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EXISTENCE AND DIFFERENTIABILITY THEOREMS
FOR SOLUTIONS OF VARIATIONAL PROBLEMS
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1. Potential Functions of their Generalized Derivatives

The notion of a "potential function $u(x^1, \dots, x^n)$ of its generalized "derivatives" was introduced by G. C. Evans[1]. For simplicity of notation we

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1) Fundamental Points of Potential Theory, Rice Institute Pamphlets, vol. 1, No. 4 (1920), pp. 274-285 particularly.

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restrict ourselves to two variables x, y, most proofs admitting immediate extensions to more variables. It will be convenient to use the following notation: Let G be an open plane region and $a > 0$. G_a shall consist, then, of all those points (x_o, y_o), for which the points

$$x_o - a \leqq x \leqq x_o + a$$
$$y_o - a \leqq y \leqq y_o + a$$

are in G. It is $G_a \subset G_b$ for $b < a$ and $G_a \to G$ for $a \to 0$.

<u>Definition.</u> A function $u(x, y)$, defined in an open region G, is called a potential function of its generalized derivatives (P. F. G. D.) if for each $\alpha > 0$ it satisfies the following conditions:

1) $u(x, y)$ is summable over G

2) two functions $v(x, y)$ and $w(x, y)$, summable over G_α exist such that

$$(1) \qquad \begin{aligned} \int_a^b [u(x, d) - u(x, c)]dx &= \int_a^b \int_c^d w(x, y)dxdy \text{ and} \\ \int_c^d [u(b, y) - u(a, y)]dy &= \int_a^b \int_c^d v(x, y)dxdy \end{aligned}$$

for almost all rectangles

$$(a, b;\ c, d):\ a \leqq x \leqq b, \quad c \leqq y \leqq d$$

interior to G, where "almost all" means: there exist sets Z_x and Z_y of measure 0 on the x - axis and y - axis respectively, such that (1) is true for all values a, b, c, d with $a \not\subset Z_x$, $b \not\subset Z_x$, $c \not\subset Z_y$, $d \not\subset Z_y$.

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We recall the concept of a derivative in the sense of Lebesgue of a set function $F(E)$[2]: One says, $F(E)$ has a derivative at P_o if for any sequence of measurable sets E_n such that E_n is contained in a square $\sigma(P_o, \alpha_n)$ with center P_o, diameter $\sqrt{2} \cdot \alpha_n$, $\alpha_n \to 0$, and

$$\frac{m(E_n)}{m\,\sigma(P_0, \alpha_n)} \geq \eta > 0$$

(η must be independent of n but may vary with the sequence $\{E_n\}$), the

$$\lim_{n \to \infty} \frac{F(E_n)}{mE_n}$$

exists.

Since in our case v is summable over G_α for each $\alpha > 0$, the set function $\int_E v\,dP$ is differentiable almost everywhere in G and, except for a new set of measure 0, its derivative is equal to v.

If $\int_E v\,dP$ is differentiable at $P_o = (x_o, y_o)$ we call its derivative $D_x u(x_o, y_o)$ the generalized derivative of u with respect to x at P_o, and define $D_y u$ correspondingly. We aim at a characterization of the P, F, G, D. For this purpose we need the following two lemmas:

<u>Lemma 1.</u> If $u(x, y)$ is a P, F, G, D, then the function

(2) $$u_h(x,y) = \frac{1}{4h^2} \int_{x-h}^{x+h} \int_{y-h}^{y+h} u(\xi, \eta)\, d\xi\, d\eta$$

is of class C' on each closed region where it is defined and one has (the subscript x meaning partial differentiation with respect to x)

(3) $$u_{hx} = \frac{1}{4h^2} \int_{x-h}^{x+h} \int_{y-h}^{y+h} D_x u(\xi, \eta)\, d\xi\, d\eta$$

and correspondingly with respect to y.

<u>Proof.</u> For small k one has

$$u_h(x+k, y) - u_h(x,y) = \frac{1}{4h^2} \int_{x+h}^{x+h+k} \int_{y-h}^{y+h} u\, d\xi\, d\eta - \frac{1}{4h^2} \int_{x-h}^{x-h+k} \int_{y-h}^{y+h} u\, d\xi\, d\eta$$

hence $\lim_{k\to 0} \frac{1}{k}\int_{x\pm h}^{x\pm h+k}\int_{y-h}^{y+h} u \, d\xi \, d\eta = \int_{y-h}^{y+h} u(x\pm h, \eta)\, d\eta$

for almost all x, for which this operation has a meaning and therefore

$$\frac{\partial u_h}{\partial x} = \frac{1}{4h^2}\int_{y-h}^{y+h} [u(x+h,\eta) - u(x-h,\eta)]\, d\eta$$

$$= \frac{1}{4h^2}\int_{x-h}^{x+h}\int_{y-h}^{y+h} v(\xi,\eta)\, d\xi\, d\eta = \frac{1}{4h^2}\int_{x-h}^{x+h}\int_{y-h}^{y+h} D_x u(\xi,\eta)\, d\xi\, d\eta .$$

The function $u_h(x, y)$ is absolutely continuous in x on each closed segment interior to the set, where it is defined; hence

(4) $$u_h(x,y) - u_h(x_0,y) = \int_{x_0}^{x} \frac{\partial u_h}{\partial x} dx = \int_{x_0}^{x}\left\{\frac{1}{4h^2}\int_{x-h}^{x+h}\int_{y-h}^{y+h} D_x u \, d\xi\, d\eta\right\} dx$$

Since the right side of (3) is continuous, we see from (4) that (3) holds everywhere. (2) is an immediate consequence of (3).

<u>Lemma 2.</u> Let $|\varphi(x,y)|^p$ $(p \geq 1)$ be summable on the open set D, and $\varphi_h(x,y)$ be defined in the same way as u_h (compare (2)). $\varphi_h(x,y)$ exists on D_h. Then one has

(5) $$\iint_{D_h} |\varphi_h(x,y)|^p \, dx\, dy \leq \iint_{D_h} |\varphi(x,y)|^p \, dx\, dy$$

(6) $$\lim_{h\to 0} \iint_{D_\alpha} |\varphi - \varphi_h|^p dx\,dy = 0 \text{ for each } \alpha > 0 \ (\alpha > h)$$

<u>Proof.</u> As a consequence of the Hölder inequality we have:

$$\iint_{D_h} \left|\frac{1}{4h^2}\int_{x-h}^{x+h}\int_{y-h}^{y+h} \varphi(\xi,\eta)\, d\xi\, d\eta\right|^p dx\, dy \leq \frac{1}{4h^2}\iint_{D_h}\left[\int_{x-h}^{x+h}\int_{y-h}^{y+h} |\varphi|^p d\xi\, d\eta\right] dx\, dy =$$

$$= \frac{1}{4h^2}\iint_{D_h}\left[\int_{-h}^{h}\int_{-h}^{h} |\varphi(x+\xi, y+\eta)|^p d\xi\, d\eta\right] dx\, dy =$$

$$= \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h}\left[\iint_{D_h} |\varphi(x+\xi, y+\eta)|^p dx\, dy\right] d\xi\, d\eta =$$

$$= \frac{1}{4h^2}\int_{-h}^{+h}\int_{-h}^{h}\left[\iint_{\Delta_{\xi,\eta}} |\varphi(x,y)|^p dx\, dy\right] d\xi\, d\eta \leq \iint_{D} |\varphi(x,y)|^p dx\, dy$$

where $\Delta_{\xi,\eta}$ is the region arising from D_h by the translation with components ξ, η. $\Delta_{\xi,\eta}$ is in D since ξ, η vary between -h and h.

In order to prove (6) let $\{\varphi_n(x, y)\}$ be a sequence of functions continuous on D and such that

$$\lim_{n\to\infty} \iint_D |\varphi - \varphi_n|^p \, dx\,dy = 0$$

For $h < \alpha$ one has, φ_{nh} being defined in the same way as u_h and φ_h

$$\left[\iint_{D_\alpha} |\varphi - \varphi_h|^p \, dx\,dy\right]^{\frac{1}{p}} \le \left[\iint_{D_\alpha} |\varphi - \varphi_n|^p \, dx\,dy\right]^{\frac{1}{p}} +$$

$$+ \left[\iint_{D_\alpha} |\varphi_n - \varphi_{nh}|^p \, dx\,dy\right]^{\frac{1}{p}} + \left[\iint_{D_\alpha} |\varphi_h - \varphi_{nh}|^p \, dx\,dy\right]^{\frac{1}{p}}.$$

It follows from (5) that the right side of this inequality is at most

$$2\left[\iint_{D_\alpha} |\varphi - \varphi_n|^p \, dx\,dy\right]^{\frac{1}{p}} + \left[\iint_{D_\alpha} |\varphi_n - \varphi_{nh}|^p \, dx\,dy\right]^{\frac{1}{p}}$$

Let now $\varepsilon > 0$ be given. We first choose n so large that the first term is less than $\frac{\varepsilon}{2}$. Since φ_n is continuous h can be chosen so small that the second term becomes less than $\frac{\varepsilon}{2}$

We can give now a characterization of the P. F. G. D. as follows:

__Theorem 1.__ u(x, y) is a P. F. G. D. on G if, and only if, for each $\alpha > 0$ there exists a sequence of functions u_n of class C' on $\mathcal{G}_\alpha$ such that

$$\text{(7)} \qquad \lim_{n\to\infty} \iint_{\mathcal{G}_\alpha} |u - u_n| \, dx\,dy = 0$$

$$\text{(8)} \qquad \lim_{\substack{m\to\infty \\ n\to\infty}} \iint_{\mathcal{G}_\alpha} |u_{nx} - u_{mx}| \, dx\,dy = 0, \quad \lim_{\substack{m\to\infty \\ n\to\infty}} \iint_{\mathcal{G}_\alpha} |u_{ny} - u_{my}| \, dx\,dy = 0.$$

When this is true one has

$$(9)\quad \begin{cases} \lim\limits_{n\to\infty} \iint\limits_{G_\alpha} |u_{nx} - D_x u|\,dx\,dy = 0 \\ \lim\limits_{n\to\infty} \iint\limits_{G_\alpha} |u_{ny} - D_y u|\,dx\,dy = 0. \end{cases}$$

Especially, as u_n one may take a sequence $u_{h_n}(x)$ with $h_n < \alpha$ and $h_n \to 0$.

Proof. A. Necessary. Let u(x, y) be a P. F. G. D. Then, according to Lemma 1, u_h is of class C' on G_α for $h < h_0$, say. By Lemma 2 we have

$$\lim_{h\to 0} \iint\limits_{G_\alpha} |u - u_h|\,dx\,dy = 0$$

which proves (7). Putting $v = D_x u$ we have, by Lemma 1,

$$\frac{\partial u_h}{\partial x} = v_h$$

and therefore by Lemma 2

$$\lim_{h\to 0} \iint\limits_{G_\alpha} |u_{hx} - D_x u|\,dx\,dy = 0.$$

This proves (9), and from (9) follows (8).

B. Sufficient. Suppose that there exists for each G_α a sequence $\{u_n\}$ of the type indicated. It follows from (8) that u_{nx} converges in the mean to a function v. Designate for a fixed x by $G_{\alpha x}$ the set of values y for which (x, y) lies in G_α. Then a subsequence $\{u_n^1\}$ of $\{u_n\}$ exists such that

$$(10)\quad \lim_{n\to\infty} \int\limits_{G_{\alpha x}} |u - u_n^1|\,dy = 0$$

for almost all x and correspondingly for y. Since for each interval (a, b; c, d) in G

$$\int_c^d [u_n^1(b, y) - u_n^1(c, y)]\,dy = \int_a^b \int_c^d u_{nx}^1\,dx\,dy,$$

one has for each such interval

$$\int_c^d [u(b,y)-u(a,y)]\,dy - \int_a^b\int_c^d v\,dx\,dy =$$

$$\int_c^d [u(b,y)-u_n^1(b,y)]\,dy - \int_c^d [u(a,y)-u_n^1(a,y)]\,dy +$$

$$-\int_a^b\int_c^d (v-u_{nx}^1)\,dx\,dy$$

(10) and the fact that u_{nx}^1 tends to v in the mean shows that the left side vanishes for almost all intervals in $\mathcal{G}_\alpha$

<u>Theorem 2.</u> A P, F, G, D, u is equivalent to a function $\bar{u}(x, y)$ ("equivalent" means: $u(x) = \bar{u}(x)$ except for a set of measure 0) with the following properties:

1) for almost every y_o the function $\bar{u}(x, y_o)$ is absolutely continuous in x on each closed segment $\gamma \le x \le \delta$ such that the points (x, y_o) are in G, and correspondingly with respect to y.

2) $\bar{u}_x = D_x u$ and $\bar{u}_y = D_y u$ almost everywhere.

3) $\bar{u}(x, y)$ is summable in $x^{(y)}$ uniformly in $y(x)$ in each closed interval (a, b, c, d) in G, i.e. the integrals

$$\int_c^d |\bar{u}(x_1, y)|\,dy \qquad \text{and} \qquad \int_a^b |\bar{u}(x, y_1)|\,dx$$

are bounded for $a \le x_1 \le b$ ~~for~~ or $c \le y_1 \le b$.

<u>Proof.</u> Let $\alpha_p > 0$ tend monotonically to 0. The proof of Theorem 1 shows that

$$\lim_{h\to 0} \iint_{\mathcal{G}_{\alpha_p}} [|u-u_h| + |D_x u - u_{hx}|]\,dx\,dy = 0.$$

Designating by $\mathcal{G}_{\alpha_\nu y_o}$ ($\mathcal{G}_{\alpha_\nu x_o}$) the set of values x (y) such that $(x, y_o)((x_o, y))$ is in $\mathcal{G}_{\alpha_\nu}$ we can choose a sequence $\{h_\nu^*\}$, $h_\nu \to 0$, such that

(11) $$\lim_{\nu\to\infty} \int_{\mathcal{G}_{\alpha_1}(y_o)} [|u-u_{h_\nu}| + |D_x u - u_{h_\nu x}|]\,dx = 0$$

for almost all y_o. We then choose a subsequence $\{h_\nu^1\}$ of $\{h_\nu\}$ such that

(12) $\lim_{v\to\infty} \int_{g_{\alpha_1 x_0}} [|u - u_{h^1_v}| + |D_y u - u_{h^1_v y}|]\,dy = 0$

for almost all x_0. This sequence $\{h^1_v\}$ corresponds to α_1. We then take a subsequence $\{h^2_v\}$ of $\{h^1_v\}$ satisfying (11) and (12) with α_2 instead of α_1; similarly we define successive subsequences $\{h^3_v\}$, $\{h^4_v\}$, ... with respect to $\alpha_3, \alpha_4, \dots$. For the sequence h^v_v (11) and (12) will hold with any α_n for almost all x_0 and y_0.

Take now a fixed value y_0 not in the exceptional set and let $a \leq x \leq b$ be any closed segment such that the points (x, y_0) are in g_{α_p}. We then have

$$\lim_{n\to\infty} \int_a^b |u(x, y_0) - u_{h^n_n}(x, y_0)|\,dx = 0$$

$$\lim_{n\to\infty} \int_a^b |u_{h^n_n x}(x, y_0) - D_x u(x, y_0)|\,dx = 0,$$

and $D_x u(x, y_0)$ is summable over $a \leq x \leq b$. We shall see in Lemma 3, following this proof, that it follows herefrom that the functions $u_{h^n_n}(x, y_0)$ tend uniformly towards a function $\bar{u}(x, y_0)$. $\bar{u}(x, y_0)$ is absolutely continuous in x, similarly one treats the case where the roles of x and y are exchanged.

In order to prove 3) it is sufficient to show that the functions $u_{h^n_n}(x,y) = u_n(x,y)$ converge in the mean in y uniformly for all x_1, i.e. that to a given $\varepsilon > 0$ a number $N(\varepsilon)$ can be found such that

$$\int_c^d |u_m(x_1, y) - u_n(x_1, y)|\,dy < \varepsilon$$

for $n, m > N(\varepsilon)$ and all x_1 with $a \leq x \leq b$. We first choose $\eta > 0$ so small that

$$\int_{x_0}^{x_1} \int_c^d |D_x u|\,dx\,dy < \frac{\varepsilon}{6}$$

as soon as $0 \leq x_1 - x_0 \leq \eta$; then a finite set of values x_0 for which $u_m(x_0, y)$ converges uniformly so densely that each value x_1 has at most distance η from a suitable value x_o. We then choose $N(\varepsilon)$ so large that

$$\int_a^b \int_c^d | D_x u - u_{nx} | \, dx \, dy < \frac{\varepsilon}{6} \qquad \text{for} \quad n > N(\varepsilon)$$

and

(13) $$\int_c^d | u_m(x_0, y) - u_n(x_0, y) | \, dy < \frac{\varepsilon}{3} \qquad \text{for} \quad m, n > N(\varepsilon)$$

and each of the finite number of values x_o. We then have for $0 \leq x_1 - x_0 < \eta$ and $r > N(\varepsilon)$, if, for instance, $x_1 > x_0$

(14) $$\int_c^d | u_r(x_1, y) - u_r(x_0, y) | \, dy \leq \int_{x_0}^{x_1} \int_c^d | u_{rx} | \, dx dy \leq$$

$$\leq \int_{x_0}^{x_1} \int_c^d | D_x u - u_{rx} | \, dx \, dy + \int_{x_1}^{x_0} \int_c^d | D_x u | \, dx dy < \frac{\varepsilon}{3}$$

Let now x_1 be arbitrary in (a, b) and choose x_o within the above set such that $0 \leq x_1 - x_0 < \eta$. Then

$$\int_c^d | u_m(x_1, y) - u_n(x_1, y) | \, dy \leq \int_c^d | u_m(x_1, y) - u_m(x_0, y) | \, dy$$

$$+ \int_c^d | u_m(x_0, y) - u_n(x_0, y) | \, dy + \int_c^d | u_n(x_0, y) - u_n(x_1, y) | \, dy$$

and the right side is smaller than ε for $m, n > N(\varepsilon)$ on ~~behalf~~ account of (13) and (14).

We finally prove the lemma used in this proof:

<u>Lemma 3.</u> Let f(x) and $\varphi(x)$ be summable over $a \leqq x \leqq b$, and let $f_n(x)$ be of class C'. If then

$$\lim_{n \to \infty} \int_a^b | f(x) - f_n(x) | \, dx = 0 \, , \quad \lim_{n \to \infty} \int_a^b | \varphi(x) - f_n'(x) | \, dx = 0$$

the functions $f_n(x)$ converge uniformly in $a \leqq x \leqq b$.

<u>Proof.</u> It is

$$\left| \int_{x_0}^{x_1} | f_n'(x) | \, dx - \int_{x_0}^{x_1} | \varphi(x) | \, dx \right| \leq \int_{x_0}^{x_1} | \varphi(x) - f_n'(x) | \, dx$$

hence

(15) $$| f_n(x_1) - f_n(x_0) | < \eta(|x_1 - x_0|) \qquad \text{for } n \geq N$$

where η , as indicated, only depends on $|x_1 - x_0|$ and tends to 0 with $|x_1 - x_0|$.

Furthermore

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \int_a^b |f_m(x) - f_n(x)|\, dx = 0$$

Designating by $S_{m,n,\eta}$ the set of points x for which

$$|f_m(x) - f_n(x)| \geq \eta$$

one has for each fixed $\eta > 0$

$$\lim_{\substack{n\to\infty \\ m\to\infty}} m\, S_{m,n,\eta} \to 0$$

Let now $\varepsilon > 0$ be given. We choose δ such that $\eta(\delta) < \frac{\varepsilon}{3}$. We then determine N such that

(16) $$m\, S_{m,n,\frac{\varepsilon}{3}} < 2\delta \text{ for } m,n > N.$$

To each point x a point x_1 of the complement of $S_{m,n,\frac{\varepsilon}{3}}$ can be found with distance less than δ from x. We then have for n, m > N

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f_m(x_1)| + |f_m(x_1) - f_n(x_1)| + |f_n(x_1) - f_n(x)| < \varepsilon,$$

for the first and third members are smaller than $\frac{\varepsilon}{3}$ on account of (15), the second is smaller than $\frac{\varepsilon}{3}$ because x_1 is not in $S_{m,n,\frac{\varepsilon}{3}}$.

Definition

We say that u(x, y) is strictly absolutely continuous in the sense of Tonelli on the rectangle (a, b; c, d) if 1) u(x, y) is continuous on (a, b; c, d); if 2)

$$\int_a^b V_c^d(u(x, y))dy_o < \infty$$

and

$$\int_c^d V_a^b(u(x, y_o))dx_o < \infty$$

where $V_c^d(u(x_o, y))$ designates the total variation of $u(x_o, y)$ on $c \leqq y \leqq d$ and $V_a^b(u(x, y_o))$ the total variation of $u(x, y_o)$ on $a \leqq x \leqq b$; 3) $u(x_o, y)$ is absolutely continuous in y for almost all x_o on (a, b) and $u(x, y_o)$ is absolutely continuous in x for almost all y_o on (c, d).

We call u(x, y) absolutely continuous in the sense of Tonelli (A.C.T.) in an open set G, if it is strictly so on each closed rectangle (a, b; c, d) in G.

Concerning these functions, we first prove the following theorem:

Theorem 3. If u(x, y) is A.C.T. on an open set G, then $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ exist almost everywhere on G and are summable over each bounded closed subset of G. Furthermore the following equations hold for each cell (a, c; b, d) in G:

$$(17)\qquad \int_a^b\int_c^d \frac{\partial u}{\partial x}\, dxdy = \int_c^d [u(b,y) - u(a,y)]dy.$$

$$(18)\qquad \int_a^b\int_c^d \frac{\partial u}{\partial y}\, dxdy = \int_a^b [u(x,d) - u(x,c)]dx.$$

Proof: It is sufficient to show that, for each cell R: (a,c; b,d) interior to G, u_x and u_y exist almost everywhere on R and are summable on R. If this is true, the equations (17) and (18) are immediate. We shall consider only u_x the proof for u_y being similar.

We first observe that the four Dini partial derivatives of u with respect to x are measurable on R. Moreover, for almost every y_0, $c \leq y_0 \leq d$, all four of these derivatives coincide for all x, $a \leq x \leq b$, not on a set of measure zero. Hence the four derivatives coincide and u_x exists almost everywhere on R. That u_x is summable follows from the existence of the repeated integral

$$\int_c^d \left[\int_a^b \left| \frac{\partial u}{\partial x} \right| dx \right] dy = \int_c^d V_a^b [u(x,y)] dy.$$

We now prove the following theorem due to G. C. Evans[2]:

2) Complements of Potential Theory II, Amer. Jour. of Math., vol. 55 (1933), pp. 42-46.

Theorem 4. A necessary and sufficient condition that a function u(x,y) be A.C.T. on G is that it be a continuous potential function of its generalized derivatives on G.

The necessity follows immediately from Theorem 3. To prove the sufficiency let $R = (a,b;\ c,d)$ be any rectangle interior to G. $u_h(x,y)$ converges uniformly to u(x,y) on (a,b; c,d) and we have by Theorem 3 and Lemma 2:

$$\lim_{h \to 0} \iint_R \{ |u_{hx} - D_x u| + |u_{hy} - D_y u| \} dx\, dy = 0$$

As in the proof of Theorem 2 we can find a sequence of values $h_\nu > 0$ tending to 0 such that

$$\lim_{\nu \to \infty} \int_a^b |u_{h_\nu x}(x, y_0) - D_x u(x, y_0)| dx = 0$$ and

$$\lim_{\nu \to \infty} \int_c^d |u_{h_\nu y}(x_0, y) - D_y u(x_0, y)| dy = 0$$

for all x_0 and y_0 not in certain linear sets of measure 0. For each such x_0 (y_0) the function $u(x_0,y)$ ($u(x,y_0)$) is absolutely continuous in y(x) and

$$\frac{\partial u(x_0,y)}{\partial y} = D_y u(x_0,y) \quad \left(\frac{\partial u(x,y_0)}{\partial x} = D_x u(x,y_0) \right)$$

for almost all y(x). The measurability of the Dini partial derivatives of u(x,y) shows, then, in the usual manner that u_x and u_y exist almost everywhere on R and are equal to the corresponding generalized derivative; u_x and u_y are therefore summable over R. This being true for each R in G, u(x,y) is A.C.T. in G.

Definition. We say u(x,y) is of "class D'_α" ($\alpha \geq 1$) on an open set G if it is a P.F.G. D on G and if $(D_x u)$ and $(D_y u)$ are summable over each bounded closed set interior to G. We say $u(x,y)$ is of "class D_α" on G if it is of class D'_α and furthermore continuous. Theorem 4 and its proof show that u(x,y) is of class D_α on G if, and only if, it is A.C.T. on G and if $|D_x u|^\alpha$ and $|D_y u|^\alpha$ are summable over each bounded closed set in G.

The property of a function u(x,y) to be of class D'_α or D_α is invariant under sufficiently smooth transformations of the coordinates. This is the content of ~~3)~~ theorem 5. This theorem is due to G. C. Evans (see reference 1).

Theorem 5. If u(x,y) is of class D'_α (or D_α) on a region G and x = x(s,t), y = x(s,t) is a topological transformation of G into a region H, and if x(s,t) and y(s,t) are of class C' and their Jacobian is different from 0 on H, then u(x(s,t), y(s,t)) is of class D' (D) on H and

$$(19) \qquad \begin{aligned} D_s u &= D_x u \frac{\partial x}{\partial s} + D_y u \frac{\partial y}{\partial s} \\ D_t u &= D_x u \frac{\partial x}{\partial t} + D_y u \frac{\partial y}{\partial t} \end{aligned}$$

at each point (s,t) corresponding to a point (x,y) where $D_x u$ and $D_y u$ both exist. (Remark: It ought to be remarked that (19) holds in general only for the generalized derivative D_s, D_{ts}, D_x, D_y and not for the partial as has been shown by S. Saks[4].)

[4] On the Surfaces without Tangent Planes, Annals of Math., vol. 34 (1933), pp. 113-124.

<u>Proof.</u> Let F be a bounded closed subset of H corresponding to a similar set $E \subset G$. Let, according to Theorem 1, $u_n(x,y)$ be of class C' on E such that

$$\lim_{n \to \infty} \iint_E [|u_n - u| + |u_{nx} - D_x u| + |u_{ny} - D_y u|]\,dxdy = 0$$

Putting

$$v = D_x u \frac{\partial x}{\partial s} + D_y u \frac{\partial y}{\partial s}, \quad w = D_x u \frac{\partial x}{\partial t} + D_y u \frac{\partial y}{\partial t}$$

we have on F (the Jacobian being bounded and bounded away from zero) that

$$\lim_{n \to \infty} \iint_F [|u_n - u| + |u_{ns} - v| + |u_{nt} - w|]\,dsdt = 0$$

But

$$\iint_F v\,dsdt = \iint_F [D_x u \frac{\partial x}{\partial s} + D_y u \frac{\partial y}{\partial s}]\,dsdt = \iint_E [D_x u \frac{\partial t}{\partial y} - D_y u \frac{\partial t}{\partial x}]\,dxdy$$

If now $\{F_n\}$ is a normal sequence closing down to (s_0, t_0), $(x_0, y_0) = (x(s_0,t_0), y(s_0,t_0))$ being a point where $D_x u$ and $D_y u$ exist, then the image $\{E_n\}$ of $\{F_n\}$ is a normal sequence with respect to (x_0, y_0), and

$$\lim_{n \to \infty} \frac{\iint_{F_n} v\,dsdt}{m F_n} = \lim \frac{m E_n}{m F_n} \frac{1}{m E_n} \iint_{E_n} [D_x u \frac{\partial t}{\partial y} - D_y u \frac{\partial t}{\partial x}]\,dx\,dy$$

exists, since $\frac{m E_n}{m F_n}$ tends to $\frac{\partial(x,y)}{\partial(s,t)}(s_0, t_0)$. Therefore the generalized derivatives of $u(x(s,t), y(s,t))$ exist almost everywhere in G and satisfy (19); the above formulae show furthermore that $|D_s u|^\alpha$ and $|D_t u|^\alpha$ are summable of each bounded closed set in H.

The next theorem which we are going to prove characterizes the functions of class D'_α and D_α.

For the proof of the next theorem we need the following lemma:

<u>Lemma 4.</u> Let $f(x)$ and $|f_n(x)|^\alpha$ $(n = 1, 2, \ldots)$ be summable on the set E, and suppose that

$$\lim_{n \to \infty} \int |f - f_n|\,dx = 0, \quad \int_E |f_n(x)|^\alpha\,dx \leq M,$$

M being independent of n. Then $|f|^\alpha$ is summable over E and

$$\int_E |f|^\alpha\,dx \leq \lim_{n \to \infty} \int_E |f_n|\,dx.$$

In particular, if f is summable over the open set D, a necessary and sufficient
bounded

condition that it be of class L_α on D is that

$$\int_{D_h} |f_h|^\alpha \, dx \leq M,$$

M being independent of n.

Proof. Since a subsequence $|f_{n_k}(x)|^\alpha$ tends almost everywhere to $|f(x)|^\alpha$ on E we have, according to Fatou's Lemma

$$\int |f(x)|^\alpha dx \leq \int \lim_{k\to\infty} |f_{n_k}(x)|^\alpha dx \leq \lim_{k\to\infty} \int |f_n(x)|^\alpha dx \leq M$$

Putting

$$f_h^* = f_h \quad \text{in } D_h$$

$$f_h^* = 0 \quad \text{in } D - D_h$$

we have

$$\lim_{h\to 0} \int_D (f - f_h^*) dx = 0$$

and

$$\int_D |f_h^*| \, dx \leq M \quad \text{for all } h;$$

hence, according to the first part of the proof, $|f|^\alpha$ is summable over D.

The proof of the next theorem being more difficult for more than two than for two variables, we formulate and prove it for n variables:

Theorem 6. A necessary and sufficient condition that $u(x^1,\dots,x^n)$ be of class D'_α ($\alpha \geq 1$) on the bounded open set G is (1) that u be of class L_α on each closed subset of G, (2) that there exist functions $v_i(x^1,\dots,x^n)$, $i = 1,\dots, n$ of class L_α on each closed subset of G, and (3) that there exists a sequence of functions $\{u_p\}$ of class C' on the set G_{α_p} where $\alpha_p \to 0$, such that for each closed subset F of G we have

$$\lim_{p\to\infty} \int_F \Big[|u_p - u|^\alpha + \sum_{i=1}^n |u_{p x^i} - v_i|^\alpha \Big] dx = 0 \tag{20}$$

A necessary and sufficient condition that u be of class D_α on G is that the above be true, the convergence of $\{u_p\}$ being uniform on each such F. In particular, we may choose $u_p = u_{h_p}$ where $h_p \to 0$.

Proof of Theorem 6: A. Necessary. By Theorem 1 and Lemma 1 we know that

$$u_{pxi} = \frac{1}{(2h)^n}\int_{x-h}^{x+h} D_{x^i} u \, d\xi , \quad \lim_{h\to 0}\int_F \Big[|u_h - u| + \sum_{i=1}^n |u_{hx^i} - D_{x^i} u| \Big] dx = 0$$

for every closed set in G. It follows from Lemma 2 that

$$\lim_{h\to 0}\int_F \sum_{i=1}^n |u_{hx^i} - D_{x^i} u|^\alpha dx = 0 \qquad \text{for every closed } F \text{ in } G.$$

If we can prove that u is of class L_α then A will follow from Lemma 2. We take u = 3. Let R = (a,b; c,d; e,f) be interior to G. For h > 0 one has

$$\iiint_R [|u_h| + |u_{hx}|^\alpha + |u_{hy}|^\alpha + |u_{hz}|^\alpha] dxdydz \leq M$$

independently of h, provided h is so small that G_h contains R. From Lemma 4 it follows that u is of class L_α on R if

(21) $$\iiint_R |u_h|^\alpha dxdydz \leq K$$

independently of h.

To prove (27) we choose h > 0 and x_0 with $a \leq x_0 \leq b$ such that

$$\int_c^d \int_e^f [|u_h(x_0,y,z)| + |u_{hy}(x_0,y,z)|^\alpha + |u_{hz}(x_0,y,z)|^\alpha] dy dz \leq \frac{M}{b-a}$$

and then y_0 such that

$$\int_e^f [|u_h(x_0,y_0,z)| + |u_{hz}(x_0,y_0,z)|^\alpha] dz \leq \frac{M}{(b-a)(c-d)}$$

and z_0 such that

$$|u_h(x_0,y_0,z_0)| < \frac{M}{(b-a)(d-c)(f-e)}$$

Then we have

$$|u_h(x_0,y_0,z) - u_h(x_0,y_0,z_0)|^\alpha \leq |z-z_0|^{\alpha-1}\int_e^f |u_{hz}(x_0,y_0,z)|^\alpha dz \leq \frac{M(f-e)^{\alpha-1}}{(b-a)(d-c)}$$

Consequently

$$(u_h(x_0,y_0,z)) \leq 2^{\alpha-1}\Big[\frac{M^\alpha}{(b-a)^\alpha(d-c)^\alpha(f-e)^\alpha} + \frac{M(f-e)^{\alpha-1}}{(b-a)(d-c)}\Big]$$

One has furthermore

$$\int_c^d |u_h(x_0,y,z) - u_h(x_0,y_0,z)|^\alpha \, dz \leq |y-y_0|^{\alpha-1} \int_c^d \int_e^f |u_{hy}(x_0,y,z)|^\alpha \, dysz \leq \frac{M(d-c)^{\alpha-1}}{b-a}$$

Continuing this process one finds (21).

B. Sufficient. If the conditions of Theorem 6 are satisfied, (20) holds with $\alpha = 1$, and hence, by Theorem 1, u is of class D_1' with $D_{x}iu = v_i$ almost everywhere. Since the v_i are each of class L_α on each closed set F in G, the function $D_{xi}u$ are also. Therefore u is of class D'_α.

The second part of Theorem 6 follows from this together with Theorem 4. We now prove:

Theorem 7. Let u and v be of class D'_α and D'_β respectively on G, with $\alpha^{-1} + \beta^{-1} = 1$. Then, for almost all rectangles R = (a,b; c,d) we have

(22) $$\int_{R^*} u\,Dv = \iint_R (D_x u \, D_y v - D_x v \, D_y u) \, dx\,dy$$

where $\int_{R^*}$ uDv means the integral of

$$u(D_x v dx + D_y v dy)$$

over the boundary of R.

Proof. Let D: (A,C,B,D) be a cell interior to G. On account of Theorem 6 there exist sequences $\{u_n\}$ and $\{v_n\}$ of class C' on D such that

$$\lim \int_A^B \int_C^D \{|u_n - u|^\alpha + |u_{nx} - D_x u|^\alpha + |u_{ny} - D_y u|^\alpha\} \, dxdy = 0 \text{ and}$$

$$\lim \int_A^B \int_C^D \{|v_n - v|^\beta + |v_{nx} - D_x v|^\beta + |v_{ny} - D_y v|^\beta\} \, dx\,dy = 0 .$$

As several times before we choose subsequences of $\{u_n\}$ and $\{v_n\}$, which we designate again by $\{u_n\}$ and $\{v_n\}$ such that

(23) $$\lim \int_A^B \{|u_n(x,y_0) - u(x,y_0)|^\alpha + |u_{nx}(x,y_0) - D_x u(x,y_0)|^\alpha + |u_{ny}(x,y_0) - D_y(u,y_0)|^\alpha\} \, dx$$

for almost all y_0 in $C \leq y \leq D$, and correspondingly for x instead of y and v instead of u.

One has, for each u, and each cell R in D, that

(24) $$\int_{R^*} u_n\, dv_n = \iint_R \frac{\partial(u_n, v_n)}{\partial(x, y)}\, dx\, dy$$

The Hölder inequality shows that the integral on the right side of (22) exists and that the right side of (24) tends to a limit. (23) shows (after a new application of the Hölder inequality) that the left side of (22) has a meaning for almost all rectangles in G, and that $\int_{R^*}$ uDv is the limit of the left side of (24) for these rectangles. As G may be written as the sum of a denumerable number of such cells D, our theorem follows.

<u>Remark.</u> In case u and v are continuous, the theorem can be given a more general form. It is then sufficient to assume that $D_x u$, $D_y u$, $D_x v$, $D_y v$ are of classes L_α L_β L_γ L_δ (on each bounded closed subset of G) respectively, with $\frac{1}{\alpha} + \frac{1}{\delta} = 1$ and $\frac{1}{\beta} + \frac{1}{\gamma} = 1$.

Before proving the final theorem on P, F, G, D we establish three lemmas which will be needed for the proof:

<u>Lemma 5.</u> Let $\{\varphi_n(x)\}$ be of class L_α $(\alpha > 1)$ on a cell R with

$$\int_R |\varphi_n|^\alpha\, dx\, dy \le M \qquad (M \text{ independent of } n).$$

Suppose furthermore that for each cell D in R, the limit

$$\lim_{n \to \infty} \int_D \varphi_n\, dx$$

exists. Then there exists a function φ of class L_α on R, such that

(25) $$\int_R |\varphi|^\alpha\, dx \le \varliminf_{n \to \infty} |\varphi_n|^\alpha\, dx$$

(26) $$\int_D \varphi\, dx = \lim_{n \to \infty} \int_D \varphi_n\, dx \qquad \text{for each } D \subset R$$

Proof: Define a function of intervals, $\phi(D)$, by

$$\phi(D) = \lim_{n\to\infty} \int_D \varphi_n \, dx$$

If $D = \sum_{i=1}^{N} D_i$, the D_i being non-overlapping intervals, we have

$$\phi(D) = \sum_{i=1}^{N} \phi(D_i)$$

Now let $D_1^*, \ldots, D_{N^*}^*$ be any set of non-overlapping cells in R. Then

$$\sum_{i=1}^{N^*} |\phi(D_i^*)| = \lim_{n\to\infty} \sum_{i=1}^{N^*} \left| \int_{D_i^*} \varphi_n \, dx \right| \le \underline{\lim}_{n\to\infty} \int_{\Sigma D_i^*} |\varphi_n| \, dx \le$$

$$\le \underline{\lim}_{n\to\infty} \left[m \left(\sum_{i=1}^{N^*} D_i^* \right) \right]^{\frac{\alpha-1}{\alpha}} \left[\int_{\Sigma D_i^*} |\varphi_n|^\alpha \, dx \right]^{\frac{1}{\alpha}}$$

This shows in the first place that the function $\phi(D)$ is of bounded variation and it can therefore be extended to a function $\phi(E)$ defined on all measurable sets in R. But this same inequality shows in the second place that $\phi(E)$ is absolutely continuous. Therefore there exists a function φ summable over R such that

(22) $$\phi(D) = \int_D \varphi \, dx = \lim_{n\to\infty} \int_D \varphi_n \, dx$$

for each cell D in R.

The functions $\varphi_{\nu h}$ are equicontinuous on R_h for each fixed $h > 0$ and they converge at each point to φ_h. They therefore converge uniformly to φ_h and we have

$$\int_{R_h} |\varphi_h|^\alpha \, dx = \lim_{\nu\to\infty} \int_{R_h} |\varphi_{\nu h}|^\alpha \, dx \le \underline{\lim}_{\nu\to\infty} \int_R |\varphi_\nu|^\alpha \, dx \le M.$$

Hence, by Lemma 4, φ is of class L_α on R. This, together with (27), proves (26). (25) follows from Lemma 2.

Lemma 6: Let the sets S_n be measurable and all situated in a bounded part of the space. Suppose furthermore

$$mS_n \geqq k > 0.$$

Designate by S^+ the set of those points which occur in infinitely many sets S_n. Then

$$mS^+ \geqq k.$$

This lemma is due to E. Borel and its proof is immediate:

$$S^+ = \prod_{n=1}^{\infty} \left\{ \sum_{p=n}^{\infty} S_p \right\}$$

The sets $\sum_{p=n}^{\infty} S_p$ are bounded and monotonically decreasing. Hence

$$mS^+ = \lim_{n\to\infty} \left(m \sum_{p=n}^{\infty} S_p \right) \geq k.$$

<u>Lemma 7</u>: Let $u(x^1, \ldots, x^n)$ be of class D'_p, $p \geqq 1$, on the open set G. Then, for each ℓ, $1 \leqq \ell \leqq n$, there exists a set Z_ℓ, of values of x^ℓ, which is of measure zero such that if x_0^ℓ is not in Z_ℓ, then $u(x^1, \ldots, x^{\ell-1}, x_0^\ell, x^{\ell+1}, \ldots, x^n)$ is of class D'_p in $(x^1, \ldots, x^{\ell-1}, x^{\ell+1}, \ldots, x^n)$ on the open set G_{x_0} of values of $(x^1, \ldots, x^{\ell-1}, x^{\ell+1}, \ldots, x^n)$ such that $(x^1, \ldots, x^{\ell-1}, x_0^\ell, x^{\ell+1}, \ldots, x^n)$ is in G.

Proof: It is evidently sufficient to prove this for each cell R interior to G. For simplicity of notation, we assume n = 3, and let $x^1 = x$, $x^2 = y$, $x^3 = z$, and R be the cell (a, c, e; b, d, f). We shall then prove, as an illustration, that $u(x, y, z_0)$ is of class D'_p in (x, y) for almost all z_0.

From the definitions concerning functions of class D'_p, it follows that

(27a) $$\int_\alpha^\beta \int_\gamma^\delta \int_\varepsilon^\zeta D_x u(\alpha, y, z)\, dx\,dy\,dz = \int_\gamma^\delta \int_\varepsilon^\zeta [u(\beta, y, z) - u(\alpha, y, z)]\, dy\,dz$$

if neither α nor β is in a certain set Z_x of measure zero, and

(27b) $$\int_\alpha^\beta \int_\gamma^\delta \int_\varepsilon^\zeta D_y u(x, y, z)\, dx\,dy\,dz = \int_\alpha^\beta \int_\varepsilon^\zeta [u(x, \delta, z) - u(x, \gamma, z)]\, dx\,dz.$$

if neither γ nor δ is in a certain set Z_y of measure zero, $u(\beta, y, z)$ and $u(\alpha, y, z)$ being of class L_p for $c \leqq y \leqq d$, $e \leqq z \leqq f$ and $u(x, \gamma, z)$ and $u(x, \delta, z)$ are of class L_p in (x,z) for $a \leqq x \leqq b$, $e \leqq z \leqq f$.

Now let x_0 be not in Z_x. Define $v_1(x,y,z)$ equal to $D_x u(x,y,z)$ when this exists and is positive, and equal to zero otherwise. Define $v_2(x,y,z)$ equal $-D_x u(x,y,z)$ when this exists and is positive, and equal to zero otherwise. Define

$$u_i(x, y, z) = \int_{x_0}^{x} v_i(\xi, y, z)\, d\xi, \quad i = 1, 2.$$

letting a u_i be zero if the integral is not defined. Clearly $u_i(x,y,z)$ is of class L_p in (y,z) for each x, $i = 1, 2$, and if β is not in Z_x, $u(\beta, y, z)$ is equivalent to the function

$$u(x_0, y, z) + u_1(\beta, y, z) - u_2(\beta, y, z)$$

as is easily seen, using (27a).

Now if z_0 is not in a certain set Z_1 of measure zero, we have

$$\text{(27c)} \qquad \lim_{h \to 0} \frac{1}{2h} \int_{x_0}^{x} \int_{\gamma}^{\delta} \int_{z_0-h}^{z_0+h} v_i(\xi, y, z)\, d\xi\, dy\, dz = \int_{x_0}^{x} \int_{\gamma}^{\delta} v_i(\xi, y, z_0)\, d\xi\, dy$$

for every x, γ, and δ, $a \leqq x \leqq b$, $c \leqq \gamma$, $\delta \leqq d$, $v_i(x,y,z_0)$ being of class L_ρ in (x,y) in $(a,c;\, b,d)$. Also if z_0 is not in a set Z_2 of measure zero

$$\text{(27d)} \qquad \lim_{h \to 0} \frac{1}{2h} \int_{\gamma}^{\delta} \int_{z_0-h}^{z_0+h} u_i(x_n, y, z)\, dy\, dz = \int_{\gamma}^{\delta} u_i(x_n, y, z_0)\, dy, \quad i = 1, 2.$$

for every (γ, δ) and x_n rational $a \leqq x_n \leqq b$. But now, each u_i is monotone non-decreasing and absolutely continuous in x for almost all (y,z) and $v_i \geqq 0$ so that the functions on both sides in (27c) and (27d) are all monotone non-decreasing in x. Hence the convergence as $h \to 0$ is uniform with respect to x. Combining all of these results we see that if z_0 is not in a certain set $\tilde{Z}_z$ and α and β are not in the set Z_x (independent of $\tilde{Z}_z$), we have

(27e) $$\int_\gamma^\delta [u(\beta, y, z_0) - u(\alpha, y, z_0)]\, dy = \int_\alpha^\beta \int_\gamma^\delta D_x u(x, y, z_0)\, dx\, dy$$

for every (γ, δ) and $D_x u(x,y,z_0)$ is of class $L_p m(x,y)$. A similar proof shows that, by excepting from Z_z a further set of measure zero, thus arriving at Z_z, we may also arrange it so that if γ and δ are not in Z_y, we have

(27f) $$\int_\alpha^\beta [u(x, \delta, z_0) - u(x, \gamma, z_0)]\, dx = \int_\alpha^\beta \int_\gamma^\delta D_y u(x, y, z_0)\, dx\, dy$$

for every (α, β) and $D_y u(x,y,z_0)$ is of class L_p in (x,y). Thus $u(x,y,z_0)$ is of class D_p' on $(a,c;\ b,d)$.

Remark: We stated this lemma (for $n \geq 3$ since the analogous statement) for $n \geq 2$ is a consequence of Theorem 2.

Theorem 8. Let $\{u_p\}$ be a sequence of functions of class D'_α defined on a cell $R \equiv (a,b)$, $\alpha > 1$. Suppose that there exists an M independent of p such that

$$\int_{a^1}^{l^1} \cdots \int_{a^n}^{l^n} \left(|u_p|^\alpha + \sum_{i=1}^n |D_{x^i} u_p|^\alpha\right) dx^1 \cdots dx^n =$$

$$= \int_{a^1}^{l^1} \cdots \int_{a^n}^{l^n} \left(|\bar{u}_p|^\alpha + \sum_{i=1}^n |\bar{u}_{p,x^i}|^\alpha\right) dx^1 \cdots dx^n \leq M$$

Then a subsequence $\{u_{p_k}\}$ may be chosen which converges in the mean of order α to a function u which is also of class D'_α and for which

(28) $$\int_R |D_{x^i} u|^\alpha dx \leq \lim_{k \to \infty} \int_R |D_{x^i} u_{p_k}|^\alpha dx, \quad i = 1, \dots, n.$$

In fact the subsequence may be chosen so that for each x_0^i, $a^i \leq x_0^i \leq l^i$ we have

(29) $$\int_{a^1}^{l^1} \cdots \int_{a^{i-1}}^{l^{i-1}} \int_{a^{i+1}}^{l^{i+1}} \cdots \int_{a^n}^{l^n} |\bar{u}_{p_k}(x^1, \dots, x^{i-1}, x_0^i, x^{i+1}, \dots, x^n) - \bar{u}(x^1, \dots, x^{i-1}, x_0^i, x^{i+1}, \dots, x^n)|^\alpha dx^1 \cdots dx^{i-1} dx^{i+1} \cdots dx^n < \varepsilon_k$$

where $\varepsilon_k \to 0$

<u>Proof.</u> We shall prove (29) first. We proceed by induction with respect to n. For $n = 1$ the theorem is known, since the u_p are A.C. and equicontinuous on (a,b), as is seen by the Hölder inequality, and uniformly bounded. This may be considered as a special case of (29) with $i = n = 1$. Suppose the whole theorem on convergence has been proved for all values

For each m, let (a^{r+1}, b^{r+1}) be divided into 2^m equal closed intervals $x^{r+1}_{m,i-1} \leq x^{r+1} \leq x^{r+1}_{m,i}$. Let $\bar{S}^{(m)}_{p,i}$ be the set of values x^{r+1} with $x^{r+1}_{m,i-1} \leq x^{r+1} \leq x^{r+1}_{m,i}$ such that

$$\int_{a^1}^{b^1} \cdots \int_{a^r}^{b^r} \Big[|\bar{u}_p(x^1, \dots, x^r, x^{r+1})|^\alpha + \sum_{i=1}^{r} |\bar{u}_{p\,x^i}(x^1, \dots, x^r, x^{r+1})|\Big]^\alpha dx^1 \dots dx^r \leq$$

$$\leq \frac{2^{m+1} M}{b^{r+1} - a^{r+1}}.$$

We have $m\bar{S}^m_{p,i} > 2^{-m-1}(b^{r+1} - a^{r+1})$. We omit in $\bar{S}^{(m)}_{p,i}$ those points of which at least one coordinate is in one of the exceptional sets of measure 0 occurring in Lemma 7; we may get in this way the set $S^m_{p,i}$. $S^m_{p,i}$ having the same measure as $\bar{S}^m_{p,i}$. Let m and i be fixed and $S^m_{p_k,i}$ be any subsequence of $S^m_{p,i}$; then we know from Lemma 6 that there is a point which occurs in infinitely many of the $S^m_{p_k i}$.

