Area and representation of surfaces

Lectures delivered by

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The invitation that I received from the Institute for Advanced Study of Princeton to come here as a Member was deeply appreciated and I considered it a great and important engagement for me owing to the high tradition that the Institute has won in these fifteen years since its foundation. Therefore I thank the Director Prof. Oppenheimer and the permanent members of the Institute and especially Prof. Marston Morse for their very kind invitation.

I, as a visiting member, wish also to express, to the permanent members and the organization my enjoyment of the charming and refined atmosphere of the Institute.

I divided the general subject that I have to develop into four parts: I. Surfaces and area; II. Plane transformations; III. Geometrical properties of surfaces; IV. Representation of surfaces.

But, first of all, that is, before passing to this general picture that cannot be complete, I wish explicitly to emphasize the great importance of the work of Prof. Tibor Radó, which is illustrated by the basic book "Length and area" which Rado recently published. It is with great pleasure that I notice that this book was written right here in this Institute a few years ago.

I. Surfaces and area.

Concept of surface

What is a surface? We say sometimes that the set of equations:

(1) $S: \quad x = x(u,v), \quad y = y(u,v) \quad z = z(u,v), \quad (u,v) \in A,$

maps a surface S of the three space E_3 upon the Jordan domain A in the plane u v. Here $x(u,v)$, $y(u,v)$, $z(u,v)$ are continuous functions in A. This is true, but with this sentence we have not yet defined the notion of surface, because a surface admits more than a map, always an infinite number of maps. Therefore we have to explain when another set of similar equations

(2) $S': \quad x = x'(u,v), \quad y = y'(u,v), \quad z = z'(u,v), \ (u,v) \in B,$

maps the same surface S and in this case we shall say that the sets of equations (1) and (2) are two different representations of the same surface or that the surface S and S' coincide or are identical. We shall say also that the sets of equations (1) and (2) are <u>equivalent</u>. When we have defined the meaning of this equivalence and it is recognized that this relation of equivalence is symmetric, reflexive and transitive, then it will be possible to divide all the possible sets of equations like (1) into classes, putting into one class all sets of equations that are equivalent to one another and putting in different classes sets of equations that are not equivalent. Each class will give all the possible maps of one surface and so the concept of surface will be defined through the class off all its possible maps upon Jordan domains.

One representation of a surface is, in such a way, an element of the class of all representations of the same surface. It is in this way that the statement "a set of equations like (1) defines a surface" is to be understood.

The equations (1) make correspond, to each point (u,v) of the Jordan domain A, one well determined point (x,y,z) of the space E_3 that we call the <u>image</u> of the point (u,v). The equations (1) make correspond to the (closed) Jordan domain A a set S(A) of E_3, that we shall call the set of points occupied by the surface S, but it may happen very well that the same point (x,y,z) in the space E_3 corresponds to two or more points of A.

Of course we cannot say that two surfaces coincide only because the sets S(A) and S'(B) occupied by them are identical. For instance it is evident that a right cylinder of a given height, and the same cylinder twice covered as by a flexible veil, are quite different surfaces, but they occupy the same set in the space E_3.

Let us observe only that not every set can be occupied by a continuous surface S. It must have certain properties due to the continuity of the functions x,y,z. These properties have been found first by Hahn and Mazurchiewicz and they can be expressed by saying that such a set S(A) is bounded, connected and locally connected. These properties are also characteristic in the sense that each set of the space E_3 having these properties is the set of points occupied by a continuous surface, but always by many different continuous surfaces.

<u>The concept of equivalence after Frèchet - McShane</u>

The definition, to day definite, of equivalence of two representations (1) and (2) of a surface is due to Frèchet with the modification of McShane.

We shall say, after Frèchet, that the two representations (1) and (2) are equivalent, that is the surfaces S and S' are identical, if for each $\varepsilon > 0$ there is a homeomorphic transformation T_ε between A and B such that to the boundary of A corresponds the boundary of B and such that, if P of A and Q of B are corresponding points under the transformation T_ε, the images S(P) and S'(Q) of P and Q have a distance not greater than ε :

$$\{S(P),\ S'(Q)\} \leq \varepsilon.$$

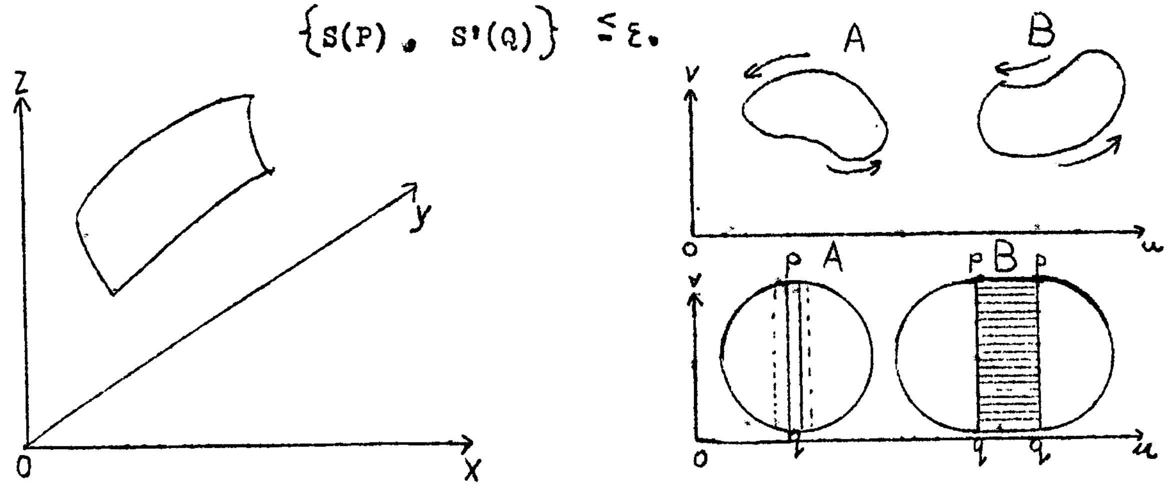


Fig. 1.

In order to understand such a definition we need to think first of all of the special case in which there exists a homeomorphic transformation T_0 between A and B such that corresponding points under T_0 have the same image in the space E_3. In this special case it is evident that the surfaces S and S' are identical.

But the necessity of the more general definition of Frèchet is shown by the simple example illustrated by the Fig.1. Let A be a circle and let us divide A into two parts by the diameter parallel to the v-axis let us separate one of the semicircles from the other by a translation parallel to the u-axis. We have a Jordan domain B. To each segment parallel to the u-axis of the rectangle that remains between the two semicircles let us make correspond upon the surface S the same point that already corresponds to the endpoints of it. A new map of the surface S has been obtained upon the domain B. We can say that the map is stationary upon each of such segments. It is evident that we have made no modification of the surface S. We have only made a modification of its map. The first map is given upon the circle A, the new map is given upon the elementary figure B, we can suppose that the given map upon A is never stationary, the new map is stationary upon all these segments. In this change of representation of the surface S there is no homeomorphic transformation between A and B with the properties that corresponding points have the same image. It would be necessary that to each of such segments of B there corresponds only one point of A and therefore the transformation would not be homeomorphic. But the general definition of Frèchet holds in this case. Indeed it is sufficient to consider in A a strip, thin enough, with sides parallel to the v-axis and to make correspond through obvious elementary transformations to the opposite circular segments of A the two semicircles of B and to the central strip of A the central rectangle of B.

The modification of McShane of the definition of Frèchet consists in asking that the homeomorphic transformation T_ε transform not only the boundary of A into the boundary of B, but also the positive sense (counterclockwise) of the boundary of A into the positive sense of the boundary of B. This observation of McShane is very important and we have to convince ourselves of the meaning of it by means of the example of a square that we can think of with one or the other orientation (Fig.2):

$$S: \quad x = u, \quad y = v, \quad z = 0, \quad (u,v) \in A = (0 \leq u \leq 1,\ 0 \leq v \leq 1)$$

$$S': \quad x = 1-u, \quad y = v, \quad z = 0, \quad (u,v) \in A = (0 \leq u \leq 1,\ 0 \leq v \leq 1)$$

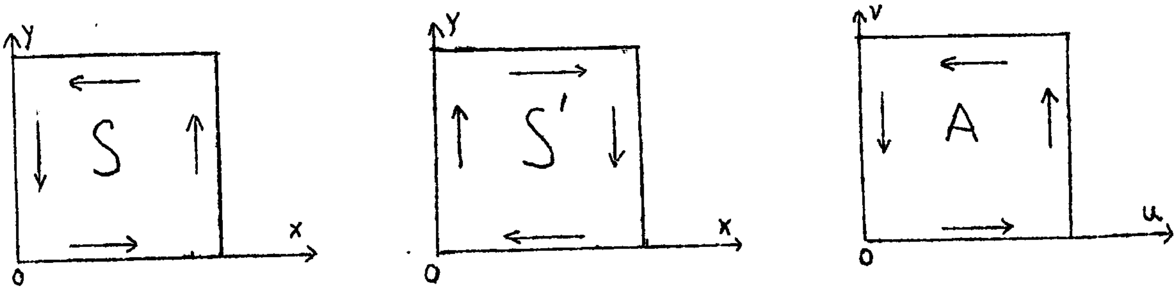


Fig. 2.

McShane has observed that there are surface integrals that, calculated upon such squares, with one or the other orientation, have different values. For instance the simplest surface integral

$$\mathcal{J}_S = \iint_S \left[\frac{\partial(x,y)}{\partial(u,v)} + \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \right] du\, dv$$

has the value 2 upon the square S and the value 0 upon the square S'. This has to convince us that a square with one and the same square with the other orientation are two different surfaces.

The Frèchet-McShane distance.

In a very similar way the notion of Frèchet-McShane distance $\delta = \| S, S' \|$ between two surfaces can be introduced. If (1) and (2) are any representations of two surfaces S and S' we say that $\delta \geq 0$ is the distance of the surfaces S and S' if δ is the greatest lower bound of the real numbers $\varepsilon > 0$ with the following property: there is a homeomorphic transformation T_ε between A and B which transforms the boundary of A into the boundary of B, the positive sense of the boundary of A into the positive sense of the boundary of B and such that for each couple of points P in A and Q in B corresponding under the transformation T_ε the images S(P) and S'(Q) have a distance not greater than ε :

$$\{S(P) , S'(Q)\} \leq \varepsilon .$$

Upper semi-continuous collections of continua

The previous examples and considerations have pointed out that in the domain A that we use for the map of a surface, there may exist certain continua whose points have the same image in the space. We have to consider the maximal continua g of A with respect to the property that the functions x,y,z are constant upon g. In addition we will consider as continua also the single points of A.

In this way we get the collection G of all maximal continua (points and proper continua) on which all three functions x,y,z are constant and this collection has the property that each point of A belongs to one and only one continuum of the collection G.

This collection G has another important property that we call upper semicontinuity which means that if we consider any sequence of continua g of the collection G it happens that all limit points of the sequence form a unique continuum, then this continuum must belong entirely to one continuum of the collection G. This collection G is important in defining the topological type of the surface. It is a point of departure of important and very refined studies upon the topological properties of the surfaces and I note here the studies of R.L.Moore, C.B. Morrey, R.L.Wilder and G.T.Whyburn.

If we have two representations of the same surface, we also have two different upper semicontinuous collections of continua. It is evident that there is a one to one correspondence between them, such that corresponding continua have the same image in the space. But this fact is only a necessary and not sufficient condition for the identity of the two surfaces, as J.T.W.Youngs has shown quite recently. A necessary and sufficient condition has been found by Youngs in connection with the concepts of Moore and Morrey and the concept of order of combinatorial topology.

Surface Area.

Very many different definitions of area of a surface were proposed in the last century, but if we demand that the area of a surface can be considered as a functional upon a surface performing in the classical Calculus of Variation for surfaces, as well as in the Morse-theory, a service similar to what Menger calls a comparison functional $\varphi(S)$, we must insist that the definition of area satisfy the fundamental principle of lower semicontinuity. This is to be understood as follows: If S, S_n, $n = 1, 2, \ldots$, are surfaces and $\|S, S_n\| \to 0$, then $\varphi(S) \leq \lim_{n\to\infty} \varphi(S_n)$.

At this moment it is very interesting to notice that all definitions of area, till now proposed and completely studied, which satisfy the principle of lower semicontinuity, coincide with the Lebesgue area. This fact holds, first of all, for the Geocze area reintroduced by McShane through the notion of topological index of a plane curve and utilized by McShane, C.B.Morrey and myself. T. Radó utilized a new notion of area the "lower area" and the identification of this area with the Lebesgue area has been proved by the theories of Radó and myself. The same statement, after recent papers of T. Radó, holds for the notions of area of Cauchy and Favard obtained by Radó by modifying preceding definitions of these authors in such a way as to make them lower semicontinuous.

All this is to be related with the conjecture that Radó expressed in 1928; each lower semicontinuous functional that coincides with the elementary area upon polyhedra, must coincide also with the Lebesgue area upon each surface, at least with very natural and weak conditions. To this problem Frèchet, Kempisty, Scorza, Zwirner, Stampacchia contributed in particular cases.

The Lebesgue definition of area.

We call Lebesgue area $L(S)$ of the surface S the inferior limit of the elementary areas $A(\Sigma)$ of the polyhedral surfaces Σ approaching S, that is

$$L(S) = \underline{\lim}\, A(\Sigma)$$

when

$$\| \Sigma, S \| \longrightarrow 0.$$

If we consider that for each lower semicontinuous functional $\varphi(S)$ that coincides with the elementary area for all polyhedral surfaces we have

$$\varphi(S) \leq \underline{\lim}\ \varphi(\Sigma) = \underline{\lim}\ A(\Sigma) = L(S),$$

$$\varphi(S) \leq L(S),$$

therefore the Lebesgue area is the largest lower semicontinuous functional upon surfaces that coincide with the elementary area upon polyhedral surfaces.

I recall here that just by means of Lebesgue area the question of the area of surfaces in non-parametric form has been settled by Tonelli in 1926. I recall also that by means of the Lebesgue area T.Radó and L. Tonelli have proved the known isoperimetric property of a sphere in the space E_3.

The first problem that is to be resolved in connection with the Lebesgue area is the following one :"If we have a surface with any representation like (1) upon a Jordan domain A, to give a characterization of the functions x,y,z in order that the Lebesgue area of the surface be finite". The way to such a charaterization has been indicated by Banach and Vitali in 1924, but their researches are in connection with a concept of area quite different from the Lebesgue area. Banach observed that a necessary and sufficient condition ought to have been sought in an opportune property not of the functions x,y,z alone, but of the couples of functions:

$$\Phi_1 : \begin{cases} y = y(u,v) \\ z = z(u,v) \end{cases} \qquad \Phi_2 : \begin{cases} z = z(u,v) \\ x = x(u,v) \end{cases} \qquad \Phi_3 : \begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases} .$$

These are plane transformations of the plane u v in the planes y z, z x, x y. These are also the projections of the surface S upon the three coordinate planes and therefore these are "flat surfaces".

