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# SEMINAR ON THE COHOMOLOGY OF DISCRETE SUBGROUPS OF SEMI-SIMPLE GROUPS

These are the Notes of a seminar held at the Institute for Advanced Study during the academic year 1976-77. They consist of 11 chapters, appendices to Chapters III and VII, and an Erratum to the latter. They are mainly devoted to the continuous, or relative Lie algebra, cohomology of a real semi-simple Lie group with coefficients in infinite dimensional representations, and to applications to the cohomology of discrete cocompact subgroups. The various chapters were written shortly after the lectures and follow them rather closely. As a consequence, some results in the earlier chapters are superseded by those in the later ones. It is planned to publish in the Annals of Mathematics Studies a revised and reorganized version. It will also include the case of p-adic groups (which was discussed in the seminar) and more generally of products of real and p-adic groups.

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# I. Relative Lie Algebra Cohomology and Ext Functors

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In this chapter, $F$ is a commutative field, $\underline{g}$ a finite dimensional Lie algebra over $F$, $\underline{k}$ a subalgebra of $\underline{g}$, $R = U(\underline{g})$ (resp. $S = U(\underline{k})$) the universal enveloping algebra of $\underline{g}$ (resp. $\underline{k}$). From 2.4 on, $F$ is of characteristic zero and $\underline{k}$ is reductive in $\underline{g}$.

## §1. Lie algebra cohomology

**1.1.** We review here the standard definitions in the cohomology of Lie algebras (see [5; 7]). A $\underline{g}$-module is a vector space over $F$ on which $\underline{g}$ acts via a homomorphism $\pi : \underline{g} \longrightarrow \underline{gl}(V)$. It will be denoted by $V$, or by the pair $(\pi, V)$. It will often be infinite dimensional. If $V$ is a $\underline{g}$-module, and $q \in \underline{\underline{N}}$, then

$$C^q = C^q(\underline{g};V) = \mathrm{Hom}_F(\Lambda^q \underline{g}, V) , \tag{1}$$

and $d : C^q \longrightarrow C^{q+1}$ is defined by

$$df(x_0,\ldots,x_q) = \Sigma_i (-1)^i x_i . f(x_0,\ldots,\hat{x}_i,\ldots,x_q) + \Sigma_{i<j} (-1)^{i+j} f([x_i,x_j],x_0,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_q) \tag{2}$$

where, as usual, $\wedge$ over an argument means that the argument should be omitted. Then $d^2 = 0$ and $H^*(\underline{g},V)$ is the cohomology of the complex $\{C^q\}$.

To $x \in \underline{g}$ there is associated an endomorphism $\theta_x$ of $C^q$ and a linear map $i_x : C^q \longrightarrow C^{q-1}$ (the interior product) defined by

$$(\theta_x f)(x_1,\ldots,x_q) = \Sigma_i \ f(x_1,\ldots,[x_i,x],\ldots,x_q) + x.f(x_1,\ldots,x_q) \tag{3}$$

$$(i_x f)(x_1,\ldots,x_{q-1}) = f(x,x_1,\ldots,x_{q-1}) \ . \tag{4}$$

The maps $d$, $i_x$, $\theta_x$ are related by

(5) $$\theta_x = d.i_x + i_x.d \quad .$$

1.2. Let $C^q(\underline{g},\underline{k},V)$ be the subspace of $C^q(\underline{g},V)$ consisting of the elements annihilated by the maps $i_x$ and $\theta_x$ for all $x \in \underline{k}$. Then $C^q(\underline{g},\underline{k};V)$ is stable under $d$ and its cohomology groups are the relative cohomology groups $H^q(\underline{g},\underline{k};V)$ of $\underline{g}$ mod $\underline{k}$, with coefficients in $V$. Note that we have

(1) $$C^q(\underline{g},\underline{k};V) = \mathrm{Hom}_{\underline{k}}(\Lambda^q(\underline{g}/\underline{k}),V) \quad ,$$

where the action of $\underline{k}$ on $\Lambda^q(\underline{g}/\underline{k})$ is induced by the adjoint representation. i.e., $C^q(\underline{g},\underline{k},V)$ may be identified with the subspace of elements $f \in \mathrm{Hom}_F(\Lambda^q(\underline{g}/\underline{k}),V)$ which satisfy the relation

(2) $$\Sigma_i\ f(x_1,\dots,[x,x_i],\dots,x_q) = x.f(x_1,\dots,x_q) \quad (x \in \underline{k};\ x_i \in \underline{g}/\underline{k},\ i = 1,\dots,q) \ .$$

We have in particular

(3) $$H^o(\underline{g},V) = H^o(\underline{g},\underline{k};V) = V^{\underline{g}} = \{v \in V \mid x.v = 0 \text{ for all } x \in \underline{g}\} \ .$$

1.3. These cohomology groups obey the Künneth rule. To simplify notation, we just consider the case of two factors. Assume then

(1) $$\underline{g} = \underline{g}_1 \oplus \underline{g}_2,\ \underline{k} = \underline{k}_1 \oplus \underline{k}_2,\ V = V_1 \otimes V_2,\ (\underline{k}_i \subset \underline{g}_i,\ V_i \text{ a } \underline{g}_i\text{-module},\ i = 1,2) \ .$$

Then, for all q's:

(2) $$H^q(\underline{g},\underline{k};V) = \oplus_{a+b=q}\ H^a(\underline{g}_1,\underline{k}_1;V_1) \otimes H^b(\underline{g}_2,\underline{k}_2;V_2) \ .$$

To see this, note that we can write

$$C^q(\underline{g},\underline{k};V) = (\Lambda^q(\underline{g}/\underline{k})' \otimes V)^{\underline{k}},\quad (\underline{g}/\underline{k})' = \Lambda(\underline{g}_1/\underline{k}_1)' \otimes \Lambda(\underline{g}_2/\underline{k}_2)' \ .$$

However, if $A_i$, $U_i$ are $\underline{k}_i$-modules $(i = 1,2)$ and $A_1 \otimes A_2$, $U_1 \otimes U_2$ are viewed as $\underline{k}$-modules in the obvious way, then

$$(3) \qquad (A_1 \otimes A_2 \otimes U_1 \otimes U_2)^{\underline{k}} = (A_1 \otimes U_1)^{k_1} \otimes (A_2 \times U_2)^{k_2} .$$

Therefore

$$C^*(\underline{g}, k; V) = C^*(g_1, k_1; V) \otimes C^*(g_2, k_2; V)$$

(graded tensor product), whence our assertion.

1.4. <u>The real case.</u> Let $F = \underline{R}$ and let $G$ be a connected Lie group with Lie algebra $\underline{g}$, $K$ a closed connected subgroup of $G$ with Lie algebra $\underline{k}$.

Given a smooth manifold $M$ and a real vector space $V$, we let $A^q(M;V)$ be the space of smooth q-forms on $M$ with values in $V$, and $A(M,V)$ the direct sum of the $A^q(M;V)$.

If $V$ is a $G$-module, then we let $G$ operate on $A(G/K;V)$ by the rule

$$(1) \qquad (g \circ \omega)(x, Y) = g(\omega(g^{-1}.x, g^{-1}.Y))$$

where $g \in G$, $x \in G/K$, and $Y$ is a q-vector at $x$. It is then readily seen that the evaluation map at the origin, which assigns to $\omega \in A(G/K;V)$ its value at $e$, defines an isomorphism of the space $A(G/K;V)^G$ of $G$-invariant differential forms onto $C(\underline{g},\underline{k};V)$, which carries the exterior differential to the differential of 1.1. Thus, $H^*(\underline{g},\underline{k};V)$ is the cohomology of the space of $G$-invariant $V$-forms on $G/K$.

Assume $G$ to be compact, $V$ to be finite dimensional and acted upon trivially by $G$. Then a standard averaging argument shows that $H^*(A(G;V)^G) =$ $= H^*(A(G;V))$, hence, by de Rham theorem

$$(2) \qquad H^*(\underline{g},\underline{k};V) = H^*(G/K;V) .$$

This is a result of E. Cartan which is in fact at the origin of the notion of Lie algebra cohomology. A bit more precisely, E. Cartan conjectured two theorems, which were proved later by de Rham, and stated that, modulo those results, the cohomology of $G/K$ could be computed using invariant differential forms. In fact, he was mainly concerned with compact symmetric spaces, for which all invariant forms are closed and even harmonic (see II, 3.2).

## §2. The Ext functors for $(\underline{g},\underline{k})$-modules

2.1. It is well-known that the groups $H^q(\underline{g};V)$ may be viewed as the derived functors of $V \longmapsto V^{\underline{g}}$ in the category of R-modules. More generally, one may define the derived functors $\mathrm{Ext}^q_R(U,V)$ of $(U,V) \longmapsto \mathrm{Hom}_{\underline{g}}(U,V)$ and we have

$$(1) \qquad \mathrm{Ext}^q_R(F,V) = H^q(\underline{g};V),\ \mathrm{Ext}^q_R(U,V) = H^q(\underline{g},\mathrm{Hom}_F(U,V)) ,$$

where $F$ is viewed as the trivial $\underline{g}$-module, (see XIII and IX, 4.3 in [5]).

We shall need similar facts in the relative case. A general theory was developed by G. Hochschild [6] in the context of relative homological algebra with respect to the pair $(R,S)$. However, in order to prove the equality

$$(2) \qquad \mathrm{Ext}^q_{R,S}(F,V) = H^q(\underline{g},\underline{k};V) ,$$

he had to assume $F$ to be of characteristic zero and $\underline{k}$ to be reductive in $\underline{g}$. This is at any rate the only case of interest in these Notes (with in fact $F = \underline{R}$). In the relative theory, one accepts only exact sequences of R-modules which split over S. We shall adopt here a slightly different point of view, using the usual absolute theory, but in a more restricted category, that of $(\underline{g},\underline{k})$-modules, defined below. In principle, this is a bit less general than Hochschild's approach, but sufficient for our purposes.

2.2. Let $V$ be a $\underline{k}$-module. An element $v \in V$ is $\underline{k}$-finite if $\underline{k}.v$ is a finite dimensional subspace. The $\underline{k}$-module $V$ is locally $\underline{k}$-finite if every element is $\underline{k}$-finite. Thus $V$ is locally $k$-finite if and only if every finite dimensional subspace is contained in a finite dimensional subspace stable under $\underline{k}$.

A vector space $V$ over $F$ is a $(\underline{g},\underline{k})$-module if it is a $\underline{g}$-module which is locally $\underline{k}$-finite and is semi-simple as a $\underline{k}$-module. In particular, every $\underline{k}$-simple submodule is finite dimensional. It suffices to require that $V$ be locally $\underline{k}$-finite and that every finite dimensional $k$-stable subspace be finite dimensional [3: §3, n° 3].

A $(\underline{g},\underline{k})$-module $V$ is admissible if the isotypic subspaces for $\underline{k}$ are all finite dimensional. If $V$ is admissible, it is clearly a direct sum of simple $\underline{k}$-modules.

Let $\underline{C}$ or $\underline{C}_{\underline{g},\underline{k}}$ be the category of $(\underline{g},\underline{k})$-modules. It is closed under direct sums. If $V \in \underline{C}$ then every $\underline{g}$-submodule of $V$ and every $\underline{g}$-module quotient of $V$ belong to $\underline{C}$.

Since $\underline{g}$-modules are canonically $R$-modules and vice versa, we get equivalent notions if we replace above $\underline{g}$ and $\underline{k}$ by $R$ and $S$. We shall use both interchangeably. Since all our modules are semi-simple for $S$, it is clear that all exact sequences in $\underline{C}$ split over $S$. If $F$ is of characteristic zero, then the tensor product over $F$ of two elements of $\underline{C}$ also belongs to $\underline{C}$. This follows from the fact that in characteristic zero, the tensor products of two finite dimensional semi-simple modules for a Lie algebra $\underline{m}$ is also semi-simple [4: §6, n° 5, Cor. 1].

Let $(\pi,V)$ be a $\underline{g}$-module. Then the subspace $V_{(\underline{k})}$ spanned by the finite dimensional $\underline{k}$-stable subspaces of $V$ is stable under $\underline{g}$. Therefore, if these subspaces are semi-simple $\underline{k}$-modules, the space $V_{(\underline{k})}$ is a $(\underline{g},\underline{k})$-module.

Now let $(\pi, V)$ be a $(\underline{g}, \underline{k})$-module. The contragredient module $(\tilde{\pi}, \tilde{V})$ is by definition the space $V'_{(\underline{k})}$ spanned by the $\underline{k}$-stable finite dimensional subspaces in the dual space $V'$ to $V$, acted upon by the usual contragredient representation, i.e., $\tilde{\pi}(x) = {}^t\pi(-x)$ $(x \in \underline{g})$, where ${}^t\pi$ is the transpose of $\pi$. If $U$ is a finite dimensional $\underline{k}$-stable subspace of $V'$, then $U$ is the dual space to the quotient of $V$ by the annihilator of $U$ in $V$, hence is a semi-simple $\underline{k}$-module. Therefore $(\tilde{\pi}, \tilde{V}) \in \underline{C}$.

As usual, the center of the universal enveloping algebra of a Lie algebra $\underline{m}$ over $F$ will be denoted $\underline{z}(\underline{m})$.

A $\underline{g}$-module $(\pi, V)$ is said to have an <u>infinitesimal character</u> if there exists a character of $\underline{z}(\underline{g})$, i.e., a unital $F$-algebra homomorphism: $\underline{z}(\underline{g}) \longrightarrow F$, to be denoted $\chi$ or $\chi_\pi$ or $\chi_V$ such that

$$\pi(z) = \chi_\pi(z).\mathrm{Id.}\ , \qquad (z \in \underline{z}(\underline{g}))\ .$$

This is the case in particular if $(\pi, V)$ is absolutely irreducible admissible.

2.3. <u>Examples</u>. Let $F = \underline{\mathbb{R}}$. Let $G$ be a connected semi-simple Lie group with finite center, $\underline{g}$ its Lie algebra, and $\underline{k}$ the Lie algebra of a compact subgroup of $G$. Then $\underline{k}$ is reductive in $\underline{g}$. Let $(\pi, V_\pi)$ be a continuous representation of $G$ in a locally convex and quasi-complete topological vector space. Let $V_\pi^\infty$ be the space of differentiable vectors and $V_{\pi,K}$ the space of $K$-finite vectors, i.e., of vectors whose translates under $K$ span a finite dimensional subspace. Then $V = V_\pi^\infty \cap V_{\pi,K}$ is a $(\underline{g}, \underline{k})$-module, (in fact a $(\underline{g}, K)$-module, see 1.12).

Assume that $G$ is semi-simple, with finite center, $K$ is a maximal compact subgroup, and $(\pi, V)$ is a topologically irreducible unitary representation of $G$ in a Hilbert space $V$. Then $V_{\pi,K} \subset V_\pi^\infty$ and $V$ is an admissible $(\underline{g}, \underline{k})$-module which is (algebraically) irreducible. For this see e.g. [1; 9].

These examples are those which have motivated the above definition, and in fact, later, besides finite dimensional modules, we shall mainly consider $(\underline{g},\underline{k})$-modules associated in this way to unitary representations.

2.4. Projective modules. We recall that from now on $F$ is of characteristic zero and $\underline{k}$ is *reductive in* $\underline{g}$, i.e., $\underline{g}$ is a semi-simple $\underline{k}$-module with respect to the adjoint representation. The algebra $\underline{k}$ operates on $\underline{g}$ by the adjoint representation, whence a representation on the tensor algebra $T(\underline{g})$ of $\underline{g}$ and on $U(\underline{g})$. Under this representation, both $T(\underline{g})$ and $U(\underline{g})$ are locally $\underline{k}$-finite and semi-simple (see [4: §6, n°. 5, Cor. 2].

LEMMA. *Let* $U$ *be a locally* $\underline{k}$-*finite semi-simple* $\underline{k}$-*module. Then the induced module* $I(U) = I_{S,R}(U) = R \otimes_S V$ *is a projective* $(\underline{g},\underline{k})$-*module.*

Although not stated in this way, this is in effect proved in [6]. We sketch the argument. First, by standard "Frobenius reciprocity" we have for every $(\underline{g},\underline{k})$-module $V$ a canonical isomorphism

$$(1) \qquad m : \mathrm{Hom}_R(I(U),V) \xrightarrow{\sim} \mathrm{Hom}_S(U,V) ,$$

defined by the restriction to $1 \otimes U$. Let now $A,B \in \underline{C}$, $f : B \longrightarrow A$ a surjective morphism, and $s : I(U) \longrightarrow A$ a morphism. We have to show the existence of $t : I(U) \longrightarrow B$ such that $f \circ t = s$. Since $A$ is a direct $S$-summand of $B$, we can find an $S$-module homomorphism $t' : U \longrightarrow B$ such that $m(s) = f \circ t'$. We then put $t = m^{-1}(t')$. There remains to see that $I(U)$ belongs to $\underline{C}$.

The $R$-module structure on $I(U)$ understood here comes from left translations on $R$. It gives by restriction an action of $S$; call it the ordinary action. On the other hand, $S$ acts on $R$ via the adjoint representation on $\underline{g}$. With respect to this action, $R$ is locally $S$-finite and semi-simple, as remarked above. Then $R \otimes_F U$, with the tensor product of these actions of $S$ is also $S$-semi-simple and locally finite. The operation of $s \in S$ is given by:

(2) $s \circ (r \otimes u) = (s.r - r.s) \otimes u + r \otimes s.u \quad (s \in \underline{k};\ r \in R,\ u \in U)$ .

It is readily seen to leave stable the kernel $M$ of the canonical map $R \otimes_F U \longrightarrow I(U)$, whence an action on $I(U)$, with respect to which $I(U)$ is locally finite and semi-simple. However, the sum of the last two terms on the right hand side of (2) belongs to $M$, hence this new $S$-action coincides with the ordinary one on $I(U)$, which proves our contention.

2.5. The functors Ext. The map $(r,u) \longmapsto r.u$ induces a surjective morphism $I(U) \longrightarrow U$. Thus every element of $\underline{C}$ is quotient of a projective one, and we can construct projective resolutions in the usual way. If

(1) $$\longrightarrow X_q \xrightarrow{\partial_q} X_{q-1} \xrightarrow{\partial_{q-1}} \cdots \longrightarrow X_0 \xrightarrow{\varepsilon} U \longrightarrow 0$$

is one for $U$, and $V \in \underline{C}$, then the groups $\mathrm{Ext}^q(U,V)$ are by definition the cohomology groups of the complex

(2) $$\mathrm{Hom}_R(X_0,V) \xrightarrow{d_0} \mathrm{Hom}_R(X_1,V) \xrightarrow{d_1} \cdots \xrightarrow{d_{q-1}} \mathrm{Hom}_R(X_q,V) \longrightarrow \cdots$$

As usual, they do not depend on the choice of the projective resolution. Moreover, it follows from [5; IX, 4.3] that:

(3) $$\mathrm{Ext}^q(F,\mathrm{Hom}_F(U,V)) = \mathrm{Ext}^q(U,V) .$$

We should check that 2.1(2) is satisfied. Let $X_q = R \otimes_S \Lambda^q(\underline{g}/\underline{k})$. Define $\partial_q : X_q \longrightarrow X_{q-1}$ by

(4) $$\partial_q(r \otimes x_1 \wedge \cdots \wedge x_q) = \Sigma(-1)^{i-1}\, x_i.r \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_q$$
$$+ \Sigma_{i<j}(-1)^{i+j}\, r \otimes [x_i,x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_q$$

and let $\varepsilon : X_0 = R \xrightarrow{\varepsilon} F$ be the augmentation. Then the $X_i$ are projective

(1.7.1) and

$$\text{(5)} \qquad \longrightarrow X_q \xrightarrow{\partial_g} \cdots \xrightarrow{\partial_1} X_0 \xrightarrow{\varepsilon} F \longrightarrow 0$$

is easily seen to be exact [6]. Hence (5) is a projective resolution of $F$. In view of 2.4(1), we have $\mathrm{Hom}_R(X_q,V) = \mathrm{Hom}_S(\Lambda^q(\underline{g}/\underline{k}),V)$, and it follows immediately from (4) that the complex $\{\mathrm{Hom}_R(X_q,V)\}$ may be identified with the one used in 1.1 to define relative Lie algebra cohomology, whence 2.1(2).

2.6. Injective modules. We have used projective resolutions in $\underline{C}$, which will suffice for our purposes. But our category also contains enough injectives. We outline briefly their construction.

Let $V$ be a locally finite semi-simple $\underline{k}$-module and

$$P^o(V) = P^o_{R,S}(V) = \mathrm{Hom}_S(R,V)$$

the usual coinduced or "produced" module from $S$ to $R$. Let

$$P(V) = P_{R,S}(V) = \mathrm{Hom}_S(R,V)_{(S)} \quad ,$$

be the subspace of $P^o(V)$ spanned by the $S$-finite elements. We claim that $P(V)$ is an injective module in $\underline{C}$.

We view $P^o(V)$ as a subspace of $\mathrm{Hom}_F(R,V)$. On the latter, $S$ acts first by left translations on $R$, the "ordinary action" and second, as above, it acts via the given action on $V$ and the operation on $R$ stemming from the adjoint representation of $\underline{k}$ on $\underline{g}$. As in 2.4, it is first checked that these two actions coincide on $P(V)$. Let us prove now that every finite dimensional subspace $M$ of $P^o(V)$ stable under $S$ is a semi-simple $S$-module. Let $\{R_j\}_{j=0,1,\ldots}$ be the usual increasing filtration of $R$ [4]. There exists $j$ such that the restriction map $\mathrm{Hom}_F(R,V) \longrightarrow \mathrm{Hom}_F(R_j,V)$ is injective on $M$, hence it identifies $M$ to a subspace of $\mathrm{Hom}_F(R_j,V)$. Since $R_j$ is finite dimensional, $\mathrm{Hom}_F(R_j,V) = R_j^* \otimes_F V$ is S-semi-simple, hence so is M. It follows that P(V) can also be defined as the subspace of $P^o(V)$ generated by the S-invariant finite dimensional subspaces of $P^o(V)$. It is then clearly an R-module, and then an (R,S)-module. Furthermore, if N is an (R,S)-module and $N \longrightarrow P^o(V)$ an R-morphism, then $\mathrm{Im}\, N \subset P(V)$. Since $P^o(V)$ is injective with respect to (R,S)-modules, (the argument is the dual to that of 2.4, see [6]) it follows that P(V) is injective in $\underline{C}$. Now let V be a $(\underline{g},\underline{k})$-module. As usual, the map which to $v \in V$ associates the homomorphism $r \longmapsto r.v$ of R into V yields an injective morphism of V into P(V). Hence every element of $\underline{C}$ is contained in an injective module in $\underline{C}$, and we can construct injective resolutions in the usual way.

## §3. Long exact sequences and Ext

3.1. Long exact sequences. We recall here the interpretation of $\mathrm{Ext}^q(U,V)$ in terms of long exact sequences. For more details, see [8: Chap. III]. Given $q \geq 1$ and $U,V \in \underline{C}$, let $S_q(U,V)$ be the set of exact sequences in $\underline{C}$ of the form

$$S : 0 \longrightarrow V \longrightarrow E_{q-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow U \longrightarrow 0 .$$

If $U',V' \in \underline{C}$ and $S' \in S_q(U',V')$, then a homomorphism $\gamma : S \longrightarrow S'$ is given by morphisms $E_i \longrightarrow E_i'$, $\gamma_U : U \longrightarrow U'$, $\gamma_V : V \longrightarrow V'$, which yield a commutative diagram

$$(1)\qquad \begin{array}{ccccccccccccc}
0 & \longrightarrow & V & \longrightarrow & E_{q-1} & \longrightarrow & \cdots & \longrightarrow & E_0 & \longrightarrow & U & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma_V} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \gamma_U} & & \\
0 & \longrightarrow & V' & \longrightarrow & E'_{q-1} & \longrightarrow & \cdots & \longrightarrow & E'_0 & \longrightarrow & U' & \longrightarrow & 0 .
\end{array}$$

In $S_q(U,V)$ we consider the smallest equivalence relation $\equiv$ such that $S \equiv S'$ if there exists either a morphism $S \longrightarrow S'$ or a morphism $S' \longrightarrow S$ which is the identity at both ends. Let $Ext'^q(U,V)$ the set of such equivalence classes. Then it is well-known that:

(i) There is an addition on $Ext'^q(U,V)$, defined by the Baer sum, with respect to which $Ext'^q(U,V)$ is a commutative group, whose zero element is represented by split exact sequences (at each stage, the kernel is a direct R-summand).

(ii) The group $Ext'^q(U,V)$ is canonically isomorphic to $Ext^q(U,V)$. This is all proved in [8: III]. We just recall some of the relevant constructions and facts in the next section.

3.2. (1) Given $S \in S_q(U,V)$ and $\gamma : V \longrightarrow V'$, there is associated an element

$$\gamma S \in S_q(U,V') : 0 \longrightarrow V' \longrightarrow E'_{q-1} \longrightarrow \dots \longrightarrow E'_0 \longrightarrow U \longrightarrow 0$$

endowed with a morphism $\alpha : S \longrightarrow S'$ such that $\alpha_U = \text{id.}$, $\alpha_V = \gamma$, called the <u>push-out</u> of $S$. The module $E'_{q-1}$ is by definition the quotient of $E_q \oplus V'$ by the subgroup generated by the elements $(\mu v, -\gamma v)$ $(v \in V$, where $\mu : V \longrightarrow E_{q-1}$ is given by $S$). The other modules $E'_i$ are constructed similarly by induction.

(2) Given $S \in S_q(U,V)$, $U' \in \underline{C}$ and $\delta : U' \longrightarrow U$, there exists $S\delta = S' \in S_q(U',V)$, the <u>pull-back</u> of $S$, endowed with a morphism $\beta : S' \longrightarrow S$ such that $\beta_V = \text{id.}$, $\beta_{U'} = \delta$. The module $E'_0$ is the pull-back of $U'$ and $E_0$ and the $E'_i$ are constructed similarly by induction.

(3) Let $S, S' \in S_q(U,V)$. Then $S \oplus S' \in S_q(U \oplus U, V \oplus V)$. Let $S_1 \in S_q(U, V \oplus V)$ be the pull-back of $S \oplus S'$ via the diagonal map $U \longrightarrow U \oplus U$. Then the Baer sum $S + S' \in S_q(U,V)$ is the push-out of $S_1$ by the map $V \oplus V \longrightarrow V$ defined by the addition in $V$.

(4) It is elementary, and follows from [8: III, 5.3], that we have

(1) $$1.S \equiv S, \qquad S.0 \equiv 0 .$$

Furthermore, if $U', V' \in \underline{C}$, $S' \in S_q(U',V')$ and $\gamma : S \longrightarrow S'$ is a morphism, then $\gamma_V . S \equiv S . \gamma_U$ [8: III, 5.1]. As a consequence, we see that if $S \in S_q(U,V)$ admits an endomorphism $\gamma$ such that $\gamma_V = 1$, $\gamma_U = 0$, then $S \equiv 0$. Indeed, we have then $0 \equiv S.0 \equiv 1.S \equiv S$.

(5) We now define the maps which yield the isomorphisms of 3.1(ii). Fix a projective resolution $(X_i)$ of U. Let $S \in S_q(U,V)$. The resolution $(X_i)$, being projective, can be mapped into S; then we get a commutative diagram:

$$\begin{array}{ccccccccccccc}
\longrightarrow & X_{q+1} & \longrightarrow & X_q & \longrightarrow & X_{q-1} & \longrightarrow & \cdots & \longrightarrow & X_0 & \longrightarrow & U & \longrightarrow 0 \\
 & \downarrow & & \downarrow{\scriptstyle a_q} & & \downarrow{\scriptstyle a_{q-1}} & & & & \downarrow{\scriptstyle a_0} & & \| & \\
 & 0 & \longrightarrow & V & \longrightarrow & E_{q-1} & \longrightarrow & \cdots & \longrightarrow & E_0 & \longrightarrow & U & \longrightarrow 0
\end{array}$$

Then $a_q \in \mathrm{Hom}_R(X_q, V)$ is zero on $\partial X_{q+1}$, hence is a cocycle. The assignment $S \longmapsto a_q$ then yields a map from $S_q(U,V)$ to the space of q-cocycles, which can be proved to induce an isomorphism $\mu$ of $\mathrm{Ext}'^q$ onto $\mathrm{Ext}^q$ [8: III, 6.4].

Conversely, a q-cocycle $z_q$ can be viewed as an R-morphism $\delta$ of $\partial X_q$ into V. To $z^q$ we associate the push-out $\delta S'$ of

$$S' : 0 \longrightarrow \partial X_q \longrightarrow X_{q-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow U \longrightarrow 0 \ .$$

This yields the inverse isomorphism to $\mu$ (cf. [8: III, 6.4])

## §4. A vanishing theorem

THEOREM. *Let* U,V *be two* $(\underline{g},\underline{k})$-*modules with infinitesimal characters* $\chi_U, \chi_V$. *If* $\chi_U \neq \chi_V$, *then* $\mathrm{Ext}^q(U,V) = 0$ *for all* q's.

To prove this theorem, we use the interpretation of $\mathrm{Ext}^q$ in terms of long exact sequences. If $\chi_U \neq \chi_V$, then we can find $z \in \underline{z}(\underline{g})$ such that $\chi_V(z) = 1$, $\chi_U(z) = 0$. Let $S \in S_q(U,V)$. Then z operates on each term of S and defines an endomorphism $\gamma(z)$ of S. By construction $\gamma(z)_V = 1$,

$\gamma(z)_U = 0$, hence $S \equiv 0$ by 3.2(4).

Remark. This theorem is an analogue of a result of D. Wigner about the continuous cohomology of real Lie groups (see [2: 2.4]). The proof is exactly the same.

4.2. COROLLARY. *Let* $U$ *be finite dimensional. If* $\chi_{\tilde{U}} \neq \chi_V$, *then* $H^q(\underline{g},\underline{k}; U \otimes V) = 0$ *for all* $q$'s.

We have $U \otimes_F V = \mathrm{Hom}_F(U',V)$. Since

$$H^q(\underline{g},\underline{k};\mathrm{Hom}_F(U',V)) = \mathrm{Ext}^q(F,\mathrm{Hom}_F(U',V)) = \mathrm{Ext}^q(U',V) ,$$

we are reduced to 4.1.

## §5. Extension to $(\underline{g},K)$-modules

In this section $F = \underline{\underline{R}}$.

5.1. In the real case, it is useful to deal with modules on which the action of $\underline{k}$ is the differential of a global action. Furthermore, it is sometimes convenient to deal with non connected groups. We shall, therefore, sketch briefly how to carry out the previous discussion in that context. As far as cohomology is concerned, this is not really necessary since the reduction to the connected case is anyhow quite easy.

Let $G$ be a Lie group with finitely many connected components, $K$ a compact subgroup of $G$, and $\underline{k}$ its Lie algebra. Then $\underline{k}$ is reductive in $\underline{g}$. A $(\underline{g},K)$-module $(\pi,V)$ is a (real or complex) vector space on which $\underline{g}$ and $K$ act, so that the following conditions are fulfilled.

i) $V$ is locally $K$-finite. If $M$ is a finite dimensional subspace of $V$ stable under $K$, then it is also stable under $\underline{k}$, the given representation of $K$ into $M$ is differentiable, and its differential coincides with the given action of $\underline{k}$.

ii) If $k \in K$ and $X \in \underline{g}$, then $\pi(\mathrm{Ad}\, k(X)) = \pi(k).\pi(X).\pi(k^{-1})$.

In particular, it follows that $V$ is a semi-simple K-module and a $(\underline{g},\underline{k})$-module.

We let $\underline{C}_{\underline{g},K}$ be the category of $(\underline{g},K)$-modules. The morphisms in $\underline{C}$ are of course homomorphisms with respect to the $\underline{g}$- and K-module structures. If $K^o$ is the identity component of $K$, then

$$\underline{C}_{\underline{g},K} \subset \underline{C}_{\underline{g},K^o} \subset \underline{C}_{\underline{g},\underline{k}} .$$

The category $\underline{C}_{\underline{g},K}$ is also closed with respect to tensor products, direct sums and subquotients.

Let $Z$ be the subgroup of elements of $K$ which act trivially on $\underline{g}$. If $G$ is connected, then $Z$ is the intersection of $K$ with the center $C(G)$ of $G$. The $(\underline{g},K)$-module $(\pi,V)$ is said to have a <u>central character</u> if there is a homomorphism $\omega_\pi = \omega_V : Z \longrightarrow \underline{\underline{C}}^*$ such that $\pi(z) = \omega_\pi(z).\mathrm{Id}.$ for $z \in Z$. This is in particular the case if $(\pi,V)$ is absolutely irreducible and admissible.

<u>Example</u>. Let $(\pi,V_\pi)$ be a continuous representation of $G$ in a quasi-complete locally convex topological vector space. Then the space $V$ of K-finite differentiable vectors of $V$ is a $(\underline{g},K)$-module, and is dense in $V_\pi$.

Let $G$ be connected semi-simple, with finite center, $K$ a maximal compact subgroup of $G$. Then it can be proved that every admissible finitely generated $(\underline{g},K)$-module is the space of K-finite vectors in some topological representation of $G$. This was proved by Lepowsky ("Algebraic results in the representations of semi-simple Lie groups," preprint) in the irreducible case, and announced by W. Casselman in general.

Note also that in this case, if $(\pi,V_\pi)$ is an irreducible unitary representation, then $V$ is an (algebraically) irreducible $(\underline{g},K)$-module.

$\underline{\underline{5}}.\underline{\underline{2}}$. <u>Cohomology</u>. Let $K^o$ be the identity component of $K$. Let $(\pi,V)$ be a $(\underline{g},K)$-module. We put

$$C^q(\underline{g},K;V) = \mathrm{Hom}_K(\Lambda^q(\underline{g}/\underline{k}),V) , \tag{1}$$

where $K$ acts on $\underline{g}/\underline{k}$ via the adjoint representation. Clearly

(2) $$C^q(\underline{g},K;V) \subset C^q(\underline{g},K^o;V) = C^q(\underline{g},\underline{k},V) .$$

Moreover, $K/K^o$ acts naturally on $C^q(\underline{g},\underline{k};V)$ and we have

(3) $$C^q(\underline{g},K;V) = C^q(\underline{g},\underline{k};V)^{K/K^o} .$$

Obviously, the $C^q(\underline{g},K;V)$ form a subcomplex of $C(\underline{g},\underline{k};V)$. The resulting cohomology groups are denoted $H^p(\underline{g},K;V)$. It follows immediately from (3) that we have

(4) $$H^q(\underline{g},K;V) = H^q(\underline{g},\underline{k};V)^{K/K^o} .$$

5.2. The functors $\mathrm{Ext}^q(U,V)$. The group $G$ also acts on the tensor algebra of $\underline{g}$ and on $R = U(\underline{g})$ by extension of the adjoint representation. As in 2.4, it is seen that $R$ thus becomes a $(\underline{g},K)$-module. It follows then that if $U$ is a locally finite semi-simple $K$-module, then $I(U) = R \otimes_S U$ ($S = U(\underline{k})$), endowed with the $K$-action stemming from the tensor product of its actions on $R$ and $U$, and the $\underline{g}$-action given by left translations on $R$, is a $(\underline{g},K)$-module, which is projective in $\underline{C}$. If $U$ and $V$ are $(\underline{g},K)$-modules then $\mathrm{Ext}^q(U,V)$ is defined as the q-th cohomology group of the complex $\{\mathrm{Hom}_{\underline{g},K}(X_i,V)\}$, where $(X_i)$ is a projective resolution of $U$. There is a natural action of $K/K^o$ on $\mathrm{Hom}_{\underline{g},K^o}(U,V)$ and on the complex $\mathrm{Hom}_{\underline{g},K^o}(X_i,V)$ and we have

(1) $$\mathrm{Hom}_{\underline{g},K}(X_i,V) = (\mathrm{Hom}_{\underline{g},K^o}(X_i,V))^{K/K^o}$$

hence

(2) $$\mathrm{Ext}^q_{\underline{g},K}(U,V) = (\mathrm{Ext}^q_{\underline{g},K^o}(U,V))^{K/K^o} .$$

Moreover, it follows from the definitions that we also have

$$(3) \qquad \mathrm{Ext}^q_{R,S}(U,V) = \mathrm{Ext}^q_{\underline{g},K^o}(U,V) \qquad (q \in \underline{\underline{N}};\ U,V \in \underline{C}_{\underline{g},K^o}) .$$

The identification with classes of long exact sequences proceeds as in §3.

5.3. THEOREM. *Let* $U,V$ *be* $(\underline{g},K)$*-modules. Assume that they have infinitesimal characters* $\chi_U,\chi_V$ *(resp. central characters* $\omega_U,\omega_V$*).*

(i) *If* $\chi_U \neq \chi_V$ *(resp.* $\omega_U \neq \omega_V$*), then* $\mathrm{Ext}^q_{\underline{g},K}(U,V) = 0$ *for all* $q$'s.

(ii) *Let* $U$ *be finite dimensional. If* $\chi_{\tilde{U}} \neq \chi_V$ *(resp.* $\omega_{\tilde{U}} \neq \omega_V$*), then* $H^q(\underline{g},K;U \otimes V) = 0$ *for all* $q$'s.

The reduction of (ii) to (i) is as in 4.2. The assertion (i) for the infinitesimal characters can be proved as in 4.1, or reduced to 4.1 using 5.2(2),(3).

Given a $(\underline{g},K)$-module $M$, any element $z \in Z$ defines an automorphism of $M$ commuting with $\underline{g}$ and $K$, hence the group algebra $\underline{\underline{R}}[Z]$ of $Z$ operates on $M$ as an algebra of endomorphisms of $(\underline{g},K)$-module. Now if $\omega_U \neq \omega_V$, there exists $z \in \underline{\underline{R}}[Z]$ such that $\omega_U(z) = 0$, $\omega_V(z) = 1$. The vanishing of $\mathrm{Ext}^q(U,V)$ then follows as in 4.1.

## REFERENCES

[1] A. Borel, Representations de Groupes Localement Compacts, Springer Lecture Notes in Mathematics #276.

[2] ————, "Cohomologie de sous-groupes discrets et représentations de groupes semi-simples," Astérisque 32-33 (1976), 73-112, Soc. Math. France.

[3] N. Bourbaki, Algèbre, 8, Modules et anneaux semi-simples, Act. Sci. Ind. 1261, Paris, 1958, Hermann éd.

[4] ————, Groupes et Algèbres de Lie I, Act. Sci. Ind. 1258, Paris, 1971, Hermann éd.

[5] H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, 1956.

[6] G. Hochschild, "Relative homological algebra," Trans. Amer. Math. Soc. 82 (1956), 246-269.

[7] J-L. Koszul, "Homologie et cohomologie des algèbres de Lie," Bull. Soc. Math. France 78 (1950), 65-127.

[8] S. Mac Lane, Homology, Grund. Math. Wiss. 114, Springer, 1963.

[9] G. Warner, Harmonic Analysis on Semi-Simple Lie Groups I, Grund. Math. Wiss. 188, Springer, 1972.

# SEMINAR ON THE COHOMOLOGY OF DISCRETE SUBGROUPS OF SEMI-SIMPLE GROUPS

These are the Notes of a seminar held at the Institute for Advanced Study during the academic year 1976-77. They consist of 11 chapters, appendices to Chapters III and VII, and an Erratum to the latter. They are mainly devoted to the continuous, or relative Lie algebra, cohomology of a real semi-simple Lie group with coefficients in infinite dimensional representations, and to applications to the cohomology of discrete cocompact subgroups. The various chapters were written shortly after the lectures and follow them rather closely. As a consequence, some results in the earlier chapters are superseded by those in the later ones. It is planned to publish in the Annals of Mathematics Studies a revised and reorganized version. It will also include the case of p-adic groups (which was discussed in the seminar) and more generally of products of real and p-adic groups.

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April 1977

# II. Scalar Product. Laplacian and Casimir Element

A. Borel

## §1. Notation and general remarks

1.1. In this chapter, $G$ is a connected semi-simple Lie group with finite center, $K$ a maximal compact subgroup of $G$, $\underline{g}$ (resp. $\underline{k}$) the Lie algebra of $\underline{g}$ (resp. $\underline{k}$), $B(\ ,\ )$ the Killing form of $\underline{g}$, and $\underline{p}$ the orthogonal complement of $\underline{k}$ with respect to $B$. We have therefore the Cartan decomposition

$$(1) \qquad \underline{g} = \underline{k} \oplus \underline{p} \quad ,$$

and $\theta : x + y \longmapsto x - y$ $(x \in \underline{k},\ y \in \underline{p})$ is an automorphism of $\underline{g}$, the Cartan involution associated to $\underline{k}$. This implies

$$(2) \qquad [\underline{k},\underline{k}] \subset \underline{k},\ [\underline{k},\underline{p}] \subset \underline{p},\ [\underline{p},\underline{p}] \subset \underline{k} \ .$$

It is also well-known that, if $\underline{g}$ has no compact factor, i.e. if $\underline{k}$ has no non zero ideal of $\underline{g}$, then $[\underline{p},\underline{p}] = \underline{k}$.

The form $B$ is negative (resp. positive) non degenerate on $\underline{k}$ (resp. $\underline{p}$) and $B(\underline{k},\underline{p}) = 0$.

In the sequel $m = \dim \underline{p}$, $n = \dim \underline{g}$, $(X_i)_{1 \leq i \leq m}$ is an orthonormal basis of $\underline{p}$ and $(X_a)_{m<a\leq n}$ a pseudo-orthonormal basis of $\underline{k}$ with respect to $B$, i.e.

$$(3) \quad B(x_i,x_j) = \delta_{ij},\ (1 \leq i,j \leq m),\quad B(x_a,x_b) = -\delta_{ab},\ (m < a,b \leq n) \ .$$

In general, we make the convention that indices $i$, $j$, $k$, $\ell$ run from 1 to $m$, and indices $a$, $b$, $c$, $d$ from $m+1$ to $n$. In view of (2), we have, with this convention

$$(4) \qquad [x_i,x_j] = \Sigma_a\, c^a_{i,j}\, x_a, \quad [x_a,x_i] = \Sigma_j\, c^j_{a,i}\, x_j \quad .$$

As usual, the structure constants are antisymmetric in the two lower indices. Moreover

**1.2.** LEMMA. *We have* $c^a_{ij} = c^i_{aj}$ $(1 \leq i,j \leq m;\ m < a \leq n)$.

In fact, since $B$ is invariant, we have

$$B([x_i,x_j],x_a) + B(x_i,[x_a,x_j]) = 0 \,. \tag{1}$$

By construction, the first term is equal to $-c^a_{ij}$ and the second one to $c^i_{aj}$.

**1.3.** We recall that if $(y_s)$ is a basis of $\underline{g}$ and $(y'_s)$ the dual basis with respect to the Killing form, then

$$C = \Sigma_{s \leq s \leq n}\, y_s . y'_s \tag{1}$$

represents an element in the center of the universal enveloping algebra $U(\underline{g})$ of $\underline{g}$, which is independent of the choice of the basis, and is called the *Casimir element* of $U(\underline{g})$. With the notation of 1.1, we have in particular

$$C = \Sigma x_j^2 - \Sigma x_a^2 \,. \tag{2}$$

**1.4.** *Relative Lie algebra cohomology.* Let $(\pi,V)$ be a $(\underline{g},\underline{k})$-module. We may write

$$C^q(V) = C^q(\underline{g},\underline{k};V) = \mathrm{Hom}_{\underline{k}}(\Lambda^q\, \underline{p}, V) = (\Lambda^q\, \underline{p}^* \otimes V)^{\underline{k}} \,. \tag{1}$$

Moreover, in view of the relation $[\underline{p},\underline{p}] \subset \underline{k}$, there are no bracket terms in the formula for the coboundary operator (I, 1.1(2)), therefore

$$d\eta(y_0,\ldots,y_q) = \Sigma_i (-1)^i\, y_i.\eta(y_0,\ldots,\hat{y}_i,\ldots,y_q), \qquad (\eta \in C^q(V)) \,. \tag{2}$$

Let

$$D^q(V) = \mathrm{Hom}_{\underline{R}}(\Lambda^q\, \underline{p}, V) \,. \tag{3}$$

Evidently, $D^q(V)$ contains $C^q(V)$. We note that (2) also makes sense on $D^q$, hence defines a linear operator $D^q(V) \longrightarrow D^{q+1}(V)$, also to be denoted $d$. The space $D^q(V)$ may be identified with the subspace of $C^q(\underline{g},V)$ whose elements are annihilated by the interior products $i_x$ $(x \in \underline{k})$. Let $d_0$ be the coboundary operator in $C^q(\underline{g};V)$. Then, of course, $d_0 f = df$ if $f \in C^q(\underline{g},\underline{k};V)$. However, if $f \in D^q(V)$, then $df$ is the restriction of $d_0 f$ to $\Lambda^{q+1}\underline{p}$, but is not equal to $df$ in general. In particular, we do not have necessarily $d^2 = 0$ on $D^q(V)$. Since $d_0$ commutes with the $\theta_x$ $(x \in \underline{g})$ and $D^q(V)$ is stable under $\theta_x$ for $x \in \underline{k}$, we have

(4) $$d \circ \theta_x = \theta_x \circ d \quad \text{on} \quad D^q(V) \quad \text{for all} \quad x \in \underline{k}\ .$$

**1.5.** Notation for cochains. If $A$ is a finite set, then $|A|$ denotes its cardinality. For a positive integer $s$, let $I_s = \{1,2,\ldots,s\}$.

We shall denote by $(\omega^a,\omega^i)$ the basis of $\underline{g}^*$ dual to $(X_a,X_i)$. The elements $\omega^i$ will also be viewed as forming a basis of $\underline{p}^*$ dual to $(X_i)$. For $I \subset I_m$ $|I| = q$, we put

(1) $$\omega^I = \omega^{j_1} \wedge \ldots \wedge \omega^{j_q}, \quad \text{if} \quad I = \{j_1,\ldots,j_q\}\ .$$

If $\eta \in D^q(V)$, let

(2) $$\eta_I = \eta_{j_1,\ldots,j_q} = \eta(x_{j_1},\ldots,x_{j_q}), \quad (x_{j_i} \in \underline{p},\ 1 \leqq i \leqq q)\ .$$

Then $\eta$ can be written

(3) $$\eta = \Sigma_{I \subset I_m, |I|=q}\, \eta_I \cdot \omega^I\ ,$$

or also

(4) $$\eta = (q!)^{-1}\, \Sigma_{j_1,\ldots,j_q \in I_q}\, \eta_{j_1,\ldots,j_q}\, \omega^{j_1} \wedge \ldots \wedge \omega^{j_q}\ .$$

If $I = (j_1, \dots, j_q)$ and $u \in I_q$, then $I(u)$ denotes $I$ with the u-th entry removed. The equality 1.3(2) can then also be written

$$(5)\quad (d\eta)_I = \Sigma_{1 \leqq u \leqq q+1} (-1)^{u-1} \pi(X_u).\eta_{I(u)}, \quad (f \in D^q(V);\ I \subset I_m,\ |I| = q+1) .$$

Note also that we have for $1 \leqq u \leqq q$:

$$(6)\quad \eta_I = \eta_{j_1,\dots,j_q} = (-1)^{u-1}\, \eta_{j_u, j_1, \dots, \hat{j}_u, \dots, j_q} = (-1)^{u-1}\, \eta_{j_u \cup I(u)} .$$

## §2. Scalar product

2.1. We shall be interested in the case where $V = H \otimes E$, where $(\rho, E)$ is a finite dimensional complex continuous representation of $G$ and $(\sigma, H)$ is a unitary $(\underline{g}, \underline{k})$-module. The latter condition means that $H$ is a complex vector space endowed with a positive non degenerate scalar product $(\ ,\ )_H$, such that $(Xu,v)_H + (u,Xv)_H = 0$ for all $u,v \in H$ and $X \in \underline{g}$. It is not required that $H$ be complete. We let $\tau = \rho \otimes \sigma$. For $x \in \mathfrak{g}$, we shall often write $\tau(x) = \sigma(x) + \rho(x)$ as a shorthand for $\sigma(x) \otimes 1 + 1 \otimes \rho(x)$.

2.2. On $E$ there is always a so-called admissible scalar product, i.e. one which is invariant under $\underline{k}$, and such that $\rho(x)$ is self-adjoint for $x \in \underline{p}$. We assume $E$ endowed with one, to be denoted $(\ ,\ )_E$, and then put on

$$(1)\qquad D^q(V) = \Lambda^q\, \underline{p}^* \otimes H \otimes E ,$$

the scalar product which is the tensor product of $(\ ,\ )_V = (\ ,\ )_H \otimes (\ ,\ )_E$ with the scalar product on $\Lambda^q\, \underline{p}^*$ defined by the Killing form. In particular, if

$$(2)\qquad \mu = \Sigma_{I \subset I_m} \mu_I . \omega^I, \quad \eta = \Sigma_I\, \nu_I . \omega^I ,$$

(notation of 1.5), then

$$(3)\qquad (\mu, \nu) = \Sigma_I (\mu_I, \eta_I)_V .$$

Since these scalar products are invariant under $\underline{k}$, we have

$$(4)\qquad (\theta_x\mu,\nu) + (\mu,\theta_x\nu) = 0, \quad (\mu,\nu \in D^q(V),\ x \in \underline{k}) .$$

For $x \in \underline{g}$, we let $\tau(x)^*$ be the adjoint of $\tau(x)$ with respect to $(\ ,\ )_V$. Thus

$$(5)\qquad \begin{aligned} \tau(x)^* &= -\tau(x) &\quad \text{if } x \in \underline{k}, \\ \tau(x) &= \rho(x) - \sigma(x) &\quad \text{if } x \in \underline{p}. \end{aligned}$$

2.3. PROPOSITION. Let $\partial : D^q(V) \longrightarrow D^{q-1}(V)$ be defined by

$$(1)\qquad (\partial\eta)_J = \Sigma_{1\leqq j\leqq m}\, \tau(x_j)^*\, \eta_{\{j\}\cup J} \qquad (J \subset I_m, |J| = q-1) .$$

Then $\partial$ commutes with the $\theta_x$ $(x \in \underline{k})$, maps $C^q(V)$ into $C^{q-1}(V)$ and is adjoint to $d$, i.e.

$$(2)\qquad (\partial\eta,\mu) = (\eta, d\mu), \qquad (\eta \in D^q(V),\ \mu \in D^{q-1}(V)) .$$

Using 1.5(5) and 2.2(3), we have

$$(\eta,\mu) = \Sigma_{|I|=q}(\eta_I,(d\mu)_I)_V = \Sigma_I(\eta_I,\Sigma_u(-1)^{u-1}\, \tau(x_{j_u})\cdot\mu_{I(u)}) ,$$

where $I = \{j_1,\ldots,j_q\}$. Hence

$$(\eta, d\mu) = \Sigma_{u,I}((-1)^{u-1}\, \tau(x_{j_u})^*\, \eta_I,\mu_{I(u)}) .$$

In view of the relation 1.5(6), this can be written

$$(\eta, d\mu) = \Sigma_{\substack{1\leqq j\leqq m \\ |J|=q-1}} (J(x_j)^*\, \eta_{j\cup J},\mu_J) = (\partial\eta,\mu) .$$

Together with 1.4(4) and 2.2(4), this implies that $\partial$ commutes with $\theta_x$ $(x \in \underline{k})$. Since $C^q(V)$ is the subspace of $D^q(V)$ annihilated by the $\theta_x$ $(x \in \underline{k})$ it follows that $\partial C^q(V) \subset C^{q-1}(V)$. This completes the proof.

**2.4.** We let $\Delta = d\partial + \partial d$ be the Laplacian. For each $q$, $\Delta$ is an endomorphism of $D^q(V)$ which leaves $C^q(V)$ stable. For $\eta \in D^q(V)$, we have by 2.3

$$(1) \qquad (\Delta\eta,\eta) = (d\eta, d\eta) + (\partial\eta,\partial\eta) \quad .$$

Since the scalar product is positive non degenerate, this implies

$$(2) \qquad \Delta\eta = 0 \Longleftrightarrow d\eta = \partial\eta = 0 \Longleftrightarrow (\Delta\eta,\eta) = 0 \quad .$$

The element $\eta$ is harmonic if it satisfies the conditions of (2). The space of harmonic forms in $C^q(V)$ will be denoted $\mathcal{H}^q(V)$. As usual $\eta$ is closed if $d\eta = 0$, coclosed if $\partial\eta = 0$ .

**2.5.** THEOREM. *Let $\pi = \sigma, \rho$ or $\tau = \sigma \otimes \rho$, and view $V$ as a $(\underline{g},\underline{k})$-module under $\pi$. Let $\Delta_\pi$ be the corresponding Laplacian. Then*

(i) $(\Delta_\pi.\eta)_I = \Sigma_{1\leq j\leq m}\pi(x_j).\pi(x_j)^*.\eta_I + \Sigma_{\substack{1\leq j\leq m\\ 1\leq u\leq q}}(-1)^{u-1}[\pi(x_{j_u}),\pi(x_j)^*].\eta_{j\cup I(u)}$, $(\eta \in D^q(V),\ I \subset I_m,\ |I| = q)$ .

(ii) *We have* $\Delta_\tau = \Delta_\sigma + \Delta_\rho$ *on* $D^q(V)$.

(iii) (Kuga) *If* $\eta \in C^q(V)$, *then*

$$(\Delta_J\eta)_I = (\rho(C) - \sigma(C)).\eta_I\,, \qquad (I \subset I_m,\ |I| = q)\,,$$

*where $C$ is the Casimir element* (1.3).

(i) We view $V$ as a $(\underline{g},\underline{k})$-module under $\pi$, but still denote $d$ the coboundary operator and $\partial$ its adjoint. In this proof, $I \subset I_m$, $|I| = q$, the index $a$ (resp. $j$, resp. $u$) runs from $m+1$ to $n$ (resp. 1 to $m$, resp. 1 to $q$).

$$(\partial d\eta)_I = \Sigma\pi(x_j)^*(d\eta)_{j\cup I} = \Sigma_j\ \pi(x_j)^*(\pi(x_j).\eta_I + \Sigma_u(-1)^u\ \pi(x_{j_u})\eta_{j\cup I(u)}) \ .$$

$$(1) \qquad (\partial d\eta)_I = \Sigma_j\ \pi(x_j)^*\ \pi(x_j)\eta_I + \Sigma_{j,u}\ (-1)^u\ \pi(x_j)^*\ \pi(x_{j_u})\eta_{j\cup I(u)} \ .$$

On the other hand

$$(d\partial\eta)_I = \sum_{1\leq u\leq q} (-1)^{u-1}\, \pi(x_{j_u}).(\partial\eta)_{I(u)}$$

$$(d\partial\eta)_I = \Sigma_{u,j}(-1)^{u-1}\, \pi(x_{j_u})\, \pi(x_j)^*\, \eta_{j\cup I(u)} ,$$

hence

$$(2) \quad (\Delta\eta)_I = \Sigma_j\, \pi(x_j)^*\, \pi(x_j)\eta_I + \Sigma_{u,j}(-1)^{u-1}[\pi(x_{j_u}),\pi(x_j)^*]\eta_{j\cup I(u)} .$$

This proves (i).

(ii) Now let $\pi = \tau$. Since $\sigma \otimes 1$ and $1 \otimes \rho$ commute, we have

$$(3) \quad \begin{aligned} [\pi(x_{j_u}),\pi(x_j)^*] &= [\sigma(x_{j_u}) + \rho(x_{j_u}),\sigma(x_j)^* + \rho(x_j)^*] \\ [\pi(x_{j_u}),\pi(x_j)^*] &= [\sigma(x_{j_u}),\sigma(x_j)^*] + [\rho(x_{j_u}),\rho(x_j)] \end{aligned}$$

Moreover, the equalities $\sigma(x_j)^* = -\sigma(x_j)$ and $\rho(x_j) = \rho(x_j)^*$ yield

$$(4) \quad \pi(x_j)\, \pi(x_j)^* = \rho(x_j)^2 - \sigma(x_j)^2 = \rho(x_j)^*\, \rho(x_j) + \sigma(x_j)\, \sigma(x_j)^* .$$

The assertion (ii) then follows from (i), (3), (4).

(iii) The first sum on the right hand side of (i) is equal to

$$\Sigma_j(\rho(x_j)^2 - \sigma(x_j)^2).\eta_I .$$

To prove (iii), there remains to show that we have

$$(5) \quad \Sigma_a(\sigma(x_a)^2 - \rho(x_a)^2)\eta_I = \Sigma(-1)^{u-1}[\tau(x_{j_u}),\tau(x_j)^*]\eta_{j\cup I(u)} .$$

Call $Q$ the right hand side. By (3) and 2.1(4);

$$[\tau(x_{j_u}),\tau(x_j)^*] = [\sigma(x_j),\sigma(x_{j_u})] - [\rho(x_j),\rho(x_{j_u})]$$

$$[\tau(x_{j_u}),\tau(x_j)^*] = \Sigma\, c^a_{j,j_u}(\sigma(x_a) - \rho(x_a)) .$$

Therefore

$$(6) \qquad Q = \Sigma_a(\sigma(x_a) - \rho(x_a))(\Sigma_{j,u}(-1)^{u-1} c^a_{j,j_u} \eta_{j\cup I(u)}) .$$

But $c^a_{j,j_u} = c^j_{a,j_u}$ by 1.2. Hence

$$L_a = \Sigma_{j,u}(-1)^{u-1} c^a_{j,j_u} \eta_{j\cup I(u)} = \Sigma c^j_{a,j_u} \eta(x_{j_1},\ldots,x_j,\ldots,x_{j_q}) ,$$

where $x_j$ is at the u-th place; this can be written

$$L_a = \Sigma_u \eta(x_{j_1},\ldots,[x_a,x_{j_u}],\ldots,x_{j_q}) .$$

Since $\eta \in C^q(V)$, it is annihilated by the $\theta_x$ $(x \in \underline{k})$ hence

$$(7) \qquad L_a = \tau(x_a).\eta(x_{j_1},\ldots,x_{j_q}) = \tau(x_a).\eta_I ,$$

and (5) follows from (6) and (7).

2.6. COROLLARY. Let $\eta \in D^q(V)$. Then $\Delta_\tau\eta = 0$ if and only if $\Delta_\rho\eta = \Delta_\sigma\eta = 0$.

This follows from 2.4(2) and 2.5(ii).

2.7. The results of this section (and the next one) have been known for sometime, but do not seem to have been formulated in this way in the literature. They have their origin in the work of Matsushima and Murakami [1, 2] on the cohomology of discrete cocompact subgroups of G, where they are proved when H is the space of K-finite smooth functions on the quotient $\Gamma\backslash G$ on G by one such subgroup. More precisely, it is shown in [2] that

$$(1) \qquad H^*(\Gamma;E) = H^*(\underline{g},\underline{k};H \otimes E)$$

(see also Chapter IV). This being granted, the computations made here are substantially those of [1], In particular, see [1: §6] for 2.3 and 2.5(iii). In that special case, 2.5(i),(ii) are also implicit in [1: §7], and are made explicit in [3: §1].

## §3. Special cases

3.1. PROPOSITION. *Assume that* $\sigma(C) = s.\mathrm{Id}.$, $\rho(C) = r.\mathrm{Id}.$

(a) *If* $r \neq s$, *then* $H^q(\underline{g},\underline{k};\ H \otimes E) = 0$ *for all* $q$'s.

(b) *If* $r = s$, *then all cochains are closed, harmonic, and we have*

$$H^q(\underline{g},\underline{k};\ H \otimes E) = C^q(\underline{g},\underline{k};\ H \otimes E) = \mathrm{Hom}_{\underline{k}}(\Lambda^q\ \underline{p}, H \otimes E)\quad \textit{for all}\quad q\text{'s}.$$

By 2.5(iii), $\Delta = (r-s).\mathrm{Id}$ on $C^q(H \otimes E)$ for all $q$'s.

Assume that $r \neq s$. Let $\eta$ be a $q$-cocycle. Then $\Delta\eta = d\partial\eta$, hence

$$\eta = (r-s)^{-1}\ \Delta\eta = (r-s)^{-1}.d\partial\eta\ ,$$

is a coboundary, whence (a).

Now let $r = s$. Then $\Delta = 0$, all cochains are harmonic, hence closed and coclosed by 2.4. This yields (b).

3.2. COROLLARY. *Let* $(\rho, E)$ *be irreducible. If* $\rho$ *is non trivial, then* $H^q(\underline{g},\underline{k};E) = 0$ *for all* $q$'s. *If* $\rho$ *is the trivial representation, then* $H^q(\underline{g},\underline{k};E) = C^q(\underline{g},\underline{k};E) = (\Lambda^q\ \underline{p}^*)^{\underline{k}}$ *for all* $q$'s.

If $\rho$ is irreducible then $\rho(C) = r.\mathrm{Id}$, and it is well-known that $r = 0$ if and only if $\rho$ is the trivial representation. 3.2 then follows from 3.1 applied to the case where $(\sigma, V)$ is the trivial representation.

*Remark.* If we identify $(\Lambda^q\ \underline{p}^*)^{\underline{k}}$ with the $G$-invariant differential forms on $G/K$, (cf. I, 1.4), the corollary in the case of the trivial representation asserts that on $G/K$ all invariant forms are harmonic, closed and co-closed. This is a well-known result of E. Cartan.

3.3. COROLLARY. *Let* $H$ *be the space of* $K$-*finite vectors in the space of an irreducible unitary representation of* $G$. *If* $\sigma(C) = 0$, *then* $H^q(\underline{g},\underline{k};H) = \mathrm{Hom}_{\underline{k}}(\Lambda^q\ \underline{p}, H)$ *and if* $\sigma(C) \neq 0$, *then* $H^q(\underline{g},\underline{k};H) = 0$, *for all* $q$'s.

Under our assumption, $\sigma(C)$ is a multiple of the identity. 3.3 then follows from 3.1, applied to the case where $\rho$ is the trivial representation.

3.4. Assume now $H$ to be an admissible $(\underline{g},\underline{k})$-module. Since $C^q(V)$ may be written as

(1) $$C^q(V) = \mathrm{Hom}_{\underline{k}}(\Lambda^q\,\underline{p}\otimes E^*, H)\ ,$$

it is finite dimensional. Our complex $C^*(V)$ is then finite dimensional and the elementary "Hodge theory" in finite dimensional vector spaces obtains: we have an orthogonal decomposition

(2) $$C^q(V) = \mathcal{H}^q(V) \oplus dC^{q-1}(V) \oplus \partial C^{q+1}(V)\ ,$$

and $\Delta$ is an invertible operator on $dC^{q-1}(V) \oplus C^{Q+1}(V)$. As usual, this implies

(3) $$H^q(\underline{g},\underline{k};V) \cong \mathcal{H}^q(V)\ ,$$

i.e. every cohomology class is represented by a unique harmonic form.

Let $A_q$ be the sum of the isotypic subspaces of $H$ corresponding to the $\underline{k}$-types occurring in $\Lambda^q\,\underline{p}\otimes E^*$. It is finite dimensional. Let $B_q = E\otimes A_q$ and $C_q$ the subspace of $B_q$ annihilated by $\rho(C) - \sigma(C)$. Then

(4) $$\mathcal{H}^q(V) \cong \mathrm{Hom}_{\underline{k}}(\Lambda^q\,\underline{p}, C_q)\ .$$

Note $\Lambda^q\,\underline{p}$ and $\Lambda^{m-q}\,\underline{p}$ are isomorphic $\underline{k}$-modules. Therefore $C_q$ and $C_{m-q}$ are isomorphic $\underline{k}$-modules and we have

(5) $$H^q(\mathfrak{g},\underline{k};V) = \mathcal{H}^q(V) = H^{m-q}(\underline{g},\underline{k};V) \qquad (q\in\underline{\underline{N}})\ .$$

Remark. As in 2.7, let $\Gamma$ be a cocompact discrete subgroup of $G$ and $H$ the space of $K$-finite smooth functions on $\Gamma\backslash G$. Then (5) is also valid, although $H$ is not admissible. Modulo 2.7(1), this is proved in [1: 6.2] using the Hodge theory of harmonic forms on $\Gamma\backslash X$. (For this [1] assumes $\Gamma$ to be torsion free so that $\Gamma\backslash X$ is a smooth manifold, but the reduction to that case is easy; one could also use Hodge theory on V-manifolds.) Another proof of (5) in this case will be given in Chapter IV.

## REFERENCES

[1] Y. Matsushima and S. Murakami, "On vector bundle valued harmonic forms and automorphic forms on symmetric spaces," Annals of Math. (2) 78 (1963), 365-416.

[2] Y. Matsushima and S. Murakami, "On certain cohomology groups attached to hermitian symmetric spaces," Osaka J. Math. 2 (1965), 1-35.

[3] M. S. Raghunathan, "Vanishing theorems for cohomology groups associated to discrete subgroups of semisimple Lie groups," Osaka J. Math. 3 (1966), 243-256.

# III. Vanishing theorems

A. Borel

In this chapter, we keep the notation and general assumptions of II. In particular $(\sigma, H)$ is a unitary $(\underline{g}, K)$-module and $(\rho, E)$ a finite dimensional representation of $G$. We want to give the analogues in our context of vanishing theorems of M. S. Raghunathan [6] and Y. Matsushima [4].

## §1. A vanishing theorem for non-trivial $\rho$

1.1. On $\mathrm{Hom}_{\mathbb{R}}(\Lambda^q \underline{p}, E) \cong \Lambda^q \underline{p} \otimes E$ we consider the scalar product

$$F_{\rho,q}(\mu, \eta) = (\Delta_\rho \mu, \eta) \quad ,$$

where $( \ , \ )$ is the positive non-degenerate scalar product used in II.

1.2. THEOREM. *Let* $q \in \mathbb{N}$. *Assume that* $F_{\rho,q}$ *is positive non-degenerate on* $\mathrm{Hom}_{\mathbb{R}}(\Lambda^q \underline{p}, E)$. *If* $(\sigma, H)$ *is admissible, then* $H^q(\underline{g}, \underline{k}; H \otimes E) = 0$.

Our scalar product on

$$D^q(H \otimes E) = \Lambda^q \underline{p}^* \otimes E \otimes H \cong \mathrm{Hom}_{\mathbb{R}}(\Lambda^q \underline{p}, E) \otimes H \quad ,$$

is the tensor product of the natural scalar products on the two factors of the last term. Therefore $\eta \longmapsto (\Delta_\rho \eta, \eta)$ is also positive non-degenerate on $D^q(V)$. But, for $\eta \in D^q(V)$, we have, by II, 2.2:

$$(\Delta\eta, \eta) = (\Delta_\rho \eta, \eta) + (\Delta_\sigma \eta, \eta) \quad ,$$

and all terms are $\geqq 0$. Thus if $\Delta\eta = 0$, then $(\Delta_\rho \eta, \eta) = 0$, hence $\eta = 0$. On the other hand, since $H$ is admissible, every cohomology class is represented by a harmonic form (II, 3.2), whence the theorem.

Remark. This argument is that of Prop. 1 in [6], although the result is stated there as a theorem on the cohomology of cocompact discrete subgroups (see IV). The main point of [6], however, is a very useful sufficient condition for $F_{\rho,q}$ to be positive non-degenerate. We shall first state this result, and then derive some consequences from it. To this effect, we need to introduce some notation.

1.3. If $\underline{m}$ is a real Lie algebra, then $\underline{m}_c$ denotes its complexification i.e. $\underline{m}_c = \underline{m} \otimes_{\mathbb{R}} \mathbb{C}$.

Let $\underline{t}$ be a Cartan subalgebra of $\underline{k}$ and $\underline{h}$ a Cartan subalgebra of $\underline{g}$ which contains it. We have then

(1) $$\underline{h} = \underline{t} \oplus \underline{u}, \quad \text{where } \underline{u} = \underline{p} \cap \underline{h} .$$

In fact, since $\underline{t}$ always contains regular elements of $\underline{g}$, $\underline{h}$ is the centralizer of $\underline{t}$ in $\underline{g}$, hence is stable under the Cartan involution $\theta$ of $\underline{g}$ with respect to $\underline{k}$.

Let $\Phi$ be the system of roots of $\underline{g}_c$ with respect to $\underline{h}_c$. For $\alpha \in \Phi$, let, as usual:

(2) $$\underline{g}_{c\alpha} = \{x \in \underline{g}_c \,|\, [h, x] = \alpha(h)x \quad (h \in \underline{h}_c)\} .$$

The space $\underline{g}_{c\alpha}$ is one-dimensional. The Cartan involution $\theta$ extends to an automorphism of $\underline{g}_c$ leaving $\underline{h}$ stable; it also induces an automorphism of $\Phi$ and we have

$$\theta(\underline{g}_{c\alpha}) = \underline{g}_{c,\theta(\alpha)}, \qquad (\alpha \in \Phi) . \tag{3}$$

Let

$$\begin{aligned} A &= \{\alpha \in \Phi \,|\, \theta(\alpha) = \alpha,\ \theta|\underline{g}_{c\alpha} = \mathrm{Id.}\} , \\ B &= \{\alpha \in \Phi \,|\, \theta(\alpha) = \alpha,\ \theta|\underline{g}_{c\alpha} = -\mathrm{Id.}\} , \\ C &= \{\alpha \in \Phi \,|\, \theta(\alpha) \neq \alpha\} . \end{aligned} \tag{4}$$

Thus $\Phi$ is the disjoint union of A, B and C. Furthermore, $\alpha \in A \cup B$ if and only if $\alpha$ is zero on $\underline{u}$. We recall that $\underline{h}_0 = i.\underline{t} \oplus \underline{u}$ is the real form of $\underline{h}_c$ on which all roots take real values.

In the following, we choose a lexicographic ordering on the dual of $\underline{h}_0$ defined by a basis $(h_i)$ of $\underline{h}_0$ such that $h_1, \ldots, h_r$ $(r = \dim \underline{t})$ span $i.\underline{t}$ and that, if the center $\underline{c}$ of $\underline{k}$ has dimension $s$, the elements $h_1, \ldots, h_s$ span $i.\underline{c}$. We let $\Phi^+$ be the set of positive roots for this ordering, put

$$A^+ = A \cap \Phi^+, \quad B^+ = B \cap \Phi^+, \quad C^+ = C \cap \Phi^+ , \tag{5}$$

and

$$C^{++} = \{\alpha \in C^+ \,|\, \theta(\alpha) > \alpha\} . \tag{6}$$

The highest weights of finite dimensional irreducible G-modules are chosen with respect to that ordering.

Let $\Delta$ be the set of simple roots in $\Phi^+$. Any linear form $\lambda$ on $\underline{h}_c$ can be written uniquely as a linear combination of elements in $\Delta$. We denote by $n_\alpha(\lambda)$ the coefficient of $\alpha$. If $\lambda \in \Phi^+$, then $n_\alpha(\lambda) \in \mathbb{N}$.

Let $\{\lambda_\alpha\}_{\alpha \in \Delta}$ be the set of fundamental highest weights. Thus

$$2(\lambda_\alpha, \beta) \cdot (\beta, \beta)^{-1} = \delta_{\alpha\beta}, \qquad (\alpha, \beta \in \Delta), \tag{7}$$

where $( \ , \ )$ is the scalar product on $\underline{h}_0^*$ defined by the Killing form. We recall that the highest weights $\lambda_\rho$ of the finite dimensional irreducible $\underline{g}_c$-modules are all the linear combinations

$$\lambda_\rho = \Sigma_{\alpha \in \Delta}\, c_\alpha(\lambda_\rho)\lambda_\alpha, \quad \text{with } c_\alpha(\lambda_\rho) \in \mathbb{N}. \tag{8}$$

1.4. THEOREM. *Assume* $(\rho, E)$ *to be irreducible and let* $\lambda_\rho$ *be its highest weight with respect to an ordering satisfying the conditions of* 1.3. *Let* $m = \mathrm{Card}\{\alpha \in B^+ \cup C^{++} \mid (\lambda_\rho, \alpha) > 0\}$. *Then the form* $F_{\rho, q}$ *is positive non-degenerate for* $q < m_\rho$. *In particular,* $H^q(\underline{g}, \underline{k}; H \otimes E)$ *for* $q < m_\rho$ *if* $(\sigma, H)$ *is admissible.*

(We recall that $H$ is unitary by our general assumption.) The second assertion follows from the first one and 1.2. The first one is due to M. S. Raghunathan [6, Thm. 1]. We refer to [6] for the proof.

*Remark.* Assume that $\underline{g} = \underline{g}' \oplus \underline{g}''$. The representation $\rho$ is then a tensor product $\rho' \otimes \rho''$, where $\rho'$ (resp. $\rho''$) is an irreducible finite dimensional representation of $\underline{g}'$ (resp. $\underline{g}''$). If we choose an ordering compatible with this direct sum decomposition, then, clearly, $m_\rho = m_{\rho'} + m_{\rho''}$. This reduces the study of $m_\rho$ to the case where $\underline{g}$ is simple non-compact. Moreover, if $\lambda_\rho$ is written as in 1.3(8), then

$$m_\rho = \Sigma_{\alpha\in\Delta}\, c_\alpha(\lambda_\rho).m_\alpha \quad ,$$

where $m_\alpha$ stands for the constant $m_\rho$ in the case where $\rho$ is the fundamental representation with highest weight $\lambda_\alpha$. Thus it suffices to compute $m_\rho$ when $\rho$ is a fundamental representation of a simple Lie algebra.

## §2. Some results on the constant $m_\rho$

In this section, we assume $(\rho, E)$ to be irreducible non-trivial and, from 2.3 on, $\underline{g}$ to be simple non-compact.

2.1. We recall that the $\mathbb{R}$-rank or split rank $\mathrm{rk}_{\mathbb{R}}(\underline{g}) = \ell_{\mathbb{R}}(\underline{g})$ is the dimension of the maximal subalgebras of $\underline{p}$. If $\underline{g} = \underline{g}' \oplus \underline{g}''$, then $\mathrm{rk}_{\mathbb{R}}(\underline{g}) = \mathrm{rk}_{\mathbb{R}}(\underline{g}') + \mathrm{rk}_{\mathbb{R}}(\underline{g}'')$.

2.2. PROPOSITION. We have $m_\rho \geqq \mathrm{rk}_{\mathbb{R}}(\underline{g})$.

By 1.4, Remark and 2.1, it suffices to prove this for $\underline{g}$ simple non-compact. This was done by N. Wallach when $\mathrm{rk}\,\underline{g} = \mathrm{rk}\,\underline{k}$, by case by case checking. Some indications on these computations are given in 2.5. The proof for $\mathrm{rk}\,\underline{g} \neq \mathrm{rk}\,\underline{k}$ is contained in 2.6, 2.7. In [6: p. 251], Raghunathan points out that he had checked it for many classical groups.

2.3. We assume first that $\underline{g}$ and $\underline{k}$ have equal ranks, i.e. that $\underline{t} = \underline{h}$, $\underline{u} = 0$, in the notation of 1.3. Then $\underline{g}_c$ is simple, $m = \dim \underline{p}$ is even, C is empty, A is the set of compact roots, i.e. the roots of $\underline{k}_c$ with respect to $\underline{h}_c$, and B the set of non-compact roots, i.e. the set of weights of $\underline{h}_c$ in $\underline{p}_c = \underline{p} \otimes \mathbb{C}$. Moreover, there is exactly one $\beta \in \Delta$ such that $B^+$ is the set of $\gamma \in \Phi$ which, when expressed as sum of simple roots, contain $\beta$ with the coefficient one, and there are two possibilities:

a) $\underline{k}$ has a one-dimensional center, $\beta$ has the coefficient one in the highest root $\delta_0$, $\Delta - \{\beta\}$ is the set of simple roots of the derived algebra $\underline{k}'$ of $\underline{k}$ with respect to $\underline{t}' = \underline{k}' \cap \underline{t}$. The space $G/K$ is equivalent to a bounded symmetric domain.

b) $\underline{k}$ is semi-simple. $\beta$ has the coefficient two in $\delta_0$ and $\Phi(\underline{k}_c, \underline{t}_c)$ is the set of elements of $\Phi$ in which $\beta$ has coefficients 0 or $\pm 2$.

The scalar product of $\lambda_\rho$ with any positive root is $\geqq 0$. Therefore, we have:

2.4. PROPOSITION. <u>We keep the notation and assumptions of</u> 2.3. <u>If</u> $(\lambda_\rho, \beta) \neq 0$, <u>then</u> $m_\rho = (\dim \underline{p})/2$.

Poincaré duality (II, 3.3(5)) then shows that $H^q(\underline{g}, \underline{k}; H \otimes E) = 0$ for $q \neq (\dim \underline{p})/2$.

2.5. We now sketch the steps allowing us to check 2.2 in the equal rank case. For $\alpha \in \Delta$, let

$$P_{\beta\alpha} = \{\gamma \in \Phi \mid n_\beta(\gamma) = 1, \; n_\alpha(\gamma) > 0\} \; .$$

Then $m_\alpha = \mathrm{Card}\, P_{\beta\alpha}$. On the other hand, $\mathrm{rk}_{\mathbb{R}}(\underline{g})$ is equal to the number of elements in a maximal set of "strongly orthogonal" elements in $B^+$. (Two roots $\alpha, \beta$ are said to be strongly orthogonal if $\alpha \pm \beta \notin \Phi$.) These ranks are also listed, together with E. Cartan's classification, in [2: p. 346, 354]. With the help of the tables of [1], one can then check that $P_{\beta\alpha} \geqq \mathrm{rk}_{\mathbb{R}}\, \underline{g}$ for all possible $\beta, \alpha$. (By 2.4, we may assume $\beta \neq \alpha$.)

This is particularly easy for the exceptional groups since in this case, [1] gives the list of the positive roots $\gamma$ together with the coefficients $n_\alpha(\gamma)$. If $\underline{g} = \underline{su}(p, q)$, the lower bound $\mathrm{rk}_{\mathbb{R}}(\underline{g})$ is indeed obtained for some $\alpha$. However, in many cases we have $m_\rho \geq \mathrm{rk}\, \underline{g}$ for all (non-trivial) $\rho$.

$\underline{\underline{2.6}}$. Assume $\underline{g}$ is not absolutely simple. Then $\underline{g}_c$ is the direct sum of two copies of $\underline{k}_c$, which are permuted by $\theta$, and $\underline{g}$ may be identified with the set of elements $(x, \bar{x})$ in $\underline{k}_c \oplus \underline{k}_c$, where $\bar{\ }$ is the complex conjugation of $\underline{k}_c$ with respect to $\underline{k}$. To avoid ambiguity, we write these two copies as $\underline{k}'_c$, $\underline{k}''_c$ and assume we are given isomorphisms $\varphi' : \underline{k} \longmapsto \underline{k}'$, $\varphi'' : \underline{k} \longmapsto \underline{k}''$ of $\underline{k}_c$ onto $\underline{k}'_c$ and $\underline{k}''_c$ respectively. With these notations, we have the identifications

$$(1) \qquad \underline{h}_c = \underline{t}'_c \oplus t''_c, \; \underline{t}_c = \{\varphi'(t), \; \varphi''(t)\}, \; \underline{u}_c = \{\varphi'(t), \; -\varphi''(t)\} \qquad (t \in \underline{t}_c) \; .$$

No root restricts trivially on $\underline{u}$, hence $A = B = \phi$, and $\Phi = C$. To define a lexicographic ordering on $\underline{h}_c^*$, we take first a basis of $\underline{t}_c$ of the form

(2) $$(h_i', h_i'')_{1 \leqq i \leqq \ell} \text{ where } (h_i) \text{ is a basis of } i.\underline{t} \subset \underline{g} .$$

Since no root restricts to zero on the diagonal, this fixes the ordering on $\Phi$, hence also on the weights. The set of roots is $\Phi$ the union of $\Phi' = \Phi(\underline{k}_c', \underline{t}_c')$ and $\Phi'' = \Phi(\underline{k}_c'', \underline{t}_c'')$. If $\gamma \longmapsto \gamma'$, $\gamma \longmapsto \gamma''$ are the isomorphisms induced by $\varphi'$ and $\varphi''$, then, with our convention, we have

(3) $$\gamma' > 0 \Longleftrightarrow \gamma'' > 0 ,$$

(4) $$\theta(\gamma') = \gamma'', \ \theta(\gamma'') = \gamma' \quad (\gamma \in \Phi(\underline{k}_c, \underline{t}_c)) .$$

The fundamental representations of $\underline{g}_c$ are those of the two factors $\underline{k}_c'$, $\underline{k}_c''$. Assume $\rho$ is a fundamental representation of the first (resp. second) factor. To define completely the lexicographic ordering on $\underline{h}_c^*$, we then complete the set $\{(h_i', h_i'')\}$ by the basis

(5) $$\{(-h_i', h_i'')\}, \quad (\text{resp. } \{(h_i', -h_i'')\}), \quad (1 \leq i \leq \ell) ,$$

of $\underline{u}$, identified to a second diagonal (see (1)). We have

(6) $$\begin{aligned} &(\theta(\gamma')-\gamma')(h_i', -h_i'') = -2\gamma(h_i) , \\ &(\theta(\gamma'')-\gamma'')(h_i, -h_i) = 2\gamma(h_i) , \qquad (\gamma \in \Phi(\underline{k}_c, \underline{t}_c), \ 1 \leq i \leq \ell) \end{aligned}$$

Therefore

(7) $$C^{++} = \Phi'^{+} \quad (\text{resp. } C^{++} = \Phi''^{+}) .$$

$\lambda_\rho$ is the highest weight of a fundamental representation of $\underline{k}'_c$ (resp. $\underline{k}''_c$). To prove that $m_\rho \geq \ell$, it suffices therefore to show that a (non-trivial) highest weight of a simple algebra of rank $\ell$ is non-orthogonal to at least $\ell$-roots. This is well-known. It follows e.g. from the fact that a Cartan subalgebra always acts faithfully on the nilpotent radical of a proper parabolic subalgebra.

2.7. We now prove 2.2 in the last remaining case, where $\underline{g}$ is absolutely simple and $\underline{u} \neq 0$. Then $\underline{k}$ is semi-simple. Let $\underline{z}$ be the centralizer of $\underline{u}$ in $\underline{g}$ and $\underline{z}'$ its derived algebra. Then $\underline{z}$ is reductive in $\underline{g}$ and $\underline{z}'$ is the semi-simple part of $\underline{z}$. The Cartan involution $\theta$ leaves $\underline{z}$ (resp. $\underline{z}'$) stable and induces on it a Cartan involution. In particular $\underline{k} \cap \underline{z}'$ is a maximal compact subalgebra of $\underline{z}'$. The algebra $\underline{h}$ is a Cartan subalgebra of $\underline{z}$, hence $\underline{t}' = \underline{t} \cap \underline{z}'$ is one of $\underline{z}'$, and therefore

(1) $$\mathrm{rk}\, \underline{z}' = \mathrm{rk}\, (\underline{k} \cap \underline{z}') \ .$$

If $\underline{a}$ is a maximal subalgebra of $\underline{p}$ containing $\underline{u}$, then $\underline{z}' \cap \underline{a}$ is a maximal subalgebra of $\underline{p} \cap \underline{z}'$, therefore

(2) $$\mathrm{rk}_{\mathbb{R}}\, \underline{z}' = \mathrm{rk}_{\mathbb{R}}\, \underline{g} - \dim \underline{u} \ .$$

We may assume the ordering on $\underline{h}^*_c$ defined by a basis whose first elements span $i.\underline{t}'$.

We claim that $\lambda' = \lambda_a \lfloor \underline{t}'$ is a non-trivial highest weight for $\underline{z}'$. The set $A \cup B$ may be identified with the set of roots of $\underline{z}'_c$ with respect to $\underline{t}'_c$, hence no element of $A \cup B$ restricts to zero on $\underline{t}'_c$. Then, our convention on the ordering implies that the elements of $A^+ \cup B^+$ restrict to the positive roots of $\underline{z}'$. The linear form $\lambda'$ is a weight, since $\lambda$ is one. If $\gamma \in A \cup B$, then $(\lambda', \gamma) = (\lambda_a, \gamma)$, hence $\lambda'$ is dominant. The highest weight $\lambda_a$ can be written as a linear combination $\lambda_a = \Sigma_\beta c_\beta . \beta$, where the $c_\beta$ are $> 0$. Since the restriction of any $\beta$ to $\underline{t}'$ is $\geqq 0$, and some $\beta$ has a non-zero restriction it follows that $\lambda' \neq 0$. This proves our claim.

Since $\theta$ restricts to a Cartan involution of $\underline{z}'$, we see that A and B are the subsets of $\Phi(\underline{z}'_c, \underline{t}'_c)$ denoted in this way in 1.3, if we start from $\underline{z}'$. In view of (1), we can apply 2.5; hence

(3) $$\mathrm{Card}\{\gamma \in B^+ \mid (\lambda_a, \gamma) \neq 0\} \geqq \mathrm{rk}_{\mathbb{R}}\, \underline{z}' = \mathrm{rk}_{\mathbb{R}}\, \underline{g} - \dim \underline{u} \ .$$

To prove 2.3 in the present case, it suffices therefore to show:

(4) $$\mathrm{Card}\{\gamma \in C^{++} \mid (\lambda_a, \gamma) \neq 0\} \geqq \dim \underline{u} \ .$$

If $\gamma \in \Phi$, we have

(5) $$(\theta(\gamma) - \gamma)|_{\underline{u}} = -2.\gamma|_{\underline{u}} \ .$$

The set of restrictions of roots to $\underline{u}$ spans $\underline{u}^*$ (otherwise $\underline{g}$ would have a non-trivial center).

Let $\Delta$ be the set of simple roots. Since $\underline{k}$ contains regular elements, no root restricts to zero on $\underline{t}$, hence $\theta$ leaves $\Phi^+$ stable. Then it also leaves $\Delta$ stable. We assume the roots numbered in such a way that $\alpha_1, \ldots, \alpha_r$ are fixed under $\theta$ and $\alpha_{r+2i-1}, \alpha_{r+2i}$ are permuted by $\theta(1 \leq i \leq (\ell-r)/2)$. Then

(6) $$\alpha_i|_{\underline{u}} = 0 (i \leq r), \ \beta_i = \alpha_{r+2i-1}|_{\underline{u}} = -\alpha_{r+2i}|_{\underline{u}} \neq 0, \ (1 \leq i \leq (\ell-r)/2) \ .$$

In fact the $\beta_i$ span $\underline{u}^*$, hence $\dim \underline{u} = (\ell-r)/2$. For $\beta, \gamma \in \Delta$ let $d(\beta, \gamma)$ be the set of vertices of the path in the Dynkin diagram $D(\Phi)$ of $\Phi$ with end points $\beta, \gamma$. We may assume further the simple roots numbered so that $\alpha_{r+2i} \notin d(\alpha, \alpha_{r+2i-1})$ $(1 \leq i \leq (\ell-r)/2)$. We recall that if $J$ is the set of vertices of some path in $D(\Phi)$, then the sum of the elements of $J$ is a root. From this it follows that it is possible to find roots $\gamma_1, \ldots, \gamma_s$, $s = \dim \underline{u}$ such that $\gamma_i$, when expressed as sum of simple roots, contains $\alpha$ and exactly one of $\alpha_{r+2i-1}, \alpha_{r+2i}$, but not both $(i = 1, \ldots, s)$. Then these roots belong to $C^+$ and are not orthogonal to $\lambda_\alpha$. Moreover $\theta(\gamma_i) - \gamma_i$ form a basis of $\underline{u}^*$. We may take the elements of the dual basis of $\underline{u}$ as the last vectors to define the lexicographic ordering on $\underline{h}_c^*$. Then the elements $\gamma_i$ are in $C^{++}$; this proves (4).

2.8. The case of $H^1$. Assume $\underline{g}$ to be simple non-compact and $(\rho, E)$ to be irreducible non-trivial. It follows from 2.2 that $H^1(\underline{g}, \underline{k}; H \otimes E) = 0$ if $\mathrm{rk}_{\mathbb{R}}\, \underline{g} \geqq 2$. This does not give any information in the split-rank one case. However, the positive definiteness of $F_{\rho, 1}$ had been studied in [5], where it is shown that $m_\rho \geqq 2$ except in the following cases:

(i) $\underline{g} = \underline{so}(n, 1)$, $(n \geqq 2)$, and $\lambda_\rho = m.\lambda_0$ $(m = 1, 2, \ldots)$, where $\lambda_0$ is the highest weight of the standard representation of $\underline{g}$.

(ii) $\underline{g} = \underline{su}(n, 1)$ $(n \geqq 1)$, and $\lambda_\rho$ is a multiple of the highest weight of the standard representation of $\underline{g}$ or of the contragredient of the standard representation.

Except in those cases, we have therefore $H^1(\underline{g}, \underline{k}; H \otimes E) = 0$ for every admissible unitary $(\underline{g}, K)$-module $H$, by 1.2. In particular, if $E = \underline{g}$ and $\rho$ is the adjoint representation, then we are not in the cases (i), (ii) except when $\underline{g} = \underline{so}(2, 1) = \underline{su}(1, 1)$, hence

$$H^1(\underline{g}, \underline{k}; H \otimes \underline{g}) = 0 \quad \text{if} \quad \underline{g} \neq \underline{so}(2, 1) \ .$$

This is the representation theoretic analogue of Weil's rigidity theorem.

## §3. A vanishing theorem for trivial $\rho$

<u>In this section, we assume that $\underline{g}$ has no compact factor.</u>

3.1. Let $L(\ ,\ )$ be the symmetric bilinear form on $\underline{k}$ defined by

$$(1) \qquad L(x, y) = \mathrm{tr}(\mathrm{ad}_{\underline{p}}\, x \circ \mathrm{ad}_{\underline{p}}\, y) \ , \qquad (x, y \in \underline{k}) \ .$$

We have

$$(2) \qquad L(x, y) = B_{\underline{k}}(x, y) + B(x, y) \ , \qquad (x, y \in \underline{k}) \ ,$$

where $B_{\underline{k}}$ (resp. $B$) is the Killing form of $\underline{k}$ (resp. $\underline{g}$). The eigenvalues of the endomorphisms $\mathrm{ad}\, x$ $(x \in \underline{k})$ are purely imaginary and our assumption on $\underline{g}$ insures that $\underline{k}$ acts faithfully on $\underline{p}$ via the adjoint representation. Hence $L(\ ,\ )$ is negative non-degenerate. We let

$$A = \min_{x \in \underline{k},\, B(x)=-1} - L(x, x) \quad . \tag{3}$$

Then $0 < A \leqq 1$.

