Records of the Office of the Director / Faculty Files / Box 10 / Einstein, Albert -- 1951 Einstein Prize Award (J. Schwinger)
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

Remarks by Dr. J. Robert Oppenheimer, Director of the Institute for Advanced Study, at the presentation of the first Albert Einstein Award, at Princeton, March 14, 1951, to Professor Julian Schwinger of Harvard University.

It will hardly be necessary to remind physicists of the debt that we owe to Julian Schwinger, and of the record of high achievement which makes it appropriate that he be one of the first recipients of the Einstein Award. Even in the field of atomic physics, and in the history of the last decade, where the close, informal collaboration between scientists has often made it hard to discern an individual contribution, we can clearly see the bright beacons lighted by Schwinger's hand.

He was the first to have insight into the deeper reasons why atomic nuclei are not little round spheres, but generally have an asymetric shape. He was the first to point out what valuable information about nuclear interactions could be obtained from experiments in which more than one nucleus affected the scattering of neutrons. He developed new and immensely powerful methods for the treatment of electro-magnetic waves, which have formed the basis for much of the practical work with micro waves, of such great civil and military significance. He has developed powerful, new mathematical tools for the analysis of collisions, on which we depend so heavily for our understanding of the relations between elementary particles. But perhaps his greatest work, as well as the most recent, has been to give us a new understanding of that old and deep problem of the interaction of light and matter, to sweep away the confusions and inhibitions of more than two decades of physics, and to give us new and correct insight into the properties of electrons and of light themselves.

This is not the place to spell out the history of quantum electrodynamics, a history no easier to write because of the follies that have so long obscured it. In his recent work in this field, Schwinger has shown in the highest measure three qualities characteristic of him and characteristic of our science at its best. The first is a steadfast preoccupation with the consequences of a logical, theoretical structure with regard to experimentally accessible predictions, of which perhaps the most brilliant example is the prediction of small anomalies in the magnetic properties of the electron itself. The second characteristic is a preoccupation with the logical meaning of physical theory, with the definition in terms of physical experience of the concepts with which the theory deals. We have learned how revolutionary the consequences of these preoccupations can be from the great work of Einstein himself. But in the reformulation of wuantum electrodynamics, that we owe so largely to Schwinger, these conceptual clarifications have played an integral and decisive part. The third quality of Schwinger's work, which we also rightly honor by an Einstein Award, is his extraordinary feeling for mathematical form, and his ability to use mathematical beauty as a tool in finding the real meaning of a theory. As a result, we have today, for the first time, an at least preliminary understanding of the implications resulting from the combined assumptions of the theory of relativity and of the quantum theory, and of the existence of charged particles in nature.

It is common, in praising the work of a scientist, to point to the good effects his guidance and example have had on the young people with whom he has worked. What Schwinger has done for physicists is to remind us again that mathematical physics can be beautiful, and that its power can be enhanced, and its truth deepened, by a man who is respectful of that beauty.

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From: Rcom 5600
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New York 20, New York

Telephone: CIrcle 7-3700, Ext. 89

FOR RELEASE IN MORNING PAPERS, MONDAY, MARCH 12, 1951

The first Albert Einstein Award for achievement in the natural sciences has been won by Professor Julian Schwinger of Harvard University, a mathematical physicist, and Professor Kurt Godel, member of the Institute for Advanced Study, Princeton, New Jersey, a mathematical logician.

The awards were made by a committee and announced today by Lewis L. Strauss, President of the Board of Trustees of the Institute for Advanced Study. Mr. Strauss established the award in memory of his parents, the late Lewis and Rosa Strauss, of Richmond, Virginia, who were interested in advancing the natural sciences.

Formal presentation of the award will take place at Princeton on Wednesday, March 14, on the occasion of Dr. Einstein's seventy-second birthday. Dr. Einstein and other noted scientists will be present.

The winners will divide the $15,000 prize and will receive a medal which has been designed by Gilroy Roberts, Philadelphia, sculptor and engraver, who has done much medallic work for the United States mint.

The award committee consisted of Dr. Einstein, Dr. Robert Oppenheimer, director of the Institute for Advanced Study; Dr. John von Neumann and Dr. Hermann Weyl, both of the Institute for Advanced Study.

The committee said it has been "persuaded of the superlative qualifications" of Professor Schwinger and Professor Godel and, under

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the circumstances, concluded the first Albert Einstein Award should be made to both candidates.

Professor Schwinger has done outstanding work in the field of atomic physics.

His most recent, and rated by many as his greatest, work has given science new understanding of the problem of interaction of light and matter and the properties of electrons and light. His calculations on the influence of self-energy on the hydrogen fine-structure and on the magnetic moment of the electron generally are regarded as a major advance in the understanding of quantum electro-dynamics.

He developed new methods for the treatment of electro-magnetic waves. This formed the basis for much of the practical work with micro waves and had great civil and military significance. He originated new mathematical tools for the analysis of collisions, on which scientists depend heavily for understanding relations between elementary particles.

During World War II, at the radiation laboratory of the Massachusetts Institute of Technology, Dr. Schwinger developed new methods for the treatment of electro-magnetic waves. These methods furnished the basis for the investigations by a substantial part of the laboratory's theoretical group.

Early in his career he contributed many important suggestions about the structure of atomic nuclei and about many decisive experimental studies of nuclear interaction.

Dr. Godel's work in mathematical logic is regarded as one of the

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greatest contributions to the sciences in recent times. He was able to prove the existence in a properly codified mathematical system of propositions inherently "undecidable."

Following a broad study of the logic of provable and disprovable propositions, he made further notable contributions to science by proving that two of the axioms generally used by mathematicians, although frequently doubted, namely the "axiom of choice" and the "cantor continuum hypothesis", are consistent with the other axioms of set theory if these axioms are consistent.

He has gone deeply into the history of logical and scientific ideas. Although he has not published much on the subject, he is an authority on Leibniz. It was part of Leibniz' program for science to work out a symbolic logic of the sort which is now being developed and of which Dr. Godel is a leading protagonist.

Dr. Schwinger was born in New York City, February 12, 1918 and received his AB and PhD at Columbia University. He has been a National Research Fellow at the University of California, instructor in physics at Purdue University and has lectured at the University of Michigan. He has been at Harvard since 1945 and Professor of Physics since 1947.

Dr. Godel was born in Brunn, Czechoslovakia, April 28, 1906. He received his PhD in 1930 at the University of Vienna and taught at the University of Vienna from 1933 to 1938, when he came to the United States. In this country, he has been connected with the Institute for Advanced Study, of which he has been a permanent member since 1946. He has taken out his first naturalization papers.

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Revised press statement (Schwinger) from Strauss 3/2/51

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Same

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He has contributed importantly to the knowledge of atomic nuclei and experimental work on nuclear interactions.

During World War II, he conducted research at the Metallurgical Laboratory at Chicago and at the Radiation Laboratory at the Massachusetts Institute of Technology. He devised methods of treating wave propogation problems which furnished the basis for investigation of a substantial part of the University's theoretical group

(Gardner told Strauss that you thought the parahydrogen work of importance; but in your statement which he received today he could not fdentify your reference. I told him I thought you had just put it in a different way. Strauss will be in his office all day to receive a call if you want to talk to him. von Neumann is preparing his statement about Godel today and I told Strauss he should have it by tomorrow morning.)

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February 28, 1951

Professor Julian Schwinger

It will hardly be necessary to remind physicists of the debt that we owe to Julian Schwinger, and of the record of high achievement which makes it appropriate that he be one of the first recipients of the Einstein Award. Even in the field of atomic physics, and in the history of the last decade, where the close, informal collaboration between scientists has often made it hard to discern an individual contribution, we can clearly see the bright beacons lighted by Schwinger's hand.

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Remarks by Dr. J. R. Oppenheimer
Einstein Prize Award Luncheon
March 14, 1951

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Julian Schwinger: Born in NewYork, New York, February 12, 1918. A.B., Columbia University, 1936, Ph.D., Columbia University, 1939. National Research Fellow, University of California, 1939-41. Instructor in Physics, Purdue University, 1941-43, Assistant Professor, Purdue University, 1943-45, Summer Lecturer, University of Michigan, 1941 and 1948, Associate Professor of Physics, Harvard University, 1945-47, Professor of Physics, Harvard University, 1947 - - -.

Schwinger's early research, published before he was twenty years old, was mainly on the magnetic scattering of neutrons, and on the scattering of neutrons by ortho and parahydrogen. A little later he investigated neutron-proton interaction in relation to the quadrupole moment of the deuteron and the tensor theory of nuclear forces. During the war his research was partly at the Metallurgical Laboratory at Chicago, but mainly at the Massachusetts Institute of Technology Radiation Laboratory. There he devised important new methods of treating propagation problems connected with wave guides which furnished the basis for the investigation of a substantial part of that laboratory's theoretical group. Recently he made calculations of the influence of self-energy on the hydrogen fine-structure and on the magnetic moment of the electron which are generally regarded as a major advance in the understanding of quantum electrodynamics.