Let $\pi_1 = (m_1, i_1)$, $\pi_2 = (m_2, i_2)$, ... be any ordering of the admissible pairs (m,i). Put $S^{m_\nu}_{p,i_\nu} = S^\nu_p$ and let $\bar{x}_1^{r+1}$ belong to $S^1_{p^1_1}, S^1_{p^1_2}, \dots, S^1_{p^1_\nu}, \dots$. Choose then a point $\bar{x}_2^{r+1}$ and a subsequence $\{p^2_\nu\}$ of $\{p^1_\nu\}$ such that $\bar{x}_2^{r+1}$ belongs to all $S^2_{p^2_\nu}$; we continue this process choosing generally a point $\bar{x}^{r+1}_{m+1}$ and a subsequence $\{p^{m+1}_\nu\}$ of $\{p^m_\nu\}$ such that $\bar{x}^{r+1}_{m+1}$ belongs to all $S^{m+1}_{p^{m+1}}$.

Then $\bar{x}_s^{r+1}$ will belong to $S^{v}_{p_v}$ for $v \geq s$ or $\bar{x}_s^{r+1} = \bar{x}^{r+1}_{m_s, i_s}$ will belong to $S^{m_s}_{p_f, i_s}$ for $f \geq k(m_s, i_s)$.

Now choose $\varepsilon > 0$ and let x_0^{r+1} be any value with $a^{r+1} \leq x_0^{r+1} \leq b^{r+1}$. Then for each p, [x,a,b standing for $(x^1, \dots, x^r)$, $(a^1, \dots, a^r)$, $(b^1, \dots, b^r)$], we have

$$\left[\int_a^b |\bar{u}_p(x, x_0^{r+1}) - \bar{u}_p(x, \bar{x}^{r+1}_{m,j})|^\alpha dx\right]^{\frac{1}{\alpha}} = \left[\int_a^b \left| \int_{\bar{x}^{r+1}_{m,j}}^{x_0^{r+1}} D_{x^{r+1}} u_p(x^1, \dots, x^{r+1})\, dx^{r+1} \right| dx\right]^{\frac{1}{\alpha}} \leq$$

$$\leq |x_0^{r+1} - \bar{x}^{r+1}_{m,j}|^{\frac{\alpha-1}{\alpha}} \left[\int_{a^1}^{b^1} \cdots \int_{a^{r+1}}^{b^{r+1}} |D_{x^{r+1}} u_p|^\alpha dx^1 \cdots dx^{r+1}\right]^{\frac{1}{\alpha}} \leq$$

$$\leq \left[2^{-m-1}(b^{r+1} - a^{r+1}) M\right]^{\frac{1}{\alpha}} < \frac{\varepsilon^{\frac{1}{\alpha}}}{3}$$

if m is large enough and $x^{r+1}_{m,j}$ denotes that one of the above points which is nearest to x_0^{r+1}. Hence, choose such an m and V(m) such that for $k, \ell > V(m)$

$$\left[\int_a^b |\bar{u}_{p_k}(x, \bar{x}^{r+1}_{m,j}) - \bar{u}_{p_\ell}(x, \bar{x}^{r+1}_{m,j})|^\alpha dx\right]^{\frac{1}{\alpha}} < \frac{\varepsilon^{\frac{1}{\alpha}}}{3}, \quad j = 1, \dots, 2^m$$

$$(p_k = p_k^k)$$

This is possible on account of the hypothesis of induction. We then have for $k, \ell > V(m)$

$$\left[\int_a^b |\bar{u}_p(x, x_0^{r+1}) - \bar{u}_{p_\ell}(x, x_0^{r+1})|^\alpha dx\right]^{\frac{1}{\alpha}} \leq \left[\int_a^b |\bar{u}_{p_k}(x, x_0^{r+1}) - \bar{u}_{p_k}(x, \bar{x}^{r+1}_{m,j})|^\alpha dx\right]^{\frac{1}{\alpha}} +$$

$$+ \left[\int_a^b |\bar{u}_{p_k}(x, \bar{x}^{r+1}_{m,j}) - \bar{u}_{p_\ell}(x, \bar{x}^{r+1}_{m,j})|^\alpha dx\right]^{\frac{1}{\alpha}} + \left[\int_a^b |\bar{u}_{p_\ell}(x, \bar{x}^{r+1}_{m,j}) - \bar{u}_{p_\ell}(x, x_0^{r+1})|^\alpha\right]^{\frac{1}{\alpha}} <$$

$$< \varepsilon^{\frac{1}{\alpha}}$$

Thus (29) is proved for $n = r+1$, $i = r+1$. By choosing further subsequences of $\{p_k = p_k^k\}$ we may arrive at one for which (29) is true for $i = 1, \dots, r+1$.

It is clear that the convergence in the mean of order α for the whole of R follows from this.

Let u be a function which is the limit in the mean of $\{u_{p_k}\}$ and choose e. There is a function u_e, equivalent to u, such that $\{u_{p_k}(x^1, -, x^{e-1}, x_0^e, x^{e+1}, \dots, x^n)$ converges in the mean of order α in $(x^1, \dots, x^{e-1}, x^{e+1}, \dots, x^n)$ to $u_e(x^1, \dots, x^{e-1}, x_0^e, x^{e+1}, \dots, x^n)$ for each x_0^e, $a^e \leq x_0^e \leq b^e$. Therefore we have for all rectangles $D = (\alpha, \beta)$ in R

$$\int_{\alpha^1}^{\beta^1}\cdots\int_{\alpha^{e-1}}^{\beta^{e-1}}\int_{\alpha^{e+1}}^{\beta^{e+1}}\cdots\int_{\alpha^n}^{\beta^n}[u(x^1,\dots,x^{e-1},\beta^e,x^{e+1},\dots,x^n)-u(x^1,\dots,x^{e-1},\alpha^e,x^{e+1},\dots,x^n)]\,dx^1\cdots dx^{e-1}\,dx^{e+1}\cdots dx^n$$

$$=\lim_{k\to\infty}\int_{\alpha^1}^{\beta^1}\cdots\int_{\alpha^{e-1}}^{\beta^{e-1}}\int_{\alpha^{e+1}}^{\beta^{e+1}}\int_{\alpha^n}^{\beta^n}[u_{p_k}(x^1,\dots,x^{e-1},\beta^e,x^{e+1},\dots,x^n)-u_{p_k}(x^1,\dots,x^{e-1},\alpha^e,x^{e+1},\dots,x^n)]\,dx^1\cdots dx^{e-1}dx^{e+1}\cdots dx^n$$

$$=\lim_{k\to\infty}\int_D u_{p_k,x^e}\,dx.$$

By Lemma 5 there exists a function v_e so that the formulas (1) hold for every interval D for u_e and v_e, and such that v_e is of class L_α and

$$\int_R |v_e|^\alpha\,dx \leq \lim_{k\to\infty}\int_R |D_{x^e}u_{p_k}|^\alpha\,dx.$$

Since this is true for each , $1 \leq \leq u$, we see that u is of class D'_α

Moreover for each e

$$\int_{\alpha^1}^{\beta^1}\cdots\int_{\alpha^{e-1}}^{\beta^{e-1}}\int_{\alpha^{e+1}}^{\beta^{e+1}}\cdots\int_{\alpha^n}^{\beta^n}\bar{u}(x^1,\dots,x^{e-1},x_0^e,x^{e+1},\dots,x^n)\,dx^1,\dots dx^{e-1}\,dx^{e+1}\cdots dx^n$$

$$\int_{\alpha^1}^{\beta^1}\cdots\int_{\alpha^{e-1}}^{\beta^{e-1}}\int_{\alpha^{e+1}}^{\beta^{e+1}}\cdots\int_{\alpha^n}^{\beta^n}u_e(x^1,\dots,x^{e-1},x_0^e,x^{e+1},\dots,x^n)\,dx^1\cdots dx^{e-1}\,dx^{e+1}\cdots dx^n$$

are both continuous in x_0^e. From this, the remainder of the statements in the theorem follow immediately.

2. Convex sets and functions

We shall not attempt a systematic development of the properties of convex functions, but shall give here only those properties which will be needed later on or which are especially fundamental.

Definition. A set s of points (in the n-dimensional euclidean space) is called convex if, whenever the points P_1 and P_2 belong to S the straight-line segment $\overline{P_1 P_2}$ belongs to S.

The statements of the following theorem are immediate consequences of this definition:

Theorem 1. If $\{S\}$ is any aggregate of convex sets, then

$$\prod_{S \subset \{S\}} S$$

is either empty or convex. If $s_1 \leqq s_2 \leqq s_3 \leqq \ldots$ and $s_1, s_2, \ldots$ are convex then

$$\sum_{\nu=1}^{\infty} S_\nu$$

is convex. The closure of a convex set is convex.

The next theorem is also almost trivial:

Theorem 2. Let S_1 be a closed convex set, S_2 a bounded closed convex set, and $S_1 S_2 = 0$. Then there exists a hyperplane separating S_1 from S_2.

We designate by $e(\mu, \nu)$ the distance of the two sets μ and ν i.e.

$$e(\mu,\nu) = \underset{x \subset \mu,\, y \subset \nu}{\text{g. L. B.}}\; e(x,y)$$

where $e(x, y)$ is the euclidean distance of x and y. Under the assumptions of the theorem there exist two points $A_1 < S_1$ and $A_2 < S_2$ such that

$$e(A_1, A_2) = e(S_1, S_2) > 0$$

Let P be an interior point of $\overline{A_1A_2}$. Then one has

$$e(P, S_1) = e(P, A_1), \quad e(P, S_2) = e(P, A_2)$$

for if there existed a point $B < S_1$ such that $e(P, B) < e(P, A_1)$ one would have

$$e(S_1, S_2) \leq e(B, A_2) \leq e(B, P) + e(P, A_2) < e(A_1, A_2)$$

Therefore the hyperplane π perpendicular to $\overline{A_1A_2}$ at P cannot contain a point of either S_1 or S_2. For if $\tau < \pi \cdot S_1$ the segment $\overline{\tau A_1}$ would belong to S_1 and $\measuredangle\, \tau PA_1$ being a right angle, a point $Q < \overline{\tau A_1}$ would exist with

$$e(P, Q) < e(P, A_2).$$

π therefore separates S_1 from S_2. We remark that π has positive distance from S_2.

<u>Definitions</u>: We generally designate the closure of a set μ by $\bar{\mu}$ and call the set $\bar{\mu} \cdot \mu' + \mu \bar{\mu'}$, where μ' is the complement of μ the <u>boundary</u> of μ. A hyperplane π separating no two points of μ is called a <u>bounding plane for μ</u>; an equivalent definition is: π is a bounding plane if one of the two closed halfspaces bounded by π wholly contains μ. Obviously any converging sequence of bounding planes for μ tends to a bounding plane for μ. A bounding plane for μ containing a point P of the boundary of μ is called a <u>supporting plane of μ at P.</u>

As a consequence of the last theorem we have:

<u>Corollary</u>: <u>If S_1 is a convex set and the point P is not in the closure $\bar{S}$ of S, then there exists a bounding plane for S through P.</u>

For applying Theorem 2 to $S = S_1$, $P = S_2$, we see that a plane exists separating P from S; the plane parallel to π through P meets the requirements of the corollary.

We prove furthermore:

Theorem 3. A convex set S either contains interior points or is wholly contained in a hyperplane. In the first case each point of S is an accumulation point of its interior points. More precisely: if Q is an interior point of S, and P any point of S, then all points of PQ except possibly P are interior points of S.

If S contains more than one point it contains infinitely many points. If it does not lie in a hyperplane, it contains n+1 points not in a hyperplane. The whole simplex with these points as vertices then belongs to S on account of the convexity.

Let S contain an interior point Q and let P be any point of S. There exists an open sphere with center Q contained in S, the segments connecting the points of this sphere to P form a set T belonging to S, and each point of $\overline{PQ}$ except P is an interior point of T and therefore of S, which proves the theorem.

If μ is any set in the hyperplane Π, then $\bar{\mu}$ is the boundary of μ and Π is a supporting plane of μ at each point of $\bar{\mu}$. This explains why we restrict ourselves in the following theorem to sets with interior points:

Theorem 4. A closed set S with interior points is convex if, and only if, there exists a supporting plane of S at each boundary point of S.

Remark. The proof of the necessity of the condition does not require that S be closed. For the sufficiency it is, however, essential, the theorem not being true otherwise.

Proof. A. Assume S is convex and P a boundary point of S. Let Q be an interior point of S. The preceding theorem shows that no point R on the prolongation of $\overline{QP}$ beyond P can belong to S. For R cannot belong to S either. Otherwise P would be an interior point of $\overline{S}$, a whole sphere around P would belong to $\overline{S}$, and S would have to be everywhere dense in this sphere. S then would contain n+1 points such that the simplex spanned by these points contains P in its interior. P would be an interior point of S. Choose a sequence of points P_ν on the prolongation of $\overline{QP}$ beyond P with $P_\nu \to P$. According to the corollary to Theorem 2, there exists a bounding plane π_ν for S through P_ν. Choose a subsequence $\{\pi_{\nu_n}\}$ of $\{\pi_\nu\}$ such that π_{ν_n} tends to plane π through P. As remarked before, π must also be a bounding plane of S, and since $\pi > P$, π is a supporting plane of S at P.

B. Assume that there exists a supporting plane of S at each boundary point of S, S being closed. If two points P_1, P_2 in S should exist such that P_1 and P_2 belong to S but an interior point C of P_1P_2 does not, one could connect C to an interior point D of S, which is not on the straight line through P_1 and P_2. Then a boundary point E of S would lie in the interior of $\overline{CE}$. According to our assumption, a supporting plane π of S at E would exist. Since D is an interior point of S, the plane cannot pass through D, but then π would separate two of the points P_1, P_2, D.

That B cannot be proved without the assumption that S is closed is shown by the following example: Take a closed tetrahedron in 3-space and leave out a boundary point which is no vertex. The remaining set would still have the property that a supporting plane exists at each boundary point, but the set is no longer convex.

We now prove a theorem which generalizes the necessary condition for convexity of the last theorem:

<u>Theorem 5.</u> <u>Let S be a closed convex set with interior points and L_{n-k} a linear subspace of dimension $0 < n-k \leq n-2$ which contains no interior point of S. Then a bounding hyperplane for S through L_{n-k} exists.</u>

<u>Proof.</u> We first take the case $n = 3 \neq \cancel{2}$. Then $n-k = 1$ and $L_{n-k} = L_1$ is a straight line. We consider the halfplanes bounded by L_1. The subset α of those halfplanes which contain interior points of S has the following properties: (1) With any halfplane the neighboring halfplanes are in α, (2) α contains no pair of opposite halfplanes, (3) two different halfplanes in α therefore always determine uniquely a convex angular space bounded by them. This whole space belongs to α (2) and (3) follow from the fact that if P_1 and P_2 are interior points of S, all points of $\overline{P_1P_2}$ are interior points of S. The set α therefore has two uniquely determined limit-halfplanes a' and a", which according to (1) do not belong to α. On account of (2) and (3) a' and a" are either opposite or α is the interior of the convex angle bounded by a' and a". In both cases it is clear that the plane containing a' is a bounding one for S through L_1.

In the higher dimensional cases it is clear that for $k = 2$ the method of the preceding proof can be applied. If $k \geq 3$ we consider an L_{n-k+2} through the L_{n-k} and an interior point of S. This point is an interior point of L_{n-k+2} S considered as a convex set of $n-k+2$ dimensions. The L_{n-k} does not contain an interior point of this convex set; therefore an L_{n-k+1} exists in the L_{n-k+2} which is a bounding plane for S L_{n-k+2} and therefore does not contain an interior point of S. Thus we have reduced the case of k to the case k-1 and it is clear that we can go on until we reach k = 2.

<u>Definition.</u> One calls convex closure of a set S the product of all closed halfspaces containing S if there are any, otherwise the whole space. The convex closure is characterized by the following

<u>Theorem 6. The convex ~~cover~~ closure $\mathcal{C}$ of a set S is identical with the product $\mathcal{C}^*$ of all closed convex sets containing S. Therefore if S is convex, the convex closure coincides with the closure of S.</u>

The <u>proof</u> is easy. Clearly we have $\mathcal{C}^* \leq \mathcal{C}$. Let $P \notin \mathcal{C}^*$. Then according to Theorem 2, a plane Π exists separating P from $\mathcal{C}$. Hence $P \notin \mathcal{C}$, which proves $\mathcal{C} \leq \mathcal{C}^*$.

<u>Definition:</u> A function $f(P)$ defined on a convex set S is said to be <u>convex</u> on S if for each P_1 and P_2 in S and each λ, $0 < \lambda < 1$, we have

(1) $$f[(1-\lambda)P_1 + \lambda P_2] \leq (1-\lambda) f(P_1) + \lambda f(P_2).$$

We first characterize the convex functions in terms of convex sets:

<u>Theorem 7. A necessary and sufficient condition that the function $f(x^1, \ldots, x^n)$ defined on the convex set S in $(x^1, \ldots, x^n)$-space be convex on S is that the set Σ of points $(x^1, \ldots, x^n, z)$ where $(x^1, \ldots, x^n) < S$ and $z \geq f(x^1, \ldots, x^n)$ be convex in the $(x^1, \ldots, x^n, z)$ space.</u>

<u>Proof.</u> As an illustration we take the case n = 2, $x^1 = x$, $x^2 = y$.

A. Suppose f convex on S and let (x_1, y_1, z_1) and (x_2, y_2, z_2) be points of Σ. Then $z_i \geq f(x_i, y_i)$, i = 1, 2, and since S is convex the point $[(1-\lambda)x_1 + \lambda x_2, (1-\lambda)y_1 + \lambda y_2]$ $(0 < \lambda < 1)$ is in S. Therefore by (1)

$$(1-\lambda) z_1 + \lambda z_2 \geq (1-\lambda) f(x_1, y_1) + \lambda f(x_2, y_2) \geq$$
$$\geq f[(1-\lambda)x_1 + \lambda x_2, (1-\lambda)y_1 + \lambda y_2].$$

Hence the point $[(1-\lambda)x_1 + \lambda x_2, (1-\lambda)y_1 + \lambda y_2, (1-\lambda)z_1 + \lambda z_2]$ is in B. Suppose Σ convex and let (x_1, y_1) and (x_2, y_2) be two points of S and suppose $z_1 = f(x_1, y_1)$, $z_2 = f(x_2, y_2)$. Then (x_1, y_1, z_1) and (x_2, y_2, z_2) are in Σ. Therefore $[(1-\lambda)x_1 + \lambda x_2, (1-\lambda)y_1 + \lambda y_2, (1-\lambda)z_1 + \lambda z_2]$ is in Σ, and hence

$$(1-\lambda) z_1 + \lambda z_2 = (1-\lambda) f(x_1, y_1) + \lambda f(x_2, y_2) \ge f[(1-\lambda) x_1 + \lambda x_2, (1-\lambda) y_1 + \lambda y_2)]$$

The next theorem is obvious, but it consists of a fact which will be useful later on:

Theorem 8. A necessary and sufficient condition that $f(x^1, \dots, x^n)$ be convex on the convex set S is that, for each pair P_1, P_2 in $(x^1, \dots, x^n)$-space, $f[(1-\lambda)P_1 + \lambda P_2)]$ be convex in λ for those values of λ (an interval, a point, or the empty set) for which $(1-\lambda)P_1 + \lambda P_2$ is in S. If $f(x^1, \dots, x^n)$ is convex, then $f(x, a)$ is convex in x on the set S_a of values x for which (x, a) lies in S; here x stands for $(x^{i_1}, \dots, x^{i_k})$ and a for $(a^{j_1 \dots j_e})$ where $(i_1, \dots, i_k, j_1, \dots, j_e)$ is a permutation of $(1, 2, \dots, n)$ and the x's and a's are supposed arranged in their natural order; the set S_a is convex or empty for each such a.

Theorem 9. The limit of a convergent sequence of convex functions is convex. The least upper bound of an aggregate of convex functions is convex (or $\equiv +\infty$).

The first statement follows at once from the definition of convex functions. The second part follows from Theorems 1 and 7 since, $\{f\}$ designating the given aggregate,

$$\prod_f (z \ge f(x,y)) = (z \ge \underset{f \subset \{f\}}{\text{L.U.B}}\, f)$$

Theorem 10. <u>Let $y(x)$ be a convex function of the single variable x on (a, b). Then</u>

(2) $$\frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1} \le \frac{\varphi(x_3)-\varphi(x_1)}{x_3-x_1} \le \frac{\varphi(x_3)-\varphi(x_2)}{x_3-x_2} \quad \text{for } x_1 < x_2 < x_3$$

(3) $$\frac{\varphi(x_4)-\varphi(x_3)}{x_4-x_3} \ge \frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1} \quad \text{for } x_1 < x_3, \quad x_2 < x_4$$

(4) If $a < x_1 \le x_2 < b$, then for $a < x < x_1$ and $x_2 \le x < b$ we have

$$\varphi(x) \ge \varphi(x_1) + \frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1}(x-x_1) = \left[1-\frac{x-x_1}{x_2-x_1}\right]\varphi(x_1) + \frac{x-x_1}{x_2-x_1}\varphi(x_2).$$

Proof. Let $x_2 = (1-\lambda)x_1 + \lambda x_3$, $0 < \lambda < 1$. Then $\varphi(x_2) \le (1-\lambda)\varphi(x_1) + \lambda\varphi(x_3)$ and (2) follows. In (4), if $a < x < x_1 < x_2$, for instance, it follows from (2) that

$$\frac{\varphi(x)-\varphi(x_1)}{x-x_1} \le \frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1}$$

Since $x - x_1 < 0$ in this case, (4) follows. The case $x > x_3$ is treated similarly.

In (3) we may assume that $x_4 > x_3$ and $x_2 > x_1$. Then there are 3 cases according as $x_3 = x_2$, $x_3 > x_2$, or $x_3 < x_2$. The first is included in (2). In either of the last cases, x_2 and x_3 are both between x_1 and x_4, so that by applying (1) we get

$$\frac{\varphi(x_4)-\varphi(x_3)}{x_4-x_3} \ge \frac{\varphi(x_4)-\varphi(x_1)}{x_4-x_1} \ge \frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1}$$

Theorem 11. <u>Let $\varphi(x)$ be a convex function of the single variable x on (a, b). Then at each interior point x_0, the derivative $D_R\varphi(x_0)$ on the right at x_0 and $D_L\varphi(x_0)$, the derivative on the left of x_0, both exist</u>

and are finite. Moreover, each is a monotone non-decreasing function of x on (a, b) and for each interior x_0, we have

(5) $$D_R \varphi(x_0) \geq D_L \varphi(x_0)$$

$$\lim_{h\to 0+} D_R \varphi(x_0 - h) \leq D_L \varphi(x_0), \quad \lim_{h \to 0+} D_L \varphi(x_0+h) \geq D_R \varphi(x_0).$$

Furthermore, $D_R \varphi(x_0) = D_L \varphi(x_0)$ except on a set of points which is at most denumerable, and $\varphi''(x)$ exists almost everywhere and is non-negative. $\varphi(x)$ satisfies a uniform Lipschitz condition on each closed interval interior to (a, b), and if $|\varphi| \leq M$ on (a, b), this Lipschitz condition takes the form

(6) $$|\varphi(x_1) - \varphi(x_2)| \leq \frac{2M}{\delta}|x_2 - x_1|, \quad \alpha \leq x_1, x_2 \leq \beta$$

where δ is the distance of (α, β) from the boundary of (a, b).

Proof. Let $a < x_0 < b$. If then $0 < h_1 < h_2$, we have, according to (2), that

$$\frac{\varphi(x_0+h_2) - \varphi(x_0)}{h_2} \geq \frac{\varphi(x_0+h_1) - \varphi(x_0)}{h_1}$$

hence $\lim \dfrac{\varphi(x_0+h) - \varphi(x_0)}{h} \geq D_R \varphi(x_0)$ exists and is $< +\infty$

In a similar way $D_L \varphi(x_0)$ exists and is $> -\infty$. Moreover, for each $h > 0$, we have (using (3)) that

$$\frac{\varphi(x_0+h) - \varphi(x_0)}{h} \geq \frac{\varphi(x_0-h) - \varphi(x_0)}{-h}$$

so that $D_R \varphi(x_0) \geq D_L \varphi(x_0)$ and both are finite. This proves the first part of (5). If $h \neq 0$ and $x_2 > x_1$, it follows from (3) that

$$\frac{\varphi(x_2+h) - \varphi(x_2)}{h} \geq \frac{\varphi(x_1+h) - \varphi(x_1)}{h}$$

so that $D_R\varphi$ and $D_L\varphi$ are monotone non-decreasing. Now take x_0 and then $x_1 > x_0$. If $0 < h < \dfrac{x_1 - x_0}{2}$ we see by Theorem 10 that

$$\frac{\varphi(x_1-h) - \varphi(x_1)}{-h} \geq \frac{\varphi(x_0+h) - \varphi(x_0)}{h}$$

If $h \to 0$ one sees that $D_L \varphi (x_1) \geqq D_R \varphi (x_0)$. From this the third of the inequalities (5) follows. The second is proved similarly. If $D_R \varphi (x)$ or $D_L \varphi (x)$ is continuous at x_0, it follows from (5) that $\varphi'(x_0)$ exists. Therefore $\varphi'(x_0)$ exists except for an at most denumerable set of values x and the monotonity of $\varphi'(x)$ shows that $\varphi''(x)$ exists almost everywhere and that $\varphi''(x) \geqq 0$.

Now let $a < \bar{\alpha} < \alpha \leqq x_1 < x_2 \leqq \beta < \bar{\beta} < b$. Then, by Theorem 10

(7) $$\frac{\varphi(\alpha) - \varphi(\bar{\alpha})}{\alpha - \bar{\alpha}} \le \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \le \frac{\varphi(\bar{\beta}) - \varphi(\beta)}{\bar{\beta} - \beta}$$

so that $\varphi (x)$ satisfies a uniform Lipschitz condition on (α, β). If we assume $|\varphi (x)| \leqq M$ on (a, b) the inequality (6) follows easily by taking $\bar{\alpha}$ arbitrarily close to a and $\bar{\beta}$ to b.

We now turn to functions of several variables and prove first:

Theorem 13. A necessary and sufficient condition that $f(x^1,\ldots,x^n)$ be convex on the open convex set S of the $(x^1,\ldots,x^n)$-plane is that for each point $x_0 = (x_0^1,\ldots,x_0^n)$ of S there exists a linear function $\sum_{\alpha=1}^{n} a_\alpha x^\alpha + b = a_\alpha x^\alpha + b$ and that

(8) $$f(x) \geq a_\alpha x^\alpha + b$$

and $$f(x_0) = a_\alpha x_0^\alpha + b$$

If f is of class C' on S, a necessary and sufficient condition that it be convex on S is that

(9) $$E(x, x_0) \equiv E(x^1,\ldots,x^n; x_0^1,\ldots,x_0^n) \equiv f(x) - (x^\alpha - x_0^\alpha) f_{x^\alpha}(x_0) \geq 0$$

for all (x, x^0) on S. If f is of class C'' on S, a necessary and sufficient condition that $f(x^1,\ldots,x^n)$ be convex on S is that

(10) $$f_{x^\alpha x^\beta}(x^1,\ldots,x^n)\,\xi^\alpha \xi^\beta \geq 0$$

for all x in S and all ξ. In two dimensions, (10) is equivalent to

(11) $$f_{xx} \geq 0, \quad f_{yy} \geq 0, \quad f_{xx} f_{yy} - f_{xy}^2 \geq 0$$

for all (x, y) in S.

Proof. (8) follows immediately by applying Theorem 4 to the set of Theorem 7. Then (9), (10), (11) follow from (8) by applying the mean value theorem.

As a consequence of Theorem 12 we have that a function f(x) which is convex on an open convex set S is continuous on S. For if that was not true a sequence of points x_i tending to a point x_0 in S would exist such that

$$\lim f(x_i) = f(x_0)$$

$f(x_0) < \lim f(x_i)$ is impossible since for each plane $z = a_\alpha x^\alpha + b$ with

$a_\alpha x_0^\alpha + b = f(x_0)$ one has for large i

$$f(x_i) < f(x_0) = a_\alpha x_0^\alpha + b$$

which contradicts Theorem 12. If $f(x_0) < \lim f(x_i)$ we could construct planes Π_i : $z = a_\alpha^i x^\alpha + b_i$ with

(12) $f(x) \geq a_\alpha^i x^\alpha + b_i$ and $f(x_i) = \sum_\alpha a_\alpha^i x_i^\alpha + b_i$

therefore in particular

(13) $a_\alpha^i x_0^\alpha + b_i \leq f(x_0)$

(12) and (13) show that the slope of Π_i tends to infinity with i, the limit plane Π of a convergent subsequence $\{\Pi_{i_s}\}$ of $\{\Pi_i\}$ contains the parallel to the z-axis through x_0. But, as limit of bounding plane of the set Σ (see Theorem 7), Π would have to be a bounding plane for Σ which is not true.

The condition that S be open is essential; for the function

$$z = \begin{cases} 0 & \text{on } x^2+y^2<1 \\ 1 & \text{on } x^2+y^2=1 \end{cases}$$

is convex on $x^2 + y^2 \leqq 1$ without admitting a plane of the type (8) at the points of $x^2 + y^2 = 1$ and without being continuous at these points.

We prove now:

Theorem 14. Let $f(x^1, \ldots, x^n)$ be convex on the convex set S. Then f satisfies a uniform Lipschitz condition on each interior closed bounded subset E of S. If $|f| \leqq M$ on S, the Lipschitz condition takes the form

(14) $|f(x_1) - f(x_2)| \leq \dfrac{2M\, e(x_1, x_2)}{\delta}$

where δ is the distance of E from the boundary S^* of S. If the functions $f_\rho(x^1, \ldots, x^n)$ are convex on S and converge to $f(x^1, \ldots, x^n)$ at each point of S, the convergence is uniform in each interior bounded closed subset E of S.

<u>Proof.</u> (14) is an immediate consequence of (6), p. 36. If one does not know that f is bounded on S, let S_1 be any bounded open convex set in S containing E and such that $\bar{S}_1$ is interior to S. Since f is continuous on $\bar{S}$, it is bounded on $\bar{S}$, and we can apply (14) to S_1 and E.

From this, the last statement follows if we know that the functions are uniformly bounded on each bounded closed set S_1 interior to S. If this were not true, there would exist a subsequence $\{p_k\}$ and a sequence x_k of points of S_1 tending to a point x_o of S_1 such that

(15) $$\lim_{k\to\infty} f_{p_k}(x_k) = \pm\infty$$

Case I. The above limit is $+\infty$. Let

(16) $$z = a^k_\alpha x^\alpha + b^k$$

be the supporting plane to $z = f_{p_k}(x)$ at $x = x_k$. Then $f_{p_k}(x) \geq a^k_\alpha x + b^k$ for every x and k and

(17) $$\lim_{k\to\infty} f_{p_k}(x_0) = f(x_0)$$

which is some finite number. The plane (16) cannot coincide with the plane $z = f_{p_k}(x_k)$ (a constant) for infinitely many k as (17) and the definition of (16) would imply that $f(x_o) = +\infty$. Hence, we assume that the two planes never coincide. Then if x is interior to the half space (of x-space)

(18) $$a^k_\alpha x^\alpha + b^k - f_{p_k}(x_k) > 0$$

the boundary of which contains x_k, we see that

(19) $$f_{p_k}(x) \geq a^k_\alpha x^\alpha + b^k > f_{p_k}(x_k)$$

Evidently (since $x_k \to x_o$) we may choose a further subsequence (still called p_k) such that the half planes (18) converge to a limit. Then x_o is on the boundary of this limiting half plane, and if x_1 is interior to this half plane, we have by (15) and (18), that

$$\lim_{k\to\infty} f_{p_k}(x_1) = +\infty.$$

which is impossible as x_o is interior to S.

Case II. The limit in (15) is $-\infty$. Let S_2 be the closed bounded set of points at a distance $\leq \rho$ from S_1; if ρ is sufficiently small, S_2 is interior to S. It is clear from (15) and (17) that x_k cannot coincide with x_o for infinitely many k. Hence, for each k, let $\bar{x}_k$ be the point on the line joining x_k and x_o which is at a distance from x_o and is such that x_o is between x_k and $\bar{x}_k$. Then, by Theorems (6) and (8) it follows that

$$f_{p_k}(\bar{x}_k) \geq (1-\lambda_k) f_{p_k}(x_k) + \lambda_k f_{p_k}(x_0)$$

where

$$\bar{x}_k = (1-\lambda_k)x_k + \lambda_k x_o, \quad \lim_{k\to\infty} \lambda_k = +\infty.$$

Hence

$$\lim_{k\to\infty} f_{p_k}(\bar{x}_k) = +\infty$$

and we have Case I as all the $\bar{x}_k$ are in S_2 and a further subsequence may be chosen so that they converge to a point $\bar{x}_o$ of S_2. ~~One has~~

~~Choose now a subsequence of p_3 such that congerges to a half plane x_o is on the boundary of and f(x) = if x is in the interior of but this is impossible, x_o being an interior point of S.~~

The next theorem is a consequence of the following

Lemma 1. Let $\sum_{i=1}^{N} \lambda_i = 1$, $\lambda_i \geq 0$, i = 1,...,N, and let P_i, i = 1,...,N be points of a convex set S on which the convex function f is defined. Then $\lambda^\alpha P_\alpha$ belongs to S and

(20) $$f(\lambda^\alpha P_\alpha) \leq \lambda^\alpha f(P_\alpha)$$

The proof is by induction on N. If N = 1, the theorem is trivial. Hence, suppose the theorem has been proved for N = k; and consider k+1 points

P_i of S and k+1 non-negative numbers λ_i whose sum is 1. Then

$$\sum_{i=1}^{k+1} \lambda_i P_i = \sum_{i=1}^{k} \lambda_i P_i + \lambda_{k+1} P_{k+1} = (1-\lambda_{k+1}) \sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} P_i + \lambda_{k+1} P_{k+1}$$

But the point

$$\sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} P_i$$

is in S according to the hypothesis of induction, and P_{k+1} is too, such that $\sum_{i=1}^{k+1} \lambda_i P_i$ is in S. We then have as a consequence of (1) and (15)

$$f\left(\sum_{i=1}^{k+1} \lambda_i P_i\right) \leq (1-\lambda_{k+1}) f\left(\sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} P_i\right) + \lambda_{k+1} f(P_{k+1}) \leq \sum_{i=1}^{k+1} \lambda_i f(P_i)$$

Theorem 15. Let $f(x^1,\ldots,x^n)$ be defined and convex for all $(x^1,\ldots,x^n)$, and let $\Phi(e)$ be a non-negative completely additive set function defined as a set E for which $\Phi(E)$ exists and is finite. Let $\varphi^1,\ldots,\varphi^n$ be functions which are summable (in the Lebesgue-Stieltjes sense) over E with respect to Φ. Finally, let $M_{\Phi,E}(\varphi)$ be defined by

$$M_{\Phi,E}(\varphi) = \frac{1}{\Phi(E)} \int_E \varphi \, d\Phi$$

Then

$$f\left[M_{\Phi,E}(\varphi^1), \ldots, M_{\Phi,E}(\varphi^n)\right] \leq M_{\Phi,E}\left[f(\varphi^1,\ldots,\varphi^n)\right]$$

Proof. This follows immediately from Lemma 1 and the usual processes of approximation by step functions.

Theorem 16. Let $f(x^1,\ldots,x^n)$ be a convex function such that

$$(16) \qquad \lim_{\Sigma (x^i)^2 \to \infty} \frac{f(x^1,\ldots,x^n)}{[(x^1)^2 + \cdots + (x^n)^2]^{\frac{1}{2}}} \to \infty$$

Then f(x) takes on its minimum, and if $a^1,\ldots,a^h$ any given numbers there exists a unique number d such that $z = a\ x + d$ is a supporting plane to $z = f(x^1,\ldots; x^n)$.

Proof. Let x_o be any point and $z_o = f(x_o)$. By Theorems 1 and 7, the set Σ_0 of $x^1,\ldots,x^n$ for which $f(x) \leqq z_0$ is a closed convex set, which is bounded on account of ($\overset{21}{\cancel{16}}$). Since f is continuous on Σ_0 it takes its minimum value c on Σ_0 and $c \leqq z_0$. For $x \notin \Sigma_0$ one has $f > z_0 \geqq c$. To prove the second part of the theorem we notice that $f(x) - a^\alpha x^\alpha$ also satisfies ($\overset{21}{\cancel{16}}$). Let d be the minimum of this function. Then we have that $z = a^\alpha x^\alpha + d$ is a supporting plane for $z = f$. It is obviously unique.

From this theorem we conclude as a

Corollary: Let $f(x^1,\ldots,x^n)$ and $g(x^1,\ldots,x^n)$ be convex and satisfy ($\overset{21}{\cancel{16}}$), and let $f \geqq g$ everywhere. Let $a^1,\ldots,a^n$ be given numbers. Then, if $a^\alpha x^\alpha + c$ and $a^\alpha x^\alpha + d$ are supporting planes to f and g, $c \geqq d$.

This follows from the proof of Theorem 16, since c and d are the minima of $f(x) - a^\alpha x^\alpha$ and $g(x) - a^\alpha x^\alpha$ respectively.

Theorem 17. Suppose that the functions $f\ (x)$ and $f(x)$ are all convex and satisfy ($\overset{21}{\cancel{16}}$). Suppose also that $f_\rho(x) \to f(x)$ at each point. Let c_p be the minimum of f_p and c that of f. Then $c_p \to c$.

Proof. Let x_0 be a point where $f(x_0) = c$. Then $f_p(x_0) \to c$, so that $c \geqq \overline{\lim}\, c_p$. Now choose $\varepsilon > 0$ and let Σ_ε be the closed convex set where $f(x) \leqq c + 3\varepsilon$ and let R_ε be the smallest sphere with center x_0 which contains Σ_ε; clearly $f \geqq c + 3\varepsilon$ on R^*_ε. Let r_ε be the radius of R_ε and choose N so large that for $p > N$ $|f_p(x) - f(x)| < \varepsilon$ for all x in $\overline{R}_\varepsilon$. Now let $x_0 + \xi$ be any point with $\xi > r_\varepsilon$. Put $\xi_\varepsilon = \frac{r_\varepsilon}{|\xi|}\xi$. Then $x_0 + \xi$ is on R^*_ε. By Theorems 8 and 10 we have for $p > N$

$$f_\rho(x_0+\xi) \geq f_\rho(x_0) + \frac{|\xi|}{r_\varepsilon}\{f_\rho(x_0+\xi_\varepsilon) - f_\rho(x_0)\} \geq c - \varepsilon + \frac{|\xi|}{r_\varepsilon}\varepsilon > c$$

Hence for $p > N$ we have $f_p(x) > c - \varepsilon$ for all x so that $c \leqq \underline{\lim}\, c_p$.

Theorem 18. Let $f(x^1,\ldots,x^n)$ be convex and satisfy (~~16~~ 21). Let $a_0 x + c_0$ be a supporting plane to $z = f$ at $(x_0^1,\ldots,x^n)$. Let $\{a_p^\alpha\}$ be a sequence such that $a_p^\alpha \to a^\alpha$, $\alpha = 1,\ldots,n$, and suppose c_p is chosen so that $a_p^\alpha x^\alpha + c_p$ is supporting to f. Then $c_p \to c_0$ so that

$$a_p^\alpha x_0^\alpha + c_p \to a_0^\alpha x_0^\alpha + c_0$$

Proof. Let $\varphi_p = f - a_p^\alpha x^\alpha$. Then $\{\varphi_p\}$ and $\varphi_0 = f - a_0^\alpha x^\alpha$ satisfy the hypotheses of Theorem 17 so that if c_p and c_0 denote the minima of φ_p and φ_0 respectively, $c_p \to \overline{c}_0$. But $\overline{c}_0 = c_0$ and, for each p, $a_p^\alpha x^\alpha + c_p$ is a supporting plane to f, which proves the theorem.

3. Lower Semicontinuity Theorems

Definition 1: Let $\varphi(R)$ be a function of cells defined on an open set G with the property that, if R is divided into the non-overlapping cells $R_1,\ldots,R_N$, we have

$$\varphi(R) \le \sum_{i=1}^{N} \varphi(R_i)$$

Such a cell function will be called normal on G*).

*) See S. Banach, "Sur une classe de fonctions d'ensembles", Fund. Math., vol. 6 (1924), pp. 170-188.

The following function of linear intervals is an example for a normal cell function: Let $x(t)$ and $y(t)$ be defined for $0 \le t \le 1$, let $[t_1,t_2]$ be an arbitrary subinterval of $[0,1]$, and put

$$\varphi([t_1,t_2]) = \left[(x(t_1) - x(t_2))^2 + (y(t_1) - y(t_2))^2\right]^{\frac{1}{2}}$$

The definition of bounded variation, variation, and absolute continuity of a normal cell function $\varphi(R)$ over cells D in G and over G are the usual ones. If $\varphi(R)$ is of bounded variation over G, the variation $V_\varphi(R)$ of φ over R is an additive cell function whose variation over G is the same as that of φ *)

*) Compare S. Saks, Théorie de l'intégrale, Warszawa 1933, Chapter I and Chapter VI, §3.

<u>Definition 2</u>: Let $\varphi(R)$ be a cell function. Then the derivative $D\varphi$ of φ at P_0 is defined by

$$D\varphi = \lim_{R \to P_0} \frac{\varphi(R)}{m(R)}$$

if it exists, R being a square containing P_0. If this limit does not exist we let $\overline{D}\varphi$ be the upper limit and $\underline{D}\varphi$ be the lower limit.

Lemma 1.**) Let $\varphi(R)$ be a non-negative normal set function of bounded variation on G. Then its derivative $D\varphi$ exists and is equal to DV_φ almost everywhere and is therefore summable, and we have

**) This has been proved by Banach (l.c.) without assuming $\varphi(R) \geq 0$. A proof can also be found in Saks (l.c.).

$$V_\varphi(G) \geq \iint_G D\varphi \, dx\, dy$$

If φ is absolutely continuous, the equality holds.

<u>Proof.</u> We know that DV_φ exists almost everywhere and is summable, and also that

$$\overline{D}V_\varphi \geq \overline{D}\varphi \geq \underline{D}\varphi \geq 0$$

at each point. Hence, let $\tilde{S}_{m,n}$ be the set of points P_0 where DV_φ exists and is $> \frac{1}{n}$ and where at the same time

$$\underline{D}\varphi < \left(1 - \frac{1}{m}\right) DV_\varphi$$

It is then sufficient to prove that $m_c(\tilde{S}_{m,n}) = 0$ for each m and n, m_e designating the exterior measure.

Suppose $m_e(S_{m,n}) > k > 0$. Let $0 < \varepsilon < \frac{k}{m\,n}$ and choose non-overlapping cells $H_1, \dots, H_k$ so that

(1) $$\sum_{i=1}^{k} \varphi(H_i) > V_\varphi(\mathcal{G}) - \varepsilon, \quad m_e(S_{m,n}) > k$$

where $S_{m,n}$ consists of those points of $\tilde{S}_{m,n}$ which are interior to some H_i. Each point P_o of $S_{m,n}$ is interior to a sequence $\{r_i\}$ of squares of arbitrary small diameter such that

(2) $$V_\varphi(r_i) > \frac{1}{n}\, m(r_i), \quad \varphi(r_i) < \left(1 - \frac{1}{m}\right) V_\varphi(r_i)$$

and we may assume that each r_i is entirely interior to the cell H_i which contains P_o. By the Vitali covering theorem, we can find a finite number $R_1, \dots, R_N$ of these squares which are non-overlapping and such that $\sum m(R_i) > k$. Then we have

(3) $$\sum_{i=1}^{N} \varphi(R_i) < \sum_{i=1}^{N} V_\varphi(R_i) - \frac{1}{m} \sum_{i=1}^{N} V_\varphi(R_i) < \sum_{i=1}^{N} V_\varphi(R_i) - \frac{k}{m \cdot n}$$

We now divide $H_1, \dots, H_k$ into further squares $R_1, \dots, R_N, R_{N+1}, \dots, R_p$, where $R_1, \dots, R_N$ are the squares previously used. From the normality of $\varphi(R)$ it follows that

(4) $$\sum_{i=1}^{p} \varphi(R_i) \geq \sum_{i=1}^{k} \varphi(H_i) > V_\varphi(\mathcal{G}) - \varepsilon > \sum_{i=1}^{p} V_\varphi(R_i) - \varepsilon$$

But from (3) it follows that

$$\sum_{i=1}^{p} \varphi(R_i) \leq \sum_{i=1}^{N} \varphi(R_i) + \sum_{i=N+1}^{p} V_\varphi(R_i) < \sum_{i=1}^{p} V_\varphi(R_i) - \frac{k}{m \cdot n}$$

which contradicts (4).

Definition 3: Let $z^{(v)}$ ($v = 1, 2, \dots$) and z be of class D' (see p. 15), on an open set G. We say that $z^{(v)} \to z$ if for every cell R interior to G we have

$$\lim_{\nu\to\infty} \iint_R |z^{(\nu)} - z|\, dx\,dy = \lim_{\nu\to\infty} \iint_R (p^{(\nu)} - p)\, dx\,dy = \lim_{\nu\to\infty} \iint_R (q^{(\nu)} - q)\, dx\,dy = 0$$

where $p^{(\nu)} = D_x z^{(\nu)}$, $q^{(\nu)} = D_y z^{(\nu)}$, $p = D_x z$, $q = D_y z$.

<u>Theorem 1.</u> <u>Let $f(p, q)$ be a convex function which is bounded below. Let $\{z^{(\nu)}\}$ and z be functions of class D_1' on the bounded open set G. Then the integrals</u>

$$I(z^{(\nu)}, \mathfrak{G}) = \iint_{\mathfrak{G}} f(p^{(\nu)}, q^{(\nu)})\, dx\,dy$$

$$I(z, \mathfrak{G}) = \iint_{\mathfrak{G}} f(p, q)\, dx\,dy$$

<u>are all finite or $+\infty$ and if $z^{(\nu)} \to z$, we have</u>

$$I(z, \mathfrak{G}) \leq \varliminf_{\nu\to\infty} I(z^{(\nu)}, \mathfrak{G})$$

<u>Proof.</u> Define the rectangle functions

$$\varphi(R, z^{(\nu)}) = m(R)\, f(p_R^{(\nu)}, q_R^{(\nu)})$$

$$\varphi(R, z) = m(R)\, f(p_R, q_R)$$

where

$$p_R = \frac{1}{m(R)} \iint_R p\, dx\,dy, \quad q_R = \frac{1}{m(R)} \iint_R q\, dx\,dy$$

and $p_R^{(\nu)}$, $q_R^{(\nu)}$ are defined correspondingly. Since G is bounded we may assume $f \geqq 0$. Now let R be a cell which is divided into the non-overlapping cells $R_1, \dots, R_N$. Then by Lemma 1 on convex functions (p. 41)

$$\frac{1}{m(R)} \varphi(R, z) = f\left[\frac{1}{m(R)} \iint_R p\, dx\,dy,\ \frac{1}{m(R)} \iint_R q\, dx\,dy\right] =$$

$$= f\left[\sum_{i=1}^{N} \lambda_i \iint_{R_i} p\, dx\,dy,\ \sum_{i=1}^{N} \lambda_i \iint_{R_i} q\, dx\,dy\right] \leq$$

$$\leq \sum_{i=1}^{N} \lambda_i f\left(\frac{1}{m(R_i)} \iint_{R_i} p\, dx\,dy,\ \frac{1}{m(R_i)} \iint_{R_i} q\, dx\,dy\right) =$$

$$= \frac{1}{m(R)} \sum_{i=1}^{N} \varphi(R_i, z), \quad \lambda_i = \frac{m(R_i)}{m(R)}.$$

Thus $\varphi(R, z)$ is normal and so is $\varphi(R, z^{(\nu)})$.

