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NOTES ON MINIMAL SURFACES

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1. Preliminaries on Minimal Surfaces

We introduce the following notations:

Vector (a_i, b_i, c_i): $\bar{a}_i$

Scalar product $a_1 \cdot a_2$: $\bar{a}_1\bar{a}_2 = a_1a_2 + b_1b_2 + c_1c_2$

Length of $\bar{a}_1$: $+\sqrt{\bar{a}_1^2}$

Vector product: $[\bar{a}_1, \bar{a}_2] = (b_1c_2 - b_2c_1, c_1a_2 - c_2a_1, a_1b_2 - a_2b_1)$

Identity of Lagrange: $[\bar{a}_1, \bar{a}_2]^2 = \bar{a}_1^2 \cdot \bar{a}_2^2 - (\bar{a}_1 \cdot \bar{a}_2)^2$

Determinant $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = |\bar{a}_1\bar{a}_2\bar{a}_3|$

In this introductory lesson we shall not worry about assumptions regarding differentiability. We consider the surface S:

$x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, or simply

$$\bar{x} = \bar{x}(u, v)$$

We designate the vector $\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$ as follows:

$$\bar{x}_u = (x_u, y_u, z_u), \quad x_u = \frac{\partial x}{\partial u}, \dots$$

$$\bar{x}_v = (x_v, y_v, z_v), \quad x_v = \frac{\partial x}{\partial v}, \dots$$

$$d\bar{x} = \bar{x}_u du + \bar{x}_v dv$$

Then we have for the

Linear element or the
First fundamental quadratic form $\Big\}$ $ds^2 = d\bar{x}^2 = Edu^2 + 2Fdudv + Gdv^2$

$E = \bar{x}_u^2, \quad F = \bar{x}_u\bar{x}_v, \quad = \bar{x}_v^2$

*

Element of area: $do = +\sqrt{EG - F^2}\, du\, dv$, $\sqrt{EG - F^2} = W$

Normal vector: $\bar{\xi} = (\xi, \eta, \zeta) = \frac{[\bar{x}_u, \bar{x}_v]}{\sqrt{[\bar{x}_u, \bar{x}_v]^2}} = \frac{[\bar{x}_u, \bar{x}_v]}{W}$

($W = \sqrt{[\bar{x}_u, \bar{x}_v]^2}$ follows from the Identity of Lagrange)

$\bar{\xi}$ is a unit vector, the point (ξ, η, ζ) is always on the sphere $x^2 + y^2 + z^2 = 1$, $\bar{\xi}$ is the spherical transform of the point $\bar{x}(u, v)$

Second fundamental quadratic form: $-d\bar{x}\, d\bar{\xi} = L\, du^2 + 2M\, du\, dv + N\, dv^2$ with

$$L = -\bar{x}_u \bar{\xi}_u = \frac{1}{W} |\bar{x}_{uu}\, \bar{x}_u\, \bar{x}_v| , \quad M = -\bar{x}_u \bar{\xi}_v = -\bar{x}_v \bar{\xi}_u = \frac{1}{W} |\bar{x}_{uv}\, \bar{x}_u\, \bar{x}_v|$$

$$N = -\bar{x}_v \bar{\xi}_v = \frac{1}{W} |\bar{x}_{vv}\, \bar{x}_u\, \bar{x}_v|$$

Then, according to the theorem of Euler, the radius of curvature $\rho(du:dv)$ of the normal section of S in $\bar{x}(u, v)$ in the direction $du:dv$ is given by

$$\frac{1}{\rho(du:dv)} = \frac{L\, du^2 + 2M\, du\, dv + N\, dv^2}{E\, du^2 + 2F\, du\, dv + G\, dv^2}$$

The Euler Theorem can be expressed in a little different manner: In the tangential plane of S at $\bar{x}$ we lay off the segment $+\sqrt{\rho(du:dv)}$ in the direction $du:dv$; then we get a conic section, the so-called Dupin Indicatrix of S at $\bar{x}$, either an ellipse or a pair of conjugate hyperbolas. The radii of curvature of S belonging to the axes of the Dupin Indicatrix are called the principal radii of curvature; we shall designate them by R_1 and R_2. The only thing we need here is the relation

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{EN - 2FM + GL}{EG - F^2}$$

Two tangential directions to S through $\bar{x}$ are conjugate when they belong to conjugate diameters of the Dupin Indicatrix, i.e. when they are divided harmonically by the asymptotes of the Dupin Indicatrix. The asymptotes, are the directions for which $\frac{1}{\rho} = 0$; therefore they are given by

$$L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0$$

and the directions $du : dv$ and $\delta u : \delta v$ are conjugate if, and only if,

$$L\,du\,\delta u + M(du\,\delta v + \delta u\,dv) + N\,dv\,\delta v = 0$$

If the parameter lines, i.e. the lines

du arbitrary $\neq 0,\ dv = 0$ and $\delta u = 0,\ \delta v$ arbitrary $\neq 0$

are conjugate, we find

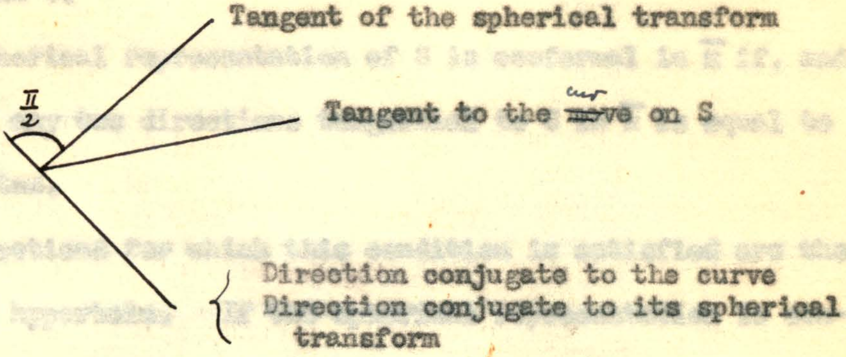
$$M = 0$$

and this is clearly sufficient for the parameter lines being conjugate at $\bar{x}$. The equation $M = 0$ can be expressed in the following form: there exist numbers A and B which satisfy the equation (in reality three equations)

$$\bar{x}_{uv} - A\bar{x}_u - B\bar{x}_v = 0$$

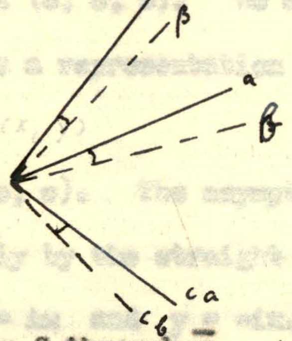
More geometrically the conjugate directions can be defined as follows: Take any ~~mo~~curve γ on S through $\bar{x}$. Its tangent may have the direction $du : dv$ Consider the intersection of the plane tangent to S at a point $P \subset \gamma$ with the plane tangent to S in $\bar{x}$. If $P \to \bar{x}$ this intersection approaches the straight line, which is conjugate to $du : dv$ This fact gives rise to a nice geometrical consideration:

$\bar{\xi}$ is the spherical transform of $\bar{x}$. The planes tangential to the sphere in $\bar{\xi}$ and to S in $\bar{x}$ are parallel. When the sphere is moved by a translation so that the point $\bar{\xi}$ falls in $\bar{x}$, the tangential planes are identical. Consider a curve on S and the corresponding curve on the sphere. Since in corresponding points the tangential planes are parallel, the directions conjugate to the tangents of the curves on S and on the sphere must be parallel in corresponding points. On the sphere the conjugate to any tangential direction in a point Q is the perpendicular tangential direction. If we make the above translation, we therefore get the following configuration:



This has many applications; for instance: The directions belonging to the principal radii of curvature are conjugate and perpendicular; therefore these directions must be parallel to their spherical transforms. This is the essential part of the formula of Olinde Rodrigues. Or the asymptotic directions are conjugate to themselves; therefore they are perpendicular to their spherical transforms.

We use the configuration for determining all the surfaces, for which the spherical representation is conformal.



Let a, b be two directions on S through $\overline{x}$, α, β their spherical transforms. When $\measuredangle(a,b) = \measuredangle(\alpha,\beta)$, then from the above configuration follows that the angle between the conjugate direction c_a and c_b of a and b must also be equal to $\measuredangle(a,b)$: $\measuredangle(a,b) = \measuredangle(c_a,c_b)$; and obviously the converse holds too:

The angle between two directions a, b on S is equal to the angle of their spherical transforms if, and only if, it is equal to the angle of the directions conjugate to a and b.

Therefore the spherical representation of S is conformal in $\bar{x}$ if, and only if, the angle between any two directions tangential to S in $\bar{x}$ is equal to the angle of their conjugates.

The only conic sections for which this condition is satisfied are the circle and the equilateral hyperbola. If the spherical representation is conformal throughout, the Dupin Indicatrix must be either always a circle or always an equilateral hyperbola. In the first case the surface is a sphere, since the sphere is the only surface all of whose points are ombilics; in the second case it is a so-called minimal surface. $\frac{1}{R_1} + \frac{1}{R_2} = 0$ therefore is ~~the~~ a necessary and sufficient condition for minimal surfaces.

Let our surface S be a minimal surface, $\bar{x}$ a point of it. We introduce a special coordinate system for which $\bar{x}$ = (o, o, o), and the (x, y) - plane is the plane tangential to S at (o, o, o). We allow the variables x, y to take complex values. ~~Then~~ S has a representation

$$z = f(x, y)$$

in the neighborhood of (o, o, o). The asymptotic directions are perpendicular. They are divided harmonically by the straight lines

(*) $y = ix$ and $y = -ix$.

It is well known that the pairs of perpendicular directions, and only these, are divided harmonically by the lines (*).

We see: the equilateral hyperbolas with center (o, o) and only these are conic sections for which the lines y = ix and y = -ix are conjugate. Since

$$z_x(0,0) = f_x(0,0) = 0 \quad , \quad z_y(0,0) = f_y(0,0) = 0$$

we have in (o, o)

$$ds^2 = dx^2 + dy^2$$

and for the directions y = ix and y = -ix

$$ds^2 = 0$$

The last equation is independent of the choice of the coordinate system. In general coordinates they are determined as the roots of the equation

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 = 0$$

and are called the isotropic directions through the point. The lines on the surface whose tangent is isotropic in every point are called the isotropic lines and we see:

On the minimal surfaces, and only on these, the isotropic lines are conjugate.

Let us take the isotropic lines as parameter lines: when α, β are the new parameters and

$$\bar{x} = \bar{x}(\alpha, \beta)$$

is the surface, then

$$E = x_\alpha^2 + y_\alpha^2 + z_\alpha^2 = \bar{x}_\alpha^2 = 0 \quad \text{and} \quad G = \bar{x}_\beta^2 = 0$$

$$ds^2 = 2F\,d\alpha\,d\beta$$

Further~~on~~more the lines α = const. and β = const. are conjugate; that means $M = 0$ or the existence of two functions $A(\alpha, \beta)$ and $B(\alpha, \beta)$ which satisfy the equation

$$\bar{x}_{\alpha\beta} - A\bar{x}_\alpha - B\bar{x}_\beta = 0$$

Now from $\bar{x}_\alpha^2 = 0$ follows

$$2\bar{x}_{\alpha\beta}\bar{x}_\alpha = 2(\bar{x}_\alpha^2 A + \bar{x}_\alpha\bar{x}_\beta B) = 2\bar{x}_\alpha\bar{x}_\beta \cdot B = 0$$

Since ds^2 is not identically 0 in du, dv, F must be different from 0; therefore

$$B = 0$$

and similarly A = 0. We find $\bar{x}_{\alpha\beta} = 0$ or

$$(1) \quad \begin{cases} x(\alpha,\beta) = f_1(\alpha) + \varphi_1(\beta) \\ y(\alpha,\beta) = f_2(\alpha) + \varphi_2(\beta) \\ z(\alpha,\beta) = f_3(\alpha) + \varphi_3(\beta) \end{cases}$$

with

$$(2) \quad E = \sum_i f_i'^2(\alpha) = 0 \ , \quad \mathfrak{G} = \sum_i \varphi_i'^2(\beta) = 0$$

Vice versa one sees immediately that any functions $x(\alpha,\beta)$, $y(\alpha,\beta)$, $z(\alpha,\beta)$ satisfying the conditions (1) and (2) give a minimal surface: it is $E = \mathfrak{G} = 0$ and on account of $\bar{x}_{\alpha\beta} = 0$ M must vanish too. Therefore $\frac{1}{R_1} + \frac{1}{R_2} = 0$.

