

Hodge Theory and Moduli

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Based in part on joint work with
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This talk will center around the questions

Q1 (Torelli): *To what extent does the Hodge structure on its cohomology $H^*(X)$ determine a smooth projective variety X ?*

Better yet: How can one reconstruct X from Hodge theoretic data associated to it?

Q2: *To what extent does the structure of a completed image of the period mapping reflect the structure of a completed moduli space $\overline{M}$?*

One knows a lot about how Hodge structures degenerate; can this be used to help understand the boundary $\overline{M} \setminus M$ of moduli spaces?

If time permits I will make a few comments about Hodge theory and fundamental groups of quasi-projective varieties and the moduli of their linear representations.

Q3: *How is the Hodge theory of the fundamental group used in studying the Shafarevich conjecture?*

That conjecture is that the universal cover of a smooth projective variety should be holomorphically convex.

Notations and terminology.*

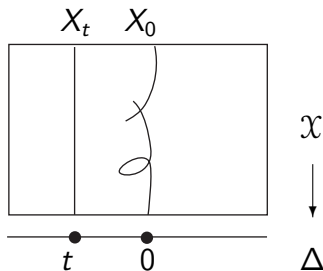
- X is a smooth projective variety $\rightsquigarrow H^r(X, \mathbb{Q})$ has a polarized Hodge structure $(V, Q, F^\bullet)$ of weight $r^\dagger$ with Hodge filtration

$$F^r \subset F^{r-1} \subset \dots \subset F^0 = V_{\mathbb{C}} \text{ where } F^p \oplus \overline{F}^{r-p+1} \xrightarrow{\sim} V_{\mathbb{C}}$$
$$V_{\mathbb{C}} = \bigoplus_{p+q=r} V^{p,q}, \quad V^{p,q} = F^p \cap \overline{F}^q = \overline{V}^{q,p}$$

- X_0 is a complete variety $\rightsquigarrow H^r(X_0, \mathbb{Q})$ has a mixed Hodge structure $(V, W_\bullet, F^\bullet)$ where $W_0 \subset W_1 \subset \dots \subset W_r = V$ is the weight filtration and where $F^\bullet$ induces on $\text{Gr}_m^W(V)$ a Hodge structure of weight m .

*The general references are for Hodge theory [V] and [PS], [GGR] and [R] for limiting mixed Hodge theory and [K] for moduli. [CM-SP] is a general reference for period mappings.

[†]The bilinear form and subsequent polarization depend on the Chern class of an ample line bundle.



- $X_t \rightarrow X_0$ is a degeneration $\rightsquigarrow H^r(X_t, \mathbb{Q})$ has a limiting mixed Hodge structure $(V, W(N), F_{\text{lim}}^\bullet)$ where after a base change T is the unipotent monodromy and $N = \log T$;

- M is a KSBA moduli space with completion $\overline{M}$; we will usually have

$$\begin{array}{ccc} B & \subset & \overline{B} \\ \downarrow & & \downarrow \\ M & \subset & \overline{M}, \end{array}$$

with $\overline{B}$ smooth and $\overline{B} \setminus B = Z = \cup Z_i$ is a normal crossing divisor; $Z_I = \cap_{i \in I} Z_i$ are the strata in Z .

- $\sigma_I \subset \mathrm{Gr}_{-2}^{W(\sigma_I)} \mathrm{End}(V)$ denotes the monodromy cone with dual cone $\check{\sigma}_I \subset \mathrm{Gr}_{+2}^{W(\sigma_I)} \mathrm{End}(V)$.

- $D = \{(V, Q, F^\bullet)\}$ is a period domain and

$$\begin{array}{ccc} B & \xrightarrow{\Phi} & \Gamma \backslash D \\ \downarrow & \nearrow \text{dashed} & \\ M & & \end{array} \quad \Phi_M$$

is the period mapping with monodromy group $\Gamma \subset \text{Aut}(V_{\mathbb{Z}}, Q)$ and image $P \subset \Gamma \backslash D$; may assume the monodromy around each Z_i is of infinite order which implies that Φ is proper.

- Study $M, \overline{M}$ via the maximal and minimal completions of Φ, P

$$\begin{array}{ccc} \overline{B} & \begin{array}{l} \nearrow \Phi_T \\ \searrow \Phi_S \end{array} & \begin{array}{c} \overline{P}_T \\ \downarrow \\ \overline{P}_S \end{array} \end{array}$$

- Are particularly interested in
 - non-classical case when $D \neq$ Hermitian symmetric domain; then P is an integral subvariety of the non-trivial horizontal distribution $I \subset TD$;‡
 - X is a surface with $p_g \geq 2$; frequently for convenience we assume $q = 0$.
- Many topics concerning Hodge theory and moduli will not be discussed; a very partial list is
 - log general type and hyperbolicity properties of M ;
 - Calabi-Yau varieties;
 - Shafaverich-Arakelov program;
 - curvature and the Chern classes of the Hodge bundles;
 - litaka conjecture;
 - algebraic cycles and Noether-Lefschetz loci in M ;
 - non-abelian Hodge theory.

‡ I is defined by $\dot{F}^p \subset F^{p-1}$.