From the convexity of f it follows furthermore, that if $R_1, \ldots, R_N$ are non-overlapping rectangles in G,

$$(5) \quad \sum_{i=1}^{N} \varphi(R_i, z) \leq \bar{I}(z, G), \quad \sum_{i=1}^{N} \varphi(R_i, z^{(\nu)}) \leq \bar{I}(z^{(\nu)}, G)$$

$$\lim \sum_{i=1}^{N} \varphi(R_i, z^{(\nu)}) = \sum_{i=1}^{N} \varphi(R_i, z)$$

$$\varphi(R_i, z) \leq \bar{I}(z; R), \quad \varphi(R_i, z^{(\nu)}) \leq \bar{I}(z^{(\nu)}, R_i)$$

If P_o is not in a set of measure 0 we have, on account of Lemma 1,

$$D_\varphi = f(p, q), \quad D_{\varphi_\nu} = f(p^{(\nu)}, q^{(\nu)})$$

Thus it follows

$$V_\varphi(G) = \bar{I}(z, G), \quad V_{\varphi_\nu}(G) = \bar{I}(z^{(\nu)}, G)$$

from which, using Lemma 1 once more, we infer our theorem.

Theorem 2: Let u(x), x = (x^1, ..., x^n), be of class D'_p ($p \geqq 1$) on a hypercube R: (a; b) of side h, then

(i) $$\int_a^b\int_a^b |u(x_1)-u(x_2)|^p dx_1\,dx_2 \leq [(2n-1)h]^p h^n \int_c^b \left(\sum D_{x^i}^2 u\right)^p dx$$

(ii) $$\int_c^b |u(x)-u_R|^p dx \leq [(2n-1)h]^p \int_c^b \left(\sum D_{x^i}^2 u\right)^p dx$$

where u_R denotes the average of u over R.

Proof: From previous theorems, it is sufficient to prove this for functions u(x) of class C' on R. For simplicity, we shall assume n = 3, $x^1 = x$, $x^2 = y$, $x^3 = z$. Let

$$M = \int_a^b\int_c^d\int_e^f [u_x^2+u_y^2+u_z^2]^{\frac{p}{2}} dx\,dy\,dz$$

There exist values $\bar{y}$, $\bar{z}$ with $c \leqq y \leqq d$, $e \leqq z \leqq f$ such that

(7) $$\int_a^b\int_c^d [u_x^2(x,y,\bar{z}) + u_y^2(x,y,\bar{z})]^{\frac{p}{2}} dx\,dy \leq \frac{M}{h_1} \qquad \int_c^b |u_x(x,\bar{y},\bar{z})\,dx \leq \frac{M}{h^2}$$

Then $$u(x_1,y_1,z_1) - u(x_2,y_2,z_2) = \int_{\bar{z}}^{z_1} u_z(x_1,y_1,\bar{z})\,dy + \int_{x_2}^{x_1} u_x(x,\bar{y},\bar{z})\,dx$$

(8) $$+ \int_{y_2}^{\bar{y}} u(x_2,y,\bar{z})\,dy + \int_{z_2}^{\bar{z}} u(x_2,y_2,z)\,dz$$

Hence $$\left[\int_c^b\int_c^d\int_e^f\int_c^b\int_c^d\int_c^f |u(x_1,y_1,z_1)-u(x_2,y_2,z_2)|^p dx_1\,dy_1\,dz_1\,dx_2\,dy_2\,dz_2\right]^{\frac{1}{p}} \leq$$

$$\leq \left[\iiint_R\iiint_R \left|\int_c^f u_z(x_1,y_1,z)\,dz\right|^p dx_1\,dy_1\,dz_1\,dx_2\,dy_2\,dz_2\right]^{\frac{1}{p}} +$$

$$+ \left[\iiint_R\iiint_R \left|\int_c^d u_y(x_1,y_1,\bar{z})\,dy\right|^p dx_1\,dy_1\,dz_1\,dx_2\,dy_2\,dz_2\right]^{\frac{1}{p}} +$$

$$+ \left[\iiint_R\iiint_R \left|\int_c^b u_x(x,\bar{y},\bar{z})\,dx\right|^p dx_1\,dy_1\,dz_1\,dx_2\,dy_2\,dz_2\right]^{\frac{1}{p}} + *$$

where the * denotes the other two terms corresponding to the last two in (8).

Using the Hölder inequality on the interior integrals and the integrating, using (7), etc., (i) follows immediately. (ii) follows from (i) using the Hölder inequality and the definition of u_R.

Lemma 2: Let $f(x,y,z,p,q)$ be defined and satisfy a uniform Lipschitz condition with constant K over the whole (x,y,z,p,q) space. Suppose further that $f(x,y,z,p,q)$ is convex in (p,q) for each fixed (x,y,z) and that there exist numbers $k > 0$ and N such that

$$f(x,y,z,p,q) \geq k(p^2+q^2)^{\frac{1}{2}} + N$$

for all (x,y,z,p,q)! Then, if $z_n(x,y) \to z(x,y)$, all being of class D_1' on the bounded open set G, we have

$$\iint_G f(x,y,z,p,q)\,dx\,dy = I(z,G) \leq \varliminf_{n\to\infty} I(z_n,G).$$

In this theorem, z may be a vector function on N variables $(x^1, \ldots, x^N)$ instead of (x,y).

Proof: First, let R be a square of side h interior to G. Then, from our hupotheses and Theorem 2, it follows that

$$\iint_R |f(x,y,z_n,p_n,q_n) - f(x_R,y_R,z_{nR},p_n,q_n)|\,dx\,dy \leq$$

$$\leq K\iint_R [|x-x_R| + |y-y_R| + |z_n - z_{nR}|]\,dx\,dy \leq K\left[\frac{h^3}{2} + 3h\iint_R (p_n^2+q_n^2)^{\frac{1}{2}}dx\,dy\right.$$

and the same holds for z.

Next, let D be any square interior to G. Then $I(z,D)$ and $I(z_n,D)$ are all finite. If $\varliminf_{n\to\infty} I(z_n,D) = +\infty$ it is clear that $I(z,D) \leq \varliminf_{n\to\infty} I(z_n,D)$. If $\varliminf_{n\to\infty} I(z_n,D)$ is finite, we may consider a subsequence (still called z_n) such that $I(z_n,D)$ tends to the above lower limit. From our hypotheses, it then follows that $D_1(z_n,D) \overset{*)}{\leq} M$, for some

*) D_1 is defined on p 53, below.

M independent of n.

Now choose $\varepsilon > 0$ and divide D into 4^{e} equal squares R_i. Then

$$\left| \iint_D f(x,y,z,p,q)\,dx\,dy - \sum_{i=1}^{4^e} \iint_{R_i} f(x_{R_i}, y_{R_i}, z_{R_i}, p, q)\,dx\,dy \right| \le \frac{Kd}{2^e}\left[\frac{m(g)}{2} + 3\Lambda\right] < \frac{\varepsilon}{2}$$

$$\left| \iint_D f(x,y,z_n,p_n,q_n)\,dx\,dy - \sum_{i=1}^{4^e} \iint_{R_i} f(x_{R_i}, y_{R_i}, z_{nR_i}, p_{nR_i}, q_{nR_i})\,dx\,dy \right| \le \frac{Kd}{2^e}\left[\frac{m(g)}{2} + 3\Lambda\right] < \frac{\varepsilon}{2}$$

if e is large enough, independently of n, d being the side of D. But now, for each e and n

$$\sum_{i=1}^{4^e} \left| f(x_{R_i}, y_{R_i}, z_{R_i}, p_n, q_n) - f(x_{R_i}, y_{R_i}, z_{R_i}, p_n, q_n) \right| dx\,dy \le$$

$$\le K \sum_{i=1}^{4^e} m(R_i) \cdot \left| z_{n,R_i} - z_{R_i} \right| \le K \iint_D |z_n - z|\,dx\,dy$$

which tends to zero as $n \to \infty$ since $z_n \to z$. Hence, for each e

$$\sum_{i=1}^{4^e} \iint_{R_i} f(x_{R_i}, y_{R_i}, z_{R_i}, p, q)\,dx\,dy \le \lim_{n\to\infty} \sum_{i=1}^{4^e} \iint_{R_i} f(x_{R_i}, y_{R_i}, z_{R_i}, p_{nR_i}, q_{nR_i})\,dx\,dy$$

Lemma 3: Let f(x,y,z,p,q) be continuous all over 5-space, convex in (p,q) for each fixed (x,y,z), and suppose $f(x,y,z,p,q) \ge \varphi(p,q)$ for all (x,y,z,p,q) where $\varphi(p,q)$ is a convex function satisfying the condition

$$\lim_{p^2+q^2\to\infty} (p^2+q^2)^{-\frac{1}{2}} \varphi(p,q) = +\infty .$$

Then there exists a monotone non-decreasing sequence $f_n(x,y,z,p,q)$ of functions of the type described in Lemma 2 which converge at each point (and hence uniformly over each closed bounded set) to f(x,y,z,p,q). It is clearly sufficient to assume f(x,y,z,p,q) lower semicontinuous.

Proof: Let a and b be any numbers. Let the function c(x,y,z; a,b) be chosen so that $w = \varphi(x,y,z,p,q;\ a,b) \equiv ap + bq + c(x,y,z;\ a,b)$ is the unique supporting plane to $w = f(x,y,z,p,q)$ determined by (a,b) we regard (a,b,x,y,z) as fixed. By Theorems 16 and 17, §2, $c(x,y,z;\ a,b)$ is continuous in (x,y,z) for each fixed (a,b) and by Theorem 16, §2, $c(x,y,z;\ a,b) \geqq c(a,b)$ for all (x,y,z), $c(a,b)$ being the corresponding function if $f = \varphi(p,q)$. Clearly, for each (a,b), we can find a non-decreasing sequence $c_n(x,y,z;a,b)$ which converges uniformly on each bounded region of (x,y,z) space to $c(x,y,z;\ a,b)$, each c_n satisfying a uniform Lipschitz condition condition over the whole (x,y,z) space. We then define

$$\varphi_n(x,y,z,p,q;a,b) = ap + bq + c_n(x,y,z,a,b) \geq ap + bq + c(a,b)$$

and $\varphi_{n+1}(x,y,z,p,q;\ a,b) \geqq \varphi_n(x,y,z,p,q;\ a,b)$ for each n,a,b, and each satisfies a uniform Lipschitz condition over the whole (x,y,z,p,q) space.

Now, we define $f_1(x,y,z,p,q)$ at each point as the largest of $\varphi_1(x,y,z,p,q;\ a,b)$ where a and b may independently take on the values -1, 0, 1. We then define $f_2(x,y,z,p,q)$ as the largest of the numbers $\varphi_2(x,y,z,p,q;\ a,b)$ where a and b may independently take on the values -2, -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, 2. In general, we define $f_n(x,y,z,p,q)$ as the largest of the numbers $\varphi_n(x,y,z,p,q;\ a,b)$ where a and b may take on the numbers, $0, 1, -1, \frac{1}{2}, 2, -\frac{1}{2}, -2, 1/3, 3, 2/3, 3/2, -1/3, -3, -2/3, -3/2, \ldots, 1/n, n, 2/n, n/2, \ldots, n-1/n, n/n-1, -1/n, -n, -2/n, -n/2, \ldots, -n-1/n, -n/n-1$. Clearly $f_n(x,y,z,p,q) \leqq f_{n+1}(x,y,z,p,2)$ for each n, each $f_n(x,y,z,p,q)$ is convex in (p,q) for each fixed (x,y,z), and satisfies a uniform Lipschitz condition over (x,y,z,p,q) space, and $f_n(x,y,z,p,q) \geqq f_1(x,y,z,p,q) \geqq N + k(p^2 + q^2)^{\frac{1}{2}}$.

Now, let $(x_0, y_0, z_0, p_0, q_0)$ be a point and let (a_0, b_0) be chosen so that

$$\varphi(x_0,y_0,z_0,p_0,q_0;a_0,b_0) \equiv a_0p_0 + b_0q_0 + c(x_0,y_0,z_0;a_0,b_0) = f(x_0,y_0,z_0,p_0,q_0)$$

$$\varphi(x_0,y_0,z_0;p,q;a_0,b_0) \leq f(x_0,y_0,z_0,p,q).$$

for all (p,q). Choose $\varepsilon > 0$. By Theorem 16 on convex functions, there exists a rational $\bar{a}$, $\bar{b}$ so that

$$\varphi(x_0, y_0, z_0, p_0, q_0; \bar{a}, \bar{b}) + \frac{\varepsilon}{2} > f(x_0, y_0, z_0, p_0, q_0)$$

Now we may choose $N(\varepsilon)$ so large that the denominators of $\bar{a}$ and $\bar{b}$ or their reciprocals are $\leq N(\varepsilon)$ and so that, if $n > N(\ )$

$$f(x_0, y_0, z_0, p_0, q_0) \geq f_n(x_0, y_0, z_0, p_0, q_0) \geq \varphi_n(x_0, y_0, z_0, p_0, q_0, \bar{a}, \bar{b}) > f(x_0, y_0, z_0, p_0, q_0) - \varepsilon .$$

Thus $f_n \to f$ at each point and hence uniformly on each bounded portion of (x,y,z,p,q) space.

__Theorem 3. Let f(x,y,z,p,q) be a function of the type described in Lemma 3. Then if $z_n(x,y) \to z(x,y)$, all being of class D' on the region G, we have__

$$I(z, G) \leq \varliminf_{n\to\infty} I(z_n, G)$$

Proof: We shall assume I(z,G) finite. If it is $+\infty$ the proof given below takes care of this case also if interpreted in the obvious way.

Choose $\varepsilon > 0$. Approximate to f(x,y,z,p,q) as in Lemma 3. Then there exists an N such that

$$I(z, G) - \varepsilon < I_N(z, G) \leq \varliminf_{n\to\infty} I_N(z_n, G) \leq \varliminf_{n\to\infty} I(z_n, G)$$

where $$I_N(z, G) = \iint_G f_N(x, y, z, p, q)\, dx\, dy .$$

__§4. Boundary value questions__

Definition 1: Let z be of class D' on a region G. We shall define $D_\alpha(z,G)$ and $\bar{D}_\alpha(z,G)$ by

$$\bar{D}_\alpha(z, G) = \int_G \left(|z|^\alpha + \sum_{i=1}^n |D_{x_i} z|^\alpha\right) dx , \quad D_\alpha(z, G) = \int_G \sum_{i=1}^n |D_{x_i} z|^\alpha dx$$

when these integrals are finite; otherwise we define them to be $+\infty$.

Clearly $D_\alpha(z,H)$ and $\overline{D}_\alpha(z,H)$ are both finite for each bounded region H with $\overline{H} < G$ if z is of class D'_α on G.

Lemma 1: Let z be of class D'_α on the open cell R: $(a^i < x^i < b^i)$, with $D_\alpha(z, R)$ finite. Then $\overline{D}_\alpha(z,R)$ is finite and there exist functions $\varphi^i(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n)$ and $\psi^i(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n)$, $i = 1, \dots, n$ of class L such that

$$\lim_{x_0^i \to a^i} \int_{a^1}^{b^1}\cdots\int_{a^{i-1}}^{b^{i-1}}\int_{a^{i+1}}^{b^{i+1}}\cdots\int_{a^n}^{b^n} |\bar z(x^1,\dots,x^{i-1},x_0^i,x^{i+1},\dots,x^n)-\varphi^i(x^1,\dots,x^{i-1},x^{i+1},\dots,x^n)|^\alpha dx^1\cdots dx^{i-1}dx^{i+1}\cdots dx^n = 0$$

$$\lim_{x_0^i \to b^i} \int_{a^1}^{b^1}\cdots\int_{a^{i-1}}^{b^{i-1}}\int_{a^{i+1}}^{b^{i+1}}\cdots\int_{a^n}^{b^n} |\bar z(x^1,\dots,x^{i-1},x_0^i,x^{i+1},\dots,x^n)-\psi^i(x^1,\dots,x^{i-1},x^{i+1},\dots,x^n)|^\alpha dx^1\cdots dx^{i-1}dx^{i+1}\cdots dx^n = 0$$

Furthermore z can be extended so as to be of class D'_α in a region which contains R in its interior. If $\alpha > n$, z is continuous on R as extended.

Proof: If k is small enough but > 0, we know that $D\ (z, R_k)$ is finite, where R_k: $a^i + k \leqq x^i \leqq b^i - k$. Now if $a^1 < x^1 \leqq a^1 + k$ or $b^1 - k \leqq x^1 < b^1$, and $a^1 + k \leqq x^1 \leqq b^1 - k$, we have

$$\int_{a^2+k}^{b^2-k}\cdots\int_{a^n+k}^{b^n-k} |z(x^{1\prime}, x^1) - z(x^{1\prime}, \bar x^1)|^\alpha dx^{1\prime} \leq |\bar x^1 - x^1|^{\alpha-1} D_\alpha(z, R)$$

$$\{x^{i\prime} = x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n,\quad (x^{i\prime}, x_0^i) = (x^1, \dots, x^{i-1}, x_0^i, x^{i+1}, \dots, x^n)\}$$

so that $|z|^\alpha$ is summable over R_{1k}: $(a^1, a^2 + k, \dots, a^n + k;\ b^1, b^2 - k, \dots, b^n - k)$. By successive steps, we see that z is of class L_α over R. This proves the first statement.

The second statement is now immediate for

$$\int_{a^{i\prime}}^{b^{i\prime}} |\bar z(x^{i\prime}, \beta^i) - \bar z(x^{i\prime}, \alpha^i)|^\alpha \leq (\beta_i - \alpha_i)^{\alpha-1} \int_{a^{i\prime}}^{b^{i\prime}}\int_{\alpha^i}^{\beta^i} |D_{x^i} z|^\alpha dx^{i\prime} dx^i, \quad \beta^i > \alpha^i, \quad i = 1, \dots, n$$

and this tends to zero as α^i and β^i tend independently to a^i or b^i.

The third statement is also immediate, for $\bar{z}$ may be extended as above to R and then to the whole space by successive reflections. The fourth statement follows immediately from Theorem 2, §3.

Lemma 2: Let $z(x^1, \ldots, x^n)$ be of class D'_α on a cell $a^i \leqq x^i \leqq b^i$, $i = 1, \ldots, n-1$, $0 < x^n \leqq b^n$ with $\bar{D}_\alpha(x, R)$ finite (R being the above cell). Suppose that

$$\lim_{x_0^n \to 0} \int_{a^{n'}}^{b^{n'}} |\bar{z}(x^{n'}, x_0^n) - \varphi(x^{n'})|^\alpha dx^{n'} = 0.$$

as in Lemma 1. Let $x^i = x^i(y^1, \ldots, y^n)$ be a transformation, of class C' with non-vanishing Jacobian, of the cell T $c^i \leqq y^i \leqq d^i$ ($i = 1, \ldots, n$, $c^n = 0$) into a subset of $\bar{R}$ in such a way that $x^n(y^1, \ldots, y^{n-1}, 0) \equiv 0$. Let w(y) be the transformed function (also of class D'_α on T with $\bar{D}_\alpha$ w is finite) of z(x) and let

$$\Psi(y^1, -, y^{n-1}) = \varphi[x^1(y^1, \ldots, y^{n-1}, 0), \ldots, x^{n-1}(y^1, \ldots, y^{n-1}, 0)]$$

Then

$$\lim_{y_0^{(n)} \to 0} \int_{c^{n'}}^{d^{n'}} |\bar{w}(y^{n'}, y_0^n) - \Psi(y^{n'})|^\alpha dy^{n'} = 0$$

Proof: Let $\bar{z}$ be extended to be of class D'_α in a region containing $\bar{R}$. Then there exists a sequence $\{z_p\}$ of class C' on R such that

$$\lim_{p \to \infty} \int_a^b (|z_p - z|^\alpha + \sum_{i=1}^n |D_{y^i}(z_p - z)|^\alpha) dy = 0$$

Clearly, we have w_p of class C' on the closed cell (c,d) and

$$\lim_{p \to \infty} \int_c^d (|w_p - w|^\alpha + \sum_{i=1}^n |D_{y^i}(w_p - w)|^\alpha) dy = 0.$$

As in the proof of Theorem 2, §1 we can choose a subsequence (still called z_p) so that

$$(1) \quad \int_{a^{n'}}^{b^{n'}} |z_p(x^{n'}, x_0^n) - \bar{z}(x^{n'}, x_0^n)|^\alpha dx^{n'} < \varepsilon_p$$

$$(2) \quad \int_{c^{n'}}^{d^{n'}} |w_p(y^{n'}, y_0^n) - \bar{w}(y^{n'}, y_0^n)| dy^{n'} < \varepsilon_p \qquad \lim \varepsilon_p = 0.$$

ε_p being independent of x_o^n and y_o^n. We include the cases $x_o^n = 0$, $y_o^n = 0$, it being understood that

(3) $\bar{z}(x^{n'}, 0) = \varphi(x^{n'})$, $\bar{w}(y^{n'}, 0) = \psi(y^{n'})$

Now, there exists a function $h(\rho) \to 0$ with ρ such that

(4) $\int_{c^{n'}}^{d^{n'}} |w_p(y^{n'}, y_o^n) - w_p(y^{n'}, 0)|^\alpha dy^{n'} \leq (y_o^n)^{\alpha-1} \int_{c^{n'}}^{d^{n'}} \int_0^{y_o^n} \left| \frac{\partial w_p}{\partial y^n} \right| dy^n dy^{n'} < h(y_o^n)$

(5) $\int_{a^{n'}}^{e^{n'}} |z_p(x^{n'}, x_o^n) - z_p(x^{n'}, 0)|^\alpha dx^{n'} \leq (x_o^n)^{\alpha-1} \int_{a^{n'}}^{e^{n'}} \int_0^{x_o^n} \left| \frac{\partial z_p}{\partial x^n} \right|^\alpha dx^n dx^{n'} < h(x_o^n)$

independently of p. The lemma follows by combining (1) to (5).

Lemma 3: Let z(x) be of class D_1' on an open set G. Let R be a subregion of G of the form $a^{e'} < x^{e'} < b^{e'}$ $f_1(x^{e'})$ $< x^e < f_2(x^{e'})$, f_1 and f_2 being continuous. Then

$$\bar{z}(x^{e'}, f_i(x^{e'})), \quad i = 1, 2$$

are summable with respect to $x^{e'}$ and

$$\int_{a'}^{e'} \{ \bar{z}[x^{e'}, f_2(x^{e'})] - \bar{z}[x^{e'}, f_1(x^{e'})] \} dx^{e'} = \int_{a^{e'}}^{e^{e'}} \int_{f_1(x^{e'})}^{f_2(x^{e'})} \frac{\partial \bar{z}}{\partial x^e} dx^{e'} dx^e$$

Proof: Evidently we may split such a region up into a finite number of similar regions in each of which either f_1 or f_2 is a constant. It is therefore sufficient to prove the theorem for such regions.

Let us suppose, for instance, that $f_1 = c$. Then we know that $\bar{z}(x^{e'}, c)$ is summable. Now we can approximate $f_2(x^{e'})$ uniformly by means of step functions $\varphi_p(x^{e'})$ constant over each of a finite number of cells of $(a^{e'}, b^{e'})$. Evidently $\bar{z}[x^{e'}, \varphi_p(x^{e'})] \to \bar{z}[x^{e'}, f_2(x^{e'})]$ almost everywhere, and $\bar{z}[x^{e'}, \varphi_p(x^{e'})]$ is measurable for each p. Thus $\bar{z}[x^{e'}, f_2(x^{e'})]$ is measurable. Now

$$\int_{a^{e'}}^{b^{e'}} |\bar{z}[x^{e'}, f_2(x^{e'})]|\, dx^{e'} \leq \int_{a^{e'}}^{b^{e'}} |\bar{z}(x^{e'}, c)|\, dx^{e'} + \int_{a^{e'}}^{b^{e'}}\int_{c}^{f_2(x^{e'})} \left|\frac{\partial \bar{z}}{\partial x^{e}}\right| dx^{e'}\, dx^{e}$$

so that $\bar{z}[x^{e'}, f_2(x^{e'})]$ is summable. From this, the theorem follows easily.

Definition 2: A transformation $x = x(y)$ of a set S consisting of an open region S_1 plus some of its boundary points into a set Σ of the same type is said to be a regular transformation of class C' if it is 1-1 and continuous, if the functions $x(y)$ are of class C' on the whole set S, and if the inverse has the same properties. It is said to be a regular transformation of class D' if all the above is true except that the functions involved are of class D'. It is said to be a regular transformation of class L if the above hold except that the functions involved satisfy uniform Lipschitz conditions.

Definition 3: A region G is said to be of class C' if there exists a $\rho > 0$ such that every point x of $\bar{G}$ at a distance $\leq \rho$ from G^* can be covered by a finite number of regions R_j into which the cells $T_j(a_j^i < y_j^i < b_j^i$, $i = 1, \ldots, n-1$, $0 \leq y_j^n < b_j^n$, $j = 1, \ldots, N)$ are carried by the regular transformations $x = x_j(y_j)$. G is said to be of class D' if the above holds except that the transformations are regular transformations of class D'. G is said to be of class L if the above holds except that the transformations are regular of class L. Clearly regions of class D' are of class L, and regions of class C' are of class D'.

Theorem 1: Let G be a region of class L and let $z(x)$ be of class D'_α in G with $D_\alpha(x,G)$ finite. Then $\bar{D}_\alpha(z,G)$ is finite and there exists a function $\varphi(P)$ of class L_α on G^* such that, if $x = x(y)$ is a regular transformation of class D' of a closed cell (a,b) $(a^n = 0)$ onto a sufficiently small portion of G, the points $y_n = 0$ corresponding to points of G^*, then

$$\lim_{y_0^n \to 0} \int_{a^n}^{b^n} |\bar{w}(y^{n'}, y_0^n) - \psi(y^{n'})|^\alpha dy^{n'} = 0$$

where w is the transform of z and $\psi(y^{n'}) = \varphi(P)$ where P is the point of G^* corresponding to $(y^{n'}, 0)$. Moreover, if $x' = x'(x)$ is a regular transformation of class D' of G onto the region H (of the same type), then the transformed function takes on the transformed boundary values in the same sense. Furthermore, if G is of class D', we have for almost every point x_o of G^* at which the tangent hyperplane to G^* exists and is not parallel to the x^ℓ axis, the function $\bar{z}(x_o^{\ell'}, x^\ell)$ is A.C. in x^ℓ and tends to $\varphi(x_o)$ as $x^\ell \to x_o^\ell$, $(x_o^{\ell'}, x^\ell)$ being in G ($\ell = 1, \ldots, n$). Finally, if $\alpha > n$, $\bar{z}$ is continuous on G of class L.

Proof: We may evidently cover G^* and the points of G at a distance $\leq \rho (> 0)$ from G^* by a finite number of regular transformations T_j of class L, where

$$T_j: \; x^i = x_j^i(y_j^1, \ldots, y_j^n), \quad a_j^i < y_j^i < b_j^i, \; i = 1, \ldots, n-1, \quad 0 \leq y_j^n < b_j^n, \quad j = 1, \ldots, N$$

We let $w_j(y_j) = z(x_j)$, and we see that w_j is of class D'_α on this cell R_j and so that $\bar{D}_\alpha(w_j, R_j)$ is finite. Hence it follows immediately that $\bar{D}_\alpha(z, G)$ is finite. Next we define $\varphi_j(P)$ on the points of G^* corresponding to $y_j^n = 0$ as the transform of the function $\psi_j(y_j^{n'})$ to which $\bar{w}_j(y_j^{n'}, y_j^n)$ converge in the mean of order α as $y_j^n \to 0$. Now let $x = x(y)$ be a regular transformation of class L of a closed cell (a,b) ($a^n = 0$) into a portion of $\bar{G}$ of diameter r the points of $y^n = 0$ corresponding to points of G^*. If is small enough, this portion of G will be entirely covered by one of the representations T_{j_o}, and if $w(y) = z[x(y)]$, then (by Lemma 2) $\bar{w}(y^{n'}, y_o^n)$ will tend in the mean of order x to the transform of $\psi_{j_o}(y_{j_o}^{n'})$, i.e. of $\varphi_{j_o}(P)$.

Of our portion of $\bar{G}$ is covered by two different T_j, it follows from this that the two different $\varphi_j(P)$ agree almost everywhere on the part of G^* in our portion. Thus the $_j(P)$ define a single function $\varphi(P)$ which is of class L_α over G^* and z takes on these boundary values in the sense described in the theorem. This proves the first statement, and the second statement is now obvious.

To prove the third statement, let x_o be a point of G^* where the tangent hyperplane to G^* exists and is not parallel to the x^n axis for instance. Then we can find a small cell $(a^{n'}, b^{n'})$ with center at $x_o^{n'}$ such that a neighborhood of x_o on G^* has the representation *) where f is of class C'. Now we make the transformation

$$y^{n'} = x^{n'}, \quad y^n = k[x^n - f(x^{n'})], \quad a^{n'} < y^{n'} < b^{n'}, \quad 0 \le y^n < b^n$$

where k is +1 or -1 according as $(x^{n'}, x^n)$ belongs to G for $x^n > f(x^{n'})$ or $x^n < f(x^{n'})$, the difference being sufficiently small. Let w(y) be the transformed function of z, and we see that $\bar{w}(y_o^{n'}, y^n)$ is absolutely continuous in y^n for each $y_o^{n'}$ for which $z(y^{n'}, x^n)$ is absolutely continuous in x^n. Thus

$$\bar{w}(y^{n'}, y^n) = \bar{z}(x(y^{n'}), z^n)$$

if $y^{n'}$ is not in a set $Z(y^n)$ of measure 0. From this the third statement follows at once. The fourth statement follows immediately from Lemma 1 as all the $\varphi_j(p)$ must be continuous with $\bar{w}_j$ continuous on $\bar{R}_j$, and $\bar{z}$ is continuous (by Theorem 2, §3) on each closed region interior to G.

*) insert: $x^n = f(x^{n'}), \quad a^{n'} < x^{n'} < b^{n'}$

Theorem 2: *Let $z_p(x)$ be of class D'_α on a region G with $D_\alpha(z_p, G)$ uniformly bounded. Suppose that there exists a cell R interior to G on which $\bar{D}_\alpha(z_p, R)$ is uniformly bounded. Then $\bar{D}_\alpha(z_p, H)$ is uniformly bounded for each bounded region H with $\bar{H} \subset G$. If G is of class L, $\bar{D}(z_p, G)$ is uniformly bounded.*

Proof: It is sufficient to prove that each point Q is interior to a cell R_Q in which $\bar{D}_\alpha(z_p, R_Q)$ is uniformly bounded. Now let Q be in G-R and let P be interior to R. Since G is open and connected we can find a finite sequence of hypercubes $R_1, \ldots, R_N$, of the same dimensions and all parallel and interior to G so that R_1 contains P in its interior and is contained in R, R_N contains Q in its interior, and R_{i-1} and R_i have a face in common, $i = 2, \ldots, N$.

Now, we have seen that, if $\bar{D}_\alpha(z_p, R)$ is uniformly bounded for some cell R interior to G, we have that z_p is of class L_α on R^* with

$\int_{R^*} |\bar{z}_p|^\alpha dS$

uniformly bounded. It is also immediate that if $D_\alpha(z_p, R)$ is uniformly bounded and the integral of $|\bar{z}_p|^\alpha$ is uniformly bounded on some face of R^*, then $\bar{D}_\alpha(z_p, R)$ is uniformly bounded. Using these two principles, we see that $\bar{D}_\alpha(z_p, R_i)$ is uniformly bounded, $i = 2, \ldots, N$.

Now suppose G is of class L, and consider a representation T_j (of definition 3). If we choose a small closed cell r interior to the cell T_j: $a_j^i < y_j^i < b_j^i$, $i = 1, \ldots, n-1$, $0 \leq y_j^n < b_j^n$, we see that $\bar{D}_\alpha(w_{jp}, r)$ is uniformly bounded. From Lemma 1 and the above, it follows that $\bar{D}_\alpha(w_{jp}, r_j)$ is uniformly bounded so that $\bar{D}_\alpha(z_p, R_j)$ is uniformly bounded. Since all the points of G at a distance $\leq \rho$ from G^* can be covered by a finite number of the R_j, the final statement follows from the above.

Theorem 3: Let $\{z_p(x)\}$ be of class D'_α on a region G, of class L, with $D_\alpha(z_p, G)$ uniformly bounded. Suppose that there exists a set E, open on G^* such that

$$\int_E |\varphi_p|^\alpha d\Sigma$$

is uniformly bounded, φ_p being the boundary values of z_p and $d\Sigma$ *) being

*) See ~~the author's paper~~ C.B. Morrey Jr, "A Class of Representations of Manifolds, Part I", American Journal of Mathematics, vol. 55 (1933), pp. 686-707

the element of hyper-area on G^*. Then $\bar{D}_\alpha(z_p, G)$ is uniformly bounded. If $\alpha > n$, the z_p are uniformly bounded and equicontinuous on G.

Proof: Evidently there exists a regular transformation $x = x(y)$ of class L of a closed cell T: $(a, b)(a^n = 0)$ into a portion of G such that all the points $(y^{n'}, 0)$ correspond to points of E. Then if $\psi(y^{n'})$ is the transform of $\varphi(P)$ and $w_p(y)$ is that of z_p, we see that

$$\int_{a^n}^{b^n} |\psi(y^{n'})|^\alpha dy^{n'}$$

is uniformly bounded so that $\bar{D}_\alpha(w_p, T)$ is uniformly bounded. We may let R be any cell interior to the transform of T and $\bar{D}_\alpha(z_p, R)$ and hence $\bar{D}_\alpha(z_p, G)$ is uniformly bounded.

If $\alpha > n$, it is clear that the z_p are equivontinuous and uniformly bounded on any region H with $\bar{H} \subset G$. If we have a representation $x = x(y)$ as above, it is clear that the z_p are equicontinuous and uniformly bounded on the transform of T. Hence the second statement follows.

Theorem 4: Let $z_p(x)$ be of class D'_α on a region G, of class L, with $\bar{D}_\alpha(z, G)$ uniformly bounded. Then

(6) $$\int_{G^*} |\varphi_p|^\alpha d\Sigma$$

is uniformly bounded. If only $D_\alpha(z_p, G)$ is assumed uniformly bounded and

$z_p \to z$ (of class D'_α) on G, then $\bar{D}_\alpha(z, G)$ is uniformly bounded and $\psi_p \to \psi$ in the mean of order α. If $\alpha > n$, $D_\alpha(z_p, G)$ is uniformly bounded, and $|z_p(x_p)|$ is uniformly bounded (x_p in G), then z_p is equicontinuous and uniformly bounded on G; if $z_p \to z$ on G, the convergence of z_p to z is uniform on $\bar{G}$.

Proof: Let $x = x(y)$ be a regular transformation of class L as above of $\bar{T}$ into a portion $\bar{R}$ of $\bar{G}$. Then $\bar{D}_\alpha(W_p, T)$ is uniformly bounded so that the integral of $|\psi_p(y^{n'})|^\alpha$ over $(a^{n'}, b^{n'})$ is uniformly bounded. Thus (6) is uniformly bounded.

If $z_p \to z$ on G, then it is clear that $\bar{D}_\alpha(z_p, R)$ is uniformly bounded for some R in G and hence $\bar{D}_\alpha(z_p, G)$ is uniformly bounded. Now, let $\{z_q\}$ be any subsequence of $\{z_p\}$, and let T_j ($j = 1, \ldots, N$) be a finite number of representations covering all the points of G at a distance $\leq \rho$ (> 0) from G^*, let $w_{pj}(y_j)$ be the transforms of z_p and $\psi_{pj}(y_j^{n'})$ those of ψ_p. Then by Theorem 8, §1, we can choose a subsequence z_n of z_q so that

$$\int_{a^{n'}}^{b_j^{n'}} |\varpi_{rj}(y_j^{n'}, y_j^n) - \varpi_j(y_j^{n'}, y_j^n)|^\alpha \, dy_j^{n'} < \varepsilon_r, \quad \varepsilon_r \to 0, \quad j = 1, \cdots N$$

ε_r being independent of y_j^n. From this, it is clear that

$$\lim_{r\to\infty} \int_{a^{n'}}^{b^{n'}} |\psi_{rj}(y_j^{n'}) - \psi_j(y_j^{n'})| \, dy_j^{n'} = 0, \qquad j = 1, \ldots, N.$$

Thus $\psi_p \to \psi$ in the mean of order α.

If $\alpha > n$ the z_p are equicontinuous on G and if $z_p(x_p)$ is uniformly bounded, it is clear that z_p is uniformly bounded. The remainder of the theorem is immediate.

Theorem 5 (a general existence theorem): Let $f(x,z,p)$ $[x = (x^1, \ldots, x^n),\ z = (z^1, \ldots, z^m),\ p = (p_1^1, \ldots, p_n^1; \ldots; p_1^m, \ldots, p_n^m)]$ be defined and continuous in (x,z,p), be convex in p for each (x,z), and satisfy

$$f(x, z, p) \geq \rho \sum_{i=1}^{n} \sum_{j=1}^{m} |p_\alpha^j|^\alpha, \quad \rho > 0, \quad \alpha > 1$$

for all (x,z,p). Let G be a region of class L, let E be a set which is open on G^*, and let $\{\psi\}$ be a closed (with respect to the metric of L_α) family of functions of L_α such that

$$\int_E |\psi|^\alpha d\Sigma$$

is uniformly bounded. Suppose that there exists a function z of class D_1' on G which takes on boundary values in $\{\psi\}$ for which

$$I(z, \mathcal{G}) = \int_{\mathcal{G}} f(x, z, p)\, dx$$

is finite.

Then there exists a function z_0 which minimizes $I(z, G)$ among all such functions. If $\alpha > n$, z_0 is continuous on G.

Proof: It is clear that if a function z is of class D_1' with $I(z, G)$ finite, then it is of class D'_α on G with $D_\alpha(z,G)$ finite. Since such a function z_1 exists which takes on boundary values in $\{\psi\}$ we may therefore consider only those functions z with $I(z,G) \leqq I(z,G)$, so that $D_\alpha(z,G)$ is uniformly bounded, where z takes on boundary values in $\{\psi\}$. Let $\{z_p\}$ be a sequence from this class such that $I(z_p,G)$ tends to its greatest lower bound k. Then, by Theorem 3, $\bar{D}_\alpha(z_p,G)$ is uniformly bounded. We may then choose a subsequence $\{z_q\}$ tending to a function z_0 which is also of class D_α' on G. By Theorem 4, the boundary values of z_q tend in the mean of order α to those of z_0. By Theorem 1 or Theorem 3 of §3, it follows that

$$k \leq I(z_0, \mathcal{G}) \leq \lim_{q \to \infty} I(z_q, \mathcal{G}) = k$$

which proves the first statement. If $\alpha > n$, it follows by Theorem 1 that z_0 is continuous on G.

Lemma 4: A necessary and sufficient condition that $z(x)$ be of class D'_α on an open set G is that for each cell $\bar{D}$: (A,B) interior to G, z be of class L_α on D, and there exist functions $\phi_i(x)$ of class L_α on D

such that

$$\int_{a^{\ell'}}^{b^{\ell'}} [z(x^{\ell'}, b^{\ell}) - z(x^{\ell'}, a^{\ell}) \, dx^{\ell'} = \int_a^b v_\ell(x) \, dx$$

provided that a^{ℓ} and b^{ℓ} do not belong to a certain set z_ℓ of measure zero.

A necessary and sufficient condition that $z(x)$ be of class D'_α on G is that each point P_o of G be interior to some cell D of the type described above.

Proof: This lemma is obvious.

Theorem 6: Let z be of class D'_α on an open set G, and let D and R be subregions of G which are of class C', and suppose that $\bar{R}$ is interior to D. Then the boundary values taken on by z considered as a function defined only in R coincide on R^* with those taken on by z considered only as a function defined in $D-\bar{R}$.

Proof: For, let P_o be any point on R^*. A cell (a,b) with center at P_o may be found such that for some ℓ, the portion of R^* which is in (a,b) may be represented in the form

$$x^{\ell} = f(x^{\ell'}), \quad a^{\ell'} < x^{\ell'} < b^{\ell'}, \quad a^{\ell} < f(x^{\ell'}) < b^{\ell}$$

where f is of class C'. By Theorem 2, we see that the boundary values of z from both sides of R^* coincide, for almost every $x_o^{\ell'}$, with the value which the function $\bar{z}(x_o^{\ell'}, x^{\ell})$, which is A.C. in x^{ℓ}, takes on for $x^{\ell} = f(x^{\ell'})$, $\bar{z}$ considered as being defined in G. This proves the theorem.

Theorem 7: Let z be of class D'_α on a region G, and suppose that D is a region of class C' such that $\bar{D}_\alpha \subset G$. Let u be of class D'_α on D and take on the same boundary values as z on D^*, and suppose that $D_\alpha(u,D)$ (and hence $\bar{D}_\alpha(u,D)$ is finite. Then the function w, which coincides with u in D, is of class D'_α on G.

F with z outside and on D^* and coincides

Proof: Let P_o be any point of D^*. For some e there exists a cell $(a^{e'}, b^{e'})$ and a number $\rho > 0$ such that a portion of D^* containing P_o has the representation $x^e = f(x^{e'})$, $a^{e'} \leqq x^{e'} \leqq b^{e'}$, where f is of class C', and all the points $(x^{e'}, x^e)$ with $a^{e'} \leqq x^{e'} \leqq b^{e'}$, $f(x^{e'}) - \rho \leqq x^e < f(x^{e'})$ are in D say, and all the points $a^{e'} \leqq x^{e'} \leqq b^{e'}$, $f(x^{e'}) < x^e \leqq f(x^{e'}) + \rho$ are in $G - \bar{D}$. Then, if we make the transformation

$$y^i = x^i, \; i \neq e, \; y^e = x^e - f(x^{e'})$$

which is of class C', we see that $\tilde{w}(y)$ (the transform of $w(x)$) is of class D' for $a^{e'} \leqq y^{e'} \leqq b^{e'}$, $-\rho \leqq y^{e'} < 0$ with $\bar{D}_\alpha(w, R_1)$ finite, and is also of class D'_α for $a^{e'} \leqq y^{e'} \leqq b^{e'}$, $0 < y^e \leqq \rho$ with $\bar{D}_\alpha(w, R_2)$ finite (R_1 and R_2 being the respective cells involved), and the two parts of $\tilde{w}$ have common boundary values of class L_x for $y = 0$. It follows easily that $\tilde{w}$ is of class D'_α for $a^{e'} \leqq y^{e'} \leqq b^{e'}$, $-\rho \leqq y^{e'} \leqq \rho$.

Hence each point P_o of D^* is interior to a cell in which $w(x)$ is of class D'_α. Obviously each P_o of $G - D^*$ is interior to a cell of G which contains no point of D^* and in which $w(x)$ is of class D'_α. Hence $w(x)$ is of class D'_α in G.

§5. The borderline case $n = 2$, $\alpha = 2$

In this section we consider continuity theorems for the case $\alpha = n$ when $n = 2$. For $n = 2$ it is clear that any region bounded by a finite number of simple closed curves is of class C_2 and conversely. We now prove

Lemma 1: Let $z(x,y)$ be of class D_2^1 on a circle $C(P,R)$ with $D_2[z, C(P,R)]$ finite. Then if we represent $z(x,y)$ on this circle by

$$(1) \quad z(r,\vartheta) = \frac{a_0(r)}{2} + \sum_{n=1}^{\infty} [a_n(r)\cos n\vartheta + b_n(r)\sin n\vartheta], \quad 0 \leq r \leq R$$

we find that the $a_i(r)$ and $b_i(r)$ are absolutely continuous for $0 \leq r \leq R$, and

$$(2) \quad D_2[z, C(P,R)] = \pi \int_0^R r\left[\frac{a_0'^2}{2} + \sum_{n=1}^{\infty}\left\{a_n'^2 + b_n'^2 + \frac{n^2(a_n^2+b_n^2)}{r^2}\right\}\right] dr$$

Proof: Let $\bar{z}(r,\vartheta)$ be the function coinciding with z almost everywhere and absolutely continuous in ϑ for almost all r and in r for almost every ϑ with

$$\bar{z}(r,\vartheta) = \bar{z}(r_0,\vartheta) + \int_{r_0}^{r} \bar{z}_r(\rho,\vartheta)\, d\rho$$

ly from this.

Lemma 2: Let $H(x,y)$ be harmonic in the circle $C(P,R)$ and let its boundary values $H(R,\vartheta)$ (polar coordinates with pole at P) be of class L_2 and be given by

$$H(R,\vartheta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\vartheta + b_n\sin n\vartheta)$$

Then

$$(3) \quad D_2[H, C(P,R)] = \pi\sum_{n=1}^{\infty} n(a_n^2+b_n^2)$$

both sides being simultaneously finite.

for every r, r_0 being a value for which $\bar{z}(r_0, \vartheta)$ is A.C. Then, for almost every r, $0 < r \leqq R$, $\bar{z}_r^2$ and $\bar{z}_\vartheta^2$ are ~~enumerable~~ summable with respect to ϑ. Using a well known*) theorem on trigonometric series, it follows that the

*) Hobson, vol. II, §360, 361, p. 553.

series for $\bar{z}_\vartheta$ is obtained by termwise differentiation of (1) with respect to ϑ for those values of r. Let $r_0 > 0$ be the above value; then

$$a_n(r) = a_n(r_0) + \frac{1}{2\pi}\int_{r_0}^{r}\int_0^{2\pi} z_r(\rho, \vartheta)\cos n\vartheta \, d\rho \, d\vartheta, \quad b_n(r) = b_n(r_0) + \cdots$$

and z_r^2 and z_ϑ^2 are summable on such a rectangle if $r > 0$. Hence we see that a_n and b_n are absolutely continuous on any interval (ε_1, R) with $\varepsilon > 0$. We see also that $a_n'(r)$ and $b_n'(r)$ are the Fourier coefficients for $\bar{z}_r$ for almost every r. By the Riess-Fischer theorem*) and its converse, it follows that

*) Hobson, vol. II, pp. 575-577.

$$(3) \qquad \int_0^{2\pi}\left(u_r^2 + \frac{1}{r^2}u_\vartheta^2\right)d\vartheta = \pi\Big[\frac{a_0'^2}{2} + \sum_{n=1}^{\infty}(a_n'^2 + b_n'^2) + r^{-2}\sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2)$$

for almost every r, the right side being finite. The lemma follows immediately from this.

Lemma 2: Let H(x,y) be harmonic in the circle C(P,R) and let its boundary values $H(R, \vartheta)$ (polar coordinates with pole at P) be of class L_2 and be given by

$$H(R, \vartheta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\vartheta + b_n \sin n\vartheta)$$

Then

$$(4) \qquad D_2[H, C(P,R)] = \pi\sum_{n=1}^{\infty} n(a_n^2 + b_n^2)$$

both sides being simultaneously finite.

Proof: This is well known and follows from Lemma 1 and the fact that

$$H(r,\vartheta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{r^n}{R^n}(a_n \cos n\vartheta + b_n \sin n\vartheta)$$

Theorem 1: **Let $z(x,y)$ [$= z^i(x,y)$, $i = 1, \ldots, N$] be of class $\mathfrak{D}_2^*$ on the circle $C(P,R)$ with**

$$(i)\quad D_2[z, C(P,R)] \le \sum_{i=1}^{N} D_2[z^i, C(P,R)] = M < \infty .$$

Suppose also that there is a number K ($\geqq 1$) such that we have

(5) $$(ii)\quad D_2[z, C(P,R)] \le K\, D_2[H\{z;\, C(P,r)\}, C(P,r)],\quad 0 < r \le R,$$

where $H\{z, C(P,r)\}$ denotes the harmonic function coinciding with z on $C^*(P,r)$. Then

(6) $$D_2[z, C(P,r)] \le K \cdot M \cdot 2^{1+\frac{1}{K}} \left(\frac{r}{R}\right)^{\frac{1}{K}},\quad 0 \le r \le R$$

If, instead of (ii), we have

(7) $$(ii')\quad D_2[z, C(P,r)] \le K \cdot D_2[H\{z, C(P,r)\}, C(P,r)] + \varphi(r),\quad \int_0^R r^{-1}\varphi(r)\,dr < \infty,$$

then we have

(8) $$\tfrac{1}{2} D_2[z, C(P, \tfrac{r}{2})] \le K \cdot M \cdot \left(\frac{r}{R}\right)^{\frac{1}{K}} + r^{\frac{1}{K}} \int_r^R \rho^{-1-\frac{1}{K}} \varphi(\rho)\,d\rho + K \int_0^r \rho^{-1}\varphi(\rho)\,d\rho$$

and the right side tends to zero with r.