Hence, if (1) is a minimal surface,

$$(1') \quad \begin{cases} x_0(\alpha,\beta) = i\,(f_1(\alpha) - \varphi_1(\beta)) \\ y_0(\alpha,\beta) = i\,(f_2(\alpha) - \varphi_2(\beta)) \\ z_0(\alpha,\beta) = i\,(f_3(\alpha) - \varphi_3(\beta)) \end{cases}$$

is also a minimal surface. It has especially strong connections to (1) and is called the surface adjoint to (1). We represent the surfaces (1) and (1') on each other by making points correspond which belong to the same (α,β). This representation is an application, for

$$d\bar{x}^2 = \bar{x}_\alpha \bar{x}_\beta \, d\alpha \, d\beta = \sum f_i' \varphi_i' \, d\alpha\, d\beta = \sum i f_i' \cdot (-i\varphi_i') \, d\alpha \, d\beta$$

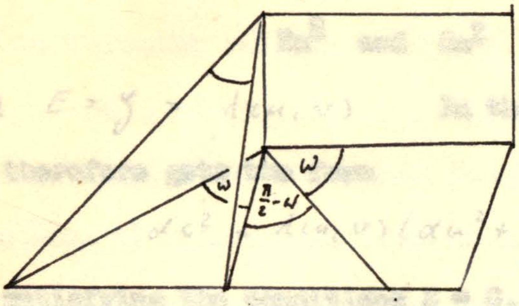
$$= (\bar{x}_0)_\alpha (\bar{x}_0)_\beta \, d\alpha \, d\beta = d\bar{x}_0^2$$

Furthermore

$$d\bar{x}\, d\bar{x}_0 = \sum_i (f_i' d\alpha + \varphi_i' d\beta)(i f_i' d\alpha - i \varphi_i' d\beta) = i \sum f_i'^2 d\alpha^2 +$$

$$+ i \sum f_i' \varphi_i' \, d\alpha\, d\beta - i \sum f_i' \varphi_i' \, d\alpha\, d\beta - i \sum \varphi_i'^2 d\beta^2 = 0$$

We see that the tangents to corresponding curves at corresponding points are perpendicular. Because the representation is an application, it is conformal; hence the tangential planes at corresponding points must be parallel (see the figure):



Hence we have

$$\bar{\xi}\,d\bar{x} = 0 \qquad \text{and} \qquad \bar{\xi}\,d\bar{x}_0 = 0$$

The vector $d\bar{x}_0$ is perpendicular to both $\bar{\xi}$ and $d\bar{x}$; it must therefore be parallel to $[\bar{\xi}, d\bar{x}]$:

$$d\bar{x}_0 = k\,[\bar{\xi}, d\bar{x}], \qquad k \text{ a scalar.}$$

On account of the Lagrange Identity we have

$$[\bar{\xi}, d\bar{x}]^2 = \bar{\xi}^2\cdot d\bar{x}^2 - (\bar{\xi}\,d\bar{x})^2 = d\bar{x}^2 = d\bar{x}_0^{\,2}$$

hence k = ± 1. It is easy to prove k = + 1, but we do not need that. What is important for us is the fact that the expressions

$\eta dz - \zeta dy,\ \zeta dx - \xi dz,\ \xi dy - \eta dx$ are <u>complete differentials</u>.

This property is independent of the choice of the parameter system.

Until now we have not used the fact that the spherical representation is conformal. The sphere can be represented conformally on the plane for ex. by stereographical projection. Let (u, v) be rectangular coordinates in the plane and $\bar{x}(u, v)$ the corresponding point of the surface. Since the curves u = const. and v = const. are perpendicular, the corresponding curves on the surface must be perpendicular; therefore

$$\bar{x}_u \cdot \bar{x}_v = F = 0 .$$

Furthermore linear elements of equal length issuing from one point must correspond to linear elements of equal length on the surface. When we take the linear elements du = a, dv = 0 and du = 0, dv = a in the plane, the corresponding elements are

$$Ea^2 \text{ and } Ga^2$$

and we find $E = G = \lambda(u,v)$ In the parameters u, v the linear element of surface therefore gets the form

$$ds^2 = \lambda(u,v)(du^2 + dv^2) ;$$

parameters satisfying the conditions E = G, F = 0 are called isothermic or isometric. We see: On a minimal surface there exist isothermic parameters in the neighborhood of each point.

It is easy to verify the following fact:

When u, v are isothermic parameters on any surface $\bar{x}(u, v)$, then the equations

$$\eta\, dz - \zeta\, dy = x_v\, du - x_u\, dv$$
$$\zeta\, dx - \xi\, dz = y_v\, du - y_u\, dv$$
$$\xi\, dy - \eta\, dx = z_v\, du - z_u\, dv$$

hold.* If the surface is minimal these expressions must be complete differentials.

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*) $E = \bar{x}_u^2 = \bar{x}_v^2 = G = W \quad ; \quad F = \bar{x}_u \bar{x}_v = 0$

$W\eta = z_u x_v - z_v x_u \quad , \quad W\zeta = x_u y_v - x_v y_u \quad , \quad d\bar{x} = \bar{x}_u du + \bar{x}_v dv$

$W(\eta\, dz - \zeta\, dy) = \alpha\, du + \beta\, dv$

$$\alpha = x_v z_u^2 - x_u z_u z_v - x_u y_u y_v + x_v y_u^2 =$$
$$= x_v z_u^2 - x_u(x_u x_v + y_u y_v + z_u z_v) + x_u^2 x_v + x_v y_u^2 =$$
$$= x_v\,(x_u^2 + y_u^2 + z_u^2) = x_v W$$
$$\beta = x_v z_u z_v - x_u z_v^2 - x_u y_v^2 + x_v y_u y_v$$
$$= x_v \cdot (\bar{x}_u \bar{x}_v) - x_u(\bar{x}_v^2) = -x_u \cdot W$$

According to a well-known theorem

$$f(u,v)\,du + g(u,v)\,dv$$

is a complete differential if, and only if,

$$f_u = g_v$$

In our case we therefore get the relations

$$x_{uu} = -x_{vv}\,,\quad y_{uu} = -y_{vv}\,,\quad z_{uu} = -z_{vv} \quad \text{or} \quad \bar{x}_{uu} = -\bar{x}_{vv}$$

the coordinates x, y, z are harmonical functions of u and v. On the other hand, if a surface $\bar{x}(u, v)$ is given in terms of isothermic parameters u, v, and the coordinates are harmonic functions of u, v, then the surface is minimal. For then $E = G$, $F = 0$,

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{E(N+L)}{EG-F^2}$$

and

$$N = \frac{1}{W}\,|\bar{x}_{uu}\ \bar{x}_u\ \bar{x}_v| = -\frac{1}{W}\,|\bar{x}_{vv}\ \bar{x}_u\ \bar{x}_v| = -L$$

We have proved:

(1) A surface $\bar{x} = \bar{x}(u, v)$, which is given in terms of isothermic parameters u, v, is minimal if, and only if, the coordinates x(u, v), y(u, v), z(u, v) are harmonic functions of u and v.

When x(u, v), y(u, v), z(u, v) are harmonic, they are the real parts of certain analytic functions $(u + iv = w)$

$$x(u,v) = R\,F_1(w)\,,\quad y(u,v) = R\,F_2(w)\,,\quad z(u,v) = R\,F_3(w)$$

Then $F_1'(w) = x_u(u,v) - i\,x_v(u,v)\,,\quad F_2' = y_u - i\,y_v\,,\quad F_3' = z_u - i\,z_v$

and

$$\sum F_i'^2 = E - G - 2iF = 0$$

$$\sum |F_i'|^2 = E + G.$$

At last we shall discuss why our surfaces have got the name "minimal surfaces". We take an arbitrary surface $\overline{x}(u, v)$ and ask when its area is minimal. We vary the surface (in a special way) by laying off on the normal at the point $\overline{x}$ the segment

$$n(u,v) = \varepsilon\, v(u,v)$$

The surface formed by the endpoints of these segments is

$$\overline{X}(u,v) = \overline{x}(u,v) + n(u,v)\,\overline{\xi}$$

By differentiation we find

$$\overline{X}_u = \overline{x}_u + n\,\overline{\xi}_u + n_u\,\overline{\xi}$$
$$\overline{X}_v = \overline{x}_v + n\,\overline{\xi}_v + n_v\,\overline{\xi}$$

$$E_1 = \overline{X}_u^{\,2} = \overline{x}_u^{\,2} + 2n\,\overline{x}_u\overline{\xi}_u + \varepsilon^2(*) = E - 2nL + \varepsilon^2(*)$$
$$F_1 = \overline{X}_u\overline{X}_v = \overline{x}_u\overline{x}_v + n(\overline{x}_u\overline{\xi}_v + \overline{x}_v\overline{\xi}_u) + \varepsilon^2(*) = F - 2nM + \varepsilon^2(*)$$
$$\mathcal{G}_1 = \overline{X}_v^{\,2} = \overline{x}_v^{\,2} + 2n\,\overline{x}_v\overline{\xi}_v + \varepsilon^2(*) = \mathcal{G} - 2nN + \varepsilon^2(*)$$

Therefore

$$W_1^2 = E_1\mathcal{G}_1 - F_1^2 = E\mathcal{G} - F^2 - 2n(EN - 2FM + \mathcal{G}L) + \varepsilon^2(*)$$
$$= (E\mathcal{G} - F^2)\left(1 - 2n\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right) + \varepsilon^2(*)$$

According to the Binomial development

$$(1+x)^{\mu} = 1 + \binom{\mu}{1}x + \binom{\mu}{2}x^2 + \cdots$$

we find

$$W_1 = W\left(1 - n\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right) + \varepsilon^2(*)$$
$$\iint W_1\, du\, dv = \iint do - \iint n\left(\frac{1}{R_1} + \frac{1}{R_2}\right) do + \varepsilon^2(*)$$

We see: the area of S is stationary only if the surface is minimal.

And: If through a simple closed curve Γ passes an analytical surface and if this surface is of the least possible area then it is minimal.

Using the Euler equations and the fact that a neighborhood of each point on a surface (of class C") can be represented in terms of isothermic parameters, the fundamental result (1), page 10, can be established very quickly as follows.

The Euler equations belonging to the variation problem: to minimize the integral

$$\iint_D \mathfrak{J}(x, y, z, x_u, x_v, y_u, y_v, z_u, z_v)\,du\,dv$$

are

$$\mathfrak{J}_x - \frac{\partial \mathfrak{J}_{x_u}}{\partial u} - \frac{\partial \mathfrak{J}_{x_v}}{\partial v} = 0$$

$$\mathfrak{J}_y - \frac{\partial \mathfrak{J}_{y_u}}{\partial u} - \frac{\partial \mathfrak{J}_{y_v}}{\partial v} = 0$$

$$\mathfrak{J}_z - \frac{\partial \mathfrak{J}_{z_u}}{\partial u} - \frac{\partial \mathfrak{J}_{z_v}}{\partial v} = 0$$

If $\mathfrak{J} = \sqrt{EG - F^2} = \sqrt{\bar{x}_u^2 \cdot \bar{x}_v^2 - (\bar{x}_u x_v)^2}$ we have

$$\frac{\partial \mathfrak{J}}{\partial x} = 0$$

$$\frac{\partial \mathfrak{J}}{\partial x_u} = \frac{x_u G - x_v F}{W}, \quad \frac{\partial \mathfrak{J}}{\partial x_v} = \frac{x_v E - x_u F}{W}$$

and similar equations in y and z. Therefore the three Euler equations belonging to $\iint W\,du\,dv$ are

$$\frac{\partial}{\partial u}\left[\frac{\bar{x}_u \cdot G - \bar{x}_v F}{W}\right] + \frac{\partial}{\partial v}\left[\frac{\bar{x}_v E - \bar{x}_u F}{W}\right] = 0$$

Introducing isothermic parameters, we have E = G, F = 0, and we find again the equation

$$\bar{x}_{uu} + \bar{x}_{vv} = 0.$$

2. The Mapping Theorem of Koebe and the Problem of Plateau

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0 In his paper with this title J. Douglas*) gives a simultaneous solution of the following two problems:

Given in the (x, y, z)-space a simple closed Jordan curve Γ.

(a) To determine a minimal surface of the type of the circular disk bounded by Γ which satisfies the further conditions: the surface admits a representation $\overline{x} = \overline{x}(u, v)$, $u^2 + v^2 \leq 1$, where x(u, v), y, z are continuous for $u^2 + v^2 \leq 1$, harmonic for $u^2 + v^2 < 1$, and satisfy for $u^2 + v^2 < 1$ the equation $E = G$, $F = 0$. Furthermore the equations $\overline{x} = \overline{x}(u, v)$ and $u^2 + v^2 = 1$ give a topological representation of $u^2 + v^2 = 1$ on Γ.

(b) To lay through Γ a surface of the type of the circular disk whose area is a minimum.

Here some remarks are to be made:

Ad (a) • E and G do not vanish identically, for with the notations of page 10 (below) that would mean

$$\sum_i |F_i'|^2 = 0$$

or F_i = const., i = 1, 2, 3. But the possibility that somewhere $E = G = 0$ is not excluded. The equation

$$\sum_i |F_i'|^2 = E + G$$

shows that the points where $E = G = 0$ are isolated. They are singularities of the surface. A closer research shows that they are similar to the branch-points which occur in study of Riemann surfaces. Furthermore (as

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*) The mapping theorem of Koebe and the problem of Plateau, Journal of Mathematics and Physics, vol. 10 (1931), pp. 106-130.

follows from this statement) self-crossings of the surface are admitted.*)

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Ad (b) The area is to be declared according to the definition which Lebesgue gave in his thesis. Let Σ be any continuous surface of the type of the circular disk bounded by Γ and $\Pi_1, \Pi_2, \cdots$ a sequence of simply connected polyhedral surfaces tending to Σ whose boundaries tend to Γ. By $S(\Pi_\nu)$ we designate the area of Π_ν in the usual sense; we put

$$S(\Pi_1, \Pi_2, \cdots) = \liminf_{\nu \to \infty} S(\Pi_\nu)$$

and understand by $S(\Sigma)$ the greatest lower bound of the numbers $S(\Pi_1, \Pi_2, \cdots)$ formed for the different sequences satisfying the above conditions.

In reality Douglas treats a slightly more general problem than (a, b). He calls minimal surfaces in the euclidean n-space with the cartesian coordinates $x_1, \dots, x_n$ surfaces which can be represented in the form

$$x_i(u, v) = R\, F_i(w), \quad i = 1, \cdots, n; \quad w = u + iv$$

where the F_i are analytic varying in a certain domain of the (u, v)-plane (the same for all F_i) and satisfy the further condition

$$\sum_{i=1}^{n} F_i'^2(w) = 0 .$$

Setting

$$E = \sum_i \left(\frac{\partial x_i}{\partial u}\right)^2, \quad F = \sum_i \frac{\partial x_i}{\partial u} \frac{\partial x_i}{\partial v}, \quad G = \sum_i \left(\frac{\partial x_i}{\partial v}\right)^2 .$$

we have (as on p. 10)

$$\sum F_i'^2 = E - G - 2iF = 0$$

hence E = G, F = 0, and

$$\sum |F_i'|^2 = E + G .$$

With this definition of a minimal surface the problems (a), (b) are solved simultaneously for the n-space ($n \geq 3$).