We will now introduce the notion of plane transformation of bounded variation.

II. Plane transformations

Total variation of a function of one variable

First we must consider a simple continuous function $f(x), a \leq x \leq b$, that is, a continuous transformation $y = f(x)$ of the interval (a,b) of the x-axis into an interval (c,d) of the y-axis. The definition of total variation of Jordan is well known

$$V(f) = \text{l.u.b.} \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right|$$

where $a = x_0 < x_1 < \dots < x_n = b$. Geometrically speaking we can say that we divide the 1-cell (a,b) in 1-cells

$$(x_0, x_1), \quad (x_1, x_2), \dots, (x_{n-1}, x_n) \quad .$$

The boundaries of these 1-cells are the couples of points here written, to which correspond upon the y-axis the couples of points

$$[f(x_0), f(x_1)], [f(x_1), f(x_2)], \dots, [f(x_{n-1}), f(x_n)] \; .$$

The expression

$$\left| f(x_i) - f(x_{i-1}) \right|$$

gives the measure of the set of points of the y-axis that separate the points of the couple $[f(x_{i-1}), f(x_i)]$. Therefore $V(f)$ is the l.u.b. of the sum of all these measures.

Another very interesting definition for us is the following:
For each number $\bar{y}$ $(-\infty < \bar{y} < +\infty)$ we call $\psi(\bar{y})$ the last upper bound of the number n of disjoint 1-cells of (a,b)

$$(\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_n, \beta_n),$$

for which, either

$$f(\alpha_i) < \bar{y} < f(\beta_i), \quad \text{or} \quad f(\alpha_i) > \bar{y} > f(\beta_i) \; .$$

This definition is a little different from that of Banach and Vitali, but, according to their papers, it is possible to prove that $\psi(y)$ is a Baire function. Let us put

$$W(f) = \int_{-\infty}^{+\infty} \psi(y)\, dy \, .$$

According to the papers of Banach and Vitali, it is possible to prove that

$$V(f) = W(f) \, .$$

Geometrically speaking $\psi(\bar{y})$ is the least upper bound of the number n of disjoint 1-cells of the x-axis such that the points of the y-axis corresponding to their endpoints are separated by $\bar{y}$. The total variation $W(f)$ is the Lebesgue integral of the function $\psi(y)$.

A third viewpoint is the following one. We consider the curve Γ: $y = f(x)$ which makes one or more, even an infinite number of oscillations and its oscillations can be more or less large. The total variation of the function $f(x)$ must give a measure of such oscillations, it must be, that is, a number depending upon the oscillations made by Γ and upon the largeness of each of them. A spontaneous idea in order to keep this measure is the following: Let us consider the projection upon the y-axis of the curve $y = f(x)$ [the curve ABCD of the figure 3]. We get a broken line, folded on itself, with straight sides [A'B'C'D' of the figure 3], of which we consider the length $L(f)$.

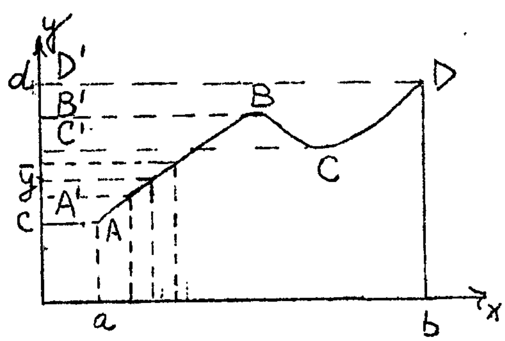


Fig. 3.

Now

$$L(f) = V(f) = W(f) \, .$$

We have then three different interpretations of the concept of total variation of a continuous function.

These definitions valuable for continuous correspondences between spaces of one dimension, must be now extended to correspondences between spaces of two dimensions.

Total variation of a plane transformation.

Let $x(u,v)$, $y(u,v)$ be continuous functions on the closed square $A \equiv (0 \leq u \leq 1,\ 0 \leq v \leq 1)$ and let Φ be the continuous plane transformation

$$\Phi : \quad x = x(u,v), \quad y = y(u,v), \quad (u,v) \in A \quad .$$

To each point P of A, Φ makes correspond one (and only one) point $Q \equiv (x,y) = \Phi(P)$ of the plane xy, which we call the image of the point P. We call B the set of the points of the plane xy, which are the image of all points of A. The set B is bounded and closed. We call K a square, with sides parallel to the axes x and y, containing in its interior the set B. Let γ be an oriented closed Jordan curve of A. To γ corresponds, owing to the transformation Φ, a continuous oriented closed curve C not necessarily simple of the plane xy, which we call the image of γ (fig.4.).

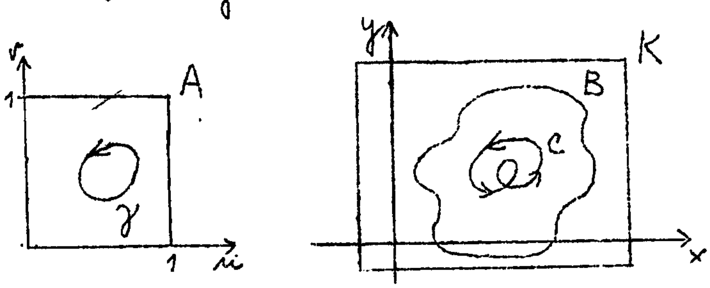


Fig. 4.

Let $O(x,y;C)$ be the topological index of the point (x,y) with respect to the curve C. As is well known, if $P \equiv (x,y)$ is a point not belonging to the curve C, then $O(x,y;C)$ is the number of complete revolutions of the vector, never zero, $\overrightarrow{PM}$, when M describes the curve C in its positive sense. Let us put $O(x,y;C) = 0$ for each point (x,y) upon C. We have now a function $O(x,y;C)$ which is zero outside of B and which is a Baire function. We can say that $O(x,y;C)$ is zero or not zero at a point (x,y) if the curve C links or not the point (x,y).

Let r be a Jordan domain of A and r^* the continuous simple closed curve forming the boundary of r. Let C be the image of r^* upon the plane xy. For each point (x,y) we put

$$o(x,y;C) = \begin{cases} 1 & \text{if } O(x,y;C) \neq 0, \\ 0 & \text{if } O(x,y;C) = 0. \end{cases}$$

the function $o(x,y;C)$ is a Baire function, zero outside of B and bounded. Let us put

$$g(r) = \iint_K |O(x,y;C)| \, dx\, dy,$$

$$u(r) = \iint_K o(x,y;C) \, dx\, dy,$$

$$t(r) = \left| \iint_K O(x,y;C) \, dx\, dy \right|,$$

where we must put $t(r) = g(r) = +\infty$ if $O(x,y;C)$ is not L-integrable.

Let $[r_i, i = 1,2,\ldots,n]$ be a subdivision of A into disjoint Jordan domains. Let C_i be the continuous closed curve image of the boundary r_i^* of r_i, $i = 1,2,\ldots, n$. Let us put

$$G(\Phi) = \text{l.u.b.} \sum_{i=1}^{n} g(r_i),$$

$$U(\Phi) = \text{l.u.b.} \sum_{i=1}^{n} u(r_i),$$

$$T(\Phi) = \text{l.u.b.} \sum_{i=1}^{n} t(r_i),$$

and, for each point $Q = (x,y)$ of K,

$$\Psi(x,y,\Phi) = \text{l.u.b.} \sum_{i=1}^{n} |O(x,y;C_i)|,$$

$$\psi(x,y,\Phi) = \text{l.u.b.} \sum_{i=1}^{n} o(x,y;C_i).$$

Evidently $0 \leq U(\Phi) \leq G(\Phi)$, $0 \leq T(\Phi) \leq G(\Phi)$, $0 \leq \psi(x,y;\Phi) \leq \Psi(x,y;\Phi)$. The functions ψ and Ψ are lower semicontinuous on K and therefore measurable.

Therefore the integrals

$$W(\Phi) = \iint_K \Psi(x,y;\Phi)\,dx\,dy, \qquad w(\Phi) = \iint_K \psi(x,y;\Phi)\,dx\,dy$$

exist finite or infinite.

It is evident that the functions $G(\Phi)$, $U(\Phi)$, $T(\Phi)$ give different but similar extensions of the variation $V(f)$ of Jordan and the functions $W(\Phi)$ and $w(\Phi)$ give different but similar extensions of the variation $W(f)$ of Banach and Vitali. In order to see this let us consider that we have divided the 2-cell A in 2-cells r_i, whose boundaries are continuous simple and closed curves. To these correspond, under Φ, the closed curves C_i of the plane xy and the functions $g(r_i)$, $u(r_i)$, $t(r_i)$ give in three different ways a measure of the set of the points that are linked by these curves.

The functions $G(\Phi)$, $U(\Phi)$, $T(\Phi)$ give the l.u.b. of the sum of such measures for all the possible subdivisions of A in 2-cells. These functions give different extensions of the total variation $V(f)$ of Jordan.

Analogously for each point $Q_o = (x_o,y_o)$ the functions $\Psi(x_o,y_o;\Phi)$ and $\psi(x_o,y_o;\Phi)$ give, in two different ways, the l.u.b. of the number $\underline{\underline{n}}$ of disjoint 2-cells whose boundaries have images linking Q_o. Finally $W(\Phi)$ and $w(\Phi)$ are the Lebesgue integrals of the functions $\Psi(x,y;\Phi)$ and $\psi(x,y;\Phi)$. Therefore $W(\Phi)$ and $w(\Phi)$ are extensions of the total variation $W(f)$ of Banach and Vitali.

If we now consider the plane transformation Φ as a "flat" surface, we shall call $L(\Phi)$ its Lebesgue area. This is the extension of the third way of introducing the concept of total variation.

I proved the following:

<u>THEOREM:</u> <u>For each plane transformation Φ we have</u>

(3) $$G(\Phi) = U(\Phi) = W(\Phi) = w(\Phi) = L(\Phi) = T(\Phi)$$

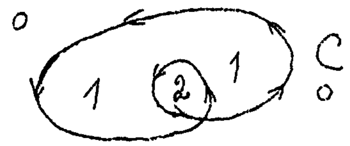
The last equality has been proved only if $L(\Phi) < +\infty$.

The essential total variation $R(\Phi)$, introduced in the theory of T. Radó in quite a different way, is equal to the preceding numbers. It seems therefore very natural to assume now anyone of these numbers as a definition of the total variation of the plane transformation Φ. We say that the plane transformation Φ is of bounded variation if anyone of these numbers is finite.

<u>Geometric considerations.</u>

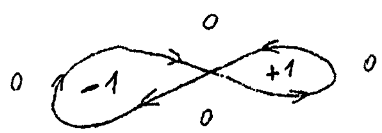
The proof of the equalities of the chain (3) is quite deep. First of all let us see its geometric meaning.

We have already noticed that for each Jordan domain r we have $0 \leq u(r) \leq g(r)$ and the sign < can hold as in the case indicated by the figure 5. Anyway we always have $U(\Phi) = G(\Phi)$. In other words, when we divide A in 2-cells small enough it happens that "statistically" the parts "linked" more than once by each curve C_i can be neglected.



<u>Fig.5.</u>

We have already noticed that for each Jordan domain r we have $0 \leq t(r) \leq g(r)$ and the sign < can hold as in the case indicated by the figure 6. Anyway it is always $T(\Phi) = G(\Phi)$. In other words when we divide A in 2-cells small enough it happens that "statistically" the parts "linked" by each curve C_i in a different sense than the remaining parts can be neglected.



<u>Fig.6.</u>

We have already seen that $\psi(x,y;\Phi) \leq \Psi(x,y;\Phi)$ and it may happen that $\psi < \Psi$ in some point of B. But the sign < holds here at most in a countable set of points and therefore we must have $w(\Phi) = W(\Phi)$. If we say that the points in which $\psi < \Psi$ are branch-points of the transformation Φ, it is sure that these branch-points are at most a countable set.

The equality between $G(\Phi)$, $U(\Phi)$, $T(\Phi)$ on one side and $w(\Phi)$ and $W(\Phi)$ on the other, corresponds perfectly to the equality $V(f) = W(f)$ for the case of the functions of one variable. All of these equalities are very hidden.

The demonstrations of the equalities $G(\Phi) = U(\Phi) = T(\Phi)$ are based on geometric considerations and on the well known theorems:

1) Almost all points of a set are points of density (in a weak meaning) of the same set;
2) In almost all points the value of a measurable function is the average (in a weak meaning) of the values of the same function in the neighborhood of the point.

I will give here in a more precise way the outline of the proof of the equality $W(\Phi) = L(\Phi)$.

First it is evident that all these functions G, T, U, W, w, L are equal to each other for each polyhedral surface. The same functions are also all lower semicontinuous, and, if we remember that the Lebesgue area is the greatest lower semicontinuous functional upon surfaces that coincide with the elementary area upon polyhedral surfaces, we have $W(\Phi) \leq L(\Phi)$.

Let us divide the square K in the plane xy into square cells Γ_j of side a. We are able to prove that it is possible to choose in each cell a point $Q_j = (x_j, y_j)$, $j = 1, 2, \ldots, \nu$, such that

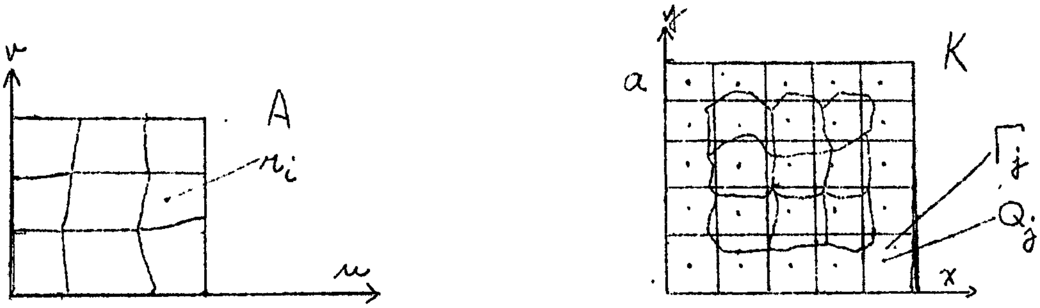


Fig. 7.

$$a^2\,\Psi(x_j,y_j;\Phi) \leq \iint_{\Gamma_j} \Psi(x,y;\Phi)\,dx\,dy$$

and, as a consequence,

$$\sum_{j=1}^{\nu} a^2\,\Psi(x_j,y_j;\Phi) \leq \sum_{j=1}^{\nu} \iint_{\Gamma_j} \Psi(x,y;\Phi)\,dx\,dy = \iint_K \Psi(x,y;\Phi)\,dx\,dy = W(\Phi).$$

Let us perform now in the fundamental square A a subdivision $[r_i,\ i=1,2,\dots,n]$ into polygonal 2-cells. Each 2-cell r_i has a boundary whose image is a curve C_i, $i=1,2,\dots,n$. It is possible to choose this subdivision, also in connection with a more precisé choice of the points Q_j, in such a way that: a) no curve C_i passes through any of the points Q_j;

b) the diameter of the curves C_i are less than the minimum distance between the points Q_j. Each point Q_j will be linked a number of times f_{ji} to the curve C_i and

$$f_{ji} = \left| \,O(x_j,y_j;\ C_i)\, \right|.$$

We have

$$\sum_{i=1}^{n} f_{ji} = \sum_{i=1}^{n}{}' \left| \,O(x_j,y_j;C_i)\, \right| \leq \Psi(x_j,y_j;\Phi).$$

It is evident that each curve C_i links at most one point Q_j. Let us continuously deform now the set of the curves C_i, $i=1,2,\dots,n$, upon the plane xy in such a way as to take it upon the sides of the cells Γ_j. This may be obtained in the simplest way by projecting from each point Q_j all points of the curves C_i that are contained in the cell Γ_j upon the sides of the same cell, as indicated in the figure 8.