1. On the Magnetic Scattering of Neutrons. Phys. Rev. 51, 544-552, 1937.
2. On the Non-Adiabatic Processes in Inhomogeneous Fields. Phys. Rev. 51, 648-651, 1937.
3. The Scattering of Neutrons by Ortho-and Parahydrogen. Phys. Rev. 52, 286-295, 1937. With E. Teller.
4. The Widths of Nuclear Energy Levels. Phys. Rev. 55, 39-46, 1939 With J. H. Manley and H. H. Goldsmith.
5. The Scattering of Thermal Neutrons by Deuterons. Phys. Rev. 58, 26-36, 1940. With L. Motz.
6. The Electromagnetic Properties of Mesotrons. Phys. Rev. 58, 953-968, 1940. With H.C. Corben.
7. The Photo-Disintegration of the Deuteron. Phys. Rev. 59, 209-211, 1941. With W. Rarita and H. A. Nye.
8. On the Neutron-Proton Interaction. Phys. Rev. 59, 436-452, 1941. With W. Rarita.
9. On the Exchange Properties of the Neutron-Proton Interaction. Phys. Rev. 59, 556-564, 1941. With W. Rarita.
10. On the Interaction of Mesotrons and Nuclei. Phys. Rev. 60, 150-152, 1941. With J. R. Oppenheimer.
11. On Tensor Forces and the Theory of Light Nuclei. Phys. Rev. 61, 138-146, 1942. With E. Gerjuoy.
12. Notes On Lectures, "Discontinuities in Wave Guides", Radiation Laboratory, Massachusetts Institute of Technology, 1945.
13. The Scattering of Slow Neutrons by Ortho- and Paradeuterium. Phys. Rev. 69, 145-154, 1946. With M. Hamermesh.
14. The Magnetic Moments of H^3 and He^3. Phys. Rev. 70, 41-43, 1946. With R. G. Sachs.
15. Neutron Scattering in Ortho- and Parahydrogen. Phys. Rev, 71, 678-680, 1947. With M. Hamermesh.
16. On the Radiation of Sound from an Unflanged Circular Pipe. Phys. Rev. 73, 389-406, 1948. With H. Levine.

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17. On the Polarization of Fast Neutrons. Phys. Rev. 73, 407-409, 1948.

18. A Note on Saturation in Microwave Spectroscopy. Phys. Rev. 73, 1020-1026, 1948. With R. Karplus.

Letters to the Editor of the Phys. Rev.

1. On the Polarization of Electrons by Double Scattering. Phys. Rev. 48, 109, 1935. With O. Halpern.

2. On the β-Radioactivity of Neutrons. Phys. Rev. 48, 704, 1935. With L. Motz.

3. On the Spin of the Neutron. Phys. Rev. 52, 1250, 1937.

4. The Neutron-Proton Scattering Cross Section. Phys. Rev. 55, 106, 1939. With V. W. Cohen and H. H. Goldsmith.

5. The Resonance Absorption of Slow Neutrons in Indium. Phys. Rev. 55, 107, 1939. With J. H. Manley and H. H. Goldsmith.

6. On Pair Emission in the Proton Bombardment of Fluorine. Phys. Rev. 56, 1066, 1939. With J. R. Oppenheimer.

7. Neutron-Deuteron Scattering Cross Section. Phys. Rev. 57, 161, 1940. With L. Motz.

8. Neutron Scattering in Ortho-and Parahydrogen and the Range of Nuclear Forces. Phys. Rev. 58, 1004, 1940.

9. On a Theory of Particles with Half-Integral Spin. Phys. Rev. 60, 61, 1941. With W. Rarita.

10. On Quantum-Electrodynamics and the Magnetic Moment of the Electron. Phys. Rev. 73, 416-41 , 1948.

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A.E. concurs
with this choice -

R.O

So would I.

H Weyl

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ON GAUGE INVARIANCE AND VACUUM POLARIZATION.

Julian Schwinger.

These notes are taken from an unfinished manuscript of Professor Schwinger. They have not been revised, edited, or officially approved by him.

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On Gauge Invariance and Vacuum Polarization.

Julian Schwinger.

Quantum electrodynamics is characterized by several formal invariance properties, notably Lorentz and gauge invariance. Yet, specific calculations by conventional methods may yield results that violate these requirements, in consequence of the divergences inherent in present field theories. Such difficulties concerning Lorentz invariance have been avoided by employing formulations of the theory that are explicitly invariant under coordinate transformations, and by maintaining this generality through the course of calculations. The preservation of gauge invariance has apparently been considered a more formidable task. It should be evident, however, that the two problems are quite analogous, and that gauge invariance difficulties naturally disappear when methods of solution are adopted that involve only gauge covariant quantities.

We shall illustrate this assertion by applying such a gauge invariant method to treat several aspects of the problem of vacuum polarization by a prescribed electromagnetic field. The calculation of the current j_μ associated with the vacuum of a charged particle field involves the construction of the Green's function for the particle field in the prescribed electromagnetic field. The vacuum current may be exhibited as the variation, with respect to the vector potential, of an action integral, which effectively adds to that of the Maxwell field in describing the behaviour of electromagnetic fields in the vacuum. We shall relate these problems to the solution of particle equations of motion with a proper time parameter. These equations of motion, which involve only electromagnetic field strengths, provide the required gauge invariant basis for our discussion. They can be solved rigorously in the two situations of constant field strengths.

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and fields propagated with the speed of light in the form of a plane wave. The modified Lagrange function for the plane wave reverts to that of the Maxwell field after a renormalization of field strengths, thus expressly denying the existence of a "light-quantum self-energy". With constant (that is, slowly varying)fields, the same field strength renormalization, with its associated charge renormalization, yields a modified Lagrange function differing from that of the Maxwell field by terms which imply a non-linear behaviour for the electromagnetic field. This result agrees with one obtained some time ago by other methods and a somewhat different viewpoint. For weak, arbitrarily varying fields, perturbation methods can be applied to the equations of motion.

The consequences thus obtained are useful in connection with a class of problem in which gauge invariance difficulties have been encountered, the multiple photon disintegration of a neutral meson. Without extensive further calculations, we shall obtain approximate gauge-invariant expressions for the interaction of a zero-spin neutral meson with two photons, where the intermediate nuclear interaction may be scalar, or the equivalent pseudoscalar and pseudovector coupling. We shall also easily obtain an approximate expression for the interaction of a spin one neutral meson with three photons, or an electron positron pair, through the interposition of a vector nuclear interaction.

The utility of the proper time technique to be exploited in this paper, apart from its value in obtaining rigorous solutions in a few special cases, lies in its isolation of the divergent aspects of a calculation in an integration with respect to the proper time, a parameter that makes no reference to the coordinate system or the gauge. Indeed, we shall show that the customary perturbation procedure of expanding in powers of the potential vector does yield gauge invariant results, provided only that the proper time integration is reserved to the last. The technique of "invariant regularization" represents a partial realization of this proper time method, through the use of specially weighted integration over the conjugate quantity, the proper mass.

Finally, in an Appendix we shall employ the Green's function for an electron in a constant field to calculate the second order electron self-energy in a weak external field, thereby providing a simple derivation of the second order correction to the electron magnetic moment.

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The current vector and energy-momentum tensor for a Dirac field are given by

$$j_\mu(x) = \frac{1}{2} e\, [\bar{\Psi}(x), \gamma_\mu \Psi(x)],$$

$$T_{\mu\nu}(x) = \frac{1}{4}[\bar{\Psi}(x), \gamma_{(\mu}\left(\frac{1}{i}\frac{\partial}{\partial x} - eA(x)\right)_{\nu)}\Psi(x)]$$

$$+ \frac{1}{4}[\left(-\frac{1}{i}\frac{\partial}{\partial x} - eA(x)\right)_{(\nu}\bar{\Psi}(x), \gamma_{\mu)}\Psi(x)].$$