The paper begins with some preparatory considerations: Let $g(\vartheta)$, $0 \leq \vartheta < 2\pi$ be a bounded real function $(|g(\vartheta)| \leq M)$ which is continuous in all but at most a countable number of points in which $g(\vartheta)$ may have discontinuities of the first kind, i.e. the limits $g(\vartheta+)$ and $g(\vartheta-)$ exist everywhere and are equal to $g(\vartheta)$ except for an enumerable set of values of ϑ. Then in the interior of the unit circle of the (u, v)-plane we consider the function (u + iv = w)

$$H(w) = H(u, v) = R\, F(w)$$

with

$$F(w) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\vartheta} + w}{e^{i\vartheta} - w}\, g(\vartheta)\, d\vartheta .$$

In the paper the fact is used that

$$H(w) \rightarrow g(\vartheta_0)$$

if $w \rightarrow e^{i\vartheta_0}$, $|w| < 1$ and $g(\vartheta)$ is continuous in ϑ_0. To prove that, let w be equal to $r e^{i\varphi}$. Then

$$\frac{e^{i\vartheta} + w}{e^{i\vartheta} - w} = \frac{(\cos\vartheta + r\cos\varphi) + i(\sin\vartheta + r\sin\varphi)}{(\cos\vartheta - r\cos\varphi) + i(\sin\vartheta - r\sin\varphi)}$$

and

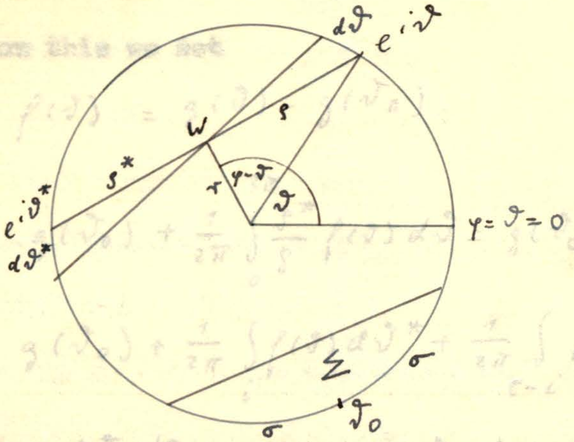
$$R\, \frac{e^{i\vartheta} + w}{e^{i\vartheta} - w} = \frac{(\cos^2\vartheta - r^2\cos^2\varphi) + (\sin^2\vartheta - r^2\sin^2\varphi)}{(\cos\vartheta - r\cos\varphi)^2 + (\sin\vartheta - r\sin\varphi)^2}$$

$$= \frac{1 - r^2}{1 + r^2 - 2r\cos(\varphi - \vartheta)}$$

Hence $R\,F(w)$ is the Poisson integral:

$$R\,F(w) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1+r^2-2\cos(\varphi-\vartheta)}\,g(\vartheta)\,d\vartheta .$$

The figure shows:



if $e^{i\vartheta^*}$ designates the point where the straight line connecting $e^{i\vartheta}$ and W crosses the unit circle the second time, and if we put

$$\rho = |W - e^{i\vartheta}| \quad , \rho^* = |W - e^{i\vartheta^*}|$$

we have the relations

$$\rho^2 = 1+r^2 - 2r\cos(\varphi-\vartheta)$$

$$\rho\rho^* = (1+r)(1-r)$$

$$\frac{\rho^*}{\rho} = \frac{1-r^2}{1+r^2-2r\cos(\varphi-\vartheta)}$$

$$\frac{d\vartheta^*}{d\vartheta} = \frac{\rho^*}{\rho}.$$

We see quite generally (if $f(\vartheta)$ is summable):

$$\int_0^{2\pi}\frac{\rho^*}{\rho}f(\vartheta)\,d\vartheta = \int_0^{2\pi}f(\vartheta)\,d\vartheta^*$$

Let D be the interior of the unit circle, $\mathcal{C}$ its boundary, and ϑ_0 any point on C; take on C an interval i of length 2σ with center ϑ_0 and put

$$\Delta(\vartheta_0,\sigma) = \underset{\vartheta\subset i}{\text{Least Upper bound}}\ |g(\vartheta)-g(\vartheta_0)|.$$

Then for all points w in the interior of the segment Σ of the unit circle bounded by i and the straight line segment connecting the end points of i, we have

(1) $$|H(w) - g(\vartheta_0)| \leq \Delta(\vartheta_0, \sigma) + \frac{2M}{\pi}\sigma.$$

To show this we set

$$f(\vartheta) = g(\vartheta) - g(\vartheta_0)$$

Then we have

$$H(w) = g(\vartheta_0) + \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^*}{\rho} f(\vartheta)\,d\vartheta = g(\vartheta_0) + \frac{1}{2\pi}\int_0^{2\pi} f(\vartheta)\,d\vartheta^*$$

$$= g(\vartheta_0) + \frac{1}{2\pi}\int_i f(\vartheta)\,d\vartheta^* + \frac{1}{2\pi}\int_{\mathfrak{C}-i} f(\vartheta)\,d\vartheta^*$$

For $w \subset \Sigma$ and $\vartheta^* \subset \mathfrak{C}-i$ we have $\vartheta \subset i$, therefore

$$\left|\int_{\mathfrak{C}-i} f(\vartheta)\,d\vartheta^*\right| \leq \Delta(\vartheta_0, \sigma)(2\pi - 2\sigma)$$

and on account of $|f(\vartheta)| \leq 2M$

$$\left|\int_i f(\vartheta)\,d\vartheta^*\right| \leq 4\sigma M.$$

From this the inequality (1) follows immediately, and from (1), that

$$H(w) \to g(\vartheta_0) \qquad \text{for } w \to \vartheta_0,\ |w| < 1,$$

if $g(\vartheta)$ is continuous in ϑ_0.

Douglas states (without using the statement later) that if $g(\vartheta)$ has at ϑ_0 a discontinuity of the first kind, the limiting values of $H(w)$ for $w \to e^{i\vartheta_0}$, $|w| < 1$, depend linearly on the direction in which w approaches $e^{i\vartheta_0}$. This can be proved in the following way.

Let $e^{i\vartheta_0}$ be the point (1, 0) of the (u, v)-plane. Then $\frac{v}{u-1}$ is defined in D. One verifies easily that arctg $\frac{v}{u-1}$ is a harmonic function in D.

The values $\lambda(\vartheta)$ of arctg $\frac{v}{u-1}$ on C have in $\vartheta=0$ a discontinuity of the first kind:

$$\lambda(0+) = +\frac{\pi}{2}, \quad \lambda(0-) = -\frac{\pi}{2}$$

If for our function $g(\vartheta)$

$$g(0+) - g(0-) = \alpha \neq 0$$

the values $\frac{\alpha}{\pi}\lambda(\vartheta)$ of

$$\Lambda(w) = \frac{\alpha}{\pi} \operatorname{arctg} \frac{v}{u-1}$$

on C have the same discontinuity as $g(\vartheta)$:

$$\frac{\alpha}{\pi}\lambda(0+) - \frac{\alpha}{\pi}\lambda(0-) = \alpha .$$

Therefore $g(\vartheta) - \frac{\alpha}{\pi}\lambda(\vartheta)$ is continuous at $\vartheta = 0$ and

$$R\,\frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{e^{i\vartheta}+w}{e^{i\vartheta}-w}\left(g(\vartheta)-\frac{\alpha}{\pi}\lambda(\vartheta)\right)\right.d\vartheta = H(w) - \Lambda(w)$$

approaches for $w \to 1$, $|w|<1$, according to our previous result, the value

$$g(0+) - \frac{\alpha}{2} = g(0-) + \frac{\alpha}{2} .$$

If we set

$$w_\nu - 1 = \bar{r} e^{i\psi_\nu}$$

and let w_ν converge to 1 in such manner that

$$\psi_\nu \to \psi ,$$

then

$$\Lambda(w_\nu) \to \frac{\alpha}{\pi}\psi$$

hence

$$H(w_\nu) \to g(0+) - \frac{\alpha}{2} + \frac{\alpha}{\pi}\psi .$$

After these remarks Douglas considers a set $g_1(\vartheta), \cdots, g_n(\vartheta)$ of functions of the same kind as our $g(\vartheta)$ and forms the corresponding functions

$$H_i(w) = H_i(u,v) = R\,F_i(w)$$

with

$$F_i(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\vartheta}+w}{e^{i\vartheta}-w}\, g_i(\vartheta)\, d\vartheta .$$

The equations

$$x_i = H_i(u,v)$$

define a surface harmonic for $u^2 + v^2 < 1$. We introduce the functionals

$$A_\rho(g) = \frac{1}{2}\iint_{D_\rho} (E+G)\,du\,dv$$

$$S_\rho(g) = \iint_{D_\rho} \sqrt{EG-F^2}\,du\,dv$$

where D_ρ designates the domain $u^2+v^2<\rho<1$. We set

$$A(g) = \lim_{\rho\to 1} A_\rho(g)$$

$$S(g) = \lim_{\rho\to 1} S_\rho(g) .$$

Then some properties of the functionals are established

I. $S(g) \le A(g)$ and $S(g) = A(g)$ if and only if

$$\sum_{i=1}^{n} F_i'^2(w) = 0$$

II. $A(g)$ is lower semicontinuous.

The functions which interest us here are uniformly bounded $(|g(\vartheta)| \le M)$. $A(g)$ is defined as soon as g is summable. Let L be the class of all measurable functions $f(\vartheta)$ with $|f(\vartheta)| \le M$. If $g_\nu(\vartheta) \in L$ and $g_\nu(\vartheta) \to g(\vartheta)$ (not necessarily uniformly), then $g(\vartheta) \in L$. This can be slightly generalized: Let $g_\nu(\vartheta)$ and $g(\vartheta)$ be defined only almost everywhere on $0 \le \vartheta < 2\pi$, i.e. except for sets N_ν, ~~resp.~~ N_0 respectively of measure 0. Then

$$N = N_0 + N_1 + N_2 + \cdots$$

has the measure 0 and we understand the relation

$$g_\nu(\vartheta) \to g(\vartheta)$$

as meaning that

$$g_\nu(\vartheta_0) \to g(\vartheta_0) \quad \text{if } \vartheta_0 \text{ is not in N.}$$

We enlarge the class L by admitting all measurable functions $f(\vartheta)$ which are defined on $0 \le \vartheta < 2\pi$ except for a set $N(f)$ of measure 0 and for which

$$|f(\vartheta)| \le M \qquad \text{if } \vartheta \text{ is not in } N(f).$$

Then we still have: if $g_\nu \subset L$ and $g_\nu \to g$ then $g \subset L$.

That $\bar{A}(g)$ is continuous on L means: if $g_\nu \to g$ $(g_\nu, g \subset L)$ then $\bar{A}(g_\nu) \to \bar{A}(g)$ and that $A(g)$ is lower semicontinuous on L means: if $g_\nu \to g$ $(g_\nu, g \subset L)$ then

$$A(g) \le \liminf_{\nu \to \infty} A(g_\nu^*)$$

The proof of Theorem II consists of two steps: the first is to prove that A(g) can be represented as limit of a non-decreasing sequence of continuous functions $\bar{A}_\nu(g)$. Douglas proves this by showing that $A_\rho(g)$ is continuous. Since

$$A_{\rho_1}(g) \ge A_{\rho_2}(g) \qquad \text{for} \quad 1 > \rho_1 > \rho_2 > 0,$$

the functions

$$\bar{A}_\nu(g) = A_{\frac{\nu-1}{\nu}}(g)$$

form a non-decreasing sequence of functions converging to $A(g)$

The second step is the proof of the following general fact:

Let $\bar{L}$ be any set of the following type: the relation of convergence is defined and satisfies the conditions: (a) if $g_\nu \to g$ $(g_\nu, g \subset \bar{L})$ then each subsequence $\{g_{n_\nu}\}$ of $\{g_n\}$ converges to g; (b) if $g_\nu = g$ v = 1, 2, ..., then g_ν converges to g. (Our class L satisfies these conditions and is even ~~compact.~~ closed.) Let the continuity and semicontinuity of a function A(g) on $\bar{L}$ be defined as above, then the theorem holds:

If $A_\nu(g)$ is lower semicontinuous on $\overline{L}$, $\nu = 1, 2, \ldots$, and $A_\nu(g) \geq A_{\nu-1}(g)$, $\nu = 2, 3, \ldots$, for each g on L, then $A_\nu(g)$ converges to a lower semicontinuous ~~f~~unction~~s~~ A(g).

Proof. $A_\nu(g)$ converges, since $A_\nu(g) \geq A_{\nu-1}(g)$. Let A(g) be the limit. If A(g) were not lower semicontinuous, a sequence $g_n^0 \to g^0$ would exist with

$$\liminf_{n\to\infty} A(g_n^0) < A(g^0)$$

or for sufficiently great ν we should have

(*)
$$\liminf_{n\to\infty} A(g_n^0) < A_\nu(g^0).$$

On the other hand

$$A_\nu(g_n^0) \leq A(g_n^0)$$

and therefore

$$\liminf_{n\to\infty} A_\nu(g_n^0) \leq \liminf_{n\to\infty} A(g_n^0).$$

This, together with (*), would give

$$\liminf_{n\to\infty} A_\nu(g_n^0) < A_\nu(g^0)$$

for great ν in contradiction to the lower semicontinuity of $A_\nu(g)$.