Proof: We shall prove (6) first. Define

$$\Psi(r) = \int_0^r \rho^{-1} \sum_{n=1}^{\infty} \sum_{i=1}^{N} n\left(a_n^{i\,2} + b_n^{i\,2}\right) d\rho,\quad z^i = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n^i \cos n\vartheta + b_n^i \sin n\vartheta .$$

Then, for almost every r, we see, using (2), (4), (5), that

$$\Psi(r) = P \cdot r \cdot \Psi'(r),\quad \Psi(R) \le D_2[z, C(P,R)] = M$$

In other words, we see that

$$\frac{d}{dr}\left(r^{-\frac{1}{P}}\Psi(r)\right) \ge 0\ \cancel{M \cdot R^{-\frac{1}{P}},\ \text{for } 0 < r \le R},\quad R^{-\frac{1}{P}}\Psi(R) \le M R^{-\frac{1}{P}}$$

so that it follows that

$$r^{-\frac{1}{P}}\Psi(r) \le M \cdot R^{-\frac{1}{P}} \quad \text{for } 0 < r \le R$$

so that

$$\Psi(r) \le M \cdot \left(\frac{r}{R}\right)^{\frac{1}{P}}$$

But now, from (4) and (5) we see that $D_2[z, C(P,r)] \leqq Pr\, \Psi'(r)$ so that we have

(9) $$\int_0^r \rho^{-1} D_2[z, C(P,\rho)]\,d\rho \le P \cdot M \cdot \left(\frac{r}{R}\right)^{\frac{1}{P}}$$

Since $D_2[z, C(P, r)]$ is monotone now decreasing in r, we see that

$$\frac{1}{2} D_2\left[z, C\left(P, \frac{r}{2}\right)\right] \leq \frac{1}{r}\int_{\frac{r}{2}}^{r} D_2[z, C(P,\rho)]\,d\rho \leq \int_{\frac{r}{2}}^{r} \rho^{-1} D_2[z, C(P,\rho)]\,d\rho \leq \int_0^r \rho^{-1} D_2\,d\rho \leq P\cdot M\cdot\left(\frac{r}{R}\right)^{\frac{1}{P}}$$

From this, (6) follows immediately.

To prove (8), we see from (2), (4), and (7), that

$$\psi(r) \leq P\cdot r\cdot \psi'(r) + \varphi(r), \quad \psi(R) \leq M.$$

In other words, we see that

$$\frac{d}{dr}\left(r^{-\frac{1}{P}}\psi\right) \geq -\frac{1}{P} r^{-\frac{1}{P}-1}\varphi(r).$$

Hence if we define $\chi(r)$ by the equations

$$\frac{d}{dr}\left(r^{-\frac{1}{P}}\chi\right) = -\frac{1}{P} r^{-\frac{1}{P}-1}\varphi(r), \quad \chi(R) = M$$

we see that $\psi(r) \leqq \chi(r)$ for $0 < r \leqq R$ and $\chi(r)$ is given by

$$r^{-\frac{1}{P}}\chi(r) - R^{-\frac{1}{P}}\chi(R) = -\frac{1}{P}\int_R^r \rho^{-1-\frac{1}{P}}\varphi(\rho)\,d\rho$$

or, by simplifying, we see that

$$\psi(r) \leq M\left(\frac{r}{R}\right)^{\frac{1}{P}} + \frac{r^{\frac{1}{P}}}{P}\int_r^R \rho^{-1-\frac{1}{P}}\varphi(\rho)\,d\rho$$

But by (7), $D_2[z, C(P, r)] \leqq Pr\,\psi'(r) + \psi(r)$ so that

$$(10) \quad \int_0^r \rho^{-1} D_2[z, C(P,\rho)]\,d\rho \leq P\cdot M\cdot\left(\frac{r}{R}\right)^{\frac{1}{P}} + r^{\frac{1}{P}}\int_r^R \rho^{-1-\frac{1}{P}}\varphi(\rho)\,d\rho + P\int_0^r \rho^{-1}\varphi(\rho)\,d\rho.$$

To see that the right side tends to zero with r, we need to confine ourselves to a discussion only of the middle term and to small values of r. Then we see that

$$r^{\frac{1}{P}}\int_r^R \rho^{-1-\frac{1}{P}}\varphi(\rho)\,d\rho = r^{\frac{1}{P}}\int_r^{r^{\frac{1}{2}}} \rho^{-1-\frac{1}{P}}\varphi(\rho)\,d\rho + r^{\frac{1}{P}}\int_{r^{\frac{1}{2}}}^R$$

each term of which clearly tends to zero. Clearly (8) follows from (10) as (6) followed from (9) above.

Definition 1: We say that $z(x, y)$ satisfies a condition $A[\lambda; M(a, d)]$ on the region G if it is of class D_2' on G and if

$$D[z, C(P, r)] \leq M(a, d)\left(\frac{r}{a}\right)^{\lambda}, \quad 0 \leq r \leq a, \quad P = (x, y) \in G, \quad \lambda > 0.$$

where $a > 0$, $d > 0$, and $a+d$ is the distance of P from G^*; $M(a, d)$ is supposed to depend only on a and d and not on (x, y).

Definition 2: We say that $z(x, y)$ satisfies a condition $B[\mu\ N(a, d)]$ on G if

$$|z(x_1, y_1) - z(x_2, y_2)| \leq N(a, d)\left(\frac{r}{a}\right)^{\mu}, \quad 0 \leq r \leq a, \quad r = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}}$$

provided that every point on the segment joining (x_1, y_1) to $x_2, y_2)$ is at a distance $\geq a/2 + d$ from G^*.

__Theorem 2: If $z(x, y)$ satisfies a condition $A[\lambda, M(a, d)]$ on G, it satisfies a condition $B[\lambda/2;\ N(a, d)]$ where__

$$N(a, d) = \frac{8}{\lambda\sqrt{3}}\ M^{\frac{1}{2}}\left(\frac{a}{2}, d\right)$$

The proof of this theorem can be found in: C. B. Morrey, Jr., On the solutions of quasi-linear elliptic partial differential equations, Transactions Am. Math. Soc., vol. 43 (1938), p. 126-166, in particular p. 134.

__Theorem 3: Let G be a region which can be mapped conformally on a region bounded by a finite number of circles. Let $z(x, y)$ be a vector function of class D_2^1 on G with $D_2(z, G) = \sum_{i=1}^{N} D_2(z_i, G)$ finite, and suppose that its boundary values are continuous. Suppose also that there exists a number $P(\geq 1)$ such that__

$$D(z, R) \leq P \cdot D[H(z, R), R] \tag{9}$$

__for every region R of class C', $H(z, R)$ being the harmonic function on R with the same boundary values as z on R^*. Then z is continuous on G.__

Proof: From our definition of boundary values, it follows that it is sufficient to prove our theorem for the case that G is a region bounded by a finite number of non-intersecting circles as G may be mapped onto such a region preserving condition (9) and then z will take on the corresponding continuous boundary values.

From Theorems 1 and 2 it follows that z satisfies a condition A[1/P, M(a,d)] and a condition B[(½P), N(a,d)] on G and is therefore continuous at each interior point of G.

Now let P_0 be a point of G^* and take polar coordinates with pole at P_0 and initial line pointing in the positive direction of the tangent to the boundary circle on which P_0 lies. If r_0 is sufficiently small, each circle $C(P_0, r)$ intersects G^* in exactly two points $[r, \vartheta(r)]$ and $[r, \pi - \vartheta(r)]$, both on the boundary circle on which P_0 lies. We have

$$\lim_{r \to 0^+} \vartheta(r) = 0$$

where $\vartheta(r) > 0$ for $r > 0$ if P_0 is on the outer boundary of G^* and $\vartheta(r) < 0$ if P_0 is on an inner boundary.

The region $0 < r < r_0$, $\vartheta(r) < \vartheta < \pi - (r)$ lies in G and $z(r, \vartheta)$ is of class D_2 in (r, ϑ) there with

$$\int_0^{2R} \int_{\vartheta(r)}^{\pi - \vartheta(r)} r\left(\bar{z}_r^2 + \frac{1}{r^2}\bar{z}_\vartheta^2\right) dr\, d\vartheta = \eta(R), \quad \lim_{R \to 0} \eta(R) = 0, \quad 0 < R < \frac{r_0}{2}$$

Thus, for almost every r, $\bar{z}(r, \vartheta)$ is A.C. in ϑ with $\bar{z}^2$ summable and converges to the continuous boundary values as ϑ tends to $\vartheta(r)$ or $\pi - \vartheta(r)$ (using Theorem 2). Thus, for each R, $0 < R < r_0/2$ there is an $\bar{r}$, $R < \bar{r} < 2R$ such that

$$\bar{r}^{-1} \int_{\vartheta(\bar{r})}^{\pi - \vartheta(\bar{r})} \bar{z}_\vartheta(\bar{r}, \vartheta)\, d\vartheta \le \frac{\eta(R)}{R}, \quad \int_{\vartheta(\bar{r})}^{\pi - \vartheta(\bar{r})} \bar{z}_\vartheta^2(\bar{r}, \vartheta)\, d\vartheta < \eta(R)$$

For this $\bar{r}$, we see that

$$\int_{\vartheta(\bar{r})}^{\pi - \vartheta(\bar{r})} |\bar{z}_\vartheta(\bar{r}, \vartheta)|\, d\vartheta < [2\pi\eta(R)]^{\frac{1}{2}} \tag{10}$$

Now, choose $\varepsilon > 0$. From (10) and the fact that the boundary values are continuous, it follows that we can choose R so small that the oscillation of $\bar{z}(\bar{r}, \vartheta)$ on the boundary of the region D: $0 < r < \bar{r}$, $\vartheta(r) < \vartheta < \pi - \vartheta(r)$, is less than $\varepsilon/2$. Now map this region conformally onto the unit circle and let w(x,y) be the transform of z. Condition (9) is preserved and w is of class D_2 in Σ and takes on the corresponding continuous boundary values. Let (r, φ) be polar coordinates in Σ. Define

$$\psi(r) = \int_0^r \int_0^{2\pi} |w_\rho(\rho,\varphi)|\, d\rho\, d\varphi, \quad \psi'(r) = r^{1/2} \int_0^{2\pi} |w_\rho(r,\vartheta)|\, d\vartheta, \quad \chi(r) = \int_0^r \rho^{-1} \psi(\rho)\, d\rho$$

in case the integrals exist. Now, by Theorem 1, we find that

$$\text{(11)} \qquad \int_0^r \rho^{-1} D[w, C(0,\rho)]\, d\rho \le P \cdot \eta(R)\, r^{\frac{1}{p}}$$

Using (11) and Schwartz's inequality, we find that

$$\psi^2(r) = \left[\int_0^r \psi'(\rho)\, d\rho\right]^2 \le r \int_0^r \psi'^2 d\rho \le 2\pi r \int_0^r \int_0^{2\pi} \rho w_\rho^2(\rho,\varphi)\, d\rho\, d\varphi \le 2\pi r\, D[w, C(0,r)]$$

so that $\psi(r)$ is certainly defined. Now, for each $\varepsilon > 0$

$$\left[\int_\varepsilon^r \rho^{-1}\psi(\rho)\, d\rho\right]^2 \le (r-\varepsilon) \int_\varepsilon^r \rho^{-2}\psi^2(\rho)\, d\rho \le 2\pi(r-\varepsilon) \int_\varepsilon^r \rho^{-1} D[w, C(0,\rho)]\, d\rho \le 2\pi P \eta(R) \cdot r^{1+\frac{1}{p}}$$

so that χ (r) is defined and

$$\chi(r) \le [2\pi P \eta(R)]^{\frac{1}{2}} r^{\frac{1}{2}+\frac{1}{2p}}, \quad \psi(1) \le [2\pi \eta(R)]^{\frac{1}{2}}.$$

Hence, for each $\varepsilon > 0$,

$$\int_0^{2\pi} |\bar{w}(1,\varphi) - \bar{w}(\varepsilon,\rho)|\, d\varphi \leq \int_\varepsilon^1 \int_0^{2\pi} |v_\rho(\rho,\varphi)|\, d\rho\, d\varphi = \int_\varepsilon^1 \rho^{-\frac{1}{2}} \psi'(\rho)\, d\rho =$$

$$= \psi(1) - \varepsilon^{-\frac{1}{2}} \psi(\varepsilon) + \frac{1}{2} \int_\varepsilon^1 \rho^{-\frac{3}{2}} \psi(\rho)\, d\rho \leq \psi(1) + \frac{1}{2} \int_\varepsilon^1 \rho^{-\frac{1}{2}} \chi'(\rho)\, d\rho =$$

$$= \psi(1) + \frac{1}{2}\chi(1) - \frac{1}{2} \varepsilon^{-\frac{1}{2}} \chi(\varepsilon) + \frac{1}{4} \int_\varepsilon^1 \rho^{-\frac{3}{2}} \chi(\rho)\, d\rho \leq$$

$$\leq \psi(1) + \frac{1}{2}\chi(1) + \frac{1}{4}\left[2\pi P \eta(R)\right]^{\frac{1}{2}} \int_\varepsilon^1 \rho^{\frac{1}{2P}-1}\, d\rho \leq \left[2\pi \eta(R)\right]^{\frac{1}{2}} \left[1 + \frac{1}{2} + \frac{P^{\frac{3}{2}}}{2}\right].$$

Since w is continuous on the interior, the inequality holds for $\varepsilon = 0$. From the fact that the oscillation of $\bar{w}(1, \varphi)$ is $< \frac{\varepsilon}{2}$ it follows that ~~max~~ $\max\limits_{0 \leq \varphi \leq 2\pi} |\bar{w}(1, \varphi) - \bar{w}(0, \varphi)| < \varepsilon$ if R is chosen small enough. Since any point (r, ϑ) of D may be carried into the origin in Σ, it follows that the oscillation in $\bar{D} < \varepsilon$ and hence that the oscillation in the region $C(P_0, R) \cdot G$ is $< \varepsilon$ if R is small enough.

The above proof has demonstrated the following theorem:

Theorem 4: Let $\{z\}$ be a family of vector functions, each defined and of class D'_α on a region G bounded by N circles, N being independent of z. Suppose that the regions G are all within some large circle, that two distinct circles of any G^* are at a distance $\geq \delta > 0$ apart (δ independent of z), and that the boundary values of the z are equicontinuous. Suppose also that (7) is satisfied for some P independent of z and that there exist numbers M and $r_0 > 0$, and a function $\eta(R) \to 0$ with R, both independent of z such that

(12) $\quad D_2(z, G) \leq M, \quad D_2[z, G \cdot C(P, R)] \leq \eta(R), \quad 0 < R < r_0$

Then the family z is equicontinuous.

Theorem 5: Let $f(x, y, z^1, \ldots, z^n, p^1, \ldots, p^n, q^1, \ldots, q^n)$ be

<u>of the type described in Theorem 5, §4, with $\alpha = 2$ and f satisfying the additional condition that there exist an M such that</u>

$$(13)\qquad f \leq M \cdot \sum_{i=1}^{n} (p_i^2 + q_i^2)$$

<u>Let G be a region bounded by a finite number of simple closed curves and suppose that z^* is a continuous vector function on G^* which is such that a z of class D'_α exists which takes on these boundary values and f or ~~with~~ which I(z,G) is finite. Then the solution z of I(z,G) = minimum, which takes on these boundary values, is continuous on G, and satisfies conditions A[λ, M(a,d)] and B[$N_2^{\lambda/2}$, N(a,d)] on G for λ = M/m, N(a,d) = M(a), and N(a,d) = N(a).</u>

If f satisfies (13), we see that z must satisfy (9) with P = M/m. For if there should exist an open set R of class C' on which (9) did not hold, we could define a function z' = z on and outside of R^* and equal in R to the harmonic function with the same boundary values as z on R^*. Then z' is of class D'_2 on G, has the same boundary values as z on G^*, and

$$\underline{I}(z', \zeta) = \underline{I}(z', \zeta - \bar{R}) + \underline{I}(z', R) = \underline{I}(z, \zeta - \bar{R}) + \underline{I}(z', R) \leq$$
$$\leq \underline{I}(z, \zeta - \bar{R}) + M D_2[H(z, R), R] < \underline{I}(z, \zeta - \bar{R}) + m D_2(z, R) \leq \underline{I}(z, \zeta - \bar{R}) + \underline{I}(z, R) \leq$$
$$= \underline{I}(z, \zeta)$$

This contradicts the fact that z minimized I(z, G) among all functions with the given boundary values. Thus z is continuous on G and satisfies conditions A[λ M(a,d)], B[/2, N(a,d)] on G, where λ = m/M, and

$$M(a, d) \leq \frac{M}{m} \cdot D_2[z, C(P, a)]$$

Thus, by Theorem 3, it follows that z is continuous on $\bar{G}$.

§6. Lemmas on potential theory

Theorem 1*): Let Φ (e) be a completely additive set function defined

*) See G.C.Evans, "Fundamental Points of Potential Theory", Rice Institute Pamphlets, vol. 7 (1920), pp. 252-329.

on a bounded region $\mathcal{G}$. Then the potential function

(1) $$V(x,y) = \frac{1}{4\pi}\iint_{\mathcal{G}} \log[(\xi-x)^2+(\eta-y)^2]\, d\Phi(e_{\xi,\eta})$$

is of class D'_α for each α $1 \leq \alpha < 2$, with $\bar{D}_\alpha$ (V, Γ) finite for each bounded region Γ. Moreover, for almost every rectangle R in the plane

(2) $$\int_{R^*} V_x\,dy - V_y\,dx = \Phi(R\cdot\mathcal{G})$$

Finally V_x and V_y are defined almost everywhere by the formulas

(3) $$V_x = -\frac{1}{2\pi}\iint_{\mathcal{G}} \frac{(\xi-x)\,d\Phi(e_{\xi\eta})}{(\xi-x)^2+(\eta-y)^2}, \quad V_y = -\frac{1}{2\pi}\iint_{\mathcal{G}} \frac{(\eta-y)\,d\Phi(e_{\xi,\eta})}{(\xi-x)^2+(\eta-y)^2}$$

Proof: It is evidently sufficient to prove this lemma under the assumption that Φ (e) $\geq$ 0 on G. For convenience, we extend the definition of Φ (e) to the whole plane by Φ(e) = Φ (e$\cdot$G).

We now define

$$h_t(x,y;\xi,\eta) = \frac{1}{2}\log[(\xi-x)^2+(\eta-y)^2], \quad (\xi-x)^2+(\eta-y)^2 \geq t^2$$

$$= \frac{1}{2t^2}[(\xi-x)^2+(\eta-y)^2] + \log t - \frac{1}{2}, \quad (\xi-x)^2+(\eta-y)^2 \leq t^2, \quad t>0,$$

and we see that h_t is of class C' in all four variables and that the function

$$V_t(x,y) = \frac{1}{2\pi}\iint_{\mathcal{G}} h_t(x,y;\xi,\eta)\, d\Phi(e_{\xi,\eta})$$

is of class C' everywhere with

$$V_{tx}(x,y) = \frac{1}{2\pi}\iint_{\mathcal{G}} h_{tx}(x,y;\xi,\eta)\,d\Phi(e_{\xi\eta}), \quad V_{ty} = \frac{1}{2\pi}\iint_{\mathcal{G}} h_{ty}(x,y;\xi,\eta)\,d\Phi(e_{\xi\eta})$$

Next, let the set functions $\Phi_n(e)$ tend weakly to $\Phi(e)$, each $\Phi_n(e)$ being of the form

$$\Phi_n(e) = \sum_{P_i \in e} \Phi_n(P_i^n), \quad \Phi_n(P_i^n) = \lambda_i^{(n)} > 0, \quad \Phi_n(\mathcal{G}) \le M, \quad P_i^n \in \mathcal{G}, \quad i = 1, \dots, N_n,$$

where $\Phi_n(e) = 0$ if e contains none of the P_i^n. Then we see that the functions (also of class C')

$$V_{tn}(x,y) = \frac{1}{2\pi}\iint_{\mathcal{G}} h_t(x,y;\xi,\eta)\,d\Phi_n(e_{\xi\eta}) = \frac{1}{2\pi}\sum_{i=1}^{N_n} \lambda_i^{(n)} h_t(x,y;x_i,y_i)$$

converge uniformly together with their derivatives to $V_t(x,y)$ on each bounded region Γ if $t > 0$. Next, we see immediately that

$$\left[\int_a^b |V_{tn}(x,y_0)|^p dx\right]^{\frac{1}{p}} \le \sum_{i=1}^{N_n} \lambda_i^{(n)} \left[\int_a^b |\log r_i|^p dx\right]^{\frac{1}{p}}, \quad \left[\int_c^d |V_{tn}(x_0,y)|^p dy\right]^{\frac{1}{p}} \le \sum_{i=1}^{N_n} \lambda_i^{(n)} \left[\int_c^d |\log r_i|^p dx\right]^{\frac{1}{p}}$$

$$r_i = (x-x_i)^2 + (y-y_i)^2$$

independently of t and n. Now, for each n

$$V_{t_1}(x,y) \le V_{t_2}(x,y), \quad V_{t_1,n}(x,y) \le V_{t_2,n}(x,y), \quad 0 < t_1 \le t_2$$

so that we see that V and V_n are summable to any power on any segment parallel to either axis and V_t and V_{tn} converge to V and V_n respectively in the mean of any order on each such segment and any rectangle. It follows also that V_n converges to V in the mean of any order on each such segment and any rectangle.

Next, let R be any cell and let α be any number, $1 \le \alpha < 2$. Then $\Phi(G) \le M$ and hence $((x,y) = P)$

$$\iint_R \left| V_{tx} + \frac{1}{2\pi}\iint_{\mathcal{G}} \frac{(\xi - x)\,d\Phi}{(\xi-x)^2+(\eta-y)^2}\right|^\alpha dx\,dy = \left(\frac{1}{2\pi}\right)^\alpha \iint_R \left| \iint_{C(P,t)} \left[\frac{\xi - x}{(\xi-x)^2+(\eta-y)^2} - \frac{\xi - x}{t^2}\right] d\Phi \right|^\alpha dx\,dy \le$$

$$\le \left(\frac{1}{2\pi}\right)^\alpha \iint_R \left[\iint_{C(P,t)} \frac{d\Phi}{[(\xi-x)^2+(\eta-y)^2]^{\frac{1}{2}}}\right]^\alpha dx\,dy \le \left(\frac{1}{2\pi}\right)^\alpha \iint_R \left\{ \Phi^{\alpha-1}(\mathcal{G}) \iint_{C(P,t)} [(\xi-x)^2+(\eta-y)^2]^{-\frac{\alpha}{2}} d\Phi \right\} dx\,dy$$

$$= \left(\frac{1}{2\pi}\right)^\alpha [\Phi(\mathcal{G})]^{\alpha-1} \iint_{R_t} \left\{ \iint_{C(\xi,\eta;t)} [(x-\xi)^2+(y-\eta)^2]^{-\frac{\alpha}{2}} dx\,dy \right\} d\Phi \le \left(\frac{\Phi(\mathcal{G})}{2\pi}\right)^\alpha \left(\frac{2\pi}{2-\alpha}\right) t^\alpha \le \left(\frac{M}{2\pi}\right)^\alpha \left(\frac{2\pi}{2-\alpha}\right) t^\alpha$$

where R_t denotes the set of all points at a distance $\le t$ from R. Evidently similar estimates hold for V_{ty}, V_{ntx} and V_{nty} independently of n. Thus we see that V and V_n are of class D'_α with $\bar{D}_\alpha(V_n, \Gamma)$ and $\bar{D}_\alpha(V, \Gamma)$ uniformly bounded for each fixed bounded Γ and each fixed α, $1 \leq \alpha < 2$. Moreover, the above also shows that $\bar{D}_\alpha(V_n - V, \Gamma)$ tends to zero for each fixed bounded Γ and each α, $1 \leq \alpha < 2$, and that formulas (3) hold.

Since $\bar{D}_\alpha(V_n - V, \Gamma)$ tends to zero as above, it follows that, for a subsequence V_{n_k} we have

$$\int_{R^*} V_x\,dy - V_y\,dx = \lim_{k\to\infty} \int_{R^*} V_{n_k x}\,dy - V_{n_k y}\,dx$$

for almost every rectangle R. Furthermore $\Phi_{n_k}(R)$ tends to $\Phi(R)$ for almost every R (as extended) and it is clear that

$$\int_{R^*} V_{n_k x}\,dy - V_{n_k y}\,dx = \Phi_{n_k}(R)$$

for almost all R and every k. From this (8) follows.

Remark: In this section, if U(x,y) is a function of class D'_α, $\alpha \geq 1$, then $\bar{U}(x,y)$ will denote the function

$$\bar{U}(x,y) = \lim_{\rho\to 0} \frac{1}{\pi\rho^2} \iint_{C(x,y;\rho)} U(\xi,\eta)\,d\xi\,d\eta$$

wherever it is defined. It is clear that $\bar{U}$ is A.C. along all lines parallel to each axis and is of class L_α on each segment parallel to an axis within its region of definition.

Lemma 1: Let U(x,y) be of class D'_2 in a region G containing the point P_0, and suppose that

(4) $$D_2[U, C(P_0, r)\cdot G] \leq M r^\lambda, \quad \lambda > 0$$

for all r. Then $\bar{U}(x,y)$ is defined at P_0. In fact, there exists a summable function $\varphi(\vartheta)$ such that

(5) $$\lim_{r\to 0} \int_0^{2\pi} |U(x_0 + r \sin\vartheta, y_0 + r\sin\vartheta) - \varphi(\vartheta)|\,d\vartheta = 0$$

if r is not allowed to assume values in a certain set of measure zero. Then

(6) $\overline{U}(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\vartheta)\,d\vartheta$, $\left|\overline{U}(x_0,y_0) - \frac{1}{\pi r^2}\iint_{C(P_0,r)} U(x,y)\,dx\,dy\right| \leq (2\pi)^{-\frac{1}{2}} M^{\frac{1}{2}}(1+\lambda^{-1})\, r^{1/2}$.

Proof: Let r and ϑ be polar coordinates with pole at P_0 and let $W(r, \vartheta)$ be the transform of $U(x,y)$ and let $\overline{W}(r, \vartheta)$ be formed as above from $W(r, \vartheta)$ in the (r, ϑ) plane. Then $\overline{W}(r, \vartheta)$ is of class D_2' in (r, ϑ) for $r \geq r_0 > 0$, no matter how small r_0 is, and $\overline{W}$ is A.C. in ϑ with $\overline{W}_\vartheta^2$ summable for almost every $r > 0$, and is A.C. in r with $\overline{W}_r^2$ summable in a region $0 \leq \vartheta \leq 2\pi$ $0 < r_0 \leq r \leq a$, if $C(P_0,a)$ lies in G, and finally

$$\int_0^r\int_0^{2\pi} \rho\left(\overline{W}_\rho^2 + \frac{1}{\rho^2}\overline{W}_\vartheta^2\right) d\rho\, d\vartheta \leq M r^\lambda, \quad 0 \leq r \leq a.$$

Define

$$k(r) = \int_0^r\int_0^{2\pi} \rho^{\frac{1}{2}} |\overline{W}_r(\rho,\vartheta)|\, d\rho\, d\vartheta; \quad \text{Then } k'(r) = \int_0^{2\pi} r^{\frac{1}{2}} |\overline{W}_r(r,\vartheta)|\, d\vartheta$$

Then

$$\cancel{k(r) \leq r\left[\int_0^r \rho\left[\int_0^{2\pi} |\overline{W}_r(r,\vartheta)|\, d\vartheta\right]^2 \leq 2\pi r\left[\int_0^r\int_0^{2\pi} \rho\, \overline{W}_r^2(\rho,\vartheta)\, d\rho\, d\vartheta\right] \leq (2\pi M) r^{1+\lambda}}$$

for almost all r and k(r) is A.C. for $0 < \varepsilon \leq r \leq a$ for each $\varepsilon > 0$ and

$$k^2(r) \leq r\left[\int_0^r \rho\left[\int_0^{2\pi} |\overline{W}_r(r,\vartheta)|\, d\vartheta\right]^2 d\rho \leq 2\pi r\left[\int_0^r\int_0^{2\pi} \rho\, \overline{W}_r^2(\rho,\vartheta)\, d\rho\, d\vartheta\right] \leq (2\pi M)\, r^{1+\lambda}\right.$$

Also, for each $\varepsilon > 0$

$$\int_0^{2\pi} |\overline{W}(r,\vartheta) - \overline{W}(\varepsilon,\vartheta)|\, d\vartheta \leq \int_0^{2\pi}\int_\varepsilon^r |W_r(\rho,\vartheta)|\, d\rho\, d\vartheta = \int_\varepsilon^r \rho^{-\frac{1}{2}} k'(\rho)\, d\rho = r^{-\frac{1}{2}}k(r) - \varepsilon^{-\frac{1}{2}}k(\varepsilon) +$$

$$+ \frac{1}{2}\int_\varepsilon^r \rho^{-\frac{3}{2}} k(\rho)\, d\rho \leq (2\pi M)^{\frac{1}{2}}(1+\lambda^{-1})\, r^{\frac{\lambda}{2}}, \quad \varepsilon < r$$

Hence the existence of $\varphi(\vartheta)$ to satisfy (10) is evident as $\overline{W}(r, \vartheta)$ is equivalent in (r, ϑ) to $U(x_0 + r\cos\vartheta,\ y_0 + r\sin\vartheta) = W(r, \vartheta)$. The existence of $\overline{U}(x_0, y_0)$ and equation (6) follow easily.

Lemma 2: Let d and e be of class L_2 on a bounded region G and satisfy

$$\iint_{C(P_0,r)\cdot G} (d^2 + e^2)\, dx\, dy \leq M r^\lambda, \quad 0 < \lambda < 1$$

for all circles $C(P_0, r)$ with center at a fixed point P_0. Then there exist sequences d_n and e_n of functions of class C'' all over the plane π, zero outside a bounded region $H \supset \overline{G}$, satisfying (7) uniformly, and such that, if $d = e = 0$ outside G, then

$$\lim_{n \to \infty} \iint_H [(d_n - d)^2 + (e_n - e)^2]\, dx\, dy = 0$$

Proof: We first observe that, if d_h and e_h are given by

$$d_h(x,y) = \frac{1}{\pi h^2} \iint_{C(0,0;h)} d(x+\xi, y+\eta)\, d\xi\, d\eta, \quad e_h(x,y) = \frac{1}{\pi h^2} \iint_{C(0,0;h)} e(x+\xi, y+\eta)\, d\xi\, d\eta$$

we have

$$\iint_{C(x_0,y_0;r)} (d_h^2 + e_h^2)\, dx\, dy \le \frac{1}{\pi h^2} \iint_{C(x_0,y_0;r)} \Big[\iint_{C(0,0;h)} \{ d^2(x+\xi, \eta+y) + e^2(x+\xi, y+\eta) \}\, d\xi\, d\eta \Big]\, dx\, dy =$$

$$= \frac{1}{\pi h^2} \iint_{C(0,0;h)} \Big[\iint_{C(x_0,y_0;r)} \{ d^2(x+\xi, y+\eta) + e^2(x+\xi, y+\eta) \}\, d\xi\, d\eta = \cancel{\frac{1}{\pi h^2} \iint}$$

$$= \frac{1}{\pi h^2} \iint_{C(0,0,h)} \Big[\iint_{C(x_0+\xi,\, y_0+\eta;\, r)} \{ d^2(x,y) + e^2(x,y) \}\, dx\, dy \Big]\, d\xi\, d\eta \le \iint_{C(x_0,y_0;r+h)} (d^2 + e^2)\, dx\, dy$$

Next we choose a sequence $r_n \to 0$ and define

$$d_{1n}(x,y) = d(x,y), \quad e_{1n}(x,y) = e(x,y), \quad (x-x_0)^2 + (y-y_0)^2 \ge r_n^2$$

$$d_{1n}(x,y) = e_{1n}(x,y) = 0, \quad (x-x_0)^2 + (y-y_0)^2 < r_n^2$$

For each n, we define a positive number h_n so small that

$$\left(\frac{r_n}{r_n - h_n} \right)^\lambda < \frac{n}{n-1}, \quad \iint_\pi \left[(d_{1n} - d_{1n,h_n})^2 + (d_{2n} - d_{2n,h_n})^2 \right] dx\, dy < \frac{1}{n}$$

d_{1n,h_n} and e_{1n,h_n} being the circular averages of d_{1n} and e_{1n}. We next let

$$d_{2n}(x,y) = \left(1 - \frac{1}{n}\right) d_{1n,h_n}, \quad e_{2n}(x,y) = \left(1 - \frac{1}{n}\right) e_{1n,h_n}$$

and we see that d_{2n} and e_{2n} are continuous, zero near infinity, and satisfy

$$\iint_{C(P_0,r)} (d_{2n}^2 + e_{2n}^2)\, dx\, dy \le \left(\frac{n-1}{n-2} \right) M r^\lambda,$$

$$\left\{ \iint_{\Pi} [(d_{2n}-d)^2+(e_{2n}-e)^2]\,dx\,dy \right\}^{\frac{1}{2}} \le \left\{ \iint_{\Pi} \left([d_{2n}-(1-\tfrac{1}{n})d_{1n}]^2 + [e_{2n}-(1-\tfrac{1}{n})e_{1n}]^2 \right) dx\,dy \right\}^{\frac{1}{2}}$$

$$+ \frac{2}{n} \left\{ \iint_{\Pi} (d_{1n}^2 + e_{1n}^2)\,dx\,dy \right\}^{\frac{1}{2}} + \left\{ \iint_{C(P_0, r_n)} (d^2+e^2)\,dx\,dy \right\}^{\frac{1}{2}}$$

which evidently tends to zero as $n \to \infty$. The determination of the d_n and e_n is now clear.

Theorem 2: Let d and e be of class L_2 on the bounded region G. Then the function

(8) $$U(x,y) = -\frac{1}{2\pi} \iint_{G} \frac{(\xi-x)d + (\eta-y)e}{(\xi-x)^2+(\eta-y)^2}\,d\xi\,d\eta$$

is defined almost everywhere (the right side existing as a Lebesgue integral) and is of class D_2' all over the plane Π satisfying

(9) $$D_2(U,\Pi) \le 4 \iint_{G} (d^2+e^2)\,dx\,dy.$$

If d and e satisfy (7) for all circles with center at some point P, then $\overline{U}(P)$ is defined by (8) and

(10) $$|\overline{U}(P)| \le 2(1+\lambda^{-1})(2\pi M)^{\frac{1}{2}} R^{\frac{1}{2}}$$

where $C(P,R)$ is the circle of smallest radius about P which contains G.

If (7) holds for every P and r, then U is continuous over Π and satisfies conditions $A[\lambda, M(a)]$ and $B[\lambda/2, N(a)]$ everywhere where

(11) $$M(a) = 4M[\lambda^{-1}+(1-\lambda)^{-1}]a^{\lambda}, \quad N(a) = 12(2\pi M)^{\frac{1}{2}}[(2-\lambda)^{-1}+\lambda^{-1}(\lambda+1)]a^{\lambda/2}$$

Proof: It is clear that we can find sequences $\{d_n\}$ and $\{e_n\}$ of functions of class C'' all over Π and zero outside a bounded region H containing G in its interior such that

$$\lim_{n\to\infty} \iint_{H} [(d_n-d)^2+(e_n-e)^2]\,dx\,dy = 0$$

If d and e satisfy (7) for every P and r, it follows from Lemma 1, §3 of D.E.,*)

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*) The author's paper, "On the Solutions of Quasi-Linear Elliptic Partial Differential Equations", Transactions of the American Mathematical Society, vol. 43 (1938), pp. 126-166. We shall hereafter refer to this paper as D.E.

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that we may require (7) uniformly with H replacing G. If (7) is satisfied only at P, then an approximation is possible in L_2 which preserves (7) at P.

Now let R be any bounded region. Then

$$\iint_R |U_n(x,y) - U(x,y)| \leq \frac{1}{2\pi}\iint_R \left[\iint_H \left\{\frac{(d_n-d)^2+(e_n-e)^2}{(\xi-x)^2+(\eta-y)^2}\right\}^{\frac{1}{2}} d\xi d\eta\right] dx\,dy$$

$$= \frac{1}{2\pi}\iint_H \left\{([d_n-d]^2+[e_n-c]^2)^{\frac{1}{2}} \iint_{C(\xi,\eta;\rho)} [(x-\xi)^2+(y-\eta)^2]^{-\frac{1}{2}} dx\,dy\right\} d\xi d\eta \leq \rho \iint_H [(d_n-d)^2+(e_n-e)^2]^{\frac{1}{2}} d\xi d\eta$$

which obviously tends to zero as n $\rightarrow \infty$, ρ being chosen large enough so that C($\xi, \eta; \rho$) contains R for each (ξ, η) in H. Thus, it is easily seen that $U_n \rightarrow$ U in the sense defined in §3 all over π so that $D_2(U,R) \leq \varliminf_{n\to\infty} D_2(U_n,R)$ for every region R. If d, d_n, e, and e_n satisfy (7) uniformly on H, it follows from Theorem 1, §3 of D.E., that the corresponding U_n and U satisfy the condition B of the theorem uniformly so that U_n tends uniformly to U over π.

Now, consider such sequences d_n, e_n and U_n. Then U_n is of class C" and satisfies

(12) $\quad \Delta U_n = d_{nx} + e_{ny} \qquad (\Delta\varphi = \varphi_{xx} + \varphi_{yy})$

all over π. Hence it is easily seen that U_n minimizes

(13) $\quad \iint_R [(\varphi_x - d_n)^2 + (\varphi_y - e_n)^2]\, dx\,dy$

among all functions φ of class D_2' on R and coinciding with U_n on R^*, R being any region of class C'. Thus if we let H_{nR} be the harmonic function coinciding with U_n on R^* and let $U_{nR} = U_n - H_{nR}$, we see that (since H_{nR} minimizes $D_2(\varphi, R)$ among all $\varphi = U_n$ on R^*'s of class D_2' on R)

(14) $$D_2(U_n, R) = D_2(H_{nR}, R) + D_2(U_{nR}; R)$$

and U_{nR} satisfies (12) and hence minimizes (13) among all φ of class D_2' on R and zero on R^*. From the remarks at the bottom of page 149 of D.E., it follows that

(15) $$D_2(U_{nR}, R) \leq 2 \iint_R [(U_{nRx} - d_n)^2 + (U_{nRy} - e_n)^2]\,dx\,dy + 2\iint_R (d_n^2 + e_n^2)\,dx\,dy \leq \cancel{\leq 4} \leq 4 \iint_R (d_n^2 + e_n^2)\,dx\,dy$$

since U_{nR} minimizes (13). Thus for each R of class C' in π we see that

(16) $$D_2(U_n, R) \leq D_2(H_{nR}, R) + 4 \iint_R (d_n^2 + e_n^2)\,dx\,dy.$$

Now, let (x,y) be a point not in $\bar{H}$. Then

$$U_{nx} = -\frac{1}{2\pi}\iint_H \frac{[(\xi-x)^2 - (\eta-y)^2]d_n + 2[(\xi-x)(\eta-y)]e_n}{[(\xi-x)^2 + (\eta-y)^2]^2}\,d\xi\,d\eta, \quad U_{ny} = -\frac{1}{2\pi}\iint_H \frac{[2(\xi-x)(\eta-y)d_n - [(\xi-x)^2 - (\eta-y)^2]e_n}{[(\xi-x)^2+(\eta-y)^2]^2}\,d\xi\,d\eta$$

Now let P_o be any point of π and let ρ be so large that $\bar{H} \subset C(P_o, \rho)$ and the distance of H from $C^*(P_o, \rho)$ is $> \rho/2$. Then, on $C^*(P_o, \rho)$, we see that

$$\left|\frac{d}{d\vartheta} U_n(x_0 + \rho\cos\vartheta, y_0 + \rho\sin\vartheta)\right|^2 \leq \frac{4K_n}{\rho^2}, \quad K_n = \frac{1}{2\pi}\iint_H (d_n^2 + e_n^2)^{\frac{1}{2}}\,dx\,dy$$

and hence

$$\sum_{p=1}^{\infty} p(a_{np}^2 + b_{np}^2) \leq \sum_{p=1}^{\infty} p^2(a_{np}^2 + b_{np}^2) = \int_0^{2\pi} U_{n\vartheta}^2\,d\vartheta \leq \frac{8\pi K_n}{\rho^2}$$

which tends to zero as $\rho \to \infty$. Here, we have placed

$$U_n(x_0 + \rho\cos\vartheta, y_0 + r\sin\vartheta) = \frac{a_{n0}}{2} + \sum_{p=1}^{\infty}(a_{np}\cos p\vartheta + b_{np}\sin p\vartheta)$$

Thus, from Lemma 2, §5, it follows that

$$\lim_{\rho \to 0} D_2[H\{U_n, C(P_0, \rho)\}, C(P_0, \rho)] = 0$$

and hence (using this and (16))

$$D_2(U_n, \pi) \leq 4 \iint_H (d_n^2 + e_n^2)\,dx\,dy$$

Now, by §4, Theorem 4, the boundary values for U_n tend in L_2 on R^* to those of U and $D_2(U_n, R)$ is uniformly bounded, R being any region of class C'. Thus $H_{nR} \to H_R$, $U_{nR} \to U_R$ also. Now, by referring to (12) and (13) and the argument at that point, we see that

$$\lim_{\substack{n\to\infty \\ m\to\infty}} D_2(u_n - u_m, \Pi) \le \lim_{\substack{m\to\infty \\ n\to\infty}} 4 \iint_H [(d_m - d_n)^2 + (e_m - e_n)^2]\, dx\, dy = 0$$

so that

$$\lim_{\substack{n\to\infty \\ m\to\infty}} D_2(H_{nR} - H_{mR}, R) = \lim_{\substack{n\to\infty \\ m\to\infty}} D_2(U_{nR} - U_{mR}, R) = 0$$

so that (by letting $m \to \infty$ first and using the lower semicontinuity of D_2) we have

$$\lim_{n\to\infty} D_2(u_n - u, \Pi) = \lim_{n\to\infty} D_2(H_{nR} - H_R, R) = \lim_{n\to\infty} D_2(U_{nR} - U_R, R) = 0$$

so that finally (9) holds, and (16) holds with U, H_R, and U_R replacing U_n, H_{nR}, and U_{nR} respectively, and d and e replacing d_n and e_n. Thus if (9) holds for a single point P and every $r > 0$, we find that (10) holds if we define $\bar{U}(P)$ by the integral*), and also we see, using Theorem 1, §5, that

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*) For the proof of this, see the proof of Theorem 1, §3, of D.E.

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(17) $D_2[U, C(P,r)] \le 4M[\lambda^{-1} + (1-\lambda)^{-1}]r^\lambda$, $D_2[U_n, C(P,r)] \le 4M[\lambda^{-1} + (1-\lambda)^{-1}]r^\lambda$.

Thus, by Lemma 1,

$$\left|\bar{U}(x_0, y_0) - \frac{1}{\pi r^2} \iint_{C(x_0, y_0; r)} U(x,y)\, dx\, dy\right| \le (2\pi)^{-1/2} \{4M[\lambda^{-1} + (1-\lambda)^{-1}]\}^{\frac{1}{2}} r^\lambda (1+\lambda^{-1})$$

$$\left|U_n(x_0, y_0) - \frac{1}{\pi r^2} \iint_{C(x_0, y_0, r)} U_n(x,y)\, dx\, dy\right| \le (2\pi)^{-\frac{1}{2}} \{4M[\lambda^{-1} + (1-\lambda)^{-1}]\}^{\frac{1}{2}} (1+\lambda^{-1}) r^\lambda$$

from which it follows immediately that $U_n(x_o, y_o) \to \bar{U}(x_o, y_o)$.

Now let

$$h_n(r) = \iint\limits_{C(P_0,r)} [(d_n - d)^2 + (e_n - e)^2]\,dx\,dy = \int_0^r \rho\left[\int_0^{2\pi} [(d_n-d)^2+(e_n-e)^2]^{\frac{1}{2}}\,d\vartheta\right]d\rho$$

Then

$$h_n(r) \leq \pi^{\frac{1}{2}} r \Big\{ \iint\limits_{C(P_0,r)} [(d_n-d)^2 + (e_n-e)^2]\,dx\,dy \Big\}^{\frac{1}{2}} \leq 2(\pi M)^{\frac{1}{2}} r^{1+\frac{\lambda}{2}}.$$

Furthermore, $h_n(r)$ tends uniformly to zero for all $r \geqq 0$. Thus, if we let $2(\pi M)^{\frac{1}{2}} r_n^{1+\lambda/2}$ be the maximum of $h_n(r)$, we see that $r_n \to 0$. Now

$$\iint\limits_{\mathfrak{H}} \left| \frac{(d_n-d)(\xi-x_0)+(e_n-e)(\eta-y_0)}{(\xi-x_0)^2+(\eta-y_0)^2} \right| d\xi\,d\eta \leq \iint\limits_{\mathfrak{H}} \left[\frac{(d_n-d)^2+(e_n-e)^2}{(\xi-x_0)^2+(\eta-y_0)^2}\right]^{\frac{1}{2}} d\xi\,d\eta = \int_0^R r^{-1} h_n'(r)\,dr =$$

$$= R^{-1} h_n(R) + \int_0^R r^{-2} h_n(r)\,dr \leq 2(\pi M)^{\frac{1}{2}} r_n^{1+\lambda/2} R^{-1} + 2(\pi M)^{\frac{1}{2}} \int_0^{r_n} r^{\lambda/2 - 1}\,dr +$$

$$+ 2(\pi M)^{\frac{1}{2}} r_n^{1+\frac{\lambda}{2}} \int_{r_n}^R r^{-2}\,dr = (\pi M)^{\frac{1}{2}} (2 + \lambda^{-1}) r_n^{\lambda/2},$$

where R was chosen large enough so that $C(P_0, R)$ contained $\bar{H}$. From this, it follows that the corresponding integrals for $U_n(x_0, y_0)$ converge to the right side of (8) so that in this case $\bar{U}(P_0)$ is defined by (8), since $U_n(x_0, y_0) \to \bar{U}(x_0, y_0)$.

Now if condition (7) is satisfied for every point P and every r, it follows from (17) that U satisfies a condition $A[\lambda, M(a)]$ where $M(a)$ is given by (11). The condition $B[\lambda, N(a)]$ where $N(a)$ is given in (11) has been proved in D.E., §3, Theorem 1. This completes the proof of the theorem.

Lemma 3: Let G be a region of class C_2, and let $u(x,y)$ be of class D_2' on G with $D_2(u, G)$ finite, and suppose u vanishes on G^*. Then we may find a sequence of functions $u_n(x,y)$, each of class C''' on $\bar{G}$ and zero near G^*, such that $\bar{D}_2(u - u_n, G) \to 0$.

Proof: Let u be of class D_2^1 on G with $D_2(u,G)$ finite. Let G be mapped conformally on the region Σ bounded by a finite number of circles and let $w(s,t)$ be the transform of $u(x,y)$; w is of class D_2^1 on Σ with $\bar{D}_2(w, \Sigma)$ finite and vanishes on Σ^*. Let $\delta(s,t)$ denote the distance of a point (s,t) of Σ from Σ^*; δ satisfies a Lipschitz condition with constant unity and is analytic near Σ^*. Now, for each $n > N$ (sufficiently large), define functions $k_n(s,t)$ and $w_n(s,t)$ as follows:

$$k_n(s,t) = 1, \quad \delta(s,t) \geq \frac{1}{n}$$

$$k_n(s,t) = 2n\left[\delta(s,t) - \frac{1}{2n}\right], \quad \frac{1}{2n} \leq \delta(s,t) \leq \frac{1}{n}$$

$$k_n(s,t) = 0, \quad \delta(s,t) \leq \frac{1}{2n}$$

$$w_n(s,t) = w(s,t)\cdot k_n(s,t)$$

Let τ_n denote the part of Σ for which $\frac{1}{2n} \leq \delta(s,t) \leq \frac{1}{n}$ and o_n that were $\delta(s,t) \leq \frac{1}{n}$. Then

$$D_2(w_n - w, \Sigma) = D_2(w_n - w, o_n) \leq 2 D_2(w, o_n) + 2 D_2(w_n, \tau_n)$$

(18) $$\leq 4 D_2(w, o_n) + 2 \iint_{\tau_n} w^2 (k_{ns}^2 + k_{nt}^2)\, ds\, dt = 4 \cdot D_2(w, o_n) + 8n^2 \iint_{\tau_n} w^2\, ds\, dt$$

Now $$\iint_{\tau_n} w^2\, ds\, dt = \sum_{i=1}^{k} \int_{r_i + \frac{1}{2n}}^{r_i + \frac{1}{n}} \int_0^{2\pi} r \bar{w}(s_i + r\cos\vartheta, t_i + r\sin\vartheta)\, dr\, d\vartheta =$$

$$= \sum_{i=1}^{k} \int_{r_i + \frac{1}{2n}}^{r_i + \frac{1}{n}} \int_0^{2\pi} \left[\int_{r_i}^{r} \bar{w}_\rho(s_i + \rho\cos\vartheta, t_i + \rho\sin\vartheta)\, d\rho\right] d\vartheta\, dr \leq$$

(19) $$\leq \sum_{i=1}^{k} \left(\frac{r_i + \frac{1}{r_i}}{r_i}\right) \int_{r_i + \frac{1}{2n}}^{r_i + \frac{1}{n}} (r - r_i) \left[\int_{r_i}^{r} \int_0^{2\pi} \rho \bar{w}_\rho^2\, d\rho\, d\vartheta\right] dr \leq 2 \sum_{i=1}^{k} \left[\int_{r_i}^{r_i + \frac{1}{n}} \int_0^{2\pi} r(\bar{w}_i^2 + w_i^2)\, dr\, d\vartheta \left(\frac{1}{2n^2} - \frac{1}{8n^2}\right)\right.$$

where r_i is the radius of the i^{th} bounding circle in Σ^*. From (18) and (19) it follows that $D_2(w_n - w, \Sigma) \to 0$ and it is also clear that $\bar{D}_2(w_n - w, \Sigma)$ 0.

