\alpha \in \exp(\mathfrak{f}^\perp)$ and β unipotent and stabilizing F, and

$$\beta\exp(\mathfrak{f}^\perp)\beta^{-1} \ = \ \exp(\mathfrak{f}^\perp)\,.$$

This implies that

$$g(\zeta\cdot\gamma) \ = \ \alpha\,\beta g(\zeta)\beta^{-1}\,.$$

Since β stabilizes F, it preserves the line $\det(F^p) \subset \bigwedge^{d_k} V$, $d_p = \dim F^p$. Since β is unipotent (§3.2.2) it acts trivially on the line. Fix a nonzero $\mu \in \det(F^p)$. Then $\beta\cdot\mu \ = \ \mu$. So the function

$$f : \tilde{\mathcal{O}}^0 \ \to \ U\,, \quad f(\zeta) \ = \ g(\zeta)\cdot\lambda$$

satisfies

$$\begin{aligned} f(\zeta\cdot\gamma) &= g(\zeta\cdot\gamma)\lambda \;=\; \alpha\,\beta g(\zeta)\beta^{-1}\cdot\lambda \\ &= \alpha\,\beta g(\zeta)\cdot\lambda \;=\; \gamma^{-1}\cdot f(\zeta)\,, \end{aligned}$$

and so defines a section of $\det(\mathcal{F}^p)\to\mathcal{O}^0$. Now this section locally extends across infinity (essentially by the same arguments as in [Del97]), and so extends to a framing of $\det(\mathcal{F}^p_e)$ over $\overline{\mathcal{O}}^0$. □

3.2.4. *Proof of Theorem 3.14.* The fact that $\Gamma_{A^0}\subset P^{-1}_{W,\mathbb{C}}$ (Remark 3.9) implies that Γ_{A^0} acts trivially on Gr^W_a. Arguing as in §3.2.3, we conclude that $\mathcal{F}^p_e(\mathrm{Gr}^W_a)$ is trivial over $\overline{\mathcal{O}}^0\cap Z_W$. □

3.3. **Divisors at infinity.** The purpose of this section is to use Lemma 3.5 and Corollary 3.6 to construct explicit sections $s_M\in H^0(\overline{\mathcal{O}}^0,L_M)$ of certain line bundles $L_M\to\overline{\mathcal{O}}^0$. We will see that the sections have divisor

$$(s_M) \;=\; \sum\kappa(M,N_i)\,(Z_i\cap\overline{\mathcal{O}}^0) \tag{3.17a}$$

for some integers $\kappa(M,N_i)$. In particular,

$$L_M \;=\; \sum\kappa(M,N_i)\;[Z_i]|_{\overline{\mathcal{O}}^0}\,. \tag{3.17b}$$

3.3.1. *Line bundles over* $\Gamma_{A^0}\backslash\mathcal{S}$. Recall the Schubert cell $\mathcal{S}$ of (3.4) and Lemma 3.5. We will construct line bundles over $\Gamma_{A^0}\backslash\mathcal{S}$ from the data:

- The left-action of Γ_{A^0} on $\mathcal{S}$ induces a right-action on the functions $f:\mathcal{S}\to\mathbb{C}$ by the prescription $(f\cdot\gamma)(\xi)=f(\gamma\cdot\xi)$.
- Let
$$\mathfrak{f}^1 \;=\; F^1(\mathfrak{g}_\mathbb{C}) \;=\; \bigoplus_{p\ge 1}\mathfrak{g}^{p,q}_{W,F}$$
be the nilpotent radical of the Lie algebra $\mathfrak{f}$ stabilizing F. The relation (B.6c) implies that the bilinear pairing
$$\kappa:\mathfrak{f}^1\times\mathfrak{f}^\perp \;\to\; \mathbb{C}$$
is nondegenerate.

Recall the biholomorphism $X:\mathcal{S}\xrightarrow{\simeq}\mathfrak{f}^\perp$ of (B.9). Given $M\in\mathfrak{f}^1$, define

$$f_M:\mathcal{S}\;\to\;\mathbb{C}\quad\text{by}\quad f_M=\exp 2\pi\mathbf{i}\,\kappa(M,X)\,.$$

Given $\gamma\in\Gamma_{A^0}$, define a holomorphic function $e^M_\gamma:\mathcal{S}\to\mathbb{C}^*$ by

$$e^M_\gamma \;=\; \frac{f_M\cdot\gamma}{f_M} \;=\; \frac{\exp 2\pi\mathbf{i}\,\kappa(M,X\cdot\gamma)}{\exp 2\pi\mathbf{i}\,\kappa(M,X)}\,. \tag{3.18}$$

Then

$$e^M_{\gamma_1\gamma_2}(\xi) \;=\; e^M_{\gamma_1}(\gamma_2\cdot\xi)\,e^M_{\gamma_2}(\xi)\,.$$

so that

$$\gamma\cdot(z,\xi) \;=\; (ze^M_\gamma(\xi),\gamma\cdot\xi)$$

defines a left action of Γ_{A^0} on $\mathbb{C} \times \mathcal{S}$. Let

$$\begin{array}{l} \mathcal{L}_M \;=\!=\; (\mathbb{C} \times \mathcal{S})/\sim \\ \downarrow \\ \Gamma_{A^0} \backslash \mathcal{S} \end{array}$$

be the associated line bundle over the quotient. Then f_M induces a section s_M

$$\begin{tikzcd} \mathcal{L}_M \arrow[d, shift left] \\ \Gamma_{A^0} \backslash \mathcal{S} \,. \arrow[u, bend left, "s_M"] \end{tikzcd}$$

3.3.2. *Line bundles over* $\mathcal{O}^0$. Pull the line bundle $\mathcal{L}_M$ back to the (punctured) neighborhood $\mathcal{O}^0$

$$\begin{tikzcd} (\Phi_{A^0})^* \mathcal{L}_M \arrow[d] & \mathcal{L}_M \arrow[d] \\ \mathcal{O}^0 \arrow[r, "\Phi_{A^0}"] \arrow[u, bend left, "\Phi^*_{A^0}(s_M)"] & \Gamma_{A^0} \backslash \mathcal{S} \,. \arrow[u, bend right, "s_M"'] \end{tikzcd}$$

The local expression for the pulled-back section $\Phi^*_{A^0}(s_M)$ is

$$\tau_M(t,w) \;=\; f_M \circ \widetilde{\Phi}_{A^0}(t,w) \;=\; \exp 2\pi \mathbf{i} \kappa(M, X \circ \widetilde{\Phi}_{A^0}(t,w)) \,. \tag{3.19}$$

If $M \in \mathfrak{g}^{1,\bullet}_{W,F}$ and $\kappa(M, N_i) \in \mathbb{Z}$ for all $i \in I_W$, then (B.10) implies

$$\tau_M(t,w) \;=\; \exp 2\pi \mathbf{i} \kappa(M, \tilde{X}(t,w)) \prod t_i^{\kappa(M,N_i)} \tag{3.20}$$

is a well-defined holomorphic function on $\mathcal{U}$. If in addition $0 \le \kappa(M, N_i) \in \mathbb{Z}$ for all $i \in I_W$, then $\tau_M(t,w)$ is holomorphic on $\overline{\mathcal{U}}$. Additionally, $\tau_M(t,w)$ vanishes along $Z^*_I \cap \overline{\mathcal{U}}$ if and only if $\kappa(M, N_i) > 0$ for some $i \in I$.

3.3.3. *Extension to* $\overline{\mathcal{O}}^0$. Define

$$\mathbf{N}^* \;=\; \{M \in \mathfrak{g}^{1,\le 1}_{W,F} \mid \kappa(M, N_i) \in \mathbb{Z} \,,\ \forall\, i \in I_W\} \,. \tag{3.21}$$

Lemma 3.22. *If $M \in \mathbf{N}^*$, then the line bundle $(\Phi_{A^0})^* \mathcal{L}_M$ is the restriction to $\mathcal{O}^0$ of a holomorphic vector bundle $L_M \to \overline{\mathcal{O}}^0$. And $(\Phi_{A^0})^* s_M$ extends to a section of L_M* (which, in a minor abuse of notation, we also denote s_M).

$$\begin{tikzcd} L_M \arrow[d] & (\Phi_{A^0})^* \mathcal{L}_M \arrow[d] & \mathcal{L}_M \arrow[d] \\ \overline{\mathcal{O}}^0 \arrow[u, bend left, "s_M"] & \tilde{\mathcal{O}}^0 \arrow[l, hook'] \arrow[r, "\Phi_{A^0}"'] \arrow[u, bend right, "(\Phi_{A^0})^* s_M"'] & \Gamma_{A^0} \backslash \mathcal{S} \,. \arrow[u, bend right, "s_M"'] \end{tikzcd}$$

The desired (3.17) now follows from (3.19) and (3.20).

Proof. Set

$$\tilde{X}_\gamma(t,w) \;=\; (X\cdot\gamma)\circ\widetilde{\Phi}_{A^0}(t,w) \;-\; \textstyle\sum \ell(t_i)\,N_i$$

Again, the key point is that it follows from (B.6d), (B.10), Lemma 3.2, (3.7), (3.8) and §3.2.2 that the component $\tilde{X}_\gamma^{-1,q}(t,w)$ taking value in $\mathfrak{g}^{-1,q}_{W,F}$ is a well-defined holomorphic function on $\overline{\mathcal{U}}$, so long as $q \ge -1$. So $\kappa(M,\tilde{X}_\gamma(t,w))$ is a holomorphic function on $\overline{\mathcal{U}}$, so long as $M \in \mathbf{N}^*$. Then

$$\begin{aligned}(\widetilde{\Phi}_{A^0})^*(f_M\cdot\gamma)(t,w) &= (f_M\cdot\gamma)\circ\widetilde{\Phi}(t,w)\\ &= \exp 2\pi\mathbf{i}\kappa(M,\tilde{X}_\gamma(t,w))\,\prod t_i^{\kappa(M,N_i)}\,,\end{aligned}$$

and

$$\begin{aligned}(\widetilde{\Phi}_{A^0})^*(e^M_\gamma)(t,w) &= \frac{(\widetilde{\Phi}_{A^0})^*(f_M\cdot\gamma)(t,w)}{(\widetilde{\Phi}_{A^0})^*(f_M)(t,w)}\\ &= \frac{\exp 2\pi\mathbf{i}\,\kappa(M,\tilde{X}_\gamma(t,w))}{\exp 2\pi\mathbf{i}\,\kappa(M,\tilde{X}(t,w))}\end{aligned} \tag{3.23}$$

is a well-defined holomorphic function on $\overline{\mathcal{U}}$. □

4. Level one extension data

In this section we restrict to the punctured neighborhood $\mathcal{O}^0 = B\cap\overline{\mathcal{O}}^0$ of $A^0 \subset Z_W$, and work with the period map $\Phi_{A^0} : \mathcal{O}^0 \to \Gamma_{A^0}\backslash D$ of (3.1). Fix a limiting mixed Hodge structure (W,F,σ_I) of the period map along $A^0\cap Z_W$. To this MHS we have associated two sets

$$\Gamma_{A^0,I}\backslash\delta_I \quad\text{and}\quad \Gamma_{A^0}\backslash\delta_W$$

of extension data (§2.1). The goal of this section is to study the level one extension data (Definitions 2.4 and 2.8) and the resulting implications for the fibre A^0.

The *level one extension data* of the MHS (W,F) is $\Gamma_{A^0}\backslash\delta^1_W$ (Definition 2.4). The σ_I–*polarized level one extension data* is $\Gamma_{A^0,I}\backslash\delta^1_I$ (Definition 2.8). The diagram (2.21) induces *level one extension data maps*

$$\begin{array}{ccc} A^0\cap Z_I & \xrightarrow{\Phi^1_I} & \Gamma_{A^0,I}\backslash\delta_I\\ \big\downarrow & & \big\downarrow\\ A^0 & \xrightarrow{\Phi^1_W} & \Gamma_{A^0}\backslash\delta_W\,. \end{array} \tag{4.1}$$

Note that both $\delta_I \subset \delta_W$ are subsets of the Schubert cell $\mathcal{S}$ of (3.4). It follows that the quotients $\Gamma_{A^0,I}\backslash\delta_I$ and $\Gamma_{A^0}\backslash\delta_W$ inherit the line bundles $\mathcal{L}_M$ of Lemma 3.22,

$$\begin{array}{ccc} \mathcal{L}_M & & \mathcal{L}_M\\ \downarrow & & \downarrow\\ \Gamma_{A^0,I}\backslash\delta_I & \longrightarrow & \Gamma_{A^0}\backslash\delta_W\,. \end{array} \tag{4.2}$$

Theorem 4.3. *Set $W = W^A$, and suppose $Z_I^* \subset Z_W$. Assume that the monodromy $\Gamma_{A^0} \subset P^{-1}_{W,\mathbb{Q}}$ is unipotent* (Remark 3.9).

(a) *The bundle* $\pi^1_W : \Gamma_{A^0}\backslash D^1_W \twoheadrightarrow D^0_W$ *admits a subbundle*

$$\begin{array}{ccccc} T_W & \hookrightarrow & \mathcal{T}_W & \subset & \Gamma_W\backslash D^1_W \\ & & \Big\downarrow\!\!\downarrow{\scriptstyle \pi^1_W} & & \\ & & \Gamma_W\backslash D^0_W & & \end{array}$$

that is fibered by compact tori T_W. *The restriction* $\Phi^1\big|_{A^0}$ *takes value in* $\mathcal{T}_W$.

(b) *The bundle* $\pi^1_I : \Gamma_{A^0,I}\backslash D^1_I \twoheadrightarrow D^0_I$ *admits a subbundle*

$$\begin{array}{ccccc} J_I & \hookrightarrow & \mathcal{J}_I & \subset & \Gamma_I\backslash D^1_I \\ & & \Big\downarrow\!\!\downarrow{\scriptstyle \pi^1_I} & & \\ & & \Gamma_I\backslash D^0_I & & \end{array}$$

that is fibered by abelian varieties J_I. *The restriction* $\Phi^1\big|_{A^0\cap Z_I}$ *takes value in* $\mathcal{J}_I$.

(c) *If* $M \in \mathfrak{g}^{1,1}_{W,F}$, *then the line bundles* (4.2) *descend*

$$\begin{array}{ccc} \mathcal{L}_M & & \mathcal{L}_M \\ \downarrow & & \downarrow \\ \Gamma_{A^0,I}\backslash\delta^1_I & \longrightarrow & \Gamma_{A^0}\backslash\delta^1_W \end{array}$$

to both $\Gamma_{A^0,I}\backslash\delta^1_I$ *and* $\Gamma_{A^0}\backslash\delta^1_W$. *In the case that* $M \in \mathbf{N}^* \cap \mathfrak{g}^{1,1}_{W,F}$, *we have*

$$L_M|_{A^0} \;=\; (\Phi^1\big|_{A^0})^*(\mathcal{L}_M) \quad \text{and} \quad L_M|_{A^0\cap Z_I} \;=\; (\Phi^1\big|_{A^0\cap Z_I})^*(\mathcal{L}_M)\,. \tag{4.4}$$

(d) *There is a nonempty subset* $\mathbf{N}^{\mathfrak{sl}_2}_I \subset \mathbf{N}^* \cap \mathfrak{g}^{1,1}_{W,F}$ *with the property that the abelian variety* J_I *is polarized by the* $\mathcal{L}^*_M$ *with* $M \in \mathbf{N}^{\mathfrak{sl}_2}_I$.

(e) *The set* $\mathbf{N}^{\mathfrak{sl}_2,+}_I \;=\; \{M \in \mathbf{N}^{\mathfrak{sl}_2}_I \mid \kappa(M,N_i) > 0\,,\ \forall\ i \in I\}$ *is nonempty. Indeed the dimension of the real span is* $\dim\sigma_I$.

Theorem 4.3 and (3.17) yield

$$(\Phi^1\big|_{A^0})^*(\mathcal{L}_M) \;=\; \sum \kappa(M,N_i)[Z_i]|_{A^0} \;=\; \sum \kappa(M,N_i)\,\mathcal{N}_{Z_i/\overline{B}}\Big|_{A^0}\,. \tag{4.5}$$

(The sum is over all i such that $Z_i^* \cap A^0 \neq \emptyset$, which is necessarily a subset of I_W.) It follows from Proposition 5.1 that *this is the central geometric information that arises when considering the variation of limiting mixed Hodge structure along* A^0.