I. Torelli results

- Global Torelli means Φ_M is 1-1;
- Generic global Torelli means $\deg \Phi_M = 1$;
- Local Torelli means Φ_M is locally 1-1, usually implied by Φ_* is injective;
- Generic local Torelli means Φ_M is locally 1-1 on a Zariski open.
- General picture is that some form of Torelli holds frequently (usually?) but not always; examples where it fails tend to be rather special (e.g., particular general type surfaces with $p_g = 0, 1$).
- Local Torelli holds for Calabi-Yau varieties.

- Since Φ is proper, the fibres of Φ are the *complete* subvarieties $Y \subset B$ with $\rho(\pi_1(Y)) = \text{finite group}$.
- If the monodromy representation ρ is faithful, then the fibres of Φ are the $Y \subset B$ with finite image of $\pi_1(Y) \rightarrow \pi_1(B)$.

Informally, *Torelli fails to hold when there are subvarieties Y of moduli for which $\pi_1(Y) \rightarrow \pi_1(M)$ is a finite mapping.*

Elliptic surfaces

- The Kuranishi space is smooth of dimension $h^1(\Theta_X) = 10p_g + 8(1 - q)$.

Theorem (M-H Saito [S])

Local Torelli holds if for the classical j function

- (i) $j \neq \text{constant}$;
- (ii) X is regular and $j = \text{constant}$ not equal $0, 1$;
- (iii) $j = \text{constant}$ not equal $0, 1$ and $p_g \geq q + 2$.

The proof is given by analyzing the differential Φ_* as expressed by

$$H^1(\Theta_X) \rightarrow \text{Hom}(H^0(K_X), H^1(\Omega_X^1)).$$

A useful form is

$$H^1(\Theta_X) \otimes H^0(\Omega_X^2) \rightarrow H^1(\Omega_X^1).$$

Theorem (Chakiris [Ch])

Generic global Torelli holds in case X is an elliptic pencil with $p_g \geq 2$.

Theorem (Shepherd-Barron [S-B])

Generic global Torelli holds for a Jacobian elliptic surface X with $p_g \geq q + 3$.

How is one going to prove such results?

In the classical case where, e.g., for a smooth curve C the Riemann theta divisor

$$\theta \subset J(C)$$

is constructed from the polarized Hodge structure on $H^1(C)$, the dual of the canonical curve is the ramification divisor of the Gauss mapping $\theta \rightarrow \mathbb{P}H^0(\Omega_C^1)$.[§]

[§]This argument is due to Andreotti. The hyperelliptic case for $g \geq 3$ requires a special treatment.

In contrast when the horizontal distribution I is non-trivial there is no “geometric object” or motive associated to a general polarized Hodge structure. There are currently three methods that have been used.

(i) Show that special Hodge structures in the image $\Phi(M)$ arise from special varieties X_m , $m \in M$. If

- the special X_m can be constructed from $H^n(X)$,
- the differential Φ_* is 1-1 at the special $m \in M$, and
- there are enough such special X_m 's,

then as in the Pyatecki-Shafarevich proof of global Torelli for polarized K3's one may conclude generic global Torelli.¶

¶For K3's since local Torelli holds everywhere and the image of $M \rightarrow \Gamma \backslash D$ contains an open set, generic global Torelli implies global Torelli.

For K3's P-S use special Kummer surfaces $E' \times E''/i' \times i''$'s. Chakiris uses *special elliptic pencils* obtained from

- elliptic curve E and $p \in E$;
- hyperelliptic curve $\tilde{C}$;
- $X =$ minimal resolution of $E \times \tilde{C}/\iota$ when $\iota = (\iota_E, \iota_{\tilde{C}})$;
- $NS(X) = \bigoplus_{j=1}^{2(p_g+1)} (G_4)_j$ and

$$\begin{cases} G_4 \cong \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3 \oplus \mathbb{Z}\beta \\ \alpha_i \circ \alpha_j = 0, \alpha_i^2 = -2, \alpha_i \circ \beta = 1; \end{cases}$$

- any elliptical pencil with this NS group is a special elliptic pencil.

The details of his beautiful argument are delicate; of course extensive use is made of the structure of elliptic surfaces due to Kodaira.

- (ii) Show that X can be reconstructed from the algebraic information in the differential, or equivalently the first order variation (1-jet), of its Hodge structure. For example, for a smooth curve C the co-differential of Φ_* may be identified with the map

$$\mathrm{Sym}^2 H^0(K_C) \rightarrow H^0(2K_C).$$

For $g \geq 4$ and C non-hyperelliptic this map is surjective and its kernel is the space of quadrics that define the canonical curve; this leads to generic global Torelli in this case.

The proof of Shepherd-Barron result is an intricate and subtle argument that shows that the rank 1 elements in the image of

$$H^1(\Theta_X) \rightarrow \mathrm{Hom}(H^0(\Omega_X^2), H^1(\Omega_X^1))$$

may be identified with the ramification points, *in the stacky sense*, of $j : C \rightarrow \overline{\mathcal{E}ll}$. From this one may proceed to recover the elliptic surface.

A prototype of this argument is yet another proof of generic global Torelli for curves: The rank 1 elements in Φ_* are the *Shiffer variations*,^{||} given by $\varphi_{2K_C}(C) \subset \mathbb{P}(H^1(\Theta_C))$ as the image of

$$\left(\frac{1}{p}\right) \frac{d}{dp} \rightarrow H^1(\Theta_C), \quad p \in C$$

arising from the cohomology sequence of

$$0 \rightarrow \Theta_C \rightarrow \Theta_C(p) \rightarrow \left(\frac{\mathbb{C}}{p}\right) \frac{d}{dp} \rightarrow 0.$$

^{||}The complex structure stays the same on $C \setminus \{p\}$ and changes by a δ -function at p .