The field equations and commutation relations of the Dirac field are given by

$$\left(\gamma\left(\frac{1}{i}\frac{\partial}{\partial x} - eA(x)\right) + m\right)\Psi(x) = 0,$$

$$\left\{\Psi(x, x_0), \bar{\Psi}(x', x_0)\right\} = \gamma_0\, \delta(x - x'),$$

where

$$\frac{1}{2}\left\{\gamma_\mu, \gamma_\nu\right\} = -\delta_{\mu\nu}$$

and

$$\gamma_0 = -i\gamma_4, \qquad \gamma_0^2 = 1.$$

The structure of the current operator,

$$j_\mu(x) = -e(\gamma_\mu)_{\beta\alpha}\, \frac{1}{2}\,[\Psi_\alpha(x), \bar{\Psi}_\beta(x)],$$

which arises from an explicit charge symmetrization, can be related to a time symmetrization by introducing chronologically ordered operators. Thus, with the notation

$$\begin{aligned}(A(x_0')B(x_0''))_+ &= A(x_0')B(x_0'') \qquad x_0' > x_0'' \\ &= B(x_0'')A(x_0') \qquad x_0' < x_0'',\end{aligned}$$

and

$$\epsilon(x_0' - x_0'') = \begin{matrix} +1 & x_0' > x_0'' \\ -1 & x_0' < x_0'', \end{matrix}$$

we have

$$\frac{1}{2}[\Psi_\alpha(x), \bar{\Psi}_\beta(x)] = (\Psi_\alpha(x')\bar{\Psi}_\beta(x''))_+ \,\epsilon(x' - x'')]_{x', x'' \to x}$$

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provided one takes the average of the forms obtained by letting x' and x'' independently approach x from the future and from the past. The quantity of actual interest here is the expectation value of $j_\mu(x)$ in the vacuum of the Dirac field,

$$\langle j_\mu(x)\rangle = ie\, \mathrm{tr}\, \gamma_\mu G(x; x'')]_{x',\, x'' \longrightarrow x}$$

where

$$G(x',x'') = i\langle(\psi(x')\bar{\psi}(x''))_+\rangle \epsilon(x'-x'')$$

and tr indicates a diagonal sum with respect to the spin indices.

The quantity $G(x', x'')$ satisfies an inhomogeneous differential equation which is obtained by noting that

$$[\gamma(\frac{1}{i}\frac{\partial}{\partial x'} - eA(x')) + m]G(x',x'') = \langle\gamma_o\{\psi(x'), \bar{\psi}(x'')\}\rangle\, \delta(x_o' - x_o'')$$

in which the right side expresses the discontinuous change in form of $G(x',x'')$ as x_o' is altered from $x_o'' - 0$ to $x_o'' + 0$. Therefore

$$[\gamma(\frac{1}{i}\frac{\partial}{\partial x'} - eA(x')) + m]G(x',x'') = \delta(x'-x''),$$

that is, $G(x',x'')$ is a Green's function for the Dirac field. We need not discuss which particular Green's function this is, as shown by the associated boundary conditions, since no ambiguity enters if actual pair creation in the vacuum does not occur, which we shall expressly assume.

It is useful to regard $G(x',x'')$ as the matrix element of an operator G in which rows and columns are labelled by space-time coordinates, as well as by the spinor indices

$$G(x',x'') = (x'|G|x'').$$

The defining differential equation for the Green's function is then considered to be a matrix element of the operator equation

$$(\gamma\pi + m)G = 1$$

where

$$\pi = p - eA$$

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is characterized by the operator properties

$$[x_\mu , \pi_\nu] = i\delta_{\mu\nu}$$

$$[\pi_\mu , \pi_\nu] = ie F_{\mu\nu} ,$$

and

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} -$$

is the antisymmetrical field-strength tensor of the external electromagnetic field.

With this symbolism, it is easy to show that the vacuum current vector

$$\langle j_\mu(x) \rangle = ie \operatorname{tr} \gamma_\mu \ (x|G|x)$$

is obtained from an action integral by variation of $A_\mu(x)$. This is accomplished by exhibiting

$$\delta W_1 = \int \langle j_\mu(x) \rangle \delta A_\mu(x) dx = ie \operatorname{Tr} \gamma \ \delta AG$$

as a total differential, subject to $\delta A_\mu(x)$ vanishing at infinity. In the second expression for δW_1, δA denotes the operator with the matrix elements

$$(x'|\delta A | x'') = \delta(x'-x'')\delta A(x'),$$

and Tr indicates the complete diagonal summation, including spinor indices and the continuous space-time coordinates. Now

$$- e\gamma \ \delta A = \delta(\gamma \pi + m)$$

and

$$G = \frac{1}{\gamma\pi + m} = i \int_0^\infty ds \ e^{-i(\gamma \pi+m)s}$$

so that

$$ie \operatorname{Tr}\gamma\delta AG = \delta[\operatorname{Tr} i \int_0^\infty \frac{ds}{s} \ c^{-i(\gamma \pi+m)s}],$$

Thus in virtue of the fundamental property of the trace,

$$\operatorname{Tr}(AB) = \operatorname{Tr}(BA),$$

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to within an additive constant

$$W_1 = i \int \frac{ds}{s} e^{-ims} \operatorname{Tr} e^{-i\gamma\pi s}$$

$$= \int L_1(x)\, dx$$

where the Lagrange function $L_1(x)$ is given by

$$L_1(x) = i\int_0^\infty \frac{ds}{s} \; e^{-ims} \operatorname{tr}(x \,|\, e^{-i\gamma\pi s} | x).$$

An alternative representation, and the one we shall actually employ for calculations, is obtained by writing

$$G = (-\gamma\pi+m) \frac{1}{m^2 - (\gamma\pi)^2} = \frac{1}{m^2 - (\gamma\pi)^2} (-\gamma\pi+m)$$

or

$$G = (-\gamma\pi + m) i \int_0^\infty ds\, e^{-i(m^2-(\gamma\pi)^2)s}$$

$$= i \int_0^\infty ds\, e^{-i(m^2 - (\gamma\pi)^2)s} (-\gamma\pi + m).$$

In virtue of the vanishing trace of an odd number of γ factors, we have

$$ie \operatorname{Tr} \gamma \,\delta A\, G = - \operatorname{Tr} \delta(\gamma\pi)\gamma\pi \int_0^\infty ds\, e^{-i(m^2 - (\gamma\pi)^2)s}$$

$$= \delta \left[\operatorname{Tr} \frac{i}{2} \int_0^\infty \frac{ds}{s} e^{-i(m^2-(\gamma\pi)^2)s}\right]$$

which again involves the fundamental symmetry property of the trace. Thus

$$L_1(x) = \frac{i}{2} \int_0^\infty \frac{ds}{s} e^{-im^2 s} \operatorname{tr} (x \,|\, e^{-iHs} \,|\, x)$$

where

$$H = - (\gamma\pi)^2 = \pi_\mu^2 - \frac{1}{2} e\sigma_{\mu\nu} F_{\mu\nu}$$

and

$$\sigma_{\mu\nu} = \frac{1}{2} i \,[\gamma_\mu, \gamma_\nu].$$

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We now see that the construction of $G(x',x'')$, and $L_1(x)$, devolves upon the evaluation of

$$(x'|\, e^{-iHs} \,|x'') = (x'(s)|x''(0)).$$

The latter notation emphasises that e^{-iHs} is to be regarded as the operator describing the development of a system governed by the "Hamiltonian" H in the "time"s, the matrix element of e^{-iHs} being the transformation function from a state in which $x(s=0)$ has the value x'' to a state in which $x(s=s)$ has the value x'. Thus we are led to an associated dynamical problem in which the space-time coordinates of a "particle" depend upon a proper time parameter, in a manner determined by the equations of motion

$$\frac{dx_\mu}{ds} = -\, i[x_\mu ,H] = 2\pi_\mu$$

$$\frac{d\pi_\mu}{ds} = -\, i[\pi_\mu ,H] = e[F_{\mu\nu}\pi_\nu + \pi_\nu F_{\mu\nu}] + \frac{e}{2}\sigma_{\lambda\nu}\frac{\partial F_{\lambda\nu}}{\partial x_\mu}$$

$$= 2eF_{\mu\nu}\pi_\nu - ie\frac{\partial F_{\mu\nu}}{\partial x_\nu} + \frac{e}{2}\sigma_{\lambda\nu}\frac{\partial F_{\lambda\nu}}{\partial x_\mu}.$$

The transformation function $(x'(s)|\,x''(0))$ is characterized by the differential equations

$$\left.\begin{aligned}
&i\frac{\partial}{\partial s}(x'(s)|x''(0)) = (x'(s)|H|x''(0))\\
&(-i\frac{\partial}{\partial x'_\mu} - eA_\mu(x'))(x'(s)|x''(0)) = (x'(s)|\pi_\mu(s)|x''(0)),\\
&(i\frac{\partial}{\partial x''_\mu} - eA_\mu(x''))(x'(s)|x''(0)) = (x'(s)|\pi_\mu(0)|x''(0))
\end{aligned}\right] \quad (1)$$

and the boundary condition

$$(x'(0)|\,x''(0)) = \delta(x' - x''). \quad (2)$$

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We shall now illustrate, for the elementary case of $F_{\mu\nu} = 0$, the procedure which will be employed in the following sections for constructing the transformation function. In this field-free situation the equations of motion read

$$\frac{d\pi_\mu(s)}{ds} = 0, \qquad \frac{dx_\mu(s)}{ds} = 2\pi_\mu(s),$$

whence

$$\pi_\mu(s) = \pi_\mu(0)$$

and

$$\frac{1}{s}(x_\mu(s) - x_\mu(0)) = 2\pi_\mu .$$

Therefore

$$H = \pi_\mu^2 = \frac{1}{4s^2}(x_\mu(s) - x_\mu(0))^2$$

$$= \frac{1}{4s^2}(x_\mu(s)^2 - 2x_\mu(s)x_\mu(0) + x_\mu^2(0)) - \frac{2i}{s} \quad ,$$

