ADELES AND ALGEBRAIC GROUPS

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FOREWORD

The present notes are based on lectures, given at the Institute for Advanced Study in 1959-1960, which, in a sense, were nothing but a commentary on various aspects of Siegel's work--chiefly his classical papers on quadratic forms, but also the later papers where the volumes of various fundamental domains are computed.

The very fruitful idea of applying the adele method to such problems comes from Tamagawa, whose work on this subject is not yet published; I was able to make use of a manuscript of his, where that idea was applied to the restatement and proof of Siegel's theorem on quadratic forms.

If the reader is able to derive some profit from these notes, he will owe it, to a large extent, to M. Demazure and T. Ono, who have greatly improved upon the oral presentation of this material as given in my lectures. At many points they have acted as collaborators rather than as note-takers. If the final product is not as pleasing to the eye as one could wish, this is not their fault; it indicates that much work remains to be done before this very promising topic reaches some degree of completion.

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A. Weil

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CHAPTER I

PRELIMINARIES ON ADELE-GEOMETRY

1.1. Adeles. We always denote by k a field of algebraic numbers or a field of algebraic functions of one variable over a finite constant field. We denote by k_v the completion of k with respect to a valuation v of k; if v is discrete, we use often the notation p and denote by $\underline{o}_p$ the ring of p-adic integers in k_p. We denote by S any finite set of valuations which contains all the non discrete valuations (infinite places).

By an adele, we mean an element $a = (a_v)$ of the product $\prod_v k_v$ such that $a \in A_S = \prod_{v \in S} k_v \times \prod_{p \notin S} \underline{o}_p$ for some S. The adeles of k form a ring A_k, addition and multiplication being defined componentwise. Each A_S has its natural product topology and $A_k = \bigcup_S A_S$ is topologized as the inductive limit with respect to S. There is an obvious embedding of k in A_k, by means of which we identify k with a subring of A_k; k is discrete in A_k and A_k/k is compact.

1.2. Adele-spaces attached to algebraic varieties. Let V be an algebraic variety defined over k and let K be a field containing k. We denote by V_K the set of points of V rational over K. We know that V admits a finite covering by Zariski-open sets each of which is isomorphic to an affine variety V_i defined over k; $V = \bigcup_i f_i(V_i)$, f_i being defined over k. We have $V_K = \bigcup_i f_i(V_{i,K})$. In particular, $V_{k_v} = \bigcup_i f_i(V_{i,k_v})$ is locally compact with respect to the v-topology. We put

$$[V, f_i, V_i]_{\underline{o}_p} = \bigcup_i f_i(V_{i,\underline{o}_p}) ,$$

where $V_{i,\underline{o}_p}$ means the (compact) subset of V_{i,k_p} formed by points with coordinates in $\underline{o}_p$. Hence $[V, f_i, V_i]_{\underline{o}_p}$ is a compact subset of V_{k_p}. Now we put

$$V_S = \prod_{v \in S} V_{k_v} \times \prod_{p \notin S} [V, f_i, V_i]_{\underline{o}_p}$$

and $V_{A_k} = \bigcup_S V_S$. The union V_{A_k} with its inductive limit topology with respect to S will be called the <u>adele-space attached to</u> V <u>over</u> k. If there is no confusion, we write simply V_A instead of V_{A_k}.

<u>Remark.</u> If V is complete, one can prove that V_{k_v} and V_A are compact and that $[V, f_i, V_i]_{\underline{o}_p} = V_{k_p}$ for almost all p, which gives $V_A = \prod_v V_{k_v}$.

<u>Theorem</u> 1.2.1. <u>Let</u> $V = \bigcup_i f_i(V_i)$, $W = \bigcup_j g_j(W_j)$ <u>be varieties defined over</u> k <u>and let</u> $F : V \to W$ <u>be a morphism defined over</u> k. <u>Then there exists an</u> S <u>such that</u> F <u>maps</u> $[V, f_i, V_i]_{\underline{o}_p}$ <u>into</u> $[W, g_j, W_j]_{\underline{o}_p}$ <u>for all</u> $p \notin S$.

To prove this, it is sufficient to consider the case of only one V_i, i.e. the case where V is affine. For each j, we set $F_j = g_j^{-1} \circ F : V \to W_j$; if S^{n_j} is the ambient space for W_j, F_j is represented by rational functions $R_{j\lambda}$ on V $(1 \leq \lambda \leq n_j)$. Let $\mathfrak{a}_j$ be the ideal in $k[X]$ consisting of all A such that $A(x)R_{j\lambda}(x) = Q_\lambda(x)$, $Q_\lambda \in k[X]$, for all λ, x being a generic point of V over k. It is then clear that F_j is defined at $x_1 \in V$ if and only if x_1 is not a zero of $\mathfrak{a}_j$. Since F is everywhere defined, at least one F_j is defined at any x_1, which means that there is no common zero of $\Sigma_j \mathfrak{a}_j$, i.e. $\Sigma_j \mathfrak{a}_j = (1)$. We write $1 = \Sigma_j A_j$, $A_j \in \mathfrak{a}_j$ and take $Q_{j\lambda} \in k[X]$ such that $A_j(x)R_{j\lambda}(x) = Q_{j\lambda}(x)$. Let S contain all p's at which some coefficients of A_j, $Q_{j\lambda}$ are not p-integral. For any $x_1 \in V_{\underline{o}_p}$, $p \notin S$, $A_j(x_1)$ is a p-unit for some $j = j_1$ and $R_{j_1\lambda}(x_1) = Q_{j_1\lambda}(x_1)/A_{j_1}(x_1)$ is in $\underline{o}_p$ for all λ; this proves the theorem.

Corollary. If F is an isomorphism: $V \cong W$, then $F([V, f_i, V_i]_{\underline{o}_p}) = [W, g_j, W_j]_{\underline{o}_p}$ for almost all p.

The definition of $[V, f_i, V_i]_{\underline{o}_p}$ is thus "almost intrinsic" and applying this Corollary to the identity map of V to itself, we see that V_{A_k} is defined independently of the choice of affine coverings. Theorem 1.2.1 also shows that F determines a continuous $F_A : V_A \to W_A$. If V is a subvariety of W, and if this is applied to the injection map, one sees that V_A can be embedded as a closed subset in W_A. V_A has the "functorial" property: if $V \xrightarrow{F} W \xrightarrow{G} U$, then $(G \circ F)_A = G_A \circ F_A$. For the product, we have $(V \times W)_A = V_A \times W_A$.

Now, we shall find a convenient criterion for the map $F_A : V_A \to W_A$ to be surjective.

Theorem 1.2.2. Let $F : V \to W$ be a morphism defined over k. If for each $P \in W$ there exists a map $\phi_P : W \to V$ defined at P, rational over k, such that $F \circ \phi_P$ = identity ("local cross-section" in the sense of algebraic geometry), then $F_A : V_A \to W_A$ is surjective.

For every P, choose ϕ_P as above; call $D(\phi_P)$ the k-open set where ϕ_P is defined; this has a covering by k-open subsets, isomorphic to affine varieties; let Ω_P be one such subset containing P. Then the Ω_P, for all P, are a k-open covering of W, over each one of which there is a global cross-section ϕ_P. By the "compactoid" property of the k-topology, we can now write $W = \bigcup_j g_j(W_j)$, with affine W_j, such that there is a global cross-section ϕ_j over each $g_j(W_j)$. Then $G_j = \phi_j \circ g_j$ is a morphism $W_j \rightarrow V$ with $F \circ G_j = g_j$. By Theorem 1.2.1, there is an S_o such that

$$G_j(W_{j,\underline{o}_p}) \subset [V, f_i, V_i]_{\underline{o}_p} ,$$

for all j, if $p \notin S_o$. Take an element $P = (P_v) \in W_A$ with $P_v \in g_{j(v)}(W_{j(v)})$. By definition of W_A, there is an $S \supset S_o$ such that $P_p \in g_{j(p)}(W_{j(p),\underline{o}_p})$ for $p \notin S$. Put $Q_v = \phi_{j(v)}(P_v)$. Then, if $p \notin S$, $Q_p \in G_{j(p)}(W_{j(p),\underline{o}_p}) \subset [V, f_i, V_i]_{\underline{o}_p}$. Thus $Q = (Q_v)$ is well defined, and we get $F(Q_v) = P_v$, i.e. $F_A(Q) = P$.

Remark. Theorem 1.2.2 is mostly applied to the group theoretic situation. Let $H = G/g$ be a homogeneous space with G connected, everything being defined over k. Then the condition of Theorem 1.2.2 is satisfied, and consequently we have $H_A = G_A/g_A$, if there exists one "generic section" (a map $\phi : H \rightarrow G$, defined over k, such that $p \circ \phi = 1_H$, p being the canonical map $G \rightarrow H$) and if G_k is Zariski-dense in G. The latter condition will be fulfilled by all groups in these lectures. As to the first condition, take for example G orthogonal, g a subgroup leaving one vector invariant. Then H is a sphere, and Witt's theorem guarantees the existence of a generic section.

1.3. Restriction of the basic field. Let K/k be a separable algebraic extension of degree d and let A_k, A_K be their adele rings. Every valuation w of K induces a valuation v on k (we write this as w/v) and there are at most d w's such that w/v; k_v will be identified with the closure of k in K_w; for discrete valuations P/p, we have $\underline{o}_p = k_p \cap \underline{O}_P$. The mapping

$a = (a_v) \to b = (b_w)$, with $b_w = a_v$ for w/v, is an injection of A_k into A_K; we identify A_k with its image by this mapping, which is a closed subset of A_K.

Suppose K/k normal with Galois group Γ; Γ acts on K continuously under the v-topology. Since K is everywhere dense in $\prod_{i=1}^{m} K_{w_i}$, w_i/v, by the approximation theorem, the operation of Γ can be extended to this product and then to A_K. Emphasizing the order with respect to K/k, we may write

$$A_K = \bigcup_S \left(\prod_{v \in S} \left(\prod_{w/v} K_w \right) \times \prod_{p \notin S} \left(\prod_{P/p} \underline{O}_P \right) \right),$$

where S runs over the finite sets of v's in k. It is easily seen that A_k is the set of invariant elements of A_K under Γ.

If a variety V is defined over k, it is so over K, and V_{A_k} is canonically embedded in V_{A_K}. If K is normal over k, its Galois group Γ operates in an obvious manner on V_{A_K}, and V_{A_k} is the set of invariant elements of A_K by Γ.

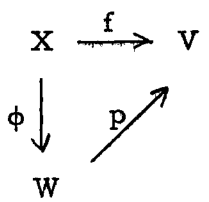
It will now be our purpose to find a variety W over k, for a given V over K, such that $V_{A_K} \cong W_{A_k}$ canonically. To do this we need an algebraic-geometric construction.

Let V, W be varieties defined over K, k respectively (K/k not necessarily normal). Let $p : W \to V$ be a map defined over K. Let $\Sigma = \{\sigma_1, \ldots, \sigma_d\}$ be the set of all distinct isomorphisms of K into $\bar{k}$. We can then define $p^\sigma : W \to V^\sigma$, and also

$$(p^{\sigma_1}, \ldots, p^{\sigma_d}) : W \to V^{\sigma_1} \times \ldots \times V^{\sigma_d},$$

this being the mapping $w \mapsto (p^{\sigma}(w))_{\sigma \in \Sigma}$. If the latter map gives an isomorphism, we call W (actually the pair $\{W, p\}$) *the variety obtained from* V *by the restriction of the field of definition from* K *to* k and write $\{W, p\} = R_{K/k}(V)$, or, by abuse of language, $W = R_{K/k}(V)$.

The *uniqueness* is a special case of the following *universal mapping property:*



let X be a variety defined over k and let $f : X \to V$ be defined over K. Then there is a unique $\phi : X \to W$, defined over k, such that $f = p \circ \phi$. In fact, put

$$\phi = (p^{\sigma_1}, \ldots, p^{\sigma_d})^{-1} \circ (f^{\sigma_1}, \ldots, f^{\sigma_d});$$

then ϕ is obviously defined over k and unique. As for the *existence* we first note the following:

Proposition 1.3.1. *Let* V, $\{W, p\}$ *be as above and let* V' *be defined over* K. *If* V' *is either a Zariski-open subset of* V *or a subvariety of* V, *then there exists* $\{W', p'\}$ *for* V'. *If* $\{W_i, p_i\}$, $i = 1, 2$, *exist for* V_i, *then* $\{W_1 \times W_2, p_1 \times p_2\} = R_{K/k}(V_1 \times V_2)$.

The proof will be left to the reader.

For $V = \underline{P}^n$ (projective space), an explicit construction is given in "The field of definition of a variety" (Am. J., 78 (1956), pp. 509-524). For an affine straight line $V = S^1$, let $K = \Sigma_{i=1}^{d} k\alpha_i$. Take $W = S^d$ and put

$$p^{\sigma}(u_1, \ldots, u_d) = \sum_i \alpha_i^{\sigma} u_i = v_{\sigma} .$$

Since K/k is separable, one has $\det \|\alpha_i^{\sigma}\| \neq 0$, i.e. $W \cong \prod V^{\sigma}$. By Proposition 1, the *existence* is settled for V affinely or projectively imbeddable; this covers all cases to be considered in these lectures.

If we apply the universal mapping property to the case where X is reduced to a point, we see that every point of V_K is an image of a point of W_k under the projection p; actually p induces on W_k a 1-1 correspondence of W_k with V_K.

If V has additional structures (group variety, algebra variety, etc.), then the morphism p should have the corresponding property (homomorphism, etc.); and the result W has the same structures as is easily seen. E.g. let $V = G_m$ (the 1-dimensional multiplicative group), considered as defined over K. Then W is a group of dimension $d = [K : k]$, defined over k; the multiplication is defined by that of K^* considered as a k-linear transformation on K; $W_k = K^*$ (regular representation!), We will consider later the "algebra varieties".

Let us return to general varieties. Let K/k be separably algebraic; let $\Sigma = \{\sigma_1, \ldots, \sigma_d\}$ be the set of distinct isomorphisms of K into $\bar{k}$. Let k' be any extension of k. Let us consider any automorphism ω of $\bar{k}'/k'$ as acting on Σ to the right, according to the formula $\xi^{\sigma\omega} = (\xi^{\sigma})^{\omega}$, $\sigma \in \Sigma$, $\xi \in K$. Then $\sigma' = \sigma\omega$ $(\sigma, \sigma' \in \Sigma)$ defines an equivalence relation, and this determines a partition of Σ into disjoint parts Σ_i. We choose representatives for the Σ_i, and assume that σ_1 = identity of K.

Let V, $\{W, p\} = R_{K/k}(V)$ be as above, with respect to K/k. As Kk' is also a field of definition for V, we can form $\{W_1, p_1\} = R_{Kk'/k'}(V)$. By the universal mapping property for W_1, there is a unique map q_1, defined

$$\begin{array}{ccc} W_1 & \xrightarrow{p_1} & V \\ {\scriptstyle q_1}\uparrow & \nearrow{\scriptstyle p} & \\ W & & \end{array}$$

over k', such that $p = p_1 \circ q_1$. If we identify (over $\bar{k}'$) W with $\prod_{\sigma \in \Sigma} V^\sigma$ by the map $(p^\sigma)_{\sigma \in \Sigma}$, W_1 with $\prod_{\sigma \in \Sigma_1} V^\sigma$ by the map $(p_1^{\sigma})_{\sigma \in \Sigma_1}$, then, by the unicity (over k') of q_1, we see that q_1 is identified with the projection on the partial product for Σ_1. Now, if we replace K, V by K^σ and V^σ, then W is not changed (up to a canonical isomorphism), but Σ_1 is replaced by Σ_i with $\sigma \in \Sigma_i$. We set $\{W_i, p_i\} = R_{K^{\sigma_i} k'/k'}(V^{\sigma_i})$, $p = p_i \circ q_i$. Then, by the same argument applied on q_i, we see that $(q_1, q_2, \ldots)$ gives a canonical isomorphism over k' : $W \cong \prod_i W_i$.

Theorem 1. 3. 1. Let K/k be separable, k'/k any extension. Let $\Sigma = \{\sigma\}$ be the set of isormorphisms of K into $\bar{k}$, let σ_i be a system of representatives of Σ with respect to the equivalence relation defined by the action on Σ of automorphisms of $\bar{k}'/k'$. Let V be a variety defined over K. Then,

$$R_{K/k}(V) \cong \prod_i R_{K^{\sigma_i} k'/k'}(V^{\sigma_i}) .$$

Let us apply the above to V = affine straight line S^1 with its algebra structure over K; then $W = S^d$ with the structure of algebra over k; $V_K = K$, $W_k = K$, $W_{k'} = K \otimes_k k'$. Since $(R_{K^{\sigma_i} \cdot k'/k'}(V^{\sigma_i}))_{k'} = (V^{\sigma_i})_{K^{\sigma_i} k'} = K^{\sigma_i} \cdot k'$, by Theorem 1. 3. 1, we have

$$K \otimes_k k' \cong \sum_i K^{\sigma_i} \cdot k' ,$$

which is well known. Let now K_i be the simple components of $K \otimes_k k'$. Then, $\sigma_i : K \otimes_k k' \to K^{\sigma_i} k'$ induces an isomorphism $\tau_i : K_i \cong K^{\sigma_i} k'$, i.e.

$\tau_i(\xi\cdot 1_i\cdot\lambda) = \xi^{\sigma_i}\cdot\lambda$, $\xi\in K$, $\lambda\in k'$, 1_i being the identity of K_i. Let again V be any variety over K; τ_i^{-1} transforms V^{σ_i} to a variety defined over K_i. If we identify K, k' with their images in K_i by $\xi\to\xi\cdot 1_i$, $\lambda\to 1_i\cdot\lambda$, K_i can be considered as an extension of K, the image of V^{σ_i} by τ_i^{-1} is identified with V, and the image of $R_{K^{\sigma_i}k'/k'}(V^{\sigma_i})$ is identified with $R_{K_i/k'}(V) : W \cong \prod_i R_{K_i/k'}(V)$. In particular, this means that

$$W_{k'} = \prod_i [R_{K_i/k'}(V)]_{k'} = \prod_i V_{K_i} .$$

<u>Theorem</u> 1.3.2. <u>Let</u> K/k, k'/k <u>be as in Theorem</u> 1.3.1, <u>and let</u> $K\otimes_k k' = \oplus_i K_i$ (<u>direct sum</u>). <u>Let</u> V <u>be a variety defined over</u> K, <u>let</u> $W = R_{K/k}(V)$. <u>Then</u> $W_{k'}$ <u>is identified canonically with</u> $\prod_i V_{K_i}$.