We use the notation and conventions of (II, §1). We have, taking (II, 1.2) into account

$$L(X_a, X_b) = \Sigma_{i,j}\, c^i_{aj} \cdot c^j_{bi} = \Sigma_{i,j}\, c^a_{ij} \cdot c^b_{ji} \ , \tag{4}$$

$$L(X_a, X_b) = -\Sigma_{ij}\, c^a_{ij} \cdot c^b_{ij} \ , \qquad (m < a,\ b \leqq n) \ . \tag{5}$$

In the sequel, we assume moreover that the $X_a$'s form an orthogonal basis with respect to $L(\ ,\ )$. Set

$$R(X, Y) = -\mathrm{ad}[x, y]_{\underline{p}} \ , \qquad (x, y \in \underline{p}) \ , \tag{6}$$

hence

$$R(x, y) . z = [[y, x], z] \ , \qquad (x, y, z \in \underline{p}) \ , \tag{7}$$

and put

$$R_{ijk\ell} = B([[x_\ell, x_k], x_j], x_i] = B([x_\ell, x_k], [x_j, x_i]) \ . \tag{8}$$

Therefore

(9) $$R_{ijk\ell} = -\Sigma_a\, c^a_{k\ell} \cdot c^a_{ij} \; .$$

As is well-known, $R(\ ,\ )$ is the curvature tensor on $G/K$, for the invariant Riemannian metric which, on $\underline{p} = T(G/K)_e$, is equal to the restriction of the Killing form [2, p. 180]. However, this interpretation will not be needed here.

3.2. The form $F^q_{\underline{g}}$. We denote by $\eta_{ij}$ the coordinates of an element $\eta \in \underline{p} \otimes \underline{p}$ with respect to the basis $x_i \otimes x_j$ $(1 \leq i, j \leq m)$, and put, for $q = 1, 2, \ldots$

(1) $$F^q_{\underline{g}}(\xi, \eta) = (A/2q).\Sigma_{ij}\, \xi_{ij} \cdot \eta_{ij} + \Sigma_{ijk\ell}\, R_{ijk\ell}\, \xi_{i\ell}\, \eta_{jk}$$

with $A$ given by 3.1(3). Let

(2) $$m(\underline{g}) = \max(\{0\} \cup \{q \,|\, F^q_{\underline{g}} > 0\}) \; .$$

3.3. THEOREM. *Let* $(\pi, V)$ *be a unitary* $(\underline{g}, \underline{k})$*-module on which the Casimir element acts by a scalar multiple of the identity and such that* $V^{\underline{g}} = 0$. *Then* $H^q(\underline{g}, \underline{k}; V) = 0$ *for* $q \leq m(\underline{g})$.

The assumptions on $V$ are satisfied if $(\pi, V)$ is irreducible, admissible and non-trivial. Therefore

3.4. COROLLARY. *If* $(\sigma, H)$ *is a non-trivial irreducible admissible unitary* $(\underline{g}, \underline{k})$*-module, then* $H^q(\underline{g}, \underline{k}; H) = 0$ *for* $q \leq m(\underline{g})$.

3.5. This theorem is the representation theoretic analogue of a theorem of Matsushima on the cohomology of cocompact discrete subgroups [4], to be discussed in IV. The proof given here is essentially the same as Matsushima's.

Theorem 3.3 also applies to any admissible unitary $(\underline{g}, K)$-module $(\pi, V)$ for which $V^{\underline{g}} = 0$. In fact, $V$ is then a direct sum of primary subspaces with respect to the Casimir element $C$. Moreover, using unitarity and admissibility, one sees that $C$ acts by scalars on each of those; this reduces us to the theorem.

3.6. Proof of 3.3. If $C$ acts non-trivially on $V$, then $H^q(\underline{g}, \underline{k}; V) = 0$ for all $q$'s by (II, 3.1). From now on, we assume that $\pi(C) = 0$. We shall prove that if there exists $q \leqq m(\underline{g})$ such that $H^q(\underline{g}, \underline{k}; V) \neq 0$, then $V^{\underline{g}} \neq 0$. If $q = 0$, this is clear. So let $q \geqq 1$. Since $\pi(C) = 0$, all cochains are closed, harmonic and $H^q(\underline{g}, \underline{k}; V) = C^q(\underline{g}, \underline{k}; V)$ (II, 3.1). We have then to show that if

(1) $$\eta = \Sigma_I \, \eta_I . \omega^I ,$$

is a $q$-cochain, then $\eta_I \in V^{\underline{g}}$, i.e.

(2) $$X_i \eta_I = X_a \eta_I = 0 ,$$

for all $i$, $a$, $I$ subject to our conditions. Since $[\underline{p}, \underline{p}] = \underline{k}$, it suffices in fact to prove

(3) $$X_i \eta_I = 0 \qquad (1 \leqq i \leqq m;\ I \subset I_m,\ |I| = q)$$

That (3) implies (2) also follows from

$$(4) \qquad 0 = (C\eta_I,\ \eta_I) = \Sigma_a \|X_a\eta_I\|^2 - \Sigma_i \|X_i\eta_I\|^2 .$$

which, incidentally, also shows that if $v \in V^{\underline{k}}$, then $v \in V^{\underline{g}}$. In the sequel, $u$ runs from 1 to $q$, $i, j, k, \ell, j_u$ from 1 to $m$, $a, b, c$ from $m+1$ to $n$, and $I$ through the subsets of $q$ elements of $I_m = \{1, 2, \ldots, m\}$. Let

$$(5) \quad \Phi(\eta) = \frac{(q-1)!}{2}\Sigma_{i,j,I}\|[X_i,\ X_j]\eta_I\|^2 = (2q)^{-1}.\Sigma_{\substack{i,j\\ j_1,\ldots,j_q}} \|[X_i,\ X_j]\eta_{j_1\ldots j_q}\|^2 .$$

We shall transform $\Phi(\eta)$ in two ways. First, using $[x_i,\ x_j] = \Sigma\, c_{ij}^a\, x_a$, we can write

$$(6) \qquad \Phi(\eta) = \frac{(q-1)!}{2}\Sigma_{a,b,i,j,I}\, c_{ij}^a . c_{ij}^b (x_a\eta_I,\ x_b\eta_I) .$$

In view of 3.1(5), this gives

$$(7) \qquad \Phi(\eta) = -\frac{(q-1)!}{-2}\Sigma_{a,b,I}\, L(x_a,\ x_b).(x_a\eta_I,\ x_b\eta_I) .$$

Since the $x_a$'s are assumed to be orthogonal with respect to $L$, the sum is in fact over $a = b$ and, by the definition of $A$ (3.1, (3)), we have

$$(8) \qquad \Phi(\eta) \geqq \frac{A.(q-1)!}{2}\Sigma_{a,I}\|X_a\eta_I\|^2 .$$

If we use the formula for $[x_i,\ x_j]$ on one term of each of the scalar products in (5), we get

$$(9) \qquad \Phi(\eta) = (2q)^{-1}\Sigma_{\substack{i,j,a\\ j_1,\ldots,j_q}}\, c_{ij}^a (x_a.\eta_{j_1\ldots j_q},\ [x_i,\ x_j].\eta_{j_1\ldots j_q}) .$$

Since $c_{ij}^{a}$ and $[x_i, x_j]$ are antisymmetric in i, j, this gives

(10) $$\Phi(\eta) = q^{-1}\Sigma_{\substack{i,j,a\\ j_1,\ldots,j_q}} c_{ij}^{a}(x_a.\eta_{j_1\ldots j_q}, x_i.x_j.\eta_{j_1\ldots j_q}) .$$

By assumption, $\eta \in C^q(\underline{g}, \underline{k}; V)$. Therefore

$$x_a.\eta_{j_1\ldots j_q} = \Sigma_u\, \eta(x_{j_1}, \ldots, [x_a, x_{j_u}], \ldots, x_{j_q}) =$$

$$= \Sigma_{a,k,u}\, c_{a,j_u}^{k}\eta(x_{j_1}, \ldots, x_k, \ldots, x_{j_q}) =$$

$$= \Sigma_{a,k,u}(-1)^{u-1}.c_{a,j_u}^{k}.\eta(x_{j_1}, x_{j_1}, \ldots, \hat{x}_{j_u}, \ldots, x_{j_q}) .$$

Then we have, using (II, 1.2)

$$q.\Phi(\eta) = \Sigma_{\substack{i,j,k,u\\ j_1,\ldots,j_q}} (-1)^{u-1}(\Sigma_a c_{ij}^{a}.c_{kj_u}^{a})(\eta_{k,j_1,\ldots,j_u,\ldots,j_q}, x_i.x_j.\eta_{j_1\ldots j_q}) .$$

Since $(\pi, V)$ is unitary, we have

$$(\eta_{k,j_1,\ldots,\hat{j}_u,\ldots,j_q}, x_i.x_j.\eta_{j_1\ldots j_q}) = -(x_i\eta_{k,j_1,\ldots,\hat{j}_u,\ldots,j_q})x_j.\eta_{j_1,\ldots,j_q}) .$$

Taking 3.1(8) into account, we get

$$q\Phi(\eta) = \Sigma_{\substack{i,j,k,u\\ j_1,\ldots,j_q}} (-1)^{u-1}R_{ijkj_u}(x_i.\eta_{k,j_1,\ldots,\hat{j}_u,\ldots,j_q}, x_j.\eta_{j_1,\ldots,j_q}) .$$

$$q\Phi(\eta) = \Sigma_{\substack{i,j,k,u\\ j_1,\ldots,j_q}} R_{ijkj_u}(x_i.\eta_{k,j_1,\ldots,\hat{j}_u,\ldots,j_q}, x_j.\eta_{j_u,j_1,\ldots,\hat{j}_u,\ldots,j_q}) .$$

This can be written

$$q\Phi(\eta) = q\Sigma_{\substack{i,j,k,\ell \\ j_2,\ldots,j_q}} R_{ijk\ell}(x_i\eta_{k,j_2,\ldots,j_q}, x_j\eta_{\ell,j_2,\ldots,j_q}) .$$

Since $R_{ijk\ell}$ is antisymmetric in the last two indices (see 3.1(8)), we get finally

$$(11) \qquad \Phi(\eta) = -\Sigma_{\substack{i,j,k,\ell \\ j_2,\ldots,j_q}} R_{ijkk}(x_i\eta_{k,j_2,\ldots,j_q}, x_j\eta_{\ell,j_2,\ldots,j_q}) .$$

Together with (8), this yields

$$(12) \quad \Sigma_{j_2,\ldots,j_q}\left\{\frac{A}{2q}\Sigma_{i,j}\|x_j\eta_{i,j_2,\ldots,j_q}\|^2 + \Sigma_{i,j,k,\ell}(x_i\eta_{k,j_2,\ldots,j_\ell}, x_j\eta_{\ell,j_2,\ldots,j}\right.$$

On $\underline{p} \otimes \underline{p} \otimes V$, we consider the tensor product $F^q_{\underline{g},V}$ of $F^q_{\underline{g}}$ and of the given scalar product on $V$. It is positive non-degenerate since $q \leqq m(\underline{g})$. The inequality (10) can now be written

$$(13) \qquad \Sigma_j F^q_{\underline{g},V}(\{\eta_{j,i\cup J}\}) \leqq 0 ,$$

where $J$ runs through the subsets of $I_m = \{1, \ldots, m\}$ having $(q-1)$-elements. Since $F^q_{\underline{g},V}$ is positive non-degenerate, we get

$$(14) \qquad x_j \cdot \eta_{i\cup J} = 0 , \qquad (1 \leqq i, j \leqq m; J \subset I_m, |J| = q-1)$$

which is just (3).

<u>3.7</u>. The value of $m(\underline{g})$ for $\underline{g}$ simple non-compact has been determined case by case (see [3; 4]). In particular, we have $m(\underline{g}) \geqq 1$ if $\underline{g}_c$ is not of type $\mathbb{G}_2$ and if the split rank of $\underline{g}$ is $\geqq 2$ [3: Thm. 4.1]. In fact, it is known that $H^1(\underline{g}, \underline{k}; V) = 0$ for $V$ irreducible unitary, non-trivial and $\underline{g}$ simple

non-compact not of type $\underline{so}(n+1, 1)$ or $\underline{su}(n, 1)$ $(n \geq 1)$. A proof and references will be given later, together with the determination of those $V$ in the exceptional cases which do produce cohomology in dimension one.

## §4. Direct products

4.1. Let $\underline{g} = \underline{g}' \oplus \underline{g}''$ be a direct product. Assume first $\underline{g}''$ to be compact. Then, if $(\pi, V)$ is any $(\underline{g}, \underline{k})$-module, we have

$$(1) \qquad H^q(\underline{g}, \underline{k}; V) = H^q(\underline{g}', \underline{k}'; V^{\underline{g}''}) , \qquad (q \geq 0)$$

where $\underline{k}' = \underline{k} \cap \underline{g}'$, and hence $\underline{k} = \underline{k}' \oplus \underline{g}''$. In fact, $\underline{g}''$ operates trivially on $\underline{g}/\underline{k} = \underline{p} = \underline{g}'/\underline{k}'$, therefore

$$(2) \qquad \mathrm{Hom}_{\underline{k}}(\Lambda^q \underline{p}, V) \xrightarrow{\sim} \mathrm{Hom}_{\underline{k}'}(\Lambda^q \underline{p}, V^{\underline{g}''}) ,$$

i.e. we have canonical isomorphisms

$$(3) \qquad C^q(\underline{g}, \underline{k}; V) \xrightarrow{\sim} C^q(\underline{g}', \underline{k}', V^{\underline{g}''}) \qquad (q \geq 0) .$$

This yields (1). Note also that, since $\underline{g}'' \subset \underline{k}$, the module $V$ is locally finite and semi-simple with respect to $\underline{g}''$, so that $V = V^{\underline{g}''} \oplus V'$, where $\underline{g}''.V' = 0$, and $V^{\underline{g}''}$, $V'$ are both stable under $\underline{g}$. This reduces us to the case where $\underline{g}$ has no compact factor.

4.2. Assume now that $\underline{g} = \underline{g}_1 \oplus \ldots \oplus \underline{g}_s$, with $\underline{g}_s$ simple non-compact. Write accordingly

(1) $\underline{k} = \underline{k}_1 + \ldots + \underline{k}_s$, $\underline{p} = p_1 \oplus \ldots \oplus p_s$, where $k_i = \underline{g}_i \cap \underline{p}$, $\underline{p}_i = g_i \cap \underline{p}$ $(1 \leqq i \leqq s)$.

Let $(\rho, E)$ be irreducible. Then

(2) $$E = E_1 \otimes \ldots \otimes E_s,\ \rho = \rho_1 \otimes \ldots \otimes \rho_s\ ,$$

where $(\rho_i, E_i)$ is an irreducible $\underline{g}_i$-module. If $H$ is also a tensor product of $(\underline{g}_i, \underline{k}_i)$-modules, then we can apply Künneth rule (I, 1.3). Let us put $m_{\rho_i} = 0$ if $\rho_i$ is trivial. Then, under the present assumptions, we have

$$H^q(\underline{g}, \underline{k}; H \otimes E) = 0 \text{ for } q < \Sigma_i\, m_{\rho_i}\ .$$

In particular, if $\rho$ is faithful, then $m_{\rho_i} \geqq 1$ for all i's, and we have vanishing at least up to $s-1$.

4.3. We keep the assumption of 4.2. Let $(\sigma, H)$ be an irreducible unitary $(\underline{g}, K)$-module. If $H^q(\underline{g}, \underline{k}; H) = 0$ for some $q$, then the infinitesimal and central character of $H$ are trivial (I, §§4, 5). There is therefore no restriction in assuming that $G$ is adjoint. Then $G = G_1 \times \ldots \times G_s$ where $G_i = \mathrm{ad}\ \underline{g}_i$. Let $K_i$ be the maximal compact subgroup of $G_i$ with Lie algebra $\underline{k}_i$. Then we have a tensor product decomposition

(1) $$H = H_1 \otimes \ldots \otimes H_s,\ \sigma = \sigma_1 \otimes \ldots \otimes \sigma_s\ ,$$

where $(\sigma_i, H_i)$ is an admissible irreducible unitary $(\underline{g}_i, K_i)$-module. To compute $H^q(\underline{g}, \underline{k}; H)$ we can then again apply Künneth rule. Define $M(\underline{g})$

to be the greatest integer such that $H^q(\underline{g}, \underline{k}; H) = 0$ for $q \leqq M(\underline{g})$ and all non-trivial irreducible unitary admissible $(\underline{g}, K)$-modules. Then the Künneth rule and (1) imply

4.4. PROPOSITION. Let $I$ be the set of indices for which $(\sigma_i, H_i)$ is not trivial. Then $H^q(\underline{g}, \underline{k}; H) = 0$ for $q < \Sigma_{i \in I} (M(\underline{g}_i)+1)$.

In particular, we see that if $H$ is faithful, then $H^q(\underline{g}, \underline{k}; H) = 0$ at least for $q < s$. This gives the vanishing of $H^1$ as soon as $\underline{g}$ has at least two non-compact factors, and $H$ is faithful. For a faithful $H$, we have then vanishing at least up to $\Sigma(m(\underline{g}_i)+1) - 1$.

## REFERENCES


[1] N. Bourbaki, Groupes et Algèbres de Lie, Chap. IV, V, VI, Act. Sci. Ind. 1337, Hermann, Paris 1968.

[2] S. Helgason, Differential geometry and symmetric spaces, Academic Press, New York 1962.

[3] S. Kaneyuki and T. Nagano, "On certain quadratic forms related to symmetric Riemannian spaces," Osaka M. J. 14 (1962), 241-252.

[4] Y. Matsushima, "On Betti numbers of compact locally symmetric Riemannian manifolds," Osaka M. J. 14 (1962), 1-20.

[5] M. S. Raghunathan, "On the first cohomology of discrete subgroups of semisimple Lie groups," Amer. J. Math. 87 (1965), 103-139.

[6] ——————, "Vanishing theorems for cohomology groups associated to discrete subgroups of semisimple Lie groups," Osaka J. Math. 3 (1966), 243-256.

## Appendix to Chapter III: "Vanishing Theorems" Using Spinors

N. R. Wallach

This appendix continues the philosophy of Chapter III. That is, the vanishing theorems proved for $L^2(\Gamma\backslash G)$ can be interpreted as vanishing theorems for relative Lie algebra cohomology. We extend the vanishing theorems of Hotta, Parthasarathy [6] to cover the case rank $K \neq$ rank $G$ (see 3.8). We also prove a mildly surprising vanishing theorem (3.2). In Section 4 we show how to prove a best possible vanishing theorem for $H^1(\underline{g}, \underline{k}; H)$ (see 4.7). Lemma 2.12 is based on ideas of Schmid [10].

### §1. Spinors.

1.1. Let $(V, \langle\,,\,\rangle)$ be a finite-dimensional, real inner product space. We will take $\langle\,,\,\rangle$ to be understood when we write $V$. Let $\underline{so}(V) = \{x \in \mathrm{End}(V) \mid \langle xv, w\rangle = -\langle v, xw\rangle\ ,\ v, w \in V\}$.

1.2. THEOREM. *There exists a unique (up to equivalence) finite-dimensional unitary $\underline{so}(V)$-module $(\sigma, S)$ satisfying the following properties.*

1) *There is an $\underline{so}(V)$-module homomorphism $V_{\mathbb{C}} \otimes S \to S$ given by*

$$x \otimes S \mapsto \gamma(x)S \quad (\gamma(x) \in \mathrm{End}(S)).$$

2) $\gamma(x)^2 = -\langle x, x\rangle I$ *for* $x \in V$.

3) *If* $Z \subset S$ *is* $\gamma(x)$ *invariant for all* $x \in V$, *then* $Z = (0)$ *or* $Z = S$.

4) *If* $u, v \in S$, $x \in V$, *then* $\langle\gamma(x)u, v\rangle = -\langle u, \gamma(x)v\rangle$.

**1.3.** To prove 1.2 we introduce a bit of notation. Let $T(V)$ be the tensor algebra on $V$ $(T(V) = \bigoplus_{k=0}^{\infty} \otimes^k V)$ and set $C(V) = T(V)/ \sum_{v,w\in V} T(V)(v \otimes w + w \otimes v + 2\langle v,w\rangle 1)T(V)$. Then $C(V)$ is called the Clifford algebra on $V$.

$C(V)$ has the following universal mapping property: If $\mathfrak{A}$ is an associative algebra with unit $1$ and if $\mu : V \to \mathfrak{A}$ is a linear map such that $\mu(x)^2 = -\langle x,x\rangle 1$; then there is a unique algebra homomorphism $\hat{\mu} : C(V) \to \mathfrak{A}$ such that $\hat{\mu} \circ j = \mu$. Here $j(x) = \bar{x}$, $x \in V$, where $u \mapsto \bar{u}$ is the natural map of $T(V)$ onto $C(V)$.

**1.4.** We first assume that $\dim V = 2r$, $r \geq 1$, $r \in \mathbb{Z}$. Let $v_1, \ldots, v_{2r}$ be an orthonormal basis of $V$. Let $\mu : V \to \otimes^r M_2(\mathbb{C})$ ($M_2(\mathbb{C})$ the $2 \times 2$-matrices over $\mathbb{C}$) be defined by

$$\mu(v_{2j-1}) = \underbrace{I' \otimes I' \otimes \ldots \otimes I'}_{j\text{-}1 \text{ factors}} \otimes J \otimes I \otimes \ldots \otimes I$$

$$\mu(v_{2j}) = \underbrace{I' \otimes I' \otimes \ldots \otimes I'}_{j\text{-}1 \text{ factors}} \otimes K \otimes I \otimes \ldots \otimes I$$

where $I' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ and extend $\mu$ to be real linear. It is not hard to check that $\mu(v_i)\mu(v_j)+\mu(v_j)\mu(v_i) = -2\delta_{ij}I$. We therefore have $\mu : V \to \otimes^r M_2(\mathbb{C})$ and $\mu(x)^2 = -\langle x,x\rangle I$. Using the universal mapping property of $C(V)$ we get that $\hat{\mu} : C(V) \to \otimes^r M_2(\mathbb{C})$ is an algebra homomorphism. Extending $\hat{\mu}$ to $C(V) \otimes_{\mathbb{R}} \mathbb{C}$ it is easy to check that $\hat{\mu}$ is surjective. On the other hand $C(V)$ is clearly spanned by the elements $\bar{v}_{i_1} \ldots \bar{v}_{i_s}$, $1 \leq i_1 < \ldots < i_s \leq 2r$. Hence $\dim C(V) \otimes_{\mathbb{R}} \mathbb{C} \leq 2^{2r} = 4^r = \dim \otimes^r M_2(\mathbb{C})$.

This clearly implies that $C(V) \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic with $\otimes^r M_2(\mathbb{C})$. Since $\otimes^r M_2(\mathbb{C})$ is isomorphic with $M_{2^r}(\mathbb{C})$ (the $2^r \times 2^r$ matrices) we have shown

1.5. LEMMA. *If* $\dim V = 2r$, *then* $C(V) \otimes_{\mathbb{R}} \mathbb{C}$ *is isomorphic with* $M_{2^r}(\mathbb{C})$.

1.6. *Proof of* 1.2 *for* $\dim V = 2r$. Let $\hat{\mu} : C(V) \otimes_{\mathbb{R}} \mathbb{C} \to M_2(\mathbb{C})$ be defined as in 1.4. If we take $S = \mathbb{C}^{2^r}$, then $\hat{\mu} : C(V) \to \mathrm{End}(S)$. Clearly $\hat{\mu}(\bar{x})^2 = -\langle x, x\rangle I$ for $x \in V$. Thus if $\gamma(x) = \hat{\mu}(\bar{x})$, 1), 2) of 1.2 are satisfied. Since $\hat{\mu}(C(V) \otimes_{\mathbb{R}} \mathbb{C})$ is the subalgebra of $M_{2^r}(\mathbb{C})$ generated by $\gamma(v)$ we see that 3) is satisfied.

Let $E_{ij}v_k = \delta_{ij}v_i$, $E_{ij} \in \mathrm{End}(V)$. Define $\sigma(E_{ij}-E_{ji}) = -\frac{1}{2}\gamma(v_i)\gamma(v_j)$; then 1) is satisfied by $\sigma$, $\gamma$.

Since $C(V) \otimes_{\mathbb{R}} \mathbb{C}$ is a simple algebra over $\mathbb{C}$ we see that $C(V) \otimes_{\mathbb{R}} \mathbb{C}$ has a unique, simple module over $\mathbb{C}$ (up to isomorphism). Taking $S = \otimes^r \mathbb{C}^2$ put on $\mathbb{C}^2$ the usual inner product and put on $S$ the tensor product inner product. Then we leave it to the reader to check that 4) of 1.2 is satisfied.

1.7. *The odd-dimensional case.* Suppose $\dim V = 2r+1$. Let $v_0 \in V$ be a unit vector. Set $V_1 = \{v \in V \mid \langle v_0, v\rangle = 0\}$. Let $(\sigma, S)$ be as in 1.6 for $so(V_1)$. Define $\gamma(v_0) = iI' \otimes \dots \otimes I'$ ( $r$ factors) (see 1.4). Then $\gamma(v_0)\gamma(v) + \gamma(v)\gamma(v_0) = 0$ for $v \in V_1$. Extend $\gamma$ by linearity to $V$. Set $E_{0i}v_j = \delta_{ij}v_0$, $E_{i0}v_j = \delta_{0j}v_i$ and $\sigma(E_{0i}-E_{i0}) = -\frac{1}{2}\gamma(v_0)\gamma(v_i)$, $i = 1, \dots, 2r$. Then $(\sigma, S)$ satisfies 1), 2), 3), 4) of 1.2.

To complete the proof of 1.2 we need only show that if $(\sigma, S)$ and $\gamma$ satisfy 1), 2), 3) of 1.2 for $V$, then $(\sigma, S)$ and $\gamma$ restricted to $so(V_1)$ and $V_1$ satisfy 1), 2), 3) of 1.2 for $V_1$. Clearly 3) is the only part that need be checked. Set $P = i\gamma(v_0)$. Then $P^2 = I$. Let $S^{\pm} = \{s \in S \mid Ps = \pm s\}$. If $x \in V_1$, then $\gamma(x) \circ P = -P \circ \gamma(x)$. Hence $\gamma(x)S^+ \subset S^-$ and $\gamma(x)S^- \subset S^+$ for $x \in V_1$. 1) implies that $\sigma(Z)S^+ \subset S^+$, $\sigma(Z)S^- \subset S^-$ for $Z \in so(V_1)$ $(Zv_0 = 0$ for $Z \in so(V_1))$. We assert that $S^+$ and $S^-$ are simple as $so(V_1)$-modules. Indeed, suppose that $0 \neq w \subset S^+$ is $so(V_1)$-invariant. If $x \in V_1$, if $w \in W$ and if $Z \in so(V_1)$, then $\sigma(Z)\gamma(x)w = \sigma(Z\cdot x)w + \gamma(x)\sigma(Z)w$. This implies that $W + \gamma(V_1)w$ is $\gamma(V_1)$-invariant. Since $W \subset S^+$, $\gamma(V_1)W \subset S^-$, this implies that $W + \gamma(V_1)W$ is $\gamma(V)$-invariant. Hence $W = S^+$ by 3) in 1.2. Similarly, $S^-$ is a simple $\underline{so}(V_1)$-module.

1) $S^+$ and $S^-$ are inequivalent as $so(V_1)$-modules.

Indeed, set $h_j = E_{2j-1,2j} - E_{2j,2j-1}$, $j = 1, \ldots, r$. Set $\underline{h} = \Sigma \mathbb{R} h_j$. Then $\underline{h}$ is a maximal abelian subalgebra of $so(V_1)$. Set $\lambda_i(h_j) = \delta_{ij}$, $1 \leq i, j \leq r$. Choose the Weyl chamber $\lambda_1 > \lambda_2 > \ldots > \lambda_r$. Let $\Lambda$ be the highest weight of $S^+$ relative to the Weyl chamber of $\underline{h}$. Let $w_0$ be a nonzero highest weight vector. Set $e_j = v_j - iv_{j+r}$, $j = 1, \ldots, r$, $e_{j+r} = v_j + iv_{j+r}$. Then $h_k e_j = i\delta_{jk} e_j$, $h_k e_{j+r} = -i\delta_{jk} e_{j+r}$, $1 \leq j \leq r$. Hence $h \cdot e_j = i\lambda_j(h) e_j$, $h \cdot e_{j+r} = -i\lambda_j(h) e_{j+r}$, $1 \leq j \leq r$. We therefore see that $\gamma(e_j) w_0 = 0$, $j = 1, \ldots, r$. Since $\gamma(v_0) w_0 = -iw_0$ we find that $S$ is spanned by the elements $w_{j_1 \ldots j_k} =$ $\gamma(e_{r+j_1}) \ldots \gamma(e_{r+j_k}) w_0$, $1 \leq j_1 < \ldots < j_k \leq r$. If $h \in \underline{h}$, then

$\underline{h} \cdot w_{j_1 \ldots j_k} = i(\Lambda(h) - \sum_{i=1}^{r} \lambda_{j_i}(h)) w_{j_1 \ldots j_k})$. Clearly, $S^+$ is spanned by the $w_{j_1 \ldots j_k}$ with $k$ even and $S^-$ is spanned by the $w_{j_1 \ldots j_k}$, $k$ odd. Hence $S^+$ and $S^-$ have distinct weights relative to $\underline{h}$. This proves the contention of 1).

2) $\sigma(E_{ij} - E_{ji}) = -\frac{1}{2}\gamma(v_i)\gamma(v_j) + c_{ij} I$ for $0 \leq i < j \leq 2r$.

Indeed, set $T_{ij} = \sigma(E_{ij} - E_{ji}) + \frac{1}{2}\gamma(v_i)\gamma(v_j)$. Then 1.2, 2) implies $T_{ij} \circ \gamma(x) = \gamma(x) \circ T_{ij}$ for $x \in V$. 1.2, 3) now implies $T_{ij} = c_{ij} I$.

Suppose now that $0 \neq W \subset S$ is $\gamma(v_1)$-invariant. Then 2) implies that $W$ is $so(v_1)$-invariant. Hence $W = S^+, S^-$ or $W = S^+ \oplus S^-$. Clearly $W = S^+$ or $S^-$ is impossible $(\gamma(v_1) S^{\pm} \subset S^{\mp})$. Thus $W = S$. The theorem is now completely proven.

$\underline{\underline{1.8.}}$ Since $[\underline{so}(v), \underline{so}(v)] = \underline{so}(v)$, it is trivial to check that the $c_{ij}$ in 1.7, 2), are all 0. We can now state many of the results used in the proof of Theorem 1.2 as properties of $(\sigma, S)$, $\gamma$. We do this in the next paragraph.

$\underline{\underline{1.9.}}$ Properties of $(\sigma, S), \gamma$. Fix $v_1, \ldots, v_n$ an orthonormal basis of $V$. Let $E_{ij} \in \mathrm{End}(V)$ be defined by $E_{ij} v_k = \delta_{jk} v_i$.

1) If $i \neq j$, then $\sigma(E_{ij} - E_{ji}) = -\frac{1}{2}\gamma(v_i)\gamma(v_j)$.

2) Set $h_j = E_{2j-1, 2j} - E_{2j, 2j-1}$, $j = 1, \ldots, [\frac{n}{2}] = r$. Set $\underline{h} = \sum \mathbb{R} h_j$. Set $\lambda_i(h_j) = \delta_{ij}$. Then the weights of $\sigma(\underline{h})$ are precisely the linear forms $i(\frac{1}{2}(\lambda_1 + \ldots + \lambda_r) - \lambda_{j_1} - \ldots - \lambda_{j_k})$ with $1 \leq j_1 < \ldots < j_k \leq r$ and each occurs with multiplicity 1.

(To see this use the proof of 1.7, 1), and the fact that $\dim S = 2^r$, which was shown in 1.6.)

1.10. We will need two other properties of $(\sigma, S)$ and $\gamma$. The first will be stated in this paragraph; the second in the next.

LEMMA. Let for $1 \leq i, j, k, \ell \leq n$, $R_{ijk\ell} \in \mathbb{C}$ and satisfy

1) $R_{ijk\ell} = R_{k\ell ij}$;

2) $R_{ijk\ell} = -R_{jik\ell}$;

3) $R_{ijk\ell} + R_{kij\ell} + R_{jki\ell} = 0$.

Then $\Sigma R_{ijk\ell}\gamma(v_i)\gamma(v_j)\gamma(v_k)\gamma(v_\ell) = 2\sum_{ij}(R_{ijji})I$.

This lemma is proved by the obvious computation.

1.11. LEMMA. Let $\mu$ be the natural representation of $so(V)$ on $\Lambda V_{\mathbb{C}}$ $(V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C})$. Then

1) if $n$ is even, $\mu$ is equivalent with $\sigma \otimes \sigma$;

2) if $n$ is odd, then $\mu$ is equivalent with $\sigma \otimes \sigma \oplus \sigma \otimes \sigma$.

Proof. Use 1.9, 2), and compare the weights of $\mu$ and $\sigma \otimes \sigma$.

## §2. The Dirac operator of a unitary representation.

2.1. In this section we use the notation of III, §3. Let $C$ be the Casimir operator of $\underline{g}$. Let $C_k = -\Sigma x_a^2$.

2.2. Let $\langle , \rangle = B|\underline{p}\times\underline{p}$. Then $(\underline{p}, \langle , \rangle)$ is a finite-dimensional inner product space. Clearly, $\tau_0 : \underline{k} \to \underline{so}(\underline{p})$ where $\tau_0(y) = \mathrm{ad}\, y|\underline{p}$, $y \in \underline{k}$. Let $(\sigma, S), \gamma$ be as in 1.2 for $\underline{so}(\underline{p})$. Set $s = \sigma \circ \tau_0$. Then $(s, S)$ is a unitary representation of $\underline{k}$.

2.3. LEMMA. $s(C_k) = cI$, $c = \frac{1}{8}\sum_{i,j} R_{ijji}$ ($R_{ijk\ell}$ is as in III, 3.1, 8)).

Proof. Let $x_1, \ldots, x_m$ be an orthonormal basis of $(\underline{p}, \langle\, ,\rangle)$. Then 1.9, 1), easily implies that if $x \in \underline{k}$, then $s(x) = \frac{1}{4}\sum_{i,j} \langle [x, x_i], x_j\rangle \gamma(x_i)\gamma(x_j)$. This implies that

$$s(C_k) = -\frac{1}{16} \sum_{a,i,j,k,\ell} \langle [x_a, x_i], x_j\rangle\langle [x_a, x_k], x_\ell\rangle \cdot \gamma(x_i)\gamma(x_j)\gamma(x_k)\gamma(x_\ell)$$

$$= \frac{1}{16}\sum_{ijk\ell} \langle [x_i, x_j], [x_k, x_\ell]\rangle \gamma(x_i)\gamma(x_j)\gamma(x_k)\gamma(x_\ell)$$

$$= \frac{1}{16}\sum_{ijk\ell} R_{ijk\ell}\, \gamma(x_i)\gamma(x_j)\gamma(x_k)\gamma(x_\ell)$$

($R_{ijk\ell}$ is as in III, 3.1, 8)). Clearly, $R_{ijk\ell}$ satisfies the hypothesis of 1.10. Hence

$$s(C_k) = \frac{1}{8}\Big(\sum_{i,j} R_{ijji}\Big)I.$$

2.4. We now derive a second formula for the constant $c$ in 2.3. Let $\underline{h}^+ \subset \underline{k}$ be a maximal abelian subalgebra of $\underline{k}$. Let $\underline{h}$ be the centralizer in $\underline{g}$ of $\underline{h}^+$. Then $\underline{h}$ is a Cartan subalgebra of $\underline{g}$. Let $\Delta$ be the root system of $(\underline{g}_{\mathbb{C}}, \underline{h}_{\mathbb{C}})$ and let $\Delta_k$ be the root system of $(\underline{k}_{\mathbb{C}}, \underline{h}^+_{\mathbb{C}})$. Fix $\Delta_k^+$, a system of positive roots for $\Delta_k$. By a compatible system of positive roots for $\Delta$ we will mean a system of positive roots $\Delta^+ \subset \Delta$ such that (see III, 2.7)

1) if $\alpha \in \Delta_k^+$ then $\alpha = \beta|_{\underline{h}_{\mathbb{C}}^+}$ for some $\beta \in \Delta^+$;

2) if $\alpha \in \Delta^+$ then $\theta\alpha \in \Delta^+$ (here $\theta$ is the Cartan involution of $(\underline{g}, \underline{k})$).

2.5. Fix $\Delta^+$ a compatible system of positive roots for $\Delta$. Set $\underline{h}_{\mathbb{C}}^- = \{h \in \underline{h}_{\mathbb{C}} | \theta h = -h\}$. We identify $(\underline{h}_{\mathbb{C}}^+)^*$ with $\{\lambda \in \underline{h}_{\mathbb{C}}^* | \theta\lambda = \lambda\}$, $(\underline{h}_{\mathbb{C}}^-)^*$, with $\{\lambda \in \underline{h}_{\mathbb{C}}^* | \theta\lambda = -\lambda\}$. We write for $\lambda \in \underline{h}_{\mathbb{C}}^*$, $\lambda = \lambda^+ + \lambda^-$ $(\lambda^{\pm} = \lambda|_{\underline{h}_{\mathbb{C}}^{\pm}})$.

2.6. Set $\delta = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. By 2.4, 2), we see that $\theta\delta = \delta$; hence $\delta = \delta^+$. Set $\delta_k = \frac{1}{2} \sum_{\alpha \in \Delta_k^+} \alpha$. Set $\delta_n = \delta - \delta_k$.

2.7. Let for $\lambda \in (\underline{h}_{\mathbb{C}}^+)^*$

$$\underline{p}_\lambda = \{x \in \underline{p}_{\mathbb{C}} | \operatorname{ad} h \cdot x = \lambda(h)x \text{ for } h \in \underline{h}_{\mathbb{C}}^+\}.$$

Set $p_{\mathbb{C}}^{\pm} = \sum_{\pm\lambda \in \Delta^+|_{(\underline{h}_{\mathbb{C}}^+)}} p_\lambda$. Using 1.9, 2), we see that the weights of $S|_{\underline{h}_{\mathbb{C}}^+}$ are of the form $\delta_n - \lambda_{i_1} - \ldots - \lambda_{i_r}$ with $\underline{p}_{\lambda_{ij}} \subset \underline{p}_a^+$.

2.8. Let for $\Lambda \in (h_{\mathbb{C}}^+)^*$ $\Delta_k^+$-dominant integral, $(\tau_\Lambda, V^\Lambda)$ denote the irreducible finite-dimensional representation of $\underline{k}_{\mathbb{C}}$ with highest weight $\Lambda$ relative to $\Delta_k^+$.

2.9. LEMMA. $(s, S)$ <u>contains the representation</u> $(\tau_{\delta_n}, V^{\delta_n})$ of $\underline{k}_{\mathbb{C}}$.

<u>Proof.</u> See 2.7.

2.10. Let $<,>$ denote the dual of the Killing form of $\underline{g}_{\mathbb{C}}$ restricted to $\underline{h}_{\mathbb{C}}$ and to $\underline{h}^{+}_{\mathbb{C}}$.

2.11. LEMMA. $S(C_k) = (\langle\delta,\delta\rangle - \langle\delta_k,\delta_k\rangle)I$. *In particular*, $\frac{1}{8}\sum_{i,j} R_{ijji} = \langle\delta,\delta\rangle - \langle\delta_k,\delta_k\rangle$.

*Proof.* $\tau_{\delta_n}(C_k) = (\langle\delta_n+\delta_k,\delta_n+\delta_k\rangle - \langle\delta_k,\delta_k\rangle)I$. Since $\delta_n+\delta_k = \delta$, the result follows from 2.8 and 2.3.

2.12. Let $(\pi, H)$ be a unitary $(\underline{g},\underline{k})$-module. We put on $H \otimes S$ the tensor product inner product. Define $D : H \otimes S \to H \otimes S$ by $D = \sum_{i=1}^{m} \pi(x_i) \otimes \gamma(x_i)$ ($x_i$ as in 2.3).

2.13. LEMMA. (Compare Schmid [10].) 1) *If* $x, y \in H \otimes S$, *then* $\langle Dx, y\rangle = \langle x, Dy\rangle$.

2) $D^2 = -\pi(C) \otimes I - (\langle\delta,\delta\rangle - \langle\delta_k,\delta_k\rangle)I + (\pi \otimes S)(C_k)$.

*Proof.* 1) is proved by the obvious computation using 1.2, 4). To prove 2) we note $D^2 = \sum_{i,j} \pi(x_i)\pi(x_j) \otimes \gamma(x_i)\gamma(x_j) =$

$= -\sum_i \pi(x_i)^2 \otimes I + \sum_{i\neq j} \pi(x_i)\pi(x_j) \otimes \gamma(x_i)\gamma(x_j) =$

$= -\pi(C) \otimes I + \pi(C_k) \otimes I + \sum_{i\neq j} \pi(x_i)\pi(x_j) \otimes \gamma(x_i)\gamma(x_j)$. Now $\gamma(x_i)\gamma(x_j) = -\gamma(x_j)\gamma(x_i)$ for $i \neq j$. Hence we find $D^2 = -\pi(C) \otimes I + \pi(C_k) \otimes I + \frac{1}{2}\sum_{i,j} \pi([x_i,x_j]) \otimes \gamma(x_i)\gamma(x_j) =$

$= -\pi(C) \otimes I + \pi(C_k) \otimes I - \frac{1}{2} \sum_{i,j,a} \beta([x_i,x_j],x_a)\pi(x_a) \otimes \gamma(x_i)\gamma(x_j) =$

$= -\pi(C) \otimes I + \pi(C_k) \otimes I - 2\sum_a \pi(x_a) \otimes S(x_a) =$ (use the formula in the proof of 2.3)

$= -\pi(C) + \pi(C_k) \otimes I + (\pi \otimes S)(C_k) - \pi(C_k) \otimes I - I \otimes s(C_k) = -\pi(C) - I \otimes s(C_k) + (\pi\otimes S)(C_k)$.

2.11 completes the proof.

2.14. Let $W^1 = \{s \in W(\Delta) \mid s\Delta^+$ is compatible with $\Delta_k^+\}$. Let $\ell_0 = \dim \underline{h}_{\mathbb{C}}^-$.

2.15. LEMMA. $S = \bigoplus_{\sigma \in W^1} 2^{[\ell_0/2]} \tau_{\sigma\delta-\delta_k}$. (*Here* $m\tau_\lambda$ *means a direct sum of* $m$ *copies of* $\tau_\lambda$.)

*Proof.* Every weight of $S$ is of the form $\delta_n - \xi$, where $\xi$ is a weight of $\underline{h}^+$ acting on $\Lambda p_{\mathbb{C}}^+$. Now $p_{\mathbb{C}}^+ \subset \sum_{\alpha \in \Delta^+} (\underline{g}_{\mathbb{C}})_\alpha$. If $Q \subset \Delta^+$, then set $\langle Q \rangle = \sum_{\alpha \in Q} \alpha$. Then every weight of $S$ is of the form $\delta_n - \langle Q \rangle|_{\underline{h}^+}$, $Q \subset \Delta^+$. Suppose that $\tau_\lambda$ is contained in s. Then $\lambda = \delta_n - \langle Q \rangle|_{\underline{h}^+}$. Hence $\lambda+\delta_k = \delta - \langle Q \rangle|_{\underline{h}^+}$. Lemma 2.11 implies that $|\lambda+\delta_k|^2 = |\delta|^2$. Thus $|\delta - \langle Q \rangle|_{\underline{h}^+}|^2 = |\delta|^2$. But $|\delta - \langle Q \rangle|_{\underline{h}^+}|^2 = |\delta|^2 - 2\langle \delta, \langle Q \rangle|_{\underline{h}^+} \rangle + \langle \langle Q \rangle|_{\underline{h}^+}, \langle Q \rangle|_{\underline{h}^+} \rangle = |\delta|^2 - 2\langle \delta, \langle Q \rangle \rangle + \langle \langle Q \rangle|_{\underline{h}^+}, \langle Q \rangle|_{\underline{h}^+} \rangle \leq |\delta - \langle Q \rangle|^2$. Thus $|\delta - \langle Q \rangle|^2 \geq |\delta|^2$.

a) Let $F^\delta$ be the finite-dimensional $\underline{g}_{\mathbb{C}}$ module with highest weight $\delta$ relative to $\Delta^+$. Then the weights of $F^\delta$ are precisely the linear forms $\delta - \langle Q \rangle$, $Q \subset \Delta^+$.

Indeed, let $ch(F^\delta)$ be the character of $F^\delta$. Let $D = \sum_{s \in W(\Delta)} \det(s) e^{s\delta} = e^\delta \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})$. Then $D \cdot ch(F^\delta) = \sum_{s \in W(\Delta)} \det(s) e^{2s\delta} = e^{2\delta} \prod_{\alpha \in \Delta^+} (1 - e^{-2\alpha})$. Hence $ch(F^\delta) = e^{2\delta} \prod_{\alpha \in \Delta^+} (1 - e^{-2\alpha}) / e^\delta \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) = e^\delta \prod_{\alpha \in \Delta^+} (1 + e^{-\alpha}) = \sum_{Q \subset \Delta^+} e^{\delta - \langle Q \rangle}$. This proves a).

In order to apply a) we prove a Scholium due to Kostant [8].

2.16. SCHOLIUM. *Let $\Lambda_1, \ldots, \Lambda_r$ be $\Delta^+$-dominant integral forms on $h_{\mathbb{C}}$. Let $F_i$ be the simple, finite-dimensional $g_{\mathbb{C}}$ module with highest weight $\Lambda_i$, $i = 1, \ldots, r$. If $\lambda$ is a weight of $F_1 \otimes \ldots \otimes F_r$, then $|\lambda| \leq |\Lambda_1 + \ldots + \Lambda_r|$ and equality occurs if and only if there is $s \in W(\Delta)$ so that $\lambda = s(\Lambda_1 + \ldots + \Lambda_r)$.*

*Proof.* Let $s \in W(\Delta)$ be such that $s\lambda$ is $\Delta^+$-dominant integral. Then $|s\lambda| = |\lambda|$. Now $\lambda = \lambda_1 + \ldots + \lambda_r$ with $\lambda_i$ a weight of $F_i$. This implies that $\lambda_i = \Lambda_i - \xi_i$, $\xi_i$ a sum of positive roots. Hence $\lambda = \Lambda_1 + \ldots + \Lambda_r - \xi_1 - \ldots - \xi_r = \Lambda_1 + \ldots + \Lambda_r - \xi$, $\xi$ a sum of positive roots. Set $\Lambda = \Lambda_1 + \ldots + \Lambda_r$. Then $\langle\lambda, \lambda\rangle = \langle\lambda, \Lambda - \xi\rangle = \langle\lambda, \Lambda\rangle - \langle\lambda, \xi\rangle \leq \langle\lambda, \Lambda\rangle = \langle\Lambda - \xi, \Lambda\rangle =$
$= \langle\Lambda, \Lambda\rangle - \langle\xi, \Lambda\rangle \leq \langle\Lambda, \Lambda\rangle$. Equality occurs if and only if $\langle\Lambda, \xi\rangle = \langle\lambda, \xi\rangle = 0$. But then $0 = \langle\lambda, \xi\rangle = \langle\Lambda - \xi, \xi\rangle = \langle\Lambda, \xi\rangle - \langle\xi, \xi\rangle = -\langle\xi, \xi\rangle$. Hence $\xi = 0$. This clearly implies the Scholium.

2.17. *The completion of the proof of* 2.15. Using a) and 2.16 we find that $\delta - \langle Q\rangle = \sigma\delta$ for some $\sigma \in W(\Delta)$. Hence $\lambda = \sigma\delta|_{\underline{h}^+} - \delta_k$. But $\lambda$ is $\Delta_k^+$-dominant integral; hence $\sigma \in W^1$. Thus if $\tau_\lambda \subset S$, then $\lambda = \sigma\delta - \delta_k$ with $\sigma \in W^1$. We note that the multiplicity of the weight $\sigma\delta - \delta_k$ with $\sigma \in W^1$ in $S$ is precisely $2^{[\ell_0/2]}$ and relative to any order gotten from $\sigma\Delta^+$, $\sigma\delta - \delta_k$ is the highest weight of $S$. The lemma now follows.

2.18. We note that 2.15 is a generalization of a lemma in Parthasarathy [9].

2.19. LEMMA. $(s, S)$ is equivalent with its contragredient.

Proof. We note that if $\lambda$ is a weight, then $-\lambda$ is a weight of $S$ (see 2.7). This implies the lemma.

2.20. We conclude this section with some results on the decomposition of tensor products of finite-dimensional representations. These results will be useful in §3 and §4. Let $\{\alpha_1, \ldots, \alpha_\ell\}$ be the simple roots in $\Delta^+$. Let $\Lambda_1, \ldots, \Lambda_\ell$ be the basic highest weights. $(2\langle\Lambda_i, \alpha_j\rangle / \langle\alpha_j, \alpha_j\rangle = \delta_{ij}.)$ Let $F^\Lambda$ be the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight $\Lambda$. Let $(F^{\Lambda_i})^* = F^{\Lambda_{i'}}$; then $i \mapsto i'$ is an involutive permutation of $\{1, \ldots, \ell\}$.

LEMMA. Let $\Lambda = \Sigma m_i \Lambda_i$, $\mu = \Sigma n_i \Lambda_i$; then

$$F^\Lambda \otimes F^\mu \supset \sum_{m_i + n_i - s_i - s_{i'} = q_i} F^{\Sigma q_i \Lambda_i} \quad \text{where,}$$

with $s_i \leq \min(m_i, n_{i'})$, $i = 1, \ldots, \ell$.

Proof. Let for $i = 1, \ldots, \ell$, $e_i \in F^{\Lambda_i}$ be a nonzero highest weight vector. Let $e_i^* \in (F^{\Lambda_i})^* = F^{\Lambda_{i'}}$ be a lowest weight vector so that $\langle e_i, e_i^* \rangle = 1$. We realize $F^\Lambda \subset X = \otimes^{m_1} F^{\Lambda_1} \otimes \otimes^{m_2} F^{\Lambda_2} \otimes \ldots \otimes \otimes^{m_\ell} F^{\Lambda_\ell}$ as the cyclic space of $\otimes^{m_1} e_1 \otimes \otimes^{m_2} e_2 \otimes \ldots \otimes \otimes^{m_\ell} e_\ell$. We realize $F^\mu$ as the cyclic space of $\otimes^{n_{1'}} e_1^* \otimes \otimes^{n_{2'}} e_2^* \otimes \ldots \otimes \otimes^{n_{\ell'}} e_\ell^*$ in $\otimes^{n_{1'}} (F^{\Lambda_1})^* \otimes \ldots \otimes \otimes^{n_{\ell'}} (F^{\Lambda_\ell})^*$. Let $C_i$ be the $\mathfrak{g}$-homomorphism of $F^{\Lambda_i} \otimes (F^{\Lambda_i})^* \to \mathbb{C}$ given by the canonical pairing. If $0 \leq s_i \leq \min(m_i, n_{i'})$, then $C_1^{s_1} \ldots C_\ell^{s_\ell}(X \otimes Y) \neq 0$ and $C_1^{s_1} \ldots C_\ell^{s_\ell}(F^\Lambda \otimes F^\mu) \neq 0$. Furthermore, $C_1^{s_1} \ldots C_\ell^{s_\ell}(F^\Lambda \otimes F^\mu)$ contains

$\otimes^{m_1-s_1} e_1 \otimes \ldots \otimes \otimes^{m_\ell - s_\ell} e_\ell \otimes \otimes^{n_{1'}-s_1} e_1^* \otimes \ldots \otimes \otimes^{n_\ell - s_\ell} e_\ell^*$.

$U(\underline{g})\cdot\eta \cong F^{\Sigma(n_i - s_{i'})\Lambda_i}$. Thus if $\underline{n}^+ = \sum_{\alpha\in\Delta^+} \underline{g}_\alpha$, then $U(\underline{n}^+)\cdot\eta$ contains a vector $\eta_0$ so that $\underline{n}^+\cdot\eta_0 = 0$ and $h\cdot\eta_0 = \Sigma(n_i - s_{i'})\Lambda_i(h)\cdot\eta_0$. Since $\underline{n}^+\cdot\xi = 0$, we see that $U(\underline{n}^+)(\xi\otimes\eta)$ contains $\xi\otimes\eta_0$. Thus $F^\Lambda\otimes F^\mu$ contains $F^\xi$ with $\xi = \Sigma(m_i - s_i)\Lambda_i + \Sigma(n_i - s_{i'})\Lambda_i = \Sigma(m_i + n_i - s_i - s_{i'})\Lambda_i$. Q.E.D.

2.21. We maintain the notation of 2.20. Perhaps the most useful corollary of 2.20 is

COROLLARY. *Let* $\Lambda = \Sigma m_i \Lambda_i$, $\mu = \Sigma n_i \Lambda_i$; *then* $F^\Lambda \otimes F^\mu \supset F^{\Sigma p_i \Lambda_i}$ *with* $p_i = m_i + n_i - \min(m_i, n_{i'}) - \min(m_{i'}, n_i)$. *In particular, if* $F^\mu = (F^\lambda)^*$ *and* $\Lambda - \lambda$ *is* $\Delta^+$*-dominant integral* (*i.e.*, $m_i \geq n_{i'}$), *then* $F^\Lambda \otimes F^\mu \supset F^{\Lambda-\lambda}$.

2.22. LEMMA. *Let* $\Lambda, \mu$ *be as above. Suppose that* $\nu$ *is a weight of* $F^\mu$ *such that*

1) $\Lambda+\nu$ *is* $\Delta^+$*-dominant integral*;

2) *if* $s \in W(\Delta)$ *and* $\xi$ *is a weight of* $F^\mu$ *such that* $s(\Lambda+\nu+\delta) = \Lambda+\delta+\xi$, *then* $\xi = \nu$, $s = 1$.

*Then* $F^\Lambda \otimes F^\mu \supset F^{\Lambda+\nu}$.

Proof. We use the notation of the proof of 2.15. Let $\pi(F^\mu)$ be the set of weights of $F^\mu$. Let $m_\xi$ be the dimension of the $\xi$ weight space in $F^\mu$. Then $D\cdot ch(F^\Lambda\otimes F^\mu) = \sum_{\xi\in\pi(F^\mu)} m_\xi \sum_{s\in W(\Delta)} \det(s) e^{s(\Lambda+\delta+\xi)} =$

$$= m_\nu \sum_{s\in W(\Delta)} \det(s) e^{s(\Lambda+\delta+\nu)} + \sum_{\substack{\xi\neq\nu \\ \xi\in\pi(F^\mu)}} m_\xi \sum_{s\in W(\Delta)} \det(s) e^{s(\Lambda+\delta+\xi)}.$$

If $s(\Lambda+\delta+\nu) = t(\Lambda+\delta+\xi)$ for $t, s \in W(\Delta)$ and some $\xi \in \pi(F^{\mu})$ with $\xi \neq \nu$, then $t^{-1}s(\Lambda+\delta+\nu) = \Lambda+\delta+\xi$. Hence by hypothesis $t = s$; hence $\xi = \nu$. This clearly implies the lemma.

2.23. COROLLARY. *Let $\Lambda, \mu$ be as in 2.27.*

1) *Suppose that $a$ is simple in $\Delta^+$ and $\Lambda - s_a\mu$ is $\Delta^+$-dominant integral; then $F^{\Lambda} \otimes (F^{\mu})^*$ contains $F^{\Lambda - s_a\mu}$.*

2) *Suppose that $a$ is $\Delta^+$-simple, $\langle\mu, a\rangle \neq 0$ and $\Lambda-\mu+a$ is $\Delta^+$-dominant integral. If $2\langle\Lambda-\mu, a\rangle/\langle a, a\rangle \neq -2$, then $F^{\Lambda-\mu+a} \subset F^{\Lambda} \otimes (F^{\mu})^*$.*

*Proof.* 1) Suppose $s \in W(\Delta)$ and $\xi \in \pi(F^{\mu})$ is such that $s(\Lambda - s_a\mu+\delta) = \Lambda+\delta-\xi$, $\pi((F^{\mu})^*) = \{-\xi \mid \xi \in \pi(F^{\mu})\}$. Now $s(\Lambda-s_a\mu) = \Lambda - s_a\mu - q_1$ with $q_1$ a sum of elements of $\Delta^+$, $s\delta = \delta - q_2$ with $q_2$ a sum of elements of $\Delta^+$. Hence $\Lambda+\delta-\xi$, $s(\Lambda-s_a\mu+\delta) = \Lambda - s_a\mu+\delta-q_1-q_2$. Hence $\xi = s_a\mu+q_1+q_2$. But $\xi = \mu - q_3$ with $q_3$ a sum of elements of $\Delta^+$. Thus $\mu = s_a\mu+q_1+q_2+q_3$. This implies that $q_i = m_i a$, $i = 1, 2, 3$, $m_i \in \mathbb{C}$, $m_i \geq 0$. Hence $s\delta = \delta - m_2 a$. But then $s = s_a$ or $s = 1$. Thus if $s \neq 1$, $s_a(\Lambda+\delta)-\mu = \Lambda+\delta-\xi$. That is, $s_a(\Lambda+\delta)-(\Lambda+\delta) = \mu-\xi$. This is impossible. Thus $s = 1$. Now 2.27 implies 1).

2) Since $\langle\mu, a\rangle \neq 0$, $-\mu+a$ is a weight of $(F^{\mu})^*$. Suppose that $\xi$ is a weight of $(F^{\mu})^*$ and that $s \in W(\Delta)$ is such that $S(\Lambda-\mu+a+\delta) = \Lambda+\delta-\xi$. Then as in the proof of 1), $S(\Lambda-\mu+a) = \Lambda-\mu+a-q_1$, $s\delta = \delta-q_2$, $\xi = \mu-q_3$. Hence $\Lambda-\mu+\delta+a-q_1-q_2 = -\mu+q_3+\delta+\Lambda$. Thus $a = q_1+q_2+q_3$. Hence $q_i = m_i a$ and $m_i \in \mathbb{Z}$, $m_i \geq 0$, $\Sigma m_i = 1$. Again, this implies that $s = s_a$ or 1. If $s = s_a$, then $q_2 = 1$; thus $q_1 = 0$. Hence $s_a(\Lambda-\mu+a) = \Lambda-\mu+a$. That is,

$s_\alpha(\Lambda-\mu)-\alpha = \Lambda-\mu+\alpha$, which implies that $2\langle\Lambda-\mu, \alpha\rangle/\langle\alpha, \alpha\rangle = -2$. Since this contradicts our assumptions, $s_\alpha = 1$.

## §3. The vanishing theorems ($F \neq \mathbb{C}$).

3.1. We maintain the notation of Section 2. In particular, we fix $\Delta_k^+$ a system of positive roots for $\Delta_k$ and $\Delta^+$ a compatible system of positive roots for $\Delta$.

3.2. THEOREM. *Let $F$ be the irreducible finite-dimensional $\underline{g}$-module with highest weight $\Lambda-\delta$ relative to $\Delta^+$. Let $(\pi, H)$ be a unitary $(\underline{g}, \underline{k})$-module with $\pi(C)$ a scalar operator. If $\theta\Lambda \neq \Lambda$, then*

$$H^r(\underline{g}, \underline{k}; H \otimes F^*) = 0$$

*for all $r$.*

3.3. Note. Of course, this theorem has no content if $\underline{h}^+ = \underline{h}$.

3.4. The proof of Theorem 3.2. Let $\pi(C) = \lambda I$. Assume $H^r(\underline{g}, \underline{k}; H \otimes F^*) \neq 0$ for some $r$. Then we must have $\lambda = |\Lambda|^2 - |\delta|^2$ (see II, 3.1) and $\mathrm{Hom}_{\underline{k}}(F \otimes \Lambda_p, H) \neq 0$. Since $\Lambda_p \equiv S \otimes S$ or $S \otimes S \oplus S \otimes S$ (see 1.11) and $S \equiv S^*$ (2.19) we have $\mathrm{Hom}_{\underline{k}}(F \otimes S, H \otimes S) \neq 0$. Let $F \otimes S = \Sigma m_\nu(\tau_\nu, V^\nu)$. We must have $\mathrm{Hom}_{\underline{k}}(V^\nu, H \otimes S) \neq 0$ for some $\nu$ with $m_\nu \neq 0$. Now 2.12 implies that if $\xi \in H \otimes S$, then

$$\langle D^2\xi, \xi\rangle = (-\lambda-\langle\delta, \delta\rangle+\langle\delta_k, \delta_k\rangle)\langle\xi, \xi\rangle + \langle(\pi \otimes S)(\Omega_k)\xi, \xi\rangle.$$

Since $\langle D^2\xi, \xi\rangle = \langle D\xi, D\xi\rangle$ (2.13), $\langle(\pi\otimes S)(\Omega_k)\xi, \xi\rangle \geq (\lambda+\delta-\langle\delta_k, \delta_k\rangle)\langle\xi, \xi\rangle$. Since $\lambda = |\Lambda|^2-|\delta|^2$, we must have $\langle(\pi\otimes S)(\Omega_k)\xi, \xi\rangle \geq (|\Lambda|^2-|\delta_k|^2)\langle\xi, \xi\rangle$. But then $|\nu+\delta_k|^2-|\delta_k|^2 \geq |\Lambda|^2-|\delta_k|^2$. That is, $|\nu+\delta_k|^2 \geq |\Lambda|^2$.

If $m_\nu \neq 0$, then $\nu = \mu^+ + \sigma\delta-\delta_k$ with $\mu$ a weight of F and $\sigma \in W^1$. Hence $|\Lambda|^2 \leq |\nu+\delta_k|^2 = |\mu^+ + \sigma\delta|^2 \leq |\mu+\sigma\delta|^2$. But $\mu$ is a weight of F and $\sigma\delta$ is a weight of $F^\delta$. Hence $|\mu+\sigma\delta| \leq |\Lambda-\delta+\delta| = |\Lambda|$ and equality occurs if and only if $\mu+\sigma\delta = t(\Lambda-\delta)+t\delta$, $t \in W(\Delta)$. That is, $t^{-1}\mu+t^{-1}\sigma\delta = \Lambda$. But $t^{-1}\mu = \Lambda-\delta-\xi$, $t^{-1}\sigma\delta = \delta-\langle Q\rangle$ with $\xi$ a sum of elements of $\Delta^+$ and $Q \subset \Delta^+$. Hence $\Lambda-\xi-\langle Q\rangle = \Lambda$. Thus $\xi = \langle Q\rangle = 0$. This implies that $t = \sigma$ and $\mu = \sigma(\Lambda-\delta)$. But then we have

$$|\Lambda|^2 \leq |(\sigma(\Lambda-\delta))^+ + \sigma\delta| \leq |\sigma\Lambda|^2 = |\Lambda|^2.$$

Since $(\sigma\delta)^+ = \sigma\delta$ we see that $|\Lambda|^2 = |(\sigma\Lambda)^+|^2 = |\Lambda^+|^2$. Finally, $\Lambda = \Lambda^+ + \Lambda^-$ and $|\Lambda|^2 = |\Lambda^+|^2+|\Lambda^-|^2$. Hence $\Lambda^- = 0$. This proves the theorem.

3.5. We now isolate one part of the proof of Theorem 3.2 which will be useful.

LEMMA. *Let* F, $\Lambda$, $(\pi, H)$ *be as in* 3.2. *Suppose that* $\pi(C) = (|\Lambda|^2-|\delta|^2)I$. *Assume that* $\theta\Lambda = \Lambda$. *Then* $\mathrm{Hom}_{\underline{k}}(F\otimes S)(H\otimes S) =$

$\sum_{t\in W^1} \mathrm{Hom}_{\underline{k}}(2^{[\ell_0/2]} V_{t\Lambda-\delta_k}, H\otimes S)$ ($\ell_0$ *is as in* 2.14). *That is,*

$\Sigma 2^{[\ell_0/2]} V_{t\Lambda-\delta_k} \subset F\otimes S$ *and* $\mathrm{Hom}_{\underline{k}}(W, H\otimes S) = 0$ *where* $W \oplus \Sigma V_{t\Lambda-\delta_k} = F\otimes S$.

This lemma has been proved in the course of the proof of 3.2.

3.6. Note that if $F \equiv \mathbb{C}$, then $\Lambda = \delta$, and the lemma in 3.5 has no content.

3.7. Let for $P \subset \Delta$ a system of positive roots compatible with $\Delta_k^+$, $p_{\mathbb{C}}^+(P) = \Sigma \underline{p}_\lambda$ the sum over all $\lambda$ so that $\underline{p}_\lambda \neq 0$ and $\lambda = \alpha|_{\underline{h}^+}$, $\alpha \in P$.

3.8. The following vanishing theorem uses ideas in Hotta, Parthasarathy [6].

THEOREM. Let $F$, $\Lambda$ and $(\pi, H)$ be as in 3.2. Suppose that $\pi(C) \doteq (|\Lambda|^2 - |\delta|^2)I$ and that $\theta\Lambda = \Lambda$. Suppose that whenever $t \in W^1$ and $\xi$ is a weight of $\Lambda p_{\mathbb{C}}^+(t\Delta^+)$ so that $t\Lambda + t\delta - 2\delta_k - \xi$ is $\Delta_k^+$-dominant integral, then $t\Lambda - \delta_k - \xi$ is $\Delta_k^+$-dominant integral. (Note that if $\Lambda$ satisfies this condition, then $\Lambda + \mu$ satisfies this condition for $\mu$ $\Delta_k^+$-dominant integral. Also if $\Lambda$ is as in 3.2, then $k\Lambda$ satisfies this condition for $k$ large.) Then $H^j(\underline{g}, \underline{k}; H \otimes F^*) = 0$ for $0 \leq j < \dim \underline{p}_{\mathbb{C}}^+$.

Proof. Proposition 3.1 of II implies that $H^j(\underline{g}, \underline{k}; H \otimes F^*) = \mathrm{Hom}_{\underline{k}}(\Lambda^j \underline{p}, H \otimes F^*) =$
$= \mathrm{Hom}_{\underline{k}}(F \otimes \Lambda^j \underline{p}, H)$.

We compute $\mathrm{Hom}_{\underline{k}}(F \otimes \Lambda \underline{p}, H)$. Now $\Lambda \underline{p} = S \otimes S$ or $S \otimes S \oplus S \otimes S$. Thus we really must complete $\mathrm{Hom}_{\underline{k}}(F \otimes S \otimes S, H) = \mathrm{Hom}_{\underline{k}}(F \otimes S, H \otimes S) =$
$\sum_{t \in W^1} \mathrm{Hom}_k(2^{[\ell_0/2]} V_{t\Lambda - \delta_k}, H \otimes S) = \sum_{t \in W^1} \mathrm{Hom}_{\underline{k}}(2^{[\ell_0/2]} V_{t\Lambda - \delta_k} \otimes S, H)$.

We look at $\mathrm{Hom}_{\underline{k}}(V_{t\Lambda - \delta_k} \otimes S, H)$. Now $\tau_{t\Lambda - \delta_k} \otimes S = \Sigma m_\lambda \tau_\lambda$ and if $m_\lambda \neq 0$, then $\lambda = t\Lambda - 2\delta_k + t\delta - \xi$ where $\xi$ is a weight of $\Lambda p_{\mathbb{C}}^+(t\Delta^+)$ Now the hypothesis of this theorem implies that $t\Lambda-\delta_k-\xi$ is $\Delta_k^+$-dominant integral if $m_\lambda \neq 0$. This implies that $\tau_\lambda \otimes S$ contains $\tau_{\lambda-\delta_n}$ since $S$ contains $\tau^*_{\delta_n}$. If $\mathrm{Hom}_{\underline{k}}(V_\lambda, H) \neq 0$, then arguing as in the proof of Theorem 3.2 we find that the lowest eigenvalue of $(\tau_\lambda \otimes S)(\Omega_k)$ is greater than or equal to $|\Lambda|^2-|\delta_k|^2$. This implies that $|t\Lambda-\xi|^2 \geq |\Lambda|^2$. Now $\xi$ is a weight of $\Lambda p^+_{\mathbb{C}}(t\Delta^+)$. Hence $\xi = t\langle Q\rangle|_{h^+}$ with $Q \subseteq \Delta^+$. Hence $|t\Lambda-t\langle Q\rangle|_{h^+}|^2 \geq |\Lambda|^2$. But then $|\Lambda-\langle Q\rangle|^2 \geq |\Lambda-\langle Q\rangle^+|^2 \geq |\Lambda|^2$. $\Lambda = \Lambda-\delta+\delta$ and $\Lambda-\delta$ is the highest weight of $F$. Hence $|\Lambda-\delta+\delta-\langle Q\rangle|^2 \geq |\Lambda|^2$. Thus $t(\Lambda-\delta)+t\delta-t\langle Q\rangle = u(\Lambda-\delta)+u\delta$ for some $u \in W(\Delta)$. Arguing as in the proof of Theorem 3.2 we see that $u = t$ and $Q = \phi$.

We have shown that if $F \otimes \Lambda p = \Sigma n_\lambda \tau_\lambda$ then $\mathrm{Hom}_{\underline{k}}(F \otimes \Lambda \underline{p}, H) =$ $= \sum_{t\in W^1} \mathrm{Hom}_{\underline{k}}(n_{t(\Lambda+\delta)-2\delta_k} V_{t(\Lambda+\delta)-2\delta_k}, H)$. To complete the proof of the theorem we must show

$$\mathrm{Hom}_{\underline{k}}(V_{t(\Lambda+\delta)-2\delta_k}, F \otimes \Lambda^j \underline{p}) = 0$$

for $0 \leq j < \dim p^+_{\mathbb{C}}$ and for $j > \dim p^+_{\mathbb{C}}+\ell_0$. By replacing $\Delta^+$ by $t\Delta^+$ we can assume $t = 1$. The weights of $F$ relative to $\underline{h}^+_{\mathbb{C}}$ are of the form $\Lambda-\delta-\xi^+$, $\xi$ a sum of elements of $\Delta^+$. The weights of $\Lambda^j \underline{p}$ are of the form $\lambda_{i_1}+\dots+\lambda_{i_j}$ with $\lambda_{i_1},\dots,\lambda_{i_j}$ a weight of $\underline{p}_{\mathbb{C}}$.

Hence the weights of $F \otimes \Lambda^j \underline{p}$ are of the form $\Lambda-\delta-\xi^++2\delta_n-\xi_1^+$ with $\xi_1$ a sum of elements of $\Delta^+$. Thus the highest possible weight is $\Lambda+\delta-2\delta_k$ and this occurs only if $2\delta_n$ is a weight of $\Lambda^j \underline{p}$. But $2\delta_n$ is a weight of $\Lambda^j \underline{p}$ only if $\dim \underline{p}^+_{\mathbb{C}} \leq j \leq \dim \underline{p}^+_{\mathbb{C}}+\ell_0$. Q. E. D.

3.9. We now assume that $(\pi, H)$ is irreducible and admissible. We assume that $\Lambda$ satisfies the conditions of Theorem 3.8.

THEOREM. *If* $H(\underline{g}, \underline{k}; H \otimes F^*) \neq 0$, *then*

1) $(\pi, H)$ *is in the fundamental series for* $\underline{g}$ *relative to* $t\Delta^+$ *for some* $t \in W^1$ *(see Enright, Wallach* [4] *for the definitions) and* $(\pi, H)$ *has lowest* $\underline{k}$*-type* $\tau_{t\Lambda + t\delta - 2\delta_k}$.

2) $\dim H^j(\underline{g}, \underline{k}; H \otimes F^*) = \binom{\ell_0}{j-q}$ *where* $q = \dim \underline{p}^+_{\mathbb{C}}$.

Proof. By the proof of Theorem 3.8, $(\pi, H)$ must contain $\tau_{t\Lambda + t\delta - 2\delta_k}$ and cannot contain any $\underline{k}$-type of the form $\tau_{t\Lambda + \tau\delta - 2\delta_k - \xi}$ with $\xi$ a weight of $\Lambda \underline{p}^+_{\mathbb{C}}(t\Delta^+)$. Theorem 6.3 of [4] now implies 1).

2) follows from the fact that if $(\pi, H)$ and $t$ is as in 1), then $\dim \mathrm{Hom}_{\underline{k}}(V_{s(\Lambda+\delta)-2\delta_k}, H) = \delta_{t,s}$ (an easy consequence of the lowest $\underline{k}$-type property).

3.10. We note that if $\operatorname{rank} \underline{k}_{\mathbb{C}} = \operatorname{rank} \underline{g}_{\mathbb{C}}$ then $\ell_0 = 0$ and $(\pi, H)$ is $\omega_{t\Lambda}$ where $\omega_{t\Lambda}$ is the element of Harish-Chandra's discrete series (see [5]) corresponding to $t\Lambda$. Since in this case the fundamental series representation associated with $t\Lambda + t\delta - 2\delta_k$ and $t\Delta^+$ is $\omega_{t\Lambda}$ (see Wallach [11]).

## §4. Vanishing theorems ($F = \mathbb{C}$).

4.1. Let $\Lambda^j \underline{p} = n_{0,j}\tau_0 + \Sigma n_{\lambda,j}\tau_\lambda$ ($\tau_0$ the trivial representation of $\underline{k}$) as a $\underline{k}$-module.

**4.2.** Let $B_j^+ = \{\lambda \mid n_{\lambda,j} \neq 0$ and $(\tau_\lambda \otimes S)(C_{\underline{k}})$ has lowest eigenvalue at least $|\delta|^2 - |\delta_{\underline{k}}|^2\}$.

**4.3.** LEMMA. *Let $(\pi, H)$ be a unitary $(\underline{g}, \underline{k})$ module with $\pi(C)$ a scalar and $H^{\underline{g}} = (0)$. If $B_j^+ = \phi$, then $H^j(\underline{g}, \underline{k}; H) = 0$.*

*Proof.* If $H^j(\underline{g}, \underline{k}; H) \neq 0$, then $\pi(C) = 0$ (II, 3.1) and $H^j(\underline{g}, \underline{k}; H) = \mathrm{Hom}_{\underline{k}}(\Lambda^j \underline{p}_{\mathbb{C}}, H)$. If $H^{\underline{g}} = 0$, then $H^{\underline{k}} = 0$. Hence $\mathrm{Hom}_{\underline{k}}(V_\lambda, H) \neq 0$ for some $\lambda \neq 0$, so that $n_{\lambda,j} \neq 0$. But then 2.13 implies that $(\tau_\lambda \otimes S)(C_{\underline{k}})$ has lowest eigenvalue at least $|\delta|^2 - |\delta_{\underline{k}}|^2$. Q.E.D.

**4.4.** We assume that $\underline{g}$ is simple as a real Lie algebra. Then there are two possibilities for $\underline{p}_{\mathbb{C}}$ as a $\underline{k}$ module.

1) $\underline{p}_{\mathbb{C}}$ is an irreducible $\underline{k}$ module.

2) $\underline{p}_{\mathbb{C}} = V_1 \oplus V_2$ with $V_1, V_2$ irreducible $\underline{k}$ submodules.

**4.5.** LEMMA. *Let $\Delta_{\underline{k}}^+$ be a system of positive roots for $\Delta_{\underline{k}}$. Let $\Delta^+$ be a compatible system of positive roots for $\Delta$. If 4.4, 1) holds, let $\lambda$ be the highest weight relative to $\Delta_{\underline{k}}^+$ of $\underline{p}_{\mathbb{C}}$ as a $\underline{k}$-module. If 4.4, 2), holds, let $\lambda_1, \lambda_2$ be the highest weights relative to $\Delta_{\underline{k}}^+$ for $V_1$ and $V_2$ respectively. Assume*

*1) if 4.4, 1) holds, there is $t \in W^1$ (see 2.14) so that $t\delta - \delta_{\underline{k}} - \lambda$ is $\Delta_{\underline{k}}^+$-dominant and $\lambda$ is not a simple root in $t\Delta^+$;*

*2) if 4.4, 2), holds, then there exists for $i = 1, 2$, $t_i \in W^1$ so that $t_i\delta - \delta_{\underline{k}} - \lambda_i$ is $\Delta_{\underline{k}}^+$-dominant and $\lambda_i = \alpha^+$ for some $\alpha \in t_i\Delta^+$, but $\lambda_i$ is not a simple root in $t_i\Delta^+$, $i = 1, 2$.*

*If* $(\pi, H)$ *is a unitary* $(\underline{g}, \underline{k})$*-module with* $\pi(C)$ *a scalar, then* $H^1(\underline{g}, \underline{k}; H) = 0$.

*Proof.* Assume 4.4, 1), holds. Then $\underline{p} = V_\lambda$. $\tau_\lambda \otimes S = \sum_{t \in W^1} 2^{[\ell_0/2]} \tau_\lambda \otimes \tau_{t\delta - \delta_k}$ (see 2.15). The lowest weight of $\tau_\lambda$ is $-\lambda$. Hence for $t$ as in 1), $\tau_\lambda \otimes S \supset \tau_{t\delta-\delta_k - \lambda}$ (see 2.21). Now $|t\delta - \delta_k - \lambda + \delta_k|^2 = |t\delta - \lambda|^2$. $\lambda = \alpha^+$ for some $\alpha \in t\Delta^+$. ($\lambda$ is the highest weight). Hence $|t\delta - \lambda|^2 = |t\delta - \alpha^+|^2 \leq |t\delta - \alpha|^2 \leq |\delta|^2$. If $|t\delta - \lambda|^2 \geq |\delta|^2$, then we must have $|t\delta - \alpha|^2 = |\delta|^2$; hence $\alpha$ is $t\Delta^+$ simple and $|t\delta - \alpha^+|^2 = |t\delta - \alpha|^2$. Hence $\alpha = \alpha^+$. This contradicts 1); hence $B_1^+ = \phi$.

2) Use the same proof for $i = 1, 2$.

$\underline{4.6}$. PROPOSITION. *Suppose that* $\underline{g}$ *is isomorphic with* $sp(n,1)$, $n \geq 2$, *the split rank* 1 *real form of* $F_4$ *or the noncompact real form of* $G_2$. *Let* $(\pi, H)$ *be a unitary* $(\underline{g}, \underline{k})$*-module with* $\pi(C)$ *a scalar. Then* $H^1(\underline{g}, \underline{k}; H) = 0$.

*Proof.* We use the notation of Bourbaki [1].

1) *sp(n,1).* $\underline{g}_{\mathbb{C}} = C_{n+1}$. We label the roots as in [1], p. 254. Then $\Delta_k^+$ has simple roots $2\varepsilon_1 = 2\sum_{i=1}^{n} \alpha_i + \alpha_{n+1}$, $\alpha_2, \ldots, \alpha_{n+1}$. We note that $s_{\alpha_1} \in W^1$. Also ([1], p. 255) $\delta = \sum_{j=1}^{n+1} (n+2-j)\varepsilon_j$. $\delta_k = \varepsilon_1 + \sum_{j=2}^{n+1} (n+2-j)\varepsilon_j$. Thus $\delta - \delta_k = n\varepsilon_1$. $s_{\alpha_1}\delta - \delta_k = n\varepsilon_1 - \alpha_1 = (n-1)\varepsilon_1 + \varepsilon_2$. $\lambda$ as in 4.5 is $\alpha_1 + 2\sum_{i=2}^{n} \alpha_i + \alpha_{n+1} = \varepsilon_1 + \varepsilon_2$. Thus $s_{\alpha_1}\delta - \delta_k - \lambda = (n-2)\varepsilon_1$. Thus if $n \geq 2$, $S_{\alpha_i}\delta - \delta_k - \lambda$ is $\Delta_k^+$-dominant

integral. The simple roots of $s_{\alpha_1}\Delta^+$ are $-\alpha_1, \alpha_1+\alpha_2, \alpha_3, \ldots, \alpha_{n+1}$. Thus if $n \geq 2$, $\lambda$ is not $s_{\alpha_1}\Delta^+$ simple. The result now follows from Lemma 4.5.

2) <u>The split rank 1 real form of</u> $F_4$. Here we use pp. 272, 273 of [1]. The simple roots of $\Delta_k^+$ are $\varepsilon_1-\varepsilon_2$, $\varepsilon_2-\varepsilon_3$, $\varepsilon_3-\varepsilon_4$, $\varepsilon_4$; those of $\Delta^+$ are $\alpha_1 = \varepsilon_2-\varepsilon_3$, $\alpha_2 = \varepsilon_3-\varepsilon_4$, $\alpha_3 = \varepsilon_4$, $\alpha_4 = \frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)$. $s_{\alpha_4} \in W^1$.
$\delta = \frac{11}{2}\varepsilon_1+5/2\,\varepsilon_2+3/2\,\varepsilon_3+\frac{1}{2}\varepsilon_4$. $\delta_k = \frac{7}{2}\,\varepsilon_1+5/2\,\varepsilon_2+3/2\,\varepsilon_3+\frac{1}{2}\varepsilon_4$ (see [1], p. 253).
$\delta-\delta_k = 2\varepsilon_1$. $s_{\alpha_4}\delta-\delta_k = 2\varepsilon_1-\frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4) = 3/2\,\varepsilon_1+\frac{1}{2}\varepsilon_2+\frac{1}{2}\varepsilon_3+\frac{1}{2}\varepsilon_4$.

Let $\lambda$ be as in 4.5. Then $\lambda = \frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)$. Thus $s_{\alpha_4}(\delta)-\delta_k-\lambda = \varepsilon_1$ which is $\Delta_k^+$-dominant integral. The simple roots of $s_{\alpha_4}\Delta^+$ are $\alpha_1$, $\alpha_2$, $\alpha_3+\alpha_4$, $-\alpha_4$; since $\lambda = \alpha_1=2\alpha_2+3\alpha_3+\alpha_4$, $\lambda$ is not $s_{\alpha_4}\Delta^+$-simple. This case now follows from Lemma 4.5.

3) <u>The noncompact real form of</u> $G_2$. (We use pp. 274, 275 of [1].) We take $\Delta_k^+ = \{\alpha_1, 3\alpha_1+2\alpha_2\}$. $s_{\alpha_2} \in W^1$. $\delta = 5\alpha_1+3\alpha_2$, $\delta_k = 2\alpha_1+\alpha_2$.
$s_{\alpha_2}\delta-\delta_k = \delta-\delta_k-\alpha_2 = 3\alpha_1+\alpha_2$. $\lambda$ (as in 4.5) $= 3\alpha_1+\alpha_2$. Hence $s_{\alpha_2}\delta-\delta_k-\lambda = 0$.
The simple roots of $s_{\alpha_2}\Delta^+$ are $\alpha_1+\alpha_2$, $-\alpha_2$ and neither is $\lambda$. Thus Lemma 4.5.applies. Q.E.D.

<u>4.7</u>. If we combine Proposition 4.6 with the results of Kaneuki, Nagano [7] (see III, 3.7) we have proved

THEOREM. <u>If g is simple and g is not isomorphic with su(n,1) or so(n,1) and if $(\pi, H)$ is a unitary $(g, k)$-module with $\pi(C)$ acting by a scalar, then</u> $H^1(g, k; H) = 0$.

4.8. The above theorem is one of the main results in Delorme [2]. We will derive this theorem in Chapter VII (without case-by-case considerations for split rank $\underline{g} > 1$). We will also show how one can use this result to derive Delorme's result on the topology of the Plancherel dual.

4.9. Actually, one can do a case-by-case check using 2.21, 2.23 to see that Lemma 4.3 implies Theorem 4.7. However, Lemma 4.5 does not always apply. (For example, the complex group $B_\ell$.) We next study $SL(n, \mathbb{R})$ for $n \geq 4$ and see $B_1^+$, $B_2^+$ are both empty. Lemma 4.5 does not apply in this case as we shall see.

4.10. PROPOSITION. Let $\underline{g} = sl(n, \mathbb{R})$, $n \geq 4$. If $(\pi, H)$ is a unitary $(\underline{g}, \underline{k})$-module with $\pi(C)$ acting by a scalar, then $H^1(\underline{g}, \underline{k}; H) = H^2(\underline{g}, \underline{k}; H) = 0$.

Proof. We use $\varepsilon_i$'s for $\underline{k}_{\mathbb{C}}$ and $\eta_j$'s for $\underline{g}_{\mathbb{C}}$.

1) $n = 2\ell+1$. Then if we take $\eta_i|_{\underline{h}^+} = \varepsilon_i$, $i \leq \ell$, and $\eta_{2\ell+2-j}|_{\underline{h}^+} = -\varepsilon_j$, $j \leq \ell$, $\eta_{\ell+1}|_{\underline{h}^+} = 0$, then $\eta_1 \geq \ldots \geq \eta_{2\ell+1}$ is a compatible order to $\varepsilon_1 \geq \ldots \geq \varepsilon_\ell$ and it is the only one. $\theta\eta_i = -\eta_{2\ell+2-i}$, $i = 1, \ldots, 2\ell+1$.

$\delta = \sum_{j=1}^{2\ell+1} (\ell+1-j)\eta_j$. Hence $\delta|_{\underline{h}^+_{\mathbb{C}}} = 2\sum_{j=1}^{\ell} (\ell+1-j)\varepsilon_j$. $\delta_k = \sum_{j=1}^{\ell} (\ell-j+\tfrac{1}{2})\varepsilon_j$. Hence $\delta_n = \delta_k + \sum_{j=1}^{\ell} \varepsilon_j$.

2) $n = 2\ell$. We take $\eta_i|_{\underline{h}^+} = \varepsilon_i$, $i \leq \ell$, $\eta_{2\ell+1-i}|_{\underline{h}^+} = -\varepsilon_i$, $i \leq \ell$. Then the compatible orders are $\eta_1 \geq \ldots \geq \eta_{2\ell}$ and $\eta_1 \geq \ldots \geq \eta_{\ell-1} \geq \eta_{\ell+1} \geq \eta_\ell \geq \eta_{\ell+2} \geq \ldots \geq \eta_{2\ell}$. There are two $\delta$'s, $\delta^+$, $\delta^-$.

$$\delta^+ = \sum_{j=1}^{2\ell} (\ell+\tfrac{1}{2}-j)\eta_j,\quad \delta^- = \sum_{j=1}^{\ell-1} (\ell+\tfrac{1}{2}-j)\eta_j + \sum_{j=\ell+2}^{2\ell} (\ell+\tfrac{1}{2}-j)\eta_j + \tfrac{1}{2}\eta_{\ell+1} - \tfrac{1}{2}\eta_\ell .$$

$$\delta_k = \sum_{j=1}^{\ell-1} (\ell-j)\varepsilon_j.\quad \delta^+|_{\underline{h}^+} = \sum_{j=1}^{\ell} (2\ell+1-2j)\varepsilon_j,\quad \delta^-|_{\underline{h}^+} = \delta^+|_{\underline{h}^+} - 2\varepsilon_\ell.\quad \text{Hence}$$

$$\delta_n^+ = \delta_k + \sum_{i=1}^{\ell} \varepsilon_i,\quad \delta_n^- = \delta_k + \sum_{i=1}^{\ell-1} \varepsilon_i - \varepsilon_\ell .$$

If the Killing form is normalized so that $\langle\eta_i, \eta_j\rangle = \delta_{ij}$ gives $\langle , \rangle$, then must take $\langle\varepsilon_i, \varepsilon_j\rangle = \frac{1}{2}\delta_{ij}$.

It is known that $\underline{p}$ has highest weight $2\varepsilon_1$. Using DoCarmo, Wallach [3], Appendix, it is not hard to show that $\Lambda^2\underline{p} = \tau_{3\varepsilon_1+\varepsilon_2} \oplus \tau_{\varepsilon_1+\varepsilon_2}$. If $n = 2\ell+1$, $\ell \geq 2$. Then $\delta_n = \Lambda_1+\Lambda_2 + \dots + 3\Lambda_\ell$. $\varepsilon_1 = \Lambda_1(\Lambda_1, \dots, \Lambda_\ell$ the basic highest weights). Now $\tau_{2\varepsilon_1} \equiv (\tau_{2\varepsilon_1})^*$. Set $\alpha = \varepsilon_1-\varepsilon_2$. Then $\alpha$ is simple $\delta_n - 2\varepsilon_1+\varepsilon_1-\varepsilon_2 = \delta_n-\varepsilon_1-\varepsilon_2$ which is $\Delta_k^+$-dominant integral (indeed, $\varepsilon_1+\varepsilon_2 = \Lambda_2$ if $\ell > 2$, $\varepsilon_1+\varepsilon_2 = 2\Lambda_2$ if $\ell = 2$). Hence 2.23, 2), implies that $\tau_{\delta_n} \otimes \tau_{2\varepsilon_1} \supset \tau_{\delta_n-\varepsilon_1-\varepsilon_2}$. If $n = 2\ell$, $\ell \geq 2$, then the same argument shows that

$\tau_{\delta_n^\pm} \otimes \tau_{2\varepsilon_1} \supset \tau_{\delta_n^\pm-\varepsilon_1-\varepsilon_2}$. If $n = 2\ell+1$, $\ell \geq 2$, then 2.21 implies $\tau_{\delta_n} \otimes \tau_{3\varepsilon_1+\varepsilon_2} \supset \tau_{\delta_n-\varepsilon_1-\varepsilon_2}$ and that $\tau_{\delta_n} \otimes \tau_{\varepsilon_1+\varepsilon_2} \supset \tau_{\delta_n-\varepsilon_1-\varepsilon_2}$. Similarly, if $n = 2\ell$, $\ell \geq 2$, then $\tau_{\delta_n^\pm} \otimes \tau_{3\varepsilon_1+\varepsilon_2} \supset \tau_{\delta_n^\pm-\varepsilon_1-\varepsilon_2}$ and $\tau_{\delta_n^\pm} \otimes \tau_{\varepsilon_1+\varepsilon_2} \supset \tau_{\delta_n^\pm-\varepsilon_1-\varepsilon_2}$.

Thus to complete the proof we need only observe that if $\ell \geq 2$, then $|\delta_n+\delta_k-\varepsilon_1-\varepsilon_2|^2 < |\delta|^2$ $(n = 2\ell+1)$ and $|\delta_n^\pm+\delta_k-\varepsilon_1-\varepsilon_2|^2 < |\delta|^2$ $(n = 2\ell)$. Q. E. D.

4.11. Proposition 4.10 has been communicated to the author by Greg Zuckerman. His proof is more in the spirit of the results of Chapter VI and Chapter VII.

## §5. A relationship with Matsushima's quadratic form.

5.1. Let $x_1, \ldots, x_m$ be an orthonormal basis of $\underline{p}$ and let $x_a$, $m < a \leq n$, be as in 3.1. Let $R_{ijk\ell}$ be as in 3.1. Set for $\xi = \Sigma \xi_{ij} x_i \otimes x_j$, $Q(\xi) = \Sigma R_{ijk\ell} \xi_{i\ell} \xi_{jk}$.

5.2. LEMMA. Let $\eta = \Sigma x_i \otimes s_i \in \underline{p} \otimes S$. Let $u_1, \ldots, u_d$ be an orthonormal basis of $S$ as a real inner product space relative to $\mathrm{Re}\langle , \rangle$. Set $\gamma(x_i)s_j = \Sigma \eta^r_{ij} u_r$. Set $\eta^r = \Sigma \eta^r_{ij} x_i \otimes x_j$. Let $\tau(y)x = [y, x]$, $y \in \underline{k}$, $x \in \underline{p}$. Then $\langle((\tau \otimes s)(C_k) - \langle\delta, \delta\rangle + \langle\delta_k, \delta_k\rangle)\eta, \eta\rangle = -\frac{1}{2} \sum_{r=1}^{d} Q(\eta^r, \eta^r) + \frac{1}{2}\langle\eta, \eta\rangle$.