Since

$$[x_\mu(s),\, x_\nu(0)] = [x_\mu(0) + 2s\pi_\mu(0),\, x_\nu(0)] = -2is\,\delta_{\mu\nu} .$$

Having ordered the coordinate operators so that x(s) everywhere stands to the left of x(0), we can immediately evaluate the matrix element of H in the equation of motion

$$i\frac{\partial}{\partial s}(x'(s)\,|\,x''(0)) = (x'(s)|\,H\,|x''(0))$$

$$= [\frac{(x'-x'')^2}{4s^2} - \frac{2i}{s}](x'(s)\,|\,x''(0)),$$

the solution of which is

$$(x'(s)\,|\,x''(0)) = \frac{C(x',x'')}{s^2}\, e^{i(x'-x'')^2/4s}. \qquad (3)$$

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To determine $C(x',x'')$ we note that

$$(x'(s)|\,\pi_\mu(s)\,|x''(0)) = (x'(s)|\,\pi_\mu(0)\,|x''(0))$$

$$= (x'(s)|\,\frac{1}{2s}(x_\mu(s) - x_\mu(0))\,|x''(0)) = \frac{x'_\mu - x''_\mu}{2s}(x'(s)|x''(0)).$$

This with the equations (1) implies that

$$(-i\frac{\partial}{\partial x'_\mu} - eA_\mu(x'))C(x',x'') = (i\frac{\partial}{\partial x''_\mu} - eA_\mu(x''))C(x',x'') = 0,$$

or

$$C(x',x'') = C\, e^{ie\int_{x''}^{x'}(A\,dx)} .$$

Since the field strength tensor $F_{\mu\nu}$ vanishes, the line integral is independent of the integration path. Finally, the constant C is fixed by the boundary condition (2). It is evident that (3) does have the character of a delta function as s approaches zero, provided we choose C so that

$$\frac{C}{s^2}\int e^{ix^2/4s}\,dx = 1$$

that is

$$C = -\frac{i}{(4\pi)^2}.$$

The Green's function is thus obtained as

$$G(x',x'') = i\int_0^\infty e^{-im^2s}ds\,(x'(s)|\,-\gamma\pi + m\,|x''(0))$$

$$= \frac{1}{(4\pi)^2}\,e^{ie\int_{x''}^{x'}Adx}\int_0^\infty\frac{ds}{s^2}\,e^{-im^2s}\quad\times$$

$$\times(-\frac{1}{2s}(\gamma(x'-x'')) + m)e^{i(x'-x'')^2/4s} .$$

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If the field strength tensor is constant, the equations of motion simplify to

$$\frac{dx_\mu}{ds} = 2\pi_\mu$$

$$\frac{d\pi_\mu}{ds} = 2eF_{\mu\nu}\pi_\nu$$

or in matrix notation

$$\frac{dx}{ds} = 2\pi, \qquad \frac{d\pi}{ds} = 2eF\pi.$$

The symbolic solution of these equations is immediately obtained,

$$\pi(s) = e^{2eFs}\,\pi(0)$$

$$x(s) - x(0) = \left[\frac{e^{2eFs} - 1}{eF}\right]\pi(0).$$

Hence

$$\pi(0) = \frac{eF}{e^{2eFs}-1}\,(x(s) - x(0)) = \frac{eFe^{-eFs}}{2\sinh eFs}\,(x(s) - x(0))$$

and

$$\pi(s) = \frac{eFe^{eFs}}{2\sinh eFs}(x(s)-x(0)).$$

We now calculate

$$H + \frac{e}{2}\,\sigma F = \pi^2$$

$$= (x(s)-x(0))\,\widetilde{\left(\frac{eFe^{eFs}}{2\sinh eFs}\right)}\left(\frac{eFe^{eFs}}{2\sinh eFs}\right)(x(s)-x(0))$$

$$= (x(s) - x(0))\,\frac{e^2F^2}{4\sinh^2 eFs}\,(x(s) - x(0))$$

in which we have used the fact that

$$\tilde{F} = -F, \quad F_{\mu\nu} = -F_{\nu\mu}.$$

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In rearranging the order of these operators, the following commutator is required,

$$[x(s),x(0)] = [x(0) + \frac{e^{2eFs} - 1}{eF} \pi(0),\ x(0)]$$

$$= - i\, \frac{e^{2eFs} - 1}{eF},$$

that is to say

$$[x_\mu(s),\ x_\nu(0)] = -i(e^{eFs}\ \frac{2\sinh eFs}{eF})_{\mu\nu}.$$

Thus

$$H + \frac{1}{2} e\sigma F = x(s)\ \frac{e^2F^2}{4\ \sinh^2 eFs}\ \ x(s) - 2x(s)\ \frac{e^2F^2}{4\ \sinh^2 eFs}\ x(0) +$$

$$+ x(0)\ \frac{e^2F^2}{4\ \sinh^2 eFs}\ x(0) - \frac{1}{2}\, i\ \mathrm{tr}\ eF\ \coth\ eFs,$$

where tr again denotes a diagonal summation, although of a quite different type of matrix, and we have used the fact that

$$\mathrm{tr}\ F = 0.$$

We now immediately obtain the differential equation

$$i\, \frac{\partial}{\partial s}\ (x'(s)\big|x''(0)) = [- \frac{1}{2}\ e\sigma F + (x'-x'')\frac{e^2\ F^2}{4\ \sinh^2 eFs}\ (x'-x'')$$

$$- \frac{1}{2}\, i\ \mathrm{tr}\ eF\ \coth\ eFs](x'(s)\big|x''(0))$$

which has the solution

$$(x'(s)\big|x''(0)) = \frac{C(x',x'')}{s^2}\ e^{- \frac{1}{2}\ \mathrm{tr}\ \log \frac{\sinh eFs}{eFs}}\ \ \times$$

$$\times\ e^{\frac{1}{2}\ ie\sigma Fs + \frac{1}{4}\ i(x'-x'')eF\ \coth\ eFs(x'-x'')}\ . \qquad (4)$$

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To calculate $C(x',x'')$, we employ

$$(x'(s)/\pi(s)|x''(0)) = (\frac{eF}{2}\coth eFs + \frac{eF}{2})(x'-x'')(x'(s)|x''(0))$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (5)

$$(x'(s)|\,\pi(0)|x''(0)) = (\frac{eF}{2}\coth eFs - \frac{eF}{2})(x' - x'')(x'(s)|x''(0))$$

in conjunction with (1) to obtain the differential equations

$$[-i\frac{\partial}{\partial x'_\mu} - eA_\mu(x') - \frac{1}{2}eF_{\mu\nu}(x'-x'')_\nu]\,C(x',x'') = 0$$

$$[i\frac{\partial}{\partial x''_\mu} - eA_\mu(x'') - \frac{1}{2}eF_{\mu\nu}(x'' - x')_\nu]C(x',x'') = 0.$$

The solution of the first equation is

$$C(x',x'') = C(x'')\,e^{ie\int_{x''}^{x'} dx(A(x)+\frac{1}{2}F(x'-x''))}$$

in which the integral is independent of the path of integration since

$$A_\mu(x) + \frac{1}{2}F_{\mu\nu}(x - x'')_\nu$$

has a vanishing curl. However, by restricting the integration path to be a straight line connecting x' and x'', we may write simply

$$C(x',x'') = C\,e^{ie\int_{x''}^{x'} Adx}$$

and, with C a constant, obtain a solution of both the equations for $C(x',x'')$. As before, C has the value

$$C = -\frac{i}{(4\pi)^2}\ ,$$

since the limiting form of $(x'(s)\,|\,x''(0))$ as $s \to 0$ is independent of the external field. The Green's function is then obtained in the two equivalent forms

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$$G(x',x'') = i\int_0^\infty ds\, e^{-im^2 s}[-\gamma(x'(s)|\,\pi(s)\,|x''(0)) + m(x'(s)|x''(0))]$$

$$= i\int_0^\infty ds\, e^{-im^2 s}[-(x'(s)|\,\pi(0)\,|x''(0))\gamma + m(x'(s)|x''(0))]$$

which will be given explicitly by substitution of (4) and (5).