Applying this to the case $k' = k_v$, $K\otimes_k k_v = \oplus_{w/v} K_w$, we get $W_{k_v} = \prod_{w/v} V_{K_w}$. Going up to adele-spaces, one gets the following

<u>Theorem</u> 1.3.3. <u>Let</u> K/k <u>be separable</u>, V <u>a variety over</u> K, $W = R_{K/k}(V)$. <u>Then</u> W_{A_k} <u>can be canonically identified with</u> V_{A_K} : $W_{A_k} = V_{A_K}$.

In what follows we give an independent proof for th. 1.3.3. and describe more explicitly the identification in it. Let K' be the Galois extension of k generated by the K^σ, let Γ be the Galois group of K'/k. By the map $(p^{\sigma_1},\dots,p^{\sigma_d})$, W is identified with $\prod V^\sigma$ over K', and W_{A_k} is identified with the part of $\prod_\sigma (V^\sigma)_{A_{K'}}$ which is invariant by Γ, where the action of Γ on this product is defined by that of Γ on $W_{A_{K'}}$ by means of that identification. Namely, let $y\in W_{A_{K'}}$, y^ω its transform by $\omega\in\Gamma$, and let $x = (x_\sigma)$, $x' = (x'_\sigma)$ be the images of y, y^ω in $\prod (V^\sigma)_{A_{K'}}$; $x_\sigma = p^\sigma(y)$, $x'_\sigma = p^\sigma(y^\omega)$.

If we transform $x_\sigma = p^\sigma(y)$ by ω, we easily see that $x_\sigma{}^\omega = p^{\sigma\omega}(y^\omega) = x'_{\sigma\omega}$, which shows that Γ acts on $\prod (V^\sigma)_{A_{K'}}$ by $(x_\sigma)^\omega = \left(x^\omega_{\sigma\omega^{-1}}\right)$. It is easily seen that the projection of $\prod (V^\sigma)_{A_{K'}}$ on the first factor $V_{A_{K'}}$ induces, on the set of invariant elements by Γ, a 1-1 mapping onto V_{A_K} : $W_{A_k} \rightleftharpoons V_{A_K}$.

CHAPTER II

TAMAGAWA MEASURES

2.1. Preliminaries.

2.1.1. We normalize the Haar measure dx_v of the complete field k_v in the following way:

$$k_v = \mathbb{R} \qquad dx_v = dx$$

$$k_v = \mathbb{C} \qquad dx_v = i\cdot dx\cdot d\bar{x}$$

$$k_p \qquad \int_{\mathfrak{o}_p} dx_p = 1$$

In general if G is a locally compact group with left Haar measure dx and if $\rho : G \to G$ is an automorphism of G, then $d(\rho(x))$ is also a left Haar measure. We define the module of ρ, denoted $|\rho|$, by $d(\rho(x)) = |\rho|dx$.

In the above case, for $a \in k_v^*$, $x \to ax$ is an automorphism of k_v, and the module of a, denoted $|a|_v$, is defined by $d(ax)_v = |a|_v dx_v$. We put $|0|_v = 0$. If $k_v = \mathbb{R}$ (resp. $\mathbb{C}$) then $|a|_v$ is the usual absolute value of a (resp. its square). In the p-adic case, $|a|_p = (Np)^{-v_p(a)}$.

2.1.2. The canonical measure of A_k. The idele-module.

The canonical Haar measure ω_A on A_k is defined as the measure which induces in each $\prod_{v\in S} k_v \times \prod_{p\notin S} \mathfrak{o}_p$ the (convergent) product measure $\prod_v dx_v$.

If I_k (ideles of k) denotes the group of invertible elements of A_k, then, for $a \in I_k$, $x \to ax$ is an automorphism of A_k, and the idele-module of a is defined: $d(ax) = |a|dx$. An alternative definition is $|a| = \prod_v |a_v|_v$. In particular if $a \in k^*$ is a principal idele, then $x \to ax$ induces an automorphism of the compact group A_k/k; this must preserve the Haar measure; hence, if a is a principal idele, $|a| = \prod_v |a_v|_v = 1$ (Artin-Whaples product-formula).

Theorem 2.1.1. *There is on A_k a character χ such that, if we put $\chi(xy) = \chi_x(y)$, the mapping $x \to \chi_x$ is an isomorphism of A_k onto its dual group, and k is orthogonal to itself in this isomorphism.*

Here, by a character, we understand a continuous homomorphism into the group $\{z \in \mathbb{C} \mid |z| = 1\}$. The last assertion means that $y \in k$ if and only if $\chi(xy) = 1$ for all $x \in k$. In the number-theoretic case, we define a character χ_o on $A_{\mathbb{Q}}$ by putting, for $x = (x_v) \in A_{\mathbb{Q}}$:

$$\chi_o(x) = \exp\left(2\pi i\left(\sum_p < x_p > - x_\infty\right)\right)$$

where $< x_p >$ denotes the rational number of the form $p^{-n}a$, $n \geq 0$, $0 \leq a < p^n$, such that $x_p - < x_p > \in \mathfrak{o}_p$; for $k = \mathbb{Q}$, this has the property asserted in our theorem. For an arbitrary k, we can identify A_k with $k \otimes_{\mathbb{Q}} A_{\mathbb{Q}}$; denote by tr the trace in k over $\mathbb{Q}$, which we extend in the obvious manner to a linear mapping of A_k into $A_{\mathbb{Q}}$; then we take $\chi(x) = \chi_o(\mathrm{tr}\, x)$. In the function-theoretic case, we proceed similarly, replacing $\mathbb{Q}$ by $k_o = F_q(t)$, where F_q is the field of constants in k, and t is so chosen in k that k is separably algebraic over k_o; thus it is enough to consider k_o. For a valuation v of degree n of k_o, and for $x_v \in k_{o/v}$, call $\rho_v(x_v)$ the trace of $\mathrm{Res}(x_v dt)$, taken in F_{q^n} over the prime field. Then, for $x = (x_v) \in A_{k_o}$, we take $\chi_o(x) = \Psi(\Sigma_v \rho_v(x_v))$, where Ψ is a non-trivial character of the additive group of the prime field.

2.1.3. The constant μ_k.

Let us compute $\mu_k = \int_{A_k/k} \omega_A$, where ω_A is the measure introduced above. Denote again by S a finite non-empty set of places, containing all the places at infinity; put $A_S = \prod_{v\in S} k_v \times \prod_{p\notin S} \underline{o}_p$, and $\underline{o}(S) = A_S \cap k$; also, call $\underline{o}'(S)$ the projection of $\underline{o}(S)$ on the first factor $k_S = \prod_{v\in S} k_v$ of A_S. By the approximation theorem, every coset of A_S in A_k contains an element of k, so that A_k/k is isomorphic to $A_S/\underline{o}(S)$. Let M be a measurable set of representatives of $k_S/\underline{o}'(S)$ in k_S; then $M' = M \times \prod_{p\notin S} \underline{o}_p$ is a measurable set of representatives of $A_S/\underline{o}(S)$ in A_S, so that we have $\mu_k = \int_{M'} \omega_A$. In view of the definition of ω_A, this gives

$$\mu_k = \int_M \prod_{v\in S} dx_v = \int_{k_S/\underline{o}'(S)} \prod_{v\in S} dx_v .$$

We now distinguish two cases:

a) k is an algebraic number-field.

Then we take for S the set of all infinite places (r_1 real, r_2 imaginary); $k_S = k \otimes_{\mathbb{Q}} \mathbb{R} = \prod_\rho k_\rho \times \prod_\iota k_\iota$ (with $k_\rho = \mathbb{R}$, $1 \leq \rho \leq r_1$, and $k_\iota = \mathbb{C}$, $r_1 + 1 \leq \iota \leq r_1 + r_2$); $\underline{o}(S)$ is the ring of integers in k; $\underline{o}'(S)$ is a lattice in the real vector-space k_S. More precisely, if (a_α), for $1 \leq \alpha \leq n = [k:\mathbb{Q}]$, is a basis for $\underline{o}(S)$, and if the σ_λ, for $1 \leq \lambda \leq r_1 + r_2$, are the embeddings $\sigma_\lambda : k \to k_\lambda$ of k into $\mathbb{C}$, the vectors $A_\alpha = (\sigma_\lambda(a_\alpha))$ are a basis for $\underline{o}'(S)$; as the set of representatives M, we can take the set of all points $\Sigma_\alpha t_\alpha A_\alpha$ in k_S with $0 \leq t_\alpha < 1$ for $1 \leq \alpha \leq n$; then we have

$$\mu_k = \int_M \prod_\rho dx_\rho \times \prod_\iota i dx_\iota d\bar{x}_\iota = i^{r_2} D \int_M dt_1 \ldots dt_n = i^{r_2} D ,$$

where D is the determinant of the matrix

$$\| \sigma_p(a_\alpha) \; \sigma_\ell(a_\alpha) \; \overline{\sigma_\ell(a_\alpha)} \| .$$

By definition, D^2 is the discriminant Δ_k of the field k; this gives:

$$\mu_k = |\Delta_k|^{\frac{1}{2}} .$$

b) k <u>is a function field over the finite field</u> F_q.

Then we take S reduced to one place p; if d is the degree of p, we have $N(p) = q^d$; and, if π is a prime element in $\underline{o}_p$ (i.e. such that $(\pi) = \pi \underline{o}_p$ is the maximal ideal in $\underline{o}_p$), we have $\int_{(\pi)} dx_p = q^{-d}$. We have $k_S = k_p$, and $\underline{o}'(S) \cap (\pi) = \{0\}$, so that we can take for M a set consisting of cosets of (π) in k_p, and μ_k is $q^{-d} \times$ the number of such cosets. The Riemann-Roch theorem shows at once that the latter number is q^{g+d-1}, where g is the genus of k. Therefore $\mu_k = q^{g-1}$.

2.2. <u>The case of an algebraic variety: the local measure.</u>

2.2.1. Let V be a variety defined over k, and ω an algebraic differential form, defined over k, of degree $n = \dim V$.

Let x^o be a simple point of V and $x_1, \ldots, x_n$ a system of local coordinates at x^o (not necessarily 0 at x^o). Suppose furthermore that ω is holomorphic and not zero at x^o.

In a neighborhood of x^o, ω can be expressed as

$$\omega = f(x) dx_1 \ldots dx_n$$

where $f(x)$ is a rational function defined at x^o, which can be written as a formal power-series:

(1) $f(x) = \Sigma\, a_{(i)}(x_1 - x_1^{\,o})^{i_1} \ldots (x_n - x_n^{\,o})^{i_n}$

Now, we take a completion k_v of k and assume the $x_i^{\,o}$ to be in k_v; then f is a power series with coefficients in k_v. By the implicit functions theorem, (1) converges in some neighborhood of the origin in $k_v^{\,n}$. If the $x_i^{\,o}$ are in A_k, the $a_{(i)}$'s are integers for almost all p, and, by a suitable linear change of coordinates, we can make them integers for all p; then for each p, (1) converges for $x_i \equiv x_i^{\,o}(p)$.

In any case, there is, for each v, a neighborhood U of x^o in V_{k_v} such that

(1) $\psi : x \rightarrow (x_1 - x_1^{\,o}, \ldots, x_n - x_n^{\,o})$ is a homeomorphism of U onto a neighborhood of the origin in $k_v^{\,n}$,

(2) the power-series expansion (1) converges in $\psi(U)$.

In $\psi(U)$ we have the positive measure $|f(x)|_v (dx_1)_v \ldots (dx_n)_v$. We can pull it back to U by ψ^{-1} and we get a positive measure ω_v on U.

A priori, this measure depends on the choice of the system of local coordinates. We shall prove now that it is actually independent of this choice. In order to prove this, it is enough to change one coordinate at a time. Consider e.g. a change of the form $(x_1, x_2 \ldots x_n) \rightarrow (y_1, x_2, \ldots x_n)$; by Fubini's theorem we are reduced to the case $n = 1$, i.e. to proving:

$$\frac{(dy)_v}{(dx)_v} = \left|\frac{dy}{dx}\right|_v$$

This formula is well known in the classical case. In the p-adic case the proof is as follows: we can by linearity suppose $y = x + a_2 x^2 + \ldots a_n x^n + \ldots$ $a_i \in \underline{o}_p$. Then $\left|\frac{dy}{dx}\right|_p = 1$, and one has to verify that $(dy)_p = (dx)_p$ in some neighborhood of 0. This follows from the fact that, for $x \equiv x' \equiv 0(p)$, or for $y \equiv y' \equiv 0(p)$, the relations $x \equiv x'(p^n)$, $y \equiv y'(p^n)$ are equivalent.

In this way, we have defined a measure ω_v on the open subset of V_{k_v} consisting of the simple points where ω is holomorphic and not zero.

2.2.2. From now on, V will be a non-singular variety subject to the following condition: <u>there exists on</u> V <u>an algebraic differential form of degree</u> n, <u>everywhere holomorphic and not zero</u>. Such a form will be called a <u>gauge-form</u>.

If ω and ω' are two gauge-forms on V, then $\omega' = \phi(x)\omega$, where ϕ is a rational function defined over k without poles or zeros, that is a morphism of the variety V into the multiplicative group in one variable, $G_m = GL(1)$.

<u>Theorem</u> 2.2.2 (Rosenlicht). <u>Let</u> V <u>be a non-singular variety. The multiplicative group of morphisms</u> $\phi : V \to G_m$ <u>is the product of the group of non-zero constants and of a finitely generated group. If</u> V <u>is an algebraic group, any such</u> ϕ <u>such that</u> $\phi(e) = 1$ <u>is a character</u> (i.e. <u>a homomorphism</u> $V \to G_m$).

For the first part, we remark that it is sufficient to prove it for an affine piece of V; it will then be true a fortiori for V. Hence we can suppose V affine. We take the normalized variety of the projective closure of V. It is a complete variety $\overline{V}$, without singularities in codimension one. The variety V is open in $\overline{V}$, the complement $\overline{V} - V$ having finitely many irreducible components of maximum dimension. The morphisms $\phi : V \to G_m$ are exactly the rational functions on $\overline{V}$ whose divisors have their support contained in $\overline{V} - V$. Up to a constant factor, such a function is uniquely determined by its divisor; and the divisors of such functions, being a subgroup of the group of all divisors with support contained in $\bar{V} - V$, make up a finitely generated group. This proves our first assertion.

If now V is an algebraic group, and x, y two independent generic points of V, then $\frac{\phi(xy)}{\phi(y)}$ as a function of x is a morphism of V into G_m. If $\phi_1, \dots \phi_r$ is a set of generators of the above group then $\frac{\phi(xy)}{\phi(y)} = \lambda \prod_i \phi_i(x)^{a_i}$, hence is independent of y. Taking $y = e$, we find $\frac{\phi(xy)}{\phi(y)} = \phi(x)$, which proves the second part.

<u>Corollary.</u> <u>Let</u> G <u>be an algebraic group, and</u> ω <u>a translation-invariant gauge-form on</u> G. <u>Any gauge-form on</u> G <u>can then be written as</u> $\lambda\chi(x)\omega$, <u>where</u> χ <u>is a character of</u> G <u>and</u> λ <u>a non-zero constant.</u>

The end of this paragraph is devoted to the reduction modulo p and will not be used in the computation of Tamagawa numbers. The reader interested only in this computation can skip it and proceed directly to 2. 3.

We begin by recalling a few facts about abstract varieties. Let V be such a variety, given, as usual, as a finite union of isomorphic images of affine varieties, $V = \bigcup_\alpha f_\alpha(V_\alpha)$; V being defined over k, take corresponding generic points $x_\alpha = f_\alpha^{-1}(M)$ of the affine varieties V_α, where M is a generic point of V over k. We may always assume that the covering of V by the $f_\alpha(V_\alpha)$ has been taken so fine that there is, on each V_α, an <u>everywhere valid system</u> of local coordinates. Then, if V_α is defined in the affine space of dimension N_α by the equations $F_\mu(X_{\alpha 1}, \dots, X_{\alpha N_\alpha}) = 0$, and if $x_{\alpha 1}, \dots, x_{\alpha n}$ are local coordinates on V_α, we can reorder the F_μ in such a way that the determinant

$$\Delta_\alpha(X_\alpha) = \det(\partial F_\mu / \partial X_{\alpha j}) \quad (1 \leq \mu \leq N_\alpha - n, n+1 \leq j \leq N_\alpha)$$

is everywhere finite and non-zero on V_α; this means that $\Delta_\alpha(x_\alpha)$ and $\Delta_\alpha(x_\alpha)^{-1}$ can be expressed as polynomials in $x_{\alpha 1}, \ldots, x_{\alpha N_\alpha}$.

Let $\mathfrak{P}_{\alpha\beta}$ be the ideal in $k[X_\alpha, X_\beta]$ which defines the graph of the birational correspondence $f_{\beta\alpha} = f_\beta^{-1} \circ f_\alpha$ between V_α and V_β; let $\mathfrak{D}_{\alpha\beta}$ be the ideal of those polynomials D in $k[X_\alpha]$ such that, for each j, $D(X_\alpha) \cdot X_{\beta j} \in k[X_\alpha] + \mathfrak{P}_{\alpha\beta}$. As the correspondence $f_{\beta\alpha}$ is biregular at every pair of points (x'_α, x'_β) of its graph, $\mathfrak{P}_{\alpha\beta}$ and $\mathfrak{D}_{\alpha\beta}$ have no common zero. By Hilbert's Nullstellensatz, we can find polynomials $D_\nu^{(\alpha,\beta)} \in \mathfrak{D}_{\alpha\beta}$ and $P_\nu^{(\alpha,\beta)} \in k[X_\beta]$ such that

$$(1) \qquad 1 - \sum_\nu D_\nu^{(\alpha,\beta)}(X_\alpha) P_\nu^{(\alpha,\beta)}(X_\beta) \in \mathfrak{P}_{\alpha\beta} ;$$

At the same time, by the definition of $\mathfrak{D}_{\alpha\beta}$, there are, for all ν and j, polynomials $F_{\nu j}^{(\alpha,\beta)}$ in $k[X_\alpha]$ such that

$$(2) \qquad D_\nu^{(\alpha,\beta)}(X_\alpha) X_{\beta j} - F_{\nu j}^{(\alpha,\beta)}(X_\alpha) \in \mathfrak{P}_{\alpha\beta} .$$

Let A be the set of all α's and for $B \subset A$ let $\mathfrak{P}_B \subset k[(X_\alpha), \alpha \in B]$ be the ideal of relations between the coordinates of the x_α's $(\alpha \in B)$ where the x_α are, as above, corresponding generic points of the V_α over k.