- (iii) Use a Hodge theoretic analysis of the blow up $\tilde{M}_0$ of a part M_0 of the boundary of moduli and a generic local Torelli theorem for the blown up locus. Generally the $\tilde{X}_{m_0}$ should be “simpler” than a general X_m . Then use a generic local Torelli for the limiting mixed Hodge structures along $\tilde{M}_0$ plus monodromy around $\tilde{M}_0$ to infer generic local Torelli for M . This method originated in Friedman [F] who used it to give other proofs of generic global Torelli both for curves** and for polarized K3’s. Usually results from both (i) and (ii) are used in this approach. It may turn out to be the most useful strategy for generic Torelli in examples; this is a part of the next topic.

**Thus giving proof #4 for the generic Torelli for curves.

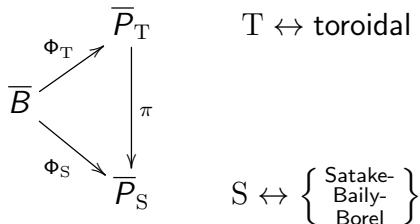
Summary: For selected special classes of varieties there is a rich interaction between Torelli questions and moduli.

Q: Do general type algebraic surfaces realizing the Noether bound (see below) satisfy generic global Torelli?

Q: Same question for Castelnuovo surfaces; i.e., non-degenerate surfaces $X \subset \mathbb{P}^n$ of fixed degree and with maximum $p_g \neq 0$?

- Some generalities concerning completions of period mappings and the boundaries of moduli spaces

Given the data $(\overline{B}, Z; \Phi)$ as above there are two natural completions



of a period mapping $\Phi : B \rightarrow P \subset \Gamma \backslash D$. The map Φ_T is related to the Kato-Usui partial completions $\overline{\Gamma \backslash D}$, which assume that a fan exists [KU]. However the above completions are of a relative character.

As a set, $\overline{P}_T$ is constructed by adding to P the equivalence classes of limiting mixed Hodge structures along the open strata $Z_I^* \subset Z_I$. Conjecturally it is a projective variety. As a set, $\overline{P}_S$ is obtained from the equivalence classes of limiting mixed Hodge structures $\overline{P}_T$ by passing to the associated graded polarized Hodge structures. The augmented Hodge line bundle

$$L := \bigotimes_{p \geq \lfloor \frac{r}{2} + 1 \rfloor} \det F^p$$

extends canonically to $\overline{B}$ and (conjecturally) descends to $\overline{P}_S$.^{††} Conjecturally $\overline{P}_S$ has the structure of a projective algebraic variety and $L \rightarrow \overline{P}_S$ is ample. This conjecture has been established in some special cases and in general when $\dim B = 2$. So far as applications to moduli are concerned one can usually proceed assuming the conjectures.

^{††}This has been verified in the examples discussed later.

- *Setup:* We assume $M \subset \overline{M}$ where over each point of $\overline{M} \setminus M$ corresponding to an irreducible variety X_0 with slc singularities we have
 - (i) a flat KSBA smoothing degeneration $\mathcal{X} \rightarrow \Delta$ where X_t is smooth for $t \neq 0$ and $\mathcal{X}$ has canonical singularities along $X_{0,\text{sing}}^*$;
 - (ii) a semi-stable-reduction $\mathcal{X}' \rightarrow \Delta$ of the above family where X'_0 has as one component a desingularization $\widetilde{X}_0 \rightarrow X_0$.

We denote by $M_f \subset \overline{M}$ the subvariety around which a smoothing of X_0 has finite monodromy.

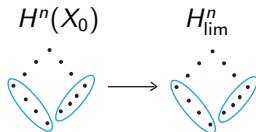
*In general X_0 will not be irreducible. A particularly interesting example due to Liu-Rollenske [LR] is the surface analogue of two $\mathbb{P}^1$'s

joined at three distinct points (dollar bill curve ) , viz. two $\mathbb{P}^2$'s joined in a special way at four lines in general position. This surface has $K_{X_0}^2 = 1$, $p_g(X_0) := h^0(\omega_{X_0}) = 2$. It may be smoothed to give an I -surface as discussed below.

- Then
 - $\Phi : M \rightarrow P$ extends to $\Phi : M_f \rightarrow P$;
 - Φ induces $\Phi : \overline{M} \rightarrow \overline{P}_S$.

For the case of surfaces a proof of the second statement may be given by checking the list in [K] of slc singularities where the associated graded to the limiting mixed Hodge structure can be determined and shown to be independent of the smoothing on a case by case checking.[†]

[†]These singularities are said to be *cohomologically insignificant*. If we picture the Hodge diamonds for the associated graded pure Hodge structures, then we should have



The blue parts should line up isomorphically; i.e., the local invariant cycle theorem should hold for the parts of $H^n(X_0)$ that are obviously birationally invariant.

The general result follows from [KL]. It is definitely *not* the case that $\Phi : M \rightarrow P$ induces a map from $\overline{M}$ to $\overline{P}_T$. We will turn to this below.

In case X_0 is normal with e simple elliptic singularities, and where for purposes of exposition we assume that $q(X_t) = 0$, we have

- $e \leq p_g(X) + 1$; if equality holds, then all of the elliptic curves in the resolved $\tilde{X}_0$ are isogenous.