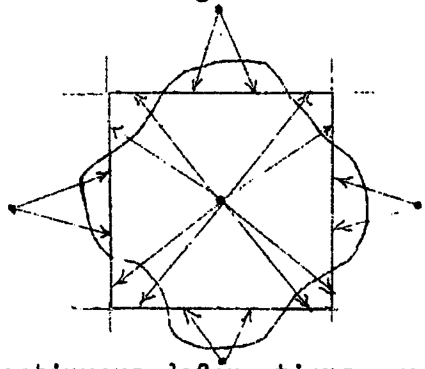


Fig.8.

With further continuous deformations, we can show that the continuous curve C_i become polygonal curves formed by sides of the cells Γ_j. But we obtain also the following situation: the curves C_i that did not link points Q_j are deformed into polygonal curves that are the boundaries of polyhedral surfaces of zero area; the curves C_i that linked points Q_j and therefore one and only one point Q_j are deformed in polygonal curves consisting of the boundary of one cell Γ_j counted a certain number of times, namely f_{ji} and such curves are the boundaries of elementary polyhedral surfaces of area $f_{ji}a^2$. All together all of these elementary polyhedrals surfaces give a unique polyhedral surface Σ whose area is

$$A(\Sigma) = \sum_{j=1}^{\nu} \sum_{i=1}^{n} a^2 f_{ji} \leq \sum_{j=1}^{\nu} a^2 \left| \Psi(x_j, y_j; \Phi) \right| \leq W(\Phi).$$

In this whole construction, $\underline{\underline{a}}$, the side of the squares in the plane xy, is arbitrary and therefore, putting $a = 1/N$, $N=1,2,\ldots$, we get a sequence of polyhedral surfaces Σ_N, $N=1,2,\ldots$, and it is easy to prove that $\|\Sigma_N, \Phi\| \to 0$. We have now

$$L(\Phi) \leq \underline{\lim}_{N \to \infty} A(\Sigma_N) \leq W(\Phi),$$

that is $L(\Phi) \leq W(\Phi)$ and since we already know that $W(\Phi) \leq L(\Phi)$, we have proved that

$$W(\Phi) = L(\Phi).$$

Note: We have outlined here the general proof of the statement

$$W(\Phi) = L(\Phi)$$

with the simplifications which result if a) the curve C, the image of the boundary of A, has measure zero; b) the flat surface Φ is open and non-degenerate. (see part IV).

On a metric approximation theorem.

We have seen before the geometrical interpretation of the equality $G \neq U \neq T$. A type of statement that gives in a more precise form, the same geometrical interpretation is the following theorem that I proved:

THEOREM: Let Φ be a plane transformation of bounded variation. For each $\varepsilon > 0$ there exists a set of simple disjoint plane polygons π_i, $i=1,2,\dots,\nu$, interior to A, such that, if C_i are the oriented curves corresponding to the boundary π_i^* of π_i under the transformation Φ, we have

a) the diameter of each curve C_i is less than ε;

b) the total measure of all the curves C_i is less than ε;

c) $\sum g(\pi_i) > G(\Phi) - \varepsilon$,

$\sum u(\pi_i) > U(\Phi) - \varepsilon$,

$\sum t(\pi_i) > T(\Phi) - \varepsilon$.

Absolutely continuous plane transformations.

In a similar way we can now introduce the notion of an absolutely continuous plane transformation.

We shall say that the transformation Φ is absolutely continuous if the two following properties are satisfied:

a) for each $\varepsilon > 0$ there exists a number $\delta > 0$ such that, if $\pi_1, \pi_2, \dots, \pi_n$ are disjoint polygons of A and $\sum|\pi_i| < \delta$, then we have $\sum g(\pi_i) < \varepsilon$;

b) if π is a polygon of A and we divide π into a finite number of polygons π_i, $i=1,2,\dots,\nu$, we have

$$G(\pi) = \sum_{i=1}^{\nu} G(\pi_i).$$

I proved that these two conditions are independent. In quite a different way T. Radó introduced the concept of essentially absolutely continuous plane transformations. The concept of T. Radó and the previous one are completely equivalent, as T. Radó has proved recently.

Generalized Jacobians

Let Φ be any plane transformation and $P(u,v)$ an interior point of A. For each square q contained in A and containing P, we consider the following ratio:

$$\frac{G(q)}{|q|}$$

between the function $G(q)$ calculated relative to the square q and the area $|q|$ of q.

If Φ is of bounded variation it is possible to prove that, for almost all points $P \equiv (u,v)$ of A there exist limits:

$$\mathcal{J}(u,v) = \lim_{\delta(q)\to 0} \frac{G(q)}{|q|}.$$

For each plane transformation of bounded variation the function $\mathcal{J}(u,v)$ is also defined a.e. in A and it is possible to prove that this function is L-integrable. In addition we can say that, if the functions x,y have first partial derivatives a.e. in A, then we have, a.e. in A,

$$\mathcal{J}(u,v) = |x_u y_v - x_v y_u| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|.$$

The proof of the existence and L-integrability of the function $\mathcal{J}(u,v)$ comes from the general theory of Banach, while on the contrary the proof of the above equality is very deep.

The following theorem holds:

<u>If</u> Φ <u>is a plane transformation of bounded variation, we have</u>

$$L(\Phi) \equiv W(\Phi) \geq \iint_A \mathcal{J}(u,v)\, du\, dv$$

and in this relation the equality holds if and only if the plane transformation Φ is absolutely continuous.

Relative generalized Jacobian.

We have observed that a surface is an oriented manifold and this holds also for a flat surface, that is a plane transformation Φ. The above generalized Jacobian $\mathcal{J}(u,v)$ (or absolute generalized Jacobian), does not depend upon the orientation of the flat surface Φ. For several applications (transformation of double integrals, geometrical properties of surfaces, integrals upon a surface) it is necessary to introduce the notion of relative generalized Jacobian.

Let $P \equiv (u,v)$ be an interior point of A, q a square contained in A and containing P as an interior point, $\delta < \delta(q)$ the diameter of q. Let $[\pi_i,\ i=1,2,\dots,n]$ be a set of disjoint simple polygons interior to q, C_i the continuous closed curves, images under Φ of the boundaries π_i^* of π_i. Let us put

$$\tau(\pi_i) = \iint_K O(x,y;C_i)\,dx\,dy, \qquad t(\pi_i) = |\tau(\pi_i)| = \left|\iint_K O(x,y;C_i)\,dx\,dy\right|$$

$$m = \frac{1}{|q|}\left|\sum_{i=1}^{n} C_i\right|, \qquad \mu = \frac{1}{|q|}\left[T(q) - \sum_{i=1}^{n} t(\pi_i)\right].$$

We have $m \geq 0$, $\mu \geq 0$ and the above metric approximation theorem assures us that there exist sets of polygons $[\pi_i,\ i=1,2,\dots,n]$ for which m and μ are as small as we like.

Let us consider now the ratio

$$\frac{1}{|q|}\sum_{i=1}^{n} \tau(\pi_i).$$

We call relative generalized Jacobian the following limit, if it exists:

$$H(u,v) = \lim_{\substack{\delta \\ m \\ \mu}\Big\}\to 0} \frac{1}{|q|}\sum_{i=1}^{n} \tau(\pi_i).$$

I proved the following theorem:

THEOREM: If the plane transformation Φ is of bounded variation, H(u,v) exists a.e. in A and we have, a.e. in A,

$$H(u,v) = \pm \mathfrak{J}(u,v).$$

If the functions x(u,v), y(u,v) have first partial derivatives a.e. in A, we have a.e. in A,

$$H(u,v) = x_u y_v - x_v y_u = \frac{\partial(x,y)}{\partial(u,v)} .$$

III. Geometrical properties of surfaces.

Analytical characterization of surfaces of finite Lebesgue area.

Let S be any continuous surface and

(1) $S: \quad x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \quad (u,v) \in A \equiv (0,1,0,1)$

any representation of S upon the fundamental square A.

Let us consider the three plane transformations:

$$\Phi_1: \begin{cases} y = y(u,v) \\ z = z(u,v) \end{cases} \qquad \Phi_2: \begin{cases} z = z(u,v) \\ x = x(u,v) \end{cases} \qquad \Phi_3: \begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases} .$$

I have proved the following:

THEOREM: The surface S has finite Lebesgue area if and only if the three plane transformations Φ_1, Φ_2, Φ_3 are of bounded variation.

This statement is contained in the following relation

(4) $W(\Phi_t) \leq L(S) \leq W(\Phi_1) + W(\Phi_2) + W(\Phi_3), \quad t = 1,2,3.$

The first part is evident because the Lebesgue area L(S) is $\geq$ the area $L(\Phi_t)$ of each plane projection of the surface, and we know that $L(\Phi_t) \geq W(\Phi_t)$ which depends only upon the lower semicontinuity of $W(\Phi_t)$. Therefore

$$L(S) \geq L(\Phi_t) \geq W(\Phi_t), \quad t = 1,2,3.$$

The second part of the above relation (4) is the essential part. The question is to prove that, if the three transformations Φ_1, Φ_2, Φ_3 are of bounded variation, it is possible to construct a sequence of polyhedral surfaces Σ_n, n=1,2,..., such that $\|\Sigma_n, S\| \to 0$ and

$$A(\Sigma_n) \leq W(\Phi_1) + W(\Phi_2) + W(\Phi_3).$$

We know that $L(S) \leq \lim_{n\to\infty} A(\Sigma_n)$ and therefore from such construction and the last relation, we have the proof of the second part of the relation (4).

It is evident that we ~~have to perform~~ a procedure quite similar to that which we have performed for the proof of the relation $L(\Phi) \leq W(\Phi)$. And really the procedure is quite similar in one case and, that is, when the set of the points of the surface S is a set S(A) in the space E_3 of measure zero.

If K is a cube in the space xyz with edges parallel to the axes, containing in its interior the whole surface S, let us divide the cube K into cubic 3-cells Γ_{ijh}, let us choose in Γ_{ijh} a point Q_{ijh}, but we must now choose these points in such a way that they are disposed upon three well determined systems r_{ij}, r_{ih}, r_{jh} of straight lines parallel to the axes and in each cube Γ_{ijh} pass three perpendicular straight lines that have the point Q_{ijh} as a common point. If S(A) has measure zero in the space E_3, it is possible to choose the lines in such a way that none of their common points lie upon the surface S. Now let us divide the square A in 2-cells, Π_s, $s = 1,2,3,\ldots,n$. To the boundary of each cell Π_s corresponds upon the surface a curve C_s and it is possible to choose the subdivision Π_s, $s=1,2,\ldots,n$ also in connection with a more precise choice of the lines r, in such a way that:

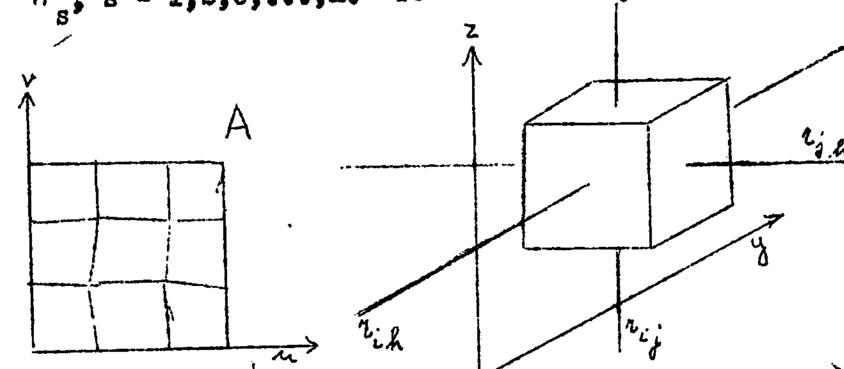


Fig. 9.

a) no curve C_i passes through any of the straight lines r_{ij}, r_{ih}, r_{jh} and therefore also through any of the points Q_{ijh}; b) the diameter of the curves C_s are less than the minimum distance between the points Q_{ijh} and the surface S. Now it is evident that if a curve C_s links a straight line r, such a curve links one and only one of these straight lines.

Introducing a convenient system of deformations similar to the previous one in the plane we now obtain deformed curves that are composed only of parts of the edges of the cells Γ_{ijh}. We have two types of deformed curves C_s: These that do not link any straight line and they are boundary of elementary polyhedral surfaces of area zero; the other that link one of the straight line r and these are boundaries of elementary polyhedral surfaces of a well determined area. All these surfaces give a unique polyhedral surface Σ of area less than $W(\Phi_1) + W(\Phi_2) + W(\Phi_3)$.

But we know that the surface S may occupy a set S(A) of positive measure. We can think of a Peano curve considered as a surface, but quite recently Besicovitch constructed a surface that is a homeomorphic transformation of a square which has finite Lebesgue area and positive measure in the space E_3.