Proof. We note that $\tau(C_k) = \frac{1}{2}I$. Also $(\tau \otimes s)(C_k) = \tau(C_k) \otimes I + {}$ $+ I \otimes s(C_k) - 2\sum_a \tau(x_a) \otimes s(x_a)$. Hence $(\tau \otimes s)(C_k) - \{\frac{1}{2} + \langle\delta, \delta\rangle - \langle\delta_k, \delta_k\rangle\} =$ $= -2\sum_a \tau(x_a) \otimes s(x_a)$. Now $s(x_a) = \frac{1}{4}\Sigma\langle[x_a, x_i], x_j\rangle \gamma(x_i)\gamma(x_j)$. Hence

$$-2\sum_a \tau(x_a) \otimes s(x_a) = -\frac{1}{2} \sum_{a,i,j} \langle[x_a, x_i], x_j\rangle \tau(x_a) \otimes \gamma(x_i)\gamma(x_j) =$$

$$= -\frac{1}{2} \sum_{a,i,j} B(x_a, [x_i, x_j])\tau(x_a) \otimes \gamma(x_i)\gamma(x_j) = \frac{1}{2} \sum_{i,j} \tau([x_i, x_j]) \otimes \gamma(x_i)\gamma(x_j).$$

This implies that

$$\langle((\tau \otimes s)(C_k) - \{\tfrac{1}{2} + \langle\delta, \delta\rangle - \langle\delta_k, \delta_k\rangle\})\eta, \eta\rangle = \frac{1}{2} \sum_{i,j,k,\ell} \langle[\tau x_i, x_j], x_k], x_\ell\rangle\langle\gamma(x_i)\gamma(x_j)s_k, s_\ell\rangle =$$

$$= -\frac{1}{2} \sum_{i,j,k,\ell} R_{ijk\ell}\langle\gamma(x_j)s_k, \gamma(x_j)s_\ell\rangle = -\frac{1}{2} \sum_{i,j,k,\ell} R_{ijk\ell} \mathrm{Re}\langle\gamma(x_j)s_k, \gamma(x_j)s_\ell\rangle =$$

$$= -\frac{1}{2} \sum_{i,j,k,\ell,r} R_{ijk\ell}\eta^r_{jk}\eta^r_{i\ell} = -\frac{1}{2}\sum_r Q(\eta^r, \eta^r).$$

Q. E. D.

5.3. We note that $F^1(\xi, \xi) = \frac{A}{2}\langle\xi, \xi\rangle + Q(\xi, \xi)$. Thus Lemma 5.2 relates the ideas of this appendix with III, §3. It also shows how different the two techniques are.

## References


[1] N. Bourbaki, Groupes et Algèbres de Lie, Chap. IV, V, VI, Act. Sci. Ind. 1337, Hermann, Paris, 1968.

[2] P. Delorme, "1-cohomologie et produits tensoriels continus de représentations", Thèse, Université Paris VI, 1975.

[3] M. DoCarmo, N. R. Wallach, "Minimal immersions of spheres into spheres", Ann. Math., 95 (1971), 43-62.

[4] T. Enright, N. R. Wallach, "The fundamental series of a real semisimple Lie algebra", to appear Acta Math.

[5] Harish-Chandra, "Discrete series for semisimple Lie groups I, II", Acta Math., 113 (1965), 241-318, 116 (1966), 1-111.

[6] R. Hotta, R. Parthasarathy, "Multiplicity formulae for discrete series", Inv. Math., 26 (1974), 133-178.

[7] S. Kaneyuki, T. Nagano, "On certain quadratic forms related to symmetric Riemannian spaces", Osaka M. J., 14 (1962), 1-20.

[8] B. Kostant, "Lie algebra cohomology and the generalized Borel-Weil theorem", Ann. of Math., 74 (1961), 329-387.

[9] R. Parthasarathy, "Dirac operator and the discrete series", Ann. of Math., 96 (1972), 1-30.

[10] W. Schmid, "Some properties of square integrable representations of semisimple Lie groups", Annals of Math., 102 (1975), 535-564.

[11] N. R. Wallach, "On the Enright, Varadarajan modules: a construction of the discrete series", Ann. Sci. Ecole Norm. Sup., 9 (1976), 81-102.

## VI Cohomology of discrete subgroups and Lie algebra cohomology

A. Borel

In this chapter, we consider the cohomology spaces $H^*(\Gamma;E)$ of a discrete subgroup $\Gamma$ of a Lie group $G$ with finitely many connected components, with coefficients in a finite dimensional complex $\Gamma$-module $(\rho,E)$ and express it in terms of relative Lie algebra cohomology. This is first done in general in §2 and yields an isomorphism

$$H^*(\Gamma;E) = H^*(\mathfrak{g}, K; I^\infty(E)) \quad , \tag{1}$$

where $K$ is a maximal compact subgroup of $G$ and

$$I^\infty(E) = I_\Gamma^G(E) = \{f \in C^\infty(G,E) \mid f(\gamma\cdot g) = \rho(\gamma)\cdot f(g) \quad (\gamma \in \Gamma;\ g \in G)\} \ . \tag{2}$$

(see 2.5). In the most important case for us, where $(\rho,E)$ is in fact a $G$-module, this takes the form

$$H^*(\Gamma;E) = H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G) \otimes E) \quad , \tag{3}$$

(see 2.7). From §4 on, we assume $\Gamma$ to be cocompact, and $E$ to be either a unitary $\Gamma$-module (§§4,5) or a $G$-module (§§6,7). The right hand side of (1) or (3) then decomposes into a finite direct sum of cohomology algebras of the type considered in the earlier chapters (see 4.2, 4.3, 6.2, 7.1). When $G^o$ is semi-simple with finite center, the results of II, III translate into properties of $H^*(\Gamma;E)$ which are discussed in §§5.7.

## §1. Manifolds

In this section we review some familiar material on manifolds, mainly to fix our notation. For more details, see for instance [15].

**1.1.** Unless otherwise stated, manifolds are $C^\infty$. Smooth is used synonymously with $C^\infty$. Let $M$ be a manifold, $L = \underline{\underline{R}}, \underline{\underline{C}}$ and $E$ a finite-dimensional vector space over $L$. Then $T(M)_m$ is the tangent space at $m \in M$, $C^\infty(M;E)$ the space of smooth functions with values in $E$, $A^q(M;E)$ the space of smooth $E$-valued differential $q$-forms on $M$ $(q = 0, 1, \ldots)$, $A_1(M)$ the space of smooth vector fields on $M$, and $A_1(M;L) = A_1(M) \otimes_{\underline{\underline{R}}} L$. If $E = L$ and $L$ is clear from the context, it will often be omitted from the notation. We have

$$(1) \qquad C^\infty(M;E) = A^o(M;E) \quad , \qquad A^q(M;E) = A^q(M;L) \otimes_L E \quad .$$

Let $\omega \in A^q(M;E)$. It associates to each $m \in M$ an element of $\mathrm{Hom}(\Lambda^q T(M)_m, E)$. The value of $\omega$ on a $q$-vector $y$ at $m$ will sometimes be denoted $\omega(m; y)$. Often, $\omega$ will be viewed as a $C^\infty(M;L)$-multilinear alternating map on $A_1(M;L)$, with values in $C^\infty(M;E)$. If $x \in A_1(M,L)$, the interior product $i_x\omega$ is defined by

$$(2) \quad i_x\omega(x_1, \ldots, x_{q-1}) = \omega(x, x_1, \ldots, x_{q-1}) \quad , \qquad (x_1, \ldots, x_{q-1} \in A_1(M;L)) \quad .$$

The exterior differential $d : A^q(M;E) \longrightarrow A^{q+1}(M;E)$ is given by

$$(3) \quad d\omega(x_o, \ldots, x_q) = \Sigma(-1)^i y_i \cdot \omega(x_o, \ldots, \hat{x}_i, \ldots, x_q) + \Sigma_{i<j}\omega([x_i, x_j], x_o, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_q) \quad ,$$

where $[\ ,\ ]$ refers to the bracket of vector fields, and $\wedge$ means omission of the corresponding argument.

1.2. If $N$ is a manifold and $\pi : M \longrightarrow N$ a smooth map, then $d\pi_m : T(M)_m \longrightarrow T(N)_{\pi(m)}$ is the differential of $\pi$ at $m$. The map $\pi$ induces a homomorphism ${}^t\pi : \omega \longmapsto \omega \circ \pi$ of $A^p(N;E)$ into $A^p(M;E)$, given by

$$(1) \qquad (\omega \circ \pi)(m,y) = \omega(\pi(m),\, d\pi_m(y)) \quad , \qquad (m \in M;\ y \in \Lambda^q T(M)_m) \quad .$$

1.3. Let $\widetilde{E}$ be the local system of coefficients on $M$ associated to a representation on $E$ of the fundamental group of $M$ (or of a subgroup of it). Then similarly, $C^\infty(M;\widetilde{E})$ denotes the space of $\widetilde{E}$-valued $C^\infty$- functions on $M$, i.e. of smooth cross-sections of $\widetilde{E}$, and $A^q(M;\widetilde{E})$ the space of smooth $\widetilde{E}$-valued $q$-forms on $M$. Since the transition functions of $\widetilde{E}$ are locally constant, the exterior differentiation still makes sense on $A^q(M;\widetilde{E})$ and 1.1(3) remains valid.

1.4. Lie derivative. Let $x \in A_1(M;L)$. Then $\theta_x$ denotes the Lie derivative in the direction $x$ [15:2.24]. In particular

$$(1) \qquad \theta_x f = x \cdot f \quad , \qquad (f \in C^\infty(M;E)) \quad ,$$

$$(2) \qquad \theta_x y = [x,y] \quad , \qquad (y \in A_1(M)) \quad ,$$

$$(3) \qquad (\theta_x \omega)(x_1,\ldots,x_q) = \theta_x(\omega(x_1,\ldots,x_q)) - \Sigma_i \omega(x_1,\ldots,[x,x_i],\ldots,x_q) \quad ,$$

$$(x,x_1,\ldots,x_q \in A_1(M);\ \omega \in A^q(M;E)) \quad .$$

The vector field $x$ defines (locally) a one-parameter group of transformations $\{\varphi_t\}$ ($t$ in a neighborhood of the origin in $\underline{R}$) and we have

$$(4) \qquad \theta_x f(m) = \frac{d}{dt}\, f(\varphi_t(m))\Big|_{t=0} \quad ,$$

$$(5) \qquad \theta_x(y)(m) = \frac{d}{dt}\, d\varphi_{-t}(y_{\varphi_t(m)})\Big|_{t=0} \quad ,$$

$$(6) \qquad (\theta_x \omega)(m,y) = \frac{d}{dt}\, \omega(\varphi_t(m),\ d\varphi_t(y(m)))\Big|_{t=0} .$$

The operators $d$, $i_x$, $\theta_x$ are related by

$$(7) \qquad d\cdot i_x + i_x\cdot d = \theta_x .$$

1.5. Let $G$ be a group. Assume it operates by diffeomorphisms on $M$ and via a linear representation $\rho$ on $E$. Then we let $G$ operate on $A^q(M;E)$ by

$$(g\ o\ \omega)(m,x_1,\dots,x_p)) = \rho(g)(\omega(g^{-1}m, g^{-1}x_1,\dots,g^{-1}x_q)) ,$$

$$(m \in M;\ x_1,\dots,x_q \in T(M)\hat{m};\ g \in G)$$

The space of invariants $q$-forms is denoted $A^q(M;E)^G$. Thus

$$\omega \in A^q(M;E)^G \Longleftrightarrow \rho(g)\ o\ \omega = \omega(g\cdot x_1,\dots,g\cdot x_q) \qquad (g \in G;\ x_1,\dots x_q \in A_1(M))$$

1.6. Assume now $M = G$ to be a Lie group. We let $l_g$ (resp. $r_g$) denote left (resp. right) translation by $g$. In particular,

$$(1) \quad l_g f(x) = f(g^{-1}\cdot x) , \quad r_g(f(x) = f(x\cdot g) \qquad (f \in C^\infty(M;E);\ g,x \in G) .$$

If $H$ is a closed subgroup, then $G/H$ (resp. $H\backslash G$) is the space of left (resp. right) cosets $x\cdot H$ (resp. $H\cdot x$), $(x \in G)$. By definition, the Lie algebra $\underline{g}$ of $G$ is the Lie algebra of left-invariant vector fields. As usual, the tangent space $T(G)_1$ is identified to $\underline{g}$ by assigning to $x \in T(G)_1$ the unique left-invariant vector field which is equal to $x$ at 1. The one-parameter group $\{\varphi_t\}$ associated to $x \in \underline{g}$ is the group of right translations by the elements $e^{tx}$ $(t \in \underline{\underline{R}})$. In particular

$$xf(g) = \frac{d}{dt}\, f(g\cdot e^{tx})\Big|_{t=0} .$$

## §2. Discrete subgroups

From now on, $G$ is a Lie group with finitely many connected components, $G^o$ its identity component, $K$ a maximal subgroup of $X = G/K$, $\Gamma$ a discrete subgroup of $G$, and $(\rho, E)$ a finite dimensional real or complex linear representation of $\Gamma$.

We recall that the maximal compact subgroups of $G$ are conjugate, and that $X$ is homeomorphic to euclidean space. If $G$ is connected, this is the well-known Cartan-Iwawasa-Maleev theorem. The extension to groups with finite component group is due to G. D. Mostow [11].

2.1. Let $M$ be any compact subgroup of $G$. Then $\Gamma$ acts properly on $G/M$ by left translations (i.e. for every compact set $C$, $\{\gamma \in \Gamma | \gamma C \cap C \neq \emptyset\}$ is finite). If $\Gamma$ has no torsion, then it acts freely (no $\gamma \neq 1$ has a fixed point). Conversely, if $\Gamma$ acts freely, then its elements of finite order act trivially, hence are contained in the intersection of all the conjugates of $K$. If $G$ is connected, these elements belong to the center of $G$.

2.2. THEOREM. The space $H^*(\Gamma;E)$ is canonically isomorphic to $H^*(A(X;E)^\Gamma)$.

This is well known. However, since it is basic for us, we recall the proof. Assume first that $\Gamma$ acts freely. Then $\Gamma\backslash X$ is a smooth manifold and, since $X$ is contractible, it is also an Eilenberg-MacLane space $K(\Gamma,1)$. We have then

$$H^*(\Gamma;E) = H^*(\Gamma\backslash X;\tilde{E}) \quad , \tag{1}$$

where $\tilde{E}$ is the local system on $\Gamma\backslash X$ defined by $(\rho,E)$. Let $\pi : X \longrightarrow \Gamma\backslash X$ be the canonical projection. Then it is immediate that $\omega \longmapsto \omega \circ \pi$ defines an isomorphism of $A(\Gamma\backslash X;\widetilde{E})$ with $A(X;E)^{\Gamma}$. Our assertion in this case follows then from de Rham's theorem (with a locally constant sheaf of coefficients).

Assume now that $\Gamma$ has a torsion-free normal subgroup $\Gamma'$ of finite index. Then $\Gamma/\Gamma'$ acts on $H^*(\Gamma';E)$ and we have

$$H^*(\Gamma;E) = (H^*(\Gamma';E))^{\Gamma/\Gamma'}, \tag{2}$$

as follows e.g. from the Hochschild-Serre spectral sequence. On the other hand

$$A(X;E)^{\Gamma} = (A(X;E)^{\Gamma'})^{\Gamma/\Gamma'} \quad . \tag{3}$$

Since taking invariants under a finite group is an exact functor in characteristic zero, this gives

$$H^*(A(X;E))^{\Gamma} = H^*(A(X;E)^{\Gamma'})^{\Gamma/\Gamma'} \quad , \tag{4}$$

and (2), (4) provide a reduction to the first case considered.

This suffices for our needs. To be complete, we treat the general case too. For $q \in \underline{\underline{N}}$, let $\underline{F}^q$ be the sheaf on $\Gamma\backslash X$ associated to the presheaf $U \longmapsto A^q(U;E)^{\Gamma}$, ($U$ open in $\Gamma\backslash X$). Since the isotropy groups of $\Gamma$ on $X$ are finite, it follows by a simple averaging process from the Poincaré lemma on $X$ that $\{\underline{F}^q\}$ is a resolution of the constant sheaf $(\Gamma\backslash X) \times L$ on $\Gamma\backslash X$. Using partition of unity, one sees moreover that it is a fine sheaf. Since $A^q(X;E)^{\Gamma}$ is just the space of global cross-sections of $\underline{F}^q$, this gives

$$H^*(\Gamma\backslash X;\widetilde{E}) = H^*(A(X;E)^{\Gamma}) \quad . \tag{5}$$

On the other hand, since the isotropy groups $\Gamma_x$ ($x \in X$) of $\Gamma$ on $X$ are finite, the groups $H^i(\Gamma_x;E) = 0$ are all zero for $i > 0$. By general principles, [6:p.204], (1) is still valid, and our assertion follows from (1) and (5).

2.3. The quotient $G \times_\Gamma E$ of $G \times E$ by the equivalence relation $(g,e) \sim (\gamma\cdot g, \rho(\gamma)\cdot e)$ $(g \in G;\ e \in E;\ \gamma \in \Gamma)$, is the total space of a (locally flat) vector bundle $\tilde{E}$ over $\Gamma\backslash G$ with typical fibre $E$ and structure group $\Gamma$. We let $I^\infty(E) = C^\infty(G;E)^\Gamma$ be the space of its smooth cross sections, i.e.

$$(1) \qquad I^\infty(E) = \{f \in C^\infty(G;E) \mid f(\gamma\cdot g) = \rho(\gamma)\cdot f(g) \quad (\gamma \in \Gamma;\ g \in G)\} .$$

Otherwise said, $I^\infty(E)$ is the space of the representation $I_\Gamma^G(E)$ induced from $(\rho,E)$ to $G$, in the $C^\infty$-sense.

Assume now that $(\rho,E)$ is the restriction to $\Gamma$ of a representation of $G$, still denoted in the same way. Then the map $f \longmapsto F$ of $C^\infty(G;E)$ into itself, given by

$$(2) \qquad F(g) = \rho(g)^{-1}\cdot f(g) \quad , \qquad (g \in G;\ f \in C^\infty(G;E))$$

is immediately seen to yield an isomorphism of $G$-modules

$$(3) \qquad I^\infty(E) \xrightarrow{\sim} C^\infty(\Gamma\backslash G;L) \otimes_L E \quad ,$$

where the $G$-module structure on the right hand side is the tensor product of the right regular representation on $C^\infty(\Gamma\backslash G;L)$ and of $\rho$.

2.4. For $g \in G$, the left translation by $g^{-1}$ provides a canonical isomorphism of $T(G)_g$ with $\underline{g} = T(G)_1$, whence an identification

$$(1) \qquad \iota : A^q(G;E) = \mathrm{Hom}(\Lambda^q\underline{g},\ C^\infty(G;E)) = C^q(\underline{g};\ C^\infty(G;E)) \ , \qquad (q \in \underline{N}) .$$

Let $\omega \in A^q(G;E)^\Gamma$. Then, for $y \in \Lambda^q\underline{g}$, we have

(2) $$\omega(\gamma\cdot g,y) = \rho(\gamma)\cdot\omega(g,y) \quad ,$$

hence $\omega$ is identified to an element of $\mathrm{Hom}(\Lambda^q\underline{g}, I^\infty(E))$. The converse is clear, so that we get an isomorphism, also to be denoted $\iota$ :

(3) $$\iota : A^q(G;E)^\Gamma \xrightarrow{\sim} C^q(\underline{g}; I^\infty(E)) \quad .$$

It follows from 1.1(3) and I, §1 and the isomorphisms $\iota$ $(q \in \underline{\underline{N}})$ commute with the differentials, hence give rise to an isomorphism

(4) $$\iota^* : H^*(A(G;E)^\Gamma) \xrightarrow{\sim} H^*(\underline{g}; I^\infty(E)) \quad .$$

Let $\widetilde{E}$ be the local system on $\Gamma\backslash G$ defined by $(\rho,E)$. Then

(5) $$A(G;E)^\Gamma \cong A(\Gamma\backslash G;\widetilde{E}) \quad ,$$

so that the left-hand side of (4) can be viewed as the cohomology of $\Gamma\backslash G$ with coefficients in the locally constant sheaf defined by $\widetilde{E}$.

If now $(\rho,E)$ comes from a representation of $G$, then, by 2.3, $\omega \longmapsto \omega^o$, where $\omega^o(g) = \rho(g)^{-1}\omega(g)$, $(g \in G)$, yields an isomorphism

(6) $$A(G;E)^\Gamma \xrightarrow{\sim} C^*(\underline{g}; C^\infty(\Gamma\backslash G;L) \otimes E) \quad ,$$

whence also

(7) $$H^*(\Gamma\backslash G;\widetilde{E}) \xrightarrow{\sim} H^*(\underline{g}; C^\infty(\Gamma\backslash G;L) \otimes E) \quad .$$

We want now to divide by $K$ on the right and relate similarly the cohomology of $\Gamma$ and relative Lie algebra-cohomology.

<u>2.5</u>. PROPOSITION. <u>Let</u> $\pi : G \longrightarrow X = G/K$ <u>be the canonical projection</u>. <u>Then</u> ${}^t\pi : \omega \longmapsto \omega \circ \pi$ <u>induces an isomorphism of graded complexes</u>

*of* $A(X;E)^{\Gamma}$ *onto* $C^*(\underline{g}, K; I^{\infty}(E))$. *In particular,* $H^*(\Gamma;E)$ *is canonically isomorphic to* $H^*(\underline{g}, K; I^{\infty}(E))$.

The map ${}^t\pi$ clearly commutes with left translations, hence sends $A(X;E)^{\Gamma}$ into $A(G;E)^{\Gamma}$. Let $A_o$ be its image. Since $\pi$ is constant along the left $K$-cosets, $A_o$ consists of the elements of $A(G;E)^{\Gamma}$ which are right invariant under $K$ and annihilated by the interior products $i_x$ $(x \in \underline{k})$. It then follows from 2.4 and the definitions that $\iota \circ {}^t\pi$ induces an isomorphism

(1) $$A(X;E)^{\Gamma} = C^*(\underline{g}, K; I^{\infty}(E)) \quad .$$

Our assertion now follows from 2.2.

2.6. Remark. If we associate to $e \in E^{\Gamma}$ the constant function on $G$ equal to $e$, then we get a map $E^{\Gamma} \longrightarrow I^{\infty}(E)^G$, which is readily seen to be bijective. The inclusion $I^{\infty}(E)^G \subset I^{\infty}(E)$ then yields a canonical homomorphism

(1) $$j^* : H^*(\Gamma;E^{\Gamma}) \longrightarrow H^*(\underline{g}, K; I^{\infty}(E)) \quad .$$

2.7. COROLLARY. *Assume that* $(\rho,E)$ *extends to a representation of* $G$. *Then the map which associates to* $\omega \in A(X;E)^{\Gamma}$ *the form* $\omega^o : g \longmapsto \rho(g^{-1})\cdot(\omega \circ \pi)(g)$ *induces an isomorphism of* $A(X;E)^{\Gamma}$ *onto* $C^*(\underline{g}, K; C^{\infty}(\Gamma\backslash G;L) \otimes E)$ *and an isomorphism of* $H^*(\Gamma;E)$ *onto* $H^*(\underline{g}, K; C^{\infty}(\Gamma\backslash G;L) \otimes E)$.

By 2.3, the map $f \longmapsto F$ given by $F(g) = \rho(g^{-1})\cdot f(g)$ induces a $G$-equivariant isomorphism of $I^{\infty}(E)$ onto $C^{\infty}(\Gamma\backslash G;L) \otimes E$. The corollary then follows from the proposition.

We note further that if we go back to the definitions, we see that the image of $A(X;E)^{\Gamma}$ in $A(G;E)$ under the map $\omega \longmapsto \omega^{o}$ consists of all the $\eta \in A(G;E)$ which satisfy the three following conditions

$$
(1) \qquad
\begin{aligned}
&\ell_{\gamma} \circ \eta = \eta , && (\gamma \in \Gamma) , \\
&r_{k} \circ \eta = \rho(k)^{-1} \cdot \eta && (k \in K) , \\
&i_{x}\eta = 0 && (x \in \underline{k}) .
\end{aligned}
$$

2.8. Remark. We have now an identification $e \longmapsto 1 \otimes e$ of $E$ onto the space of constant $E$-valued functions on $\Gamma\backslash G$, whence a natural homomorphism

$$
(1) \qquad j^{*} : H^{*}(\Gamma;E) \longrightarrow H^{*}(\underline{g}, K; C^{\infty}(\Gamma\backslash G;L) \otimes E) .
$$

2.9. As remarked in [9:§3], the case of 2.7 could be subsumed to that of 2.5 by adding a compact factor to $G$.

2.7 is in substance proved in [9, 10], although stated there under narrower assumptions.

## §3. Remarks on unitary representations

In order not to interrupt the discussion of cocompact subgroups, we collect here some known facts and notions about unitary representations.

3.1. Let $(\pi,H)$ be a unitary representation of $G$ in a Hilbert space $H$. We let $H^{\infty}$ (resp. $H_{K}$) be the space of $C^{\infty}$-vectors (resp. $K$-finite vectors) of $H$. The space $H_{K}$ is dense in $H$. The space $H^{\infty}$ is endowed with the topology defined by the semi-norms $v \longmapsto ||xv||$ $(x \in U(\underline{g}), v \in H)$. It is locally convex, complete and a Fréchet space, and $H_{K} \cap H^{\infty}$ is dense in it. The latter is a $(\underline{g},K)$-module. If it is admissible, then $H_{K} \subset H^{\infty}$. This is in particular the case if $G$ is semi-simple, $G^{o}$ has finite center, and $\pi$ is (topologically) irreducible. The $(\underline{g},K)$-module $H_K$ is then algebraically irreducible. We note that we always have

$$(1) \qquad C^*(\underline{g}, K; H^\infty) = C^*(\underline{g}, K; H_K^\infty) , \quad \text{where } H_K^\infty = H_K \cap H^\infty ,$$

since the image of any element of $\mathrm{Hom}_K(\Lambda^q(\underline{g}/\underline{k}), H^\infty)$ consists of K-finite elements, hence

$$(2) \qquad H^*(\underline{g}, K; H^\infty) = H^*(\underline{g}, K; H_K^\infty) .$$

3.2. The set of equivalence classes of irreducible unitary representations of a locally compact group $M$ is denoted $\hat{M}$. Often, we shall make no notational distinction between $\pi \in \hat{M}$ and a representative of $\pi$. In particular, $H_\pi$ will denote the space of a representative of $\pi$. Objects attached to an irreducible unitary representation, but which depend only on the equivalence class of the representation, such as the infinitesimal or central character, will often be labeled by the equivalence class.

The infinitesimal (resp. central) character of the trivial representation is denoted $\chi_o$ (resp. $\omega_o$).

3.3. Let $G = G_1 \times \ldots \times G_t$ be a direct product. Let $(\pi,H)$ be an irreducible unitary representation. Then there exist irreducible unitary representations $(\pi_i,H_i)$ of $G_i$ such that $\pi$ is the Hilbert tensor product of the $\pi_i$. We write this

$$(1) \qquad \pi = \pi_1 \hat{\otimes} \ldots \hat{\otimes} \pi_t , \qquad H = H_1 \hat{\otimes} \ldots \hat{\otimes} H_t .$$

The $\pi_i$ are determined up to equivalence. Hence there is a canonical identification $\hat{G} = \hat{G}_1 \times \ldots \times \hat{G}_t$. The decomposition of $\pi \in \hat{G}$ will be written as in (1).

The group $K$ is the direct product of its intersections $K_i = K \cap G_i$ with the $G_i$'s. If $\pi$ is admissible then we have an ordinary tensor product decomposition:

(2) $$H_K = H_{1,K_1} \otimes \cdots \otimes H_{t,K_t} .$$

## §4. $\Gamma$ cocompact, $E$ a unitary $\Gamma$-module

4.1. We keep the notation and assumptions of §2, and assume moreover $\Gamma$ to be cocompact, and $E$ to be a unitary $\Gamma$-module, $L = \underline{\underline{C}}$. The group is then necessarily unimodular. Let $dx$ denote a Haar measure on $G$, and the associated measure on $\Gamma\backslash G$, $(\ ,\ )_E$ the scalar product on $E$. If $u, v \in I^\infty(E)$ then

(1) $$(u(\gamma\cdot x), v(\gamma\cdot x))_E = (u(x), v(x))_E \quad , \qquad (x \in G;\ \gamma \in \Gamma) \quad ,$$

hence this scalar product $f_{u,v}$ is a function on $\Gamma\backslash G$, and we can define a global scalar product $(u,v)$ by integrating it over $\Gamma\backslash G$. The completion of $I^\infty(E)$ under this scalar product is the space $I_2(E)$ of square integrable cross-sections of the bundle $G \times_\Gamma E \longrightarrow \Gamma\backslash G$ (see 2.3). The space $I_2(E)$ is a unitary $G$-module with respect to right translations. It follows from Sobolev's lemma that $(I_2(E))^\infty = I^\infty(E)$ and that this identification carries the topology of 3.1 onto the $C^\infty$-topology.

By a theorem of Gelfand and Piateckii-Šapiro [5:1; §3], $I_2(E)$ decomposes into a discrete Hilbert direct sum with finite multiplicities of irreducible $G$-modules. We can write

(2) $$I_2(E) = \widetilde{\bigoplus}_{\pi \in \hat{G}} m(\pi,\Gamma,E)\ H_\pi \quad ,$$

where the $m(\pi,\Gamma,E)$ are natural numbers. We have then also a topological direct sum decomposition

$$(3) \qquad I^{\infty}(E) = \widetilde{\bigoplus}_{\pi \in \hat{G}} m(\pi,\Gamma,E)\ H_{\pi}^{\infty} .$$

<u>4.2</u>. THEOREM. <u>We have</u>

$$(1) \qquad H^{*}(\Gamma,E) = \bigoplus_{\pi \in \hat{G}} m(\pi,\Gamma,E)\ H^{*}(\mathfrak{g}, K; H_{\pi,K}^{\infty}) \ ,$$

<u>where the sum is finite and may be restricted to the</u> $\pi \in \hat{G}$ <u>which have trivial infinitesimal and central characters</u>. <u>The natural homomorphism</u> $j^{*} : H^{*}(\mathfrak{g}, K; E^{\Gamma}) \longrightarrow H^{*}(\Gamma,E)$ <u>of</u> 2.6 <u>is injective</u>. <u>Its image is the contribution of the trivial representation</u> $\pi_o$ <u>of</u> $G$ <u>to</u> (1) <u>and we have</u> $m(\pi_o,\Gamma,E) = \dim E^{\Gamma}$.

By 2.5 and 4.1(3), we have

$$(2) \qquad H^{*}(\Gamma;E) = H^{*}(\mathfrak{g}, K; \widetilde{\bigoplus}_{\pi} m(\pi,\Gamma,E)\ H_{\pi}^{\infty}) \ .$$

The main point of the proof is to show that the topological direct sum in the right hand side can be replaced by an algebraic direct sum. For $\pi \in \hat{G}$, $q \in \underline{\mathbb{N}}$, let

$$(3) \qquad C_{\pi}^{q} = C^{q}(\mathfrak{g}, K; m(\pi,\Gamma,E)\ H_{\pi}^{\infty}) \ , \qquad C_{\pi}^{*} = \bigoplus_{q} C_{\pi}^{q} \ .$$

Let $S \subset G$ be finite; put

$$(4) \qquad C_{S}^{*} = \bigoplus_{\pi \in S} C_{\pi}^{*} \ , \qquad C_{S'}^{*} = C^{*}(\mathfrak{g}, K; \widetilde{\bigoplus}_{\pi \in \hat{G}-S} m(\pi,\Gamma,E)\ H_{\pi}^{\infty}) \ .$$

Then $C^{*}(\mathfrak{g}, K; I^{\infty}(E)) = C_{S}^{*} \oplus C_{S'}^{*}$, hence

$$(5) \qquad H^{*}(\Gamma,E) = \bigoplus_{\pi \in S} m(\pi,\Gamma,E)\ H^{*}(\mathfrak{g}, K; H_{\pi}^{\infty}) \oplus H^{*}(C_{S'}^{*}) \ .$$

The space $H^*(\Gamma;E)$ is the cohomology of $\Gamma\backslash X$ with coefficients in a local system. The space $\Gamma\backslash X$ is compact, locally contractible, in fact may be triangulated, hence

(6) $$\dim H^*(\Gamma;E) < \infty \quad .$$

In view of (5), there exists then $S \subset G$ such that $H^*(C^*_\pi) = 0$ for $\pi \notin S$. Assuming $S$ to be so chosen, we want to prove that $H^*(C^*_{S'}) = 0$.

The map $d : C^{q-1}_{S'} \longrightarrow C^q_{S'}$ is continuous. This follows directly from its definition (I, §1) and that of the topology on $I^\infty(E)$ ~~(4.1)~~. Therefore

(7) $$Z^q_{S'} = C^q_{S'} \cap \ker d \quad ,$$

is closed. Moreover, since $H^*(C^*_{S'})$ is finite dimensional (by (5) and (6)), the existence of the exact sequence

(8) $$0 \longrightarrow dC^{q-1}_{S'} \longrightarrow Z^q_{S'} \longrightarrow H^q(C^*_{S'}) \longrightarrow 0 \quad ,$$

implies that $dC^{q-1}_{S'}$ has a closed supplement in $Z^q_{S'}$, hence is closed (as follows e.g. from Cor.1, p.25 in [4]).

Let now $z \in Z^q_{S'}$. It can be written as a convergent sum

(9) $$z = \widetilde{\bigoplus}_{\pi \in S'} z_\pi \quad , \qquad \text{with} \quad z_\pi \in Z^q_\pi = C^q_\pi \cap \ker d \quad .$$

By our assumption on $S$, there exists for each $\pi \in S'$ an element $c_\pi \in C^{q-1}_\pi$ such that $z_\pi = dc_\pi$. Thus every finite sum of the $z_\pi$'s is a coboundary, hence $z_\pi$ is in the closure of $dC^{q-1}_{S'}$. Since the latter has been shown to be closed, it follows that $z_\pi$ is a coboundary, hence $H^*(C^*_{S'}) = 0$. In view of 3.1(2), this proves (1). The second assertion then follows from

(6) and (I, 5.3); in view of 2.6, the third one is then clear.

4.3. COROLLARY. *Assume* G *to be semi-simple with finite center. Then*

$$H^q(\Gamma;E) = \bigoplus_{\pi \in \hat{G},\ \chi_\pi = \chi_o,\ \omega_\pi = \omega_o} m(\pi,\Gamma,E)\ \mathrm{Hom}(\Lambda^q(\underline{g}/\underline{k}),\ H_{\pi,K}) \ , \qquad (q \in \underline{\underline{N}}) \ .$$

Indeed, we have $H_{\pi,K} = H^{\infty}_{\pi,K}$ (3.1). The corollary follows then from 4.2 and (II, 3.1).

4.4. Remarks. (I) If G is connected, and $E = \underline{\underline{C}}$, this relation is due to Y. Matsushima [8], except for the fact that the sum in [8] is over the $\pi$ which map the Casimir element into zero.

(2) The proof in 4.2 shows that d has closed image in $C^*_{S'}$, hence also in $C^*$, since each $C^*_\pi$ is finite dimensional (II, 3.4); it also applies to the dual operator $\partial$ (II, 2.3). We have therefore a Hodge decomposition

$$C^q = C^q(\underline{g}, K;\ I^{\infty}(E)) = \mathcal{H}^q \oplus dC^{q-1} \oplus \partial C^{q+1} \ , \qquad \text{where } \mathcal{H}^q = \ker \partial \cap \ker d \cap C^q \ ,$$

as in the case of an admissible module (II, 3.4), so that $H^q(\Gamma,E)$ may be identified to the space of harmonic q-forms in $C^q$. In this case, the isomorphisms of 2.4 identify harmonic forms in $C^*(g, K;\ I^{\infty}(E))$, in the sense of (II, §2), with $\widetilde{E}$-valued harmonic forms in $\Gamma\backslash X$ (say if $\Gamma$ is torsion-free, otherwise one has to invoke the theory of harmonic forms on V-manifolds). This the above yields a proof of the Hodge theorem in this case.

(3) Assume $(\rho,E)$ is irreducible. Then the center $C(\Gamma)$ of $\Gamma$ acts by scalars. If it does not act trivially, then $H^*(\Gamma;E) = 0$. This follows by the same argument used in §4 of I, in the category of modules over the group algebra of $\Gamma$. If N is a finite central subgroup of $\Gamma$ which acts trivially on E, then the Hochschild-Serre spectral sequence of

$\Gamma$ mod N shows that $H^*(\Gamma;E) = H^*(\Gamma/N;E)$.

Assume now G to be connected. Then the formula of 4.3 involves effectively only representations of G which are trivial on the center C(G) of G, i.e., only representations of the adjoint group Ad $\mathfrak{g}$ of G. If C(G) is finite, the computation of $H^*(\Gamma;E)$ may therefore be reduced to the case where G is its own adjoint group.

## §5. G semi-simple, $\Gamma$ cocompact, E a unitary $\Gamma$-module

We assume now that $G^o$ is semi-simple with finite center. $\Gamma$ and $(\rho,E)$ are as in §4.

5.1. We say that $G^o$ has no compact factor if it has no infinite normal compact subgroup. A discrete subgroup L of G is said to be irreducible if the image of $\Gamma \cap G^o$ under any surjective morphism $f : G^o \longrightarrow G'$ with non trivial image and non-compact kernel is non-discrete. If G/L has finite invariant volume, and $G^o$ has no compact factor, then this condition implies in fact that $f(L')$ is dense in $G'$ [1].

5.2. LEMMA. Assume that G is connected with no compact factor, has a direct product decomposition $G = G_1 \times \ldots \times G_t$ and that $\Gamma$ is irreducible in G. Let $(\pi,H)$ be a unitary irreducible representation of G which occurs in $I_2(E)$, and $\pi = \pi_1 \hat{\otimes} \ldots \hat{\otimes} \pi_t$ its canonical decomposition (3.3(1)). If $\pi$ is not trivial, then no $\pi_i$ is.

If E is a direct sum of unitary $\Gamma$-modules, then $I_2(E)$ decomposes accordingly, so we may assume E to be simple. Assume that one of the $\pi_i$'s, say $\pi_1$, is trivial. We have to show that $\pi$ is trivial, too. Since $\pi_1$ is Trivial, $H_\pi^\infty$ consists of functions which are right-invariant under $G_1$. Since $G_1$ is normal in G, they are also left-invariant under $G_1$.

Let $G' = G_2 \times \ldots \times G_t$, $\pi' : G \longrightarrow G'$ the natural projection and $\Gamma' = \pi'(\Gamma)$. Then $\Gamma'$ is dense in $G'$ (see above). By definition, $H_\pi^\infty$ consists of smooth functions $f : G \longrightarrow E$ such that

$$(1) \qquad f(\gamma \cdot g) = \rho(\gamma) \cdot f(g) \quad , \qquad (\gamma \in \Gamma;\ g \in G) \ .$$

If $\gamma \in \Gamma \cap G_1$, then $f(\gamma \cdot g) = f(g)$, hence $\rho(\gamma)$ is the identity on any element $e \in E$ of the form $f(g)$, for some $g \in G$, $f \in H_\pi^\infty$. Since $E$ is assumed to be irreducible, (1) implies that the set of such $e$'s spans $E$, hence $\rho(\gamma) = Id$, and $\rho$ may be viewed as a representation of $\Gamma'$. Assume that $\gamma_n \in \Gamma$ is a sequence such that $\pi'(\gamma_n) \longrightarrow 1$. Then for $g \in G'$, $f \in H_\pi^\infty$, we have $f(\gamma_n \cdot g) \longrightarrow f(g)$ by continuity and left $G_1$-invariance. By (1), this shows that $\rho(\gamma_n) \cdot f(g) \longrightarrow f(g)$. Since the $f(g)$'s span $E$, we see that $\rho(\gamma_n) \longrightarrow 1$, i.e., the representation $\rho$ of $\Gamma'$ is continuous for the topology induced by that of $G'$. But then $\rho$ extends to a finite dimensional unitary representation of $G'$, hence is trivial. By (1), the elements of $H_\pi^\infty$ are then left-invariant under $\Gamma$, hence under $G_1 \cdot \Gamma = G_1 \times \Gamma'$, which is dense in $G$. Then $H_\pi^\infty$ is the space of constant functions.

5.3. PROPOSITION. Assume that $G$ is connected and has no compact factor. Let $\underline{g} = \underline{g}_1 \times \ldots \times \underline{g}_t$ be the decomposition of $\underline{g}$ into simple ideals. Assume $\Gamma$ to be irreducible in $G$. Then the natural homomorphism $j^* : H^q(\underline{g}, K; E^\Gamma) \longrightarrow H^q(\Gamma;E)$ (see 2.6) is an isomorphism for $q < \Sigma_i (M(\underline{g}_i) + 1)$, where $M(\underline{g}_i)$ is as in III, 4.3. [illegible] for [illegible].

(We recall that $M(\underline{g}_i)$ is the greatest integer such that

$$H^q(\underline{g}_i, \underline{k}_i; V) = 0 \ , \quad \text{for} \quad q \le M(\underline{g}_i) \ ,$$

and any non-trivial irreducible admissible unitary $(\underline{g}_i, \underline{k}_i)$-module $V$.

In particular $M(\mathfrak{g}_i) \geq m(\mathfrak{g}_i)$, where $m(\mathfrak{g}_i)$ is Matsushima's constant (III, 3.2).

Using 4.4(3), we see that it suffices to prove 5.3 when $G = \mathrm{Ad}\,\mathfrak{g}$. By the theorem, $H^q(\Gamma;E)$ is the sum of $H^q(\mathfrak{g}, \underline{k}; E^\Gamma)$ and of the spaces $H^q(\mathfrak{g}, \underline{k}; H_{\pi,K})$, with $\pi \in \hat{G}$, $\pi$ non-trivial. Since $G = \mathrm{Ad}\,\mathfrak{g}$, the decomposition of $\mathfrak{g}$ into simple ideals corresponds to one of $G$ as a product of simple groups and 5.2 obtains; we can therefore apply III, 4.4, which shows that those groups vanish in the range indicated.

5.4. COROLLARY. a) *If* $(\rho,E)$ *is irreducible and non trivial, then* $H^q(\Gamma;E) = 0$ *for* $q < \Sigma_i(M(\mathfrak{g}_i) + 1)$.

b) *The homomorphism* $j^*$ *is an isomorphism of* $H^q(\mathfrak{g}, \underline{k}; \underline{C})$ *onto* $H^q(\Gamma;\underline{C})$ *for* $q < \Sigma_i(M(\mathfrak{g}_i) + 1)$.

These are in fact special cases of 5.3. Since $M(\mathfrak{g}_i) \geq m(\mathfrak{g}_i)$ we see in particular that if $E = \underline{C}$ is the trivial $\Gamma$-module and $\mathfrak{g}$ is simple, then $j^*$ is an isomorphism at least up to $m(\mathfrak{g})$, a result due to Y. Matsushima [7:Thm 1].

5.5. The space $H^1(\mathfrak{g}, K; L)$ is zero for any finite dimensional $(\mathfrak{g},K)$-module $L$. Thus, in particular $H^1(\Gamma;E) = 0$ for any $E$ if $t \geq 2$. Assume now that $t = 1$, i.e. $\mathfrak{g}$ is simple non compact. Then 4.3 and the results quoted in (III, 3.7) imply that $H^1(\Gamma;E) = 0$ if $\mathfrak{g}$ is not of type $\underline{so}(n,1)$ or $\underline{su}(n,1)$, in particular if the split rank $\mathrm{rk}_{\underline{R}}\mathfrak{g}$ of $\mathfrak{g}$ is $> 1$. This proves the first assertion of the following corollary:

5.6. COROLLARY. *Let* $G$ *and* $\Gamma$ *be as in* 5.3. *Assume that* $\mathrm{rk}_{\underline{R}}\mathfrak{g} \geq 2$. *Then* $H^1(\Gamma;E) = 0$. *Let* $Q$ *be a compact connected Lie group. Then, up to inner automorphisms of* $Q$, *there are only finitely many homomorphisms of* $\Gamma$ *into* $L$.

Since $\Gamma$ is finitely generated, the second assertion is a consequence of the first one and of the following lemma. (See [2:1.1] for a similar proof).

5.7. LEMMA. *Let* $L$ *be a finitely generated group and* $Q$ *a compact connected Lie group. Assume that for every finite dimensional unitary representation* $(\rho, E)$ *of* $L$, *the group* $H^1(\Gamma;E)$ *is zero. Then, up to inner automorphisms of* $Q$, *there are only finitely many homomorphisms of* $L$ *into* $Q$.

The space $R(L,Q)$ of homomorphisms of $L$ into $Q$ may be viewed as the set of real points of an affine algebraic variety defined over $\mathbb{R}$, namely the space $R(L,Q_c)$ of homomorphisms of $L$ into the complexification of $Q_c$ of $Q$ (see [16]). Let $f \in \mathrm{Hom}(L,Q)$, and let $\rho = \mathrm{Ad} \circ f$ be the representation of $L$ into the Lie algebra $\mathfrak{q}$ of $Q$ defined by composing $f$ with the adjoint representation of $Q$. Our assumption insures that $H^1(L;\mathfrak{q}) = 0$. Then we also have $H^1(L;\mathfrak{q}_c) = 0$. By [16], the irreducible component of $R(L,Q_c)$ passing through $f$ is the orbit of $Q_c$, acting by inner automorphisms. Thus $R(L,Q)$ is contained in finitely many orbits of $Q_c$. But then it is also the union of finitely many orbits of $Q$ [3:6.4].

5.8. In particular, we see that, up to equivalence, $\Gamma$ has only finitely many unitary representations of a given degree $m$. As is known, this is false if $G = \mathbf{SL}_2(\mathbb{R})$. In fact, if $S$ is a compact Riemann surface of genus $\geq 2$, then equivalence classes of certain holomorphic bundles on $S$ correspond canonically to equivalence classes of unitary representations of suitable fuchsian group (see [12]). The results recalled above show that the only possible exceptions to 5.6 would occur when $\mathfrak{g} = \mathfrak{so}(n+1,1)$, $\mathfrak{su}(n,1)$. We ~~I~~ do not know whether they do for $n \geq 2$.

## §6. $\Gamma$ *cocompact*, $E$ *a* $G$*-module*

6.1. In this section, $\Gamma$ is a cocompact subgroup, and $E$ a finite dimensional G-module. As a special case of 4.1(2)(3), we have discrete sum decompositions with finite multiplicities

(1) $$L^2(\Gamma\backslash G) = \widetilde{\bigoplus}_{\pi \in \hat{G}} m(\pi,\Gamma)\, H_\pi \ ,$$

(2) $$C^\infty(\Gamma\backslash G) = (L^2(\Gamma\backslash G))^\infty = \widetilde{\bigoplus}_{\pi \in \hat{G}} m(\pi,\Gamma)\, H_\pi^\infty \ .$$

Moreover, the canonical isomorphism 2.3(3) yields

(3) $$I^\infty(E) \cong \widetilde{\bigoplus}_{\pi \in \hat{G}} m(\pi,\Gamma)\, H_\pi^\infty \otimes E \ .$$

The summand corresponding to the trivial representation $\pi_o$ represents the constant E-valued functions on G. Obviously

(4) $$m(\pi_o,\Gamma) = 1 \ .$$

6.2. THEOREM. *We have*

(1) $$H^*(\Gamma;E) = \bigoplus_{\pi \in \hat{G}} m(\pi,\Gamma)\, H^*(\mathfrak{g}, K; H_{\pi,K}^\infty \otimes E) \ .$$

*The natural homomorphism* $j^* : H^*(\mathfrak{g}, K; E) \longrightarrow H^*(\Gamma;E)$ (*see* 2.8) *is injective. Its image is the contribution of the trivial representation of* G *to* (1).

By 2.6,

(2) $$H^*(\Gamma;E) = H^*(\mathfrak{g}, K; \widetilde{\bigoplus}_\pi m(\pi,\Gamma)\, H_\pi^\infty \otimes E) \ .$$

The proof that we can replace the topological sum on the right hand side by an algebraic direct sum is then the same as in the case of 4.2, and will not be repeated. By ~~3.1~~, we can substitute $H_{\pi,K}^\infty$ for $H_\pi^\infty$. The last assertion is then obvious.

6.3. Assume E to be a simple G-module. Then it is also a simple

$\Gamma$-module [1], and the center $C(\Gamma)$ of $\Gamma$ acts by scalars. If it acts non-trivially, then $H^*(\Gamma;E) = 0$ (4.4(3)). So assume it acts trivially.

Let $C_\rho(G) = C(G) \cap \ker \rho$. Then 4.4(3) also shows that $H^*(\Gamma;E) = H^*(\Gamma';E)$ where $\Gamma' = \Gamma/(\Gamma \cap C_\rho(G))$. Thus we may replace $G$ by $G'$ $(G' = G/C_\rho(G))$ and $\Gamma$ by $\Gamma'$. Now $G'$ admits a faithful linear representation, namely the sum of its adjoint representation and of $\rho$. We may therefore assume $G$ to be linear. Let $G_o$ be the analytic group generated by $\underline{g}$ in the simply connected complex Lie group with Lie algebra the complexification $\underline{g}_c$ of $\underline{g}$. Then $G$ is a quotient of $G_o$. Let $\alpha : G_o \longrightarrow G'$ be the natural projection, and $\Gamma' = \rho' = \alpha^{-1}(\Gamma)$. We may view $E$ as a $G_o$-module on which $\ker \alpha$ acts trivially. Therefore, $H^*(\Gamma;E) = H^*(\Gamma';E)$.

In conclusion, the computation of $H^*(\Gamma;E)$ may be reduced to the case where $G$ is a real form of a simply connected complex semi-simple Lie group $G_c$. In particular $G$ may be assumed to be linear, and to have a global direct product decomposition $G = G_1 \times \ldots \times G_t$ corresponding to the decomposition of $\underline{g}$ into simple ideals $\underline{g}_i$ $(1 \leq i \leq t)$.

## §7. $G$ semi-simple, $\Gamma$ cocompact, $E$ a $G$-module

In this section, $G$ is connected semi-simply with finite center and no compact factor. $(\rho,E)$ and $\Gamma$ are as in §6.

7.1. THEOREM. *Assume* $(\rho,E)$ *to be irreducible. Let* $(\rho^v,E^*)$ *be the contragredient representation. Then, in the notation of* 6.1, *we have*

$$H^*(\Gamma;E^*) = \oplus_{\pi \in \hat{G}}\, m(\pi,\Gamma)\, H^*(\underline{g}, K; H_{\pi,K} \otimes E^*) \quad , \tag{1}$$

*where the sum is finite and restricted to those* $\pi$ *such that* $\chi_\pi = \chi_\rho$ *and* $\omega_\pi = \omega_\rho$. *For those* $\pi$'s, *we have* $H^q(\underline{g}, K; H_{\pi,K} \otimes E^*) = \mathrm{Hom}_K(\Lambda^q(\underline{g}/\underline{k}), H_{\pi,K} \otimes E^*)$ $(q \in \underline{N})$.

By ~~3.1~~, $H_{\pi,K} = H^{\infty}_{\pi,K}$. The first assertion then follows from 6.2 and (I, 5.3), the second one from the first one and (II, 3.1). Note that since $G$ is assumed to be connected, we could replace $K$ by $\underline{k}$.

7.2. PROPOSITION. (Raghunathan [14: Prop. 1]). Let $q \in \underline{N}$. Assume that the quadratic form $F_{\rho,q}$ of III, 1.1 is positive non-degenerate. Then $H^q(\Gamma;E) = 0$.

This follows from 7.1 and III, 1.2.

7.3. In analogy with the definition of $M(\underline{g})$, let $M_\rho$, or, if a more explicit notation is needed $M(\underline{g},\rho)$, be the greatest integer such that

$$H^q(\underline{g}, \underline{k}; V \otimes E) = 0$$

for $q \leq M_\rho$, and any irreducible unitary, admissible non-trivial $(\underline{g},\underline{k})$-module $V$. If $E$ is simple, non-trivial, then $M_\rho + 1 \geq m_\rho$, where $m_\rho$ is Raghunathan's constant (III, 1.4). Clearly, $M(\underline{g}) = M(\underline{g},\rho_o)$, where $\rho_o$ is the trivial representation.

7.4. Let $\underline{g} = \underline{g}_1 \oplus \cdots \oplus \underline{g}_t$ be the decomposition of $\underline{g}$ into simple ideals. Assume $E$ to be a simple $G$-module. Write accordingly

$$(1) \qquad E = E_1 \otimes \cdots \otimes E_t \quad , \qquad \rho = \rho_1 \otimes \cdots \otimes \rho_t \quad ,$$

where $(\rho_i, E_i)$ is a simple $\underline{g}_i$-module $(1 \leq i \leq t)$.

7.5. PROPOSITION. We keep the previous assumptions and notation and assume moreover $E$ to be non-trivial and $\Gamma$ to be irreducible. Then

$$(1) \qquad H^q(\Gamma;E) = 0 \quad \text{for} \quad q < \Sigma_{1 \leq i \leq t} (M(\underline{g}_i,\rho_i) + 1) \quad .$$

By the reductions described in 6.3, we may assume that $G = G_1 \times \cdots \times G_t$, where $G_i$ has Lie algebra $\underline{g}_i$ $(1 \leq i \leq t)$. By 6.1,

(2) $$H^*(\Gamma;E) = H^*(\underline{g}, \underline{k}; E) \oplus \oplus' m(\pi,\Gamma,E)\, H^*(\underline{g}, \underline{k}; H_{\pi,K} \otimes E) \quad ,$$

where $\oplus'$ extends over those $\pi$ which have the same infinitesimal and central characters as $\rho^\vee$. In particular $\pi$ is not trivial. By II, 3.2,

(3) $$H^q(\underline{g}, \underline{k}; E) = 0 \quad .$$

Any $\pi \in \hat{G}$ decomposes as $\pi = \pi_1 \tilde{\otimes} \ldots \tilde{\otimes} \pi_t$ $(\pi_i \in \hat{G}_i,\ i = 1, \ldots, t)$, as recalled in 3.3. By the Künneth rule (I, 1.3), we have, taking 7.3 into account

(4) $$H^q(\underline{g}, \underline{k}; H_{\pi,K} \otimes E) = \oplus_{q_1+\ldots+q_t=q} \left(\otimes_i\ H^{q_i}(\underline{g}_i, \underline{k}_i; H_{\pi_i,K_i} \otimes E_i)\right) \quad ,$$

where $K_i = K \cap G_i$, $\underline{k}_i = \underline{k} \cap \underline{g}_i$ $(1 \leq i \leq t)$. By 5.2, if $\pi$ is non-trivial, then no $\pi_i$ is trivial; therefore, for the left-hand side of (4) to be non-zero, it is necessary that $q_i > M(\underline{g}_i,\rho_i)$ for all $i$, whence our assertion.

<u>7.6</u>. PROPOSITION. (Raghunathan [13]). <u>Assume</u> $\Gamma$ <u>to be irreducible and</u> $(\rho,E)$ <u>to be simple non trivial</u>. <u>Then</u> $H^1(\Gamma;E) = 0$ <u>except possibly when</u> $\underline{g} = \underline{so}(n+1,1)$ (<u>resp</u>. $\underline{g} = \underline{su}(n,1)$) <u>and the highest weight of</u> $\rho$ <u>is a multiple of the highest weight of the standard representation of</u> $\underline{so}(n,1)$ <u>(resp. of the standard representation of</u> $\underline{su}(n,1)$ <u>or of its contragredient representation)</u> $(n \geq 1)$.

If $\underline{g}$ is not simple, this is a consequence of 7.5. If $\underline{g}$ is simple, it follows ~~from~~ the results of [13] recalled in III, 2.8.

<u>7.7</u>. <u>Remarks on</u> III. The argument of 7.5 shows in fact that if $V$ is an irreducible unitary $(\underline{g},K)$-module which is faithful as a $\underline{g}$-module, then

(1) $$H^q(\underline{g}, \underline{k}; H \otimes E) = 0 \quad \text{for} \quad q < \Sigma_i (M(\underline{g}_i,\rho_i) + 1) \quad .$$

This contains 4.2 and 4.4 of III in that case.

Erratum. In the second line of III, 4.3, replace $= 0$ by $\neq 0$.

## REFERENCES


[1] A. Borel, "Density properties of certain subgroups of semi-simple groups", Annals of Math. (2) 72 (1962), 179-188.

[2] --------, "On the automorphisms of certain subgroups of semi-simple Lie groups", Proc. Bombay Colloquium on Algebraic Geometry 1968, 43-73.

[3] A. Borel et J-P. Serre, "Théorèmes de finitude en cohomologie galoisienne", Comm. Math. Helv. 39 (1964), 111-164.

[4] N. Bourbaki, Espaces vectoriels topologiques, Chap. 1 et 2, Act. Sci. Ind. 1189, Paris 1966, Hermann éd.

[5] I. M. Gelfand, M. I. Graev, I. I. Piateckii-Shapiro, Representation theory and automorphic functions, Saunders Math. Books 1969.

[6] A. Grothendieck, "Sur quelques points d'algèbre homologique", Tohoku Math. Jour. 9 (1957), 119-221.

[7] Y. Matsushima, "On Betti numbers of compact, locally symmetric Riemannian manifolds", Osaka Math. J. 14 (1962), 1-20.

[8] -------------, "A formula for the Betti numbers of compact locally symmetric Riemannian manifolds". Jour. Diff. Geom. 1 (1967), 99-109.

[9] Y. Matsushima and S. Murakami, "On vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds", Annals of Math. (2) 78 (1963), 365-416.

[10] ------------- and -----------, "On certain cohomology groups attached to hermitian symmetric spaces", Osaka Jour. Math. 2 (1965), 1-35.

[11] G. D. Mostow, "Self-adjoint group", Annals of Math. (2) 62 (1955), 44-55.

[12] M. S. Narasimhan and C. S. Seshadri, "Stable and unitary vector bundles on a compact Riemann surface", Annals of Math. (2) 82 (1965), 540-567.

[13] M. S. Raghunathan, "On the first cohomology of discrete subgroups of semi-simple Lie groups", Amer. J. Math. 87 (1965), 103-139.

[14] -----------------, "Vanishing theorems for cohomology groups associated to descrete subgroups of semisimple Lie groups", Osaka J. Math. 3 (1966), 243-256.

[15] F. W. Warner, "Foundations of differentiable manifolds and Lie groups", Scott, Foreman and Co., Glenview, IL. 1971.

[16] A. Weil, "Remarks on the cohomology of groups", Annals of Math. (2) 80 (1964), 149-177.

# V. An interpretation of $H^1(\underline{g}, \underline{k}; V \otimes F^*)$

A. Borel and N. Wallach

In this chapter, $\underline{g}$ is a semi-simple Lie algebra over $\mathbb{R}$, $\underline{g} = \underline{k} \oplus \underline{p}$ a Cartan decomposition of $\underline{g}$ (see II, 1.1), $G$ a connected Lie group with finite center, Lie algebra $\underline{g}$, and $K$ the maximal compact subgroup of $G$ with Lie algebra $\underline{k}$. Given a finite dimensional simple $G$-module, we shall define a $(\underline{g}, K)$-module $F$ such that $H^1(\underline{g}, \underline{k}; V \otimes F^*)$ is canonically isomorphic to $\mathrm{Hom}_{\underline{g}}(Y^F, V)$ for every simple $(\underline{g}, K)$-module $V$ (see 2.2 if $F$ is the trivial $G$-module and 4.4 in the general case).

## §1. The modules $X^{\mathbb{C}}$ and $Y^{\mathbb{C}}$

1.1. Let $U = U(\underline{g})$ be the universal enveloping algebra of $\underline{g}$, and $U(\underline{k}) \subset U(\underline{g})$ the universal enveloping algebra of $\underline{k}$. Let $U^{\underline{k}}$ denote the centralizer of $\underline{k}$ in $U$. Set $V(\underline{g}, \underline{k}) = U^{\underline{k}} \cdot U(\underline{k})$.

1.2. We look at $U$ as a left $\underline{g}$-module under left multiplication and as a right $V(\underline{g}, \underline{k})$-module under right multiplication. We set $X^{\mathbb{C}} = U \otimes_{V(\underline{g}, \underline{k})} \mathbb{C}$ where $\mathbb{C}$ is the trivial $V(\underline{g}, \underline{k})$-module. It is a quotient of the induced module $U \otimes_{U(\underline{k})} \mathbb{C}$, hence it is a $(\underline{g}, \underline{k})$-module (I, 2.4), or also a $(\underline{g}, K)$-module (I, §5).

1.3. LEMMA. *If $V$ is a $\underline{g}$-module then $\mathrm{Hom}_{\underline{g}}(X^{\mathbb{C}}, V)$ is naturally isomorphic with $\mathrm{Hom}_{V(\underline{g}, \underline{k})}(\mathbb{C}, V)$.*

<u>Proof</u>. If $f \in \mathrm{Hom}_{\underline{g}}(X^{\mathbb{C}}, V)$ set $\check{f}(z) = f(1 \otimes z)$ for $z \in \mathbb{C}$. Then $\check{f} \in \mathrm{Hom}_{V(\underline{g}, \underline{k})}(\mathbb{C}, V)$. If $g \in \mathrm{Hom}_{V(\underline{g}, \underline{k})}(\mathbb{C}, V)$ set $\hat{g}(u \otimes 1) = u \cdot g(1)$ then $\hat{g} \in \mathrm{Hom}_{\underline{g}}(X^{\mathbb{C}}, V)$. Clearly $\hat{\check{f}} = f$, $\check{\hat{g}} = g$.

<u>1.4</u>. In the notation of the proof of Lemma 1.3 we can form $\hat{i} : X^{\mathbb{C}} \longrightarrow \mathbb{C}$, where $i : \mathbb{C} \longrightarrow \mathbb{C}$ is the identity map. It is a surjective $\underline{g}$-module homomorphism. Set $Y^{\mathbb{C}} = \ker \check{i}$. Then we have the exact sequence of $(\underline{g}, K)$-modules

$$0 \longrightarrow Y^{\mathbb{C}} \longrightarrow X^{\mathbb{C}} \longrightarrow \mathbb{C} \longrightarrow 0 . \tag{1}$$

<u>1.5</u>. PROPOSITION. <u>We have</u> $\dim(X^{\mathbb{C}})^{\underline{k}} = 1$, $(Y^{\mathbb{C}})^{\underline{k}} = (0)$, <u>and</u> $(X^{\mathbb{C}})^{\underline{k}}$ <u>is a trivial</u> $V(\underline{g}, \underline{k})$-<u>module</u>.

$X^{\mathbb{C}}$ is a quotient of the induced module $M = U \otimes_{U(\underline{k})} \mathbb{C}$. On the latter the natural action of $U(\underline{k})$ by left translations on $U$ is the same as the tensor product of the trivial action on $\mathbb{C}$ and the action stemming from the adjoint representation on $U$ (I, 2.4). Call the latter the ad-action. It follows that the map $\psi : U \longrightarrow X^{\mathbb{C}}$ defined by $\psi(u) = u(1 \otimes 1)$, $(u \in U)$, is a $U(\underline{k})$-homomorphism, if $U$ is viewed as a $\underline{k}$-module under the ad-action. The fixed point set of $\underline{k}$ in $U$ under this action is $U^{\underline{k}}$. Since $\psi$ is surjective and both modules are semi-simple $\underline{k}$-modules, we have $\psi(U^{\underline{k}}) = (X^{\mathbb{C}})^{\underline{k}}$. Thus $(X^{\mathbb{C}})^{\underline{k}} = U^{\underline{k}}(1 \otimes 1) = \mathbb{C}(1 \otimes 1)$, which proves the first assertion. The others follow from the exactness of the sequence of $V(\underline{g}, \underline{k})$-modules

(1) $$0 \longrightarrow (Y^{\mathbb{C}})^{\underline{k}} \longrightarrow (X^{\mathbb{C}})^{\underline{k}} \longrightarrow \mathbb{C} \longrightarrow 0 \quad ,$$

derived from 1.4(1).

## §2. An interpretation of $H^1(\underline{g}, \underline{k}; V)$

2.0. Notation. Let A be an algebra with unit over a field F. A character of A is unital homomorphism of A into F. We let $X(A)$ denote the set of characters of A. If $(\pi, M)$ is a A-module and $\chi \in X(A)$, then $M_\chi$ denotes the primary subspace associated to $\chi$ [1: VII, §1]. Thus, $m \in M_\chi$ if and only, for every $a \in A$, the element m is annihilated by some power of $(\pi(a)-\chi(a).\mathrm{Id})$. If A is commutative, M finite-dimensional, then $M = \oplus_\chi M_\chi$ [1: VII, §1, n° 1, Thm. 1

2.1. PROPOSITION. Let V be a non-trivial simple admissible $(\underline{g}, \underline{k})$-module. Then

$$\mathrm{Hom}_{\underline{g}}(X^{\mathbb{C}}, V) = 0 = \mathrm{Ext}^1_{\underline{g},\underline{k}}(X^{\mathbb{C}}, V) \quad .$$

Assume there exists a non-trivial $\underline{g}$-morphism $\psi : X^{\mathbb{C}} \longrightarrow V$. It is then surjective, and $\psi(X^{\mathbb{C}})^{\underline{k}} = V^{\underline{k}}$. Since $X^{\mathbb{C}} = U.(X^{\mathbb{C}})^k$, we have $V^{\underline{k}} \neq 0$. But $(Y^{\mathbb{C}})^k = 0$ by 1.5, hence $\psi(Y^{\mathbb{C}}) \neq V$, and therefore $\psi(Y^{\mathbb{C}}) = 0$. Thus V is the trivial one-dimensional $\underline{g}$-module, a contradiction. This proves the first equality of 2.1.

Let now

(1) $$0 \longrightarrow V \xrightarrow{\alpha} W \xrightarrow{\beta} X^{\mathbb{C}} \longrightarrow 0 \quad ,$$

be an exact sequence of $(\underline{g}, \underline{k})$-modules. Then

(2) $$0 \longrightarrow V^{\underline{k}} \xrightarrow{\alpha} W^{\underline{k}} \xrightarrow{\beta} (X^{\mathbb{C}})^{\underline{k}} \longrightarrow 0$$

is also exact. In particular, $W^{\underline{k}}$ is finite dimensional. The sequence (2) is also an exact sequence of $V(\underline{g}, \underline{k})$-modules, or, equivalently, of $A = U^{\underline{k}}/(U^{\underline{k}} \cap U.\underline{k})$ modules. The algebra $A$ is commutative [3: Thm. 2.9, p. 396], hence, for any character $\chi$ of $A$, the sequence

(3) $$0 \longrightarrow (V^{\underline{k}})_\chi \longrightarrow (W^{\underline{k}})_\chi \longrightarrow (X^{\mathbb{C}})^{\underline{k}}_\chi \longrightarrow 0 \ ,$$

(notation of 2.0) is exact. In particular, let $\varepsilon$ be the trivial character of $A$. Then $(X^{\mathbb{C}})^{\underline{k}}_\varepsilon = \mathbb{C}$, hence $\beta((W^{\underline{k}})_\varepsilon) \neq 0$. There exists then $z_0 \in W^{\underline{k}}$, $z_0 \neq 0$, suc that $\mathbb{C}.z_0$ is the trivial $A$-module, and therefore also the trivial $V(\underline{g}, \underline{k})$-module. By 1.3 and our first assertion, $z_0 \notin V$. But then, $\beta(U.z_0) = X^{\mathbb{C}}$. Moreover, by 1.3, the map $j : u \longmapsto u.z_0$ induces a homomorphism of $X^{\mathbb{C}}$ into $W$. Clearly $\beta \circ j = \mathrm{id.}$, assuming, as we may, that $\beta(z_0) = 1 \otimes 1$, hence (2) splits, which proves the second part of the proposition.

2.2. THEOREM. Let $V$ be an admissible simple $(\underline{g}, \underline{k})$-module. Then $H^1(\underline{g}, \underline{k}; V)$ is naturally isomorphic to $\mathrm{Hom}_{\underline{g}}(Y^{\mathbb{C}}, V)$.

The long exact sequence in cohomology derived from the exact sequence 1.4(1) yields the exact sequence

$$\ldots \longrightarrow \mathrm{Hom}(X^{\mathbb{C}}, V) \longrightarrow \mathrm{Hom}(Y^{\mathbb{C}}, V) \xrightarrow{\delta} \mathrm{Ext}^1_{\underline{g},\underline{k}}(\mathbb{C}, V) \longrightarrow \mathrm{Ext}^1_{\underline{g},\underline{k}}(X^{\mathbb{C}}, V) \longrightarrow \ldots$$

If $V$ is non trivial, the two extreme terms are zero by 2.1, hence $\delta$ is an isomorphism, and our assertion follows from the isomorphism

$$\mathrm{Ext}^1_{\underline{g},\underline{k}}(\mathbb{C}, V) = H^1(\underline{g}, \underline{k}; V) \quad ,$$

(see I, 2.5). This proves our assertion in this case. Let $V$ be the trivial one-dimensional $\underline{g}$-module. Then $\mathrm{Hom}_{\underline{g}}(Y^{\mathbb{C}}, V) = 0$ since $(Y^{\mathbb{C}})^{\underline{k}} = (0)$. On the other hand, $\underline{g}$ being semi-simple, $\mathrm{Ext}^1_{\underline{g},\underline{k}}(\mathbb{C}, \mathbb{C})$ is obviously zero.

Remark. We have $\dim V^{\underline{k}} \leq 1$ by the commutativity of $A = U^{\underline{k}}/(U^{\underline{k}} \cap U.\underline{k})$. If $\mathrm{Hom}_{\underline{g}}(Y^{\mathbb{C}}, V) \neq 0$, then $V^{\underline{k}} = 0$, since $(Y^{\mathbb{C}})^{k} = 0$. In fact, we shall see later that $V^{\underline{k}} = 0$ for any non-trivial simple admissible $(\underline{g}, K)$-module $V$.

## §3. Some remarks on cohomology and infinitesimal characters

3.1. We let $z(\underline{g})$ be the center of $U$, $X$ the set of characters of $z(\underline{g})$, and denote by $\chi_\delta$ the trivial character of $z(\underline{g})$, i.e. the character of the action of $z(\underline{g})$ on the trivial one-dimensional $\underline{g}$-module.

Let $V$ be a $(\underline{g}, \underline{k})$-module. For any positive integer $g$ and $\chi \in X$

$$(1) \qquad V_{\chi, q} = \{v \in V \mid (\pi(z) - \chi(z).\mathrm{Id.})^q . v = 0, \ (z \in z(\underline{g})\}$$

is a $\underline{g}$-submodule, and the primary subspace $V_\chi$ is the union of the $V_{\chi, q}$ $(q = 1, 2, \ldots)$. The isotypic subspaces for $\underline{k}$ are of course stable under $z(\underline{g})$. Assume that $V$ is admissible. Then $V$ is a union of $z(\underline{g})$-stable finite dimensional subspaces, hence is the direct sum of the submodules $V_\chi$ $(\chi \in X)$ [1: VII, §1, n° 1, Thm. 1]. Moreover, for every $\chi \in X$, the $\underline{g}$-module $V_\chi$ has a finite Jordan-Hölder series. This follows from Harish-Chandra's result that, up to equivalence, there are only finitely many simple admissible $(\underline{g}, K)$-modules with a given infinitesimal character (see e.g. [4: 9.15]). This also shows that $V_\chi = V_{\chi, q}$ for $q$ big enough, and that $V_\chi$ is a finitely generated $\underline{g}$-module.

3.2. LEMMA. Let $(\pi, V)$ be an admissible $(\underline{g}, K)$-module. Then,

$$H^q(\underline{g}, \underline{k}; V) = H^q(\underline{g}, \underline{k}; V_{\chi_\delta}) , \qquad (q = 0, 1, 2, \ldots) .$$

$V$ is the direct sum of the $V_\chi$ $(\chi \in X)$ (3.1), hence the left-hand side is the direct sum of the groups $H^q(\underline{g}, \underline{k}; V_\chi)$. It suffices therefore to show that $H^q(\underline{g}, \underline{k}; V_\chi) = 0$ if $\chi \neq \chi_\delta$. If $V$ is simple, then it has the infinitesimal character $\chi$, and our assertion follows from (I, §4). The general case then follows by induction on the length of a Jordan-Hölder series for $V_\chi$ (3.1) using the cohomology exact sequence.

3.3. Let $Z$ be the subgroup of elements of $K$ which act trivially on $\underline{g}$ and let $A = \mathbb{C}[Z]$ be its group algebra. If $(\pi, V)$ is a $(\underline{g}, K)$-module, the action of $Z$ defines one of $A$, and $V$ is a semi-simple $A$-module. If $V$ is admissible, we have then

$$V = \oplus_{\chi \in X(A)} V_\chi , \tag{1}$$

where now

(2) $$V_\chi = V_{\chi, 1} = \{v \in V \mid \pi(z).v = \chi(z).v, \ (z \in Z)\} .$$

By (I, 5.3), we have then, in analogy with 3.2,

(3) $$H^q(\underline{g}, \underline{k}; V) = H^q(\underline{g}, \underline{k}; V_{\chi_0}) , \qquad (q = 0, 1, \dots) ,$$

where $\chi_0$ is the trivial character of A.

## §4. An interpretation of $H^1(\underline{g}, \underline{k}; V \otimes F^*)$

4.1. PROPOSITION. The module $X^{\mathbb{C}}$ has the infinitesimal character $\chi_\delta$ and is admissible. More precisely, if W is a finite-dimensional simple $\underline{k}$-module, then $\dim \operatorname{Hom}_{\underline{k}}(W, X^{\mathbb{C}}) \leq \dim W$.

$z(\underline{g})$ is contained in $V(\underline{g}, \underline{k})$, hence acts trivially on $(X^{\mathbb{C}})^{\underline{k}}$. Since $X^{\mathbb{C}} = U.(X^{\mathbb{C}})^{\underline{k}}$, it follows that $X^{\mathbb{C}}$ has the trivial infinitesimal character.

In [3], it is shown that U, viewed as a $\underline{k}$-module under the ad-action, is isomorphic to a direct sum $U = H \oplus (U.U^{\underline{k}} + U.\underline{k})$ and that

(1) $$\dim \operatorname{Hom}_{\underline{k}}(W, H) = \dim W^{\underline{m}}$$

for every finite dimensional simple $\underline{k}$-module W, where $\underline{m}$ is the centralizer in $\underline{k}$ of a maximal subalgebra of $\underline{p}$. This implies the second assertion and shows in fact that (1) is also valid with H replaced by $X^{\mathbb{C}}$.

4.2. Let F be a finite-dimensional G-module, $F^*$ the contragredient

module. If $V$ is an admissible $(\underline{g}, \underline{k})$-module, then $V \otimes F^*$ is also admissibl
since, for any $\underline{k}$-module $W$, we have

(1) $$\mathrm{Hom}_{\underline{k}}(W, V \otimes F^*) = \mathrm{Hom}_{\underline{k}}(W \otimes F, V) .$$

Assume now that $F$ and $V$ are simple $(\underline{g}, K)$-modules with the same infinitesimal character $\chi = \chi_F = \chi_V$. We shall need the two following facts, which are special cases of results of Zuckerman [5]:

(i) The $\underline{g}$-module $(V \otimes F^*)_{\chi_\delta}$ is simple.

(ii) Let $W$ be an admissible $(\underline{g}, K)$-module with the trivial infinitesimal character. Then $\mathrm{Hom}_{\underline{g}}((W \otimes F)_{\chi_F}, V)$ and $\mathrm{Hom}_{\underline{g}}(W, (V \otimes F^*)_{\chi_\delta})$ are canonically isomorphic. [This follows directly from (1) and 3.1.]

4.3. Let $X^F = (X^{\mathbb{C}} \otimes F)_{\chi_F}$ and $Y^F = (Y^{\mathbb{C}} \otimes F)_{\chi_F}$. If we tensor 1.4(1) by $F$ and pass to the $\chi_F$-components, then we get the exact sequence

(1) $$0 \longrightarrow Y^F \longrightarrow X^F \longrightarrow F \longrightarrow 0 .$$

4.4. THEOREM. Let $V$ and $F$ be simple admissible $(\underline{g}, K)$-modules, with $F$ finite dimensional. Then $H^1(\underline{g}, \underline{k}; V \otimes F^*)$ is naturally isomorphic to $\mathrm{Hom}_{\underline{g}}(Y^F, V)$.

Proof. If $\chi_F \neq \chi_V$, then both terms are zero by I, §4. So assume $\chi_F = \chi_V$. By 3.2 we have

(1) $$H^1(\underline{g}, \underline{k}; V \otimes F^*) = H^1(\underline{g}, \underline{k}; (V \otimes F^*)_{\chi_\delta}) .$$

By 4.2(i) and 2.2:

(2) $$H^1(\underline{g}, \underline{k}; (V \otimes F^*)_{\chi_\delta}) = \mathrm{Hom}_{\underline{g}}(Y^{\mathbb{C}}, (V \otimes F^*)_{\chi_\delta}) \; .$$

Moreover, by 4.1, 4.2(ii) and the definition of $Y^F$, we have

(3) $$\mathrm{Hom}_{\underline{g}}(Y^{\mathbb{C}}, (V \otimes F^*)_{\chi_\delta}) = \mathrm{Hom}_{\underline{g}}((Y^{\mathbb{C}} \otimes F)\chi_F, V) = \mathrm{Hom}_{\underline{g}}(Y^F, V) \; ,$$

whence the theorem.

4.5. In Chapter VI the $(\underline{g}, K)$-modules $X^F$, where $F$ is a simple, finite dimensional $G$-module, will be identified with certain elements of the (non-unitary) principal series, and $Y^F$ will be resolved by certain canonical principal series representations.

REFERENCES

[1] N. Bourbaki, Groupes et algèbres de Lie, Chap. VII, VIII, Act. Sci. Ind. 1364, Hermann, Paris 1975.

[2] S. Helgason, Differential geometry and symmetric spaces, Academic Press, New York 1962.

[3] B. Kostant and S. Rallis, "Orbits and representations associated to symmetric spaces," Amer. J. Math. 93 (1971), 753-809.

[4] J. Lepowsky, "Algebraic results on representations of semisimple Lie groups," Trans. Amer. Math. Soc. 176 (1973), 1-44.

[5] G. Zuckerman, "Tensor products of finite and infinite dimensional representations of semi-simple Lie groups" (to appear).

# VI. The simple $(\underline{g}, K)$-modules with $H^1(\underline{g}, \underline{k}; V \otimes F^*) \neq 0$

Nolan R. Wallach

In this chapter we show how to use the results in chapter V to determine the simple $(\underline{g}, K)$-modules with $H^1(\underline{g}, \underline{k}; V \otimes F^*) \neq 0$. We show how these modules fit in the classification of admissible $(\underline{g}, K)$-modules in Langlands [7].

## 1. The principal series.

1.1. Let $G$ be a connected semi-simple Lie group with finite center. Let $G = KAN$ be an Iwasawa decomposition of $G$ (here we follow the notation of [2]). In particular, $A$ is a maximal, split, vector subgroup of $G$, $N$ is a connected simply connected nilpotent subgroup of $G$ and $A$ normalizes $N$. Let $\underline{g}_0$, $\underline{k}_0$, $\mathfrak{a}_0$, $\mathfrak{n}_0$ denote the respective Lie algebras of $G$, $K$, $A$ and $N$. As usual, if we delete the subscript "0" we mean complexification.

1.2. Let $M$ be the centralizer of $A$ in $K$. Then $MAN = P$ is a minimal parabolic subgroup of $G$ (cf. [10]). Let $\log : A \longrightarrow \underline{a}_0$ be the inverse map to $\exp : \underline{a}_0 \longrightarrow A$. Let for $H \in \underline{a}_0$, $\rho(H) = \frac{1}{2}\operatorname{tr}(\operatorname{ad} H|_{\mathfrak{n}_0})$.

1.3. The definition of the principal series. Let $(\xi, H_\xi)$ be an irreducible, unitary, representation of $M$. Set $X_\infty^\xi$ equal to the space of all $C^\infty$ functions $f : K \longrightarrow H_\xi$ so that $f(km) = \xi(m)^{-1} f(k)$ for $m \in M$, $k \in K$. If $x, y \in K$, $f \in X_\infty^\xi$ define $(\pi_\xi(x)f)(y) = f(x^{-1}y)$. Set $X^\xi$ equal to the $\pi_\xi(K)$-finite elements of $X_\infty^\xi$.

Let $\underline{a}^*$ denote the space of all complex valued, real linear forms on $\underline{a}_0$. If $\nu \in \underline{a}^*$ and if $f \in X^\xi$ define $f_\nu(kan) = e^{-(\rho+\nu)(\log a)}f(k)$ for $k \in K$, $a \in A$, $n \in N$. Define for $x \in \underline{g}_0$, $f \in X^\xi$, $\nu \in \underline{a}^*$ and $k \in K$

$$(\pi_{\xi,\nu}(x)f)(k) = \frac{d}{dt} f_\nu(\exp(-tx)k)\Big|_{t=0} .$$

Then it is easy to see that $(\pi_{\xi,\nu}, X^\xi)$ defines an admissible $(\underline{g}, K)$-module with $\underline{g}$ acting by $\pi_{\xi,\nu}(x)$, $x \in \underline{g}$ and $K$ acting by $\pi_\xi(k)$, $k \in K$.

1.4. Let $\underline{m}_0$ be the Lie algebra of $M$. Let $\underline{h}_0^-$ be a maximal abelian subalgebra of $\underline{m}_0$. Set $\underline{h}_0 = \underline{h}_0^- \oplus \underline{a}_0$. Then $\underline{h}_0$ is well-known to be a Cartan subalgebra of $\underline{g}_0$. Let $\Delta$ denote the root system of $(\underline{g}, \underline{h})$. If $\alpha \in \Delta$ let $\underline{g}_\alpha$ denote the $\alpha$ root space. Fix $\Delta^+$ a system of positive roots for $\Delta$ so that $\underline{n}_0 = (\sum_{\alpha\in\Delta^+} \underline{g}_\alpha) \cap \underline{g}_0$. (Such a system always exists, cf, [8].)

1.5. Let $\Delta_m = \{\alpha \in \Delta \mid \alpha|_{\underline{a}} = 0\}$. Then $\Delta_m$ is the root system of $(\underline{m}, \underline{h}^-)$. Set $\Delta_m^+ = \Delta_m \cap \Delta^+$. Then $\Delta_m^+$ is a system of positive roots for $\Delta_m$.

1.6. If $\lambda \in (\underline{h}^-)^*$ and $\mu \in \underline{a}^*$ define $\lambda+\mu \in \underline{h}^*$ by $(\lambda+\mu)(H_1+H_2) = \lambda(H_1) + \mu(H_2)$, $H_1 \in \underline{h}^-$, $H_2 \in \underline{a}$. Set $\delta = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$, $\delta_m = \frac{1}{2}\sum_{\alpha\in\Delta_m^+} \alpha$. Then $\delta = \delta_m + \rho$.

<u>1.7.</u> THEOREM (Harish-Chandra, cf. Dixmier [1]). <u>Let $z(g)$ denote the center of the universal enveloping algebra of $\underline{g}$. Then to each $\Lambda \in \underline{h}^*$ there exists a homomorphism, $\chi_\Lambda : \underline{z}(\underline{g}) \longrightarrow \mathbb{C}$ such that $\chi_\Lambda(1) = 1$. Furthermore</u>

$\chi_\Lambda = \chi_{\Lambda'}$ <u>if and only if there is</u> $s \in W(\Delta)$ (<u>the Weyl group of</u> $\Delta$) <u>so that</u> $s\Lambda = \Lambda'$. <u>If</u> $\chi : \underline{z}(\underline{g}) \longrightarrow \mathbb{C}$ <u>is a homomorphism with</u> $\chi(1) = 1$ <u>then</u> $\chi = \chi_\Lambda$ <u>for some</u> $\Lambda$.

<u>1.8.</u> The definition of $\chi_\Lambda$ is not too hard so we recall it here. Let $\underline{n}^+ = \sum_{\alpha\in\Delta^+} \underline{g}_\alpha$, $\underline{n}^- = \sum_{\alpha\in\Delta^+} \underline{g}_{-\alpha}$. Then $U(\underline{g}) = U(\underline{h}) \oplus (\underline{n}^- U + U\underline{n}^+)$. Let $P : U(\underline{g}) \longrightarrow U(\underline{h})$ be the corresponding projection. Then $\chi_\Lambda(z) = (\Lambda-\delta)(Pz)$ for $z \in \underline{z}(\underline{g})$.

<u>1.9.</u> It is easy to see from this definition that if $F$ is the simple $\underline{g}$-module with highest weight $\Lambda$ relative to $\Delta^+$ then $\chi_F = \chi_{\Lambda+\delta}$.

<u>1.10.</u> LEMMA. <u>Let</u> $(\xi, H_\xi)$ <u>be an irreducible unitary representation of</u> $M$. <u>Let</u> $\Lambda_\xi$ <u>be the highest weight of</u> $(\xi, H_\xi)$ <u>relative to</u> $\Delta_m^+$. <u>Set for</u> $\nu \in \underline{a}^*$, $\Lambda(\xi, \nu) = \Lambda_\xi + \delta_m + \nu$. Then $\pi_{\xi,\nu}(z) = \chi_{\Lambda(\xi,\nu)}(z)I$ for $z \in \underline{z}(\underline{g})$.

<u>1.11.</u> The proof of 1.10 is a fairly simple consequence of the definition in 1.8. In any event, the result is well-known.

## 2. <u>The identification of</u> $\underline{X}^F$.

<u>2.1.</u> Let $\xi_0$ denote the trivial representation of $M$. Let $1 \in X^{\xi_0}$ be the function constantly equal to 1.

LEMMA. 1) $X^{\xi_0} = \pi_{\xi_0,\rho}(U)\cdot 1$.

2) <u>If</u> $j_0(f) = \int_K f(k)dk$ <u>for</u> $f \in X^{\xi_0}$ <u>then</u> $j_0 : (\pi_{\xi_0,\rho}, X^{\xi_0}) \longrightarrow \mathbb{C}$ <u>is a</u> $(\underline{g}, K)$-<u>module homomorphism</u>.

2.2. This result is well-known. A proof of 2.1.1) can be found in [3]. 2.1.2) follows from the basic integral formula of Harish-Chandra:

(1) $$\int_K e^{-2\rho(H(gk))} f(k(gk))dk = \int_K f(k)dk \;,$$

where $g \in G$ is written $k(g)\exp H(g)n(g)$ with $k(g) \in K$, $H(g) \in \underline{a}_0$, $n(g) \in N$.

2.3. PROPOSITION. $X^{\mathbb{C}}$ is isomorphic with $(\pi_{\xi_0,\rho}, X^{\xi_0})$.

Proof. Clearly, $\dim(X^{\xi_0})^K = 1$. Lemma 2.1.2) implies that $U_+^{\underline{k}} = U^{\underline{k}} \cap U\underline{g}$ acts trivially on $(X^{\xi_0})^K$. Lemma 2.1.2) and [V: 1.3] now imply that there is a surjective $(\underline{g}, K)$-module homomorphism $\alpha : X^{\mathbb{C}} \longrightarrow (\pi_{\xi_0,\rho}, X^{\xi_0})$. As a consequence of a result of Kostant-Rallis [6] (see remark below), one has for any simple $K$-module $V$

(1) $$\dim \mathrm{Hom}_K(V, X^{\mathbb{C}}) = \dim V^M \;, \qquad (V^M = \{v \in V \mid m.v = v, \ (m \in M$$

By Frobenius reciprocity, the same equality is valid for $X^{\xi_0}$. Therefore $\alpha$ is injective on each isotypic subspace of $K$. Q.E.D.

REMARK. In V, 4.1, the result 4.1(1) of Kostant-Rallis [6] is quoted incorrectly. It should read

(1) $$\dim \mathrm{Hom}_K(W, H) = \dim W^M \;, \qquad (W^M = \{w \in W \mid m.w = w \ \ (m \in$$

for every simple $K$-module, where $M$ is the centralizer in $K$ of a maximal subalgebra of $\underline{p}$ (and $H$ is a $K$-module supplement in $U = U(\underline{g})$ to $U.U^{\underline{k}} + U.\underline{k}$). Since $X^{\mathbb{C}}$ is isomorphic to $H$ as a $K$-module, this yields

2.3(1) above.

<u>2.4.</u> Let $F$ be a simple finite dimensional $G$-module. We look upon $F$ as a $(\underline{g}, K)$-module in the obvious way. Let $F^N = \{v \in F \mid n.v = v \text{ for } n \in N\}$. Then $F^N$ is an irreducible $MA$-module. Let $\xi_F \otimes e^{\lambda_F - \rho}$ denote the action of $MA$ on $F^N$. (It is given by $ma.v = e^{\lambda_F - \rho(\log a)} \xi_F(m)v$).

<u>2.5.</u> THEOREM. <u>The</u> $(g, K)$-<u>modules</u> $X^F$ (<u>see</u> V, 4.3) <u>and</u> $(\pi_{\xi_F, \lambda_F}, X^{\xi_F})$ <u>are isomorphic</u>.

<u>Proof.</u> In view of 2.3, we need only show

$$((\pi_{\xi_0, \rho}, X^{\xi_0}) \otimes F)_{\chi_{\Lambda+\delta}} \equiv (\pi_{\xi_F, \lambda_F}, X^{\xi_F}) . \tag{1}$$

Here $\Lambda$ is the highest weight of $F$ relative to $\Delta^+$. We prove first

$$\Lambda(\xi_F, \lambda_F) = \Lambda+\delta . \tag{2}$$

Indeed, $\Lambda_{\xi_F} + \lambda_F - \rho = \Lambda$. Thus $\Lambda + \delta = \Lambda_{\xi_F} + \delta_m + \lambda_F = \Lambda(\xi_F, \lambda_F)$.