Example 4.6. Suppose that $A^0 \subset Z_i^*$ and $N_i \neq 0$. Taking $I = \{i\}$, we may choose $M \in \mathbf{N}^{\mathfrak{sl}_2,+}_i$, so that $\mathcal{L}^*_M \to J_i$ is ample and $\kappa(M,N_i) > 0$. Then $\mathcal{N}^*_{Z_i/\overline{B}}\Big|_{A^0}$ is ample if the differential of $\Phi^1\big|_{A^0}$ is injective.

More generally, we have

Corollary 4.7. *Suppose the differential of* $\Phi^1\big|_{A^0\cap Z_I}$ *is injective and* $M \in \mathbf{N}^{\mathfrak{sl}_2}_I$. *Then the line bundle* $\sum \kappa(M,N_i)\mathcal{N}^*_{Z_i/\overline{B}}\big|_{A^0\cap Z_I}$ *is ample.*

Remark 4.8. The sum in Corollary 4.7 is over those j with $Z_j \cap (A^0 \cap Z_I)$ nonempty. Theorem 4.3(e) asserts that we may choose M so that the integers $\kappa(M, N_j)$ are positive when $j \in I$; we are not able to say the same when $j \notin I$.

The remainder of §4 is occupied with the proof of Theorem 4.3. In outline, the argument is as follows:

- To begin, we review the structure of $\Gamma_{A^0} \backslash \delta^1_W$ and $\Gamma_{A^0,I} \backslash \delta^1_I$ in §4.1. The compact torus $J_I \subset \Gamma_{A^0,I} \backslash \delta^1_I$ is identified in §4.3.
- The action of Γ_{A^0} on $\delta_W \subset \mathcal{S}$ was analyzed in §3.2.2. This action preserves δ_W, and the restricted action is further analyzed in §4.4.
- The line bundle $\mathcal{L}_M \to \Gamma_{A^0} \backslash \delta_W$ descends to $\Gamma_{A^0} \backslash \delta^1_W$ if and only if the functions e^M_γ of (3.18) are constant on the fibres of $\delta_W \twoheadrightarrow \delta^1_W$. In §4.5 it is shown that the bundles parameterized by $M \in \mathfrak{g}^{1,1}_{W,F}$ have this property. If, in addition, $M \in \mathbf{N}^* \cap \mathfrak{g}^{1,1}_{W,F}$ then we also have $L_M|_{A^0}$ (Lemma 3.22). In order to see that (4.4) holds, we must show that the associated systems of multipliers coincide.
- We then restrict to a subset $\mathbf{N}^1 \subset \mathfrak{g}^{1,1}_{W,F} \cap \mathbf{N}^*$ (which may be thought as imposing an integrality condition on M) and compute the Chern forms ω_M in §4.6.
- We restrict to a final subset $\mathbf{N}^{\mathfrak{sl}_2}_I \subset \mathbf{N}^1$ (which may be thought of as a positivity condition) and confirm that $-\omega_M$ is positive on J_I. It then follows that the line bundle $\mathcal{L}^*_M \to J_I$ is ample and J_I is an abelian variety.

4.1. **Lie theoretic description.** The level one extension data has the following structure. First note that $P^{-1}_{W,\mathbb{C}}/P^{-2}_{W,\mathbb{C}}$ is an abelian group. Since the exponential map $\exp : \mathfrak{p}_{W,\mathbb{C}} \to P_{W,\mathbb{C}}$ is a biholomorphism, and

$$\mathfrak{p}^{-a}_{W,\mathbb{C}} \;=\; \bigoplus_{p+q\le -a} \mathfrak{g}^{p,q}_{W,F}\,,$$

we see that there is a canonical identification

$$P^{-1}_{W,\mathbb{C}}/P^{-2}_{W,\mathbb{C}} \;\simeq\; \bigoplus_{p+q=-1} \mathfrak{g}^{p,q}_{W,F}\,.$$

Setting

$$\mathbb{L} \;=\; \bigoplus_{\substack{p+q=-1\\ p<0}} \mathfrak{g}^{p,q}_{W,F}\,,$$

we have

$$P^{-1}_{W,\mathbb{C}}/P^{-2}_{W,\mathbb{C}} \;\simeq\; \mathbb{L} \oplus \overline{\mathbb{L}}\,.$$

Additionally $\mathfrak{p}^{-a}_{W,\mathbb{C}} = (\mathfrak{f} \cap \mathfrak{p}^{-a}_{W,\mathbb{C}}) \oplus (\mathfrak{f}^\perp \cap \mathfrak{p}^{-a}_{W,\mathbb{C}})$, and the map $\mathfrak{f} \cap \mathfrak{p}^{-1}_{W,\mathbb{C}} \to \delta_W$ given by $x \mapsto \exp(x) \cdot F$ is a biholomorphism. It follows that we have a canonical identification

$$P^{-2}_{W,\mathbb{C}} \backslash (P^{-1}_{W,\mathbb{C}} \cdot F) \;=\; \mathbb{L}\,.$$

Taking Λ to be the discrete image of Γ_{A^0} under the projection $P^{-1}_{W,\mathbb{C}} \to \mathbb{L}$, we obtain

$$\Gamma_{A^0} \backslash \delta^1_W \;=\; \Lambda \backslash \mathbb{L} \tag{4.9a}$$

In particular,

$$\Gamma_{A^0}\backslash\delta^1_W \;=\; \mathbb{C}^{d_1}\,\times\,(\mathbb{C}^*)^{d_2}\,\times\,T_W \tag{4.9b}$$

is biholomorphic to the product of an affine space $\mathbb{C}^{d_1}$ with a complex torus $(\mathbb{C}^*)^{d_2}\,\times\,T_W$ having compact factor T_W.

Setting

$$\mathbb{L}_I \;=\; \bigoplus_{\substack{p+q=-1\\ p<0}} \mathfrak{c}^{p,q}_{I,F}$$

and letting Λ_I be the discrete image of $\Gamma_{A^0,I}$ under the projection $C^{-1}_{\sigma_I}\to\mathbb{L}_I$, we have

$$\Gamma_{A^0,I}\backslash\delta^1_I \;=\; \Lambda_I\backslash\mathbb{L}_I \;=\; \mathbb{C}^{d_{I,1}}\,\times\,(\mathbb{C}^*)^{d_{I,2}}\,\times\,J_I\,, \tag{4.10}$$

with $(\mathbb{C}^*)^{d_{I,2}}\,\times\,J_I$ a complex torus having compact factor J_I. Note the obvious map

$$\Lambda_I \;\hookrightarrow\; \Lambda\,.$$

4.2. **The IPR along fibres.** Consider the restriction $F_{I,A}:A^0\,\cap\,Z^*_I\,\cap\,\overline{\mathfrak{U}}\to\delta_I$ of (B.3). Let

$$\xi_{I,\beta} \;=\; \xi|_{A^0\cap Z^*_I\cap\overline{\mathfrak{U}}} \;=\; \xi_I|_{A^0\cap Z^*_I\cap\overline{\mathfrak{U}}}\,.$$

The infinitesimal period relation (B.13) and the discussion of §B.5 imply that the Maurer-Cartan form

$$\xi^{-1}_{I,\beta}\,\mathrm{d}\xi_{I,\beta}\;\;\textit{takes value in}\;\;\bigoplus_{q\le 0}\mathfrak{c}^{-1,q}_{\sigma,F}\,. \tag{4.11}$$

We have a well-defined logarithm

$$\log\xi_{I,\beta}:A^0\,\cap\,\overline{\mathfrak{U}}\;\to\;\mathfrak{c}^{-1}_{\sigma,\mathbb{C}}\,\cap\,\mathfrak{f}^{\perp}\,.$$

Let $(\log\xi^a_{I,\phi})^{p,q}$ denote the component taking value in $\mathfrak{c}^{p,q}_{\sigma,F}$. Then (4.11) implies

$$(\log\xi_{I,\beta})^{p,q}\quad\text{is locally constant for all}\quad p+q=-1\,,\;p\le-2\,. \tag{4.12}$$

4.3. **Compact torus: Proof of Theorem 4.3(a).** It follows from (3.8) that

$$\Lambda\;\subset\;\mathfrak{g}^{-1,0}_{W,F}\;\subset\;\mathbb{L}\quad\text{and}\quad\Lambda_I\;\subset\;\mathfrak{c}^{-1,0}_{\sigma,F}\;\subset\;\mathbb{L}_I\,.$$

In particular, the torus factor $(\mathbb{C}^*)^{d_2}\times T^{d_3}$ of $\Gamma_{A^0}\backslash\delta^1_W$ of (4.9) is contained in the image of $\mathfrak{g}^{-1,0}_{W,F}\to\Lambda\backslash\mathbb{L}$; likewise, the torus factor $(\mathbb{C}^*)^{d_{I,2}}\,\times\,J_I$ of $\Gamma_{A^0}\backslash\delta^1_I$ is contained in the image of $\mathfrak{c}^{-1,0}_{I,F}\to\Lambda_I\backslash\mathbb{L}_I$. It follows from the IPR (4.12) and the compactness of A^0 that the image of $\Phi^1_{A^0,W}:A^0\to\Gamma_{A^0}\backslash\delta^1_W$ is contained in the compact torus T^{d_3} of (4.9b). Likewise, the image of $\Phi^1:A^0\cap Z_I\to\Gamma_{A^0}\backslash\delta^1_I$ is contained in the compact torus J_I of (4.10). We will show that J_I is abelian by exhibiting ample Lie bundles $\mathcal{L}_M\to J_I$.

4.4. **Action on LMHS of the fibre.** When restricted to $\delta_W \subset \mathcal{S}$, the map $X : \mathcal{S} \to \mathfrak{f}^\perp$ of §3.3.1 takes value in

$$X : \delta_W \ \to \ \mathfrak{p}^{-1}_{W,\mathbb{C}} \cap \mathfrak{f}^\perp\,.$$

Set $\xi = \exp(X)$, so that $X = \xi \cdot F = \exp(X)\cdot F$. In anticipation of the arguments to follow, it will be helpful to work out some formula. To begin, recall the Deligne splitting (§B.3) of $\mathfrak{g}_\mathbb{C}$. Given any $x \in \mathfrak{g}_\mathbb{C}$, there are unique $x^{p,q} \in \mathfrak{g}^{p,q}_{W,F}$ so that

$$x \ = \ \sum x^{p,q}\,.$$

Recall the notation and observations of §3.2.2. Given $\gamma = \alpha\beta \in \Gamma_{A^0} \subset P^{-1}_{W,\mathbb{Q}}$, one may verify that the logarithms satisfy

$$\begin{aligned}
c^{-1,0} &= a^{-1,0}\\
c^{0,-1} &= b^{0,-1}\\
c^{-1,-1} &= a^{-1,-1} + \tfrac{1}{2}[a^{-1,0}, b^{0,-1}]\,.
\end{aligned}$$

The action of γ on $\xi = \exp(X)\cdot F \in \delta_W$ satisfies

$$(\log \alpha\, \beta\xi\beta^{-1})^{-1,0} \ = \ X^{-1,0} + a^{-1,0} \tag{4.13a}$$

$$(\log \alpha\, \beta\xi\beta^{-1})^{-1,-1} \ = \ X^{-1,-1} + a^{-1,-1} + [b^{0,-1}, X^{-1,0}]\,. \tag{4.13b}$$

The containment (3.11) implies

$$(\log \alpha\, \beta\xi\beta^{-1})^{p,q} \ = \ X^{p,q}\,, \quad \forall\ p+q = -1 > p\,. \tag{4.13c}$$

Under the identifications of §4.1 we have

$$\lambda \ = \ a^{-1,0} \quad \text{and} \quad \overline{\lambda} \ = \ b^{0,-1}\,,$$

and $(X^{p,-1-p})_{p\le -1} = X^{-1,0} + X^{-2,1} + X^{-3,2} + \cdots$ parameterizes a point in $\mathbb{L}$. So (4.13) is describing the action of Λ on $\mathbb{L}$.

Consider $\gamma_i = \alpha_i\beta_i \in \Gamma_{A^0}$, with $\gamma_i = e^{c_i}$, $\alpha_i = e^{a_i}$ and $\beta_i = e^{b_i}$, as above. Suppose that $\gamma = \gamma_1\,\gamma_2$. Then one may verify that

$$\begin{aligned}
a^{-1,0} &= a_1^{-1,0} + a_2^{-1,0}\\
b^{0,-1} &= b_1^{0,-1} + b_2^{0,-1}\\
c^{-1,-1} &= c_1^{-1,-1} + c_2^{-1,-1} + \tfrac{1}{2}[a_1^{-1,0}\,,\, b_2^{0,-1}] + \tfrac{1}{2}[b_1^{0,-1}\,,\, a_2^{-1,0}]\\
a^{-1,-1} &= a_1^{-1,-1} + a_2^{-1,-1} + [b_1^{0,-1}, a_2^{-1,0}]\,.
\end{aligned}$$

4.5. **Proof of Theorem 4.3(c).** The line bundle $\mathcal{L}_M \to \Gamma_{A^0}\backslash\delta_W$ descends to $\Gamma_{A^0}\backslash\delta^1_W$ if and only if the functions e^M_γ of (3.18) are constant on the fibres of $\delta_W \twoheadrightarrow \delta^1_W$. If $M \in \mathfrak{g}^{1,1}_{W,F}$, then (B.6c), (3.16), (3.18), and (4.13) yield

$$e^M_\gamma(X) \ = \ \exp 2\pi\mathbf{i}\kappa(M, a^{-1,-1} + [b^{0,-1}, X^{-1,0}]) \tag{4.14}$$

on δ_W. These functions are constant on the fibres of $\delta_W \twoheadrightarrow \delta^1_W$, and so descend to well-defined on functions on δ^1_W. There they induce line bundles (also denoted)

$$\begin{array}{ccc} \mathcal{L}_M & & \mathcal{L}_M \\ \downarrow & & \downarrow \\ \Gamma_{A^0,I}\backslash\delta^1_I & \longrightarrow & \Gamma_{A^0}\backslash\delta^1_W \end{array}$$

over the level one extension data.

Additionally, if $M \in \mathbf{N}^* \cap \mathfrak{g}^{1,1}_{W,F}$, then (3.23), (4.13) and (4.14) yield

$$(\widetilde{\Phi}_{A^0})^*(e^M_\gamma)\Big|_{A^0} \;=\; (\Phi^1_{A^0,W})^* e^M_\gamma(X)\,;$$

establishing (4.4).

4.6. **Chern classes.** We now wish to compute the first Chern class $c_1(\mathcal{L}_M)$ of $\mathcal{L}_M \to \Gamma_{A^0}\backslash\delta^1_W = \Lambda\backslash\mathbb{L}$ for $M \in \mathfrak{g}^{1,1}_{W,F}$. We have

$$H^1(\Lambda\backslash\mathbb{L},\mathbb{C}) \;=\; (\mathbb{L}\oplus\overline{\mathbb{L}})^* \;\simeq\; \bigoplus_{p+q=-1} \mathfrak{g}^{p,q}_{W,F}\,,$$

and

$$\begin{aligned} H^2(\Lambda\backslash\mathbb{L},\mathbb{C}) &\;=\; \textstyle\bigwedge^2 H^1(\Lambda\backslash\mathbb{L},\mathbb{C}) \;=\; \bigwedge^2(\mathbb{L}\oplus\overline{\mathbb{L}})^*\,,\\ H^{1,1}(\Lambda\backslash\mathbb{L}) &\;=\; \mathbb{L}^*\otimes\overline{\mathbb{L}}^*\,. \end{aligned}$$

We have a map

$$\omega : \mathfrak{g}^{1,1}_{W,F} \;\hookrightarrow\; \mathbb{L}^*\otimes\overline{\mathbb{L}}^* \;\simeq\; H^{1,1}(\Lambda\backslash\mathbb{L})\,,$$

defined by sending $M \in \mathfrak{g}^{1,1}_{W,F}$ to the form $\omega_M \in H^{1,1}(\Lambda\backslash\mathbb{L})$ defined by

$$\omega_M(u,\bar v) \;:=\; \kappa(M,[u,\bar v]) \;=\; -\kappa(u,\mathrm{ad}_M(\bar v))$$

with $u, v \in \mathbb{L}$.