The Lagrange function $L_1(x)$ is now computed as

$$L_1(x) = \frac{1}{2} i\int_0^\infty \frac{ds}{s} e^{-im^2 s}\, \mathrm{tr}(x'(s)|x''(0))]_{x',x''\to x}$$

$$= \frac{1}{32\pi^2}\int_0^\infty \frac{ds}{s^3} e^{-im^2 s}\, e^{-\frac{1}{2}\mathrm{tr}\log\frac{\sinh eFs}{eFs}}\, \mathrm{tr}\, e^{\frac{1}{2} ie\sigma Fs}.$$

We may exhibit this more explicitly as a real quantity by a deformation of the integration path, which is effectively the substitution $s \longrightarrow -is$. Thus

$$L_1(x) = -\frac{1}{32\pi^2}\int_0^\infty \frac{ds}{s^3} e^{-m^2 s}\, e^{-\frac{1}{2}\mathrm{tr}\log\frac{\sin eFs}{eFs}}\, \mathrm{tr}\, e^{\frac{1}{2} e\sigma Fs}.$$

To evaluate the Dirac matrix trace, we employ the following spin matrix property

$$\frac{1}{2}\{\sigma_{\mu\nu}, \sigma_{\lambda\kappa}\} = \delta_{\mu\lambda}\delta_{\nu\kappa} - \delta_{\mu\kappa}\delta_{\nu\kappa} + i\epsilon_{\mu\nu\lambda\kappa}\gamma_5$$

where

$$\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_4, \qquad \gamma_5^2 = -1,$$

and $\epsilon_{\mu\nu\lambda\kappa}$ denotes +1 or -1 if $\mu\nu\lambda\kappa$ forms an even or an odd permutation of 1234, and zero otherwise. Then

$$\left(\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}\right)^2 = \frac{1}{2}F_{\mu\nu}^2 + \frac{1}{2}\gamma_5 F_{\mu\nu}F_{\mu\nu}^*,$$

where

$$F_{\mu\nu}^* = \frac{1}{2} i\epsilon_{\mu\nu\lambda\kappa}F_{\lambda\kappa} -$$

is the dual field strength tensor. We encounter the scalar

$$\mathcal{F} = \frac{1}{4}F_{\mu\nu}^2 = \frac{1}{2}(H^2 - E^2)$$

and pseudoscalar

$$G = \frac{1}{4}F_{\mu\nu}F_{\mu\nu}^* = E\cdot H$$

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constructed from the field strengths. Since

$$\left(\tfrac{1}{2}\sigma F\right)^2 = 2(\mathcal{F} + \gamma_5 G)$$

and $\gamma_5^2 = -1$, it follows that $\frac{1}{2}\sigma F$ has the four eigenvalues

$$\left(\tfrac{1}{2}\sigma F\right)' = \pm\sqrt{2(\mathcal{F} \pm iG)}$$

and therefore

$$\mathrm{tr}\; e^{\frac{1}{2}e\sigma Fs} = 4\mathrm{Re}\cosh es\sqrt{2(\mathcal{F} + iG)},$$

where Re denotes the real part of the following expression. Note incidentally that

$$2(\mathcal{F} + iG) = (H + iE)^2.$$

We shall require the eigenvalues of the matrix $F = (F_{\mu\nu})$ to construct

$$e^{-\frac{1}{2}\;\mathrm{tr}\log\frac{\sin eFs}{eFs}}\;.$$

The evaluation is accomplished with the aid of the easily verifiable relations

$$\begin{aligned} F_{\mu\lambda}F^*_{\lambda\nu} &= -\delta_{\mu\nu}G, \\ F^*_{\mu\lambda}F^*_{\lambda\nu} - F_{\mu\lambda}F_{\lambda\nu} &= 2\delta_{\mu\nu}\mathcal{F}. \end{aligned} \qquad (6)$$

From the eigenvalue equation

$$F_{\mu\nu}\psi_\mu = F'\psi_\mu$$

and its equivalent

$$F^*_{\mu\nu}\psi_\nu = -\frac{1}{F'}G\psi_\mu$$

according to (6), we obtain by iteration

$$\begin{aligned} F_{\mu\lambda}F_{\lambda\nu}\psi_\nu &= F'^2\psi_\mu \\ F^*_{\mu\lambda}F^*_{\lambda\nu}\psi_\nu &= \frac{1}{F'^2}G^2\psi_\mu\;. \end{aligned}$$

Then (6) yields the eigenvalue equation

$$F'^4 + 2\mathcal{F}\,F'^2 - G^2 = 0,$$

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which has the solutions $\pm F^{(1)}$, $\pm F^{(2)}$, where

$$F^{(1)} = i\sqrt{\tfrac{1}{2}(\mathcal{F} + iG)} + i\sqrt{\tfrac{1}{2}(\mathcal{F} - iG)}$$

$$F^{(2)} = i\sqrt{\tfrac{1}{2}(\mathcal{F} + iG)} - i\sqrt{\tfrac{1}{2}(\mathcal{F} - iG)}.$$

Expressed in terms of these eigenvalues

$$e^{-\frac{1}{2}\,\mathrm{tr}\,\log\frac{\sin eFs}{eFs}} = (es)^2\,\frac{F^{(1)}F^{(2)}}{\sin eF^{(1)}s\,\sin eF^{(2)}s}$$

$$= 2(es)^2\,\frac{F^{(1)}F^{(2)}}{\cos es(F^{(1)}-F^{(2)}) - \cos es(F^{(1)}+F^{(2)})}$$

or

$$e^{-\frac{1}{2}\,\mathrm{tr}\,\log\frac{\sin eFs}{eFs}} = \frac{(es)^2 G}{\mathrm{Im}\,\cosh es\sqrt{2(\mathcal{F} + iG)}}$$

where Im denotes the imaginary part of the following expression.

Finally, then

$$L_1 = -\frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\, e^{-m^2 s} \times$$

$$\times\left\{(es)^2 G\,\frac{\mathrm{Re}\,\cosh es\sqrt{2(\mathcal{F}+iG)}}{\mathrm{Im}\,\cosh es\sqrt{2(\mathcal{F}+iG)}} - 1\right\},$$

in which we have supplied the additive constant necessary to make L_1 vanish in the absence of a field. The first term in the expression of L_1 for weak fields is

$$L_1 \cong -\frac{e^2}{12\pi^2}\int_0^\infty \frac{ds}{s}\, e^{-m^2 s}\,\mathcal{F}.$$

On separating this explicitly, and adding the Lagrange function of the Maxwell field

$$L_0 = -\mathcal{F},$$

we obtain the total Lagrange function

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$$L = -\left[1 + \frac{e^2}{12\pi^2} \int_0^\infty \frac{ds}{s} e^{-m^2 s}\right]\mathcal{F}$$

$$- \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-m^2 s} \left\{ (es)^2 G \frac{\mathrm{Re}}{\mathrm{Im}} - 1 - \frac{2}{3}(es)^2 \mathcal{F} \right\}.$$

The logarithmically divergent factor that multiplies the Maxwell Lagrange function may be absorbed by a change of scale for all fields, and a corresponding scale change, or renormalization, of charge. If we identify the quantities thus far employed by a zero subscript, and introduce new units of field strength and charge, according to

$$(1 + Ce_0^2)(\mathcal{F}_0 + iG_0) = \mathcal{F} + iG,$$

$$e^2 = \frac{e_0^2}{1 + Ce_0^2}, \qquad C = \frac{1}{12\pi^2} \int_0^\infty \frac{ds}{s} e^{-m^2 s} \tag{7}$$

we obtain the finite, gauge invariant result

$$L = -\mathcal{F} - \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-m^2 s} \left[(es)^2 G \frac{\mathrm{Re} \cosh es\sqrt{2(\mathcal{F} + iG)}}{\mathrm{Im} \cosh es\sqrt{2(\mathcal{F} + iG)}} - 1 - \frac{2}{3}(es)^2 \mathcal{F}\right]$$

$$= \frac{E^2 - H^2}{2} + \frac{2\alpha^2}{45}\left(\frac{\hbar}{mc}\right)^3 \frac{1}{mc^2}\left\{(E^2 - H^2)^2 + 7(E\cdot H)^2\right\} + \ldots$$

In the latter expression the conventional rational units have been reinstated, and α stands for the fine structure constant $(e^2/4\pi\hbar c)$.