Consider the $MAN$-module $e^{\delta} \otimes F = U_1$. That is $(man) \circ v = e^{\delta(\log a)} man.v$ ($\circ$ the new action, dot the old). Then $U_1$ has a composition series $U_1 \supset U_2 \supset \ldots \supset U_k \supset U_{k+1} = (0)$ where $U_i/U_{i+1}$ is a simple $MAN$-module. This implies that the action of $MAN$ on $U_i/U_{i+1}$ is $\xi_i \otimes e^{\lambda_i}$. It is clear that $\Lambda(\xi_i, \lambda_i) = \Lambda_i + \delta$ where $\Lambda_i$ is a weight of $F$. The number of $j$ with $\Lambda(\xi_j, \lambda_j) = \mu + \delta$ is less than the dimension of the $\mu$ weight space in $F$. Using, say, (10.6), p. 191 in [8] we see that $Z = (\pi_{\xi_0, \rho}, X^{\xi_0}) \otimes F$ has a composition series $Z = Z_1 \supset Z_2 \supset \ldots \supset Z_k \supset Z_{k+1} = (0)$

with $Z_i/Z_{i+1} \equiv (\pi_{\xi_i, \lambda_i}, X^{\xi_i})$, $i = 1, \ldots, k$. Now $((\pi_{\xi_0, \rho}, X^{\xi_0}) \otimes F)_{\chi_{\Lambda+\delta}}$ has a composition series whose subquotients are subquotients of $\pi_{\xi_i, \lambda_i}$ with $\chi_{\Lambda_i+\delta} = \chi_{\Lambda+\delta}$. If $\chi_{\Lambda_i+\delta} = \chi_{\Lambda+\delta}$ then there is $s \in W(\Delta)$ so that $s(\Lambda+\delta) = \Lambda_i + \delta$. That is, $s^{-1}(\Lambda_i) + s^{-1}\delta = \Lambda + \delta$. But $s^{-1}\Lambda_i$ is a weight of $F$ and $s^{-1}\delta$ is a weight of the finite, dimensional, simple $\underline{g}$-module with highest weight $\delta$. Thus $s^{-1}\Lambda_i = \Lambda - Q_1$, $s^{-1}\delta = \delta - Q_2$ with $Q_j$ a sum of elements of $\Delta^+$, $j = 1, 2$. But then $\Lambda + \delta - Q_1 - Q_2 = \Lambda + \delta$. Hence $Q_1 = Q_2 = 0$. Thus $s = 1$. Since the dimension of the $\Lambda$ weight space in $F$ is $1$, this proves (1).

<u>2.6.</u> Theorem 2.5 implies that there exists $j_F : X^{\xi_F} \longrightarrow F$ so that $j_F \circ \pi_{\xi_F, \lambda_F}(x) = x \circ j_F$ for $x \in \underline{g}$. We leave it to the reader to show that

$$j_F(f) = \int_K k.\, f(k)dk$$

where $f \in X^{\xi_F}$ and we look upon $H_{\xi_F}$ as $F^N$ and $k$ acts on $F$ by the restriction of the $G$ action to $K$.

## 3. <u>The resolution of</u> $Y^F$ <u>by principal series representations.</u>

<u>3.1.</u> Let $G_{\mathbb{C}}$ be the connected, simply connected Lie group with Lie algebra $\underline{g}$. Let $G_0$ be the connected subgroup of $G_{\mathbb{C}}$ corresponding to $\underline{g}_0$. Let $K_0$ be the connected subgroup of $G_0$ corresponding to $\underline{k}_0$. Then every quotient of $Y^F$ (see chapter V) defines a $(\underline{g}, K_0)$-module as well as a $(\underline{g}, K)$-module. Thus for our purposes, nothing is lost by assuming that $G_0 = G$. We make this assumption for the remainder of this chapter.

3.2. Let $H \subset G_{\mathbb{C}}$ be the connected subgroup with Lie algebra $\underline{h}$. If $\mu \in \underline{h}^*$ and $\mu$ is integral $(2\langle \mu, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}$ for $\alpha \in \Delta)$ then we can define a quasi-character of $H$ by the formula $(\exp h)^{\mu} = e^{\mu(h)}$ for $h \in \underline{h}$.

3.3. If $x \in \underline{g}_0$ (resp. $x \in \underline{m}_0$) and if $U$ is a finite dimensional vector space over $\mathbb{C}$ and if $f \in C^{\infty}(G; U)$ (resp. $C^{\infty}(M; U)$) then define $(R_x f)(g) = \frac{d}{dt} f(g \exp tx)|_{t=0}$ for $g \in G$ (resp. $M$). Define $R_{x+iy} = R_x + iR_y$.

3.4. Set $T = M \cap H$. Then $H \cap G = TA$. If $\mu \in \underline{h}^*$ and $\mu$ is integral then $\mu$ defines a quasi-character on $TA = H \cap G$.

3.5. LEMMA. Let $\mu \in \underline{h}^*$ be integral. Set $W_{\mu}$ equal to the space of all $f \in C^{\infty}(M)$ such that

1) $f(mt) = t^{\mu} f(m)$, $m \in M$, $t \in T$

2) $R_z f = 0$ for $z \in \sum_{\alpha \in \Delta_m^+} \underline{g}_{\alpha}$.

Then $W_{\mu} \neq 0$ if and only if $2\langle \mu, \alpha \rangle / \langle \alpha, \alpha \rangle$ is a non-negative integer for $\alpha \in \Delta_m^+$. Set for $f, g \in W_{\mu}$, $\langle f, g \rangle = \int_M f(m)\overline{g(m)}dm$ ($dm$ is Haar measure on $M$ normalized so that $\int_M dm = 1$). Set for $m \in M$, $f \in W_{\mu}$ $(\gamma_{\mu}(m)f)(x) = f(m^{-1}x)$, $x \in M$. If $W_{\mu} \neq 0$ then $(\gamma_{\mu}, W_{\mu})$ is an irreducible, unitary, representation of $M$. $(\gamma_{\mu}, W_{\mu})$ has lowest weight (as a $T$-representation) $-\mu$ relative to $\Delta_m^+$.

3.6. This result is substantially the Borel-Weil theorem, see e.g. [8], 6.3.7, p. 155.

3.7. Set $\underline{n}^+ = \sum_{\alpha\in\Delta^+_{\gamma_\mu}} \underline{g}_\alpha$. Let $\mu \in \underline{h}^*$ be integral and such that $W_\mu \neq 0$. Let $\lambda_\mu = \mu|_{\underline{a}_0}$. Let $C : X^{\gamma_\mu} \longrightarrow C^\infty(G)$ be defined by $C(f)(g) = f_{-\lambda_\mu - \rho}(g)(1)$. If $\varphi = C(f)$ for $f \in X^{\gamma_\mu}$ then we leave it to the reader to check that

a) $\varphi(gh) = h^\mu \varphi(g)$ for $h \in TA$, $g \in G$,

b) $R_z \varphi = 0$ for $z \in \underline{n}^+$.

3.8. Set $\mathcal{H}^\mu_\infty$ equal to the space of all $\varphi \in C^\infty(G)$ satisfying a), b) in 3.7. If $\varphi \in \mathcal{H}^\mu_\infty$ and $g, x \in G$ define $(T_\mu(x)\varphi)(g) = \varphi(x^{-1}g)$. Let $\mathcal{H}^\mu$ be the space of all $T_\mu(K)$-finite elements of $\mathcal{H}^\mu_\infty$. If $\varphi \in \mathcal{H}^\mu$ define for $x \in \underline{g}_0$, $(T_\mu(x)\varphi)(g) = \frac{d}{dt}\varphi(\exp(-tx)g)|_{t=0}$.

3.9. LEMMA. C (defined as in 3.7) is a bijection between $X^{\gamma_\mu}$ and $\mathcal{H}^\mu$. Furthermore, $C \circ \pi_{\gamma_\mu, -\lambda_\mu - \rho}(x) = T_\mu(x) \circ C$ for $x \in \underline{g}$.

Proof. $C(X^{\gamma_\mu}) \subset \mathcal{H}^\mu$ by construction. The intertwining property is also clear. If $f \in X^{\gamma_\mu}$ and if $C(f) = 0$ then $f_{-\lambda_\mu - \rho}(g)(1) = 0$ for all $g \in G$. Hence $f_{-\lambda_\mu - \rho}(gm)(1) = 0$ for all $g \in G$, $m \in M$. But then $f_{-\lambda_\mu - \rho}(g)(m) = 0$ for all $g \in G$, $m \in M$. Thus $f_{-\lambda_\mu - \rho} = 0$. This says that C is injective.

If $\varphi \in \mathcal{H}^\mu$ set $\underset{\sim}{\varphi}(g)(m) = \varphi(gm)$. Then $\underset{\sim}{\varphi}(g) \in W_\mu$ by the definition of $W_\mu$. $\underset{\sim}{\varphi}(gm_1)(m) = \varphi(gm_1m) = \underset{\sim}{\varphi}(g)(m_1m) = \gamma_\mu(m_1)^{-1}\underset{\sim}{\varphi}(g)(m)$. Clearly, $\underset{\sim}{\varphi}$ is left K-finite and $C^\infty$ as a function from G to $W_\mu$. Also, $\underset{\sim}{\varphi}(gan) = e^{\lambda_\mu(\log a)}\underset{\sim}{\varphi}(g)$ for $g \in G$, $a \in A$, $n \in N$ (use a), b) of 3.7). Hence $\underset{\sim}{\varphi} = (\underset{\sim}{\varphi}|_K)_{-\lambda_\mu - \rho}$. Finally, $\underset{\sim}{\varphi}(g)(1) = \varphi(g)$. Hence $\varphi = C(\underset{\sim}{\varphi}|_K)$. This proves that C is bijective.

3.10. Let $\pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be the simple roots of $\Delta^+$. We assume that we have labeled the elements of $\pi$ so that $\pi \cap \Delta_m^+ = \{\alpha_{\ell_0+1}, \ldots, \alpha_\ell\}$ $(\{\alpha_{\ell+1}, \ldots, \alpha_\ell\} = \phi)$. Let for $\alpha \in \Delta$, $E_\alpha \in \underline{g}_\alpha$ be chosen so that $[E_\alpha, E_{-\alpha}] = H_\alpha$ with $\alpha(H_\alpha) = 2$.

3.11. Let for $\alpha \in \Delta$, $s_\alpha$ be the Weyl reflection about the hyperplane in $\underline{h}$, $\alpha = 0$. Let $s_\alpha \mu = \mu \circ s_\alpha$. Set for $i = 1, \ldots, \ell$, $s_i = s_{\alpha_i}$. Set for $i = 1, \ldots, \ell$, $E_{\alpha_i} = Y_i$.

3.12. LEMMA. Let $\mu \in \underline{h}$ be integral and such that $W_\mu \neq (0)$. Fix $1 \leq i \leq \ell_0$. Suppose that $2\langle \mu, \alpha_i \rangle / \langle \alpha_i, \alpha_i \rangle = n_i$ is a non-negative integer. Set $A_i \varphi = R_{Y_i}^{n_i+1} \varphi$ for $\varphi \in \mathcal{H}^\mu$. Then

1) $A_i : \mathcal{H}^\mu \longrightarrow \mathcal{H}^{s_i(\mu+\delta)-\delta}$

2) $A_i \neq 0$

3) $A_i \circ T_\mu(x) = T_{s_i(\mu+\delta)-\delta}(x) \circ A_i$

for $x \in \underline{g}$.

Proof. We note that $s_i(\mu+\delta)-\delta = \mu - (n_i+1)\alpha_i$. It is therefore clear that if $\varphi \in \mathcal{H}^\mu$ then $(A_i\varphi)(gh) = h^{s_i(\mu+\delta)-\delta}\varphi(g)$ for $g \in G$, $h \in TA$.

a) If $\alpha \in \Delta^+$ and $\alpha \neq \alpha_i$ then $R_{E_\alpha} R_{Y_i}^m \varphi = 0$ for all $m \geq 0$, $m \in \mathbb{Z}$, $\varphi \in \mathcal{H}^\mu$.

We prove a) by induction on $m$. If $m = 0$ then a) is clear. Assume a) for $m$. Then $R_{E_\alpha} R_{Y_i}^{m+1}\varphi = R_{E_\alpha} R_{E_{-\alpha_i}} R_{Y_i}^m \varphi = R_{[E_\alpha, E_{-\alpha_i}]} R_{Y_i}^m \varphi + R_{E_{-\alpha_i}} R_{E_\alpha} R_{Y_i}^m \varphi$. Now $R_{E_\alpha} R_{Y_i}^m \varphi = 0$ by the inductive hypothesis. $(E_\alpha, E_{-\alpha_i}) \in \underline{g}_{\alpha-\alpha_i}$. If

$\alpha - \alpha_i \in \Delta$ then $\alpha - \alpha_i \in \Delta^+$ and $\alpha - \alpha_i \neq \alpha_i$. Hence $R_{(E_\alpha, E_{-\alpha_i})} R^m_{Y_i}\varphi = 0$ by the inductive hypothesis. This proves a).

b) $R_{E_{\alpha_i}} R^m_{E_{-\alpha_i}} = mR^{m-1}_{E_{-\alpha_i}}(R_{H_{\alpha_i}} - m+1) + R^m_{E_{-\alpha_i}} R_{E_{\alpha_i}}$.

b) can be proved by a straightforward induction (cf. [1], 7.1.14, p. 223).

b) implies that $R_{E_{\alpha_i}} R^{m_i+1}_{Y_i}\varphi = (n_i+1)(\mu(H_{\alpha_i})-n_i)R^{n_i}_{Y_i}\varphi$ for $\varphi \in \mathcal{H}^\mu$. Since $\mu(H_{\alpha_i}) = 2\langle\mu, \alpha_i\rangle / \langle\alpha_i, \alpha_i\rangle = n_i$, we see that $R_{E_{\alpha_i}} A_i\varphi = 0$. But then $R_z A_i \varphi = 0$ for $z \in \underline{n}^+$.

We have shown that $A_i(\mathcal{H}^\mu) \subset \mathcal{H}^{s_i(\mu+\delta)-\delta}$.

If $A_i(f) = 0$ for all $f \in \mathcal{H}^\mu$ then it is easy to see that $A_i(\mathcal{H}^\mu_\infty) = 0$.

If $\varphi \in C^\infty_c(\overline{N})$ $(\overline{N} = \theta(N))$ and if $h \in W_\mu$ then we can define $f(\overline{n}man) = \varphi(\overline{n})h(m)$, $\overline{n} \in \overline{N}$, $m \in M$, $a \in A$, $n \in N$ and $f(g) = 0$ if $g \notin \overline{N}MAN$. Then $f \in \mathcal{H}^\mu_\infty$. Now $A_i f|_{\overline{N}} = R^{n_i+1}_{Y_i}\varphi$. Since $R_{Y_i}$ is a non-zero vector field on $\overline{N}$ we see that there exists $\varphi$ so that $A_i f \neq 0$.

3.13. If $(\xi, H_\xi)$ is an irreducible finite dimensional representation of M and if $(\xi^*, H^*_\xi)$ is the contragredient representation of $\xi$ we can define for $f \in X^\xi$, $g \in X^{\xi^*}$,

$$(1) \qquad (f, g) = \int_K \langle f(k), g(k)\rangle\, dk$$

where $\langle\, ,\rangle$ is the canonical pairing between $H_\xi$ and $H^*_\xi$.

The integral formula 2.2.1) implies

$$(2) \qquad (\pi_{\xi,\nu}(x)f, g) = -(f, \pi_{\xi^*,-\nu}(x)g)$$

for $f \in X^{\xi}$, $g \in X^{\xi^*}$, $x \in \underline{g}$, $\nu \in \underline{a}^*$.

<u>3.14.</u> Let $F$ be an irreducible, finite dimensional $G$-module. Let $\Lambda$ be the highest weight of $F$.

<u>3.15.</u> LEMMA. <u>$(\pi_{\xi_F^*, -\lambda_F}, X^{\xi_F^*})$ is equivalent with $(T_\Lambda, \mathcal{H}^\Lambda)$ under $C$ (see 3.7 and 2.4 for notation).</u>

<u>Proof.</u> $\xi_F$ has highest weight $\Lambda|_T$ relative to $\Delta_m^+$. Hence $\xi_F^*$ has lowest weight $-\Lambda|_T$, $\lambda_F = (\Lambda+\delta)|_{\underline{a}}$. Hence $-\lambda_F = -\Lambda|_{\underline{a}} - \rho = -\lambda_\Lambda - \rho$. Lemma 3.9 now implies the lemma.

<u>3.16.</u> We have seen that $X^F$ is isomorphic with $(\pi_{\xi_F, \lambda_F}, X^{\xi_F})$. Let $j_F : X^{\xi_F} \longrightarrow F$ be as in 2.6. Let

$$i_F = j_F^* : F^* \longrightarrow X^{\xi_F^*} \subset (X^{\xi_F})^*$$

(see 3.13) be the dual mapping. Then $i_F(x.\nu) = \pi_{\xi_F^*, -\lambda_F}(x)i_F(\nu)$ for $\nu \in F^*$, $x \in \underline{g}$.

<u>3.17.</u> Since $\Lambda$ is $\Delta^+$-dominant integral, all of the operators $A_i : \mathcal{H}^\Lambda \longrightarrow \mathcal{H}^{s_i(\Lambda+\delta)-\delta}$, $1 \le i \le \ell_0$ are defined and non-zero. Set for $1 \le i \le \ell_0$

$$(\xi_{i,F}^*, H_{\xi_i,F}^*) = (\gamma_{s_i(\Lambda+\delta)-\delta}, W_{s_i(\Lambda+\delta)-\delta}), \quad \lambda_{i,F} = s_i(\Lambda+\delta)|_{\underline{a}_0} .$$

Then if $B_i = C^{-1} \circ A_i \circ C$, $B_i : X^{\xi_F^*} \longrightarrow X^{\xi_{i,F}^*}$ is non-zero and $B_i \circ \pi_{\xi_F^*, -\lambda_F}(x) = \pi_{\xi_{i,F}, -\lambda_{i,F}}(x) \circ B_i$ for $x \in \underline{g}$ and $1 \le i \le \ell_0$.

<u>3.18.</u> LEMMA. $i_F(F^*) = \{f \in X^{\xi_F^*} \mid B_i(f) = 0,\ i = 1, \ldots, \ell_0\}$.

<u>Proof.</u> It is enough to show that $C(i_F(F^*)) = \{f \in \mathcal{H}^\Lambda \mid A_i(f) = 0,\ i = 1, \ldots, \ell_0\}$. Set $n_i = 2\langle \Lambda, \alpha_i\rangle / \langle \alpha_i, \alpha_i\rangle$, $i = 1, \ldots, \ell$. We note that if $\varphi \in W_\Lambda$ then $R_{Y_i}^{n_i+1}\varphi = 0$, $i \geq \ell_0 + 1$ ($W_\Lambda$ has lowest weight $W_\Lambda$). Hence if $f \in X^{\xi_F^*}$ and $A_i(f) = 0$, $i = 1, \ldots, \ell_0$ then $R_{Y_i}^{n_i+1} f = 0$ for $i = 1, \ldots, \ell$. The space of all solutions to these equations is finite dimensional and irreducible (cf. 7.2.5, p. 225 in [8]).

<u>3.19.</u> Using the pairing between $X^{\xi_F^*}$ and $X^{\xi_F}$ (see 3.13) we can define $B_i^* : X^{\xi_{i,F}} \longrightarrow X^{\xi_F}$ and $B_i^* \circ \pi_{\xi_{i,F},\lambda_{i,F}}(x) = \pi_{\xi_F,\lambda_F}(x) \circ B_i^*$ for $i = 1, \ldots, \ell_0$. Set $D_i = B_i^*$, $i = 1, \ldots, \ell_0$.

<u>3.20.</u> THEOREM. <u>We identify</u> $X^F$ <u>with</u> $(\pi_{\xi_F,\lambda_F}, X^{\xi_F})$.

1) $Y^F = \sum_{i=1}^{\ell_0} D_i(X^{\xi_{i,F}})$.

2) <u>If</u> $1 \leq i \leq \ell_0$ <u>is such that</u> $\alpha_i|_{\underline{h}^-} = 0$ <u>then</u> $C \circ D_i \circ C^{-1}$ <u>is given by</u> $R_{Y_i}^{n_i+1}$.

<u>Proof.</u> 1) is a direct consequence of Lemma 3.8. 2) is not hard and is left to the reader.

## 4. <u>The linear functionals</u> $\lambda_{i,F}$.

<u>4.1.</u> Let $\theta$ be the Cartan involution of $\underline{g}$ corresponding to $\underline{k}$. Then $\theta\underline{h} = \underline{h}$. Let $\theta$ act on $\underline{h}^*$ by $\theta\lambda = \lambda \circ \theta$. Then $\theta\Delta = \Delta$. Also if $\alpha \in \Delta^+$ and $\alpha|_{\underline{a}_0} \neq 0$ then $-\theta\alpha \in \Delta^+$. (Here $\Delta^+$, $\{\alpha_1, \ldots, \alpha_\ell\}$, $\{\alpha_1, \ldots, \alpha_{\ell_0}\}$ are as

in §3).

4.2. LEMMA. *Fix* $1 \leq i \leq \ell_0$. *Then there are three possibilities for* $\alpha_i$ *and* $-\theta\alpha_i$.

1) $-\theta\alpha_i = \alpha_i$.

2) $\langle \alpha_i, -\theta\alpha_i \rangle = 0$.

3) $2\langle \alpha_i, -\theta\alpha_i \rangle / \langle \alpha_i, \alpha_i \rangle = 1$.

(*In cases* 2), 3) $-\theta\alpha_i \neq \alpha_i$).

*Proof.* Let $\Delta_i = (Z\alpha_i + Z(-\theta\alpha_i)) \cap \Delta$. We assume $-\theta\alpha_i \neq \alpha_i$. Set $\underline{g}_i = \sum_{\alpha\in\Delta_i} \mathbb{C}H_\alpha + \sum_{\alpha\in\Delta_i} \underline{g}_\alpha$. Then $\theta\underline{g}_i = \underline{g}_i$ and $\underline{g}_i = (\underline{g}_0 \cap \underline{g}_i) \otimes_{\mathbb{R}} \mathbb{C}$. From this we see that there is no loss in generality by assuming that $\underline{g} = \underline{g}_i$. We relabel so that $i = 1$. In particular, $\underline{g}$ is of rank 2, $\underline{g}_0$ is of split rank 1. Let $\Delta^+$ be the simple roots of $\Delta$. Then $\Delta^+ = \{\alpha_1, \gamma\}$. Then there are four possibilities for the Dynkin diagram of $\Delta^+$:

i) $\underset{\alpha_1}{\circ} \quad \underset{\gamma}{\circ}$

ii) $\underset{\alpha_1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{\gamma}{\circ}$

iii) $\underset{\alpha_1}{\circ}\!\Rightarrow\!\underset{\gamma}{\circ}$

iv) $\underset{\alpha_1}{\circ}\!\Leftarrow\!\underset{\gamma}{\circ}$

In case i) $-\theta\alpha_1 = a\alpha_1 + b\gamma$. $a, b \geq 0$. Hence since $a$ or $b$ must be zero and $-\theta\alpha_1 \neq \alpha_1$ we see that $-\theta\alpha_1 = \gamma$. Thus i) corresponds to 1). In case ii), $\underline{g} \equiv s\ell(3, \mathbb{C})$ and hence $\underline{g}_0 \equiv su(2, 1)$. This implies that

$\gamma = \alpha_2$ ($\ell_0 = 2$). $-\theta\alpha_1 \neq \alpha_1$ hence since $\frac{1}{2}(H_{\alpha_1} - \theta H_{\alpha_1}) = H_0$ satisfies $\alpha_1(H_0) = 1$ we see that $-\theta\alpha_1 \neq \alpha_1 + \alpha_2$ (otherwise $-\theta\alpha_1(H_0) = 2$). Thus $-\theta\alpha_1 = \alpha_2$.

iii), iv). In these cases $\underline{g} \equiv so(5, \mathbb{C}) \equiv sp(2, \mathbb{C})$. This implies that $\underline{g}_0 \equiv so(4, 1)$. Only case iii) is possible (cf. Warner [10], p. 30). Now $-\theta\alpha_1 = s_\gamma \alpha_1 = \alpha_1 + 2\gamma$. (This is because $s_\gamma \circ (-\theta)\Delta^+ = \Delta^+$ and o$\Longrightarrow$o has no automorphisms.) But then $\langle -\theta\alpha_1, \alpha_1 \rangle = 0$. Hence iii), iv) correspond to 1) in the statement of the lemma.

__4.3.__ Let $\Sigma(P, A)$ denote the roots of $\underline{a}_0$ on $\underline{n}_0$. We note then $\Sigma(P, A) = \{\alpha|_{\underline{a}_0} \mid \alpha \in \Delta^+, \alpha|_{\underline{a}_0} \neq 0\}$. Let $\langle \, , \rangle$ denote the dual to the Killing form restricted to $\underline{a}_0$.

__4.4.__ LEMMA. __Let__ F __be a simple, finite dimensional__ G-__module.__ __Let__ $\lambda_{i, F}$, $1 \leq i \leq \ell_0$ __be as in__ §3. __If__ $\lambda \in \Sigma(P, A)$ __is such that__ $\langle \lambda_{i, F}, \lambda \rangle \leq 0$ __then__

1) $\lambda = \alpha_i|_{\underline{a}_0}$.
2) If $\alpha \in \Delta^+$ and $\alpha|_{\underline{a}} = m\lambda$ with $m > 0$ then $\alpha = \alpha_i$ or $-\theta\alpha_i$.
3) If $-\theta\alpha_i \neq \alpha_i$ then $-\theta\alpha_i = \alpha_{i'}$ with $1 \leq i' \leq \ell_0$ and $\langle \alpha_i, \alpha_{i'} \rangle = 0$.

__Proof.__ If $\lambda \in \Sigma(P, A)$ then $\lambda = \alpha|_{\underline{a}_0}$, $\alpha \in \Delta^+$. Hence

$$\langle \lambda_{i, F}, \lambda \rangle = \tfrac{1}{2} \langle \lambda_{i, F}, \alpha - \theta\alpha \rangle = \tfrac{1}{2} \langle s_i(\Lambda+\delta), \alpha - \theta\alpha \rangle = \tfrac{1}{2} \langle \Lambda+\delta, s_i(\alpha) + s_i(-\theta\alpha) \rangle .$$

If $\alpha|_{\underline{a}_0} \neq \alpha_i|_{\underline{a}_0}$ then $\alpha \neq \alpha_i$ or $-\theta\alpha_i$. Thus $s_i\alpha$ and $s_i(-\theta\alpha)$ are in $\Delta^+$. Hence $\langle \lambda_{i, F}, \lambda \rangle > 0$. This proves 1) and 2).

To prove 3) we assume $-\theta\alpha_i \neq \alpha_i$ and if $\lambda = \alpha_i|_{\underline{a}_0}$ then $\langle \lambda_{i, F}, \lambda \rangle = 0$.

Now $4\langle\lambda_{i,F}, \lambda\rangle / \langle\alpha_i, \alpha_i\rangle = 2\langle\Lambda+\delta, -\alpha_i\rangle / \langle\alpha_i, \alpha_i\rangle$ $+ 2\langle\Lambda+\delta, s_i(-\theta\alpha_i)\rangle / \langle -\theta\alpha_i, -\theta\alpha_i\rangle$. .

If case 3) of lemma 4.2 is satisfied then $s_i(-\theta\alpha_i) = \alpha_i - \theta\alpha_i$. Hence $4\langle\lambda_{i,F}, \lambda\rangle / \langle\alpha_i, \alpha_i\rangle > 0$. Hence only case 2) of lemma 4.2 is possible. Thus $s_i(-\theta\alpha_i) = -\theta\alpha_i$. Now if $1 \leq j \leq \ell_0$ then it is not hard to see that $-\theta\alpha_j = \alpha_{j'} + \sum_{i \geq \ell_0+1} c_i^j \alpha_i$, $c_i^j \geq 0$, $c_i^j \in \mathbb{Z}$. If $i = i'$ and $-\theta\alpha_i \neq \alpha_i$. Since we are in case 2) of lemma 4.2, $-\theta\alpha_i \pm \alpha_i \notin \Delta$. Hence $\sum_{i \geq \ell_0+1} c_i^j \alpha_i \notin \Delta$. Hence $-\theta\alpha_i + \alpha_j \in \Delta$ for some $j \geq \ell_0 + 1$ and $-\theta\alpha_i + \alpha_j \neq \alpha_i$ or $-\theta\alpha_i$. But $(-\theta\alpha_i + \alpha_j)|_{\underline{a}_0} = \lambda$. This contradicts 2) in the lemma. We therefore know that $i \neq i'$. If $-\theta\alpha_i \neq \alpha_{i'}$ then $\alpha_{i'}|_{\underline{a}_0} = \lambda$. We have again contradicted 2). Hence $-\theta\alpha_i = \alpha_{i'}$. This proves 3).

## 5. Some remarks on parabolic induction.

5.1. Let G, K, M, A, N be as in §3. Then $P = MAN$ is a minimal parabolic subgroup of G. Let $P_1$ be a parabolic subgroup of G with Langlands composition $P_1 = M_1A_1N_1$. We assume that $A_1 \subset A$, $N_1 \subset N$ and $M_1 \supset M$, $\theta(M_1) = M_1$. Then the pair $(P_1, A_1)$ is called a standard parabolic pair (standard p-pair for short). $A_1$ is called the split component of $P_1$ (cf. Warner [10], 1.2.4 for a discussion of parabolic subgroups).

5.2. Fix for the moment, $(P_1, A_1)$, a standard p-pair. Let $\underline{a}_{1,0}$ be the Lie algebra of $A_1$ and let $\underline{n}_{1,0}$ denote the Lie algebra of $N_1$. Then $\mathrm{ad}(\underline{a}_{1,0})\underline{n}_{1,0} \subset \underline{n}_{1,0}$. Set for $\lambda \in \underline{a}_{1,0}^*$, $(\underline{n}_{1,0})_\lambda = \{x \in \underline{n}_{1,0} \mid \mathrm{ad}\,h.x = \lambda(h)x \text{ for } h \in \underline{a}_{1,0}\}$.

Set $\Sigma(P_1, A_1) = \{\lambda \in \underline{a}^*_{1,0} | (\underline{n}_{1,0})_\lambda \neq (0)\}$.

5.3. Let $\langle\, ,\rangle$ be the dual of the Killing form restricted to $\underline{a}_0$. We also use the notation $\langle\, ,\rangle$ for the dual of the Killing form restricted to $\underline{a}_{1,0}$.

5.4. Set $\underline{a}^+_{1,0} = \{h \in \underline{a}_{1,0} | \lambda(h) > 0$ for $\lambda \in \Sigma(P, A)$.

5.5. Set $K_1 = K \cap M_1$. Let $(\omega, H_\omega)$ be an irreducible unitary representation of $M_1$. Then $(\omega, H_\omega)$ is called tempered if the $K_1$-finite matrix entries of $\omega$ satisfy the weak inequality. This means that the character of $\omega$ defines a tempered distribution on $M_1$ (cf. [11], 9.3.1, 8.3.7).

5.6. Set $\rho_{P_1}(h) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}\, h)|_{\underline{n}_{1,0}}$ for $h \in \underline{a}_{1,0}$. Let $\underline{m}_{1,0}$ denote the Lie algebra of $M_1$. We will (as we have throughout this chapter) denote the complexification of a real Lie algebra $\underline{u}_0$ by $\underline{u}$.

5.7. Let $(\sigma, H)$ be an admissible representation of $M_1$. This means that $H$ is a Hilbert space, $\sigma$ is a strongly continuous representation of $M_1$ on $H$ and that $\sigma|_{K_1}$ splits into a countable direct sum of irreducible representations of $K_1$ each appearing with finite multiplicity.

5.8. Let $H_\infty$ denote the space of $C^\infty$ vectors of $(\sigma, H)$ and let $W$ be the space of $K_1$-finite vectors in $H$. Then as is well-known (cf. Warner [9]), $W \subset H_\infty$. $W$ is an $(\underline{m}_1, K_1)$-module under the differential of $\sigma$.

5.9. Let $I^\sigma_\infty$ denote the space of all $f : K \longrightarrow H$ such that

1) $f$ is of class $C^\infty$.

2) $f(K) \subset W$.

3) $f(kk_1) = \sigma(k_1)^{-1} f(k)$ for $k \in K$, $k_1 \in K_1$.

<u>5.10.</u> If $f \in I^{\sigma}_{\infty}$ set $(\pi_{\sigma}(k)f)(x) = f(k^{-1}x)$ for $k, x \in K$. Let $I^{\sigma}$ denote the space of $\pi_{\sigma}(K)$-finite elements of $I^{\sigma}_{\infty}$.

<u>5.11.</u> If $f \in I^{\sigma}_{\infty}$ and if $\nu \in \underline{a}_1^*$ define $f_{\nu}(kman) = \sigma(m)^{-1} e^{-(\rho_{P_1}+\nu)(\log a)} f(k)$ for $k \in K$, $m \in M_1$, $a \in A_1$, $n \in N_1$. Since $P_1 \cap K = M_1 \cap K = K_1$ we see that $f_{\nu} : G \longrightarrow H$ is of class $C^{\infty}$ and $f_{\nu}(G) \subset H_{\infty}$.

<u>5.12.</u> If $x \in \underline{g}_0$ and $f \in I^{\sigma}_{\infty}$ define $(\pi_{\sigma,\nu}(x)f)(g) = \frac{d}{dt} f(\exp(-tx)g)\big|_{t=0}$. Then it can be shown that $(\pi_{\sigma,\nu}, I^{\sigma})$ defines an admissible $(\underline{g}, K)$-module. We denote this $(\underline{g}, K)$-module by $\mathrm{Ind}_{P_1}^{G}(\sigma \otimes \nu \otimes I)$ and the action by $\pi_{P_1,\sigma,\nu}$.

<u>5.13.</u> If $f_1, f_2 \in I^{\sigma}$ then we define $\langle f_1, f_2 \rangle = \int_K \langle f_1(k), f_2(k) \rangle \, dk$ where $\langle \, , \rangle$ also denotes the Hilbert space structure on $H$.

<u>5.14.</u> Let $(P_1, A_1)$ and $(P_2, A_2)$ be standard p-pairs with $P_1 \supset P_2$, $A_2 \supset A_1$. $P_1 = M_1A_1N_1$, $P_2 = M_2A_2N_2$. Then $P_2 \cap M_1 = M_2{}^*A{}^*N = {}^*P$ is a parabolic subgroup of $M_1$. Also ${}^*AA_1 = A_2$. Let $\nu \in \underline{a}_1^*$. Then $\nu = \nu_1 + {}^*\nu$ with $\nu_1 = \nu|_{\underline{a}_{1,0}}$, ${}^*\nu = \nu|_{{}^*\underline{a}_{1,0}}$ (${}^*\underline{a}_{1,0}$ the Lie algebra of $\underline{a}_{1,0}$).

<u>5.15.</u> Let $(\sigma, H)$ be an admissible representation of $M_2$. Let $\nu \in \underline{a}_2^*$.

LEMMA. $\mathrm{Ind}_{P_1}^{G}(\mathrm{Ind}_{{}^*P}^{M_1}(\sigma \otimes {}^*\nu \otimes 1) \otimes \nu_1 \otimes 1)$ <u>is isomorphic with</u> $\mathrm{Ind}_{P_2}^{G}(\sigma \otimes \nu \otimes 1)$.

This result is well-known, cf. Wallach [9].

## 6. Some results of Langlands.

6.1. Let $(P_1, A_1)$ be a standard p-pair. Let $(\sigma, H)$ be an admissible representation of $M_1$. Let $\nu \in \underline{a}_1^*$. Then a matrix entry of $\pi_{P_1,\sigma,\nu}$ is by definition a function on $G$ of the form $g \mapsto \langle \pi_{P_1,\sigma,\nu}(g)f_1, f_2 \rangle$ where $f_1, f_2 \in I^\sigma$. If $W \subset I^\sigma$ is a $\pi_{P_1,\sigma,\nu}$-invariant subspace of $I^\sigma$ then $I^\sigma/W$ defines a $(\underline{g}, K)$-module under the quotient action $\bar{\pi}$ of $\pi_{P_1,\sigma,\nu}$ on $I^\sigma/W$.

6.2. A matrix entry of $\bar{\pi}$ will be by definition a function of the form $g \mapsto \langle \pi_{P_1,\sigma,\nu}(g)f_1, f_2 \rangle$ where $f_1, f_2$ are orthogonal to $W$. If $(\bar{\pi}, I^\sigma/W)$ is simple then it can be shown that the space of matrix entries of $\bar{\pi}$ depends only on the isomorphic class of $\bar{\pi}$ and not on the particular realization.

6.3. THEOREM (Langlands [7]). Let $(P_1, A_1)$ be a standard p-pair. Let $\omega$ be an irreducible tempered representation of $M_1$. Let $\nu \in \underline{a}_1^*$ be such that $\mathrm{Re}\langle \nu, \lambda \rangle > 0$ for $\lambda \in \Sigma(P_1, A_1)$. Then $(\pi_{P_1,\omega,\nu}, I^\omega)$ has a unique, non-zero, simple quotient $J(P_1, \omega, \nu)$. If $(P_2, A_2)$ is a standard p-pair and if $\omega'$ is an irreducible, tempered representation of $M_2$, $\nu_2 \in \underline{a}_2^*$ and if $\mathrm{Re}\langle \nu_2, \lambda \rangle > 0$ for $\lambda \in \Sigma(P_2, A_2)$ then $J(P_1, \omega, \nu)$ is isomorphic with $J(P_2, \omega', \nu')$ if and only if $(P_1, A_1) = (P_2, A_2)$, $\nu = \nu'$ and $\omega'$ is equivalent with $\omega$.

6.4. THEOREM (Langlands [7]). Let $(P_1, A_1)$ be a standard p-pair. Let $(\omega, H_\omega)$ be an irreducible tempered representation of $M_1$. Let $\nu \in \underline{a}_1^*$ be such that $\mathrm{Re}\langle \nu, \lambda \rangle > 0$ for $\lambda \in \Sigma(P_1, A_1)$. If $\varphi$ is a matrix entry of

$J(P_1, \omega, \nu)$ <u>(see</u> 6.2), <u>then</u>

$$\lim_{a \xrightarrow[P_1]{} \infty} e^{(\rho_{P_1} - \nu)(\log a)} \varphi(a) = L_\varphi$$

<u>exists (here</u> $a \xrightarrow[P_1]{} \infty$ <u>means</u> $B(\log a, \log a) = \|\log a\|^2$ <u>goes to</u> $\infty$ <u>and there is</u> $\varepsilon > 0$ <u>so tha</u> $\lambda(\log a) > \varepsilon \|\log a\|$ <u>for</u> $\lambda \in \Sigma(P_1, A_1)$). <u>Furthermore, if</u> $\varphi$ <u>is a non-zero matrix entry of</u> $J(P_1, \omega, \nu)$ <u>then there is</u> $u \in U(g)$ <u>so that if</u> $\varphi_1 = \ell_u \varphi$ <u>then</u> $L_{\varphi_1} \neq 0$ <u>(here</u> $(\ell_x \varphi)(g) = \frac{d}{dt} \varphi(\exp(-tx)g)|_{t=0}$ <u>for</u> $x \in \underline{g}_0$).

<u>6.5.</u> The importance of Theorems 6.3 and 6.4 is that the $(\underline{g}, K)$-modules $J(P_1, \omega, \nu)$ exhaust the set of all simple $(\underline{g}, K)$-modules. (We allow $(G, 1)$ as a standard p-pair.) Langlands proves this fact by a deep analysis of the asymptotics of admissible representations using Harish-Chandra's (unpublished) results on the asymptotics of admissible representations in parabolic directions (an important special case of this theory can be found in Warner [11], chapter 9). The tempered representations are just the J's associated with $(G, 1)$. This means in particular that if $(P_1, A_1) \neq (G, 1)$ then $J(P_1, \omega, \nu)$ is non-tempered.

## 7. <u>The classification.</u>

<u>7.1.</u> In this section we complete the line of ideas of this chapter by giving a list of the simple $(\underline{g}, K)$-modules, $V$, such that $H^1(\underline{g}, \underline{k}; V \otimes F^*) \neq 0$ where $F$ is a simple finite dimensional $G$-module.

<u>7.2.</u> Fix $F$ a simple finite dimensional $G$-module. Let $\Lambda$ be the

highest weight of $F$ relative to $\Delta^+$. Let $(\xi_F, \lambda_F)$ and $(\xi_{i,F}, \lambda_{i,F})$, $i = 1, \ldots, \ell_0$ be as in §3. Then we have

$$\bigoplus_{i=1}^{\ell_0} (\pi_{\xi_{i,F}, \lambda_{i,F}}, X^{\xi_{i,F}}) \xrightarrow{\bigoplus_{i=1}^{\ell_0} D_i} Y^F \longrightarrow 0$$

(see 3.20).

7.3. Applying the results of chapter V we see that if $V$ is a simple $(\underline{g}, K)$-module then $\mathrm{Hom}_{\underline{g}}(Y^F, V) \equiv H^1(\underline{g}, \underline{k}; V \otimes F^*)$.

7.4. LEMMA. *Let $V$ be a simple $(\underline{g}, K)$-module. Then*

$$\dim H^1(\underline{g}, \underline{k}; V \otimes F^*) \leq \sum_{i=1}^{\ell_0} \dim \mathrm{Hom}_{\underline{g}}(D_i(X^{\xi_{i,F}}), V) .$$

*Proof.* This is just a recapitulation of 7.2, 7.3.

7.5. We are left with computing the simple quotients of $D_i(X^{\xi_{i,F}})$, $1 \leq i \leq \ell_0$. The work of §4 (see lemma 4.4) says that the indices $1 \leq i \leq \ell_0$ naturally fall into three sets:

I) $-\theta\alpha_i \neq \alpha_i$ and $-\theta\alpha_i \neq \alpha_{i'}$ with $\langle \alpha_i, \alpha_{i'} \rangle = 0$.

II) $-\theta\alpha_i = \alpha_{i'}$ with $\langle \alpha_i, \alpha_{i'} \rangle = 0$.

III) $-\theta\alpha_i = \alpha_i$.

We will study the cases I), II), III) individually.

7.6. In case I) of 7.5, lemma 4.4 says that $\langle \lambda_{i,F}, \lambda \rangle > 0$ for $\lambda \in \Sigma(P, A)$. Theorem 6.3 implies that $(\pi_{\xi_{i,F}, \lambda_{i,F}}, X^{\xi_{i,F}})$ has a unique non-zero quotient $J(P, \xi_{i,F}, \lambda_{i,F})$. Hence since $D_i \neq 0$ we have

7.7. LEMMA. *If* $1 \leq i \leq \ell_0$ *and* I) 7.5 *is satisfied and if* $V$ *is a simple* $(\underline{g}, K)$-*module with* $\mathrm{Hom}_{\underline{g}}(D_i(X^{\xi_{i,F}}), V) \neq 0$ *then* $V$ *is isomorphic with* $J(P, \xi_{i,F}, \lambda_{i,F})$.

7.8. Set for $1 \leq i \leq \ell_0$, $\lambda_i = \alpha_i|_{\underline{a}_0}$. Set $\underline{a}_{i,0} = \{h \in \underline{a}_0 \mid \lambda_i(h) = 0\}$. Set

$$\underline{n}_{i,0} = \sum_{\substack{\mu \in \Sigma(P,A) \\ \mu|_{\underline{a}_{i,0}} \neq 0}} (\underline{n}_0)_\mu .$$

Set $A_i = \exp \underline{a}_{i,0}$, $N_i = \exp(\underline{n}_{i,0})$. Let $\tilde{M}_i = \{g \in G \mid gag^{-1} = a \quad a \in A_i\}$. Then $\tilde{M}_i = M_iA_i$ with $M_i$ a reductive subgroup of $G$ with compact center and $M_i \times A_i \longrightarrow \tilde{M}_i$, $m, a \longmapsto ma$ a Lie isomorphism. Set $P_i = M_iA_iN_i$. Then $(P_i, A_i)$ is a standard p-pair.

7.9. Of course, if $\dim A = 1$ then $(P_i, A_i) = (G, 1)$ for $1 \leq i \leq \ell_0$.

7.10. Let $1 \leq i \leq \ell_0$ be fixed (for the moment). Let ${}^*P_i = M_i \cap P$. Then ${}^*P_i = M\,{}^*A_i\,{}^*N_i$ and ${}^*A_i = M_i \cap A$, ${}^*N_i = M_i \cap N$. ${}^*P_i$ is a minimal parabolic subgroup of $M_i$.

7.11. Now suppose that $1 \leq i \leq \ell_0$ is as in II) in 7.5. Let $(\xi, H_\xi)$ be an irreducible unitary representation of $M$. Let $\nu \in {}^*\underline{a}_i^*$. Let $(\sigma_{\xi,\nu}, Z^\xi) = \mathrm{Ind}_{{}^*P_i}^{M_i} (\xi \otimes \nu \otimes 1)$ (see 5.14, 5.15). Set $\nu_{i,F} = \lambda_{i,F}|_{{}^*\underline{a}_{i,0}}$.

LEMMA. 1) $(\sigma_{\xi_{i,F}}, \nu_{i,F}, Z^{\xi_{i,F}})$ is a simple $(\underline{m}_i, K_i)$ $(K_i = M_i \cap K)$ module.

2) $(\sigma_{\xi_{i,F}}, \nu_{i,F}, Z^{\xi_{i,F}})$ is isomorphic with $(\sigma_{\xi_{i',F}}, \nu_{i',F}, Z^{\xi_{i',F}})$.

Proof. $M_i$ is locally isomorphic with a product of a compact subgroup of M and SL(2, $\mathbb{C}$). This lemma follows from the theory of the principal series of SL(2, $\mathbb{C}$) (cf. Jacquet, Langlands [4]).

7.12. LEMMA. Let i, i' be as in 7.11. Then $(\pi_{\xi_{i,F}}, \lambda_{i,F}, X^{\xi_{i,F}})$ is isomorphic with $(\pi_{\xi_{i',F}}, \lambda_{i',F}, X^{\xi_{i',F}})$.

Proof. Combine the lemma in 7.11 with the lemma in 5.14.

7.13. LEMMA. Let i, i' be as in 7.11. That is, $-\theta\alpha_i = \alpha_{i'}$ and $\langle \alpha_i, \alpha_{i'} \rangle = 0$. By interchanging i and i' we may assume that

$$\frac{2\langle \Lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \le \frac{2\langle \Lambda, \alpha_{i'} \rangle}{\langle \alpha_{i'}, \alpha_{i'} \rangle} .$$

1) If $2\langle \Lambda, \alpha_i \rangle / \langle \alpha_i, \alpha_i \rangle < 2\langle \Lambda, \alpha_{i'} \rangle / \langle \alpha_{i'}, \alpha_{i'} \rangle$. Then $\langle \lambda_{i,F}, \lambda \rangle > 0$ for $\lambda \in \Sigma(P, A)$. If V is a simple $(\underline{g}, K)$-module with $\mathrm{Hom}_{\underline{g}}(D_i(X^{\xi_{i,F}}), V)$ or $\mathrm{Hom}_{\underline{g}}(D_{i'}(X^{\xi_{i',F}}), V)$ non-zero, then V is isomorphic with $J(P, \xi_{i,F}, \lambda_{i,F})$.

2) If $2\langle \Lambda, \alpha_i \rangle / \langle \alpha_i, \alpha_i \rangle = 2\langle \Lambda, \alpha_{i'} \rangle / \langle \alpha_{i'}, \alpha_{i'} \rangle$ then $\nu_{i,F} = 0$ (see 7.11). $(\sigma_{\xi_{i,F}}, 0, Z^{\xi_{i,F}})$ is a unitary principal series representation of $M_i$ (hence it is tempered). If V is a simple $(\underline{g}, K)$-module and if $\mathrm{Hom}_{\underline{g}}(D_i(X^{\xi_{i,F}}), V) \neq 0$ or $\mathrm{Hom}_{\underline{g}}(D_{i'}(X^{\xi_{i',F}}), V) \neq 0$, then V is isomorphic

with $J(P_i, \omega_{i,F}, \lambda_{i,F}|_{\underline{a}_{i,0}})$. Here $\omega_{i,F}$ is the class of $(\sigma_{\xi_{i,F},0}, Z^{\xi_{i,F}})$.

Proof. 1) $2\langle \Lambda, \alpha_i\rangle / \langle \alpha_i, \alpha_i\rangle < 2\langle \Lambda, \alpha_{i'}\rangle / \langle \alpha_{i'}, \alpha_{i'}\rangle$. Then $\langle \lambda_{i,F}, \lambda_i\rangle = \frac{1}{2}\langle S_i(\Lambda+\delta), \alpha_i + \alpha_{i'}\rangle = \frac{1}{2}\langle \Lambda+\delta, \alpha_{i'}\rangle - \frac{1}{2}\langle \Lambda+\delta, \alpha_i\rangle = \frac{1}{2}\langle \Lambda, \alpha_{i'}\rangle - \frac{1}{2}\langle \Lambda, \alpha_i\rangle > 0$. Thus lemma 4.4 implies that $\langle \lambda_{i,F}, \lambda\rangle > 0$ for $\lambda \in \Sigma(P, A)$. We can now argue as in the proof of 7.7 to prove 1).

To prove 2) we note that the lemma in 5.15 implies that $(\pi_{\xi_{i,F}, \lambda_{i,F}}, X^{\xi_{i,F}})$ is isomorphic with $\mathrm{Ind}_{P_i}^G(\omega_{i,F} \otimes \lambda_{i,F}|_{\underline{a}_{i,0}} \otimes I)$. It is easy to see that if $\mu_{i,F} = \lambda_{i,F}|_{\underline{a}_{i,0}}$ then $\langle \mu_{i,F}, \lambda\rangle > 0$ for $\lambda \in \Sigma(P_i, A_i)$. Hence $(\pi_{\xi_{i,F}, \lambda_{i,F}}, X^{\xi_{i,F}})$ has a unique, non-zero, quotient (theorem 6.3), $J(P_i, \omega_{i,F}, \lambda_{i,F}|_{\underline{a}_{i,0}})$. The proof of 2) now is the same as that of 7.7.

7.14. We are now left with case III) in 7.5. We use the notation of 7.8 and 7.10. Set $\nu_{i,F} = \lambda_{i,F}|_{{}^*\underline{a}_{i,0}}$. Set $\mu_{i,F} = \lambda_{i,F}|_{\underline{a}_{i,0}}$. Then $(\pi_{\xi_{i,F}, \lambda_{i,F}}, X^{\xi_{i,F}})$ is isomorphic with $\mathrm{Ind}_{P_i}^G(\mathrm{Ind}_{{}^*P_i}^{M_i}(\xi_{i,F} \otimes \nu_{i,F} \otimes 1) \otimes \mu_{i,F} \otimes 1)$ (see 5.15).

7.15. Since $-\theta\alpha_i = \alpha_i$, $\alpha_i|_{\underline{n}^-} = (0)$. Hence 3.20.2) applies. We note that $R_{Y_i}^{n_i+1}$ (notation as in 3.20) defines a non-zero intertwining operator, $T_i$, from $\mathrm{Ind}_{{}^*P_i}^{M_i}(\xi_{i,F} \otimes \nu_{i,F} \otimes 1)$ to $\mathrm{Ind}_{{}^*P_i}^{M_i}(\xi_F \otimes -\nu_F \otimes 1)$. This can be seen by observing that $M_i$ is locally isomorphic to the product of a compact subgroup of $M$ and $SL(2, \mathbb{R})$. Now the representation theory of $SL(2, \mathbb{R})$ (cf. Jacquet, Langlands [4]) implies that $\ker T_i$ is finite dimensional and $\mathrm{Im}\, T_i = E_{i,F}$ is either simple or $E_{i,F} = E_{i,F}^+ \oplus E_{i,F}^-$ with $E_{i,F}^+$ and $E_{i,F}^-$ simple

$(\underline{m}_i, K_i)$-modules. Also $E_{i,F}$ is the $(\underline{m}_i, K_i)$-module of $K_i$-finite vectors of a square integrable $M_i$-representation $\omega_{i,F}$. If $E_{i,F} = E^+_{i,F} \oplus E^-_{i,F}$ then $\omega_{i,F} = \omega^+_{i,F} \oplus \omega^-_{i,F}$.

7.16. Using 3.20.1), 2) we see that $D_i(X^{\xi_{i,F}})$ is isomorphic with either $\mathrm{Ind}_{P_i}^G(\omega_{i,F} \otimes \mu_{i,F} \otimes 1)$ or $\mathrm{Ind}_{P_i}^G(\omega^+_{i,F} \otimes \mu_{i,F} \otimes 1) \oplus \mathrm{Ind}_{P_i}^G(\omega^-_{i,F} \otimes \mu_{i,F} \otimes 1)$. We also note that if $\lambda \in \Sigma(P_i, A_i)$ then $\langle \mu_{i,F}, \lambda \rangle > 0$. We can now argue as in 7.7 to prove

7.17. LEMMA. *If* $i$ *is as in case* III) *of* 7.5 *and if* $V$ *is a simple* $(\underline{g}, K)$-*module with* $\mathrm{Hom}_{\underline{g}}(D_i(X^{\xi_{i,F}}), V) \neq 0$, *then* $V$ *is isomorphic with* $J(P_i, \omega_{i,F}, \lambda_{i,F}|_{\underline{a}_{i,0}})$ *if* $\omega_{i,F}$ *is simple or* $V$ *is isomorphic with one of* $J(P_i, \omega^+_{i,F}, \lambda_{i,F}|_{\underline{a}_{i,0}})$ *or* $J(P_i, \omega^-_{i,F}, \lambda_{i,F}|_{\underline{a}_{i,0}})$.

7.18. Combining 7.7, 7.13 and 7.17 we have completed our classification of simple $(\underline{g}, K)$-modules with $\mathrm{Hom}_{\underline{g}}(D_i(X^{\xi_{i,F}}), V) \neq 0$ for some $i$.

## 8. Some immediate implications of the classification.

8.1. In the next chapter we will show how the results of §7 can be used to prove vanishing theorems for cohomology of unitary representations. In this section we point out two immediate consequences of the results of §7.

8.2. THEOREM. *Let* $F$ *be a simple finite dimensional* $G$-*module. Let* $V$ *be an admissible, simple* $(\underline{g}, K)$-*module, then*

1) $\dim H^1(\underline{g}, \underline{k}; V \otimes F^*) \leq 2$.

2) *If there is no* $1 \leq i \leq \ell_0$ *so that* $-\theta\alpha_i = \alpha_{i'}$ *and* $\langle \alpha_i, \alpha_{i'} \rangle = 0$,

*then* $\dim H^1(\underline{g}, \underline{k}; V \otimes F^*) \leq 1$.

*Proof.* This is an exercise using 7.4, 7.7, 7.13, 7.17, and 6.3.

8.3. THEOREM. *Assume that* $G$ *is simple. Let* $F$ *be a simple, finite dimensional* $G$-*module. Let* $(\pi, H)$ *be an irreducible, admissible representation of* $G$. *Let* $V$ *be the* $(\underline{g}, K)$-*module of* $K$-*finite vectors of* $H$. *If* $H^1(\underline{g}, \underline{k}; V \otimes F^*) \neq 0$ *and if* $(\pi, H)$ *is tempered, then* $G$ *is locally isomorphic with* $SL(2, \mathbb{R})$ *or* $SL(2, \mathbb{C})$.

*Proof.* 7.4, 7.7, 7.13, 7.17, and 6.5 imply that if $(\pi, H)$ is tempered, then $V$ is a quotient of $D_i(X^{\xi_{i,F}})$, $i$ must be in case II or III of 7.5, and $P_i = G$. But then $M_i = G$. This implies that $G$ has a normal subgroup isomorphic to $SL(2, \mathbb{R})$ or $SL(2, \mathbb{C})$. Q.E.D.

8.4. We note that 8.3 is a generalization of a result of Hotta, Wallach for $G = SO(n, 1)$.

## REFERENCES

[1] J. Dixmier, Algèbres Enveloppantes, Gauthier-Villars, Paris, 1974.

[2] S. Helgason, Differential Geometry and Symmetric Spaces, Academic Press, New York, 1962.

[3] ——————, Lie groups and symmetric spaces, Battelle Rencontres, 1967 Lectures in Mathematics and Physics, W. A. Benjamin, Inc., New York, 1968, 1-72.

[4] H. Jacquet and R. P. Langlands, Automorphic Forms on GL(2), Springer-Verlag, Lecture Notes in Mathematics, vol. 114, Berlin, 1970.

[5] A. Knapp and N. R. Wallach, Szegö kernels associated with the discrete series, Invent. Math. 34 (1976), 163-200.

[6] B. Kostant and S. Rallis, Orbits and representations associated to symmetric spaces, Amer. Jour. Math. 93 (1971), 753-809.

[7] R. P. Langlands, On the classification of irreducible representations of real algebraic groups, Mimeographed Article, Institute for Advanced Study.

[8] N. R. Wallach, Harmonic Analysis on Homogeneous Spaces, Marcel Dekker, New York, 1973.

[9] ——————, On Harish-Chandra's generalized c-functions, Amer. J. Math. 97 (1975), 386-403.

[10] G. Warner, Harmonic Analysis on Semisimple Lie Groups I, Springer-Verlag, Berlin, 1973.

[11] ——————, Harmonic Analysis on Semisimple Lie Groups II, Springer-Verlag, Berlin, 1973.

# VII. Unitary $(\underline{g}, \underline{K})$-modules with $H^1(\underline{g}, \underline{K}; V \otimes F^*) \neq 0$

Nolan R. Wallach

In this chapter we complete our discussion of the first cohomology of $(\underline{g}, \underline{K})$-modules by studying the unitary case. We also give a new proof of Delorme's theorem that relates first cohomology with the topology of $\hat{G}$.

## 1. Some results of Harish-Chandra

1.1. Let $G$ be a connected semi-simple Lie group with finite center. Let $(\pi, H)$ be an irreducible, admissible representation of $G$. We recall several (unpublished) results of Harish-Chandra and Langlands [9] on the asymptotic behavior of the matrix entries of $\pi$.

1.2. Let $(P, A)$ be the minimal parabolic pair of $G$ corresponding to $G = KAN$ (we retain the notation of VI, §§1, 5, 6). Let for $t \geq 0$, $\underline{a}_t^+ = \{h \in \underline{a} | \lambda(h) \geq t$ for $\lambda \in \Sigma(P, A)\}$. (Here we use $\underline{a}_{\mathbb{C}}$ for what was $\underline{a}_0$ in VI, §5 and $\underline{a}_{\mathbb{C}}$ for $\underline{a}$).

1.3. THEOREM (Harish-Chandra, cf. Warner [13], Chapter 9). Let $H_0$ be the space of all K-finite vectors in H. There is a countable set $\mathcal{E}(\pi)$ of elements of $\underline{a}_{\mathbb{C}}^*$ and a collection of non-zero functions $P_\Lambda : \underline{a} \times H_0 \times H_0 \longrightarrow \mathbb{C}$, for $\Lambda \in \mathcal{E}(\pi)$ satisfying the following properties

1) If $h \in \underline{a}$ then $P_\Lambda(h; v_1, v_2)$ is linear in $v_1$, conjugate linear in $v_2$,

2) If $Z \subset H_0$ is a finite dimensional subspace of $H_0$ and if $v_1, v_2 \in H_0$ then $h \longmapsto P_\Lambda(h; v_1, v_2)$ is a polynomial function of degree

less than or equal to $d_Z$ (a constant depending only on $Z$).

3) $\langle \pi(\exp h)v_1, v_2 \rangle = \sum_{\Lambda \in \mathcal{E}(\pi)} e^{\Lambda(h)} P_\Lambda(h; v_1, v_2)$ with convergence uniform on $\underline{a}_t^+$ for each $t > 0$.

<u>1.4</u>. The set $\mathcal{E}(\pi)$ is called the set of exponents of $\pi$. There is one more property of the $\mathcal{E}(\pi)$ which we record. If $\Lambda_1, \Lambda_2 \in \underline{a}_{\mathbb{C}}^*$ then we say that $\Lambda_1 > \Lambda_2$ if $\Lambda_1 - \Lambda_2$ is a sum of (not necessarily distinct) elements of $\Sigma(P, A)$.

THEOREM. (Harish-Chandra, cf. Warner [13], Chapter 9) There is a finite set $\mathcal{E}^0(\pi) \subset \mathcal{E}(\pi)$ so that

1) If $\Lambda \in \mathcal{E}(\pi)$ then there is $\mu \in \mathcal{E}^0(\pi)$ so that $\Lambda < \mu$.

2) If $\Lambda_1, \Lambda_2 \in \mathcal{E}^0(\pi)$ and $\Lambda_1 \neq \Lambda_2$ then $\Lambda_1 - \Lambda_2$ is not a sum of elements of $\Sigma(P, A)$.

<u>1.5</u>. The set $\mathcal{E}^0(\pi)$ is called the set of leading exponents of $\pi$.

<u>1.6</u>. Let now $(P_1, A_1)$ be a standard p-pair. $P_1 = M_1A_1N_1$ as in VI, §5. Then $A = {}^*A_1 \cdot A_1$ with ${}^*A_1 = A \cap M_1$. Set $(\underline{a}_1)_t^+ = \{h \in \underline{a}_1 | \lambda(h) \geq t$ for $\lambda \in \Sigma(P_1, A_1)\}$ for $t > 0$. The following result is an unpublished theorem of Harish-Chandra. A discussion of this theorem can be found in Langlands [9].

THEOREM. There is a countable set $\mathcal{E}_{P_1}(\pi) \subset (\underline{a}_1)_{\mathbb{C}}^*$ and a collection $\left\{q_{\mu,P_1}\right\}_{\mu \in \mathcal{E}_{P_1}(\pi)}$ of non-zero functions $q_{\mu,P_1} : {}^*A_1 \times \underline{a}_1 \times H_0 \times H_0 \longrightarrow \mathbb{C}$ satisfying

1) $q_{\mu,P_1}(a; h; v_1, v_2)$ is linear in $v_1$, anti-linear in $v_2$ and for fixed $v_1, v_2$ is analytic in $a \in {}^*A_1$, and a polynomial in $h$.

2) If $a \in {}^*A_1$ is fixed then $\langle \pi(a \exp h)v_1, v_2 \rangle = \sum_{\mu \in \mathcal{E}_{P_1}(\pi)} e^{\mu(h)} q_{\mu,P_1}(a;h;v_1,$

with convergence uniform on $(\underline{a}_1)_t^+$ for $t > 0$.

3) If ${}^*h \in {}^*\underline{a}_1$, and $h \in \underline{a}_1$, (${}^*\underline{a}_1$ the Lie algebra of ${}^*A_1$) and if ${}^*h + h \in \underline{a}_t^+$ for some $t > 0$ then $e^{\mu(h)} q_{\mu,P_1}(\exp {}^*h; h; v_1, v_2) = \sum_{\substack{\Lambda \in \mathcal{E}(\pi) \\ \Lambda|_{\underline{a}_1} = \mu}} e^{\Lambda(h)} p_\Lambda(h; v_1, v_2)$. Thus in particular $\mathcal{E}_{P_1}(\pi) = \{\Lambda|_{\underline{a}_1} | \Lambda \in \mathcal{E}(\pi)\}$.

## 2. A necessary condition for unitarizability

2.1. Let $(\pi, H)$ be a representation of $G$ on a Hilbert space $H$. Then $(\pi, H)$ is said to be uniformly bounded if there is a constant $C > 0$ so that $||\pi(g)v|| \leq C||v||$ for $g \in G$, $v \in V$.

2.2. Clearly a unitary representation is uniformly bounded (take $C = 1$). The following theorem is inspired by a result of Howe [4]. A similar result has been communicated to the author by Peter Trombi (for $(\pi, H)$ unitary).

2.3. THEOREM. Suppose that $G$ is simple. Let $(P_1, A_1)$ be a standard p-pair. Let $\omega$ be a tempered representation of $M_1$ and let $\nu \in (\underline{a}_1)_{\mathbb{C}}^*$ be such that $\mathrm{Re}\langle \nu, \lambda\rangle > 0$ for $\lambda \in \Sigma(P, A)$. If there exists $(\pi, H)$ a non-trivial uniformly bounded representation of $G$ so that $J(P_1, \omega, \nu)$ (see VI, §6) is isomorphic with the $(\underline{g}, K)$-module of K-finite vectors in $H$ then

$$\mathrm{Re}(\rho_{P_1} - \nu)(h) > 0$$

for all $h \in \underline{a}_1$ so that $h \neq 0$ and $\lambda(h) \geq 0$ for $\lambda \in \Sigma(P_1, A_1)$ (i.e. $h \neq 0$, $h \in (\underline{a}_1)_0^+$).

Proof. Assume that $(\pi, H)$ is uniformly bounded and $H_0$ is isomorphic with $J(P_1, \omega, \nu) \neq \mathbb{C}$. We first show that $\mathrm{Re}(\rho_{P_1} - \nu)(h) > 0$ for $h \in (\underline{a}_1)_t^+$, $t > 0$. Indeed, fix such an $h$. Theorem 6.4, VI §6 implies that if $v_1, v_2 \in H_0$ then ($a_t = \exp th$)

1) $\lim\limits_{\substack{t \to \infty \\ t > 0}} e^{t(\rho_{P_1} - \nu)(h)} \langle \pi(a_t)v_1, v_2 \rangle = L(v_1, v_2)$ with $L$ linear in $v_1$, anti-linear in $v_2$ and not identically zero.

Now 2) $|\langle \pi(a_t)v_1, v_2 \rangle| \leq C||v_1|| \, ||v_2||$ for $t \in \mathbb{R}$. This implies that $\mathrm{Re}(\rho_{P_1} - \nu)(h) \geq 0$. Suppose $\mathrm{Re}(\rho_{P_1} - \nu)(h) = 0$. Then 1), 2) imply that

3) $(L(v_1, v_2)) \leq C||v_1|| \, ||v_2||$, $v_1, v_2 \in H_0$. Now 3) implies that $L(v_1, v_2) = \langle Bv_1, v_2 \rangle$ with $B : H \longrightarrow H$ a bounded operator.

Set $ic = (\rho_{P_1} - \nu)(h)$, $c \in \mathbb{R}$.

4) $\lim\limits_{\substack{t \to \infty \\ t > 0}} e^{ict} \langle \pi(a_t)v_1, v_2 \rangle = L(v_1, v_2)$ for all $v_1, v_2 \in H$.

This is proved by an easy "$\mathcal{E}/3$" argument. 4) immediately implies that

5) $B^2 = B$ and $B \circ \pi(a_t) = \pi(a_t)B = e^{-ict}B$, $t \in \mathbb{R}$.

Now if $n \in N_1$ then $\pi(n)B = e^{ict}\pi(n)\pi(a_t)B$ (for $t > 0$) $= e^{ict}\pi(a_t)\pi(a_t^{-1}na_t)B$. Let $v \in H$. Then if $\varepsilon > 0$ is given there is $T$ so that if $t \geq T$ then $||\pi(a_t^{-1}na_t)Bv - Bv|| < \varepsilon$. Hence $||\pi(n)Bv - Bv|| = ||e^{ict}\pi(a_t)\pi(a_t^{-1}na_t)Bv - Bv|| = ||e^{ict}\pi(a_t)(\pi(a_t^{-1}na_t)Bv - Bv) + e^{ict}\pi(a_t)Bv - Bv|| = ||e^{ict}\pi(a_t)(\pi(a_t^{-1}na_t)Bv - Bv)|| \leq \varepsilon\, C$ for $t \geq T$. Since $\varepsilon > 0$ is arbitrary we have

6) $\pi(n)B = B$ for $n \in N_1$.

Set $\bar{N}_1 = \theta(N_1)$. Arguing the same way for $\bar{n} \in \bar{N}_1$ using $\pi(\bar{n})B = e^{-ict}\pi(\bar{n})\pi(a_t)^{-1}B$ we see that

7) $\pi(\bar{n})B = B$ for $\bar{n} \in \bar{N}_1$.

Now our assumptions imply $(P_1, A_1) \neq (G, 1)$. Hence $N_1$ and $\bar{N}_1$ generate $G$. Thus 6), 7) combine to prove that

8) $\pi(g)B = B$ for $g \in G$.

Since $J(P_1, \omega, \nu)$ was assumed non-trivial and $B \neq 0$ we have a contradiction.

Combining the above results we have

9) If $h \in (\underline{a}_1)_0^+$ then $\mathrm{Re}(\rho_{P_1} - \nu)(h) \geq 0$ and if $\mathrm{Re}(\rho_{P_1} - \nu)(h) = 0$ then $\lambda(h) = 0$ for some $\lambda \in \Sigma(P_1, A_1)$.

Suppose now $h \in (\underline{a}_1)_0^+$, $h \neq 0$ and $\mathrm{Re}(\rho_{P_1} - \nu)(h) = 0$. Set $\Sigma_h = \{\lambda \in \Sigma(P_1, A_1) \mid \lambda(h) = 0\}$. 9) implies that $\Sigma_h \neq \emptyset$. We note that if $\alpha, \beta \in \Sigma_h$ and if $\alpha + \beta \in \Sigma(P_1, A_1)$ then $\alpha + \beta \in \Sigma_h$ and if $\alpha, \beta \in \Sigma(P_1, A_1)$, $\alpha + \beta \in \Sigma_h$ then $\alpha, \beta \in \Sigma_h$. This implies that there is a standard p-pair $(P_2, A_2)$ so that

i) $P_2 \supset P_1$ ; $A_2 \subset A_1$ .

ii) Let $\underline{a}_2$ be the Lie algebra of $A_2$ then $\underline{a}_2 = \{h_1 \in \underline{a}_1 \mid \lambda(h_1) = 0 \ , \ \lambda \in \Sigma_h\}$ .

iii) $\Sigma(P_2, A_2) = \{\lambda|_{\underline{a}_2} \mid \lambda \in \Sigma(P_1, A_1) \ , \ \lambda \notin \Sigma_h\}$ .

In particular, $h \in \underline{a}_2$ and $\lambda(h) > 0$ for $\lambda \in \Sigma(P_2, A_2)$. We apply Theorem 1.7 first of all to $(P_1, A_1)$. Let ${}^*A_1 = M_1 \cap A$ as usual. Let $h_1 \in (\underline{a}_1)_t^+$, $t > 0$. If ${}^*a_1 \in {}^*A_1$ then

10) $$\langle \pi({}^*a_1 \exp h_1)v_1, v_2 \rangle = \sum_{\mu \in \mathcal{E}_{P_1}(\pi)} e^{\mu(h_1)} q_{\mu,P_1}({}^*a_1; h_1; v_1, v_2).$$

$v_1, v_2 \in H_0$.

Set $q_{\mu,P_1}(1; h_1; v_1, v_2) = q_{\mu,P_1}(h_1; v_1, v_2)$.

Since the right hand side of 10) converges uniformly on the sets $(\underline{a}_1)_t^+$ for $t > 0$ and for fixed $v_1, v_2$, $h_1 \longrightarrow q_{\mu,P_1}(h_1; v_1, v_2)$ is a polynomial of degree less than or equal to $d$ (see 1.3, 1.7. $d$ depends on $v_1, v_2$). 1) implies

11) If $\mu \in \mathcal{E}_{P_1}(\pi)$ and $\mu \neq \nu - \rho_{P_1}$ then $\mu = \nu - \rho_{P_1} - d\mu$ with $\mathrm{Re}\, d\mu(h_1) > 0$ for $h_1 \in \bigcap_{t > 0} (a_1)_t^+$ .

11) implies

12) $q_{\nu-\rho_{P_1}}(h_1; v_1, v_2) = L(v_1, v_2)$.

We now use Theorem 1.7 applied to $(P_2, A_2)$. We have if $h_2 \in \bigcap_{t>0} (a_2)_t^+$ then if $v_1, v_2 \in H_0$

13) $\langle \pi(\exp h_2)v_1, v_2 \rangle = \sum_{\mu \in \mathcal{E}_{P_2}(\pi)} e^{\mu(h)} q_{\mu,P_2}(1; h_2; v_1, v_2)$.

If $\mu \in \mathcal{E}_{P_2}(\pi)$ then $\mu = \xi|_{\underline{a}_2}$ with $\xi \in \mathcal{E}_{P_1}(\pi)$ (see Theorem 1.7). Let $\mu_0 = (\rho_{P_1} - \nu)|_{\underline{a}_2}$. If $\mu \in \mathcal{E}_{P_2}(\pi)$ and $\xi \in \mathcal{E}_{P_1}(\pi)$, $\xi|_{\underline{a}_2} = \mu$ and if $\xi \neq \rho_{P_1} - \nu$ then $\xi = \rho_{P_1} - \nu - d_\xi$ as in 11) and $d_\xi(h_1) > 0$ for $h_1 \in \bigcap_{t>0} (a_1)_t^+$. Thus if $d_\xi|_{\underline{a}_2} \neq 0$ then $d_\xi(h_2) > 0$ for $h_2 \in \bigcap_{t>0} (a_2)_t^+$. We therefore find for ${}^*a_1 \in {}^*A_2$

14) $\lim_{\substack{t \to \infty \\ t > 0}} (e^{-t\mu_0(h)} \langle \pi({}^*a_2)\pi(\exp th)v_1, v_2 \rangle - q_{\mu_0,P_2}(a_2^*; th; v_1, v_2)) = 0$

(here $h$ is as in the discussion preceeding 10)). Now $\operatorname{Re} \mu_0(h) = 0$ by assumption and $|\langle \pi(\exp th)v_1, v_2 \rangle| \leq C||v_1|| \, ||v_2||$ by assumption. Hence 14) implies that $q_{\mu_0,P_2}(a_2^*; th; v_2, v_2)$ is independent of $t$.

Set $q_{\mu_0,P_2}(a_2^*; th; v_1, v_2) = q(a_2^*; v_1, v_2)$. If $q(a_2^*; v_1, v_2) \equiv 0$ then applying 1.7 again we would have $L = 0$. Hence there is $a \in {}^*A_2$ so that $q(a; v_1, v_2) \neq 0$. We therefore have

15) $\lim_{\substack{t \to \infty \\ t > 0}} e^{-t\mu_0(h)} \langle \pi(\exp th)\pi(a)v_1, v_2 \rangle = q(a; v_1, v_2)$ for $v_1, v_2 \in H_0$.

Arguing as before we see that $|q(a; v_1, v_2)| \leq C||v_1|| \, ||v_2||$ hence $v_1, v_2 \longmapsto q(a; v_1, v_2)$ extends to $H \times H$. Arguing by an "$\mathcal{E}/3$" argument it is easy to see that 15) is true for $v_1, v_2 \in H$. But then we have

16) $q(a; v_1, v_2) = L_{P_2}(\pi(a)v_1, v_2)$ with $L_{P_2} : H \times H \longrightarrow \mathbb{C}$ continuous and sesquilinear. We argue as before to see that $L_{P_2}(v_1, v_2) = \langle B_{P_2}v_1, v_2\rangle$ with $B_{P_2} : H \longrightarrow H$ bounded $B_{P_2} \neq 0$ and $B_{P_2}^2 = B_{P_2}$, $B_{P_2}\pi(\exp th) = \pi(\exp th)B_{P_2} = e^{-t\mu_0(h)}B_{P_2}$. Finally, arguing as in 6), 7) we see $\pi(n)B_{P_2} = \pi(\bar{n})B_{P_2} = B_{P_2}$ for $n \in N_2$, $\bar{n} \in \theta(N_2)$. Hence $\pi(g)B_{P_2} = B_{P_2}$ for $g \in G$. We have (finally) reached a contradiction.

## 3. <u>The vanishing theorem for</u> $H^1(\underline{g}, \underline{k}; V \otimes F^*)$.

<u>3.1</u>. In this section we give a proof of the theorem in the Appendix to Chapter III, §4, 4.7 which uses no case by case checking for split rank $G \geq 1$.

<u>3.2</u>. Let $G$ be a connected, simple Lie group with finite center. Let $F$ be an irreducible, finite dimensional $G$-module.

THEOREM. Let $(\pi, H)$ be an irreducible, admissible, uniformly bounded representation of $G$. Let $V$ be the $(\underline{g}, \underline{k})$-module of $K$-finite vectors of $H$. If $G$ has split rank greater than 1 then $H^1(\underline{g}, \underline{k}; V \otimes F^*) = 0$.

<u>Proof</u>. If $H^1(\underline{g}, \underline{k}; V \otimes F^*) \neq 0$ then $V$ must be one of the $(\underline{g}, \underline{k})$-modules described in Chapter VI, §7. As in VI, 7.5 we seperate our analysis into cases I, II, III.

I) $V$ is isomorphic with $J(P, \xi_{i,F}, \lambda_{i,F})$ (see VI, 7.7). Clearly $\lambda_{i,F} = S_{\alpha_i}(\delta)|_{\underline{a}}$. Thus if $h \in \underline{a}$ and $\alpha_i(h) = 0$ then $\lambda_{i,F}(h) = \delta(h)$. If $G$ has split rank $> 1$ then there is $h \in \underline{a}_0^+$ with $\alpha_i(h) = 0$, $h \neq 0$. Since $\delta|_{\underline{a}} = \rho$ we have $(\rho - \lambda_{i,F})(h) = 0$. Hence we have contradicted Theorem 2.3.

II) In this case $V$ is isomorphic to $J(P, \xi_{i,F}, \lambda_{i,F})$ if $2\langle\Lambda, \alpha_i\rangle/\langle\alpha_i, \alpha_i\rangle < 2\langle\Lambda, \alpha_{i'}\rangle/\langle\alpha_{i'}, \alpha_{i'}\rangle$ or $J\left(P_i, \omega_{i,F}, \lambda_{i,F}|_{\underline{a}_{i,0}}\right)$ if $2\langle\Lambda, \alpha_i\rangle/\langle\alpha_i, \alpha_i\rangle = 2\langle\Lambda, \alpha_{i'}\rangle/\langle\alpha_{i'}, \alpha_{i'}\rangle$ (see VI, 7.13). Since $\lambda_{i,F}(h) = \rho(h)$ for $\alpha_i(h) = 0$ we have the same contradiction to Theorem 2.3 as in I).

III) In this case $V$ is isomorphic with $J\left(P_i, \omega, \lambda_{i,F}|_{\underline{a}_{i,0}}\right)$ with $\omega = \omega_{i,F}$ or $\omega^{\pm}_{i,F}$ (see VI, 7.17). Argue as above.

<u>3.3</u>. See A. III, 4.8 for a discussion of this theorem for $F = \mathbb{C}$. If $F \neq \mathbb{C}$ this result is due to Raghunathan [10] (see Chapter III).

<u>3.4</u>. In section 5 we will study the case when $G$ is of split rank 1.

## 4. <u>The relationship with the topology of</u> $\hat{G}$.

<u>4.1</u>. In this section we will derive a theorem of Delorme that relates the vanishing theorems for $H^1$ and the topology of $\hat{G}$.

<u>4.2</u>. Let $G$ be a Lie group. Let $\hat{G}$ denote the set of all equivalence classes of irreducible unitary representations of $G$. Let $\omega \in \hat{G}$ and $(\pi, H) \in \omega$. Let $\mathcal{P}(\omega) = \{\varphi_{\xi,\pi}(g) \mid \xi \in H\}$ here $\varphi_{\xi,\pi}(g) = \langle\pi(g)\xi, \xi\rangle$. Then $\mathcal{P}(\omega)$ is called the set of all positive definite functions associated with $\omega$.

<u>4.3</u>. If $S \subset \hat{G}$ set $\mathcal{P}(S) = \bigcup_{\omega \in S} \mathcal{P}(\omega)$.

<u>4.4</u>. Let $C(G)$ denote the space of all continuous complex valued functions on $G$. Topologize $C(G)$ with the topology of uniform convergence on compacta.

4.5. Clearly $\mathcal{P}(\omega) \subset C(G)$. If $S \subset \hat{G}$ and $\omega \in \hat{G}$ we say that $\omega$ is in $\overline{S}$ (the closure of $S$) if there is $\varphi \in \mathcal{P}(\omega)$, $\varphi \neq 0$ so that $\varphi \in \overline{\mathcal{P}(S)}$ (the closure of $\mathcal{P}(S)$). It can be shown that if $\omega \in \overline{S}$ and $\varphi \in \mathcal{P}(\omega)$ then $\varphi \in \overline{\mathcal{P}(S)}$ (see Dixmier [2], 18.1.5 p.315). This definition of closure in $\hat{G}$ defines a topology on $\hat{G}$. We therefore look upon $\hat{G}$ as a topological space.

4.6. Let $1 \in \hat{G}$ denote the class of the trivial representation. The following theorem is due to Delorme [1]. The proof below is new.

4.7. THEOREM. Suppose that $G$ is a connected semi-simple Lie group with finite center. If $1$ is not isolated in $\hat{G}$ then there exists a non-trivial irreducible unitary representation $(\pi, H)$ of $G$ so that if $\omega$ is the class of $(\pi, H)$ in $\hat{G}$ then

1) $\omega$ cannot be seperated from $1$ in $\hat{G}$ (i.e. every neighborhood of $1$ in $\hat{G}$ contains $\omega$).

2) If $V$ is the $(\underline{g}, \underline{K})$-module of K-finite vectors in $H$ then $H^1(\underline{g}, \underline{k}; V) \neq 0$.

Proof. Suppose that $1$ is not isolated in $\hat{G}$. Then $1$ is in the closure of $\hat{G} - \{1\}$. This implies that there exists a sequence $\omega_j \in \hat{G}$, $\omega_j \neq 1$ and $\varphi_j \in \mathcal{P}(\omega_j)$ so that $\lim_{j\to\infty} \varphi_j = 1$ uniformly on compacta.

If $f \in C(G)$ define ${}^0f(g) = \int_{K\times K} f(k_1gk_2)dk_1\, dk_2$ ($dk$ is Haar measure on $K$ normalized so that $\int_K dk = 1$).

Let $(\pi_j, H_j) \in \omega_j$. Let $E_j : H_j \longrightarrow H_j$ be defined by $E_j = \int_K \pi_j(k)dk$. Now $\varphi_j(g) = \langle \pi_j(g)v_j, v_j \rangle$, $j = 1, 2, \ldots$ . $g \in G$. Hence ${}^0\varphi_j(g) = \langle \pi_j(g)E_jv_j, E_jv_j \rangle$. Hence ${}^0\varphi_j \in \mathcal{P}(\omega_j)$. Furthermore, it is clear that

$\lim_{j\to\infty} {}^0\varphi_j = 1$ uniformly on compacta. This implies that for $j \geq j_0'$, $E_jH_j \neq 0$. We therefore suppose that $E_jH_j \neq 0$ for $j = 1, 2, \ldots$ .

As is well known $\dim E_jH_j = 1$ ($E_jH_j \neq 0$ by assumption). Using Harish-Chandra's parametrization of the zonal spherical functions (cf. Helgason [3], Chapter 10) we see that

$$ {}^0\varphi_j(g) = C_j \left\langle \pi_{\xi_0,\nu_j}(g)1, 1\right\rangle $$

with $\nu_j \in \underline{a}_{\mathbb{C}}^*$, $\operatorname{Re}\left\langle \nu_j, \lambda\right\rangle \geq 0$ $\lambda \in \Sigma(P, A)$ (here we use the notation of Chapter VI). Since $\lim_{j\to\infty} {}^0\varphi_j(1) = 1$ we may assume $C_j = 1$. We first prove

I) By taking a subsequence we may assume $\lim_{j\to\infty} \nu_j = \rho$. Applying the results of §2 we see that $\left\langle \operatorname{Re}\nu_j, \operatorname{Re}\nu_j\right\rangle \leq \left\langle \rho, \operatorname{Re}\nu_j\right\rangle$. Hence $\|\operatorname{Re}\nu_j\| \leq \|\rho\|$. Thus by taking a subsequence we may assume that $\lim_{j\to\infty} \operatorname{Re}\nu_j = \mu_0$.

We note that ${}^0\varphi_j(g)$. $\int_K e^{-(\rho + \nu_j)(H(g^{-1}k))} dk$. If $f \in C_i^\infty(\underline{a})$ define $\hat{f}(\nu) = \int_{\mathfrak{a}} f(h) e^{-i\nu(h)} dk$, $\nu \in \underline{a}_{\mathbb{C}}^*$ (dh Lebesgue measure on $\underline{a}$). If $f \in C_c^\infty(G)$, define $F_f(h) = \int_N f(\exp h\, n) dn$ where $dn$ is some fixed normalization of Haar measure on $N$. Using standard integration formulas it can be shown that if $\varphi_\nu(g) = \int_K e^{-(\rho + \nu)(H(g^{-1}k))} dk$ for $\nu \in \underline{a}_{\mathbb{C}}^*$ then $dy$ can be normalized so that

$$ \text{a)} \quad \int_G \varphi_\nu(g) f(g) dg = \left(F_{{}^0f}\right)^\wedge(-i\nu) $$

Using a) we observe that $\|\operatorname{Im}\nu_j\|$ is bounded. Indeed, if $f \in C_c^\infty(G)$, ${}^0f \neq 0$ and $\int_G f(g) dg = 1$ then $\lim_{j\to\infty} \int_G \varphi_{\nu_j}(g) f(g) dg = \int_G f(g) dg.$

Hence $\lim_{j\to\infty} \left(F_{0_f}\right)^{\wedge}(-i\nu_j) = 1$. If $||\mathrm{Im}\nu_j||$ were not bounded then the (easy part) of the classical Paley-Wiener theorem would imply that $\lim_{j\to\infty} \left(F_{0_f}\right)^{\wedge}(-i\nu_j) = 0$.

This implies that we can choose a subsequence of the $j$'s so that

$$\text{b)} \quad \lim_{j\to\infty} \nu_j = \nu_0 \in \underline{a}_{\mathbb{C}}^{*} .$$

But now it is clear that $\nu_0 = \rho$. Indeed, $\lim_{j\to\infty} \left(F_{0_f}\right)^{\wedge}(\nu_j) = \left(F_{0_f}\right)^{\wedge}(-i\nu_0)$. $\left(F_{0_f}\right)^{\wedge}(-i\nu_0) = \left(F_{0_f}\right)^{\wedge}(-i\rho)$ for all $f \in C_c^{\infty}(G)$. Since $\mathrm{Re}\langle\nu_0, \lambda\rangle \geq 0$ for $\lambda \in \Sigma(P, A)$ we see that if $\nu_0 \neq \rho$ then $S\Lambda(\xi_0, \nu_0) \neq \delta$ for any $S \in W(\Delta)$. Hence there is $z \in \underline{Z}(\underline{g})$ so that $\chi_{\Lambda(\xi_0,\nu_0)}(z) = 0$ and $\chi_{\delta}(z) = 1$. Let $f \in C_c^{\infty}(G)$ be such that $\int_G f(g)dg = 1$. If $z \cdot f = h$ then $F_{0_h}(-i\rho) = F_{0_f}(-i\rho) = 1$ and $F_{0_h}(-i\nu_0) = \chi_{\Lambda(\xi_0,\nu_0)}(z)F_{0_f}(-i\nu_0) = 0$. This contradiction implies that $\nu_0 = \rho$. We have proved I).

Now I) implies that for $j$ sufficiently large $\mathrm{Re}\langle\nu_j, \lambda\rangle > 0$ for $\lambda \in \Sigma(P, A)$. We take a subsequence and assume that this is true for all $j$. Let $V_j$ be the $(\underline{g}, \underline{K})$-module of K-finite vectors of $(\pi_j, H_j)$. If for $\nu \in \underline{a}_{\mathbb{C}}^{*}$ and $\mathrm{Re}\langle\nu, \lambda\rangle > 0$ for $\lambda \in \Sigma(P, A)$ then Harish-Chandra's C-function at $\nu$ is not zero (cf. Wallach [12], 8.10.16). This implies that $V_j$ is isomorphic with $J(P, \xi_0, \nu_j)$. Hence every K-finite matrix entry of $\pi_j$ is a matrix entry of $\pi_{\xi_0,\nu_j}$. Let $Z_j \subset X^{\xi_0}$ be such that $J(P, \xi_0, \nu_j) = X^{\xi_0}/Z_j$. Let $Z_j^{\perp} = \{\varphi \in X^{\xi_0} | \langle\varphi, Z_j\rangle = 0\}$. Then as a K-module $Z_j^{\perp}$ is isomorphic with $J(P, \xi_0, \nu_j)$. Let $\alpha_j : \mathrm{Hom}_K(\underline{p}, Z_j^{\perp}) \otimes \underline{p} \longrightarrow Z_j$ be defined by $\alpha(A \otimes X) = A(X)$. Since $\omega_j \neq 1$ for all $j$ we must have $\mathrm{Im}\alpha_j \neq 0$ for all $j$. (Otherwise $\pi_{\xi_0,\nu_j}(X)\cdot 1 = 0$ $X \in \underline{g}$). Let $U_j = \mathrm{Im}\alpha_j$. The inner product on $U_j$ gotten from the inner product on $H_j$ is of the form $\langle x, y\rangle_j = \langle B_j x, y\rangle$ where $B_j \in \mathrm{End}\, U_j$ and $\langle f, g\rangle = \int_K f(k)\overline{g(k)}dk$. Hence if $v_j \in \mathrm{Im}\alpha_j$ then $h_j(g) = \langle B_j \pi_{\xi_0,\nu_j}(g)v_j, v_j\rangle$ is in $\mathcal{P}(\omega_j)$. Hence $\langle \pi_{\xi_0,\nu_j}(g)v_j, B_j v_j\rangle$ is in $\mathcal{P}(\omega_j)$. Since $B_j$ is self adjoint and positive definite there is $v_j \in U_j$, $||v_j|| = 1$ so that $B_j v_j = \lambda_j v_j$, $\lambda_j > 0$. Hence if $\psi_j(g) = \langle \pi_{\xi_0,\nu_j}(g)v_j, v_j\rangle$ then $\psi_j \in \mathcal{P}(\omega_j)$. Finally, if $\alpha : \mathrm{Hom}_K(\underline{p}, X^{\xi_0}) \otimes \underline{p} \longrightarrow X^{\xi_0}$ is defined by $\alpha(A \otimes X) = A(X)$. Then $v_j \in \mathrm{Im}\alpha$ for each $j = 1, 2, \ldots$ and $\dim \mathrm{Im}\alpha < \infty$. Since $||v_j|| = 1$ we may choose a subsequence of the $j$'s so that $\lim_{j\to\infty} {}_j = v_0$. Clearly $||v_0|| = 1$. Set $\psi_0(g) = \langle \pi_{\xi_0,\rho}(g)v_0, v_0\rangle$. Then $\lim_{j\to\infty} \psi_j = \psi_0$ uniformly on compacta. Hence $\psi_0$ is a positive definite function on $G$.

This implies that $\pi_{\xi_0,\rho}$ contains an irreducible unitarizable subquotient, $V$, so that $\mathrm{Hom}_K(\underline{p}, V) \neq 0$. Let $(\pi, H)$ be the unitary, irreducible representation of $G$ so that $V$ is the $(\underline{g}, K)$-module of $K$-finite vectors in $H$. Then if $\omega$ is the class of $(\pi, H)$ we have seen that $\lim_{j\to\infty} \omega_j = \omega$ and $\lim_{j\to\infty} \omega_j = 1$. This proves 1) of Theorem 4.7. 2) follows from the fact that $\chi_{\Lambda(\xi_0,\rho)} = \chi_\delta$. Hence $\pi(C) = 0$. Thus $H^1(\underline{g}, \underline{k}; V) \doteq \mathrm{Hom}_K(\underline{p}, H) \neq 0$ (see III, 3.1) by construction. Q.E.D.