Sketch of proof. From the cohomology sequence of

$$0 \rightarrow \Omega_{\tilde{X}_0}^2 \rightarrow \Omega_{\tilde{X}_0}^2(C) \rightarrow \omega_C \rightarrow 0$$

where $C = \sum_{i=1}^e C_i$ we obtain

$$e = p_g(X) + q(\tilde{X}_0) - p_g(\tilde{X}_0).$$

By Castelnuovo's lemma, for α and $\beta \in H^0(\Omega_{\tilde{X}_0}^1)$ if $\alpha \wedge \beta = 0$ then α, β are pulled back from a map to a curve A of genus ≥ 2 . If this does not happen, then $\alpha_1 \wedge \alpha_2, \dots, \alpha_1 \wedge \alpha_q$ are linearly independent and

$$\begin{aligned} p_g(\tilde{X}_0) &\geq q(\tilde{X}_0) - 1 \\ \implies e &\leq p_g(X) + q(\tilde{X}_0) - (q(\tilde{X}_0) - 1) \leq p_g(X) + 1. \end{aligned}$$

If there is a map $\tilde{X}_0 \rightarrow A$, then analyzing it gives that again $e \leq p_g(X) + 1$ and if equality holds, then all C_i are isogeneous to A .

From the dimension of the vanishing cohomology associated to each p_i one may determine the degrees $d_i = -C_i^2$ (cf. [A]).

Extension data: The mapping Φ_S on a stratum Z_I^* looks like an ordinary period mapping. Thus the new information needed to describe the period mapping on the boundary of moduli arises when an open stratum Z_I^* is a fibre of Φ_S ; i.e., when the limiting mixed Hodge structures along Z_I^* have locally constant associated graded Hodge structures. What is then varying is the *extension data*.[‡]

- Given Hodge structures $H^0, H^1, \dots, H^m$ we denote by $E = E\{H^0, H^1, \dots, H^m\}$ the set of mixed Hodge structures $(V, W_\bullet, F^\bullet)$ with $\mathrm{Gr}_k^W(V) = H^k$.

[‡]This gives a polarizable, admissible variation of equivalence classes of mixed Hodge structure.

Then E has the structure of an iterated fibre space

$$\begin{array}{c} E_m \\ \downarrow \\ \vdots \\ \downarrow \\ E_2 \\ \downarrow \\ E_1 \end{array}$$

of length m where $E_k =$ set of at most k -fold extensions in E . Thus E_1 is the set of extensions

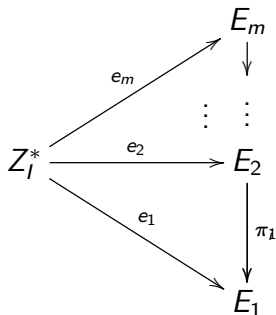
$$0 \rightarrow H^{\ell-1} \rightarrow A \rightarrow H^{\ell} \rightarrow 0$$

where $A \in \text{Ext}_{\text{MHS}}^1(H^{\ell}, H^{\ell-1})$ is a mixed Hodge structure. This particular $\text{Ext}_{\text{MHS}}^1$ is a compact complex torus. E_2 is the set of mixed Hodge structures of length 3 whose associated graded is $\{H^{\ell-2}, H^{\ell-1}, H^{\ell}\}$, and so forth.

If we denote by $E_{k,k-1}$ the fibre of $E_k \rightarrow E_{k-1}$, then

$$E_{k,k-1} \cong \bigoplus^{\ell} \mathrm{Ext}_{\mathrm{MHS}}^1(H^{k+\ell}, H^{\ell}).^{\S}$$

It is the quotient of a $\mathbb{C}^a$ by a discrete abelian subgroup.
There are mappings



of Z_I^* to the above tower.

^{\S}We recall that $\mathrm{Ext}_{\mathrm{MHS}}^q(A, B) = 0$ for $q \geq 2$.

Theorem

- (i) *The mapping e_1 extends to Z_I to give*

$$\begin{array}{ccc}
 & \text{Alb } Z_I & \\
 \alpha \nearrow & & \downarrow \\
 Z_I & & J_I \subset E_1 \\
 e_1 \searrow & &
 \end{array}$$

where J_I is an abelian variety polarized by the line bundles $L_M \rightarrow J_I$ where $M \in \check{\sigma}_I$;

- (ii) *The image of e_2 lies in a variety which the direct sum of the $\{L_M \setminus \text{zero section}\}$;*
- (iii) *"The mapping $e_m(Z_I^*) \rightarrow e_2(Z_I^*)$ is a finite morphism of algebraic varieties."*¶

¶The " " means that only an informal argument for this has been written down.

Part (iii) says that up to finite data (integration constants), the totality of the extension data is determined by that of levels 1,2.

Example: The discrete data in the fibres carries interesting information. If we have a variation of limiting mixed Hodge structure of Hodge-Tate type whose associated graded is $\{\mathbb{Q}^a, \mathbb{Q}(-1)^{a+b}, \mathbb{Q}(-2)^a\}$, the period matrices are of the form

$$F^2 = \begin{pmatrix} I \\ A \\ B \end{pmatrix} \begin{matrix} a \\ a+b \\ a \end{matrix}, \quad F^2/F^1 = \begin{pmatrix} 0 \\ I \\ {}^tA \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{pmatrix}$$

where

$$B + {}^tB = {}^tA \cdot A \quad (\text{Hodge-Riemann I}).$$

From horizontality we have

$$dB = {}^tA dA.$$

Thus B is determined by A up to integration constants.