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We shall now discuss the approximate evaluation of

$$W_1 = \frac{1}{2} i \int_0^\infty \frac{ds}{s} e^{-im^2 s} \operatorname{Tr} e^{-iHs},$$

$$H = (p - eA)^2 - \frac{1}{2} e\sigma F,$$

by an expansion in A_μ and $F_{\mu\nu}$. For this purpose we write

$$H = H_o + H_1$$

where

$$H_o = p^2$$

and

$$H_1 = -e(pA + Ap) - \frac{1}{2} e\sigma F + e^2 A^2.$$

To obtain the expansion of $\operatorname{Tr} e^{-is(H_o+H_1)}$ in powers of H_1, we introduce

$$U(s) = e^{-is(H_o+H_1)},$$

which obeys the differential equation

$$i\frac{\partial}{\partial s} U(s) = (H_o + H_1)U(s).$$

The related operator

$$V(s) = e^{isH_o} U(s)$$

is determined by

$$i\frac{\partial}{\partial s} V(s) = e^{isH_o} H_1 e^{-isH_o} V(s)$$

and

$$V(o) = 1.$$

$V(s)$ thus satisfies the integral equation

$$V(s) = 1 - i\int_o^s e^{is'H_o} H_1 e^{-is'H_o} V(s')ds'$$

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which is solved by iteration

$$V(s) = 1 - i\int_0^s ds' e^{is'H_0} H_1 e^{-is'H_0} ds'$$

$$+(-i)^2 \int_0^s ds' e^{is'H_0} H_1 e^{-is'H_0} \int_0^{s'} ds'' e^{is''H_0} H_1 e^{-is''H_0} + \dots .$$

On introducing new variables of integration $u_1, u_2, \dots$ according to

$$s' = su_1, \quad s'' = s'u_2, \dots$$

we obtain the expansion

$$e^{-is(H_0 + H_1)} = e^{-isH_0} + (-is)\int_0^1 du_1 e^{-is(1-u_1)H_0} H_1 e^{-isu_1H_0}$$

$$+(-is)^2 \int_0^1 u_1 du_1 \int_0^1 du_2 e^{-is(1-u_1)H_0} H_1 e^{-is(1-u_2)u_1H_0} H_1 e^{-isu_1u_2H_0}$$

$$+ \dots .$$

the n' th term being

$$(-is)^n \int_0^1 u_1^{n-1} du_1 \int_0^1 u_2^{n-2} du_2 \dots \int_0^1 du_n \times$$

$$\times e^{-is(1-u_1)H_0} H_1 e^{-isu_1(1-u_2)H_0} H_1 \dots \times$$

$$\times e^{-isu_1u_2\dots u_{n-1}(1-u_n)H_0} H_1 e^{-isu_1u_2\dots u_nH_0} .$$

Instead of directly taking the trace of this expression, which would require further simplification, we remark that

$$\mathrm{Tr}\, e^{-is(H_0+H_1)} - \mathrm{Tr}\, e^{-isH_0} = (-is)\int_0^1 d\lambda \mathrm{Tr} H_1 e^{-is(H_0+\lambda H_1)}$$

and insert the expansion for $e^{-is(H_0+\lambda H_1)}$. Thus

$$\mathrm{Tr}\, e^{-is(H_0+H_1)} = \mathrm{Tr}\, e^{-isH_0} + (-is)\mathrm{Tr} H_1 e^{-isH_0}$$

$$+ \tfrac{1}{2}(-is)^2 \int_0^1 du_1 \mathrm{Tr}(H_1 e^{-is(1-u_1)H_0} H_1 e^{-isu_1H_0}) + \dots$$

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$$..+\frac{1}{n}(-is)^n\int_0^1 u_1^{n-2}\,du_1..\int_0^1 du_{n-1}\,x$$

$$x\mathrm{Tr}(H_1 e^{-is(1-u_1)H_o}\,H_1...H_1 e^{-isu_1u_2..u_{n-1}\,H_o}).$$

We shall be content with the first non-vanishing field dependent term in the expansion

$$W_1 \cong \frac{1}{2}i\int_0^\infty \frac{ds}{s}e^{-im^2s}\Big\{-is\,\mathrm{Tr}(e^2A^2\,e^{-isp^2})$$

$$+\frac{1}{2}(-is)^2\int_{-1}^1\frac{dv}{2}\,\mathrm{Tr}(e(pA+Ap)e^{-is(\frac{1-v}{2})p^2}e(pA+Ap)e^{-is(\frac{1+v}{2})p^2})$$

$$+\frac{1}{2}(-is)^2\int_{-1}^1\frac{dv}{2}\,\mathrm{Tr}(\frac{1}{2}\,e\sigma F\,e^{-is(\frac{1-v}{2})p^2}\,\frac{1}{2}\,e\sigma Fe^{-is(\frac{1+v}{2})p^2})\Big\}.$$

For convenience the variable u has been replaced by $(\frac{1+v}{2})$. The evaluation of these terms is naturally performed in a momentum representation. The matrix elements of the coordinate-dependent field quantities depend only on momentum differences,

$$(p+\frac{1}{2}k\,|\,A_\mu|p-\frac{1}{2}k)=\frac{1}{(2\pi)^4}\int e^{-ik\cdot x}A_\mu(x)dx=(\frac{1}{(2\pi)^2})\,A_\mu(k),$$

and

$$(p\,|\,A^2\,|\,p)=\frac{1}{(2\pi)^4}\int A_\mu^2(x)dx=(\frac{1}{(2\pi)^4})\int A_\mu(-k)A_\mu(k)dk.$$

Therefore

$$W_1 \cong \frac{2ie^2}{(2\pi)^4}\int_0^\infty\frac{ds}{s}\,e^{-im^2s}\Big\{-is\int dp\,e^{-isp^2}\int dkA(-k)A(k)$$

$$+\frac{1}{2}(-is)^2\int_{-1}^1\frac{dv}{2}\int dp\int dk[2pA(-k)e^{-is(\frac{1-v}{2})(p+\frac{1}{2}k)^2}2pA(k)e^{-is(\frac{1+v}{2})(p-\frac{1}{2}k)^2}$$

$$+\frac{1}{4}\,\mathrm{tr}\,\frac{1}{2}\sigma F(-k)e^{-is(\frac{1-v}{2})(p+\frac{1}{2}k)^2}\,\frac{1}{2}\sigma F(k)e^{-is(\frac{1+v}{2})(p-\frac{1}{2}k)^2}]\Big\}.$$

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We encounter the elementary integrals

$$\int dp\, e^{-isp^2} = \frac{\pi^2}{is^2},$$

$$\int dp\, e^{-is(p^2+\frac{1}{4}k^2)+ispkv} = \frac{\pi^2}{is^2}\, e^{-is\frac{1}{4}k^2(1-v^2)}$$

and

$$\int dp p_\mu p_\nu\, e^{-is(p^2+\frac{1}{4}k^2)+ispkv}$$

$$= e^{-is\frac{1}{4}k^2}\left(\frac{i}{sv}\right)^2 \frac{\partial}{\partial k_\mu}\frac{\partial}{\partial k_\nu}\int dp\, e^{-ip^2 s+ispkv}$$

$$= \frac{\pi^2}{is^2}\, e^{-is\frac{1}{4}k^2(1-v^2)}\left(-\frac{i}{2s}\delta_{\mu\nu}+\frac{1}{4}v^2 k_\mu k_\nu\right).$$

It is useful to replace the $\delta_{\mu\nu}$ of the last integral by an expression equivalent to it in virtue of the integration with respect to v. Now

$$\int_{-1}^{1}\frac{dv}{2}\, e^{-is\frac{1}{4}k^2(1-v^2)} = 1 - is\,\frac{1}{2}k^2\int\frac{dv}{2}\, v^2\, e^{-is\frac{1}{4}k^2(1-v^2)}$$

so that effectively

$$\int dp\, p_\mu p_\nu\, e^{-is(p^2+\frac{1}{4}k^2)+ispkv}$$

$$= -\frac{\pi^2}{2s^3}\delta_{\mu\nu} + \frac{\pi^2}{2s^2}\, e^{-is\frac{1}{4}k^2(1-v^2)}\,\frac{v^2}{4}(k_\mu k_\nu - \delta_{\mu\nu}k^2).$$

On inserting the values of the various p integrals and noting that

$$(k_\mu k_\nu - \delta_{\mu\nu}k^2)A_\mu(-k)A_\nu(k) = -\frac{1}{2}F_{\mu\nu}(-k)F_{\mu\nu}(k)$$

we immediately find the gauge invariant form

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$$W_1 \cong -\frac{e^2}{(4\pi)^2} \int dk \, \frac{1}{2} F_{\mu\nu}(-k)F_{\mu\nu}(k) \times$$

$$\times \int_{-1}^{1} \frac{dv}{2} (1-v^2) \int_0^\infty \frac{ds}{s} e^{-s(m^2+\frac{1}{4}k^2(1-v^2))} .$$

This has been achieved without any special device, other than reserving to the last the proper time integration.