Then
$$\begin{cases} \mathfrak{P}_B \cap k[X_\alpha]_{\alpha \in B'} = \mathfrak{P}_{B'}, \quad B' \subset B. \\ \mathfrak{P}_{\{\beta,\alpha\}} = \mathfrak{P}_{\beta\alpha} \\ \mathfrak{P}_{\{\alpha\}} = \text{ideal in } k[X_\alpha] \text{ defining } V_\alpha . \end{cases}$$

That being so, the conjunction of (1) and (2) is equivalent to

$$(3)\quad \begin{cases} 1 - \sum_\nu D_\nu^{(\alpha,\beta)}(X_\alpha)P_\nu^{(\alpha,\beta)}(X_\beta) \in \mathfrak{p}_A & \text{for all } \alpha,\beta \\ D_\nu^{(\alpha,\beta)}(X_\alpha)X_{\beta j} - F_{\nu,j}^{(\alpha,\beta)}(X_\alpha) \in \mathfrak{p}_A & \text{for all } \alpha,\beta,\nu,j, \end{cases}$$

where $\mathfrak{p}_A$ is the absolutely prime ideal of $k[(X_\alpha), \alpha \in A]$ defining the locus of $(x_\alpha)_{\alpha \in A}$ over k.

Conversely, whenever an absolutely prime ideal $\mathfrak{p}_A$ is given in $k[(X_\alpha), \alpha \in A]$ such that there exist polynomials D, F, P satisfying the above relations, this defines an abstract variety defined over k.

Let there be given, now, a gauge-form ω on V; on V_α, ω can be written $\omega = \phi_\alpha(x_\alpha)\, dx_{\alpha 1} \dots dx_{\alpha n}$, where $\phi_\alpha(x_\alpha)$ and $\phi_\alpha(x_\alpha)^{-1}$ are everywhere defined rational functions, hence polynomials,

We recall the definition of reduction of an ideal modulo p. Let $\mathfrak{p}$ be an absolutely prime ideal in $k[Y]$, $Y = (Y_1, \dots, Y_n)$. Let us take a finite set of generators $P_\mu(Y)$ of $\mathfrak{p}$. For a finite place p where all the coefficients of the P_μ's are integral, the reduction $\overline{P}_\mu^{(p)}$ of P_μ modulo p is defined and we define $\overline{\mathfrak{p}}^{(p)}$ as the ideal generated by the $\overline{P}_\mu^{(p)}$ in $(\mathfrak{o}_p/p)[Y]$. If we take another set of generators, then for almost all p, the reduction $\overline{\mathfrak{p}}^{(p)}$ will be the same as above ($\overline{\mathfrak{p}}^{(p)}$ is "almost intrinsic"). A well known theorem of E. Noether (Math. Annalen 1923) asserts:

Theorem 2.2.2. For almost all p, $\overline{\mathfrak{p}}^{(p)}$ is absolutely prime.

Let now S be a finite set of primes such that for all p not in S

(I) all the $\overline{\mathfrak{p}}_B^{(p)}$ $(B \subset A)$ are absolutely prime

(II) all the polynomials $D_\nu^{(\alpha,\beta)}$, $P_\nu^{(\alpha,\beta)}$, $F_{\nu,j}^{(\alpha,\beta)}$, Δ_α, Δ_α^{-1}, ϕ_α, ϕ_α^{-1} have integral coefficients.

Then for all p not in S we shall define a non-singular variety $\bar{V}^{(p)}$ defined over the field $\underset{-}{o}_p/p = F_q$ (finite field with q = Np elements) called the <u>reduction of</u> V <u>modulo</u> p.

Let $\bar{V}_\alpha^{(p)}$ be the variety defined by $\bar{\wp}_\alpha^{(p)} \in F_q[X_\alpha]$. Let $\bar{f}_{\beta\alpha}^{(p)}$ be the birational correspondence between $\bar{V}_\alpha^{(p)}$ and $\bar{V}_\beta^{(p)}$ defined by $\bar{\wp}_{\{\alpha,\beta\}}^{(p)}$. By the hypothesis (II), the polynomials D, P, F can be reduced modulo p, and the equations (1), (2), (3) remain true. Hence we can glue together the varieties $\bar{V}_\alpha^{(p)}$ by the correspondences $\bar{f}_{\beta\alpha}^{(p)}$, and find an abstract variety $\bar{V}^{(p)}$ defined over F_q.

Moreover the Δ_α are p-adic units and the n first coordinates are uniformizing parameters on $\bar{V}_\alpha^{(p)}$ which is therefore non-singular.

2.2.3. We want now to interpret the rational points of $\bar{V}^{(p)}$ over F_q. We have already defined $[V; f_\alpha, V_\alpha]_{\underset{-}{o}_p} = \bigcup_\alpha f_\alpha(V_{\alpha/\underset{-}{o}_p})$; we denote it here by $V_{\underset{-}{o}_p}$ (though it is only "almost intrinsic"), and we introduce an equivalence relation on $V_{\underset{-}{o}_p}$. Let μ be a fixed integer > 0.

<u>We say that</u> $a \equiv a'(p^\mu)$, $a, a' \in V_{\underset{-}{o}_p}$ <u>if one can find an</u> $\alpha \in A$ <u>such that</u> $a = f_\alpha(a_\alpha)$, $a' = f_\alpha(a'_\alpha)$, $a_\alpha, a'_\alpha \in V_{\alpha/\underset{-}{o}_p}$, $a_\alpha \equiv a'_\alpha(p^\mu)$.

To justify this definition we have to prove

<u>Lemma</u> 2.2.1. <u>If</u> $a = f_\alpha(a_\alpha) = f_\beta(a_\beta)$, $a_\alpha \in V_{\alpha/\underset{-}{o}_p}$, $a_\beta \in V_{\beta/\underset{-}{o}_p}$, <u>and if</u> $a' = f_\alpha(a'_\alpha)$ <u>with</u> $a'_\alpha \equiv a_\alpha(p^\mu)$ <u>then one can find</u> $a'_\beta \in V_{\beta/\underset{-}{o}_p}$ <u>such that</u> $a' = f_\beta(a'_\beta)$ <u>and</u> $a'_\beta \equiv a_\beta(p^\mu)$.

As (a_α, a_β) is in the graph of $f_{\beta,\alpha}$, we have

$$1 = \Sigma\, D_\nu^{(\alpha,\beta)}(a_\alpha) P_\nu^{(\alpha,\beta)}(a_\beta) \ .$$

In view of (II), at least one of the $D_\nu^{(\alpha,\beta)}(a_\alpha)$ must be a p-adic unit; then, by (2), we have, for this value of ν:

$$a_{\beta,j} = \frac{F_{\nu,j}^{(\alpha,\beta)}(a_\alpha)}{D_\nu^{(\alpha,\beta)}(a_\alpha)} .$$

But, if we replace a_α by a'_α and define $a'_{\beta,j} = \frac{F_{\nu,j}^{(\alpha,\beta)}(a'_\alpha)}{D_\nu^{(\alpha,\beta)}(a'_\alpha)}$, then $D_\nu^{(\alpha,\beta)}(a'_\alpha)$ is still a p-adic unit, $a'_\beta \equiv a_\beta (p^\mu)$ and a'_β is in $V_{\beta/\mathfrak{o}_p}$.

We are now ready to prove;

<u>Theorem</u> 2.2.3. <u>The reduction modulo</u> p <u>gives a one-to-one correspondence between the equivalence classes modulo</u> p <u>of</u> $V_{\mathfrak{o}_p}$ <u>and the rational points of</u> $\bar{V}^{(p)}$ <u>over</u> F_q.

Two equivalent points lie on the same affine V_α by definition, hence it is sufficient to prove the theorem for an affine V. In this case, V is given by an absolutely prime ideal $\mathfrak{p}$. Obviously a zero of $\mathfrak{p}$ in $(\mathfrak{o}_p)^N$ reduces modulo p to a zero of $\bar{\mathfrak{p}}^{(p)}$. The equivalent points modulo p reduce to the same zero. It remains only to prove the following generalization of Hensel's lemma:

<u>Lemma</u> 2.2.4. <u>Let</u> k_p <u>be a p-adic field and</u> V <u>an affine variety of dimension</u> n <u>defined over</u> k_p <u>by an ideal</u> $\mathfrak{p} \in k_p[X_1, \ldots X_N]$. <u>Let</u> $\bar{a}$ <u>be a zero in</u> $F_q^N (q = N(p))$ <u>of the reduction</u> $\bar{\mathfrak{p}}^{(p)}$. <u>Suppose that there exists</u> $F_1, \ldots F_{N-n}$ <u>in</u> $\mathfrak{p} \cap \mathfrak{o}_p[X]$ <u>such that the matrix</u> $\| \partial \bar{F}_\mu / \partial \bar{a}_\nu \|$ <u>is of rank</u> N - n. <u>Then there exists a zero</u> a <u>of</u> $\mathfrak{p}$ <u>in</u> $(\mathfrak{o}_p)^N$ <u>reducing modulo</u> p <u>to</u> $\bar{a}$.

(It is not necessary, for the lemma to be valid, that $\bar{\mathfrak{p}}^{(p)}$ should be an absolutely prime ideal.) For a proof, see P. Samuel, Int. Congress Amsterdam 1954, vol. 2, p. 63.

Let us now compute

$$\int_{V_{\underset{-}{o}_p}} \omega_p$$

We decompose $V_{\underset{-}{o}_p}$ into equivalence classes:

$$\int_{V_{\underset{-}{o}_p}} \omega_p = \sum_{\bar{a} \in \bar{V}(p)} \int_{x \equiv a(p)} \omega_p$$

But $\{x \mid x \equiv a(p)\}$ is by definition in the image of one affine V_α; on V_α, ω can be written as $\phi_\alpha(x) dx_{\alpha 1} \ldots dx_{\alpha n}$. But ϕ_α is a p-adic unit; therefore, for a in $V_{\alpha/\underset{-}{o}_p}$:

$$\int_{x \equiv a(p)} \omega_p = \int_{x \equiv a(p)} dx_{\alpha 1} \ldots dx_{\alpha n}$$

The projection of V_α on the space $(x_{\alpha 1}, \ldots x_{\alpha n})$ is an isomorphism on the subset $x \equiv a(p)$; therefore

$$\int_{x \equiv a(p)} dx_{\alpha 1} \ldots dx_{\alpha n} = q^{-n} ,$$

This completes the proof of

<u>Theorem</u> 2.2.5. <u>If</u> V <u>is non-singular and</u> ω <u>is a gauge-form on</u> V, <u>then, for almost all</u> p, V <u>reduces modulo</u> p <u>to a non-singular variety which has</u> $q^n \int_{V_{\underset{-}{o}_p}} \omega_p$ <u>rational points over</u> F_q, <u>where</u> $q = N(p)$.

2.3. The global measure and the convergence factors

A set (λ_v) of strictly positive real numbers indexed by the valuations

v of k is called a set of factors for k. Let V be a non-singular variety, defined over k, and ω a gauge-form on V, rational over k. For each place v of k, we have defined the positive measure ω_v on V_{k_v}. For every p, we put $\mu_p(V) = \int_{V_{\underline{o}_p}} \omega_p$.

Definition. A set of factors (λ_v) is called a set of convergence factors for V if the product $\prod_p(\lambda_p^{-1}\mu_p)$ is absolutely convergent.

This definition is valid because

(1) $V_{\underline{o}_p}$ is almost intrinsic

(2) If ω' is another form, $\int_{V_{\underline{o}_p}} \omega_p = \int_{V_{\underline{o}_p}} \omega'_p$ for almost all p.

(This follows from Theorem 2.2.5. Alternative proof: $\omega' = f\omega$ where f is a morphism of V into G_m. For almost all p, $f(V_{\underline{o}_p}) \subset (G_m)_{\underline{o}_p} = U_p$ and $|f|_p = 1$.)

Definition. If (λ_v) is a set of convergence factors for V, the Tamagawa measure $\Omega = (\omega, (\lambda_v))$ on V_{A_k} derived from ω by means of (λ_v) is defined as the measure on V_{A_k} inducing in each product $\prod_{v\in S} V_{k_v} \times \prod_{p\notin S} V_{\underline{o}_p}$ the product measure $\mu_k^{-\dim V} \prod_v(\lambda_v^{-1}\omega_v)$ (μ_k is the constant introduced in 2.1.3).

Theorem 2.3.1. If $c \in k^*$, $(\omega, (\lambda_v)) = (c\omega, (\lambda_v))$.

In fact, $(c\omega)_v = |c|_v\omega_v$ and $\prod_v |c|_v = 1$.

Let now K/k be a finite and separable algebraic extension. Let V be a variety defined over K and $(W, p) = R_{K/k}(V)$ (1.3). If ω is a differential form on V defined over K, we define the differential form

$p^*(\omega)$ on W as follows. Let θ be such that $K = k(\theta)$. Let (σ_i) be the isomorphisms of K into $\bar{k}$ and let $\sqrt{\Delta} = \prod_{i<j}(\theta^{\sigma_i} - \theta^{\sigma_j})$. Then we define $p^*(\omega) = (\sqrt{\Delta})^{\dim V} \bigwedge_{\sigma} (p^{\sigma})^* \omega^{\sigma}$. We see easily that $p^*(\omega)$ is defined over k. Then one has

<u>Theorem</u> 2.3.2. <u>Let</u> K/k <u>be finite and separable and</u> $(W, p) = R_{K/k}(V)$. <u>Let</u> (λ_w) <u>be a set of factors for</u> K <u>and</u> $\mu_v = \prod_{w/v} \lambda_w$. <u>Then</u>

(1) (μ_v) <u>is a set of convergence factors for</u> W <u>if and only if</u> (λ_w) <u>is a set of convergence factors for</u> V.

(2) <u>If</u> (1) <u>is satisfied,</u> $(\omega, (\lambda_w))$ <u>corresponds to</u> $(p^*(\omega), (\mu_v))$ <u>in the canonical isomorphism</u> $V_{A_K} \approx W_{A_k}$ (Theorem 1.3.3).

This follows at once from Theorems 1.3.2 and 1.3.3 and from the definition of W.

This theorem gives the reason for the introduction of the factor μ_k in the definition of the Tamagawa measures.

2.4. <u>Algebraic groups and Tamagawa numbers</u>

By an algebraic group, we shall always mean in this course an algebraic group isomorphic to a subgroup of a linear group.

If G is an algebraic group defined over k, the product morphism $G \times G \to G$ induces a group law on G_{A_k} which makes G_{A_k} a topological locally compact group: the <u>adele-group</u> of G.

If ω is a gauge-form on G, invariant by left-translations on G and if (λ_v) is a set of convergence factor for G, then the Tamagawa measure $\Omega = (\omega, (\lambda_v))$ is a <u>left Haar measure</u> on G_{A_k}. This measure is independent of the choice of ω (Theorem 2.3.1) and will be called *the Tamagawa measure for* G *derived from the convergence factors* (λ_v). If (1) is a set of convergence factors, then $(\omega, (1))$ will be called *the Tamagawa measure for* G.

The group G_k is a discrete subgroup of G_{A_k} and we consider the left homogeneous space G_{A_k}/G_k. We shall be interested in the following questions:

(I) *Is* G_{A_k}/G_k *compact?*

(II) *Is* G_{A_k}/G_k *of finite measure for any left Haar measure on* G_{A_k}?

(III) *If* G *is unimodular, if* (II) *is true, and if* (1) *a set of convergence factors for* G, *then what is the number* $\tau(G) = \int_{G_{A_k}/G_k} (\omega, (1))$?

Under the assumptions in (III), $\tau(G)$ will be called the *Tamagawa number of* G.

For instance, if $G = G_a^n$ (additive group in n variables), then (I) is true (a fortiori (II)), and, by definition of μ_k, $\tau(G) = 1$.

If G is an algebraic group and ω a left-invariant algebraic gauge-form, $\omega(xs)$ is also left-invariant for all $s \in G$, hence must be of the form $\omega(xs) = \Delta(s)\omega(x)$ where Δ is a character of G, called the *module* of G. As in the topological case, $\omega(x^{-1}) = \Delta(x^{-1})\omega(x)$. If $\Delta = 1$, G is said to be *unimodular* (example: all reductive groups).

If G is defined over k, then we may take ω defined over k; therefore the module Δ also is defined over k and extends to a homomorphism $\Delta_{A_k} : G_{A_k} \to I_k = (G_m)_{A_k}$. Then $|\Delta_{A_k}|$ where $|\ |$ denotes the idele-module is a homomorphism of G_{A_k} into R_+^* which coincides obviously with the (topological) module of G_{A_k}. Hence, if G is unimodular, G_{A_k} is unimodular.

Let now $H = G/g$ be an algebraic homogeneous space of G. A gauge-form ω on H is called a relatively invariant form belonging to the character χ of G if

$$\omega(s\cdot P) = \chi(s)\cdot\omega(P) \qquad s \in G,\ P \in H .$$

We have, as in the topological case, the theorem:

Theorem 2.4.1. Denote by Δ and δ the modules of G and g. There exists a relatively invariant form on G/g belonging to the character χ of G, if and only if the character $\delta(\sigma)\Delta(\sigma^{-1})$ of g coincides with χ on g.

Corollary. If G and g are unimodular, there exists on G/g a gauge-form invariant by G.

Let dx, $d\sigma$, dP be respectively a left-invariant gauge-form on G, a left-invariant gauge-form on g, and a relatively invariant gauge-form on G/g belonging to the character χ of G; let ϕ be the canonical mapping of G onto G/g, and put $P = \phi(x) = xg$ for $x \in G$. Then $\phi^*(dP)$ is a differential form on G. Put $\alpha(x) = \phi^*(dP)$, and let $\beta(x)$ be any differential form on G such that, for every $s \in G$, $\beta(s\sigma) = d\sigma$ for $\sigma \in g$, i.e. such that $\beta(sx)$ induces on g the form $d\sigma$. It is easily seen that the form $\alpha(x)\wedge\beta(x)$ on G is a gauge-form which does not depend upon the choice of β; this will be denoted symbolically by $dP\cdot d\sigma$. We have $dP\cdot d\sigma = \lambda\chi(x)dx$, where λ is a constant. If $\lambda = 1$, we say that dx, $d\sigma$, dP match together algebraically.