Typically A is a linear combination with holomorphic coefficients of $\log t_i$'s; B will then involve dilogarithms $li_2(t_i)$'s and horizontality gives an ODE for the dilogarithm terms.

Geometric interpretation: Let $\mathcal{X} \rightarrow \Delta$ be a degeneration where X_0 has an elliptic singular point p with resolution $(\tilde{X}_0, \tilde{C}) \rightarrow (X_0, p)$. Using the above and the fact that semi-stable-reductions exist, we may infer that the limiting mixed Hodge structure has associated graded

$$H^1(\tilde{C}), H^2, H^1(\tilde{C})(-1)$$

where $h^{2,0} = p_g(X_t) - 1$. Arguing heuristically we may assume that the central fibre in a semi-stable reduction $\mathcal{X}' \rightarrow \Delta'$ may be taken to be

$$X'_0 = \tilde{X}_0 \cup_{\tilde{C}} Y$$

for a smooth surface Y .*

*The reason is that when $N^2 = 0$ the Clemens-Schmid exact sequence only requires double curves.

Denoting by Hg^1 the integral Hodge classes in H^2 , the algorithm for computing H^2 as the cohomology of

$$H^0(\tilde{C})(-1) \xrightarrow{\mathrm{Gy}} H^2(\tilde{X}_0) \oplus H^2(Y) \xrightarrow{R} H^2(\tilde{C})$$

suggests that generically there exists a class $\xi \in \mathrm{Hg}^1(\tilde{X}_0)$ and classes $\eta_i \in \mathrm{Hg}^1(Y)$ such that the $\xi \oplus \eta_i$ give a basis for $\mathrm{Hg}^1 \subset H^2$.[†] Then $\xi - \eta_i|_{\tilde{C}} \in \mathrm{Pic}^o(\tilde{C})$ and the level 1 extension data is given in

$$\mathrm{Ext}_{\mathrm{MHS}}^1(\mathrm{Hg}^1, H^1(\tilde{C}))$$

by

$$\xi \oplus \eta_i \rightarrow \mathrm{AJ}_{\tilde{C}}(\xi - \eta_i).$$

[†]The η_i will correspond to a basis for the vanishing cohomology.

Realizing $\tilde{C}$ as a plane cubic, we may take for Y the del Pezzo surface given by the blow up at points $p_i \in \tilde{C}$ to have the Friedman condition

$$N_{\tilde{C}/\tilde{X}_0} \otimes N_{\tilde{C}/Y} \cong \mathcal{O}_{\tilde{C}}$$

for first order smoothability of $\tilde{X}_0 \cup_{\tilde{C}} Y$.

Thought example: $\overline{M}$ = moduli space of general type surfaces and we have a Zariski open set $\overline{U} \subset \overline{M}$ such that $U = \overline{U} \cap M$ is smooth and where $\overline{U}$ is a normal variety with a smooth singular locus $\Sigma = \overline{U} \setminus U$ along which the singular surfaces have simple elliptic singularities. Then the period mapping on U extends to

$$\Phi_{\overline{U}} : \overline{U} \rightarrow \overline{P}_S$$

where each point $\overline{u}$ of Σ maps to the associated graded of the limiting mixed Hodge structure as above. We note that taking *any* normal disc $\Delta \subset \overline{U}$ with $\overline{u} = \{0\}$ and $\Delta \cap U = \Delta^*$ we get the same point $\Phi_{\overline{U}}(\overline{u}) \in \overline{P}_S$. The above construction suggests that we attempt to desingularize $\overline{M}$ along Σ by blowing up using as parameters the extension data to the limiting mixed Hodge structure along normal discs.

Specific example ([FPR]). Recall the Noether bound

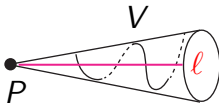
$$p_g \leq \frac{1}{2}(K_X^2 + 3)$$

for X smooth and regular:

- First non-classical surface that achieves the bound is an I -surface; general type with
 $K_X^2 = 1$, $p_g(X) = 2$, $q(X) = 0$ (in fact $\pi_1(X) = \{e\}$);
- much studied classically; for Gorenstein X the pluricanonical ring R_X is well known; setting $\varphi_m = \varphi|_{mK_X}$ we have

$$\varphi_2 : X \xrightarrow{2:1} \mathbb{P}(1, 1, 2)$$

where $\mathbb{P}(1, 1, 2)$ is isomorphic to the singular quadric $Q \subset \mathbb{P}^3$ and the branch locus $B = P + V$ where V is $Q \cap \{\text{quintic}\}$



- $\varphi_1 : X \dashrightarrow \mathbb{P}^1$ is a **pencil** of $g = 2$ hyperelliptic curves with base point P ;
- $\varphi_5 : X \longrightarrow \mathbb{P}(1, 1, 2, 5)$ is an embedding and a general X has an equation

$$F = z^2 - f_{10}(x_0, x_1, y) = 0$$

where $f_{10} = 0$ is the branch locus;[‡]

- M_I has dimension 28 and is unirational;
- for X corresponding to a point in $\overline{M}_I^{\text{Gor}}$, V does not pass through P and R_X has the same structure (syzygies) as in the smooth case; M_I is smooth at X .