Recall the definition of $\mathbf{N}^*$ in (3.21) and consider the subset

$$\mathbf{N}^1 \;=\; \left\{ M \in \mathfrak{g}^{1,1}_{W,F} \;\middle|\; \begin{array}{l} \kappa(M\,,\,[a^{-1,0},b^{0,-1}]) \in \mathbb{Z}\,,\ \forall\,\gamma\in\Gamma_{A^0}\,;\\ \kappa(M,N_i)\in\mathbb{Z}\,,\ \forall\, i\in I_W \end{array} \right\}.$$

Remark 4.15. (i) When $\gamma = \exp(N_i)$, we have $a^{-1,-1} = N$ and $a^{-1,0}, b^{0,-1} = 0$.

(ii) The fact that κ is defined over $\mathbb{Q}$ implies that $\mathbf{N}^1$ is non-empty; in fact, $\mathbf{N}^1$ spans $\mathfrak{g}^{1,1}_{W,F}$.

Lemma 4.16. *If $M \in \mathbf{N}^1$, then the form ω_M represents the Chern class $c_1(\mathcal{L}_M)$.*

Proof. Define a smooth function $h_M : \mathbb{L} \to \mathbb{R}$ by

$$h_M(z) \;:=\; \exp 2\pi\mathbf{i}\,\kappa\,(M,[z,\bar z])\;.$$

With the formulæ of §§4.4–4.5, is straightforward to confirm

$$h_M(z+\lambda) \;=\; |e^M_\gamma(z)|^{-2}\, h_M(z)\,.$$

So h_M defines a metric on $\mathcal{L}_M \to \Lambda\backslash\mathbb{L}$ with curvature form $-\partial\bar\partial \log h_M$, cf. [GH94, p. 310–311]. It follows that the Chern form of $\mathcal{L}_M$ is

$$c_1(\mathcal{L}_M) \;=\; -\frac{\mathbf{i}}{2\pi}\partial\bar\partial \log h_M \;=\; \partial\bar\partial\kappa\,(M,[z,\bar z]) \;=\; \kappa\,(M,[\mathrm{d}z,\mathrm{d}\bar z]) \;=\; \omega_M\,.$$

□

4.7. $\mathfrak{sl}_2$**–triples.** The ample line bundles $\mathcal{L}_M \to J_I$ are constructed from $\mathfrak{sl}_2$–triples $\{M,Y,N\}$ constructed from the data of a LMHS (W,F,N), $N\in\sigma_I$. Here we briefly review this well-known construction (see, for example, [CM93] or [Sch73]), and discuss those properties that we will use later.

Define $Y\in\mathrm{End}(\mathfrak{g}_\mathbb{C})$ by specifying that Y acts on $\mathfrak{g}^{p,q}_{W,F}$ by the eigenvalue $(p+q)$. Then $Y\in\mathfrak{g}^{0,0}_{W,F}\cap\mathfrak{g}_\mathbb{R}$, and

$$\mathrm{ad}_Y(N) \;=\; [Y,N] \;=\; -2\,N\,.$$

Notice that Y depends only on (W,F); in particular Y is independent of N. The pair $\{Y,N\}$ may be uniquely completed to a triple $\{M,Y,N\}\subset\mathfrak{g}_\mathbb{R}$ with the properties that

$$[M,N] \;=\; Y \quad\text{and}\quad [Y,M] \;=\; 2\,M\,; \tag{4.17}$$

In particular, $\{M,Y,N\}$ spans a subalgebra of $\mathfrak{g}_\mathbb{R}$ that is isomorphic to $\mathfrak{sl}_2\mathbb{R}$. We have

$$M \;\in\; \mathfrak{g}^{1,1}_{W,F}\cap\mathfrak{g}_\mathbb{R}\,.$$

From $[M,N]=Y$ and $\kappa(Y,Y)>0$ it follows that

$$0 \;<\; \kappa(Y,Y) \;=\; \kappa([M,N],Y) \;=\; \kappa(M,[N,Y]) \;=\; 2\kappa(M,N)\,. \tag{4.18}$$

We regard (W,F), and hence Y, as fixed. And consider $M=M(N)$ as a function of $N\in\sigma_I$.

Remark 4.19. The map $N\mapsto M(N)$ is the restriction to σ_I of a diffeomorphism $M:\mathcal{N}\to\mathcal{M}$ from an open cone $\mathcal{N}\subset\mathfrak{g}^{-1,-1}_{W,F}$ onto an open cone $\mathcal{M}\subset\mathfrak{g}^{1,1}_{W,F}$. This is a well-known and classical result in the theory of nilpotent elements of semisimple Lie algebras, cf. [CM93] and the references therein, and is discussed in the context of Hodge theory and polarized mixed Hodge structures in [BPR17, §3.2]. In general the map is not linear; in particular, while the image $M(\sigma_I)$ is a cone, it need not be convex.

Notice that the first equation of (4.17) implies that

$$M(\lambda N) \;=\; \tfrac{1}{\lambda}M(N)\,, \tag{4.20}$$

for all $\lambda>0$. We claim that

$$\mathrm{ad}_N^2(\mathrm{d}M) \;=\; 2\mathrm{d}N\,. \tag{4.21}$$

To see this note that the fact that $Y=[M,N]$ is constant implies

$$[N,\mathrm{d}M] \;=\; [M,\mathrm{d}N]\,.$$

Since elements of the vector subspace $\mathrm{span}_\mathbb{R}\,\sigma_I\subset\mathfrak{g}^{-1,-1}_{W,F}\cap\mathfrak{g}_\mathbb{R}$ commute, we also have

$$[N,\mathrm{d}N] \;=\; 0\,. \tag{4.22}$$

Thus

$$\mathrm{ad}_N^2(\mathrm{d}M) \;=\; [N\,,\,[M\,\mathrm{d}N]] \;=\; [\mathrm{d}N\,,\,[M,N]] \;=\; 2\,\mathrm{d}N\,.$$

In particular, the differential $\mathrm{d}M$ of $N \mapsto M(N)$ is injective.

Notice that (4.21) and (4.22) imply that

$$\mathrm{ad}_N^3(\mathrm{d}M) \;=\; 0\,.$$

Since $N \in \sigma_I$ polarizes the MHS (W,F) on $(\mathfrak{g}, -\kappa)$, we have

$$0 \;\le\; -\tfrac{1}{2}\kappa(\mathrm{d}M, \mathrm{ad}_N^2(\mathrm{d}M)) \;=\; -\kappa(\mathrm{d}M, \mathrm{d}N)\,, \tag{4.23}$$

with equality if and only if $\mathrm{d}N = 0$.

Lemma 4.24. *Fix $0 \neq N' \in \mathrm{span}_{\mathbb{R}}\, \sigma_I$. The set*

$$\sigma_0' \;=\; \{N \in \sigma_I \mid \kappa(M(N), N') = 0\}$$

is contained in the closure of

$$\sigma_+' \;=\; \{N \in \sigma_I \mid \kappa(M(N), N') > 0\}\,.$$

Proof. Suppose that $N \in \sigma_0'$. Fix a smooth curve $\nu(t)$ in σ_I with the property that $\nu(0) = N$ and $\nu'(0) = -N'$. Set $\mu(t) = M(\nu(t))$. Then (4.23) implies

$$0 \;<\; \kappa(\mu'(0), N')\,.$$

In particular, $\nu(t) \in \sigma_+'$ for small $t > 0$. □

4.8. **Ample line bundles.** Define

$$\mathbf{N}_I^{\mathfrak{sl}_2} \;=\; \{M \in \mathbf{N}^1 \mid M = M(N) \text{ for some } N \in \sigma_I\}\,.$$

The fact that both σ_I and κ are defined over $\mathbb{Q}$ implies that $\mathbf{N}_I^{\mathfrak{sl}_2}$ is nonempty.

We have $NMu = u$ for all $u \in \mathfrak{c}_{I,F}^{p,q}$ with $p+q = -1$. The fact that that $N \in \sigma_I$ polarizes the MHS (W,F) on $(\mathfrak{g}, -\kappa)$ implies that

$$\begin{aligned} -\mathbf{i}\,\omega_M(u,\bar u) \;&=\; -\mathbf{i}\,\kappa(M, [u,\bar u]) \;=\; \mathbf{i}\,\kappa(u, \mathrm{ad}_M \bar u)\\ &=\; \mathbf{i}\,\kappa(\mathrm{ad}_N \mathrm{ad}_M u\,,\, \mathrm{ad}_M \bar u) \;=\; -\mathbf{i}\,\kappa(\mathrm{ad}_M u\,,\, \mathrm{ad}_N \mathrm{ad}_M \bar u) \;<\; 0 \end{aligned}$$

for all $0 \neq u \in \mathfrak{c}_{I,F}^{-1,0} \subset \mathbb{L}_I$. It follows that the line bundle $\mathcal{L}_M^* \to \Gamma_{A^0,I}\backslash \delta_I^1$ has positive Chern form $-\omega_M$ for every $M \in \mathbf{N}_I^{\mathfrak{sl}_2}$ (Lemma 4.16). Thus $\mathcal{L}_M^* \to J_I$ is ample.

4.9. **Positivity.** It remains to establish Theorem 4.3(e); this is a consequence of Remark 4.19 and Lemma 4.25.

Lemma 4.25. *The cone*

$$\sigma_I^+ \;=\; \{N \in \sigma_I \mid \kappa(M(N), N_i) > 0\,,\ \forall\, i \in I\}$$

is open and nonempty.

Proof. In the case that $\dim \sigma_I = 1$, (4.20) and (4.26) yield $\sigma_I^+ = \sigma_I$.

For the general case $\dim \sigma_I \geq 1$, with $I = \{1, \ldots, k\}$, set

$$\mathbb{R}_+^k \;=\; \{y = (y^1, \ldots, y^k) \in \mathbb{R}^k \mid y^i > 0\}$$

so that

$$\sigma_I \;=\; \{N(y) = y^i N_i \mid y \in \mathbb{R}_+^k\}\,.$$

Set $M(y) = M(N(y))$ and $\kappa_i(y) = \kappa(M(y), N_i)$. Then it suffices to show that the cone

$$S^+ \;=\; \{y = (y^1, \dots, y^k) \in \mathbb{R}^k_+ \;|\; \kappa_i(y) > 0\}$$

is open. From (4.17) and (4.18) we see that

$$0 \;<\; \kappa(M(y), N(y)) \;=\; y^i\, \kappa_i(y)\,. \tag{4.26}$$

Since the y^i are all positive, this forces some $\kappa_i(y)$ to be positive (with i depending on y).

Decompose

$$\mathbb{R}^k_+ \;=\; S \cap S' \cap S''$$

with

$$\begin{aligned} S_1 &= \left\{ y \in \mathbb{R}^k_+ \;\middle|\; \kappa_1(y) \geq 0\,,\ \textstyle\sum_{i=2}^k y^i \kappa_i(y) \geq 0 \right\} \\ S_1' &= \left\{ y \in \mathbb{R}^k_+ \;|\; \kappa_1(y) < 0 \right\} \\ S_1'' &= \left\{ y \in \mathbb{R}^k_+ \;\middle|\; \textstyle\sum_{i=2}^k y^i \kappa_i(y) < 0 \right\}. \end{aligned}$$

The inequality (4.26) forces the open sets S_1' and S_1'' to be disjoint. Since $\mathbb{R}^k_+$ is open and connected, this in turn forces S to be nonempty. Then Lemma 4.24 implies that the cone

$$S_1^+ \;=\; \left\{ y \in \mathbb{R}^k_+ \;\middle|\; \kappa_1(y) > 0\,,\ \textstyle\sum_{i=2}^k y^i \kappa_i(y) > 0 \right\} \;\subset\; S$$

is nonempty and open in $\mathbb{R}^k_+$. This proves Theorem 4.3(e) in the case that $|I| \leq 2$.

For the general case $|I| = k$ we induct. Assume that the cone

$$S_a^+ \;=\; \left\{ y \in \mathbb{R}^k_+ \;\middle|\; \kappa_i(y) > 0\,,\ 1 \leq i \leq a\,; \quad \textstyle\sum_{i=a+1}^k y^i \kappa_i(y) > 0 \right\}$$

is nonempty (and therefore open) for some $1 \leq a \leq k-1$. Define a decomposition

$$S_a^+ \;=\; S_{a+1} \cup S_{a+1}' \cup S_{a+1}''$$

by

$$\begin{aligned} S_{a+1} &= \left\{ y \in S_a^+ \;\middle|\; \kappa_{a+1}(y) \geq 0\,,\ \textstyle\sum_{i=a+2}^k y^i \kappa_i(y) \geq 0 \right\} \\ S_{a+1}' &= \left\{ y \in S_a^+ \;|\; \kappa_{a+1}(y) < 0 \right\} \\ S_{a+1}'' &= \left\{ y \in S_a^+ \;\middle|\; \textstyle\sum_{i=a+2}^k y^i \kappa_i(y) < 0 \right\}. \end{aligned}$$

The definition of S_a^+ forces the open sets S_{a+1}' and S_{a+1}'' to be disjoint. Since S_a^+ is open, every connected component of S_a^+ must have nonempty intersection with S_{a+1}. Then Lemma 4.24 implies that the cone S_{a+1}^+ is nonempty and open in $\mathbb{R}^k_+$. This completes the inductive step. $\square$

5. Higher level extension data

The goal here is to study the higher level extension data along a connected component

$$A^1 \subset A^0$$

of a $\Phi^1_{A^0,W}$–fibre. We will see that the monodromy around A^1 takes value in $\exp(\mathbb{C}\sigma_{I(A^1)}) \cap P_{W,\mathbb{Q}}$ (Proposition 5.1). This is the essential structural result that will be used to prove Theorem 1.3 (§5.2).

5.1. **Extension data along Φ^1–fibres.** Set

$$I(A^1) = \{i \mid Z_i^* \cap A^1 \neq \emptyset\}.$$

Consider the period map

$$\Phi_{A^1} : \mathcal{O}^1 \to \Gamma_{A^1}\backslash D$$

induced by $\mathcal{V}|_{\mathcal{O}^1}$. Set $W = W^{I(A^1)}$, so that

$$A^1 \subset A^0 \subset Z_W .$$

Given $Z_I^* \subset Z_W$, let

$$\Phi_{A^1,I} : Z_I^* \cap \overline{\mathcal{O}}^1 \to (\exp(\mathbb{C}\sigma_I)\Gamma_{A^1})\backslash D_I$$

be the map induced by Φ_{A^1} (as Φ_I in (2.1) is induced by Φ). While Φ_I does not in general extend to the weight closure $Z_I \cap Z_W$ (§C.4.1), the map $\Phi_{A^1,I}$ does admit an extension if we replace the quotient of $\exp(\mathbb{C}\sigma_I)$ with the quotient by the larger $\exp(\mathbb{C}\sigma_{I(A^1)})$.