<u>4.8</u>. COROLLARY. If $G$ is simple and of split rank greater than 1 or if $G$ is a real form of $F_4$ or $C_n$, $n \geq 3$ then 1 is isolated in $\hat{G}$.

<u>Proof</u>. Use Theorem 4.7, Theorem 3.2 and A.III Theorem 4.7.

4.9. The above result is due to Kazhdan [6] and Wang [12] for the groups of split rank $> 1$. The result for $F_4$ and $C_n$ is essentially due to Kostant [8]. Delorme's proof of Theorem 3.2 uses Theorem 4.7 and Corollary 4.8 (obviously proved independently of each other).

4.10. We note that Corollary 4.9 says that if $1$ is not isolated in $\hat{G}$ and $G$ is simple then $G$ is locally isomorphic with $SO(n, 1)$ or $SU(n, 1)$.

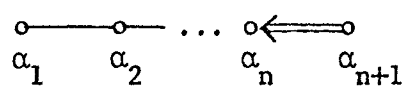
## 5. Groups of split rank 1

5.1. In this section we assume that $G$ has split rank $1$. We study the irreducible, admissible, $(\underline{g}, \underline{k})$-modules, $V$, with $H^1(\underline{g}, \underline{k}; V \otimes F^*) = 0$ for $F$ a simple, finite dimensional $G$-module.

5.2. THEOREM. Suppose that $G$ is locally isomorphic with $Sp(n, 1)$, $n \geq 2$ or a real form of $F_4$. Let $F$ be a simple $G$-module. Let $(\pi, H)$ be an irreducible, unitary representation of $G$ and let $V$ be the $(\underline{g}, \underline{k})$-module of $k$-finite vectors in $H$. Then $H^1(\underline{g}, \underline{k}; V \otimes F^*) = 0$.

Proof. We have seen in §3 that this result is true for $F$ the trivial $G$-module. We also note that the results in Johnson, Wallach [5] imply that the unique quotient of $Y^{\mathbb{C}}$ cannot be unitarizable. We may thus assume that $F$ is non-trivial.

a) $G$ is locally $Sp(n, 1)$, $n \geq 2$. Then $\Delta^+$ has Dynkin diagram



There is a unique real root $\alpha = \alpha_1 + 2\sum_{i=2}^{n} \alpha_i + \alpha_{n+1}$ (this means $\alpha(n^-) = 0$).

$S_{\alpha_1}\alpha = \alpha_1 + \alpha_2 + 2\sum_{i=3}^{n} \alpha_i + \alpha_{n+1}$ if $n \geq 3$, $S_{\alpha_1}\alpha = \alpha_1 + \alpha_2 + \alpha_3$ if $n = 3$.

In any event $\langle \lambda_{1,F} - \rho, \alpha \rangle = \langle \Lambda + \delta, s_{\alpha_1}\alpha \rangle - \langle \delta, \alpha \rangle$ where $\Lambda$ is the highest weight of $F$ relative to $\Delta^+$. Now $\alpha$ is short. Hence $2\langle \delta, \alpha \rangle / \langle \alpha, \alpha \rangle = 2n + 1$, $2\langle \delta, s_{\alpha_1}\alpha \rangle / \langle \alpha, \alpha \rangle = 2n$. If $\Lambda \neq 0$ then $2\langle \Lambda, s_{\alpha_1}\alpha \rangle / \langle \alpha, \alpha \rangle \geq 1$. Thus $\langle \lambda_{1,F} - \rho, \alpha \rangle \geq 0$. This implies that $J(P, \xi_{1,F}, \lambda_{1,F})$ is not unitarizable. Since $\ell_0$ is easily seen to be 1 in this case the theorem now follows for $Sp(n, 1)$.

b) $G$ <u>a real form of</u> $F_4$. Then $\Delta^+$ has Dynkin diagram ($\Pi_m = \{\alpha_1, \alpha_2, \alpha_3\}$ hence $\Pi - \Pi_m = \{\alpha_4\}$ with this labeling)

$$\underset{\alpha_1}{\circ} \!-\!\!-\!\!-\!\!-\! \underset{\alpha_2}{\circ} \Longrightarrow \underset{\alpha_3}{\circ} \!-\!\!-\!\!-\!\!-\! \underset{\alpha_4}{\circ} \quad .$$

The real root is $\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ which is short. $2\langle \lambda_{4,F} - \rho, \alpha \rangle / \langle \alpha, \alpha \rangle = 2\langle \Lambda + \delta, s_4\alpha \rangle / \langle \alpha, \alpha \rangle - 2\langle \delta, \alpha \rangle / \langle \alpha, \alpha \rangle$ where $\Lambda$ is the highest weight of $F$ relative to $\Delta^+$. Now $2\langle \delta, \alpha \rangle / \langle \alpha, \alpha \rangle = 11 \therefore s_{\alpha_4}\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$. Hence $2\langle \delta, s_{\alpha_4}\alpha \rangle / \langle \alpha, \alpha \rangle = 10$. Again $2\langle \Lambda, s_{\alpha_4}\alpha \rangle / \langle \alpha, \alpha \rangle \geq 1$ if $\Lambda \neq 0$. Thus $J(P, \xi_{4,F}, \lambda_{4,F})$ is not unitarizable. This completes the proof of the theorem.

<u>5.3</u>. We now study the cases where $G$ is locally isomorphic with $SO(n, 1)$ or $SU(n, 1)$. As is well known if $G$ is simple and of split rank 1 and if $G$ is not covered by Theorem 5.2 then $G$ is locally isomorphic with $SO(n, 1)$ or $SU(n, 1)$. We will deal with the orthogonal group and unitary groups separately.

<u>5.4</u>. THEOREM. Suppose that $G$ is locally isomorphic with $SO(n, 1)$, $n \geq 3$ (the case $n = 2$ fits more naturally with $SU(n, 1)$, $n = 1$). Let $\Lambda_0$ be the highest weight of the standard (matrix) representation of $G$ on

$\mathbb{C}^{n+1}$. Let $F$ be the simple, finite dimensional $G$-module with highest weight $\Lambda$ relative to $\Delta^+$. If $\Lambda \neq k\Lambda_0$, $k \geq 0$, $k \in \mathbb{Z}$ then $H^1(\underline{g}, \underline{k}; V \otimes F^*) = 0$ for $V$ a simple, admissible, unitary $(\underline{g}, \underline{k})$-module. If $\Lambda = k\Lambda_0$ then there exists a unique (up to isomorphism) unitary, simple $(\underline{g}, \underline{k})$-module, $V$, so that $H^1(\underline{g}, \underline{k}; V \otimes F^*) \neq 0$ furthermore $\dim H^1(\underline{g}, \underline{k}; V \otimes F^*) = 1$.

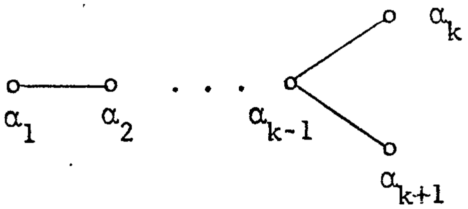
Proof. Let $\xi$ be an irreducible unitary representaion of $M$. Let $\nu \in \underline{a}^*_{\mathbb{C}}$ be such that $\mathrm{Re}\langle \nu, \lambda\rangle > 0$ for $\lambda \in \Sigma(P, A)$. Let $s \in W(A)$ be the unique non-trivial element $W(A)$. A simple consequence of the results of Knapp, Stein [7] is

1) If $\xi^s \neq \xi$ and $\langle \nu, \lambda\rangle \notin \mathbb{R} \cup i\mathbb{R}$ for $\lambda \in \Sigma(P, A)$ then $J(P, \xi, \nu)$ is not unitarizable.

We will use 1) and Theorem 2.3 to prove Theorem 5.4. We divide the proof up into three cases, $n = 3$, $n = 2k + 1$ and $n = 2k$.

a) $n = 3$. Then $\ell_0 = 2 = \ell$. By interchanging 1, 2 we may assume that $2\langle \Lambda, \alpha_1\rangle/\langle \alpha_1, \alpha_1\rangle \leq 2\langle \Lambda, \alpha_2\rangle/\langle \alpha_2, \alpha_2\rangle$. If $2\langle \Lambda, \alpha_1\rangle/\langle \alpha_1, \alpha_1\rangle <$ $2\langle \Lambda, \alpha_2\rangle/\langle \alpha_2, \alpha_2\rangle$ then $\xi_{1,F} \neq \xi_{2,F}$ and since $\xi^s_{1,F} = \xi_{2,F}$ we see that $J(P, \xi_{1,F}, \lambda_{1,F})$ is not unitarizable in this case. Thus if there exists a simple, unitarizable, $(\underline{g}, \underline{k})$-module, $V$, with $H^1(\underline{g}, \underline{k}; V \otimes F^*) \neq 0$ we must have $2\langle \Lambda, \alpha_1\rangle/\langle \alpha_1, \alpha_1\rangle = 2\langle \Lambda, \alpha_2\rangle/\langle \alpha_2, \alpha_2\rangle$. This means $\Lambda = k\Lambda_0$ with $k \geq 0$, $k \in \mathbb{Z}$. If $\Lambda = k\Lambda_0$ then $\lambda_{i,F} = 0$, $i = 1, 2$. Hence $\pi_{\xi_{i,F},0}$ defines a simple, unitary $(\underline{g}, \underline{k})$-module, $V$, with
(*)$H^1(\underline{g}, \underline{k}; V \otimes F^*) \neq 0$. $\dim H^1(\underline{g}, \underline{k}; V \otimes F^*) = 1$ and $V$ is determined up to isomorphism by (*) (by the results of VI, §7). The theorem is now proved in case a).

b) $n = 2k + 1$, $k \geq 2$. The Dynkin diagram of $\Delta^+$ is




and $\pi_m = \{\alpha_2, \ldots, \alpha_{k+1}\}$. Here $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i \leq k$, $\alpha_{k+1} = \varepsilon_k + \varepsilon_{k+1}$.
Also, $\alpha_1 - \theta\alpha_1 = 2\varepsilon_1 = 2\alpha_i + \ldots + 2\alpha_{k-1} + \alpha_k + \alpha_{k+1}$.

$$2\langle s_1(\Lambda + \delta), \alpha_1 - \theta\alpha_1\rangle/\langle\alpha_1, \alpha_1\rangle = 2\langle\Lambda + \delta, -\alpha_1 - \theta\alpha_1\rangle/\langle\alpha_1, \alpha_1\rangle =$$
$$2\sum_{i=2}^{k-1} 2\langle\Lambda, \alpha_i\rangle/\langle\alpha_i, \alpha_1\rangle + 2\langle\Lambda, \alpha_k\rangle/\langle\alpha_k, \alpha_k\rangle + 2\langle\Lambda, \alpha_{k+1}\rangle/\langle\alpha_{k+1}, \alpha_{k+1}\rangle +$$
$$2\langle\delta, \alpha_1 - \theta\alpha_1\rangle/\langle\alpha_1, \alpha_1\rangle - 2.$$

Using Theorem 2.3 we see that $J(P, \xi_{1,F}, \lambda_{1,F})$ is unitarizable only if

i) $2\langle\Lambda, \alpha_i\rangle/\langle\alpha_i, \alpha_i\rangle = 0, \quad 2 \leq i \leq k - 1$

ii) $2\langle\Lambda, \alpha_h\rangle/\langle\alpha_k, \alpha_k\rangle + 2\langle\Lambda, \alpha_{k+1}\rangle/\langle\alpha_{k+1}, \alpha_{k+1}\rangle \leq 1$.

Now a simple computation shows that $\xi^s_{1,F} = \xi_{1,F}$ only if $2\langle\Lambda, \alpha_k\rangle/\langle\alpha_k, \alpha_k\rangle = 2\langle\Lambda, \alpha_{n+1}\rangle/\langle\alpha_n, \alpha_n\rangle$. Hence if $J(P, \xi_{1,F}, \lambda_{1,F})$ is unitarizable then ii) implies $2\langle\Lambda, \alpha_n\rangle/\langle\alpha_n, \alpha_n\rangle = 2\langle\Lambda, \alpha_{n+1}\rangle/\langle\alpha_{n+1}, \alpha_{n+1}\rangle = 0$.

Now $2\langle\Lambda_0, \alpha_j\rangle/\langle\alpha_j, \alpha_j\rangle = \delta_{1,j}$, $j = 1, \ldots, k + 1$. If $\Lambda = k\Lambda_0$ then $\xi_F$ is the trivial representation. The results in Johnson, Wallach [5] now imply that $J(P, \xi_{1,F}, \lambda_{1,F})$ is unitarizable.

c) $n = 2k$, $k \geq 2$. The Dynkin diagram of $\Delta^+$ is


,

$\pi_m = \{\alpha_2, \ldots, \alpha_k\}$. $\alpha_1 - \theta\alpha_1 = 2(\alpha_1 + \ldots + \alpha_k)$. Thus $-\theta\alpha_1 = \alpha_1 + 2\left(\sum_{i=2}^{k} \alpha_i\right)$.
$\alpha = \alpha_1 + \ldots + \alpha_k$ is the real root in $\Delta^+$. $2\langle s_1(\Lambda + \delta), \alpha\rangle/\langle\alpha, \alpha\rangle =$

$2\langle \Lambda, \alpha_2 + \dots + \alpha_n\rangle/\langle \alpha, \alpha\rangle + 2\langle \delta, \alpha\rangle/\langle \alpha, \alpha\rangle - 2\langle \delta, \alpha_1\rangle/\langle \alpha, \alpha_F\rangle$ . Now $\langle \alpha, \alpha\rangle = \frac{1}{2}\langle \alpha_1, \alpha_1\rangle$ . Hence $2\langle \delta, \alpha_1\rangle/\langle \alpha, \alpha\rangle = 2$. Arguing as above, we find that if $J(P, \xi_{1,F}, \lambda_{1,F})$ is unitarizable then

i) $2\langle \Lambda, \alpha_i\rangle/\langle \alpha_i, \alpha_i\rangle = 0, \quad i \le k - 1$

ii) $2\langle \Lambda, \alpha_k\rangle/\langle \alpha_k, \alpha_k\rangle \le 1$ .

If $2\langle \Lambda, \alpha_k\rangle/\langle \alpha_k, \alpha_k\rangle = 1$ then the center of $M$ acts non-trivially under $\xi_{1,F}$. The results of Knapp, Stein [7], now imply that $J(P, \xi_{1,F}, \lambda_{1,F})$ is not unitarizable. Hence if $J(P, \xi_{1,F}, \lambda_{1,F})$ is unitarizable then $\Lambda = k\Lambda_0$ as above. The rest of the argument is the same as in case b).

The proof of the theorem is now complete.

5.5. We now look at $SU(n, 1)$, $n \ge 2$. In this case $\ell_0 = 2$. We label the Dynkin diagram for $\Delta^+$ as follows:

$$\overset{\circ}{\alpha_1} - \overset{\circ}{\alpha_3} \;\cdots\; \overset{\circ}{\alpha_{n-1}} - \overset{\circ}{\alpha_2}$$

Let $2\langle \Lambda_i, \alpha_j\rangle/\langle \alpha_j, \alpha_j\rangle = \delta_{ij}$.

5.6. THEOREM. Suppose that $G$ is locally isomorphic with $SU(n, 1)$. Let $F$ be the simple, finite dimensional $G$-module with highest weight $\Lambda$. Let $V$ be an admissible, simple, unitary $(\mathfrak{g}, \mathfrak{k})$-module. If $\Lambda \ne k\Lambda_1$ or $k\Lambda_2$ for $k \ge 0$, $k \in \mathbf{Z}$ then $H^1(\mathfrak{g}, \mathfrak{k}; V \otimes F^*) = 0$.

Suppose that $\Lambda = k\Lambda_i$ for $k \ge 0$, $k \in \mathbf{Z}$ and $i = 1$ or $2$.

a) If $k = 0$ (i.e. $F = \mathbb{C}$) then $J(P, \xi_{1,F}, \lambda_{1,F})$ and $J(P, \xi_{2,F}, \lambda_{1,F})$ are unitarizable.

b) If $k > 0$ then $J(P, \xi_{i,F}, \lambda_{i,F})$ is unitarizable, but $J(P, \xi_{i,F}, \lambda_{j,F})$, $i \ne j$ is not (see VI, §7 for notation).

Proof. The real root in $\Delta^+$ is $\alpha = \alpha_1 + \ldots + \alpha_n$. Then $2\langle s_1(\Lambda + \delta), \alpha\rangle/\langle\alpha, \alpha\rangle - 2\langle\delta, \alpha\rangle/\langle\alpha, \alpha\rangle = 2\langle\Lambda, \alpha_2 + \ldots + \alpha_n\rangle/\langle\alpha, \alpha\rangle - 1$. Thus Theorem 2.3 implies that if $J(P, \xi_{1,F}, \lambda_{1,F})$ is unitarizable then $\Lambda = k\Lambda_1$. Similarly, if $J(P, \xi_{2,F}, \lambda_{2,F})$ is unitarizable then $\Lambda = k\Lambda_2$. This proves the first assertion and the last assertion of b) of the theorem. We defer the proof of a) and the first assertion of b) to Chapter VIII where the representations $J(P, \xi_{i,F}, \lambda_{i,F})$, $\Lambda = k\Lambda_i$ are realized in the restriction of the oscillator (Weil, Metaplectic, ...) representation to $SU(n, 1)$.

5.7. The vanishing parts of the theorems of this section are due to Raghunathan [10] (see Chapter III). Of course, the proofs we have given are quite different from those of Raghunathan.

5.8. The above results (finally) complete our discussion of $H^1(\underline{g}, \underline{k}; V \otimes F^*)$ for $V$ unitarizable.

REFERENCES

[1] P. Delorme, "1-cohomology et produits tensoriels continus de représentations", Thèse, Université Paris VI, 1975.

[2] J. Dixmier, Les $C^*$-algèbres et leurs représentations, Gauthier-Villars, Paris, 1969.

[3] S. Helgason, Differential Geometry and Symmetric Spaces, Academic Press, New York, 1962.

[4] R. Howe, "On the asymptotic behavior of matrix coefficients", preprint.

[5] K. Johnson and N. R. Wallach, "Composition series and intertwining operators for the spherical principal series, I", to appear, Trans. A.M.S.

[6] D. A. Kazdan, "Connection of the dual space of a group with the structure of its closed subgroups", Func. Anal. Appl. 1(1967), 63-65.

[7] A. Knapp and E. Stein, "Intertwining operators for semisimple groups", Annals of Math. 93(1971), 489-577.

[8] B. Kostant, "On the existence and irreducibility of certain series of representations", Lie Groups and Their Representations, Edited by I. M. Gelfand, John Wiley, New York, 1975, 231-330.

[9] R. P. Langlands, "On the classification of irreducible representations of real algebraic groups", Mimeographed article, Institute for Advanced Study, 1973.

[10] M. S. Raghunathan, "Vanishing theorems for cohomology groups associated with discrete subgroups of semisimple Lie groups", Osaka J. Math. 3(1966), 243-256.

[11] N. R. Wallach, Harmonic Analysis on Homogeneous Spaces, Marcel Dekker, New York, 1973.

[12] S. P. Wang, "The dual space of semisimple Lie groups", Amer. J. Math. 89(1967), 124-132.

[13] G. Warner, Harmonic Analysis on Semisimple Lie groups II, Springer-Verlag, New York, 1973.

# Appendix to Chapter VII. Additional results on the growth of matrix entries of principal series representations.

Nolan R. Wallach

In this appendix we prove a result about the growth of matrix entires of the representations that occur as subquotients (but not quotients) in the module $\pi_{P,\omega,\nu}$ (as in VI, 6.3). Our main result (Theorem 3.2) is also known to G. Zuckerman.

## 1. The basic lemma.

1.1. As usual on $\mathbb{R}^n$, set $\langle x, y\rangle = \Sigma x_i y_i$. Let $(\mathbb{Z}^+)^n$ be the set of n-tuples of non-negative integers. If $m = (m_1, \ldots, m_n) \in (\mathbb{Z}^+)^n$ set $|m| = m_1 + \ldots + m_n$.

1.2. LEMMA. Let $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ be distinct. Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$, $\mu_i > 0$, $i = 1, \ldots, n$. Let for each $m \in (\mathbb{Z}^+)^n$, $p_{i,m}(t)$ be a polynomial of degree $\leq d$. Suppose that

$$\varphi(t) = \sum_{i=1}^{k} e^{\lambda_i t} \sum_{m\in(\mathbb{Z}^+)^n} e^{-\langle m,\mu\rangle t} p_{i,m}(t),$$

with convergence absolute and uniform for $t \geq 1$. Suppose also that $p_{i,0} \not\equiv 0$ for $i = 1, \ldots, k$. Then $\lim_{t\to+\infty} \varphi(t) = 0$ if and only if $\operatorname{Re} \lambda_i < 0$, $i = 1, \ldots, k$.

Proof. Set

$$\psi_i(t) = \sum_{\substack{m\in(\mathbb{Z}^+)^n \\ |m|>0}} e^{-\langle m,\mu\rangle t} p_{i,m}(t).$$

for $i = 1, 2, \ldots, k$.

1) $\lim_{t\to+\infty} \psi_i(t) = 0$, $i = 1, \ldots, k$.

To prove 1) we note that if $\varepsilon > 0$ is given there is $M$ so that

a) $$\sum_{|m|\geq M} e^{-\langle m, \mu\rangle t} |p_{i,m}(t)| < \varepsilon \quad \text{for} \quad t \geq 1 .$$

Also, since $p_{i,m}$ is a polynomial there is a constant $C$ so that

b) $$\sum_{0<|m|\leq M} e^{-\langle m, \mu\rangle t} |p_{i,m}(t)| \leq C \sum_{0<|m|\leq M} e^{-\frac{1}{2}\langle m, \mu\rangle t} .$$

b) implies that there is $T > 1$ so that if $t \geq T$ then

$$\sum_{0<|m|\leq M} e^{-\langle m, \mu\rangle t} |p_{i,m}(t)| \leq \varepsilon .$$

Hence if $t \geq T$ then $|\psi_i(t)| \leq 2\varepsilon$. This proves 1).

Now

$$\varphi(t) = \sum_{i=1}^{k} e^{\lambda_i t} (p_{i,0}(t) + \psi_i(t)) .$$

If $\operatorname{Re} \lambda_i < 0$ for $i = 1, \ldots, k$ then

$$\lim_{t\to+\infty} e^{\lambda_i t} = 0, \quad \lim_{t\to+\infty} e^{\lambda_i t} p_{i,0}(t) = 0 .$$

Thus by 1) $\lim_{t\to+\infty} \varphi(t) = 0$.

Thus to complete the proof we need only show that if $\lim_{t\to+\infty} \varphi(t) = 0$, $\operatorname{Re} \lambda_i < 0$, $i = 1, \ldots, k$. By relabeling the $\lambda_i$ we have $\operatorname{Re} \lambda_1 \geq \operatorname{Re} \lambda_2 \geq \ldots \geq \operatorname{Re}$

We assume that $\operatorname{Re} \lambda_1 \geq 0$. Then for $i = 1, 2, \ldots, k$.

1) $\lim_{t\to+\infty} \psi_i(t) = 0$, $i = 1, \ldots, k$.

To prove 1) we note that if $\varepsilon > 0$ is given there is $M$ so that

a) $$\sum_{|m|\geq M} e^{-\langle m,\mu\rangle t}|p_{i,m}(t)| < \varepsilon \quad \text{for} \quad t \geq 1 \quad .$$

Also, since $p_{i,m}$ is a polynomial there is a constant $C$ so that

b) $$\sum_{0<|m|\leq M} e^{-\langle m,\mu\rangle t}|p_{i,m}(t)| \leq C \sum_{0<|m|\leq M} e^{-\frac{1}{2}\langle m,\mu\rangle t} \quad .$$

b) implies that there is $T > 1$ so that if $t \geq T$ then

$$\sum_{0<|m|\leq M} e^{-\langle m,\mu\rangle t}|p_{i,m}(t)| \leq \varepsilon \quad .$$

Hence if $t \geq T$ then $|\psi_i(t)| \leq 2\varepsilon$. This proves 1).

Now

$$\varphi(t) = \sum_{i=1}^{k} e^{\lambda_i t}(p_{i,0}(t) + \psi_i(t)) \quad .$$

If $\operatorname{Re}\lambda_i < 0$ for $i = 1, \ldots, k$ then

$$\lim_{t\to+\infty} e^{\lambda_i t} = 0, \ \lim_{t\to+\infty} e^{\lambda_i t} p_{i,0}(t) = 0 \quad .$$

Thus by 1) $\lim_{t\to+\infty} \varphi(t) = 0$.

Thus to complete the proof we need only show that if $\lim_{t\to+\infty} \varphi(t) = 0$, $\operatorname{Re}\lambda_i < 0$, $i = 1, \ldots, k$. By relabeling the $\lambda_i$ we have $\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_2 \geq \ldots \geq \operatorname{Re}\lambda_k$. We assume that $\operatorname{Re}\lambda_1 \geq 0$. Then

$$|e^{-\lambda_1 t}\varphi(t) - \sum_{i=1}^{k} e^{(\lambda_i-\lambda_1)t} p_{i,0}(t)| = |\sum_{i=1}^{k} e^{(\lambda_i-\lambda_1)t} \psi_i(t)| \ .$$

Since $\mathrm{Re}\ \lambda_1 \geq 0$ we see that

$$\lim_{t\to+\infty} e^{-\lambda_1 t}\varphi(t) = 0 \ .$$

Hence by 1) we have

$$\lim_{t\to+\infty} |\sum_{i=1}^{k} e^{(\lambda_i-\lambda_1)t} p_{i,0}(t)| = 0 \ .$$

Let $\mathrm{Re}\ \lambda_1 = \mathrm{Re}\ \lambda_2 = \ldots = \mathrm{Re}\ \lambda_{k_0} > \mathrm{Re}\ \lambda_{k_0+1}$. Then

$$\lim_{t\to+\infty} |\sum_{i>k_0} e^{(\lambda_i-\lambda_1)t} p_{i,0}(t)| = 0 \ .$$

We therefore have

i) $\lim\limits_{t\to+\infty} |\sum\limits_{i=1}^{k_0} e^{(\lambda_i-\lambda_1)t} p_{i,0}(t)| = 0.$

Now $p_{i,0}(t) = \sum\limits_{j=0}^{q} a_{i,j}t^j$ with $a_{i,q} \neq 0$ for some $(q \leq d)$, $i \leq k_0$.

Multiplying through in i) by $t^{-q}$ we see

ii) $\lim\limits_{t\to+\infty} |\sum\limits_{i=1}^{k_0} e^{(\lambda_i-\lambda_1)t} a_{i,q}| = 0.$

Now Lemma A. 3. 2. 1, p. 428 of [2], implies $a_{i,q} = 0$ for $i \leq k_0$.

This contradiction implies the lemma.

## 2. Some results on exponents.

2.1. In this section we use the notation of Chapter VII, §1. There is a misstatement in Theorem 1.6, Chapter VII, which we wish to remedy before we go on (it causes no difficulty in the rest of that chapter). If $(P_1, A_1)$ is a standard p-pair and if $t > 0$, $\eta > 0$ set $(\underline{a}_1)^+_{t,\eta} = \{H \in (\underline{a}_1)^+_t \mid \lambda(H) \geq \eta B(H, H)^{\frac{1}{2}}$ for $\lambda \in \Sigma(P_1, A_1)\}$. In Theorem 1.6, Chapter VII, 2), the uniform and absolute convergence is on the sets $(\underline{a}_1)^+_{t,\eta}$ for $t > 0$, $\eta > 0$ not on $(\underline{a}_i)^+_t$ for $t > 0$.

We note also that on the last line of p. 5 and line 3 of p. 6, $\cap$ should be replaced by $\cup$.

2.2. We will also need a sharper version of a result of Langlands (see Chapter VI, Theorem 6.4, for one version). Fix $(P_1, A_1)$ a standard p-pair. Let $(\omega, H_\omega)$ be an irreducible tempered representation of $M_1$ and $\nu \in (\underline{a}_1)^*_{\mathbb{C}}$ so that $\mathrm{Re}\langle \nu, \lambda \rangle > 0$ for $\lambda \in \Sigma(P_1, A_1)$.

2.3. THEOREM (Langlands [1], Lemma 3.12 and Lemma 3.13).

1) If $\varphi, \psi \in I^\omega$ (see Chapter VI, §5) are K-finite, and $m \in M_1A_1$ then (with $\pi_\nu = \pi_{P_1,\omega,\nu}$)

$$\lim_{a \xrightarrow[P_1]{} \infty} e^{(\rho_{P_1} - \nu)(\log a)} \langle \pi_\nu(ma)\varphi, \psi \rangle = L(\varphi, \psi)(m) .$$

2) Let $Z^{\omega,\nu} = \{\varphi \in I^\omega \mid L(\varphi, \psi) = 0$ for all $\psi \in I^\omega\}$. Then $J(P_1, \omega, \nu) = I^\omega / Z^{\omega,\nu}$.

<u>2.4.</u> LEMMA. <u>Retain the notation of</u> 2.2 <u>and</u> 2.3. <u>Let</u> $(\pi, H)$ <u>be an irreducible subquotient of</u> $Z^{\omega,\nu}$. <u>If</u> $\lambda \in \mathcal{E}_{P_1}(\pi)$ <u>then</u>

$$\mathrm{Re}(\lambda + \rho_{P_1} - \nu)\big|_{\underline{a}_1^+} < 0$$

<u>(here</u> $\underline{a}_1^+ = \cup_{t>0} (\underline{a}_1)_t^+$).

<u>Proof.</u> Let $\varphi$ be a right and left K-finite matrix entry of $\pi$. Then $\varphi$ is a finite linear combination of matrix entries of $Z^{\omega,\nu}$. The definition of $Z^{\omega,\nu}$ in Theorem 2.3, 2), implies that

$$\lim_{a \xrightarrow[P_1]{} \infty} e^{(\rho_{P_1} - \nu)(\log a)} \varphi({}^*aa) = 0$$

for ${}^*a \in {}^*A_1 = M_1 \cap A$. Fix $h \in \underline{a}_1^+$. Then $th \xrightarrow[P_1]{} \infty$ if $t \to \infty$, $t > 0$. Also for suitable $\eta > 0$ we have $th \in (\underline{a}_1)_{1,\eta}^+$ for $t \geq T$ with $T > 0$ fixed. We therefore have

$$1)\ \lim_{t \to +\infty} e^{t(\rho_{P_1} - \nu)(h)} \varphi({}^*a\, \mathrm{exp}\, th) = 0.$$

Also

$$2)\ \varphi({}^*a\, \mathrm{exp}\, th) = \sum_{\lambda \in \mathcal{E}_{P_1}(\pi)} e^{t\lambda(h)} q_\lambda({}^*a; th)$$

(as in Theorem 1.6, Chapter VII).

The hypotheses of Lemma 1.2 are easily seen to be satisfied for

$$\psi(t) = e^{t(\rho_{P_1} - \nu)(h)} \varphi({}^*a\, \mathrm{exp}\, th) = \sum_{\lambda \in \mathcal{E}_{P_1}(\pi)} e^{t(\rho_{P_1} - \nu + \lambda)(h)} q_\lambda({}^*a; th) .$$

Hence Lemma 1.2 implies that if $t \longrightarrow q_\lambda({}^*a; th)$ is not identically zero then $\mathrm{Re}(\rho_{P_1} - \nu + \lambda)(h) < 0$. This proves the lemma.

2.5. LEMMA. Let $(\pi, H) = J(P', \omega', \nu')$ with $(P', A')$ a standard p-pair.

1) If $\lambda \in \mathcal{E}_{P'}(\pi)$ then $\mathrm{Re}(-\rho_{P'} + \nu' - \lambda)|_{(\underline{a}')^+} \geq 0$.

2) There exists $\lambda \in \mathcal{E}_{P'}(\pi)$ with $\mathrm{Re}\,\lambda = \mathrm{Re}(-\rho_{P'} + \nu')$.

Proof. 1) is proved in exactly the same way as Lemma 2.4.

2) Suppose there is no $\lambda \in \mathcal{E}_{P'}(\pi)$ with $\mathrm{Re}\,\lambda = \mathrm{Re}(-\rho_{P'} + \nu)$. Set $S = \{h \in \underline{a}' \,|\, \mathrm{Re}(\rho_{P'} - \nu' - \lambda)(h) = 0\}$. Since $\mathcal{E}_{P'}(\pi)$ is countable and $\mathrm{Re}(\rho_{P'} - \nu' - \lambda) \neq 0$ for all $\lambda \in \mathcal{E}_{P'}(\pi)$ the set $(\underline{a}')^+ \cap S$ has measure zero in $(\underline{a}')^+$. Hence there is $h \in (\underline{a}')^+$ with $\mathrm{Re}(\rho_{P'} - \nu' - \lambda)(h) < 0$ for all $\lambda \in \mathcal{E}_{P'}(\pi)$. Now Lemma 1.2 implies a contradiction (again one argues as in the proof of Lemma 2.4).

## 3. The main result.

3.1. We retain the notation of §2.

3.2. THEOREM. Let $(P_1, A_1)$ be a standard p-pair. Let $(\omega, H_\omega)$ be an irreducible tempered representation of $M_1$. Let $\nu \in (\underline{a}_1)^*_{\mathbb{C}}$ be such that $\mathrm{Re}\langle \nu, \lambda \rangle > 0$ for $\lambda \in \Sigma(P_1, A_1)$. Let $Z^{\omega,\nu}$ be as in Theorem 2.3. If $(P', A')$ is a standard p-pair, if $\omega'$ is an irreducible tempered representation of $M'$, if $\nu' \in (\underline{a}')^*_{\mathbb{C}}$ with $\mathrm{Re}\langle \nu', \lambda \rangle > 0$ for $\lambda \in \Sigma(P', A')$ and if $J(P', \omega', \nu')$ is a subquotient of $Z^{\omega,\nu}$ then $P' \supset P_1$ and $\mathrm{Re}(\nu - \nu')|_{(\underline{a}')^+} \geq 0$. If $P' = P_1$ then $\mathrm{Re}(\nu - \nu')|_{(\underline{a}_1)^+} > 0$.

Hence Lemma 1.2 implies that if $t \longrightarrow q_\lambda({}^*a; th)$ is not identically zero then $\mathrm{Re}(\rho_{P_1} - \nu + \lambda)(h) < 0$. This proves the lemma.

2.5. LEMMA. Let $(\pi, H) = J(P', \omega', \nu')$ with $(P', A')$ a standard p-pair.

1) If $\lambda \in \mathcal{E}_{P'}(\pi)$ then $\mathrm{Re}(-\rho_{P'} + \nu' - \lambda)|_{(\underline{a}')^+} \geq 0$.

2) There exists $\lambda \in \mathcal{E}_{P'}(\pi)$ with $\mathrm{Re}\,\lambda = \mathrm{Re}(-\rho_{P'} + \nu')$.

Proof. 1) is proved in exactly the same way as Lemma 2.4.

2) Suppose there is no $\lambda \in \mathcal{E}_{P'}(\pi)$ with $\mathrm{Re}\,\lambda = \mathrm{Re}(-\rho_{P'} + \nu)$. Set $S = \{h \in \underline{a}' \,|\, \mathrm{Re}(\rho_{P'} - \nu' - \lambda)(h) = 0\}$. Since $\mathcal{E}_{P'}(\pi)$ is countable and $\mathrm{Re}(\rho_{P'} - \nu' - \lambda) \neq 0$ for all $\lambda \in \mathcal{E}_{P'}(\pi)$ the set $(\underline{a}')^+ \cap S$ has measure zero in $(\underline{a}')^+$. Hence there is $h \in (\underline{a}')^+$ with $\mathrm{Re}(\rho_{P'} - \nu' - \lambda)(h) < 0$ for all $\lambda \in \mathcal{E}_{P'}(\pi)$. Now Lemma 1.2 implies a contradiction (again one argues as in the proof of Lemma 2.4).

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Proof. a) $P' = P_1$. If $(\pi, H) = J(P_1, \omega', \nu')$ we can apply Lemma 2.4 to see that if $\lambda \in \mathcal{E}_{P_1}(\pi)$ then $\mathrm{Re}(\lambda+\rho_{P_1}-\nu)|_{(\underline{a}_1)^+} < 0$. Now Lemma 2.5 implies there is $\lambda \in \mathcal{E}_{P_1}(\pi)$ with $\mathrm{Re}\,\lambda = \mathrm{Re}(\rho_{P_1}-\nu')$. Hence $\mathrm{Re}(\nu'-\nu)|_{(\underline{a}_1)^+} < 0$. This proves the result in the case $P' = P_1$.

b) If we can show that if $P' \neq P$ then $P' \supset P$ then we can prove the result.

Indeed, if $P' \supset P$ and $(\pi, H) = J(P', \omega', \nu')$ then $\mathcal{E}_{P'}(\pi) = \subset \{\lambda|_{\underline{a}'} \mid \lambda \in \mathcal{E}_{P_1}(\pi)\}$. If $\lambda \in \mathcal{E}_{P_1}(\pi)$ then $\mathrm{Re}(\lambda+\rho_{P_1}-\nu)|_{(\underline{a}_1)^+} < 0$ (Lemma 2.4). Hence $\mathrm{Re}(\lambda+\rho_{P_1}-\nu)|_{(\underline{a}')^+} \leq 0$. Since $\rho_{P'} = \rho_{P_1}|_{\underline{a}'}$ the contention of b) follows using Lemma 2.5 as in a).

c) Let $\Lambda$ be the linear form on $\underline{a}$ which is zero on $^*\underline{a}_1$ and is equal to $\mathrm{Re}\,\nu$ on $\underline{a}_1$. Let, as usual, $\varphi_\Lambda$ be the spherical function

$$\varphi_\Lambda(g) = \int_K e^{-(\rho_P+\Lambda)H(g^{-1}.k)}\,dk \ .$$

Then, given K-finite elements $v, w \in I^\omega$, there exists a constant $c > 0$ such that

$$|\langle \pi(g)v, w\rangle| \leqq c.\varphi_\Lambda(g) \text{ for all } g \in G \ .$$

This is contained in Lemma 3.8 of [1].

d) If $h \in \underline{a}^+$ then $\varphi_\Lambda(\exp h) \leq e^{\Lambda(h)}\varphi_o(\exp h)$.

This is (essentially) Lemma 3.6 of [1].

e) If $h \in \underline{a}^+$ and $\varepsilon > 0$ is given then

$$\lim_{t \to +\infty} e^{t(\rho_P - (\varepsilon\rho_P + \Lambda))(h)} \varphi_\Lambda(\exp th) = 0 .$$

This follows from the (standard) fact that if $h \in \underline{a}^+$ then $\varphi_o(\exp h) \leq Ce^{\rho_P(h)}(1 + \|h\|)^d$ for some $d$.

f) Let $(\pi, H) = J(P', \omega', \nu')$. If $\lambda \in \mathcal{E}^o(\pi)$ then $\mathrm{Re}(\Lambda - \rho_P - \lambda)|_{(\underline{a})^+} \geq 0$. Indeed, our usual argument using Lemma 1.2 implies that $\mathrm{Re}(\Lambda + \varepsilon\rho_P - \rho_P - \lambda)|_{(\underline{a})^+} > 0$ for all $\varepsilon > 0$. Take the limit as $\varepsilon \longrightarrow 0$.

We now complete the proof of the fact that $P' \supset P$. We use the notation of Langlands [1], pp. 86-91. In particular, his $E(\pi)$ is our $\mathcal{E}^o(\pi)$. Let $F = F(\pi)$ as in Langlands' construction of the standard p-pair corresponding to $(\pi, H)$ (see p. 89 [1]). Then f) implies that if $(P_1, A_1) = (P, A)_{F_1}$ then $F \subset F_1$. But then $(P_1, A_1) \prec (P, A)_F = (P', A')$ (see Lemma 3.14 of [1]). Thus $P' \supset P_1$. Q.E.D.

References

[1] R. P. Langlands, "On the classification of irreducible representations of real algebraic groups," Mimeographed article, Institute for Advanced Study, 1973.

[2] G. Warner, Harmonic Analysis on Semisimple Lie Groups II, Springer-Verlag, New York, 1973.

# ERRATUM TO APPENDIX TO CHAPTER VII

N. Wallach

In §3, Theorem 3.2 should say

3.2. THEOREM. *Let* $(P_1, A_1)$ *be a standard* p-*pair. Let* $(\omega, H_\omega)$ *be an irreducible tempered representation of* $M_1$. *Let* $\nu \in (\underline{a}_1)^*_{\mathbb{C}}$ *be such that* $\mathrm{Re}\langle \nu, \lambda \rangle > 0$ *for* $\lambda \in \Sigma(P_1, A_1)$. *Let* $Z^{\omega,\nu}$ *be as in Theorem* 2.3. *If* $(P', A')$ *is a standard* p-*pair, if* $\omega'$ *is an irreducible tempered representation of* $M'$, *if* $\nu' \in (\underline{a}')^*_{\mathbb{C}}$ *with* $\mathrm{Re}\langle \nu', \lambda \rangle > 0$ *for* $\lambda \in \Sigma(P', A')$ *and if* $J(P', \omega', \nu')$ *is a subquotient of* $Z^{\omega,\nu}$ *then*

a) *If* $P' = P_1$ *then* $\mathrm{Re}(\nu - \nu')|_{(\underline{a}_1)^+} > 0$.

b) *If* $P' \neq P_1$ *and if* $\Lambda$ (*resp.* $\Lambda'$) *is the extension of* $\mathrm{Re}\,\nu$ (*resp.* $\mathrm{Re}\,\nu'$) *to* $\underline{a}$ *such that* $\Lambda|_{{}^*\underline{a}_1} = 0$ (*resp.* $\Lambda'|_{{}^*\underline{a}'} = 0$), *then* $(\Lambda - \Lambda')|_{\underline{a}^+} > 0$.

Proof. The proof of a) is identical with part a) of the proof of Theorem 3.2 in the Appendix to Chapter VII.

To prove b) we note that everything in the original proof is correct up to and including statement (f). Now (f) combined with Corollary 4.6 in Langlands [1] implies that if $\alpha_1, \ldots, \alpha_\ell$ are the simple roots of $(P, A)$, then $\Lambda - \Lambda' = \sum x_i \alpha_i$ with $x_i \geq 0$. To complete the proof we need only show that $\Lambda \neq \Lambda'$. But if $\Lambda = \Lambda'$ then $P_1 = P'$ by the assumptions on $\nu$ and $\nu'$.

3.3. COROLLARY. *Let the notation and assumptions on* $(P_1,A_1)$, $(P',A')$ *and* $\omega$, $\nu$, $\omega'$, $\nu'$ *be as in Theorem* 3.2. *Suppose that* $\mathrm{Re}(\rho_{P_1}-\nu)(h) > 0$ *for all* $h \in \underline{a}_1$ *so that* $h \neq 0$ *and* $\lambda(h) \geq 0$ *for all* $\lambda \in \Sigma(P_1,A_1)$. *Then* $\mathrm{Re}(\rho_{P'}-\nu')(h') > 0$ *for all* $h' \in \underline{a}'$ *so that* $h' \neq 0$ *and* $\lambda'(h') \geq 0$ *for all* $\lambda' \in \Sigma(P',A')$.

Proof. Clearly we can assume that $P_1 \neq P'$. Let $\Lambda,\Lambda'$ be as in 3.2(b). Let $\alpha_1,\ldots,\alpha_\ell$ be the simple roots of $(P,A)$. Let $\beta_i \in \underline{a}^*$ be defined by $\langle \beta_i,\alpha_j\rangle = \delta_{ij}$. We assert that $\langle \rho_P-\Lambda,\beta_i\rangle > 0$ for $i = 1,\ldots,\ell$. Indeed, let $F \subset \{1,\ldots,\ell\}$ be such that $\underline{a}_1 = \{h \in \underline{a}\,|\,\alpha_i(h) = 0,\ i \notin F\}$. Now $\rho_P = {}^*\rho + \rho_{P_1}$ with ${}^*\rho = \rho_{P\cap M_1} = \sum_{i\notin F} x_i\alpha_i$, $x_i > 0$. Let $\alpha_i^F$ be the projection of $\alpha_i$ on $\underline{a}_1$ for $i \in F$. Then it is easy to show that $\langle \alpha_i^F,\beta_j\rangle = \delta ij$ for $j,i \in F$, $\langle \alpha_i^F,\beta_j\rangle \geq 0$ for $i \in F$, $j \notin F$. But our hypothesis implies that $\rho_{P_1} - \mathrm{Re}\,\nu = \sum_{i\in F} z_i\alpha_i^F$ with $z_i > 0$. Hence $\rho_P - \Lambda = {}^*\rho + \rho_{P_1} - \mathrm{Re}\,\nu = \sum_{i\notin F} x_i\alpha_i + \sum_{i\in F} z_i\alpha_i^F$. Hence $\langle \rho_P-\Lambda,\beta_i\rangle = z_i > 0$ if $i \in F$ and $\langle \rho_P-\Lambda,\beta_i\rangle \geqq x_i > 0$ for $i \notin F$. This proves our assertion.

Now $\rho_P - \Lambda' = \rho_P - \Lambda + \Lambda - \Lambda'$ and $\langle \Lambda-\Lambda',\beta_i\rangle \geq 0$ for $1 \leq i \leq \ell$ by 3.2(b). Thus

$$\langle \rho_P-\Lambda',\beta_i\rangle = \langle \rho_P-\Lambda,\beta_i\rangle + \langle \Lambda-\Lambda',\beta_i\rangle$$

$$\geq \langle \rho_P-\Lambda,\beta_i\rangle > 0, \quad i = 1,\ldots,\ell.$$

Let now $F'$ be such that $\underline{a}' = \{h \in \underline{a}\,|\,\alpha_i(h) = 0,\ i \notin F'\}$. Then if $i \in F'$, $\langle \rho_{P'}-\mathrm{Re}\,\nu',\beta_i\rangle = \langle \rho_P-\Lambda',\beta_i\rangle > 0$. This implies the corollary.

3.4. We are grateful to Brigit Speh for giving us a counterexample in the case of $SL(4,\mathbb{R})$ to the original Theorem 3.2 of the Appendix to Chapter VII.

3.5. It is of interest to note that we can assign to $J(Q,\omega,\nu)$ the parameter $\Lambda = \Lambda_{Q,\omega,\nu}$ as defined in Theorem 3.2. If $\Lambda_1,\Lambda_2 \in \underline{a}^*$ then we say $\Lambda_1 \geq \Lambda_2$ if $(\Lambda_1-\Lambda_2)|_{a^+} \geq 0$. Let $\chi$ be a homomorphism of the center of $U(\underline{g}_{\mathbb{C}})$ to $\mathbb{C}$. Set $S_\chi = \{J(\overline{Q},\omega,\nu) | J(Q,\omega,\nu)$ has infinitesimal character $\chi$ and $Q \neq G\}$. Then $S_\chi$ is finite. If $J(Q,\omega,\nu) \in S_\chi$ and $\Lambda(Q,\omega,\nu)$ is minimal among the $\Lambda(P',\omega',\nu')$ with $J(P',\omega',\nu') \in S_\chi$ then Theorem 3.2 says that if $J(P',\omega',\nu') \in S_\chi$ is a subquotient of $I_{Q,\omega,\nu}$ then $\Lambda(P',\omega',\nu') \leq \Lambda(Q,\omega,\nu)$ hence $\Lambda(P',\omega',\nu') = \Lambda(Q,\omega,\nu)$. This implies that $P = P'$, $\omega = \omega'$, $\nu = \nu'$. But then we see that any subquotient other than $J(Q,\omega,\nu)$ of $Z^{\omega,\nu}$ is tempered.

# VIII. The construction of irreducible unitary representations with cohomology.

Nolan R. Wallach

In this chapter the oscillator representation is used to construct non-trivial unitary representations of $SU(p, q)$ $(p \geq q > 0)$ with cohomology in dimension $q$. This is of interest since $q$ is the split rank of $SU(p, q)$. Using Weil's ideas on the relationship between theta functions and automorphic forms and some recent ideas of Kazhdan we give a generalization of Kazhdan's theorem on the first Betti number for $SU(n, 1)$. Also, in this chapter, we compute the relative Lie algebra cohomology of a tensor product of a discrete series representation and a finite dimensional representation (see §5).

## 1. The metaplectic group.

1.1. We look at $\mathbb{R}^{2n}$ as the space of all columns $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x, y \in \mathbb{R}^n$. We define a symplectic form on $\mathbb{R}^{2n}$ by the formula $\beta\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix}\right) = \langle x, y' \rangle - \langle y, x' \rangle$ where $\langle x, y \rangle = \sum x_i y_i$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

1.2. The Heisenberg group of demension $2n + 1$ is the group with underlying space $\mathbb{R}^{2n} \times \mathbb{R}$ and multiplication given by $(z, t) \cdot (w, s) = (z + w, t + s + \frac{1}{2}\beta(z, w))$. We denote by $H_n$ this Lie group.

1.3. It is easy to see that the 1-parameter subgroups of $H_n$ are of the form $s \mapsto (sz, st)$. Thus the Lie algebra of $H_n$, $\mathfrak{h}_n$, is the vector space $\mathbb{R}^{2n} \times \mathbb{R}$ and $[(x, t), (y, s)] = (0, \beta(x, y))$.

1.4. The Stone-Von Neumann theorem says that $H_n$ has (up to dilation and duality) one infinite dimensional, irreducible, unitary representation $(\pi, L^2(\mathbb{R}^n))$ with

$$\left(\pi\left(\begin{bmatrix} x \\ y \end{bmatrix}, t\right)\varphi\right)(z) = \exp(i(t + \langle x, z - \tfrac{1}{2} y\rangle))\varphi(z - y)$$

for $\varphi \in L^2(\mathbb{R}^n)$, $z, x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$.

1.5. Here, of course, if $G$ is a Lie group then a unitary representation $(\pi, H)$ of $G$ on a Hilbert space is assumed to be strongly continuous. That is, we give $U(H)$ the group of unitary operators on $H$ the strong topology and $\pi : G \longrightarrow U(H)$ is continuous. A neighborhood basis for the strong topology in $U(H)$ is the collection of sets $N(x_1, \ldots, x_m; \varepsilon) = \{u \in U(H) \mid \|ux_i - x_i\| < \varepsilon, \ i = 1, \ldots, m\}$, $x_1, \ldots, x_m \in H$ and $\varepsilon > 0$.

1.6. LEMMA. $\pi : H_n \longrightarrow U(L^2(\mathbb{R}^n))$ is a homeomorphism of $H_n$ onto $\pi(H_n)$ (with the subspace topology).

A proof of this lemma can be found in Igusa [9] and in Wallach [14].

1.7. An important ingredient in the proof of 1.6 is the Fourier transform. Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of $\mathbb{R}^n$. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ set $(\mathcal{F}\varphi)(z) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \varphi(x)e^{-i\langle x,z\rangle}dx$. Then $\mathcal{F}$ extends to a unitary operator of $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$. A simple computation gives

$$\text{1)} \qquad \mathcal{F}\pi\left(\begin{bmatrix} x \\ y \end{bmatrix}, t\right)\mathcal{F}^{-1} = \pi\left(\begin{bmatrix} -y \\ x \end{bmatrix}, t\right)$$

1.8. Let $G = \{g \in U(L^2(\mathbb{R}^n)) \mid g\, \pi(z, t) g^{-1} = \pi(z', t)$ for each $z \in \mathbb{R}^{2n}$, $t \in \mathbb{R}\}$. In this definition $z'$ clearly depends on $z$. 1.6 implies that $z' = \nu(g)(z)$ and $\nu(g) : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ is continuous.

1.9. Let $Sp(n, \mathbb{R}) = \{g \in GL(2n, \mathbb{R}) \mid \beta(gz, gw) = \beta(z, w)$ for all $z, w \in \mathbb{R}^{2n}\}$.

LEMMA. $\nu(g) \in Sp(n, \mathbb{R})$ for $g \in G$.

Proof. $\pi(z, t) \cdot \pi(w, s) = \pi(z + w, t + s + \frac{1}{2}\beta(z, w))$ now use the definition of $G$.

1.10. PROPOSITION. Let $T^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$. Then we have the exact sequence

$$1 \longrightarrow T^1 \cdot I \longrightarrow G \xrightarrow{\nu} Sp(n, \mathbb{R}) \longrightarrow 1$$

We will need some notation to sketch a proof of this result. We first note that if $g \in \operatorname{Ker} \nu$ then $g\pi(h)g^{-1} = \pi(h)$ for $h \in H_n$. Since $(\pi, L^2(\mathbb{R}^n))$ is irreducible it is clear that $g = \lambda I$, $|\lambda| = 1$. We therefore must only show that $\nu(G) = Sp(n, \mathbb{R})$.

1.11. Let $M = \left\{ \left[\begin{array}{c|c} A & 0 \\ \hline 0 & {}^tA^{-1} \end{array}\right] \middle| A \in GL(n, \mathbb{R}) \right\}$. Set $N = \left\{ \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \middle| X \in M_n(\mathbb{R}),\ {}^tX = X \right\}$. Set $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. Then it is well known that $Sp(n, \mathbb{R})$ is generated by the set $N \cup M \cup \{J\}$. 1.7 (1) says that

1) $$\nu(\mathcal{F}) = J .$$

1.12. Define for $A \in GL(n, \mathbb{R})$, $f \in L^2(\mathbb{R}^n)$

1) $$(\alpha(A)f)(z) = |\det A|^{1/2} f({}^tAz) .$$

A simple computation gives

2) $\alpha(A) \in G$ and $\nu(\alpha(A)) = \begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix}$.

<u>1.13</u>. Let $\underline{P}_n(\mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid {}^tX = X\}$.

1) If $X \in \underline{P}_n(\mathbb{R})$ and $f \in L^2(\mathbb{R}^n)$ set $(\mu(X)f)(z) = e^{i\langle xz, z\rangle/2} f(z)$.

A simple computation gives

2) If $x \in \underline{P}_n(n)$ then $\mu(X) \in G$ and $\nu(\mu(X)) = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$.

<u>1.14</u>. Combining 1.11(1), 1.12(2), 1.13(2) we see that $\nu(G) = Sp(n, \mathbb{R})$. This completes the sketch of the proof of Proposition 1.10.

<u>1.15</u>. Using the fact that an extension of a Lie group by a Lie groups is a Lie group we see that $G$ is the underlying topological group of a Lie group which we also denote by $G$. Since $\nu : G \longrightarrow Sp(n, \mathbb{R})$ is a continuous homomorphism it is a Lie homomorphism.

<u>1.16</u>. As is well known $Sp(n, \mathbb{R})$ is a simple Lie group. Let be the Lie algebra of $G$. Let $\nu_*$ denote the differential of $\nu$. Let $\underline{sp}(n, \mathbb{R})$ denote the Lie algebra of $Sp(n, \mathbb{R})$. Then $\nu_* : [\mathfrak{g}, \mathfrak{g}] \longrightarrow \underline{sp}(n, \mathbb{R})$ is an isomorphism.

<u>1.17</u>. We <u>define</u> $Mp(n, \mathbb{R})$ (the metaplectic group) to be the commutator subgroup of $G$. We set $j = \nu|_{Mp(n, \mathbb{R})}$.

LEMMA. $j : Mp(n, \mathbb{R}) \longrightarrow Sp(n, \mathbb{R})$ is a finite covering.

<u>Proof</u>. Since $j_*$ is bijective, $\operatorname{Ker} j$ is discrete. But $\operatorname{Ker} j \subset T^1 I$. Hence $\operatorname{Ker} j$ is finite.

## 2. The oscillator representation.

2.1. We look upon $(Mp(n, \mathbb{R}), j)$ as an abstract covering group of $Sp(n, \mathbb{R})$. The realization of $Mp(n, \mathbb{R}) \subset U(L^2(\mathbb{R}^n))$ is then denoted $(W, L^2(\mathbb{R}^n))$. $(W, L^2(\mathbb{R}^n))$ is called the oscillator representation of $Mp(n, \mathbb{R})$.

2.2. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Let $(\pi, H)$ be a unitary representation of $G$. Then $\varphi \in H$ is said to be a $C^\infty$-vector if the map $G \longrightarrow H$ given by $g \longmapsto \pi(g)\cdot\varphi$ is $C^\infty$. Let $H_\infty$ denote the space of all $C^\infty$ vectors in $H$. As is well known $H_\infty$ is dense in $H$ (cf. Warner [16], Chapter 4). If we define for $X \in \mathfrak{g}$ and $\varphi \in H_\infty$, $\pi(X)\varphi = \frac{d}{dt}\pi(\exp tX)\varphi|_{t=0}$ then $(\pi, H_\infty)$ becomes a representation of $\mathfrak{g}$. We topologize $H_\infty$ using the semi-norms $\mu_p(\varphi) = ||\pi(p)\varphi||$, $p \in U(\mathfrak{g})$, $\varphi \in H_\infty$. Then with this topology $H_\infty$ is a Frechet space (cf. Warner [16], Chapter 4) and $G$ acts continuously on $H_\infty$.

2.3. LEMMA. Let $L^2(\mathbb{R}^n)_\infty$ denote the space of $C^\infty$ vectors for $(W, L^2(\mathbb{R}^n))$. Then $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)_\infty$.

Proof. It is enough to show that if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then map $\underline{P}_n \times GL(n, \mathbb{R}) \times \underline{P}_n \longrightarrow L^2(\mathbb{R}^n)$ given by

$$(X, A, Y) \longmapsto \mathfrak{F}\mu(X)\mathfrak{F}^{-1}\alpha(A)\mu(Y)\varphi$$

is $C^\infty$. We leave this to the reader.

2.4. We identify the Lie algebra of $Mp(n, \mathbb{R})$ with $\underline{sp}(n, \mathbb{R})$. We denote by Exp the exponential map of $\underline{sp}(n, \mathbb{R})$ into $Mp(n, \mathbb{R})$ and exp the exponential map of $\underline{sp}(n, \mathbb{R})$ into $Sp(n, \mathbb{R})$.

2.5. Let $E_{ij}$ denote the $n \times n$ matrix with a 1 in the $i, j$ position at zeros elsewhere. Set

1) $$h_j = i\left[\begin{array}{c|c} 0 & E_{jj} \\ \hline -E_{jj} & 0 \end{array}\right].$$

Then $ih_j \in \underline{sp}(n, \mathbb{R})$. A simple computation gives

2) If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then $\frac{d}{dt} W(\mathrm{Exp}(tih_j))\varphi|_{t=0} = iH_j\varphi$ with $2H_j\varphi = \left(\frac{\partial^2}{\partial x_j^2} - x_j^2\right)\varphi$.

2.6. Set $T = \{\mathrm{Exp}(\Sigma t_j(ih_j)) \mid t_j \in \mathbb{R}\}$, $T_0 = \nu(T)$. Then $T$ and $T_0$ are tori in $Mp(n, \mathbb{R})$ and $Sp(n, \mathbb{R})$ respectively.

2.7. Set $P(\mathbb{R}^n)$ equal to the space of all complex valued polynomials on $\mathbb{R}^n$. Set $P^j(\mathbb{R}^n)$ equal to the space of all polynomials of degree $\leq j$.

2.8. LEMMA. Set $\psi_0(X) = (2\pi)^{-n/2} e^{-\langle x,x\rangle/2}$ for $x \in \mathbb{R}^n$. Then $W(T)\psi_0 P^j(\mathbb{R}^n) \subset \psi_0 P^j(\mathbb{R}^n)$ for all $j$. (We note that $\psi_0$ is a unit vector in $L^2(\mathbb{R}^n)$.)

Proof. We note that if $\varphi \in P(\mathbb{R}^n)$ then

1) $H_j(\psi_0\varphi) = -\frac{1}{2}\psi_0\left(\varphi - 2x_j\frac{\partial\varphi}{\partial x_j} + \frac{\partial^2\varphi}{\partial x_j^2}\right)$. Hence $H_j\psi_0 P^k(\mathbb{R}^n) \subset \psi_0 P^k(\mathbb{R}^n)$

for each $k$. Now if $( \, , \, )$ denotes the $L^2$ inner product on $L^2(\mathbb{R}^n)$ then it is clear that if $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ then $(H_j\varphi, \psi) = (\varphi, H_j\psi)$. We therefore see that each $H_j$ diagonalizes on $\psi_0 P^k(\mathbb{R}^n)$ with real eigenvalues. If $h \in \psi_0 P^k(\mathbb{R}^n)$ and $H_j h = \lambda h$. Then if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have

2) $\frac{d}{dt}\left\langle W(\mathrm{Exp}(ith_j))h,\ \varphi\right\rangle = i\lambda\left\langle W(\mathrm{Exp}(ith_j))h,\ \varphi\right\rangle.$

1) implies that $W(\mathrm{Exp}(ith_j))h \doteq e^{it\lambda}h$. This in turn implies the Lemma.

2.9. We note that 1) implies that

1) $H_j\psi_0 = -\frac{1}{2}\psi_0$ for $j = 1, \ldots, n$.

Using this observation we can complete our discussion of the structure of $Mp(n, \mathbb{R})$.

2.10. PROPOSITION. $(Mp(n, \mathbb{R}), j)$ is a twofold covering of $Sp(n, \mathbb{R})$.

Proof. $j^{-1}(T_0) = T$. Thus $\mathrm{Ker}\ j \subset T$. If $t \in T$ and $j(t) = I$ then $W(t) = \xi(t)I$. If $t \in T$ then $t = \mathrm{Exp}(\sum_j t_j(ih_j))$. If $\nu(t) = I$ then $t_j = 2\pi k_j$, $k_j \in \mathbb{Z}$. Now $W(t)\cdot\psi_0 = \exp(-\frac{i}{2}\sum_{j=1}^{n} 2\pi k_j)\psi_0$ (by 2) in 2.11 and 2.9). This implies that $\xi(t)^2 = 1$. Hence $\xi(t) = \pm 1$. Obviously there is a $t \in \mathrm{Ker}\ j$ so that $\xi(t) = -1$. This proves the proposition.

2.11. Define $A_j^{\pm} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} \pm x_j\right)$. Then

1) $[A_i^+, A_j^-] = -\frac{1}{2}\delta_{ij}I$ .

2) $A_j^+A_j^- + A_j^-A_j^+ = H_j$ .

3) $[H_i, A_j^+] = \delta_{ij}A_j^+$

$[H_i, A_j^-] = -\delta_{ij}A_j^-$ .

4) $A_j^+\psi_0 = 0$ .

5) $[A_j^+, A_i^+] = [A_j^-, A_i^-] = 0$ .

1) through 5) are direct computations we leave to the reader. We also note

6) If we set for $m = (m_1, \dots, m_n)$ $m_i \in Z^+$, $\psi_m = (\underline{m}!)^{-1/2}(A_1^-)^{m_1} \dots (A_n^-)^{m_n}\,\psi_0$ $(m! = m_1! \dots m_n!)$ then $\sum_{|m|\leq k} \mathbb{C}\,\psi_m = \psi_0 P^k(\mathbb{R}^n)$ $(|m| = \Sigma\, m_i)$.

2.12. LEMMA. 1) $\psi_k$, $k \in (Z^+)^n$, is an orthonormal basis of $L^2(\mathbb{R}^n)$.

2) $W(\mathrm{Exp}(i\Sigma t_j h_j))\psi_k = \exp(-i\Sigma \frac{(2k+1)}{2}\, t_j)\psi_k$.

Proof. 1) Compute and use 2.11(6).

2) Use 2.11(3), 2.8(2) and 2.9.

2.13. Using 2.12 1), 2) and the classical theory of Fourier series we see that if $h = \Sigma\, a_{\underline{m}}\psi_{\underline{m}} \in L^2(\mathbb{R}^n)$ is a $C^\infty$ vector for $W|_T$ then we must have for each $r > 0$

1) $$|a_{\underline{m}}| \leq C_r(1 + \langle \underline{m}, \underline{m}\rangle)^{-r} .$$

Condition 1) is precisely the condition that $\Sigma\, a_{\underline{m}}\psi_{\underline{m}} \in \mathcal{S}(\mathbb{R}^n)$ (cf. Reed, Simon [11]). Furthermore if we set $||h||_s^2 = \Sigma(1 + \langle \underline{m}, \underline{m}\rangle)^s |a_{\underline{m}}|^2$ then the norms $||\cdot||_s$ define the Schwartz topology on $\mathcal{S}(\mathbb{R}^n)$. The following result is now clear.

2.14. PROPOSITION. The space of $C^\infty$ vectors for $W|_T$ is $\mathcal{S}(\mathbb{R}^n)$. In particular, $L^2(\mathbb{R}^n)_\infty = \mathcal{S}(\mathbb{R}^n)$ with the Schwartz topology.

## 3. The theta distributions.

3.1. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ define

$$(F\varphi)(z) = \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi i \langle x, z\rangle} \, dx \quad .$$

Then $F : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ and $F$ extends to a bijective unitary operator on $L^2(\mathbb{R}^n)$.

3.2. Set $A = \alpha((2\pi)^{1/2} I)$ (see 1.12). Then $A\mathcal{F}A^{-1} = F$. Define for $g \in Mp(n, \mathbb{R})$, $\mathcal{W}(g) = AW(g)A^{-1}$. Set $\Phi_{\underline{m}} = A\psi_{\underline{m}}$ for $\underline{m} \in (\mathbb{Z}^+)^n$.

3.3. In particular, we note that if $X \in \underline{P}_n(R)$,
$\mathcal{W}\left(\mathrm{Exp}\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\right)\varphi(z) = e^{\pi i \langle Xz, z\rangle} \varphi(z)$.

3.4. If $L \subset \mathbb{R}^n$ is a lattice define $L^* = \{r \in \mathbb{R}^n | \langle r, \tau\rangle \in \mathbb{Z}$ for all $\tau \in L\}$. If $L$ is a lattice then $T_L = \mathbb{R}^n/L$ is a torus. If $\gamma \in L^*$ and $x \in \mathbb{R}^n$ set $e_\gamma(x) = \exp(2\pi i\langle \gamma, x\rangle)$. Then $e_\gamma(x + \tau) = e_\gamma(x)$ for $\tau \in L$. Thus $e_\gamma \in \hat{T}_L$. It is easy to see that $\hat{T}_L = \{e_\gamma | \gamma \in L^*\}$. On $T_L$ we put the invariant measure that satisfies $\int_{\mathbb{R}^n} f(x)dx = \int_{T_L} f_L(t)dt$ where $f_L(x + L) = \sum_{\gamma \in L} f(x + \gamma)$ for, say, $f \in \mathcal{S}(\mathbb{R}^n)$. Let $m(L) = \mathrm{vol}(T_L)$ relative to $dt$.

3.5. THEOREM. (Poisson summation). If $f \in \mathcal{S}(\mathbb{R}^n)$ then

$$\sum_{\gamma \in L^*} (Ff)(\gamma) = m(L) \sum_{\gamma \in L} f(\gamma).$$

Proof. (sketch). If $f \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi \in L^2(T_L)$ define $(\lambda(f)\varphi)(z)$

$$= \int_{\mathbb{R}^n} \varphi(z - x) f(x)dx = \int_{T_L} \varphi(z - t) f_L(t)dt = \int_{T_L} \varphi(t) f_L(z - t)dt.$$

The standard theory of Fourier series (or the Peter-Weyl theorem) implies that $\lambda(f)$ is of trace class and $\operatorname{tr} \lambda_L(f) = m(L) f_L(0) = m(L) \sum_{\gamma \in L} f(\gamma)$. On the other hand if $\gamma \in L^*$, $(\lambda(f)e_\gamma)(z) = e_\gamma(z) \int_{T_L} e_\gamma(t)^{-1} f_L(t)\,dt =$

$e_\gamma(z)(Ff)(\gamma)$ by the normalization of measures. Hence $\operatorname{tr} \lambda(f) = \sum_{\gamma \in L^*} (Ff)(\gamma)$.

<u>3.6</u>. Define for $L \subset \mathbb{R}^n$ a lattice $\delta_L(f) = f_L(0) = \sum_{\gamma \in L} f(\gamma)$ for $f \in \mathcal{S}(\mathbb{R}^n)$. Clearly $\delta_L \in \mathcal{S}'(\mathbb{R}^n)$ (i.e. $\delta_L$ is a tempered distribution).

<u>3.7</u>. If $L$ is a lattice, if $S \subset L$ is a sublattice and if $\chi \in (L/S)^\wedge$ define

$$\delta_{L,S,\chi}(f) = \sum_{\tau \in L/S} \chi(\tau) f_S(\tau)$$

(see the proof of 3.5 for notation).

<u>3.8</u>. LEMMA. Let $L$, $S$ be as in 3.7. Then

1) $S^* \supset L^*$

2) If $\chi \in (L/S)^\wedge$ then $\chi(\gamma + S) = e^{2\pi i \langle \gamma, \mu \rangle}$ with $\mu \in S^*$, $\mu + L^*$ depending only on $\chi$.

3) If $f \in \mathcal{S}(\mathbb{R}^n)$ then $\delta_{L,S,\chi}(f) = \frac{1}{m(L)} (Ff)_{L^*}(-\mu)$ where $\mu$ is as in 2).

<u>Proof</u>. 1) is clear. 2) is a simple counting argument. To prove 3) we note that $\delta_{L,S,\chi}(f) = \sum_{\tau \in L/S} e^{2\pi i \langle \tau, \mu \rangle} f_S(\tau) = \sum_{\tau \in L/S} e^{2\pi i \langle \tau, \mu \rangle} \delta_S(\varphi_{\tau'})$ (here $\tau' \in L$ is so that $\tau' + S = \tau$ and $\varphi_{\tau'}(x) = f(x + \tau')$). Now $\delta_S(\varphi_{\tau'}) = \frac{1}{m(S)} \delta_{S^*}(F\varphi_{\tau'})$. $(F\varphi_{\tau'})(z) = e^{2\pi i \langle \tau', z \rangle}(Ff)(z)$. Hence

$\delta_{S^*}(F\varphi_{\tau'}) = \sum_{\gamma \in S^*} e^{2\pi i\langle\tau',\gamma\rangle} Ff(\gamma) = \sum_{\gamma \in S^*/L^*} e^{2\pi i\langle\tau,\gamma\rangle}(Ff)_{L^*}(\gamma)$. Hence we have

$$\delta_{L,S,\chi}(f) = \sum_{\tau \in L/S} \sum_{\gamma \in S^*/L^*} e^{2\pi i\langle\tau,\mu\rangle} e^{2\pi i\langle\tau,\gamma\rangle} (Ff)_{L^*}(\gamma) .$$

Clearly $\sum_{\tau \in L/S} e^{2\pi i\langle\tau,\mu\rangle} e^{2\pi i\langle\tau,\gamma\rangle} = 0$ unless $\mu = -\gamma$ in $S^*/L^*$. If $\mu = -\gamma$ then $\sum_{\tau \in L/S} e^{2\pi i\langle\tau,\mu\rangle} e^{2\pi i\langle\tau,\gamma\rangle} = [L : S]$. Since $m(S) = [L : S]m(L)$ the lemma follows.

<u>3.9</u>. If $S$ is a sublattice of $\mathbb{Z}^n$ and $L = \mathbb{Z}^n$ then we denote $\delta_{\mathbb{Z}^n,S,\chi}$ by $\delta_{S,\chi}$, $\chi \in (\mathbb{Z}^n/S)^\wedge$. Set $\Gamma_{S,\chi} = \{\gamma \in Mp(n, \mathbb{R}) \mid \delta_{S,\chi} \circ \mathcal{W}(\gamma) = \delta_{S,\chi}$ and $\nu(\gamma) \in Sp(n, \mathbb{Z})\}$.

<u>3.10</u>. THEOREM. (Bass, Milnor, Serre [1]). If $S \subset \mathbb{Z}^n$ is a sublattice and $\chi \in (\mathbb{Z}^n/S)^\wedge$ then $\nu(\Gamma_{S,\chi})$ contains a congruence subgroup of $Sp(n, \mathbb{Z})$.

<u>Proof</u>. Let $m \in \mathbb{Z}$, $m > 0$ be such that $mS^* \subset \mathbb{Z}^n$. Let $\underline{P}_n(\mathbb{Z}) \circ \{X \in \underline{P}_n(\mathbb{R}) \mid X$ has integral matrix entries$\}$. If $X \in \underline{P}_n(\mathbb{Z})$ and $\gamma_1, \gamma_2 \in S^*$ then $\langle 2m^2X\gamma_1, \gamma_2\rangle \in 2\mathbb{Z}$. We compute $\delta_{S,\chi} \circ \left(\mathcal{W}\left(\operatorname{Exp}\begin{bmatrix} 0 & 2m^2X \\ 0 & 0 \end{bmatrix}\right)f\right) =$
$\sum_{\tau \in \mathbb{Z}^n/S} \chi(\tau) \sum_{\gamma \in S} e^{\pi i\langle(2m^2X(\gamma+\tau), \gamma+\tau\rangle} f(\gamma + \tau) = \sum_{\tau \in \mathbb{Z}^n/S} \chi(\tau) \sum_{\gamma \in S} f(\gamma + \tau) = \delta_{S,\chi}(f)$.
Also $\delta_{S,\chi} \circ \mathcal{W}\left(\operatorname{Exp}\begin{bmatrix} 0 & 0 \\ -2m^2X & 0 \end{bmatrix}\right)f = \delta_{S,\chi} \circ F \circ \mathcal{W}\begin{bmatrix} 0 & 2m^2X \\ 0 & 0 \end{bmatrix} F^{-1}f =$
$\sum_{\tau \in \mathbb{Z}^n} e^{\pi i\langle(2m^2X)(\tau+\mu), (\tau+\mu)\rangle} (F^{-1}f)(\tau + \mu) = \sum_{\tau \in \mathbb{Z}^n} (Ff)(\tau - \mu) = \delta_{S,\chi}(f)$
(here $\mu$ is as in Lemma 3.8).

This implies that $\nu(\Gamma_{S,\chi})$ contains the group generated by the elements of the form $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$ with $X \in 2m^2\underline{P}_n(\mathbb{Z})$. It is shown in [1], p.130, that these matrices generate a congruence subgroup of $Sp(n, \mathbb{Z})$. Q.E.D.

<u>3.11</u>. LEMMA. If $f \in \mathcal{S}(\mathbb{R}^n)$ and $\delta_{S,\chi}(f) = 0$ for all $S \subset \mathbb{Z}^n$, $S$ a lattice and $\chi \in \mathbb{Z}^n/S$ then $f(\tau) = 0$ for all $\tau \in \mathbb{Z}^n$.

<u>Proof</u>. Let $\mu \in \mathbb{Q}^n$ ($\mathbb{Q}$ the rationals). Then there is $j$ so that $\mu \in (j\mathbb{Z}^n)^*$. If $\chi_\mu(\tau) = e^{2\pi i\langle\mu,\tau\rangle}$ then $\chi_\mu \in (\mathbb{Z}^n/mj\mathbb{Z}^n)^\wedge$ for all $m = 1, 2, \ldots.$ Set $S_m = mj\mathbb{Z}^n$.

1) If $f \in \mathcal{S}(\mathbb{R}^n)$ then $\delta_{S_m,\chi_\mu}(f) = \sum_{\tau\in\mathbb{Z}^n} e^{2\pi i\langle\mu,\tau\rangle} f(\tau)$.

This is clear since $e^{2\pi i\langle\mu, \tau+\gamma\rangle} = e^{2\pi i\langle\mu,\tau\rangle}$ for $\gamma \in \Gamma_m$.

2) If $\lim_{i\to\infty} \mu_i \longrightarrow \mu_0$, $\mu_0 \in \mathbb{R}^n$ then

$$\lim_{j\to\infty} \sum_{\tau\in\mathbb{Z}^n} e^{2\pi i\langle\mu_j,\tau\rangle} f(\tau) = \sum_{\tau\in\mathbb{Z}^n} e^{2\pi i\langle\mu_0,\tau\rangle} f(\tau).$$

2) follows from the dominated convergence theorem.

1), 2) imply that

3) $\sum_{\tau\in\mathbb{Z}^n} e^{2\pi i\langle\tau,\mu\rangle} f(\tau) = 0$ for all $\mu \in \mathbb{R}^n$.

Using standard estimates $(|f(\tau)| \le C_k(1 + ||\tau||^2)^{-k}$ for $k = 1, 2, \ldots)$ it is easy to see that if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then $\int_{\mathbb{R}^n} \varphi(\mu) \sum_{\tau\in\mathbb{Z}^n} e^{2\pi i\langle\tau,\mu\rangle} f(\tau)d\mu$

$= \sum_{\tau\in\mathbb{Z}^n} (F\varphi)(-\tau)f(\tau)$.

Take $\varphi \in \mathcal{S}(\mathbb{R}^n)$ so that $(F\varphi)(-X) = \overline{f(X)}$. Then we see that $\sum_{\tau\in\mathbb{Z}^n} |f(\tau)|^2 = 0$. This proves the lemma.

<u>3.12</u>. THEOREM. If $\Gamma \subset Mp(n, \mathbb{R})$ is a discrete subgroup so that $\nu(\Gamma) \subset Sp(n, \mathbb{Z})$ set $\mathcal{S}'(\mathbb{R}^n)^\Gamma = \{\lambda \in \mathcal{S}'(\mathbb{R}^n) | \lambda \circ \mathcal{H}(\gamma) = \lambda,\ \gamma \in \Gamma\}$. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is such

that $\lambda(\varphi) = 0$ for all $\lambda \in \mathcal{S}'(\mathbb{R}^n)^\Gamma$ and all $\Gamma \subset Mp(n, \mathbb{R})$ so that $\nu(\Gamma)$ contains a congruence subgroup of $Sp(n, \mathbb{Z})$ then $\varphi(\tau) = 0$ for all $\tau \in \mathbb{Z}^n$.

Proof. This follows from 3.10 and 3.11.

3.12. The discussion in this section is strongly influenced by the many conversations the author has had with Roger Howe about the Oscillator representation. In particular, the term theta distribution is due to Roger Howe.

## 4. The decomposition of the oscillator representation under certain subgroups.

4.1. Set for $p + q = n$

$$I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$$

where $I_p$ (resp. $I_q$) is the $p \times p$ (resp. $q \times q$) identity matrix. If $g \in M_n(\mathbb{C})$ we denote by $g^*$ the conjugate transpose of $g$.

4.2. We look upon $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ by writing $z \in \mathbb{C}^n$ as $x + iy$, $x, y \in \mathbb{R}^n$ and $z$ corresponds to $\begin{bmatrix} x \\ y \end{bmatrix}$. If $g \in M_n(\mathbb{C})$ then we look upon $g \in M_{2n}(\mathbb{R})$ by forgetting complex linearity.

4.3. Let $U(p, q)$ $(p + q = n)$ be the group of all $g \in M_n(\mathbb{C})$ so that $gI_{p,q}g^* = I_{p,q}$. Set $Z_{p,q} = \begin{bmatrix} I_{p,q} & 0 \\ 0 & I_n \end{bmatrix}$. Define for $g \in U(p, q)$, $\psi(g) = Z_{p,q}\, gZ_{p,q}$ as an element of $GL(2n, \mathbb{R})$.

4.4. It is easily checked that $\psi : U(p, q) \longrightarrow Sp(n, \mathbb{R})$ where $\beta\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix}\right) = \langle x, y'\rangle - \langle x', y\rangle$ as usual (see §1). We will also use $\psi$ to denote the Lie algebra homomorphism of $\underline{u}(p, q)$ into $\underline{sp}(n, \mathbb{R})$.

4.5. Let $h_j \in \underline{sp}(n, \mathbb{C})$ be defined as in 2.5. Set $\mathfrak{h} = \Sigma\, \mathbb{C}h_j$. Then $\mathfrak{h} \cap \underline{sp}(n, \mathbb{R})$ is a Cartan subalgebra. Define $\varepsilon_i(\Sigma\, z_j h_j) = z_j$. The roots, $\Delta$, of $\mathfrak{h}$ on $\underline{sp}(n, \mathbb{C})$ consist of the functionals $\varepsilon_i - \varepsilon_j$, $i \neq j$, $\pm(\varepsilon_i + \varepsilon_j)$, $1 \leq i, j \leq n$. We take $\Delta^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}$ $\cup \{\varepsilon_i + \varepsilon_j \mid 1 \leq i \leq j \leq n\}$.

4.6. $\underline{u}(p, q)$ denote the Lie algebra of $U(p, q)$. Let $\underline{t}$ be the subalgebra of diagonal elements of $\underline{u}(p, q)$. Set $z_i = E_{ii}$ (see 2.5). Then $\psi(z_i) = h_i$, $j \leq p$ $\psi(z_{i+p}) = -h_{p+i}$, $1 \leq i \leq q$.

4.7. Define $\eta_i(z_j) = \delta_{ij}$, $\eta_i \in \underline{t}_{\mathbb{C}}^*$. Then $\psi^*\varepsilon_i = \eta_i$, $1 \leq i \leq p$, $\psi^*\varepsilon_{i+p} = -\eta_{i+p}$, $1 \leq i \leq q$. We note that $\psi : \underline{t}_{\mathbb{C}} \longrightarrow \mathfrak{h}_{\mathbb{C}}$ is a linear isomorphism.

4.8. Let $\Phi$ be the root system of $\underline{u}_{\mathbb{C}} = \underline{u}(p, q) \otimes_{\mathbb{R}} \mathbb{C}$ relative to $\underline{t}_{\mathbb{C}}$. Set $\Phi^+ = \psi^*\Delta^+ \cap \Phi$. Then

$$\Phi^+ = \{\eta_i - \eta_j \mid 1 \leq i < j \leq p \text{ or } 1 \leq i \leq p < j \leq n\} \cup \{\eta_j - \eta_i \mid p < i < j \leq n\} \quad .$$

4.9. Let $\underline{g} = \underline{sp}(n, \mathbb{R})$, $\underline{g}_{\mathbb{C}} = \underline{sp}(n, \mathbb{C})$. Let $(W, L^2(\mathbb{R}^n))$ as usual be the oscillator representation of $Mp(n, \mathbb{R})$. Let $(W, \mathcal{S}(\mathbb{R}^n))$ be the representation of $\underline{g}$ on $L^2(\mathbb{R}^n)_\infty = \mathcal{S}(\mathbb{R}^n)$.

4.10. LEMMA. $W((\underline{g}_{\mathbb{C}})_{\varepsilon_i+\varepsilon_j}) = \mathbb{C}A_i^+ A_j^+$, $1 \leq i \leq j \leq n$

$$W((\underline{g}_{\mathbb{C}})_{\varepsilon_i-\varepsilon_j}) = \mathbb{C}A_i^+ A_j^-, \quad 1 \leq i < j \leq n.$$

Proof. Using the results of §2 it is not hard to see that $W(\underline{g}_{\mathbb{C}})$ is spanned by the following elements: multiplication by $ix_r x_s$, $r \leq s$, $i \dfrac{\partial}{\partial x_r \partial x_s}$, $r \leq s$, $x_r \dfrac{\partial}{\partial x_s} + \dfrac{1}{2}\delta_{rs}$, $1 \leq r, s \leq n$. Now, checking which of these elements has the right weight relative to $\mathfrak{h}$ gives the result.

4.11. By going to a two fold covering $\widetilde{U}(p, q)$ of $U(p, q)$ we can lift $\psi : U(p, q) \longrightarrow Sp(n, \mathbb{R})$ to $\widetilde{\psi} : \widetilde{U}(p, q) \longrightarrow Mp(n, \mathbb{R})$. Let $(V, L^2(\mathbb{R}^n))$ denote the representation of $\widetilde{U}(p, q)$ given by $(W \circ \widetilde{\psi}, L^2(\mathbb{R}^n))$.

We will also denote by $V$ the action of the differential of $V$ on $\mathcal{S}(\mathbb{R}^n)$. We note that $\widetilde{\psi}(\widetilde{U}(p, q))$ contains $T$. This implies that the space $C^\infty$ vectors of $(V, L^2(\mathbb{R}^n))$ is precisely $\mathcal{S}(\mathbb{R}^n)$ (see Proposition 2.14).

4.12. Set $\underline{u}^+_{\mathbb{C}} = \sum_{\alpha\in\Phi^+} (\underline{u}_{\mathbb{C}})_\alpha$.

4.13. LEMMA. 1) $(V, L^2(\mathbb{R}^n))$ splits into a countable direct sum of inequivalent, irreducible, invariant subspaces.

2) If $H \subset L^2(\mathbb{R}^n)$ is a closed invariant subspace under $V$ then $H \cap \psi_0 \cdot P(\mathbb{R}^n)$ is dense in $H$ and if $H \neq (0)$ then $H_\infty^{\underline{u}^+} = \{f \in H_\infty | V(\underline{u}^+_{\mathbb{C}})f = 0\} \neq (0)$.

Proof. 1) is already true for $W(T) \subset V(\widetilde{U}(p, q))$.

2) Since $W(T) \subset V(\widetilde{U}(p, q))$ we see that $H \cap \psi_0 P(\mathbb{R}^n)$ is dense in $H$. Assuming $H \neq (0)$, $H \cap \psi_0 P(\mathbb{R}^n) \neq (0)$. Lemma 4.10 easily implies that

$$\text{I)}\quad V(\underline{u}^+) = \sum_{1\le i<j\le p} \mathbb{C}\, A_i^+ A_j^- + \sum_{1\le i\le p<j\le n} \mathbb{C}\, A_i^+ A_j^+ + \sum_{p<i<j\le n} \mathbb{C}\, A_i^+ A_j^- .$$

Order $(\mathbb{Z}^+)^n$ as follows: $m > m'$ if $|m| > |m'|$ or if $|m| = |m'|$ then $m_i = m_i'$ for $i > j$ and $m_j > m_j'$. Using I) it is easily seen that

$$\text{II)}\quad V(\underline{u}^+)\psi_m \subset \sum_{m>m'} \mathbb{C}\, \psi_{m'} .$$

Now $H \cap \psi_0 \cdot P(\mathbb{R}^n) = \sum_{m\in S(H)} \mathbb{C}\, \psi_m$. Let $m \in S(H)$ be a minimal element of $S(H)$. Then II) implies that $V(\underline{u}^+)\psi_m = 0$. This proves the lemma.

4.14. LEMMA. Set $(\psi_0 P(\mathbb{R}^n))^{\underline{u}^+} = \{f \in \psi_0 P(\mathbb{R}^n) | V(\underline{u}^+)f = 0\}$. Then

$$(\psi_0 P(\mathbb{R}^n))^{\underline{u}^+} = \sum_{k\geq 0} \mathbb{C}\, \psi_{0,\ldots,0,\underset{p\text{th position}}{k},0,\ldots,0} + \sum_{k>0} \mathbb{C}\, \psi_{0,\ldots,0,k}.$$

Proof. We leave it to the reader to check that $\psi_0 f \in (\psi_0 P(\mathbb{R}^n))^{\underline{u}^+}$, $f \in P(\mathbb{R}^n)$, if and only if

$$1)\quad \frac{\partial^2 f}{\partial x_i \partial x_j} = 2x_j \frac{\partial f}{\partial x_i} \qquad 1 \leq i < j \leq p$$

$$2)\quad \frac{\partial^2 f}{\partial x_i \partial x_j} = 2x_j \frac{\partial}{\partial x_i} \qquad p < i < j \leq n$$

$$3)\quad \frac{\partial^2 f}{\partial x_i \partial x_j} = 0 \qquad 1 \leq i \leq p < j \leq n\ .$$

Write $f = \sum_{\ell\leq k} f_\ell(x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_n)x_p^\ell$. Then $f$ satisfies 1) for $1 \leq i \leq p$ implies that

$$\text{III)}\quad \frac{\partial^2 f}{\partial x_i \partial x_p} = \sum_{\ell\leq k} \ell \frac{\partial f_\ell}{\partial x_i} x_p^{\ell-1} = 2x_p \sum_{\ell\leq k} \frac{\partial f_\ell}{\partial x_i} x_p^\ell.$$

Comparing coefficients of $x_p$ in III) we see that $\frac{\partial f_k}{\partial x_i} = 0$, $i \leq p - 1$. Hence $f_k$ is independent of $x_1, \ldots, x_{p-1}$. Arguing by downward induction using III) we see that $f_\ell$ is independent of $x_1, \ldots, x_{p-1}$ for $\ell = 0,\ldots,k$.

Arguing the same way expanding in terms of $x_n$ and using 2) we see that $f$ is a polynomial in $x_p$ and $x_n$. 3) implies that $\frac{\partial^2 f}{\partial x_p \partial x_n} = 0$. Write $f(x_p, x_n) = \sum_{j=0}^{r} f_j(x_p)x_n^j$. Then $0 = \sum j \frac{\partial f_j(x_p)}{\partial x_p} x_n^{j-1}$. This implies that $\frac{\partial f_j(x_p)}{\partial x_p} = 0$ for $j > 0$. Hence $f(x_p, x_n) = h_1(x_p) + h_2(x_n)$. Clearly, if $h_1, h_2 \in \mathbb{C}[x]$ then $h_1(x_p) + h_2(x_n)$ satisfies 1), 2), 3). The lemma now follows.

4.15. Set $J_{p,q} = \psi(-iI) = -Z_{p,q}JZ_{p,q} = \sum_{j=1}^{p} ih_j - \sum_{j=p+1} ih_j$.

Then

$$1) \qquad W(\mathrm{Exp}\ tJ_{p,q})\psi_m = \exp\left(-i\left(\frac{p-q}{2} + \sum_{i=1}^{p} m_i - \sum_{i=1}^{q} m_{i+p}\right)t\right)\psi_m .$$

4.16. Set $\xi_k(\mathrm{Exp}(tJ_{p,q})) = e^{-i\left(\frac{p-q}{2} + k\right)t}$, $k \in \mathbb{Z}$. Set $L^2_k(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) \mid W(\mathrm{Exp}(tJ_{p,q}))f = \xi_k(\mathrm{Exp}\ tJ_{p,q})f\}$. Then since $W(\mathrm{Exp}(tJ_{p,q})) \circ V(g) = V(g) \circ W(\mathrm{Exp}(tJ_{p,q}))$ for $g \in \widetilde{U}(p, q)$ we see that $V(g)L^2_k(\mathbb{R}^n) \subset L^2_k(\mathbb{R}^n)$ for $k \in \mathbb{Z}$, $g \in \widetilde{U}(p, q)$. Set $V_k(g) = V(g)|_{L^2_k(\mathbb{R}^n)}$.

4.17. LEMMA. 1) $L^2(\mathbb{R}^n) = \bigoplus_{k=-\infty}^{\infty} L^2_k(\mathbb{R}^n)$ (orthogonal direct sum).

2) $(V_k, L^2_k(\mathbb{R}^n))$ is an irreducible representation of $\widetilde{U}(p, q)$.

Proof. 1) is clear. To prove 2) we note that $W(\mathrm{Exp}\ tJ_{p,q})\psi_{0,\ldots,0,\underset{\text{pth position}}{k},0,\ldots,0} = \xi_k(\mathrm{Exp}\ tJ_{p,q})\psi_{0,\ldots,0,k,0,\ldots,0}$ and $W(\mathrm{Exp}\ tJ_{p,q})\psi_{0,\ldots,0,k} = \xi_{-k}(\mathrm{Exp}\ tJ_{p,q})\psi_{0,\ldots,0,k}$. This implies that $\dim(L^2_k(\mathbb{R}^n) \cap \psi_0 P(\mathbb{R}^n))^{\underline{u}^+} = 1$ for $k \in \mathbb{Z}$. Lemma 4.13 now implies 2).