In the general case we are obliged to choose the points Q_{ijh} upon the surface. This compels us to choose points Q_{ijh} and polygons π_s in a very refined manner. In this way it is still possible to obtain that the curves C_s, that must be utilized for the construction of elementary polyhedral surfaces of positive area, link still one and only one of the straight lines r_{ij}, r_{ih}, r_{jh} and the procedure can be repeated. But the remaining curve C_s can be arbitrarily near the point Q_{ijh}. They link no straight line r but this is not sufficient to prove that they are boundaries of surfaces without common points with such straight lines, that is, nulhomotop in the space E_3 outside of the straight lines r. But it is possible to substitute for the three straight lines r a convenient collection of straight lines parallel to the lines r and not linked by the curves C_s. We are in condition to utilize the following linkage theorem:

THEOREM: If x,y,z are three coordinate axes; if g_1, g_2, g_3, g_4 are four straight lines g_1, $g_2 // x$, $g_3 // y$, $g_4 // z$, no two of which have common points and such that g_3 and g_4 intersect the strip formed by the two parallel lines g_1 and g_2 ; if C is a continuous closed curve whose distance from the x,y,z axes is greater then the distance of g_1, g_2, g_3, g_4 from the axes, then C is nulhomotop with respect to the set E - (x+y+z) if and only if C is nulhomotop with respect to $E-(g_1+g_2+g_3+g_4)$.

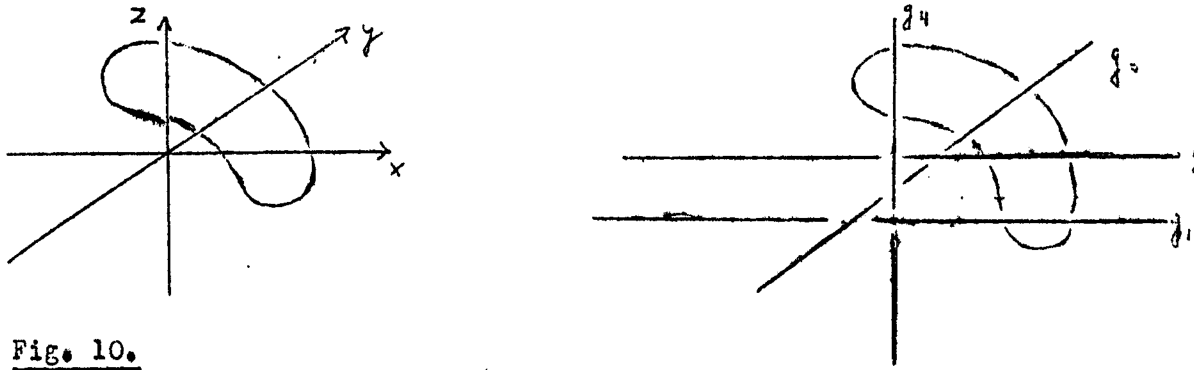


Fig. 10.

Of this theorem I gave an elementary proof and now S. Eilenberg gave a quite short proof with the methods of combinatorial topology.

Through the use of this theorem it is possible to complete the proof, that we have outlined, of the relation (4), and therefore to obtain the desired analytical characterization of surfaces of finite Lebesgue area.

The classical integral for surface area.

Let S be a surface of finite Lebesgue area and let (1) be any representation of S. Then the plane transformation Φ_1, Φ_2, Φ_3 are of bounded variation, the three generalized Jacobians $J_1(u,v)$, $J_2(u,v)$, $J_3(u,v)$ exist and are L-integrable functions. Therefore the classical integral for the area exists and T.Rado and I have proved that always

$$L(S) \geq \iint_A \sqrt{J_1^2(u,v) + J_2^2(u,v) + J_3^2(u,v)}\, du\, dv.$$

In addition we have proved that in this relation the sign = holds, that is the Lebesgue area is given by the classical integral, if and only if the three plane transformations Φ_1, Φ_2, Φ_3 are absolutely continuous.

The Geocze area.

Let S be any continuous surface and

(1) $S;\ x = x(u,v),\quad y = y(u,v),\quad z = z(u,v),\qquad (u,v) \in A \equiv (0,1,0,1),$

any representation of S. Let Φ_1, Φ_2, Φ_3 be the three relative plane transformations. If r is any Jordan domain of A and r^* the oriented Jordan curve formed by the boundary of r, then to r^* corresponds upon the surface S a certain oriented continuous closed curve C. Let C_1, C_2, C_3 be the plane continuous closed curves projections of C upon the coordinate planes, and $O(y,z;C_1)$, $O(z,x;C_2)$, $O(x,y;C_3)$ be their topological indices. We can put

$$g_s(r) = \iint_{K_s} |\, O(x,y;C_s)|\, dxdy\ ,$$

$$u_s(r) = \iint_{K_s} O(x,y;C_s)\ dxdy\ , \qquad s=1,2,3$$

$$t_s(r) = \left| \iint_{K_s} O(x,y;C_s)\ dxdy \right|,$$

where K is a cube whose edges are parallel to the axes x,y,z containing in its interior the whole surface S and K_s, s=1,2,3, the projections of K upon the coordinate planes. We put now

$$g(r) = [g_1^2 + g_2^2 + g_3^2]^{\frac{1}{2}},\quad u(r) = [u_1^2 + u_2^2 + u_3^2]^{\frac{1}{2}},\quad t(r) = [t_1^2 + t_2^2 + t_3^2]^{\frac{1}{2}}\ ;$$

We have $0 \leq u(r) \leq g(r)$, $0 \leq t(r) \leq g(r)$.

If $[r_i,\ i=1,2,\dots,n]$ is any subdivision of the fundamental square A into disjoint Jordan domains r_i, then we can put

$$G(S) = \text{l.u.b.} \sum g(r_i)$$

$$U(S) = \text{l.u.b.} \sum u(r_i)$$

$$T(S) = \text{l.u.b.} \sum t(r_i)$$

It is possible to prove the following:

THEOREM: For each surface S we have

(5) $$U(S) = G(S) = L(S) = T(S) .$$

It is to be noticed that G(S) is the so-called Geocze area of the surface S and so we have the result: for each surface S the Geocze area and the Lebesgue area are always equal to each other. The functionals U(S) and T(S) are notions of area of the type of the Geocze one. All of these functionals are lower semicontinuous and equal to each other.

The chain (5) of equalities has a geometrical interpretation that is similar to that for the plane transformations and that we can now outline saying that each surface is "in the small" and "statistically speaking" almost plane.

A metric approximation theorem.

Another type of statement that gives us the same geometrical interpretation is the following one that we call "a metric approximation theorem", which is similar to that for plane transformations:

Let S be a surface of finite Lebesgue area and (1) any representation of S. For each number $\varepsilon > 0$ there exists a set of simple disjoint plane polygons π_i, $i=1,2,\dots,\nu$, interior to A, such that if C_i are the oriented curves corresponding to the boundary of π_i upon S, if C_{is}, $s=1,2,3$, are the projections of C_i upon the coordinate planes, then

a) the diameter of each curve C_i is less than ε ;

b) the total measure of all the curves C_{is}, $i=1,2,\dots,\nu$, is less then ε $(s=1,2,3)$.

c) $$\sum_{i=1}^{\nu} t(\pi_i) > T(S) - \varepsilon ,$$
$$\sum_{i=1}^{\nu} u(\pi_i) > U(S) - \varepsilon ,$$
$$\sum_{i=1}^{\nu} g(\pi_i) > G(S) - \varepsilon .$$

Tangential properties of surfaces.

a) We have recalled that a surface of finite Lebesgue area may be without tangent planes at any point. Nevertheless certain properties hold that we call tangential properties of surfaces.

Let S be a continuous surface of finite Lebesgue area, let $Oxyz$, $O\xi\eta\zeta$ two sets of similarly oriented orthogonal carthesian axes, let

(1) $S: x = x(u,v),\ y = y(u,v),\ z = z(u,v),\ (u,v) \in A \equiv (0,1,0,1),$

be any representation of S relative to the xyz axes, and

(6) $S: \xi = \xi(u,v),\quad \eta = \eta(u,v),\quad \zeta = \zeta(u,v),\quad (u,v) \in A \equiv (0,1,0,1),$

the representation of S relative to the $\xi\eta\zeta$ -axes, which we obtain with the elementary formulas:

$$(7)\qquad \begin{cases} \xi = x \cos x\xi + y \cos y\xi + z \cos z\xi \\ \eta = x \cos x\eta + y \cos y\eta + z \cos z\eta \\ \zeta = x \cos x\zeta + y \cos y\zeta + z \cos z\zeta \end{cases}$$

We know that the Lebesgue area $L(S)$ is independent of the axes. We have now three plane transformations Φ_1, Φ_2, Φ_3 relative to the representation (1) and three plane transformations $\Phi'_1, \Phi'_2, \Phi'_3$ relative to the representation (6) and all the transformations are of bounded variation. In addition there exist a.e. in A, the absolute and relative generalized Jacobians J_r, H_r, $r=1,2,3$, and J'_r, H'_r, $r=1,2,3$, and we have a.e. in A, $H_r = \pm J_r$, $H'_r = \pm J'_r$, $r=1,2,3$. The following theorem holds:

THEOREM: If S is a surface of finite Lebesgue area, then we have a.e. in A,

$$H_{\xi\eta}(u,v) = \cos x\zeta \cdot H_{yz}(u,v) + \cos y\zeta \cdot H_{zx}(u,v) + \cos x\zeta \cdot H_{xy}(u,v),$$
$$H_{\eta\zeta}(u,v) = \cos x\xi \cdot H_{yz}(u,v) + \cos y\xi \cdot H_{zx}(u,v) + \cos y\xi \cdot H_{xy}(u,v),$$
$$H_{\zeta\xi}(u,v) = \cos x\eta \cdot H_{yz}(u,v) + \cos y\eta \cdot H_{zx}(u,v) + \cos z\eta \cdot H_{xy}(u,v)$$

and therefore

$$H^2_{\xi\eta} + H^2_{\eta\zeta} + H^2_{\zeta\xi} = H^2_{xy} + H^2_{yz} + H^2_{zx} \quad)$$

where we have written H_{yz}, H_{zx}, H_{xy}, $H_{\eta\zeta}$, $H_{\zeta\xi}$, $H_{\xi\eta}$ instead of H_1, H_2, H_3, H'_1, H'_2, H'_3.

b) Let S be a surface of finite Lebesgue area and (1) any representation of S; let $P \equiv (u,v)$ be an intirior point of A, q a square contained in A and containing P as an interior point. Let us consider the ratio $G(q)/|q|$. We call $D(u,v)$ the limit, if it exists,

$$D(u,v) = \lim_{\delta|q|\to 0} \frac{G(q)}{|q|} .$$

We know that all the following definitions are equivalent:

$$D(u,v) = \lim_{\delta|q|\to 0} \frac{G(q)}{|q|} = \lim_{\delta|q|\to 0} \frac{U(Q)}{|q|} = \lim_{\delta|q|\to 0} \frac{T(q)}{|q|} = \lim_{\delta|q|\to 0} \frac{L(q)}{|q|} .$$

I proved the following:

THEOREM: If S is a surface of finite Lebesgue area, then the limit D(u,v) exists a.e. in A and we have, a.e. in A,

$$D(u,v) = [J_1^2 + J_2^2 + J_3^2]^{\frac{1}{2}} \equiv [H_1^2 + H_2^2 + H_3^2]^{\frac{1}{2}} .$$

c) Let S be a surface of finite Lebesgue area and (1) any representation of S; let P be an interior point of A and $Q \equiv (x,y,z)$ the image of P upon the surface S; let $Q\,\xi\eta\zeta$ be a set of ortogonal cartesian axes, oriented like Oxyz and having their origin at point Q; Let (6) be the representation that we obtain from (1) through the elementary relations like (7). Now let Φ_1, Φ_2, Φ_3, Φ'_1, Φ'_2, Φ'_3 be the plane transformations relative to the xyz axes and $\xi\eta\zeta$ axes, let G, J, H be the known functions relative to the xyz-axes and G', J', H' be the ones relative to the $\xi\eta\zeta$-axes.

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I proved the following:

THEOREM: Let S be a surface of finite Lebesgue area and (1) any representation of S. For almost each point $P \equiv (u,v)$ interior to A, there exists a system of orthogonal axes $Q\xi\eta\zeta$, whose origin is the point $Q = S(P)$, the image of the point P upon the surface and such that

$$H'_{\xi\eta}(u,v) = J'_{\xi\eta}(u,v) = \sqrt{J^2_{xy} + J^2_{yz} + J^2_{zx}} = D(u,v)$$

$$H'_{\xi\zeta}(u,v) = J'_{\xi\zeta}(u,v) = 0, \quad H'_{\eta\zeta}(u,v) = J'_{\eta\zeta}(u,v) = 0.$$

For these properties of these special directions ξ, η, ζ, we call the plane $Q\xi\eta$ an almost tangent plane at the point Q and the oriented direction ζ an almost normal direction at the point Q for the general surface S.

IV. Representation of surfaces.

The problem of representation

We know that each continuous rectifiable curve has at least one representation for which the length is equal to the classical integral and such a representation is to be obtained, for instance, in the simplest way, if we choose as a parameter, the length of the arc from a fixed point to any point of the curve. We know also that for non-rectifiable curves the problem of a special map has been resolved, in the general case, by Marston Morse.

Now let S be a continuous surface of finite Lebesgue area and

(1) $S: \quad x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \quad (u,v) \in A \equiv (0,1,0,1),$

any representation of S upon the fundamental square A. We know that the classical integral calculated with generalized Jacobians exists and we have

$$L(S) \geq \iint_A [J_1^2 + J_2^2 + J_3^2]^{\frac{1}{2}} \, dudv \quad . \tag{8}$$

The problem arises of determining whether each surface S of finite Lebesgue area admits at least one representation (1) upon the fundamental square A for which the = sign holds in the relation (8), that is, the area is given by the classical integral.

Open non degenerate surfaces.

In order to study this problem we must study, first, a particular type of surface. We have observed that the representation (1) of the surface S defines on A the collection G of maximal continua g upon which all three functions x,y,z are constant. This collection characterizes the surface S and we say that the surface S is open non degenerate if it has the following property:

each continuum g of the collection G separates neither the plane uv, nor the square A.

This condition excludes the possibility that the surface S can have narrowings of any kind, but the surface can very well have multiple lines. It is possible to prove that such surfaces have a representation upon a square which is never stationary. With the terminology used also by Radó, we can say that such surfaces have a "light" representation upon a square.

C.B.Morrey proved that each open non degenerate surface admits a representation upon a square for which the area is given by the classical integral. The representation obtained by Morrey is very remarkable because it is "almost conformal". We shall say here that a map (1) of the surface S upon A is "almost conformal" if

(i) the functions x, y, z have first partial derivatives a.e. in Q;

(ii) the first partial derivatives $x_u, x_v, \dots, z_v$ are L^2-integrable functions in A;

(iii) at almost all points of A the classical equalities of conformal maps are satisfied:

$$E = G, \quad F = 0 \tag{9}$$

where

$$E = x_u^2 + y_u^2 + z_u^2, \quad G = x_v^2 + y_v^2 + z_v^2, \quad F = x_u x_v + y_u y_v + z_u z_v.$$

The theorem of Morrey says that each open non degenerate surface whose Lebesgue area is finite, admits an almost conformal map upon the fundamental square (or upon a circle) for which

a) the functions x, y, z are absolutely continuous in the sense of Tonelli;

b) the first partial derivatives $x_u, \dots, z_v$ are L^2-integrable in A;

c) the Lebesgue area is given by the classical integral.