Proposition 5.1. (a) *The neighborhood $A^1 \subset \overline{\mathcal{O}}^1 \subset \overline{B}$ of Corollary 2.32 may be chosen so that the restriction of $\mathcal{V}$ to $\mathcal{O}^1 = \overline{\mathcal{O}}^1 \cap B$ has monodromy $\Gamma_{A^1} \subset \Gamma_{A^0}$ with unipotent radical $\Gamma_{A^1} \cap P_W^{-1} \subset \exp(\mathbb{C}\sigma_{I(A^1)}) \subset P_W^{-2}$. In particular,*

$$\Gamma_{A^1} \subset G_{I(A^0)} \ltimes \exp(\mathbb{C}\sigma_{I(A^1)}) \subset C_{I(A^0)} .$$

(b) *There is a well-defined holomorphic map*

$$\Phi'_{A^1,W} : Z_W \cap \overline{\mathcal{O}}^1 \to (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash D_W ,$$

and commutative diagram

$$\begin{array}{ccc} Z_I^* \cap \overline{\mathcal{O}}^1 & \xrightarrow{\Phi_{A^1,I}} & (\exp(\mathbb{C}\sigma_I)\Gamma_{A^1})\backslash D_I \\ \hookdownarrow & & \downarrow \\ Z_W \cap \overline{\mathcal{O}}^1 & \xrightarrow{\Phi'_{A^1,I}} & (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash D_W . \end{array}$$

(c) *The map $\Phi'_{A^1,W}$ is locally constant on the fibres of Φ^1.*

Remark 5.2. The information contained in $\exp(\mathbb{C}\sigma_{I(A^1)})$ is level two extension data. So the content of Proposition 5.1(c) is that *the full extension data is determined by the level ≤ 2 extension data*, up to constants of integration.[1] The level 2 extension data contained in $\exp(\mathbb{C}\sigma_{I(A^1)})$ is not truly lost; it is encoded in the sections $s_M \in H^0(\overline{\mathcal{O}}^0, L_M)$, with

[1] In the case that D is Hermitian, all extension data is level ≤ 2; that is, $D_W = D_W^2$. So here we find here another example of the ansatz that horizontality (the IPR) forces period maps and their images to behave "as if they were Hermitian".

$M \in \mathfrak{g}^{1,1}_{W,F}$, of (3.17). These sections are essentially discrete data as their restriction to the Φ^0–fibres is determined up to a constant factor.

As discussed in Remark 3.9, Γ_{A^1} is neat if and only if it is unipotent; equivalently, $\Gamma_{A^1} \subset \exp(\mathbb{C}\sigma_{I(A^1)})$.

Corollary 5.3. *Assume that the monodromy Γ_{A^1} neat. Then the Hodge filtrations $\mathcal{F}^p_{\mathrm{e}}$ are trivial over $\overline{\mathcal{O}}^1$.*

Proof of Corollary 5.3. Let $W = W^A$ be the weight filtration of $A^0 \supset A^1$. Since $\sigma_{I(A^1)} \subset \mathfrak{g}^{-1,-1}_{W,F} \subset \mathfrak{f}^\perp$, the proposition implies $\Gamma_{A^1} \subset \exp(\mathfrak{f}^\perp)$. The theorem now follows from Remark 3.15. □

5.1.1. *Outline of the proof of Proposition 5.1.* The proposition is proved by an inductive analysis of the higher level extension data along A^1. We begin with the level ≤ 2 extension data. Applying the discussion of §C.4.1 to the period map Φ_{A^0} yields a commutative diagram

$$\begin{tikzcd}
Z_I^* \cap \overline{\mathcal{O}}^0 \arrow[r, "\Phi^2_{A^0,I}"] \arrow[d, hook] & (\exp(\mathbb{C}\sigma_I)\Gamma_{A^0,I})\backslash D^2_I \arrow[d, two heads] \\
Z_W \cap \overline{\mathcal{O}}^0 \arrow[r, "\Phi^2_{A^0,W}"] \arrow[dr, "\Phi^1_{A^0,W}"'] & (\exp(\mathbb{C}\sigma_{I(A^0)})\Gamma_{A^0})\backslash D^2_W \arrow[d, two heads] \\
 & \Gamma_{A^0}\backslash D^1_W\,.
\end{tikzcd}$$

Lemma 5.4. *The map $\Phi^2_{A^0,W}$ is locally constant on $\Phi^1_{A^0,W}$–fibres.*

A straightforward modification of the proof of Lemma 3.2 establishes

Corollary 5.5. *There is a neighborhood $\overline{\mathcal{O}}^1$ of A^1 in $\overline{B}$ with the property that the restriction of $\mathcal{V}$ to $\mathcal{O}^1 = \overline{\mathcal{O}}^1 \cap B$ has monodromy $\Gamma_{A^1} \subset \Gamma_{A^0}$ taking value in $G_{I(A^0)} \ltimes (\exp(\mathbb{C}\sigma_{I(A^1)})P^{-3}_W)$.*

It then follows that the action of $\exp(\mathbb{C}\sigma_{I(A^1)})$ on D^3_W does descend to $\Gamma_{A^1}\backslash D^3_W$ (§C.4.1), yielding a well-defined map

$$\Phi^3_{A^1,W} : Z_W \cap \overline{\mathcal{O}}^1 \ \to\ (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash D^3_W\,.$$

The inductive step for $a \ge 3$ is

Lemma 5.6. *If the monodromy Γ_{A^1} about A^1 takes value in $G_{I(A^0)} \ltimes (\exp(\mathbb{C}\sigma_{I(A^1)})P^{-a}_W)$, then the action $\exp(\mathbb{C}\sigma_{I(A^1)})$ on D^a_W does descend to the $\Gamma_{A^1}\backslash D^a_W$, yielding a well-defined map*

$$\Phi^a_{A^1,W} : Z_W \cap \overline{\mathcal{O}}^1 \ \to\ (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash \delta^a_W\,.$$

This map is constant on A^1, implying $\Gamma_{A^1} \subset G_{I(A^0)} \ltimes (\exp(\mathbb{C}\sigma_{I(A^1)})P^{-a-1}_W)$.

Note that Proposition 5.1 follows directly from Lemma 5.6. The remainder of §5 is occupied with the proof of Lemma 5.6 (which subsumes Lemma 5.4 and Corollary 5.5).

5.1.2. *Lie theoretic description.* Fix $a \geq 2$. The fibres of $\Gamma_{A^1}\backslash\delta^a_W \twoheadrightarrow \Gamma_{A^1}\backslash\delta^{a-1}_W$ are the *level a extension data* (Definition 2.4). We begin by observing that these fibres are biholomorphic to the quotient $\Lambda^a\backslash\mathbb{L}^a$ of a vector space $\mathbb{L}^a$ by a discrete subgroup $\Lambda^a \subset \mathbb{L}^a$. To see this, first note that the fibre is

$$\begin{array}{ccc} \dfrac{P^{-a}_{W,\mathbb{C}} \cdot F}{(\Gamma_{A^1} \cap P^{-a}_{W,\mathbb{C}}) \cdot P^{-a-1}_{W,\mathbb{C}}} & \hookrightarrow & \Gamma_{A^1}\backslash\delta^a_W \\ & & \downarrow \\ & & \Gamma_{A^1}\backslash\delta^{a-1}_W \,. \end{array}$$

We have

$$\begin{aligned} P^{-a-1}_{W,\mathbb{C}}\backslash P^{-a}_{W,\mathbb{C}} &\simeq \bigoplus_{p+q=-a} \mathfrak{g}^{p,q}_{W,F} \\ P^{-a-1}_{W,\mathbb{C}}\backslash(P^{-a}_{W,\mathbb{C}} \cdot F) &\simeq \bigoplus_{\substack{p+q=-a \\ p<0}} \mathfrak{g}^{p,q}_{W,F} \;=\; \mathbb{L}^a \,. \end{aligned}$$

The latter is an abelian group, with discrete subgroup

$$\Lambda^a \;=\; \frac{P^{-a}_{W,\mathbb{C}} \cap \Gamma_{A^1}}{P^{-a-1}_{W,\mathbb{C}} \cap \Gamma_{A^1}} \,.$$

We now see that the level a extension data of (W, F) is biholomorphic to the product

$$\Lambda^a\backslash\mathbb{L}^a \;\simeq\; \mathbb{C}^{d_1} \times (\mathbb{C}^*)^{d_2} \times \mathbb{T}^{d_3} \tag{5.7}$$

of an affine space $\mathbb{C}^{d_1}$ with a complex torus $(\mathbb{C}^*)^{d_2} \times \mathbb{T}^{d_3}$ having compact factor $\mathbb{T}^{d_3}$. (The dimensions d_i depend on a.)

Since $\sigma_{I(A^1)} \subset \mathfrak{g}^{-1,-1}_{W,F}$, it then follows that the fibres of

$$(\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash\delta^a_W \;\twoheadrightarrow\; (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash\delta^{a-1}_W$$

are, for $a = 2$:

$$\begin{array}{ccc} (\Lambda^2 \cdot \sigma_{I(A^1)})\backslash\mathbb{L}^2 & \hookrightarrow & (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash\delta^2_W \\ & & \downarrow \\ & & \Gamma_{A^1}\backslash\delta^1_W \,, \end{array}$$

and, for $a \geq 3$:

$$\begin{array}{ccc} \Lambda^a\backslash\mathbb{L}^a & \hookrightarrow & (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash\delta^a_W \\ & & \downarrow \\ & & (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash\delta^{a-1}_W \,. \end{array}$$

Note that $(\Lambda^2 \cdot \sigma_{I(A^1)})\backslash\mathbb{L}^2$ inherits (5.7) in the sense that it is also biholomorphic to the product

$$(\Lambda^2 \cdot \sigma_{I(A^1)})\backslash\mathbb{L}^2 \;\simeq\; \mathbb{C}^{d_1} \times (\mathbb{C}^*)^{d_2} \times \mathbb{T}^{d_3} \tag{5.8}$$

of an affine space $\mathbb{C}^{d_1}$ with a complex torus $(\mathbb{C}^*)^{d_2} \times \mathbb{T}^{d_3}$ having compact factor $\mathbb{T}^{d_3}$. (We abuse notation by continuing to denote the dimensions by d_i.)

5.1.3. *The IPR along fibres.* The local version of $\Phi^a_{A^1,W}$ is the map

$$\widetilde{\Phi}^a_W : Z_W \cap \overline{\mathfrak{U}} \ \to \ \exp(\mathbb{C}\sigma_{I(A^1)})\backslash D^a_W$$

defined by (C.22). If the level $\le a$ extension data map $\Phi^a_{A^0,W}$ is constant along A^1, then the restriction

$$\xi^a_{A^1} \ = \ \xi|_{A^1 \cap \overline{\mathfrak{U}}}$$

takes value in the affine space

$$\delta^a_W \ = \ P^{-a-1}_{W,\mathbb{C}} \cdot F \ \simeq \ \exp(\mathfrak{p}^{-a-1}_{W,\mathbb{C}} \cap \mathfrak{f}^\perp) \ \simeq \ \mathfrak{p}^{-a-1}_{W,\mathbb{C}} \cap \mathfrak{f}^\perp \ = \ \bigoplus_{b \ge a} \mathbb{L}^{b+1} \,.$$

Recall the discussion of the IPR in §4.2, and note that (4.11) implies

$$(\xi^a_{A^1})^{-1}\,\mathrm{d}\xi^a_{A^1} \ \ \textit{takes value in} \ \ \bigoplus_{b\ge a} \mathfrak{g}^{-1,-b}_{W,F} \ \subset \ \bigoplus_{b \ge a} \mathbb{L}^{b+1} \,. \tag{5.9}$$

Additionally, we have well-defined logs

$$\log \xi^a_{A^1} : A^1 \cap \overline{\mathfrak{U}} \ \to \ \mathfrak{p}^{-a-1}_{W,\mathbb{C}} \cap \mathfrak{f}^\perp \,.$$

Let $(\log \xi^a_{A^1})^{p,q}$ denote the component taking value in $\mathfrak{g}^{p,q}_{W,F}$. Then (5.9) implies

$$(\log \xi^a_{A^1})^{p,q} \quad \text{is locally constant for all} \quad p+q = -a-1\,,\ p \le -2\,. \tag{5.10}$$

5.1.4. *Proof of Lemma 5.6.* The argument is inductive. Assume that $a \ge 1$ and that we have a well-defined

$$\Phi^{a+1}_{A^1,W} : Z_W \cap \overline{\mathfrak{U}} \ \to \ (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash D^a_W \,.$$

We will show that $\Phi^{a+1}_{A^0,W}$ is constant along A^1.

Recalling (5.9), let η^a be the component of the Maurer-Cartan form $(\xi^a_{A^1})^{-1}\,\mathrm{d}\xi^a_{A^1}$ taking value in

$$\mathfrak{g}^{-1,-a}_{W,F} \ \hookrightarrow \ \mathbb{L}^{a+1} \ \simeq \ \mathcal{P}^{-a-2}_W \backslash \left(P^{-a-1}_{W,\mathbb{C}} \cdot F \right) \,. \tag{5.11}$$

Then fixing a point $z_0 \in A^1$ we may define a holomorphic map

$$A^1 \ \to \ \begin{cases} (\Lambda^2 \cdot \sigma_{I_W})\backslash \mathbb{L}^2\,, & a = 1\,,\\ \Lambda^{a+1}\backslash \mathbb{L}^{a+1}\,, & a \ge 2\,, \end{cases} \tag{5.12a}$$

by integration

$$z \ \mapsto \ \int_{z_0}^{z} \eta^a \tag{5.12b}$$

along a curve $\delta : [0,1] \to A^1$ joining $z_0 = \delta(0)$ and $z = \delta(1)$.

The key point is that when $b \ge 2$, the complex conjugate

$$\overline{\mathfrak{g}^{-1,-b}_{W,F}} \ = \ \mathfrak{g}^{-b,-1}_{W,F}$$

is contained in $\mathfrak{p}^{-b-1}_{W,\mathbb{C}} \cap \mathfrak{f}^\perp$ and has trivial intersection with $\mathfrak{g}^{-1,-b}_{W,F}$. This implies that the image of $\mathfrak{g}^{-1,-b}_{W,F}$ under the composition of (5.11) with the projection

$$\mathbb{L}^{a+1} \ \twoheadrightarrow \ \begin{cases} (\Lambda^2 \cdot \sigma_{I_W})\backslash \mathbb{L}^2 \,, & a = 1\,, \\ \Lambda^{a+1}\backslash \mathbb{L}^{a+1}\,, & a \geq 2\,, \end{cases}$$

lies in the noncompact factors $\mathbb{C}^{d_1} \times (\mathbb{C}^*)^{d_2}$ of (5.7) and (5.8). Since η^a takes value in $\mathfrak{g}^{-1,a}_{W,F}$, it follows that (5.12) defines a holomorphic map $A^1 \to \mathbb{C}^{d_2} \times (\mathbb{C}^*)^{d_3}$. Since A^1 is compact, this map must be locally constant. This forces $\eta^a = 0$. Equivalently, the Maurer–Cartan form $(\xi^a_{A^1})^{-1}\,\mathrm{d}\xi^a_{A^1}$ takes value in $\mathfrak{p}^{-a-2}_{W,\mathbb{C}}$ along A^1. This is precisely the statement that $\Phi^{a+1}_{A^0,W}$ is locally constant along A^1. □

5.2. **Proof of Theorem 1.3.** It suffices to prove

Proposition 5.13. *There exists a proper holomorphic map $f : \overline{\mathcal{O}}^1 \to \mathbb{C}^d$ with the following properties:*

(a) *The map $f|_{\mathcal{O}^1}$ is constant on the fibres of $\Phi|_{\mathcal{O}^1}$.*
(b) *Conversely, $\Phi|_{\mathcal{O}^1}$ is locally constant on the fibres of $f|_{\mathcal{O}^1}$.*

The proposition is proved in §§5.2.1–5.2.3. Assume for the moment that Proposition 5.13 holds. Let

$$\begin{tikzcd} \overline{\mathcal{O}}^1 \arrow[r, "\hat f"'] \arrow[rr, bend left, "f"] & \hat{\mathcal{O}}^1 \arrow[r] & \mathbb{C}^m \end{tikzcd}$$

be the Stein factorization. This completes the proof of Theorem 1.3 as outlined in the Introduction (§1).