4.18. Before we go on we have one piece of unfinished business. We state it as a lemma.

LEMMA. $\psi : SU(p, q) \longrightarrow Sp(n, \mathbb{R})$ lifts to an injective homomorphism

$\widetilde{\psi} : SU(p, q) \longrightarrow Mp(n, \mathbb{R})$. (That is, the connected subgroup of $\widetilde{U}(p, q)$ corresponding to $\underline{su}(p, q)$ is $SU(p, q)$.)

<u>Proof</u>. Let $B$ denote the group of diagonal elements of $SU(p, q)$. Then $B$ has Lie algebra ${}^{0}\underline{t} = \{z \in \underline{t} \mid \operatorname{tr} z = 0\}$. Now

$$v(\exp(\Sigma\, i\theta_j z_j))\psi_m = \exp\left(-i\left(\sum_{j=1}^{p} \frac{(2m_i+1)}{2}\,\theta_j - \sum_{j=1}^{q} \frac{(2m_{i+p}+1)}{2}\theta_{j+p}\right)\psi_m\right).$$ Now if

$$\sum_{j=1}^{n} \theta_j = 0 \quad \text{then} \quad \sum_{j=1}^{p}\left(\frac{2m_i+1}{2}\right)\theta_j - \sum_{j=1}^{q}\left(\frac{2m_{j+p}+1}{2}\right)\theta_{j+p} = \sum_{j=1}^{p} m_j\theta_j - \sum_{j=1}^{q} m_{j+p}\theta_{j+p} + \sum_{j=1}^{p} \theta_j.$$

This implies that the weights of $W\circ\widetilde{\psi}$ on ${}^{0}\underline{t}$ are $SU(p, q)$ integral. Since $\exp({}^{0}\underline{t})$ contains the center of $SU(p, q)$ the lemma follows.

<u>4.19</u>. The center $\widetilde{U}(p, q)$ acts on $(V_k, L^2_k(\mathbb{R}^n))$ by scalars. Hence the restriction of $V_k$ to $SU(p, q)$ is still irreducible. We will look upon $(V_k, L^2_k(\mathbb{R}^n))$ as a representation of $SU(p, q)$.

# 5. The computation of some relative Lie algebra cohomology.

5.1. In this section we wish to give cohomological interpretations of the representations $(V_k, L^2_k(\mathbb{R}^n))$ of $SU(p, q)$ given in Section 4. The technique of the proofs naturally leads to the computation of the cohomology of discrete series representations tensored with finite dimensional representations. Although the results about the discrete series are a digression from the material of §1-4, the applications of these results in later chapters are of sufficient importance to justify the digression.

5.2. Let $G$ be a connected, linear, semisimple Lie group. Let $K$ be a maximal compact subgroup of $G$. Let $T \subset K$ be a maximal torus of $G$. We assume that $G$ is contained in its simply connected complexification.

5.3. Let $(\pi, H)$ be an irreducible unitary representation of $G$. Then $\pi$ is said to be square integrable if every matrix entry of $H$ is in $L^2(G)$. Let $\mathcal{E}_2(G)$ denote the set of equivalence classes of irreducible, square integrable unitary representations of $G$.

5.4. THEOREM (Harish-Chandra [6]). $\mathcal{E}_2(G) \neq \emptyset$ if and only if $T$ is a Cartan subgroup of $G$.

5.5. In light of 5.4 we assume that $T$ is a Cartan subgroup of $G$. Let $\underline{g}$ be the Lie algebra of $G$. Let $\underline{k}$ be the Lie algebra of $K$ and let $\underline{t}$ be the Lie algebra of $T$. Let $\Delta$ be the root system of $(\underline{g}_{\mathbb{C}}, \underline{t}_{\mathbb{C}})$. Let $\Delta_k$ be the root system of $(\underline{k}_{\mathbb{C}}, \underline{t}_{\mathbb{C}})$. Let $W(\Delta)$ and $W(\Delta_k)$

denote respectively the Weyl groups of $\Delta$ and $\Delta_k$.

5.6. Let $\langle , \rangle$ denote the dual of the Killing form of $\underline{g}_{\mathbb{C}}$ restricted to $\underline{t}_{\mathbb{C}}$. If $\Lambda \in \underline{t}_{\mathbb{C}}^*$ then there is a character $t \longrightarrow t^\Lambda$ of $T$ defined by $(\exp h)^\Lambda = \exp \Lambda(h)$ for $h \in \underline{t}$ if and only if $2\langle \Lambda, \alpha\rangle/\langle \alpha, \alpha\rangle \in \mathbb{Z}$ for $\alpha \in \Delta$. We will call such $\Lambda \in \underline{t}_{\mathbb{C}}^*$ integral.

5.7. If $\Lambda \in \underline{t}_{\mathbb{C}}^*$ we denote by $\chi_\Lambda$ the homomorphism $\chi_\Lambda : \mathfrak{Z} \longrightarrow \mathbb{C}$ ($\mathfrak{Z}$ the center of $U(\underline{g}_{\mathbb{C}})$) defined by Harish-Chandra. (cf. VI, Theorem 1.7).

5.8. THEOREM (Harish-Chandra [6]). If $\Lambda \in \underline{t}_{\mathbb{C}}^*$ is integral and regular ($\langle \Lambda, \alpha\rangle \neq 0$, $\alpha \in \Delta$) then there exists $\omega_\Lambda \in \mathcal{E}_2(G)$ with infinitesimal character $\chi_\Lambda$. Furthermore, $\omega_\Lambda = \omega_{\Lambda'}$ if and only if there is $s \in W_k$ such that $s\Lambda = \Lambda'$. Finally, $\mathcal{E}_2(G) = \{\omega_\Lambda | \Lambda \in \underline{t}_{\mathbb{C}}^*, \ \Lambda$ integral and regular$\}$.

5.9. As is well known, this theorem of Harish-Chandra is one of the most profound theorems in the theory of semisimple Lie groups. Of course, Harish-Chandra proves much more than Theorem 5.8 in [6]. He in fact, gives a "recipe" for computing the characters of the $\omega_\Lambda$. Using Harish-Chandra's theory of characters and ideas of Zuckerman [18] the next Theorem is not too hard to prove. First we need some notation.

5.10. Let $\Lambda \in \underline{t}_{\mathbb{C}}^*$ be regular and integral. Let $P = \{\alpha \in \Delta | \langle \Lambda, \alpha\rangle > 0\}$. Set $P_k = P \cap \Delta_k$. If $\lambda \in \underline{t}_{\mathbb{C}}^*$ is integral and $P_k$-dominant integral let $(\tau_\lambda, V_\lambda)$ denote the irreducible unitary representation of $K$ with highest weight $\lambda$ relative to $P_k$. Let $\delta = \frac{1}{2} \sum_{\alpha \in P} \alpha$, $\delta_k = \frac{1}{2} \sum_{\alpha \in P_k} \alpha$.

5.11. THEOREM (Schmid [13], Wallach [15]). Let $\Lambda$, $P$, $P_k$, $\tau_\lambda$ be as in 5.10. Fix $(\pi, H) \in \omega_\Lambda$.

1) $\dim \mathrm{Hom}_K(V_{\Lambda+\delta-2\delta_k}, H) = 1$.

2) If $\mathrm{Hom}_K(V_\lambda, H) \neq 0$ then $\lambda = \Lambda + \delta - 2\delta_k + Q$ where $Q$ is a sum of elements of $P$.

5.12. Theorem 5.11 is a qualitative form of Blattner's conjucture. The proof that Blattner's formula for $\mathrm{Hom}_K(V_\lambda, H)$ is true is in Schmid [12] (Hermitian symmetric case) and Hecht, Schmid [7] (general case). We will only need the simpler result in 5.11.

5.13. THEOREM. Let $\Lambda$, $P$, $P_k$ be as in 5.10. Let $(\pi, H) \in \omega_\Lambda$. Let $V$ be the $(\underline{g}, K)$-module of $K$-finite vectors in $H$. Let $(\sigma, F)$ be an irreducible finite dimensional representation of $G$.

1) If the highest weight of $(\sigma, F)$ relative to $P$ is not $\Lambda - \delta$ then $H^i(\underline{g}, K; V \otimes F^*) = 0$ for all $i$.

2) If the highest weight of $(\sigma, F)$ is $\Lambda - \delta$ then $\dim H^i(\underline{g}, K; V \otimes F^*) = \delta_{i,q}$ where $q = \frac{1}{2} \dim G/K$.

Proof. 1) is a direct consequence of Corollary 4.2 in chapter I.

2) The results of chapter II, imply that $\dim H^i(\underline{g}, K; V \otimes F^*) = \dim \mathrm{Hom}_K(\Lambda^i \underline{p}_{\mathbb{C}} \otimes F, V)$ here $\underline{p}$ is the orthogonal complement to $\underline{k}$ in $\underline{g}$ relative to the killing form. Thus we must compute $\mathrm{Hom}_K(\Lambda^i \underline{p}_{\mathbb{C}} \otimes F, V)$.

Let $\Delta_n = \Delta - \Delta_k$. Then the weights of $T$ on $\Lambda^i \underline{p}_{\mathbb{C}}$ are of the form $\alpha_1 + \ldots + \alpha_i$ with $\alpha_1, \ldots, \alpha_i$ distinct elements of $\Delta_n$. Set $\delta_n = \delta - \delta_k$. Let $P_n = \Delta_n \cap P$. Then the weights of $\Lambda^i \underline{p}_{\mathbb{C}}$ are of the form $\alpha_1 + \ldots + \alpha_j - (\alpha_{j+1} + \ldots + \alpha_i)$, $\alpha_1, \ldots, \alpha_j$ distinct elements of $P_n$ and $\alpha_{j+1}, \ldots, \alpha_i$ distinct elements of $P_n$. Now $\alpha_1 + \ldots + \alpha_j = 2\delta_n - \gamma_1 - \ldots - \gamma_t$ with $\{\gamma_1, \ldots, \gamma_t\} \cup \{\alpha_1, \ldots, \alpha_j\} = P_{ng}$ $\gamma_1, \ldots, \gamma_t$ distinct. Hence the weigts of $\Lambda^i p_{\mathbb{C}}$ are of the form $2\delta_n - Q$ with $Q$ a sum of elements of $P_n$. Furthermore, if $2\delta_n$ is a weight of $\Lambda^i p_{\mathbb{C}}$ then $i = q$ and $2\delta_n$ is a weight in $\Lambda^q p_{\mathbb{C}}$ of multiplicity 1. The weights of $(\sigma, F)$ relative to $T$ are of the form $\Lambda - \delta - Q$ with $Q$ a sum of elements of $P$ and $\Lambda - \delta$ is a weight of multiplicity 1 (this is the theorem of the highest weight). This implies:

a) The weights of $T$ on $\Lambda^i p_{\mathbb{C}} \otimes F$ are of the form $2\delta_n + \Lambda - \delta - Q$ with $Q$ a sum of elements of $P$. If $2\delta_n + \Lambda - \delta + Q'$ is a weight of $\Lambda^i p_{\mathbb{C}} \otimes F$ then $Q' = 0$, $i = q$ and the weight $2\delta_n + \Lambda - \delta = \Lambda + \delta - 2\delta_k$ has multiplicity 1 in $\Lambda^q p_{\mathbb{C}} \otimes F$.

a) implies

b) If $\lambda$ is $P_k$-dominant integral and $\mathrm{Hom}_K(V_\lambda, \Lambda^i p_{\mathbb{C}} \otimes F) \neq 0$ then $\lambda = \Lambda + \delta - 2\delta_k - Q$ with $Q$ a sum of elements of $P$. If $\lambda = \Lambda + \delta - 2\delta_k + Q$ with $Q$ a sum of elements of $P$ then $\mathrm{Hom}_K(V_\lambda, \Lambda^i p_{\mathbb{C}} \otimes F) \neq 0$ implies $Q = 0$ and $i = q$. Furthermore, $\dim \mathrm{Hom}_K(V_{\Lambda+\delta-2\delta_k}, \Lambda^q p_{\mathbb{C}} \otimes F) = 1$.

b) combined with Theorem 5.11 implies 2).

5.14. For later applications it will be necessary to extend Theorem 5.13 to a larger class of groups. We now assume that $M$ is a reductive linear Lie group with compact center satisfying the following conditions:

1) If $M^0$ is the identity component of $M$ then $M$ contains a finite subgroup $\Omega$ consisting of elements of order 2 so that $\Omega M^0 = M$.

2) $M$ contains a maximal compact subgroup $K$ so that $\Omega \subset K$ and $M^0 \cap K$ (resp. $[M^0, M^0] \cap K$) is maximal compact in $M^0$ (resp. $[M^0, M^0]$).

5.15. If $G$ is as above, if $P \subset G$ is a parabolic subgroup of $G$ and if $P = MAN$ is a Langland's decomposition of $P$ then $M$ satisfies the conditions of 5.14.

5.16. If $(\pi, H)$ is an irreducible unitary representation of $M$, then $\pi|_{M^0}$ splits into a finite direct sum of irreducible representations of $M^0$. If $(\eta, W)$ is an irreducible representation of $M^0$ then since the center of $M^0$ must act by scalars under $\eta$, $\eta|_{[M^0, M^0]}$ is irreducible. It is now easily seen that $(\pi, H)$ is a square integrable representation of $M$ if and only if $\pi|_{[M^0, M^0]}$ splits into a direct sum of irreducible, square integrable representations of $[M^0, M^0]$.

5.17. Let $\underline{m}$ be the Lie algebra of $M$. Let $\underline{k}$ be the Lie algebra of $K$ and let $\underline{m}_1 = [\underline{m}, \underline{m}]$ be the Lie algebra of $[M^0, M^0]$. Set $\underline{k}_1 = \underline{k} \cap \underline{m}_1$. Let $\underline{p}$ be the orthogonal complement to $\underline{k}_1$ in $\underline{m}_1$ relative to the Killing form of $\underline{m}_1$. Then $\mathrm{Ad}(K)\underline{p} \subset \underline{p}$ and $\underline{k} \oplus \underline{p} = \underline{m}$.

5.18. Using the results for $G$ as above (think of $[M^0, M^0]$ as covered by $G$) we see that $\mathcal{E}_2(M) \neq \emptyset$ if and only if there is a maximal torus $T$ of $K$ so that $T \cap [M^0, M^0]$ is a Cartan subgroup of $[M^0, M^0]$. That is, $M$ has a compact Cartan subgroup.

5.19. PROPOSITION. Let $(\pi, H)$ be an irreducible square integrable representation of $M$. Let $V$ be the $(\underline{m}, K)$-module of $K$-finite vectors of $H$. Let $(\sigma, F)$ be an irreducible finite dimensional representation of $M$. Let $q = \frac{1}{2} \dim M/K = \frac{1}{2} \dim \underline{p}$. Then $H^i(\underline{m}, K; V \otimes F^*) = 0$ if $i \neq q$.

Proof. Let $C$ be the Casimir operator of $\underline{m}_1$. If $\sigma(C) = \lambda I$, $\pi(C) = \mu I$

and $\lambda \neq \mu$ then $H^i(\underline{m}, K; V \otimes F^*) = 0$ for all $i$ (see chapter II). If $\lambda = \mu$ then $\dim H^i(\underline{m}, K; V \otimes F^*) = \dim \mathrm{Hom}_K(\Lambda^i p \otimes F; V)$. Now $H = H_1 \oplus \dots \oplus H_r$, $H_j$ closed, $[M^0, M^0]$ invariant and $[M^0, M^0]$ irreducible. Let $\pi_j(m) = \pi(m)|_{H_j}$, $j = 1, \dots, r$, $m \in [M^0, M^0]$. Then $\pi_j \in \omega_j \in \mathcal{E}_2([M^0, M^0])$. Also $F = F_1 \oplus \dots \oplus F_s$ with $F_j$, $[M^0, M^0]$ invariant and $[M^0, M^0]$ irreducible. Hence ($V_\ell = H_\ell \cap V$) $\dim H^i(\underline{m}, K; V \otimes F^*) \leq \sum_{j,\ell} \mathrm{Hom}_{K \cap [M^0, M^0]}(\Lambda^i \underline{p} \otimes F_j; V_\ell) =$

$\sum_{j,\ell} H^i(\underline{m}_1, K \cap [M^0, M^0]; V_\ell \otimes F_j^*)$ (see chapter II). Now apply Theorem 5.13.

<u>5.20</u>. We now look at the specific groups $G = SU(p, q)$, $p \geq q > 0$, As in §4. We return to the notation of §4. Let $B$ denote the group of diagonal elements of $G$. Then $B$ is a maximal compact torus of $G$ and $B$ is a Cartan subgroup of $G$. Let $K = G \cap U(p+q)$. Let $\underline{g}$, $\underline{k}$, $\underline{b}$ denote the respective Lie algebras of $G$, $K$ and $B$.

<u>5.21</u>. Let $\Phi$ denote the root system of $(\underline{g}_{\mathbb{C}}, \underline{b}_{\mathbb{C}})$. Let $\Phi^+$ be defined as in 4.8. Let $(V_\ell, L^2_\ell(\mathbb{R}^n))$ $(n = p + q)$ be as in §4. Let $H_\ell = L^2_\ell(\mathbb{R}^n) \cap \psi_0 \cdot P(\mathbb{R}^n)$. Then $H_\ell$ is easily seen to be the space of K-finite vectors in $L^2(\mathbb{R}^n)_\ell$ relative to $V_\ell$.

<u>5.22</u>. Let $\Phi_k$ be the root system of $(\underline{k}_{\mathbb{C}}, \underline{b}_{\mathbb{C}})$ and set $\Phi_k^+ = \Phi_k \cap \Phi^+$.

<u>5.23</u>. LEMMA. Set $\Lambda_\ell = -\ell\eta_p + \eta_{p+1} + \dots + \eta_{p+q}$ if $\ell \geq 0$ and $\Lambda_\ell = \eta_{p+1} + \dots + \eta_{p+q-1} + (1 - \ell)\eta_{p+q}$ for $\ell < 0$. ($\eta_i$ are as in §4.) Then the weights of $\underline{b}$ on $H_\ell$ are all of multiplicity 1 and are of the form $\Lambda_\ell - Q$ with $Q$ a sum of elements of $\Phi^+$.

<u>Proof</u>. $H_{\ell}^{\underline{u}^+} = \mathbb{C}\,\psi_{0,\ldots,0,\ell,0,\ldots,0}$ if $\ell \geq 0$
pth position

$H_{\ell}^{\underline{u}^+} = \mathbb{C}\,\psi_{0,\ldots,0,-\ell}$ if $\ell < 0$

(see the proof of 4.17). Let $\underline{u}^- = \sum_{\alpha\in-\Phi^+} (\underline{g}_{\mathbb{C}})_{-\alpha}$ (where $[\underline{u}, \underline{u}] = \underline{g}$).
Then $H_\ell = U(\underline{u}^-)\cdot H_{\ell}^{\underline{u}^+}$ by the Poincaré-Birkhoff-Witt theorem. The now lemma follows by computing that $\Lambda_\ell$ is the action of $\underline{b}$ on $H_{\ell}^{\underline{u}^+}$.

<u>5.24</u>. Let $\underline{b}_{\mathbb{R}} = \{h \in \underline{b} | \alpha(h) \in \mathbb{R},\ \alpha \in \Phi\}$. Then the Weyl chamber in $\underline{b}_{\mathbb{R}}$ defined by $\Phi^+$ is defined by the inequalities: $\eta_1 > \ldots > \eta_p > \eta_{p+q-1} > \ldots > \eta_{p+1}$.

<u>5.25</u>. Let $\underline{p}$ be the $\underline{k}$ invariant complement to $\underline{k}$ in $\underline{g}$. Let $\Phi_n = \Phi - \Phi_k$, $\Phi_n^+ = \Phi_n \cap \Phi^+$. Then $\underline{p}_{\mathbb{C}} = \underline{p}^+ \oplus \underline{p}^-$ with $\underline{p}^+ = \sum_{\alpha\in\Phi_n^+} (\underline{g}_{\mathbb{C}})_\alpha$, $\underline{p}^- = \sum_{\alpha\in\Phi_n^+} (\underline{g}_{\mathbb{C}})_{-\alpha}$. It is easily checked that $\underline{p}^+$ and $\underline{p}^-$ are Ad(K)-invariant.

<u>5.26</u>. LEMMA. Let $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$. Then $V_\ell$ has infinitesimal character $\chi_{\Lambda_\ell+\delta}$.

<u>Proof</u>. This follows from the fact $V_\ell$ has highest weight $\Lambda_\ell$ relative to $\Phi^+$.

<u>5.27</u>. For future reference we note that $\delta = \sum_{i=1}^{p} (p+q-i)\eta_i + \sum_{i=1}^{q} (i-1)\eta_{i+p}$. Hence we see

1) If $\ell \geq 0$ then $\Lambda_\ell + \delta = \sum_{i=1}^{p-1} (p+q-i)\eta_i + (q-\ell)\eta_p + \sum_{i=1}^{q} i\eta_i$

2) If $\ell < 0$ then $\Lambda_\ell + \delta = \sum_{i=1}^{p} (p + q - i)\eta_i +$

$$\sum_{i=1}^{q-1} i\eta_{i+p} + (q - \ell)\eta_{p+q} \quad .$$

We are now ready to give a cohomological interpretation of the $V_\ell$, $\ell \in \mathbb{Z}$.

<u>5.28.</u> PROPOSITION. Let $(\sigma_\ell, F_\ell)$ be the irreducible finite dimensional representation of $G$ with highest weight $-\ell\eta_{p+1}$, for $\ell \geq 0$. (That is, $F_\ell$ is the $\ell$<u>th</u> symmetric power of the dual of the standard representation of $G$ on $\mathbb{C}^{p+q}$). Then if $\ell \geq q$

$$H^q(\mathfrak{g}, K; H_\ell \otimes F^*_{\ell-q}) \neq (0) \quad .$$

<u>Proof</u>. Let $S$ be the permutation of $1, \ldots, p + q$ given by $(p\ p{+}1 \ldots p{+}q)$. The Weyl group of $\Phi$ acts on $\underline{b}^*_{\mathbb{C}}$ as $S_{p+q}$, the permutations of $p + q$ letters, by $t \cdot \eta_j = \eta_{t^{-1}j}$. Then $S^{-1}(\Lambda_\ell + \delta) = (q - \ell)\eta_{p+1} + \delta$ by the obvious computation. This implies that $\chi_{\Lambda_\ell + \delta} = \chi_{F_{\ell-q}}$ for $\ell \geq q$. Using the results of chapter II we find that if $\ell \geq q$ then $\dim H^j(\mathfrak{g}, K; H_\ell \otimes F^*_{\ell-q}) = \dim \mathrm{Hom}_K(\Lambda^j \underline{p} \otimes F_{\ell-q}, H_\ell)$.

1) $$\dim \mathrm{Hom}_K(V_{S\delta-\delta}, \Lambda^q \underline{p}^-) = 1 \quad .$$

To prove 1) we will need to use some notation. Let $\ell$ be the length function on $W(\Phi)$ defined by $\Phi^+$. Then if $t \in W(\Phi)$, $\ell(t)$ is the number of $\alpha \in \Phi^+$ so that $t\alpha \in -\Phi^+$. We note that $\ell(t) = \ell(t^{-1})$. We leave it to the reader to compute that $\ell(S) = \ell(S^{-1}) = q$. We also assert that $S\Phi^+ \supset \Phi^+_k$. Indeed, we must only show that $S^{-1}\alpha \in \Phi^+$ for $\alpha \in \Phi^+_k$. If $\alpha \in \Phi^+_k$ then $\alpha = \eta_i - \eta_j$, $1 \leq i < j \leq p$ or $\alpha = \eta_j - \eta_i$,

$p < i < j \leq p + q$. If $1 \leq i < j \leq p - 1$ then clearly $S^{-1}\alpha = \alpha$. If $1 \leq i \leq p - 1$ and $\alpha = \eta_i - \eta_p$ then $S^{-1}\alpha = \eta_i - \eta_{p+1} \in \Phi^+$. If $\alpha = \eta_j - \eta_i$, $p \leq i < j \leq p + q - 1$ then $S^{-1}\alpha = \eta_{j+1} - \eta_{i+1} \in \Phi^+$. If $\alpha = \eta_{p+q} - \eta_i$, $p < i \leq p + q$ then $S^{-1}\alpha = \eta_p - \eta_{i+1} \in \Phi^+$. This proves the assertion. Now Lemma 3.5 in Hotta, Wallach [8] implies 1).

2) $$\dim \mathrm{Hom}_K(\tau_{(q-\ell)S\eta_{p+1}}, F_{\ell-q}) = 1 .$$

Indeed, $S(q - \ell)\eta_{p+1})$ is the highest weight of $F_{\ell-q}$ relative to $S\Phi^+$. Since $S\Phi^+ \supset \Phi_k^+$ (see the proof of 1)), 2) follows.

We now prove the Proposition.

1) and 2) imply that $\Lambda^q\underline{p} \otimes F_{\ell-q} \supset V_{S\delta-\delta} \otimes V_{S((q-\ell)\eta_{p+1})}$ as K-submodule. Thus $\dim \mathrm{Hom}_K(V_{S((q-\ell)\eta_{p+1}+\delta)-\delta}, \Lambda^q\underline{p} \otimes F_{\ell-q}) \geq 1$. Now $S((q - \ell)\eta_{p+1} + \delta) - \delta = \Lambda_\ell$ by the first part of this proof. Thus $\mathrm{Hom}_K(\tau_{\Lambda_\ell}, \Lambda^q\underline{p} \otimes F_{\ell-q}) \neq 0$. But $\Lambda_\ell$ is the highest weight of $H_\ell$. Thus $\mathrm{Hom}_K(\tau_{\Lambda_\ell}, H_\ell) \neq 0$. Combining these observations gives:

$$\mathrm{Hom}_K(\Lambda^q\underline{p} \otimes F_{\ell-q}, H^\ell) \neq 0 .$$

The proposition now follows.

<u>5.29</u>. We observe that we have actually shown that $\dim \mathrm{Hom}_K(\Lambda^q\underline{p}^- \otimes F_{\ell-q}, H^\ell) \neq 0$ for $\ell \geq q$. Thus $V_\ell$ will contribute to $(0, q)$ cohomology of the locally constant sheaf over $\Gamma\backslash G|K$ for $\Gamma \subset G$ a cocompact discrete subgroup.

## 6. The reciprocity formula

6.1. Let $G$ be a connected Lie group. Let $(\pi, H)$ be a unitary representation of $G$.

6.2. Let $H_\infty$ denote the space of $C^\infty$ vectors of $H$. That is $H_\infty$ is the space of all $v \in H$ so that $g \mapsto \pi(g)v$ is a $C^\infty$ function from $G$ to $H$. Then it is well known that $H_\infty$ is dense in $H$ (cf. Warner [16], chapter 4). Let $\mathfrak{g}$ be the Lie algebra of $G$ and let $U = U(\mathfrak{g}_{\mathbb{C}})$. If $v \in H_\infty$ and $u \in U$ define $P_u(v) = ||\pi(u)v||$. Here if $X \in \mathfrak{g}$, $\pi(X)v = \frac{d}{dt}\pi(\exp tX)v|_{t=0}$, $v \in H_\infty$. $(\pi, H_\infty)$ defines a representation of $\mathfrak{g}$ and thus $\pi(u)$ makes sense on $H_\infty$ for $u \in U$.

6.3. We topologize $H_\infty$ using the semi norms $P_u$. Then $H_\infty$ is a locally convex, complete, topological vector space. If $(\pi_i, H_i)$, $i = 1, 2$ are unitary representations of $G$ we set $\mathrm{Hom}_G(H_1, H_2)$ equal to the space of all bounded operators $A : H_1 \longrightarrow H_2$ so that $A \circ \pi_1(g) = \pi_2(g) \circ A$ for $g \in G$.

6.4. If $A \in \mathrm{Hom}_G(H_1, H_2)$ and $v \in (H_1)_\infty$ then $g \longmapsto A\pi_1(g)v = \pi_2(g)Av$ is $C^\infty$. Thus $A : (H_1)_\infty \longrightarrow (H_2)_\infty$. By the definition of the topology on $(H_i)_\infty$ in 6.2 we see that $A : (H_1)_\infty \longrightarrow (H_2)_\infty$ is continuous and $A \circ \pi_1(u) = \pi_2(u) \circ A$ on $(H_1)_\infty$ for $u \in U$.

6.5. Let $\Gamma \subset G$ be a cocompact, discrete subgroup of $G$. Let $\pi_\Gamma$ be the right regular representation of $G$ on $L^2(\Gamma\backslash G)$ (Note that $G$ is automatically unimodular if $\Gamma$ exists).

6.6. LEMMA. The space $C^\infty$ vectors of $(\pi_\Gamma, L^2(\Gamma\backslash G))$ is $C^\infty(\Gamma\backslash G)$ with the $C^\infty$ topology.

Proof. This follows from standard Sobolev theory on compact manifolds and is left to the reader.

6.7. Let $(\pi, H)$ be a unitary representation of $G$. Let $H_\infty^*$ be the space of all continuous linear functionals on $H_\infty$. If $g \in G$ and $\lambda \in H_\infty^*$ set $(\pi^*(g)\lambda)(v) = \lambda(\pi(g)^{-1}v)$ for $v \in H_\infty$. Then $\pi^*(g)\lambda \in H_\infty^*$ for $\lambda \in H_\infty^*$ and $g \in G$.

6.8. Let $\Gamma$ be as in 6.5. Let $(\pi, H)$ be an irreducible unitary representation of $G$. If $A \in \mathrm{Hom}_G(H, L^2(\Gamma\backslash G))$ then we have seen that $A : H_\infty \longrightarrow C^\infty(\Gamma\backslash G)$ is continuous. Define $\lambda_A(v) = A(v)(\Gamma\cdot 1)$ for $A \in \mathrm{Hom}_G(H, L^2(\Gamma\backslash G))$. Then $\lambda_A \in (H_\infty)^*$. Furthermore, if $\gamma \in \Gamma$, and if $v \in H_\infty$ then $(\pi^*(\gamma)\lambda_A)(v) = \lambda_A(\pi(\gamma)^{-1}v) = A(\pi(\gamma)^{-1}v)(\Gamma\cdot 1) = (\pi_\Gamma(\gamma)^{-1}A(v))(\Gamma\cdot 1) = A(v)(\Gamma\cdot 1) = \lambda_A(v)$. That is, $\pi^*(\gamma)\lambda_A = \lambda_A$ for $A \in \mathrm{Hom}_G(H, L^2(\Gamma\backslash G))$. Set $(H_\infty^*)^\Gamma = \{\lambda \in H_\infty^* | \pi^*(\gamma)\lambda = \lambda, \ \gamma \in \Gamma\}$.

6.9. THEOREM (Gelfand, Graev, Pyateckii-Shapiro [5]). The map $\mathrm{Hom}_G(H, L^2(\Gamma\backslash G)) \longrightarrow (H_\infty^*)^\Gamma$ $A \longmapsto \lambda_A$ is bijective.

Proof. Let $\lambda \in (H_\infty^*)^\Gamma$. Define for $v \in H_\infty$, $A_\lambda(v)(g) = \lambda(\pi(g)v)$. Then the definition of the topology on $H_\infty^*$ and the invariance of $\lambda$ under $\Gamma$ imply that $A_\lambda(v) \in C^\infty(\Gamma\backslash G)$. Clearly, $A_\lambda(\pi(g)v) = \pi_\Gamma(g)A_\lambda(v)$ for $v \in H_\infty$. We show that $A_\lambda$ extends to a bounded operator from $H$ to $L^2(\Gamma\backslash G)$.

Define for $v, w \in H_\infty$, $(v, w) = \langle A_\lambda(v), A_\lambda(w)\rangle_\Gamma$ (here $\langle\, , \rangle$ denotes the Hilbert space structure on $H$ and $\langle\, , \rangle_\Gamma$ defines the Hilbert space structure on $L^2(\Gamma\backslash G)$). If $v, w \in H_\infty$ then $(v, w) = \langle Bv, w\rangle$

with $B : H_\infty \longrightarrow H_\infty$ and $\langle Bv, w\rangle = \langle v, Bw\rangle$ for $v, w \in H_\infty$. We also note that $B \circ \pi(g) = \pi(g) \circ B$ on $H_\infty$. We need:

<u>6.10</u>. LEMMA. Let $(\pi, H)$ be an irreducible unitary representation of $G$. If $B : H_\infty \longrightarrow H_\infty$ is linear, $B \circ \pi(g) = \pi(g) \circ B$ on $H_\infty$ and if $\langle Bv, w\rangle = \langle v, Bw\rangle$ for $v, w \in H_\infty$ then $B = \lambda I$.

<u>Proof</u>. Let $\mathcal{A}$ denote the algebra of operators on $H$ spanned by the $\pi(g)$, $g \in G$. If $\mathcal{B} \subset L(H, H)$ is a subspace (the bounded operators on $H$) define $\mathcal{B}' = \{X \in L(H, H) \mid XY = YX,\ Y \in \mathcal{B}\}$. The irreducibility of $(\pi, H)$ implies that $\mathcal{A}' = \mathbb{C}I$. The Von Newmann density theorem (cf. Bruhat [4], p. 87) implies that if $T \in L(H, H)$, $v_1, \ldots, v_n \in H$ and $\varepsilon > 0$ is given then there is $A \in \mathcal{A}$ so that $\sum \|Av_i - Tv_i\|^2 < \varepsilon$ (this is because it is clear that $A \in \mathcal{A}$ implies $A^* \in \mathcal{A}$). If $A \in \mathcal{A}$ then $AB = BA$ on $H_\infty$. Suppose that $v \in H_\infty$ and $v$, $Bv$ are linearly independent. Then there is $T \in L(H, H)$ so that $Tv = v$ and $TBv = v$. There is thus a sequence $A_j \in \mathcal{A}$ so that $\lim_{j \to \infty} A_j v = v$ and $\lim_{j\to\infty} A_j Bv = v$. If $w \in H_\infty$ then $\langle BA_j v, w\rangle = \langle A_j Bv, w\rangle$ and $\langle BA_j v, w\rangle = \langle A_j v, Bw\rangle$. Hence $\lim_{j\to\infty}\langle A_j Bv, w\rangle = \lim_{j\to\infty}\langle A_j v, Bw\rangle$. Thus $\langle v, w\rangle = \langle v, Bw\rangle$ for all $w \in H_\infty$. Hence $\langle Bv - v, w\rangle = 0$ for all $w \in H_\infty$. This implies $Bv = v$ which contradicts the assumption that $v$ and $Bv$ are linearly independent. We therefore see that if $v \in H_\infty$, $Bv = c(v)v$, $c(v) \in \mathbb{C}$. This clearly implies that $B$ is a scalar operator. Q.E.D.

<u>6.11</u>. We now continue the proof of 6.9. By 6.10, $B = cI$. Clearly $c \geq 0$ and $c = 0$ implies $\lambda = 0$. Thus we may assume $c > 0$. But then

1) $$\langle A_\lambda(v), A_\lambda(w)\rangle_\Gamma = c\langle v, w\rangle .$$

We therefore see that $||A_\lambda v|| = c^{1/2}||v||$. Thus $A_\lambda$ extends to an element of $\mathrm{Hom}_G(H, L^2(\Gamma\backslash G))$. Clearly $\lambda_{A_\lambda} = \lambda$ and $A_{\lambda_A} = A$. This proves 6.9.

<u>6.12</u>. In [5], 6.9 is called the Duality theorem. It is proved for $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})$ by explicitly computing certain analogues of $H_\infty$. 6.9 is substantially the same as the Duality theorem of [5].

## 7. The imbedding of $V_\ell$ into $L^2(\Gamma\backslash G)$.

7.1. Let $k$ be a totally real finite extension of $\underline{\underline{Q}}$, and denote by $r+1$ its degree. We assume $r \geq 1$, fix an imbedding of $k$ into $\underline{\underline{R}}$, and view $k$ as a subfield of $\underline{\underline{R}}$. Let $\Sigma = \{\sigma_1, \ldots, \sigma_{r+1}\}$ be the set of isomorphisms of $k$ into $\underline{\underline{R}}$, where $\sigma_{r+1} = \mathrm{id}$. Let $k' = k(i)$. We extend $\sigma \in \Sigma$ to the imbedding of $k'$ into $\underline{\underline{C}}$ which leaves $i$ fixed.

Let $n$ be a positive integer, $h$ a non-degenerate hermitian form on $V_{k'} = k'^n$ of signature $(p, q)$ $(p \geq q > 0;\ p + q = n)$. We assume that for $\sigma \in \Sigma$, $\sigma \neq 1$, the form ${}^\sigma h$, given by $z, w \mapsto \sigma^{-1}(h(\sigma z, \sigma w))$ is definite.

7.2. Let $H_k$ be the group of all $g \in GL(n, k')$ so that $h(g\cdot z, g\cdot w) = h(z, w)$, $z, w \in V_k$ and $\det(g) = 1$. It is the group of points over $k$ of a $k$-form $\underline{H}$ of $SL_n$. Now $h(z, w) = \mu(z, w) + \sqrt{-1}\,\beta(z, w)$, with $\mu$ a symmetric, $k$-bilinear form with values in $k$ and $\beta$ a skew symmetric $k$-bilinear form with values in $k$. We look at $V_{k'}$ as being a $2n$-dimensional vector space over $k$ and write $V_k$ instead. Using a symplectic basis for $\beta$ we see that $H_k \subset Sp(n, k)$, or more precisely that we have an imbedding defined over $k$ of $\underline{H}$ in the symplectic group $\underline{Sp}_n$, viewed as a $k$-group.

7.3. Let $\mathrm{Res}_{k/\underline{Q}}$ denote restriction of scalars from $k$ to $\underline{Q}$ (see Weil [17], Chap. 1). Then $\underline{G} = \mathrm{Res}_{k/\underline{Q}}(H)$ and $\mathrm{Res}_{k/\underline{Q}}(\underline{Sp}_n)$ are defined over $\underline{Q}$ and we have a canonical imbedding $\underline{G} = \mathrm{Res}_{k/\underline{Q}}\underline{H} \to \mathrm{Res}_{k/\underline{Q}}\underline{Sp}_n$. Moreover, the group $\underline{G}_{\underline{Q}}$ of rational points of $\underline{G}$ is equal to $\mathrm{Res}_{k/\underline{Q}}(H_k)$. Let $V_{\underline{Q}}$ be $V_k$ viewed as a $2n(r+1)$-dimensional vector space over $\underline{Q}$, and $\beta_{\underline{Q}}$ the bilinear form on $V_{\underline{Q}}$ defined by $\beta$. It is antisymmetric non-degenerate and we have $V_{\underline{Q}} \doteq \mathrm{Res}_{k/\underline{Q}} V_k$, $\beta_{\underline{Q}} = \mathrm{Res}_{k/\underline{Q}}\beta$. Therefore $\underline{G}$ is naturally embedded in the group of automorphisms of $V_{\underline{Q}} \otimes_{\underline{Q}} \mathbb{C}$ leaving $\beta_{\underline{Q}}$ invariant, i.e. in $\underline{Sp}_N$, where $N = n(r + 1)$.

Over $\underline{R}$, the group $\underline{G}$ is isomorphic to the product of the groups ${}^{\sigma}H$ $(\sigma \in \Sigma)$, where ${}^{\sigma}H$ is the group of automorphisms of $V_{k'} \otimes \underline{C}$ preserving ${}^{\sigma}h$. Therefore the group $G_{\underline{R}}$ of real points of $\underline{G}$ is isomorphic to the product of $SU(p, q)$ by $r$ copies of $SU(n)$. Of course ${}^{\sigma}\underline{Sp}_n$ is again $\underline{Sp}_n$, hence the group of real points of $\mathrm{Res}_{k/\underline{Q}}\underline{Sp}_n$ is the direct product of $r + 1$ copies of $Sp(n, \underline{R})$. The imbedding $\underline{H} \hookrightarrow \underline{Sp}_n$ yields $\psi_{r+1} : SU(n) \hookrightarrow Sp(n, R)$ and, for $\sigma_i \in \Sigma$, $i \neq r + 1$, the corresponding imbedding of ${}^{\sigma}\underline{H}$ into $\underline{Sp}_n$ yields $\psi_i : SU(n) \hookrightarrow Sp(n, R)$. The direct product of $(r + 1)$ copies of $Sp(n, \underline{R})$ is naturally contained in $Sp(N, \underline{R})$; our given embedding $G_R \hookrightarrow Sp(N, \underline{R})$ is, up to conjugation over $\underline{R}$, the product $\psi$ of the $\psi_i$, followed by that inclusion.

<u>7.4</u>. Let $e_1, \ldots, e_{2N}$ be a basis of $V_{\underline{Q}}$ so that $\beta_{\underline{Q}}$ is in standard form. $G_{\mathbb{Z}} = \{\gamma \in G_{\mathbb{Q}} \mid \psi(\gamma) \in Sp(N, \mathbb{Z})\}$. Then $G_Z$ is an arithmetic subgroup of $G_{\mathbb{R}}$ (see Borel [2], 7.11, 7.12). Also $\psi : G_{\mathbb{Z}} \to Sp(N, \mathbb{Z})$.

<u>7.5</u>. THEOREM (Borel, Harish-Chandra [3]). $G_{\mathbb{Z}}$ is a cocompact discrete subgroup of $G_{\mathbb{R}}$.

<u>Proof</u>. We have $G_{\mathbb{R}} = \prod_{i=1}^{r+1} G_i$, where $G_i = SU(n)$, $i \leq r$ and $G_{r+1} = SU(p, q)$. Let $p_i : G_R \to G_i$ be the <u>ith</u> projection. The definition of $G_{\mathbb{Q}}$ implies that $p_i|_{G_{\mathbb{Q}}}$ is injective for each $i$.

If $\gamma \in G_{\mathbb{Z}}$ were unipotent then $p_i(\gamma)$ would be for each $1 \leq i \leq r$. But $p_i(\gamma) \in G_i = SU(n)$. Thus there are no non-trivial unipotent elements of $G_{\mathbb{Z}}$. The result now follows from [3].

7.6. Let $p_i : G_{\mathbb{R}} \longrightarrow G_i$, $i \leq r+1$ be as in the proof of 7.5. Set $p_{r+1}(G_{\mathbb{Z}}) = \Gamma$. If $\omega \subset SU(p, q)$ is a compact subset then $p_{r+1}^{-1}(\omega) \subset G_{\mathbb{R}}$ is compact. Thus $\Gamma$ is a cocompact, discrete subgroup of $SU(p, q)$.

7.7. Lemma 4.18 implies that $\psi : G_{\mathbb{R}} \longrightarrow Sp(N, \mathbb{R})$ lifts to $\tilde{\psi} : G_{\mathbb{R}} \longrightarrow Mp(N, \mathbb{R})$. Indeed, we have $\tilde{\alpha} : \mathop{\times}\limits_{i=1}^{r+1} Mp(n, \mathbb{R}) \longrightarrow Mp(N, \mathbb{R})$ and $\psi_i : G_i \longrightarrow Sp(n, \mathbb{R})$ lifts to $\hat{\psi}_i : G_i \longrightarrow Mp(n, \mathbb{R})$ and we set $\tilde{\psi} = \alpha \circ \mathop{\times}\limits_{i=1}^{r+1} \hat{\psi}_i$. Using this observation we see that if $W^j$ is the oscillator representation of $Mp(n, \mathbb{R})$, $j = 1, \ldots, r+1$ and $W$ is the oscillator representation of $Mp(N, \mathbb{R})$ then $W \circ \tilde{\psi}$ is equivalent with $(W^1 \circ \hat{\psi}_1) \hat{\otimes} (W^2 \circ \hat{\psi}_2) \hat{\otimes} \cdots \hat{\otimes} (W^{r+1} \circ \hat{\psi}_{r+1})$ $((g_1, \ldots, g_{r+1}) \longmapsto \bigotimes_{i=1}^{r+1} (W^i \circ \hat{\psi}_i)(g_i))$. Set $V^i = W^i \circ \hat{\psi}_i$. The $V^i$ acts on the coordinates $x_{(i-1)n+1}, \ldots, x_{in}, y_{(i-1)n}, \ldots, y_{in}$.

7.8. It should be noted that the basis that splits $\psi$ into a product is <u>not</u> the same as the basis for which our $Sp(N, \mathbb{Z})$ is defined.

7.9. If $1 \leq j \leq r$ then $V^j = \bigoplus_{\ell \geq 0} V_\ell^j$ with $\dim V_\ell^j < \infty$. (This corresponds to the case $q = 0$ in §4).

7.10. This implies that

$$V = \bigoplus_{\substack{\ell_1, \ldots, \ell_r \geq 0 \\ \ell_i \in \mathbb{Z}}} V_{\ell_1}^1 \hat{\otimes} \cdots \hat{\otimes} V_{\ell_{r+1}}^{r+1}$$

as a representation of $G_{\mathbb{R}}$. Set $V_{(\ell_1, \ldots, \ell_{r+1})}$ equal to $V_{\ell_1}^1 \hat{\otimes} \cdots \hat{\otimes} V_{\ell_{r+1}}^{r+1}$ and let $L^2(\mathbb{R}^N)_{(\ell_1, \ldots, \ell_{r+1})}$ be the representation space for $V_{\ell_1, \ldots, \ell_{r+1}}$.

7.11. The results of §2 and §4 easily imply that

1) The space of $C^\infty$ vectors of $V_{(\ell_1,\ldots,\ell_{r+1})}$ is precisely $L^2(\mathbb{R}^N)_{(\ell_1,\ldots,\ell_{r+1})} \cap \mathcal{S}(\mathbb{R}^N)$ with the subspace topology.

2) $L^2(\mathbb{R}^N)_{(\ell_1,\ldots,\ell_{r+1})} \cap \psi_0 P(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)_{(\ell_1,\ldots,\ell_{r+1})}$ and in the $C^\infty$ vectors for $V_{(\ell_1,\ldots,\ell_{r+1})}$.

7.12. THEOREM. If $\ell \in \mathbb{Z}$ then there exists $\Gamma' \subset \Gamma$ a subgroup of finite index (indeed, a congruence subgroup) so that

$$\left(\left(V_\ell^{r+1}\right)_\infty^*\right)^{\Gamma'} \neq (0) \quad .$$

(Here, of course, $V_\ell^{r+1}$ is the representation of $SU(p, q)$ denoted by $V_\ell$ in §4.)

Proof. Fix $\ell_1, \ldots, \ell_r \in \mathbb{Z}$, $\ell_i \geq 0$. Let $H = L^2(\mathbb{R}^N)_{(\ell_1,\ldots,\ell_r,\ell)}$. Then $H_\infty = H \cap \mathcal{S}(\mathbb{R}^N)$ and $H \cap \psi_0 P(\mathbb{R}^N)$ is dense in $H$ and $H_\infty$. Let $\mathbb{Z}^N$ be the lattice associated with the basis for the $Sp(N, \mathbb{Z})$ we are considering in this section (see 7.8). Let $A$ be as in 3.2. If $\varphi \in H \cap \psi_0 P(\mathbb{R}^N)$ and $A\varphi(\tau) = 0$ for $\tau \in \mathbb{Z}^N$ then $\varphi = 0$. (i.e. $\mathbb{Z}^N$ is Zariski dense in $\mathbb{R}^N$). Thus if $\varphi \neq 0$ there exists $\Delta \subset Mp(N, \mathbb{R})$ so that:

1) $\nu(\Delta)$ contains a congruence subgroup of $Sp(N, \mathbb{Z})$.

2) There is $\lambda \in \mathcal{S}'(\mathbb{R}^N)^\Delta$ so that $\lambda(A\varphi) \neq 0$.

See Theorem 3.12 for this result and the notation.

Set $\mu = \lambda \circ A$ restricted to $H_\infty$. Let $\Omega$ be a congruence subgroup of $G_{\mathbb{Z}}$ so that $\psi(\Omega) \subset \nu(\Delta)$. Then $\tilde{\psi} : \Omega \longrightarrow \Delta$. Hence $\mu \in (H_\infty^*)^\Omega$. Let $\Gamma' = \pi(\Omega)$. Then $\Gamma' \subset \Gamma$ is a congruence subgroup. Fix

$\varphi = \varphi_1 \otimes \dots \otimes \varphi_{r+1} \in \psi_0 P(\mathbb{R}^N) \cap H$ so that $\mu(\varphi) \neq 0$. Define $\xi(f) = \mu(\varphi_1 \otimes \dots \otimes \varphi_r \otimes f)$ for $f \in (V_\ell^{r+1})_\infty^*$. Then $\xi \in \left(\left(V_\ell^{r+1}\right)_\infty^*\right)^{\Gamma'}$, $\xi \neq 0$. Q.E.D.

7.13. We now revert to our old notation. $G = SU(p, q)$, $p \geq q > 0$. $\psi : G \longrightarrow Sp(n, \mathbb{R})$ $(n = p + q)$ and $\tilde{\psi} : G \longrightarrow Mp(n, \mathbb{R})$ the lift of $\psi$. $V = W \circ \tilde{\psi}$. $V_\ell$ is as in §4. However, we fix $\Gamma$ as constructed above.

7.14. COROLLARY. If $\ell \in \mathbb{Z}$ then there is $\Gamma' \subset \Gamma$ a congruence subgroup (possibly depending on $\ell$) so that

$$\mathrm{Hom}_G(V_\ell, L^2(\Gamma\backslash G)) \neq 0 \quad .$$

Proof. This is just 7.12 combined with 6.9.

7.15. COROLLARY. Let for $\ell \geq 0$, $\ell \in \mathbb{Z}$, $F_\ell$ be as in 5.28. Let $\Gamma$ be as above. Then there is a congruence subgroup $\Gamma' \subset \Gamma$ so that $H^q(\Gamma', F_{\ell-q}) \neq (0)$ for $\ell \geq q$.

Proof. This follows from Theorem 4.2, chapter IV and 5.28.

7.16. We note that in this case the cohomology of $\Gamma$ is bigraded in the same way as the cohomology of $\Gamma\backslash G|K$. We actually have $H^{0,q}(\Gamma', F_{\ell-q}) \neq 0$ for $\ell \geq q$. We also note that $\dim H^q(\Gamma', \mathbb{C}) =$ qth Betti number of $\Gamma\backslash G|K$.

7.17. The results of this section are substantially due to Kazhdan [10]. He concentrated on the case $SU(n, 1)$ and $V_1$. He also studied the significance of the $V_{-j}$, $j > 0$ for $SU(2, 1)$ for $\Gamma$ not necessarily cocompact. Kazhdan's proof of the pertinent results uses the global oscillator representation and Strong approximation rather than Theorem 3.10.

## References


[1] H. Bass, J. Milnor, J. Serre, "Solution of the congruence subgroup problem for $SL_n$, $Sp_n$", Inst. Hautes Études Sci. Publ. Math., 33 (1967),

[2] A. Borel, "Introduction aux groupes arithmétiques", Hermann, Paris, 1969.

[3] A. Borel and Harish-Chandra, "Arithmetic subgroups of algebraic groups", Ann. of Math. 75 (1962), 485-535.

[4] F. Bruhat, "Lectures on Lie groups and representations of locally compact groups", Tata Institute of Fundamental Research, 1958 (Reissued 1968).

[5] I. M. Gelfand, M. I. Graev, I. I. Pyateckii-Shapiro, "Representation theory and automorphic functions", W. B. Saunders Co., Philadelphia, 1969.

[6] Harish-Chandra, "Discrete series for semi-simple Lie groups II", Acta. Math. 116 (1966), 1-111.

[7] H. Hecht and W. Schmid, "A proof of Blattner's conjecture", Inventiones Math. 31 (1975), 129-154.

[8] R. Hotta and N. R. Wallach, "On Matsushima's formula for the Betti numbers of a locally symmetric space", Osaka J. Math., 12 (1975), 419-431.

[9] J. I. Igusa, "Theta functions", Grundlehren der Mathematischen Wissen chaften, 194, Springer-Verlag, NY, 1973.

[10] D. A. Kazhdan, "Some applications of the Weil representation", mimeographed lecture notes.

[11] M. Reed and B. Simon, Methods of Mathematical Physics I, Functional Analysis, Academic Press, NY, 1972.

[12] W. Schmid, "On the characters of the discrete series (the Hermitian symmetric case)", Inventiones Math. 30 (1975), 47-144.

[13] W. Schmid, "Some properties of square-integrable representaions of semi-simple Lie groups", Ann. of Math. 102 (1975), 535-564.

[14] N. R. Wallach, "Symplectic geometry and Fourier analysis", Math. Sci. Press, Boston, 1977.

[15] N. R. Wallach, "On the Enright-Varadarajan modules, a construction of the discrete series", Ann. Sci. Ec. Norm. Sup. 9 (1976), 81-102.

[16] G. Warner, "Harmonic Analysis on semisimple Lie groups I", Springer-Verlag, New York, 1973.

[17] A. Weil, Adeles and algebraic groups, Notes by M. Demazure and T. Ono, Institute for Advanced Study, Princeton, 1961.

[18] G. Zuckerman, "Tensor products of finite and infinite dimensional representations", to appear, Ann. of Math.

# IX: CONTINUOUS COHOMOLOGY AND DIFFERENTIABLE COHOMOLOGY

A. Borel

## Introduction

In most of the previous lectures we have been studying the relative Lie algebra cohomology spaces $H^*(\underline{g}, K; V)$ with coefficients in a $(\underline{g}, K)$-module. Our only case of interest is when $V$ is the set of $K$-finite vectors in the space $V^\infty$ of $C^\infty$-vectors of a continuous $G$-module. In that case, by the van Est theorem, this space is also the space $H^*_d(G; V^\infty)$ of continuous (or differentiable) Eilenberg-Mac Lane cohomology of $G$ with coefficients in $V^\infty$. The relationship between cohomology of discrete subgroups and cohomology with coefficients in infinite dimensional representations described in IV can also be expressed in terms of continuous cohomology (and obtained directly by use of a suitable Shapiro lemma). Moreover, this relationship is also valid in the $p$-adic case, where there is no direct analog of the Lie algebra cohomology.

This lecture is devoted to the basic notions and results on continuous or differentiable cohomology. This is not a completely self-contained exposition, since, when convenient, we have referred to [3] or [10]. But a number of proofs have been included.

At this point, we are mainly interested in real Lie groups. However, in preparation for the p-adic or mixed case, we shall first develop continuous cohomology for locally compact groups (§§1 to 4), in the framework of [3] or [10: §2]. §§5, 6 are concerned with differentiable cohomology, which was initiated by W. T. van Est (see [4, 5] and earlier references given there). We shall largely follow the exposition of [10]. We have also borrowed from three lectures given by G. D. Mostow at the Institute in Spring 1975, in particular for 5.4, 6.2, 6.3 and the proof of 5.2.

Initially, the reason to discuss differentiable cohomology at this point was the existence of a Hochschild-Serre spectral sequence and a suitable Shapiro lemma. After this was done, however, we noticed that these could also be obtained in the algebraic framework of Chapter I, slightly modified. We shall sketch this briefly in §7. More details will be given in the final version of these Notes. In any case, as far as real Lie groups are concerned, this makes it possible to bypass the van Est theorem and to get all what we need in these lectures in the context of relative Lie algebra cohomology.

In this chapter, locally compact groups are assumed to be countable at infinity, topological vector spaces are Hausdorff, over $\mathbb{R}$ and locally convex.

## §1. Continuous cohomology for locally compact groups.

1.1. Let $G$ be a locally compact group. By a topological $G$-module, or simply a $G$-module $(\pi, V)$, we mean a topological vector space on which $G$ acts via a continuous representation $\pi$. A $G$-morphism of two such

G-modules is a continuous linear map which commutes with G. We let $\underline{C}_G$ or simply $\underline{C}$, if G is clear from the context, denote the category of topological G-modules and G-morphisms.

For some of the main theorems, we shall assume that the G-modules under consideration are Fréchet spaces, i.e. are complete and metrizable. In fact, for our needs, it would be no essential loss in generality to assume this from the start, as far as real Lie groups are concerned.

1.2. If X, F are topological spaces, we let C(X, F) denote the space of continuous maps of X into F, endowed with the compact open topology. It is a topological vector space if F is one, and a Fréchet space if F is one and X is locally compact, countable at infinity, as follows from [1: X 21, Cor. to Prop. 1]. If A and B are topological vector spaces, then Hom(A, B) denotes the space of continuous linear maps from A to B, endowed with the compact open topology. If $A, B \in \underline{C}_G$ then Hom(A, B) will be given the G-module structure defined by

$$(1)\qquad (xf)(a) = x(f(x^{-1}.a)), \qquad (x \in G;\ a \in A;\ f \in \mathrm{Hom}(A, B)).$$

$\mathrm{Hom}_G(A, B)$ denotes the set of homomorphisms which commute with G. Both Hom(A, B) and $\mathrm{Hom}_G(A, B)$ are closed subspaces of C(A, B).

We recall that if $f : A \longrightarrow B$ is a surjective continuous linear map of Fréchet spaces, then f induces a topological isomorphism of A/ker f, endowed with the quotient topology, onto B [2: I, §3, n°2, Thm 1]. In particular, if

(2) $$0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0 ,$$

is an exact sequence of Fréchet spaces and continuous homomorphisms, then $u$ is an isomorphism of $A$ onto $u(A)$ and $v$ induces an isomorphism of $B/u(A)$ onto $C$. Moreover, <u>if</u> $X$ <u>is a topological space, then the associated sequence</u>

(3) $$0 \longrightarrow C(X; A) \xrightarrow{u'} C(X; B) \xrightarrow{v'} C(X; C) \longrightarrow 0$$

<u>is exact.</u> This is obvious at $C(X; A)$ and $C(X; B)$. The surjectivity of $v'$ follows from the fact that $v$ admits a continuous (not necessarily linear) cross-section (Bourbaki, yet unpublished). In particular, if $X$ is locally compact, countable at infinity (our only case of interest), (3) is again an exact sequence of Fréchet spaces. The surjectivity of $v'$ in that case has already been pointed out, without proof, by A. Grothendieck in the footnote on p. 84 of [7].

<u>1.3</u>. Let $V \in \underline{C}_G$ and $q \in \mathbb{N}$. We let $C^q(G; V) = C(G^{q+1}; V)$, viewed as a $G$-module by means of the action

(1) $$(x.f)(x_o, \ldots, x_q) = x(f(x^{-1}.x_o, \ldots, x^{-1}.x_q)) , \qquad (x, x_o, \ldots, x_q \in G) .$$

We let $F^q(G; V)$ be the same space, but with the action of $G$ defined by right translations on $G$, i.e.

(2) $$(g.f)(x_o, \ldots, x_q) = f(x_o.x, \ldots, x_q.x) , \qquad (x, x_o, \ldots, x_q \in G) .$$

Since $G$ is assumed to be countable at infinity, these spaces are Fréchet spaces if $V$ is one (1.2).

The map $\mu : F^{o}(G; V) \longrightarrow C^{o}(G; V)$ defined by

(3) $$\mu(f)(x) = x.f(x^{-1}), \qquad (x \in G;\ f \in F^{o}(G; V)),$$

is readily seen to be a $G$-isomorphism. Since the canonical map

(4) $$C(G; C(G^{q}; V)) \longrightarrow C(G^{q+1}; V)$$

is a topological isomorphism [1: X 29, Thm. 3, Cor. 2], we get by iteration a $G$-isomorphism of $F^{q}(G; V)$ onto $C^{q}(G; V)$ $(q = 0, 1, \ldots)$ (see [10: §2]).

We let $\varepsilon$ denote the maps $V \longrightarrow F^{o}(G; V)$ and $V \longrightarrow C^{o}(G; V)$ which assign respectively to $v$ the function $x \longmapsto x.v$ and the constant function equal to $v$ on $G$. These two injections are $G$-morphisms, which correspond to each other under $\mu$.

<u>1.4.</u> The <u>standard homogeneous resolution</u> of $V \in \underline{C}$ is the (augmented) complex

(1) $$0 \longrightarrow V \xrightarrow{\varepsilon} A^{o}(V) \xrightarrow{d_o} A^{1}(V) \longrightarrow \cdots \longrightarrow A^{q}(V) \xrightarrow{d_q} A^{q+1}(V) \longrightarrow \cdots,$$

where $A^{q}(V) = C^{q}(G; V)$ and $d_q$ is given by

(2) $$(d_q f)(x_o, \ldots, x_{q+1}) = \Sigma_i (-1)^i \, f(x_o, \ldots, \hat{x}_i, \ldots, x_{q+1}), \qquad (x_i \in G;\ i = 0, \ldots, q+1).$$

The $q$-th continuous cohomology group $H^{q}_{ct}(G; V)$ of $G$ with coefficients in $V$

is then, by definition, the q-th cohomology group of the complex

$$(3) \qquad A^o(V)^G \longrightarrow \ldots \longrightarrow A^q(V)^G \longrightarrow \ldots \ .$$

The topological vector space $A^q(V)^G$ is isomorphic to $F^{q-1}(G; V)$ via the map $f \longmapsto f'$, where

$$(4) \qquad f'(x_1, \ldots, x_q) = f(1, x_1, x_1.x_2, \ldots, x_1 \ldots x_q) \ .$$

(By definition $F^{-1}(G; V) = V$.) The complex (3) can then be written

$$(5) \quad V \xrightarrow{d'_o} F^o(G; V) \xrightarrow{d'_1} \ldots \longrightarrow F^q(G; V) \xrightarrow{d'_{q+1}} F^{q+1}(G; V) \longrightarrow \ldots \ ,$$

where $F^q(G; V)$ is viewed as the space of elements of degree $q+1$, and the differential $d'_q$ is given by

$$(6) \quad (d'_q f)(x_o, \ldots, x_q) = x_o.f(x_1, \ldots, x_q) + \Sigma'_{0 \leq i < q}(-1)^{i+1} f(x_o, \ldots, x_i.x_{i+1}, \ldots, x_q) + (-1)^{q+1} f(x_o, \ldots, x_{q-1}) \ .$$

(5) is the complex of <u>non-homogeneous continuous cochains</u>. For all this, see [10: §2]. This is of course just the continuous analog of standard notions concerning the Eilenberg-Mac Lane cohomology of abstract groups.

<u>1.5.</u> We want next to define these groups in the context of relative homological algebra [9; 11]. For this, as usual, we keep the objects of <u>C</u> but restrict the morphisms. We shall say that a G-morphism $f : A \longrightarrow B$ is an s-morphism (strong morphism) if: (i) ker f and im f are closed

topological direct summands; (ii) $f$ induces an isomorphism of $A/\ker f$ onto $f(A)$. The facts recalled in 1.2 imply that if $A$ and $B$ are Fréchet spaces, then (ii) follows from (i). In fact, for (ii) to hold, it suffices then that $f(A)$ be closed in $B$. A sequence of morphisms in $G$ is strong (or an s-sequence) if all the morphisms are s-morphisms.

An element $U \in \underline{C}$ is s-<u>injective</u> if, given a strong injection $A \longrightarrow B$, every $G$-morphism $f : A \longrightarrow U$ extends to a $G$-morphism $B \longrightarrow U$ (neither is required to be strong). A continuously s-injective resolution or, simply, an s-injective resolution of $V \in \underline{C}_G$, is an exact sequence:

$$(1) \qquad 0 \longrightarrow V \longrightarrow A^1 \longrightarrow \ldots \longrightarrow A^q \longrightarrow \ldots$$

in which the $A^i$'s are s-injective. Given such a resolution of $V$, and $U \in \underline{C}$, one defines as usual $\mathrm{Ext}_G^q(U, V)$ to be the q-th cohomology group of the complex $\{\mathrm{Hom}_G(U, A^i)\}$. In particular, $\mathrm{Ext}_G^q(\mathbb{R}, V)$, where $\mathbb{R}$ is viewed as the trivial $G$-module, is the q-th cohomology group of the complex $\{A^{iG}\}$ $(q = 0, 1, \ldots)$. Clearly,

$$(2) \qquad \mathrm{Ext}_G^o(U; V) = \mathrm{Hom}_G(U, V), \ \mathrm{Ext}_G^o(\mathbb{R}; V) = V^G .$$

It is standard that these groups do not depend on the s-injective resolution chosen, up to natural isomorphisms [10; §2]. That s-injective resolutions exist follows from the following lemma:

1.6. LEMMA. *Let* $V \in \underline{C}$. *Then* $F^o(G; V)$ *is* s-*injective and* $\varepsilon : V \longrightarrow F^o(G; V)$ (*see* 1.3) *is a strong injection. The homogeneous resolution of* $V$ (1.4(1)) *is* s-*injective. It consists of Fréchet spaces if* $V$ *is one.*

The last assertion follows from 1.2. The others are proved in [10: §2], see also [3]. In view of 1.4, this implies in particular

$$\mathrm{Ext}^q_G(\mathbb{R}; V) = H^q_{ct}(G; V) , \qquad (q = 0, 1, 2, \dots) . \tag{1}$$

Since the topology of $V$ is uniform, the natural bijections

$$\mathrm{Mp}(U \times G^q, V) \xrightarrow{\sim} \mathrm{Mp}(U, \mathrm{Mp}(G^q, V)) , \quad \mathrm{Mp}(U \times G^q, V) \xrightarrow{\sim} \mathrm{Mp}(G^q, \mathrm{Mp}(U, V)) \tag{2}$$

where Mp refers to arbitrary maps, induce topological isomorphisms

$$C(U \times G^q; V) \xrightarrow{\sim} C(U; C(G^q; V)) , \quad C(U \times G^q; V) \longrightarrow C(G^q; C(U; V)) , \tag{3}$$

[1: X §1, n°4, Prop. 2]. From this follows that we have a canonical isomorphism of topological vector spaces

$$\mathrm{Hom}(U, C^q(G; V)) = C^q(G; \mathrm{Hom}(U, V)) , \tag{4}$$

which is easily checked to commute with $G$. Consequently, (1) generalizes to

$$\mathrm{Ext}^q_G(U, V) = H^q_{ct}(G; \mathrm{Hom}(U, V)) , \qquad (U, V \in \underline{C}_G; q \in \mathbb{N}) . \tag{5}$$

1.7. LEMMA. *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 , \tag{1}$$

be an exact sequence in C and let, for $q \in \mathbb{N}$,

(2) $$0 \longrightarrow F^q(G; A) \longrightarrow F^q(G; B) \xrightarrow{u} F^q(G; C) \longrightarrow 0 ,$$

be the canonically associated sequence of G-modules.

(i) If (1) is s-exact, then so is (2).

(ii) If A, B, C are Fréchet spaces, then (2) is an exact sequence of Fréchet spaces.

(iii) In both cases, u induces a topological isomorphism

(3) $$F^q(G; B)/F^q(G; A) \xrightarrow{\sim} F^q(G; C) .$$

Clearly, if $B = B' \oplus B''$ is the topological direct sum of two closed subspaces, then $F^q(G; B)$ is isomorphic to the topological direct sum of $F^q(G; B')$ and $F^q(G; B'')$, whence (i) and (iii) in this case. The other assertions follow from 1.2.

We note that, in both cases, we can identify in (1) A with its image in B and B/A with C, hence (3) can also be written

(4) $$F^q(G; B)/F^q(G; A) \xrightarrow{\sim} F^q(G; B/A) .$$

1.8. Lemma 1.3 implies, under either set of assumptions, that the sequence

(1) $$0 \longrightarrow F^*(G; A) \longrightarrow F^*(G; B) \longrightarrow F^*(G; C) \longrightarrow 0 ,$$

of non-homogeneous complexes is exact. Therefore, in either case, there is associated to 1.7(1) a long exact sequence in continuous cohomology. Note also that, by 1.5,

$$H^q_{ct}(G; V) = 0 \quad \text{for} \quad q \geq 1 \quad , \tag{2}$$

if $V$ is s-injective.

1.9. As usual, [11: III] we can interpret the groups $\mathrm{Ext}^q_G(U, V)$ as equivalence classes of long exact s-sequences from $V$ to $U$, as in I, §3. [There, we did not have to introduce strong morphisms: by definition the $(\underline{g}, \underline{k})$-modules are locally finite and semi-simple with respect to $\underline{k}$, hence all morphisms of $(\underline{g}, \underline{k})$-modules are strong, with respect to the $\underline{k}$-module structure.] As a result, there are vanishing theorems as in I, §§4, 5. In particular, if there exists in the (abstract) group algebra of $G$ over $\mathbb{R}$ an element which acts as the identity on $U$, and as the zero-morphism on $V$, then $\mathrm{Ext}^q_G(U, V) = 0$ for all $q$'s. The proof is the same as that of the theorem in I, §4.

1.10. PROPOSITION. *Let* $U, V \in \underline{C}$. *Assume* $G$ *to be compact,* $U$ *a Fréchet space and* $V$ *quasi-complete. Then* $\mathrm{Ext}^q_G(U, V) = 0$ *for* $q \geq 1$. *In particular* $H^q_{ct}(G; V) = 0$ *for* $q \geq 1$.

Under our assumptions $\mathrm{Hom}(U, V)$ is quasi-complete (cf. [2], III, §1, n°1, Cor. and §3, n°7, Cor. 2). By 1.6(5), it suffices therefore to prove the second assertion. Let $dx$ be a Haar measure of total measure one on $G$. Then one checks that the map $J : F^q(G; V) \longrightarrow F^{q-1}(G; V)$ defined by

$$(1) \qquad (Jf)(x_1, \ldots, x_q) = \int_G f(x_1, \ldots, x_q, x)dx \ ,$$

satisfies $d.J - J.d = Id$, i.e., is a homotopy operator, whence the result. For another argument, see [3: lemma 7].

1.11. Let $H$ be a group which operates on $G$ by continuous automorphisms. Let $\underline{C}_{G,H}$ be the category of those $V \in \underline{C}_G$ on which $H$ acts as a group of operators for the $G$-module structure, i.e. every $h \in H$ defines an automorphism of $V$ such that $h(g.v) = h(g).h(v)$. In particular, $H$ leaves $V^G$ stable. Then, <u>if</u> $V \in \underline{C}_{G,H}$, <u>the group</u> $H$ <u>operates canonically on</u> $H^*_{ct}(G; V)$. In fact, $H$ operates in the obvious way on the homogeneous complex, 1.4(1), hence on $C^*(G; V)^G$ and on the cohomology. The complex $C^*(G; V)$ may also be viewed as a continuously $s$-injective resolution of $V$ in $\underline{C}_{G,H}$. Therefore, we may compute $H^*_{ct}(G; V)$ using any continuously $s$-injective resolution in the smaller category $\underline{C}_{G,H}$. It follows also that the action of $H$ on the cohomology introduced above may be defined similarly starting from any continuously $s$-injective resolution of $V$ in $\underline{C}_{G,H}$. As usual, $G$ (acting on itself by inner automorphisms) operates trivially on the cohomology.

## §2. Shapiro's lemma.

2.1. Let $H$ be a closed subgroup of $G$ and $U \in \underline{C}_H$. As usual, we put

(1) $$I(U) = \mathrm{Ind}_H^G U = \{f \in C(G; U) \,|\, f(gh) = h^{-1}.f(g), \quad (g \in G; h \in H)\} .$$

It is a closed subspace of $C(G, U)$, hence a Fréchet space if $U$ is one, (1.2). If $G$ acts trivially on $U$, and $H = \{1\}$, then $\mathrm{Ind}_H^G U = C(G; U)$. If $U$ is a $G$-module, then $I(U)$ is canonically isomorphic to the space $C(G/H; U)$, endowed with the compact-open topology [3: lemma 1].

2.2. LEMMA. *Let $H$, $U$ be as above and $V \in \underline{C}_G$. Then the map*

$$\mathrm{Hom}_G(V, I(U)) \longrightarrow \mathrm{Hom}_H(V, U) ,$$

*associated to the map $I(U) \longrightarrow U$ given by $f \longmapsto f(1)$, is a topological isomorphism.*

For the proof, cf. [3: lemma 2].

2.3. PROPOSITION. *Let $H$ be a closed subgroup of $G$. Assume that the fibration of $G$ by $H$ admits a continuous local cross-section.*

(i) *Every s-injective G-module is s-injective as an H-module.*

(ii) ("Shapiro's lemma") *Given $U \in \underline{C}_H$ and $V \in \underline{C}_G$, there are canonical isomorphisms*

(1) $$\mathrm{Ext}_G^q(V, I(U)) = \mathrm{Ext}_H^q(V; U) , \qquad (q \in \mathbb{N}) .$$

In particular

$$H^q_{ct}(G; I(U)) = H^q_{ct}(H; U) \quad , \qquad (q \in \mathbb{N}) \quad . \tag{2}$$

Since $G$ is by assumption countable union of compact subsets, the space $G/H$ is paracompact; hence (i) is lemma 3.4 of [10].

(ii) is proved in exactly the same way as Prop. 3 of [3]: one starts from the homogeneous resolution $C^*(G; I(U))$ of $I(U)$ (see 1.4) and shows that

$$C^n(G; I(U)) = I(C^n(G; U)) \quad , \qquad (n \in \mathbb{N}) \quad . \tag{3}$$

By 2.2, we have then

$$\mathrm{Hom}_G(V; C^n(G; I(U))) = \mathrm{Hom}_H(V, C^n(G; U) \quad , \qquad (n \in \mathbb{N}) \quad . \tag{4}$$

Since these isomorphisms are natural, they yield an isomorphism of complexes

$$\{\mathrm{Hom}_G(V; C^n(G; I(U)))\} \xrightarrow{\sim} \{\mathrm{Hom}_H(V, C^n(G; U)))\} \quad .$$

By definition, the $q$-th cohomology group of the left-hand side is $\mathrm{Ext}^q_G(V; I(U))$. Since $C^n(G; U)$ is s-injective with respect to $H$ (by (i)), the complex $\{C^n(G; U)\}$ provides an s-injective resolution of $U$ in $\underline{C}_H$, hence the $q$-th cohomology group of the right-hand side is $\mathrm{Ext}^q_H(V, U)$. This proves (1).

## §3. Hausdorff cohomology.

3.1. Let $C^*$ be a complex in $\underline{C}_G$ (we do not exclude trivial action, i.e., $C^*$ may just be a complex of topological vector spaces, with continuous linear differentials). Then $Z^q$ is closed in $C^q$ and $H^q(C) = Z^q/d(C^{q-1})$ may be given the quotient topology. It is Hausdorff if and only if $d(C^{q-1})$ is closed in $Z^q$ or, equivalently, in $C^q$. If so, we shall view $H^q(C)$ as a topological vector space in this way, and shall say that $H^q(C)$ is Hausdorff, or that $C$ has Hausdorff cohomology in dimension $q$. If this is true for all $q$'s, then we say that $H^*(C)$ is Hausdorff or that $C^*$ has Hausdorff cohomology. Since $Z^q$ and $d(C^{q-1})$ are stable under $G$, $H^q(C)$ inherits an action of $G$, which is continuous with respect to the quotient topology. Thus, if $H^q(C)$ is Hausdorff, it is canonically in $\underline{C}_G$. Of course, $H^o(C)$ is always Hausdorff.

3.2. LEMMA. Let $V \in \underline{C}_G$ and $q \in \mathbb{N}$. Assume that there exists an s-injective resolution $E^*$ of $V$ such that $H^q(E^{*G})$ is Hausdorff. Then any s-injective resolution $F^*$ of $V$ has the same property. The canonical isomorphism of $H^q(E^{*G})$ onto $H^q(E^{*G})$ associated to the identity map of $V$ is topological.

The identity map of $V$ extends to a $G$-morphism $\tilde{u}$ of $F^*$ into $E^*$ [10: §2], whence also a morphism $u : F^{*G} \longrightarrow E^{*G}$, which induces an isomorphism $u^*$ of $H^*(F^{*G}) \longrightarrow H^*(E^{*G})$ (loc. cit). Let $Z^q$ and $B^q$ (resp. $Z'^q$ and $B'^q$) the cocycles and coboundaries in $F^{qG}$ (resp. $E^{qG}$), and $t$ be the projection of $Z'^q$ onto $H^q(E^{*G})$. Since $u^*$ is an isomorphism, it follows that $t \circ u$ is surjective, with kernel $B^q$. Since $t \circ u$ is continuous, this shows first that $B^q$ is closed in $Z^q$ and then that $u^*$ is a continuous bijective map of $H^q(F^{*G})$ onto $H^q(E^{*G})$. Similarly, a lifting of the identity of $V$ to a map $E^* \longrightarrow F^*$ yields to a bijective continuous map $v^* : H^q(E^{*G}) \longrightarrow H^q(F^{*q})$. Since $u^* \circ v^*$ and $v^* \circ u^*$ are the identity, this proves the lemma.

3.3. In fact, the proof of the lemma shows that $u^*$ is a topological isomorphism of $H^q(F^{*G})$ onto $H^q(E^{*G})$, both spaces being endowed with the quotient topology, regardless of whether they are Hausdorff or not, whence the existence of a canonical topology on $H^q_{ct}(G; V)$. If the condition of 3.2 is fulfilled, then we shall say that $H^q_{ct}(G; V)$ is Hausdorff. $H^*_{ct}(G; V)$ will be said to be Hausdorff if $H^q_{ct}(G; V)$ is so for all $q$'s.

3.4. LEMMA. Assume that $C^*$ is a complex of Fréchet spaces (and continuous linear maps) and that $H^q(C)$ is finite dimensional. Then $d_{q-1}(C^{q-1})$ is closed in $C^q$.

The proof is the same as that of Prop. 6 in [3]. We repeat it for the sake of completeness. Let $E$ be a subspace of $Z^q$ which maps bijectively onto $H^q(C)$ under the natural projection $Z^q \longrightarrow H^q(C)$. It is finite dimensional, hence closed in $Z^q$ [2: I, §2, n°3]. The obvious map $B^q \oplus E \longrightarrow Z^q$, where $B^q = C^{q-1}/Z^{q-1}$ is endowed with the quotient topology, is continuous, bijective, hence is an isomorphism (1.2), whence the lemma.

Remark. The proof shows more precisely that the sequence

$$0 \longrightarrow C^{q-1}/Z^{p-1} \xrightarrow{d_{q-1}} Z^q \longrightarrow H^q(C) \longrightarrow 0 \ ,$$

is s-exact.

3.5. PROPOSITION. Let $V \in \underline{C}_G$ and $q \in \mathbb{N}$. Assume that $V$ is a Fréchet space and that $H^q(G; V)$ is finite dimensional. Then $H^q(G; V)$ is Hausdorff.

In fact the standard homogeneous resolution consists of Fréchet spaces, and the condition of 3.2 is satisfied in view of 3.4.

## §4. Spectral sequences.

We shall assume familiarity with standard material on spectral sequences (cf., e.g., [11: XI] or [6: I, §4]). The spectral sequences considered here are all "first quadrant" spectral sequences associated to double complexes with positive degrees.

4.1. THEOREM. Let

(1) $$A^* : A^o \xrightarrow{e_o} A^1 \longrightarrow \ldots \longrightarrow A^q \xrightarrow{e_q} A^{q+1} \longrightarrow \ldots \ ,$$

be a complex of acyclic $G$-modules and $G$-morphisms. If either (i) (1) is an s-sequence, or (ii) $A^*$ consists of Fréchet spaces and has Hausdorff cohomology (3.1), then there exists a spectral sequence $(E_r)$ which abuts to the cohomology of the complex $A^{*G} = \{A^{q^G}\}$ and where

(2) $$E_2^{p,q} = H^p_{ct}(G; H^q(A)) \ , \qquad (p, q \geqq 0) \ .$$

We note first that in both cases $H^q(A)$ is in $\underline{C}_G$ in a canonical way (3.1). It is this G-module structure which underlies (2).

Let $F^*(G; A^q)$ be the non-homogeneous complex of continuous $A^q$-values cochains (see 1.4 (5), (6)). Then the direct sum $C^*$ of the $F^*(G; A^q)$ is a double complex in the usual way, with differentials induced by 1.4(6) and by the differentials of $A^*$. We have (see 1.4)

(1) $$C^{p,q} = F^{p-1}(G; \underline{A}^q), \qquad (p, q \in \mathbb{N})$$

and $C^{p,q} = 0$ otherwise. We consider the two spectral sequences $('E_r)$, $(''E_r)$ associated to the filtrations defined by the partial degrees. If the degree in $A$ is used (giving the "second filtration"), then

(2) $$''E_o^{*,q} = F^*(G; A^q) ,$$

and the differential $d_o''$ of the spectral sequence is that of $F^*(G; A^q)$. Therefore, $''E_1^{p,q} = H^p_{ct}(G; A^q)$. Since the $A^q$'s are acyclic, we have $''E_1^{p,q} = 0$ if $p \neq 0$ and $''E_1^{o,q} = A^{q^G}$. Then $d_1''$ is induced by the differentials of $A^*$, whence

(3) $$''E_2^{o,p} = H^q(A^{*^G}) = ''E_\infty^{o,q} = H^q(C^*), \quad ''E_r^{p,q} = 0 \qquad (r \geqq 1; p \neq 0) .$$

We now consider the spectral sequence $('E_r)$ associated to the filtration by the degree in $F^*$ (the "first filtration"). We have then

(4) $$'E_o^{p,*} = F^p(G; A^*) , \qquad (p \in \mathbb{N}) .$$

We want to prove

$$(5)\qquad {}'E_1^{p,q} = F^p(G;\, H^q(A)) , \qquad (p,\ q \in \mathbb{N}) .$$

Let

$$(6)\qquad Z^q = \ker e_q ,\ B^q = A^q/Z^q .$$

By 1.7 and our assumptions, the exact sequence

$$(7)\qquad 0 \longrightarrow Z^q \longrightarrow A^q \longrightarrow B^q \longrightarrow 0 , \qquad (q \in \mathbb{N})$$

yields a topological isomorphism

$$(8)\qquad F^p(G;\, A^q)/F^p(G;\, Z^q) \xrightarrow{\ \sim\ } F^p(G;\, B^q) , \qquad (p,\ q \in \mathbb{N}) .$$

If (1) is strong, then the injection $e_{q-1} : B^q \longrightarrow Z^q$ is strong and

$$(9)\qquad 0 \longrightarrow B^q \longrightarrow Z^q \longrightarrow H^q(A) \longrightarrow 0$$

is an exact s-sequence, where $H^q(A)$ is endowed with the quotient topology. Under assumption (ii) the subspace $e_{q-1}(B^q)$ of $Z^q$ is closed, hence $e_{q-1}$ is an isomorphism of $B^q$ onto its image, and (9) is again an exact sequence of Fréchet spaces, $H^q(A)$ being endowed with the quotient topology. Lemma 1.7 then yields

$$(10)\qquad F^p(G;\, Z^q) = (\ker d_o) \cap {}'E_o^{p,q} ,$$

(11) $$F^p(G;\, B^q) \cong F^p(G;\, e_{q-1}(B^q)) = d_o({}'E_o^{p,\,q-1}) \ ,$$

(12) $$F^p(G;\, H^q(A)) = F^p(G;\, Z^q)/F^p(G;\, B^q) \ .$$

This proves (5). The differential $d_1'$ of $'E_1$ is then the one of $F^*$, given by 1.4(6), whence

(13) $$'E_2^{p,\,q} = H^p_{ct}(G;\, H^q(A)) \ , \qquad (p,\ q \in \mathbb{N}) \ .$$

Since $('E_r)$ abuts to $H^*(C)$, and the latter is equal to $H^*(A^{*G})$ by (3), the spectral sequence $('E_r)$ satisfies our conditions.

4.2. COROLLARY. Let $V \in \underline{C}$ and let

(1) $$0 \longrightarrow V \xrightarrow{e} A^o \xrightarrow{e_o} A^1 \longrightarrow \dots \longrightarrow A^q \xrightarrow{e_q} A^{q+1} \longrightarrow \dots$$

be a resolution of $V$ by acyclic $G$-modules. Assume that (1) is strong or consists of Fréchet spaces. Then

(2) $$H^q_{ct}(G;\, V) = H^q(A^{*G}) \ , \qquad (q \in \mathbb{N}) \ .$$

We have $H^q(A^*) = 0$ for $q \geqq 1$ hence the complex $A^* = \{A^i\}$ is Hausdorff. Moreover, $e$ is an isomorphism of $V$ onto $\ker e_o = H^o(A^*)$: this is clear if $e$ is strong, and follows from 1.2 if $V$ and $A^o$ are Fréchet spaces. Therefore, we can apply 4.1. We have then

$$E_2^{p,\,q} = 0 \ \text{ for } \ q \neq 0,\ E_2^{p,\,o} = H^p_{ct}(G;\, V) \ , \qquad (p,\ q \in \mathbb{N}) \ ,$$

and our assertion follows.

4.3. THEOREM. *Let* $N$ *be a closed normal subgroup of* $G$. *Assume that the fibra of* $G$ *by* $N$ *admits a continuous local cross section. Let* $V \in \underline{C}_G$ *be such that* $H^*_{ct}(N; V)$ *is Hausdorff* (3.3). *Assume either that* $V$ *is a Fréchet space or that there exists an* s-*injective resolution* $A^*$ *of* $V$ *in* $\underline{C}_G$ *such that* $A^{*N}$ *is a strong complex. Then* $H^*_{ct}(N; V)$ *admits a natural structure of topological* (G/N)-*module* (1.11) *and there exists a spectral sequence* $(E_r)$ *abuting to* $H^*_{ct}(G; V)$, *in which*

$$E_2^{p,q} = H^*_{ct}(G/N; H^*_{ct}(N; V)) , \qquad (p, q \in \mathbb{N}) . \tag{1}$$

We let $A^*$ be any s-injective resolution of $V$ in $\underline{C}_G$ if $V$ is a Fréchet space and be as in the statement of the theorem otherwise. It is s-injective in $\underline{C}_N$ (2.3), therefore

$$H^q(A^{*N}) = H^q_{ct}(N; V) , \qquad (q \in \mathbb{N}) . \tag{2}$$

Moreover, $A^{*N}$ has Hausdorff cohomology, in view of 3.2 and our assumption. Now, if a $G$-module $B$ is s-injective in $\underline{C}_G$ then, clearly, $B^N$ is s-injective in $\underline{C}_{G/N}$. Therefore $A^{qN}$ is s-injective in $\underline{C}_{G/N}$ $(q \in \mathbb{N})$. A fortiori it is $(G/N)$-acyclic (1.8(2)). Thus $A^{*N}$ is a complex of $(G/N)$-acyclic modules, which either is strong or consists of Fréchet spaces. In both cases, we may apply 4.1, with $G/N$ and $A^{*N}$ playing the role of $G$ and $C^*$. There exists therefore a spectral sequence $(E_r)$ abuting to

$H^*((A^{*^N})^{G/N})$, in which

(3) $$E_2^{p,q} = H_{ct}^p(G/N; H^q(A^{*^N}))) , \qquad (p, q \in \mathbb{N}) .$$

In view of (2) and of the obvious equality $(A^{*^N})^{G/N} = A^{*^G}$, this spectral sequence has the required properties.

4.4. COROLLARY. Assume $G/N$ to be compact and $V$ quasi-complete. Then

$$H_{ct}^q(G; V) = H^q(N; V)^{G/N} , \qquad (q \in \mathbb{N}) .$$

$H^*(\mathbb{N}; V)$ is also quasi-complete. By 1.10, we have in 4.3(1)

(1) $$E_2^{p,q} = 0 \text{ if } p \neq 0, \ E_2^{o,q} = H^q(N; V)^{G/N} , \qquad (q \in \mathbb{N}) .$$

This implies $H^q(G; V) \cong E_2^{o,q}$, $(q \in \mathbb{N})$ and the corollary.

## §5. Differentiable cohomology and continuous cohomology for Lie groups.

In this section, $G$ is a Lie group with finitely many connected components and $K$ a maximal compact subgroup of $G$. We let $\underline{C}_G^\infty$ be the category of differentiable $G$-modules [13: p. 259] and continuous $G$-maps. All manifolds are assumed to be smooth and countable at infinity.

5.1. If $X$ is a manifold and $V$ a topological vector space, then $C^\infty(X; V)$ is the space of $C^\infty$-maps from $X$ to $V$, endowed with the $C^\infty$-topology, and $A^q(X; V)$ the space of smooth $V$-valued differential forms on $X$, also endowed with the $C^\infty$-topology $(q \in \mathbb{N})$. If $V$ is quasi-complete (resp. a Fréchet space), then $C^\infty(X; V)$ and $A^q(X; V)$ are so. If $V = \mathbb{R}$, we shall also denote these spaces $C^\infty(X)$ and $A^q(X)$.

If $V \in \underline{C}^\infty_G$, then, in agreement with 1.3, we let $C^\infty(G; V)$ be endowed with the $G$-module structure defined by $(x.f)(g) = x.f(x^{-1}.g)$ $(g, x \in G)$, while $F^\infty(G; V)$ denotes the same space, but with $G$ acting by right translations on the first argument, i.e., $xf(g) = f(g.x)$ $(g, x \in G)$. They are differentiable $G$-modules, isomorphic under the map $\mu$ of 1.3.

An element $V \in \underline{C}^\infty_G$ is differentiably or smoothly (resp. continuously) s-injective if it is s-injective in $\underline{C}^\infty_G$ (resp. $\underline{C}_G$). Of course, the latter implies the former.

If $V \in \underline{C}^\infty_G$, then $F^\infty(G; V)$ (or, equivalently, $C^\infty(G; V)$) is smoothly s-injective [10: 5.1]. (Note that since $V$ is Hausdorff by our standing assumption, the separability condition in 5.1 of [10] is automatically fulfilled.) As in 1.4, it follows that smoothly s-injective resolutions exist. In fact, the standard homogeneous resolution of 1.4, computed with smooth cochains, is one. We can then define the q-th differentiable cohomology group $H^q_d(G; V)$ as in 1.4(5)(6), using smooth cochains, and, as in 1.5, it can be computed by means of any smoothly s-injective resolution. Since smooth cochains are in particular continuous cochains, there is a natural map

$$\mu : H^*_d(G; V) \longrightarrow H^*_{ct}(G; V)\,, \qquad (V \in \underline{C}^\infty_G)\,,$$

which is natural in $V$. If $V$ is quasi-complete, then $j^*$ is an isomorphism [10: Thm. 5.1]. This follows from the following lemma, which implies that the smooth standard resolution is also continuously s-injective.

5.2. LEMMA [10: lemma 5.2]. *Let* $V \in \underline{C}_G^\infty$ *be quasi-complete. Then* $C^\infty(G; V)$ *is continuously* s-*injective*.

Put $A = C^\infty(G; V)$. Since $C(G; A)$ is continuously s-injective (1.6), it suffices to show that the canonical injection $\varepsilon : A \longrightarrow C(G; A)$ maps $A$ onto a topological direct $G$-summand, or, equivalently, that there exists a continuous $G$-map $\mu : C(G; A) \longrightarrow A$ such that $\mu \circ \varepsilon = id_A$.