[‡]For any smooth general type surface X , φ_{mK_X} is an embedding for some $m \leq 5$; this bound is sharp and is realized by an I -surface.

Hodge theoretic aspects of the normal locus in $\overline{M}_I^{\text{Gor}}$

- two main results
 - (i) local Torelli where X is smooth [PZ];[§]
 - (ii) the extension data in the limiting mixed Hodge structure desingularizes general points in boundary strata.
- $\dim D = 57$, horizontality is a contact structure;

$$\Phi : M_I \rightarrow \Gamma \backslash D$$

it is an immersion where X is smooth and $\Phi(X) \subset \Gamma \backslash D$ is a contact subvariety.

- M_I is log general type and is hyperbolic possibly modulo the proper subvariety where X is nodal.[¶]
- For X_0 with simple elliptic singularities and smoothable as above we have given a general method for mapping the first order extension data to the blow-up of the locus of such X_0 's in the boundary of moduli.

[§]Below we will sketch two other proofs of this result.

[¶]These are both general consequences of generic local Torelli.

For I surfaces an informal statement of the result of this process is

For I -surfaces this mapping is generically locally 1-1, and for each such stratum in $\overline{M}_I^{\text{Gor}}$ the level one extension data in the limiting mixed Hodge structure desingularizes $\overline{M}_I$ on an open set in the boundary.

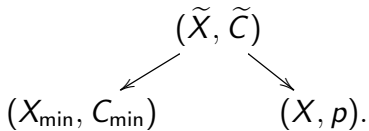
For an idea of the proof from [\[FPR\]](#) we have Table 1 below. This table shows that the codimension in moduli of the locus of X 's with k simple elliptic singularities p_α of degrees $d_\alpha = -\widetilde{C}_\alpha^2$ is exactly 1 less than the number of parameters in the level 1 extension data of the limiting mixed Hodge structures. Further details will be given below.

stratum	dimension	minimal resolution $\tilde{X}$	$\sum_{i=1}^k (9 - d_\alpha)$	k	codim in $\overline{M}_I$
I_0	28	canonical singularities	0	0	0
I_2	20	blow up of a K3-surface	7	1	8
I_1	19	minimal elliptic surface with $\chi(\tilde{X}) = 2$	8	1	9
$II_{2,2}$	12	rational surface	14	2	16
$II_{1,2}$	11	rational surface	15	2	17
$II_{1,1,R}$	10	rational surface	16	2	18
$II_{1,1,E}$	10	blow up of an Enriques surface	16	2	18
$III_{1,1,2}$	2	ruled surface with $\chi(\tilde{X}) = 0$	23	3	26
$III_{1,1,1}$	1	ruled surface with $\chi(\tilde{X}) = 0$	24	3	27

Note that $9 - d_\alpha$ is the Milnor number μ_α of the singularity.
 For I_2 with the notation

- $(X, p) = I$ -surface with a simple elliptic singularity of degree 2;
- $(\tilde{X}, \tilde{C}) \rightarrow (X, p)$ the resolution.

The picture is



- $X_{\min}$ is a K3 with a degree 2 polarization given by $X_{\min} \xrightarrow{\pi} \mathbb{P}^2$ with branch curve a sextic D and $C_{\min}$ is π^{-1} (tangent line to D); it has $p_a(C_{\min}) = 2$ and $\tilde{C} \rightarrow C_{\min}$ is the normalization.

LMHS has $\left\{ \begin{array}{l} \text{Gr}_2 \text{ contains a canonical sub-Hodge structure} \\ \cong H^2(X_{\min})_{\text{prim}} \\ \text{Gr}_3 \cong H^1(\tilde{C})(-1). \end{array} \right.$

This will give for a general boundary point that Gr_2 contains the primitive cohomology of a polarized K3 surface and Gr_3 gives $\tilde{C}$. The remaining parameters are given by the extension data.

The parameter count is

$$\begin{array}{ccccc}
 X_{\min} & & C_{\min} & & p_i \\
 | & & \uparrow & & \\
 19 \text{ parameters} & & 1 \text{ parameter} & & 7 \text{ parameters} \\
 \underbrace{\hspace{10em}} & & & & = 27 \text{ parameters} \\
 \text{this gives the blowup of the} & & & & \\
 \text{singular boundary component in } \overline{M}_I & & & &
 \end{array}$$

The remaining normal parameter is the scaling one for

$$N : H^1(\tilde{C})(-1) \xrightarrow{\sim} H^1(\tilde{C}).$$

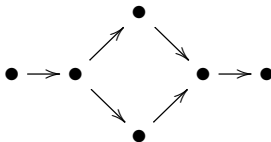
Denoting by (V, p) the local elliptic singularity we have a surjection

$$T \operatorname{Def}(X_0) \twoheadrightarrow T \operatorname{Def}(V).$$

In particular up to a finite covering the versal deformation of (V, p) is captured by the versal deformation of the global surface X .

- A (very) coarse stratification of limiting mixed Hodge structures is given by the associated graded together with degeneration arrows (cf. [R]); for weight 2 the degeneration diagram is

(*)



- A natural question is whether the strata of $\overline{M}_l^{\text{Gor}}$ surject onto the strata given by $(*)$? In [CFPR] it is proved that for l -surfaces this is indeed the case; in other words “all possible Hodge-theoretic degeneration diagrams are realized algebro-geometrically.”
- A related natural question is whether this correspondence is faithful; in other words “does the type of Hodge-theoretic degeneration capture the algebro-geometric type of the degeneration?” So far this seems to not be known.