Now let $g_i(\vartheta)$ be again a bounded function with an almost innumerable set of discontinuities of the first kind and no others. We shall obtain later on by a limiting process functions $g_i(\vartheta),\ i = 1, \dots, n,$ which clearly have these properties, and it will be important to exclude the possibility that $g_i(\vartheta)$ has discontinuities or ~~is~~ all $g_i(\vartheta)$ are constant on certain arcs of $\mathfrak{C}$. To prepare this conclusion Douglas proves the following facts:

If one of the functions $g_i(\vartheta)$ (at least) has a discontinuity, then

III. $A(g) = +\infty$ and

IV. $\sum_i F_i'^2(w) \not\equiv 0$

V. If all $g_i(\vartheta)$ are constant on an arc γ of $\mathfrak{C}$ but not all $g_i(\vartheta)$ are constant on the whole circle $\mathfrak{C}$, then

$$\sum_i F_i'^2(w) \not\equiv 0.$$

The proof of V given in the paper uses a theorem of Fatou. Professor Morse gave the following proof recurring to more elementary means. According to the identity of Lagrange we have

$$EG - F^2 = \frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial x_i}{\partial u}\frac{\partial x_j}{\partial v} - \frac{\partial x_i}{\partial v}\frac{\partial x_j}{\partial u}\right)^2$$

(cf. pp. 1, 2, where this is proved for n = 3). Hence, if $EG - F^2 \neq 0$ at (u_0, v_0) one of the above Jacobians, say

$$\frac{D(x_i, x_j)}{D(u, v)} \quad (i \neq j)$$

does not vanish at (u_0, v_0) and the equations

$$x_i = H_i(u, v),$$
$$x_j = H_j(u, v)$$

set forth a topological mapping of a neighborhood $|w-w_0|<\eta$, $(u_0+iv_0=w_0)$ of (u_0, v_0) on a neighborhood of the point $\{x_i(u_0,v_0), x_j(u_0,v_0)\}$ in the (x_i, x_j)-plane. Therefore the equations

$$x_i = H_i(w), \quad \cancel{|w-w_0|<\eta}$$

give a one-to-one correspondence between the points

$$x_i = H_i(w), \quad |w-w_0|<\eta$$

of the surface and the circle $|w-w_0|<\eta$ of the w-plane. Let now

$$g_i(\vartheta) = k_i = \text{const. on } \gamma$$

but not all $g_i(\vartheta)$ be constant on the whole of $\mathcal{C}$. If V were false we should have

$$\sum F_i'^2(w) \equiv 0$$

hence (see p. 15)

(1) $$E \equiv G, \quad F \equiv 0.$$

Since $H_i(u, v)$ is the harmonic function defined by the Poisson integral with the boundary values $g_i(\vartheta)$ (cf. pp. 18-19), we have

$$H_i(w) \to k_i$$

if w approaches an interior point of γ. Therefore $H_i(w)$ can be continued across γ by the method of reflection.*) The mapping of our surface

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*) As is well known, this process is defined as follows: We set

$$w = re^{i\varphi}, \quad w^*(w) = \frac{1}{r}e^{i\varphi} \qquad (w^*(0)=\infty)$$

and for $|w|>1$

$$H_i(w) = -H_i(w^*) + 2k_i\,;$$

to prove that $H_i(w)$ is harmonic for all w not on the closed arc $\mathcal{C}-\gamma$ consider a circle k perpendicular to $\mathcal{C}$ intersecting $\mathcal{C}$ at two interior points of γ. Under the transformation $w \to w^*$ the part of k in the interior D of C is transformed into the part of k outside C. The values of $H_i(w)$ on k are continuous and define (by means of the Poisson integral) a function $\overline{H}_i(w)$ harmonic in the interior of k with the

$$\overline{H}_i(\omega) = H_i(\omega) \qquad \text{for} \qquad w < k$$

The transformation

$$\omega \to \omega^*(\omega)$$

transforms $\overline{H}_i(\omega)$ into the function

$$\overline{\overline{H}}_i(\omega) = \overline{H}_i(\omega^*(\omega))$$

Since on k

$$\overline{\overline{H}}_i(\omega) = -\overline{H}_i(\omega) + 2k_i$$

this equation must also hold in the interior of k, especially

$$\overline{H}_i(\omega) = k_i \qquad \text{for} \qquad \omega < \gamma .$$

Hence on the whole boundary of the domain inside of both k and γ we have

$$\overline{H}_i(\omega) = H_i(\omega)$$

and therefore $\overline{H}_i(\omega) = H_i(\omega)$ everywhere in this domain (we use the fact that a function harmonic in a domain bounded by a piecewise analytic simple closed Jordan curve with continuous values on this curve is uniquely determined by these boundary values). Concerning the method of reflection, see Hurwitz-Courant, Funktionentheorie, 3d edition, Berlin 1929, p. 372 ff. and p. 463.

S can also be continued over γ. Since $g_i(\vartheta) = k_i$ on γ this mapping cannot be one-to-one in the neighborhood of any point of γ. Our preliminary remark shows that

$$EG - F^2 = 0 \quad \text{on } \gamma$$

and on account of (1)

$$E = G = 0, \quad F^2 = 0 \quad \text{on } \gamma .$$

or (see the definitions of E and G)

$$\frac{\partial x_i}{\partial u} = \frac{\partial x_i}{\partial v} = 0 \ , \quad i = 1, 2, 3, \dots, n \quad \text{on } \gamma$$

This would involve

$$F_i'(u) = \frac{\partial x_i}{\partial u} - i\,\frac{\partial x_i}{\partial v} = 0 \quad \text{on } \gamma$$

and since $F_i'(u)$ is analytic in a neighborhood of a subarc of γ we should have

$$F_i'(u) \equiv 0 \ , \quad i = 1, \dots, n$$

or $F_i(u)$ const. and $g_i(\vartheta)$ = const. on the whole of C contrary to our hypothesis.

The last preparatory consideration concerns the set of all topological mappings of the unit circle $\mathcal{C}$ onto itself, which leave three points of $\mathcal{C}$ fixed. As coordinate on $\mathcal{C}$ we take again the angle ϑ. Let

$$\vartheta \rightarrow f(\vartheta)$$

be a topological mapping of $\mathcal{C}$ onto itself with the three fixed points $\vartheta_1, \vartheta_2, \vartheta_3$ $\quad (0 \leq \vartheta_1 < \vartheta_2 < \vartheta_3 < 2\pi)$

$$f(\vartheta_i) = \vartheta_i \ , \quad i = 1, 2, 3.$$

If then

$$\vartheta_1 \leq \vartheta' < \vartheta'' < \vartheta_2$$

we must have

$$\vartheta_1 \leq f(\vartheta') < f(\vartheta'') \leq \vartheta_2$$

for if $f(\vartheta') > f(\vartheta'')$ $\left(f(\vartheta') = f(\vartheta'') \right.$ is impossible since $\left. \vartheta' \neq \vartheta'' \right)$ the function

$$\varphi(t) = f((1-t)\,\vartheta_1 + t\vartheta') - f(t\vartheta'' + (1-t)\,\vartheta_2)$$

continuous for $0 \leq t \leq 1$ and satisfying the conditions

$$\varphi(0) = f(\vartheta_1) - f(\vartheta_2) = \vartheta_1 - \vartheta_2 < 0 \quad \text{and}$$
$$\varphi(1) = f(\vartheta') - f(\vartheta'') > 0$$

would vanish for some value $\bar{t}$ between 0 and 1:

$$f((1-\bar{t})\vartheta_1+\bar{t}\vartheta')=f(\bar{t}\vartheta''+(1-\bar{t})\vartheta_2)$$

This equation would involve necessary equality of the arguments of $f(\ )$ on the right and on the left sides, but

$$(1-\bar{t})\vartheta_1+\bar{t}\vartheta'\leq\vartheta'<\vartheta''\leq\bar{t}\vartheta''+(1-\bar{t})\vartheta_2 .$$

We see that the function $f(\vartheta)$ is increasing in each of the intervals

$$(*)\quad \vartheta_1\leq\vartheta\leq\vartheta_2\ ,\quad \vartheta_2\leq\vartheta\leq\vartheta_3\ ,\quad \vartheta_3\leq\vartheta\leq\vartheta_1+2\pi .$$

Now let $\vartheta\to f_\nu(\vartheta)$ be any sequence of topological mappings of C onto itself with the three fixed points $\vartheta_1, \vartheta_2, \vartheta_3$. According to a well known theorem (which we shall prove later) in $\{f_\nu(\vartheta)\}$ a subsequence $\{f_{1\nu}(\vartheta)\}$ can be chosen which converges for $\vartheta_1\leq\vartheta\leq\vartheta_2$ to a monotonic (non-decreasing) function, according to the same theorem in $\{f_{1\nu}(\vartheta)\}$ a subsequence $\{f_{2\nu}(\vartheta)\}$ exists converging in $\vartheta_2\leq\vartheta\leq\vartheta_3$ to a monotonic function, and finally (choosing a subsequence of $f_{2\nu}(\vartheta)$) we find a sequence $f_{3\nu}(\vartheta)$ converging to a function $f(\vartheta)$ non-decreasing in each of the intervals (*) with

$$f(\vartheta_i)=\vartheta_i\ ,\quad i=1,2,3 .$$

In general $f(\vartheta)$ will have singularities of the two following kinds: there may be arcs on C on which $f(\vartheta)$ is constant, and there may be discontinuities. These are of the first kind since the monotony of $f(\vartheta)$ involves the existence of the limits

$$f(\vartheta+)\quad\text{and}\quad f(\vartheta-)$$

at each point ϑ. The set σ of all these discontinuities is countable. For to every discontinuity belongs a positive number

$$\alpha=f(\vartheta+)-f(\vartheta-)$$

Denoting by σ_ν the subset of those points for which

$$\alpha>\frac{1}{\nu}$$

we have

$$\sigma = \Sigma \sigma_\nu$$

and if σ were not countable at least one of the sets σ_ν , say σ_{ν_0} , must be infinite. Then one of the intervals (*) must contain an infinite subset of σ_{ν_0} and $f(\vartheta)$ would not be bounded.*) The set of (greatest) arcs, where $f(\vartheta)$ is constant, is countable (this can be deduced from the preceding investigation, since these arcs are the discontinuities of the function inverse to $f(\vartheta)$). We see the mapping

$$\vartheta \to f(\vartheta)$$

has in general the following singularities: countably many subarcs of C are transformed into points; to a countable set of points on C correspond whole arcs.

To prove the compactness of the monotonic functions, we start with the

Lemma: Let $h_\nu(P)$, $\nu = 1, 2, \dots$ be any sequence of uniformly bounded functions defined on a point set μ To any arbitrary countable subset $\bar{\mu}$ of μ a subsequence of $\{ h_\nu(P) \}$ can be found converging at the points of $\bar{\mu}$.

Proof. Let $A_1, A_2, \dots$ be the points of $\bar{\mu}$. The sequence $h_\nu(A_1)$ is bounded; therefore a converging subsequence $h_{1\nu}(A_1)$ of $h_\nu(A_1)$ exists. Since $h_{1\nu}(A_2)$ is bounded, a converging subsequence $h_{2\nu}(A_2)$ of $h_{1\nu}(A_2)$ exists. Going on in the same

---

*) If σ is not countable, one of the sets σ_ν must be even non-countable; using this fact one sees easily that any function which has only discontinuities of the first kind has only countably many points of discontinuity.

manner we get a sequence

$$h_{11}(P),\quad h_{12}(P),\ \dots$$
$$h_{21}(P),\quad h_{22}(P),\ \dots$$
$$\cdot\ ,\quad \cdot\ ,\ \dots$$

of subsequences of $\{h_\nu(P)\}$, where $h_{n1}(P), h_{n,2}(P), \dots$ converges at the points $A_1, \dots, A_n$ and is a subsequence of the preceding sequences. Then

$$h_{11}(P),\ h_{22}(P),\ \dots,\ h_{nn}(P),\ \dots$$

converges at each point A_m of $\bar{\mu}$, because

$$h_{mm}(P),\quad h_{m+1,m+1}(P),\ \dots$$

is a subsequence of $h_{m,1}(P), h_{m,2}(P), \dots$

We now prove the theorem we used above:

Let the functions $f_\nu(\vartheta)$ be defined and non-decreasing for $\vartheta_1 \le \vartheta \le \vartheta_2$ and uniformly bounded: $|f(\vartheta)| \le M$. Then a subsequence $\{f_{2n}(\vartheta)\}$ of $f(\vartheta)$ exists converging to a non-decreasing function $f(\vartheta)$.