Now for each $n > N$, we may choose h_n so small that w_{n,h_n^3} is defined on Σ and zero near Σ^*; here w_{n,h_n^3} denotes the third iterated average of w_n and is of class $C''{}'$ on $\overline{\Sigma}$; h_n may also be chosen so that

$$D_2(w_n - \overset{++}{w}_{n,h_n^3}; \Sigma) < \frac{1}{n}$$

If we let u_n denote the transform of w_{n,h_n^3} we see that we have the desired approximation fo u. It is clear that u is of class D_2' over Π and u_n of class $C''{}'$ over Π, and hence it is evident that $\overline{D}_2(\overset{u-u_n}{\cancel{wu_n}}, \Pi) \to 0$ as well as $D_2(u - u_n, \Pi) \to 0$.

<u>Theorem 3: Let $u(x,y)$ be of class D_2' on a region G of class C_2 with $D_2(u,G)$ finite, and suppose that u vanishes on G^*. Then, if we define $u = 0$ for (x,y) not in G, u, as extended, is of class D_2' over Π and is given almost everywhere by</u>

$$\overline{u}(x,y) = -\frac{1}{2\pi}\iint_{G} \frac{(\xi - x)\,\overline{u}_x(\xi,\eta) + (\eta - y)\,\overline{u}_y(\xi,\eta)}{(\xi - x)^2 + (\eta - y)^2}\, d\xi\, d\eta \tag{20}$$

<u>In fact (20) holds at each point (x,y) where the integrand on the right is summable.</u>

<u>If $u(x,y)$ also satisfies</u>

$$D_2[u, G\cdot C(P,r)] \le M r^{\lambda}, \quad 0 < \lambda < 1 \tag{21}$$

<u>for every circle with center at some point P_0, then $\overline{u}(x_0,y_0)$ and (20) exist, $\overline{u}$ is given by (20), and</u>

$$|\overline{u}(x_0,y_0)| \le (2\pi M)^{\frac{1}{2}}(1+\lambda^{-1})\,\delta^{\frac{1}{2}} \tag{22}$$

<u>where δ is the diameter of G. If $u(x,y)$ satisfies (21) for every P and r, then u, as extended satisfies (22) and a condition $B[\lambda/2, N(a)]$ everywhere, where</u>

$$N(a) = \frac{8M^{\frac{1}{2}}}{\lambda\sqrt{3}}\, a^{\lambda/2} \tag{23}$$

Proof: According to Lemma 3, we may find a sequence $\{u_n(x,y)\}$, each of class C''' and zero near G^* such that $\bar{D}_2(u_n - u, G) \to 0$. If we define $u_n = 0$ for all (x,y) outside of G we see that u_n is of class C''' all over Π and $\bar{D}_2(u_n - u, \Pi) \to 0$. Also, for each $u_n(x,y)$, we have

$$-\frac{1}{2\pi}\iint_{\mathcal{G}} \frac{(\xi - x_0)u_{nx}(\xi,\eta) + (\eta - y_0)u_{ny}(\xi,\eta)}{(\xi - x_0)^2 + (\eta - y_0)^2}\, d\xi\, d\eta = -\frac{1}{2\pi}\int_0^R\int_0^{2\pi} \frac{\partial u_n(x_0 + r\cos\vartheta, y_0 + r\sin\vartheta)}{\partial r}\, dr\, d\vartheta$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} [u_n(x_0 + R\cos\vartheta, y_0 + R\sin\vartheta) - u_n(x_0, y_0)]\, d\vartheta = u_n(x_0, y_0)$$

R being chosen large enough so that $C(P_0, R)$ contains $\bar{G}$ in its interior. From the above, $u_n \to u$ in the mean of order 2 over Π and from the proof of Theorem 2 it follows that the corresponding integrals tend in the mean of order 1 to the right side of (20). Thus (20) holds almost everywhere. From this the existence of $u(x_0, y_0)$ follows at each point where (21) holds for every r.

Now suppose (21) holds at (x_0, y_0). Let (r, ϑ) be polar coordinates with pole at (x_0, y_0) and let $w(r, \vartheta) = u(x_0 + r\cos\vartheta, y_0 + r\sin\vartheta)$. From Lemma 1 and its proof, we see that there exists a summable function $\varphi(\vartheta)$ such that

$$\lim_{r\to 0}\int_0^{2\pi} |\bar{w}(r,\vartheta) - \varphi(\vartheta)|\, d\vartheta = 0, \quad \frac{1}{2\pi}\int_0^{2\pi}[\bar{w}(r,\vartheta) - \varphi(\vartheta)]\, d\vartheta = \frac{1}{2\pi}\int_0^{2\pi}\bar{w}(r,\vartheta)\, d\vartheta - \bar{u}(x_0, y_0)$$

$$(24) \qquad = \frac{1}{2\pi}\int_0^r\int_0^{2\pi}\bar{w}_r(\rho,\vartheta)\, d\rho\, d\vartheta, \quad \frac{1}{2\pi}\int_0^r\int_0^{2\pi}|\bar{w}_r(\rho,\vartheta)|\, d\rho\, d\vartheta \le (2\pi\lambda)^{1/2}(1+\lambda^{-1})\, r^{\lambda/2}.$$

Formula (22) follows from (24) if one notes that $\bar{u}(x_0, y_0) = 0$ if $(x_0, y_0) \varepsilon\ \Pi - \bar{G}$ and if $(x_0, y_0) \varepsilon\ \bar{G}$, then $\bar{w}(r, \vartheta) = 0$ if $r > \delta$. The continuity of $u(x,y)$ follows from the fact that $u(x_0, y_0)$ is given by (20) and that the right side of (20) is continuous everywhere by Theorem 2 in case (21) holds for every P and r. The condition $B[\lambda/2, N(a)]$ as given follows from Theorem 2, §5, if (21) holds everywhere.

To show in general that $u(x_0, y_0)$ exists at each point where (20) exists, we merely observe that

$$\frac{1}{2\pi}\iint_{\mathcal{G}}\left|\frac{(\xi-x_0)\bar{u}_x(\xi,\eta)+(\eta-y_0)\bar{u}_y(\xi,\eta)}{(\xi-x_0)^2+(\eta-y_0)^2}\right|d\xi\,d\eta=\frac{1}{2\pi}\int_0^R\int_0^{2\pi}|\bar{w}_r(r,\vartheta)|\,dr\,d\vartheta$$

if R is chosen large enough. Since $\bar{w}(r,\vartheta)$ is of class D_2' in (r,ϑ) for $\varepsilon \leq r \leq R$, $0 \leq \vartheta \leq 2\pi$ and A.C. in r for almost all ϑ, and in ϑ for almost all r in any such region, $\varepsilon > 0$, and since $\bar{w}_r(r,\vartheta)$ is summable for $0 \leq r \leq R$, $0 \leq \vartheta \leq 2\pi$ we see immediately that a summable function $\varphi(\vartheta)$ exists such that

$$\lim_{r\to 0}\int_0^{2\pi}|\bar{w}(r,\vartheta)-\varphi(\vartheta)|\,d\vartheta=0,\quad \int_0^{2\pi}[\bar{w}(r,\vartheta)-\varphi(\vartheta)]\,d\vartheta=\int_0^r\int_0^{2\pi}\bar{u}_r(\rho,\vartheta)\,d\rho\,d\vartheta$$

Thus if we let u_0 = the mean value of $\varphi(\vartheta)$, we see that

$$\left|\frac{1}{\pi r^2}\int_0^r\int_0^{2\pi}\rho\,\bar{w}(\rho,\vartheta)\,d\rho\,d\vartheta-u_0\right|=\left|\frac{2}{r^2}\int_0^r\rho\left\{\frac{1}{2\pi}\int_0^{2\pi}[\bar{w}(\rho,\vartheta)-\varphi(\vartheta)]\,d\vartheta\right\}d\rho\right|\leq$$

$$\leq\frac{1}{\pi r^2}\int_0^r\rho\left[\int_0^\rho\int_0^{2\pi}|\bar{w}_r(\sigma,\vartheta)|\,d\sigma\,d\vartheta\right]d\rho\leq\frac{1}{2\pi}\int_0^r\int_0^{2\pi}|\bar{w}_r(\rho,\vartheta)|\,d\rho\,d\vartheta$$

which tends to zero with r. Thus $\bar{u}(x_0, y_0)$ exists and

$$\bar{u}(x_0,y_0)=u_0=\frac{1}{2\pi}\int_0^{2\pi}\varphi(\vartheta)\,d\vartheta=-\frac{1}{2\pi}\int_0^{2\pi}[\bar{w}(R,\vartheta)-\varphi(\vartheta)]\,d\vartheta=-\frac{1}{2\pi}\int_0^R\int_0^{2\pi}\bar{w}_r(r,\vartheta)\,dr\,d\vartheta$$

$$=-\frac{1}{2\pi}\iint_{\mathcal{G}}\frac{(\xi-x_0)\bar{u}_x(\xi,\eta)+(\eta-y_0)\bar{u}_y(\xi,\eta)}{(\xi-x_0)^2+(\eta-y_0)^2}\,d\xi\,d\eta$$

as we observed above.

<u>Theorem 4:</u> <u>Let G be a bounded region, let $\Phi(e)$ be a completely additive set function on G, and let ζ be of class D_2 on G, be zero on G^*, and satisfy (21). Then</u>

(25) $$\iint_{\mathcal{G}}\zeta\,d\Phi(e)=-\iint_{\mathcal{G}}(\zeta_x V_x+\zeta_y V_y)\,dx\,dy$$

where the right side exists as a Lebesgue integral, V being the potential of ϕ (e) as defined by (1).

Proof: We first observe that, for any such ϕ and ζ, we have

$$\iint_{\zeta}\left[\iint_{\zeta}\left|\frac{(\xi-x)\zeta_x+(\eta-y)\zeta_y}{(\xi-x)^2+(\eta-y)^2}\right|dx\,dy\right]|d\,\phi(e_{\xi\eta})| \qquad (|d\,\phi(e_{\xi\eta})| = d\,V_{\phi}(e_{\xi\eta}))$$

exists and is finite by (10) so that the four dimensional Lebesgue Stieltjes integral without the absolute value signs exists and may be evaluated by means of the iterated double integrals in either order. Now let V be defined by (1) and then, by using formulas (3) and (20), we see that

$$-\iint_{\zeta}(\zeta_x V_x+\zeta_y V_y)\,dx\,dy = +\frac{1}{2\pi}\iint_{\zeta}\left[\iint_{\zeta}\frac{(\xi-x)\zeta_x+(\eta-y)\zeta_y}{(\xi-x)^2+(\eta-y)^2}\,d\,\phi(e_{\xi\eta})\right]dx\,dy =$$

$$= -\frac{1}{2\pi}\iint_{\zeta}\left[\iint_{\zeta}\frac{(x-\xi)\zeta_x+(\eta-y)\zeta_y}{(x-\xi)^2+(y-\eta)^2}\,dx\,dy\right]d\,\phi(e_{\xi\eta}) = \iint_{\zeta}\zeta(\xi,\eta)\,d\,\phi(e_{\xi\eta})$$

which proves the theorem.

Lemma 4: Let P_1 and P_2 be two points of the plane with $|P_1P_2| = \delta > 0$. Let r_1 and r_2 be the distances of P_1 and P_2 from a variable point (x,y). Then

$$(26) \qquad 0 \le \varphi(r,\delta) = \max_{P_0 \varepsilon \pi}\iint_{C(P_0,r)}\frac{dx\,dy}{r_1 r_2} \le \begin{cases}\dfrac{4\pi r}{\delta}, & 0 \le r \le \dfrac{\delta}{4}\\[2mm] \dfrac{4\pi r\sqrt{2}}{\delta}, & \dfrac{\delta}{4} \le r \le \delta\\[2mm] \dfrac{4\pi\sqrt{2}}{\delta}+\dfrac{8\pi}{3}\log\dfrac{r}{\delta}, & r \ge \delta.\end{cases}$$

Proof: First of all, if new coordinates (x',y') are taken in with x' = kx, y' = ky, we see that $r_1' = kr_1$, $r_2' = kr_2$, $\delta' = k\delta$ so that $\varphi(kr, k\delta) = \varphi(r,\delta)$, k > 0. Thus $\varphi = h(\frac{r}{\delta})$. Hence we choose $\delta = 1$, P_1: $(-\frac{1}{2}, 0)$, $P_2(\frac{1}{2}, 0)$, $P_0(x_0, y_0)$, and try to dominate the maximum of the integral. Clearly we may take $y_0 = 0$. Now

$$(27) \qquad r_1^2 r_2^2 = \left[(x+\tfrac{1}{2})^2+y^2\right]\left[(x-\tfrac{1}{2})^2+y^2\right] = g(x,y)$$

$$g_x = 4x\left(x^2+y^2-\tfrac{1}{4}\right)$$

Now let r be fixed and $< \frac{1}{2}$ and let

$$f(x_0) = \iint\limits_{C(x_0,0;r)} \left[(x+\tfrac{1}{2})^2 + y^2\right]^{-\frac{1}{2}} \left[(x-\tfrac{1}{2})^2 + y^2\right]^{-\frac{1}{2}} dx\,dy$$

Then

$$f'(x_0) = \int\oint\limits_{C^*(x_0,0;r)} \left[(x+\tfrac{1}{2})^2 + y^2\right]^{-\frac{1}{2}} \left[(x-\tfrac{1}{2})^2 + y^2\right]^{-\frac{1}{2}} dy$$

It is seen that if $x_0 < -\frac{1}{2} - r$, then $f'(x_0) > 0$, and if $-\frac{1}{2} + r < x_0 < 0$, then $f'(x_0) < 0$ (by referring to (27)), and as $x_0 \to -\frac{1}{2} - r$ from either side $f'(x_0) \to +\infty$ and as $x_0 \to -\frac{1}{2} + r$ from either side $f'(x_0) \to -\infty$; also $f'(0) = 0$. Thus $f(x_0)$ has symmetrically placed maxima at $x_0 = \pm \bar{x}_0$, $\frac{1}{2} - r < \bar{x}_0 < \frac{1}{2} + r$, and a relative minimum at $x_0 = 0$. In fact it is also easy to see that $f'(x_0) > 0$ for $-\frac{1}{2} - r < x_0 \leqq -\frac{1}{2}$, so $\frac{1}{2} - r < \bar{x}_0 < \frac{1}{2}$.

If $r > \frac{1}{2}$, we see that $f'(x_0) \to +\infty$ as $x_0 \to -\frac{1}{2} - r$ or $\frac{1}{2} - r$, and it is obvious that $f(\frac{1}{2} - r) > f(-\frac{1}{2} - r)$. Also $f'(0) = 0$ and $f'(x_0) > 0$ for $\frac{1}{2} - r < x_0 < 0$. Thus $f(x_0)$ has its max for $x_0 = 0$. This last is evidently also true for $r = \frac{1}{2}$. Clearly if there is only one point P_3: (0, 0), the maximum of

$$\iint\limits_{C(P_0,r)} (x^2+y^2)^{-\frac{1}{2}}\, dx\,dy$$

is taken on for $P_0 = P_3$, and equals $2\pi r$.

Thus, for $r < \frac{1}{4}$, $h(r) < 4\pi r$. Now let $\frac{1}{4} \leqq r \leqq 1$, and let P_0: $(\bar{x}_0, 0)$. Then the line $x = 0$ divides $C(P_0, r)$ into two parts of measure πr_1^2 and πr_2^2 respectively, which are at a distance $\geqq \frac{1}{2}$ from P_2 and P_1 respectively. Since it is obvious that

$$\iint\limits_{E} \frac{dx\,dy}{(x^2+y^2)^{1/2}}$$

is a maximum among all E with m(E) constant if E is a circle with center at (0, 0), we see that

$$h(r) \leq 4\pi r\sqrt{2}, \quad \frac{1}{4} \leq r \leq 1.$$

If $r > 1$, then $\bar{x}_0 = 0$ and $(r_1 r_2)^{-1} \leq \frac{4}{3}(x^2 + y^2)^{-1}$

$$h(r) \doteq h(1) + \frac{4}{3} \int_1^r \int_0^{2\pi} \rho^{-2}\, d\rho\, d\vartheta \leq 4\pi\sqrt{2} + \frac{8\pi}{3} \log r .$$

From this, the lemma follows.

<u>Theorem 5:</u> <u>Let Φ (e) be a completely additive set function, defined on a bounded region G, satisfying</u>

(28) $$V_\Phi[C(P,r)\cdot G] \leq M r^\lambda, \quad 0 < \lambda < 1$$

for every circle $C(P,r)$, $V_\Phi(E)$ denoting the variation of Φ over E. Let V be the potential of Φ. Then V is of class D_2 everywhere, and satisfies conditions $A[\lambda, M(a)]$ and $B[\lambda/2, N(a)]$ where

(29) $$M(a) = \frac{M\cdot V_\Phi(G)}{\pi}\left(\sqrt{2} + \frac{2}{3\lambda} + \frac{\sqrt{2}}{1-\lambda}\right)a^\lambda, \quad N(a) = \frac{8a^{\lambda/2}}{\lambda(3\pi)^{1/2}}\left[M V_\Phi(G)\right]^{\frac{1}{2}}\left(\sqrt{2} + \frac{2}{3\lambda} + \frac{\sqrt{2}}{1-\lambda}\right)^{\frac{1}{2}}$$

Moreover if $h_t(x,y;\xi,\eta)$ and $V_t(x,y)$ are defined as in the proof of Theorem 1, then

(30) $$|V_t(x,y) - V(x,y)| \leq \frac{M}{2\pi\lambda} t^\lambda$$

On G we have (31) : $|V(x,y)| \leq \frac{1}{2\pi}\left[M\delta^\lambda(\log \frac{1}{\delta} + \lambda^{-1})\right]$

(31) where δ is the diameter of G. Moreover if G is of class C_2 and ζ is any function of class D_2 on G and zero on G^*, then

(32) $$\iint_G \bar{\zeta}\, d\Phi = -\iint_G (\zeta_x V_x + \zeta_y V_y)\, dx\, dy,$$

where the integral on the left exists as a Lebesgue Stieltjes integral.

Proof: We define (as before V_Φ (E) is the variation of ϕ over E)

$$H(s,t;r) = V_\Phi[C(s,t;r)\cdot G], \quad h_r(\xi,\eta;s,t) = \begin{cases} 4\pi\sqrt{2} + \frac{8\pi}{3}\log\frac{r}{d} & \text{if } d \leq r \\ 4\pi\sqrt{2}\left(\frac{r}{d}\right) & \text{if } d \geq r . \\ d = [(\xi-s)^2 + (\eta-t)^2]^{\frac{1}{2}} \end{cases}$$

and we see that

$$H(s,t;r) \leq M r^{\lambda} \quad , \quad H(s,t;\infty) = V_{\phi}(\mathfrak{F})$$

$$\iint_{C(P,r)} [(\xi-x)^2+(\eta-y)^2]^{-\frac{1}{2}} [(s-x)^2+(t-y)^2]^{-\frac{1}{2}} \, dx\, dy \leq h_r(\xi,\eta;s,t)$$

for every circle $C(P,r)$. Then it follows that

$$\frac{1}{4\pi^2} \iint_{\mathfrak{F}} \iint_{\mathfrak{F}} \left[\iint_{C(P_0,r)} \left| \frac{(\xi-x)(s-x)+(\eta-y)(t-y)}{[(\xi-x)^2+(\eta-y)^2][(s-x)^2+(t-y)^2]^2} \right| dx\, dy \right] d V_{\phi}(e_{\xi\eta})\, d V_{\phi}(e_{st}) \leq$$

$$\leq \frac{1}{4\pi^2} \iint_{\mathfrak{F}} \left[\iint_{\mathfrak{F}} h_r(\xi,\eta;s,t)\, d V_{\phi}(e_{\xi\eta}) \right] d V_{\phi}(e_{st}) \leq \frac{M\, V_{\phi}(\mathfrak{F}) \sqrt{2}}{\pi} r^{\lambda} +$$

$$+ \frac{2}{3\pi} \iint_{\mathfrak{F}} \left[\int_0^r \log \frac{r}{\rho}\, d H(s,t;\rho) \right] d V_{\phi}(e_{st}) + \frac{r\sqrt{2}}{\pi} \iint_{\mathfrak{F}} \left[\int_{r}^{\infty} \rho^{-1}\, d H(s,t;\rho) \right] d V_{\phi}(e_{st}) \leq$$

$$\leq \frac{M\, V_{\phi}(\mathfrak{F}) \sqrt{2}}{\pi} r^{\lambda} + \frac{2 M\, V_{\phi}(\mathfrak{F})}{3\pi \lambda} r^{\lambda} + \frac{M\, V_{\phi}(\mathfrak{F}) \sqrt{2}}{\pi(1-\lambda)} r^{\lambda} \leq \frac{M\, V_{\phi}(\mathfrak{F})}{\pi} \left[\sqrt{2} + \frac{2}{3\lambda} + \frac{\sqrt{2}}{1-\lambda} \right] r^{\lambda}$$

for every circle $C(P,r)$.

Hence

$$\iint_{C(P,r)} (V_x^2 + V_y^2)\, dx\, dy = \frac{1}{4\pi^2} \iint_{C(P,r)} \left[\iint_{\mathfrak{F}} \iint_{\mathfrak{F}} \frac{(\xi-x)(s-x)+(\eta-y)(t-y)}{[(\xi-x)^2+(\eta-y)^2][(s-x)^2+(t-y)^2]} d\Phi(e_{\xi\eta})\, d\Phi(e_{st}) \right] dx\, dy$$

$$\leq \frac{1}{4\pi^2} \iint_{\mathfrak{F}} \iint_{\mathfrak{F}} \left[\iint_{C(P,r)} \left| \frac{(\xi-x)(s-x)+(\eta-y)(t-y)}{[(\xi-x)^2+(\eta-y)^2][(s-x)^2+(t-y)^2]} \right| dx\, dy \right] d V_{\phi}(e_{\xi\eta})\, d V_{\phi}(e_{st}) \leq \frac{M\, V_{\phi}(\mathfrak{F})}{\pi} \left(\sqrt{2} + \frac{2}{3\lambda} + \frac{\sqrt{2}}{1-\lambda} \right) r^{\lambda}$$

for every circle $C(P,r)$, all the 6-dimensional integrals existing as Lebesgue Stieltjes integrals. Thus (29) follows.

Now

$$|V(x,y)| = \left| \frac{1}{4\pi} \iint_{\mathfrak{F}} \log [(\xi-x)^2+(\eta-y)^2]\, d\Phi(e_{\xi\eta}) \right| \leq \frac{1}{2\pi} \int_0^R \log \frac{1}{r}\, d H(x,y;r) =$$

$$= \frac{1}{2\pi} \left[H(x,y;R) \log \frac{1}{R} + \int_0^R r^{-1} H(x,y;r)\, dr \right] \leq \frac{1}{2\pi} \left[M R^{\lambda} \log \frac{1}{R} + \frac{M}{\lambda} R^{\lambda} \right]$$

for every (x,y) in the plane, R being chosen so large that $C(x,y;R) \geq G$.

Thus (31) follows.

Finally

$$|V_t(x,y) - V(x,y)| = \left| \frac{1}{4\pi} \iint\limits_{\xi \cdot C(x,y;t)} \left\{ \frac{1}{t^2}\left[(\xi-x)^2 + (\eta-y)^2\right] + 2\log t - 1 - \log\left[(\xi-x)^2 + (\eta-y)^2\right] \right\} d\Phi(e_{\xi\eta})\right|$$

$$\leq \frac{1}{2\pi}\int_0^t \log\frac{t}{s}\, dH(x,y;s) = \frac{1}{2\pi}\int_0^t s^{-1} H(x,y;s)\, ds \leq \frac{M}{2\pi\lambda} t^{\lambda},$$

which proves (30).

To prove (32), we observe that the functions

$$B = \frac{(\xi-x)\bar{\xi}_x + (\eta-y)\bar{\xi}_y}{(\xi-x)^2 + (\eta-y)^2}, \qquad \frac{C^{\frac{1}{2}}}{D^{\frac{1}{2}}} = \frac{\left[\bar{\xi}_x^{\,2} + \bar{\xi}_\eta^{\,2}\right]^{\frac{1}{2}}}{\left[(\xi-x)^2 + (\eta-y)^2\right]^{\frac{1}{2}}} \qquad \begin{array}{l} C = \bar{\xi}_x^{\,2} + \bar{\xi}_\eta^{\,2} \\ D = (\xi-x)^2 + (\eta-y)^2 \end{array}$$

are measurable with respect to the completely additive set functions in $(\xi,\eta\,;\,x,y)$-space generated by $m(e_{\xi\eta})$ and $\Phi(e_{xy})$ or $m(e_{\xi\eta})$ and $V_\Phi(e_{xy})$. Now, define

$$Z(x,y;r) = \iint\limits_{\xi \cdot C(x,y;r)} C^{\frac{1}{2}}\, d\xi\, d\eta$$

Then

$$|Z(x,y;r)| \leq \left[\pi D_2(\bar{\xi},\xi)\right]^{\frac{1}{2}} r.$$

Then, for each $r > 0$,

$$\iint\limits_{\xi}\left[\iint\limits_{\xi - C(x,y;r)} |B|\, d\xi\, d\eta\right] dV_\Phi(e_{xy}) \leq \iint\limits_{\xi}\left[\iint\limits_{\xi - C(x,y,r)} \left[\frac{C}{D}\right]^{\frac{1}{2}} d\xi\, d\eta\right] dV_\Phi(e_{xy}) =$$

$$= \iint\limits_{\xi}\left[\int_r^\delta s^{-1}\frac{dZ(x,y;s)}{ds}\, ds\right] dV_\Phi(e_{xy}) = \iint\limits_{\xi}\left[\delta^{-1} Z(x,y;\delta) - r^{-1} Z(x,y;r) + \int_r^\delta s^{-2} Z(x,y;s)\, ds\right] \cdot dV_\Phi(e_{xy})$$

$$\leq \left[\pi D_2(Z,\xi)\right]^{\frac{1}{2}} V_\Phi(\xi) + \int_r^\delta s^{-2} \iint\limits_{\xi}\left[\iint\limits_{C(x,y;s)} C^{\frac{1}{2}}\, d\xi\, d\eta\right] dV_\Phi(e_{xy}) \leq \left[\pi D_2(Z,\xi)\right]^{\frac{1}{2}} V_\Phi(\xi)$$

$$+ \int_r^\delta s^{-2}\left[\pi s^2 V_\Phi(\xi)\right]^{\frac{1}{2}} \left\{\iint\limits_{\xi}\left[\iint\limits_{C(x,y;s)\cdot\xi} C\, d\xi\, d\eta\right] dV_\Phi(e_{xy})\right\}^{\frac{1}{2}} =$$

$$= \left[\pi D_2(Z,\xi)\right]^{\frac{1}{2}} V_\Phi(\xi) + \left[\pi V_\Phi(\xi)\right]^{\frac{1}{2}} \int_r^\delta s^{-1}\left\{\iint\limits_{\xi}\left[\iint\limits_{C(\xi,\eta;s)\xi} dV_\Phi(e_{xy})\right] C\, d\xi\, d\eta\right\}^{\frac{1}{2}} =$$

$$\leq \left[\pi D_2(Z,\xi)\right]^{\frac{1}{2}} V_\Phi(\xi) + \pi M V_\Phi(\xi)\, D_2(\bar{\xi},\xi)^{\frac{1}{2}} \frac{2\delta^{\lambda/2}}{\lambda}.$$

Thus the four-dimensional Lebesgue-Stieltjes integral

$$\iint_{\mathcal{Z}} \iint_{\mathcal{Z}} B \, d\xi \, d\eta \, d\Phi(e_{xy})$$

exists and may be evaluated by repeated double integrals in either order. Hence, by Theorem 1,

$$-\iint_{\mathcal{Z}} (\bar{\xi}_\xi V_\xi + \bar{\xi}_\eta V_\eta)\, d\xi d\eta = -\frac{1}{2\pi} \iint_{\mathcal{Z}} \left[\iint_{\mathcal{Z}} B \, d\xi d\eta \right] d\Phi(e_{xy}) = \iint \bar{\xi} \, d\Phi(e_{xy})$$

since $\bar{\xi}$ exists and is given by the inside bracket whenever the inside bracket exists on a Lebesgue integral.

<u>Theorem 6: Let $\Phi_n(e)$ be completely additive set-functions defined on the bounded region G which tend weakly to the set function $\Phi(e)$ on G and satisfy (28) uniformly. Then $\Phi(e)$ satisfies (28), the corresponding potential functions V_n and V satisfy (29), (30), and (31) uniformly and</u>

$$\lim_{n\to\infty} D_2(V_n - V, \mathcal{Z}) = 0$$

Proof: From Theorem 5, the V_n and V are equicontinuous and

$$|V_{nt}(x,y) - V_n(x,y)| \le \frac{M}{2\pi\lambda} t^\lambda, \quad |V_t(x,y) - V(x,y)| \le \frac{M}{2\pi\lambda} t^\lambda$$

the right sides being independent of n. From the definition and standard theorems on weak convergence, it follows that $V_{nt} \Rightarrow V_t$ on each bounded portion of the plane. Thus $V_n \Rightarrow V$ on each bounded portion of the plane.

Now $D_2(V_n, G)$ and $D_2(V, G)$ are uniformly bounded. Let $U_n = V_n - V$, $\Psi_n(e) = \Phi_n(e\cdot G) - \Phi(e\cdot G)$ and we see that $U_n \Rightarrow 0$ on $\overline{C(0,R)}$ and $\Psi_n(e) \to 0$ weakly on C(0,R) which circle is chosen to include G in its interior. Now let H_n be the function harmonic on C(0,R) and coinciding with U_n on $C^*(0,R)$, and let $W_n = U_n - H_n$. Now for (x,y) on $C^*(0,R)$ so that $\frac{d}{d\vartheta} U_n(R\cos\vartheta, R\sin\vartheta)$ tends uniformly to zero, so that $D_2[H_n, C(0,R)] \to 0$. Also, by Theorem 5,

$$-\iint_{C(0,R)} W_n\, d\Psi_n(e) = \iint_{C(0,R)} (W_{nx}U_{nx} + W_{ny}U_{ny})\,dx\,dy = D_2[W_n, C(0,R)] + \iint_{C(0,R)} (H_{nx}W_{nx} + H_{ny}W_{ny})\,dx\,dy$$
$$= D_2[W_n, C(0,R)]$$

since $W_n = 0$ on $C^*(0,R)$ and H_n is harmonic on $C^*(0,R)$. Thus

$$\lim_{n\to\infty} D_2(U_n, \xi) \le \lim_{n\to\infty} D_2[U_n, C(0,R)] = \lim_{n\to\infty} \{D_2[H_n, C(0,R)] + D_2[W_n, C(0,R)]\} = 0$$

§7. Haar's Lemmas

Definition 1: By "almost all rectangles" in a region G we mean all rectangles $(a,c;\, b,d)$ in G where $(a,c;\, b,d)$ is not in a four-dimensional set of measure zero.

Lemma 1: Let $A(x,y)$ and $B(x,y)$ be of class L_α ($\alpha \geqq 1$) over the simply connected region G and suppose that

$$\int_{R^*} A\,dx + B\,dy = 0$$

for almost all rectangles in G. Then there exists a function v of class D'_α on G such that

$$D_x v = A, \quad D_y v = B$$

almost everywhere on G. The function is uniquely determined up to an additive constant plus a null function.

Proof: Let D: $(\alpha, \gamma;\, \beta, \delta)$ be a cell interior to G. If a, $\alpha < a < \beta$, is not in a set Z_x of measure zero $B(a,y)$ is summable for $\gamma < y < \delta$ and if c is not in a set of measure zero, Z_y, $\gamma < c < \delta$, then

$A(x,c)$ is summable for $\alpha < x < \beta$. Also if (a,c) is not in a set Z of measure zero, the set $Z_1(a,c)$ of points (b,d) such that

$$\int_{R^*} A\,dx + B\,dy \equiv \int_c^d [B(b,y) - B(a,y)]\,dy - \int_a^b [A(x,d) - A(x,c)]\,dx$$

fails to exist and be zero is of measure zero.

Hence, suppose a not in Z_x, c not in Z_y, (a,c) not in Z, and define

$$v_1(x,y) = \int_a^x A(\xi,c)\,d\xi + \int_c^y B(x,\eta)\,d\eta, \qquad x \notin Z_x$$

$$v_2(x,y) = \int_c^y B(a,\eta)\,d\eta + \int_a^x A(\xi,y)\,d\xi, \qquad y \notin Z_y$$

Then it is easy to see that $v_1(x,y)$ is summable and summable in x for each y on D and that

$$\int_a^b [v_1(x,d) - v_1(x,c)]\,dx = \int_a^b \int_c^d B(x,y)\,dx\,dy$$

for every $(a,c;\ b,d)$ and also that $v_2(x,y)$ is summable and summable in y for each x in D such that

$$\int_c^d [v_2(b,y) - v_2(a,y)]\,dy = \int_a^b \int_c^d A(x,y)\,dx\,dy$$

for every $(a,c;\ b,d)$ on D. Moreover v_1 and v_2 coincide except possibly on a set of measure zero so that either one satisfies the condition of the theorem.

Now if v_1 and v_2 are any two functions of class D_1' satisfying the above conditions, then $V = v_1 - v_2$ os si,,ab;e pm D and has the property that

$$\int_a^b [V(x,d) - V(x,c)]\,dx = 0, \qquad \int_c^d [V(b,y) - V(a,y)]\,dy = 0$$

the first for c and d not in Z_y and any (a,b), and the second for a and b not in Z_x and any (c,d). Thus our theorem follows for the cell D. It is clear how to extend the function to the whole of G.

Lemma 2: Let f(x) be summable on (a,b) and suppose that

(1) $$\int_a^b f(x)\cdot\varphi(x)\,dx = 0$$

for every bounded measurable function $\varphi(x)$ for which

(2) $$\int_a^b \varphi(x)\,dx = 0$$

Then f is constant, except possibly for a set of measure zero. If (1) holds for every bounded measurable $\varphi(x)$, then $f(x) = 0$ except possibly for a set of measure zero.

Proof: The second statement is obvious, for we may take $\varphi(x) = 0$ wherever $f(x) = 0$ and

$$\varphi(x) = \frac{f(x)}{|f(x)|}$$

wherever $f(x) \neq 0$.

Now any bounded measurable function $\varphi(x)$ may be written as

$$\varphi(x) = \varphi^*(x) + \frac{1}{b-a}\int_a^b \varphi(x)\,dx$$

and

$$\int_a^b \varphi^*(x)\,dx = 0$$

Thus

$$\int_a^b f(x)\left[\varphi(x) - \frac{1}{b-a}\int_a^b \varphi(x)\,dx\right]dx = 0 = \int_a^b \left\{f(x) - \frac{1}{b-a}\int_a^b f(y)\,dy\right\}\varphi(x)\,dx$$

for every bounded measurable φ so that

$$f(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

almost everywhere.

Lemma 3: Let u(x,y) and v(x,y) be of class D_p' and D_q' respectively on a cell R: (a,c; b,d) with $p^{-1} + q^{-1} = 1$; we include the case $p = 1$, $q = \infty$

(or $p = \infty$, $q = 1$) by interpreting this to mean that v (or u in the second case) satisfies a uniform Lipschitz condition on R. Suppose that v is absolutely continuous with $|\bar{v}_s|^q$ summable along R^*, s denoting arc length. Then

$$\iint_R (\bar{u}_x \bar{v}_y - \bar{u}_y \bar{v}_x)\, dx\, dy = \int_{R^*} \bar{u}\, \bar{v}_s\, ds$$

Proof: First, we may extend the definitions of u and v so that they are of class D'_p and D'_q respectively over the whole plane. Now let u_h and v_k be the usual average functions. Then

$$(3) \qquad \iint_R (u_{hx} v_{ky} - u_{hy} v_{kx})\, dx\, dy = \int_{R^*} u_h\, dv_k = -\int_{R^*} v_k\, du_h = -\int_{R^*} v_k u_{hs}\, ds$$

Now we know that

$$\lim_{h \to 0} \iint_R (|u_h - u|^p + |u_{hx} - \bar{u}_x|^p + |u_{hy} - \bar{u}_y|^p)\, dx\, dy = \lim_{k \to 0} \iint_R (|v_k - v|^q + |v_{kx} - \bar{v}_x|^q + |v_{ky} - \bar{v}_y|^q)\, dx\, dy =$$

Also there exist subsequences h_n and k_n tending to zero so that

$$\lim_{n \to \infty} \int_a^b |v_{k_n}(x, y_0) - \bar{v}(x, y_0)|^q\, dx = \lim_{n \to \infty} \int_c^d |v_{k_n}(x_0, y) - \bar{v}(x_0, y)|^q\, dy = 0$$

$$\lim_{n \to \infty} \int_a^b |u_{h_n}(x, y_0) - \bar{u}(x, y_0)|^p\, dx = \lim_{n \to \infty} \int_c^d |u_{h_n}(x_0, y) - \bar{u}(x_0, y)|^p\, dy = 0$$

uniformly in (x_0, y_0). Thus in (3) we may choose $\{k_n\}$ and let $n \to \infty$ obtaining

$$\iint_R (u_{hx} \bar{v}_y - u_{hy} \bar{v}_x)\, dx\, dy = -\int_{R^*} \bar{v}\, u_{hs}\, ds = \int_{R^*} u_h\, \bar{v}_s\, ds.$$

In this we may take $h = h_n$ and let $n \to \infty$ and we obtain our result.

Theorem 1: Let G be a bounded region and let A and B be functions of class L_α on G ($\alpha \geq 1$). Suppose that

$$(4) \qquad \iint_G (A \zeta_x + B \zeta_y)\, dx\, dy = 0$$

for every ζ satisfying a uniform Lipschitz condition over G and vanishing on G^*. Then

$$(5) \qquad \int_{R^*} A\, dy - B\, dx = 0$$

for almost every rectangle R in G. If G is simply connected, there exists a

<u>function v of class D'_α on G such that</u>

(6) $\quad \bar{v}_y = D_y v = A, \quad \bar{v}_x = D_x v = -B$

<u>almost everywhere. If (3) is satisfied for every ζ which satisfies a uniform Lipschitz condition on G and if G is also of class L, v may be chosen to vanish on G^*.</u>

Proof: Choose (x_0, y_0) in G and choose $\alpha > 0$ and $h > 0$ so that the rectangle $(x_0-\alpha, y_0-h\alpha; x_0+\alpha, y_0+h\alpha)$ is interior to G. In this rectangle we choose coordinates (r,s) as follows: Given (x_1, y_1), r_1 is the value of r for which the rectangle R_r: $(x_0-r, y_0-hr; x_0+r, y_0+hr)$ contains (x_1, y_1) on its boundary, and s_1 is the arc length along R_r^* measured in the positive direction from the point (x_0+r, y_0-hr). Thus our rectangle contains all points (r,s) for which $0 \leqq r \leqq \alpha$, $0 \leqq s \leqq 4r(1+h)$. Now, if we let ζ be a function of r alone defined and satisfying a uniform Lipschitz condition for all r and vanishing for $r \geqq \alpha$ we see that ζ satisfies a uniform Lipschitz condition in (x,y) on $\bar{G}$ and vanishes on G^*. Our integral (4) takes the form

$$\iint_G (A\zeta_x + B\zeta_y)\,dx\,dy = \int_0^\alpha \zeta_r \varphi(r)\,dr, \quad \varphi(r) = \int_{R_r^*} A\,dy - B\,dx$$

and φ (r) is certainly summable. Since $\zeta(0)$ may have any value, ζ_r is perfectly arbitrary, and it follows from Lemma 1 that φ (r) = 0 for almost all r, $0 \leqq r \leqq \alpha$.

Now if we determine a rectangle R in G as above by the numbers (x_0, y_0, r, h) we see first that

$$\psi(x_0, y_0, r, h) = \int_{R^*} A\,dy - B\,dx$$

is measurable in (x_0, y_0, r, h) and hence also in (a,c; b,d) since we have

$$a = x_0 - r, \quad b = x_0 + r, \quad c = y_0 - hr, \quad d = y_0 + hr, \quad x_0 = \frac{a+b}{2}, \quad y_0 = \frac{c+d}{2}, \quad r = \frac{b-a}{2}, \quad h = \frac{d-c}{b-a}$$

Also, for each fixed (x_0, y_0, h), we have seen above that the set of values of r where Ψ is not defined or is different from zero is of measure zero. Hence except for a four-dimensional set of (a,c; b,d) of measure zero, we see that (5) is defined and equal to zero. If G is simply connected, the existence of the function v satisfying (6) follows from Lemma 1.

To prove the last statement, let G be also of class L, let v be a function of class D'_α on G satisfying (6), and suppose that (4) is zero for every ζ satisfying a uniform Lipschitz condition on $\bar{G}$. Let $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ be a regular representation of class L of a portion of $\bar{G}$ on a cell R: $a \leqq \xi \leqq b$, $0 \leqq \eta \leqq d$ such that the points $\eta = 0$ correspond to points of G^*. Let $w(\xi, \eta)$ be the transform of v and let $\varphi(\xi, \eta)$ be that of a given Now, let $\varphi(\xi, \eta)$ be any function satisfying a uniform Lipschitz condition on R and zero on the whole boundary except on the points $\eta = 0$, $a < \xi < b$, where it may be arbitrary. Evidently the function $\zeta(x,y)$ defined as the transform of φ on the transform of $\bar{R}$ and zero elsewhere satisfies a uniform Lipschitz condition on G. Our integral (4) now takes the form

$$\iint_G (A\zeta_x + B\zeta_y)\,dx\,dy = \iint_G (\zeta_x \bar{v}_y - \zeta_y \bar{v}_x)\,dx\,dy = \iint_R (\varphi_\xi \bar{w}_\eta - \varphi_\eta \bar{w}_\xi)\,d\xi\,d\eta = -\int_{R^*} \bar{w}\,\varphi_s\,ds = -\int_a^b \bar{w}(\xi,0)\,\varphi_\xi(\xi,0)\,d\xi = 0.$$

for every bounded measurable function $\varphi_\xi(\xi, 0)$ such that

$$\int_a^b \varphi_\xi(\xi,0)\,d\xi = \varphi(b,0) - \varphi(a,0) = 0$$

Thus $\bar{w}(\xi, 0)$ is a constant almost everywhere. It follows that v takes on constant boundary values on G^*.

<u>Theorem 2:</u> <u>Let G be a bounded region, let $\Phi(e)$ be a completely additive set function on G, and let A and B be summable on G. Suppose that, for each ζ which satisfies a uniform Lipschitz condition on $\bar{G}$ and vanishes on G^*, we have</u>

(7) $$\iint_{\mathcal{S}} (A\zeta_x + B\zeta_y)\,dx\,dy + \iint_{\mathcal{S}} \zeta\,d\phi(e) = 0$$

Then, for almost all rectangles R in G, we have

(8) $$\int_{R^*} A\,dy - B\,dx = \phi(R)$$

Proof: Let V be the potential of ϕ(e). Then

$$\iint_{\mathcal{S}} \zeta\,d\phi = -\iint_{\mathcal{S}} (V_x\zeta_x + V_y\zeta_y)\,dx\,dy$$

for every ζ satisfying a uniform Lipschitz condition on G and vanishing on G^*. Thus for such ζ

$$\iint_{\mathcal{S}} [(A - V_x)\zeta_x + (B - V_y)\zeta_y]\,dx\,dy = 0$$

Thus, for almost all rectangles R, we have

$$\int_{R^*} A\,dy - B\,dx = \int_{R^*} V_x\,dy - V_y\,dx = \phi(R)$$

Theorem 3: Let G be a region of class C_2, let A and B be of class L_2 on G, let ϕ (e) be completely additive on G, and suppose that ϕ satisfies the condition

$$V_\phi[C(P,r)\cdot\mathcal{S}] \le M r^\lambda, \quad \lambda > 0$$

where V_ϕ (E) denotes the variation of ϕ over E. Suppose that (8) holds for almost every R in G. Then (7) holds for every ζ of class D_2' on G which vanishes on G^*.

Proof: By Lemma 3, §1, we can find a sequence $\{\zeta_n\}$ of class C"' on $\bar{G}$, zero near G^* and such that $\bar{D}_2(\zeta_n - \zeta, G) \to 0$. Let V be the potential of ϕ (e). Then V is of class D_2 everywhere and

$$\int_{R^*} (A - V_x)\,dy - (B - V_y)\,dx = 0$$

for almost every R in G. Thus if ζ is of class D_2' on G with $D_2(\zeta, G)$ finite,

(9) $$\iint_{\mathcal{S}} [(A - V_x)\zeta_x + (B - V_y)\zeta_y]\,dx\,dy$$

exists as a Lebesgue integral. Hence if a sequence $\zeta_n \to \zeta$ as above, then (9) will be the limit of the corresponding integrals for ζ_n. Moreover, by Theorem 5, §6, we see that

$$(10) \quad \iint_{\mathcal{G}} [(A - V_x)\zeta_x + (B - V_y)\zeta_y]\, dx\,dy = \iint_{\mathcal{G}} (A\zeta_x + B\zeta_y)\, dx\,dy + \iint_{\mathcal{G}} \zeta \, d\Phi(\mathcal{E})$$

Now we can find a region G_n of class C' for each n with $\bar{G}_n \subset G$ such that $\zeta_n = 0$ on G_n^* and

$$(11) \quad \iint_{\mathcal{G}} [(A - V_x)\zeta_{nx} + (B - V_y)\zeta_{ny}]\, dx\,dy = \iint_{\mathcal{G}_n} [(A - V_x)\zeta_{nx} + (B - V_y)\zeta_{ny}]\, dx\,dy$$

Now if Σ is any simply connected subregion of G, there exists a function v of class D_2' in Σ such that

$$\bar{v}_x = -(B - V_y), \quad \bar{v}_y = A - V_x$$

almost everywhere in Σ. Hence if we define A_{h2} and B_{h2} as the iterated average functions for $(A - V_x)$ and $(B - V_y)$, we see that they are of class C' on $\bar{G}_n$ (if h is small enough) with $A_{h^2x} + B_{h^2y} = 0$. Thus

$$\iint_{\mathcal{G}_n} (A_{h^2}\zeta_{nx} + B_{h^2}\zeta_{ny})\, dx\,dy = -\iint_{\mathcal{G}_n} \zeta_n (A_{h^2x} + B_{h^2y})\, dx\,dy = 0$$

By letting $h \to 0$ it follows that (11) is zero for ζ_n and hence (10) is zero or (25) holds.

§8. Differentiability properties of functions minimizing a double integral

In this section we consider the functions z which minimize an integral

$$I(z, \mathcal{G}) = \iint_{\mathcal{G}} f(x, y, z^1, \dots, z^N, p^1, \dots, p^N, q^1, \dots, q^N)\, dx\,dy$$

among all functions of class D_2' which take on given boundary values on G^*, G^* being assumed to be of class C_2. In case the given boundary values are continuous and there exist numbers m and M with $0 < m \leqq M < \infty$ which are independent of (x,y,p,q) such that

$$m \sum_{i=1}^{N} (p^{i2} + q^{i2}) \leq f \leq M \sum_{i=1}^{N} (p_i^2 + q_i^2) \tag{1}$$

and if z^* is of class D_2' with $D_2(z,G)$ finite and takes on these boundary values, we have seen that a function z exists which minimizes $I(z,G)$ among all functions of class D_2' on G and coinciding with z^* on G^*, and that z is continuous on G and satisfies conditions $A[\lambda, \bar{M}]$ and $B[\lambda/2, \bar{N}]$ on G, where $\lambda = m/M$, $\bar{M} = D_2(z,G)$. The latter conditions are satisfied in G whether the boundary values are continuous or not. It is assumed also that f is continuous in its arguments and convex in (p,q) for each fixed (x,y,z).