5.2.1. *Preliminaries.* Consider the lift

$$\begin{array}{ccc} \tilde{\mathcal{O}}^1 & \xrightarrow{\widetilde{\Phi}_{A^1}} & \mathcal{S} \cap D \\ \downarrow & & \downarrow \\ \mathcal{O}^1 & \xrightarrow{\Phi_{A^1}} & \Gamma_{A^1}\backslash D\,. \end{array} \tag{5.14}$$

Recall notations of §B.4. Let W index the weight strata Z_W containing A^1, so that

$$\mathfrak{f}^\perp \ = \ \bigoplus_{p<0} \mathfrak{g}^{p,q}_{W,F}\,,$$

and set

$$I(A^1) \ = \ \{i \mid A^1 \cap Z_i^* \neq \emptyset\} \ = \ \cup\{I \mid A^1 \cap Z_I^* \neq \emptyset\} \ \subset \ I_W\,.$$

5.2.2. *Horizontal entries of the period matrix.* Fix a basis $\{M_\mu\}$ of $\mathfrak{g}^{1,\bullet}_{W,F} = \oplus_q\, \mathfrak{g}^{1,q}_{W,F}$. Keeping (B.6c) in mind, the

$$\varepsilon_\mu \ = \ \kappa(X \circ \widetilde{\Phi}_{A^1}, M_\mu) : \tilde{\mathcal{O}}^1 \ \to \ \mathbb{C} \tag{5.15}$$

are the *horizontal coefficients of the period matrix.* We may choose the basis $\{M_\mu\}$ so that for each $I \subset I(A^1)$ there is a disjoint union $\{M_\mu\} = \mathbf{N}_I^* \cup \mathbf{N}_I^\perp$ so that

$$\mathrm{span}_{\mathbb{C}}\{M_\mu \in \mathbf{N}_I^\perp\} \ = \ \mathrm{Ann}(\sigma_I) \ \subset \ \mathfrak{g}^{1,\bullet}_{W,F}\,.$$

It follows that

$$\varepsilon_\mu \ \in \ \mathcal{O}(\overline{\mathcal{O}}^1)\,, \quad \forall\, M_\mu \in \mathbf{N}^\perp_{I(A^1)}\,.$$

We think of the ε_μ indexed by $M_\mu \in \mathbf{N}^\perp_{I(A^1)}$ as the *smooth horizontal coefficients of the period matrix.*

We may additionally suppose the basis $\{M_\mu\}$ is chosen so that

$$0 \ \leq \ \kappa(N_i, M_\mu) \ \in \ \mathbb{Z}\,, \quad \forall\, i \in I(A^1)\,. \tag{5.16}$$

Then, for the $M_\mu \in \mathbf{N}^*_{I(A^1)} \supset \mathbf{N}^*_I$, we have

$$\tau_\mu \ = \ \exp(2\pi\mathbf{i}\,\varepsilon_\mu) \ \in \ \mathcal{O}(\overline{\mathcal{O}}^1)\,.$$

We think of the ε_μ indexed by $M_\mu \in \mathbf{N}^*_{I(A^1)}$ as the *logarithmic horizontal coefficients of the period matrix.*

5.2.3. *Proof of Proposition 5.13.* Let $f : \overline{\mathcal{O}}^1 \to \mathbb{C}^m$ be the holomorphic map defined by the $\{\varepsilon_\mu\}_{M_\mu \in \mathbf{N}^\perp_{I(A^1)}}$ and $\{\tau_\mu\}_{M_\mu \in \mathbf{N}^*_{I(A^1)}}$. The IPR implies f has the desired properties.

5.2.4. *Zero locus of the τ_μ.* Note that (3.20) is the local expression for τ_μ. In particular,

$$A_I \ = \ Z_I \cap \overline{\mathcal{O}}^1 \ \subset \ \{\tau_\mu = 0\} \quad \text{if and only if} \quad M_\mu \in \mathbf{N}^*_I\,.$$

Reciprocally, τ_μ is nowhere vanishing on $A^*_I = Z^*_I \cap \overline{\mathcal{O}}^1$ if and only if $M_\mu \in \mathbf{N}^\perp_I$. More generally, the function τ_μ is nowhere vanishing on the weight strata $Z_W \cap \overline{\mathcal{O}}^1$ if and only if $M_\mu \in \mathbf{N}^\perp_{I_W}$.

Suppose that $j \notin I_W$, and set $J = I_W \cup \{j\}$. Then $W \neq W^J$ (Corollary C.10). So there exists $M_{\mu_j} \in \mathbf{N}^\perp_{I_W}$ such that $\kappa(M_{\mu_j}, N_j) > 0$. The associated τ_{μ_j} is nowhere vanishing on $Z_W \cap \overline{\mathcal{O}}^1$, but vanishes along $Z_j \cap \overline{\mathcal{O}}^1$. Whence

$$\tau_W \ = \ \prod_{j \notin I_W} \tau_{\mu_j}\,.$$

is nowhere vanishing on $Z_W \cap \overline{\mathcal{O}}^1$, but vanishes on every $j \notin I_W$. In particular, τ_W vanishes along every $Z_J \cap \overline{\mathcal{O}}^1$ with $J \not\subset I_W$.

In the case that $I = \emptyset$ (the weight filtration $W^\emptyset$ is trivial and), we have

$$Z \cap \overline{\mathcal{O}}^1 \ = \ \{\tau_{W^\emptyset} = 0\}\,.$$

5.3. **Logarithmic differentials and a local Torelli condition.** In [GGR21a] we will discuss a map $\Psi : T_{\overline{B}}(-\log Z) \to F^{-1}\mathrm{End}(\mathcal{E}_{\mathrm{e}})$ that is induced by the Gauss–Manin connection on $\mathcal{V} \to B$. In anticipation of that discussion it is convenient to close §5.2 with a discussion of the algebra $\Omega^\bullet_{\overline{\mathcal{O}}^1}(Z \cap \overline{\mathcal{O}}^1)$ of logarithmic differentials on $(\overline{\mathcal{O}}^1, Z \cap \overline{\mathcal{O}}^1)$.

5.3.1. *Logarithmic differentials on* $(\overline{\mathcal{O}}^1, Z \cap \overline{\mathcal{O}}^1)$. It is evident from the discussions of §5.2.2 and §5.2.4 that

$$\mathrm{d}\varepsilon_\mu \;\in\; \Omega^1_{\overline{\mathcal{O}}^1}(\log Z \cap \overline{\mathcal{O}}^1)\,, \tag{5.17}$$

and

$$\{\mathrm{d}\varepsilon_\mu \mid M_\mu \in \mathbf{N}^\perp_{I(A^1)}\} \;\subset\; \Omega^1_{\overline{\mathcal{O}}^1}\,.$$

The differentials define a map

$$\Psi_1 : T_{\overline{\mathcal{O}}^1}(-\log Z \cap \overline{\mathcal{O}}^1) \;\to\; \overline{\mathcal{O}}^1 \,\times\, \mathbb{C}^m \tag{5.18}$$

by mapping $v \in T_{\overline{\mathcal{O}}^1}(-\log Z \cap \overline{\mathcal{O}}^1)$ to $(\mathrm{d}\varepsilon_\mu(v)) \in \mathbb{C}^m$. (Here we suppress the base point $b \in \overline{\mathcal{O}}^1$ of v.)

5.3.2. *Local Torelli condition for* $(\overline{\mathcal{O}}^1, Z \cap \overline{\mathcal{O}}^1; \Phi_{A^1})$. Since the coordinates of $\Psi_1|_{\mathcal{O}^1}$ are the horizontal period matrix entries of $\widetilde{\Phi}_{A^1}$, we see that the differential of $\Phi|_{\mathcal{O}^1}$ is injective if and only if the differential of $\Psi_1|_{\mathcal{O}^1}$ is injective. More generally, we have

Lemma 5.19. *The sheaf map* Ψ_1 *is injective at points* $b \in \overline{\mathcal{O}}^1 \cap Z_I^*$ *if and only if*

(i) *The differential* $\mathrm{d}\Phi^1_{A^1,I} : T(Z_I^* \cap \overline{\mathcal{O}}^1) \to T(\Gamma_{A^1,I}\backslash D^1_I)$ *is injective.*

(ii) *The* $\{N_i \mid i \in I\}$ *are linearly independent.*

Proof. It will be convenient to write

$$\Psi_1 = (\Psi_1^{\mathrm{hol}}, \Psi_1^{\log})$$

with

$$\Psi_1^{\mathrm{hol}}(v) \;=\; (\mathrm{d}\varepsilon_\mu(v))_{M_\mu \in \mathbf{N}^\perp_{I(A^1)}}$$

given by the holomorphic differentials, and

$$\Psi_1^{\log}(v) \;=\; (\mathrm{d}\varepsilon_\mu(v))_{M_\mu \in \mathbf{N}^*_{I(A^1)}}$$

given by the log differentials. It follows from the IPR and Remark 5.2 that the following are equivalent:

(a) The restriction of Ψ_1^{hol} to $T_b(Z_I^* \cap \overline{\mathcal{O}}^1)$ is injective.

(b) The restriction of $\mathrm{d}\Phi^1_{A^1,I}$ to $T_b(Z_I^* \cap \overline{\mathcal{O}}^1)$ is injective.

Fix a coordinate chart $(t,w) \in \overline{\mathcal{U}} \subset \overline{\mathcal{O}}^1$ centered at a point $b \in Z_I^* \cap \overline{\mathcal{O}}^1$, as in §B.2.2. Then

$$\{\mathrm{d}\log t_i\,,\, \mathrm{d}w_a\}$$

is a local framing of $\Omega^1_{\overline{B}}(\log Z)$ over $\overline{\mathcal{U}}$,

$$\{t_i\partial_{t_i}\,,\, \partial_{w_a}\}$$

is a local framing of $T_{\overline{B}}(-\log Z)$ over $\overline{\mathcal{U}}$, and $\{\partial_{w_a}\}$ is a local framing of $T(Z_I^* \cap \overline{\mathcal{O}}^1)$ over $\overline{\mathcal{U}} \cap Z_I^* = \{t = 0\}$. We have

$$\Psi_1^{\mathrm{hol}}(t_i\partial_{t_i})\Big|_{t_i=0} \;=\; 0\,.$$

Following the notation of (3.20), the logarithmic differentials are

$$\mathrm{d}\varepsilon_\mu \;=\; \frac{\mathrm{d}\tau_\mu}{2\pi\mathbf{i}\tau_\mu} \;=\; \kappa(M_\mu, \mathrm{d}\tilde{X}(t,w)) \;+\; \sum \frac{\kappa(M_\mu, N_i)\mathrm{d}t_i}{2\pi\mathbf{i}\, t_i}\,, \quad M_\mu \in \mathbf{N}^*_{I(A^1)}\,.$$

Recalling that $\tilde{X}(t,w)$ is holomorphic on $\overline{\mathcal{U}}$, we see that

$$\mathrm{d}\varepsilon_\mu\,(t_i\partial_{t_i})\big|_{t=0} \;=\; \frac{\kappa(M_\mu, N_i)}{2\pi\mathbf{i}}\,. \tag{5.20}$$

□

Informally we express (5.20) as

$$\Psi_1^{\log}\left(t_i\partial_{t_i}\big|_{t=0}\right) \;=\; 2\pi\mathbf{i}\, N_i\,.$$

Appendix A. Summary of notation

- period domain D parameterizing pure, Q–polarized HS on V of weight n
- compact dual $\check{D}$
- algebraic automorphism group $G = \mathrm{Aut}(V,Q)$, with Lie algebra $\mathfrak{g} = \mathrm{End}(V,Q)$
- smooth projective $\overline{B}$ with reduced normal crossing divisor $Z \subset \overline{B}$
- polarized variation of Hodge structure $\mathcal{V} = \tilde{B} \times_{\pi_1(B)} V$ over $B = \overline{B}\backslash Z$ with monodromy representation $\pi_1(B) \twoheadrightarrow \Gamma \subset G$
- the induced period map $\Phi : B \to \Gamma\backslash D$
- Hodge filtration $\mathcal{F}^p \subset \mathcal{V}$ and Hodge line bundle
$$\Lambda \;=\; \det(\mathcal{F}^n) \otimes \det(\mathcal{F}^{n-1}) \otimes \cdots \otimes \det(\mathcal{F}^{\lceil (n+1)/2\rceil})$$
- extensions $\mathcal{F}^p_{\mathrm{e}} \subset \mathcal{V}_{\mathrm{e}}$ and Λ_{e} to $\overline{B}$
- graded quotients $\mathcal{E}^p_{\mathrm{e}} = \mathcal{F}^p_{\mathrm{e}}/\mathcal{F}^{p+1}_{\mathrm{e}} = \mathrm{Gr}^p_{\mathcal{F}_{\mathrm{e}}}$, and $\mathcal{E}_{\mathrm{e}} = \oplus \mathcal{E}^p_{\mathrm{e}}$
- bundle map $\Psi : T_{\overline{B}}(-\log Z) \to \mathrm{Gr}^{-1}_{\mathcal{F}_{\mathrm{e}}}(\mathrm{End}(\mathcal{E}_{\mathrm{e}}))$ induced by flat connection
- algebraic subgroup $P_W \subset G$ stabilizing weight filtration W, filtered by normal subgroups
$$P_W^{-a} \;:=\; \{g \in P_W \mid g \text{ acts trivially on } W_\ell/W_{\ell-a}\ \forall\, \ell\}\,,$$
$a \geq 0$, $P_W = P^0_W$
- algebraic subgroup $C_I \subset P_W$ centralizing cone $\sigma_I \subset \mathfrak{g}$, with $W = W^I = W(\sigma_I)$, filtered by normal subgroups
$$C_I^{-a} \;:=\; C_I \,\cap\, P_W^{-a}\,.$$
- reference filtration $F^\bullet_0 \in \check{D}$, Hodge numbers $f^p_\ell := \dim F^p_0(\mathrm{Gr}^W_\ell)$
- $D_W \;:=\; \{F \in \check{D} \mid (W,F) \text{ is a MHS}, \dim F^p(\mathrm{Gr}^W_\ell) = f^p_\ell\}\,.$
- $\Gamma_W = \Gamma \cap P^{-1}_{W,\mathbb{Q}}$
- $G_W := (P_{W,\mathbb{R}}/P^{-1}_{W,\mathbb{R}}) \ltimes P^{-1}_{W,\mathbb{C}}$ acts transitively on D_W
- $D^a_W := P^{-a-1}_{I,\mathbb{C}}\backslash W_I$ with automorphism group $G^a_W := G_W/P^{-a-1}_{W,\mathbb{C}}$. [2]
- projections $D_W \twoheadrightarrow D^a_W \twoheadrightarrow D^0_W$ and $G_W \twoheadrightarrow G^a_W \twoheadrightarrow G^0_W$

[2] We think of this as indicating that G_W acts on D^a_W, with the normal subgroup $P^{-a-1}_{W,\mathbb{C}}$ acting trivially.

- $D_I := \{F \in D_W \mid (W,F) \text{ is polarized by } \sigma_I\}$
- $G_I := (C_{I,\mathbb{R}}/C_{I,\mathbb{R}}^{-1}) \ltimes C_{I,\mathbb{C}}^{-1}$ acts transitively on D_I
- $D_I^a := C_{I,\mathbb{C}}^{-a-1}\backslash D_I$ with automorphism group $G_I^a := G_I/C_{I,\mathbb{C}}^{-a-1}$. [3]
- projections $D_I \twoheadrightarrow D_I^a \twoheadrightarrow D_I^0$ and $G_I \twoheadrightarrow G_I^a \twoheadrightarrow G_I^0$
- D_I^0 is a Mumford–Tate domain with Mumford–Tate group $G_I^0 = C_I/C_I^{-1}$. [4]
- $\Gamma_I = \Gamma \cap C_{I,\mathbb{Q}}$
- period map to various quotients of LMHS

$$
\begin{tikzcd}
Z_I^* \arrow[r, "\Phi_I"] \arrow[dr, bend right, "\Phi_I^a"] \arrow[ddr, bend right, "\Phi_I^2"] \arrow[dddr, bend right, "\Phi_I^1"] \arrow[ddddr, bend right, "\Phi_I^0"] & (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I \arrow[d, two heads] \\
& (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I^a \arrow[d, two heads] \\
& (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I^2 \arrow[d, two heads] \\
& \Gamma_I\backslash D_I^1 \arrow[d, two heads] \\
& \Gamma_I\backslash D_I^0
\end{tikzcd}
$$

- $D_I^a \hookrightarrow D_W^a$ and $\Gamma_I \subset \Gamma_W$ induce $\Gamma_I\backslash D_I^a \to \Gamma_W\backslash D_W^a$
- *weight-strata* $Z_W = \bigcup_{W^I = W} Z_I^*$
- $Z_I \cap Z_W$ is the *weight-closure* of Z_I^*
- unique maximal I_W such that $W^{I_W} = W$
- if $Z_J^* \subset Z_I \cap Z_W$, then $D_J \hookrightarrow D_I$ and $\Gamma_J \subset \Gamma_I$ induce $\Gamma_J\backslash D \to \Gamma_I\backslash D_I$
- Φ_I^0 and Φ_I^1 extend to proper holomorphic maps on the weight-closure, and the extensions are compatible with Φ_J^a on $Z_J^* \subset Z_I \cap Z_W$

$$
\begin{tikzcd}
Z_J^* \arrow[r, hook] \arrow[d, "\Phi_J^1"] \arrow[dd, bend right=60, "\Phi_J^0"'] & Z_I \cap Z_W \arrow[d, "\Phi_I^1"'] \arrow[dd, bend left=60, "\Phi_I^0"] \\
\Gamma_J\backslash D_J^1 \arrow[r] \arrow[d, two heads] & \Gamma_I\backslash D_I^1 \arrow[d, two heads] \\
\Gamma_J\backslash D_J^0 \arrow[r] & \Gamma_I\backslash D_I^0
\end{tikzcd}
$$

[3]We think of this as indicating that G_I acts on D_I^a, with the normal subgroup $C_{I,\mathbb{C}}^{-a-1}$ acting trivially.