Let G' be a locally compact group, g' a closed subgroup; if dx',

$d\sigma'$, dP' are respectively a Haar measure on G', a Haar measure on g', and a relatively invariant measure on G'/g' belonging to the character χ' of G', we will say that dx', $d\sigma'$, dP' match together topologically if the integral formula

$$\int_{G'/g'} dP' \int_{g'} f(x'\sigma')d\sigma' = \int_{G'} f(x')\chi'(x')dx'$$

holds for every f in $L^1(G', dx')$; in this formula, P' means the image $x'g'$ of x' in G'/g'. This formula will be abbreviated symbolically by $dP' \cdot d\sigma' = \chi'(x')dx'$.

Theorem 2.4.2. Let G and $g \subset G$ be algebraic groups defined over k. Suppose that the map $G \to G/g = H$ admits local cross-sections (in the sense of Theorem 1.2.2), and that the density condition of Theorem 1.2.2 is fulfilled. Then $G_A/g_A = H_A$.

Straightforward application of Theorem 1.2.2. We note that the density condition of Theorem 1.2.2 is automatically verified if k is a number-field (Rosenlicht). For the function fields it is easily verified in each of the particular cases treated in Chapters III and IV.

Theorem 2.4.3. Let G, g be as in Theorem 2.4.2. Let dx, $d\sigma$, dP be gauge-forms defined over k on G, g, G/g, matching together algebraically. Let (λ_v), (μ_v), (ν_v) be three sets of factors for k with $\lambda_v = \mu_v \cdot \nu_v$. Then:

(1) If two of three sets (λ_v), (μ_v), (ν_v) are sets of convergence factors (for G, g, G/g respectively), so is the third one.

(2) If (1) is satisfied, denoting $d_Ax = (dx, (\lambda_v))$, $d_A\sigma = (d\sigma, (\mu_v))$, $d_AP = (dP, (\nu_v))$, then d_Ax, $d_A\sigma$, d_AP match together topologically.

Moreover if $dP\cdot d\sigma = \chi(x)dx$, then $d_A P\cdot d_A\sigma = |\chi_A(x)| d_A x$ where $|\ |$ is the idele-module and $\chi_A : G_A \to I_k$ the adele extension of $\chi : G \to G_m$.

The proof is straightforward and is left to the reader.

Lemma 2.4.1. Let G be a locally compact unimodular group, g a unimodular subgroup of G, γ a discrete subgroup of g. Let $d_G x$, $d_g u$, $d_{G/g}x$ be Haar measures of G, g, G/g such that $d_G x = d_{G/g}x\cdot d_g u$. Let $d_{g/\gamma}u$ and $d_{G/\gamma}x$ be the measures induced by $d_g u$ and $d_G x$ in the local isomorphism $g \to g/\gamma$, $G \to G/\gamma$. Then we have $d_{G/\gamma}x = d_{G/g}x\cdot d_{g/\gamma}u$ in the sense that

$$(1) \qquad \int_{G/\gamma} f(x)d_{G/\gamma}x = \int_{G/g} d_{G/g}x \int_{g/\gamma} f(xu)d_{g/\gamma}u \quad \text{for } f \in L^1(G/\gamma).$$

Slight modification of Fubini's theorem.

Lemma 2.4.2. Let furthermore Γ be a discrete subgroup such that $G \supset \Gamma \supset \gamma$ and let $d_{G/\Gamma}x$ be the Haar measure on G/Γ locally isomorphic to $d_G x$. Let $\mu_g = \int_{g/\gamma} d_{g/\gamma}u \leq +\infty$. Then for $f \in L^+(G/g)$ (positive integrable continuous functions), we have:

$$(2) \qquad \mu_g\cdot \int_{G/g} f(x)d_{G/g}x = \int_{G/\Gamma}\Big[\sum_{\xi\in\Gamma/\gamma} f(x\xi)\Big]d_{G/\Gamma}x$$

where the two sides of (2) are both infinite or both finite and equal (as usual $\infty\cdot 0 = 0$).

From (1) $\int_{G/\gamma} f(x)d_{G/\gamma}x = \int_{G/g} d_{G/g}x \int_{g/\gamma} f(xu)d_{g/\gamma}u = \mu_g\cdot \int_{G/g} f(x)\cdot d_{G/g}x;$

on the other hand:

$$\int_{G/\gamma} f(x)d_{G/\gamma}x = \int_{G/\Gamma} d_{G/\Gamma}x \left[\sum_{\xi\in\Gamma/\gamma} f(x\xi)\right] d_{G/\gamma}x \ .$$

Lemma 2.4.3. With the same notations, suppose g normal and $\Gamma \cap g = \gamma$. Put $H = G/g$, $\Gamma_H = \Gamma g/g \approx \Gamma/\gamma$. Assume furthermore that $d_G x = d_H y \cdot d_g u$, where $y \in H$ is the image of $x \in G$ by the canonical map $p : G \to H$. Then for $f \in L^+(H/\Gamma_H)$,

$$\mu_g \cdot \int_{H/\Gamma_H} f(y) d_{H/\Gamma_H} y = \int_{G/\Gamma} f(p(x)) d_{G/\Gamma} x \ . \tag{3}$$

In particular, if $\mu_H = \int_{H/\Gamma_H} d_{H/\Gamma_H} y$, $\mu_G = \int_{G/\Gamma} d_{G/\Gamma} x$, then: if two of the three numbers μ_g, μ_H, μ_G are finite, so is the third one, and $\mu_g \cdot \mu_H = \mu_G$.

By assumption we have $d_H y = d_{G/g} y$. Then by formula (2),

$$\mu_g \cdot \int_H h(y) d_H y = \int_{G/\Gamma} \sum_{\xi\in\Gamma/\gamma} h(p(x\xi)) d_{G/\Gamma} x$$

$$= \int_{G/\Gamma} \sum_{\xi\in\Gamma_H} h(p(x)\cdot\xi) d_{G/\Gamma} x, \text{ for } h \in L^+(H, d_H y)$$

But by Fubini,

$$\int_H h(y) d_H y = \int_{H/\Gamma_H} d_{H/\Gamma_H} y \sum_{\xi\in\Gamma_H} h(y\cdot\xi) \ .$$

If we put $f(y) = \sum_{\xi\in\Gamma_H} h(y\xi)$, then we have:

$$\mu_g \int_{H/\Gamma_H} f(y) d_{H/\Gamma_H} y = \int_{G/\Gamma} f(p(x)) d_{G/\Gamma} x \ .$$

But there are "sufficiently many" functions $\sum_{\xi\in\Gamma_H} h(y\xi)$ in $L^+(H/\Gamma_H)$. This proves (3). Putting $f = 1$ in (3), we get the second part.

Theorem 2.4.4. Let G and $g \subset G$ be unimodular algebraic groups defined over k; assume that g is normal; call ϕ the canonical mapping of G onto $H = G/g$. Put $H'_A = \phi_A(G_A)$, $H'_k = \phi(G_k)$; assume that H'_A is an open subgroup of H_A; also, assume that (1) is a set of convergence factors for g. Then every set (λ_v) of convergence factors for G is such a set for H; and, if $d_A x$, $d_A y$ are the corresponding Tamagawa measures for G_A, H_A, we have, for $f \in L^+(H'_A/H'_k)$:

$$\tau(g) \int_{H'_A/H'_k} f(y) d_A y = \int_{G_A/G_k} f(\phi(x)) d_A x \, . \tag{4}$$

($\tau(g)$ = Tamagawa number of g, if it is defined; otherwise $+\infty$.)

As the kernel of the mapping ϕ_A of G_A onto H'_A is g_A, we can identify H'_A with G_A/g_A. As H'_A is open in H_A, the image of G_{k_v} by ϕ is open in H_{k_v} for every v, and the image of $G_{\underset{-p}{o}}$ by ϕ is $H_{\underset{-p}{o}}$ for almost all p. Let dx, dy, du be gauge-forms for G, H and g, matching together algebraically. By the theory of analytic groups over complete valued fields, there are, locally, analytic cross-sections for g_{k_v} in G_{k_v} for each v; this is enough to guarantee (just as in Theorem 2.4.3) that $(dx)_v$, $(dy)_v$, $(du)_v$ match together topologically for every v, with the same conclusion as in Theorem 2.4.3 about sets of convergence factors; in particular, if (1) is such a set for g, the sets of convergence factors for G and H are the same. Then $d_A x$, $d_A y$ and the Tamagawa measure for g match together topologically on G_A, H'_A and g_A; (4) follows now from Lemma 2.4.3.

Corollary. If at the same time G and g satisfy the conditions of Theorem 2.4.2 (existence of a cross-section and the density condition), then (4) holds with $H'_A = H_A$, $H'_k = H_k$; and, if (1) is a set of convergence factors

<u>for</u> G (<u>or for</u> H), $\tau(G) = \tau(g)\tau(H)$.

<u>Remark</u> 1. If, instead of assuming that (1) is a set of convergence factors for g, we merely denote by (λ_v), (μ_v), (ν_v) sets of convergence factors for G, g and H satisfying $\lambda_v = \mu_v \nu_v$ for all v (as we may by Theorem 2.4.3), then the proof of Theorem 2.4.4 remains valid and shows that (4) holds when we take for $d_A x$, $d_A y$ the Tamagawa measures for G_A and H_A corresponding to the sets (λ_v), (ν_v), provided we replace $\tau(g)$ by the "modified Tamagawa number"

$$\tau'(g) = \int_{g_A/g_k} d_A u$$

where $d_A u$ is the Tamagawa measure for g_A corresponding to the set (μ_v).

<u>Remark</u> 2. In the proofs of Theorems 2.4.2 (identification of H_A with G_A/g_A) and 2.4.4 (identification of H'_A with G_A/g_A), we have made implicit use of the fact that, whenever a group such as G_A acts on a locally compact space, then the mapping of G_A onto any locally closed orbit in that space is open, so that the orbit can be identified with the quotient of G_A by the stability group of any one of its points. This is a special case of a known theorem (cf. Montgomery-Zippin, Top. Transf.-Groups, p. 65), proved by an elementary category argument. The same theorem will be used occasionally, sometimes without reference, in the next chapters.

CHAPTER III

THE LINEAR, PROJECTIVE AND SYMPLECTIC GROUPS

3.1. The zeta-function of a central division algebra

As formerly, whenver V is a variety, defined over a field k, we denote by V_k the set of points of V, rational over k; a vector-space of dimension d over k can always be denoted by R_k, where R is an affine space of dimension d in the sense of algebraic geometry. In particular, any algebra over k can be so written; the obvious extension to R of the multiplication-law on the algebra R_k makes R into an algebra-variety, defined over k (which means that the multiplication-law on R is defined over k). The given algebra R_k over k is absolutely semisimple if and only if R is so, i.e. if and only if R is isomorphic (over the universal domain) to a direct sum of matrix algebras.

Let D_k be a central division algebra of dimension n^2 over k. This algebra defines an algebra-variety D of center $Z(Z_k = k)$. Over $\bar{k}$, D is isomorphic to $M_n(\Omega)$ (total matrix algebra over the universal domain Ω).

The reduced norm is a multiplicative mapping $N : D \rightarrow Z$; it is a homogeneous polynomial function of degree n.

Let $D^{(1)}$ be the subvariety of dimension $n^2 - 1$ defined in D by the equation $N(x) = 1$. This is an algebraic group defined over k, the special linear group of D.

In the projective space $\underline{\underline{P}}(D)$ derived from D considered as an affine space, the multiplication on D induces on the Zariski-open subset defined by the homogeneous relation $N(x) \neq 0$, a structure of algebraic group defined over k, the projective group of D. We can describe it alternatively in this manner: denote by D^* the multiplicative (algebraic) group of the elements of D with non-zero norm. Then $G = D^*/Z^*$ is isomorphic to the projective group of D. Furthermore, the restriction to $D^{(1)}$ of the canonical mapping $D^* \to G$ identifies $D^{(1)}/(\text{center of } D^{(1)})$ with G.

Let α_i, $1 \leq i \leq n^2$, be a basis of D_k over k, and let $x = \Sigma_i \alpha_i x_i$ be a generic point of D. The set $D_{o_p} = \Sigma_i \alpha_i o_p$ forms a ring for almost every p. Furthermore if p does not divide the discriminant of D then D_{o_p} is a maximal order of D_{k_p}, and one has $D_{k_p} = M_n(k_p)$. Since every maximal order of $M_n(k_p)$ is conjugate to $M_n(o_p)$, there is a ring isomorphism $D_{o_p} \cong M_n(o_p)$ if $p \notin S$, S being suitably chosen.

As an invariant measure on D^*, we can take

$$\omega = N(x)^{-n} dx_1 \ldots dx_{n^2} ,$$

and if $p \notin S$, we have

$$\mu_p = \int_{D^*_{o_p}} \omega_p = \int_{GL_n(o_p)} |\det X|_p^{-n} (dX)_p = \int_{GL_n(o_p)} (dX)_p$$

$$= q^{-n^2}(q^n - 1)(q^n - q) \ldots (q^n - q^{n-1}) = (1-q^{-n})(1-q^{-(n-1)}) \ldots (1-q^{-1}) ,$$

$$X = (x_{ij}), \quad dX = \prod_{i,j} dx_{ij}, \quad q = N(p) ,$$

From this formula follows that the $\lambda_p = 1 - N(p)^{-1}$ are a set of convergence factors for D^*. The Tamagawa measure $(\omega, (\lambda_p))$ will be called

ω'_A:

$$\omega'_A = \mu_k^{-n^2} \prod_{\lambda \text{ infinite}} \omega_\lambda \prod_p \omega'_p, \text{ where } \omega'_p = (1-N(p)^{-1})^{-1} \omega_p$$

The mapping $N : D^* \to Z^* \cong G_m$ can be extended to $N : D_A^* \to (G_m)_A = I_k$. We denote by $| \ |$ the idele-module (2.1.1).

We take on D_A the Tamagawa measure $dx_A = dx_S \times \prod_{p \notin S} dx_p$, where $dx_S = \mu_k^{-n^2} \prod_{v \in S} dx_v$. Then $\int_{D_A/D_k} dx_A = 1$,

<u>Lemma</u> 3.1.1 (Fujisaki). <u>If</u> $0 < m \leq M$, <u>the image in</u> D_A^*/D_k^* <u>of the subset of</u> D_A^* <u>determined by</u> $m \leq |N(x)| \leq M$ <u>is compact</u>.

Let X be that subset. Take a compact subset C of D_A whose measure (for the measure dx_A) is $> M^n$ and $> m^{-n}$; put $C' = C + (-C) = \{c_1 - c_2 | c_1, c_2 \in C\}$. If $a \in D_A^*$, $x \to ax$ and $x \to xa$ are automorphisms of D_A whose module is $|N(a)|^n$ (idele-module of the regular norm $N(a)^n$); therefore, if $a \in X$, the images $a^{-1}C$, Ca of C under the automorphisms $x \to a^{-1}x$, $x \to xa$ of D_A have a measure > 1 and are not mapped in a one-to-one manner onto their images in D_A/D_k by the canonical homorphism of D_A onto D_A/D_k, since the measure of D_A/D_k is 1. This is the same as to say that $a^{-1}C' \cap D_k^*$ and $C'a \cap D_k^*$ are not empty, i.e. that there are $\alpha, \beta \in D_k^*$ such that $c' = a\alpha \in C'$, $c'' = \beta a^{-1} \in C'$. Then $c''c' = \beta\alpha$ is in $C'^2 \cap D_k^*$, which is a finite set $\{\xi_1, \ldots, \xi_h\}$ since D_k is discrete in D_A and C'^2 is compact. Assume that $\beta\alpha = \xi_i$; as $c' = a\alpha$, this gives $c'^{-1} = \xi_i^{-1}c''$. As D^* can be identified with its image in $D \times D$ under the mapping $x \to (x, x^{-1})$ (since that image is closed in $D \times D$), D_A^* can be identified with its image in $D_A \times D_A$ under the corresponding adele-mapping; so the set Y_i of the points $x \in D_A^*$ such that $(x, x^{-1}) \in C' \times (\xi_i^{-1} C')$ is compact. We have thus proved that, whenever $a \in X$, there is $\alpha \in D_k^*$ such that $a\alpha$ is in the union of the compact sets Y_i. This proves the lemma.

Remark. Lemma 3.1.1 shows that Problem (I) of 2.4 ("is G_A / G_k compact?") has an affirmative answer for the group $D^{(1)}$.

Lemma 3.1.2. There is a constant $c > 0$ such that, in the number-field case

$$\int_{D_A^*/D_k^*} F(|N(x)|)\omega'_A = c \int_0^{+\infty} F(t)dt/t \quad ,$$

resp., in the function-field case

$$\int_{D_A^*/D_k^*} F(|N(x)|)\omega'_A = c \sum_{\nu=-\infty}^{+\infty} F(q^\nu) \quad ,$$

whenever F is a function on $\mathbb{R}_+$ (resp. on the group $\{q^\nu \mid \nu \in \mathbb{Z}\}$) such that the integral (resp. the sum) in the right-hand side converges absolutely. (N.B. In the function-field case, q denotes the number of elements of the field of constants of k.)

Let first F be a continuous function with compact support on the multiplicative group $\mathbb{R}_+$ (resp. on the group $\{q^\nu\}$); then lemma 3.1.1 shows that $F(|N(x)|)$ has compact support in D_A^*/D_k^*, so that the integral in the left-hand side is convergent. For any $t_o \in \mathbb{R}_+$ (resp. for any $t_o = q^\nu$) there is $x_o \in D_A^*$ such that $|N(x_o)| = t_o$; replacing then, in the left-hand side, x by $x_o x$, we see that it does not change if $F(t)$ is replaced by $F(t_o t)$; in other words, as a function of F, it is translation-invariant in the group $\mathbb{R}_+$

(resp. $\{q^\nu\}$). This proves the lemma, for F continuous with compact support; the general case follows from this in the usual manner. (For the value of c, cf. Theorem 3.1.1.)