The theorem of Morrey leads us to two well known theorems of the theory of conformal representation. One is the theorem of Weierstrass: any Jordan domain admits a conformal map upon a circle. The other is the theorem of Schwarz: any polyhedral surface, of the topological type of a 2-cell, admits a map upon a circle which is conformal everywhere except at the vertices.

The theorem of Morrey is a great extension of the theorem of Schwarz to any surface (open non degenerate and of finite Lebesgue area). It is quite natural that now the exceptional points are no longer finite in number but a quite general set of measure zero.

We have already recalled that McShane proved that there are surfaces, of this topological type, of finite Lebesgue area without a tangent plane at any point. Anyway, according to the theorem of Morrey, they have an almost conformal map upon a circle.

It is important to notice that the almost conformal representations of an open non degenerate surface satisfying the conditions a) and b) of the theorem of Morrey correspond to a minimum principle:

<u>each almost conformal map (1) of an open non degenerate surface S, satisfying the conditions a) and b), is minimal for the integral</u>

$$I = \frac{1}{2} \iint_A (x_u^2+y_u^2+ z_u^2+ x_v^2+ y_v^2+ z_v^2)\, dudv = \frac{1}{2} \iint_A (E + G)\, dudv \ ;$$

<u>conversely, each minimal map, with the properties a) and b), is almost conformal. The minimum is the value of the Lebesgue area L(S).</u>

I succeded in proving this and in finding therefore again in a simple way the theorem of Morrey, making use of the direct method of the calculus of variations. It is the question of a problem that is rather similar to the problems of Dirichlet and Plaseau.

Surfaces of zero Lebesgue area.

The notion of surface that results from the definition of Frèchet-McShane is of great generality. Let us see the main types of surfaces. We begin by observing that a surface can be reduced to a curve. For instance this is the case if the functions x,y,z are constant upon each segment parallel to the v-axis or upon the boundary of each square concentric and homothetic with A (fig.11). The upper semicontinuous collection is, in the first case, the set of all segments of A parallel to the v-axis, in the second case, the set of the boundaries of all such squares.

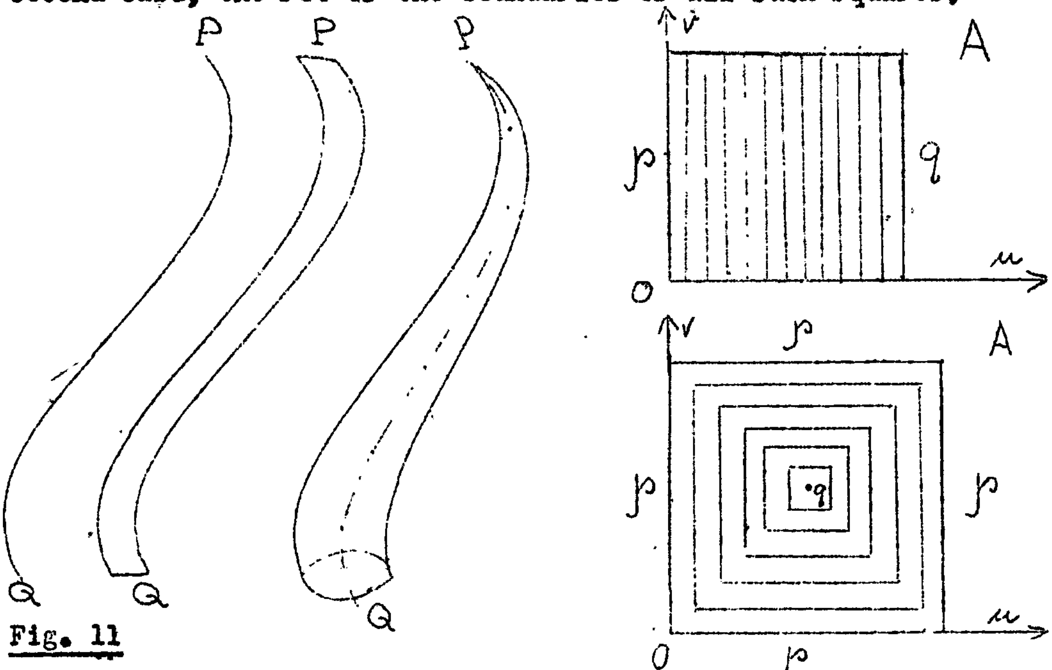


Fig. 11

In both cases the surface is reduced to a curve $\widehat{PQ}$. This curve corresponds also to a side of the square A in the first case, to a semidiagonal in the second case. Even if the curve $\widehat{PQ}$ is the same, these two kind of surfaces are quite different. And this corresponds also to the intuition. They are indeed two different limit cases of the common intuitive concept of surface. In the first case we must think of a strip getting narrower and narrower that reduces itself to the curve $\widehat{PQ}$; in the second case we must think of a kind of cone having for axis the curve $\widehat{PQ}$, vertex in P, and closed in Q; whose opening gets narrower and narrower and that reduces itself to the axis $\widehat{PQ}$.

In both cases we shall say that the surface is a thread. It is evident that its Lebesgue area is zero if we think that such surfaces can be approached as closely as we like, by a polygonal path of elementary area zero, or by polyhedral surfaces with elementary area as small as we like. It is therefore evident that a thread has Lebesgue area zero but this is a very strong result if we think that there are threads that fill a square or a cube.

A surface can reduce itself not only to a thread but to a ramified set of threads, as we can see in the fig. 12 in which we have indicated also a possible map of them upon a square. Such a set of threads may be also composed of a countable number of threads, but, naturally, the conditions of Hahn and Mazurkiewicz are always to be satisfied which requires that for each positive $\varepsilon > 0$ there is only a finite number of threads whose diameter is greater than ε. We shall say that the surface is a tree, or better, a tree of threads. These figures are only an outline because each curve may be such a complicated curve as we have pointed out before and they can cover the same set several times. Radó has proved that these surfaces give the most general type of surface of Lebesgue area zero.

An observation of J.T.Youngs gives the possibility of having an almost conformal map for each surface of zero Lebesgue area. If we consider the familiar Cantor ternary function $f(\alpha)$ in the interval $0 \leq \alpha \leq 1$, then $f(\alpha)$ is continuous, non decreasing in $(0,1)$, $f(0) = 0$, $f(1) = 1$, and $f(\alpha)$ is constant upon a set of subintervals, whose total measure is one.

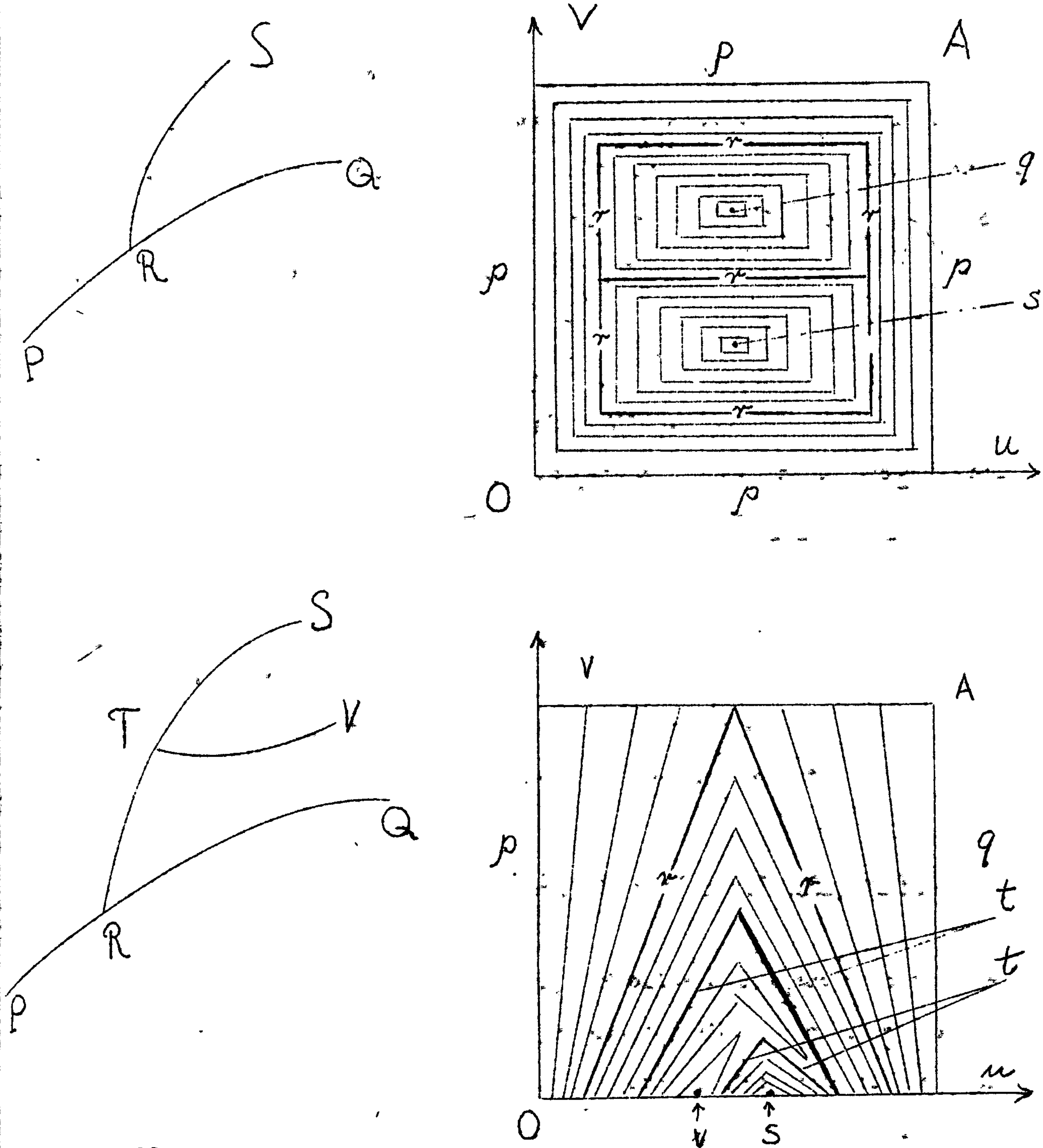


Fig. 12.

Then the equations

$$(10) \qquad S' \equiv S: \quad x = x[f(\alpha), f(\beta)], \quad y = y[f(\alpha), f(\beta)], \quad z = z[f(\alpha), f(\beta)],$$

$$(\alpha, \beta) \in B = [\, 0 \leq \alpha \leq 1,\ 0 \leq \beta \leq 1],$$

give still a representation of the surface S, but x,y,z are now functions of α and β that are constant upon a set of rectangles in B whose total measure is one. Therefore the functions x,y,z have first partial derivatives zero almost everywhere in B, and we have, a.e. in B,

$$E = G = F = 0 .$$

In addition

$$L(S) = 0, \qquad \iint_B \sqrt{EG - F^2}\, du dv = 0 .$$

In such a way we have proved that each surface of zero Lebesgue area has an almost conformal map for which the area is given by the classical integral.

Surfaces of the "type A" or "base surfaces".

We proceed to a more general type of surface, that of type A or, according to a nomenclature introduced by Morrey, "base surface". We say that the surface S is of the "type A" or a "base surface" if it has the following property:

each continuum g of the collection G may separate the square A, but not the plane.

For instance the surfaces composed of pieces, that are open non degenerate, joined through points of their boundary directly or by means of threads are base surfaces. In fig. 13 there are outlined some of the simplest cases and there is also indicated one of their possible maps upon a square. The junction points and the points of the threads are the images of continua that must separate the square A, but these do not separate the plane uv.

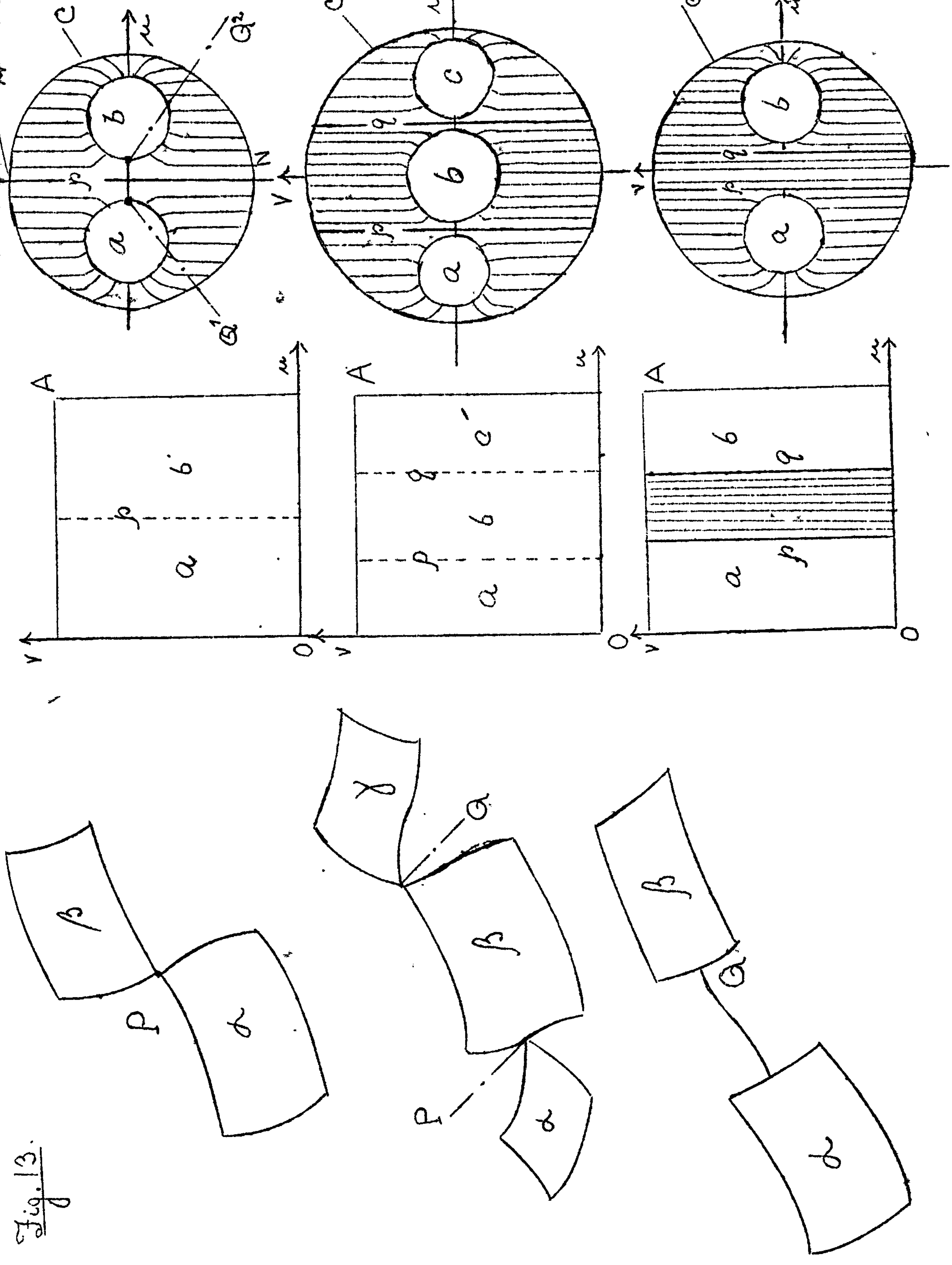


Fig. 13.