Let $\vartheta^1, \vartheta^2, \dots$ be the rational values of ϑ contained in the interval $\vartheta_1 \le \vartheta \le \vartheta_2$. According to our Lemma we can find a subsequence $f_{11}(\vartheta), f_{12}(\vartheta), \dots$ of $\{f_\nu(\vartheta)\}$ converging at the points $\vartheta^1, \vartheta^2, \dots$. We set

$$\lim_{\nu\to\infty} f_{1\nu}(\vartheta) = f(\vartheta^i),\quad i = 1, 2, \dots$$

Since

$$f_{1\nu}(\vartheta^i) \le f_{1\nu}(\vartheta^k) \qquad \text{for} \quad \vartheta^i < \vartheta^k$$

we have

$$f(\vartheta^i) \le f(\vartheta^k) \qquad \text{for} \quad \vartheta^i < \vartheta^k.$$

Hence the limits

$$f(\vartheta+) = \lim f(\vartheta'), \quad \vartheta' > \vartheta, \ \vartheta' \to \vartheta$$

and

$$f(\vartheta-) = \lim f(\vartheta''), \quad \vartheta'' < \vartheta, \ \vartheta'' \to \vartheta$$

exist for $\vartheta_1 \leq \vartheta \leq \vartheta_2$. (for $\vartheta = \vartheta_1, \vartheta_2$ respectively, only one of these limits has a sense). The functions

$$\left.\begin{array}{l} \bar{f}(\vartheta) = f(\vartheta+) \\ \underline{f}(\vartheta) = f(\vartheta-) \end{array}\right\} \vartheta_1 \leq \vartheta \leq \vartheta_2$$

are monotonic and we have for all ϑ

$$\underline{f}(\vartheta) = \underline{f}(\vartheta-) = f(\vartheta-) = \bar{f}(\vartheta-) \quad \text{and}$$

$$\bar{f}(\vartheta) = \bar{f}(\vartheta+) = f(\vartheta+) = \underline{f}(\vartheta+).$$

The discontinuities of $\bar{f}(\vartheta)$ therefore are at the same time those of $\underline{f}(\vartheta)$; as proved on p. 26 they form at most a countable set and except for these points we have $\underline{f}(\vartheta) = \bar{f}(\vartheta)$. Let ϑ_0 be a point where $\underline{f}(\vartheta)$ and $\bar{f}(\vartheta)$ are continuous and let the rational numbers $\underline{\vartheta}_\nu$ and $\bar{\vartheta}_\nu$ be chosen in such manner that

$$\underline{\vartheta}_\nu < \vartheta_0 < \bar{\vartheta}_\nu$$

and

$$\underline{\vartheta}_\nu \to \vartheta_0, \quad \bar{\vartheta}_\nu \to \vartheta_0.$$

Then we have

$$f_{1n}(\underline{\vartheta}_\nu) \leq f_{1n}(\vartheta_0) \leq f_{1n}(\bar{\vartheta}_\nu)$$

and therefore

$$\underline{f}(\underline{\vartheta}_\nu) \leq f(\underline{\vartheta}_\nu) \leq \liminf_{n\to\infty} f_{1n}(\vartheta_0) \leq \limsup_{n\to\infty} f_{1n}(\vartheta_0) \leq f(\bar{\vartheta}_\nu) \leq \bar{f}(\bar{\vartheta}_\nu);$$

since ϑ_0 is a point of continuity for $\underline{f}(\vartheta)$ as well as $\bar{f}(\vartheta)$ we have

$$\lim \underline{f}(\underline{\vartheta}_\nu) = \underline{f}(\vartheta_0) = \bar{f}(\vartheta_0) = \lim \bar{f}(\bar{\vartheta}_\nu)$$

Consequently

$$\liminf f_{1n}(\vartheta_0) = \limsup f_{1n}(\vartheta_0).$$

or $f_{1n}(\vartheta)$ converges at ϑ_0. Then our Lemma shows that in $\{f_{1n}(\vartheta)\}$ a subsequence $\{f_{2n}(\vartheta)\}$ exists converging at all points where $\bar{f}(\vartheta)$ and $\underline{f}(\vartheta)$ are discontinuous. The sequence $\{f_{2n}(\vartheta)\}$ has all properties required in the assertion.

Returning a moment to our previous discussion where we applied this theorem, we see: if $f(\vartheta_0-) < f(\vartheta_0+)$ by fixing the value $f(\vartheta_0)$ we determine the point $f(\vartheta_0)$ as image of ϑ_0. The mapping $\vartheta \to f(\vartheta)$ then does not cover any point between $f(\vartheta_0-)$ and $f(\vartheta_0+)$ except $f(\vartheta_0)$. Instead of doing this we can agree to consider all the points between $f(\vartheta_0-)$ and $f(\vartheta_0+)$ (eventually including one or both of these points) as images of ϑ_0. Then the function $f(\vartheta)$ is not one-valued, but $\vartheta \to f(\vartheta)$ gives a mapping of C onto the whole of itself. That is what we meant by saying on p. 27 that whole arcs may correspond to single points.

We now come to the solution of the problem of Plateau. The area of a continuous surface S of the type of the circular disk has been defined on p. 14. Thereby sequences of polyhedrons $\Pi_1, \Pi_2, \dots$ occurred whose boundaries P_1, P_2, ... tend to the given curve Γ. We did not ~~perceive~~ rigourize then the sense in which P_1, P_2, ... has to approach Γ. We mean "approaching" in the sense of Fréchet: Γ and P_m are topological images of the unit circle C. Let Γ be given by

$$x_i = f_i(t), \quad 0 \leq t \leq 2\pi, \quad i = 1, \dots, n$$

and P_m by

$$x_i = \varphi_i^m(t).$$

If $t \to \tau(t)$ is a topological representation of C onto itself,

$$x_i = \varphi_i^m(\tau) = \psi_i(t)$$

still represents P_m. We set

$$d_\tau(\Gamma, P_m) = \max_{0 \leq t \leq 2\pi} \sqrt{\sum_{i=1}^{n} \left(f_i(t) - \psi_i(t)\right)^2}$$

and define as distance of Γ and P_m the number

$$d(\Gamma, P_m) = \text{g. l. b. } d_\tau(\Gamma, P_m);$$

where $\tau(t)$ traverses all topological mappings $t \to \tau(t)$ of C into itself. We say that P_m tends to Γ if

$$d(\Gamma, P_m) \to 0.$$

At first sight this definition seems to be complicated, but it is only an exact (and for more complicated cases than Jordan curves ~~and~~ the only useful) formulation of the requirement: there must be parametric representations of P_m, for which "corresponding" points of P_m and Γ are near to each other uniformly along Γ. Indeed we have:

$P_m \to \Gamma$ holds if and only if such parametrizations $x_i = \varphi_i^m(t)$ and $x_i = f_i(t)$ of P_m and Γ can be found that $\varphi_i^m(t)$ tends to $f_i(t)$ uniformly in t.

The proof is obvious since the uniform convergence is equivalent to

$$\max_{0 \le t \le 2\pi} |f_i(t) - \varphi_i^m(t)| \to 0$$

From the uniform convergence of $\varphi_i^m(t)$ follows: For $t_m \to t$ we have

(*) $$f_i(t) - \varphi_i^m(t_m) \to 0$$

for

$$|f_i(t) - \varphi_i^m(t_m)| \le |f_i(t) - f_i(t^m)| + |f_i(t^m) - \varphi_i^m(t^m)| .$$

Now let $\Pi_1, \Pi_2, \dots$ be any sequence of simply connected polyhedrons whose boundaries tend to Γ and put

$$s(\Pi_1, \Pi_2, \dots) = \liminf_{n \to \infty} s(\Pi_n),$$

where $s(\Pi_n)$ is the area of Π_n. The greatest lower bound $m(\Gamma)$ of all simply-connected surfaces spanned by Γ is then equal to the greatest lower bound of all the numbers $s(\Pi_1, \Pi_2, \dots)$ formed for the diferent sequences $\overline{\pi}_1, \overline{\pi}_2, \dots$. Then there exists a sequence ~~$\{\overline{\pi}_1, \overline{\pi}_2, \dots\}$~~ $\pi_1, \pi_2, \dots$ of simply-connected polyhedrons whose boundaries $P_1, P_2, \dots$ tend to Γ with

(**) $S(\pi_\nu) \to m(\Gamma)$.

The expression "simply connected" requires an interpretation. For if, for instance, Γ is a knotted curve in the 3-space, the curves P_m (for great m) will be knotted too, and there exists no polyhedron of the type of the circular disk spanned by P_m. We mean that π_m is of the type of the circular disk in the combinatorial sense: Take a finite set of triangles in the plane with certain incidental relations for the sides. These relations must be such that they define a complex π_m^* which is topologically equivalent to the circular disk. We may suppose that identified sides have equal lengths. Then in a space of sufficiently ~~large~~ high dimension a realization of π_m^* exists whose faces are congruent to the given triangle and which is homeomorphic to the circular disk. In the three-space in general we only get a polyhedron π_m whose faces are congruent to the given triangle and cross each other in certain straight-line segments. π_m is only a continuous, not a one-to-one continuous, image of the abstract complex π_m^* unless we agree (as one usually does in the theory of Riemann surfaces) to count a point of π_m r times if it corresponds to r different points of π_m^*.

We now apply the so-called mapping theorem of Koebe:

Let π^* be a simply-connected polyhedron bounded by the polygon Γ^*. Then there exists a topological map of $\pi^* + \Gamma^*$ on the closed unit circle D + C, which is conformal at every interior point not being a vertex.*)

Regarding the interior points of a face of Π^* the meaning of the expression "conformal" is clear. For the points on an edge η of Π^* which are not corners the word "conformal" means: we move the two faces touching at η so that they fall into the same plane. As metric on Π^* in the neighborhood of an interior point P of η we take the metric of the plane neighborhood of P after the movement. The mapping in the theorem of Koebe is conformal with respect to this metric.

We apply the theorem to our polyhedron Π_m (see (**), p. 32) The boundary P_m of Π_m may be given by

$$x_i = \bar{f}_i^m(t), \quad 0 \le t \le 2\pi,$$

where the functions $\bar{f}_i^m(t)$ may be chosen in such manner that $\bar{f}_i^m$ approaches $f_i(t)$ uniformly in t, where

$$x_i = f_i(t)$$

is a representation of Γ. The mapping of Π_m on D + C induces a topological mapping of P_m on C, which may be given by

(1) $\quad t = \varpi_m(\vartheta)$.

Fixing on P_m the three points t_1, t_2, t_3 ($0 \le t_1 < t_2 < t_3 < 2\pi$) and taking on C the three fixed points $\vartheta_1 = t_1$, $\vartheta_2 = t_2$, $\vartheta_3 = t_3$ we may suppose that

$$t_i = \vartheta_i = \varpi_m(\vartheta_i), \quad i = 1, 2, 3; \quad m = 1, 2, \dots$$

For we can compose our conformal mapping with a linear mapping of D + C onto itself which transforms the original images of t_1, t_2, t_3 into the points $\vartheta_1, \vartheta_2, \vartheta_3$.

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*) In this form the theorem has already been proved by H. A. Schwarz. (Koebe treated much more general cases.) For a proof see C. Caratheodory, Conformal representation (No. 28 of the Cambridge Tracts), Chap. VII.

(1) defines a topological mapping of C onto itself (considered as parameter on C):

$$\vartheta \to \bar{\omega}_m(\vartheta) = t$$

According to our statement on p. 27, we can choose a subsequence $\{\omega_m(\vartheta)\}$ of $\{\bar{\omega}_m(\vartheta)\}$ converging to a function $\omega(\vartheta)$. We cannot exclude *a priori* that the mapping

$$\vartheta \to \omega(\vartheta)$$

might have singularities of the types mentioned on p. 27 and 30. The equations

$$x_i = f_i^m(\omega_m(\vartheta)) = g_i^m(\vartheta) \qquad (f_i^m = \bar{f}_i^{\bar{m}} \text{ if } \omega_m = \bar{\omega}_{\bar{m}})$$

still represent P_m, but by setting

$$x_i = f_i(\omega(\vartheta)) = g(\vartheta)$$

we are not sure to have a proper parametric representation of Γ. However, from the uniform convergence of the $f_i^m(t)$ it follows that

(2) $$\lim g_i^m(\vartheta_0) = g_i(\vartheta_0)$$

if ϑ_0 is point of continuity for $\omega(\vartheta)$. For then

$$\omega_m(\vartheta_0) \to \omega(\vartheta_0) \qquad \text{(cf. p. 29)}$$

and (2) is a consequence of (*), p. 31.

The mapping of Π_n on $u^2+v^2 \le 1$ sets forth a parametric representation

$$x_i = x_i^m(u,v), \quad u^2+v^2 \le 1$$

of Π_m; since this mapping is conformal for $u^2+v^2 < 1$ except for a finite number of points we have

$$E \equiv \sum_i \left(\frac{\partial x_i}{\partial u}\right)^2 \equiv \sum_i \left(\frac{\partial x_i}{\partial v}\right)^2 \equiv G, \quad F \equiv \sum_i \frac{\partial x_i}{\partial u}\frac{\partial x_i}{\partial v} \equiv 0 .$$

Therefore

$$(3) \quad \mathfrak{D}(x^m) = \frac{1}{2}\iint_D (E+G)\,du\,dv = \iint_D \sqrt{EG-F^2}\,du\,dv = S(\Pi_m)$$

Now let $H_i^m(u,v)$ be the function harmonic in D with the boundary values $g_i^m(\vartheta)$. Then

$$D(H_i^m) = \iint_D \left[\left(\frac{\partial H_i^m}{\partial u}\right)^2 + \left(\frac{\partial H_i^m}{\partial v}\right)^2\right] du\,dv \leq \iint_D \left[\left(\frac{\partial x_i^m}{\partial u}\right)^2 + \left(\frac{\partial x_i^m}{\partial v}\right)^2\right] du\,dv.$$

For, omitting the subscript i and the superscript m, and considering that

$$\frac{1-r^2}{1+r^2-2r\cos(\varphi-\vartheta)} = 1 + \frac{1}{\pi}\sum_{n=1}^{\infty} r^n \cos n(\varphi-\vartheta)$$

and that this series is uniformly convergent in ϑ for r < 1, we have $(u+iv = re^{i\varphi})$

$$H(u,v) = \bar{H}(r,\varphi) = \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\varphi-\vartheta)}\, g(\vartheta)\,d\vartheta = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\varphi + b_n \sin n\varphi)$$

with

$$a_n = \frac{1}{\pi}\int_0^{2\pi} g(\vartheta)\cos n\vartheta\,d\vartheta, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} g(\vartheta)\sin n\vartheta\,d\vartheta$$

Putting

$$\bar{H}_n = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos k\vartheta + b_k \sin k\vartheta)$$

we have for r = 1

$$\frac{\partial \bar{H}_n(1,\vartheta)}{\partial r} = \cancel{k}\sum_{k=1}^{n} k(a_k \cos k\vartheta + b_k \sin k\vartheta)$$

and

$$\int_0^{2\pi}(x-\bar{H}_n)\cos k\vartheta\,d\vartheta = 0, \quad \int_0^{2\pi}(x-\bar{H}_n)\sin k\vartheta\,d\vartheta = 0 \qquad k=1,\dots,n$$

hence for r = 1

$$(4) \quad \int_0^{2\pi}(x-\bar{H}_n)\frac{\partial \bar{H}_n}{\partial r}\,d\vartheta = 0.$$