A significant separation of terms is produced by an integration by parts with respect to v, according to

$$\int_{-1}^{1} \frac{dv}{2}(1-v^2) \int_0^\infty \frac{ds}{s} e^{-(m^2+\frac{1}{4}k^2(1-v^2))s}$$

$$= \frac{2}{3}\int_0^\infty \frac{ds}{s} e^{-m^2 s} - \frac{1}{2} k^2 \int_0^1 dv(v^2 - \frac{1}{3}v^4)\int_0^\infty ds \, e^{-(m^2+\frac{1}{4}k^2(1-v^2))s} .$$

Adding the action integral of the Maxwell field, which is given in momentum space by

$$W_0 = -\int dk \, \frac{1}{4} F_{\mu\nu}(-k)F_{\mu\nu}(k),$$

we obtain the modified action integral in the form

$$W = -[1 + \frac{e^2}{12\pi^2}\int_0^\infty \frac{ds}{s} e^{-m^2 s}] \int dk \, \frac{1}{4} F_{\mu\nu}(-k)F_{\mu\nu}(k)$$

$$+ \frac{e^2}{(4\pi)^2} \int dk \, \frac{1}{4} F_{\mu\nu}(-k)F_{\mu\nu}(k)k^2 \int_0^1 dv \, \frac{v^2(1-\frac{1}{3}v^2)}{m^2+\frac{1}{4}k^2(1-v^2)} .$$

The field strength and charge renormalizations contained in (7) then produce the finite, gauge invariant result,

$$W = -\int dk \, \frac{1}{4} F_{\mu\nu}(-k)F_{\mu\nu}(k) \times$$

$$\times \left\{1 - \frac{\alpha}{4\pi}\frac{k^2}{m^2}\int_0^1 dv \, \frac{v^2(1-\frac{1}{3}v^2)}{1+\frac{k^2}{4m^2}(1-v^2)}\right\} . \qquad (8)$$

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The restriction which we have thus far imposed, that no actual pair creation occurs, corresponds to the assumption that $1 + \frac{k^2}{4m^2}(1 - v^2)$ never vanishes. This will be true if $-k^2 < 4m^2$, for all k_μ contained in the Fourier representation of the field. Indeed, it is evident from energy and momentum considerations that to produce a pair by the absorbtion of a single particle, the momentum vector of the latter must be time-like and have a magnitude exceeding 2m. We now simply remark that to extend our results to pair-producing fields, it is merely necessary to add an infinitesimal negative imaginary constant to the denominator of (8) and understand the positive imaginary contribution to W thus obtained with the statement that

$$\left| e^{iW} \right|^2 = e^{-2\mathrm{Im}W}$$

represents the probability that no actual pair creation occurs during the history of the field.

The infinitesimal imaginary constant, as employed in

$$\lim_{\epsilon \to +0} \frac{1}{x - i\epsilon} = P\left(\frac{1}{x}\right) + \pi i \delta(x)$$

represents a familiar device for dealing with real processes. We obtain from (8) that

$$2\mathrm{Im}W = \frac{\alpha}{2} \int dk \, \frac{1}{4} F_{\mu\nu}(-k) F_{\mu\nu}(k) \frac{k^2}{m^2} \int_0^1 dv\, v^2 \left(1 - \frac{1}{3}v^2\right) \delta\left[1 + \frac{k^2}{4m^2}(1 - v^2)\right]$$

$$= \alpha \int_{-k^2 > 4m^2} dk \left(-\frac{1}{4}\right) F_{\mu\nu}(-k) F_{\mu\nu}(k) \left(1 - \left(\frac{4m^2}{-k^2}\right)\right)^{\frac{1}{2}} \quad \frac{1}{3}\left(2 + \left(\frac{4m^2}{-k^2}\right)\right). \tag{9}$$

In the weak fields that are being considered, this is just the probability that a pair is created by the field. It should be noticed incidentally that

$$-\frac{1}{4} F_{\mu\nu}(-k) F_{\mu\nu}(k) = \frac{1}{2} \left| E(k) \right|^2 - \frac{1}{2} \left| H(k) \right|^2$$

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is actually positive for a pair-generating field. This follows, for example, from the vanishing of the magnetic field in the special coordinate system where k_μ has only a temporal component. An alternative version of (9) is obtained by replacing the field with the current required to generate this field, according to the Maxwell equations

$$ik_\mu F_{\mu\nu}(k) = -J_\nu(k),$$

$$k_\mu F_{\nu\lambda}(k) + k_\nu F_{\lambda\mu}(k) + k_\lambda F_{\mu\nu}(k) = 0.$$

Now

$$\begin{aligned} k_\lambda^2 F_{\mu\nu}(-k)F_{\mu\nu}(k) &= 2k_\lambda F_{\lambda\mu}(k)k_\nu F_{\nu\mu}(-k) \\ &= 2J_\mu(k)J_\mu(-k) \end{aligned}$$

so that

$$\begin{aligned} 2\,\mathrm{Im}\,W = \frac{\alpha}{8m^2} \int_{-k^2 > 4m^2} dk\, J_\mu(-k)J_\mu(k)\ x \\ x\left(\frac{4m^2}{-k^2}\right)\left(1 - \left(\frac{4m^2}{-k^2}\right)\right)^{\frac{1}{2}}\ \frac{1}{3}\left(2 + \left(\frac{4m^2}{-k^2}\right)\right). \end{aligned}$$

We must now consider the connection, within the frame work of this special problem, between the proper time method and that of "invariant regularization" devised by Pauli and Villars. The vacuum polarization addition to the action integral has the general structure

$$W_1 = \int dk A_\mu(-k)K_{\mu\nu}(k,m^2)A_\nu(k).$$

The proper time technique yields the coefficient $K_{\mu\nu}(k,m^2)$ in the form

$$K_{\mu\nu}(k,m^2) = \int_0^\infty ds\ e^{-im^2 s}\ K_{\mu\nu}(k,s)$$

where $K_{\mu\nu}(k,s)$ is a finite gauge invariant quantity. Infinities appear only in the final stage of integrating s to the origin. In effect, this method substitutes a lower limit s_0 in the proper time integration, and reserves the limit $s_0 \rightarrow 0$ to the end of the calculation. If, on the contrary, the proper time

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technique is not explicitly introduced, $K_{\mu\nu}(k,m^2)$ will be represented by divergent integrals, leading in general to non-gauge-invariant results. The regulator technique avoids this difficulty by introducing a suitably weighted integration with respect to the square of the proper mass, thus substituting for $K_{\mu\nu}(k,m^2)$ the quantity

$$K_{\mu\nu}(k,m^2)_R = \int_{-\infty}^{\infty} dk\, \rho(k) K_{\mu\nu}(k,k).$$

The "regulator" $\rho(k)$ must, in an appropriate limit, reduce to $\delta(k-m^2)$, and will produce gauge invariant results in this problem if the integral conditions

$$\int_{-\infty}^{\infty} \rho(k)dk = 0$$

$$\int_{-\infty}^{\infty} k\, \rho(k)dk = 0$$

are satisfied. Expressed in terms of the Fourier transformed quantities

$$R(s) = \int_{-\infty}^{\infty} \rho(k)e^{-iks}\, dk$$

$$K_{\mu\nu}(k,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{iks}K_{\mu\nu}(k,k),$$

we have

$$K_{\mu\nu}(k,m^2)_R = \int_{-\infty}^{\infty} R(s)\, K_{\mu\nu}(k,s)ds \tag{10}$$

while the conditions on $\rho(k)$ appear as

$$R(0) = R'(0) = 0, \qquad R(s) \to e^{-im^2 s} .$$

Now observe that the proper time method yields $K_{\mu\nu}(k,m^2)$ in the form (10), with $K_{\mu\nu}(k,s) = 0$, $s<0$ and

$$\begin{aligned} R(s) &= e^{-im^2 s} & s &> s_0 \\ &= 0 & s &< s_0 \end{aligned}$$

This $R(s)$ and all its derivatives vanish at the origin, thus satisfying the regulator conditions as $s_0 \to 0$. It appears, then, that regularization is a procedure for inserting, into a calculation that does not employ it, enough of the structure provided by the proper time representation to ensure gauge invariant results.