The open non degenerate pieces that belong to this kind of surface may also be a countable set, but for each positive ε there is only a finite set of pieces that have a diameter larger than ε.

The example of fig. 14 is very important. It is composed of a closed set of points of a curve $\widehat{PQ}$. For each complementary interval of this closed set we substitute an open non degenerate surface, two of whose boundary points are endpoints of such intervals. We can call such a surface a "thread of leaves".

The most general surface of this kind (base surface) is composed of a tree of curves as the previous $\widehat{PQ}$. The fig. 14 gives the outline of such a surface and the case, that some threads are without open surfaces, is not excluded. We could call these surfaces "trees of threads of leaves".

These leaves are open non degenerate surfaces S_i and if $L(S_i)$ is their Lebesgue area, C.B.Morrey has proved that

$$L(S) = \sum_i L(S_i) .$$

Exceptional and proper sets.

We must find a representation for such surfaces for which the area is equal to the classical integral.

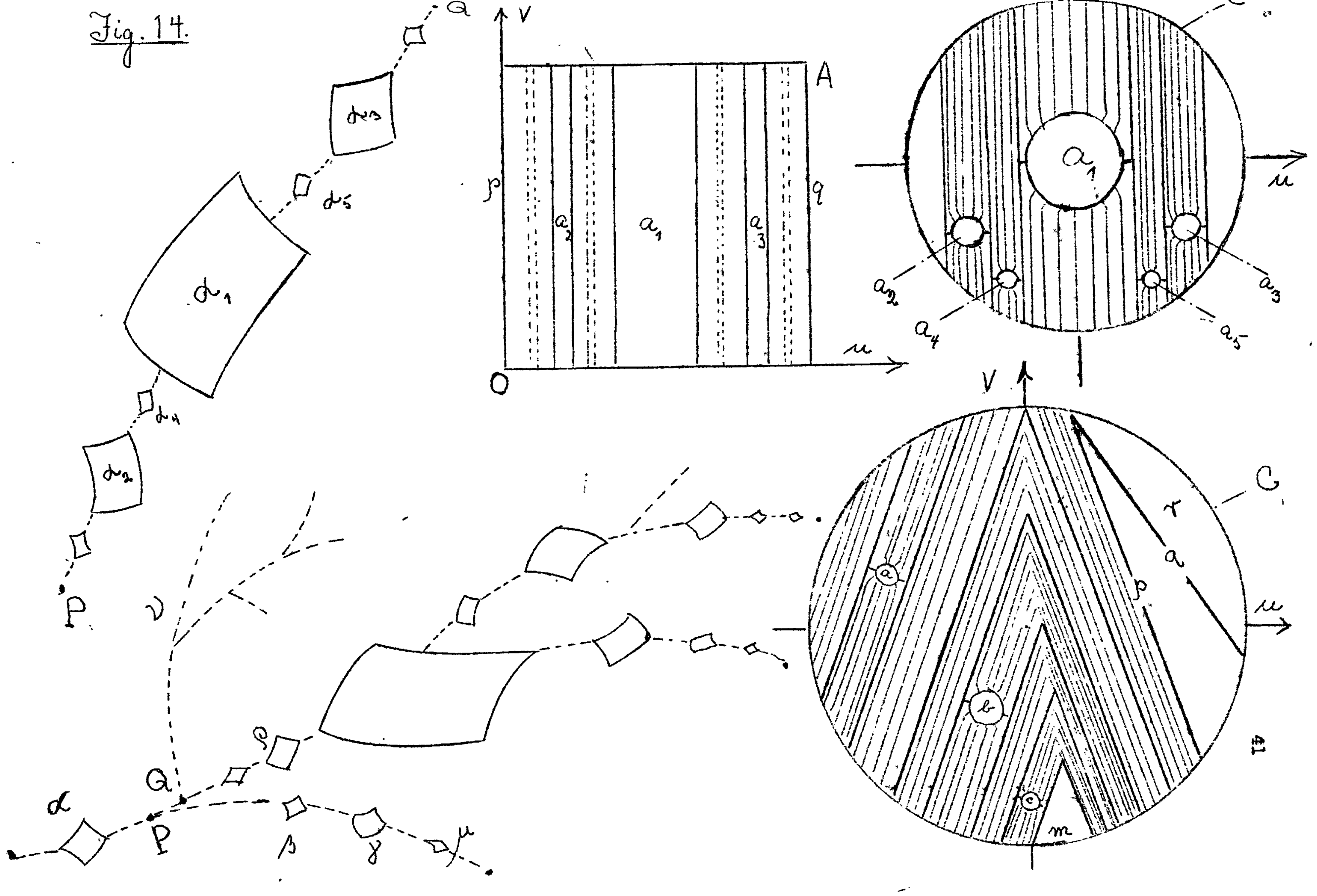
Let us first give some definitions:

We say a point of A is exceptional for the map (1) of a surface S if there exists a neighborhood U of P upon which the map (1) defines a surface of Lebesgue area zero.

We say a subset of A is exceptional for the map (1) if it is open and consists only of exceptional points.

We say a subset of A is proper for the map (1) if its complementary set is exceptional. Each proper set is closed.

Fig. 14.



We say the map (1) is almost conformal upon a measurable set M of A if:

i) the functions x, y, z have first partial derivatives almost everywhere in M;

ii) the first partial derivatives x_u, x_v, ..., z_v are L^2-integrable in M;

iii) almost everywhere in M we have $E = G$, $F = 0$.

Representation of surfaces of type A.

Let us consider now the simplest of the base surfaces that we have illustrated, the first of the fig. 13. The two parts α and β forming it are open non-degenerate surfaces and therefore each of them has an almost conformal representation upon the circles a and b inside the large circle C. The boundary of α is mapped upon the boundary of a, the boundary of β upon the boundary of b. We can always suppose that the point P comes from the points Q^1 of a and Q^2 of b. We define the functions x, y, z upon the cross-shape continuum p giving to them anywhere the same constant value that they have already upon Q^1 and Q^2. Than to the whole continuum p corresponds upon the surface the same point P and upon the other lines joining a point of the boundary of a and b with the boundary of C, let us give to the functions x, y, z the same constant value, they have already at their starting point upon a and b. In this way the whole surface S has a map upon C for which the proper set is formed by the two circles a and b, the exceptional set is the complementary set C - a - b. The representation is almost conformal upon the proper set and the Lebesgue area is given by the classical integral, calculated with ordinary Jacobians upon the proper set, with generalized Jacobians upon the exceptional set, which are everywhere zero.

In a similar way the other surfaces of the figure 13 must be dealt with.

A little more difficult is the case of a surface like that of figure 14 where there are infinite open non.degenerate parts. Here the closure set of the infinte circles such as $a_1, a_2, a_3, \ldots$, may contain points other than those of the circles themselves, that is points that are neither interior, nor of the boundary, of the circles. It is possible to place these circles in the interior of C but nearer and nearer the boundary of C. In such a way only points of this boundary may be limit points of the circles and the boundary of C is a set of measure zero.

Therefore we obtain the statement: each base surface admits a representation which is almost conformal upon the proper set and for which the Lebesgue area is given by the classical integral, calculated with ordinary Jacobians upon the proper set, with generalized Jacobians, everywhere zero, upon the exceptional set.

It is evident though that in the points of the exceptional set it is not possible to assure the existence of partial derivatives, but only of the generalized Jacobians that are certainly zero. Now it is possible to modify the map obtained of the surface S, only upon the open exceptional set*) in such a way that the functions x,y,z have first partial derivatives almost everywhere zero and therefore the map is almost conformal also in the exceptional set, for E = G = F = 0.

We get in such a way the following:

<u>THEOREM:</u> <u>Each surface of the"type A"(base surface) of finite Lebesgue area, admits an almost conformal map upon a square (or circle), the Lebesgue area being equal to the classical integral.</u>

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*) see appendix.

Closed non degenerate surfaces

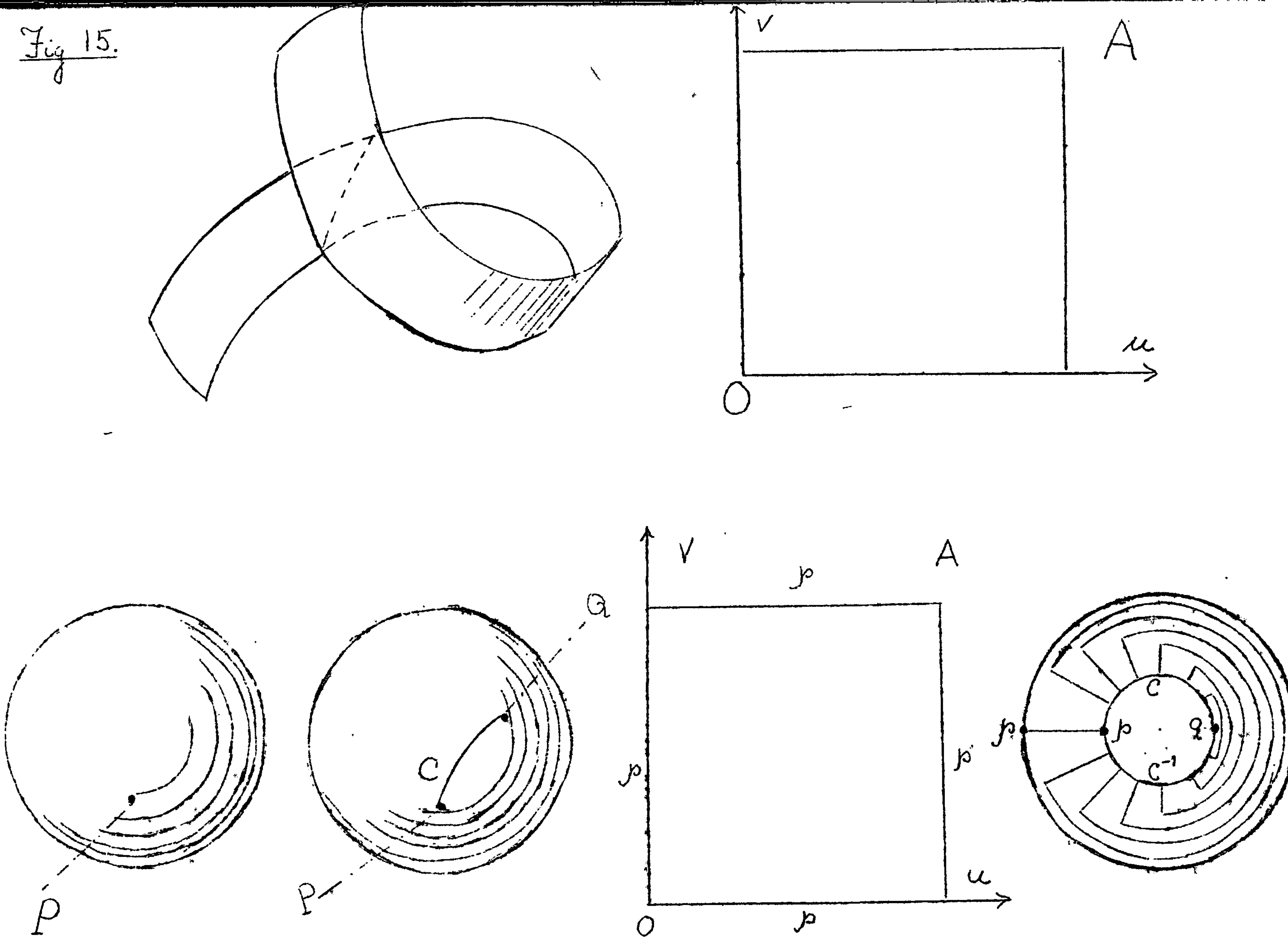
We shall call closed non degenerate surfaces, or surfaces of the topological type of a 2-sphere, those surfaces that have a never stationary map, that is, a light map, upon a sphere. These surfaces have the following property: in any map of them upon a square A, the proper continua on which the functions x,y,z are constant, separate neither the plane uv, nor the domain A, with exception of one continuum that contains the whole boundary of A.

In such a map to the boundary p of A corresponds a unique well determined point P of the surface. The functions $\bar{x},y,z$ are constant upon the boundary of A.

Let us cut the surface S along a curve $c = \widehat{PQ}$. It can be shown that under the conditions mentioned it is always possible to make a cut along a continuous curve c which is rectifiable or at least whose three projections upon the coordinate planes have measure zero. We get a new surface, or "cut surface" and we will call it S_c. The surface S_c may be "open non degenerate" or a "base surface". In order to simplify matters let us suppose here that S_c is an open non degenerate surface. Then the cut surface S has a map almost conformal upon a circle (the interior circle in the fig.15) and now we must build a map for the given closed surface S.

We can separate upon the boundary of this interior circle(fig.15) two arcs having for image the two lips of the cut: c and c^{-1}. Let us join through continua (fig.15) the points of the two arcs that have the same image upon the two lips of the cut, the continua filling the annular region between the considered circle and another bigger and concentric circle(fig.15). In such a way we have upon this new circle a representation of the

Fig 15.



given closed surface S. The closed interior circle is the proper set, the annular domain is the exceptional set. As a consequence of the properties of the closed curve c along which we made the cut, the Lebesgue area of the closed surface is equal to the Lebesgue area of the cut surface S_c and now it is evident that in such a way we have a representation of the closed surface that is almost conformal upon a proper set and for which the Lebesgue area $L(S) = L(S_c)$ is given by the classical integral.