Now denote by Tr the reduced trace in D_k over k, which we extend in the usual manner to a linear function on D, and also to a linear mapping of $D_A = D_k \otimes_k A_k$ into A_k; $Tr(xy)$ is then a non-degenerate bilinear form on $D \times D$. If χ is the character of A_k defined in Theorem 2.1.1, we put $\chi_D(x) = \chi(Tr\, x)$ for $x \in D_A$; then χ_D has the properties corresponding to those stated for χ in Theorem 2.1.1: $\chi_D(xy)$ determines an isomorphism of D_A onto its dual group, and D_k is self-orthogonal for this isomorphism. As a consequence, if we define the Fourier transform $\Psi(y)$ of a function $\Phi(x)$ on D_A by the formula

$$\Psi(y) = \int_{D_A} \Phi(x)\chi_D(xy)dx_A \quad ,$$

we have the inversion formula

$$\Phi(x) = \int_{D_A} \Psi(y)\, \overline{\chi_D(xy)}dy_A$$

and the Poisson summation formula

$$\sum_{\alpha \in D_k} \Phi(\alpha) = \sum_{\beta \in D_k} \Psi(\beta)$$

under suitable conditions for Φ and Ψ. For instance, these formulas are valid if both Φ and Ψ are continuous, absolutely integrable, and such that the series $\Sigma_\alpha \Phi(x+\alpha)$, $\Sigma_\beta \Psi(y+\beta)$ are absolutely and uniformly convergent; when that is so, we say that Φ, Ψ are "of Poisson type". If, in the definition for Φ, we substitute xa, $a^{-1}y$ for x, y, where $a \in D_A^*$, we see that the Fourier transform of $\Phi(xa)$ is $|N(a)|^{-n}\Psi(a^{-1}y)$; similarly (and in view of the fact that $\chi_D(axa^{-1}) = \chi_D(x)$, by the defintion of χ_D) the Fourier transform of $\Phi(ax)$ is $|N(a)|^{-n}\Psi(ya^{-1})$.

Now we say that Φ is <u>of standard type</u> if, for every $a \in D_A^*$, $\Phi(ax)$ and $\Phi(xa)$ are of Poisson type and if the following conditions are satisfied:

(a) There is an S (i.e. a finite set of valuations of k, including all the infinite places) such that, for $x = (x_v) \in D_A$, $x_S = (x_v)_{v\in S}$:

$$\Phi(x) = \Phi_S(x_S) \prod_{p\notin S} \phi_p(x_p)$$

where ϕ_p is the characteristic function of $D_{\underline{o}_p}$, and Φ_S is continuous and absolutely integrable in $D_S = \prod_{v\in S} D_{k_v}$. If this condition is satisfied for some S, it is clearly also satisfied for every $S' \supset S$. Also, it implies that the Fourier transform Ψ of Φ satisfies a similar condition (but possibly with another S).

(b) The integral

$$(1) \qquad \int_{D_S} \prod_{v\in S} |N(x_v)|_v^s \, \Phi_S(x_S) dx_S$$

and the similar integral for Ψ, converge absolutely and uniformly for $s \geq 0$.

One special method for constructing functions of standard type is as follows. In the function-theoretic case, take any set S as in (i), and take for Φ_S the characteristic function of any compact open subset of D_S, or a finite linear combination of such functions. In the number-theoretic case, let S_o be the set of the infinite places of k, so that $D_{S_o} = D_k \otimes_k \underline{R}$; on D_{S_o}, considered as a vector-space over $\underline{\underline{R}}$, let P be a polynomial function and F a positive-definite quadratic form; for $x_o \in D_{S_o}$, take $\Phi_o(x_o) = P(x_o)e^{-F(x_o)}$; on the other hand, let S' be any finite set of valuations of k, disjoint from S_o, and let Φ' be the characteristic function of a compact open subset of $D_{S'}$, or a finite linear combination of such functions; take $S = S_o \cup S'$, so that $D_S = D_{S_o} \times D_{S'}$, and, for $x_o \in D_{S_o}$, $x' \in D_{S'}$, take $\Phi_S(x_o, x') = \Phi_o(x_o)\Phi'(x')$; then define Φ as in (i). It is easily seen that, in both cases, Φ is of standard type, and that its Fourier transform is another function of the same nature.

Definition. Let Φ be a function of standard type on D_A; the function

$$Z^{\Phi}(s) = \int_{D_A^*} |N(x)|^s \Phi(x)\omega'_A \tag{2}$$

will be called the zeta-function of D with respect to Φ.

This definition will be justified by proving, firstly, by a multiplicative calculation, that the integral for $Z^{\Phi}(s)$ is absolutely convergent for $Re(s) > n$ (and uniformly so for $Re(s) \geq n + \epsilon$ for every $\epsilon > 0$), and, secondly, by an additive calculation, that it can be continued analytically in the whole s-plane. For the multiplicative calculation, take S as in (a) and put, for every $S' \supset S$:

$$D^*_{S'} = \prod_{v \in S'} D^*_{k_v}, \quad D^*_{(S')} = D^*_{S'} \times \prod_{p \notin S'} D^*_{\underline{O}_p} .$$

Then, by definition of the adele-space, we have

$$Z^{\Phi}(s) = \lim_{S'} \int_{D^*_{(S')}} |N(x)|^s \Phi(x)\omega'_A ,$$

where the limit is taken over the filter of the sets S', ordered by inclusion. Now call $Z_S^{\Phi}(s)$ the integral (1); this is the same as the same integral taken over D_S^*, since $\prod_{v\in S}|N(x_v)|_v$ is 0 on $D_S - D_S^*$. Also, put, for $p \notin S$

$$Z_p(s) = \int_{D_{k_p}^*} |N(x_p)|_p^s \, \phi_p(x_p)\omega'_p \ ,$$

and denote by $Z_p^o(s)$ the same integral taken over $D_{o_p}^*$; then:

$$Z^\Phi(s) = (\prod_{v\in S}\lambda_v^{-1})Z_S^\Phi(s-n)\lim_{S'}(\prod_{p\in S'-S} Z_p(s)\prod_{p\notin S'} Z_p^o(s))$$

where, as before, $\lambda_p = 1 - N(p)^{-1}$ for a p-adic valuation, and $\lambda_v = 1$ for an infinite place. Now, on $D_{o_p}^*$, we have $|N(x_p)|_p = 1$ (provided S has been taken large enough) and $\phi_p(x_p) = 1$, so that $Z_p^o(s)$ is the ω'_p-measure of $D_{o_p}^*$; by the definition of convergence factors, this implies that $\prod_p Z_p^o(s)$ is absolutely convergent. Therefore

$$Z^\Phi(s) = (\prod_{v\in S}\lambda_v^{-1})Z_S^\Phi(s-n)\prod_{p\notin S} Z_p(s) \ .$$

As we have seen, we can identify D_{o_p} with $M_n(o_p)$ for $p \notin S$ provided S has been taken large enough. Write $M_n(o_p)^*$ for the set of the matrices in $M_n(o_p)$ with a non-zero determinant, and $U_{n,p}$ for the set of the invertible matrices in $M_n(o_p)$, i.e. those whose determinant is a unit in o_p; we have, for $p \in S$ and $q = N(p)$:

$$Z_p(s) = \int_{M_n(o_p)^*} |\det X|_p^s \, (1-q^{-1})^{-1}|\det X|_p^{-n}(dX)_p \ .$$

For a given p, call π a prime element in o_p (this means that πo_p is the maximal prime ideal in o_p). It is well-known that $M_n(o_p)$ is the disjoint union of the sets $U_{n,p}A$ when one takes for A the following matrices:

$$A = \begin{pmatrix} \pi^{d_1} & & \alpha_{ij} & \cdot \\ & \pi^{d_2} & & \\ & & \ddots & \\ 0 & & & \pi^{d_n} \end{pmatrix}$$

where $(d_1, \ldots, d_n)$ run through all n-tuples of integers ≥ 0, and, for each choice of the d_i, each α_{ij} runs through a complete set of representatives of $\underline{o}_p$ mod π^{d_j}. This gives:

$$(1-q^{-1})Z_p(s) = \sum_A |\det A|_p^s \int_{U_{n,p}A} |\det X|_p^{-n}(dX)_p \quad .$$

As the integrand in the right-hand side is (by construction) invariant under translations in $M_n(k_p)$, the integral is independent of A; taking $A = 1_n$ (the unit-matrix), we see that its value is the measure of $U_{n,p}$ for the measure $(dX)_p$, in the space $M_n(k_p)$; this is $q^{-n^2}\nu$ if ν is the number of matrices of non-zero determinant in the ring $M_n(F_q)$, where F_q is the finite field with q elements; this gives

$$\int_{U_{n,p}} (dX)_p = (1-q^{-n})(1-q^{-n+1}) \ldots (1-q^{-1}) \quad .$$

On the other hand, for given values of $d_1, \ldots, d_n$, there are

$$q^{d_2 + 2d_3 + \ldots + (n-1)d_n}$$

matrices A. Therefore:

$$Z_p(s) = (1-q^{-n}) \ldots (1-q^{-2}) \sum_{(d_i)} q^{\Sigma_i (i-1-s)d_i}$$

$$= \prod_{i=2}^{n} (1-q^{-i}) \prod_{i=1}^{n} (1-q^{i-1-s})^{-1} \quad .$$

From the elementary theory of the zeta-function for k, we borrow the fact that the product $\zeta_k(s) = \prod_p (1-q^{-s})^{-1}$ is absolutely convergent for s real and > 1, and that $(s-1)\zeta_k(s)$ tends to a finite limit ρ_k for $s \to 1$; this implies that $\prod_{p \notin S} Z_p(s)$ converges absolutely for $\mathrm{Re}(s) > n$ and is $\sim \rho(s-n)^{-1}$, with ρ a constant, for $s \to n$. This proves that, as asserted, the integral which defines $Z^{\Phi}(s)$ is absolutely convergent for $\mathrm{Re}(s) > n$. Also, we have

$$(3) \qquad \lim_{s \to n} (s-n) Z^{\Phi}(s) = \rho_k Z_S^{\Phi}(0) = \rho_k \int_{D_A} \Phi(x) dx_A \quad ,$$

where ρ_k depends only upon the field k.

For the <u>additive</u> calculation, introduce on $\underline{\underline{R}}_+$ the function $\lambda(t)$ defined by $\lambda(t) = 1$ for $0 < t < 1$, $\lambda(t) = 0$ for $1 < t$, and $\lambda(1) = \frac{1}{2}$. For $x \in D_A^*$, put:

$$f_+(x) = \lambda(|N(x)|^{-1}) \ , \quad f_-(x) = \lambda(|N(x)|) \quad ,$$

so that we have $f_+ + f_- = 1$; and write

$$Z_+^{\Phi}(s) = \int_{D_A^*} f_+(x) |N(x)|^s \Phi(x) \omega'_A \quad ,$$

$$Z_-^{\Phi}(s) = \int_{D_A^*} f_-(x) |N(x)|^s \Phi(x) \omega'_A \quad ,$$

so that $Z^{\Phi} = Z_+^{\Phi} + Z_-^{\Phi}$. Clearly, if the integral for Z_+^{Φ} converges absolutely for some s_o, it converges absolutely and uniformly for $\mathrm{Re}(s) \leq \mathrm{Re}(s_o)$; as the multiplicative calculation has shown that the integral for Z^{Φ}, hence also the integral for Z_+^{Φ}, converges absolutely for $\mathrm{Re}(s) > n$, we conclude that $Z_+^{\Phi}(s)$ is an entire function of s. On the other hand, we have, for $\mathrm{Re}(s) > n$

$$Z_-^{\Phi}(s) = \int_{D_A^*/D_k^*} f_-(x)\,|N(x)|^s \left(\sum_{\alpha\in D_k^*} \Phi(x\alpha)\right)\omega'_A$$

$$= \int_{D_A^*/D_k^*} f_-(x)\,|N(x)|^s \left(\sum_{\alpha\in D_k} \Phi(x\alpha) - \Phi(0)\right)\omega'_A$$

(D_A^*/D_k^* is the space of <u>right</u> cosets xD_k^* of D_k^* in D_A^*). By Poisson summation, we have

$$\sum_{\alpha\in D_k} \Phi(x\alpha) = |N(x)|^{-n} \sum_{\beta\in D_k} \Psi(\beta x^{-1}) \ ,$$

where Ψ is as before the Fourier transform of Φ; hence

$$Z_-^{\Phi}(s) = \int_{D_A^*/D_k^*} f_-(x)\,|N(x)|^s \left(|N(x)|^{-n} \sum_{\beta\in D_k} \Psi(\beta x^{-1}) - \Phi(0)\right)\omega'_A \ ,$$

this still being absolutely convergent for $\mathrm{Re}(s) > n$. Now, in the integral defining $Z_+^{\Psi}(s)$, which converges absolutely for all s, substitute $n - s$ for s and x^{-1} for x; as the latter substitution does not affect the Haar measure, and as $f_+(x^{-1}) = f_-(x)$, we get:

$$Z_+^{\Psi}(n-s) = \int_{D_A^*} f_-(x)\,|N(x)|^{s-n}\, \Psi(x^{-1})\omega'_A$$

$$= \int_{D_A^*/D_k^*} f_-(x)\,|N(x)|^{s-n} \left(\sum_{\beta\in D_k^*} \Psi(\beta x^{-1})\right)\omega'_A \ .$$

Combining our two last formulas, we get

$$Z_-^{\Phi}(s) - Z_+^{\Psi}(n-s) = \int_{D_A^*/D_k^*} \mu(|N(x)|)\omega'_A$$

where $\mu(t)$ is the function defined on $\underline{\underline{R}}_+$ by

$$\mu(t) = [\Psi(0)t^{s-n} - \Phi(0)t^{s}]\lambda(t) \quad .$$

By lemma 3.1.2, this gives

$$Z_{-}^{\Phi}(s) - Z_{+}^{\Psi}(n-s) = \begin{cases} c\left(\frac{\Psi(0)}{s-n} - \frac{\Phi(0)}{s}\right) \\ \frac{c}{2}\left(\Psi(0)\,\frac{q^{s-n}+1}{q^{s-n}-1} - \Phi(0)\,\frac{q^{s}+1}{q^{s}-1}\right) \end{cases}$$

(c is the constant in Lemma 3.1.2). This shows that Z_{-}^{Φ} and Z^{Φ} are meromorphic in the whole plane, with a residue at $s = n$ equal to $c\Psi(0)$ (resp. $c\Psi(0)/\log q$). As we have $\Psi(0) = \int_{D_A} \Phi(x)dx_A$, a comparison with (3) gives $c = \rho_k$ (resp. $c = \rho_k \log q$). This proves:

<u>Theorem</u> 3.1.1. (i) $Z^{\Phi}(s)$ <u>is the sum of an entire function of</u> s <u>and of</u>

$$\rho_k\left(\frac{\Psi(0)}{s-n} - \frac{\Phi(0)}{s}\right) \text{ resp. } \rho_k \log q \left(\frac{\Psi(0)}{q^{s-n}-1} - \frac{\Phi(0)}{q^{s}-1}\right)$$

<u>where</u> $\rho_k = [(s-1)\zeta_k(s)]_{s=1}$, <u>and</u> Ψ <u>is the Fourier transform of</u> Φ.

(ii) $Z^{\Phi}(s) = Z^{\Psi}(n-s)$.

(iii) <u>We have the formula</u>

$$\int_{D_A^*/D_k^*} F(|N(x)|)\omega'_A = \begin{cases} \rho_k \int_0^{+\infty} F(t)dt/t \\ \text{resp. } \rho_k \log q \sum_{\nu=-\infty}^{+\infty} F(q^{\nu}) \end{cases}$$

<u>provided</u> F <u>is such that the right-hand side is absolutely convergent</u> (N.B. <u>In the function-field case,</u> q <u>is the number of elements of the field of constants of</u> k).

3.2. <u>The projective group of a central division algebra.</u>

<u>Lemma</u> 3.2.1. <u>Let</u> G <u>be a locally compact unimodular group,</u> g <u>a closed subgroup of the center of</u> G; <u>put</u> $G' = G/g$, <u>and let</u> dx, $d'x'$, $d_g z$ <u>be Haar measures matching together topologically on</u> G, G', g. <u>Let</u> Δ <u>be a discrete subgroup of</u> G; <u>put</u> $H = G/\Delta$, $\delta = \Delta \cap g$, $\Delta' = g\Delta/g = \Delta/\delta$, $H' = G'/\Delta' = G/g\Delta$, $\gamma = g/\delta = g\Delta/\Delta$. <u>Then</u> γ <u>is a commutative group operating continuously on</u> H; <u>the quotient</u> H/γ <u>is canonically isomorphic to</u> H'; <u>and we have, for</u> $f \in L^+(H)$

$$\int_H f(u)du = \int_{H'} d'u' \int_\gamma f(tu)d_g t$$

<u>where</u> tu, <u>for</u> $u \in H$, $t \in \gamma$, <u>is the transform of</u> u <u>by</u> t <u>acting on</u> H <u>as we have said, and</u> u' <u>is the image</u> γu <u>of</u> u <u>in the canonical mapping of</u> H <u>onto</u> $H' = H/\gamma$.

The first assertion follows algebraically from the fact that g is in the center, and topologically from the fact that Δ is discrete in G. Now we have, under obvious assumptions on ϕ, ϕ', ϕ'':

$$\int_G \phi(x)dx = \int_{G'} d'x' \int_g \phi(xz)d_g z \qquad (x' = xg),$$

$$\int_{G'} \phi'(x')d'x' = \int_{H'} \left(\sum_{\xi' \in \Delta'} \phi'(x'\xi') \right) d'x'' \qquad (x'' = x'\Delta'),$$

$$\int_g \phi''(z)d_g z = \int_\gamma \left(\sum_{\zeta \in \delta} \phi''(z\zeta) \right) d_g z' \qquad (z' = z\delta).$$

Combining these, and using the fact that g and δ are in the center, we get

(1) $$\int_G \phi(x)dx = \int_{H'} d'u' \int_\gamma \left(\sum_{\xi \in \Delta} \phi(xz\xi) \right) d_g z' \qquad (z' = z\delta,\ u' = xg\Delta).$$

On the other hand, we have

(2) $$\int_G \phi(x)dx = \int_H \left(\sum_{\xi \in \Delta} \phi(x\xi) \right) du \qquad (u = x\Delta) .$$

The comparison of (1) and (2) concludes the proof, since there are "sufficiently many" functions $\Sigma_{\xi\in\Delta}\ \phi(x\xi)$ on H.