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The Lagrange function for a zero spin neutral meson field in scalar interaction with the proton-antiproton field is given by

$$L = -\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x_\mu}\right)^2 + \mu^2\phi^2\right] + g\phi\frac{1}{2}[\bar{\psi},\psi].$$

To find an approximate expression for the resulting coupling between the neutral mesons and the electromagnetic field we replace $\frac{1}{2}[\bar{\psi},\psi]$ by its vacuum expectation value calculated in the presence of a known electromagnetic field. In using the latter to represent the photons emitted in the spontaneous neutral meson decay, we are introducing an approximation which neglects terms in the square of the meson-nucleon mass ratio, (μ/M). Now

$$\begin{aligned}\frac{1}{2}\langle[\bar{\psi}(x),\psi(x)]\rangle &= i \,\mathrm{tr}\, G(x,x)\\ &= -M\int_0^\infty ds\, e^{-iM^2 s}\,\mathrm{tr}\,(x|\,e^{-iHs}|x)\\ &= -\frac{\partial}{\partial M} L_1(x)\end{aligned}$$

according to the formulae of page (6). Keeping only the first term in the power series expansion of L_1 we find

$$\begin{aligned}\frac{1}{2}\langle[\bar{\psi}(x),\psi(x)]\rangle &\cong -\frac{e^2}{6\pi^2} M\int_0^\infty ds\, e^{-M^2 s}\,\mathcal{F}\\ &= -\frac{2\alpha}{3\pi}\,\frac{1}{M}\,\mathcal{F}.\end{aligned}$$

Therefore the effective coupling term between neutral meson and electromagnetic field is

$$g\phi\frac{1}{2}\langle[\bar{\psi},\psi]\rangle = \frac{\alpha}{3\pi}\frac{g}{M}\phi(E^2 - H^2)$$

which describes the decay of the meson into two photons with parallel polarizations at the rate

$$\frac{1}{\tau} = \frac{\alpha^2}{144\pi^3}\,\frac{g^2}{\hbar c}\left(\frac{\mu}{M}\right)^2\frac{\mu c^2}{\hbar}.$$

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A pseudoscalar interaction between the spin-zero neutral meson field and the proton field is described by the term

$$g\phi\,\tfrac{1}{2}[\bar{\psi},\gamma_5\psi]$$

in the Lagrange function. For our purpose, $\frac{1}{2}\langle[\bar{\psi},\gamma_5\psi]\rangle$ is to be replaced by

$$\tfrac{1}{2}\langle[\bar{\psi}(x),\gamma_5\psi(x)]\rangle = i\ \mathrm{tr}\,\gamma_5 G(x,x)$$

$$= -M\int_0^\infty ds\ e^{-iM^2 s}\ \mathrm{tr}\,\gamma_5(x|\,e^{-iHs}\,|x).$$

Inserting the transformation function (4) and replacing s by -is, this becomes

$$\tfrac{1}{2}\langle[\bar{\psi}(x),\gamma_5\psi(x)]\rangle$$

$$= -\frac{M}{16\pi^2}\int_0^\infty \frac{ds}{s^2}\ e^{-M^2 s}\ e^{-\frac{1}{2}\,\mathrm{tr}\,\log\frac{\sin eFs}{eFs}}\ \mathrm{tr}(\gamma_5 e^{\frac{1}{2}e\sigma Fs}).$$

The eigenvalues of σF, related to γ_5 by

$$(\tfrac{1}{2}\sigma F)^2 = 2(\mathcal{F}+\gamma_5 G)$$

give

$$\mathrm{tr}(\gamma_5 e^{\frac{1}{2}e\sigma Fs}) = -4\ \mathrm{Im}\cosh es\sqrt{2(\mathcal{F}+iG)},$$

while it was proved before that

$$e^{-\frac{1}{2}\,\mathrm{tr}\,\log\frac{\sin eFs}{eFs}} = (es)^2 G/\mathrm{Im}\cosh es\sqrt{2(\mathcal{F}+iG)}.$$

Therefore without any further approximation

$$\tfrac{1}{2}\langle[\bar{\psi}(x),\gamma_5\psi(x)]\rangle = \frac{e^2 M}{4\pi^2}\int_0^\infty ds\ e^{-M^2 s}\ G \qquad (12)$$

$$= \frac{\alpha}{\pi}\frac{1}{M}\,E\cdot H,$$

Therefore the effective coupling term between pseudoscalar neutral mesons and the electromagnetic field is

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$$g\phi\langle\tfrac{1}{2}[\bar\Psi(x),\gamma_5\Psi(x)]\rangle = \frac{\alpha}{\pi}\,\frac{g}{M}\phi\; E\cdot H,$$

which describes the decay of the meson into two photons perpendicularly polarized at the rate

$$\frac{1}{\tau} = \frac{\alpha^2}{64\pi^3}\,\frac{g^2}{\hbar c}\,\left(\frac{\mu}{M}\right)^2\frac{\mu c^2}{\hbar}.$$

The pseudovector coupling term

$$\frac{g}{2M}\,\frac{\partial\phi}{\partial x_\mu}\,\frac{1}{2i}[\bar\Psi,\gamma_5\gamma_\mu\Psi] \tag{12}$$

is formally equivalent to the pseudoscalar interaction for the problem under discussion, in the approximation to which it is being treated. This is demonstrated by an integration by parts, combined with the use of the Dirac equation. Yet it has been found difficult to verify this equivalence in the actual results of calculations. However, this apparent ambiguity is not an indication of a fundamental defect in the theory. We shall demonstate that it is caused by insufficient attention to the limiting process implicit in the formalism. On introducing the vacuum expectation value of the proton field, (12) becomes

$$\begin{aligned} &-\frac{g}{2M}\,\frac{\partial\phi(x)}{\partial x_\mu}\,\mathrm{tr}\,\gamma_5\gamma_\mu G(x,x) \\ &= \frac{g}{2M}\,\frac{\partial\phi(x)}{\partial x_\mu}\,\mathrm{tr}\,\gamma_5\gamma_\mu (x|\,M-\gamma\pi\,|i\textstyle\int_0^\infty ds\; e^{-iM^2 s\,-\,iHs}\,|\,x). \end{aligned}$$

The integration by parts replaces (12) with

$$\begin{aligned} &\frac{g}{2M}\phi(x)\,\mathrm{tr}\,\gamma_5(x|\{\gamma\pi,\ (M-\gamma\pi)\textstyle\int_0^\infty ds\; e^{-iM^2 s\,-\,iHs}\}|\,x) \\ &= \frac{g}{M}\,\phi(x)\,\mathrm{tr}\gamma_5(x|\ \ H\textstyle\int_0^\infty ds\; e^{-iM^2 s\,-\,iHs}|\,x) \\ &= i\,\frac{g}{M}\phi(x)\,\mathrm{tr}\gamma_5(x\,|\textstyle\int_0^\infty ds\; e^{-iM^2 s}\frac{\partial}{\partial s}\,e^{-iHs}|\,x). \end{aligned}$$

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A formal verification of the equivalence theorem is now obtained on integrating by parts with respect to s. This gives for (12)

$$-g\phi(x)M\int_0^\infty ds\, e^{-iM^2 s}\, \mathrm{tr}\,\gamma_5(x|e^{-iHs}|x)$$

$$+ i\frac{g}{M}\phi(x)\left[e^{-iM^2 s}\, \mathrm{tr}\,\gamma_5(x)|e^{-iHs}|x)\right]_{s=0}^{s=\infty}. \qquad (13)$$

The integrated term vanishes for $s\longrightarrow\infty$ in virtue of the convergence factor implicit in the integral representation, and vanishes for s = 0 since $\mathrm{tr}\,\gamma_5=0$. This exhibits the identity of the pseudo-scalar and pseudo-vector couplings.

However, we could also evaluate (12) by writing it as

$$i\frac{g}{M}\phi(x)\int_0^\infty ds\, e^{-iM^2 s}\frac{\partial}{\partial s}\,\mathrm{tr}\,\gamma_5(x|e^{-iHs}|x)$$

and use the already calculated value

$$\mathrm{tr}\,\gamma_5(x|e^{-iHs}|x) = -i\frac{e^2}{4\pi^2}G. \qquad (14)$$

This is independent of s and therefore yields zero for (12). The same result would be obtained from (13) since (14) does not vanish as $s\rightarrow 0$.

It is the second method that is at fault, of course. The error lies in overlooking the fact that the divergent matrix element $(x|e^{-iHs}|x)$ is actually the limit of $(x'|e^{-iHs}|x'')$ as x' and x'' approach x. As long as $(x'-x'')^2$ is finite, though arbitrarily small, (14) will be dependent on s and so the basis of the second, null, result is invalidated. On referring to (3) we see that

$$\mathrm{tr}\,\gamma_5(x'|e^{iHs}|x'') = -i\frac{e^2}{4\pi^2}G\, e^{\frac{i(x'-x'')^2}{4s}}.$$

The additional factor is effectively field-independent since, with (x'-x'')

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arbitrarily small, only its form for correspondingly small s is of significance. We now evaluate (12) as

$$\frac{e^2}{4\pi^2} \frac{g}{M} \phi(x)\, G \lim_{x',x''\to x} \int_0^\infty ds \frac{\partial}{\partial s} e^{i\frac{(x'-x'')^2}{4s}}$$

$$= \frac{e^2}{4\pi^2 M} \phi(x) G \lim_{x',x''\to x} \left[1 - \lim_{s\to 0} e^{i\frac{(x'-x'')^2}{4s}}\right].$$

With the limits taken in this order the result is just

$$\frac{e^2 g}{4\pi^2 M} \phi(x) G$$

as given by the pseudo-scalar coupling. Reversing the order of the limits would give the spurious result zero.