A modification of the map obtained upon the exceptional set, similar to that we used for the base surfaces, gives now the following statement:

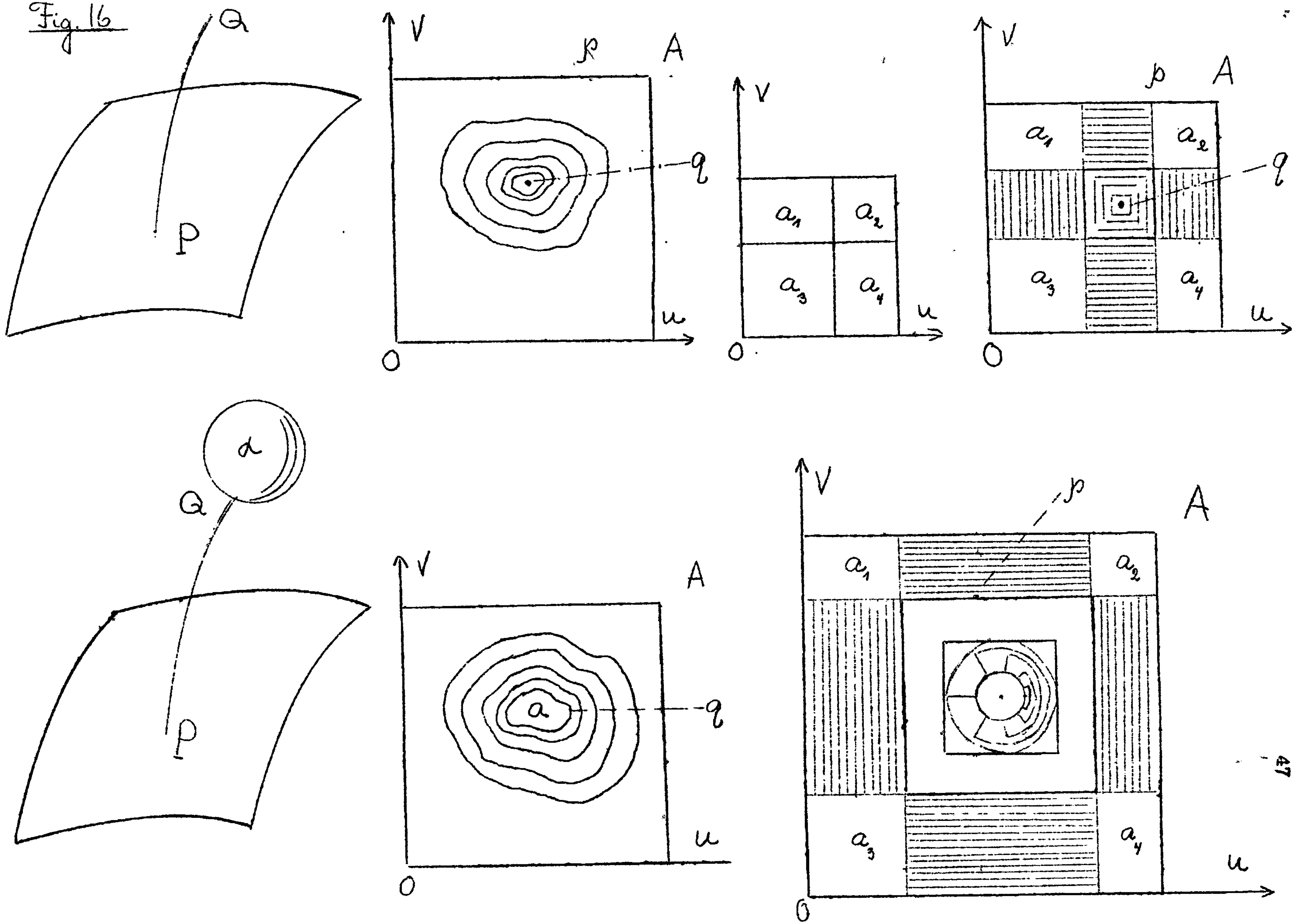
Each closed non degenerate surface of finite Lebesgue area admits an almost conformal map upon a circle (or a square) for which the Lebesgue area is given by the classical integral.

Surfaces of general kind.

Let us pass to the most general type of surface. Now the continua $\mathcal{E}$ of the collection G may separate the square A and the plane uv. The figures 16, 17, 18 give some simple examples of them: one open surface with one thread that has its point of departure P in any point of the open surface; one open surface with one thread and one sphere, one thread with two spheres; one base surface with three threads; one open surface with one thread and one sphere and, from here, two threads of which one has another sphere and another thread. In the figures 16,17,18 there is also outlined a possible map of such surfaces upon a square.

The closed non degenerate parts that belong to this kind of surface may be also a countable set, but for each positive ε there is only a finite set of parts that have a diameter larger than ε. The example of the figure 19 is very important. It is composed of a closed set of points of

Fig. 16



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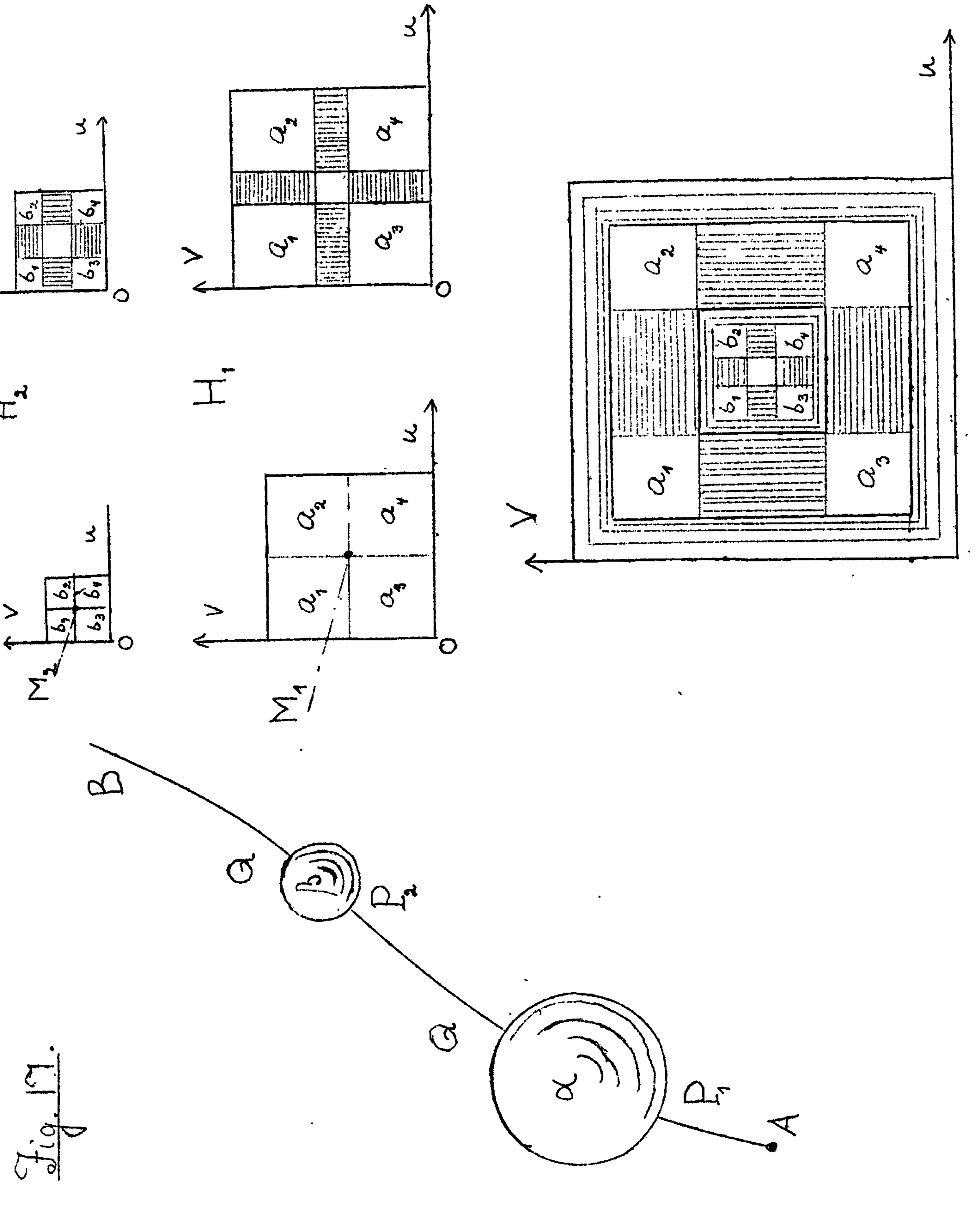
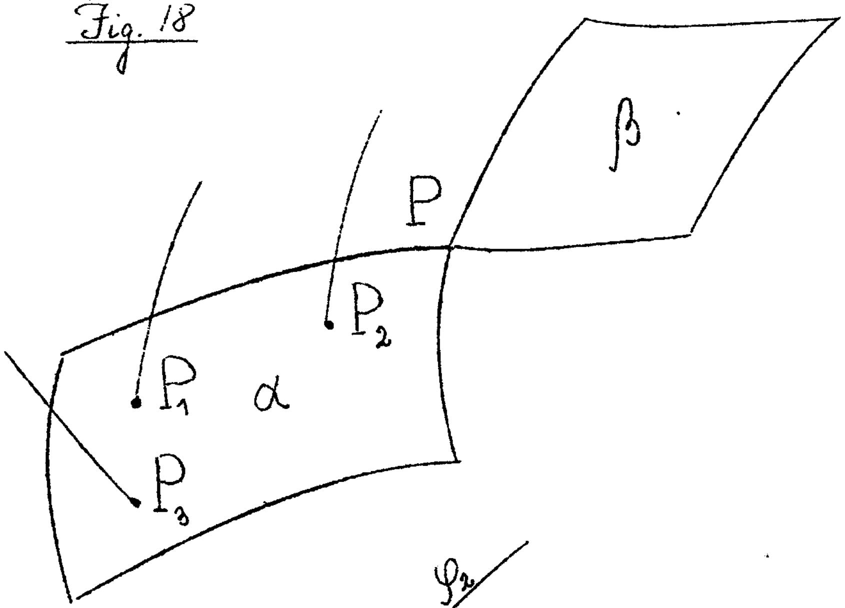


Fig. 17.

Fig. 18


β
P
P_2
P_1
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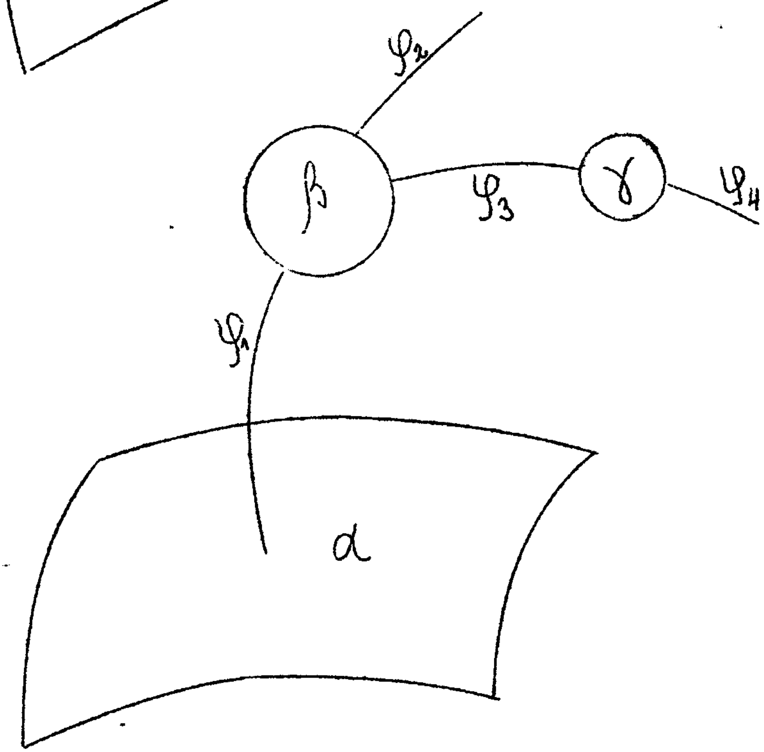
P_3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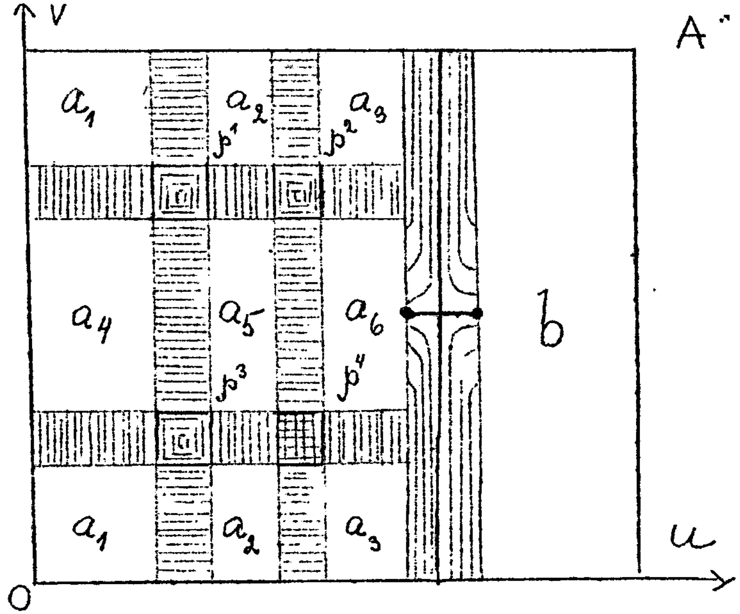
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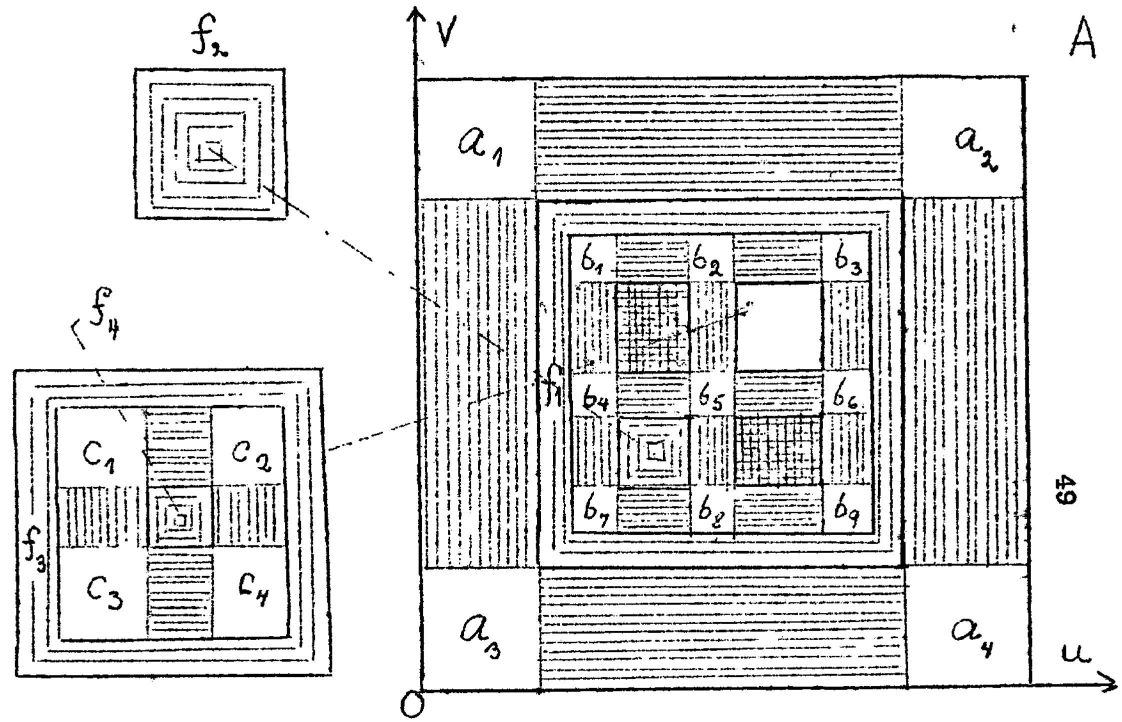
f_2
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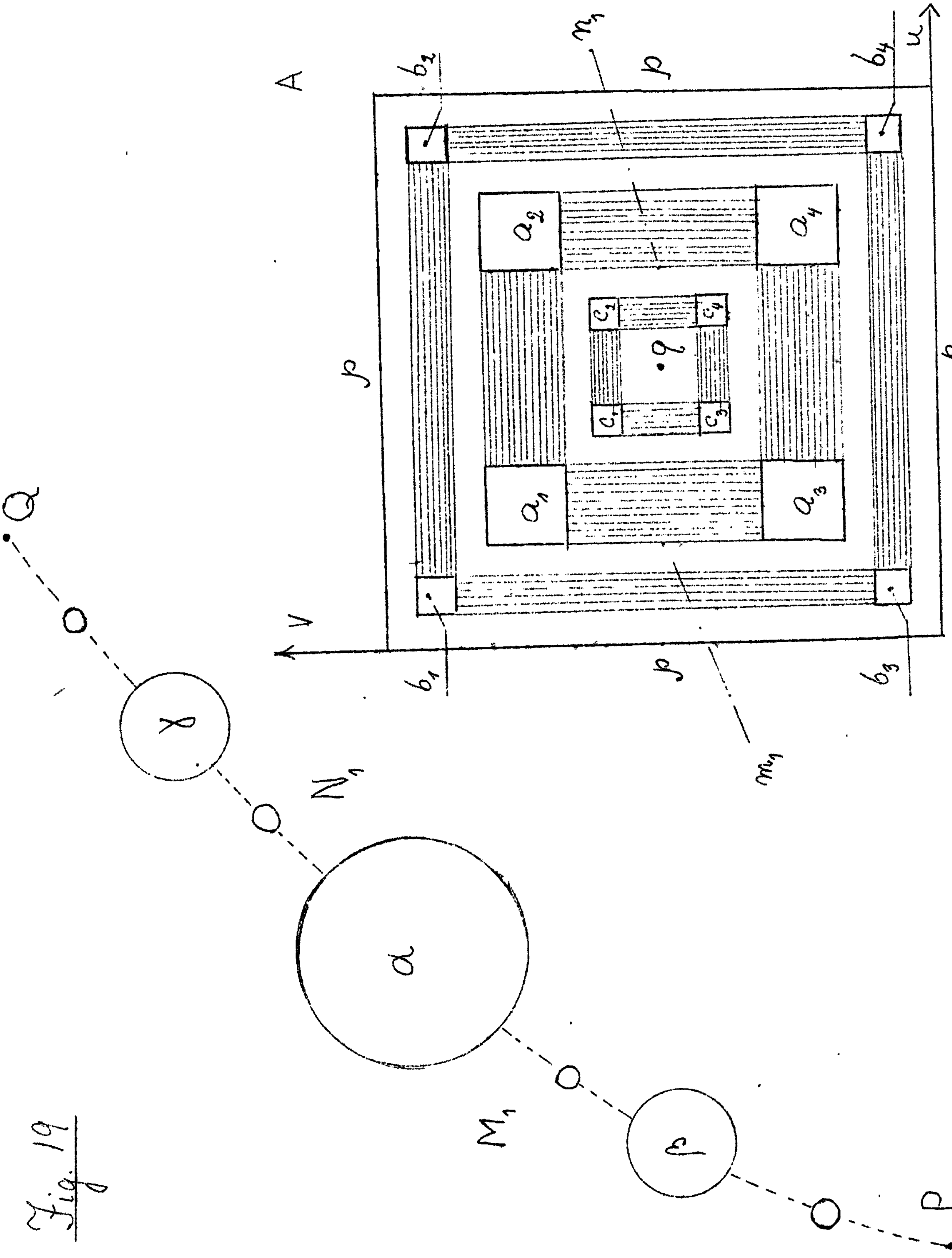