A sketch of an algebraic proof of local Torelli goes as follows:

- using the above cohomological expression for the differential of Φ we want to show the surjectivity of

$$H^0(K_X) \otimes H^1(\Theta_X) \rightarrow H^1(\Omega_X^1)_{\text{prim}};$$

- using the standard descriptions of these groups in terms of the equation defining X , for $\mathbb{P} = \mathbb{P}(1, 1, 2, 5)$ we need the surjectivity of

$$\frac{H^0(\mathcal{O}_{\mathbb{P}}(1))}{\text{Im } dF} \otimes \frac{H^0(\mathcal{O}_{\mathbb{P}}(10))}{\text{Im } dF} \rightarrow \frac{H^0(\mathcal{O}_{\mathbb{P}}(11))}{\text{Im } dF};$$

for ordinary projective space the map on numerators above is surjective. In our case the image has codimension 1 and is spanned by zy^3 .

- From the equation of X we see that dF surjects onto the right-hand term.^{||}

^{||} This argument shows that in contrast to the case for ordinary projective space we cannot expect generic local Torelli for all smooth surfaces in $\mathbb{P}(1, 1, 2, 5)$.

Alternate proof of generic Torelli.

Let $\overline{U}$ be a neighborhood of a general point, as described above, in the blown up boundary component of I -surfaces having a simple elliptic singularity of degree 2. Then $U = \overline{U} \cap M_I$ is an open set in M_I with ∂U a smooth hypersurface, and the differential of the period mapping in U extends smoothly to the differential of the limiting mixed Hodge structure along ∂U ; here the normal component of that differential is the monodromy logarithm N . By Torelli for polarized K3's and the interpretation of the extension data the limiting mixed Hodge determines $X_{\min}$ and, up to a finite number of possibilities, the tangent line to D . The details require the computation of the differential of the period mapping ([GG]).

Non-Gorenstein example.

- Let $\mathcal{X} \rightarrow \Delta$ be a non-Gorenstein KSBA degeneration where X_0 has a normal singular point p ;
- The singularity is a quotient singularity of type $\frac{1}{dn^2}(1, dna - 1)$, $(a, n) = 1$; p is a rational singularity whose resolution is a tree of $\mathbb{P}^1$'s.
- The monodromy is of finite order; even though X_0 is singular the Hodge structures on $H^2(X_t)$, $t \neq 0$, fill in at $t = 0$ to a pure Hodge structure.
- For $\overline{U} \subset \overline{M}_I$ a neighborhood of the point corresponding to X_0 the period mapping on U then extends to

$$\Phi : \overline{U} \rightarrow P \subset \Gamma \backslash D.$$

- In contrast to the simple elliptic and cusp singularities the locus where the surfaces are normal and have a quotient singularity may form a divisor in $\overline{M}$.
- A natural question is whether this divisor can be recognized Hodge theoretically as in the case for ordinary nodes?*
- For I -surfaces there are two divisorial boundary components in $\overline{M}$ ([FPR]); these correspond to a $\frac{1}{18}(1, 5)$ and to a $\frac{1}{4}(1, 1)$ (Wahl) singularity.

**Of course this question makes sense for any quotient singularity no matter what the expected codimension in moduli of such surfaces is. The divisorial case is already quite interesting. When p is a node on X_0 the expected codimension is 1 and this is the case if p is not a base point of $|K_{X_0}|$. In general the expected codimension in moduli is described by the (suitably interpreted) number of conditions imposed by the canonical series.

- For (X_0, p) an I -surface with a Wahl singularity and resolution $(\tilde{X}, E) \rightarrow (X_0, p)$, $\tilde{X}$ is an elliptic surface with a bisection and $\tilde{E}^2 = -4$.
- By computing the suitably defined differential of the period mapping at the point of $\overline{M}_I$ corresponding to (X_0, p) it is shown in [GG] that the condition that surfaces X close to X_0 have a Wahl singularity is a divisor that is defined Hodge theoretically.

Hodge theory and the fundamental group

- For X a smooth quasi-projective variety the unipotent completion $\widehat{\pi_1(X, x)}$ has a mixed Hodge structure ([M], [H1]).

- For X projective the associated graded of the dual

$$\mathrm{Gr} \left(\widehat{\pi_1(X, x)}^* \right) \cong \mathcal{L}H^1(X)/\mathcal{I}_2$$

where “ $\mathcal{L}$ ” denotes the free Lie algebra and $\mathcal{I}_2$ is the ideal generated by

$$\ker \{ \wedge^2 H^1(X) \rightarrow H^2(X) \}.$$

- This result extends a to general quasi-projective X ;^{††} in general $\widehat{\pi_1(X, x)}$ gives a unipotent variation of mixed Hodge structure ([HZ]).
- Given $\rho : \pi_1(X) \rightarrow G \subset \mathrm{Aut}(V)$ we denote by $\mathcal{K}_\rho$ the Kuranishi space $\mathrm{Def}(\rho)$ with Zariski tangent space a subspace in $H^1(X, \mathrm{End} \mathbb{V})$ where $\mathbb{V} \rightarrow X$ is the local system (flat vector bundle) associated to ρ .