For the purposes of this paper we shall assume in addition to the above that f is of class C' everywhere and of class C", except possibly where $p^i = q^i = 0$, $i = 1, \ldots, N$, where the second derivatives need not be continuous, and that there exist functions $m(R)$ and $M(R)$ defined and positive for all values of R such that

$$\begin{gathered} m(R) \sum_{i=1}^{N} (\xi^{i2} + \eta^{i2}) \leq f_{p^\alpha p^\beta} \xi^\alpha \eta^\beta + f_{p^\alpha q^\beta} \xi^\alpha \eta^\beta + f_{q^\alpha p^\beta} \eta^\alpha \xi^\beta + f_{q^\alpha q^\beta} \eta^\alpha \eta^\beta \leq M(R) \sum_{i=1}^{N} (\xi^{i2} + \eta^{i2}) \\ \sum_{i=1}^{N} \left(f_{p^i x}^2 + f_{p^i y}^2 + f_{q^i x}^2 + f_{q^i y}^2 + |f_{z^i x}| + |f_{z^i y}| \right) + \sum_{i=1}^{N} \sum_{j=1}^{N} \left(f_{p^i z^j}^2 + f_{q^i z^j}^2 + |f_{z^i z^j}| \right) \leq M(R) \sum_{i=1}^{N} (\xi_i^2 + \eta_i^2) \end{gathered} \tag{2}$$

for all ξ^i, η^i, p^i, q^i and for all x, y, z^i with

$$x^2 + y^2 + \sum_{i=1}^{N} z^{i2} \leq R^2.$$

Clearly any function f which is of class C" in all its arguments except possibly where $p^i = q^i = 0$, which is homogeneous of degree 2 in (p,q) for each fixed (x,y,z) and regular (i.e. with

$$f_{p^\alpha p^\beta} \xi^\alpha \xi^\beta + f_{p^\alpha q^\beta} \xi^\alpha \eta^\beta + f_{q^\alpha p^\beta} \eta^\alpha \xi^\beta + f_{q^\alpha q^\beta} \eta^\alpha \eta^\beta$$

a positive definite quadratic form in ξ^i, η^i for each (x,y,z,p,q) except possibly where $p^i = q^i = 0$) satisfies (2) automatically and hence satisfies all of our conditions if it satisfies (1).

Lemma 1: Let $f(x^1, \ldots, x^n)$ be continuous for all values of x and of class C' except possibly where $x^{k+1} = \ldots = x^n = 0$, in the neighborhood of which points the first derivatives of f remain bounded provided $(x^1, \ldots, x^k)$ are in a bounded region of k-space. Then if $x^i + \xi^i$ and x^i, $i = k+1, \ldots, n$, are not simultaneously zero, we have

$$f(x^1+\xi^1, \ldots, x^n+\xi^n) = f(x^1, \ldots, x^n) + A_\alpha(x,\xi)\cdot\xi^\alpha$$

where

$$A_i(x,\xi) = \int_0^1 f_{x^i}(x^1+t\xi^1, \ldots, x^n+t\xi^n)\,dt, \quad i = 1, \ldots, N$$

the function of t under the integral sign being continuous except for at most one value of t in the interval and bounded throughout. Moreover $A_i(x,\xi)$ is continuous except possibly at a point (x,ξ) such that $x^i + \xi^i \# = x^i = 0$, $i = k+1, \ldots, n$.

Proof: Let $\varphi(t)$ satisfy a uniform Lipschitz condition on (0,1) and suppose $\varphi'(t)$ is continuous except possibly at one point in this interval. Then

$$\varphi(1) = \varphi(0) + \int_0^1 \varphi'(t)\,dt$$

Hence, let

$$\varphi(t) = f(x^1+t\xi^1, \ldots, x^n+t\xi^n)$$

and we see that the above conditions on $\varphi(t)$ are fulfilled with

$$\varphi'(t) = \xi^\alpha f_{x^\alpha}(x+t\xi)$$

The stated continuity properties of the $A_i(x,\xi)$ are evident from its form.

Lemma 2: Let $f(x^1, \ldots, x^n)$ be of class C' everywhere and suppose that each of its first partial derivatives satisfies the conditions of the f of Lemma 1. Then, if $x^i + \xi^i$ and x^i, $i = k+1, \ldots, n$, are not simultaneously zero, then

$$f(x^1+\xi^1,\cdots,x^n+\xi^n) = f(x^1,\cdots,x^n) + \xi^\alpha f_{x^\alpha}(x^1,\cdots,x^n) + A_{\alpha\beta}(x,\xi)\,\xi^\alpha\xi^\beta$$

where

$$A_{ij}(x,\xi) = \int_0^1 (1-t)\, f_{x^i x^j}(x^1+t\xi^1,\cdots,x^n+t\xi^n)\,dt$$

the function under the integral sign being bounded in t throughout and continuous except possibly for one value of t. The functions $A_{ij}(x, \xi)$ are continuous except possibly for points (x, ξ) for which

$$x^i+\xi^i = x^i = 0\,, \quad i = k+1,\cdots,n.$$

Proof: Let $\varphi(t)$ be of class C' on $(0, 1)$ with $\varphi'(t)$ satisfying a uniform Lipschitz condition and $\varphi''(t)$ continuous except possibly at one point. Then we verify immediately that

$$\varphi(s) = \varphi(0) + s\varphi'(0) + s^2\int_0^1 (1-t)\,\varphi''(st)\,dt = \varphi(0) + s\varphi'(0) + \int_0^s (s-\tau)\,\varphi''(\tau)\,d\tau$$

for

$$\varphi'(s) = \varphi'(0) + \int_0^s \varphi''(\tau)\,d\tau\,, \quad \varphi(s) = \varphi(0) + \int_0^s \varphi'(\tau)\,d\tau$$

Thus, if we let

$$\varphi(t) = f(x^1+t\xi^1,\cdots,x^n+t\xi^n)$$

we see that φ satisfies the above conditions with

$$\varphi'(t) = \xi^\alpha f_{x^\alpha}(x+t\xi)\,, \quad \varphi''(t) = \xi^\alpha\xi^\beta f_{x^\alpha x^\beta}(x+t\xi).$$

The stated continuity properties of the A_{ij} are evident from their forms.

Theorem 1: Let z be of class D_2' on a region G of class C_2 with $D_2(z,G)$ finite, and suppose that it minimizes $I(z,G)$ among all such functions which take on the same boundary values. Then z satisfies Haar's Equations

(3) $$\int_{R^*} f_{p^i}\,dy - f_{q^i}\,dx = \iint_R f_{z^i}\,dx\,dy\,, \quad i = 1,\cdots,N,$$

on almost all rectangles R in G.

Proof: Let ζ satisfy a uniform Lipschitz condition on $\bar{G}$, vanishing on G^*. Then

$$\varphi(\lambda) = I(z+\lambda\zeta, G) = \iint_G f(x,y,z+\lambda\zeta,\ p+\lambda\pi,\ q+\lambda\kappa)\,dx\,dy =$$

$$= \iint_G \Big[f(x,y,z,p,q) + \lambda\big\{\zeta^\alpha f_{z^\alpha}(x,y,z,p,q) + \pi^\alpha f_{p^\alpha}(x,y,z,p,q) + \kappa^\alpha f_{q^\alpha}(x,y,z,p,q)\big\} +$$

$$+ \lambda^2\big\{A_{\alpha\beta}(x,y;\lambda)\pi^\alpha\pi^\beta + B^1_{\alpha\beta}(x,y;\lambda)\pi^\alpha\kappa^\beta + B^2_{\alpha\beta}(x,y)\kappa^\alpha\pi^\beta + \cancel{A}C_{\alpha\beta}\kappa^\alpha\kappa^\beta + D^1_{\alpha\beta}\pi^\alpha\zeta^\beta +$$

$$+ D^\alpha_{\alpha\beta}\zeta^\alpha\pi^\beta + E^1_{\alpha\beta}\kappa^\alpha\zeta^\beta + E^2_{\alpha\beta}\zeta^\alpha\kappa^\beta + F_{\alpha\beta}(x,y;\lambda)\zeta^\alpha\zeta^\beta\big\}\Big]\,dx\,dy$$

where we define

$$A_{\alpha\beta} = \int_0^1 (1-t) f_{p^\alpha p^\beta}[x,y,z^i(x,y)+t\lambda\zeta^i(x,y),\ p^i(x,y)+t\lambda\pi^i(x,y),\ q^i(x,y)+t\lambda\kappa^i(x,y)]\,dt$$

$$B^1_{\alpha\beta} = \int_0^1 (1-t) f_{p^\alpha q^\beta}[\ ]\,dt,\quad C_{\alpha\beta} = \int_0^1 (1-t) f_{q^\alpha q^\beta}\,dt,\quad D^1_{\alpha\beta} = \int_0^1 f_{p^\alpha z^\beta}\,dt,\quad E^1_{\alpha\beta} = \int_0^1 (1-t) f_{q^\alpha z^\beta}\,dt$$

$$F_{\alpha\beta} = \int_0^1 (1-t) f_{z^\alpha z^\beta}\,dt$$

at those points (x,y) where we do not have $p^i(x,y) = q^i(x,y) = \pi^i(x,y) = \kappa^i(x,y) = 0$ or where these functions are not defined [$z^i(x,y)$ and $\zeta^i(x,y)$ are continuous] all such points constituting a measurable set; at these points, we define $A_{ij} = C_{ij} = \delta_{ij}$ and $B^1_{ij} = D^1_{ij} = E^1_{ij} = F_{ij} = 0$. We then define

$$B^2_{ij}(x,y) = B^1_{ji}(x,y),\quad D^2_{ij}(x,y) = D^1_{ji}(x,y),\quad E^2_{ij}(x,y) = E^1_{ji}(x,y)$$

throughout G. Clearly all these functions as well as f_{p^i}, f_{q^i}, and f_{z^i} are measurable functions of (x,y,λ), and since p and q are of class L_2 f_{z^i} is summable and f_{p^i} and f_{q^i} are each of class L_2 in (x,y). Also A_{ij}, B^1_{ij}, B_{ij}, and C_{ij} are uniformly bounded, and

$$\iint_G \sum_{i=1}^N \sum_{j=1}^N \left[(D^1_{ij})^2 + (D^2_{ij})^2 + (E^1_{ij})^2 + (E^2_{ij})^2 + |F_{ij}| \right] dx\,dy$$

is uniformly bounded if $|\lambda| \leq 1$. Thus

$$\left| I(z+\lambda\zeta, G) - I(z,G) - \lambda \iint_G (\zeta^\alpha f_{z^\alpha} + \pi^\alpha f_{p^\alpha} + \kappa^\alpha f_{q^\alpha})\,dx\,dy \right| < K\lambda^2$$

for all λ with $|\lambda| \leqq 1$. Thus we see that $\varphi'(0)$ exists and

$$\varphi'(0) = \iint_{G} (\zeta^{\alpha} f_{z\alpha} + \pi^{\alpha} f_{p\alpha} + \kappa^{\alpha} f_{q\alpha})\, dx\, dy$$

But $\varphi(\lambda)$ has a minimum for $\lambda = 0$ so that $\varphi'(0)$ must be zero for every ζ which we have admitted. From this, the theorem follows, using Haar's second lemma.

Now let H be a region of class C_2 with $\bar{H} \subset G$. Then if h is a sufficiently small positive number, we see from (3) that

$$(3) \quad \int_{R^*} f_{p^i}[x, y, z^i(x,y), p^i(x,y), q^i(x,y)]\,dy - f_{q^i}[x, y, z^i, p^i, q^i]\,dx = \iint_R f_{z^i}[x, y, z^i, p^i, q^i]\,dx\,dy$$

$$(4) \quad \int_{R^*} f_{p^i}[x+h, y, z^i(x+h,y), p^i(x+h,y), q^i(x+h,y)]\,dy - f_{q^i}(x+h, y, z^i(x+h,y), p^i(x+h,y), q^i(x+h,y)]\,dx$$
$$= \iint_R f_{z^i}[x+h, y, z^i(x+h), p^i(x+h,y), q^i(x+h,y]\,dx\,dy$$

$$(5) \quad \int_{R^*} f_{p^i}(x, y+h, z^i(x,y+h), p^i(x,y+h), q^i(x,y+h)]\,dy - f_{q^i}[x, y+h, z^i(x,y+h), p^i(x,y+h), q^i(x,y+h)]\,dx$$
$$\iint_R f_{z^i}[x, y+h, z^i(x,y+h), p^i(x,y+h), q^i(x,y+h)]\,dx\,dy$$

hold simultaneously for almost all rectangles R in H, (4) and (5) being derived from (3) merely by translation.

Next, define

$$p_h^i(x,y) = \frac{z^i(x+h,y) - z^i(x,y)}{h}, \quad q_h^i(x,y) = \frac{z^i(x,y+h) - z^i(x,y)}{h}, \quad i = 1, \cdots, N$$

Then, if h is sufficiently small but positive, we see that p_h^i and q_h^i are continuous on $\bar{H}$ and of class D_2 on H with

$$(6) \quad \begin{aligned} p_{hx}^i &= \frac{p^i(x+h,y) - p^i(x,y)}{h}, & p_{hy}^i &= \frac{q^i(x+h,y) - q^i(x,y)}{h} \\ q_{hx}^i &= \frac{p^i(x,y+h) - p^i(x,y)}{h}, & q_{hy}^i &= \frac{q^i(x,y+h) - q^i(x,y)}{h} \end{aligned}$$

almost everywhere. If we now subtract (3) from (4) and (5) in turn, dividing by h each time, we see, using Lemma 1 and (6), that p_h^i and q_h^i satisfy (7) and (8) below, respectively:

$$(7) \quad \int_{R^*} (a_{i\beta}^h p_{hx}^\beta + b_{i\beta}^{1(h)} p_{hy}^\beta + d_{i\beta}^{1(h)} p_h^\beta + g_i^{(h)})\,dy - (b_{i\beta}^{2(h)} p_h^\beta + g_{i\beta} p_y^\beta + e_{i\beta}^{1(h)} p^\beta + k_i^{(h)})\,dx =$$
$$= \iint_R (d_{i\beta}^{2(h)} p_{hx}^\beta + e_{i\beta}^{2(h)} p_{hy}^\beta + f_{i\beta}^h p_h^\beta + \ell_i^{(h)})\,dx\,dy$$

$$\int_{R^*}\left(\bar{a}^{h}_{i\beta}q^{\beta}_{hx}+\bar{b}^{1(h)}_{i\beta}q^{\beta}_{hy}+\bar{d}^{1(h)}_{i\beta}q^{\beta}_{h}+\bar{g}^{(h)}_{i}\right)dy-\left(\bar{b}^{2(h)}_{i\beta}q^{\beta}_{x}+\bar{c}^{(h)}_{i\beta}q^{\beta}_{y}+\bar{e}^{1(h)}_{i\beta}q^{\beta}+\bar{k}^{(h)}_{i}\right)dx=$$

(8) $$=\iint_R\left(\bar{d}^{2(h)}_{i\beta}q^{\beta}_{hx}+\bar{e}^{2(h)}_{i\beta}q^{\beta}_{hy}+\bar{f}^{h}_{i\beta}q^{\beta}_{h}+\bar{l}^{(h)}_{i}\right)dx\,dy$$

on almost all rectangles in H (if h is small enough), where we may define

$$a^{(h)}_{ij}(x,y)=\int_0^1 f_{p^ip^j}[x+th,y,z^k(x,y)+t\{z^k(x+h,y)-z^k(x,y)\},p^k(x,y)+$$
$$+t\{p^k(x+h,y)-p^k(x,y)\},q^k(x,y)+t\{q^k(x+h,y)-q^k(x,y)\}]\,dt$$

(9) $$b^{1(h)}_{ij}=\int_0^1 f_{p^iq^j}\,dt,\quad c^{(h)}_{ij}=\int_0^1 f_{q^iq^j}\,dt,\quad d^{1(h)}_{ij}=\int_0^1 f_{p^iz^j}\,dt,\quad e^{1(h)}_{ij}=\int_0^1 f_{q^iz^j}\,dt,$$
$$f^{(h)}_{ij}=\int_0^1 f_{z^iz^j}\,dt,\quad g^{(h)}_{i}=\int_0^1 f_{p^ix}\,dt,\quad k^{(h)}_{i}=\int_0^1 f_{q^ix}\,dt,\quad l^{(h)}_{i}=\int_0^1 f_{z^ix}\,dt$$

at all points not of the measurable set S_h of H where we have $p^i(x+h,y)=p^i(x,y)$ $=q^i(x+h,y)=q^i(x,y)=0$, $i=1,\dots,N$, or at least one of these is not defined; at points of S_h we may define

$$a^{(h)}_{ij}=c^{(h)}_{ij}=\frac{M+m}{2}\,\delta_{ij},\quad b^{1(h)}_{ij}=d^{1(h)}_{ij}=e^{1(h)}_{ij}=f^{(h)}_{ij}=g^{(h)}_{i}=k^{(h)}_{i}=l^{(h)}_{i}=0$$

We then define

$$b^{2(h)}_{ij}=b^{1(h)}_{ji},\quad d^{2(h)}_{ij}=d^{1(h)}_{ji},\quad e^{2(h)}_{ij}=e^{1(h)}_{ji}$$

all over H. Clearly the $\bar{a}^{(h)}_{ij}$, etc., may be defined analogously.

It is clear, first, that all of these functions are measurable. Next, let $\xi^1,\dots,\xi^N,\eta^1,\dots,\eta^N$ be any values, and let (x,y) be any point of $H-S_h$. Then $p^i(x+h,y)$, $p^i(x,y)$, $q^i(x+h,y)$, and $q^i(x,y)$ are all defined but not all zero. Then we see that

$$m\sum_{i=1}^N(\xi^{i2}+\eta^{i2})\le\int_0^1\left(f_{p^\alpha p^\beta}\xi^\alpha\xi^\beta+f_{p^\alpha q^\beta}\xi^\alpha\eta^\beta+f_{q^\alpha p^\beta}\eta^\alpha\xi^\beta+f_{q^\alpha q^\beta}\eta^\alpha\eta^\beta\right)dt$$

(10) $$=a^{(h)}_{\alpha\beta}\xi^\alpha\xi^\beta+b^{1(h)}_{\alpha\beta}\xi^\alpha\eta^\beta+b^{2(h)}_{\alpha\beta}\eta^\alpha\xi^\beta+c^{(h)}_{\alpha\beta}\eta^\alpha\eta^\beta\le M\sum_{i=1}^N(\xi^{i2}+\eta^{i2})$$

the argument in the $f_{p^\alpha p^\beta}$ etc., in the integral being that occuring in (9). If (x,y) is in S_h, it is obvious that (10) holds. Similarly (10) holds also for the $\bar{a}^h_{ij}$, etc. We can choose $\bar{m}$ and $\bar{M}$, independently of h and everything else occurring in (10) as z is continuous and hence bounded in a regino D containing $\bar{H}$, and, of course, x and y are in the bounded region G. For the same reason, it is easy to see that we may find an $\bar{N}$, independent of x,y, h such that

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\left[(d^{1(h)}_{ij})^2+(d^{2(h)}_{ij})^2+(e^{1(h)}_{ij})^2+(e^{2(h)}_{ij})^2+|f^{(h)}_{ij}|\right]+\sum_{i=1}^{N}\left[(g^{(h)}_{i})^2+(k^{(h)}_{i})^2+|l^{(h)}_{i}|\right]$$

(11)
$$\leq \bar{N}\int_0^1\sum_{i=1}^{N}\left\{[(1-t)p^i(x,y)+tp^i(x+h,y)]^2+[(1-t)q^i(x,y)+tq^i(x+h,y)]^2\right\}dt$$

$$\leq \frac{\bar{N}}{2}\int_0^1\sum_{i=1}^{N}\left\{[p^i(x,y)]^2+[p^i(x+h,y)]^2+[q^i(x,y)]^2+[q^i(x+h,y)]^2\right\}dt$$

Thus, for $0<h<h_0$, sufficinetly small, we see that there exists a K and a $\lambda = m/M$, such that

(12)
$$\iint_{H\cdot C(P,r)}\left\{\sum_{i=1}^{N}\sum_{j=1}^{N}\left[(d^{1(h)}_{ij})^2+(d^{2(h)}_{ij})^2+(e^{1(h)}_{ij})^2+(e^{2(h)}_{ij})^2+|f^{(h)}_{ij}|\right]+\right.$$
$$\left.+\sum_{i=1}^{N}\left[(g^{(h)}_{i})^2+(k^{(h)}_{i})^2+|l^{(h)}_{i}|\right]\right\}dx\,dy \leq K r^{\lambda}$$

Obviously, inequalities analogous to (11) and (12) hold for the $\bar{d}^{1(h)}_{ij}$, etc.

Now, suppose that H is also simply connected. Let $V^i(x,y)$ be the potential of the set function

$$\phi_i(e) = \iint_{S\cdot e} f_{z^i}(x,y,z^i,p^i,q^i)\,dx\,dy$$

Since $|f_{z^i}| \leq M(R)\sum_{j=1}^{N}(p^{j2}+q^{j2})$ and since z satisfies condition A $[\lambda, M']$

on G, we see that

$$V_{\phi_i}[C(P,r)] \le \bar{K}\left(\frac{r}{a}\right)^{\lambda}, \quad \lambda = \frac{m}{M}, \quad 0 \le r \le a, \quad P \le g$$

where a is the distance of P from G*. Thus, we see that V is of class D all over G and satisfies a condition $[A\ \lambda, M'']$ all over G. Then, it follows from (3) and theorem 1, § 6, that

$$\int_{R^*} (f_{pi} - V^i_x)\,dy - (f_{gi} - V^i_y)\,dx = 0$$

around almost all rectangles R in G so that there exists a function v which is of class D and is easily seen to satisfy a condition $^*[A, M''']$ on G such that

(13) $$V^i_x + v^i_y = f_{pi}, \quad V^i_y - v^i_x = f_{gi}$$

almost everywhere. Now if we let

$$\pi^i_h(x,y) = \frac{v^i(x+h,y) - v^i(x,y)}{h}, \quad \kappa^i_h(x,y) = \frac{v^i(x,y+h) - v^i(x,y)}{h},$$

$$\rho^i_h(x,y) = \frac{V^i(x+h,y) - V^i(x,y)}{h}, \quad Q^i_h(x,y) = \frac{V^i(x,y+h) - V^i(x,y)}{h},$$

$$h > 0,$$

we see that all of these functions are of class D and continuous on $\bar{H}$ if h is small enough and their derivatives are given almost everywhere by formulas analogous to (6). Now, if we translate (13) to (x+h,y) and (x,y+h) and subtract (13) from each of these and divide by h, we see that

(14) $$\rho^i_{hx} + \pi^i_{hy} = a^{(h)}_{i\beta}\rho^\beta_{hx} + b^{1(h)}_{i\beta}\rho^\beta_{hy} + d^{1(h)}_{i\beta}\rho^\beta_h + g^{(h)}_i$$

$$\rho^i_{hy} - \pi^i_{hx} = b^{2(h)}_{i\beta}\rho^\beta_{hx} + c^{(h)}_{i\beta}\rho^\beta_{hy} + e^{1(h)}_{i\beta}\rho^\beta_h + k^{(h)}_i$$

$$(15)\quad \begin{cases} Q^i_{hx} + K^i_{hy} = \bar{a}^{(h)}_{i\beta}\, q^\beta_{hx} + \bar{b}^{1(h)}_{i\beta}\, q^\beta_{hy} + \bar{d}^{1(h)}_{i\beta}\, q^\beta_h + \bar{g}^{(h)}_i \\ Q^i_{hy} - K^i_{hy} = \bar{b}^{2(h)}_{i\beta}\, q^\beta_{hx} + \bar{c}^{(h)}_{i\beta}\, q^\beta_{hy} + \bar{c}^{1(h)}_{i\beta}\, q^\beta_h + \bar{k}^{(h)}_i \end{cases}$$

$$(16)\quad \int_{R^*} P^i_{hx}\,dy - h^i_{hy}\,dx = \iint_R \left(d^{2(h)}_{i\beta} P^\beta_{hx} + e^{2(h)}_{i\beta} P^\beta_{hy} + f^{(h)}_{i\beta} P^\beta_h + l^{(h)}_i\right) dx\,dy$$

$$(17)\quad \int_{R^*} Q^i_{hx}\,dy - Q^i_{hy}\,dx = \iint_R \left(\bar{d}^{2(h)}_{i\beta} q^\beta_{hx} + \bar{e}^{2h}_{i\beta} q^\beta_{hy} + \bar{f}^{(h)}_{i\beta} q^\beta_h + \bar{l}^{(h)}_i\right) dx\,dy$$

almost everywhere, etc. It is also important to notice that htere exists an L such that

$$(18)\quad \iint_{H\cdot C(P,r)} \sum_{i=1}^{N} \left[(p^i_h)^2 + (q^i_h)^2 + (\Pi^i_h)^2 + (K^i_h)^2 + (P^i_h)^2 + (Q^i_h)^2\right] dx\,dy \le L r^\lambda \qquad \lambda = \frac{m}{M}$$

L independent of h if h is sufficiently small.

In the sequel, we shall prove that a family of vector functions which satisfy (18) uniformly and satisfy a set of equations of the type of (14) and (16) or (15) and (17) where the coefficients are all measurable and satisfy (10) and (12) uniformly on a simply connected region H of class C ,also satisfy a condition A [μ, M(a,d)] uniformly on H (i.e. the same μ and M(a,d) holds for all the functions) where μ is less than the smaller of $\bar{M}/\bar{M}$ and m/M (from (10) and M(a,d) depends only on a,d (as in definition 1, 5), M,m,$\bar{M}$,$\bar{m}$,K, and L (the last two depending on the first four). From this and theorem 2, 5, it follows that p (x,y) and q (x,y) are also of class D and satisfy uniform Holder conditions on each closed subregion of G.

It is shown further concerning the solutions of the above systems [(14) and (16), etc.] that if the $a^{(h)}_{ij}$, etc. also tend to limit functions

$a_{ij}(x,y)$ almost every where and if the solutions p_h^i and q_h^i tend on subregions uniformly to limit functions p^i and q^i, there the limiting functions satisfy the limiting equations. Now, suppost that we assume, in addition to what we have assumed about $f(x,y,z,p,q,)$ that there is a point $(x_0, y_0, z_0, p_0, q_0)$, where $(x_0, y_0) \in G$ and $z_0 = z(x_0, y_0)$, $p_0 = p(x_0, y_0)$, $q_0 = q(x_0, y_0)$ for our solution, in the neighborhood of which the second derivatives of f satisfy uniform Holder conditions. Then we see that, in (7) and (8), $p_h^i \Rightarrow p^i$ and $q_h^i \Rightarrow q$ on each closed subregion of G sufficiently near (x_0, y_0) ; and the a_{ij}^h, etc. converge uniformly on such sets to the functions $a^i(x,y)$ given by

$$a_{ij} = f_{p^i p^j}(x, y, z^k, p^k, q^k), \quad b^1_{ij} = f_{p^i q_j}, \quad b^2_{ij} = f_{q^i p^j}, \quad c_{ij} = f_{q^i q^j}$$

$$d^1_{ij} = f_{p^i z^j}, \quad d^2_{ij} = f_{z^i p^j}, \quad e^1_{ij} = f_{q^i z^j}, \quad e^2_{ij} = f_{z^i q^j}, \quad f_{ij} = f_{z^i z^j}$$

$$g_i = f_{p^i x}, \quad k_i = f_{q^i x}, \quad l_i = f_{z^i x}$$

and the $\bar{a}_{ij}$ are defined analogously and all satisfy uniform Holder conditions on some small circle with center at (x_0, y_0). It is also shown in the sequel that any solution of a system of equations of the type (7) in which the coefficients satisfy uniform Holder condition must be of class C' with its first derivatives satisfying uniform Holder conditions on interior regions. From this it follows that the first derivatives of p^i and q^i, i.e. the second derivatives of z^i, satisfy uniform Holder conditions near the point (x_0, y_0). In this case, it is clear that the z^i satisfy the Euler differential equations for a minimizing function z of class C".

9 Some Preliminary Existence Theorems.

Lemma 1: Let

(1) $$\varphi \equiv A_{\alpha\beta} x^\alpha x^\beta + 2 B_\alpha x^\alpha$$

be a quadratic function such that there exist numbers m and M such that

(2) $$m \sum_{i=1}^{n} x_i^{\,2} \le A_{\alpha\beta} x^\alpha x^\beta \le M \sum_{i=1}^{n} x^{i2}, \qquad 0 \le m \le M$$

Then there exist numbers k(m,M) and K(m,M), a number C, and numbers a_{ij} and b_i such that

(3) $$|a_{ij}| \le K(m,M), \quad k(m,M) \sum_{i=1}^{n} B_i^2 \le C \le K(m,M) \sum_{i=1}^{n} B_i^2$$
$$k(m,M) \sum_{i=1}^{n} B_i^{\,2} \le \sum_{i=1}^{n} b_i^{\,2} \le K(m,M) \sum_{i=1}^{n} B_i^2$$

and such that

(4) $$A_{\alpha\beta} x^\alpha x^\beta + 2 B_\alpha x + C = \sum_{i=1}^{n} \left(a_{i\alpha} x^\alpha + b_i\right)^2.$$

Furthermore if $A_{\alpha\beta}$ and B_α are measurable functions of parameters (s,t), C, a_{ij} and b_i may be chosen to be measurable functions of (s,t) over the same domain of definition.

Proof: If, in (1) we let $x^i = y^i + h^i$, (1) becomes

(5) $$A_{\alpha\beta} y^\alpha y^\beta + 2(A_{\alpha\beta} h^\beta + B_\alpha) y^\alpha + A_{\alpha\beta} h^\alpha h^\beta + B_\alpha h^\alpha$$

Let us choose h^i and C successively to satisfy

(6) $\quad A_{i\beta} h^\beta + B_i = 0, \quad A_{\alpha\beta} h^\alpha h^\beta + 2 B_\alpha h^\alpha + C = 0$

and we see that (1) reduces to

(7) $$A_{\alpha\beta} y^\alpha y^\beta = \varphi + C$$

Clearly C satisfies (3) and the h^i satisfy

$$k(m,M) \sum_{i=1}^n B_i^2 \leq \sum_{i=1}^n h^{i2} \leq K(m,M) \sum_{i=1}^n B_i^2$$

for some k and K.

Next we observe first, by letting $x^i = 1$, $x^j = 0$, $j \neq i$, that

$$m \leqslant A_{ii} \leqslant M \quad .$$

Next, by letting $x^k = 0$ if $k \neq i$ or j, then

$$m(x^{i2} + x^{j2}) \leq A_{ii} x^{i2} + 2A_{ij} x^i x^j \leq A_{jj} x^{j2} \leq M(x^{i2} + x^{j2})$$

so that

$$A_{ii} A_{jj} - A_{ij}^2 \geqslant 0,$$

and hence

$$|A_{ij}| \leqslant M.$$

Now, define

$$\varphi_1(y) \equiv A_{\alpha\beta} y^\alpha y^\beta - \frac{1}{A_{11}} (A_{1\alpha} y^\alpha)^2 \equiv \frac{A_{11} A_{\alpha\beta} - A_{1\alpha} A_{1\beta}}{A_{11}} y^\alpha y^\beta$$

and one sees that φ_1 is independent of y^1 as the above coefficient is zero if either α or $\beta = 1$. Hence, if we define

$$\bar{y}^1 = (A_{11})^{-\frac{1}{2}} (A_{1\alpha} y^\alpha), \quad \bar{y}^i = y^i, \quad i = 2, \ldots, n,$$

we see that

$$\varphi + C \equiv (\bar{y}^1)^2 + \varphi_1(\bar{y}^2, \cdots, \bar{y}^n)$$

where φ_1 has to be positive definite in $(y^2, \ldots, y^n)$ as the normal form of φ is a sum of squares.

Now, first, we see that each coefficient of φ_1 is in absolute value $\leq 2M^2/m$, and hence there exists a number M (M,m) > 0 such that

(8) $\quad \varphi_1(y^2, \cdots, y^n) \leq M \sum_{i=2}^{n} (y^i)^2$

Now for $(y^2)^2 + \ldots + (y^n)^2 = 1$, each φ_1 arising as above from a $\varphi + C$ has a minimum d which is > 0. Let us choose a sequence φ_β such that

(9) $$m \sum_{i=1}^{n} (y^i)^2 \leq \varphi_\rho(y) + C_\rho \leq M \sum_{i=1}^{n} y_i^2$$

and a sequence of points y_ρ such that $\varphi_{1\rho}(y_\rho)$ tends to the greatest lower bound of all the minima d. We may choose a subsequence so that the points $y_\rho \rightarrow y_0$ on the unit sphere $(y_0^2)^2 + \ldots + (y_0^n)^2 = 1$ and such that all the coefficients of φ_ρ and $\varphi_{1\rho}$ tend to limits, those for $\varphi_{1\rho}$ being derived from the limiting form $\varphi_0 + C_0$, say. But $\varphi_0 + C_0$ again satisfies (9) so that $\varphi_{10}(y)$ must be > 0. In other words there exists a number m (m,M) > 0 such that

(10) $$\varphi_1(y^2, \cdots, y^n) \geq m_1(m, M) \sum_{i=2}^{n} (y^i)^2$$

Evidently, we may repeat the process starting with φ and our lemma follows in a finite number of steps. The last statement is obvious from the above specified method given for obtaining the functions in question.

Lemma 2*) Let $\{\varphi_n(x,y)\}$, n = 1,2, ..., and $\varphi(x,y)$ be of class L_p on a rectangle R:(a,c;b,d), with

(11) $$\iint_R |\varphi_n(x,y)|^p dx\,dy \leq G, \quad p>1,$$

G being independent of n. Let

(12) $$\lim_{n\to\infty} \iint_D \varphi_n(x,y)\,dx\,dy = \iint_D \varphi(x,y)\,dx\,dy$$

for each cell D in R. Suppose that A(x,y) and $\{A_n(x,y)\}$, n=1,2, ..., are of class L_q on R, with q = p/(p-1), and that

$$\lim_{n\to\infty} \iint_R |A_n - A|^q dx\,dy = 0$$

Then

$$\lim_{n\to\infty} \iint_R A_n \varphi_n \,dx\,dy = \iint_R A\varphi\,dx\,dy$$

Proof: From the definition of weak convergence in the space L_p, and from a well known theorem**) characterizing this type of convergence, it follows that hypotheses (11) and (12) are equivalent to weak convergence. Hence

$$\lim_{n\to\infty} \iint_R A\varphi_n\,dx\,dy = \iint_R A\varphi\,dx\,dy$$

Also, we have

$$\left|\iint_R (A_n - A)\varphi_n\,dx\,dy\right| \leq \left[\iint_R |\varphi_n|^p dx\,dy\right]^{\frac{1}{p}} \left[\iint_R |A_n - A|^q dx\,dy\right]^{\frac{1}{q}}$$

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*) For a direct proof, see the author's paper D.E., pp. 129 - 131.

**) S. Banach, "Theorie des operations lineaires", pp. 133 - 135.

which tends to zero as $n \to \infty$. From this the lemma follows.

Lemma 3[*]: Let $\{u_p(x,y)\}$ and $u(x,y)$ be of class D_2' on the bounded region G with $D_2(u_p, G)$ uniformly bounded, and suppose $u_p \to u$ (in our previous sense) on G. Let the functions $a_{ij}^{(p)}(x,y)$ and $b_{ij}^{(p)}(x,y)$ be of class L_2 on G and converge strongly in L_2 to the functions $a_{ij}(x,y)$ and $b_{ij}(x,y)$. Suppose that the functions $c_i^{(p)}(x,y)$ are of class L_2 on G and converge weakly in L_2 to $c_i(x,y)$. Then

$$\iint_G \sum_{i=1}^{n} \left(a_{i\beta} u_x^\beta + b_{i\beta} u_y^\beta + c_i \right)^2 dx\,dy \le \varliminf_{p\to\infty} \iint_G \sum_{i=1}^{n} \left(a_{i\beta}^{(p)} u_x^\beta + b_{i\beta}^{(p)} u_y^\beta + c_i^p \right)^2 dx\,dy$$

Now, let $a_{ij}(x,y)$, $b_{ij}(x,y)$, $c_{ij}(x,y)$ be uniformly bounded and measurable on the region G of class C_2 and suppose thatthere exist numbers m and M, independent of (x,y) such that (13)

$$(13)\quad m \sum_{i=1}^{n} (\xi^{i2} + \eta^{i2}) \le a_{\alpha\beta}\xi^\alpha\xi^\beta + 2 b_{\alpha\beta}\xi^\alpha\eta^\beta + c_{\alpha\beta}\eta^\alpha\eta^\beta \le M \sum_{i=1}^{n} (\xi^{i2} + \eta^{i2})$$

$$a_{\alpha\beta} = a_{\beta\alpha}, \quad c_{\alpha\beta} = c_{\beta\alpha}$$

for all (ξ, η) and almost every (x,y) in G. Let $d_i(x,y)$ and $e_i(x,y)$ be of class L_2 on G and let $f(x,y)$ and $A_{ij}(x,y)$ and $B_{ij}(x,y)$ and $C_i(x,y)$ be determined, as in lemma 1 so that

$$(14)\quad a_{\alpha\beta}\xi^\alpha\xi^\beta + 2 b_{\alpha\beta}\xi^\alpha\eta^\beta + c_{\alpha\beta}\eta^\alpha\eta^\beta + 2 d_\alpha \xi^\alpha + 2 e_\alpha \eta^\alpha + f \equiv \sum_{i=1}^{n} \left(A_{i\alpha}\xi^\alpha + B_{i\alpha}\eta^\alpha + C_i \right)^2$$

the identity being in (x, y, ξ, η). Then the A_{ij} and B_{ij} are bounded and measurable, the C_i are of class L_2, and f is summable over G, and we have two numbers $k(m,M)$ and $K(m,M)$ such that

$$(14')\quad \begin{aligned} k(m,M) \sum_{i=1}^{n} (d_i^2 + e_i^2) &\le f \le K(m,M) \sum_{i=1}^{n} (d_i^2 + e_i^2) \\ k(m,M) \sum_{i=1}^{n} (d_i^2 + e_i^2) &\le \sum_{i=1}^{n} C_i^2 \le K(m,M) \sum_{i=1}^{n} (d_i^2 + e_i^2) \end{aligned}$$

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*) See D.E., p. 132.

We now define the integrals I(u,g) and J(u,G) for functions of class D_2' by

$$I(u,G)=\iint_G (a_{\alpha\beta}u_x^\alpha u_y^\beta+2b_{\alpha\beta}u_x^\alpha u_y^\beta+c_{\alpha\beta}u_y^\alpha u_y^\beta)\,dx\,dy$$

$$J(u,G)=\iint_G (a_{\alpha\beta}u_x^\alpha u_y^\beta+2b_{\alpha\beta}u_x^\alpha u_y^\beta+c_{\alpha\beta}u_y^\alpha u_y^\beta+2d_\alpha u_x^\alpha+2e_\alpha u_y^\alpha+f)\,dx\,dy$$

$$I(u,v;G)=\iint_G [u_x^\alpha(a_{\alpha\beta}u_x^\beta+b_{\alpha\beta}u_y^\beta)+u_y^\alpha(b_{\beta\alpha}u_x^\beta+c_{\alpha\beta}u_y^\beta)]\,dx\,dy$$

$$J(u,v;G)=\iint_G [v_x^\alpha(a_{\alpha\beta}u_x^\beta+b_{\alpha\beta}u_y^\beta+d_\alpha)+v_y^\alpha(b_{\beta\alpha}u_x^\beta+c_{\alpha\beta}u_y^\beta+e_\alpha)]\,dx\,dy$$

Then it is immediately clear that I(u,G) = I(u,u;G) and that

$$I(\lambda u+\mu v,G)\equiv\lambda^2 I(u,G)+2\lambda\mu\, I(u,v;G)+\mu^2 I(v,G)$$

(15) $$J(u+\lambda v,G)\equiv J(u,G)+2\lambda\, J(u,v;G)+\lambda^2 I(u,G)$$

$$I^2(u,v;G)\le I(u,G)\cdot \frac{I(v,G)}{\cancel{J(u,G)}}$$

Using (14), we see first that

(16) $$I(u,G)\le 2J(u,G)+2\iint_G f\,dx\,dy,\quad J(u,G)\le 2I(u,G)+2\iint_G f\,dx\,dy$$

Secondly, it follows from (14) and lemma 3 that I(u,G) and J(u,G) are lower semi-continuous in u among all functions n of class D_2' on G with convergence defined as in § 3.

Theorem 1: Let u^* be of class D'_2 on G with $D_2(u^*,G)$ finite. Then there exist unique functions u_I and u_J which are of class D'_2 on G, coincide with u^* on G^*, and minimize $I(u,G)$ and $J(u,G)$ respectively, among all such functions. If ζ is any function of class D'_2 on G, with $D_2(\zeta,G)$ finite, which vanishes on G^*, then

$$\bar{I}(u_I,\zeta,\zeta)=J(u_J,\zeta,\zeta)=0,\quad I(u_I+\zeta,\zeta)=\bar{I}(u_I,\zeta)+I(\zeta,\zeta), \tag{17}$$
$$J(u_J+\zeta,\zeta)=J(u_J,\zeta)+I(\zeta,\zeta)$$

Hence u_I and u_J satisfy

$$\int_{R^*}(a_{i\beta}u^\beta_{Ix}+b_{i\beta}u^\beta_{Iy})\,dy-(b_{\beta i}u^\beta_{Ix}+c_{i\beta}u^\beta_{Iy})\,dx=0 \tag{18}$$

$$\int_{R^*}(a_{i\beta}u^\beta_{Jx}+b_{i\beta}u^\beta_{Jy}+d_i)\,dy-(b_{\beta i}u^\beta_{Jx}+c_{i\beta}u^\beta_{Jy}+e_i)\,dx=0 \tag{19}$$

on almost all rectangles of G. The function u satisfies conditions $A[\lambda,\bar{M}]$ and $B[\lambda/2,\bar{N}]$ on G with $\lambda=m/M$ and $\bar{M}=D_2(u_I,G)$, and is continuous on G if the boundary values are continuous.

Proof: The existence of u_I and u_J and the fact that u_I satisfies conditions $A[\lambda,\bar{M}]$ and $B[\lambda/2,\bar{N}]$ as stated above follows immediately from the fact that I and J are lower semi-continuous in u, and from the theorems of § 4 and 5 and equation (14); this part of the proof and that of the continuity of u_I on $\bar{G}$ whenever its boundary values are continuous parallels the proof of theorem 5, 1 .

Now, let ζ be any function of class D'_2 on G with $D_2(\zeta, G)$ finite and zero on G^*. There $u_I+\lambda\zeta$ and $u_J+\lambda\zeta$ are admissable functions for the integrals I and J, and we have

(20) $$\begin{aligned} I(u_I+\lambda\zeta, G) &= I(u_I, G) + 2\lambda I(u_I, \zeta, \zeta) + \lambda^2 I(\zeta, \zeta) \\ J(u_J+\lambda\zeta, G) &= J(u_J, G) + 2\lambda J(u_J, \zeta, G) + \lambda^2 I(\zeta, G) \end{aligned}$$

all the integrals being finite. Since u_I and u_J minimize I and J, the middle terms in (20) on the right must vanish. This gives us formulas (17); and formulas (18) and (19) follow from (17) on account of theorem 2, 7 (Haar's second lemma).

Theorem 2: Let $u_J{}^*$ be any function of class D_2' on G (of class C_2) with $D_2(u_J, G)$ finite, which satisfies (19) on almost all rectangles of G. There u_J minimizes $J(u,G)$ among all functions u of class D_2' on G and coinciding with u_J on G*. Clearly the analogous statement for u_I and equation (18) holds, being a special case of the above.

Proof: It follows immediately from theorem 3, § 7(the converse of Haar's lemma with $\phi(c) \equiv 0$) that $J(u_J, \zeta, G) = 0$ for every ζ which is of class D_2' on G and zero on G*. There, from equations (20) with $\lambda = 1$, we obtain

$$J(u_J+\zeta, G) = J(u_J, G) + I(\zeta, G)$$

for each ζ of class D_2' on G and zero on the boundary. But $I(\zeta, G) > 0$ for every such ζ which is not essentially zero. Thus u_J has the stated minimum property.

Theorem 3: If, in theorem 1, the d_i and e_i satisfy

(21) $$\iint_{C(P,r)\cdot G} (d^2+e^2)\,dx\,dy \equiv \iint_{C(P,r)\cdot G} \sum_{i=1}^{N} (d_i^2+e_i^2)\,dx\,dy \leq Q r^{\mu}, \quad 0<\mu<\lambda = \frac{m}{M}$$

for every P and r, then u_J satisfies conditions $A[\mu, \tilde{M}]$ and $B[\mu/2, \tilde{N}]$

on G; in fact

(22) $$D_2[u_\gamma, C(\bar P, r)] \le \frac{4KQ}{m\lambda}\left(\delta^\mu + \frac{a^\mu}{\mu}\right)\left(\frac{r}{a}\right)^\mu, \quad 0 \le r \le a$$

where a is the distance of $\bar P$ from G*, is the diameter of G and K is the K(M,m) of equation (14'). If the given boundary values are continuous, u is continuous on $\bar G$.

If (21) holds only for circles with center at a fixed point P_o, interior to G, then

(23) $$|\bar u_{\gamma_0}(P_0)| \le \left[\frac{8\pi KQ}{m\lambda}\left(\delta^\mu + \frac{a^\mu}{\mu}\right)\right]^{\frac{1}{2}}\left(1 + \frac{1}{\mu} + \frac{1}{2}\log\frac{\delta}{a}\right)$$

where u_{γ_o} is the solution of (19) which vaishes on G*.

If we assume merely that d and e are of class L_2 on G, then

(24) $$D_2(u_{\gamma_0}, \mathcal{G}) \le \frac{4K}{m}\iint_{\mathcal{G}}(d^2+e^2)\,dx\,dy, \quad \iint_{\mathcal{G}} u_{\gamma_0}^2\,dx\,dy \le \delta^2 D_2(u_{\gamma_0}, \mathcal{G})$$

Proof: We first observe that, for each R of class C' in G, we have (using theorem 2)

$$u_\gamma(x,y) = u_{\gamma R_0}(x,y) + u_{\Gamma R}(x,y), \quad (x,y) \in R$$

where $u_{\gamma R_0}$ is the u_γ defined in R and vainishing on R* and $u_{\Gamma R}$ is the $u_{\Gamma R}$ defined in R and coinciding with u_Γ on R*. Then from (16) and (17) it follows that

$$I(u_\gamma, R) = I(u_{\Gamma R}, R) + \bar I(u_{\gamma R_0}, R) \le \bar I[H(u_\gamma, R), R] + 2\mathcal{J}(u_{\gamma R_0}, R) + 2\iint_{\mathcal{G}} f\,dx\,dy$$

$$\le M\, D_2[H(u_\gamma, R), R] + 4\iint_R f\,dx\,dy$$

$$\le M\, D_2[H(u_\gamma, R), R] + 4K\iint_R (d^2+e^2)\,dx\,dy$$

H(γ,R), having its significance in §5, and K being the K(m,M) of equation

(14'). Thus

$$D_2(u_g, R) \leq \frac{M}{m} D_2[H(u_g, R), R] + \frac{4K}{m} \iint_R (d^2 + e^2)\,dx\,dy \tag{25}$$

Evidently (22) follows from theorem 1, § 5. Now suppose the boundary values are continuous. Let H be a region of class C_2 including G in its interior and let $b_{ij} = d_i = e_i = 0$ in H − G and $a_{ij} = c_{ij} = \frac{M+m}{2}\delta_{ij}$ in H − G. Then the function u_{gH} for H which vanishes on H* is continuous on $\bar{G}$ so that the function u_I which coincides on G* with the given boundary values minus those of u_{gH} is continuous on $\bar{G}$. Thus $u_g = u_{gH} + u_I$ is continuous on G.