[4]One may define analogous spaces D_W^a for MHS. In the absence of the polarization, these spaces have less structure. For example, the analog D_W^0 of D_I^0 is a flag domain in Wolf's sense [FHW06, Wol69], but not a Mumford–Tate domain – the isotropy group is not compact.

- $\Phi_W \in \{\Phi_W^0, \Phi_W^1\}$ defined by

$$\begin{tikzcd} Z_I^* \arrow[r, hook] \arrow[d, "\Phi_I^1"] \arrow[dd, bend right=60, "\Phi_I^0"'] & Z_W \arrow[d, "\Phi_W^1"'] \arrow[dd, bend left=60, "\Phi_W^0"] \\ \Gamma_I\backslash D_I^1 \arrow[r] \arrow[d, two heads] & \Gamma_W\backslash D_W^1 \arrow[d, two heads] \\ \Gamma_I\backslash D_I^0 \arrow[r] & \Gamma_W\backslash D_W^0 \end{tikzcd}$$

- $\wp_W^0 = \Phi_W^0(Z_W) \subset \Gamma_W\backslash D_W^0$ and $\wp_W^1 = \Phi_W^1(Z_W) \subset \Gamma_W\backslash D_W^1$,

$$\overline{\wp}^0 \;=\; \bigcup \wp_W^0 \quad \text{and} \quad \overline{\wp}^1 \;=\; \bigcup \wp_W^1\,,$$

- two proper topological extensions of $\Phi : B \to \Gamma\backslash D$ defined strata-wise

$$\begin{tikzcd} \overline{B} \arrow[r, "\Phi^1"'] \arrow[rr, bend left, "\Phi^0"] & \overline{\wp}^1 \arrow[r, two heads] & \overline{\wp}^0 \end{tikzcd}$$

- $A \subset Z_W$ compact Φ^0–fibre
- $A^0 \subset A$ (compact) connected component of A
- $A^1 \subset A^0$ (compact) connected component of Φ^1–fibre
- $\delta_W \subset D_W$ is the preimage of $\Phi^0(A^0) \in \Gamma_W\backslash D_W^0$ under the projection $D_W \twoheadrightarrow \Gamma_W\backslash D_W^0$; these are pairs (W, F) with the same $F(\mathrm{Gr}^W)$
- $\delta_I \subset D_I$ is the preimage of $\Phi^0(A^0 \cap Z_I) \in \Gamma_I\backslash D_I^0$ under the projection $D_I \twoheadrightarrow \Gamma_I\backslash D_I^0$; these are pairs (W, F) that are polarized by σ_I and with the same $F(\mathrm{Gr}^W)$.
- neighborhood $A^0 \subset \overline{\mathcal{O}}^0 \subset \overline{B}$, Schubert cell $\mathcal{S} \subset \check{D}$, period map

$$\Phi_{A^0} : B \cap \overline{\mathcal{O}}^0 \;\to\; \Gamma_{A^0}\backslash(D \cap \mathcal{S})$$

- $\Gamma_{A^0} \subset \Gamma \cap P_W$ monodromy about A^0, $\Gamma_{A^0}^{-1} = \Gamma_{A^0} \cap P_W$ monodromy acting trivially on Gr^W, *finite* quotient $\Gamma_{A^0}/\Gamma_{A^0}^{-1}$ acting on Gr^W is contained in $G_{I(A^0)} = C_{I(A^0)}^0/C_{I(A^0)}^{-1}$, $I(A^0) = \{i \mid A^0 \cap Z_i^* \neq 0\}$
- $A^1 \subset A^0$ (compact) connected component of Φ^1–fibre, neighborhood $A^1 \subset \overline{\mathcal{O}}^1 \subset \overline{\mathcal{O}}^0 \subset \overline{B}$, period map

$$\Phi_{A^1} : B \cap \overline{\mathcal{O}}^1 \;\to\; \Gamma_{A^1}\backslash(D \cap \mathcal{S})$$

- $\Gamma_{A^1} \subset \Gamma \cap P_W$ monodromy about A^0, $\Gamma_{A^0}^{-1} = \Gamma_{A^0} \cap P_W^{-1}$ monodromy acting trivially on Gr^W, *finite* quotient $\Gamma_{A^0}/\Gamma_{A^0}^{-1}$ acting on Gr^W

Appendix B. Asymptotics of period maps: review of local properties

Here we set notation and review well-known properties of period maps and their local behavior at infinity. Good references for this material include [CMSP17, CKS86, GGK12, GS69, PS08, Sch73].

B.1. **Notation.**

B.1.1. *Groups.* Given a $\mathbb{Q}$–algebraic group G, the Lie groups of real and complex points will be denoted by $G_\mathbb{R}$ and $G_\mathbb{C}$, respectively. The associated Lie algebras are denoted $\mathfrak{g}_\mathbb{R}$ and $\mathfrak{g}_\mathbb{C}$, respectively.

Let $V = V_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$ is a rational vector space, with underlying lattice $V_\mathbb{Z}$. Let $\mathrm{End}(V) = V \otimes V^*$ denote the Lie algebra of linear maps $V \to V$, and let $\mathrm{Aut}(V) \subset \mathrm{End}(V)$ denote the $\mathbb{Q}$–algebraic group of invertible linear maps.

Fix $n \in \mathbb{Z}$, and suppose that , and $Q : V \times V \to \mathbb{Q}$ is a nondegenerate (skew-)symmetric bilinear form satisfying

$$Q(u,v) \ = \ (-1)^n Q(v,u)\,, \quad \text{for all} \quad u,v \in V\,.$$

From this point on, G will denote the $\mathbb{Q}$–algebraic group

$$G \ = \ \mathrm{Aut}(V,Q) \ = \ \{g \in \mathrm{Aut}(V) \mid Q(gu,gv) = Q(u,v)\,,\ \forall\ u,v \in V\}\,.$$

with Lie algebra

$$\mathfrak{g} \ = \ \mathrm{End}(V,Q) \ = \ \{X \in \mathrm{End}(V) \mid 0 = Q(Xu,v) + Q(u,Xv)\,,\ \forall\ u,v \in V\}\,.$$

B.1.2. *Period domains.* Let $D = G_\mathbb{R}/K^0$ be the period domain parameterizing effective weight $n > 0$, Q–polarized Hodge structures on V with Hodge numbers $\mathbf{h} = (h^{n,0}, \ldots, h^{0,n})$. Given $\varphi \in D$, let

$$V_\mathbb{C} \ = \ \bigoplus_{p+q=n} V^{p,q}_\varphi$$

be the Hodge decomposition; let

$$F^n_\varphi \ \subset \ F^{n-1}_\varphi \ \subset \cdots \subset \ F^1_\varphi \ \subset \ F^0_\varphi \ = \ V_\mathbb{C}$$

be the Hodge filtration. The weight zero Hodge decomposition

$$\mathfrak{g}_\mathbb{C} \ = \ \oplus\, \mathfrak{g}^{p,-p}_\varphi \tag{B.1}$$

induced by φ, is polarized by $-\kappa$, where $\kappa \in \mathrm{Sym}^2\mathfrak{g}^*_\mathbb{C}$ is the Killing form. The isotropy group $K^0 = \mathrm{Stab}_G(\varphi)$ stabilizing $\varphi \in D$ is compact, with complexified Lie algebra

$$\mathfrak{k}^0_\mathbb{C} \ = \ \mathfrak{k}^0_\mathbb{R} \otimes \mathbb{C} \ = \ \mathfrak{g}^{0,0}_\varphi\,.$$

Let $\check D = G_\mathbb{C}/P_\varphi$ denote the compact dual of D. Here P_φ is the complex parabolic stabilizer of the Hodge filtration F_φ, and has Lie algebra $\mathfrak{p}_\varphi = \oplus_{p\ge 0}\, \mathfrak{g}^{p,-p}_\varphi$.

B.2. **Period maps at infinity.**

B.2.1. Unit disc

$$\Delta \ = \ \{t \in \mathbb{C} \mid |t| < 1\}$$

and punctured unit disc

$$\Delta^* \ = \ \{t \in \mathbb{C} \mid 0 < |t| < 1\}\,.$$

Upper half plane

$$\mathcal{H} \ = \ \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$$

and covering map

$$\mathcal{H} \ \to \ \Delta^* \quad \text{sending} \quad z \ \mapsto \ t = e^{2\pi \mathbf{i} z}\,.$$

Multivalued inverse

$$\ell(t) \;=\; \frac{\log t}{2\pi\mathbf{i}}\,,$$

and (well-defined) differential $\mathrm{d}\ell \;=\; \dfrac{\mathrm{d}t}{2\pi\mathbf{i}\,t}$.

B.2.2. Fix a point $b \in Z_I^* \subset \overline{B}$. Choose a coordinate chart

$$(t,w) : \overline{\mathcal{U}} \;\subset\; \overline{B} \;\xrightarrow{\simeq}\; \Delta^{k+r}$$

centered at a point b with

$$(t,w) : \mathcal{U} \;=\; B \cap \overline{\mathcal{U}} \;\xrightarrow{\simeq}\; (\Delta^*)^k \times \Delta^r\,.$$

Reindexing the Z_i if necessary, we may assume that

$$\overline{\mathcal{U}} \cap Z_i \;=\; \{t_i = 0\}\,, \quad \text{for all} \quad 1 \le i \le k\,,$$

and $\overline{\mathcal{U}} \cap Z_\mu = \emptyset$ for all $k+1 \le \mu \le \nu$. (We are assuming, as we may by shrining $\overline{\mathcal{U}}$ if necessary, that $\overline{\mathcal{U}} \cap Z_I = \overline{\mathcal{U}} \cap Z_I^*$.)

B.2.3. The counter-clockwise generator $\alpha_i \in \pi_1(\Delta^*) \hookrightarrow \pi_1((\Delta^*)^k) = \pi_1(\mathcal{U})$ induces a quasi-unipotent monodromy operator $\gamma_i \in \mathrm{Aut}(V,Q)$, $1 \le i \le k$ [Sch73]. Passing to a finite cover of B if necessary, we may assume without loss of generality that γ_i is unipotent; let

$$N_i \;=\; \log \gamma_i \;\in\; \mathfrak{g}$$

be the nilpotent logarithm of monodromy, and

$$\sigma_I \;=\; \mathrm{span}_{\mathbb{R}_{>0}}\{N_1, \ldots, N_k\} \;\subset\; \mathfrak{g}_\mathbb{R} \;=\; \mathrm{Aut}(V_\mathbb{R}, Q)\,,$$

the *monodromy cone (for the coordinate chart centered at b).*

B.2.4. The universal cover of $\mathcal{U}$ is

$$\widetilde{\mathcal{U}} \;=\; \mathcal{H}^k \,\times\, \Delta^r\,.$$

The local lift

$$\widetilde{\Phi} : \widetilde{\mathcal{U}} \;\to\; D$$

of $\Phi|_{\mathcal{U}}$ is of the form

$$\widetilde{\Phi}(t,w) \;=\; \exp(\textstyle\sum \ell(t_i) N_i)\xi(t,w) \cdot F\,. \tag{B.2}$$

Here, $F \in \check{D}$,

$$\xi : \overline{\mathcal{U}} \;\to\; G_\mathbb{C}$$

is a holomorphic map, and we abuse notation by regarding the multi-valued $\ell(t_i)$ as giving coordinates on $\mathcal{H}$. Additionally, if $F(w) = \xi(0,w) \cdot F$, then $(W, F(w))$, is a mixed Hodge structure (MHS) polarized by the local monodromy cone σ_I. We say (W, F, σ_I) is a *limiting mixed Hodge structure* (LMHS).

The infinitesimal period relation implies that the restriction $\xi_I = \xi|_{\overline{\mathcal{U}} \cap Z_I^*}$ takes value in the centralizer

$$C_{I,\mathbb{C}} \;=\; \{g \in G_\mathbb{C} \mid \mathrm{Ad}_g N = N\,,\ \forall\ N \in \sigma_I\}$$

of the nilpotent cone σ_I. The map

$$F_I : Z_I^* \cap \overline{\mathcal{U}} \;\to\; D_I\,, \quad w \mapsto F_I(w) = \xi(0,w) \cdot F \tag{B.3}$$

defines a variation of limiting mixed Hodge structure $(W, F_I(w), \sigma_I)$ over $Z_I^* \cap \overline{\mathfrak{U}}$. The map (B.3) is not well-defined; it depends on our choice of coordinates. What is well-defined is the composition

$$Z_I^* \cap \overline{\mathfrak{U}} \xrightarrow{F_I} D_I \twoheadrightarrow \exp(\mathbb{C}\sigma_I)\backslash D_I\,. \tag{B.4}$$

(That is, it is the nilpotent orbit that is well-defined.) This yields the map Φ_I of §1.1 and (2.1).

The fact that $\exp(\mathbb{C}\sigma_I) \subset P^{-2}_{W,\mathbb{C}}$ implies that $(\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I^\alpha = \Gamma_I\backslash D_I^\alpha$ for $\alpha = 0, 1$. So (B.3) *does* induce well-defined maps

$$F_I^\alpha : Z_I^* \cap \overline{\mathfrak{U}} \ \to\ D_I^\alpha\,. \tag{B.5}$$

The maps (B.5) are local lifts of the maps Φ_I^α of (2.1).

B.3. **Deligne bigrading.** Given a mixed Hodge structure (W, F) on (V, Q), we have a Deligne splitting

$$V_\mathbb{C} \ = \ \oplus V^{p,q}_{W,F}$$

satisfying

$$W_\ell \ = \ \bigoplus_{p+q\le\ell} V^{p,q}_{W,F} \quad\text{and}\quad F^k \ = \ \bigoplus_{p\ge k} V^{p,q}_{W,F}\,.$$

The induced splitting

$$\mathfrak{g}_\mathbb{C} \ = \ \oplus\, \mathfrak{g}^{p,q}_{W,F}\,, \tag{B.6a}$$

of the Lie algebra $\mathfrak{g}_\mathbb{C}$ is defined by

$$\mathfrak{g}^{p,q}_{W,F} \ = \ \{x \in \mathfrak{g}_\mathbb{C} \mid x(V^{r,s}_{W,F}) \subset V^{p+r,q+s}_{W,F}\,,\ \forall\ r,s\}\,, \tag{B.6b}$$

satisfies

$$\kappa(\mathfrak{g}^{p,q}_{W,F}\,,\, \mathfrak{g}^{r,s}_{W,F}) \ = \ 0 \quad\text{if}\quad (p,q)+(r,s) \ne (0,0)\,, \tag{B.6c}$$

and is compatible with the Lie bracket in the sense that

$$[\mathfrak{g}^{p,q}_{W,F}\,,\, \mathfrak{g}^{r,s}_{W,F}] \ \subset\ \mathfrak{g}^{p+r,q+s}_{W,F}\,. \tag{B.6d}$$

It follows that $\mathfrak{g}_\mathbb{C} = \mathfrak{f} \oplus \mathfrak{f}^\perp$ with

$$\mathfrak{f} \ = \ \oplus_{p\ge 0}\, \mathfrak{g}^{p,q}_{W,F}$$

the parabolic Lie algebra of the stabilizer $\mathrm{Stab}_{G_\mathbb{C}}(F)$ of F, and

$$\mathfrak{f}^\perp \ = \ \oplus_{p<0}\, \mathfrak{g}^{p,q}_{W,F} \tag{B.7}$$

a nilpotent subalgebra of $\mathfrak{g}_\mathbb{C}$. The holomorphic $\xi : \overline{\mathfrak{U}} \to G_\mathbb{C}$ is determined by the property

$$\xi(t,w) \ \in\ \exp(\mathfrak{f}^\perp)\,.$$

Remark B.8. Without loss of generality, we may assume that (W, F) is $\mathbb{R}$–split

$$\overline{V^{p,q}_{W,F}} \ = \ V^{q,p}_{W,F}\,,$$

which implies

$$\overline{\mathfrak{g}^{p,q}_{W,F}} \ = \ \mathfrak{g}^{q,p}_{W,F}\,.$$

Then $\xi(0,0) \in P^{-2}_{W,\mathbb{C}}$.