Now

$$D(x) = D(\bar H_n + (x-\bar H_n)) = D(\bar H_n) + D(x-\bar H_n) +$$
$$+ \iint_D \left(\frac{\partial \bar H_n}{\partial u}\, \frac{\partial (x-\bar H_n)}{\partial v} + \frac{\partial \bar H_n}{\partial v}\, \frac{\partial (x-\bar H_n)}{\partial v} \right) du\, dv$$

According to Green's formula the last integral on the right side is equal to

$$- \iint_D (x-\bar H_n)\, \Delta \bar H_n \, du\, dv + \int_{\mathcal{C}} (x-\bar H_n) \frac{\partial \bar H_n}{\partial r}\, d\vartheta$$

From $\Delta \bar H_n = 0$ and (4) follows

$$D(x) = D(\bar H_n) + D(x-\bar H_n)$$

and

$$\iint_{D_\rho} \left[\left(\frac{\partial \bar H_n}{\partial u}\right)^2 + \left(\frac{\partial \bar H_n}{\partial v}\right)^2 \right] du\, dv \le D(\bar H_n) \le D(x),$$

where D_ρ designates the domain $0 \le u^2+v^2 < \rho < 1$. In D_ρ the functions $\bar H_n$ and their derivatives converge uniformly to $\bar{H}$; therefore for $n \to \infty$ the integral on the left side tends to the Dirichlet integral of H over D_ρ and we have

$$\iint_{D_\rho} \left[\left(\frac{\partial H}{\partial u}\right)^2 + \left(\frac{\partial H}{\partial v}\right)^2 \right] du\, dv \le D(x)$$

and

$$D(H) \le D(x)$$

Now, returning to our previous notation

$$A(g^m) = \sum_{i=1}^{n} D(H_i^m)$$

we find on account of (3), p. 35

$$A(g^m) \le S(\Pi_m)$$

Since $A(g^m)$ is lower semi-continuousm we have

$$A(g) \le \liminf A(g^m) \le \lim S(\Pi_n) = m(\Gamma)$$

If $m(\Gamma) < \infty$, theorem III, p. 22, shows that the functions $g_i(\vartheta)$ are continuous. We then consider the harmonic surface H defined by the boundary values $g_i(\vartheta)$ on C. In a sense defined by Douglas, H may be considered as bounded by Γ (though the $g_i(\vartheta)$ might be constant on certain arcs of C). Therefore

$$m(\Gamma) \leq S(H) = S(g)$$

and the theorems I, IV, V easily lead to the conclusion that H is a minimal surface with the area $m(\Gamma)$ bounded by Γ and so that the equations

$$x_i = g_i(\vartheta)$$

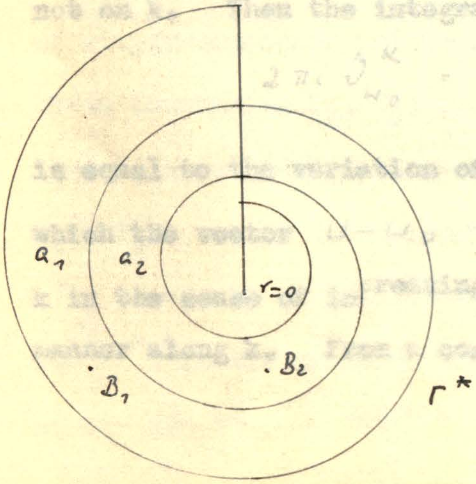
set forth a topological mapping of C on Γ.

Finally Douglas treats the case $m(\Gamma) = \infty$ and proves that a minimal surface through Γ exists.

3. Examples

The case $m(\Gamma) = \infty$ can really occur for a suitable simple closed Jordan curve in the three-space. The example which we shall discuss here has been indicated by R. Rado.*) Professor S.S.Cairns gave the details of the proof.

Consider the spiral $r = \frac{1}{\sqrt{\varphi}}, \ \frac{\pi}{2} \leq \varphi \leq \infty$ in the plane z = 0 of a 3-space with the cylindrical coordinates r, φ, z. This spiral, together with the straight line segment connecting the end-points of the spiral, forms a curve Γ^* which divides the plane z = 0 into an infinite and enumerably-many simply-



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*) On the problem of Plateau, Ergebnisse der Mathematik, vol. 2 (1933), fasc. 2, p. 92.

connected open regions. We designate these latter regions by $a_1, a_2, \dots,$ where a_ν is the region which contains the point

$$r = \frac{1}{\sqrt{\pi(2\nu-1)+1}}, \quad \varphi = \pi$$

Denoting by $|a_\nu|$ the content of a_ν, one readily verifies that

$$|a_1| + 2|a_2| + 3|a_3| + \cdots = \infty.$$

We determine in a_ν a simple closed polygon β_ν bounding a (simply connected) region b_ν with

$$|b_\nu| \geq \frac{|a_\nu|}{2}.$$

β_ν has a positive distance from the boundary of a_ν. We have

(1)
$$|b_1| + 2|b_2| + 3|b_3| + \cdots = \infty$$

Let Γ be defined as follows: Γ consists of the curve

$$r = \frac{1}{\sqrt{\varphi}}, \quad z = \frac{1}{\varphi}, \quad \frac{\pi}{2} \leq \varphi \leq \infty$$

and the straight line segment connecting the end-points of the curve. Γ will not span any simply-connected surface of finite area. In order to prove this we need an auxiliary consideration:

In the w-plane let

$$w = w(t), \quad 0 \leq t \leq 2\pi$$

be any continuous closed curve k $(w(0) = w(2\pi))$ and w_0 a point not on k. Then the integral

$$2\pi i J^{k}_{w_0} = \int_k \log(w - w_0)\, dw$$

is equal to the variation of the argument of $w - w_0$ (i.e. the angle which the vector $w - w_0$ forms with a fixed direction), when w traverses k in the sense of increasing t, and this angle is continued in a continuous manner along k. From a consideration familiar in the theory of complex functions it follows that

(1) $$J_{w_0}^{k} = J_{w_1}^{k}$$

if w_1 can be connected with w_0 by an arc which has no common points with k.

Let k' be another continuous curve so near k that their distance in the sense of Fréchet is less than the (usual) distance of either curve from w_0. Then there exist such parametrizations $w(\tau)$ and $w'(\tau)$ of k and k' that

$$|w(\tau) - w'(\tau)| < \mathrm{Min}\,(|w(\tau)-w_0|, |w'(\tau)-w_0|), \quad 0 \leq \tau \leq 2\pi .$$

We shall prove that

(2) $$J_{w_0}^{k} = J_{w_0}^{k'} .$$

The angle (in the common sense) between the vectors $w(\tau) - w_0$ and $w'(\tau) - w_0$ is less than $\frac{\pi}{2}$. Fixing the arguments of $w(0)-w_0$ and $w'(0) - w_0$ so that their difference is less than $\frac{\pi}{2}$ and continuing these arguments for $0 \leq \tau \leq 2\pi$ in a continuous manner, we see that

$$|\arg(w(\tau)-w_0) - \arg(w'(\tau)-w_0)| < \frac{\pi}{2}, \text{ for } 0 \leq \tau \leq 2\pi,$$

therefore

$$|[\arg(w(2\pi)-w_0) - \arg(w(0)-w_0)] - [\arg(w'(2\pi)-w_0) - \arg w'(0)-w_0]| < \pi;$$

from this follows (2) immediately.

Let now $\mathfrak{B}_\nu$ be a point in b_ν and $\pi_1, \pi_2, \dots$ any sequence of simply-connected polyhedrons (in the combinatorial sense, cf. p. 32) whose boundaries P_1, P_2, ... tend to Γ (in the sense of Fréchet). We have to show that the areas $S(\pi_\nu)$ of π_ν tend to ∞. Let ~~P_r^*~~ P_1^*, P_2^*, ... be the projections of P_1, P_2, ... on the plane z = 0. Since P_ν tends to Γ in the sense of Fréchet, one easily sees that P_ν^* tends to Γ^* in the sense of Frechet (using the fact that the ratio of the lengths of any arc on Γ^* and the corresponding arc of Γ is bounded). Therefore on account of (2), to given N a number $\nu(N)$ can be found such that for $n > \nu(N)$ and m = 1, ..., N,

(a) P_n^* does not intersect the boundary β_m of b_m,

(b) $J_{B_m}^{P_n^*} = J_{B_m}^{\Gamma^*}$.

From (1) it follows that we have

(c) $J_{B_m}^{P_n^*} = J_{Q}^{P_n^*}, \quad m = 1, \dots, N, \quad n > \nu(N),$

for each ~~e~~ point $Q \subset b_m$.

$J_{B_m}^{\Gamma^*}$ can easily be determined. Let ϱ_m be the sum of the spiral from $\varphi = \frac{\pi}{2}$ to $\varphi = \frac{\pi}{2} + 2(m+1)\pi$ and the straight line segment connecting the end-points of this arc, and σ_m the spiral $r = \frac{1}{\sqrt{\varphi}}$ for $\frac{\pi}{2} + 2m\pi \leq \varphi \leq \infty$ plus the segment connecting its end-points. (ϱ_m is the boundary of $a_1 + a_2 + \dots + a_m$ and σ_m is that of $a_{m+1} + a_{m+2} + \dots$) We then have

$$J_{B_m}^{\varrho_m} = m \qquad \text{and}$$

$$J_{B_m}^{\sigma_m} = 0$$

therefore

$$J_{B_m}^{\Gamma^*} = m,$$

and on account of (b) and (c)

$$J_{Q}^{P_n^*} = m \qquad \text{for} \quad Q \subset b_m, \; m = 1, \dots, N, \; n > \nu(N).$$

We orient the curve Γ so that the part over the spiral is traversed positively if φ increases. The orientation of Γ induces orientations of Γ^* and P_1, P_2, ... (at least for great subscripts) and herewith of $P_1^*, P_2^*, \dots$. We may suppose that the faces of Π_n are triangles and we can assume without loss of generality that the plane of no face of Π_n is perpendicular to $z = 0$. The orientation of P_n induces in the well-known manner a ("coherent") orientation of the triangles forming Π_n. We consider the projection Π_n^* of Π_n as a continuous image of Π_n. This induces as above positive orientations of the triangles forming Π_n^*; the oriented polygon P_n^* is the boundary of Π_n^*. Let A be any point of the plane $z = 0$ which is not the projection of a point on an edge of Π_n or a line where Π_n crosses itself. A is covered by certain triangles of Π_n^*; let $\bar{\rho}$ be the number of those among them which are oriented in the same way as the plane $z = 0$ itself (positively if φ increases), and $\bar{n}$ the number of those oriented in the opposite manner. We then apply the theorem that

$$J_A^{P_n^*} = \bar{\rho} - \bar{n}.$$

From our formula (d) it follows that

$$J_Q^{P_n^*} = \bar{\rho} - \bar{n} = m \quad \text{for} \quad Q < b_m, \quad m = 1, \dots, N, \quad n > \nu(N).$$

Hence each point of b_m not under an edge or a self-crossing of Π_n is a projection of at least m different points of Π_m. Since the content of a triangle does not increase by perpendicular projection of the triangle on a plane, we see that the area $S(\Pi_n)$ of Π_n for $n > \nu(N)$ satisfies the relation

$$S(\Pi_n) \geq |b_1| + 2\,|b_2| + \dots + N\,|b_N|$$

hence

$$S(\Pi_n) \to \infty.$$

We shall now consider examples concerning the local behavior of minimal surfaces, and examples showing that the solution of Plateau's problem generally is not unique. We start with a new and more general definition of a minimal surface.

Let D^* be any domain in the (u^*, v^*)-plane bounded by a simple closed Jordan curve C^* and D' a domain in the (u', v')-plane bounded by the simple closed curve C'. Then it is well known that $D' + C'$ can be mapped topologically on $C^* + D^*$:

$$u^* = f(u', v'), \quad v^* = g(u', v')$$

Let $\overline{x} = \overline{x}_0(u^*, c^*)$ be any continuous surface defined on $D^* + C^*$. Then the equations

$$\overline{x} = \overline{x}_0(u^*, v^*) = \overline{x}_0(f(u', v'), g(u', v')) = \overline{x}_1(u', v'), \quad (u', v') \subset \Gamma' + D'.$$

represent the same point set S and we regard the two systems of equations

$$\overline{x} = \overline{x}_0(u^*, v^*) \quad \text{and} \quad \overline{x} = \overline{x}_1(u', v')$$

merely as two different parametrizations of the same surface. Since $D^* + C^*$ can be mapped topologically on the unit-circle D plus its boundary C of a (u, v)-plane, in a suitable parametrization the domain of the parameter space is given by $u^2 + v^2 \leq 1$.

We generalize this process a little introducing local parameters. Let V be any neighborhood of a point (u_0, v_0), $u_0^2 + v_0^2 < 1$, and W the points of S corresponding to V. We transform V topologically into a vicinity V_0 of the point (α_0, β_0) in a (α, β)-plane

$$\alpha = \alpha(u, v), \quad \beta = \beta(u, v), \quad (u, v) \subset V$$

or

$$u = u(\alpha, \beta), \quad v = v(\alpha, \beta), \quad (\alpha, \beta) \subset V_0$$

and introduce in W the parameters α, β by setting

(1) $$\overline{x} = \overline{x}(u, v) = \overline{x}(u(\alpha, \beta), v(\alpha, \beta)) = \overline{x}_0(\alpha, \beta), \quad (\alpha, \beta) \subset V_0$$

We call α, β local parameters for the neighborhood W of the point $\overline{x}(u_0, v_0)$.