To prove (24), we use equations (13), (14'), and (16) as follows:

$$m\,D_2(u_{g_0}, G) \leq I(u_{g_0}, G) \leq 2J(u_{g_0}, G) + 2\iint_G f\,dx\,dy \leq 4\iint_G f\,dx\,dy \leq 4K \iint_G (d^2 + e^2)\,dx\,dy$$

Next, let P_0 be a point of G and (r, ϑ) be polar coordinates with pole at P_0 and define $u = u_{g_0}$ in G and $u = 0$ outside G. We then define

$$H(r) = \int_0^r \int_0^{2\pi} \rho\, u_r^2(\rho, \vartheta)\,d\rho\,d\vartheta$$

and $H(r)$ is A.C. with $H'(r) \geq 0$ and $H(r) \leq D_2(u_{g_0}, G)$. Then

$$\iint_G u_{g_0}^2\,dx\,dy = \int_0^\delta \int_0^{2\pi} r\,u^2(r, \vartheta)\,dr\,d\vartheta = \int_0^\delta r \int_0^{2\pi} [u(r,\vartheta) - u(\delta,\vartheta)]^2\,dr\,d\vartheta$$

$$\leq \int_0^\delta r \int_0^{2\pi} \left[\int_r^\delta u_r(\rho, \vartheta)\,d\rho\right]^2 d\vartheta\,dr \leq \delta \int_0^\delta r \left[\int_r^\delta \int_0^{2\pi} u_r^2(\rho,\vartheta)\,d\rho\,d\vartheta\right] dr$$

$$= \delta \int_0^\delta r \left[\int_r^\delta \rho^{-1} H'(\rho)\,d\rho\right] dr \leq \delta \int_0^\delta [H(\delta) - H(r)]\,dr \leq \delta^2 D_2(u_{g_0}, G)$$

To prove (23), we observe that

$$D_2[u, \mathcal{C}(P_0, r)] \le K_1 \left(\frac{r}{a}\right)^{\mu}, \quad 0 \le r \le a$$

$$D_2[u, C(P_0, r)] \le K_1 \qquad , \quad r \ge 0$$

$$K_1 = \frac{4K}{m\lambda}\left(\delta^{\mu} + \frac{a^{\mu}}{\mu}\right)$$

where u will hereafter denote u_{ρ_0} . Define u = 0 outside G and let

$$H(r) = \int_0^r\int_0^{2\pi} \rho \bar{u}_r(\rho, \vartheta)\, d\vartheta, \quad h(r) = \int_0^r\int_0^{2\pi} \rho^{-\frac{1}{2}} |u_r(\rho, \vartheta)|\, d\rho\, d\vartheta$$

Then

$$h^2(r) \le 2\pi r H(r), \qquad H(r) \le K\left(\frac{r}{a}\right)^{\mu}, \quad 0 \le r \le a$$
$$H(r) \le K, \quad r \ge a$$

Now, we know from theorem , § 6 that a function $\bar{u}(0, \vartheta)$ exists so that

$$\bar{u}(P_0) = \frac{1}{2\pi}\int_0^{2\pi} \bar{u}(0, \vartheta)\, d\vartheta, \quad \lim_{r \to 0} \int_0^{2\pi} |\bar{u}(r, \vartheta) - \bar{u}(0, \vartheta)|\, d\vartheta \le$$
$$\le \lim_{r \to 0} \int_0^r\int_0^{2\pi} |\bar{u}_r(\rho, \vartheta)|\, d\rho\, d\vartheta = 0$$

Then, we see that

$$|\bar{u}(P_0)| \le \int_0^{\delta}\int_0^{2\pi} |\bar{u}_r(\rho, \vartheta)|\, d\rho\, d\vartheta = \int_0^{\delta} \rho^{-\frac{1}{2}} h'(\rho)\, d\rho = \delta^{-\frac{1}{2}} h(\delta) + \frac{1}{2}\int_0^{\delta} \rho^{-\frac{3}{2}} h(\rho)\, d\rho$$
$$\le (2\pi K)^{\frac{1}{2}} + \frac{1}{2}\left(2\pi K/a^{\mu}\right)^{\frac{1}{2}} \int_0^a \rho^{\frac{\mu}{2}-1}\, d\rho + \frac{1}{2}(2\pi K)^{\frac{1}{2}} \int_a^{\delta} \rho^{-1}\, d\rho$$
$$\le (2\pi K)^{\frac{1}{2}}\left(1 + \frac{1}{\mu} + \frac{1}{2}\log\frac{\delta}{a}\right)$$

which proves (23).

Theorem 4: Let d and e be of class L_2 on G (of class C_2) and let ϕ (e) be completely additive on G satisfying

(26) $$V_{\phi}[C(P, r) \cdot \mathfrak{G}] \le T r^{\mu}, \quad \mu > 0 .$$

Let u^* be of class D_2' on G with $D_2(u^*,G)$ finite. Then there exists a unique function u of class D_2' on G and coinciding with u^* on G^* which satisfies

$$\int_{R^*}(a_{i\alpha}u_x^\alpha + b_{i\alpha}u_y^\alpha + d_i)\,dy - (b_{\alpha i}u_x^\alpha + c_{i\alpha}u_y^\alpha + e_i)\,dx = \phi_i(R) \tag{27}$$

on almost all rectangles R of G.

If d_n and e_n tend strongly in L_2 to d and e, and if the set functions $\phi_n(e)$ are completely additive on G, satisfy (26) uniformly, and tend weakly to $\phi(e)$, then

$$\lim_{n\to\infty} \overline{D}_2(u_n - u, G) = 0$$

u_n being the solution of (27) for d_n, e_n, and $\phi_n(e)$, which coincides with u^* on G^*.

Proof: To demonstrate the first statement, let $V(x,y)$ be the potential of $\phi(e)$. Then V is of class D_2 on G and

$$\int_{R^*} V_x^i\,dy - V_y^i\,dx = \phi^i(R)$$

for almost every R. We then choose a function u^* of class D_2' on G to coincide with u^* on G^* and to satisfy

$$\int_{R^*}(a_{i\alpha}u_x^\alpha + b_{i\alpha}u_y^\alpha + d_i - V_{ix})\,dy - (b_{\alpha i}u_x^\alpha + c_{i\alpha}u_y^\alpha + e_i - V_{iy})\,dx = 0$$

on almost all rectangles and u is a desired solution. To see that u is unique, suppose u_1 is another solution of (27) of class D_2' on G with $D_2(u_1,G)$ finite, and coinciding with u^* on G^*. If $U = u_1 - u$, we see that U is of class D'_2 on G, is zero on G*, and satisfies (18) on almost rectangles. Thus by theorem 2, U is the unique minimizing function for I(u,G) which vanishes on G*, and is therefore essentially zero.

To prove the second statement, we may assume $u^* \equiv 0$. By theorem 6, § 6, it follows that $D_2(V_n - V, G) \to 0$. Thus $d_n - V_{nx}$ and $e_n - V_{ny}$ tend strongly in L_2 to $d - V_x$ and $e - V_y$ respectively. The result follows immediately from this and theorem 3.

10 The existence theory for a general linear system.

In this section, we shall develop our general existence theory for circular regions Σ_A of radius A, Greater generality may be attained but this is sufficient for our purposes. We shall consider the system

$$(1) \quad \int_{R^*} (a_{i\alpha} u^\alpha_x + b_{i\alpha} u^\alpha_y + \rho d_{i\alpha} u^\alpha + g_i)\, dx - (b_{\alpha i} u^\alpha_x + c_{i\alpha} u^\alpha_y + \rho e_{i\alpha} u^\alpha + k_i)\, dx = \iint_R (\rho d_{\alpha i} u^\alpha_x + \rho e_{\alpha i} u^\alpha_y + \rho f_{i\alpha} u^\alpha + l_i)\, dx\, dy$$

to be satisfied on almost all R in Σ_A. We shall assume that d_{ij}, e_{ij}, g_i, and k_i are of class L_2 on Σ_A, that f_{ij} and l_i are summable on Σ_A, that

$$(2) \quad \iint_{C(P,r)\cdot\Sigma_A} \sum_{i=1}^{N} \Big\{ |l_i| + \sum_{j=1}^{N} (d_{ij}^2 + e_{ij}^2 + |f_{ij}|) \Big\}\, dx\, dy \le M_1 r^\mu, \quad 0 < \mu < 1$$

and that the a_{ij}, b_{ij}, and c_{ij} satisfy the hypotheses of the last section. Further restrictions on g_i and k_i will be added as desired.

For this section, we shall define

$$\bar{D}_2(u, \Sigma_A) = D_2(u, \Sigma_A) + \frac{1}{A^2\pi} \iint_{\Sigma_A} u^2\, dx\, dy$$

Lemma 1: Let u be of class D'_2 on Σ_A with $\overline{D}_2(u, \Sigma_A)$ finite. Then, we may extend u to be of class D'_2 on Σ_{3A} and vanish on Σ^*_{3A}, and so that

$$D_2(u, \Sigma_{3A}) \le (2 + \log 3 + 10\pi)\, \overline{D}_2(u, \Sigma_A)$$

Proof: Since $\overline{D}_2(u, \Sigma_A)$ is finite and

$$\overline{D}_2(u, \Sigma_A) = \int_0^A \int_0^{2\pi} r\left[\frac{\bar u^2(r,\vartheta)}{\pi A^2} + \bar u_r^2(r,\vartheta) + \frac{1}{r^2}\bar u_\vartheta^2(r,\vartheta)\right] dr\, d\vartheta = K_2$$

say, we see that we can find an $\bar r$ with $A/2 \le \bar r \le A$ such that $\bar u(\bar r, \vartheta)$ is A.C. in ϑ and

$$\int_0^{2\pi} \bar u^2(\bar r,\vartheta)\, d\vartheta \le 4\pi K_2, \quad \int_0^{2\pi} \bar u_\vartheta(\bar r,\vartheta)\, d\vartheta \le 2K_2$$

We then extend u to Σ_{3A} as follows:

$$\bar u(r,\vartheta) = \bar u\left(\frac{A^2}{r}, \vartheta\right) \qquad A \le r \le \frac{A^2}{\bar r}$$

$$\bar u(r,\vartheta) = \frac{3A - r}{3A - \frac{A^2}{\bar r}}\, \bar u(\bar r, \vartheta), \qquad \frac{A^2}{\bar r} \le r \le A$$

and we see that $\bar u(r,\vartheta)$ as extended is of class D'_2 on Σ_{3A} and zero Σ^*_{3A}.

Moreover

$$\bar u_r(r,\vartheta) = -\frac{1}{3A - \frac{A^2}{\bar r}}\, \bar u(\bar r,\vartheta), \quad \bar u_\vartheta(r,\vartheta) = \frac{3A - r}{3A - \frac{A^2}{\bar r}}\, \bar u_\vartheta(\bar r, \vartheta)$$

$$\left(\frac{A^2}{\bar r} \le r \le A\right)$$

Thus

$$D_2(u, \Sigma_{3A}) = D_2(u, \Sigma_A) + D_2\left(u, \Sigma_{\frac{A^2}{\bar r}} - \Sigma_A\right) + D_2\left(u, \Sigma_{3A} - \Sigma_{\frac{A^2}{\bar r}}\right)$$

$$= D_2(u, \Sigma_A) + D_2(u, \Sigma_A - \Sigma_{\bar r}) + D_2\left(u, \Sigma_{3A} - \Sigma_{\frac{A^2}{\bar r}}\right)$$

$$\le 2K_2 + \left(3A - \frac{A^2}{\bar r}\right)\int_{\frac{A^2}{\bar r}}^{3A}\left[r\int_0^{2\pi} \bar u^2(\bar r,\vartheta)\, d\vartheta\right] dr + \int_{\frac{A^2}{\bar r}}^{3A}\left[\frac{1}{r}\left(\frac{3A - r}{3A - \frac{A^2}{\bar r}}\right)^2 \int_0^{2\pi} \bar u_\vartheta^2(\bar r,\vartheta)\, d\vartheta\right] dr \le$$

$$\le 2K_2 + 10\pi K_2 + 2K_2 \log 3$$

Lemma 2: Let $\rho(x,y)$ be summable on Σ_A and satisfy

(3) $$\iint_{C(P,r)\cdot\Sigma_A} |\varphi|\,dx\,dy \le k(r), \quad \int_0^t \rho^{-1} k(\rho)\,d\rho = K(t) < \infty$$

for every C(P,r) where k(r) is monotone non-decresing. Let

(4) $$V_r(x,y;x_0,y_0) = \frac{1}{4\pi}\iint_{C(P_0,r)\cdot\Sigma_A} \log\left[(\xi-x)^2+(\eta-y)^2\right]\varphi(\xi,\eta)\,d\xi\,d\eta, \quad 0<r\le A$$

Then

(5) $$D_2(V_r,\Sigma_A) \le k^2(r)\left(\frac{2\sqrt{2}}{\pi}+\log\frac{A}{r}\right)+\frac{8}{3\pi}k(r)K(r)$$

Proof: To prove this lemma, we simply rewrite the first part of the proof of theorem 5, §6. We define

$$\begin{aligned}\varphi_r(x,y) &= \varphi(x,y), && (x,y)\in C(P_0,r)\Sigma_A; \quad H_r(s,t;\rho) = \iint_{C(s,t;r)} |\varphi_r(x,y)|\,dx\,dy\\ \varphi_r(x,y) &= 0, && (x,y)\notin C(P_0,r)\Sigma_A.\end{aligned}$$

$$h_A(\xi,\eta;s,t) = \begin{cases} 4\pi\sqrt{2}+\frac{8\pi}{3}\log\frac{A}{d} & \text{if } d\le A\\ 4\pi\sqrt{2}\left(\frac{A}{d}\right) & \text{if } d\ge 0\end{cases}, \quad d=\left[(\xi-s)^2+(\eta-t)^2\right]^{\frac{1}{2}}$$

It is then clear that

$$H_r(s,t;\rho)\le k(\rho), \qquad H_r(s,t;\rho)\le k(r)$$

for all ρ, and it follows from lemma 4, §6 that putting $E=\dfrac{(\xi-x)(s-x)+(\eta-y)(t-y)}{\left[(\xi-x)^2+(\eta-y)^2\right]\left[(s-x)^2+(t-y)^2\right]}$

$$\iint_{\Sigma_A} |E|\,dx\,dy \le h_A(\xi,\eta;s,t)$$

for all $(\xi,\eta;s,t)$. Then we see that (using theorem 5, §6)

$$\iint\limits_{\Sigma_A} (v_{rx}^2 + v_{ry}^2)\,dx\,dy = \frac{1}{4\pi^2}\iint\limits_{\Sigma_A}\Big[\iint\limits_{C(P_0,r)}\iint\limits_{C(P_0,r)} E\,\varphi_r(\xi,\eta)\,\varphi_r(s,t)\,d\xi\,d\eta\,ds\,dt\Big]\,dx\,dy$$

$$\le \frac{1}{4\pi^2}\iint\limits_{C(P_0,r)}\iint\limits_{C(P_0,r)}\Big[\iint\limits_{\Sigma_A}|E|\,|\varphi(\xi,\eta)|\,|\varphi(s,t)|\,d\xi\,d\eta\,ds\,dt$$

$$\le \frac{1}{4\pi^2}\iint\limits_{C(P_0,r)}\Big[\iint\limits_{C(s,t;A)\,C(P_0,r)}\Big(4\pi\sqrt{2} + \frac{8\pi}{3}\log\frac{A}{\alpha}\Big)|\varphi(\xi,\eta)|\,d\xi\,d\eta + \iint\limits_{\Sigma_A\,C(P_0,r) - C(s,t;A)\,C(P_0,r)} 4\pi\sqrt{2}\Big(\frac{A}{\alpha}\Big)|\varphi(\xi,\eta)|\,d\xi\,d\eta\Big]\,|\varphi(s,t)|\,ds\,dt$$

$$\le \frac{\sqrt{2}}{\pi}\,[k(r)]^2 + \frac{2}{3\pi}\iint\limits_{C(P_0,r)}\Big[\int_0^A \log\frac{A}{\rho}\,dH_r(s,t;\rho)\Big]|\varphi(s,t)|\,ds\,dt + \frac{\sqrt{2}}{\pi}\iint\limits_{C(P_0,r)}\Big[\int_A^{2A}\frac{A}{\rho}\,dH(s,t;\rho)\Big]|\varphi(s,t)|\,ds\,dt$$

We find that

$$\int_0^A \log\frac{A}{\rho}\,dH_r(s,t;\rho) = \int_0^A \rho^{-1}H_r(s,t;\rho)\,d\rho \le \int_0^r \rho^{-1}k(\rho)\,d\rho + k(r)\int_r^A \rho^{-1}\,d\rho \le k(r) + k(r)\log\frac{A}{r}$$

$$\int_A^{2A}\frac{A}{\rho}\,dH_r(s,t;\rho) = \frac{1}{2}H_r(s,t;2A) - H_r(s,t;A) + A\int_A^{2A}\rho^{-2}H_r(s,t;\rho)\,d\rho$$

$$\le \frac{1}{2}k(r) + k(r)\cdot A\Big(\frac{1}{A} - \frac{1}{2A}\Big) = k(r)$$

Thus we see, finally, that (5) follows.

Lemma 3: Let $\varphi(x,y)$ be summable on Σ_A and satisfy (3) for every $C(P,r)$, where $k(r)$ satisfies the hypotheses of lemma 2. Let $\{u_n\}$ be a sequence of functions of class D'_2 on Σ_A with $\overline{D}_2(u_n, \Sigma_A)$ uniformly bounded. Then all the integrals

$$\iint\limits_{\Sigma_A}\varphi u_n\,dx\,dy$$

exist and we have

$$(6)\quad \iint\limits_{C(P_0,r)\,\Sigma_A}|\varphi u_n|\,dx\,dy \le (K_1 L)^{\frac{1}{2}}\Big(\frac{2\sqrt{2}}{\pi} + \log\frac{3A}{r} + \frac{2k(r)}{3\pi k(r)}\Big)^{\frac{1}{2}}\cdot k(r)$$

where K_1 is the constant of lemma 1 and L is a uniform bound for $\bar{D}_2(u_n, \Sigma_c)$. Moreover, a subsequence $\{u_{n_k}\}$ may be chosen to converge to a function u, which is of class D_2' on Σ_a with $\bar{D}_2(u, \Sigma_a) \leq L$ and satisfies (6), so that φu_{n_k} converges weakly in L_1 to φu.

Proof: According to lemma 1, we may extend each u_n to Σ_{3A} so that $u_n = 0$ on Σ^*_{3A} and

$$D_2(u_n, \Sigma_{3A}) \leq (2 + \log 3 + 10\pi)\bar{D}_2(u_n, \Sigma_A), \quad D_2(u, \Sigma_{3A}) \leq (2 + \log 3 + 10\pi)\bar{D}_2(u, \Sigma_A)$$

Now, evidently $|u_n|$ is of class D_2' on Σ_{3A} with

$$D_2(|u_n|, \Sigma_{3A}) = D_2(u_n, \Sigma_{3A})$$

To prove (6), let u simply denote any function of class D_2' on Σ_{3A} and zero on Σ^*_{3A} with $D_2(u, \Sigma_{3A}) < K_1 L$ where K_1 is the constant of lemma 1 and L is the uniform bound for $D_2(u_n, \Sigma_A)$, clearly $K_1 L$ is a uniform bound for $D_2(u_n, \Sigma_{3A})$. We let φ satisfy (3) and define

$$V_r(x, y; x_0, y_0) = \frac{1}{4\pi} \iint_{C(P_0, r)} \log \sigma \, |\varphi(\xi, \eta)| \, d\xi \, d\eta, \quad \sigma = (\xi - x)^2 + (\eta - y)^2$$

$$V(x, y) = \frac{1}{4\pi} \iint_{\Sigma_{3A}} \log \sigma \, |\varphi(\xi, \eta)| \, d\xi \, d\eta, \quad \varphi = 0, \ (x, y) \in \Sigma_{3A} - \Sigma_A$$

Then, by lemma 2, and theorem 5, § 6, the integrals below exist and are given by

$$\iint_{\Sigma_{3A}} |\varphi(x, y)| \, |\bar{u}(x, y)| \, dx \, dy = -\iint_{\Sigma_{3A}} (V_x |\bar{u}|_x + V_y |\bar{u}|_y) \, dx \, dy \leq$$

$$\leq [D_2(V, \Sigma_{3A})]^{\frac{1}{2}} \cdot [\bar{D}_2(u, \Sigma_{3A})]^{\frac{1}{2}} \leq (K_1 L)^{\frac{1}{2}} \left[\frac{2\sqrt{2}}{\pi} + \log \frac{3A}{3A} + \frac{2K(3A)}{2k(3, A)}\right]^{\frac{1}{2}}_{k(3, A)}$$

Also, it has been found legitimate in §6 to proceed as follows:

$$\iint_{C(P_0,r)\cdot\Sigma_A} |\varphi|\cdot|u|\,dx\,dy = \iint_{C(P_0,r)} |\varphi|\cdot|u|\,dx\,dy = -\frac{1}{2\pi}\iint_{C(P_0,r)}\left[|\varphi(x,y)|\iint_{\Sigma_{3A}} \frac{(\xi-x)\,\overline{|u|}_\xi+(\eta-y)\,\overline{|u|}_\eta}{\sigma}\,d\xi\,d\eta\right]dx\,dy$$

$$= -\frac{1}{2\pi}\iint_{\Sigma_{3A}}\left[\overline{|u|}_\xi\iint_{C(P_0,r)}\frac{(\xi-x)|\varphi(x,y)|\,dx\,dy}{\sigma}+\overline{|u|}_\eta\iint_{C(P_0,r)}\frac{(\eta-y)|\varphi(x,y)|\,dx\,dy}{\sigma}\right]d\xi\,d\eta$$

$$= -\iint_{\Sigma_{3A}}\left[V_{rx}(\xi,\eta;x_0,y_0)\cdot\overline{|u|}_\xi+V_{ry}(\xi,\eta;x_0,y_0)\,\overline{|u|}_\eta\right]d\xi\,d\eta \le$$

$$\le (K_1L)^{\frac{1}{2}}\left[D_2(V_r,\Sigma_{3A})\right]^{\frac{1}{2}} \le (K_1L)^{\frac{1}{2}}\left(\frac{2\sqrt{2}}{\pi}+\log\frac{3A}{r}+\frac{2K(r)}{3\pi k(r)}k(r)\right)$$

which demonstrates (6).

Now, let $\{u_{n_k}\}$ be chosen to converge to u on Σ_{3A}. To show that φu_{n_k} tends weakly in L_1 to φ u, we need only to show that, for each bounded function ψ, we have

$$\lim_{k\to\infty}\iint \varphi\psi u_{n_k}\,dx\,dy = \iint_{\Sigma_A}\varphi\psi u\,dx\,dy$$

Obviously $\varphi\psi$ is just another function φ if $|\psi| < 1$, so we need to show only that

$$\lim_{k\to\infty}\iint_{\Sigma_A}\varphi u_{n_k}\,dx\,dy = \iint_{\Sigma_A}\varphi u\,dx\,dy$$

Evidently we may proceed as above and find that

$$(7)\quad \iint_{\Sigma_A}\varphi u_{n_k}\,dx\,dy = -\iint_{\Sigma_{3A}}(V_x\bar{u}_{n_kx}+V_y\bar{u}_{n_ky})\,dx\,dy,\quad \iint_{\Sigma_A}\varphi u\,dx\,dy = -\iint_{\Sigma_{3A}}(V_x\bar{u}_x+V_y\bar{u}_y)\,dx\,dy$$

Obviously the $\bar{u}_{n_kx}$ and $\bar{u}_{n_ky}$ tend weakly in L_2 to $\bar{u}_x$ and $\bar{u}_y$ on Σ_{3A} which means that the right sides in (7) for $\bar{u}_{n_k}$ tend to the right side for $\bar{u}$. This proves the lemma.

Lemma 4: Let $\{u_n\}$ be a sequence of functions of class D'_2 on Σ_A with $\bar{D}_2(u_n, \Sigma_A)$ uniformly bounded ($\leq L$). Let d be of class L_2 on Σ_A with

$$(8)\qquad \iint_{C(P_0,r)\,\Sigma_A} d^2\,dx\,dy \leq M_2 r^{\nu}, \quad \nu > 0$$

Then a subsequence of $\{u_n\}$ may be chosen to converge to a function u of class D'_2 on Σ_A so that the functions du_n converge strongly in L_2 to du. Furthermore the u_n and u satisfy

$$(9)\qquad \iint_{C(P_0,r)} d^2 u_n^2\,dx\,dy \leq K_1 L\left(\frac{2\sqrt{2}}{\pi} + \frac{2}{3\pi\nu} + \log\frac{3A}{r}\right)\left(\frac{3\pi+2}{2\pi\nu} + \frac{2\sqrt{2}}{\pi} + \log\frac{3A}{r}\right) M_2 r^{\nu}$$

where K_1 is the constant of lemma 1.

Proof: Let each u_n be extended as in lemma 1 to Σ_{3A} and let d = 0 in $\Sigma_{3A} - \Sigma_A$. Then the functions $d^2 u_n$ are all summable on Σ_{3A} and satisfy (6) uniformly with $k(r) = M_2 r^{\nu}$. We choose a subsequence u_{n_k} so that u_{n_k} converges to a function u on Σ_{3A} and then $d^2 u_n$ converges weakly on Σ_{3A} to $d^2 u$. Evidently the functions $d^2 u_n$ are of the type of the φ of lemma3 with $k_n(r) \leq (K_1 L)^{\frac{1}{2}}\left(\frac{2\sqrt{2}}{\pi} + \frac{2}{3\pi\nu} + \frac{\log 3A}{r}\right)^{\frac{1}{2}}$. Hence the potentials V_n of $d^2 u_n$ are all of class D_2 on Σ_{3A}, and by theorem 6, §6, we see that $\bar{D}_2(V_n - V, \Sigma_{3A}) \to 0$. Also it is clear that the functions $d^2 u_n^2 = (d^2 u_n)\, u_n$ are all summable and we see by theorem 5, §6, that

$$\iint_{\Sigma_{3A}} d^2 (u_n - u)^2\,dx\,dy = \iint_{\Sigma_{3A}} \left[d^2(u_n - u)\right]\cdot(u_n - u)\,dx\,dy = -\iint_{\Sigma_{3A}} \left[(V_{nx} - V_x)(u_{nx} - u_x) + (V_{ny} - V_y)(u_{ny} - u_y)\right] dx\,dy$$

which is easily seen by the Hölder inequality to converge to zero. Formula (8) follows from an application of (6) with $\varphi = d^2 u_n$, and theorem 5, §6.

We now define

$$\|u\| = \left[\bar{D}_2(u, \Sigma_A)\right]^{\frac{1}{2}}$$

for functions u of class D'_2 on Σ_A with $\overline{D}_2(u, \Sigma_A)$ finite. Evidently the space of such functions with this norm is a (complete) Banach space, which we shall call B.

Theorem 1: Let $u \varepsilon B$ and let $U = Tu$ be the unique solution of

$$(10) \quad \int_{R^*} (a_{i\alpha} U^\alpha_x + b_{i\alpha} U^\alpha_y + d_{i\alpha} u^\alpha)\,dy - (b_{\alpha i} U^\alpha_x + c_{i\alpha} U^\alpha_y + e_{i\alpha} u^\alpha)\,dx = \iint_R (d_{\alpha i} u^\alpha_x + e_{\alpha i} u^\alpha_y + f_{i\alpha} u^\alpha)\,dx\,dy$$

which vanishes on Σ^*_A. Then Tu is a linear completely continuous transformation on B.

Proof: Let $\{u_n\}$ be a sequence of functions in B with $\|u_n\|$ uniformly bounded $\leq G$. From lemmas 2 and 3, a subsequence u_{n_k} may be chosen to converge (in our previous sense, not according to our norm) to a function u in B with $\|u\| \leq G$, so that $d_{ij} u^j_n$ converges strongly in L_2 to $d_{ij} u^j$ for each i and j, and sothat $f_{ij} u^j_n$ and also, clearly, $d_{ji} u^j_{nx}$ and $e_{ji} u^j_{nx}$, converge weakly in L_1 to $f_{ij} u^j$, $d_{ji} u^j_x$ and $e_{ji} u^j_y$ respectively where we have (using (2) and (6))

$$(11) \quad \iint_{C(P_0, r)} |d_{\alpha i} u^\alpha_{nx} + e_{\alpha i} u^\alpha_{ny} + f_{i\alpha} u^\alpha_n|\,dx\,dy \leq G\left(M_1^{\frac{1}{2}} r^{\frac{1}{2}} + N K_1 M_1^{\frac{1}{2}} \left[\frac{2\sqrt{2}}{\pi} + \frac{2}{\pi 3\mu} + \log\frac{3A}{r}\right] r^\mu\right) = G P r^{\frac{\mu}{2}}$$

uniformly, where P defends only on M_1, μ, and any given upper bound for A, $\ddagger$ K_1 being the constant of lemma 1. Thus, from theorem 4, §9, it follows that $\|U_n - U\| \to 0$ where U is formed as above from the limiting u.

Theorem 2: Let u^* be of class D'_2 on Σ_A with $D_2(u^*, \Sigma_A)$ finite. Then there exists a unique solution u_ρ in B of (1) which coincides with u^* on Σ^*_A provided that ρ is not one of a denumerable isolated set $\{\rho_n\}$ of characteristic values. If $\rho = \rho_n$ for some n, there exist solutions of the homogeneous equation (1) ($g_i = k_i = l_i = 0$) which vanish on Σ_A^* and are not essentially zero.

Proof: Let be the solution in B of

$$\int_{R^*} (a_{i\alpha}\varphi_x^\alpha + b_{i\alpha}\varphi_y^\alpha + g_i)\,dy - (b_{\alpha i}\varphi_x^\alpha + c_{i\alpha}\varphi_y^\alpha + k_i)\,dx = \iint_R l_i\,dx\,dy$$

which coincides with u* on Σ_A^*. By a well known theorem of F. Riesz, the equation

$$u - \rho\, Tu = \varphi$$

has a unique solution u for each φ in B provided that ρ is not one of a set discrete characteristic values. We observe that the function u so determined by our above φ is the desired solution.

We now define $\|T\|$ for linear transformations T defined from B to B as the greatest lower bounded of all numbers G such that

$$\|Tu\| \le G\,\|u\|$$

for every u in B. Concerning this concept, we now prove

Theorem 3: There exists a number $A_0 > 0$ which depends only on m, M, M_1, and μ (m and M from (13), § 9, and M and μ from (2) of this section) such that $\|T\| < 1/2$ for each a, $0 < A < A_0$, T being the transformation of theorem 1. Thus, for such A, $\rho = 1$ is not a characteristic value.

Proof: Let u be any function of B with $\|u\| \le 1$. From inequalities (6) and (9) and theorem 5, § 6, it follows that U = Tu is the solution of

$$\int_{R^*} (a_{i\alpha}U_x^\alpha + b_{i\alpha}U_y^\alpha + g_i)\,dy - (b_{\alpha i}U_x^\alpha + c_{i\alpha}U_y^\alpha + k_i)\,dx = 0$$

where U' is the potential of $d_{\alpha i}u_x^\alpha + e_{\alpha i}u_y^\alpha + f_{i\alpha}u^\alpha$ and

$$g_i = d_{i\alpha}u^\alpha - V^i_x, \quad k_i = e_{i\alpha}u^\alpha - V^i_y$$

$$\iint_{C(P,r)\cdot\Sigma_A} (g_i^2 + k_i^2)\,dx\,dy \le g^2 \bar{P} r^{\frac{\mu}{2}} \le g^2 \bar{P} A^{\frac{\mu}{2}}$$

where $\bar{P}$ depends only on M_1 and μ and any upper bound for A. Now, by theorem 9, §3, wehave

$$D_2(u, \Sigma_A) \le \frac{4K(M,m)}{m} \iint_{\Sigma_A} (g^2 + k^2)\,dx\,dy \le \frac{4K(M,m)}{m} \cdot \bar{P} g^2 A^{\frac{\mu}{2}}$$

$$\frac{1}{\pi A^2} \iint_{\Sigma_A} u^2\,dx\,dy \le \frac{4}{\pi} D_2(u, \Sigma_A)$$

so that

$$\bar{D}_2(u, \Sigma_A) \le \bar{P}_1 g^2 A^{\frac{\mu}{2}}$$

where $\bar{P}_1$ depends only on M, m, M_1, μ, and any upper bound (say 1) for A. Thus the existence of A_0 follows. Clearly, for $A \le A_0$, we have

$$\frac{1}{2}\|u\| \le \|u - Tu\| \le \frac{3}{2}\|u\|$$

so that the transformation u - Tu has an inverse whose norm is ≤ 2, so that $\rho = 1$ is not a characteristic value.

Theorem 4: Let $A \le A_0$, $l = 0$, and let g and k be of class L_2 on Σ_A. Let u be the solution of (1) with $\rho = 1$ which vanishes on Σ_A^*. Then

(22) $$\|u\|^2 \le 4\left(1 + \frac{4}{\pi}\right)\left(\frac{4K}{m}\right) \iint_{\Sigma_A} (g^2 + k^2)\,dx\,dy$$

the K being the K(M,m) of equation (14'), §9. If g and k satisfy

(13) $$\iint_{C(P_0,r)\,\Sigma_A} (g^2 + k^2)\,dx\,dy \le M_3 r^2, \quad 0 < r < \frac{\mu}{2}$$

for all circles with center at a fixed point P_0, then $\bar{u}(P_0)$ exists and

(14) $$|\bar{u}^i(P_0)| \le P_3 M_3^{\frac{1}{2}} r^{-\frac{3}{2}} A^{\frac{r}{2}}, \quad i = 1, \dots, N$$

where P_3 depends only on m, M, μ_1, M_1, N, and a, the distance of P_0

from Σ_A^*. If g and k satisfy (13) for every P_0 and r, then u is continuous on $\bar{\Sigma}_A$ and satisfies conditions $A[v,\bar{M}]$ and $B[\bar{v}/2,\bar{N}]$ on Σ_A where

(15) $$\bar{M} = P_4 M_3 A^v$$

where P depends only on m, M, μ_1, M_1 and N.

Proof: Since $\|T\| < 1/2$, it is clear that $\|(I - T)^{-1}\| \leq 2$ so that $\|u\| \leq \|\psi\|$, ψ being the solution of (1) with $\rho = 0$ and taking on the desired boundary values. From theorem 3, §9, we see that

$$D_2(\psi, \Sigma_A) \leq \frac{4K}{m} \iint_{\Sigma_A} (g^2 + k^2)\,dx\,dy, \quad \frac{1}{\pi A^2} \iint_{\Sigma_A} \psi^2\,dx\,dy \leq \frac{4}{\pi} D_2(\psi, \Sigma_A)$$

so that

$$\|\psi\|^2 = D_2(\psi, \Sigma_A) + \frac{1}{\pi A^2} \iint_{\Sigma_A} \psi^2\,dx\,dy \leq \left(1 + \frac{4}{\pi}\right) \frac{4K}{m} \iint_{\Sigma_A} (g^2 + k^2)\,dx\,dy$$

from which (12) follows.

Now, from theorem 3, §9, and (12), we see that

(16) $$|\bar{\psi}(P_3)| \leq \left[\frac{8\pi K M_3}{m\lambda}\left\{(2A)^v + \frac{a^v}{v}\right\}\right]^{\frac{1}{2}} \left(1 + \frac{1}{v} + \frac{1}{2}\log\frac{2A}{a}\right)$$

where a is the distance of P_0 from Σ_A^*. Now, let $u \in B$ with $\|u\| \leq S$, let $V^i(x,y)$ be the potential of $d_{\alpha i} u_x^\alpha + e_{\alpha i} u_y^\alpha + f_{i\alpha} u^\alpha$, and we see from (11) and theorem 5, §6 that

$$\iint_{\Sigma_A \cdot C(P_0, r)} (V_x^{i\,2} + V_y^{i\,2})\,dx\,dy \leq (SP)^2 \left[\frac{\sqrt{2}}{\pi} + \frac{4}{3\pi\mu} + \frac{\sqrt{2}}{\mu}\left(\frac{1}{2} + \frac{2}{2-\mu}\right)\right] A^{\frac{\mu}{2}} r^{\frac{\mu}{2}}$$

where P depends only on m, M, M_1, μ_1, and N. Moreover, by inspecting (9) we see that

$$\iint_{\Sigma_A \cdot C(P_0, r)} d_{ij}^2\, u_x^{j\,2}\,dx\,dy \leq K_1 S^2 \cdot P_1 r^{\frac{\mu}{2}}$$

where P_1 depends only on m, M, M_1, μ_1, and N, since $A \leq A_0(m, M, M_1, \mu_1)$,

K_1 being the constant of lemma 1. Finally, if we let

$$g_i = d_{i\alpha} u^{\alpha} - V^i_x, \quad k_i = l_{i\alpha} u^{\alpha} - V^i_y$$

we see that

$$\iint\limits_{C(P_0, r)\,\Sigma_A} (g^2 + k^2)\, dx\, dy \leq P_2 G^2 r^{\frac{\mu}{2}}$$

where P_2 depends only on m, M, M_1, μ_1, and N. Hence if U = Tu, there U is continuous on $\overline{\Sigma}_A$ and satisfies a condition $A[\frac{\mu}{2}, M']$ on Σ_A, where M' depends only on m, M, M_1, μ_1, N, and G, being linear in G^2. Furthermore, by theorem 3,

$$(17) \quad |\overline{U}(P_0)| \leq \left[\frac{8\pi K P_2}{m\lambda}\left\{(2A)^{\frac{\mu}{2}} + \frac{2a^{\frac{\mu}{2}}}{\mu}\right\}\right]^{\frac{1}{2}} \left(1 + \frac{2}{\mu} + \frac{1}{2}\log\frac{2A}{a}\right) G$$

Thus, we see, by combining (16) and (17) and observing from (12) and (13) that

$$(18) \quad G^2 \leq 4\left(1 + \frac{4}{\pi}\right)\left(\frac{4K}{m}\right) M_3 A^{\nu}$$

we find that

$$|\overline{u}(P_0)| \leq P_3 \cdot M^{\frac{1}{2}} r^{-\frac{3}{2}} A^{\frac{\nu}{2}}$$

where P_3 depends only on m, M, μ_1, M_1, N, and a(a<A).

Finally, by theorem 3, §9, we see that if g and k satisfy (13) for every P_0 and r, we see that φ is continuous on $\overline{\Sigma}_A$ and satisfies a condition $A[\nu, M'']$, where

$$(19) \quad M'' = \frac{4KM_3}{m\lambda}\left[(2A^{\mathrm{II}})^{\nu} + \frac{a^{\nu}}{\nu}\right], \quad a \leq A$$

Combining the results of the previous paragraph with equations (18) and 919) we see that u satisfies a condition $A[\nu, \overline{M}]$ where $\overline{M}$ satisfies (15).

§11. The Green's Function

In this section we shall consider the equation (1) of §10 where we assume $A \leqq A_0$, $\rho = 1$, $b_0 = 0$, and where the coefficients a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij}, and f_{ij} satisfy the conditions there set forth. We shall assume also that g_i and k_i are of class L_2 and will add supplementary conditions on g and k as desired. We rewrite our equation as

$$\int_{R^*} (a_{i\alpha} u_x^\alpha + b_{i\alpha} u_y^\alpha + d_{i\alpha} u^\alpha) dy - (b_{\alpha i} u_x^\alpha + c_{i\alpha} u_y^\alpha + e_{i\alpha} u^\alpha) dx =$$

$$(1) \qquad - \int_{R^*} (g_i dy - h_i dx) + \iint_R (d_{\alpha i} u_x^\alpha + e_{\alpha i} u_y^\alpha + f_{i\alpha} u^\alpha) dx\,dy .$$

From an examination of the existence theory which we have developed for (1), we shall derive the existence of a Green's function for (1). We can then derive a double integral representation for u in terms of the values of u and certain "conjugate functions" v and V and G and its conjugate functions. From this, the first main result stated in §8 will be derived.

Let g and k be of class L_p with $p > 2$ on Σ_A. Then (13) of §10 holds for every P_0 and r for a certain M_3 and $\gamma(p) > 0$. Then the solution u of (1) which vanishes on Σ_A^* is continuous on $\bar{\Sigma}_A$ and

$$|u(P_0)| \leq M_4 \left[\iint_{\Sigma_A} (g^2 + k^2)^{\frac{p}{2}} dx\,dy \right]^{\frac{1}{p}}$$

In other words $u(P_0)$ is a bounded and obviously linear functional on the space $(g_i, k_i) \in L_p$, for each $p > 2$. From this we may conclude *) that

$$(2) \qquad \bar{u}^{(i)}(P_0) = \iint_{\Sigma_A} \left[\Delta_{1i\alpha}(x_0, y_0; x, y) g^\alpha(x,y) + \Delta_{2i\alpha}(x_0, y_0; x, y) k^\alpha(x,y) \right] dx\,dy$$

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*) See S. Banach, "Theorie des operations lineaires", Chapter 4.

where Δ_{1ij} and Δ_{2ij} are of class $L_q (p^{-1} + q^{-1} = 1)$ on Σ_A.

Now, let P_o be any point interior to Σ_A, and let $\rho > 0$. Let g and k be of class L_2 in $\Sigma_A - C(P_o, \rho) \cdot \Sigma_A$ and zero in $C(P_o, \rho) \cdot \Sigma_A$. Then it is clear that g and k satisfy (13) of §10 for each ν, $0 < \nu < \mu/2$, if we take

(3) $$M_3 = \rho^{-\nu} \iint_{\Sigma_A - \Sigma_A C(P_0, \rho)} (g^2 + k^2)\, dx\, dy$$

In other words, $u^i(P_o)$ is abounded linear functional over the space of functions g and k of class L_2 on $\Sigma_A - \Sigma_A \cdot C(P_o, \rho)$ and we have

(4) $$|\bar{u}^{(i)}(P_0)| \le P_3 \left(\frac{2A}{\rho}\right)^{\frac{\nu}{2}} \cdot \nu^{-\frac{3}{2}} \| g, k \|$$

where $\| g, k \|$ is that on $\Sigma_A - \Sigma_A \cdot C(P_o, \rho)$, i.e.

$$\| g, k \| = \left[\iint_{\Sigma_A - \Sigma_A C(P_0, \rho)} (g^2 + k^2)\, dx\, dy \right]^{\frac{1}{2}}$$

and P_3 depends only on m, M, μ, M_1, B, A, and a. Thus, from the above chapter in Banach, we conclude that

$$\iint_{\Sigma_A - \Sigma_A C(P_0, \rho)} \sum_{j=1}^{N} (\Delta_{1ij}^2 + \Delta_{2ij}^2)\, dx\, dy \le P_3^2 \left(\frac{2A}{\rho}\right)^{\nu} \nu^{-3}, \quad i = 1, \cdots N$$

for each ν, $0 < \nu < \mu/2$. By investigating the function on the right, we see that

(5) $$\iint_{\Sigma_A - \Sigma_A C(P_0, \rho)} \sum_{j=1}^{N} (\Delta_{1ij}^2 + \Delta_{2ij}^2)\, dx\, dy \le \begin{cases} P_3^2 \left(\frac{2A}{\rho}\right)^{\frac{\mu}{2}} \left(\frac{2}{\mu}\right)^3 & \text{if } \rho \ge 2A e^{-\frac{6}{\mu}} \\ P_3^2 \left(\frac{e}{3}\right)^3 \left(\log \frac{2A}{\rho}\right)^3 & \text{if } \rho \le 2A e^{-\frac{6}{\mu}} \end{cases}$$

We first prove:

Theorem 1: <u>Let D and E be of class L_2 on Σ_A and satisfy</u>

(6) $$\iint_{\Sigma_A \cdot C(P, r)} (D^2 + E^2)\, dx\, dy \le M_5 r^{\nu}, \quad 0 < \nu < \frac{\mu}{2}$$

<u>for every P and r. Then the integrals</u>

(7) $$\iint_{\Sigma_A} (\Delta_{1i\alpha} D^{\alpha} + \Delta_{2i\alpha} E^{\alpha})\, dx\, dy, \quad i = 1, \cdots, N$$

<u>exist as Lebesgue integrals and define functions $U^i(x_o, y_o)$ which are con-</u>

<u>tinuous on $\bar{\Sigma}_A$, and satisfy conditions $A[\nu, \bar{M}]$ and $B[\nu/2, \bar{N}]$ on Σ_A, where $\bar{M}$ and $\bar{N}$ depend only on m, M, μ, M_1, ν, M_5, and A. Moreover</u>

(8) $$\iint_{\Sigma_A \cdot C(P, r)} |\Delta_{1i\alpha} D^\alpha + \Delta_{2i\alpha} E^\alpha| \, dx\,dy \le M_6 \, r^{\frac{\nu}{2}} \left(\log \frac{A}{r}\right)^{\frac{3}{2}}, \quad 0 \le r \le A e^{-\frac{6}{\mu}}$$

<u>for every P and r, where M_6 depends only on m, M, μ, M_1, ν, M_5, A, and a.</u>

Proof: Define $\Delta_{1ij} = \Delta_{2ij} = D^i = E^i = 0$ outside Σ_A and choose $P_0 \in \Sigma_A$. Now, let $0 < \rho < r$, and we see that

$$\iint_{C(P_0, r) - C(P_0, \rho)} \left[\sum_{j=1}^N (\Delta_{1ij}^2 + \Delta_{2ij}^2)\right]^{\frac{1}{2}} \left[\sum_{j=1}^N (D^{j2} + E^{j2})\right]^{\frac{1}{2}} dx\,dy \le \left[\iint_{C(P_0, r)} \sum_{j=1}^N (D^{j2} + E^{j2})\, dx\,dy\right]^{\frac{1}{2}} \cdot$$

Thus, if $0 < r \leqq 2Ae^{-\frac{6}{\mu}}$ we have

$$\iint_{C(P_0, r)} |\Delta_{1i\alpha} D^\alpha + \Delta_{2i\alpha} E^\alpha| \, dx\,dy \le \sum_{k=1}^\infty \iint_{C(P_0, 2^{1-k} r) - C(P_0, 2^{-k} r)} \left[\sum_{j=1}^N (\Delta_{1ij}^2 + \Delta_{2ij}^2)\right]^{\frac{1}{2}} \left[\sum (D^{j2} + E^{j2})\right]^{\frac{1}{2}} dx\,dy$$

$$\le P_3 M_5^{\frac{1}{2}} \left(\frac{e}{3}\right)^{\frac{3}{2}} r^{\frac{\nu}{2}} \sum_{k=1}^\infty \left[\left(\frac{1}{2}\right)^{\frac{\nu}{2}}\right]^{k-1} \left(\log \frac{2^{k+1} A}{r}\right)^{3/2}$$

$$\le P_3 M_5^{\frac{1}{2}} \left(\frac{e}{3}\right)^{\frac{3}{2}} r^{\frac{\nu}{2}} \left(\log \frac{2A}{r}\right)^{\frac{3}{2}} \sum_{k=1}^\infty \left\{\left(\left[\frac{1}{2}\right]^{\nu/2}\right)^{k-1} \left(1 + \frac{k}{\log \frac{2A}{r}} \log 2\right)^{3/2}\right.$$

$$\le P_3 M_5^{\frac{1}{2}} \left(\frac{e}{3}\right)^{\frac{3}{2}} r^{\frac{\nu}{2}} \left(\log \frac{2A}{r}\right)^{\frac{3}{2}} \sqrt{2} \left[\frac{1}{1 - \left(\frac{1}{2}\right)^{\nu/2}} + \left(\frac{\log 2}{\log \frac{2A}{r}}\right)^{\frac{3}{2}} \varphi\left(\left[\frac{1}{2}\right]^\nu\right)\right.$$

$$\le M_5' \, r^{\nu/2} \left(\log \frac{2A}{r}\right)^{\frac{3}{2}}$$

where $\varphi(x) = \sum_{k=1}^{\infty} k^{\frac{3}{2}} x^{k-1}$ and M_5' depends only on m, M, μ, M_1, γ, M_5, A, and a. Thus all the integrals (7) exist as Lebesgue integrals.

Now let C(P, r) be any circle. If $|PP_o| \leq 2r$, then

$$\text{(9)} \qquad \iint_{C(P,r)} |\Delta_{1i\alpha} D^\alpha + \Delta_{2i\alpha} E^\alpha| \, dx\,dy \leq \iint_{C(P_0, 3r)} |\Delta_{1i\alpha} D^\alpha + \Delta_{2i\alpha} E^\alpha| \, dx\,dy \leq 3^{\frac{\sqrt{}}{2}} M_5' \, r^{\frac{\sqrt{}}{2}} \left(\log \frac{2A}{r}\right)^{\frac{3}{2}}, \quad r \leq Ae^{-1}.$$

If $|PP_o| \geq 2r$, then

$$\text{(10)} \qquad \iint_{C(P,r)} |\Delta_{1i\alpha} D^\alpha + \Delta_{2i\alpha} E^\alpha| \, dx\,dy \leq \left[\iint_{C(P,r)} (D^2 + E^2) \, dx\,dy\right]^{\frac{1}{2}} \left[\iint_{\Pi - C(P_0, \delta)} \sum_{j=1}^{N} (\Delta_{1ij}^2 + \Delta_{2ij}^2) \, dx\,dy\right]^{\frac{1}{2}}$$
$$\leq M_5^{\frac{1}{2}} \cdot P_3 \left(\frac{e}{3}\right)^{\frac{3}{2}} \left(\log \frac{A}{r}\right)^{\frac{3}{2}}, \quad 0 \leq r \leq Ae^{-1}$$

It is clear that (8) follows from (9) and (10).