Now, in Lemma 3.2.1, we replace G, Δ, g by D^*_A, D^*_k, Z^*_A, respectively, where Z is, as before, the center of the algebra variety D. As in 3.1, we write G for the algebraic group D^*/Z^*, i.e. for the projective group of D; Z^* may be identified with G_m; as it is well known that every fibering by G_m has local cross-sections, we can apply Theorem 2.4.2 to D^* and Z^*, and therefore identify D^*_A/Z^*_A with G_A; also, if we identify Z^* with G_m, Z^*_A gets identified with $(G_m)_A = I_k$, the idele-group of k. On D^*_A and D^*_A/D^*_k, we take the measure denoted above by ω'_A. The proof in 3.1, applied to the case n = 1, shows that the set of convergence factors (λ_p) which was used to define ω'_A is also a set of convergence factors for $Z^* = G_m$; we denote by ω'_Z the Tamagawa measure determined by this set on $Z^*_A = I_k$, and also on $Z^*_A/Z^*_k = I_k/k^*$. Then, by Theorem 2.4.3, (1) is a set of convergence factors for $G = D^*/Z^*$; we denote by ω_G the Tamagawa measure on G_A, and the corresponding measure on G_A/G_k. In view of Theorem 2.4.3, we can now apply to this situation our Lemma 3.2.1, and get

$$\int_{D^*_A/D^*_k} F(|N(x)|)\omega'_A = \int_{G_A/G_k} \omega_G \int_{Z^*_A/Z^*_k} F(|N(xz)| \cdot \omega'_Z(z) ,$$

where F is any function such that the left-hand side converges absolutely. The left-hand side can be expressed by means of Theorem 3.1.1 (iii); on the other hand, the second integral in the right-hand side can be written as

$$\int_{Z_A^*/Z_k^*} F(|N(x)|\cdot |z|^n)\omega'_Z(z) \ ,$$

which can be expressed by Theorem 3.1.1 (iii) applied to Z instead of D and is thus seen to have the value

$$\rho_k \int_0^{+\infty} F(|N(x)|t^n)dt/t = \frac{\rho_k}{n} \int_0^{+\infty} F(t)dt/t$$

in the number-field case, and

$$\rho_k \log q \sum_{\nu=-\infty}^{+\infty} F(|N(x)|q^{\nu n})$$

in the function-field case. Comparing both results, we see at once, in the former case, that $\tau(G) = \int_{G_A/G_k} \omega_G$ has the value n; we get the same conclusion in the latter case by taking, for instance, F such that $F(q^\nu) = 1$ for $0 \leq \nu \leq n-1$, and $= 0$ otherwise. Thus:

Theorem 3.2.1. *The Tamagawa number of the projective group of a division algebra of dimension* n^2 *over its center is* n.

3.3. Isogenies.

We recall that an isogeny is a homomorphism of an algebraic group onto another of the same dimension; two groups G, G' are called isogenous if G'' can be found so that there are isogenies of G'' onto G and onto G'. In this section, we consider Tamagawa numbers of groups isogenous to projective groups of simple algebras, and products of such groups; this, combined with Theorem 3.2.1, will give for instance the Tamagawa number of the special linear group of a division algebra.

Lemma 3.3.1. *If two groups* G, G' *are isogenous over* k, *every* set of convergence factors for G is a set of convergence factors for G'.

Assume that there is an isogeny f of G onto G' over k; by means of representations of G, G' into special linear groups, we can consider them as affine varieties; then, if $x' = f(x)$, the coordinates of x' can be written as polynomials in those of x. As in 2.2, we see that G, G' and f can be reduced modulo p for almost all p. Our lemma is now an immediate consequence of Theorem 2.2.5 and of Lang's theorem, according to which two isogenous groups over a finite field have the same number of rational points (Am. J. of Math. 78; see last five lines of p. 561).

Any simple algebra R can be written as $M_m(D)$, where D, as in 3.1, is a division algebra. The same calculation as in 3.1 shows that the $\lambda_p = 1 - N(p)^{-1}$ is a set of convergence factors for R^*; as it is also a set of convergence factors for Z^*, where Z is the center of R, Theorem 2.4.3 shows that (1) is such a set for R^*/Z^*, hence also for every group isogenous to R^*/Z^* (in particular, for $R^{(1)}$) and that (λ_p) is such a set for every group isogenous to $(R^*/Z^*) \times G_m$. In what follows, we shall use these facts freely. Tamagawa measures in the strict sense (derived from the set (1) of convergence factors) will be denoted by ω, ω_A, dx, etc.; by ω', ω'_A, $d'x$, etc., we denote Tamagawa measures derived from the set of convergence factors $(1-N(p)^{-1})$; for instance, on I_k, we use the Tamagawa measure $(dt/t)'$.

As in 3.1, let D_k be a division algebra of dimension n^2 over its center $Z_k = k$; D being the algebra variety defined by D, with the center Z, take $R = M_m(D)$; call N the reduced norm in R over its center. Take any divisor ν of mn $(1 \leq \nu \leq mn)$. Let Γ be the group

$$\Gamma = \left\{ (x, v) \in R^* \times G_m \,|\, v^\nu = N(x) \right\} ,$$

i.e. the algebraic subgroup of $R^* \times G_m$ determined by $v^\nu = N(x)$; it is connected and defined over k. The connected component of the identity in its center is

$$\Gamma_o = \left\{ (t \cdot 1_R, t^{mn/\nu}) \,|\, t \in G_m \right\}$$

and is obviously isomorphic to G_m; here 1_R is the unit-element in R. Put $G = \Gamma/\Gamma_o$; this is an algebraic group, defined over k; for $\nu = 1$, Γ is isomorphic to R^* and G can be identified with R^*/Z^*, the projective group of R; for $\nu = mn$, the mapping $(x, v) \to v^{-1}x$ determines a homomorphism of Γ onto $R^{(1)}$ with the kernel Γ_o, so that G can be identified with $R^{(1)}$. For any ν, there are obvious isogenies $R^{(1)} \to G \to R^*/Z^*$ such that the composite isogeny $R^{(1)} \to R^*/Z^*$ is the canonical one.

Theorem 3.3.1. We have $\tau(G) = mn/\nu$.

Consider the homomorphism ϕ of Γ onto G_m given by $\phi(x, v) = v$; its kernel is

$$\Gamma' = \left\{ (x, 1) \in R^{(1)} \times G_m \right\}$$

and is isomorphic to $R^{(1)}$; by means of ϕ, we can identify Γ/Γ' with G_m; as usual, we extend ϕ to a homomorphism of Γ_A into $(G_m)_A = I_k$.

Lemma 3.3.2. $\phi(\Gamma_A)/\phi(\Gamma_k) = I_k/k^*$.

It is known (Eichler, Math. Zeitschr. 1938) that an element λ of k^* is the norm of an element of R_k if and only if it is the norm of an element of R_{k_v} for every v; the latter condition is equivalent to saying that λ must be the norm of an element of R_A. Now, for $\alpha \in k^*$, we have $\alpha \in \phi(\Gamma_A)$ if and only if α^ν is the norm of an element of R_A; then, by Eichler's theorem, α^{ν} is the norm of an element of R_k. This proves that $k^* \cap \phi(\Gamma_A) = \phi(\Gamma_k)$. Now we show that $I_k = k^* . \phi(\Gamma_A)$. In fact, if $x = (x_v) \in I_k$, it is known that x_p is the norm of an element of R_{k_p} for every p, and that x_v is the norm of an element of R_{k_v} whenever v is a complex place (i.e. $k_v = \underline{\underline{C}}$), and also whenever v is a real place (i.e. $k_v = \underline{\underline{R}}$) and $x_v > 0$. Moreover, for almost all p, $R_{\underline{o}_p} = M_{mn}(\underline{o}_p)$, so that the image of $\Gamma_{\underline{o}_p}$ by ϕ is $(G_m)_{\underline{o}_p} = U_p$ (the unit-group of $\underline{o}_p$). Algebraically, our conclusion follows at once from these facts; the same holds topologically (so that we may identify $\phi(\Gamma_A)/\phi(\Gamma_k)$ with I_k/k^*) in view of the final remark of Chapter II.

As $(x, v) \to x$ is an isogeny of Γ onto R^*, $(1-N(p)^{-1})$ is a set of convergence factors for Γ; let $d'(x, v)$ be the corresponding measure. We shall discuss the number-field case; the function-field case can be treated quite similarly. We compute in two ways the integral

$$\int_{\Gamma_A/\Gamma_k} F(|v|)d'(x, v)$$

where F is an arbitrary function (say, continuous with compact support) on the group $\underline{\underline{R}}_+$. We first use the decomposition $G = \Gamma/\Gamma_o$; as $\Gamma_o \cong G_m$, it has cross-sections in Γ, so that we can apply Theorem 2.4.3. Then, by Lemma 3.2.1, we have

$$\int_{\Gamma_A/\Gamma_k} F(|v|)d'(x, v) = \int_{G_A/G_k} \omega_G \int_{I_k/k^*} F(|t^{mn/\nu} v|)(dt/t)' \;;$$

the second integral in the right-hand side can be computed by Theorem 3.1.1 (iii) applied to G_m, which shows that it is independent of v (this would have to be modified in the function-field case, just as in the last part of the proof of Theorem 3.2.1), and gives

$$\int_{\Gamma_A/\Gamma_k} F(|v|)d'(x,v) = \tau(G)\frac{\rho_k{}^{\nu}}{mn}\int_0^{\infty} F(t)dt/t \quad .$$

On the other hand, applying Theorem 2.4.4 and Lemma 3.3.2 to the decomposition $G_m = \Gamma/\Gamma'$, we get:

$$\int_{\Gamma_A/\Gamma_k} F(|v|)d'(x,v) = \tau(\Gamma')\int_{I_k/k^*} F(|v|)(dv/v)' \quad ,$$

where the right-hand side can again be computed as above. This shows that, if one of the numbers $\tau(G)$, $\tau(\Gamma')$ is finite, the other is so, and that $\tau(\Gamma') = \nu\tau(G)/mn$.

Take $m = \nu = 1$; then $\Gamma' = D^{(1)}$, G is the projective group of D, and $\tau(G) = n$ by Theorem 3.2.1; therefore $\tau(D^{(1)}) = 1$. Take $m = 1$, and take for ν any divisor of n; then $\Gamma' = D^{(1)}$, $\tau(\Gamma') = 1$; this gives $\tau(G) = n/\nu$, and proves Theorem 3.3.1 in the case $n = 1$.

In the general case, we have $\tau(G) = mn\tau(\Gamma')/\nu$, with $\Gamma' \cong R^{(1)}$. Thus, in order to complete the proof of Theorem 3.3.1, it only remains to show that $\tau(R^{(1)}) = 1$ for $R = M_m(D)$; as we know that this is so for $m = 1$, we shall proceed by induction on m.

3.4. <u>End of proof of Theorem</u> 3.3.1: <u>central simple algebras.</u>

We change our notations slightly. From now on, D will be as before; we write $R_m = M_m(D)$. We denote by D^m the space of $(m,1)$-matrices (i.e., column-vectors of order m) over D, and let R_m act on D^m by $(X, x) \to Xx$ for $X \in R_m$, $x \in D^m$. Put

$$e = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(we write 1 for the unit-element of D). Call H the orbit of e under the action of R_m^* in D^m; as Xe is generic over k in D^m if X is so in R_m, H is a Zariski-open subset of D^m over k. More precisely, if K is any field containing k, H_K can be determined as follows. As $D_K = D_k \otimes K$ is a simple central algebra over K, there is an isomorphism ρ of D_K onto a matrix algebra $M_r(D')$ over a central division algebra D' over K; then H_K consists of the column-vectors $x = (x_i)_{1 \leq i \leq m}$, $x_i \in D_K$, such that the (mr, r)-matrix

$$\rho(x) = \begin{pmatrix} \rho(x_1) \\ \rho(x_2) \\ \vdots \\ \rho(x_m) \end{pmatrix}$$

over D' has the rank r.

From now on, we assume that $m \geq 2$, and we put $G = R_m^{(1)}$, $G' = R_{m-1}^{(1)}$; it is easily seen that H is also the orbit of e under G. Call g the subgroup of G leaving e fixed; it consists of the matrices

$$X = \begin{pmatrix} 1 & {}^t u \\ 0 & X' \end{pmatrix} \qquad (u \in D^{m-1},\ X' \in G')$$

(${}^t u$ is the transpose of u). If $x = (x_i)_{1 \leq i \leq m}$ is a generic point of D^m over k, it is the image $M(x)e$ of e under the element

$$M(x) = \begin{pmatrix} x_1 & 0 & 0 & 0 & \dots & 0 \\ x_2 & x_1^{-1} & 0 & 0 & \dots & 0 \\ x_3 & 0 & & & & \\ \vdots & \vdots & & 1_{m-2} & & \\ x_m & 0 & & & & \end{pmatrix}$$

of G, where 1_{m-2} is the unit element of R_{m-2}. This is a "generic cross-section" (in the sense of Theorem 1.2.2) for the mapping $X \to Xe$ of G into $H = G/g$; also, G_k is obviously Zariski-dense in G. We can therefore apply Theorem 2.4.2 (cf. Remark 2 at the end of Chapter II), and therefore identify G_A/g_A, and, for every v, G_{k_v}/g_{k_v}, with the orbits H_A, H_{k_v} of e under G_A and under G_{k_v}, respectively; also, for almost every p, we can identify $G_{\mathfrak{o}_\mathfrak{p}}/g_{\mathfrak{o}_\mathfrak{p}}$ with the orbit $H_{\mathfrak{o}_\mathfrak{p}}$ of e under $G_{\mathfrak{o}_\mathfrak{p}}$. For every v, there is an isomorphism ρ_v of D_{k_v} onto a matrix algebra $M_{r_v}(D'_v)$ over a division algebra D'_v with the center k_v; this can be extended to an isomorphism which we also call ρ_v, of $R_{k_v} = M_m(D_{k_v})$ onto $M_{mr_v}(D'_v)$, and also to an isomorphism of $(D_{k_v})^m$ onto the space of (mr_v, r_v)-matrices over D'_v; as we have seen, H_{k_v} consists of the elements $x \in (D_{k_v})^m$ such that $\rho_v(x)$ has the rank r_v. For almost all p, we have $r_p = n$, $D'_p = k_p$, and, for $x \in (D_{\mathfrak{o}_\mathfrak{p}})^m$, $\rho_p(x)$ is an (mn, n)-matrix over $\mathfrak{o}_\mathfrak{p}$; this matrix, by reduction modulo p, determines an (mn, n)-matrix $\bar{\rho}_p(x)$ over the finite field F_q with $q = N(p)$ elements; it is easily seen that we can choose S such that this is so for $p \notin S$, and also that, for $p \notin S$, $H_{\mathfrak{o}_\mathfrak{p}}$ consists of the elements x of $(D_{\mathfrak{o}_\mathfrak{p}})^m$ such that $\bar{\rho}_p(x)$ is of rank n.

The additive group D_A^m is isomorphic to $(A_k)^{mn^2}$; it will now be shown that $D_A^m - H_A$ is of measure 0. This is a special case of the following:

<u>Lemma</u> 3.4.1. <u>Let</u> ω <u>be a gauge-form on a non-singular variety</u> V, <u>defined over</u> k; <u>let</u> V' <u>be a k-open subset of</u> V; <u>assume that there is a common set</u> (λ_v) <u>of convergence factors for</u> V <u>and for</u> V'. <u>Then the measure</u> $(\omega, (\lambda_v))$ <u>for</u> V'_A <u>is the measure induced on</u> V'_A <u>by the measure</u> $(\omega, (\lambda_v))$ <u>for</u> V_A; <u>and, for that measure,</u> $V_A - V'_A$ <u>has the measure</u> 0.

Let j be the injection mapping of V' into V; then j_A is an injective mapping of V'_A into V_A, which we use to identify V'_A with a subset of V_A (note that the topology of V'_A is in general <u>not</u> that induced by V_A, as shown by the case of G_m considered as the complement of $\{0\}$ in G_a, which corresponds to the "natural" embedding of I_k in A_k; in that case the two varieties have no common set of convergence factors, and I_k, not $A_k - I_k$, is of measure 0 in A_k). The first assertion follows from the definition of $(\omega, (\lambda_v))$. Now define $V_{\underset{=}{o}_p}$, $V'_{\underset{=}{o}_p}$ ("almost intrinsically", cf. 2.2.3 and 2.3) by means of suitable coverings of V and V'; by Theorem 1.2.1, there is S such that $V'_{\underset{=}{o}_p} \subset V_{\underset{=}{o}_p}$ for $p \notin S$; let μ_p, μ'_p be the measures of $V_{\underset{=}{o}_p}$, $V'_{\underset{=}{o}_p}$ for the "local measure" ω_p; by assumptions, $\prod_p \lambda_p^{-1}\mu_p$ and $\prod_p \lambda_p^{-1}\mu'_p$ are absolutely convergent. For each v, put $X_v = V_{k_v} - V'_{k_v}$; this is an analytic subset of V_{k_v} and is therefore of measure 0 for the local measure ω_v (by the theory of analytic varieties over complete valued fields). Take $S'' \supset S' \supset S$; put $V_{S'} = \prod_{v \in S'} V_{k_v}$, $P_{S'} = \prod_{p \notin S'} V_{\underset{=}{o}_p}$, and

$$P_{S'}(S'') = \prod_{p \in S''-S'} V_{\underset{=}{o}_p} \times \prod_{p \notin S''} V'_{\underset{=}{o}_p} .$$

Then, for the product-measure $\prod_p(\lambda_p^{-1}\omega_p)$ on $P_{S'}$, $P_{S'}(S'')$ has the measure

$$\prod_{p\in S''-S'}(\lambda_p^{-1}\mu_p)\cdot\prod_{p\notin S''}(\lambda_p^{-1}\mu_p') .$$

Therefore, if we put $Q_{S'} = \bigcup_{S''} P_{S'}(S'')$, $P_{S'} - Q_{S'}$ has the measure 0. We have thus shown that, in the product $V_{S'}\times P_{S'}$, the set of the points $x = (x_v)$ such that either $x_v \in X_v$ for some v or $(x_p)_{p\in S'} \notin Q_{S'}$ is of measure 0; since the complement of that set is contained in V'_A, this proves the lemma.