Fix a left invariant Haar measure $dg$ on $G$. Let $\varphi \in C_c^\infty(G)$ be a compactly supported smooth real valued function on $G$ such that $\int_G \varphi(g^{-1})dg = 1$. Given $f \in C(G; A)$, let $\alpha(f) = f * \varphi$, i.e.

$$\alpha(f)(x) = \int_G \varphi(y^{-1}.x)f(y)dy \ , \qquad (x \in G) \ .$$

This defines a continuous $G$-map $\alpha : C(G; A) \longrightarrow C(G; A)$ with image in $C^\infty(G; A)$, which is the identity on $\varepsilon(A)$. Let then $\beta : C^\infty(G; A) \longrightarrow A$ be defined by $(\beta f)(g) = f(g)(g)$. Then $\mu = \beta \circ \alpha$ satisfies our conditions.

5.3. Let $X$ be a space on which $G$ operates continuously. $G$ is said to *operate properly* on $X$ if the map $G \times X \longrightarrow X \times X$ defined by $(g, x) \longmapsto (g.x, x)$ is proper [1: III]. This implies in particular that the isotropy groups $G_x$ $(x \in X)$ are compact and that the orbit space $X/G$ is Hausdorff if $X$ is (loc. cit.)

Let now $M$ be a manifold on which $G$ operates smoothly. A <u>differentiable slice</u> $S$ at a given point $m \in M$ is a closed submanifold in a neighborhood of $m$ with the following properties:

(i) $S \cap G.m = \{m\}$; $G_m(S) = S$; $G_m = \{g \in G \mid g.S \cap S \neq \phi\}$.

(ii) The map $(g, s) \longmapsto g.s$ induces a diffeomorphism of $G \times_{G_m} S$ ($G$ operating on the right on itself) onto $G.S$, and $G.S$ is an open neighborhood of $G.m$ in $M$.

(iii) The map $(g, s) \longmapsto g.m$ induces a smooth $G$-equivariant retraction $r_m$ of $G.S$ onto $G.m = G/G_m$.

Note that the definition of $r_m$ makes good sense since, by (i), if $s \in S$, then $G_s \subset G_m$.

If $f \in C^\infty(G.S)^G$, then its restriction $\bar{f}$ to $S$ is in $C^\infty(S)^{G_m}$. We claim that the map $f \longmapsto \bar{f}$ of $C^\infty(G.S)^G$ into $C^\infty(S)^{G_m}$ is bijective. It is clearly injective. Let now $\bar{f} \in C^\infty(S)^{G_m}$. By (ii), $GS$ is the total space of a $C^\infty$-fibration over $G/G_m$ with structural group $G_m$ and typical fibre $S$, which is locally trivial. In any local chart of the form $U \times S$ we extend $\bar{f}$ to a function $f_U$ constant on the sets $U \times \{s\}$. Then these functions match to a $G$-invariant smooth function on $G.S$ which restricts to $\bar{f}$ on $S$.

If $G$ operates properly on $M$, then there is a differentiable slice at every point of $M$ [12: 2.2.2]. In fact, $M$ always has a smooth

G-invariant Riemannian metric [12: 4.3.1], and we may take $S$ such that $GS$ is a tubular neighborhood of $G.m$ [12: 2.2.3].

5.4. PROPOSITION (G. D. Mostow). Let $M$ be a smooth manifold on which $G$ operates smoothly and properly. Let $V \in \underline{C}_G^\infty$ be a Fréchet space. Then the space $A^q(M; V)$ (cf. 5.1) is a continuously s-injective Fréchet G-module $(q \in \mathbb{N})$.

We already pointed out that $A^q(M; V)$ is a Fréchet space.

If $M$ is the quotient of $G$ by a compact subgroup, this is shown in [10: p. 385-6]. This case suffices in fact to prove van Est theorem (5.6). We sketch Mostow's argument in the general case.

Assume first that there exists $m \in M$ and a differentiable slice $S$ at $m$ such that $G.S = M$. Put $A = A^q(M; V)$. We know that $C^\infty(G; A)$ is continuously s-injective (5.2). It suffices therefore to show that $A$ is a topological direct G-summand of $C^\infty(G; A)$, i.e., that there exists a continuous G-map $\mu : C^\infty(G; A) \longrightarrow A$ such that $\mu \circ \varepsilon = \mathrm{id}_A$. Let $dy$ be a Haar measure on $G_m$ with total mass 1. Given $f \in C^\infty(G; A)$, define $\alpha(f)$ by

$$\alpha(f)(x) = \int_{G_m} f(x.y)dy \ .$$

Then, $\alpha$ is a continuous G-map: $C^\infty(G; A) \longrightarrow C^\infty(G/G_m; A)$.

Given $x \in M$, $Y_x \in T_x(M)$, choose $g \in G$ such that $x \in g.S$; put

$$\beta(f)(x, \ Y_x) = f(g)(x, \ Y_x) \ , \qquad (f \in C^\infty(G; A)) \ .$$

(This is well defined in view of 5.3(i).)

It is then immediately checked that $\beta$ maps $C^\infty(G/G_m; A)$ into $A$, and that $\mu = \beta \circ \alpha$ has the required properties.

We consider now the general case and let $\pi : M \longrightarrow G\backslash M$ be the canonical projection. Since $M$ is paracompact, and the action is proper, so is $G\backslash M$. In view of 5.2, we can find a countable subset $Q \subset M$ and a differentiable slice $S_m$ at $m \in Q$ such that the sets $\pi(S_m)$ $(m \in Q)$ form a locally finite open cover of $G\backslash M$. Then the sets $M_m = G.S_m$ form a locally finite open cover Ⓤ of $M$ by $G$-stable sets. By making use of a continuous partition of unity on $G\backslash M$ subordinated to the cover $\{\pi(S_m)\}_{m\in Q}$, we get first a continuous partition of unity on $M$ by $G$-invariant functions, subordinated to the cover Ⓤ. But then, using the bijection $C^\infty(M_m)^G \longrightarrow C^\infty(S_m)^{G_m}$ (cf. 5.2) we see that we can change it slightly to get a <u>smooth</u> partition of unity $(t_m)_{m\in Q}$ by $G$-invariant smooth functions subordinated to the cover Ⓤ. Then, the map $v : f \longmapsto (t_m f)_{m\in Q}$ is a continuous bijective $G$-map of $A^q(M; V)$ onto the direct product of the space $A^q(M_m; V)$ $(m \in Q)$. Since $Q$ is countable, the latter is Fréchet, hence $v$ is an isomorphism (1.2). Since each $A^q(M_m; V)$ is continuously s-injective by the first part of the proof, it follows that $A^q(M; V)$ also is.

5.5. PROPOSITION. <u>Let</u> $M$ <u>and</u> $V$ <u>be as in</u> 5.4. <u>Assume that</u> $M$ <u>is diffeomorphic to a euclidean space</u>. <u>Then</u> $0 \longrightarrow V \longrightarrow A^0(M; V) \longrightarrow A^1(M; V) \longrightarrow$ <u>is a continuously</u> s-<u>injective resolution</u> (1.5) <u>of</u> $V$ <u>by Fréchet modules in</u> $\underline{C}^\infty_G$.

We already know that each $A^q(M, V)$ is continuously s-injective (5.4).

Since $V$ is a Fréchet space (in fact quasi-complete would suffice) the usual proof of the Poincaré lemma in euclidean space (see e.g. [14: 4.18] works also for $V$-valued forms and provides, for each $q$, a continuous linear map $h_q : A^q(M; V) \longrightarrow A^{q-1}(M; V)$ such that $h_{q+1} \circ d + d \circ h_q = \mathrm{id}$. This implies that our sequence is a strong resolution.

Remark. In the case where $M = G/K$, this is proved in [10: p. 385-6].

5.6. COROLLARY. (i) $H_d^*(G; V)$ is isomorphic to $H^*(A^*(M; V)^G)$.

(ii) $H_d^*(G; V)$ is isomorphic to $H^*(\underline{g}, K; V)$.

(i) follows from 5.5 and the definitions (see 5.1).

Let now $M = G/K$. Then $A^*(M; V)^G = C^*(\underline{g}, K; V)$ (see I, 1.4), hence (ii) is a special case of (i).

Remark. (ii) is the well-known van Est theorem (see [4: Thm. 2], or [10: Thm. 6.1] where it is in fact stated under somewhat more general assumptions on $V$.

5.7. COROLLARY. Let $E$ be a finite dimensional $G$-module, and assume that $G$ operates trivially on $V$. Then $H_d^*(G; E \otimes V)$ is Hausdorff and isomorphic to $H_d^*(G; E) \otimes V$.

(In this statement, the tensor product of a finite dimensional vector space $E$ and of a topological vector space $F$ is endowed with the obvious topology, such that for any basis $(e_i)$ of $E$ $(1 \leqq i \leqq n = \dim E)$, the isomorphism of $E \otimes F$ onto the direct sum of $n$ copies of $F$ associated to $(e_i)$ is topological.)

Let $M = G/K$. Then $A^*(M; E \otimes V)$ defines an s-injective resolution of $E \otimes V$ (5.5) and we have

$$A^*(M; E \otimes V)^G = C^*(\underline{g}, \underline{k}; E \otimes V)^{K/K^o} = (\Lambda(\underline{g}/\underline{k})^* \otimes E \otimes V)^{K/K^o} ,$$

where $K^o$ is the identity component of $K$ (I, 1.4), hence, since $K/K^o$ acts trivially on $V$:

$$A^*(M; E \otimes V)^G = (\Lambda(\underline{g}/\underline{k})^* \otimes E)^{K/K^o} \otimes V = C^*(g, K; E) \otimes V .$$

Since $C^*(\underline{g}, K; E)$ is finite dimensional, it is then clear that $C^*(\underline{g}, K; E) \otimes V$ has Hausdorff cohomology. Since we started from a continuously s-injective resolution, this means, by definition (3.3), that $H^*_d(G; E \otimes V)$ is Hausdorff, and implies also that it is equal to $H^*(g, K; E) \otimes V$, i.e., to $H^*_d(G; E) \otimes V$.

5.8. THEOREM. Let $N$ be a closed normal subgroup of $G$ which has finitely many connected components. Let $E$ be a finite dimensional $G$-module and $V \in C^\infty_G$ a Fréchet differentiable $G$-module on which $N$ acts trivially. Then $H^*(N; E \otimes V)$ is Hausdorff, isomorphic to $H^*(N; E) \otimes V$, admits a natural structure of differentiable Fréchet $(G/N)$-module and there exists a spectral sequence $(E_r)$ abuting to $H^*(G; E \otimes V)$ and in which

$$E_2^{p,q} = H^p_d(G/N; H^q_d(N; E) \otimes V) , \qquad (p, q \in \mathbb{N}) .$$

Let $M = G/K$. Then $A^*(M; E \otimes V)$ provides a continuously s-injective resolution of $E \otimes V$ (5.5). The fibration of $G$ by $N$ has local cross sections, therefore $A^q(M; V)^N$ is continuously s-injective in $\underline{C}_N$ (2.3).

By 5.7, $H^*(N; E \otimes V)$ is Hausdorff. By 2.3 it is the cohomology of $C^{*N}$, where $C^*$ is a continuously s-injective resolution of $E \otimes V$ in $\underline{C}_G$. The $G/N$-module structure on $H^*(N; E \otimes V)$ then stems from the natural action of $G/N$ on $C^{*N}$ (1.11). Theorem 5.8 then follows from 5.7, applied to $N$, and $E \otimes V$, and from 4.3.

Remark. To determine the action of $G/N$ on $H^*(N; E \otimes V)$ we may use any continuously s-injective resolution with respect to $N$ of $V$ in $\underline{C}_G$ (1.11). In particular, take as resolution $A^*(X; E \otimes V)$, where $X$ is the space of maximal compact subgroups of $N$. Then

$$A^*(X; E \otimes V)^N \cong A^*(X; E)^N \otimes V \ , \tag{1}$$

as follows from the equality

$$A^*(X; E \otimes V)^N = C^*(\underline{u}, L; E \otimes V) = C^*(u, L; E) \otimes V \ ,$$

where $\underline{u}$ is the Lie algebra and $L$ a maximal compact subgroup of $N$. As a consequence, the action of $G/N$ on $H^*(N; E) \otimes V$ is the tensor product of its actions on the two factors.

5.9. Induced modules, Shapiro's lemma. Let $H$ be a closed subgroup of $G$ and $U \in \underline{C}_H^\infty$. The induced module in the differentiable category is the space

$$I^{\infty}(U) = \mathrm{Ind}_H^G(U)^{\infty} = \{f \in C^{\infty}(G; U) \mid f(gh) = h^{-1}f(g)(h \in H,\ g \in G)\} .$$

It is a differentiable G-module with respect to left translation and a Fréchet space if $U$ is one.

If we consider Fréchet modules, there is no difficulty in seeing that §2 remains true if continuous functions are replaced by smooth ones, and the compact open topology by the $C^{\infty}$ topology. One has only to use 1.2 and to remark that if $X, Y$ are manifolds, and $V$ a Fréchet space, then the canonical map $C^{\infty}(X, Y; V) \longrightarrow C^{\infty}(X; C^{\infty}(Y; V))$ is a continuous linear bijection of Fréchet spaces, hence an isomorphism.

## §6. Further results on differential cohomology.

To complete this discussion of differential cohomology, we prove the existence of a spectral sequence relating continuous cohomology and cohomology of invariant differential forms, which generalizes one of [5], and give a further relation between continuous and differentiable cohomology However, the results of this section will not be needed in the sequel.

6.1. We need some facts on topological tensor products, for which we refer to [7, 8]. If $E$, $F$ are topological vector spaces (we recall that only locally convex Hausdorff spaces are considered here) then $E \otimes F$ will be endowed with the "projective tensor product topology" [8: I, §1, n°3], and $E \overline{\otimes} F$ will denote the completion of $E \otimes F$ with respect to that topology. We recall that the latter is the finest locally convex topology such that the canonical bilinear map $E \times F \longrightarrow E \otimes F$ is continuous, where $E \times F$ is endowed with the product topology. If $u : E \longrightarrow E'$, $v : F \longrightarrow F'$ are continuous linear maps of topological vector spaces, then $u \otimes v$ is continuous. Its unique continuous extension to a map $E \overline{\otimes} F \longrightarrow E' \overline{\otimes} F'$ is denoted $u \overline{\otimes} v$.

6.1.1. If $E$ and $F$ are Fréchet spaces, then so is $E \overline{\otimes} F$ [8: I, §1, n°3, Prop. 5]. If $E$ is finite dimensional then $E \overline{\otimes} F = E \otimes F$, and the topology is the one used in 5.7.

6.1.2. Let $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ be an exact sequence of Fréchet spaces and $F$ a Fréchet space. If either $E$ or $F$ is nuclear [8: II, §2, n°1], then

$$0 \longrightarrow E' \overline{\otimes} F \xrightarrow{u \overline{\otimes} 1} E \overline{\otimes} F \xrightarrow{v \overline{\otimes} 1} E'' \overline{\otimes} F \longrightarrow 0 ,$$

is exact. Without nuclearity assumption, it follows from [8: I, §1, n°2, Prop. 3] that $u \overline{\otimes} 1$ is injective, $v \overline{\otimes} 1$ is surjective and that $\mathrm{Im}\,(u \overline{\otimes} 1) \subset \ker\,(v \overline{\otimes} 1)$. The equality $\mathrm{Im}\,(u \overline{\otimes} 1) = \ker\,(v \overline{\otimes} 1)$, when either $E$ or $F$ is nuclear, follows from the cor. to Prop. 10 in [8: II, §3, n°1]. Note that if $E$ is nuclear, then so are $E'$ and $E''$ [7: II, Thm. 3].

6.1.3. Let $F$ be a Fréchet space. If $C^* : C^0 \longrightarrow C^1 \longrightarrow \ldots$ is a complex of nuclear Fréchet spaces with Hausdorff cohomology (3.1), then so is $C^* \overline{\otimes} F : C^0 \overline{\otimes} F \longrightarrow C^1 \overline{\otimes} F \longrightarrow \ldots$, and we have $H^*(C^* \overline{\otimes} F) = H^*(C^*) \overline{\otimes} F$.

This follows from 6.1.2 by splitting $C^*$ into short exact sequences, and using 1.2. It follows also that if $C^*$ is acyclic, then so is $C^* \overline{\otimes} F$.

6.1.4. Let $M$ be a smooth manifold, $E$ a finite dimensional real or complex vector space, and $q \in \mathbb{N}$. Then $A^q(M; E)$ is a nuclear Fréchet space [8: II, §2, n°3, Thm. 10]. If $V$ is a Fréchet space, then Exemple 1 in [8: II, §3, n°3] implies that the natural map

$$A^q(M; E) \overline{\otimes} V \longrightarrow A^q(M; E \otimes V) \tag{1}$$

is an isomorphism.

6.1.5. Let $M$ be a smooth manifold. Then the assignment $V \longmapsto C^\infty(M; V)$ is an exact functor from Fréchet spaces to Fréchet spaces.

By 1.2, it suffices to prove that if

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0 ,$$

is an exact sequence of Fréchet spaces, then

$$0 \longrightarrow C^\infty(M; V') \longrightarrow C^\infty(M; V) \longrightarrow C^\infty(M; V'') \longrightarrow 0 ,$$

is exact. This follows from 6.1.2 (with $F = C^\infty(M)$) and 6.1.4.

6.2. PROPOSITION. Let $M$ be a manifold with finite dimensional real cohomology. Let $V$ be a Fréchet space and $E$ a finite dimensional real vector space. Then $A^*(M; V \otimes E)$ has Hausdorff cohomology and we have

(1) $H^*(M; E) \otimes V = H^*(M; E \otimes V) = H^*(A^*(M; E \otimes V)) = H^*(A^*(M; E)) \otimes V$ .

(The main point here is the second equality, which says that de Rham theorem is valid for forms with values in a Fréchet space.)

The first equality follows from the universal coefficient theorem and the finite dimensionality of $H^*(M; \mathbb{R})$.

The space $H^*(M; E) = H^*(M; \mathbb{R}) \otimes E$ is finite dimensional, therefore $A^*(M; E) = A^*(M) \otimes E$ is a complex of Fréchet spaces with Hausdorff cohomology (3.5). By 6.1(3), so is the complex $A^*(M; E) \overline{\otimes} V$; this proves our first assertion, and shows that we have $H^*(A(M; E) \overline{\otimes} V) = H^*(A(M; E)) \otimes V$. Of course, $H(A^*(M; E)) = H^*(M; E)$ by the usual de Rham theorem, hence the first and fourth terms of (1) are equal. Finally, it follows from 6.1.4 that we have

$$A^q(M; E \otimes V) = A^q(M) \overline{\otimes} (E \otimes V) = (A^q(M) \otimes E) \overline{\otimes} V = A^q(M; E) \overline{\otimes} V , \qquad (q \in \mathbb{N}) ,$$

whence the last equality in 1).

6.3. THEOREM (G. D. Mostow). Let $M$ be a manifold with finite dimensional real cohomology, on which $G$ operates smoothly and properly. Let $V$ be a Fréchet $G$-module. Then there exists a spectral sequence $(E_r)$ which abuts to $H^*(A(M; V)^G)$ and in which

$$E_2^{p,q} = H_d^p(G; H^q(M; \mathbb{R}) \otimes V) \qquad (p, q \in \mathbb{N}) .$$

We consider the sequence

$$(1) \qquad 0 \longrightarrow V \longrightarrow A^{o}(M; V) \longrightarrow A^{1}(M; V) \longrightarrow \dots \ .$$

It is an augmented complex of Fréchet spaces, with Hausdorff cohomology (6.2). Each $A^{q}(M; V)$ is a continuously s-injective G-module (5.4), in particular it is G-acyclic. Taking 6.2 into account, we see that the spectral sequence of 4.1 has the required properties.

Remark. If $G$ is connected and $M$ is a smooth principal $G$-bundle, this result is due to van Est [5: Thm. 4].

6.4. COROLLARY. Assume that $M$ is acyclic over $\mathbb{R}$. Then

$$H_{d}^{*}(G; V) = H^{*}(A^{*}(M; V)^{G}) \ .$$

In fact, the complex $A^{*}(M; V)$ is also acyclic (6.2), hence 6.3(1) yields a resolution of $V$ by a complex of $G$-acyclic Fréchet spaces. We may apply 4.2 and 5.1, or remark that we have $E_{2}^{p,q} = 0$ for $q \neq 0$ in the spectral sequence of 6.3.

6.5. Given $V \in \underline{C}_{G}$ we let $V^{\infty}$ be the subspace of $C^{\infty}$ vectors of $V$, endowed with the usual topology induced from that of $C^{\infty}(G; E)$ (see [13: 4.4]). It is a smooth $G$-module. The injection $V^{\infty} \longrightarrow V$ is continuous with dense image. If $V$ is a Fréchet space, so is $V^{\infty}$. If $U, V \in \underline{C}_{G}$ and $f : U \longrightarrow V$ is a $G$-morphism, then $f$ induces a $G$-morphism $f_{\infty} : U^{\infty} \longrightarrow V^{\infty}$.

6.6. LEMMA. Let $V \in \underline{C}_G$.

(i) If $V$ is s-injective in $\underline{C}_G$, then $V^\infty$ is s-injective in $\underline{C}_G^\infty$.

(ii) If $V$ is quasi-complete, differentiable, and s-injective in $\underline{C}_G^\infty$, then $V$ is continuously s-injective in $\underline{C}_G$.

(iii) The functor $V \longmapsto V^\infty$ is exact in the category of Fréchet $G$-modules.

Proof. (i) Let $u : A \longrightarrow B$ be a strong injection of differentiable Fréchet G-modules, and $f : A \longrightarrow V^\infty$ a G-morphism. By assumption, $f$ extends to a G-morphism $g : B \longrightarrow V$. The latter induces a continuous G-morphism $g_\infty : B^\infty \longrightarrow V^\infty$. Since $B$ is differentiable, we have $B = B^\infty$ set theoretically and topologically by definition [13: p. 259]. Hence $\operatorname{Im} g \subset V^\infty$ and $g$, viewed as a map of $B$ into $V^\infty$, where $V^\infty$ is endowed with its topology of differentiable G-modules (which is finer than the topology induced from $V$) is also continuous.

(ii) Since $V$ is s-injective in $\underline{C}_G^\infty$, it is a topological direct G-summand in any differentiable G-module containing it, in particular in $C^\infty(G; V)$. Our assertion then follows from 5.2.

(iii) In view of 1.2, one sees that it suffices to show that $V \longmapsto V^\infty$ preserves short exact sequences, and that the only non-obvious part is the exactness on the right, i.e. if $f : U \longrightarrow V$ is a surjective G-morphism of Fréchet G-modules, then $f_\infty : U^\infty \longrightarrow V^\infty$ is also surjective. But this is proved in [13] (see 4.4.1.11, p. 260 in [13], taking into account that $f$ induces an isomorphism of $U/\ker u$ onto $V$).

6.7. PROPOSITION (P. Blanc, P. Delorme). Let $V$ be a Fréchet $G$-module. Then the natural map $H^q_{ct}(G; V^\infty) \longrightarrow H^q_{ct}(G; V)$ is an isomorphism $(q \in \mathbb{N})$.

Let $0 \longrightarrow V \longrightarrow F^o \longrightarrow F^1 \longrightarrow \ldots$ be an s-injective resolution of $V$ by Fréchet $G$-modules (see 1.5). Then by 6.6, $0 \longrightarrow V^\infty \longrightarrow F^{o\infty} \longrightarrow F^{1\infty} \longrightarrow \ldots$ is a resolution of $V^\infty$ by Fréchet modules, which are s-injective in $\underline{C}^\infty_G$, and in particular acyclic. (It is not necessarily s-exact, though). By 4.2 and 5.2, $H_d(G; V^\infty)$ is the cohomology of the complex $\{F^{\infty q^G}\}$. But $F^{q^G} \subset F^{\infty q}$, hence $F^{q^G} = F^{\infty q^G}$ $(q \in \mathbb{N})$, and therefore $H^*_{ct}(G; V)$ is the cohomology of the same complex as $H^*_d(G; V^\infty)$. In view of 5.1, this proves our assertion.

Remark. This result is due to P. Blanc and P. Delorme, and was pointed out to me some months ago by P. Delorme. Both he and P. Blanc had proved it independently. It may be that their proof, which I do not know, is valid for a wider class of $G$-modules.

## §7. Lie algebra cohomology.

In this section we sketch a proof of the existence of a Hochschild-Serre spectral sequence in relative Lie algebra cohomology. We shall limit ourselves to our main case of interest, that of $(\underline{g}, K)$-modules over $\mathbb{R}$, but there are obvious variations in other contexts (see 7.8). In the final version of these Notes, this will be expanded and incorporated in Chapter I.

7.1. The category of $(\underline{g}, \underline{k}, L)$-modules. Let $\underline{g}$ be a Lie algebra over $\mathbb{R}$, $\underline{k}$ a subalgebra reductive in $\underline{g}$. Let $L$ be a compact Lie group, whose Lie algebra $\underline{\ell}$ contains an ideal isomorphic to $\underline{k}$ (also to be denoted $\underline{k}$), $\alpha : L \longrightarrow \mathrm{Aut}\, \underline{g}$ a continuous representation of $L$ in $\mathrm{Aut}\, \underline{g}$ by automorphisms which leave $\underline{k}$ stable.

Let $K$ be the analytic subgroup of $L$ with Lie algebra $\underline{k}$.

A real vector space $V$ is a $(\underline{g}, \underline{k}, L)$-module if the following conditions are fulfilled:

(i) $\underline{g}$, hence $U(\underline{g})$, and $L$ operate on $V$. With respect to $L$, the space $V$ is locally finite and semi-simple. The representation of $L$ on any finite dimensional $L$-stable subspace is differentiable.

(ii) $L$ is a group of operators for the $U(\underline{g})$-module structure, i.e.

$$x(u.v) = x(u).x(v)\ , \qquad (x \in L;\ u \in U(\underline{g});\ v \in V)\ .$$

(iii) Any finite dimensional $K$-stable subspace $M$ of $V$ is stable under $\underline{k}$, and the differential of the representation of $K$ in $M$ is the representation of $\underline{k}$ obtained by restriction of the representation of $\underline{g}$.

Thus, $V$ is a $(\underline{g}, K)$-module (I, §5) with an additional group of operators $L$. We let $\underline{C}_{\underline{g}, \underline{k}, L}$ be the category of $(\underline{g}, \underline{k}, L)$-modules, the morphisms being the linear maps commuting with both $\underline{g}$ and $L$. It is a subcategory of $\underline{C}_{\underline{g}, K}$.

7.2. Cohomology spaces. The complex $C^*(\underline{g}, \underline{k}; V) = \mathrm{Hom}_{\underline{k}}(\Lambda(\underline{g}/\underline{k}), V)$ has a natural $L$-module structure, stemming from the actions on $\underline{g}/\underline{k}$ and $V$, which commutes with the differentials, whence a $L$-module structure on $H^*(\underline{g}, \underline{k}; V)$, with respect to which this space is locally finite and semi-simple. Furthermore, we define $H^q(\underline{g}, L; V)$ to be the $q$-th cohomology space of the complex $\mathrm{Hom}_L(\Lambda(\underline{g}/\underline{k}), V)$, $(q = 0, 1, 2, \ldots)$. Since the $L$ action is semi-simple, taking fixed points is an exact functor hence

$$H^q(\underline{g}, L; V) = H^q(\underline{g}, \underline{k}; V)^{L/K} \quad (1)$$

The case considered in I, §5 is the one where $K$ is open in $L$. However, our main reason for introducing this greater generality is to be able to consider also the case where $\underline{k} = (0)$.

7.3. Ext functors. In $\underline{C}_{\underline{g}, \underline{k}, L}$ we may consider projective and injective modules, and derived functors of $\mathrm{Hom}_{\underline{g}, \underline{k}}$ and of $\mathrm{Hom}_{\underline{g}, L}$. Let again $R = U(\underline{g})$, $S = U(\underline{k})$. The actions of $L$ on $\underline{g}$ and $\underline{k}$ extend to representations of $L$ in $R$ and $S$, with respect to which these are locally finite and semi-simple $L$-modules. The argument of I, 2.4 shows that if $V \in \underline{C}_{\underline{g}, \underline{k}, L}$, then

$I(V) = R \otimes_S V$, endowed with the L-module structure given by the tensor product of the actions on the two factors, is a projective $(\underline{g}, \underline{k}, L)$-module. It follows that there is at least one projective resolution of $V$ in $\underline{C}_{\underline{g}, \underline{k}, L}$ which is at the same time a projective resolution in $\underline{C}_{\underline{g}, \underline{k}}$. In fact, the projective resolution $\{X_q\}$ of the groundfield given in I, 2.5 is one in $\underline{C}_{\underline{g}, \underline{k}, L}$. Consequently, the derived functors of $\mathrm{Hom}_{\underline{g}}$ in $\underline{C}_{\underline{g}, \underline{k}, L}$ are the same as in $\underline{C}_{\underline{g}, \underline{k}}$; but they are endowed moreover with a canonical structure of locally finite and semi-simple L-module, which may be defined from the action of $L$ on any projective resolution in $\underline{C}_{\underline{g}, \underline{k}, L}$; the standard arguments show it to be independent of the resolution (as in 1.11).

As in I, 2.5, let $P^o(V) = \mathrm{Hom}_S(R, V)$, where $V \in \underline{C}_{\underline{g}, \underline{k}, L}$. It is a L-module in the obvious way. Let $P(V) = \mathrm{Hom}_S(R, V)_{(L)}$ be the space of L-finite vectors. By an argument similar to the one of I, 2.5, one sees that the representation of $L$ on any finite dimensional L-stable subspace is differentiable, and therefore semi-simple. Thus $P(V) \in \underline{C}_{\underline{g}, \underline{k}, L}$ and is again injective. Hence there are injective resolutions of $V$ in $\underline{C}_{\underline{g}, \underline{k}, L}$ which are injective resolutions in $\underline{C}_{\underline{g}, \underline{k}}$.

We denote again by $\mathrm{Ext}_{\underline{g}, \underline{k}}$ the derived functors of $\mathrm{Hom}_{\underline{g}}$ in $\underline{C}_{\underline{g}, \underline{k}, L}$, and let moreover $\mathrm{Ext}_{\underline{g}, L}$ be the derived functors of $\mathrm{Hom}_{\underline{g}, L}$ in that category. If $U, V \in \underline{C}_{\underline{g}, \underline{k}, L}$, and if $0 \longrightarrow V \longrightarrow C^o \longrightarrow \dots$ is an injective resolution of $V$ in $\underline{C}_{\underline{g}, \underline{k}, L}$, then $\mathrm{Ext}_{\underline{g}, \underline{k}}(U, V)$ is again the q-cohomology of $\mathrm{Hom}_{\underline{g}}(U, C^q)$, while $\mathrm{Ext}^q_{\underline{g}, L}(U, V)$ is the q-th cohomology space of the complex $\{\mathrm{Hom}_{\underline{g}, L}(U, C^i)\}$.

7.4. LEMMA. *Let* $\underline{n}$ *be an ideal of* $\underline{g}$ *which is stable under* $L$. *Let* $V$ *be an injective* $(\underline{g}, \underline{k}, L)$*-module. Then* $V$ *is also injective as a* $(\underline{n}, \underline{k} \cap \underline{n}, L)$*-module.*

There are $L$-invariant subspaces $\underline{m}$, $\underline{m}'$ of $\underline{g}$ such that $\underline{g} = \underline{n} \oplus \underline{m}$ and $\underline{m} = \underline{m}' \oplus \underline{k}_2$. Using the Poincaré-Birkhoff-Witt theorem, we see that we can write

$$(1) \qquad R = U(\underline{n}) \otimes M, \quad M = M' \otimes U(\underline{k}_2) \ ,$$

with $M$ and $M'$ stable under $L$. Also, $M$ is invariant under right translations by $U(\underline{k}_2)$. By the so-called adjoint associativity between Hom and $\otimes$ (see e.g. [8: VI, (8.7)]), we have

$$(2) \qquad \mathrm{Hom}_S(R, U) = \mathrm{Hom}_{S_1}(U(\underline{n}), \mathrm{Hom}_{S_2}(M, U)) \ ,$$

where $S_1$ acts on $U(\underline{n})$ by left translations, $S_2$ acts on $M$ by right translations, and $S_1$ acts on $\mathrm{Hom}_{S_2}(M, U)$ by the given action on $U$ (this is compatible with the $S_2$-action since $S_1$ and $S_2$ commute). Moreover, this isomorphism is compatible with the natural operations of $L$, whence an isomorphism

$$(3) \qquad \mathrm{Hom}_S(R, U)_{(L)} = \mathrm{Hom}_{S_1}(U(\underline{n}), \mathrm{Hom}_{S_2}(M, U)))_{(L)} \ .$$

Thus $\mathrm{Hom}_S(R, U)_{(L)}$ can be written in $\underline{C}_{\underline{n}, \underline{k}_1, L}$ in the form $P(U')$, for some $U' \in \underline{C}_{\underline{n}, \underline{k}_1, L'}$ hence it is injective in that category.

7.5. THEOREM. *Let $\underline{n}$ be an ideal in $\underline{g}$ stable under $L$, $k_1 = \underline{k} \cap \underline{n}$ and $V \in \underline{C}_{\underline{g}, \underline{k}, L}$. Then there exists a spectral sequence which abuts to $H^*(\underline{g}, \underline{k}; V)$, in which $E_2^{p,q} = H^p(\underline{g}/\underline{n}, \underline{k}/\underline{k}_1; H^q(\underline{n}, \underline{k}_1; V))$, and a spectral sequence which abuts to $H^*(\underline{g}, L; V)$ and in which $E_2^{p,q} = H^p(\underline{g}/\underline{n}, L; H^q(\underline{n}, \underline{k}_1; V))$.*

The argument is the standard one. Start from an injective resolution

$$0 \longrightarrow V \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \tag{1}$$

of $V$ in $\underline{C}_{\underline{g}, \underline{k}, L}$ and consider the complex $C^{*\underline{n}} = \{C^{i\underline{n}}\}$ of $\underline{n}$-fixed elements. By 7.4, (1) is also an injective resolution of $V$ in $\underline{C}_{\underline{n}, k_1, L'}$ therefore,

$$H^q(C^{*\underline{n}}) = H^q(\underline{n}, \underline{k}_1; V) \ . \tag{2}$$

The complex $C^{*\underline{n}}$ is a complex of $(\underline{g}/\underline{n}, \underline{k}/\underline{k}_1, L/K_1)$-modules, where $K_1$ is the analytic subgroup of $L$ with Lie algebra $K_1$, whence a natural structure of $(\underline{g}/\underline{n}, \underline{k}/\underline{k}_1, L/K_1)$-module on the right hand side of (2). It follows immediately from the definitions that if $M$ is injective in $\underline{C}_{\underline{g}, \underline{k}, L'}$ then $M^{\underline{n}}$ is injective in $\underline{C}_{\underline{g}/\underline{n}, \underline{k}/\underline{k}_1, L/K_1}$. Thus the $C^{i\underline{n}}$'s are injective in the latter category. In particular they are acyclic. The standard double complex argument used in 4.1 then yields the existence of a spectral sequence abuting to $H^*((\underline{C}^{*\underline{n}})^{\underline{g}/\underline{n}})$, in which

$$E_2^{p,q} = H^*(\underline{g}/\underline{n}, \underline{k}/\underline{k}_1; H^q(C^{*\underline{n}})) \ . \tag{3}$$

But $(C^{\underline{in}})^{\underline{g}/\underline{n}} = C^{i\underline{g}}$, hence $H^*((C^{\underline{n}})^{\underline{g}/\underline{n}}) = H^*(\underline{g}, \underline{k}; V)$. In view of (2), we get the first spectral sequence mentioned in our theorem. Furthermore, it is clear that $L$ acts as a group of operators on the whole situation, and that the $E_r$'s are locally finite and semi-simple $L$-modules. The second spectral sequence is then obtained by taking $L$-invariants.

7.6. In the next chapter, we shall use 5.8 and also 5.9 for induced representation of a semi-simple group induced from a parabolic subgroup. 7.5 supplies an algebraic analogue of 5.8. If we want to remain in an algebraic context, we should also have a substitute for 5.9 and an essentially algebraic description of $K$-finite vectors in an induced representation. We now state without proof a result due to N. Wallach which supplies this.

We take the notation of VI. Let $(P, A)$ be a standard p-pair. Write $P = M.N$, where $M$ is a Levi subgroup and $N$ the unipotent radical of $P$. Let $(\sigma, H)$ be an admissible irreducible representation of $M$ on a Hilbert space. Let

$$I_\sigma = \{f \in C^\infty(G, H) \,|\, f(g.p) = \sigma(p)^{-1} f(g), \quad (g \in G, \; p \in P)\}$$

be the representation induced from $\sigma$, and $I_{\sigma,(K)}$ the $(\underline{g}, K)$-module of $K$-finite vectors in $I_\sigma$. On the other hand, consider

$$U_o = \mathrm{Hom}_{U(\underline{p})}(U(\underline{g}), H_o) = \{f : U(\underline{g}) \longrightarrow H_o \,|\, f(u.p) = \check{\sigma}(p).f(u), \quad (u \in U(\underline{g});\; p \in U(\underline{p}))\},$$

where $H_o$ is the space of $(K \cap M)$-finite vectors in $H$, $\underline{p}$ the Lie algebra of P, and $^\vee$ the standard involution (induced from $X \longmapsto -X$, $X \in \underline{p}$). Let $U_1$ be the space of $\underline{k}$-finite vectors in $U_o$. The representation of $\underline{k}$ in $U_1$ integrates to one of the universal covering $\widetilde{K}$ of K. We now want to single out a subspace on which this representation of $\widetilde{K}$ goes down to one of K. The center Z(G) is contained in the center Z(M) of M. Since $(\sigma, H)$ is irreducible, there exists a character $\chi$ of Z(G) (in fact of Z(M)) such that $\sigma(z) = \chi(z).\mathrm{Id}$. for $z \in Z(G)$. Let $\widetilde{G}$ be the universal covering group of G. It contains $\widetilde{K}$, its center $Z(\widetilde{G})$ is contained in $\widetilde{K}$ and maps onto Z(G) under the natural projection $\widetilde{G} \longrightarrow G$. Therefore $\chi$ may be viewed as a character of $Z(\widetilde{G})$, whose kernel contains the kernel of $\tau : \widetilde{K} \longrightarrow K$. We let then

$$U_2 = \{f \in U_1 \mid z.f = \chi(z)^{-1} f, \qquad (z \in Z(\widetilde{G}))\} .$$

Then $U_2$ is a $(\underline{g}, K)$-module. Finally, let

$$U = \{x \in U_2 \mid f(k.x) = \sigma(k)^{-1}.x, \qquad (x \in U(\underline{g}),\ k \in K \cap M)\} .$$

Given $f \in I_{(\sigma)}$, let $Tf : U(\underline{g}) \longrightarrow H_o$ be defined by

$$Tf(x) = x.f(1), \qquad (x \in U(\underline{g})) .$$

It is readily seen that T maps $I_{\sigma,(K)}$ into U and commutes with $\underline{g}$ and K. We have then

<u>7.7</u>. PROPOSITION (N. Wallach). (i) <u>The map</u> T <u>is an isomorphism of</u> $(\underline{g}, K)$-<u>modules</u>.

(ii) $H^*(\underline{p}, K \cap P; H_o) \xrightarrow{\sim} H^*(\underline{g}, K; U)$.

A proof will be given in the final version of the Notes.

7.8. The point of 7.3 (also of 1.11) is that one can under suitable assumptions, without changing the cohomology, impose additional structures on the modules under consideration, which are inherited by the cohomology groups. There are of course many other instances of that. For instance in the situation of I, 2.2, 2.4, let $\underline{\ell}$ be a Lie algebra of derivations of $\underline{g}$ leaving $\underline{k}$ stable, under which $\underline{g}$ is fully reducible. Let us define a $(\underline{g}, \underline{k}, \underline{\ell})$-module $V$ to be a $(\underline{g}, \underline{k})$-module and a $\underline{\ell}$-module, which is locally finite and semi-simple with respect to $\underline{\ell}$, such that $x(y.v) = (x.y).z + y.(x.z)$ $(x \in \underline{\ell},\ y \in \underline{g},\ v \in V)$. Again the derived functors of $\mathrm{Hom}_{\underline{g}}$ in the category of $(\underline{g}, \underline{k}, \underline{\ell})$-modules are the same as in $\underline{C}_{\underline{g}, \underline{k}}$, but are $\underline{\ell}$-modules in a natural way. Of course, $\underline{C}_{\underline{g}, \underline{k}}$ may be identified with $\underline{C}_{\underline{g}, \underline{k}, \underline{k}}$. Using this, one deduces, exactly as in 7.5, the existence of a Hochschild-Serre spectral sequence in $\underline{C}_{\underline{g}, \underline{k}}$.

## REFERENCES

[1] N. Bourbaki, Topologie générale, Chap. 1 à 10, 2 vol., Hermann, Paris.

[2] ————, Espaces vectoriels topologiques, Chap. 1 à 5, 2 vol., Act. Aci. Ind. 1189, 1229, Hermann, Paris.

[3] W. Casselman and D. Wigner, "Continuous cohomology and a conjecture of Serre's," Inv. Math. 25 (1974), 199-211.

[4] W. T. van Est, On the algebraic concepts in Lie groups I, II, Proc. Koninkl. Ned. Ak. v. Wet.-Amsterdam, Series A, 58 (1955), 225-233, 286-294.

[5] ————, "A generalization of the Cartan-Leray spectral sequence I, II," ibid. 61 (1958), 399-413.

[6] R. Godement, Théorie des Faisceaux, Act. Sci. Ind. 1252, Hermann, Paris 1958.

[7] A. Grothendieck, "Résumé des résultats essentiels dans la théorie des produits tensoriels topologiques et des espaces nucléaires," Annales Inst. Fourier Grenoble 4 (1952), 49-112.

[8] ————, "Produits tensoriels topologiques et espaces nucléaires," Memoir Amer. Math. Soc. 16 (1955), 331 p.

[9] G. Hochschild, "Relative homological algebra," Trans. Amer. Math. Soc. 82 (1956), 246-269.

[10] ———— and G. D. Mostow, "Cohomology of groups," Ill. Jour. Math. 6 (1962), 367-401.

[11] S. Mac Lane, "Homology," Grund. Math. Wiss. 114, Springer, 1963.

[12] R. Palais, "On the existence of slices for actions of non-compact Lie groups," Annals of Math. (2) 73 (1961), 295-323.

[13] G. Warner, Harmonic analysis on semi-simple Lie groups I, Grund. Math. Wiss. 188, Springer, Berlin 1972.

[14] S. Warner, "Foundations of differentiable manifolds and Lie groups," Scott, Foresman and Co., 1971.

# X. COHOMOLOGY WITH RESPECT TO AN INDUCED REPRESENTATION

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This chapter is mainly devoted to the computation of $H_d^*(G; V \otimes F)$, where $G$ is semi-simple, $F$ is a finite dimensional representation of $G$ and $V$ is induced from a representation of a parabolic subgroup $P$ of $G$. It is expressed essentially in terms of the cohomology of a Levi subgroup $M$ of $P$ with respect to the tensor product of the representation of $M$ induced from and of a suitable finite dimensional representation of $M$ (3.4). Together with the results of VIII, §5, this yields an essentially complete description of the cohomology space $H_d^*(G; V \otimes F)$ when $V$ is tempered (5.2). In particular it is concentrated in an interval of length $\ell_o(G) = \mathrm{rk}\, G - \mathrm{rk}\, K$ ($K$ a maximal compact subgroup) around $(\dim G/K)/2$, and is zero if $P$ is not fundamental (in the sense of 4.1). If $V$ is induced from a tempered irreducible representation, then the cohomology is zero outside an interval of length at most the $\mathbb{R}$-rank of $G$ (6.1).

After this work was done I was informed that G. Zuckerman had obtained independently similar results. My own starting point was a formula proved by P. Delorme and describing the cohomology of complex semi-simple groups with coefficients in certain degenerate principal series. I thank him very much for having communicated it to me.

## §1. General remarks

1.1. If $G$ is a Lie group with finitely many connected components and $(\pi, V)$ a differentiable $G$-module, then the continuous cohomology and differential cohomology of $G$ with respect to $V$ are canonically isomorphic (IX, 5.2) and are also isomorphic to the relative Lie algebra cohomology with coefficients in $V$ or in the module of K-finite vectors of $V$ (IX, 5.6; I, §5). In the sequel, to fix the ideas, we usually consider differentiable cohomology, but implicitely use these isomorphisms when referring to a result stated in another context.

1.2. If $V$ is a $G$-module and $G'$ a finite covering of $G$, then the Hochschild-Serre spectral sequence shows readily that $H^*(G;V) = H^*(G';V)$.

1.3. Let $G$ be connected semi-simple, with finite center, $V$ an irreducible admissible differentiable $G$-module and $F$ a finite dimensional irreducible $G$-module. Then $H^*_d(G; V\otimes F) = 0$ if the central character of $V$ is not equal to that of the contragredient module to $F$ (I, 5.3). If that is the case, then $G \to \mathrm{Aut}\, V$ factors through a linear quotient of $G$. In view of 1.2, this means that, in computing differentiable cohomology of $G$, we may without loss of generality assume $G$ to be linear, or even to be embedded as a real form in a simply connected complex semi-simple group, as we shall usually do.

1.4. For induction purposes, it would be more convenient to use a broader class of groups, e.g. the one introduced in [3: §3]. Since we need not go beyond linear groups, a sufficiently general and "hereditary" notion is that of a "reductive linear group of connected type": an open subgroup $L$ of the group $\underline{L}(\mathbb{R})$ of real points of an algebraic group $\underline{L}$ defined over $\mathbb{R}$, whose identity component $\underline{L}^{o}$ is reductive, such that $\mathrm{Ad}\, L \subset \mathrm{Ad}\, \underline{L}^{o}$ (this is the condition "of connected type"). We shall occasionally use it. We note that the notions of Cartan involution and Cartan decomposition are well defined for such groups [1: 1.6], or [3: §3]. The $\mathbb{R}$-rank $\mathrm{rk}_{\mathbb{R}}\, L$ is by definition the $\mathbb{R}$-rank of $\underline{L}^{o}$, and is also equal to the common dimension of the maximal commutative $\mathbb{R}$-split subalgebras of the Lie algebra of $L$.

## §2. Notation and conventions

2.1. Unless otherwise stated, the Lie algebra of a real Lie group $G, A, K, \ldots$ is denoted by the underlined lower case letter, and the subscript $_c$ denotes complexification. If $G$ is a topological group, then $G^{o}$ is the connected component of the identity of $G$.

2.2. In the sequel, $G$ denotes a connected linear semi-simple Lie group whose complexification $G_c$ is simply connected. Then $G_c$ may be viewed as an algebraic group over $\mathbb{R}$, whose set of real points is $G$. For the standard notions on parabolic subgroups used below see e.g. [5: 1.2.4] or [2; 3].

Let $(P, A)$ be a p-pair. We have the decompositions

(1) $$P = M.N = {}^oM.A.N ,$$

where $N = R_uP$ is the unipotent radical of $P$, $M = Z_G(A)$ the centralizer in $G$ of $A$ and a Levi subgroup of $P$, and

$${}^oM = \bigcap_{\chi \in X(M)_{\mathbb{R}}} |\chi| ,$$

where $X(M)_{\mathbb{R}}$ is the group of morphisms of $M$ into $\mathbb{R}^*$.

If more precision seems required, we shall write ${}^oM_P, A_P, M_P, N_P$ for ${}^oM, A, N, P$. The decomposition $P = M.N$ is the standard Levi decomposition of $(P, A)$.

We recall that ${}^oM$ contains a finite group $F$, which is a direct product of groups of order two, such that ${}^oM$ is the semi-direct product of ${}^oM^o$ and $F$, (a direct product if $P$ is minimal). This follows immediately from [2: 14.4].

2.3. Fix a maximal connected commutative $\mathbb{R}$-split subgroup $A_o$ of $G$. A p-pair $(P, A)$ is said to be semi-standard if $A \subset A_o$.

We fix a Cartan subalgebra $\underline{h}$ of $\underline{g}$ contains $\underline{a}_o$ and let $H = Z_G(\underline{h})$ be the corresponding Cartan subgroup. If $(P, A)$ is semi-standard, then

(1) $$\underline{h} = \underline{b} \oplus \underline{a}, \quad \text{where } \underline{b} = \underline{b}_P = \underline{h} \cap {}^o\underline{m} ,$$

and $\underline{b}$ is a Cartan subalgebra of ${}^o\underline{m}$. We also have

(2) $H = B \times A$, where $B = {}^{o}M \cap H$ is a Cartan subgroup of ${}^{o}M$.

We have then a canonical isomorphism

(3) $$\underline{h}_c^* = \underline{b}_c^* + \underline{a}_c^*,$$

where $\underline{b}_c^*$ (resp. $\underline{a}_c^*$) is identified to the space of linear forms on $\underline{h}_o$ which are zero on $\underline{a}$ (resp. $\underline{b}$).

2.4. Let $\Phi = \Phi(\underline{g}_c, \underline{h}_c)$ (resp. ${}_{\mathbb{R}}\Phi = \Phi(\underline{g}_c, \underline{a}_{oc})$) be the set of roots of $\underline{g}_c$ with respect to $\underline{h}_c$ (resp. $\underline{a}_{oc}$). Its elements will also be viewed as roots of $G_c$ with respect to $H$ (resp. $A$), i.e. we make no distinction between a "global" root and its differential at the origin. The elements of ${}_{\mathbb{R}}\Phi$ are the $\mathbb{R}$-roots. The value of a root $\alpha$ on an element a is denoted $\alpha(a)$ or $a^{\alpha}$. If $(P, A)$ is a p-pair, then $\Phi(P, A)$ is the set of roots of $P$ with respect to $A$, i.e. the characters of $A$ in $\underline{n}$ with respect to the adjoint action, and $\Delta(P, A)$ the set of "simple" elements in $\Phi(P, A)$. We recall that $\Delta(P, A)$ is a basis of $\underline{a}^*$ and that every element in $\Phi(P, A)$ is linear combination with coefficients in $\mathbb{N}$ of elements in $\Delta(P, A)$. The dimension of $A$ is the parabolic rank prk $P$ of $P$. As usual we let $\rho_P \in \underline{a}^*$ be defined by

(1) $$\rho_P(a) = (1/2) \det \mathrm{Ad}\, a|_{\underline{n}_P}, \qquad (a \in A).$$

If an ordering on $\Phi$ (resp. ${}_{\mathbb{R}}\Phi$) is fixed, then $\Delta$ (resp. ${}_{\mathbb{R}}\Delta$) denotes the set of simple roots (resp. $\mathbb{R}$-roots) and $\Phi^+$ (resp. ${}_{\mathbb{R}}\Phi^+$) the set of positive roots (resp. $\mathbb{R}$-roots). Orderings on $\Phi$ and ${}_{\mathbb{R}}\Phi$ are compatible if the restriction of a positive element is positive. The choice of an ordering on ${}_{\mathbb{R}}\Phi$ is equivalent to that of a minimal parabolic subgroup $P_o \supset A_o$ and then ${}_{\mathbb{R}}\Phi^+ = \Phi(P_o, A_o)$. If this is fixed, then a p-pair $(P, A)$ is said to be standard if $A \subset A_o$ and $P \supset P_o$, i.e. if $(P, A)$ dominates $(P_o, A_o)$.

Fix an ordering on $\Phi$. The fundamental highest weights $\bar{\omega}_\alpha$ $(\alpha \in \Delta)$ are then defined by

$$(2) \qquad (\bar{\omega}_\alpha, \beta) = \delta_{\alpha\beta}(\alpha, \beta)/2, \qquad (\alpha, \beta \in \Delta),$$

where $(\ ,\ )$ is a scalar product invariant under the Weyl group. We recall that

$$(3) \qquad \bar{\omega}_\alpha = \Sigma_{\gamma \in \Delta}\, d_{\alpha\gamma}\, \gamma, \qquad \text{with } d_{\alpha\gamma} \geqq 0,\ d_{\alpha\alpha} > 0,$$

(and more precisely $d_{\alpha\gamma} > 0$ if and only if $\alpha, \gamma$ belong to the same simple factor of $\underline{g}_c$).

If $(P, A)$ is a semi-standard p-pair, then $\Phi(\underline{m}_c, \underline{h}_c) = \Phi({}^o\underline{m}_c, \underline{b}_c)$ may be identified to the set of roots which are zero on $\underline{a}$, and $\Delta_M = \Delta \cap \Phi(\underline{m}_c, \underline{h}_c)$ is the set of simple roots for the ordering induced from the given one on $\Phi$. Moreover, if we let

(4) $$2\rho = \Sigma_{\alpha\in\Phi^+} \alpha, \quad 2\rho_{{}^oM} = \Sigma_{\alpha\in\Phi({}^o\underline{m}_c, \underline{b}_c)^+} \alpha,$$

then

(5) $$\rho|_{\underline{b}} = \rho_{{}^oM}.$$

If $(P, A)$ is a standard p-pair, then

(6) $$\rho_P(a) = \rho(a) = (1/2)\prod_{\alpha\in\Phi(\underline{p}_c, \underline{h}_c)} \alpha(a), \qquad (a\in A_P).$$

2.5. Weyl groups. Let $W = W(\underline{g}_c, \underline{h}_c)$ be the Weyl group of $\underline{g}_c$ with respect to $\underline{h}_c$ and similarly $W_M = W(\underline{m}_c, \underline{h}_c) = W({}^o\underline{m}_c, \underline{b}_c)$. We put

(1) $$W^M = \{w \in W \mid w^{-1}(\alpha) > 0, \ (\alpha\in\Delta_M)\}.$$

Then $W^M$ is a set of representatives for the right cosets $W_M \cdot w$ in $W$. As usual, the length $\ell(w)$ of $w \in W$ is meant with respect to the set $S$ of reflections $s_\alpha \in W$ $(\alpha\in\Delta)$. We recall that if $t \in W$, the minimum of $\ell(w)$ on $W_M \cdot w$ is attained on $W^M \cap \{W_m \cdot w\}$, and only on that element [4: 5.13].

2.6. Infinitesimal characters. If $(\pi, V)$ is an irreducible admissible representation of a linear reductive group of connected type (1.4) $L$ then $\chi_\pi$ or $\chi_V$ denotes its infinitesimal character. We shall use the standard parametrization of the infinitesimal characters by $\underline{q}_c^*$ modulo the Weyl group, where $\underline{q}_c$ is a Cartan subalgebra of $\underline{\ell}_c$ : if $V$ is finite dimensional, with highest weight $\mu$, then $\chi_\pi = \chi_{\mu+\rho}$.

## §3. Principal series representations

In this section, we fix $G$, $A_o$, $\underline{h}$, compatible orderings on $\Delta$ and $\mathbb{R}^\Delta$ as in §2, and let $(P_o, A_o)$ be the standard minimal p-pair.

We first state, in the form needed below, a special case of a theorem of Kostant [4: Thm 5.14]:

3.1. THEOREM (B. Kostant). *Let* $\lambda$ *be a dominant weight of* $G_c$ *and* $F_\lambda$ *a finite dimensional* $G$-*module with highest weight* $\lambda$. *Let* $(P, A)$ *be a standard* p-*pair*, $P = M.N$ *its standard Levi decomposition. For* $\mu \in \underline{b}_c^*$ ($\underline{b} = {}^o\underline{m} \cap \underline{h}$, *cf.* 2.3), *let* $E_\mu$ *denote an irreducible* $M_c$ *module with extreme weight* $\mu$. *Let* $j \in \mathbb{N}$. *Then, there is an isomorphism of* $M_o$-*modules*

$$H^j(\underline{n}_P, F) = \bigoplus_{s \in W^M,\, \ell(s)=j} E_{s(\rho+\lambda)-\rho} \quad .$$

Note that the weights $s(\rho+\lambda) - \rho$ are all dominant and distinct, as $s$ ranges through $W^M$ (loc. cit.), hence the decomposition of $H^*(\underline{n};F_\lambda)$ as a $M_c$-module is multiplicity free.

3.2. If $R$ is a closed subgroup of a Lie group $Q$ and $(\pi, V_\pi)$ a differentiable $R$-module, then the representation of $Q$ induced from $\pi$ is the representation defined by left translations on

$$\text{(1)} \qquad \mathrm{Ind}_R^Q(\pi) = \mathrm{Ind}_R^Q(V_\pi) = \{f \in C^\infty(Q;V_\pi) \,|\, f(q.r) = \pi(r)^{-1}.f(q),\ (q \in Q,\ r \in R)\} .$$

If $(\tau, U_\tau)$ is a finite dimensional continuous representation of $Q$, then there is a canonical isomorphism

(2) $$\zeta : \mathrm{Ind}_R^Q(V_\pi \otimes U_\tau) \xrightarrow{\sim} (\mathrm{Ind}_R^Q(V_\pi)) \otimes U_\tau ,$$

given by

(3) $$\zeta(f)(q) = (q).f(q), \qquad (f \in \mathrm{Ind}_R^Q(V_\pi \otimes U_\tau); q \in Q).$$

3.3. We fix a p-pair $(P, A)$ of $G$ and let $P = H.N$ be the standard Levi decomposition of $P$. Let $(\sigma, H_\sigma)$ be a differentiable admissible Fréchet ${}^oM$-module with an infinitesimal character $\chi_\sigma$, and let $\nu \in \underline{a}_c^*$. Then the induced representation $(\pi_{P,\sigma,\nu}, I_{P,\sigma,\nu})$ is the representation defined by left translations on

(1) $$I_{P,\sigma,\nu} = \{ f \in C^\infty(G;H_\sigma) \,|\, f(g.man) = a^{-(\rho_P+\nu)}.\sigma(m)^{-1}.f(g)\},$$

$(g \in G,\ m \in {}^oM,\ a \in A,\ n \in N)$. Thus, in the notation of 3.2:

(2) $$I_{P,\sigma,\nu} = \mathrm{Ind}_P^G(H_\sigma \otimes \mathbb{C}_{\rho_P+\nu}),$$

where, for $\mu \in \underline{a}_c^*$, we let $\mathbb{C}_\mu$ denote $\mathbb{C}$ acted upon via $\mu$ by $A$.

It is an admissible finitely generated Fréchet G-module whose infinitesimal character is $\chi_{\lambda_\sigma+\nu}$ (cf. 2.5) if $\lambda_\sigma \in \underline{b}_c^*$ is such that $\chi_\sigma = \chi_{\lambda_\sigma}$.

3.4. THEOREM. *Let* $(P, A)$ *be a standard* p-*pair*, $P = M.N$ *the standard Levi decomposition of* P. *Let* $\sigma$ *and* $\nu$ *be as in* 3.3 *and write* I *for* $I_{P,\sigma,\nu}$. *Let* $\lambda \in \underline{h}_c^*$ *be a dominant weight and* $F_\lambda$ *a simple* $G_c$-*module with highest weight* $\lambda$.

(i) *If* $H_d^*(G; I \otimes F_\lambda) \neq 0$, *then there exists* $s \in W^M$ *such that*

(1) $s(\rho+\lambda)|_A + \nu = 0$,

(2) $\chi_\sigma = \chi_{-s(\rho+\lambda)|\underline{b}_c}$;

*Such an* s *is unique*.

(ii) *If* $s \in W^M$ *satisfies* (1) *and* (2) *then, for every* $q \in \mathbb{N}$, *we have*

(3)
$$H_d^{q+\ell(s)}(G; I \otimes F_\lambda) = (H_d^*({}^oM; H_\sigma \otimes E_{(s(\rho+\lambda)-\rho)}) \otimes \Lambda \underline{a}_c^*)^q .$$

*Remarks*. 1) The conditions (1) and (2) are equivalent to

$$\rho + \lambda \in -W(\lambda_\sigma + \nu) .$$

Condition (1) implies that $\nu$ is real valued.

2) In (3), $E_{s(\rho+\lambda)-\rho}$ is viewed as a ${}^oM$-module by restriction. Since M is the direct product of ${}^oM$ by a commutative groups, $E_{s(\rho+\lambda)-\rho}$ is an irreducible ${}^oM$-module. Its restriction to ${}^oM^o$ is a multiple of the irreducible representation with highest weight $(s(\rho+\lambda)-\rho)|\underline{b}_c = s(\rho+\lambda)|\underline{b}_c - \rho_{{}^oM}$.

3.5. Proof of the theorem. By 3.2(2) and 3.3(2) we have

(1) $$I \otimes F_\lambda = \mathrm{Ind}_P^G(F_\lambda \otimes H_\sigma \otimes \mathbb{C}_{\lambda+\rho}) .$$

By Shapiro's lemma (IX, § 2, and 5.9), we have then

(2) $$H_d^*(G;I\otimes F_\lambda) = H_d^*(P;F_\lambda \otimes H_\sigma \otimes \mathbb{C}_{\lambda+\rho}) .$$

By definition, $N$ acts trivially on $H_\sigma \otimes \mathbb{C}_{\lambda+\rho}$, and $F_\lambda$ is finite dimensional. Therefore $H_d(N;F\otimes H_\sigma \otimes \mathbb{C}_{\rho+\lambda})$ is Hausdorff and equal to $H_d(N;F_\lambda) \otimes H_\sigma \otimes \mathbb{C}_{\rho+\lambda}$ (IX, 5.7). We can then apply IX, 5.8. There exists therefore a spectral sequence $(E_r)$ abuting to $H_d^*(P;F_\lambda \otimes H_\sigma \otimes \mathbb{C}_{\rho+\lambda})$ and in which

(3) $$E_2^{p,q} = H_d^p(M;H_d^q(N;F_\lambda)\otimes H_\sigma \otimes \mathbb{C}_{\rho+\lambda}) .$$

The group $N$ is unipotent; therefore it has no compact subgroup $\neq \{1\}$, hence by van Est theorem (IX, 5.6), $H_d(N;U) = H(\underline{n};U)$ for any differentiable $N$-module $U$; Kostant's theorem (3.1) yields then

(4) $$H_d^q(N;F) = \oplus_{s\in W^M,\, \ell(s)=q} L_s, \quad \text{where } L_s = E_{s(\lambda+\rho)-\rho} .$$

Therefore

(5) $$E_2^{p,q} = \oplus_{s\in W^M,\, \ell(s)=q} H_d^p(M;H_\sigma \otimes \mathbb{C}_{\rho+\lambda} \otimes L_s) .$$

Since $M = {}^{O}M \times A$, the M-module $L_s$ may be viewed as the tensor product of an irreducible ${}^{O}M$-module by the one-dimensional A-module $\mathbb{C}_{(s(\rho+\lambda)-\rho)|A}$.

Let

(6) $$\nu_s = s(\rho+\lambda)|_A - \rho|_A + \rho_P + \nu.$$

Since $(P, A)$ is assumed to be standard, we have $\rho|_A = \rho_P$, hence

(7) $$\nu_s = s(\rho+\lambda)|_A + \nu.$$

(2.4). Using I, 1.3 and I, 5.1(4) we can apply the Künneth formula and get

(8) $$H_d^*(M;L_s \otimes H_\sigma \otimes \mathbb{C}_{\nu_s}) = H^*(M^O;L_s \otimes H_\sigma) \otimes H^*(A, \mathbb{C}_{\nu_s}) .$$

If $\nu_s \neq 0$, then $H^*(A;\mathbb{C}_{\nu_s}) = 0$ by IX, 1.9, and then $E_2 = 0$ in view of (8), (5), which proves the necessity of (1).

If now $\nu_s = 0$, then

(9) $$H^*(A;\mathbb{C}_{\nu_s}) = H^*(\underline{a};\mathbb{C}) = \Lambda \underline{a}_c^*,$$

and we have

(10) $$H_d^*(M;L_s \otimes H_\sigma \otimes \mathbb{C}_{\nu_s}) = H_d^*({}^{O}M;L_s \otimes H_\sigma) \otimes \Lambda \underline{a}_c^* .$$

By I, 5.3, the space $H^*({}^{O}M;L_s \otimes H_\sigma)$ is zero if $\chi_\sigma$ is not equal to the infinitesimal character of the representation $\widetilde{L}_s$ contragredient to $L_s$. Since the highest weight of $L_s$ is $(s(\rho+\lambda)-\rho)|_{\underline{b}_c}$ and $\rho|_{\underline{b}_c} = \rho_{{}^{O}M}$, the infinitesimal character of $\widetilde{L}_s$ is $\chi_{-(s(\lambda+\rho))|\underline{b}_c}$. This proves the necessity of (2) in (i).

These two conditions determine $s(\rho+\lambda)$ uniquely; but $\rho + \lambda$ is regular, therefore they fix $s \in W$ as well, and the uniqueness assertion of (i) follows.

Let now $s \in W^M$ satisfy those conditions. By the previous argument, we have

(11) $$H^*(M;L_t \otimes H_\sigma \otimes \mathbb{C}_{\rho+\lambda}) = 0, \qquad \text{if } t \in W^M,\ t \neq s.$$

Then (5) and (11) imply

(12) $$E_2^{p,q} = 0, \qquad \text{if } q \neq \ell(s),$$

and (5) and (10) yield:

(13) $$E_2^{p,\ell(s)} = (H^*({}^oM;L_s \otimes H_\sigma) \otimes \Lambda \underline{a}_c^*)^p, \qquad (p \in \mathbb{N}).$$

(12) and (13) show that the spectral sequence $(E_r)$ degenerates and that we have

(14) $$H_d^j(G;I \otimes F_\lambda) = E_2^{j-\ell(s),\ell(s)}, \qquad (j \in \mathbb{N});$$

(3) now follows from (13) and (14).

3.5. Let $(\tau, U_\tau)$ be a continuous finite dimensional representation of $A$ which is quasi-unipotent: i.e., there exists $\nu \in \underline{a}_c^*$, called the weight of $\nu$, such that $(\tau(a) - a^\nu . \mathrm{Id.})$ is nilpotent for every $a \in A$. Any continuous finite dimensional representation of $A$ is direct sum of quasi-unipotent ones. Theorem 3.4 holds true if $\mathbb{C}_\nu$ is replaced by $U_\tau$, provided that in (3), the factor $\Lambda \underline{a}_c^*$ is replaced by $H^*(\underline{a}_c; U_\tau)$. The proof of (i) is reduced to the case considered above by using a Jordan-Hölder decomposition of $(\tau, U_\tau)$. The proof of (ii) is then the same as above.

## §4. Fundamental parabolic subgroups

From now on, $K$ is a maximal compact subgroup of $G$ whose Lie algebra is orthogonal to that of $A_o$.

4.1. Let $L$ be a reductive group of connected type (1.4) and $L_1$ the greatest connected normal semi-simple group of $L^o$. A Cartan subgroup $C$ of $L$ is fundamental if and only if it contains a maximal torus of $L$. This condition is equivalent to $C \cap L_1$ being fundamental in $L_1$. The fundamental Cartan subgroups of $L$ form one conjugacy class [5: 1.4.1.4, p. 110]. A parabolic subgroup $P$ of $L$ is fundamental if it is minimal among those which contain a fundamental Cartan subgroup. $P$ is fundamental if and only if $P \cap L_1$ is fundamental in $L_1$. Those parabolic subgroups form one class of associated parabolic subgroups: if $C$ is a fundamental Cartan subgroup of $L^o$ and $C_d^o$ its greatest connected $\mathbb{R}$-split subgroup, then $Z_{L^o}(C_d^o)$ is a Levi subgroup of $P$ for all fundamental parabolic subgroups of $L^o$ containing $C$. In particular, prk $P$ is equal to the difference rk $L$ - rk $Q$, where $Q$ is a maximal compact subgroup of $L$. If rk $L$ = rk $Q$, i.e., if $L$ has a discrete series, then $L$ is its own fundamental parabolic subgroup.

Recall that a parabolic pair $(P, A)$ is cuspidal if ${}^oM_P$ has a compact Cartan subgroup. If so, the center of ${}^oM_P$ is compact. A fundamental parabolic subgroup is cuspidal.

4.2. LEMMA. Let $(P, A)$ be a cuspidal p-pair in $G$, $M = Z(A)$, $N = R_uP$.

(i) If $P$ is fundamental, then all root spaces in $\underline{n}$ are even dimensional. In particular $\dim \underline{n}$ is even. Moreover, $\dim \underline{n} \geq 2.\dim A$, $2.\mathrm{rk}\, K$.

(ii) If $P$ is not fundamental, then the Cartan subalgebras of ${}^o\underline{m}_c$ are singular in $\underline{g}_c$.

(i) Assume $P$ to be fundamental. Let $S$ be a maximal torus of ${}^oM_P$. Then $S$ is also a Cartan subgroup in a maximal compact subgroup of $G$, hence it contains elements which are regular in $\underline{g}_c$ [5: 1.3.3.2], and the Cartan subgroup $S.A$ is the centralizer of some element in $S$. In particular, the representation of $S$ in $\underline{n}$ given by the adjoint representation does not contain any trivial representation. It is therefore a sum of two-dimensional real irreducible representations. Since $S$ leaves all root spaces stable, this proved the first assertion of (i), and also shows that $\dim \underline{n} \geq 2.\dim S = 2.\mathrm{rk}\, K$. Since $A$ acts faithfully on $\underline{n}$, there are at least $\dim A$ linearly independent roots, hence $\dim \underline{n} \geq 2\dim A$.

(ii) Assume now $P$ is not fundamental. Let $T$ be a maximal torus of $G$ containing $S$. Then $T \subset Z(S)$ and $T \nsupseteq S$. The group $R = Z(S)/S$ is reductive. The group $A$ maps isomorphically onto the identity component of a Cartan subgroup of $R$. It is $\mathbb{R}$-split. But $R$ contains a non-trivial torus, namely $T/S$, hence its Cartan subgroups are not all conjugate to each other. As a consequence, $R$ is not commutative; therefore $Z(S)$ has a non-trivial semi-simple subgroup. But then $\underline{s}$ is singular. Since $\underline{s}_c$ is a Cartan subalgebra of ${}^o\underline{m}_c$, this proves (ii).

4.3. Let $L$ be a Lie group with finitely many connected components and $Q$ a maximal compact subgroup of $L$. We put

(1) $$2.\,q(L) = \dim L - \dim Q\,.$$

Assume the Lie algebra of $L$ to be reductive. Then we let

(2) $$\ell_o(L) = \operatorname{rk} L - \operatorname{rk} Q\,, \qquad 2.\,q_o(L) = 2q(L) - \ell_o(L)\,.$$

Since the rank and the dimension of a reductive Lie algebra are congruent mod 2, $q_o(L)$ is an integer.

4.4. LEMMA. *Let $L$ be a reductive group with compact center (1.4). Then $q_o(L) \geqq \operatorname{rk}_{\mathbb{R}} L$ and $q_o(L) + \ell_o(L) \leqq 2.\,q(L) - \operatorname{rk}_{\mathbb{R}} L$.*

We may assume $L$ to be connected. Then $L = L'.S$, with $S$ central compact, $L'$ semi-simple, and $L' \cap S$ finite. $q_o(\ )$, $\ell_o(\ )$, $\operatorname{rk}_{\mathbb{R}}$

q( ), are the same for $L$ and $L'$; this reduces us to the case where $L$ is connected, semi-simple. Passing to a finite covering does not change these constants, so we may assume $L = G$ and use our standard notation.

The set ${}_{\mathbb{R}}\Delta$ has $\dim A_o$ elements, hence $\dim N_o \geqq \dim A_o$. By the Iwasawa decomposition $G = K.A_o.N_o$, we have then

(1) $$2q(G) = \dim A_o + \dim N_o \geqq 2 \dim A_o = 2\mathrm{rk}_{\mathbb{R}} G .$$

This proves the lemma when $\ell_o(G) = 0$. Let now $(P, A)$ be a standard fundamental p-pair of $G$, $P = MN$ the standard Levi decomposition of $P$ and $S$ a maximal torus of ${}^oM$. The group ${}^oM$ has compact center, hence (1) also yields

(2) $$q({}^oM) \geqq \mathrm{rk}_{\mathbb{R}}({}^oM) .$$

We have

(3) $$\mathrm{rk}_{\mathbb{R}} G = \mathrm{rk}_{\mathbb{R}} {}^oM + \dim A, \qquad \dim A = \ell_o(G) ,$$

Since $P$ is standard, the Iwasawa decomposition $G = K.A_o.N_o$ induces one on ${}^oM$, whence

(4) $$2q(G) = 2q({}^oM) + \dim N + \dim A = 2q({}^oM) + \dim N + \ell_o(G) ,$$

(5) $$2q_o(G) = 2q({}^oM) + \dim N .$$

Using (2), 4.2 and (3), we get

(6) $$q_o(G) \geqq rk_{\mathbb{R}}\, {}^oM + (\dim N)/2 \geqq rk_{\mathbb{R}}\, {}^oM + \dim A = rk_{\mathbb{R}}\, G.$$

On the other hand, by (4), (5):

(7) $$2q(G) - rk_{\mathbb{R}}\, G = 2.\,q({}^oM) + \dim N - rk_{\mathbb{R}}\, {}^oM = 2.\,q_o(G) - rk_{\mathbb{R}}\, {}^oM;$$

(6) then yields:

(8) $$2q(G) - rk_{\mathbb{R}}\, G \geqq q_o(G) + rk_{\mathbb{R}}\, G - rk_{\mathbb{R}}\, {}^oM = q_o(G) + \ell_o(G).$$

4.5. LEMMA. *Let* $L$ *be a reductive group with compact center.*

(i) *We have* $q(L) = rk_{\mathbb{R}}\, L$ *if and only if the non-compact normal subgroups of* $L^o$ *are of type* $SL_2(\mathbb{R})$.

(ii) *We have* $q_o(L) = rk_{\mathbb{R}}\, L$ *if and only if every non-compact simple factor of* $L^o$ *is of type* $SL_2(\mathbb{R})$, $SL_2(\mathbb{C})$ *or* $SL_3(\mathbb{R})$.

*Proof.* The reduction to the case where $L$ is connected, simple non-compact is immediate and left to the reader. So assume $L$ to be so. Fix a minimal p-pair $(P_o, A_o)$ and let $P_o = M_o.N_o$ be the standard Levi decomposition of $P_o$.

(i) By 4.4(1), the condition $q(L) = rk_{\mathbb{R}}\, L$ is equivalent to

(1) $$\dim N_o = rk_{\mathbb{R}}\, G = \dim A_o.$$

Since $L$ is simple, and $\dim N_o \geqq (\mathrm{Card}\, {}_{\mathbb{R}}\Delta)$, this is possible only if

$\dim A_o = 1$. Also, $\mathrm{rk}\, L = 1$, because any maximal torus of $M_o$ acts necessarily trivially on the one-dimensional space $N_o$, hence is reduced to $\{1\}$. Then $L$ is locally isomorphic to $SL_2(\mathbb{R})$. The converse is clear.

(ii) Let now $q(L) \neq q_o(L)$ and $q_o(L) = \mathrm{rk}_{\mathbb{R}}\, G$. Let $(P, A)$ be a standard fundamental p-pair, $P = M.N$ the standard Levi decomposition of $P$. The group $P$ is cuspidal, hence $q({}^oM) = q_o({}^oM)$. By 4.4(5), 4.4 and 4.2:

(2)
$$q_o(L) = q({}^oM) + (\dim N)/2, \qquad q({}^oM) \geqq \mathrm{rk}_{\mathbb{R}}\, {}^oM, \qquad \dim N \geqq 2 \dim A.$$

In view of 4.4(6),

(3)
$$q_o(L) = \mathrm{rk}_{\mathbb{R}}\, L \Leftrightarrow q({}^oM) = \mathrm{rk}_{\mathbb{R}}\, {}^oM, \quad \dim N = 2.\dim A.$$

By (i) the first equality on the right hand side is equivalent to ${}^oM$ having all its non-compact simple factors of type $SL_2(\mathbb{R})$. In view of 4.2, the second one yields

(4)
$$\Phi(P, A) = \Delta(P, A)\ .$$

Assume now $L$ to be absolutely simple. Then (4) implies, by standard facts on roots that $\dim A = 1$, hence, by 4.2, $\mathrm{rk}({}^oM) \leqq 1$. If $\mathrm{rk}({}^oM) = 0$, then $L$ is of type $SL_2(\mathbb{R})$, and $q(L) = q_o(L)$, in contradiction with our present assumption. Hence $\mathrm{rk}({}^oM) = 1$ and therefore $\mathrm{rk}\, L = 2$. The representation of ${}^oM$ in $\underline{n}$ given by the adjoint representation has finite kernel, hence ${}^oM^o$ is either a circle group or locally isomorphic to $SL_2(\mathbb{R})$. In the former case, no root of $L$ would restrict to zero on $\underline{a}$, and $\Phi(P,A)$ would have at least two elements. Therefore ${}^oM^o$ is of type $SL_2(\mathbb{R})$. We have a semi-direct product decomposition $N_o = N.({}^oM \cap N_o)$ where ${}^oM \cap N$ is one-dimensional, hence $\dim N_o = 3$. From this it follows readily that $L$ is locally isomorphic to $SL_3(\mathbb{R})$.

Finally, assume $L$ not to be absolutely simple. Then there exists an absolutely simple complex group $R$ such that $L$ is $R$, viewed as a real Lie group. In this case, $\Phi(P,A)$ may be viewed as the set of positive roots in the root system $\Phi(R)$ of $R$, for some ordering. Then (4) shows that $R$ has rank 1, i.e., $R$ is locally isomorphic to $SL_2(\mathbb{C})$.

## §5. Tempered representations

5.1. THEOREM. *Let* $(P,A)$ *be a standard cuspidal p-pair of* $G$, $(\sigma, H_\sigma)$ *a discrete series representation of* ${}^oM$ *and* $\nu \in \underline{a}_c^*$ *purely imaginary. Let* $I = I_{P,\sigma,\nu}$ (3.3) *and* $F_\lambda$ *be a finite dimensional irreducible G-module with highest weight* $\lambda$. *Assume* $H_d^*(G; I \otimes F_\lambda) \neq 0$. *Then* $\nu = 0$, $P$ *is fundamental* (4.1), *the length* $\ell(s)$ *of the element* $s \in W^M$ *satisfying* 3.4(1), (2) *is equal to* $(\dim N)/2$ *and there exists a strictly positive integer* $d$ *such that*

(1) $$\dim H_d^q(G; I \otimes F) = d.\binom{\ell_o}{q-q_o}, \quad (q \in N; q_o = q_o(G), \ell_o = \ell_o(G)).$$

<u>In particular</u>, $H_d^q(G;I\otimes F_\lambda) = 0$ if $q \notin [q_o, q_o + \ell_o]$.

The non-vanishing of the cohomology implies that $\nu$ is real (3.4), hence $\nu = 0$. We must then have, by 3.4(1)

(2) $$s(\rho+\lambda)|_A = 0 ,$$

which means that $s(\rho+\lambda) \in \underline{b}_c^*$ (notation of §2). Since $s(\rho+\lambda)$ is regular, it follows that $\underline{b}_c^*$ is not orthogonal to any root or, equivalently, that, $\underline{b}$ contains regular elements of $\underline{g}_c$. Then 4.2(ii) shows that P is fundamental. Consequently,

(3) $$\dim A = \ell_o(G) .$$

We now use 3.4(3), writing $L_s$ for $E_{s(\rho+\lambda)-\rho}$. By assumption, $\sigma$ belongs to the discrete series of ${}^oM$. By VIII, §5, $H^*({}^oM;H\otimes L_s)$ is concentrated in dimension $q({}^oM)$. Let U be that cohomology space. Then, for $q \in \mathbb{N}$, we have

(4) $$H^{q+\ell(s)}(G;I\otimes F_\lambda) = U \otimes \Lambda^j \underline{a}_c^* , \qquad \text{with } j = q - q({}^oM).$$

In particular, the lowest and highest dimensions in which the left-hand group is not zero are $q({}^oM) + \ell(s)$ and $q({}^oM) + \ell(s) + \ell_o(G)$. The representation $\pi_{P,\sigma,\gamma}$ is unitary since $\sigma$ is, and $\nu$ is purely imaginary. Therefore $H^*(G;I\otimes F_\lambda)$ satisfies Poincaré duality (II, 3.4), and we have

$$2.q({}^{o}M) + 2.\ell(s) + \ell_o(G) = 2.q(G).$$

Then, (3) and 4.4(4) show that $2.\ell(s) = \dim N$ and the theorem follows, with $d = \dim U$.

5.2. COROLLARY. (i) $H_d^q(G;I\otimes F_\lambda) = 0$ *if* $q < \mathrm{rk}_{\mathbb{R}}\, G$ or $q > 2q(G) - \mathrm{rk}_{\mathbb{R}}\, G$.

(ii) *If* $H_d^q(G;I\otimes F_\lambda) \neq 0$ for $q = \mathrm{rk}_{\mathbb{R}}\, G$, *then each non-compact simple factor of* $G$ *is isomorphic to* $SL_2(\mathbb{R})$, $SL_2(\mathbb{C})$ *or* $SL_3(\mathbb{R})$.

(iii) *Let* $(\pi, V)$ *be an irreducible tempered representation of* $G$. *Then*

$$H_d^q(G;V\otimes F_\lambda) = 0 \quad \text{if } q \notin [q_o(G), q_o(G) + \ell_o(G)] .$$

*If* $V$ *is not a fundamental principal series representation, then* $H_d^*(G;V\otimes F_\lambda) = 0$.

(i) follows from 5.1 and 4.4, (ii) from 5.1 and 4.5. If $V$ is as in (iii), then it is a direct $G$-summand of a representation $I = I_{P,\sigma,\nu}$ with $\sigma$ and $\nu$ as in 5.1, hence $H_d^*(G;V\otimes F_\lambda)$ is a direct summand of $H_d^*(G:I\otimes F_\lambda)$ and (iii) is a consequence of 5.1.

5.3. PROPOSITION. *Let* $L$ *be a reductive group of connected type* (1.4) *with compact center. Let* $(\pi, V)$ *be an irreducible tempered representation of* $L$ *and* $(\sigma, F)$ *a finite dimensional rational representation of* $L$. *Then*

(1) $H_d^q(L;V\otimes F) = 0$ for $q \notin [q_o(L), q_o(L) + \ell_o(L)]$, $q < \mathrm{rk}_{\mathbb{R}} L$, $q > 2q(L) - \mathrm{rk}_{\mathbb{R}} L$.

We have $H_d^*(L;V\otimes F) = H_d^*(F^o;V\otimes F)^{L/L^o}$. Moreover, the restriction of $(\pi, V)$ to $L^o$ is direct sum of finitely many tempered irreducible representations (cf. 5.5). This reduces us to the case where $L$ is connected. There is a finite covering $L' \to L$, where $L'$ is reductive, $L' = L_1 \times L_2$, with $L_1$ compact, $L_2$ semi-simple. In view of 1.2, we may assume that $L = L'$. We may pass to Lie algebra cohomology and Q-finite vectors ($Q$ maximal compact in $L$), hence write $(\pi, V)$ as a tensor product of irreducible $(\underline{\ell}_i, Q \cap L_i)$-modules. Since the rational representations of $L$ are fully reducible, we may assume $F$ to be irreducible, and then write it as a tensor product. $F = F_1 \otimes F_2$, where $F_i$ is an irreducible representation of $L_i$ $(i=1, 2)$. We have then, by Künneth rule (I, 1.3)

$$H_d^*(L;V\otimes F) = H_d^*(L_1;V_1\otimes F_1) \otimes H_d^*(L_2;V_2\otimes F_2)\ .$$

Since $L_1$ is compact, the first factor is trivial (IX, 1.10); the corollary applies to the second factor. The proposition follows immediately from this and 4.4.

5.4. COROLLARY. *Let* $L$ *be a reductive group of connected type with compact center. Let* $(\pi, V)$ *be an irreducible tempered representation of* $L$ *and* $(\sigma, F)$ *a finite dimensional rational representation of* $L$. *If* $H_d^q(L;V\otimes F) \neq 0$ *for* $q = \mathrm{rk}_{\mathbb{R}} L$, *then each non-compact simple factor of*

$L^{\circ}$ <u>is locally isomorphic to</u> $SL_2(\mathbb{R})$, $SL_3(\mathbb{R})$ <u>or</u> $SL_2(\mathbb{C})$.

This follows from 4.5 and 5.3.

5.5. In the above proof, we have used the following fact:

Let $L$ be a reductive group, $L'$ an open normal subgroup of $L$, $(\pi, V)$ an irreducible admissible $L$-module. Then $V$ is the direct sum of finitely many irreducible admissible $L'$-modules.

This is well-known. Not knowing a good reference for it, I include a proof for the sake of completeness. Let $Q$ be a maximal compact subgroup of $L$ and $Q' = Q \cap L$. Then $Q'$ is a maximal compact subgroup of $L'$ and is a normal subgroup of finite index of $Q$. It follows from Frobenius reciprocity that if $\sigma \in \hat{L}'$, then there exist only finitely many $\tau \in \hat{L}$ whose restriction to $L'$ contains $\sigma$. Therefore $V$ is an admissible $L'$-module. It suffices then to show the existence of one irreducible $L'$-submodule $U \subset V$, because then $V$ is the sum of the transforms $x(U)$ $(x \in L/L')$, hence the direct sum of finitely many of them. To prove the existence of $U$, one may use the fact (proved by Harish-Chandra) that, up to infinitesimal equivalence, there are only finitely many representations with a given infinitesimal character. A simpler argument, suggested by H. Jacquet, is the following: Let $(\tilde{\pi}, \tilde{V})$ be the contragredient representation to $(\pi, V)$. It is a simple $L$-module, hence a finitely generated $L'$-module. Consequently, it has a proper simple quotient. But, since we deal with admissible representations, $(\pi, V)$ is infinitesimally equivalent to the contragredient of $(\tilde{\pi}, \tilde{V})$. As a consequence, it has a proper simple $L'$-submodule.

## §6. Representations induced from tempered ones.

For later reference, we draw a consequence of the previous results.

6.1. THEOREM. Let $(P, A)$ be a standard p-pair in $G$, $M = Z_G(A)$, $(\sigma, H_\sigma)$ an irreducible admissible tempered representation of ${}^oM$ and $\nu \in \underline{a}_c^*$. Let $F_\lambda$ be a finite dimensional irreducible representation of $G$ with highest weight $\lambda$, and $I = I_{P,\sigma,\nu}$. Let $s \in W^M$ satisfy 3.4(1), (2). Then

(1) $\quad H_d^q(G; I \otimes F_\lambda) = 0$ for $q \notin [q_o({}^oM) + \ell(s), q_o({}^oM) + \ell(s) + \ell_o({}^oM) + \dim A]$.

By 3.4, there exists a finite dimensional representation $L_s$ of $M$ such that $H^*(G; I \otimes F_\lambda)$ is equal to the tensor product of $H_d^*({}^oM; H \otimes L_s)$ by $\Lambda \underline{a}_c^*$, up to a shift of degrees by $\ell(s)$. By 5.3, the first factor has cohomology concentrated in the interval $[q_o({}^oM), q_o({}^oM) + \ell_o({}^oM)]$, whence our assertion.

### REFERENCES

[1] A. Borel and J-P. Serre, "Corners and arithmetic groups", Comm. Math. Helv. 48 (1973), 436-491.

[2] --- et J. Tits, "Groupes réductifs" Publ. Math. I.H.E.S. 27 (1965), 55-151.

[3] Harish-Chandra, "Harmonic analysis on real reductive groups I. The theory of the constant term" J. of func. anal. 19 (1975), 104-204.

[4] B. Kostant, "Lie algebra cohomology and the generalized Borel-Weil theorem" Annals of Math. (2) 74 (1961), 329-387.

[5] G. Warner, Harmonic analysis on semi-simple Lie groups I, Grund. Math. Wiss. 188, Springer, 1972.

# XI: COHOMOLOGY WITH COEFFICIENTS IN $\mathcal{E}_0(G)$

A. Borel and N. Wallach

In this chapter, we prove some results on the cohomology with coefficients in certain irreducible admissible representations of the semi-simple group $G$. We shall proceed by induction, starting from the results of X on induced or tempered representations and using Langlands' classification. Although we are mainly interested in unitary representations, we consider more generally those representations whose coefficients satisfy the necessary condition for unitarizability of VII, 2.3, and denote by $\mathcal{E}_0(G)$ the set of infinitesimal equivalence classes of such representations (see §2). In §3 it is shown that if $(\pi,V) \in \mathcal{E}_0(G)$ and $(\sigma,F)$ is a finite dimensional representation of $G$, then

$$H_d^q(G; V \otimes F) = 0 \quad \text{for} \quad q < \mathrm{rk}_{\mathbb{R}}\, G, \quad q > \dim(G/K) - \mathrm{rk}_{\mathbb{R}}\, G .$$

The vanishing of $H_d^q(G; V)$ below the $\mathbb{R}$-rank has also been proved by G. Zuckerman (in fact he informed one of us that he could prove it, shortly after having been told about Theorem 2.3 of VII, before we had done the work described here).

We then describe, when $G = \mathbf{SO}(n,1)^0$ or $\mathbf{SU}(n,1)$, the infinitesimal equivalence classes of irreducible admissible representations with infinitesimal character $\chi_\rho$ and the associated cohomology groups. If $G = \mathbf{SO}(n,1)^0$ there exists, for each $i < [n/2]$, up to infinitesimal equivalence, exactly one representation $J_i$ such that $H_d^i(G; J_i) \neq 0$. We have then $H^q(G; J_i) = \mathbb{C}$ for $q = i, n-i$, $H^q(G; J_i) = 0$ otherwise. If $n = 2m$ is even, and $i = m$, there are two discrete series representations $D_m^{\pm}$ with $H_d^m(G; D_m^{\pm}) \neq 0$ (see 4.5, 4.6, 4.10). If $G = \underline{\underline{SU}}(n,1)$, these representations are parametrized by a pair of natural integers $i,j$ such that $0 \leqq i+j \leqq n$. The cohomology with coefficients in these representations is determined in 5.13. These results were also known to G. Zuckerman. We also give an interpretation of primitive cohomology in this case (5.22).

The theorems of §§3, 4, and 5, combined with some results of IV, translate into cohomological properties of discrete cocompact subgroups (cf. 3.6, 4.12, 5.24).

The results of §§4 and 5 display a close connection between cohomology, the rate of decay of coefficients of representations and the composition series of principal series representations with Langlands quotients in $\mathcal{E}_0(G)$. In particular, one can say roughly that the slower the decay is, the lower the degree of the first non-zero cohomology group is. For more general groups, the pattern will of course be much more complicated, but it is to be expected that some of these features will obtain, though in a weaker form. §6 gives some further illustrations of such relations, pertaining to cohomology at the $\underline{\underline{R}}$-rank (6.4), cohomology close to the middle dimension (6.2) and the irreducibility of certain principal series representations (6.3). This is just meant as examples, without any attempt to get comprehensive results.

## §1. Preliminaries.

The notation of Chapter X is freely used.