Fig. 19

a curve $\widehat{PQ}$. For each complementary interval of this closed set we substitute a closed non degenerate surface two of whose points are the end points of such intervals. In fig. 19 the curve is a simple elementary arc and the closed set is the Cantor "ternary" set. We shall say that such a surface is a "thread of pearls".

One more general kind of surface is composed of a tree of threads like such a thread $\widehat{PQ}$. The fig. 20 (first part) gives the outline of such a surface and it may happen that some threads are without closed surfaces. We call this a "tree of threads of pearls".

And now the most general kind of surface (fig.20, second part): A base surface and a countable set of trees of threads of pearls which have their point of departure at any point of the base-surface. Naturally, owing to the continuity of the functions x,y,z of the map, the following property is satisfied: for each positive ε there is only a finite set of such trees of threads of pearls that have a diameter larger than ε.

We have emphazised with the previous considerations the topological structure of a general surface. This structure has been studied especially by Moore, Morrey and Youngs and I think this is a very fine result of research in set-topology. It is shown that each surface contains a countable number of open or closed non degenerate surfaces S_i and Morrey has demonstrated that the Lebesgue area of the whole surface S is exactly the sum of the Lebesgue areas of all these non degenerate surfaces

$$L(S) = \sum_i L(S_i) .$$

<u>Representation of surfaces of general kind</u>

We must find now an almost conformal map for such a general kind of surface.

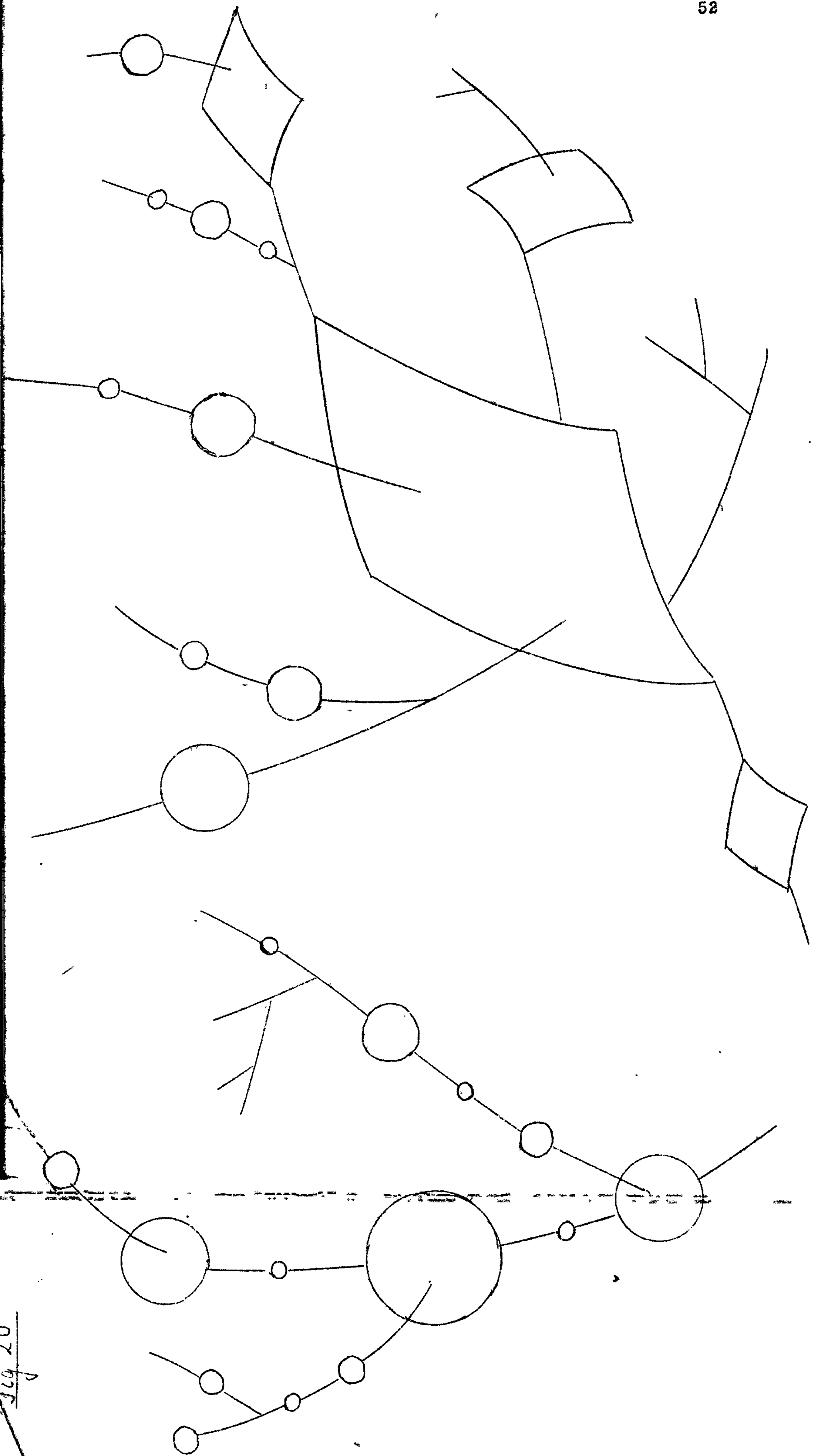

Fig 20

We must begin from the simplest of them, those of figure 16. The open non degenerate part of the first admits an almost conformal map upon a square. There is a point that has for its image the point P, starting point of the thread $\widehat{PQ}$. Let us cut the square along two straight lines parallel to the u,v axes and let us separate the four rectangles a_1, a_2, a_3, a_4, from one another parallel to the u,v axes. Let us define the functions x,y,z upon the four arms of the cross to be constant upon each segment parallel to the u or v axis. Upon the boundary of the central square the functions x,y,z are constant, their values being the coordinates of the point P. It is easy to map the thread $\widehat{PQ}$ upon the central square (fig.16). The map of the whole surface S that we have got upon the square A is almost conformal upon the proper set $M = a_1+a_2+a_3+a_4$ formed by the four corner rectangles. The cross is the open exceptional set. It is clear that the classical integral calculated upon the proper set and also upon the whole square is equal to the Lebesgue area of the whole surface.

Similar considerations lead to a representation of the second case of the fig. 16 in which there is also indicated an outline of a map that is almost conformal in a proper set, the Lebesgue area being equal to the classical integral.

As a second example let us consider a surface like the one of fig. 17 and let us begin to map separately the closed surfaces α and β as we have already said, almost conformally upon certain closed proper sets H_1 and H_2 of given squares. Let us fix the points M_1 and M_2 whose images are the points Q_1 and Q_2. Let us enlarge the map through strips parallel to the axes, as before. Finally combining the two maps we can get a unique map of the whole surface as is outlined in the same fig.17. Here the proper set is the closed set formed of the subsets of H_1 and H_2 that are contained in some of the rectangles a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4. The exceptional set of the map is the complementary set and, as is evident, this exceptional set is required in order to map the threads of the surface and the cuts that we have made upon the surface.

Analogously for the cases of the fig. 18.

Here also we can pass to the general case but the difficulties are much greater. Let us consider for instance the case of a surface S formed by one part S_o open non degenerate and by an infinite number of threads whose starting points are everywhere dense upon S_o. Of course, owing to the continuity of the map, for each positive ε there is only a finite set of threads whose diameter is larger than ε. In this case an almost conformal map upon a proper set may still be obtained, but this proper set is the perfect set that we get by taking away from a square a countable and everywhere dense set of strips parallel to the u and v axes. The proper set is therefore a completely disconnected closed set.

We can apply the same procedure that we used for the case of fig. 17 in the mapping of a thread of infinite pearls, like the one of the fig. 19, but in this case we have an infinite number of squares one in the interior of the other. It is evident that the infinite closed proper sets may now have new limit points, which are upon the diagonals of the exterior square. The proper set of the whole map is therefore formed by all the proper sets relative to the different pearls and the four semidiagonals. In such a way we obtain for the whole surface a map on the square A, which is almost conformal upon a proper set.

In the same way it is possible also to get a map on a square which is almost conformal upon a closed proper set for a surface of the most general type as outlined in fig. 20. It is very difficult to prove that the limit points of the infinite proper set of the different non degenerate parts, that do not belong to one of these, form a set of measure zero.

We have already recalled that, in these general cases, the proper set may be completely disconnected. Here also with one modification of the map only upon the open exceptional set, it is possible to get representations which have first partial derivatives zero a.e. upon the exceptional set. But now every point of the proper set is a limit point of points of the exceptional set and therefore if we perform a modification in the exceptional set we may have as a consequence that we are no longer sure that there are ordinary first derivatives at almost all points of the proper set, but only asymptotic derivatives. However, in a recent paper, I succeeded in showing that it is possible to define the modification described in a manner so refined that the functions x,y,z still have ordinary first derivatives at almost all points of the proper set (see Appendix).

We have, therefore, the following :

THEOREM: Every surface of finite Lebesque area admits an almost conformal map upon a circle, the Lebesgue area being given by the classical integral.

Appendix

We say that a continuous plane transformation

$$\Phi : u = u(\alpha, \beta), \quad v = v(\alpha, \beta), \quad (\alpha, \beta) \in A,$$

is monotone if for each point $Q \equiv (u,v)$, the set $\Phi(Q)$ of all points $P \equiv (\alpha, \beta)$ of A whose image is Q, if it is not empty, is a continuum (or a single point) of A.

I proved the following theorem, which gives an extension of the familiar Cantor ternary function to the plane transformations:

THEOREM: A transformation Φ exists which transforms a square g into itself, which is continuous upon g, monotone, the identity upon the boundary g^* of g and constant upon a countable set of disjoint squares of g, whose total measure is equal to the measure of g.

This theorem is utilized in the proof of the previous general theorem for the representation of surfaces.

ADDENDUM

We add the following foot-note at page 13, line 16:

We indicate $G(\Phi)$ also with $G(A)$. We can define $G(r)$ for each Jordan domain r of A in an analogous way.

Likewise for $U(\Phi)$ and $T(\Phi)$.