In order to apply this to D^m and H, all we need show is that (1) is a set of convergence factors for $H = G/g$, or (in view of Theorem 2.4.3) that this is so for g. In fact, g is the semi-direct product of the groups consisting respectively of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & X' \end{pmatrix} \quad (X'\in G') , \quad \begin{pmatrix} 1 & {}^tu \\ 0 & 1_{m-1} \end{pmatrix} \quad (u\in D^{m-1}) ,$$

these groups being respectively isomorphic to G' and to D^{m-1}; our assertion follows now from Theorem 2.4.3 applied to g, D^{m-1} and $G' = g/D^{m-1}$. Also, this shows that g is unimodular; and, in view of the fact that $\tau(D^{m-1}) = 1$ (since $\tau(G_a) = 1$), and that $\tau(G') = 1$ by the induction assumption, Theorems 2.4.3 and 2.4.4 show that $\tau(g) = 1$. Now, applying Lemma 2.4.2 to the groups G_A, g_A, G_k, g_k, we get

$$\int_{H_A} f(x)dx = \int_{G_A/G_k}\left(\sum_{\xi\in H_k} f(X\xi)\right)dX$$

where dx and dX are the Tamagawa measures in H_A and G_A respectively; as $D_A^m - H_A$ is of measure 0, we may, in the left-hand side, replace the integral on H_A by the integral on D_A^m, dx being then (in view of Lemma 3.4.1) the Tamagawa measure on D_A^m. On the other hand, since D_k is a division algebra, we have $H_k = D_k^m - \{0\}$; therefore

$$(1) \qquad \int_{D_A^m} f(x)dx = \int_{G_A/G_k} \Big(\sum_{\xi \in D_k^m} f(X\xi) - f(0) \Big) dX \ .$$

Here f may be taken as any function in D_A^m such that the left-hand side is absolutely convergent. If at the same time we assume f to be such that f(Xx), for every $X \in G_A$, is of Poisson type (one can select such a function by a procedure similar to that described in 3.1), the right-hand side can be transformed by Poisson's formula. Let g be the Fourier transform of f:

$$g(y) = \int_{D_A^m} f(x)\chi_D({}^t y \cdot x)dx$$

(as before, ${}^t y$ is the transpose of the column-vector y); replacing x, y by Xx, ${}^t X^{-1} y$, with $X \in G_A$, we see that the Fourier transform of f(Xx) is $g({}^t X^{-1} y)$ since det X = 1; (1) gives now:

$$\int_{D_A^m} f(x)dx = \int_{G_A/G_k} \Big(\sum_{\eta \in D_k^m} g({}^t X^{-1}\eta) - f(0) \Big) dX \ .$$

If in (1) we substitute g for f and ${}^t X^{-1}$ for X (which does not change dX), and combine the two formulas, we get

$$\int_{D_A^m} [f(x) - g(x)]dx = \int_{G_A/G_k} [g(0) - f(0)]dX \ .$$

Here the left-hand side has the value g(0) - f(0), and all integrals are absolutely convergent; choosing f so that $g(0) \neq f(0)$, we see that $\tau(G) = 1$. This completes the proof of Theorem 3. 3.1.

3.5. The symplectic group.

Using the same method as in 3.4, we prove:

Theorem 3.5.1. $\tau(Sp(2n)) = 1$.

As usual, $Sp(2n)$ is the symplectic group in $2n$ variables, i.e. the subgroup of $GL(2n)$ which leaves invariant the exterior form

$$x_1 \wedge x_2 + \dots + x_{2n-1} \wedge x_{2n} .$$

We have $Sp(2) = SL(2)$, so that $\tau(Sp(2)) = 1$ is a special case of Theorem 3.3.1. Now we proceed by induction on n. Take $n \geq 2$; put $G = Sp(2n)$, $G' = Sp(2n-2)$; call g the subgroup of G leaving the column-vector $e = (1, 0, \dots, 0)$ invariant. An easy calculation shows that g consists of the matrices

$$X = \begin{pmatrix} 1 & x & {}^t u \cdot J'X' \\ 0 & 1 & 0 \\ 0 & u & X' \end{pmatrix}$$

where $X' \in G'$, u is a column-vector of order $2n - 2$, x is arbitrary, and J' is the alternating matrix invariant under G'. The subgroup g' of g, consisting of the matrices X for which $X' = 1_{2n-2}$, is normal; more precisely, g is the semidirect product of g' and of $g/g' \cong G'$; moreover, the subgroup g'' of g', consisting of the matrices X for which $X' = 1_{2n-2}$ and $u = 0$, is normal, and g' is the semidirect product of $g'' \cong G_a$ and of $g'/g'' \cong (G_a)^{2n-2}$. From these facts and from the induction assumption $\tau(G') = 1$, one concludes, by applying twice Theorem 2.4.3, that (1) is a set of convergence factors, first for g', and then for g, that $\tau(g') = 1$, and that $\tau(g) = 1$. Now the orbit of e under G is $H = S^{2n} - \{0\}$, where S^{2n}

is the affine space of all column-vectors of order $2n$; just as in 3.4, we see that we can identify H with G/g, H_A with G_A/g_A, etc.; also, $H_{\underline{o}_p}$ consists of the vectors in $\underline{o}_p^{2n}$ which are not $\equiv 0 \bmod p$; an easy calculation shows then that (1) is a set of convergence factors for H as well as for S^{2n}, so that we can apply Lemma 3.4.1; also, by Theorem 2.4.3, (1) is a set of convergence factors for G. Now we can apply Lemma 2.4.2 to G_A, g_A, G_k, g_k; this gives

$$\int_{H_A} f(x)dx = \int_{G_A/G_k} \left(\sum_{\xi \in H_k} f(X\xi) \right) dX \ .$$

In the left-hand side, we can replace H_A by $(A_k)^{2n}$; also, H_k is the same as $k^{2n} - \{0\}$. Proceeding now exactly as in 3.4, we get $\tau(G) = 1$.

3.6. Isogenies for products of linear groups.

The method of 3.3 can be applied to a more general situation. Let R be any absolutely semisimple algebra-variety over k; this is the same as to say that R_k is absolutely semisimple, or also that R, over the universal domain, is isomorphic to a direct sum of matrix-algebras. We can write $R = \oplus R_i$, where the R_i, for $1 \leq i \leq r$, are the simple components of R; for each i, we have $R_i = M_{m_i}(D_i)$, where D_i is such that $(D_i)_k$ is a division-algebra. If Z_i is the center of D_i, which we identify in an obvious manner with the center of R_i, $(Z_i)_k = k_i$ is a finite, separably algebraic extension of k, and $(D_i)_k$ may be considered as a central division algebra over k_i, which can be written as $(D_i')_{k_i}$, where D_i' is an algebra-variety over k_i; then we have, with the notations of 1.3, $D_i = R_{k_i/k}(D_i')$, hence $R_i = R_{k_i/k}(R_i')$ with $R_i' = M_{m_i}(D_i')$; also, if Z_i' is the center of D_i'

and R_i', we have $Z_i = R_{k_i/k}(Z_i')$; and the reduced norm N_i', taken in R_i' over its center, determines (by applying to its graph the operation $R_{k_i/k}$) a norm mapping N_i of R_i into Z_i. We write $R_i^{(1)}$ for the special linear group of R_i, i.e. the algebraic subgroup of R_i^* determined by $N_i(x) = 1$; this is the same as $R_{k_i/k}(R_i'^{(1)})$, and we have, by Theorems 1.3.2, 1.3.3:

$$(R_i'^{(1)})_{k_i} = (R_i^{(1)})_k \quad , \quad (R_i'^{(1)})_{A_{k_i}} = (R_i^{(1)})A_k \quad ,$$

which, in view of Theorem 2.3.2, implies $\tau(R_i^{(1)}) = 1$. By the definition of the operation $R_{k_i/k}$, the algebraic group $R_i^{(1)}$, over the universal domain, is isomorphic to $(SL(m_i n_i))^{d_i}$, where $d_i = [k_i:k]$ and n_i^2 is the dimension of D_i'.

Write N for the mapping

$$x = (x_1, \dots, x_r) \rightarrow N(x) = (N_1(x_1), \dots, N_r(x_r))$$

of R into its center $Z = \oplus_i Z_i$. This determines a homomorphism of the multiplicative group R^* of the invertible elements of R into the corresponding group Z^* for Z; we have $Z^* = \prod_i Z_i^*$, with $Z_i^* = R_{k_i/k}(Z_i'^*) \cong R_{k_i/k}(G_m)$; this is a torus of dimension $d = \Sigma_i d_i$; we have $(Z^*)_k \cong \prod_i k_i^*$; and the mapping N induces on Z^* an isogeny of Z^* onto itself, given by

$$z = (z_1, \dots, z_r) \rightarrow N(z) = (z_1^{m_1 n_1}, \dots, z_r^{m_r n_r}) \; .$$

In the special case treated in 3.3, the norm mapping $N ; z \rightarrow z^{mn}$ of Z^* into itself was factored into $z \rightarrow z^{mn/\nu} \rightarrow z^{mn}$. Similarly, we assume now that we have factored N as follows:

$$\begin{array}{ccc} Z^* & \xrightarrow{\quad N \quad} & Z^* \\ & {}_{\mu}\searrow \quad \nearrow_{\nu} & \\ & T & \end{array}$$

where T is a commutative group-variety, defined over k, and μ, ν are isogenies of Z^* onto T and of T onto Z^*, also defined over k; then T is a torus, isogenous to Z^*. We introduce the algebraic groups:

$$\Gamma = \{(x, t) \in R^* \times T \mid \nu(t) = N(x)\}$$
$$\Gamma_o = \{(z, \mu(z)) \mid z \in Z^*\} \quad .$$

As in 3.3, Γ_o is the connected component of the identity in the center of Γ; it is isomorphic to Z^*; the group $G = \Gamma/\Gamma_o$ is isogenous to $R^{(1)}$ (it is <u>perhaps</u> the most general group isogenous to $R^{(1)}$ over k, but this will not be discussed here), and we investigate $\tau(G)$.

Call ϕ the homomorphism of Γ into T given by $\phi(x,t) = t$; its kernel is the group $\Gamma' = \{(x, e) \mid x \in R^{(1)}\}$, where e is the neutral element in T, and is isomorphic to $R^{(1)}$. Lemma 3.3.2 can be generalized as follows:

<u>Lemma</u> 3.6.1. $\phi(\Gamma_A)/\phi(\Gamma_k)$ <u>is canonically isomorphic to an open subgroup of</u> T_A/T_k <u>of finite index</u> 2^i.

Just as in the proof of Lemma 3.3.2, one sees, by using Eichler's theorem, that $\phi(\Gamma_A) \cap T_k$ is the same as $\phi(\Gamma_k)$, and also that any $t \in T_A$ is in $\phi(\Gamma_A)$ whenever $t_v \in \phi(\Gamma_{k_v})$, i.e. whenever $\nu(t_v) \in N(R^*_{k_v})$, for every infinite place v of k. This proves the lemma, with $i = 0$, in the function-field case. In the number-field case, call S the set of the infinite places of k; put $Z^*_S = \prod_{v \in S} Z^*_{k_v}$; call $Z^*_{S,o}$ the connected component of the identity in Z^*_S, and put

$$Z^o_A = Z^*_{S,o} \times \prod_{p \notin S} Z^*_{k_p} \quad ;$$

as Z_S^* is a product of factors $\underline{\underline{R}}^*$, $\underline{\underline{C}}^*$, the group $\gamma = Z_A^*/Z_A^o$ is a finite abelian group of type (2, 2, . . . , 2); call σ the canonical homomorphism of Z_A^* onto γ. From the remarks made above and in the proof of Lemma 3. 3. 2, it follows that, if $t \in T_A$ is such that $\nu(t) \in Z_A^o$, t is in $\phi(\Gamma_A)$; therefore $\phi(\Gamma_A)$ contains the kernel of the homomorphism $\sigma \circ \nu$ of T_A into γ; as the index in our lemma is equal to the index of $\phi(\Gamma_A)T_k$ in T_A, this completes the proof.

(N. B. It has been shown by Serre and Tate that, if T is any torus over $\underline{\underline{Q}}$ (the rational numbers), and $T_{\underline{\underline{R}}}^o$ is the connected component of the identity in $T_{\underline{\underline{R}}}$, then $T_{\underline{\underline{R}}} = T_{\underline{\underline{R}}}^o \cdot T_{\underline{\underline{Q}}}$; from this one easily concludes, firstly (using the operation $R_{k/\underline{\underline{Q}}}$) that, if T is a torus over a number-field k, and T_A^o is defined from T as Z_A^o was defined above from Z^*, $T_A = T_A^o \cdot T_k$, and, secondly, that the index 2^i in Lemma 3. 6. 1 has <u>always</u> the value 1).

Now we apply to the groups Γ, Γ', Γ_o the method used in 3. 3. For each i, let ν_i be the norm in $(Z_i)_k = k_i$ over k (this is both the reduced norm and the regular norm in the sense of the theory of algebras); this can be extended to a norm mapping ν_i in Z_i (which is a polynomial function of degree $d_i = [k_i : k]$), which induces on Z_i^* a character, i. e. a representation of Z_i^* into G_m, rational over k; moreover, the group of all such characters of Z_i^* is generated by ν_i, and the group of all characters of $Z^* = \prod_i Z_i^*$, rational over k, is generated by the characters

$$z = (z_1, \ldots, z_r) \rightarrow \nu_i(z_i) \ .$$

As μ is an isogeny of Z^* into T, one concludes from this (by well-known elementary arguments in the theory of algebraic toruses) that the group of the characters of T, rational over k, is generated by r characters χ_i, and that the characters $\chi_i \circ \mu$ of Z^* generate a subgroup of finite index of the similar group for Z^*. In other words, if we write

$$\chi_i(\mu(z)) = \prod_j \nu_j(z_j)^{a_{ij}} ,$$

with integers a_{ij}, we have $\det(a_{ij}) \neq 0$.

Now we need a Tamagawa measure for Γ_A. As Γ is isogenous to $R^* = \prod_i R_i^*$, we can use for Γ any set of convergence factors $(\lambda_v) = (\prod_i \lambda_v^{(i)})$, where $(\lambda_v^{(i)})$ is a set of convergence factors for $R_i^* = R_{k_i/k}(R_i'^*)$; such a set can be chosen at once, by Theorem 2. 3. 2 and the results of 3. 3, by putting $\lambda_v^{(i)} = 1$ whenever v is an infinite place of k, and otherwise:

$$\lambda_p^{(i)} = \prod_{P/p} (1-N(P)^{-1})$$

where the product is extended to all the prime divisors P of p in k_i, and the norm N(P) is the absolute norm (equal to $N(p)^f$ if P is of relative degree f over p). For the same reasons, this same set of convergence factors can also be used for Z^*, hence also for Γ_o, and (by Lemma 3. 3.1) for the torus T. On the other hand, the same argument shows that (1) is a set of convergence factors for $R^{(1)}$, hence for Γ', and also for G which is isogenous to $R^{(1)}$. We denote by $d''(x,t)$ the Tamagawa measure for Γ_A, with the set of convergence factors (λ_v), and use similar notations for Z^* and T.

Now we compute in two different ways the integral

$$I = \int_{\Gamma_A/\Gamma_k} F(|\chi_1(t)|, \ldots, |\chi_r(t)|) d''(x,t) \quad ;$$

here $|\ |$ denotes as usual the idele-module, and F is an arbitrary function (say, continuous with compact support) on the group $(\underline{\underline{R}}_+)^r$; we do this by using the two decompositions $G = \Gamma/\Gamma_o$ and $T = \Gamma/\Gamma'$; this will be carried out in the number-field case (the function-field case can be treated similarly, mutatis mutandis). The first decomposition gives:

$$\text{(1)} \qquad I = \tau(G) \int_{Z_A^*/Z_k^*} F((\prod_j |\nu_j(z_j)|^{a_{ij}})_{1 \leq i \leq r}) d''z$$

while the second one gives, since $\tau(\Gamma') = 1$:

$$\text{(2)} \qquad I = \int_{\phi(\Gamma_A)/\phi(\Gamma_k)} F(|\chi_1(t)|, \ldots, |\chi_r(t)|) d''t \; .$$

For each i, we can identify $(Z_i^*)_A = (R_{k_i/k}(G_m))_A$ with $(G_m)_{A_{k_i}} = I_{k_i}$; therefore Z_A^* is the same as $\prod_i I_{k_i}$, while Z_k^* is the same as $\prod_i k_i^*$; and the idele-module $|\nu_i(z_i)|$, taken in I_k, is the same as the idele-module $|z_i|_i$ taken in I_{k_i}. From these facts, and from Theorem 3.1.1 (iii), we conclude that we have

$$\int_{Z_A^*/Z_k^*} f(|\nu_1(z_1)|, \ldots, |\nu_r(z_r)|) d''z$$

$$= (\prod_{i=1}^{r} \rho_{k_i}) \int_{(\underline{\underline{R}}_+)^r} f(\theta_1, \ldots, \theta_r) d\theta_1 \ldots d\theta_r / \theta_1 \ldots \theta_r$$

whenever f is such that the right-hand side is absolutely convergent. This gives for (1) the value:

$$\text{(3)} \qquad I = \tau(G) |\det(a_{ij})|^{-1} (\prod_i \rho_{k_i}) \int_{(\underline{\underline{R}}_+)^r} F(\theta_1, \ldots, \theta_r) d\theta_1 \ldots d\theta_r / \theta_1 \ldots \theta_r \; .$$

As to (2), we apply Lemma 3.6.1, together with the following remark: for every $t \in T_A$, there is $(x', t') \in \Gamma_A$ such that $|\chi_i(t^{-1}t')| = 1$ for $1 \leq i \leq r$ (in fact, it is easily seen that we can take $x' = z$, $t' = \mu(z)$, for a suitable $z \in Z_A^*$); therefore, we can choose, as representatives of the 2^i cosets of $\phi(\Gamma_A)T_k$ in T_A, elements t such that $|\chi_i(t)| = 1$ for all i. From this, one concludes that the integral in the right-hand side of (2), taken over every one of the 2^i cosets of $\phi(\Gamma_A)/\phi(\Gamma_k)$ in T_A/T_k, has always the same value. Therefore:

$$(4) \qquad I = 2^{-i} \int_{T_A/T_k} F(|\chi_1(t)|, \ldots, |\chi_r(t)|) d''t \ .$$

The comparison between (3) and (4) shows that (apart from the determination of the index 2^i, which is effected by the result of Serre and Tate) the computation of $\tau(G)$ has been reduced to a purely commutative problem concerning the torus T, viz. the computation of the integral in (4). This problem has been studied by Ono (Ann. of Math. 1961) but is not yet completely solved. Before making some applications of the results obtained above to the orthogonal groups, we insert a few general remarks about toruses.