1.1. Let $r_0$ be the restriction mapping $\underline{h}_c^* \longrightarrow \underline{a}_{0c}^*$, or also $X(H) \longrightarrow X(A_0)$. We fix compatible orderings on $\underline{h}^*$ and $\underline{a}_0^*$ and let $\Delta$ (resp. ${}_{\underline{\underline{R}}}\Delta$) be the corresponding set of simple roots in $\Phi(\underline{g}_c, \underline{h}_c)$

(resp. ${}_{\underline{\underline{R}}}\Phi = \Phi(\underline{g}_c, \underline{a}_{0c})$). We have then

(1) $$ {}_{\underline{\underline{R}}}\Delta \subset r_0(\Delta) \subset {}_{\underline{\underline{R}}}\Delta \cup \{0\} . $$

Let

(2) $$ \Delta = \Delta_0 \cup \bigcup_{\beta \in {}_{\underline{\underline{R}}}\Delta} \Delta_\beta , $$

where

(3) $$ \Delta_0 = \{\alpha \in \Delta \mid r_0\alpha = 0\}, \quad \Delta_\beta = \{\alpha \in \Delta \mid r_0\alpha = \beta\} \quad (\beta \in {}_{\underline{\underline{R}}}\Delta) . $$

In particular,

(4) $$ \Delta_0 = \Delta_M $$

is the set of simple roots of $\Phi(\underline{m}_c, \underline{h}_c) = \Phi({}^o\underline{m}_c, \underline{b}_c)$.

$\underline{\underline{1.2}}$. For the standard parabolic subgroups of $G$ (resp. $G_c$) we use the usual indexing by subsets of ${}_{\underline{\underline{R}}}\Delta$ (resp. $\Delta$) (see [11: 1.2]). If $P$ is a standard parabolic subgroup of $G$, there is a unique subset $J = J(P)$ of ${}_{\underline{\underline{R}}}\Delta$ such that $P = P_J$. Then $A_P$ is the intersection of the kernels of the $\alpha \in J$. The complexification $P_c$ of $P$, viewed as a standard parabolic subgroup of $G_c$, is then $P_{\tilde{J}}$, where

(1) $$ \tilde{J} = r_0^{-1}(J) \cap \Delta = \Delta_0 \cup \bigcup_{\beta \in J(P)} \Delta_\beta = \Delta_{M_P} . $$

Let $(P,A)$ be a standard p-pair, and $r_P : X(A_0) \longrightarrow X(A)$ the restriction mapping then

(2) $$ \Delta(P,A) \subset r_P({}_{\underline{\underline{R}}}\Delta) \subset \Delta(P,A) \cup \{0\} . $$

More precisely,

(3) $$r_P(J) = 0; \quad r_P : {}^cJ \simeq \Delta(P,A) \quad \text{is a bijection.}$$

In particular,

(4) $$\mathrm{prk}(P) = \dim A_P = \mathrm{Card}\ {}^cJ.$$

1.3. Weyl chambers. On $\underline{a}$ and $\underline{a}^*$ we use the scalar product induced by the Killing form. We put

(1) $$\underline{a}^+ = \{a \in \underline{a} \,|\, \beta(a) > 0, \ (\beta \in \Delta(P,A))\}, \quad A^+ = \exp \underline{a}^+,$$

(2) $$\underline{a}^{*+} = \{\lambda \in \underline{a}^* \,|\, (\lambda,\beta) > 0, \ (\beta \in \Delta(P,A))\},$$

(3) $$^+\underline{a}^* = \{\lambda \in \underline{a}^* \,|\, \lambda = \Sigma_{\beta \in \Delta(P,A)}\, x_\beta \cdot \beta \quad (x_\beta > 0 \ \text{ for all } \ \beta)\}.$$

If

(4) $$C\ell(\underline{a}^+) = \{a \in \underline{a} \,|\, \beta(a) \geqq 0 \quad (\beta \in \Delta(P,A))\},$$

then

(5) $$^+\underline{a}^* = \left\{\lambda \in \underline{a}^* \,|\, \lambda(a) > 0 \quad \text{for all} \quad a \in C\ell(\underline{a}^+) - \{0\}\right\}.$$

As is well-known

(6) $$\underline{a}^{*+} \subset {}^+\underline{a}^*, \quad {}^+\underline{a}^* = \{\lambda \in \underline{a}^* \,|\, (\lambda,\mu) > 0 \quad \text{for all} \quad \in \underline{a}^{*+}\}.$$

1.4. Let $(P,A)$ be a standard p-pair, $P = M.N$ the standard Levi decomposition of $P$. Let $w_G$ (resp. $w_M$) be the longest element in $W$ (resp. $W_M$). Then $s \longmapsto s' = w_M . s . w_G$ is an involution of $W^M$, and we have

(1) $$\ell(s) + \ell(s') = \dim N \qquad (s \in W^M).$$

The proof is elementary and left to the reader. As is well-known, the lengths of $w_G$ and $w_M$ are respectively equal to the number of positive roots in $\Phi$ and $\Phi_M$. As a consequence, $\ell(s)$ takes all values between 0 and $\dim N$ when $s$ ranges through $W^M$. The longest element is $w_M \cdot w_G$.

We have $w_G\rho = -\rho$, therefore

$$(2) \qquad s\rho|_A + s'\rho|_A = 0, \qquad (s \in W^M).$$

Let $\underline{b} = \underline{h} \cap {}^o\underline{m}$ (cf. X, 2.3). Let $s \in W^M$. We have

$$(3) \qquad s'\rho = -w_M s\rho, \quad (s'\rho-\rho)|_{\underline{b}} = -w_M(s\rho-\rho)_{\underline{b}}.$$

The automorphism $-w_M$ of $\underline{b}_c^*$ transforms the highest weight of an irreducible representation of ${}^o\underline{m}_c$ into that of the contragredient one [2 : VIII, §7, n° 5]. In particular, the irreducible representations of ${}^o\underline{m}_c$ with highest weights $(s\rho-\rho)|_{\underline{b}}$ and $(s'\rho-\rho)|_{\underline{b}}$ are contragredient to one another.

We note that if we replace the given ordering on $\Phi$ by the opposite one, then $W^M$ and the length function on $W^M$ are unchanged.

The group $W_M$ acts trivially on A, therefore, if $\lambda \in \underline{h}_c^*$, then $s\lambda|_A = t\lambda|_A$ whenever $t \in W_M(s)$, $(s,t \in W)$, hence

$$(4) \qquad \{s\lambda|_A\}_{s\in W} = \{s\lambda|_A\}_{s\in W^M}, \qquad (\lambda \in \underline{h}_c^*).$$

1.5. PROPOSITION. Let $(P,A)$ be a standard p-pair, $P = M.N$ the standard Levi decomposition of P, and $(\overline{P},A)$ the p-pair opposite to $(P,A)$. Let $(\sigma,H_\sigma)$ be a unitary representation of ${}^oM$ and $\nu \in \underline{a}^*$. Let $I = I_{P,\sigma,\nu}$, $I' = I_{\overline{P},\sigma,\nu}$ (X, 3.3). Then $H_d^q(G; I) = H_d^{N-q}(G; I')$ for all $q$'s, where $N = 2q(G)$ (X, 4.3).

If $\nu$ is not real, then both cohomology groups are zero (X, 3.4), so assume $\nu \in \underline{a}^*$. Let $\tilde{\rho}$ be the sum of the positive roots for the order opposite to the given one on $\Phi$. Then $\tilde{\rho} = -\rho$, and $(\overline{P},A)$ is standard for that new ordering. Let $s \in W^M$. Then the conditions

$$(1) \qquad s\rho|_A + \nu = 0, \quad \chi_\sigma = \chi_{-s(\rho)}|\underline{b}\ ,$$

are equivalent to

$$(2) \qquad s'\tilde{\rho}|_A + \nu = 0, \quad \chi_\sigma = \chi_{-s'(\tilde{\rho})}|\underline{b}\ ,$$

as follows immediately from 1.4. Also, the representations $L_s$, $L_{s'}$ of $M$ with highest weights $s\rho - \rho$ (in the given ordering) and $s'\tilde{\rho} - \tilde{\rho}$ (in the opposite ordering) have equivalent restrictions to ${}^oM$. We have then, by X, 3.4

$$(3) \quad H_d^{q+\ell(s)}(G;\ I) = H^{q+\ell(s')}(G;\ I') = (H^*({}^oM;\ H \otimes L_s \otimes \Lambda\underline{a}_c)^q, \quad (q \in \underline{\underline{N}})\,.$$

But it follows from II, 3.4, and I, §5, that the first factor on the right-hand side satisfies Poincaré duality, the top dimension being $2q({}^oM) + \dim A$. Since $\ell(s) + \ell(s') = \dim N$, and $2q(G) = 2q({}^oM) + \dim A + \dim N$, our assertion follows.

## §2. The class $\mathcal{E}_0(G)$.

2.1. We let $\mathcal{E}_0(G)$ denote the infinitesimal equivalence classes of irreducible admissible smooth representations $(\pi,V)$ of $G$ which are either tempered or represented by a Langlands quotient $J(P,\omega,\nu)$ (see VI, §5), where $(P,A)$ is a standard p-pair and

$$(1) \qquad \operatorname{Re}\nu \in \underline{a}^{*+}, \quad \rho_P - \operatorname{Re}\nu \in {}^+\underline{a}^*\,.$$

Often we shall say that $(\pi,V)$ belongs to $\mathcal{E}_0(G)$ if its infinitesimal equivalence class does so. Let $G$ be simple non compact. As is shown in VII,

§6, all non trivial unitarizable (in fact uniformly bounded) Langlands quotients belong to $\mathcal{E}_0(G)$, therefore $\mathcal{E}_0(G)$ contains all non-trivial simple unitarizable representations of $G$. If $G = G' \times G''$, then $\mathcal{E}_0(G) = \mathcal{E}_0(G') \times \mathcal{E}_0(G'')$, via tensor product. It follows that, in general, $\mathcal{E}_0(G)$ contains all simple unitarizable representations of $G$ with compact kernel.

<u>2.2.</u> PROPOSITION. <u>Let</u> $(P,A)$ <u>be a</u> p-<u>pair,</u> $\omega$ <u>an irreducible tempered representation of</u> ${}^oM_P$ <u>and</u> $\nu \in \underline{a}_c^*$. <u>Assume that</u> 2.1(1) <u>is satisfied. Then all constituents of the induced representation</u> $I_{P,\omega,\nu}$ (see X, 3.3) <u>belong to</u> $\mathcal{E}_0(G)$.

After conjugation, we may assume $(P,A)$ to be semi-standard. Let $(P',A)$ be the standard p-pair associated to $(P,A)$. Then $I_{P,\omega,\nu}$ and $I_{P',\omega,\nu}$ have the same character [4: §21, Lemma 3], hence the same constituents. We may therefore assume $(P,A)$ to be standard. Moreover, it suffices to prove 2.2 for $G$ simple. But then, this is just Corollary 3.3 in the Erratum to AVII.

<u>2.3.</u> Let $(P,A)$ be a standard p-pair, $(\overline{P},A)$ the opposite p-pair. Let $\nu \in \underline{a}_c^*$ be such that $\operatorname{Re} \nu \in \underline{a}^{*+}$, and $\omega$ as in 2.2. Then there is an intertwining operator

$$A : I_{P,\omega,\nu} \longrightarrow I_{\overline{P},\omega,\nu}, \tag{1}$$

whose image is the Langlands quotient $J(P,\omega,\nu)$. We have therefore two exact sequences of admissible finitely generated $G$-modules

$$0 \longrightarrow U \longrightarrow I_{P,\omega,\nu} \longrightarrow J(P,\omega,\nu) \longrightarrow 0, \tag{2}$$

$$0 \longrightarrow J(P,\omega,\nu) \longrightarrow I_{\overline{P},\omega,\nu} \longrightarrow U' \longrightarrow 0. \tag{3}$$

## §3. A vanishing theorem.

3.1. LEMMA. *Let* $(P,A)$ *be a standard* $p$-*pair*, $J = J(P)$, ${}^cJ = {}_{\underline{R}}\Delta - J$, *and* $\lambda \in \underline{h}^*$ *a dominant weight of* $\underline{g}_c$. *Let* $\nu \in \underline{a}^*$ *be such that* $\rho_P + \nu \in {}^+\underline{a}^*$. *Let* $s \in W^M$ *be such that* $s(\rho+\lambda)|_A + \nu = 0$. *Then* $\ell(s) \geq \dim A$. *More precisely, if* $s = s_{\alpha_1} \cdots s_{\alpha_m}$ $(m = \ell(s))$ *is a reduced decomposition of* $s$, *then* $\{\alpha_i\}_{1\leq i\leq m}$ *contains at least one element of each set* $\Delta_\beta$ $(\beta \in {}^cJ)$, (cf. 1.1(2)).

We may write,

(1) $$\lambda = \Sigma\, c_\alpha \overline{\omega_\alpha}\,, \qquad (c_\alpha \in \underline{N})\,.$$

The $\overline{\omega_\alpha}$ are positive linear combinations of the simple roots, therefore (1.1, 1.2)

(2) $$\lambda|_A \in C\ell({}^+\underline{a}^*) = \{\mu\in\underline{a}^* \mid \mu = \Sigma_{\beta\in\Delta(P,A)}\, y_\beta\cdot\beta\,,\ (y_\beta \geq 0)\}\,.$$

Since $\rho|_A = \rho_P$, our assumption on $\nu$ then implies

(3) $$(\rho+\lambda)|_A + \nu \in {}^+\underline{a}^*\,.$$

On the other hand, $s(\rho+\lambda)$ is a weight of the finite dimensional irreducible representation of $G_c$ with highest weight $\rho + \lambda$, therefore

(4) $$s(\rho+\lambda) = \rho + \lambda - \Sigma_{\alpha\in\Delta}\, m_\alpha(s)\alpha\,, \quad \text{with } m_\alpha(s) \in \underline{N}\,,$$

hence

(5) $$s(\rho+\lambda)|_A + \nu = (\rho+\lambda)|_A + \nu - \Sigma_{\beta\in{}^cJ}\, r_P(\beta)(\Sigma_{\alpha\in\Delta_\beta}\, m_\alpha(s))\,.$$

The left-hand side of (5) is zero by assumption, therefore (3) implies

(6) $$\Sigma_{\alpha\in\Delta_\beta}\, m_\alpha(s) > 0 \qquad \text{for every } \beta \in {}^cJ\,.$$

Now if $\gamma \in \Delta$, and $\mu \in \underline{h}_c^*$, then $s_\gamma(\mu) - \mu$ is an integral multiple of $\gamma$. Therefore, since the $\gamma \in \Delta$ are linearly independent, we see that if a reduced decomposition of $s$ does not contain $\gamma$, then $m_\gamma(s) = 0$. The lemma then follows from (6).

3.2. LEMMA. *Let* $(P,A)$, $\lambda$ *be as in* 3.1 *and* $\nu \in \underline{a}^*$ *be such that* $\rho_P - \nu \in {}^+\underline{a}^*$. *Let* $s \in W^M$ *be such that* $s(\rho+\lambda)|_A + \nu = 0$. *Then* $\ell(s) \leqq \dim N - \dim A$.

We shall reduce this to 3.1 by using the involution $t \longmapsto t'$ of $W^M$ introduced in 1.4.

Let $\lambda' = -w_G(\lambda)$. It is also a dominant weight. We have

$$s'(\rho+\lambda') = w_M s w_G(\rho+\lambda') = w_M s(-\rho-\lambda) = -w_M s(\rho+\lambda) ,$$

therefore, since $W_M$ acts trivially on $A$

$$s'(\rho+\lambda')|_A = -s(\rho+\lambda)|_A = \nu .$$

Thus, $s', \lambda'$ and $\nu' = -\nu$ satisfy the conditions of 3.1. Hence $\ell(s') \geqq \dim A$. But then $\ell(s) = \dim N - \ell(s') \leqq \dim N - \dim A$.

3.3. THEOREM. *Let* $(\sigma, F)$ *be a finite dimensional representation of* $G$. *Let* $(\pi, V)$ *be an irreducible admissible representation whose class belongs to* $\mathcal{E}_0(G)$. *Then*

(1) $H^q(G; V \otimes F) = 0$ *for* $q < \mathrm{rk}_{\underline{\underline{R}}}\, G$ *and* $q > 2q(G) - \mathrm{rk}_{\underline{\underline{R}}}\, G$ .

In this proof, we shall write $H^q(U)$ instead of $H_d^q(G; U)$, if $U$ is a differentiable $G$-module.

(a) Let $j \in \underline{\underline{N}}$. Assume that $H^j(U \otimes F) = 0$ for all $U \in \mathcal{E}_0(G)$. Then we have $H^j(U \otimes F) = 0$ for every admissible $G$-module of finite length whose constituents belong to $\mathcal{E}_0(G)$.

In fact, if

$$0 \longrightarrow U' \longrightarrow U \longrightarrow U'' \longrightarrow 0 \tag{2}$$

is an exact sequence of $G$-modules, then the long exact sequence associated to the exact sequence

$$0 \longrightarrow U' \otimes F \longrightarrow U \otimes F \longrightarrow U'' \otimes F \longrightarrow 0 \quad , \tag{3}$$

yields the exact sequence

$$H^j(U' \otimes F) \longrightarrow H^j(U \otimes F) \longrightarrow H^j(U'' \otimes F) \quad , \tag{4}$$

therefore, if the two extreme terms are zero, so is the middle one. Our assertion then follows by induction on the length of $U$.

(b) If $V$ is tempered, then our theorem follows from X, 5.3. There remains therefore to consider the case where $V = J(P,\omega,\nu)$ is a Langlands quotient with $\nu$ satisfying 2.1(1).

(c) We now prove the vanishing below the split rank by induction on $q$. It is obvious for $q < 0$, so let $q < \mathrm{rk}_{\mathbb{R}}\, G$, $q \geqq 0$, and assume our assertion proved for $q - 1$. The exact sequence 2.3(3) gives rise to the exact sequence

$$0 \longrightarrow V \otimes F \longrightarrow I_{\bar{P},\omega,\nu} \otimes F \longrightarrow U' \otimes F \longrightarrow 0 \ , \tag{5}$$

whence an exact sequence

$$H^{q-1}(U' \otimes F) \longrightarrow H^q(V \otimes F) \longrightarrow H^q(I_{\bar{P},\omega,\nu} \otimes F) \ . \tag{6}$$

The constituents of $U'$ all belong to $\mathcal{E}_0(G)$ by 2.2, hence the first term of (6) is zero by (a). In view of X, 6.1,

(7) $$H^j(I_{\overline{P},\omega,\nu} \otimes F) = 0 \quad \text{for} \quad j < q_0({}^oM) + \ell(s) ,$$

where $s \in W^M$ is such that

(8) $$s(\rho+\lambda)|_{A_{\overline{P}}} + \nu = 0 ,$$

and the ordering on $\underline{h}_c^*$ such that $(\overline{P}, A_{\overline{P}})$ is standard. We have

(9) $$A_P = A_{\overline{P}} , \quad {}^+\underline{a}_P^* = -{}^+\underline{a}_{\overline{P}}^* , \quad \rho_P = -\rho_{\overline{P}} ,$$

therefore the condition $\rho_P - \mathrm{Re}\,\nu \in {}^+\underline{a}_P^*$ of 2.1(1) can be written

(10) $$\rho_{\overline{P}} + \mathrm{Re}\,\nu \in {}^+\underline{a}_{\overline{P}}^* .$$

But then 3.1 obtains for $\overline{P}$ and shows that $\ell(s) \geqq \dim A_P$. Since $q_0({}^oM) \geqq \mathrm{rk}_{\underline{\underline{R}}}\, {}^oM$ (X, 4.4) and $\mathrm{rk}_{\underline{\underline{R}}}\, G = \mathrm{rk}_{\underline{\underline{R}}}\, {}^oM + \dim A_P$, it follows from (7) that the last term of (6) is also zero. But then so is the second one.

(d) The second part of (1) will be proved similarly by descending induction on $q$. It is trivial for $q > 2q(G)$, so we let $q > 2q(G) - \mathrm{rk}_{\underline{\underline{R}}}\, G$ and assume our assertion is true for $q + 1$. We now consider the exact sequence

(11) $$0 \longrightarrow U' \otimes F \longrightarrow I_{P,\omega,\nu} \otimes F \longrightarrow V \otimes F \longrightarrow 0 ,$$

associated to 2.3(2) and the exact sequence

(12) $$H^q(I_{P,\omega,\nu} \otimes F) \longrightarrow H^q(V \otimes F) \longrightarrow H^{q+1}(U' \otimes F) ,$$

By (a), 2.2 and the induction assumption, the last term is zero. By X, 6.1 we have

(13) $$H^j(I_{P,\omega,\nu} \otimes F) = 0 \quad \text{for} \quad j > 2q({}^oM) - \mathrm{rk}_{\underline{\underline{R}}}\, {}^oM + \dim A_P + \ell(s) ,$$

where $s \in W^M$ satisfies the condition

(14) $$s(\rho+\lambda)|_{A_P} + \nu = 0 .$$

In view of 2.1(1), we can apply 3.2, hence $\ell(s) \leqq \dim N_P - \dim A_P$. We have then

(15) $$2q({}^oM) - rk_{\underline{\underline{R}}}\, {}^oM + \dim A_P + \ell(s) \leqq 2q({}^oM) + \dim N_P - rk_{\underline{\underline{R}}}\, {}^oM .$$

But

$$2q({}^oM) + \dim N_P + \dim A_P = 2q(G) ,$$

and $rk_{\underline{\underline{R}}}\, G = rk_{\underline{\underline{R}}}\, {}^oM + \dim A_P$. Therefore the right-hand side of (15) is equal to $2q(G) - rk_{\underline{\underline{R}}}\, G$, hence, by (13), the first term of (12) is also zero, and our assertion follows.

3.4. COROLLARY. *Let $(\pi, V)$ be an irreducible unitary representation of $G$ with compact kernel. Then $H^q(G; V \otimes F) = 0$ for $q < rk_{\underline{\underline{R}}}\, G$ and $q > 2q(G) - rk_{\underline{\underline{R}}}\, G$.*

In fact, the equivalence classes of such representations all belong to $\mathcal{E}_0(G)$.

*Remark.* A reduction similar to that of X, 3.4 shows that the corollary also holds for reductive groups of connected type (X, 1.4) with compact center.

3.5. Let $G$ be simple. Let $(\sigma, E)$ be a finite dimensional representation of $G$. Then Theorem 3.3 implies:

(1) $$M(\underline{g}, \sigma) \geqq rk_{\underline{\underline{R}}}\, G - 1 ,$$

where $M(\underline{g}, \sigma)$ is the constant introduced in IV, 7.3. If $\sigma$ is irreducible, non trivial, this had been proved earlier, starting from a result of Raghunathan (see III, 1.2, 2.2). If $\sigma$ is the trivial representation, then $M(\underline{g}, \sigma)$ is the constant $M(\underline{g})$ considered in III, 4.3. The minoration given there was

$M(\underline{g}) \geq m(\underline{g})$, where $m(\underline{g})$ is Matsushima's constant (III, 3.2, 3.3). This constant is in general $< \mathrm{rk}_{\mathbb{R}}\, G$, so that III, 3.3 is in fact contained in Theorem 3.3 of this chapter.

If we now use (1) together with 5.3 and 7.5 of Chapter IV, we get

3.6. PROPOSITION. *Assume that* $G$ *has no compact factor. Let* $\Gamma$ *be an irreducible* (IV, 5.1) *discrete cocompact subgroup of* $G$.

(a) *Let* $(\sigma, E)$ *be a finite dimensional unitary representation of* $\Gamma$. *Then the natural homomorphism* $H^*(\underline{g}, K; E^\Gamma) \longrightarrow H^q(\Gamma; E)$ *is an isomorphism for* $q < \mathrm{rk}_{\mathbb{R}}\, G$.

(b) *Let* $(\sigma, E)$ *be a finite dimensional representation of* $G$. *Then the natural isomorphism* $H^q(\underline{g}, K; E) \longrightarrow H^q(\Gamma; E)$ *is an isomorphism for* $q < \mathrm{rk}_{\mathbb{R}}\, G$.

In (a), $E^\Gamma$ is viewed as a trivial $\underline{g}$-module. In (b), we have $H^q(\underline{g}, K; E) = H^q(\underline{g}, K; E^G)$ for all $q$'s, as follows from II, 3.2 and the full reducibility of $\sigma$.

## §4. The group $\underline{SO}(n,1)^0$.

4.1. In this section, $G = \underline{SO}(n,1)^0$ is the identity component of the special orthogonal group of $\mathbb{R}^{n+1}$, endowed with the quadratic form $\Sigma_1^n x_i^2 - x_{n+1}^2$. We let $\{e_i\}_{1 \leq i \leq n+1}$ be the canonical basis of $\mathbb{R}^{n+1}$.

$K = \underline{SO}(n)$ is the isotropy group of $e_{n+1}$, $P = {}^oM.A.N$ the stabilizer of the hyperplane spanned by $e_1, \ldots, e_{n-1}$ and $A$ the identity component of the group leaving $e_1, \ldots, e_{n-1}$ fixed. Thus $K$ is a maximal compact subgroup and $P$ a minimal parabolic subgroup. We have

(1) $\quad 2q(G) = \dim G - \dim K = n, \qquad \dim N = n - 1, \qquad {}^oM = \underline{SO}(n-1), \ \mathrm{rk}_{\mathbb{R}}\, G = 1,$

(2) $$\Phi(P,A) = \Delta(P,A) = \{\beta\} \qquad \rho_P = ((n-1)/2)\beta \ .$$

We fix a Cartan subalgebra $\underline{h} \supset \underline{a}$, an ordering on $\Phi$ compatible with the given one on $_{\underline{\underline{R}}}\Phi = \{\pm\beta\}$, and use the numbering of simple roots of [1: p.252, 256]. We have

(3) $$\Delta = \{\alpha_1,\ldots,\alpha_r\} , \qquad (r = [n+1/2])$$

(4) $$\Delta_0 = \Delta_M = \{\alpha_2,\ldots,\alpha_r\} , \quad \beta = \alpha_1\Big|_A \ .$$

<u>4.2.</u> We assume now that $n = 2m$ is even. Then

(1) $$W^M = \{s_i\}_{0 \leqq i \leqq 2m-1} ,$$

where

(2) $$s_0 = 1, \quad s_i = s_{\alpha_1} \cdots s_{\alpha_i} \qquad (1 \leqq i \leqq m) ,$$

(3) $$s_{m+i} = s_m . s_{\alpha_{m-1}} \cdots s_{\alpha_{m-i}} , \qquad (1 \leqq i \leqq m-1) .$$

In particular,

(4) $$\ell(s_i) = i , \qquad (1 \leqq i \leqq 2m-1) .$$

It is easily checked that one has

(5) $$s_i\rho = \rho - i\alpha_1 \qquad \text{modulo } \alpha_2,\ldots,\alpha_r , \quad (1 \leqq i \leqq 2m-1) ;$$

more precisely

(6) $$s_i\rho = \rho - \alpha_i - 2.\alpha_{i-1} - \cdots - i.\alpha_1 , \quad (1 \leqq i \leqq m) .$$

Therefore

(7) $$s_i\rho\big|_A = \nu_i = \rho_P - i\beta = (m-i-(1/2)) , \quad (1 \leqq i \leqq 2m-1) .$$

As in X, §2, let $\underline{b} = \underline{h} \cap {}^o\underline{m}$. Then

(8) $\mu_i = (s_i\rho-\rho)|_{\underline{b}} = (s_{2m-i}\rho-\rho)_{\underline{b}} = (-\alpha_i-2\alpha_{i-1}-\ldots-i\alpha_1)|_{\underline{b}}$, $(1 \leqq i < m)$.

Let $\iota$ be the standard representation of ${}^oM$. Then $(\sigma_i, F_i)$, where $\sigma_i = \Lambda^i\iota$, is irreducible, self-contragredient, with highest weight $\mu_i$, $(1 \leqq i < m)$.

4.3. PROPOSITION. *Let* $I_i = I_{P,\sigma_i,\nu_i}$ *and* $I_i' = I_{\bar{P},\sigma_i,\nu_i}$ $(1 \leqq i < m)$ (cf. X, 3.3 *for the notation*). *Then*

(1) $$H_d^q(G;\ I_i') = \begin{cases} \underline{\underline{C}}, & q = i, i+1, \\ 0, & q \neq i, i+1 \end{cases} ; \quad H_d^q(G;\ I_i) = \begin{cases} \underline{\underline{C}}, & q = n-i-1, n-i, \\ 0, & q \neq n-i-1, n-i. \end{cases}$$

Fix $i$ $(1 \leqq i < m)$. It follows from 4.2(7) that $s_{2m-i}$ is the only element $s \in W^M$ such that $s\rho|_A + \nu_i = 0$. By 4.2(8) $\mu_i = (s_{2m-i}\rho-\rho)|_{\underline{b}}$ is the highest weight of $\sigma_i$. Since $\sigma_i$ is self-contragredient, and ${}^oM$ is compact, connected, we have

$$H^0({}^oM;\ F_i \otimes F_i) = \underline{\underline{C}}, \qquad H^q({}^oM;\ F_i \otimes F_i) = 0, \qquad (q \geqq 1).$$

Since $\ell(s_{2m-i}) = 2m - i = n - 1 - i$, the second part of (1) is a consequence of X, 3.4(3) and 4.2(4).

The first part of (1) can be proved similarly, or deduced from the above and 1.5.

4.4. If we restrict the elements of $I_i$ or $I_i'$ to K, then we get an isomorphism of K-modules of $I_i$ or $I_i'$ onto

(1) $$\mathrm{Ind}_{{}^oM}^K(F_i) = \{f \in C^\infty(K;\ F_i) \mid f(km) = \sigma_i(m)^{-1}.f(k), \qquad (k \in K;\ m \in {}^oM)\}.$$

If V is a G-module, let us write $C^q(V)$ for the space $C^q(\underline{g}, K;\ V)$ of elements of degree q in the relative Lie algebra complex (I, §§1, 5). We have then

(2) $$C^q(I_i) = C^q(I_i'), \qquad (1 \leqq i \leqq 2m-1)\ ,$$

and, by (1) and Frobenius reciprocity,

(3) $$C^q(I_i) = \mathrm{Hom}_K(\Lambda^q(\underline{g}/\underline{k}),\ I_i) = \mathrm{Hom}_{{}^oM}(\Lambda^q(\underline{g}/\underline{k},\ F_i), \qquad (1 \leqq i \leqq 2m-1)\ .$$

As a K-module, $\underline{g}/\underline{k}$ is equivalent to the standard representation of K; therefore, as a ${}^oM$-module, $\underline{g}/\underline{k}$ decomposes as $\underline{\underline{R}} \oplus \iota$, and we have ${}^oM$-module isomorphisms

(4) $$\Lambda^q(\underline{g}/\underline{k}) \xrightarrow{\sim} F_q \oplus F_{q-1}\ , \qquad (1 \leqq q \leqq 2m-1)\ .$$

The two representations $F_q$, $F_{q-1}$ are inequivalent if $q \neq m$, equivalent if $q = m$. From (3) and (4) we get

(5) $$\dim C^q(I_i) \begin{cases} 1, & q = m-1, m+1, \\ 2, & q = m, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{if } i = m-1, m+1;$$

(6) $$\dim C^q(I_i) \begin{cases} 1, & q = i, i+1, n-i-1, n-i, \\ 0, & \text{otherwise}, \end{cases} \qquad (1 \leqq i \leqq 2m-1,\ i \neq m-1, m, m+1)\ .$$

Together with 4.3, this implies the following relations, where $Z^q$ stands for cocycles:

(7) $$C^q(I_i') = Z^q(I_i'), \qquad (q = i, i+1;\ 1 \leqq i < m-1),$$

(8) $$dC^{n-i-1}(I_i') = C^{n-i}(I_i'), \qquad (1 \leqq i < m-1)\ .$$

4.5. The index of $W(\underline{g},\underline{h})$ in $W(\underline{g}_c,\underline{h}_c)$ is two; therefore there are, up to equivalence, two discrete series representations, say $D_m^{\pm}$, with the infinitesimal character $\chi_\rho$. By VIII, §5, these representations satisfy

(1) $$H_d^m(G;\ D_m^{\pm}) = \underline{\underline{C}}, \qquad H_d^q(G;\ D_m^{\pm}) = 0, \qquad (q \neq m).$$

4.6. THEOREM. *Let* $n = 2m$ *be even. Let* $J_i$ *be the simple Langlands quotient of* $I_i$ $(1 \leqq i < m)$.

(i) *We have*

(1) $$C^q(\underline{g},K;\ J_i) = \underline{\underline{C}}, \text{ for } q = i,\ n-i, \quad C^q(\underline{g},K;\ J_i) = 0 \text{ for } q \neq i,\ n-i,\ (1 \leqq i < m);$$

*in particular*

(2) $$H_d^q(G;\ J_i) = \underline{\underline{C}} \text{ for } q = i,\ n-i, \quad H_d^q(G;\ J_i) = 0 \text{ for } q \neq i,\ n-i, \quad (1 \leqq i \leqq m).$$

(ii) *We have the exact sequences*

(3) $$0 \longrightarrow J_{i+1} \longrightarrow I_i \longrightarrow J_i \longrightarrow 0, \qquad (i = 1,\ldots,m-2)\ ,$$

(4) $$0 \longrightarrow D_m^+ \oplus D_m^- \longrightarrow I_{m-1} \longrightarrow J_{m-1} \longrightarrow 0.$$

(iii) *If the irreducible admissible* $G$*-module* $(\pi, V)$ *has infinitesimal character* $\chi_\rho$, *then* $V$ *is infinitesimally equivalent to* $J_i$ $(1 \leqq i < m)$, $D_m^+, D_m^-$ *or the trivial representation.*

(a) We prove (iii) first. Let $s \in W(\underline{g}_c,\underline{h}_c)$. Then $s\rho|_A \neq 0$, since $P$ is not fundamental (X, 4.2), hence the only tempered representations with infinitesimal character $\chi_\rho$ belong to the discrete series (a result originally proved by Harish-Chandra, for $G$ semi-simple with finite center, [3: Lemma 73]). In view of 4.5, there remains to consider the case where $V = J(P,\sigma,\nu)$ is a Langlands quotient, with $\nu \in \underline{a}^{*+}$, i.e. $\nu > 0$. The formula for the infinitesimal character of $I_{P,\sigma,\nu}$ (X, 3.3) shows that $\nu = s\rho|_A$ for some $s \in W(\underline{g}_c,\underline{h}_c)$. But then (1.4(4)), we also have $\nu = s\rho|_A$ for some $s \in W^M$, hence $\nu$ satisfies 2.1(1) and $V \in \mathcal{E}_0(G)$. We have then also, in the notation of 4.2, $s = s_i$ for some $i$, $(1 \leqq i < m)$. Moreover, $s_i$ must satisfy the condition $s\rho|_A \in W_M(\lambda_\sigma)$, hence $\lambda_\sigma \in W_M(s_i\rho|_{\underline{b}})$, which implies that $\sigma = \sigma_i$. Therefore $V = J_i$, up to infinitesimal equivalence. This proves (iii).

(b) In the sequel, we write $H^q(V)$ instead of $H^q_d(G; V)$ and $C^q(V)$ for $C^q(g,K; V)$. We consider the exact sequences (2.3)

$$(5)_i \qquad 0 \longrightarrow U_i \longrightarrow I_i \longrightarrow J_i \longrightarrow 0,$$

$$(6)_i \qquad 0 \longrightarrow J_i \longrightarrow I'_i \longrightarrow U'_i \longrightarrow 0, \qquad (1 \leqq i < m).$$

First let $i = m - 1$. It follows from the discussion in (a) and from the Erratum to AVII, 3.5, that the constituents of $U_{m-1}$ or $U'_{m-1}$ belong to the discrete series. Therefore

$$(7) \qquad H^q(U_{m-1}) = H^q(U'_{m-1}) = 0, \qquad q \neq m.$$

The exact cohomology sequences associated to $(5)_{m-1}$ and $(6)_{m-1}$ and 4.3 then give

$$(8) \qquad H^{m-1}(J_{m-1}) = H^{m-1}(I'_{m-1}) = H^{m+1}(I_{m-1}) = H^{m+1}(J_{m-1}) = \underline{\underline{C}},$$

$$(9) \qquad H^q(J_{m-1}) = 0, \qquad q \neq m-1,m,m+1.$$

The exact sequence

$$(10)\qquad 0 \longrightarrow H^m(J_{m-1}) \longrightarrow H^m(I'_{m-1}) \longrightarrow H^m(U'_{m-1}) \longrightarrow H^{m+1}(J_{m-1}) \longrightarrow 0,$$

shows then that there are only two possibilities

$$(11)\qquad H^m(J_{m-1}) = 0,\ H^m(U'_{m-1}) = \underline{\underline{C}}^2,$$

or

$$(12)\qquad H^m(J_{m-1}) = H^m(U'_{m-1}) = \underline{\underline{C}}.$$

Assume now that $i = m - 2$. It follows from the Erratum to AVII, and (a) that the constituents of $U_{m-2}$ or $U'_{m-2}$ are infinitesimally equivalent to $D^{\pm}_m$ or $J_{m-1}$, hence

$$(13)\qquad H^q(U_{m-2}) = H^q(U'_{m-2}) = 0, \quad (q < m-1,\ q > m+1)\ .$$

By 4.3 and the cohomology sequences associated to $(1)_{m-2}$ and $(2)_{m-2}$, we get then

$$(14)\qquad H^{m-2}(I_{m-2}) = H^{m-2}(J_{m-2}) = H^{m+2}(I_{m-2}) = H^{m+2}(J_{m-2}) = \underline{\underline{C}}\ ,$$

$$(15)\qquad H^q(J_{m-2}) = 0, \quad (q < m-2;\ q > m+2),$$

$$(16)\qquad H^{m+1}(U'_{m-2}) = H^{m+2}(J_{m-2}) = \underline{\underline{C}}\ .$$

In view of (8) and 4.4, this shows that $J_{m-1}$ is a constituent of $U'_{m-2}$, with multiplicity one. Therefore $C^q(J_{m-1}) \subset C^q(I'_{m-2})$ for all $q$'s. But $C^m(I'_{m-2}) = 0$ by 4.4(6), hence $C^m(J_{m-1}) = 0$ and, a fortiori, $H^m(J_{m-1}) = 0$. This shows that (11) holds, proves (2) for $i = m - 1$, and also that $U_{m-1}$ consists of two discrete series. Then 4.4(5) for $i = m - 1$ shows that $C^*(\underline{g},K;\ J_{m-1})$ is at most two-dimensional. But then (8) implies that it is $\neq 0$

in dimensions $m \pm 1$, whence (1) for $i = m - 1$. We already know that $J_{m-1}$ is a constituent of $U'_{m-2}$. Since $C^*(I_{m-2})$ is four-dimensional (4.4(6)), $C^*(J_{m-2})$ is then at most two-dimensional, and (1) for $i = m - 2$ then follows from (14). $U_{m-1} = D_m^+ \oplus D_m^-$ since $I_{m-1}$ is K-multiplicity free.

We now proceed by descending induction on $i$. Fix $i \geqq 1$, $i < m - 2$, and assume (i), (ii) to be proved for $J_j$ $(i < j \leqq m-1)$. By the Erratum to AVII, and (iii) the constituents of $U_i$ or $U_i'$ are infinitesimally equivalent to $J_j$ $(i < j \leqq m-1)$ or $D_m^{\pm}$, hence,

$$H^q(U_i) = H^q(U_i') = 0, \quad \text{for } q < i+1, \quad q > n-i-1, \tag{17}$$

and then the above arguments show that (14), (15), (16) remain valid if $m - 2$ is replaced by $i$. Therefore $J_{i+1}$ is a constituent of $U_i$ with multiplicity one. From (1) for $i + 1$ and 4.4(6) we see that $C^*(J_i)$ is at most two-dimensional, and then (1) and (3) for $i$ follow from (14), 4.4(6).

4.7. Now let $n = 2m + 1$ be odd. Then

$$\ell_0(G) = 1, \ q_0(G) = m, \quad \text{Card } W^M = 2(m+1), \tag{1}$$

and $P$ is a fundamental parabolic subgroup (see X, §4 for the notation). Let

(2) $s_0 = 1$, $s_j = s_{\alpha_1} \cdots s_{\alpha_j}$, $(1 \leqq j \leqq m)$, $t_m = s_{m-1} \cdot s_{\alpha_{m+1}}$, $s_i' = w_M \cdot s_i \cdot w_G$ $(0 \leqq i < m)$,

(cf. 1.4.) Then

$$\ell(s_i) = i \ (0 \leqq i \leqq m), \quad \ell(t_m) = m, \quad \ell(s_i') = 2m-i, \quad (0 \leqq i < m). \tag{3}$$

$$W^M = \{1, s_1, \ldots, s_m, t_m, s_0', s_1', \ldots, s_{m-1}'\}. \tag{4}$$

Then one checks easily

$$(5) \qquad s_j\rho = \rho - \alpha_j - 2\alpha_{j-1} - \dots - j.\alpha_1 \,, \qquad (1 \leqq j \leqq m),$$

$$(6) \qquad t_m\rho = \rho - \alpha_{m+1} - 2.\alpha_{m-1} - \dots - m.\alpha_1 \,,$$

hence

$$(7) \qquad \nu_j = s_j\rho|_A = (m-j)\beta, \quad (1 \leqq j \leqq m), \quad t_m\rho|_A = 0 \,.$$

Let $\iota$ be the standard representation of ${}^oM = \underline{SO}(2m)$, $(\sigma_j, F_j)$ the j-th exterior power of $\iota$, $(1 \leqq j \leqq 2m)$. Of course, $\sigma_j$ is isomorphic to $\sigma_{2m-j}$ $(1 \leqq j \leqq 2m)$. We recall that $\sigma_j$ is irreducible and self-contragredient for $j \neq m$, fundamental for $j \leqq m-2$, and that $(\sigma_m, F_m) = (\sigma_m^+, F_m^+) \oplus (\sigma_m^-, F_m^-)$ is direct sum of two irreducible representations which are contragredient to each other, with highest weights the double of those of the half-spinor representations, (see e.g. [2: VIII, §13, n$^o$ 4]). It follows from (5), (6) that

$$(8) \qquad \mu_j = (s_j\rho-\rho)|_{\underline{b}} \quad (1 \leqq j < m), \qquad \mu_m = (t_m\rho-\rho)|_{\underline{b}}$$

are the highest weights of $\sigma_j$ $(1 \leqq j < m)$, $\sigma_m^+$ $(j = m)$ and $\sigma_m^-$ respectively.

**4.8.** PROPOSITION. *Let* $I_i = I_{P_i,\sigma_i,\nu_i}$, $I_i' = I_{\overline{P},\sigma_i,\nu_i}$ $(1 \leqq i < m)$, $I_m = I_{P,\sigma_m^+,0}$.

$$(1) \quad H_d^q(G; I_i') = \begin{cases} \underline{C}, & q = i, i+1 \\ \\ 0, & q \neq i, i+1 \end{cases} \quad (1 \leqq i < m); \quad H_d^q(G; I_i) = \begin{cases} \underline{C}, & q = n-i-1, n-i \,, \dots \\ \\ 0, & q \neq n-i-1, n-i \,, \dots \end{cases} \quad (1 \leqq i \leqq m).$$

*The representation* $I_m$ *is tempered, irreducible, equivalent to* $I_{P,\sigma_m^-,0}$ *and represents the only tempered class with infinitesimal character* $\chi_\rho$.

Fix $i$ $(1 \leqq i < m)$. Since $\sigma_i$ is self-contragredient we have, by 1.4 (1), (2)

(2) $$s_i'\rho|_A + \nu_i = 0, \quad (s_i'\rho-\rho)|_{\underline{b}} = \mu_i ,$$

moreover, by IX, 1.10,

(3) $$H_d^0({}^oM; F_i \boxtimes F_i) = \underline{\underline{C}}, \qquad H_d^q({}^oM; F_i \boxtimes F_i) = 0 \quad (q > 0) .$$

Then (1) for $I_i$ follows from X, 3.4. For $H_d^q(G; I_i')$ we proceed similarly, using the order on $\Phi$ opposite to the given one. Then the element of $W^M$ satisfying (1), (2) of X, 3.4 is $s_i$, and X, 3.4 yields our assertion.

Now let $i = m$. There exists an element $t$ in the normalizer of $A$ such that Int $t$ induces the inversion on $A$ and an outer automorphism on ${}^oM$. It operates in a natural way on $({}^oM)^\wedge$, and it permutes $\sigma_m^+, \sigma_m^-$. This implies the equivalence of $I_m$ and $I_{P,\sigma_m^-,0}$. They are irreducible [6: Prop. 49], tempered. This proves the last assertion. Moreover, in view of 4.7 (7), (8), $t_m \in W^M$ satisfies the conditions (1), (2) of X, 3.4 for $I_m$. Since $\ell(t_m) = m$, the equality (1) for $I_m$ follows.

Finally, if $I_{P,\sigma,\nu}$ is tempered with infinitesimal character $\chi_\rho$, then $\nu = 0$ and $\sigma$ must have as infinitesimal character $\chi_\lambda$, where $\lambda \in \underline{b}_c^*$ is of the form $s\rho|_{\underline{b}}$ for some $s \in W$ such that $s\rho|_A = 0$. We may take $s \in W^M$, and then have $s = s_m, t_m$, hence $\lambda$ is in the $W_M$-orbit of either $\mu_m$ or $\mu_m^-$, and $\sigma$ is equivalent to $\sigma_m^+$ or $\sigma_m^-$. This completes the proof of 4.8.

4.9. Let $C^q(I_i)$, $C^q(I_i')$ the space of q-th cochains in $C(\underline{g},K; I_i)$ and $C(\underline{g},K; I_i')$ respectively. As in 4.4., we have

(1) $$C^q(I_i) = C^q(I_i') = \mathrm{Hom}_{{}^oM}(\Lambda^q(\underline{g}/\underline{k}), F_i) , \qquad (0 \leqq q \leqq n;\ 1 \leqq i < m) ,$$

(2) $$C^q(I_m) = \mathrm{Hom}_{{}^oM}(\Lambda^q(\underline{g}/\underline{k}),\ F_m^+)\ , \qquad (0 \leqq q \leqq n)\ .$$

We have again ${}^oM$-modules isomorphisms

(3) $$\Lambda^q(\underline{g}/\underline{k}) = F_q \oplus F_{q-1}\ , \qquad (1 \leqq q \leqq 2m)\ .$$

From this and 4.7, we deduce

(4) $$\dim C^q(I_i) = \begin{cases} \underline{\underline{C}}, & q = i, i+1, n-i-1, n-i, \\ 0, & \text{otherwise}, \end{cases} \qquad (1 \leqq i < m)\ ,$$

(5) $$\dim C^q(I_m) = \begin{cases} \underline{\underline{C}}, & q = m, m+1, \\ 0, & \text{otherwise}. \end{cases}$$

4.10. THEOREM. *Let* $n = 2m+1$, *be odd. Let* $J_i$ *be the simple Langlands quotient of* $I_i$ $(1 \leqq i < m)$ *and put* $J_m = I_m$. *Then*

(i) *We have, for* $i = 1, \ldots, m$,

(1) $$C^q(\underline{g}, K;\ J_i) = \underline{\underline{C}},\ \textit{if}\ q = i, n-i,\quad C^q(\underline{g}, K;\ J_i) = 0,\ \textit{if}\ q \neq i, n-i,$$

*in particular*

(2) $$H_d^q(G;\ J_i) = \underline{\underline{C}},\ \textit{if}\ q = i, n-i,\quad H_d^q(G;\ J_i) = 0,\ \textit{if}\ q \neq i, n-i.$$

(ii) *We have the exact sequences*

(3) $$0 \longrightarrow J_{i+1} \longrightarrow I_i \longrightarrow J_i \longrightarrow 0, \qquad (1 \leqq i < m)\ .$$

(iii) *If the irreducible admissible G-module* $(\pi, V)$ *has infinitesimal character* $\chi_\rho$, *then* $V$ *is infinitesimally equivalent to some* $J_i$ $(1 \leqq i \leqq m)$, *or the trivial representation.*

(a) Proof of (iii). If $V$ is tempered, this assertion is contained in 4.8. If not, $V$ is a Langlands quotient $J(P,\sigma,\nu)$ with $\nu > 0$, hence of the form $s_i\rho|_A$ $(1 \leqq i < m)$ (see 4.7), and the argument is the same as the one given for 4.6(iii).

(b) The proofs of (i) and (ii) parallel those of 4.6(i), (ii), and are in fact slightly simpler. We describe them briefly, writing again $H^q(V)$ and $C^q(V)$ for $H^q_d(G; V)$ and $C^q(\underline{g},K; V)$.

For $i = m$, (i) and (ii) follow from 4.8 and 4.9(5). Fix $i$ $(1 \leqq i < m)$ and assume (i) and (ii) true for $J_j$ $(i < j \leqq m)$. We have the exact sequences

$$(4) \qquad 0 \longrightarrow U_i \longrightarrow I_i \longrightarrow J_i \longrightarrow 0 ,$$

$$(5) \qquad 0 \longrightarrow J_i \longrightarrow I'_i \longrightarrow U'_i \longrightarrow 0 .$$

By (iii) and the Erratum to AVII, the constituents of $U_i$ or $U'_i$ are infinitesimally equivalent to $J_j$ $(i < j \leqq m)$. The induction assumption then implies

$$(6) \qquad H^q(U) = H^q(U') = 0 \quad \text{for} \quad q \leqq i, \quad q \geqq n-i .$$

Therefore, by the cohomology sequences associated to (4), (5) and 4.8, we get

$$(7) \qquad H^i(J_i) = H^i(I'_i) = H^{n-i}(I_i) = H^{n-i}(J_i) = \underline{C} ,$$

$$(8) \qquad H^q(J_i) = 0, \quad (q < i;\ q > n-i) ,$$

$$(9) \qquad H^{n-i-1}(U'_i) = H^{n-i}(J_i) = \underline{C} .$$

In view of the induction assumption, (9) shows that $J_{i+1}$ is a constituent of $U'_i$, hence of $U_i$. By (1) for $J_{i+1}$ and 4.9.(4), $C^*(J_i)$ is at most 2-dimensional.

But then, (1) and (3) for $i$ follow from (7) and the induction assumption.

4.11. It is known that the Langlands quotients $J_i$ in 4.6, 4.10 are unitarizable. This follows from Prop. 44, 45 in [5]. For another approach see the remark to 4.12 below. Therefore, by II, §3, (1) and (2) in 4.6, 4.10 are in fact equivalent. Also, this shows a priori that $H^*(J_i)$ satisfies Poincaré duality. If this is assumed, then the above proofs can be slightly simplified. One need not consider both $I_i$ and $I_i^!$. This is how we will proceed in §5.

4.12. Let $\Gamma$ be a cocompact discrete subgroup of $G$, and $(\sigma, E)$ be a finite dimensional unitary representation of $\Gamma$. As in IV, 4.1, let $I_2(E)$ be the space of square integrable sections of the bundle $G \times_\Gamma E \longrightarrow \Gamma\backslash G$. Then 4.6, 4.10 and IV, 4.2 imply

PROPOSITION. *Let $\Gamma$ be a discrete cocompact subgroup of $G = SO(n,1)^0$ and $q \in \underline{\mathbb{N}}$. If $q < n/2$ (resp. $q = n/2$) then $\dim H^q(\Gamma; E)$ is equal to the multiplicity of $J_q$ (resp. to the sum of the multiplicities of $D_m^+$ and $D_m^-$) in $I_2(E)$.*

*Remark.* Millson and Raghunathan have shown the existence of such groups with $H^q(\Gamma; \underline{\mathbb{C}}) \neq 0$ for $q = 0,\ldots,n$ [9]. Since the representations $J_q$ are the only ones which can give rise to cohomology, this result, together with 4.6, 4.10, and IV, 4.2 imply in fact that the $J_q$'s are unitarizable.

## §5. The group $\underline{\underline{SU}}(n,1)$

5.1. In this section we compute the cohomology of the irreducible unitary representations of $\underline{\underline{SU}}(n,1)$ $n \geq 2$, using methods similar to those of §4. It should be pointed out that Zuckerman has an independent proof of Theorem 5.13.

5.2. We first give a classification of the irreducible unitary representations of $\underline{\underline{SU}}(n,1)$ (using results of Kraljevic [7] and Langlands [8]). We will state the results without proofs. The complete proofs of the results on the composition series of the principal series will appear in Wallach [10].

5.3. Let $(P, A)$ be a minimal p-pair, $P = MN$ the standard Levi decomposition of $P$. Then $\dim A = 1$. Let $K$ equal $\underline{U}(n)$ imbedded in $\underline{\underline{SU}}(n,1)$ in the obvious fashion

$$\left( u \mapsto \left[\begin{array}{c|c} u & 0 \\ \hline 0 & (\det u)^{-1} \end{array}\right] \right).$$

Let $\theta$ be the Cartan involution of $(G, K)$. We assume (as we can) that

$$\theta a = a^{-1}, \quad a \in A.$$

5.4. We leave it to the reader to check:

(1) $2q(G) = \dim G - \dim K = 2n.$

(2) $\dim N = 2n - 1.$

(3) ${}^0M$ is connected, locally isomorphic with $\underline{\underline{U}}(n-1)$ (in fact a 2-fold covering of $\underline{\underline{U}}(n-1)$).

(4) $\Phi(P, A) = \{\beta, 2\beta\}$, $\Delta(P, A) = \{\beta\}$, $\rho_P = n\beta$.

5.5. Fix a $\theta$-stable Cartan subalgebra $\underline{h} \supset \underline{a}$. Let $\Phi$ be the root system of $(\underline{g}_c, \underline{h}_c)$. Let $W = W(\Phi) = W(\underline{g}_c, \underline{h}_c)$. Fix a system of positive roots $\Phi^+$ of $\Phi$ compatible with $\Phi(P, A)$. Using the numbering of $\Delta$ the simple roots of $\Phi^+$ as in Bourbaki [1: p. 250] we have

(1) $\Delta = \{\alpha_1, \ldots, \alpha_n\}$.

(2) $\Delta_0 = \Delta_M = \phi$ if $n = 2$. $\Delta_0 = \{\alpha_2, \ldots, \alpha_{n-1}\}$ if $n \geq 3$.

(3) $\beta = \alpha_1|_A = \alpha_n|_A$.

5.6. Let $\alpha_0 = \alpha_1 + \cdots + \alpha_n$. Then $H_{\alpha_0} \in \underline{a}$. That is $\alpha_0$ is a "real root" for $(\underline{g}, \underline{h})$. Let $\Phi_M^+ = \Phi_0^+$ be the set of positive roots spanned by $\Delta_0$. Let $\widetilde{W}^M = \{s \in W \mid s\Phi^+ \supset \Phi_M^+ \cup \{\alpha_0\}.\}$. Let $\widetilde{W}_M$ equal to the group generated by $s_{\alpha_0}$ and the $s_{\alpha_j}$, $2 \leq j \leq n-1$. Then $W = \widetilde{W}_M \cdot \widetilde{W}^M$.

5.7. As usual we write $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$. Then $\Phi^+$ corresponds to the Weyl chamber $\varepsilon_1 > \ldots > \varepsilon_{n+1}$. Let for $i, j \geq 0$, $i + j \leq n - 1$, $\Phi_{ij}^+$ be the Weyl chamber given by

$$\varepsilon_{r_1} > \ldots > \varepsilon_{r_{n+1}}$$

with $r_{i+1} = 1$, $r_{n+1-j} = n+1$ and $\varepsilon_2 > \ldots > \varepsilon_n$. It is easy to check that if

$s \in \widetilde{W}^M$ then $s\Phi^+ = \Phi^+_{ij}$ for some pair $i, j$ with $i, j \geq 0$, $i + j = n - 1$. Set $s = s_{ij}$ if $s\Phi^+ = \Phi^+_{ij}$. Then $s_{0,0} = 1$ and $\widetilde{W}^M = \{s_{ij}\}$.

5.8. It is easy to check that:

(1) $\Phi^+_{i,j}$ and $\Phi^+_{i+1,j}$ (resp. $\Phi^+_{i,j+1}$) are related by a simple reflection to $\Phi^+_{ij}$.

(2) (1) implies that $\ell(s_{ij}) = i + j$.

(3) $\ell(s_{\alpha_0}) = 2n - 1$.

5.9. Set $\underline{b} = \underline{h} \cap {}^0\underline{m}$. Then, since ${}^0M$ is connected, an irreducible representation of ${}^0M$ is uniquely determined by its highest weight on $\underline{b}$ relative to $\Phi^+$. Let $\xi_{ij}$ be the irreducible unitary representation of ${}^0M$ with highest weight $\mu_{ij} = (s_{ij}\rho - \rho)\big|_{\underline{b}}$. Set $\nu_{ij} = s_{ij}\rho\big|_{\underline{a}}$. Set $I_{ij} = I_{P, \xi_{ij}, \nu_{ij}}$. Since $i + j \leq n - 1$ we see that $\nu_{ij}$ is in the positive chamber relative to $\Delta(P, A)$. Hence the Langlands quotient $J(P, \xi_{ij}, \nu_{ij}) = J_{ij}$ is defined.

5.10. THEOREM. 1) (Kraljević [7]) $J_{ij}$ is unitarizable for all $i, j \geq 0$, $i + j \leq n - 1$.

2) Let $D_0, \ldots, D_n$ be the discrete series representations of $G$ with infinitesimal character $\chi_\rho$. Then after possible relabeling there is the following $\underline{g}$-module exact sequence

$$0 \to D_i \oplus D_{i+1} \to I_{i, n-1-i} \to J_{i, n-1-i} \to 0,$$

for $0 \leq i \leq n - 1$.

(2) of Theorem 5.10 follows from the results of Kraljević [7] and Knapp-Wallach [6]. The following theorem can be proved by methods analogous to the techniques in §4. The details will be in Wallach [10].

5.11. THEOREM. *Let* $i, j$ *be natural integers,* $i + j \leqq n - 1$ *and* $U_{ij} = \ker(I_{ij} \to J_{ij})$.

(a) *If* $i + j = n - 2$, *then there is an exact sequence*

$$0 \to D_{i+1} \to U_{ij} \to J_{i+1, j} \oplus J_{i, j+1} \to 0 ;$$

(b) *if* $i + j < n - 2$, *then there is an exact sequence*

$$0 \to J_{i+1, j+1} \to U_{ij} \to J_{i+1, j} \oplus J_{i, j+1} \to 0 .$$

5.12. LEMMA. *Let* $i, j \geq 0$, $i + j \leq n - 1$, *and* $p = i + j$. *Then*

$$H_d^r(G, I_{ij}) = \begin{cases} 0, \text{ if } r \neq 2n - 1 - p \text{ or } 2n - p, \\ \mathbb{C}, \text{ if } r = 2n - 1 - p \text{ or } 2n - p. \end{cases}$$

*Proof.* This follows from X, 3.4 and the following facts:

a) ${}^0M$ is compact hence $H_d^*({}^0M, H_\xi) = 0$ if $H_\xi$ is irreducible and non-trivial and $H_d^*({}^0M, \mathbb{C}) = H_d^0({}^0M, \mathbb{C}) = \mathbb{C}$ for $\mathbb{C}$ the trivial representation of ${}^0M$.

b) $\xi_{ij}^* = \xi_{ji}$.

c) If $s \in W$ is as in X, 3.4, then $s = s_{\alpha_0} \cdot s_{j,i}$ and $\ell(s) = 2n-1-i-j$.

5.13. THEOREM. (i) *If* $(\pi, V)$ *is an irreducible admissible* $G$-*module with infinitesimal character* $\chi_\rho$, *then* $(\pi, V)$ *is infinitesimally equivalent with* $D_i$ $(0 \leq i \leq n)$, $J_{ij}$ $(i, j \geq 0,\ 1 \leq i + j \leq n-1)$ *or the trivial representation.*

(ii) $$H^q_d(G; D_i) = \begin{cases} \mathbb{C}, & q = n, \\ 0, & q \neq n, \end{cases} \qquad (0 \leq i \leq n).$$

(iii) *If* $i, j \in \mathbb{N}$, $(1 \leq i+j \leq n-1)$, $p = i + j$, *then*

$$H^q_d(G; J_{ij}) = \begin{cases} \mathbb{C}, & \text{for } q = p + 2\ell, \quad (0 \leq \ell \leq n-p), \\ 0, & \text{otherwise.} \end{cases}$$

The proof of (i) is identical with that of 4.6(iii). Assertion (ii) was proved in VIII, 5.2. Assertion (iii) will be proved in 5.14 for $p = n-1$, 5.15 for $p = n-2$ and 5.16 for $p < n-2$.

5.14. We first compute the cohomology of $J_{ij}$ for $i + j = n - 1$. To simplify notation we write $H^r(H)$ for $H^r_d(G, H)$.

Theorem 5.10 implies that we have the exact sequence:

(1) $$\begin{aligned} &\to H^r(D_i) \oplus H^r(D_{i+1}) \to H^r(I_{i,\,n-1-i}) \to H^r(J_{i,\,n-1-i}) \to \\ &H^{r+1}(D_i) \oplus H^{r+1}(D_{i+1}) \to H^{r+1}(I_{i,\,n-1-i}) \to \end{aligned}$$

If $r \leq n - 2$ or $r \geq n + 2$ then, by 5.13(ii) and 5.12,

(2)

$$H^r(D_i) \oplus H^r(D_{i+1}) = H^{r+1}(D_i) \oplus H^{r+1}(D_{i+1}) = H^r(I_{i,n-1-i}) = H^{r+1}(I_{i,n-1-i}) = 0.$$

Hence

(3) $$H^r(J_{i,n-1-i}) = 0 \quad \text{if} \quad r \leq n-2, \; r \geq n+2 .$$

Using 5.10 again and (3) we find

(4) $$\dim H^{n-1}(J_{i,n-1-i}) - \dim H^n(J_{i,n-1-i}) + \dim H^{n+1}(J_{i,n-1-i}) = 2 .$$

Poincaré duality implies

(5) $$\dim H^{n-1}(J_{i,n-1-i}) = \dim H^{n+1}(J_{i,n-1-i}) .$$

By (1) for $r = n-1$ and 5.12 we get an exact sequence

(6) $$0 \to H^{n-1}(J_{i,n-1-i}) \to H^n(D_{i+1}) \oplus H^n(D_{i+2}) \to H^n(I_{i,n-1-i}) \to H^n(J_{i,n-1-i}) \to 0$$

which, by 5.13(ii) and 5.12, can be written

(7) $$0 \to H^{n-1}(J_{i,n-1-i}) \to \underline{\underline{C}} \oplus \underline{\underline{C}} \to \underline{\underline{C}} \to H^n(J_{i,n-1-i}) \to 0 .$$

There are therefore two possibilities:

(a) $$H^{n-1}(J_{i,n-1-i}) = \underline{\underline{C}} \oplus \underline{\underline{C}} \quad \text{and} \quad H^n(J_{i,n-1-i}) = \underline{\underline{C}} ,$$

(b) $$H^{n-1}(J_{i,n-1-i}) = \underline{\underline{C}} \quad \text{and} \quad H^n(J_{i,n-1-i}) = 0 .$$

But (a) contradicts (4) and (5), hence (b) holds. This proves Theorem 5.13(iii) for $i + j = n - 1$.

5.15. We now look at the case $i + j = n - 2$. We must study the exact sequences of Theorem 5.11. They yield the following long exact sequences:

(1) $$\to H^r(D_{i+1}) \to H^r(U_{ij}) \to H^r(J_{i+1,j}) \oplus H^r(J_{i,j+1}) \to H^{r+1}(D_{i+1}) \to$$

(2) $$\to H^r(I_{ij}) \to H^r(J_{ij}) \to H^{r+1}(U_{ij}) \to H^{r+1}(I_{ij}) \to$$

By 5.12, $H^r(I_{ij}) = 0$ if $r \neq n+1,\ n+2$, hence

(3) $$H^r(J_{ij}) = H^{r+1}(U_{ij}), \quad \text{if} \quad r \leqq n-1 \quad \text{or} \quad r \geqq n+3 \,.$$

By (1), 5.3(iii) for $i + j = n - 1$, and 5.13(ii), we have

(4) $$H^r(U_{ij}) = 0 \quad \text{if} \quad r \leqq n-2,\ r \geqq n+c\,,\ H^{n+1}(U_{ij}) = \underline{\underline{C}} \oplus \underline{\underline{C}}\,,$$

whence

(5) $$H^r(J_{ij}) = 0 \quad (r \leqq n-3;\ r \geqq n+3),\ H^{n+2}(J_{ij}) = \underline{\underline{C}}\,.$$

The sequence (1) for $r = n - 1$ gives rise to the exact sequence

(6) $$0 \to H^{n-1}(U_{ij}) \to \underline{\underline{C}} \oplus \underline{\underline{C}} \to \underline{\underline{C}} \to H^n(U_{ij}) \to 0\,.$$

The relations (4), (6) show that the Euler characteristic $\chi(U_{ij})$ satisfies

(7) $$\chi(U_{ij}) = \Sigma_q (-1)^q \dim H^q(U_{ij}) = 3(-1)^{n+1} .$$

By 5.12, $\chi(I_{ij}) = 0$, hence,

(8) $$\chi(U_{ij}) + \chi(J_{ij}) = 0, \quad \chi(J_{ij}) = 3(-1)^n .$$

Taking (2), (4) and 5.12 into account, we get an exact sequence

(9) $$0 \to H^n(J_{ij}) \to \underset{=}{C} \oplus \underset{=}{C} \to \underset{=}{C} \to H^{n+1}(J_{ij}) \to 0 .$$

By (5) and Poincaré duality, we have

$$(-1)^n \cdot \chi(J_{ij}) = 2 - 2 \cdot \dim H^{n+1}(J_{ij}) + \dim H^n(J_{ij}) ,$$

hence, by (8)

(10) $$\dim H^n(J_{ij}) - 2 \cdot \dim H^{n+1}(J_{ij}) = 1 .$$

Together with (9), this readily implies

(11) $$H^n(J_{ij}) = \underset{=}{C} , \quad H^{n+1}(J_{ij}) = 0 .$$

5.13(iii) for $i + j = n - 2$ now follows from (5), (11) and Poincaré duality.

5.16. We now suppose that we have proved Theorem 5.13(iii) for $i + j \geq p + 1$ and $p \leq n - 3$. We look at the case $i + j = p$. Again we have two long exact sequences:

(1) $$\to H^r(I_{ij}) \to H^r(J_{ij}) \to H^{r+1}(U_{ij}) \to H^{r+1}(I_{ij}) \to$$

(2) $$\to H^r(J_{i+1,\,j+1}) \to H^r(U_{ij}) \to H^r(J_{i+1,\,j}) \oplus H^r(J_{i,\,j+1}) \to H^{r+1}(J_{i+1,\,j+1}) \to$$

Suppose $r \geq 2n - p$; then by the inductive hypothesis $H^r(J_{i+1,\,j+1}) = 0$ and $H^r(J_{i+1,\,j}) \oplus H^r(J_{i,\,j+1}) = 0$. Hence (2) implies $H^r(U_{ij}) = 0$ for $r \geq 2n - p$. Similarly $H^r(U_{ij}) = 0$ for $r \leq p$.

Lemma 5.16 says $H^{2n-p}(I_{ij}) = \underline{\underline{C}}$. Thus (1) implies:

(3) $$\dim H^{2n-p}(J_{ij}) \leqq 1 .$$

Hence also, by Poincaré duality, $\dim H^p(J_{ij}) \leqq 1$.

Now $H^r(I_{ij}) = 0$ for $r < 2n - i - j - 1$ and $r > 2n - i - j$. Thus (1) implies

(4) $$H^r(J_{ij}) = H^{r+1}(U_{ij}), \quad \text{for } r < 2n - i - j - 2 .$$

This, combined with the above observations, gives

(5) $$H^r(J_{ij}) = H^{2n-r}(J_{ij}) = 0 \quad \text{for } r \leq p - 1 .$$

We now look at (2) in the case $r = p + 1$, and get the exact sequence

(6) $$0 \to H^{p+1}(U_{ij}) \to \underline{\underline{C}} \oplus \underline{\underline{C}} \to \underline{\underline{C}} \to H^{p+2}(U_{ij}) \to 0 \ .$$

Hence there are two possibilities

(a) $$H^{p+1}(U_{ij}) = \underline{\underline{C}} \oplus \underline{\underline{C}} \quad \text{and} \quad H^{p+2}(U_{ij}) = \underline{\underline{C}}$$

(b) $$H^{p+1}(U_{ij}) = \underline{\underline{C}} \quad \text{and} \quad H^{p+2}(U_{ij}) = 0 \ .$$

Now in Case (a) we would have by (4) $\dim H^p(J_{ij}) = 2$. But this contradicts (3). We therefore see that (b) is true. Hence (4) implies that

(7) $$H^p(J_{ij}) = \underline{\underline{C}} \ , \quad H^{p+1}(J_{ij}) = 0 \ .$$

We apply (2) to $r = 2n - p - 1$ and find,

$$H^{2n-p-1}(U_{ij}) = \underline{\underline{C}} \oplus \underline{\underline{C}} \ .$$

Hence (1), Lemma 5.12 and (7) imply the exactness of the sequence:

$$0 \to H^{2n-p-2}(J_{ij}) \to \underline{\underline{C}} \oplus \underline{\underline{C}} \to \underline{\underline{C}} \to 0 \ .$$

We therefore see that

(8) $$H^{2n-p-2}(J_{ij}) = \underline{\underline{C}} = H^{p+2}(J_{ij}).$$

Using (8) and (4) we get $H^{p+3}(U_{ij}) = \underset{=}{C}$. Now (2) implies the exactness of

$$0 \to \underset{=}{C} \to \underset{=}{C} \oplus \underset{=}{C} \to \underset{=}{C} \to H^{p+4}(U_{ij}) \to 0 .$$

Hence $H^{p+4}(U_{ij}) = 0$ and therefore

(9) $$H^{p+3}(J_{ij}) = H^{2n-3-p}(J_{ij}) = 0 .$$

Suppose that we have shown that

$$H^{p+2s+1}(J_{ij}) = 0, \ H^{p+2s}(J_{ij}) = \underset{=}{C}$$

for $0 \leq s \leq \ell - 1$, $\ell \geq 2$ and $p + 2(\ell-1) \leq 2n - p$. If $p + 2(\ell-1) = 2n - p$ our assertion is proved. Assume $p + 2(\ell-1) < 2n - p$.

Using $s = \ell - 1$ we find $H^{p+2\ell-1}(J_{ij}) = 0$ and $H^{p+2\ell-2}(J_{ij}) = \underset{=}{C}$. Hence $H^{2n-p-2\ell+1}(J_{ij}) = \underset{=}{C}$. Since $2\ell - 1 \geq 3$ we see that (4) applies. Hence $H^{2n-p-2\ell+2}(U_{ij}) = 0$. Applying (2) with $r = 2n - p - 2\ell + 2$ we find the exact sequence

(10) $$0 \to H^{2n-p-2\ell+1}(U_{ij}) \to \underset{=}{C} \oplus \underset{=}{C} \to \underset{=}{C} \to 0 ,$$

hence $H^{2n-p-2\ell+1}(U_{ij}) = \underset{=}{C}$ and thus (4) implies $H^{2n-p-2\ell}(J_{ij}) = \underset{=}{C}$.

We therefore see that $H^{p+2\ell}(J_{ij}) = \underset{=}{C}$.

By Poincaré duality we may assume that $p + 2\ell \leq n$, which implies that $p + 2\ell + 1 < 2n - p - 2$ (since $p \leq n - 3$). Applying (4) to $r = p + 2\ell$, $p + 2\ell + 1$ and (2) we get successively $H^{p+2\ell+1}(U_{ij}) = \underline{\underline{C}}$,

$$H^{p+2\ell+2}(U_{ij}) = 0 \quad \text{and} \quad H^{p+2\ell+1}(J_{ij}) = 0 .$$

This completes the induction.

5.17. We now give an interpretation of the "periodicity" in Theorem 5.13. Let $\underline{g} = \underline{k} \oplus \underline{p}$ be the Cartan decomposition of $\underline{g}$ corresponding to $\underline{k}$. (This notation is unfortunately inconsistent with our previous conventions.) As is well-known $\underline{p}$ has a complex structure $J$ so that $\mathrm{Ad}(K)|_{\underline{p}}$ acts by complex linear transformations. $B|_{\underline{p}}$ is then the real part of a Hermitian $\mathrm{Ad}(k)$ - invariant form $\sigma$ on $\underline{p}$ (relative of $J$). Let $\sigma = B|_{p} + i\omega$. Then $\omega \in \Lambda^2\underline{p}^*$. It has the following properties:

(1) $\Lambda^2\mathrm{Ad}(k) \cdot \omega = \omega$ for $k \in K$.

(2) Set $Lu = \omega \wedge u$ for $u \in \Lambda^r\underline{p}^*$. Then $L : \Lambda^r\underline{p}^* \to \Lambda^{r+2}\underline{p}^*$ is injective for $r \leq n - 1$.

(1) is clear. A proof of (2) can be found in [12: p. 28].

Now if $(\pi, H)$ is a smooth representation of $G$, then we can identify $\mathrm{Hom}_K(\Lambda^j(\underline{g}/\underline{k}), H) = C^j(H)$ with $\mathrm{Hom}_K(\Lambda^j\underline{p}, H) = (\Lambda^j\underline{p}^* \otimes H)^K$. Now we map $C^j(H) \to C^{j+2}(H)$ as follows: If $u = \Sigma\, \eta_i \otimes v_i \in C^s(H)$ then $Lu = \Sigma\, \omega \wedge \eta_i \otimes v_i \in C^{s+2}(H)$. Using (2) we see that

(3) if $j \leq n - 1$ then $L : C^j(H) \to C^{j+2}(H)$ is injective.

5.18. Now let $H = J_{ij}$ for some $i, j \geq 0$, $i + j \leq n - 1$. Then $H^r_d(G; H) = H^r(\underline{g}, K; H) = C^r(H)$, by IX, 5.6 and II, §3. We therefore have an injective mapping

(1) $L : H^r_d(G; H) \to H^{r+2}_d(G; H)$ for $r \leq n - 1$.

Theorem 5.13 now says

(2) $H^{i+j+2r}_d(G; J_{ij}) = L^r H^{i+j}_d(G; J_{i,j})$ for $r \leq \frac{1}{2}(n-i-j-1)$.

5.19. Following Weil [12] we call $\eta \in H^r_d(G; J_{i,j})$ primitive if $r \leq n$ and $L^{n-r+1}\eta = 0$. Let $H^*_{prim}(G; J_{ij})$ be the space of all primitive classes. By [12:I, n°4] and 5.18(2), we get the following result:

5.20. LEMMA. $H^r_{prim}(G; J_{ij}) = 0$ *if* $r \neq i + j$ *and* $H^{i+j}_{prim}(G; J_{ij}) = \mathbb{C}$. *Furthermore* $H^*_d(G; J_{ij}) = \Sigma\, L^r H^{i+j}_{prim}(G; J_{ij})$.

5.21. Let $\Gamma \subset \underline{SU}(n, 1)$ be a discrete cocompact subgroup without non-central elements of finite order. The space $G/K$ may be identified with $B^n = \{z \in \mathbb{C}^n \mid \Sigma\, |z_i|^2 < 1\}$, with $G$ acting holomorphically and leaving invariant the Kähler structure on $B^n$ corresponding to the Bergmann metric on $B^n$. Set $X = \Gamma\backslash G/K = \Gamma\backslash B^n$. Then $X$ is a compact Kähler manifold.

5.22. Let $L : H^r(X, \mathbb{C}) \to H^{r+2}(X, \mathbb{C})$ correspond to multiplication by the Kähler form. Then $H^r(X, \mathbb{C}) = \sum_j L^j H^{r-2j}_{Prim}(X, \mathbb{C})$ where $H^r_{prim}(X, \mathbb{C})$

is the primitive cohomology of $X$. That is, $\eta \in H^r_{prim}(X, \underset{=}{C})$ if and only if $r \leq n$ and $L^{n-r+1}\eta = 0$. (See Weil [12]) .

5.23. If $(\pi, H)$ is an irreducible unitary representation of $G$ then we use the notation $H^r(H)$ for $H^r(\underline{g}, K;H)$ and $H^r_{prim}(H)$ for the primitive cohomology.

Unwinding the mapping of Chapter IV, we see that there is a map

(1) $$\psi : H^r(H) \otimes \mathrm{Hom}_G(H, L^2(\Gamma\backslash G)) \to H^r(X;\underset{=}{C}),$$

such that

(2) $$\psi \circ (L \otimes I) = L \circ \psi .$$

This can also be seen easily if we represent cohomology by harmonic forms and expand the harmonic forms in terms of the irreducible representations of $G$. Combining the above results, we get

5.24. THEOREM. (a) If $r = n$ then

$$\dim H^n_{prim}(X, \underset{=}{C}) = \sum_{i=0}^{n} \dim \mathrm{Hom}_G(D_i, L^2(G)) .$$

(b) If $0 < r \leq n - 1$ then

$$\dim H^r_{prim}(X, \underset{=}{C}) = \sum_{\substack{i, j \geq 0 \\ i+j=r}} \dim \mathrm{Hom}_G(J_{i,j}, L^2(\Gamma\backslash G)) .$$

5.25. It can be shown that if $H^{i,j}(X, \mathbb{C})$ is the space of harmonic $(i,j)$-forms on $X$ then $\dim H^{i,j}_{prim}(X, \mathbb{C}) = \dim \mathrm{Hom}_G(J_{i,j}, L^2(\Gamma\backslash G))$. (after possibly reversing the complex structure).

## §6. Some further special cases

In this section $(P, A)$ is a standard p-pair in $G$, $(\bar{P}, A)$ the opposite p-pair, and $P = M.N$ the standard Levi decomposition of $P$. If $V$ is a $G$-module and $q \in \mathbb{N}$, then $H^q(V)$ stands for $H^q_d(G;V)$.

6.1. We assume that $P$ is proper maximal (hence $\dim A = 1$), cuspidal, ${}^0M$ connected and that $G$ has a discrete series. Then $q(G)$, $q({}^0M)$ (see X, 4.4.) are integers. We have

$$(1) \qquad q(G) = q({}^0M) + 1/2 + (\dim N)/2.$$

In particular, $\dim N$ is odd. Moreover, $P$ is not fundamental, hence X, 4.2 implies that

$$(2) \qquad s\rho\big|_A \neq 0, \qquad \text{for all } s \in W.$$

Note also that we have

$$(3) \qquad -\rho_P = w_M . w_G(\rho)\big|_A \leqq s\rho\big|_A \leqq \rho_P = \rho\big|_A, \qquad (s \in W).$$

6.2. PROPOSITION. We keep the assumptions of 6.1. Let $\nu_0 \in \underline{a}^*$ be

$> 0$ *and of the form* $s\rho|_A$ *for some* $s \in W$. *Assume that the canonical extension* $\Lambda(P, \nu_0)$ *of* $\nu_0$ *to a linear form on* $\underline{a}_0^*$ *defined in the Erratum to* AVII *is minimal with respect to the partial ordering defined there. Fix* $s \in W^M$ *such that* $s\rho|_A = \nu_0$. *Let* $(\sigma, H_\sigma)$ *be a discrete series representation of* ${}^0M$ *such that* $\chi_\sigma = \chi_{s\rho|\underline{b}}$, *(see* (X, § 2) *for* $\underline{b}$). *Let* $J = J(P, \sigma, \nu_0)$. *Then* $\ell(s) = (\dim N\text{-}1)\ 2$ *and we have*

(1) $$H_d^q(G;J) = \underline{\underline{C}} \text{ for } q = q(G) \pm 1,\quad H_d^q(G;J) = 0 \text{ for } q \neq q(G) \pm 1,\ q(G).$$

Let $I = I_{P,\sigma,\nu_0}$, $I' = I_{\overline{P},\sigma,\nu_0}$. As in §4, we consider the two exact sequences of 2.3

(1) $$0 \to U \to I \to J \to 0\,,$$

(2) $$0 \to J \to I' \to U' \to 0\,.$$

It follows from the Erratum to AVII, and our assumptions that the constituents of $U$ or, equivalently, of $U'$, are tempered, and therefore square integrable, since they have infinitesimal character $\chi_p$ [3: Lemma 73]. By VIII, § 5, we have then

(3) $$H^q(U) = H^q(U') = 0\ (q \neq q(G)),\ H^q(U) \neq 0,\ H^q(U') \neq 0,\ (q = q(G))\,.$$

(In fact, for $q = q(G)$, $\dim H^q(U)$ and $\dim H^q(U')$ equal the number of constituents of $U$.) The cohomology sequences associated to (1) and (2) then yield

$$H^q(I) = H^q(J), \quad (q \neq q(G)-1, q(G)), \tag{4}$$

$$H^q(J) = H^q(I'), \quad (q \neq q(G), q(G)+1). \tag{5}$$

Let $s' = w_M \cdot s \cdot w_G$ (see 1.4). Then, by 1.4(2) and the assumption on $s$:

$$s'\rho\big|_A + \nu_0 = 0. \tag{6}$$

If $L_t$ denotes the irreducible finite dimensional representation of ${}^0M_c$ with highest weight $(t\rho-\rho)\big|_{\underline{b}}$ $(t \in W^M)$, then $L_s$ and $L_{s'}$ are contragredient of one another (1.4(3)), hence $s'$ satisfies the relation

$$\chi_\sigma = \chi_{-s'(\rho)|_{\underline{b}}}. \tag{7}$$

(6) and (7) are the conditions (1), (2) of X, 3.4. Moreover, by VIII, §5, we have

$$H_d^q({}^0M; H_\sigma \otimes L_{s'}) = \begin{cases} \underline{\underline{C}}, & \text{if } q = q({}^0M) \\ 0, & \text{otherwise}. \end{cases} \tag{8}$$

Therefore, X, 3.4 yields

$$H^q(I) = \begin{cases} \underline{\underline{C}}, & q = q({}^0M) + \ell(s'),\ q({}^0M) + \ell(s') + 1. \\ 0, & \text{otherwise}. \end{cases} \tag{9}$$

Similarly, or by 1.5, we get

$$(10)\qquad H^q(I') = \begin{cases} \underline{\underline{C}}, & q = q({}^0M) + \ell(s),\ q({}^0M) + \ell(s) + 1, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

We want to prove that $\ell(s) \leqq (\dim N-1)/2$. Assume the contrary. Then $\ell(s) \geqq (\dim N+1)/2$ (since $\dim N$ is odd, cf. 6.1), hence $q({}^0M) + \ell(s) \geqq q(G)$, and we get, by (5) and (10), $H^q(J) = 0$ for $q < q(G)$. On the other hand $\ell(s') \leqq (\dim N-1)/2$, hence $q({}^0M) + \ell(s') + 1 \leqq q(G)$ and, by (4), (9), $H^q(J) = 0$ for $q > q(G)$. Thus $H^q(J)$ could be $\neq 0$ only for $q = q(G)$. But then (3) and the exact cohomology sequence associated to (1) would imply the same for $H^q(I)$, which contradicts (9). Therefore $\ell(s) \leqq (\dim N-1)/2$. But then (4), (9) imply

$$(11)\qquad H^q(J) = \begin{cases} \underline{\underline{C}}\,, & q = q({}^0M) + \ell(s) + 1, \\ 0\,, & q < q(G) - 1,\ q > q({}^0M) + \ell(s') + 1, \end{cases}$$

and (5), (10) yield

$$(12)\qquad H^q(J) = \begin{cases} \underline{\underline{C}}\,, & q = q({}^0M) + \ell(s), \\ 0\,, & q > q(G) + 1,\ q < q({}^0M) + \ell(s). \end{cases}$$

Comparing (11), (12), we see that we must have

(13) $$q({}^0M) + \ell(s) = q(G) - 1,$$

hence, in view of 6.1(1)

(14) $$\ell(s) = (\dim N-1)/2 .$$

We have then $\ell(s') = (\dim N+1)/2$, $q({}^0M) + \ell(s') = q(G)$, and (1) follows from (11), (12).

6.3. Remarks. (1) As in §4, we have *a priori* two possibilities for the cohomology in middle dimension $q = q(G)$. Either $H^q(J) = H^q(U) = \mathbb{C}$, or $H^q(J) = 0$, $H^q(U) = \mathbb{C}^2$. We do not know whether the former one does occur.

(2) The assumptions of 6.1 have been made largely for convenience, to avoid technicalities and get a simple statement, but they are not all essential. One could e.g. dispense with the assumption that $G$ has a discrete series, (or even that $P$ is maximal), and still get under the minimality assumption of 6.2 on $\Lambda(P,\nu_0)$, cohomology in dimension $q_0(G) - 1$, i.e. closest to the range for the cohomology with respect to tempered representations (X, 6.1). The connectedness of ${}^0M$ has been assumed mainly to insure the non-vanishing part of (8). If on the other hand, the cohomology spaces in (8) are all zero, then $I$ is irreducible, as follows from the following proposition.

6.4. PROPOSITION. *Assume* $G$ *has a discrete series. Let* $J = J(P, \sigma, \nu)$ *be a Langlands quotient in* $\mathcal{E}_0(G)$ *and* $I = I_{P,\sigma,\nu}$. *Assume that the extension* $\Lambda(P, \nu)$ *of* $\nu$ *is minimal (cf. Erratum to AVII), and that* $H_d^*(G;I) = 0$. *Then* $I$ *is irreducible.*

Let $I' = I_{\overline{P},\sigma,\nu}$. Then $H^*(I') = 0$ by 1.5.

Assume that $I$ is not simple. Then we have the exact sequences

$$(1) \qquad 0 \to U \to I \to J \to 0, \qquad 0 \to J \to I' \to U' \to 0,$$

Since $H^*(I) = H^*(I') = 0$, the associated cohomology sequences yield the isomorphisms

$$(2) \qquad H^q(J) = H^{q+1}(U), \qquad H^q(U') = H^{q+1}(J), \qquad (q \in \mathbb{N}).$$

As in 6.2, the constituents of $U$ or $U'$ belong to the discrete series, and 6.2(3) holds. But then the two equalities of (2) cannot be true simultaneously, whence a contradiction.

*Remark.* Here again, these results hold, with a similar proof, if $G$ has no discrete series.

6.5. We give now a necessary condition for the cohomology with respect to a Langlands quotient in $\mathcal{E}_0(G)$ to be non-zero at the $\mathbb{R}$-rank.

Let $r = \mathrm{rk}_{\mathbb{R}} G$ and $c = \dim A$. We identify $\Delta(P, A)$ with a subset of $\mathbb{R}^\Delta$ as in 1.2(3). Let $J = J(P, \sigma, \nu)$ be a Langlands quotient in $\mathcal{E}_0(G)$.

PROPOSITION. <u>Assume that</u> $H_d^r(G;J) \neq 0$. <u>Then the following conditions are satisfied:</u>

(i) $\nu = s\rho|_A$, <u>where</u> $\ell(s) = c$ <u>and, if</u> $s = s_{\alpha_1} \cdots s_{\alpha_c}$ <u>is a reduced decomposition of</u> $s$, <u>then</u> $\{\alpha_i\}_{1 \leq i \leq c}$ <u>contains one representative of each set</u> $\Delta_\beta$ $(\beta \in \Delta(P, A))$ (cf. 1.2(3)).

(ii) <u>The non-compact simple factors of</u> ${}^0M$ <u>are locally isomorphic to</u> $SL_2(\mathbb{R})$, $SL_3(\mathbb{R})$ <u>or</u> $SL_2(\mathbb{C})$.

We consider the exact sequence 2.3(2):

$$0 \to J \to I' \to U' \to 0 , \qquad \text{where } I' = I_{\overline{P}, \sigma, \nu} . \tag{1}$$

Let $q < r$. By 3.3 $H^q(V) = 0$ for every constituent of $U'$, hence $H^q(U') = 0$ (see 3.3(a)). Therefore $H^r(J) \to H^r(I')$ is injective, whence $H^r(I') \neq 0$. We consider the ordering of $\Phi$ opposite to the given one and let $\tilde{\rho}$ be half the sum of the positive roots. Thus $\tilde{\rho} = -\rho$. By X, 3.4, there exists $s \in W^M$ such that

$$s\tilde{\rho}|_A + \nu = 0 . \tag{2}$$

By 2.1(1), we have $\nu \in \underline{a}_P^{*+}$, $\rho_P - \nu \in {}^+\underline{a}_P^*$; the second condition can be written

$$\rho_{\overline{P}} + \nu \in {}^+\underline{a}_{\overline{P}}^* . \tag{3}$$

Lemma 3.1 implies then $\ell(s) \geq \dim A$. By X, 6.1, $H^q(I) = 0$ for $q < q_0({}^0M) + \ell(s)$. We have

$$(4) \qquad q_0({}^0M) \geqq \mathrm{rk}_{\underline{\underline{R}}}\, {}^0M,$$

(X, 4.4), hence

$$(5) \qquad q_0({}^0M) + \ell(s) \geqq \mathrm{rk}_{\underline{\underline{R}}}\, {}^0M + \dim A = r,$$

with equality if and only if

$$(6) \qquad q_0({}^0M) = \mathrm{rk}_{\underline{\underline{R}}}\, {}^0M, \quad \ell(s) = \dim A .$$

The assertions (i), (ii) now follow from 3.1 and X, 4.5.

## REFERENCES


[1] N. Bourbaki, Groupes et algèbres de Lie, Chap. IV, V, VI, Act. Sci. Ind. 1337, Hermann, Paris 1968.

[2] __________, Groupes et algèbres de Lie, Chap. VII, VIII, Act. Sci. Ind. 1364, Hermann, Paris 1975.

[3] Harish-Chandra, "Discrete series for semi-simple Lie groups II" Acta Math. 116 (1966), 1-111.

[4] ______________, "Harmonic analysis of real reductive groups III. The Maass-Selberg relations and the Plancherel formula", Annals of Math. (2) 104(1976), 117-201.

[5] A. Knapp and E. Stein, "Intertwining operators for semisimple groups" Annals of Math. (2) 93 (1971), 489-578.

[6] A. Knapp and N. Wallach, "Szegö kernels associated with the discrete series" Inv. Math. 34 (1976), 163-200.

[7] H. Kraljević, "On representations of the group $SU(n,1)$, to appear.

[8] R. P. Langlands, "On the classification of representations of real algebraic groups" mimeographed Notes, The Institute for Advanced Study, 1973.

[9] J. J. Millson and M. S. Raghunathan, "Geometric construction of homology for arithmetic groups, I. Unit groups of anisotropic quadratic forms", preprint.

[10] N. Wallach, "Harmonic analysis on homogeneous spaces II," to appear.

[11] G. Warner, "Harmonic analysis on semi-simple Lie groups I", Grund. Math. Wiss. 188, Springer, 1972.

[12] A. Weil, Variétés kähleriennes, Act. Sci. Ind. 1267, Hermann, Paris 1958.