^{††}However the weights are no longer given by degree.

- If ρ comes from a variation of mixed Hodge structure, then the completed local ring $\widehat{\mathcal{O}}_\rho$ has a mixed Hodge structure ([ES]).
- If X is projective and ρ comes from a variation of Hodge structure, then $\mathcal{K}_\rho$ is *formal* ([GM]); i.e.,

$$\mathcal{K}_\rho \cong \left\{ \begin{array}{l} \text{subvariety of } H^1(X, \text{End}(\mathbb{V})) \text{ defined by the quadratic} \\ \text{equations } S^2 H^1(X, \text{End}(\mathbb{V})) \rightarrow H^2(X, \text{End}(\mathbb{V})) \end{array} \right\}$$

(this means that $\mathcal{K}_\rho$ is the intersection of the quadrics given by the kernel of this mapping; only the first obstructions to deforming ρ count).

- Given $\mathbb{V} \rightarrow X$ there is a pro-unipotent completion $\mathcal{G}$ of $\pi_1(X, x)$ relative to ρ ; if ρ underlies a variation of mixed Hodge structure then the completed algebra $\widehat{\mathcal{O}(\mathcal{G})}$ of regular functions on $\mathcal{G}$ has a pro-mixed Hodge structure ([H2]).

Application to the Shafarevich conjecture

- The conjecture is that for X smooth and projective its universal cover $\tilde{X}$ is holomorphically convex.^{‡‡}
- In concrete terms, if we contract all positive dimensional, compact connected subvarieties of $\tilde{X}$ to points, then the resulting set is a complex analytic Stein variety $\text{Sh}(\tilde{X})$ and the resulting mapping $\text{Sh}_{\tilde{X}} : \tilde{X} \rightarrow \text{Sh}(\tilde{X})$ is proper and holomorphic.
- We note that the fibres of this map are the inverse images in $\tilde{X}$ of positive dimensional connected subvarieties $Y \subset X$ such that the image of $\pi_1(Y) \rightarrow \pi_1(X)$ is finite; using the mixed Hodge structure on $\widehat{\pi_1(X, x)}$ this is equivalent to $H_1(Y) \rightarrow H_1(X)$ being trivial.

^{‡‡}Holomorphic convexity means that for $K \subset \tilde{X}$ compact

$$\hat{K} := \{x \in \tilde{X} : |f(x)| \leq |f|_K\}$$

is compact. Then

Holomorphic convexity $\iff \tilde{X} \rightarrow \text{Sh}(\tilde{X})$ is proper and the image is Stein.

- The conjecture requires both an existence theorem — namely that the algebra $\mathcal{O}(\tilde{X})$ of global holomorphic functions on $\tilde{X}$ contains sufficiently many functions — and that the image of $\tilde{X} \rightarrow \mathbb{C}^N$ be Stein.

Theorem ([EKPR])

The Shafarevich conjecture is true if X is projective and if $\pi_1(X)$ has a faithful linear representation.

- To prove such a result it is sufficient to have a pluri-sub-harmonic exhaustion function

$$\varphi : \tilde{X} \rightarrow \mathbb{R} \cup \{-\infty\}.$$

This means that the sets $\varphi^{-1}[-\infty, r)$ are relatively compact and that

$$\left(\frac{i}{2}\right) \partial\bar{\partial}\varphi \geq 0$$

with strict inequality outside a compact set.

- In the special case when ρ underlies a variation of Hodge structure such that Φ_* is injective at one point, we have the period mapping

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\Phi}} & D \subset \check{D} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Phi} & \Gamma \backslash D \end{array}$$

which outside a compact set immerses $\tilde{X}$ as a complex manifold. Here D is the period domain of all filtrations $F^p \subset V_{\mathbb{C}}$ giving polarized Hodge structures, i.e., satisfying Hodge-Riemann I and II, and $\check{D}$ is the compact dual of all F^p satisfying only Hodge-Riemann I (think of the upper half plane $\mathcal{H}$ or equivalently the unit disc Δ inside $\mathbb{P}^1$).

Then

$$\begin{aligned} D &= G_{\mathbb{R}}/H && (\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)) \\ \check{D} &= M/H && (U(2)/\mathrm{SO}(2)) \end{aligned}$$

Set

$$\varphi = \log \frac{\Omega_{\check{D}}}{\Omega_D}$$

where $\Omega_{\check{D}}, \Omega_D$ are the $M, G_{\mathbb{R}}$ -invariant volume forms; e.g., taking $\Delta \subset \mathbb{P}^1$

$$\frac{\Omega_{\check{D}}}{\Omega_D} = \frac{1 + |z|^2}{1 - |z|^2} = \frac{\text{Fubini-Study}}{\text{Poincaré}}.$$

Then φ gives a psh exhaustion function and using L^2 $\bar{\partial}$ -methods leads to the desired result for the variation of Hodge structure case.

- Results in [ES] are then used to deform any linear representation to a variation of mixed Hodge structure; an extension of the above construction to the mixed case leads to the result.
- Roughly speaking the extended construction uses the above one for the map to a VHS given by the associated graded to the mixed Hodge structures and the Albanese map along the fibres of this mapping.
- There is an extensive and ongoing literature centered around the Shafarevich conjecture and the related topics of general type, hyperbolicity etc.; cf. [CDY] and the references cited therein.

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