Now, by a simple limiting process in (2), we see that (7) defines the functions $U^i(x,y)$ which are of class D_2' on Σ_A, vanish on Σ_A^*, and satisfy

$$\int_{R^*} (a_{i\alpha} U_x^\alpha + b_{i\alpha} U_y^\alpha + d_{i\alpha} U^\alpha + D^i)\,dy - (b_{\alpha i} U_x^\alpha + c_{i\alpha} U_y^\alpha + e_{i\alpha} U^\alpha + E^i)\,dx$$
$$= \iint_R (d_{\alpha i} U_x^\alpha + e_{\alpha i} U_y^\alpha + f_{i\alpha} U^\alpha)\,dx\,dy$$

on almost all R. These functions U^i are seen by Theorem 4, §10, to satisfy the conditions A and B as stated and to be continuous on $\bar{\Sigma}_A$.

We next prove:

Theorem 2: <u>There exist functions G_{ij} which are of class D_q' on Σ_A for each $q < 2$, vanish on Σ_A^*, and are of class D_2' on $\Sigma_A - P_o$, such that</u>

$$\text{(11)} \qquad \Delta_{1ij} = G_{ijx}, \quad \Delta_{2ij} = G_{ijy}$$

<u>Moreover, if F^i are summable on Σ_A and satisfy</u>

$$\text{(12)} \qquad \iint_{\Sigma_\alpha} G_{i\alpha} F^\alpha \, dx\,dy, \quad i = 1, \dots, N$$

then the integrals

(13) $$\iint_{\Sigma_A} g_{i\alpha} F^{\alpha}\, dx\, dy \qquad i = 1, \ldots, N,$$

exist as Lebesgue integrals, and we have

(14) $$\iint_{\Sigma_A C(P_0, r)} |g_{i\alpha} F^{\alpha}|\, dx\, dy \le P_6 M_7\, b^{r} + P_7 M_7\, r^{\frac{1}{2}}\, r^{-3}\, b^{r} \left(\log \frac{2A}{b}\right)^3$$

for every $C(P, r)$ where P_6 and P_7 depend only on m, M, μ, M_1, N, A, and a. Also, if V^i is the potential of F^i, we have

(15) $$\iint_{\Sigma_A} g_{i\alpha} F^{\alpha}\, dx\, dy = -\iint_{\Sigma_A} (g_{i\alpha x} V^{\alpha}_x + g_{i\alpha y} V^{\alpha}_y)\, dx\, dy$$

Proof: Let $V(x,y)$ be any function satisfying a uniform Lipschitz condition on $\overline{\Sigma}_A$, and define

$$g^i = V^i_y, \quad k^i = -V^i_x$$

and we see that

$$\int_{R^*} (g^i dy - k^i dx) = 0$$

on almost all rectangles R. Since $u = 0$ on Σ^*_A, and since the solution of (1) is unique and given by (2), it follows that

(6) $$u(P_0) = \iint_{\Sigma_A} [\Delta_{1i\alpha}(x_0, y_0; x, y)\, g^{\alpha}(x, y) + \Delta_{2i\alpha}(x_0, y_0; x, y)\, k^{\alpha}(x, y)]\, dx\, dy = \iint_{\Sigma_A} [\Delta_{1i\alpha} V^{\alpha}_y - \Delta_{2i\alpha} V^{\alpha}_x]\, dx\, dy = 0$$

for each such V and each (x_0, y_0). From §7, Theorem 1, it follows that there exist functions $G_{ij}(x_0, y_0; x, y)$ which are of class D'_q in (x, y) on Σ_A for each $q < 2$, which vanish on Σ^*_A, are of class D'_2 on $\Sigma_A - P_0$, and satisfy (11) almost everywhere.

Now, let (x, y) be a point where

$$\iint_{C(x,y;r)} \sum_{\nu=1}^{N} (\xi_{ijx}^2 + \xi_{ijy}^2)\,dx\,dy \leq M_q r^{\pi}, \quad \pi > 0, \quad 0 < r < r_0, \quad i = 1, \dots, N$$

this being true for every (x, y) not in a certain set of measure zero. We know from Lemma 1, §6, that $\bar{G}_{ij}(x_0, y_0;\ x, y)$ exists and that there is a function $\varphi(\vartheta)$ such that

$$\bar{\xi}_{ij}(x_0, y_0; x, y) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\vartheta)\,d\vartheta, \quad \lim_{r\to 0}\int_0^{2\pi} |\bar{\xi}_{ij}(x_0, y_0; x + r\cos\vartheta, y + \sin r\vartheta) - \varphi(\vartheta)|\,d\vartheta = 0$$

$$\int_0^{2\pi}[\bar{\xi}_{ij}(x_0, y_0; x + r\cos\vartheta, y + r\sin\vartheta) - \varphi(\vartheta)]\,d\vartheta = \int_0^r\int_0^{2\pi} \xi_{ijr}(x_0, y_0; x + r\cos\vartheta, y + r\sin\vartheta)\,dr\,d\vartheta$$

Thus, define $\bar{G}_{ij} = 0$ for (x, y) not in Σ_A and G_{ij} is of class D'_α in $\Pi - P_0$ and D'_q in Π for each $q < 2$. Thus

$$\bar{\xi}_{ij}(x_0, y_0; x, y) = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\vartheta)\,d\vartheta = -\frac{1}{2\pi}\int_0^{2\pi}[\bar{\xi}(x_0, y_0; 2A, \vartheta) - \varphi(\vartheta)]\,d\vartheta =$$

$$= -\frac{1}{2\pi}\int_0^{2A}\int_0^{2\pi}\bar{\xi}_{rij}(x_0, y_0; r, \vartheta) = -\frac{1}{2\pi}\iint_{\Sigma_A} \frac{(\xi - x)\,\xi_{ijx}(P_0, \xi, \eta) + (\eta - y)\,\xi_{ijy}(P_0, \xi, \eta)}{(\xi - x)^2 + (\eta - y)^2}\,d\xi\,d\eta$$

almost everywhere.

Now, let $G(x_0, y_0;\ x, y)$ be any function of class D'_2 in (x, y) for $(x, y) \in \Sigma_A - (x_0, y_0)$, and zero on Σ_A^*, and let F(x, y) be any summable function on Σ_A where F and G satisfy

$$\iint_{\Sigma_A - \Sigma_A \cdot C(P_0, r)} (\xi_x^2 + \xi_y^2)\,dx\,dy \leq \begin{cases} P_3^2 \left(\frac{2A}{r}\right)^{\frac{\mu}{2}} \left(\frac{\varepsilon}{\mu}\right)^3, & \text{if } r \geq 2A e^{-\frac{6}{\mu}} \\ P_3^2 \left(\frac{\varepsilon}{3}\right)^2 \left(\log\frac{2A}{r}\right)^3, & \text{if } r \leq 2A e^{-\frac{6}{\mu}} \end{cases}$$

$$\iint_{\Sigma_A \cdot C(P, r)} |F|\,dx\,dy \leq M_7 r^{r}, \quad 0 < r < \frac{\mu}{2}$$

for every P and r, where P_3 depends only on m, M, μ, M_1, N, A, and a. Let

G = 0 outside Σ_A and define

$$H(r) = -\iint_{\Sigma_A - \Sigma_A \cdot C(P_0, r)} (g_x^2 + g_y^2)\, dx\, dy$$

Then H(2A) = 0 and H(r) increases with r. Now

$$\iint_{\Sigma_A} (g_x^2 + g_y^2)\left[1 + \log \frac{2A}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}\right]^{-4} dx\,dy = \lim_{\rho \to 0} \int_\rho^{2A} \left(1 + \log \frac{2A}{r}\right)^{-4} H'(r)\, dr$$

$$= \lim_{\rho \to 0}\left[-\left(1 + \log \frac{2A}{\rho}\right)^{-4} \cdot H(\rho) - 4\int_\rho^{2A} r^{-1}\left(1 + \log \frac{2A}{r}\right)^{-5} H(r)\, dr\right] \le$$

$$\le \lim_{\rho \to 0}\left[4 P_3^2 \left(\frac{e}{3}\right)^2 \int_\rho^{2Ae^{-\frac{6}{\mu}}} \left(\log \frac{2A}{r}\right)^{-2} \frac{dr}{dr} + 4 P_3^2 \int_{2Ae^{-6/\mu}}^{2A} \left(\frac{2A}{r}\right)^{\frac{\mu}{2}-1} r\left(1 + \log \frac{2A}{r}\right)^{-5} dr\right] \le P_5$$

where P_5 depends only on m, M, μ, M_1, N, A, and a. Finally we see that, for each $\rho > 0$, we have

$$\iint_{\Sigma_A}\left[\iint_{\Sigma_A} (g_\xi^2 + g_\eta^2)^{\frac{1}{2}} [(\xi - x)^2 + (\eta - y)^2] \,|F(x,y)|\, d\xi\, d\eta\right] dx\,dy = \iint_{\Sigma_A}\left[\int_\rho^{2A} r^{-1} \frac{dZ(x,y,r)}{dr} dr\right] |F(x,y)|\, dx\,dy$$

$$\le \iint_{\Sigma_A}\left[(2A)^{-1} \iint_{\Sigma_A} (g_\xi^2 + g_\eta^2)^{\frac{1}{2}} d\xi\, d\eta\right] |F|\, dx\,dy + \int_\rho r^{-2} \iint_{\Sigma_A} \iint_{C(x,y,r)\cdot\Sigma_A} (g_\xi^2 + g_\eta^2)^{\frac{1}{2}} |F|\, d\xi\, d\eta\, dx\,dy$$

$$\le \iint_{\Sigma_A} |F|\, dx\,dy\, (2A)^{-1}\left[\iint_{\Sigma_A} (g_\xi^2 + g_\eta^2)\left\{\log \frac{2A}{\sqrt{(\xi-x_0)^2 + (\eta - y_0)^2}}\right\}^{-4} d\xi\, d\eta\right]^{\frac{1}{2}} \left[\iint_{\Sigma_A} \left\{\log \frac{2A}{\sqrt{(\xi-x_0)^2 + \cdots}}\right\}^4 d\xi\, d\eta\right]^{\frac{1}{2}}$$

(16) $$+ \int_\rho^{2A} \left\langle r^{-2}\left[\iint_{\Sigma_A} \left\{\iint_{C(\xi,\eta,r)\Sigma_A} |F|\, dx\,dy\right\} (g_\xi^2 + g_\eta^2)\left\{\log \frac{2A}{\sqrt{(\xi - x_0)^2 + \cdots}}\right\}^{-4} d\xi\, d\eta\right]^{\frac{1}{2}} \cdot\right.$$

$$\left. \cdot \left[\iint_{\Sigma_A} \left\{\iint_{\Sigma_A C(x,y,r)} \left[\log \frac{2A}{\sqrt{(\xi-x_0)^2 + \cdots}}\right]^4 d\xi\, d\eta\right\} |F|\, dx\,dy\right]^{\frac{1}{2}} \right\rangle dr$$

$$\le P_5^{\frac{1}{2}} \left(\frac{3}{4}\right)^{\frac{1}{2}} \iint_{\Sigma_A} |F|\, dx\,dy + \int_\rho^{2A} r^{-2}\left[\frac{11 r^2}{2}\left(\log \frac{2A}{r}\right)^4\right]^{\frac{1}{2}} \left[\iint_{\Sigma_A} |F|\, dx\,dy\right]^{\frac{1}{2}} \cdot P_5^{\frac{1}{2}} [h(r)]^{\frac{1}{2}} dr$$

$$\le P_5^{\frac{1}{2}} \left(\frac{3}{4}\right)^{\frac{1}{2}} \iint_{\Sigma_A} |F|\, dx\,dy + P_5^{\frac{1}{2}} \left(\frac{11}{2}\right)^{\frac{1}{2}} \left[\iint_{\Sigma_A} |F|\, dx\,dy\right]^{\frac{1}{2}} \int_\rho^{2A} r^{-1} \left(\log \frac{2A}{r}\right)^2 [h(r)]^{\frac{1}{2}} dr$$

where we have put

$$h(r) = \max_{(x,y) \in \Sigma_A} \iint_{C(P,r)\cdot\Sigma_A} |F| dx\,dy \le M_7 r^\nu$$

The right side of (16) is thus evidently bounded independently of ρ. Hence the integral

$$\iint_{\Sigma_A} \iint_{\Sigma_A} d^{-\frac{1}{2}}[(\xi-x)\, g_\xi + (\eta-y)\, g_\eta]\, |F|\, d\xi\, d\eta\, dx\, dy \quad , d = [(\xi-x)^2 + (\eta-y)^2]$$

exists as a 4-dimensional Lebesgue integral. Moreover, if we let $F_{P_1, b}(x,y) = F(x,y)$ if $(x,y) \in C(P_1, b)\cdot \Sigma_A$, and let it be zero otherwise, we see that

$$h_{F_{P_1,b}}(r) \le M_7 r^\nu, \quad \cancel{h_{P_1 b}(r)}, \quad b \le A$$

for all r. Then, in this case we see, by substituting in (16), that

(17) $$\iint_{\Sigma_A} \iint_{\Sigma_A} (g_\xi^2 + g_\eta^2)\, d^{\frac{1}{2}} |F_{P_1,b}(x,y)|\, dx\, dy\, d\xi\, d\eta \le (P_5 M_7)^{\frac{1}{2}} b^\nu \left[\left(\tfrac{3}{4}\right)^{\frac{1}{2}} M_7^{\frac{1}{2}} + \right.$$

$$\left. + \left(\tfrac{11}{2}\right)^{\frac{1}{2}} \left(\tfrac{1}{3} + \tfrac{2}{\nu} + \tfrac{8}{\nu^2} + \tfrac{16}{\nu^3}\right)\left(\log \tfrac{2A}{b}\right)^3\right] \le \tilde{P}_6 M_7 b^\nu + \tilde{P}_7 \nu^{-3} M_7^{\frac{1}{2}} b^\nu \left(\log \tfrac{2A}{b}\right)^3$$

where $\tilde{P}_6$ and $\tilde{P}_7$ depend only on m, M, μ, M_1, N, A, and a.

Now evidently (14) follows immediately from (17). Moreover, the above shows that the following is legitimate:

$$\iint_{\Sigma_A} g_{i\alpha} F^\alpha dx\,dy = -\frac{1}{2\pi}\iint_{\Sigma_A}\iint_{\Sigma_A} d^{\prime}[(\xi-x)\, g_{i\alpha\xi} + (\eta-y)\, g_{i\alpha\eta}]\, F^\alpha(x,y)\, d\xi\, d\eta\, dx\, dy$$

$$= -\iint_{\Sigma_A}\left\{ g_{i\alpha\xi}\left[-\frac{1}{2\pi}\iint_{\Sigma_A} d(\xi-x)\, F^\alpha(x,y)\, dx\,dy\right] + g_{i\alpha\eta}\left[-\frac{1}{2\pi}\iint_{\Sigma_A} d(\eta-y)\, F^\alpha(x,y)\, dx\,dy\right]\right\} d\xi\, d\eta$$

$$= -\iint_{\Sigma_A} (g_{i\alpha x} V_x^\alpha + g_{i\alpha y} V_y^\alpha)\, dx\,dy$$

which proves (15).

Theorem 3: <u>The functions $G_{ij}(x_0, y_0; x, y)$ satisfy the equations</u>

(18) $$\int_{R^*} (a_{i\beta}\, g_{j\beta x} + b_{i\beta}\, g_{j\beta y} + d_{i\beta}\, g_{j\beta})\, dy - (b_{\beta i}\, g_{j\beta x} + c_{i\beta}\, g_{j\beta y} + e_{i\beta}\, g_{j\beta})\, dx$$

$$= \iint_R (d_{\beta i}\, g_{j\beta x} + e_{\beta i}\, g_{j\beta y} + f_{i\beta}\, g_{j\beta})\, dx\,dy + \delta_{ij}\, \Phi_0(x_0, y_0, R)$$

where

(19) $$\phi_0(x_0, y_0; e) = \begin{cases} 1 & \text{if } e \text{ contains } (x_0, y_0) \\ 0 & \text{if } e \text{ does not contain } (x_0, y_0), \end{cases}$$

and the double integral on the right exists as a Lebesgue integral. The functions G_{ij} satisfy conditions $A[\gamma, M(a,d)]$ and $B[\gamma/2, N(a,d)]$ on $\Sigma_A - P_0$, where $M(a,d)$ and $N(a,d)$ depend only on $a, d, m, M, \mu, M_1, N, r$, and A. Moreover there exist functions K_{1ij} and H_{ij} with these same properties such that

$$K_{1ijx} + K_{2ijx} + H_{ijy} = a_{i\beta} \mathcal{G}_{j\beta x} + b_{i\beta} \mathcal{G}_{j\beta y} + d_{i\beta} \mathcal{G}_{j\beta}$$

(20) $$K_{1ijy} + K_{2ijy} - H_{ijx} = b_{\beta i} \mathcal{G}_{j\beta x} + c_{i\beta} \mathcal{G}_{j\beta y} + e_{i\beta} \mathcal{G}_{j\beta}$$

$$\int_{R^*} (K_{1ijx}\, dy - K_{1ijy}\, dx) = \iint_R (d_{\beta i} \mathcal{G}_{j\beta x} + e_{\beta i} \mathcal{G}_{j\beta y} + f_{i\beta} \mathcal{G}_{j\beta})\, dx\, dy$$

$$K_{2ij} = \frac{\delta_{ij}}{4\pi} \log[(x-x_0)^2 + (y-y_0)^2]$$

hold with the usual limitation.

Proof: Let V be any function which satisfies a uniform Lipschitz condition on $\overline{\Sigma}_A$, and let W^i be the potential function of $d_{\alpha i} V^\alpha_x + e_{\alpha i} V^\alpha_y + f_{i\alpha} V^\alpha$. Clearly, from (2) of §10, it follows that

$$\iint_{C(P,r)\,\Sigma_A} |d_{\alpha i} V^\alpha_x + e_{\alpha i} V^\alpha_y + f_{i\alpha} V^\alpha|\, dx\, dy \le K_3 r^\mu$$

so that W is of class D_2 over the whole plane and satisfies a condition $A(\mu, M''')$ there. We next define

$$g^i = W^i_x - (a_{i\alpha} V^\alpha_x + b_{i\alpha} V^\alpha_y + d_{i\alpha} V^\alpha); \quad k^i = W^i_y - (b_{\alpha i} V^\alpha_x + c_{i\alpha} V^\alpha_y + e_{i\alpha} V^\alpha)$$

and we see that g^i and k^i satisfy (6). We then see, by substitution in (1) that u satisfies

$$\int_{R^*} (a_{i\alpha} u^\alpha_x + b_{i\alpha} u^\alpha_y + d_{i\alpha} u^\alpha)\, dy - (b_{\alpha i} u^\alpha_x + c_{i\alpha} u^\alpha_y + e_{i\alpha} u^\alpha)\, dx - \iint_R (d_{\alpha i} u^\alpha_x + e_{\alpha i} u^\alpha_y + f_{i\alpha} u^\alpha)\, dx\, dy$$

$$= \int_{R^*} (a_{i\alpha} V^\alpha_x + b_{i\alpha} V^\alpha_y + d_{i\alpha} V^\alpha)\, dy - (b_{\alpha i} V^\alpha_x + c_{i\alpha} V^\alpha_y + e_{i\alpha} V^\alpha)\, dx - \iint_R (d_{\alpha i} V^\alpha_x + e_{\alpha i} V^\alpha_y + f_{i\alpha} V^\alpha)\, dx\, dy$$

on almost all R, since

$$\int_{R^*} W^i_x\,dy - W^i_y\,dx = \iint_R (d_{\alpha i} V^\alpha_x + e_{\alpha i} V^\alpha_y + f_{i\alpha} V^\alpha)\,dx\,dy$$

for almost all R. Since V = 0 on Σ^*_A and since u is unique, it follows that u = V on Σ^*_A. Thus

$$V^i(x_0,y_0) = -\iint_{\Sigma_A} [g_{i\alpha x}(a_{\alpha\beta} V^\beta_x + b_{\alpha\beta} V^\beta_y + d_{\alpha\beta} V^\beta) + g_{i\alpha y}(b_{\beta\alpha} V^\beta_x + c_{\alpha\beta} V^\beta_y + e_{\alpha\beta} V^\beta)]\,dx\,dy$$
$$+ \iint_{\Sigma_A} (g_{i\alpha x} W^\alpha_x + g_{i\alpha y} W^\alpha_y)\,dx\,dy$$

where the second integral on the right exists as a Lebesgue integral. We then see, using Theorem 2, and the fact that W satisfies a condition A(ν, M '') all over the plane, that

$$\iint_{\Sigma_A} (g_{i\alpha x} W^\alpha_x + g_{i\alpha y} W^\alpha_y)\,dx\,dy = -\iint_{\Sigma_A} g_{i\alpha}(d_{\beta\alpha} V^\beta_x + e_{\beta\alpha} V^\beta_y + f_{\alpha\beta} V^\beta)\,dx\,dy$$

so that

$$\iint_{\Sigma_A} [(a_{\alpha\beta} g_{i\beta x} + b_{\alpha\beta} g_{i\beta y} + d_{\alpha\beta} g_{i\beta}) V^\alpha_x + (b_{\beta\alpha} g_{i\beta x} + c_{\alpha\beta} g_{i\beta y} + e_{\alpha\beta} g_{i\beta}) V^\alpha_y]\,dx\,dy \tag{21}$$
$$+ \iint_{\Sigma_A} V^\alpha\, d\phi_{i\alpha} = 0$$

for each V satisfying a uniform Lipschitz condition on $\bar{\Sigma}_A$ and zero on Σ^*_A, where

$$\phi_{ij}(x_0,y_0;e) = \iint_e (d_{\beta j} g_{i\beta x} + e_{\beta j} g_{i\beta y} + f_{j\beta} g_{i\beta})\,dx\,dy + \delta_{ij}\,\phi_0(x_0,y_0,e) \tag{22}$$

where we see from Theorems 1 and 2 that $\phi_{ij}(x_0, y_0; e)$ is completely additive on Σ_A and

$$\iint_{\Sigma_A \cdot C(P_0,r)} |d_{\beta i} g_{j\beta x} + e_{\beta i} g_{j\beta x} + f_{i\beta} g_{j\beta}|\,dx\,dy \le K_4 r^{\nu\mu}, \quad r > 0 \tag{23}$$

for every P and P_0 in Σ_A. From (23) and Theorem 2, §7, it follows that (18) holds.

Now, let K_{1ij} be the potential of $d_{\beta i}G_{j\beta x} + e_{\beta i}G_{j\beta y} + f_{i\beta}G_{j\beta}$ and we see that K_{1ij} satisfies a condition $A[\nu, \overline{M}]$ over the whole plane. Using Theorems 1 and 2 several times, we see that G_{ij} satisfies

(24) $$\int_{R^*} (a_{i\beta} g_{j\beta x} + b_{i\beta} g_{j\beta y} + \gamma_{ij})\,dx - (b_{\beta i} g_{j\beta x} + c_{i\beta} g_{j\beta y} + \kappa_{ij})\,dx = 0$$

where

(25) $$\iint_{C(P,r)\,\Sigma_A} \sum_{i=1}^{N} (\gamma_{ij}^2 + \delta_{ij}^2)\,dx\,dy \le K_5 r^{\nu'}, \quad \gamma_{ij} = d_{i\beta} g_{i\beta} - K_{1ijx}, \quad \delta_{ij} = e_{i\beta} g_{j\beta} - K_{1ijy}$$

so that G_{ij} satisfy conditions A and B as indicated. From (24) and (25) and the fact that K_{1ij} is the potential of the equations (20) follow.

Theorem 4: <u>Let u, v, and V be of class D_2 on $\Sigma_A = C(P_1, A)$</u>

$$V_x^i + v_y^i = a_{i\beta} u_x^\beta + b_{i\beta} u_y^\beta + d_{i\beta} u^\beta, \quad V_y^i - v_x^i = b_{\beta i} u_x^\beta + c_{i\beta} u_y^\beta + e_{i\beta} u^\beta$$

(26) $$\int_{R^*} V_x^i\,dy - V_y^i\,dx = \iint_R (d_{\beta i} u_x^\beta + e_{\beta i} u_y^\beta + f_{i\beta} u^\beta)\,dx\,dy$$

Then u(x,y) is given in $C(P_1, \frac{A}{2})$ by ~~$+dy$~~

$$u^j(x_0,y_0) = K_1 \int_{\frac{A}{2}}^{\frac{3A}{4}} \int_{\frac{3A}{4}}^{A} r_1 r_2 \Bigg[\iint_{C(P_1 r_2) - C(P_1, r_1)} \Bigg\{ \frac{u^\alpha[(x-x_0)(a_{\alpha\beta} g_{j\beta x} + b_{\alpha\beta} g_{j\beta y} + d_{\alpha\beta} g_{j\beta}) + (y-y_0)\boxed{(b_{\beta\alpha} g_{j\beta x} + c_{\alpha\beta} g_{j\beta y} + e_{\alpha\beta} g_{j\beta})}]}{(x-x_0)^2 + (y-y_0)^2}$$

(27) $$+ \frac{v^\alpha[(x-x_0) g_{j\alpha y} - (y-y_0) g_{j\alpha x}] + V^\alpha[(x-x_0) g_{j\alpha x} + (y-y_0) g_{j\alpha y}]}{(x-x_0)^2 + (y-y)^2} \Bigg\}\,dx\,dy \Bigg]\,dr_1\,dr_2$$

$$- K_2 \iint_{C(P_1,A) - C(P_1, \frac{3A}{4})} g_{j\alpha} V^\alpha\,dx\,dy + K_3 \iint_{C(P_1, \frac{3A}{4}) - C(P_1, \frac{A}{2})} g_{j\alpha} V^\alpha\,dx\,dy$$

$$K_1 = \frac{128}{\pi A^4 \sigma}, \quad K_2 = \frac{20}{\pi A^2 \sigma}, \quad K_3 = \frac{28}{\pi A^2 \sigma}, \quad \sigma = 49 \log 2 - 27 \log 3$$

Proof: By the methods of §§1 and 6, it may be shown that there exist sets Z_1 and Z_2 of linear measure zero such that if r is not in Z_1, and is not in Z_2, and $\overline{C(P_0, \rho)} \subset C(P_1, r)$, we have

$$\int_{C^*(P_1,r)-C^*(P_0,\rho)} u^\alpha (K_{1\alpha jx}\,dy - K_{1\alpha jy}\,dx) = \int_{C^*(P_1,r)} u^\alpha r K_{1\alpha jr}\,d\vartheta - \int_{C^*(P_0,\rho)} u^\alpha \rho K_{1\alpha j\rho}\,d\vartheta =$$

$$= \iint_{C(P_1,r)-C(P_0,\rho)} (u^\alpha_x K_{1\alpha jx} + u^\alpha_y K_{1\alpha jy} + u^\alpha (d_{\beta\alpha} g_{j\beta x} + e_{\beta\alpha} g_{j\beta y} + f_{\alpha\beta} g_{j\beta})\,dx\,dy,$$

$$\int_{C^*(P_1,r)-C^*(P_0,\rho)} u^\alpha (K_{2\alpha jx}\,dy - K_{2\alpha jy}\,dx) = \int_{C^*(P_1,r)} u^j \frac{(\xi-x_0)\,dy - (\eta-y_0)\,dx}{(\xi-x_0)^2+(\eta-y_0)^2} - \int_0^{2\pi} u^j(x_0+\rho\cos\vartheta, y_0+\rho\sin\vartheta)\,d\vartheta$$

$$= \iint_{C(P_1,r)-C(P_0,\rho)} (K_{2\alpha jx} u^\alpha_x + K_{2\alpha jy} u^\alpha_y)\,dx\,dy$$

$$\int_{C^*(P_1,r)-C^*(P_0,\rho)} u^\alpha\, dH_{\alpha j} = \iint_{C(P_1,r)-C(P_0,\rho)} (u^\alpha_x H_{\alpha jy} - u^\alpha_y H_{\alpha jx})\,dx\,dy, \quad \int_{C^*(P_1,r)-C^*(P_0,\rho)} g_{i\alpha}\,dv^\alpha = \iint_{C-C} (g_{i\alpha x} v^\alpha_y - g_{i\alpha y} v^\alpha_y)\,dx\,dy$$

$$\int_{C^*(P_1,r)-C^*(P_0,\rho)} g_{i\alpha}(V^\alpha_x\,dy - V^\alpha_y\,dx) = \int_{C^*(P_1,r)} r\, g_{j\alpha} V^\alpha_r\,d\vartheta - \int_{C^*(P_0,\rho)} \rho\, g_{i\alpha} V^\alpha_\rho\,d\vartheta =$$

$$= \iint_{C(P_1,r)-C(P_0,\rho)} [g_{j\alpha x} V^\alpha_x + g_{j\alpha y} V^\alpha_y + g_{j\alpha}(d_{\beta\alpha} u^\beta_x + e_{\beta\alpha} u^\beta_y + f_{\alpha\beta} u^\beta)]\,dx\,dy$$

Adding all of the above equations and using (20) and (26), we see that

$$\int_{C^*(P_1,r)} [r(u^\alpha K_{1\alpha jr} + u^\alpha K_{2\alpha jr} - g_{j\alpha} V^\alpha_r) + u^\alpha H_{\alpha j\vartheta} - g_{i\alpha} v^\alpha_\vartheta]\,d\vartheta$$

(28)
$$- \int_{C^*(P_0,\rho)} [\rho(u^\alpha K_{1\alpha j\rho} + g_{j\alpha} V^\alpha_\rho) + u^\alpha H_{\alpha j\varphi} - g_{i\alpha} v^\alpha_\varphi]\,d\varphi = \int_0^{2\pi} u^j(x_0+\rho\cos\vartheta, y_0+\rho\sin\vartheta)\,d\vartheta$$

if r is not in Z_1 and ρ is not in Z_2. We may prove also that for almost all we have

$$\int_{C^*(P_0,\rho)} K_{1ij\,x}\,dy - K_{1ij\,y}\,dx = \iint_{C(P_0,\rho)} (d_{\beta i}\,\zeta_{j\beta x} + e_{\beta i}\,\zeta_{j\beta y} + f_{i\beta}\,\zeta_{j\beta})\,dx\,dy$$

and the right side evidently tends to zero with ρ ; we shall call this $\varepsilon_{ij}(x_o, y_o; \rho)$. We then see that if r is not in Z_1 and ρ is not in $\tilde{Z}_2$, we habe

$$\int_{C^*(P_0,r)} [r(u^\alpha K_{1\alpha j r} + u^\alpha K_{2\alpha j r} - \zeta_{j\alpha} V^\alpha_r) + u^\alpha H_{\alpha j \vartheta} - \zeta_{j\alpha} v^\alpha_\vartheta]\,d\vartheta - \varepsilon_{\alpha j}(x_0,y_0,\rho)\,u^\alpha(x_0,y_0)$$

(29) $$- \int_{C^*(P_0,\rho)} [\rho\{(u^\alpha - u^\alpha_0) K_{1\alpha j\rho} + \zeta_{j\alpha} V^\alpha_\rho\} + (u^\alpha - u^\alpha_0) H_{\alpha j\varphi} - \zeta_{j\alpha} v^\alpha_\varphi]\,d\varphi$$

$$= \int_0^{2\pi} u^j(x_0 + \rho\cos\vartheta,\ y_0 + \rho\sin\vartheta)\,d\vartheta$$

where we have set $u^\alpha_o = u^\alpha(x_o, y_o)$.

Next we observe that , putting $d_0 = (x-x_0)^2 + (y-y_0)^2$,

$$\int_0^{\rho_0}\Big|\int_{C^*(P_0,\rho)} [\rho\{(u^\alpha - u^\alpha_0) K_{1\alpha j\rho} + \zeta_{j\alpha} V^\alpha_\rho\} + (u^\alpha - u^\alpha_0) H_{\alpha j\varphi} - \zeta_{j\alpha} V^\alpha_\varphi]\,d\varphi\Big|\,d\rho \leq$$

$$\leq \iint_{C(P_0,\rho_0)} \Big| \frac{u^\alpha - u^\alpha_0}{\sqrt{d_0}} [(x-x_0)(K_{1\alpha jx} + H_{\alpha jy}) + (y-y_0)(K_{1\alpha jy} - H_{\alpha jx})] - \frac{\zeta_{j\alpha}}{\sqrt{d_0}}((x-x_0)(V^\alpha_x + v^\alpha_y) + (y-y_0)(V^\alpha_y - v^\alpha_x))\Big|\,dx\,dy$$

$$= \iint_{C(P_0,\rho_0)} \Big| \frac{(u^\alpha - u^\alpha_0)}{\sqrt{d_0}} [(x-x_0)(a_{\alpha\beta}\,\zeta_{j\beta x} + b_{\alpha\beta}\,\zeta_{j\beta y} + d_{\alpha\beta}\,\zeta_{j\beta}) + (y-y_0)(b_{\beta\alpha}\,\zeta_{j\beta x} + c_{\alpha\beta}\,\zeta_{j\beta y} + e_{\alpha\beta}\,\zeta_{j\beta})]$$

$$+ \frac{u^j - u^j_0}{\sqrt{d_0}} - \frac{\zeta_{j\alpha}}{\sqrt{d_0}} [(x-x_0)(a_{\alpha\beta} u^\beta_x + b_{\alpha\beta} u^\beta_y + d_{\alpha\beta} u^\beta) + (y-y_0)(b_{\alpha\beta} u^\beta_x + c_{\alpha\beta} u^\beta_y + e_{\alpha\beta} u^\beta)]\Big|\,dx\,dy$$

$$\leq$$

$$\leq \iint_{C(P_0,\rho)} \frac{|u^i - u_0^i|}{\sqrt{d_0}}\, dx\,dy + M \iint_{C(P_0,\rho)} \left[\sum_{k=1}^{N}(u^k - u_0^k)^2\right]^{\frac{1}{2}} \left[\sum_{k=1}^{N}(g_{jkx}^2 + g_{jky}^2)\right]^{\frac{1}{2}} dx\,dy +$$

$$+ \iint_{C(P_0,\rho)} \left\{\sum_{i=1}^{N}\sum_{k=1}^{N} |u^i - u_0^i| \left(|\alpha_{ik}| + |\alpha_{ki}|\right) |g_{jk}|\right\} dx\,dy + M \iint_{C(P_0,\rho)} \left[\sum_{k=1}^{N} g_{jk}^2\right]^{\frac{1}{2}} \left[\sum_{k=1}^{N}(u_x^{k^2} + u_y^{k^2})\right]^{\frac{1}{2}} dx\,dy$$

$$\leq \frac{2\pi \bar{N}\rho^{\nu'}}{\alpha^{\nu'}} + M \left\{\iint_{C(P_0,\rho)} \left[\sum_{k=1}^{N}(u^k - u_0^k)^2\right]^{\frac{p}{2}} dx\,dy\right\}^{\frac{1}{p}} \left\{\iint_{C(P_0,\rho)} \left[\sum_{k=1}^{N}(g_{jkx}^2 + g_{jky}^2)\right]^{\frac{q}{2}} dx\,dy\right\}^{\frac{1}{q}}$$

$$+ \sum_{i=1}^{N}\sum_{j=1}^{N} \left[\iint_{C(P_0,\rho)} (u^i - u_0^i)^2\right]^{\frac{1}{2}} \left[\iint_{C(P_0,\rho)} (|\alpha_{ik}| + |\alpha_{ki}|)^2 g_{jk}^2\, dx\,dy\right]^{\frac{1}{2}} + M\left[D_2(u, C(P_0,\rho))\right]^{\frac{1}{2}} \left[\iint_{C(P_0,\rho)} \sum_{k=1}^{N} g_{jk}^2\, dx\,dy\right]^{\frac{1}{2}}$$

where the M is our original bound M. If we remember that u satisfies conditions A[ν', $\bar{M}$] and B[ν', $\bar{N}$], and choose p properly and slightly > 2 with $p^{-1} + q^{-1} = 1$, and remember the fact, sketched in the proof of Theorem 3, that $d^2 G_{jk}^2$ is summable under our hypotheses on d, and if we apply Theorems 1 and 2 several times, we see that the above tends to zero more rapidly than ρ_0. Hence, for each r not in Z_1, we can pick a sequence of values of tending to zero so that the integral over $C^*(P_0, \rho)$ in (29) tends to zero. We therefore see that, for each P_0 in $C(P_1, \frac{A}{2})$ say and each r not in a set $Z_1(P_0)$ of measure zero with $r \geqq \frac{A}{2}$, we have

$$u^j(x_0, y_0) = \frac{1}{2\pi} \int_{C^*(P_1,r)} \left[r(u^\alpha K_{1\alpha jr} + u^\alpha K_{2\alpha jr} - g_{j\alpha} V_r^\alpha) + u^\alpha H_{\alpha j\vartheta} - g_{j\alpha} v_\vartheta^\alpha\right] d\vartheta$$

(30)

$$= \frac{1}{2\pi} \int_{C^*(P_1,r)} \left[r(u^\alpha K_{1\alpha jr} + u^\alpha K_{2\alpha jr} - g_{j\alpha} V_r^\alpha) + u^\alpha H_{\alpha j\vartheta} + v^\alpha g_{j\alpha\vartheta}\right] d\vartheta$$

We now choose r_1 and r_2 with $\frac{A}{2} \leq r_1 \leq \frac{3A}{4}$ and $\frac{3A}{4} \leq r_2 \leq A$. We then divide (30) by r and integrate between r_1 and r_2, and we obtain

$$\left(\log \frac{r_2}{r_1}\right) u^\alpha(x_0, y_0) = \frac{1}{2\pi} \int_{r_1}^{r_2} \int_0^{2\pi} \left[u^\alpha \left(K_{1\alpha j r} + K_{2\alpha j r} + \frac{H_{\alpha j \vartheta}}{r} \right) + v^\alpha \frac{\xi_{j\alpha\vartheta}}{r} \right] dr\, d\vartheta$$

$$- \frac{1}{2\pi} \int_0^{2\pi} \xi_{j\alpha}(P_0, r_2, \vartheta) V^\alpha(r_2, \vartheta)\, d\vartheta + \frac{1}{2\pi} \int_0^{2\pi} \xi_{j\alpha}(P_0, r_1, \vartheta) V^\alpha(r_1, \vartheta) + \frac{1}{2\pi} \int_{r_1}^{r_2} \int_0^{2\pi} V^\alpha \xi_{j\alpha r}\, dr\, d\vartheta$$

(31)

$$= \frac{1}{2\pi} \iint_{C(P_1, r_2) - C(P_1, r_1)} d_0^{-1} \Big\{ u^\alpha \left[(x - x_0)(K_{1\alpha j x} + K_{2\alpha j x} + H_{\alpha j y}) + (y - y_0)(K_{1\alpha j y} + K_{2\alpha j y} - H_{\alpha j x}) \right]$$

$$+ v^\alpha \left[(x - x_0) \xi_{j\alpha y} - (y - y_0) \xi_{j\alpha x} + V^\alpha \left[(x - x_0) \xi_{j\alpha x} + (y - y_0) \xi_{j\alpha y} \right] \right\} dx\, dy$$

$$- \frac{1}{2\pi} \int_0^{2\pi} \left[\xi_{j\alpha}(P_0, r_2, \vartheta) V^\alpha(r_2, \vartheta) - \xi_{j\alpha}(P_0, r_1, \vartheta) V^\alpha(r_1, \vartheta) \right] d\vartheta$$

Then (27) follows by multiplying (31) by $r_1 r_2$ and integrating with respect to r_1 and r_2 over the region $\frac{A}{2} \leq r_1 \leq \frac{3A}{2} \leq r_2 \leq A$.

Theorem 5: Let u, v, and V be of class D_2 on a region G and satisfy

$$V_x^i + v_y^i = a_{i\beta} u_x^\beta + b_{i\beta} u_y^\beta + d_{i\beta} u^\beta + g_i, \quad V_y^i - v_x^i = b_{\beta i} u_x^\beta + c_{i\beta} u_y^\beta + e_{i\beta} u^\beta + k_i,$$

(32)

$$\int_{R^*} V_x^i\, dy - V_y^i\, dx = \iint_R (d_{\beta i} u_x^\beta + e_{\beta i} u_y^\beta + f_{i\beta} u^\beta + l_i)\, dx\, dy$$

where the coefficients a, b, c, d, e, and f satisfy their previous conditions on G, and where

(33) $$\iint_{G \cdot C(P, r)} (g^2 + k^2 + |l|)\, dx\, dy \leq M_8\, r^\nu, \quad \iint_{G \cdot C(P, r)} (u^2 + v^2 + V^2)\, dx\, dy \leq M_9\, r^\sigma$$

$$\nu > 0, \quad \sigma > 0.$$

Then the functions u, v, and V satisfy conditions $A[\tau, M(a,d)]$ and $B[\tau/2, N(a,d)]$ on G where the choice of τ, $M(a,d)$, and $N(a,d)$ depends only on m, M, μ, M_1, ν, M_8, σ, M_9, and N. In particular, u, v, and V satisfy a uniform Hölder condition on each bounded closed subregion $\bar{H}$ of G where this Hölder condition depends only on m, M, μ, M_1, ν, M_8, σ, M_9, N, and the distance of H from G^*.

Proof: Let P_1 be any point of G, and let its distance from G^* be A ($\leq A_0$). Let u_0 be the solution of (1) which vanishes on Σ_A^*, let V_0^1 be the potential of $d_{\beta i} u_{0x}^{\beta} + e_{\beta i} u_{0y}^{\beta} + f_{i\beta} u_0^{\beta} + \ell_i$, and then choose v_0 to satisfy (32) with u_0 and V_0. From our previous theorems, u_0, v_0, and V_0 satisfy conditions $A[\tau_1, \bar{M}]$ and $B[\tau_1/2, \bar{N}]$ on $\bar{\Sigma}_A$ and satisfy conditions like (33) on Σ_A where the τ_1, $\bar{M}$, $\bar{N}$, and corresponding M_9' of (33) depend only on m, M, μ, M_1, ν, M_8, N, and A[$\bar{M}$, $\bar{N}$, and M_9' tending to zero with A].

Thus, if we let $u_1 = u - u_0$, $v_1 = v - v_0$, $V_1 = V - V_0$, we see that u_1, v_1, and V_1 satisfy (26) almost everywhere on Σ_A, and u_1, v_1, and V_1 satisfy (33) with a new M_9' which depends on the indicated quantities. Thus a representation of u_1 on $C(P_1, \frac{A}{2})$ by (27) is valid. We observe that this representation gives

$$u_1^j(x_0, y_0) = \iint_{\Sigma_A} (g_{j\alpha x} G^\alpha + g_{j\alpha y} K^\alpha + g_{j\alpha} L^\alpha)\, dx\, dy$$

where G^α, K^α, and L^α are so related to u_1, v_1, and V_1, that we have

$$\iint_{C(P_1,A)\cdot C(P_1,r)} (G^2 + K^2 + |L|)\, dx\, dy \leq M_{10}\, r^{\tau_1}, \quad \tau_1 > 0$$

where M_{10} depends on A as well as the other quantities and τ_1 depends on m, M, μ, ν, σ. By letting W^{α} be the potential of L^{α}, we may obtain

$$u_i^1(x_0, y_0) = \iint_{\Sigma_A} (g_{j\alpha x} g_i^{\alpha} + g_{j\alpha y} k_i^{\alpha})\,dx\,dy$$

where

$$\iint_{C(P_1,A)\cdot C(P,r)} (g_1^2 + k_1^2)\,dx\,dy \le M_{11} r^{\tau_1}$$

Thus, by Theorem 1, u_1 satisfies conditions A and B as indicated. Since V_1 is the potential of $d_{\beta i} u_{1x}^{\beta} + e_{\beta i} u_{1y}^{\beta} + f_{i\beta} u_1^{\beta}$ plus a harmonic function on Σ_A, the same holds for V_1 on $C(P_1, \frac{A}{2})$, and hence also for v_1. This proves the theorem.

Theorem 6: Let u_n, v_n, and V_n satisfy a sequence of equations of the type of (32) on G, and suppose the coefficients a_n, b_n, c_n, d_n, e_n and f_n satisfy our previous conditions uniformly, and suppose g_n, k_n, ℓ_n, u_n, v_n, and V_n satisfy (33) uniformly. Suppose also that u_n, v_n, and V_n converge uniformly on each bounded closed subset of G to functions u, v, and V, and suppose that the a_n, b_n, c_n converge almost everywhere to a, b, c, that d_n, e_n, g_n, and k_n converge strongly in L_2 to d, e, g, and k, and that f_n and ℓ_n converge strongly in L_1 to f and ℓ. Then our coefficients satisfy the same conditions as the a_n, etc., the u, v, and V are of class D_2 on G and satisfy the limiting equations (32) almost everywhere.

Proof: Since the u_n, v_n, and V_n all satisfy certain conditions A[τ, M(a,d)] and B[$\tau/2$, N(a,d)] uniformly, it is clear that the u, v, and V will satisfy the same conditions. That the limiting coefficients satisfy the conditions of boundedness satisfied uniformly by the a_n, etc., is obvious. It is easily seen further that the first derivatives of u_n, v_n, and V_n converge weakly in L_2 to those of u, v, and V, respectively, on any bounded closed subset of G. Thus the functions $d_{\beta i}^{(n)} u_{nx}^{\beta} + e_{\beta i}^{(n)} u_{ny}^{\beta} + f_{i\beta}^{(n)} u_n^{\beta}$ tend weakly in L_1 to $d_{\beta i} u_x^{\beta} + e_{\beta i} u_y^{\beta} + f_{i\beta} u^{\beta}$ on any such bounded closed subset of G.

Now let D be a circle with $\bar{D} \subset G$. On D, $V_n^i = H_n^i + W_n^i$, where W_n^i is the potential of $d_{\beta i}^{(n)} u_{nx}^{\beta} + e_{\beta i}^{(n)} u_{ny}^{\beta} + f_{i\beta}^{(n)} u_n^{\beta}$ on D only. From Theorem 6, §6, W_n^i tends uniformly to W^i on D and $\lim D_2(W_n - W, D) = 0$. Thus, clearly, H_n^i tends uniformly on $\bar{D}$ to H^i. Thus, if D' is a circle with $\bar{D}' \subset D$, $D_2(V_n - V, D') \to 0$. Thus, we see that on D' we have

$$v_{ny}^i = a_{i\beta}^{(n)} u_{nx}^{(\beta)} + b_{i\beta}^{(n)} u_{ny}^{\beta} + \zeta_{in}, \quad v_{nx}^i = -\left(b_{\beta i}^{(n)} u_{nx}^{\beta} + c_{i\beta}^{(n)} u_{ny}^{\beta} + \kappa_{in}\right)$$

where

$$\begin{cases} \zeta_{in} = d_{i\beta}^{(n)} u_n^{\beta} + g_i^{(n)} - V_{nx}^i, & \zeta_i = d_{i\beta} u^{\beta} + g_i - V_x^i \\ \kappa_{in} = e_{i\beta}^{(n)} u_n^{\beta} + k_i^{(n)} - V_y^i, & \kappa_i = e_{i\beta} u^{\beta} + k_i - V_y^i \end{cases}$$

and

$$\lim_{n \to \infty} \iint_{D'} \left[(\zeta_n - \zeta)^2 + (\kappa_n - \kappa)^2 \right] dx\,dy = 0$$

Then, from Lemma 3, §9, it follows that

$$\iint_{D'} \sum_{i=1}^{N} \left[(v_y^i - a_{i\beta} u_x^{\beta} - b_{i\beta} u_y^{\beta} - \zeta_i)^2 + (v_x^i + b_{\beta i} u_x^{\beta} + c_{i\beta} u_y^{\beta} + \kappa_i)^2 \right] dx\,dy$$

$$\leq \varliminf_{n \to \infty} \iint_{D'} \sum_{i=1}^{N} \left[(v_{ny}^i - a_{i\beta}^{(n)} u_{nx}^{\beta} - b_{i\beta}^{(n)} - \zeta_i^{(n)})^2 + (v_{nx}^i + b_{\beta i}^{(n)} u_{nx}^{\beta} + c_{i\beta}^{(n)} u_{ny}^{\beta} + \kappa_i^{(n)})^2 \right] dx\,dy$$

$$= 0$$

which shows that the limiting equations (32) hold on D'. The theorem follows easily from this.

Theorem 7: Let u, v, and V be of class D_2 on a region G and satisfy (32) almost everywhere on G. Suppose, in addition to the previous conditions, the coefficients all satisfy uniform Hölder conditions on each bounded closed subset of G. Then u, v and V are of class C' on G and their first partial derivatives satisfy uniform Hölder conditions on each bounded closed subset of G.

Proof: This theorem has been proved in essence by E. Hopf*) and we

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*) E. Hopf, Zum analytischen Charakter der Lösungen regulärer zweidimensionaler Variationsprobleme, Mathematische Zeitschrift, vol. 30, pp. 404-413.

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shall not include a proof here.

Theorems 5, 6, and 7 give a complete demonstration of the statements made in §8 concerning the solutions of equations of the type of (32).