By means of this we define again what we mean by a minimal surface S in the three-space bounded by the simple closed Jordan curve Γ. We require that S can be represented by a continuous vector function $\bar{x}(u, v)$:

$$\bar{x} = \bar{x}(u, v) \quad , \quad u^2 + v^2 \leq 1$$

such that the equations

$$\bar{x} = \bar{x}(u, v) \quad , \quad u^2 + v^2 = 1$$

yield a topological mapping of Γ on the circle C and that a suitable neighborhood W of each point $\bar{x}(u, v)$ of S with $u^2 + v^2 < 1$ can be parametrized by equations of the form (1) in such manner that the functions $x_0(\alpha, \beta)$, $y_0(\alpha, \beta)$, $z_0(\alpha, \beta)$ are harmonic in V_0 and satisfy the conditions

(2) $$E = (\bar{x}_0)_\alpha^2 = (\bar{x}_0)_\beta^2 = G \quad , \quad F = (\bar{x}_0)_\alpha (\bar{x}_0)_\beta = 0$$

The representation of W on V_0 by (1) therefore is conformal. The parameters α, β are called local typical parameters for W. Putting

$$\varphi_1 = x_\alpha - i x_\beta \, , \quad \varphi_2 = y_\alpha - i y_\beta \, , \quad \varphi_3 = z_\alpha - i z_\beta$$

(φ_1 corresponds to our previous F_i' cf. p. 14). The conditions (2) are equivalent to

(3) $$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$$

and the surface is given by $(\zeta = \alpha + i\beta)$

(4) $$\begin{cases} x(\zeta) = R \int^{\zeta} \varphi_1(\bar{\zeta}) \, d\bar{\zeta} \\ y(\zeta) = R \int^{\zeta} \varphi_2(\bar{\zeta}) \, d\bar{\zeta} \\ z(\zeta) = R \int^{\zeta} \varphi_3(\bar{\zeta}) \, d\bar{\zeta} \end{cases}$$

The point $\bar{x}(\alpha_0, \beta_0)$ is a regular point of S in the sense of differential geometry if

$$\sum |\varphi_i(\alpha_0, \beta_0)|^2 = EG - F^2 \neq 0$$

If $\bar{x}(\alpha_0, \beta_0)$ is not regular and all partial derivatives of x, y, z up to the order n vanish, but not all those of order n+1, we call $\bar{x}(\alpha_0, \beta_0)$

a branch point of order n. $\bar{X}(\alpha_0, \beta_0)$ is a branch point of order n if, and only if, at $\alpha_0 + i\beta_0$

$$\varphi_i' = \varphi_i'' = \cdots = \varphi_i^{(n-1)} = 0 , \quad i = 1, 2, 3$$

and one of the functions $\varphi_i^{(n)}$ does not vanish at $\alpha_0 + i\beta_0$. Introducing other local typical parameters α', β' for W, which may be defined by

$$\alpha' = \alpha'(u, v) , \quad \beta' = \beta'(u, v) , \quad (u, v) < V$$

or

$$u = u'(\alpha', \beta') , \quad v = v'(\alpha', \beta') , \quad (\alpha', \beta') < V_0'$$

the equations

(5) $$\alpha' = \alpha'(u(\alpha, \beta), v(\alpha, \beta)) , \quad \beta' = \beta'(u(\alpha, \beta), v(\alpha, \beta)) , \quad (\alpha, \beta) < V_0$$

set forth a topological mapping of V_0 on V_0'. The representation of W onto V_0 (resp. V_0') is conformal on account of $E = G$, $F = 0$ except for the points where $EG - F^2 = E^2 = 0$. Hence the mapping (5) of V_0 on V_0' is conformal except for an isolated set of points, and since it is topological it must be conformal throughout. From this it follows that the definition of regularity and the order of a branch point does not depend on the choice of the local typical parameters.

Let $\bar{X}(\alpha_0, \beta_0)$ be a regular point of S and α, β local typical parameters. It is natural to use the equation (3) in order to eliminate one of the functions φ_i. We have either $\varphi_1(\zeta_0) \neq 0$ ($\zeta_0 = \alpha_0 + i\beta_0$) or $\varphi_2(\zeta_0) \neq 0$ therefore either

(a) $$\varphi_1(\zeta_0) + i\varphi_2(\zeta_0) \neq 0$$ or

(b) $$\varphi_1(\zeta_0) - i\varphi_2(\zeta_0) \neq 0$$

In case (a) we set

$$\psi = \sqrt{-\frac{\varphi_1 + i\varphi_2}{2}}$$

where $\sqrt{\ }$ means any one of the two dastinations of ψ, and $\Phi = \frac{\varphi_3}{2\psi}$, In case (b) we put

$$\phi = \sqrt{\frac{\varphi_1 - i\varphi_2}{2}}$$

$$\psi = \frac{\varphi_3}{2\phi}$$

In both cases we have

$$(6) \qquad \begin{cases} \varphi_1 = \phi^2 - \psi^2 \\ \varphi_2 = i(\phi^2 + \psi^2) \\ \varphi_3 = 2\phi\psi \end{cases}$$

Conversely for any two functions ϕ, ψ regular in a neighborhood of ζ_0 the functions φ_i defined by (6) satisfy (3); therefore the equations

$$(7) \qquad \begin{cases} x = R\int^{\zeta} (\phi^2 - \psi^2)\, d\zeta \\ y = R\int^{\zeta} i(\phi^2 + \psi^2)\, d\zeta \\ z = R\int^{\zeta} 2\phi\psi\, d\zeta \end{cases}$$

always yield a minimal surface and the point $\bar{x}(\alpha_0, \beta_0)$ is regular, unless

$$\phi(\zeta_0) = \psi(\zeta_0) = 0$$

We have

$$\varphi_1' = x_{uu} - i x_{uv} = 2(\phi\phi' - \psi\psi')$$

$$\varphi_2' = y_{uu} - i y_{uv} = 2i(\phi\phi' + \psi\psi')$$

$$\varphi_3' = z_{uu} - i z_{uv} = 2(\phi\psi' + \phi'\psi)$$

If $\bar{x}(\alpha_0, \beta_0)$ is singular (= not regular) therefore all partial derivatives of x, y, z of the first and second order (at least) vanish, and we see:

A branch point of a minimal surface represented in the form (7) is at least of order 2.

There exists another local representation of minimal surfaces which is especially well suited for the treatment of branch points. We start with a surface given in the form (4). If $\bar{x}(\alpha_0, \beta_0)$ is regular, one of the inequalities (a), (b), p. 44 holds; in the case (a) we put

(8') $$\mu = \frac{\varphi_1 - i\varphi_2}{2}, \quad \lambda = \frac{\varphi_3}{2\mu}$$, in case (b)

(8") $$\mu = \frac{\varphi_1 + i\varphi_2}{2}, \quad \lambda = \frac{\varphi_3}{2\mu};$$

in both cases we have

(8) $$\begin{cases} \varphi_1 = (1-\lambda^2)\mu \\ \varphi_2 = i(1+\lambda^2)\mu \\ \varphi_3 = 2\lambda\mu \end{cases}$$

and

(9) $$\begin{cases} x = R\int^{\zeta}(1-\lambda^2)\mu\, d\bar{\zeta} \\ y = R\int^{\zeta} i(1+\lambda^2)\mu\, d\bar{\zeta} \\ z = R\int^{\zeta} 2\lambda\mu\, d\bar{\zeta} \end{cases}$$

Conversely, if λ, μ are regular analytic in a neighborhood of ζ_0 the functions φ_i defined by (8) satisfy (3), and (9) represents a minimal surface. If $\mu(\zeta_0) = 0$ the point $\bar{x}(\alpha_0, \beta_0)$ is singular. But, as Professor Morse remarked, admitting (usual orthogonal) transformations of the coordinates x, y, z, we can represent the neighborhood of any (not only a regular) point $\bar{x}(\alpha_0, \beta_0)$ of a minimal surface in the form (9), and even so, that $\lambda(\zeta_0) = 0$ (The advantage of this last condition will appear later.)

By a change of the local typical parameters we can transform into the origin of the ζ-plane. Let

$$\varphi_i = a^{i}_{m^{(i-1)}} z^{m^{(i-1)}} + \cdots, \quad i = 1, 2, 3,$$

be the Taylor expansions of $\varphi_1, \varphi_2, \varphi_3$. The relation $\sum \varphi_i^2 = 0$ implies that the two least among the numbers m, m', m" must be equal. If not,

$$m = m' = m'' \qquad (m^{(0)} = m)$$

the assertion is obvious, for if, for instance,

$$m > m' = m''$$

we simply make the transformation

$$z' = x, \quad x' = y, \quad y' = z;$$

then the surface gets the form

$$x' = R \int \varphi_2 \, d\zeta$$

$$y' = R \int \varphi_3 \, d\zeta$$

$$z' = R \int \varphi_1 \, d\zeta .$$

We have either $a^2_{m'} - i a^3_{m'} \neq 0$ or $a^2_{m'} + i a^3_{m'} \neq 0$. Putting correspondingly either

$$\mu = \frac{\varphi_2 - i\varphi_3}{2}, \quad \lambda = \frac{\varphi_1}{2\mu} \qquad \text{or}$$

$$\mu = \frac{\varphi_2 + i\varphi_3}{2}, \quad \lambda = \frac{\varphi_1}{2\mu}$$

we get a representation of the form (9) with $\lambda(0) = 0$. If m = m' = m" we must have

$$\sum (a^i_m)^2 = 0$$

on account of (8). Setting

$$a^i_m = b_i + i c_i, \quad i = 1, 2, 3$$

we must have

$$\sum b_i^2 = \sum c_i^2 = k^2, \quad \sum b_i c_i = 0 \qquad (k > 0)$$

Choosing d_i so that

(10) $$\sum d_i^2 = k^2, \quad \sum d_i b_i = \sum d_i c_i = 0$$

the transformation

$$x_1 = \frac{1}{k}(b_1 x + b_2 y + b_3 z)$$

$$y_1 = \frac{1}{k}(c_1 x + c_2 y + c_3 z)$$

$$z_1 = \frac{1}{k}(d_1 x + d_2 y + d_3 z)$$

is a rigid motion. In this coordinate system the surface has the equations

$$(11) \quad \begin{cases} x_1 = R \int \bar{\varphi}_1 \, d\zeta \\ y_1 = R \int \bar{\varphi}_2 \, d\zeta \\ z_1 = R \int \bar{\varphi}_3 \, d\zeta \end{cases}$$

where

$$(12) \quad \begin{cases} \bar{\varphi}_1 = \frac{1}{k} \sum b_i \varphi_i = \cancel{\frac{1}{k}} 2k \zeta^m + \cdots \\ \bar{\varphi}_2 = \frac{1}{k} \sum c_i \varphi_i = \cancel{\frac{1}{k}} 2ik \zeta^m + \cdots \\ \bar{\varphi}_3 = \frac{1}{k} \sum d_i \varphi_i = \frac{1}{k^2} \sum (d_i b_i + i d_i c_i) \zeta^m + \cdots \end{cases}$$

But (10) shows that

$$\sum (d_i b_i + i d_i c_i) = 0$$

therefore the functions

$$\mu = \frac{\bar{\varphi}_1 - i\bar{\varphi}_2}{2}, \quad \lambda = \frac{\bar{\varphi}_3}{2\mu}$$

are regular in a neighborhood of 0, we have $\lambda(0) = 0$ and the $\bar{\varphi}_i$ are expressed in terms of λ and μ as the φ_i in (8). We then get the representation (9) with the properties desired.

The equations (11) and (12) show that the plane $z_1 = 0$ is the uniquely determined tangential plane of the surface at $\bar{x}(0, 0)$, specially:

At a branch point a minimal surface has a uniquely determined tangential plane.

The chief advantage of the representation (11) is that one recognizes immediately if $\overline{x}(0, 0)$ is regular or a branch point of a certain order: since $\lambda(0) = 0$ the point $\overline{x}(0, 0)$ can only be regular if $\mu(0) \neq 0$

If

$$\mu(\zeta) = a_{m_1}\zeta^{m_1} + a_{m_2}\zeta^{m_2} + \cdots \quad a_{m_\nu} \neq 0, \; m_\nu < m_{\nu+1},$$

then $x(0, 0)$ is a branch point of order m_1. But it must be remarked: if the numbers $m_1, m_2, \ldots$ have the greatest common divisor r, then the equations (9) represent the same point set as the analogous equations with the functions $\bar{\lambda}, \bar{\mu}$ where

$$\bar{\lambda} = \lambda, \quad \bar{\mu} = a_{m_1}\zeta^{\frac{m_1}{r}} + a_{m_2}\zeta^{\frac{m_2}{r}} + \cdots$$

and $\overline{x}(0, 0)$ is a branch point of order $\frac{m_1}{r}$ for this minimal surface. Our original minimal surface is this one covered r times. We therefore see:

The point $\overline{x}(0, 0)$ of the surface in the representation (9) with $\lambda(0) = 0$ is a (proper) branch point of order $m_1 \geq 0$, if

$$\mu(\zeta) = a_{m_1}\zeta^{m_1} + a_{m_2}\zeta^{m_2} + \cdots$$

and the numbers $m_1, m_2, \ldots$ are relatively prime.

It follows that for

$$\lambda = \zeta, \; \mu = \zeta^{m} + \zeta^{m+1}, \quad m = 0, 1, \ldots$$

we have a branch point of order m.

We now state (following T. Rado, l.c., p. 32) the different forms in which the problem of Plateau has been treated:

P_1. Given Γ, to determine a minimal surface S bounded by Γ in the general sense as defined on p. 43

P_2. The same problem with the side-condition that suitable parameters u, v may be found which are local typical parameters for each interior point of S i.e. $x(u, v)$, $y(u, v)$, $z(u, v)$ have to be harmonic for $u^2 + v^2 < 1$ and to satisfy the condition

$$E = G, \quad F = 0.$$

P_3. The same as P_2, but requiring furthermore that functions ϕ, ψ can be found such that equations of the form (7) hold for $u^2 + v^2 < 1$.

P_4. The same as P_3, but requiring that the surface has no singularities, i.e. that never $\phi = \psi = 0$ for $u^2 + v^2 < 1$.

P_5. When Γ projects in a one-to-one fashion onto an (x, y)-plane to obtain a minimal surface in the form $z = f(x, y)$, the function $z(x, y)$ being analytic inside the projection of Γ and one-valued and continuous in and on the projection of Γ.