B.4. **Period matrices and Schubert cells.** Since the *period matrix*

$$\exp(\textstyle\sum \ell(t_i)N_i)\xi(t,w)$$

of the local lift (B.2) takes value in $\exp(\mathfrak{f}^\perp)\cdot F$, the local lift $\widetilde{\Phi}(t,w)$ takes value in the open Schubert cell $\mathcal{S}$

$$\mathcal{S} \;=\; \exp(\mathfrak{f}^\perp)\cdot F \;=\; \left\{ E\in\check{D} \mid \dim\left(E^a\cap\overline{F^b_\infty}\right) = \dim\left(F^a\cap\overline{F^b_\infty}\right),\ \forall\, a,b \right\},$$

defined by

$$\overline{F^b_\infty} \;=\; \bigoplus_{c\le n-b} V^{c,a}_{W,F}\,.$$

The map $\mathfrak{f}^\perp\to\mathcal{S}$ sending $X\mapsto\exp(X)\cdot F$ is a biholomorphism. Let

$$X:\mathcal{S}\;\xrightarrow{\simeq}\;\mathfrak{f}^\perp\,. \tag{B.9}$$

denote the inverse. The obvious analogs of (B.6) hold with $\mathrm{End}(V_\mathbb{C})$ in place of $\mathfrak{g}_\mathbb{C}$. Given $X\in\mathrm{End}(V_\mathbb{C})$, let $X^{p,q}$ denote the component taking value in $\mathrm{End}(V_\mathbb{C})^{p,q}_{W,F}$. Recalling the notation of §B.4, we have

$$(\log\xi(t,w))^{-1,q} \;=\; \xi(t,w)^{-1,q}\,,$$

and

$$\begin{aligned}(X\circ\widetilde{\Phi}_{A^0})(t,w)^{-1,-1} &\;=\; \textstyle\sum_{i=1}^k \ell(t_i)N_i \;+\; \xi(t,w)^{-1,-1}\\ (X\circ\widetilde{\Phi}_{A^0})(t,w)^{-1,q} &\;=\; \xi(t,w)^{-1,q}\,,\quad q\neq -1\,.\end{aligned}$$

We say

$$(X\circ\widetilde{\Phi}_{A^0})^{-1,\bullet} \;=\; \textstyle\sum\,(X\circ\widetilde{\Phi}_{A^0})^{-1,q}$$

is the *horizontal component of the* (logarithm of the) *period matrix.*

In general, the function $\tilde{X}:\tilde{\mathcal{U}}\to\mathfrak{f}^\perp$ defined by

$$\tilde{X}(t,w) \;=\; X\circ\widetilde{\Phi}_{A^0}(t,w) \;-\; \textstyle\sum \ell(t_i)\,N_i$$

is well-defined on $\widetilde{\mathcal{U}}$, but multi-valued over $\mathcal{U}$. But the discussion above implies

$$\tilde{X}^{-1,\bullet}(t,w) \;\in\; \mathcal{O}(\overline{\mathcal{U}})\,. \tag{B.10}$$

B.5. **Extension data.** The fibre $\delta_I=\delta_{I,F}$ of $D_I\twoheadrightarrow D^0_I$ through $F\in D_I$ is the set of $\tilde{F}\in D_I$ inducing the same pure, weight ℓ Hodge filtrations on the $H^{n-a}(-a)$ as F. It is a complex affine space. To see this, first note that $\delta^1_{I,F}=C^{-1}_{I,\mathbb{C}}\cdot F$. As a unipotent group $C^{-1}_{I,\mathbb{C}}=\exp(\mathfrak{c}^{-1}_{I,\mathbb{C}})$ is biholomorphic to its Lie algebra $\mathfrak{c}^{-1}_{I,\mathbb{C}}$. The Lie algebra of $C^{-a}_{I,\mathbb{C}}$ is

$$\mathfrak{c}^{-a}_{I,\mathbb{C}} \;=\; \bigoplus_{p+q\le -a}\mathfrak{c}^{p,q}_{I,F}\,. \tag{B.11}$$

Since

$$\mathfrak{c}^{-1}_{I,\mathbb{C}} \;=\; \left(\mathfrak{c}^{-1}_{I,\mathbb{C}}\cap\mathfrak{f}\right)\;\oplus\;\left(\mathfrak{c}^{-1}_{I,\mathbb{C}}\cap\mathfrak{f}^\perp\right)$$

with

$$\mathfrak{c}^{-1}_{I,\mathbb{C}}\cap\mathfrak{f} \;=\; \bigoplus_{\substack{p\ge 0\\ p+q\le -1}}\mathfrak{c}^{p,q}_{I,F}\,.$$

the stabilizer F in $\mathfrak{c}^{-1}_{I,\mathbb{C}}$ and

$$\mathfrak{c}^{-1}_{I,\mathbb{C}} \cap \mathfrak{f}^{\perp} \;=\; \bigoplus_{\substack{p<0\\ p+q\le -1}} \mathfrak{c}^{p,q}_{I,F}\,,$$

we see that

$$\delta^1_{I,F} \;=\; \exp(\mathfrak{c}^{-1}_{\sigma,\mathbb{C}} \cap \mathfrak{f}^{\perp}) \cdot F\,,$$

and the map $\mathfrak{c}^{-1}_{\sigma,\mathbb{C}} \cap \mathfrak{f}^{\perp} \to \delta^1_{I,F}$ is a biholomorphism.

Likewise, $\mathbb{C}\sigma_I \subset \mathfrak{g}^{-1,-1}_{W,F}$ is an abelian ideal of the nilpotent algebra $\mathfrak{c}^{-1}_{I,\mathbb{C}} \cap \mathfrak{f}^{\perp}$, and we have a well-defined induced biholomorphism

$$\frac{\mathfrak{c}^{-1}_{I,\mathbb{C}} \cap \mathfrak{f}^{\perp}}{\mathbb{C}\sigma_I} \;\xrightarrow{\simeq}\; \exp(\mathbb{C}\sigma_I)\backslash\delta_{I,F}\,.$$

An identical argument establishes analogous statements for the fibre $\delta_W = \delta_{W,F}$ of $D_W \twoheadrightarrow D^0_W$ through $F \in D_W$.

B.6. **Infinitesimal period relation.** Finally we note that the infinitesimal period relation implies that the map (B.3) satisfies

$$\mathrm{d}F^p_I \;\subset\; F^{p-1}_I \otimes \Omega^1(Z^*_I \cap \overline{\mathfrak{U}})\,.$$

Equivalently, the pull-back $\xi_I^{-1}\,\mathrm{d}\xi_I$ of the Maurer-Cartan form on $\exp(\mathfrak{c}_{I,\mathbb{C}} \cap \mathfrak{f}^{\perp})$ under the map

$$\xi_I \;=\; \xi|_{Z^*_I\cap\overline{\mathfrak{U}}} \quad \text{sending} \quad w \;\mapsto\; \xi_I(w) = \xi(0,w) \tag{B.12}$$

takes value in $\mathfrak{c}_{I,\mathbb{C}} \cap (\oplus_p\, \mathfrak{g}^{-1,q}_{W,F})$. Since the centralizer inherits the Deligne splitting

$$\mathfrak{c}_{I,\mathbb{C}} \;=\; \bigoplus_{p+q\le 0} \mathfrak{c}^{p,q}_{I,F}\,, \quad \text{with} \quad \mathfrak{c}^{p,q}_{I,F} \;=\; \mathfrak{c}_{\sigma_I,\mathbb{C}} \cap \mathfrak{g}^{p,q}_{W,F}\,,$$

we may write this as

$$\xi_I^{-1}\,\mathrm{d}\xi_I \;\in\; \Omega^1(Z^*_I \cap \overline{\mathfrak{U}}\,,\, \mathfrak{c}^{-1,\bullet}_{I,F})\,. \tag{B.13}$$

Appendix C. Compatibility of weight closures

The purpose of this section is to review compatibility properties between the weight filtrations $W^I = W(\sigma_I)$, and discuss some of the implications for local lifts of period maps. These local results will have global consequences, including the following corollary of Lemma C.21.

Lemma C.1. *The maps Φ^0_W and Φ^1_W defined by* (2.21) *are proper and holomorphic.*

C.1. **The commuting $\mathfrak{sl}(2)$'s.** Our constructions are defined over the open strata Z_I^*. We will need to see that these strata-wise constructions satisfying certain compatibility conditions in order to obtain the properties asserted in the lemmas above. The key technical result here is the SL(2) orbit theorem [CKS86]. We briefly review the theorem, and then discuss consequences.

Suppose that $Z_J \subset Z_I$; equivalently, $I \subset J$. To begin we assume that we have a local coordinate chart centered at $b \in Z_J^*$ with local monodromy cone $\sigma = \sigma_J$ generated by $N_1, \ldots, N_k$ as in §B.2. Given $I \subset J = \{1, \ldots, k\}$, let σ_I be the face of σ_J generated by the N_i, with $i \in I$. Define

$$N_I \;=\; \sum_{i \in I} N_i \quad \text{and} \quad N_J \;=\; \sum_{j \in J} N_j \,.$$

Given this data, the SL(2) orbit theorem [CKS86] produces "commuting $\mathfrak{sl}_2$–pairs

$$N_I, Y_I \,;\; \hat{N}_J, \hat{Y}_J \;\in\; \mathfrak{g}_{\mathbb{R}} \,.$$

These pairs have following properties: N_I and Y_I commute with $\hat{N}_J$ and $\hat{Y}_J$; and there is a $(Y_I, \hat{Y}_J)$–eigenspace decomposition $\mathfrak{g}_{\mathbb{C}} = \oplus\, \mathfrak{g}_{a,b}$,

$$\mathfrak{g}_{a,b} \;=\; \{\xi \in \mathfrak{g}_{\mathbb{C}} \mid [Y_I, \xi] = a\xi \,,\; [\hat{Y}_J, \xi] = b\xi\} \,,$$

with integer eigenvalues a, b that splits the weight filtrations

$$W_\ell^I(\mathfrak{g}_{\mathbb{C}}) \;=\; \bigoplus_{a \le \ell} \mathfrak{g}_{a,b} \quad \text{and} \quad W_\ell^J(\mathfrak{g}_{\mathbb{C}}) \;=\; \bigoplus_{a+b \le \ell} \mathfrak{g}_{a,b} \,. \tag{C.2}$$

We have

$$N_I \;\in\; \mathfrak{g}_{-2,0}$$

and

$$N_J \;\in\; \bigoplus_{a \le 0} \mathfrak{g}_{a,-a-2} \,.$$

If we write

$$N_J \;=\; \sum_{a \le 0} N_{J,a} \,, \tag{C.3a}$$

with $N_{J,a} \in \mathfrak{g}_{a,-a-2}$, then

$$N_{J,0} \;=\; \hat{N}_J \,. \tag{C.3b}$$

C.2. **Consequences local lifts of Φ^α.** Recall the two maps $F_I^\alpha : Z_I^* \cap \overline{\mathfrak{U}} \to D_I^\alpha$ of (B.5). Since $I \subset J$, we have $Z_J^* \subset Z_I$. Fix a coordinate neighborhood $(t, w) \in \overline{\mathfrak{U}} \subset \overline{B}$ so that $Z_J^* \cap \overline{\mathfrak{U}} = \{t = 0\}$.

Lemma C.4. *Suppose that (t_m, w_m) and (t_m', w_m') are two sequences in $Z_I^* \cap \overline{\mathfrak{U}}$ converging to points $(0, w_\infty)$ and $(0, w_\infty') \in Z_J^* \cap \overline{\mathfrak{U}}$, respectively. If $F_I^\alpha(t_m, w_m) = F_I^\alpha(t_m', w_m')$ for all m, then $F_J^\alpha(0, w) = F_J^\alpha(0, w')$.*

This lemma is the analog of Proposition 2.25 for the "local lift" of Φ^α, and implies that this lift is continuous.

Proof. Given $(t,w) \in Z_I^* \cap \overline{\mathfrak{U}}$, recall that $F_I^\alpha(t,w)$ is the composition of $F_I(t,w) = \xi(t,w)\cdot F$ with the projection $D_I \twoheadrightarrow D_I^\alpha = C_{I,\mathbb{C}}^{-\alpha-1}\backslash D_I$ (§B.2.4). Moreover, $\xi(t,w)$ is holomorphic (and therefore continuous) on Δ^{k+r}, and takes value in $C_{I,\mathbb{C}}$ when restricted to $Z_I^* \cap \overline{\mathfrak{U}}$. So to prove the lemma, it suffices to show that

$$W_\ell^I(\mathfrak{c}_J) \ \subset\ W_\ell^J(\mathfrak{c}_J)\,. \tag{C.5}$$

It is a general fact that the centralizers satisfy

$$\mathfrak{c}_J \ \subset\ W_0^J(\mathfrak{g}_\mathbb{C}) \quad\text{and}\quad \mathfrak{c}_J \ \subset \mathfrak{c}_I \ \subset\ W_0^I(\mathfrak{g}_\mathbb{C})\,.$$

So (C.2) implies

$$\mathfrak{c}_J \ \subset \bigoplus_{\substack{a\,\le\,0\\ a+b\,\le\,0}} \mathfrak{g}_{a,b}\,. \tag{C.6}$$

Note that (C.6) implies the desired (C.5) for $\ell \ge 0$.

Suppose that $X \in W_\ell^I(\mathfrak{c}_J)$ for some $\ell < 0$. Then (C.2) and (C.6) imply that there exists unique $X_{a,b} \in \mathfrak{g}_\mathbb{C}$ so that

$$X \ =\ \sum_{\substack{a\,\le\,\ell\\ a+b\,\le\,0}} X_{a,b}\,.$$

In order to establish (C.5), we need to show

$$X_{a,b} \ =\ 0 \quad\text{for all}\ \ a+b>\ell\,. \tag{C.7}$$

From $N_J(X)=0$ and (C.3) we see that $\hat N_J(X_{\ell,b}) = 0$. Since $\{\hat N_J, \hat Y_J\}$ is an $\mathfrak{sl}_2$–pair, the centralizer $\mathfrak{c}(\hat N_J)$ of $\hat N_J$ satisfies

$$\mathfrak{c}(\hat N_J) \ \subset\ \bigoplus_{b\le 0}\mathfrak{g}_{a,b}\,. \tag{C.8}$$

This forces $X_{\ell,b} = 0$ for all $b > 0$, and yields the desired (C.7) for $a = \ell$.

Working inductively, fix $m < \ell < 0$ and assume that (C.7) holds for all $m < a \le \ell$. Again, $N_J(X) = 0$ and (C.3) implies $\hat N_J(X_{m,b}) = 0$ for all $m+b > \ell$. Since, $b > \ell - m > 0$, (C.8) implies $X_{m,b} = 0$ for all $m + b > \ell$. This establishes the desired (C.7) for $a = m$ and completes the induction. $\square$

C.3. **When weight filtrations coincide.** The properties (C.2) and (C.3b) yield

Lemma C.9. *Suppose that $I \subset J$. The following are equivalent:*

(i) *The weight filtrations coincide $W^I = W^J$.*
(ii) *We have $\hat Y_J = 0$.*
(iii) *We have $\hat N_J = 0$.*
(iv) *The cone $\sigma_J \subset \mathfrak{c}_I^{-1}$.*

Corollary C.10. (a) *If $I \subset I' \subset J$ and $W^I = W^J$, then $W^I = W^{I'} = W^J$.*
(b) *If $W^{I_1} = W^{I_2}$, then $W^{I_i} = W^{I_1\cup I_2}$.*

(c) *The union*

$$I_W \;=\; \bigcup_{W^I=W} I$$

is the unique maximal set I_W *such that* $W = W^{I_W}$.

If $W^I = W^J$, then $\mathfrak{g}_{a,\bullet} = \mathfrak{g}_{a,0}$ implies

$$\mathfrak{c}_J^{-a} \;\subset\; \mathfrak{c}_I^{-a}\,, \tag{C.11a}$$

and

$$\frac{\mathfrak{c}_J^{-a}}{\mathfrak{c}_J^{-a-1}} \;\hookrightarrow\; \frac{\mathfrak{c}_I^{-a}}{\mathfrak{c}_I^{-a-1}}\,. \tag{C.11b}$$

In the case $a = 1$, the inclusion (C.11a) yields the striking implication (known to the experts)

Lemma C.12. *If* $\sigma_J \subset \mathfrak{c}_I^{-1}$, *then* $\sigma_J \subset \mathfrak{c}_I^{-2}$.

Corollary C.13. *We have* $\exp(\mathbb{C}\sigma_{I_W}) \subset C_{I,\mathbb{C}}^{-2}$.

C.4. **Consequences for LMHS.** Note that $Z_J^* \subset Z_I$ if and only if $I \subset J$. In this case, $\Gamma_J \subset \Gamma_I$. We will also see that $D_J \subset D_I$, cf. (C.20). In particular, we have an induced $\Gamma_J\backslash D_J \to \Gamma_I\backslash D_I$. When $W^I = W^J$ (equivalently, $Z_J^* \subset Z_I \cap Z_W$), then this map descends to $\Gamma_J\backslash D_J^a \to \Gamma_I\backslash D_I^a$.

Lemma C.14. *The maps* Φ_I^0 *and* Φ_I^1 *of* (2.1) *extend to proper holomorphic maps on* $Z_I \cap Z_W$. *These extensions are compatible with the* Φ_J^0 *and* Φ_J^1 *on* $Z_J^* \subset Z_I \cap Z_W$ *in the sense that the we have a commutative diagram*

$$\begin{tikzcd}
Z_J^* \arrow[r, hook] \arrow[d, "\Phi_J^1"] \arrow[dd, bend right=60, "\Phi_J^0"'] & Z_I \cap Z_W \arrow[d, "\Phi_I^1"'] \arrow[dd, bend left=60, "\Phi_I^0"] \\
\Gamma_J\backslash D_J^1 \arrow[r] \arrow[d, two heads] & \Gamma_I\backslash D_I^1 \arrow[d, two heads] \\
\Gamma_J\backslash D_J^0 \arrow[r] & \Gamma_I\backslash D_I^0\,.
\end{tikzcd} \tag{C.15}$$

Lemma C.14 is a corollary of Lemma C.17.

Recall (§B.2.4) that the local lift of $\Phi_I : Z_I \to (\Gamma_I \exp(\mathbb{C}\sigma_I))\backslash D_I$ is

$$\nu_I \circ F_I : Z_I^* \cap \overline{\mathfrak{U}} \to \; \exp(\mathbb{C}\sigma_I)\backslash D_I\,. \tag{C.16}$$

Lemma C.17. *There is a well-defined holomorphic map*

$$\widetilde{\Phi}_I : Z_I \,\cap\, Z_W \,\cap\, \overline{\mathfrak{U}} \;\to\; \exp(\mathbb{C}\sigma_{I_W})\backslash D_I \tag{C.18}$$

that, when restricted to $Z_J^* \subset Z_I \cap Z_W$, *coincides with the composition* $\nu_{I_W} \circ F_J$.

Proof of Lemma C.14. Given $a = 0, 1$, Corollary C.13 implies that

$$(\exp(\mathbb{C}\sigma_{I_W})C_{I,\mathbb{C}}^{-a-1})\backslash D_I \;=\; C_{I,\mathbb{C}}^{-a-1}\backslash D_I \;=\; D_I^a\,.$$

So the composition

$$Z_I \cap Z_W \cap \overline{\mathfrak{U}} \xrightarrow{\widetilde{\Phi}_I} \exp(\mathbb{C}\sigma_{I_W})\backslash D_I \longrightarrow\!\!\!\!\rightarrow (\exp(\mathbb{C}\sigma_{I_W})C_{I,\mathbb{C}}^{-a-1})\backslash D_I \;=\; D_I^a$$

is the local coordinate expression for the extension $\Phi_I^a : Z_I \cap Z_W \to \Gamma_I\backslash D_I^a$ of (C.15). Thus Lemma C.14 follows directly from Lemma C.17. □

Proof of Lemma C.17. Suppose that $I \subset J$ and $W^I = W^J$. Consider a local lift $\widetilde{\Phi}(t,w)$ over a chart $\overline{\mathfrak{U}}$ centered at $b \in Z_J^*$ (as in §B.2). Along

$$Z_J \cap \overline{\mathfrak{U}} \;=\; \{t_j = 0 \;\forall\; j \in J\} \;=\; \{0\} \times \Delta^r \;\ni\; (0,w)$$

we have the map $F_J : Z_J^* \cap \overline{\mathfrak{U}} \to D_J$ of (B.3)

$$F_J(w) \;=\; \xi(0,w)\cdot F\,. \tag{C.19a}$$

Along $Z_I^* \cap \overline{\mathfrak{U}} = \{t_i = 0 \text{ iff } i \in I\}$ we may choose a well-defined branch of $\ell(t_j)$ for all $j \in J\backslash I$. Then the map $F_I : Z_I^* \cap \overline{\mathfrak{U}} \to D_I$ is given by

$$F_I(t,w) \;=\; \exp\left(\sum_{j\in J\backslash I} \ell(t_j)N_j\right)\xi(t,w)\cdot F\,. \tag{C.19b}$$

Comparing the expressions (C.19) for F_J and F_I, and keeping $C_J \subset C_I$ and (C.11a) in mind, we see that

$$F \;\in\; D_J \;\subset\; D_I \tag{C.20}$$

and F_J takes value in D_I. (Note that the containment $F \in D_I$ is nontrivial, as F arises from the LMHS along Z_J^*.) It follows from (C.19) and (C.20) that

$$\nu_J \circ F_J : Z_J^* \cap \overline{\mathfrak{U}} \;\to\; \exp(\mathbb{C}\sigma_J)\backslash D_I$$

also takes value in (a quotient of) D_I. The lemma now follows from (C.19). □

If follows from Corollary C.10(c) and (C.20) that the orbit

$$D_W \;=\; P_{W,\mathbb{C}}\cdot F \;\supset\; D_I$$

is independent of our choice of D_I and $F \in D_I$ so long as $W^I = W$. It is straightforward to verify

Lemma C.21. *There is a well-defined holomorphic map*

$$\widetilde{\Phi}_W : Z_W \cap \overline{\mathfrak{U}} \;\to\; \exp(\mathbb{C}\sigma_{I_W})\backslash D_W \tag{C.22}$$

that, when restricted to Z_I^**, coincides with* $\nu_{I_W} \circ F_I$*.*

Proof of Lemma C.1. By essentially the same argument as given for Φ_I^a in the proof of Lemma C.14, the composition

$$Z_W \cap \overline{\mathfrak{U}} \xrightarrow{\widetilde{\Phi}_I} \exp(\mathbb{C}\sigma_{I_W})\backslash D_W \longrightarrow\!\!\!\!\rightarrow D_W^a$$

is the local coordinate expression for Φ_W^a. So it follows immediately that Φ_W^a is holomorphic.

To see that Φ_W^1 is proper, it suffices to show that Φ_W^0 is proper. And to see that Φ_W^0 is proper, it suffices to show that the extension $\Phi_I^0 : Z_I \cap Z_W \to \Gamma_I\backslash D^0$ of (C.15) is proper. The latter is due to [Gri70, §9]. □

C.4.1. *Remark on the extension question.* Given Lemmas C.14 and C.17, it is natural to ask if the extension (C.18) is global; that is, does there exist an extension of $\Phi_I : Z_I^* \to (\Gamma_I \exp(\mathbb{C}\sigma_I))\backslash D_I$ to the weight closure $Z_I \cap Z_W$? The answer in general is no, because the action of $\exp(\mathbb{C}\sigma_{I_W})$ on D_I does not descend to a well-defined action on $\Gamma_I\backslash D_I$. (Likewise, while the quotient $\exp(\mathbb{C}_{I_W})\backslash D_I$ is well-defined, the action of Γ_I on D_I does not descend to the quotient.) In general, to obtain such an extension, one would need at the very least for $\Gamma_I \exp(\mathbb{C}\sigma_{I_W}) \subset G_I$ to be a subgroup. (In general it is not. The product $\Gamma_I \exp(\mathbb{C}\sigma_I)$ is a subgroup because $\Gamma_I \subset C_I$ centralizes σ_I.) The ideal circumstance here would be for Γ_I to centralize the larger cone σ_{I_W}. If it is the case that the image of $\Gamma_I \exp(\mathbb{C}\sigma_{I_W})$ under the projection $G_I \twoheadrightarrow G_I^a$ is a subgroup, then one does obtain an extension of Φ_I^a. For example, since $\exp(\mathbb{C}\sigma_{I_W}) \subset C_{I,\mathbb{C}}^{-2}$, and the C_I^{-a} are normal subgroups of C_I, the image is always a subgroup when $a = 0, 1, 2$. In particular, in the case $a = 2$, we have

$$\begin{array}{ccc}
Z_I^* & \xrightarrow{\Phi_I^2} & (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I^2 \\
\big\downarrow & & \downarrow \\
Z_I \cap Z_W & \longrightarrow & (\exp(\mathbb{C}\sigma_{I_W})\Gamma_I)\backslash D_I^2 \\
\big\downarrow & & \downarrow \\
Z_W & \xrightarrow{\Phi_W^2} & (\exp(\mathbb{C}\sigma_{I_W})\Gamma_W)\backslash D_W^2\,.
\end{array}$$

C.5. **Implications for polarizations.** We close §C with two results on polarizations. These are consequences of: (i) the fact that $W(N)$ is independent of our choice of $N \in \sigma_{I_W}$ [CK82], and (ii) the classification of $\mathrm{Ad}(G_\mathbb{R})$–orbits of nilpotent $N \in \mathfrak{g}_\mathbb{R}$ [CM93] (not the SL(2) orbit theorem).

Lemma C.23. *Suppose that (W, F) is a MHS and $W = W^I = W^J = W^{I\cup J}$. If (W, F) is polarized by both σ_I and σ_J, then the MHS is also polarized by $\sigma_{I\cup J}$. In particular, $D_I \cap D_J \subset D_{I\cup J}$.*

Proof. Let

$$\overline{\sigma}^W_{I\cup J} \;=\; \bigcup_{\substack{W = W^K \\ K \subset I \cup J}} \sigma_K$$

denote the "weight-closure" of $\sigma_{I\cup J}$; note that each of the σ_I, σ_J and $\sigma_{I\cup J}$ is contained in $\overline{\sigma}^W_{I\cup J}$. Suppose that $N \in \sigma_{I\cup J}$. The definition of $W = W(N)$ implies that $N^k : \mathrm{Gr}^W_{n+k} \to \mathrm{Gr}^W_{n-k}$ is an isomorphism. Standard $\mathfrak{sl}(2)$–representation theory implies that

$$Q^N_{n+k} \;=\; Q(\cdot, N^k\cdot)$$

defines a nondegenerate, $(-1)^{n+k}$–symmetric bilinear form on Gr^W_{n+k}, and that the restriction of this bilinear form to

$$\mathrm{Prim}^N_{n+k} \;=\; \ker\{N^{k+1} : \mathrm{Gr}^W_{n+k} \to \mathrm{Gr}^W_{n-k-2}\}$$

is also nondegenerate. The mixed Hodge structure (W, F) is polarized by N if and only if the Hodge–Riemann bilinear relations are satisfied by the Hodge filtration $F(\mathrm{Prim}^N_{n+k})$ and Q^N_{n+k}. The first Hodge–Riemann bilinear relation follows directly from $\overline{\sigma}^W_{I\cup J} \subset \mathfrak{g}^{-1,-1}_{W,F}$ and the fact that $Q(V^{p,q}_{W,F}, V^{r,s}_{W,F}) = 0$ unless $(p+q)+(r+s) = 2n$ and $p - q = s - r$.

Consider the adjoint action of G on $\mathfrak{g}$, and let $G_{\mathbb{R}}^{0,0} \subset G_{\mathbb{R}}$ be the subgroup preserving the Deligne splitting $\mathfrak{g}_{\mathbb{C}} = \oplus\, \mathfrak{g}^{p,q}_{W,F}$. The weight-closure $\overline{\sigma}^W_{I\cup J} \subset \mathfrak{g}^{-1,-1}_{W,F}$ is contained in a $G_{\mathbb{R}}^{0,0}$–orbit [BPR17, Lemma 3.5]. The second Hodge–Riemann bilinear relation is then a consequence of the representation theoretic classification [CM93] of $\mathrm{Ad}(G_{\mathbb{R}})$–orbits of nilpotent $N \in \mathfrak{g}_{\mathbb{R}}$ and the discussion of [BPR17, §2.5]. $\square$

Lemma C.24. *Suppose that (W, F_1) and (W, F_2) are MHS polarized by σ_{I_1} and σ_{I_2}, respectively, and that $F_1(\mathrm{Gr}^W) = F_2(\mathrm{Gr}^W)$. Set $J = I_1 \cup I_2$. Given $N \in \sigma_J$, the bilinear form Q^N_{n+k} is nondegenerate on Gr^W_{n+k}, and the restriction to Prim^N_{n+k} polarizes the Hodge structure defined by $F_1(\mathrm{Prim}^N_{n+k}) = F_2(\mathrm{Prim}^N_{n+k})$.*

Remark C.25. Note that the lemma does not assert that σ_J *polarizes* the MHS (W, F_a), $a = 1, 2$: a priori, it need not be the case that $N(F_a^p) \subset F_a^{p-1}$. So, given the hypothesis of Lemma C.24, it would be interesting to know if there exists a MHS (W, F) that is polarized by σ_J and such that $F(\mathrm{Gr}^W) = F_a(\mathrm{Gr}^W)$, $a = 1, 2$? Equivalently, are the $F_a(\mathrm{Gr}^W) \in D^0_J$?

Proof. Corollary C.10 asserts that $W = W^J$. As in the proof of Lemma C.23, the fact that $W(N)$ is independent of the choice of $N \in \overline{\sigma}^W_J$ implies that $\overline{\sigma}^W_J$ is contained in an $\mathrm{Ad}(G_{\mathbb{R}})$–orbit. Additionally, $\sigma_{I_a} \subset \mathfrak{g}^{-1,-1}_{W,F_a} \subset W_{-2}(\mathfrak{g}_{\mathbb{C}})$ and $F_1(\mathrm{Gr}^W) = F_2(\mathrm{Gr}^W)$ imply that

$$\overline{\sigma}_J \ \subset\ \mathfrak{g}^{-1,-1}_{W,F_a} \quad \text{modulo} \quad W_{-3}(\mathfrak{g}_{\mathbb{C}})\,, \quad a = 1, 2\,.$$

The lemma then follows from the representation theoretic classification [CM93] of $\mathrm{Ad}(G_{\mathbb{R}})$–orbits of nilpotent $N \in \mathfrak{g}_{\mathbb{R}}$ and the discussion of [BPR17, §2.5]. $\square$

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