In the paper of Douglas dealt with in §2, problem P_2 (and therefore P_1) is solved under the side condition, that the area of the surface is a minimum. Until now problems P_1 and P_2 have not been solved, when they are modified by requiring that <u>all</u> minimal surfaces through Γ (instead of one) shall be found. But it has been shown that P_1 and P_2 also in this modified form are equivalent, namely Rado and Beckenbach have proved that for any minimal surface, parameters u, v can be found which are locally typical in the neighborhood of each interior point.*)

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*) E.F.Beckenbach and T.Rado, Subharmonic functions and minimal surfaces, Trans.Am.Math.Soc. 35 (1933)

That the solution of P_1 and P_2 is not unique can be shown by the following simple example due to N. Wiener:

We introduce cylinder coordinates r, φ, z related to x, y, z by

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z$$

For a surface of revolution $r = f(z)$ the parallel circle z = const. and the meridians y = const. are the lines of curvature. The center of curvature $(\bar{r}_0, \bar{\varphi}_0, \bar{z}_0)$ for a line of curvature being a parallel circle at a point $(r_0 = f(z_0), \varphi_0, z_0)$ is the point where the normal to the plane curve

(13) $$r = f(z), \quad \varphi = \varphi_0$$

at $(\bar{\bar{r}}_0, \bar{\bar{y}}_0, \bar{\bar{z}}_0)$ intersects the z-axis (hence $\bar{\bar{r}}_0 = 0$). Hence the two principal radii of curvature of $r = f(z)$ at (r_0, φ_0, z_0) are equal if, and only if, the curvature of (13) at the point (r_0, φ_0, z_0) is equal to the distance of this point to $(r=0,\ z=\bar{z}_0)$. The only plane curves for which this relation holds everywhere are the circle and the catenary; the only minimal surface of revolution therefore is the catenary. Who does not know this geometrical property of the catenary verifies easily that the radius of curvature of

$$r(z) = \frac{a}{2}\left(e^{\frac{z}{a}} + e^{-\frac{z}{a}}\right), \quad a > 0 \tag{14}$$

for $z = z_0$ is equal to

$$\frac{a}{4}\left(e^{\frac{z_0}{a}} + e^{-\frac{z_0}{a}}\right)^2 = \frac{r^2(z_0)}{a}$$

and that the segment of the normal to (14) at $(r_0 = r(z_0), z_0)$ from this point to the intersection of the normal with $z = 0$ has length

$$r(z_0)\cdot\sqrt{1 + r'^2(z_0)} = \frac{r^2(z_0)}{a}$$

The area of the part C_q of the catenoid (14) between the planes $z = \pm q$ is equal to

$$2\pi\int_{-q}^{q} r(z)\sqrt{1 + r'^2(z_A)}\,dz = 2\pi\int_{-q}^{q}\frac{a}{4}\left(e^{\frac{z}{a}} + e^{-\frac{z}{a}}\right)^2 dz =$$

$$= \frac{a^2\pi}{2}\left(e^{\frac{2q}{a}} - e^{-\frac{2q}{a}} + \frac{2q}{a}\right). \tag{15}$$

Each of the boundary circles of C_q bounds a plane domain k_q^1 resp. k_q^2 with the area

$$\frac{a^2\pi}{4}\left(e^{\frac{2q}{a}} + e^{-\frac{2q}{a}} + 2\right)$$

For large q the double of this number is less than (15). Taking out from

c_q the piece $|\varphi| < \varepsilon$ the rest $\bar{c}_q$ is homeomorphic to the closed circular disk and bounded by a piecewise analytic Jordan curve Γ. The two circular disks k^1_q, k^2_q together with the part $|\varphi| < \varepsilon$ of the catenoid form a surface homeomorphic to the circular disk and equally bounded by Γ. For large q and small ε this surface has a smaller area than $\bar{c}_q$. Since we know that there exists a minimal surface (of the type of the circular disk) bounded by Γ whose area is a minimum, we see that two different minimal surfaces of the type of the circular disk pass through Γ. By a closer research one can show that if q and ε are properly chosen these two minimal surfaces are different and have the same area, so that the solution of P_1 and P_2, even under the restriction of minimizing area, is not unique.*)

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*) If one admits as geometrically evident that one of the minimal surfaces for small ε must be very near to the second surface through Γ which we constructed above, one can prove this easily, using the fact that up to a certain q the first minimal surface through Γ actually minimizes the area.

The solution of problem P_3 would imply that there always exist minimal surfaces through a given curve Γ which have only branch points of order greater than 1. Professor Morse stated that there are examples which show that P_3 is not always solvable. (P_4 is then naturally not solvable either.) It has long been known that P_5 is not always solvable, but different sufficient conditions can be given under which P_5 has an (even unique) solution; for instance, if the projection of Γ on the (x, y)-plane is convex.

Some of these last results can easily be derived from the following

<u>Lemma</u>: If $h(u, v)$ is continuous for $u^2 + v^2 \leqq 1$ and harmonic for $u^2 + v^2 < 1$, and if at (u_o, v_o) $(u_o^2 + v_o^2 < 1)$ h and all its partial derivatives up to the order n-1, but not all those of order n, vanish (we then call (u_o, v_o) a zero of type u) then h vanishes on $u^2 + v^2 = 1$ in at least 2n different points.

<u>Proof.</u> We consider the curve $(h(u, v) = 0$ at first in the neighborhood of (u_o, v_o). We represent h in polar coordinates r, φ

$$r e^{i\varphi} = (u - u_0) + i(v - v_0)$$

$$h = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} r^\nu (a_\nu \cos \nu\varphi + b_\nu \sin \nu\varphi).$$

Since the expansion of $h(u, v)$ in terms of $u - u_o$ and $v - v_o$ begins with term of order u, we must have

$$a_0 = a_1 = \cdots = a_{n-1} = b_1 = \cdots b_{n-1} = 0$$

and $a_n \neq 0$ or $b_n \neq 0$. Therefore $h(u, v) = 0$ has a representation of the form

$$a_n \cos n\varphi + b_n \sin n\varphi + r F(r, \varphi) = 0$$

where $F(r, \varphi)$ is a Fourier series converging for small r. Hence the tangents to $h(u, v) = 0$ at (u_o, v_o) are given by

$$a_n \cos n\varphi + b_n \sin n\varphi = 0$$

and we see that there are n different tangents, the angle between two consecutive tangents being equal to $\frac{\pi}{n}$. The curve $h(u, v) = 0$ has at (u_o, v_o) n different branches $b_1 \ldots b_n$ with n different tangents. Let b_1^+ and b_1^- be the two parts of b_1 issuing from (u_o, v_o). We traverse b_1^+ starting at (u_o, v_o). If we pass a point $(\overline{u}, \overline{v})$ where $h(u, v)$ has a zero of type $q > 1$, b_1^+ has (as we have just proved) a tangent different from the tangents of the other branches of $h(u, v) = 0$ at $(\overline{u}, \overline{v})$. If we require b_1 to have a tangent everywhere the continuation is therefore uniquely determined. Continuing b_1 in this manner in $u^2 + v^2 < 1$ as far as possible we must approach the unit circle C. We either approach a definite point P_i^+ on C or all points of a whole subarc C' of C. Since h is continuous in $u^2 + v^2 \leqq 1$ we have in this latter case h = 0 on C' and the Lemma is proved. If to each of the branches b_j^+, b_j^- belongs a definite point P_j^+, P_j^- all these 2n points must be different. Otherwise, if for instance $P_s^+ = P_t^-$ the two branches b_s^+ and b_t^- would bound one or more simply connected domains on the boundary of each of which we should have h = 0 Hence h ≡ 0. Since $h(P_j^\pm) = 0$, the lemma is proved.

We now consider minimal surfaces s of the type of the circular disk in the three-space bounded by a simple closed Jordan curve Γ as defined on p. 43. We suppose that s is represented in the form

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

where x(u, v), y(u, v), z(u, v) are harmonic for $u^2 + v^2 < 1$, and continuous for $u^2 + v^2 \leqq 1$ and satisfy the condition

$$E = G, \qquad F = 0.$$

We know on account of the paper of Beckenbach and Rado, quoted on p. 50, that this assumption does not mean a loss of generality. Besides, the following considerations can easily be met made using only local typical parameters. An immediate consequence of our lemma is the theorem

(1) If π is a plane through a branch point $P = \bar{x}(u_0, v_0)$ of order u on s (the regular case included as $\mu = 0$), then π intersects Γ in at least $2(\mu+1)$ different points; if π is the tangent plane of s at P (this plane exists according to our statement on p. 48), then π has at least 2(n+2) different intersections with Γ.

Let π be the plane

$$a x + b y + c z + d = 0.$$

The intersections of π with Γ are given by

$$h(u,v) = a\,x(u,v) + b\,y(u,v) + c\,z(u,v) + d = 0\,,\quad u^2+v^2=1$$

$h(u, v)$ is harmonic. Since π passes $\overline{\overline{x}}(u_0, v_0)$ we have $h(u_0, v_0) = 0$ and from the very definition of the order of a branch point (cf. pp. 43-44) follows that all partial derivatives of h up to the order n vanish at (u_0, v_0). If π is the tangent plane of s at P in the sense used on p. 48 (which is easily seen to be equivalent to the definition used in algebraic geometry), we see that all partial derivatives of $h(u, v)$ up to the $(n+1)^{st}$ order vanish at (u_0, v_0). Hence in both cases our lemma proves the theorem (1).

From (1) we conclude:

(2) If a convex region K contains Γ the minimal surface s is contained in K.

For if there were a point P of s exterior to K a plane π through P would exist having no ~~corners or~~ common points with K and therefore with Γ, whereas according to (1) π must intersect Γ at least twice.

A further consequence of (1) is

(3) If there exists a straight line g such that no plane through g intersects Γ in more than two different points, s has no branch points.

For a plane through g and a branch point would intersect Γ in at least 4 distinct points.

The assumption of (3) is obviously satisfied if there exists a simply covered bounded and star-shaped central or parallel projection Γ^* of Γ upon some plane. For if P^* is the center of the star (or one such center if Γ^* can be regarded as star in different manners) we chose as the straight line g of (13) the line connecting P^* with the center of projection (eve~~rywhere~~ntually on the plane at infinity).

If Γ has a simply covered parallel projection Γ' on some plane which is convex, then the orthogonal projection Γ^* of Γ upon a plane π^* orthogonal to the direction of projection is identical to the orthogonal projection of Γ' on π^*. Therefore Γ^* is convex ~~to~~ and a one-to-one image of Γ. As proved before, S has no branch points. Furthermore there cannot exist any tangent plane of S perpendicular to π^*, for such a plane would intersect Γ and therefore Γ^* in at least 4 distinct points, in contradiction to the convexity of Γ^*. Hence the orthogonal projection of S on π^* sets forth a one-to-one continuous mapping of a neighborhood of each interior point $\bar{x}(u_o, v_o)$ of S on a neighborhood of a point of π^*. Choosing π^* as (x, y)-plane, we see that S can be represented locally in the form $z = f(x, y)$.

The cylinder consisting of the straight lines perpendicular to π^* at the points of Γ^* bounds a convex region, whose closure contains Γ. From (2) it follows that s does not contain any point outside this cylinder. We can even say that this cylinder does not contain other points of s than those of Γ. For a supporting plane of the cylinder at such a point would be a supporting plane of s*) at that point and therefore a tangent plane of s since s is analytic, whereas s has no tangent planes perpendicular to π^*.

We designate the convex region bounded by Γ^* with S^*. Now we can show that the orthogonal projection of S on S^* gives a one-to-one mapping of S on S^*.

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*) One also sees easily that minimal surfaces, which are no planes, have no supporting planes.

Let

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad u^2 + v^2 \leq 1,$$

be the equation of S. The fact that S can be represented locally in the form $z = f(x, y)$ together with our last statement, means that by

$$(*) \qquad (u, v) \rightarrow (x(u, v), y(u, v))$$

a neighborhood of each point $\overline{x}(u_o, v_o)$, $u_o^2 + v_o^2 < 1$ of the (u, v)-plane is mapped topologically on a neighborhood of the point $x(u_o, v_o), y(u_o, u_o)$ of Π^*; furthermore by (*) $u^2 + v^2 = 1$ is mapped topologically on Γ^*. From a topological theorem *) it follows that (*) sets forth a topological mapping of $u^2 + v^2 \leq 1$ on $S^* + \Gamma^*$, i.e., u, v are single-valued functions of x and y. Putting these values of u, v in the above equations of S we get a representation of S as a whole in the form

$$z = f(x, y).$$

Since S has no branch point and no tangent plane perpendicular to the (x, y)-plane, z is analytic in the interior of Γ^*. We have (cf. pp. 1-2)

$$E = 1 + f_x^2, \quad F = f_x f_y, \quad G = 1 + f_y^2$$

$$W = \quad EG - F^2 = \quad 1 + f_x^2 + f_y^2$$

$$L = \frac{f_{xx}}{W}, \qquad M = \frac{f_{xy}}{W}, \qquad N = \frac{f_{yy}}{W}$$

The condition

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{EN - 2MF + GL}{W^2} = 0$$

for minimal surfaces gives us

$$(1 + f_x^2)f_{yy} - 2f_x f_y f_{xy} + (1+f_y^2)f_{xx} = 0.$$

From the theory of partial differential equations one knows that the solution of this equation, if it exists, is unique for given values of f on the boundary of S^*. We see:

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*) See Kerekjarto, Vorlesungen über Topologie I, p. 175.

Problem P_5 (p. 50) is solvable in a unique manner if the projection of Γ on the (x, y)-plane occurring in the formulation of P_5 is convex.

4. The area of a surface $z = f(x, y)$

The next subject discussed in the seminar was the theory of the area of surfaces, which can be represented in the form

$$z = f(x, y).$$

The lectures followed closely the 6th chapter of the book: Théorie de l'Intégrale by Stanislaw Saks (Warszawa, 1933).