Let T be any torus over k; by Theorem 2.2.2, its characters (i.e., its representations into G_m over the universal domain) make up a finitely generated free abelian group (free, because T is here assumed to be algebraically connected), on which the Galois group $\mathfrak{G}$ of $\bar{k}/k$ ($\bar{k}$ = algebraic closure of k) operates in an obvious manner (actually, every character of T is defined over a separably algebraic extension of k). This group, written additively and considered as a representation-module for $\mathfrak{G}$, is denoted by $\hat{T}$

and is known as the dual module of T; Tate has shown that there is a duality of the usual type between such modules and toruses over k. We write $\hat{T}_k$ for the group of the elements of $\hat{T}$ which are invariant under $\mathfrak{g}$ (these correspond to the characters of T which are defined over k). Let Ψ be the trace of the representation of $\mathfrak{g}$ (with coefficients in $\mathbb{Q}$) given by the operation of $\mathfrak{g}$ on the vector-space $\hat{T} \otimes \mathbb{Q}$ over $\mathbb{Q}$; according to Artin, there belongs to the "character" Ψ of $\mathfrak{g}$ an L-function $L(s, \Psi)$, which has at $s = 1$ a pole of order r if r is the number of generators for $\hat{T}_k$; this L-function is given by an infinite product

$$L(s, \Psi) = \prod_p L_p(s, \Psi)$$

taken over all the p-adic valuations of k. Now, combining Theorems 2.3.2 and 2.4.3 with a theorem of Artin on rational representations of finite groups, we see that, by putting $\lambda_v = 1$ for $v \in S$, $\lambda_p = L_p(1, \Psi)$ for $p \notin S$ (as usual, S is a finite set of valuations of k containing all the infinite places), we define a set of convergence factors for T. If $\tilde{d}t$ is the Tamagawa measure constructed by means of that set, we have a formula

$$\int_{T_A/T_k} F(|\chi_1(t)|, \ldots, |\chi_r(t)|)\tilde{d}t = c \int_{(\mathbb{R}_+)^r} F(\theta_1, \ldots, \theta_r)d\theta_1 \ldots d\theta_r/\theta_1 \ldots \theta_r$$

in the number-field case (the integral in the right-hand side is to be replaced by a series, in the usual manner, in the function-field case); here $\chi_1, \ldots, \chi_r$ are generators for the group of characters of T, defined over k; moreover, if we put

$$\rho = [(s-1)^r \prod_{p \notin S} L_p(s, \Psi)]_{s=1}$$

the constant $r(T) = c/\rho$ is independent of the choice of S and of the generators χ_i and is invariantly attached to the torus T; it has obvious "functorial" properties such as $r(T_1 \times T_2) = r(T_1)r(T_2)$, and $r(R_{k'/k}T') = r(T')$ if T' is a torus over k'.

If now notations are again as in (3) and (4), our results (taking into account the theorem of Serre and Tate) give $\tau(G) = r(T)|\det(a_{ij})|$. This raises the question whether, at least for toruses isogenous to Z^*, the number r(T) is always an integer.

3.7. Application to some orthogonal and hermitian groups.

In view of the well-known "canonical isomorphisms" between classical groups, the Tamagawa numbers for the orthogonal groups in 3 and 4 variables and for the hermitian groups in 2 variables can be calculated by means of the above results; this will provide the starting point for the consideration of the orthogonal groups, by induction on the number of variables, in Chapter IV. We avoid complications by excluding once for all the case of characteristic 2 (there is, however, no reason for thinking that our main results do not remain true even in that case). A quadratic form F, with coefficients in k, is said to be of index 0 if it "does not represent" 0, i.e. if $F(x) = 0$ has no solution in k, other than 0.

(a) Orthogonal group in 3 variables: let F be a quadratic form in 3 variables, and G the "special" orthogonal group of F ("special" = determinant 1); then G is isomorphic to the projective group of a simple central algebra of dimension 4, viz. a quaternion algebra if F is of index 0, and the matrix algebra M_2 otherwise; by Theorem 3.2.1 in the former case, and by Theorem 3.3.1 in the latter case, we have $\tau(G) = 2$.

(b) Hermitian group in 2 variables: let k' be a quadratic extension of k; take Z' over k', such that $Z'_{k'} = k'$; take $D' = Z'$, $R' = M_2(Z')$, $Z = R_{k'/k}(Z')$, $R = R_{k'/k}(R') = M_2(Z)$, so that $R_k = (M_2)_{k'}$. The non-trivial automorphism $z \to \bar{z}$ of k' over k can be extended in an obvious manner to an automorphism of the algebra variety Z, defined over k, which we also denote by $z \to \bar{z}$, and then to an automorphism $X \to \overline{X}$ of R, defined over k. We can identify G_m with the subgroup of Z^* defined by $z = \bar{z}$; the norm mapping of Z^* into G_m can then be written as $z \to z\bar{z}$, and its kernel is the subgroup U of Z^* defined by $z\bar{z} = 1$. Now let S be an invertible hermitian matrix over k', i.e. an element of $R_k^* = (M_2)_{k'}$ such that ${}^t\bar{S} = S$; this determines the hermitian form $F(x) = {}^t\bar{x} \cdot S \cdot x$ on the space Z^2 of vectors $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ over Z; F is said to be of index 0 if $F(x) = 0$ has no solution in $Z_k^2 = k'^2$, other than 0. The hermitian (or "unitary") group attached to S, or to $F(x)$, is the subgroup G of R^* given by:

$$G = \left\{ X \in R^* \mid {}^t\overline{X} \cdot S \cdot X = S, \; N(X) = 1 \right\}$$

where $N(X) = \det(X)$ is the reduced norm taken in R over Z.

It is known that G is isomorphic to the special linear group $R_1^{(1)}$ of a simple central algebra R_1 of dimension 4 over k, viz. a quaternion algebra if F is of index 0, and the matrix algebra M_2 otherwise. Therefore $\tau(G) = 1$.

(c) Orthogonal group in 4 variables: let F be a quadratic form over k, in 4 variables, Δ its discriminant, G the "special" orthogonal group for F. If $\Delta^{\frac{1}{2}} \in k$, there is an algebra R_1 of dimension 4 (a quaternion algebra if F is of index 0, otherwise the matrix algebra M_2) such that $G = (R_1^{(1)} \times R_1^{(1)})/\gamma$, where γ is the subgroup of order 2 consisting of the elements $(1,1)$, $(-1,-1)$. With the notations used in 3.6, this can be written as $G = \Gamma/\Gamma_o$ when we take $R_1 = R_2$, $R = R_1 \oplus R_2$, hence $Z^* = G_m \times G_m$, $N(z) = z^2$, $T = G_m \times G_m$, $\mu(z) = (z_1 z_2, z_1^{-1} z_2)$ for $z = (z_1, z_2) \in Z^*$, and $\nu(t) = (t_1 t_2^{-1}, t_1 t_2)$ for $t = (t_1, t_2) \in T$. The integral in (4) can be calculated by Theorem 3.1.1 (iii) and has the value

$$\rho_k^2 \int_{(\underline{\underline{R}}_+)^2} F(\theta_1, \theta_2) d\theta_1 d\theta_2 / \theta_1 \theta_2 \quad ;$$

also, one finds at once that $2^i = 1$, $|\det (a_{ij})| = 2$. This gives $\tau(G) = 2$.

Now assume that $k' = k(\Delta^{\frac{1}{2}})$ is a quadratic extension of k. Then G is isogenous to $R_{k'/k}(R'^{(1)})$, where R' is a central simple algebra of dimension 4 over k' (again a quaternion algebra or M_2 according as F is of index 0 or not); over k', it becomes isomorphic to $(R'^{(1)} \times R'^{(1)})/\gamma$, with γ as above. Let Z', Z, U and the mapping $z \rightarrow \bar{z}$ be defined as in (b). Then we can write G, with the notations of 3.6, as $G = \Gamma/\Gamma_o$ for $r = 1$, $k_1 = k'$, $R_1' = R'$, $R_1 = R$, $N(z) = z^2$, $T = G_m \times U$, $\mu(z) = (z\bar{z}, z^{-1}\bar{z})$ for $z \in Z^*$, $\nu(t) = (t_1 t_2^{-1}, t_1 t_2)$ for $t = (t_1, t_2)$, $t_1 \in G_m$, $t_2 \in U$. We have now $a_{11} = 1$, $2^i = 1$. Now we have to calculate (4) for $T = G_m \times U$. As we take our Tamagawa measure for G_m by means of the factors $\lambda_p = 1 - N(p)^{-1}$, and for Z^* and T by means of the factors

$$\lambda'_p = \prod_{p'/p} (1 - N(p')^{-1})$$

where the product is taken over the prime divisors p' of p in k', we have to take the Tamagawa measure on $U = T/G_m$ by means of the factors

$$\lambda''_p = \lambda_p^{-1}\lambda'_p = 1 - \chi(p)N(p)^{-1}$$

where χ is the character associated with the quadratic extension k' of k. Applying now Theorem 2.4.3 and Theorem 2.4.4 (the latter, in the modified form explained in the Remark following it) to the groups Z^*, U, $G_m = Z^*/U$ and to the norm mapping $z \to z\bar{z}$ of Z^* onto G_m, we get

$$(5) \qquad \tau'(U) \cdot \int_{H'_A/H'_k} F(|y|)d'_A y = \int_{Z^*_A/Z^*_k} F(|z\bar{z}|)d'_A z$$

where H'_A, H'_k are the images of Z^*_A, Z^*_k under the norm mapping, $\tau'(U) = \int_{U_A/U_k} d'_A u$, and $d'_A z$, $d'_A y$, $d'_A u$ are the Tamagawa measures for Z^*, G_m, U constructed by means of the sets λ'_p, λ_p, λ''_p defined above. In the right-hand side of (5), Z^*_A, Z^*_k may be identified with $I_{k'}$ and with k'^*; as $|\ |$ is the idele-module taken in I_k, $|z\bar{z}|$ is then the same as the idele-module taken in $I_{k'}$; therefore the right-hand side of (5), computed by Theorem 3.1.1 (iii), is

$$\rho_{k'} \int_0^\infty F(t)dt/t \ .$$

On the other hand, by class-field theory, H'_A/H'_k, in the left-hand side, is nothing else than the open subgroup of I_k/k^* of index 2 determined by $\chi(y) = 1$, where χ is the character of I_k/k^* belonging to the quadratic extension k'/k. Therefore the integral in the left-hand side has the value

$$\tfrac{1}{2}\rho_k \int_0^\infty F(t)dt/t \ .$$

This gives $\tau'(U) = 2\rho_{k'}/\rho_k$ (which can also be written as $\tau'(U) = 2L(1,\chi)$). As the integral in (4) has the value

$$\tau'(U) \int_{I_k^{\cdot}/k^*} F(|x|)(dx/x)' = \tau'(U)\rho_k \int_0^\infty F(t)dt/t \quad ,$$

we get, as before, $\tau(G) = 2$:

Theorem 3.7.1. The Tamagawa number of all special orthogonal groups in 3 and 4 variables has the value 2.

3.8. The zeta-function of a central simple algebra.

We have already twice made use of results in class-field theory (in the latter part of 3.8, and implicitly by using Eichler's norm theorem in the proof of Lemmas 3.3.2 and 3.6.1); we shall also use such results freely in our treatment of the classical groups in Chapter IV, both directly and by our use of Hasse's theorem on quadratic forms (which can be derived formally from the norm theorem for quaternion algebras). It is known, on the other hand, that most of these results can be derived from Hasse's theorem according to which "a central simple algebra which splits locally everywhere splits globally". This will now be proved by a more precise calculation of "the" zeta-function of an algebra (independently of our use of class-field theory in 3.3, 3.6, 3.8). It is therefore likely that, by following up this idea, our treatment could be rendered completely self-contained.

The multiplicative calculation of $Z^\Phi(s)$ for a division algebra in 3.1 can be extended to any central simple algebra; this will be done (following Fujisaki) for a special choice of Φ.

Let R be an algebra-variety over k, with the center Z, such that $Z_k = k$ and that R_k is a simple algebra; let n^2 be the dimension of R (equal to the dimension of R_k over k). Take a basis $(\alpha_i)_{1 \leq i \leq n^2}$

of R_k over k. As we have observed before, there is S such that, for $\mathfrak{p} \notin S$, $R_{k_\mathfrak{p}}$ is isomorphic to $M_n(k_\mathfrak{p})$ and $\underline{O}_\mathfrak{p} = \Sigma_i \alpha_i \underline{o}_\mathfrak{p}$ is a maximal order of $R_{k_\mathfrak{p}}$, isomorphic to $M_n(\underline{o}_\mathfrak{p})$ (this follows from the consideration of the discriminant). For every v, R_{k_v} is isomorphic to a matrix algebra $M_{m_v}(D'_v)$ over a central division algebra D'_v over k_v; we call r_v^2 the dimension of D'_v over k_v, so that $n = m_v r_v$. For each $\mathfrak{p} \in S$ we select a maximal order $\underline{O}_\mathfrak{p}$ of $R_{k_\mathfrak{p}}$; this is isomorphic to $M_{m_\mathfrak{p}}(\underline{o}'_\mathfrak{p})$, where $\underline{o}'_\mathfrak{p}$ is the maximal order of $D'_\mathfrak{p}$. For every $\mathfrak{p}$, we denote by π' a prime element of $\underline{o}'_\mathfrak{p}$ (a generator of the maximal two-sided ideal in $\underline{o}'_\mathfrak{p}$); by the theory of $\mathfrak{p}$-adic algebras, this can be done in such a way that $\pi = \pi'^{r_\mathfrak{p}}$ is a prime element of $\underline{o}_\mathfrak{p}$.

<u>The</u> zeta-function of R is defined as

$$(1) \qquad Z(s) = \int_{R_A^*} |N(x)|^s \Phi(x) \omega'_A$$

where ω'_A, as before, is the Tamagawa measure for R^* with the convergence factors $1 - N(\mathfrak{p})^{-1}$, and $\Phi(x) = \prod_v \phi_v(x_v)$ for $x = (x_v) \in R_A$, the ϕ_v being defined as follows. For $\mathfrak{p} \notin S$, $\phi_\mathfrak{p}$ is the characteristic function of $\underline{O}_\mathfrak{p}$. For $\mathfrak{p} \in S - S_o$, where S_o is the set of the infinite places of k (the empty set in the function-field case), $\phi_\mathfrak{p}$ is the characteristic function of $\pi'^{a(\mathfrak{p})} \underline{O}_\mathfrak{p}$, where $a(\mathfrak{p})$ is any integer (for $\mathfrak{p} \notin S$, we put $a(\mathfrak{p}) = 0$). For $v = v_\lambda \in S_o$, we have $k_\lambda = \underline{\underline{R}}$ or $\underline{\underline{C}}$ (we write k_λ instead of k_{v_λ}, etc.), with $k_\rho = \underline{\underline{R}}$ for $1 \le \rho \le r_1$, $D'_\rho \cong \underline{\underline{R}}$ or $\underline{\underline{K}}$ ($\underline{\underline{K}}$ = quaternion algebra over $\underline{\underline{R}}$), $k_\iota = \underline{\underline{C}}$ for $r_1 + 1 \le \iota \le r_1 + r_2$, $D'_\iota \cong \underline{\underline{C}}$. Let Tr be the reduced trace in R_k over k, which we extend as usual to R, and to R_{k_v} for each v. For each $v_\lambda \in S_o$, choose a positive involution $x \to \tilde{x}$ in R_λ, i.e. an involutory antiautomorphism such that $\mathrm{Tr}(\bar{x}x) > 0$ for $x \neq 0$; this can be done by transporting to R_λ, by means of some isomorphism of R_λ with $M_{m_\lambda}(D'_\lambda)$, the involution $X \to {}^t\bar{X}$ (here ${}^tX$ is the transpose, and $z \to \bar{z}$ is the identity in $\underline{\underline{R}}$, the complex conjugate in $\underline{\underline{C}}$, the quaternion conjugate in $\underline{\underline{K}}$). Now we take

$$\phi_\lambda(x) = \exp(-A_\lambda \mathrm{Tr}(\bar{x}x)) \quad ,$$

where A_λ is any constant > 0. It will be seen that the choice of S, of the integers $a(p)$, and of the constants A_λ does not affect "the" zeta-function except for an inessential factor.

We have, by definition,

$$\omega'_A = \mu_k^{-n^2} \prod_v \lambda_v^{-1} \omega_v$$

with $\lambda_v = 1$ for $v \in S_o$, $\lambda_p = 1 - q^{-1}$, $q = N(p)$, for $p \notin S_o$, and

$$\omega_v = |N(x_v)|_v^{-n} \left(\prod_i dt_i\right)_v$$

for $x_v = \Sigma_i \alpha_i t_i$, $t_i \in k_v$. Just as in 3.1, we find

$$(2) \qquad Z(s) = \mu_k^{-n^2} \prod_v Z_v(s) \quad , \quad Z_v(s) = \lambda_v^{-1} \int_{R^*_{k_v}} |N(x_v)|_v^s \phi_v(x_v) \omega_v \ .$$

In order to calculate $Z_p(s)$ for a given $p \notin S_o$, write a, r, m instead of $a(p)$, r_p, m_p. Identifying R_{k_p} with $M_m(k_p)$, we have

$$(3) \quad Z_p(s) = (1-q^{-1})^{-1} \int_\Omega |N(x)|_p^s \omega_p \quad , \quad \Omega = M_m(D'_p)^* \cap \pi'^a M_m(\underline{o}'_p) \ .$$

Let U be the set of the invertible matrices in the ring $M_m(\underline{o}'_p)$; Ω is the disjoint union of cosets $U.\pi'^a A$, if one takes for A the same matrices as in 3.1, except that π' has now to be substituted for π, and that α_{ij} has now to run through a complete set of representatives of $\underline{o}'_p$ modulo π'^{d_j}. As ω_p is an invariant measure in the multiplicative group $R^*_{k_p}$, the integral in (3), taken over $U\cdot\pi'^a A$, has the value $|N(\pi'^a A)|_p^s \int_U \omega_p$. Here $|N(\pi'^a A)|_p$ is easily computed by remembering that the regular norm in R is $N(x)^n$, and that π'^r is a prime element of $\underline{o}_p$; one finds $|N(\pi'^a A)|_p = q^{-N}$ with $N = \Sigma_i d_i + ma$. Also, the number of matrices A for given values of the d_i is q^M with $M = r\Sigma_i(i-1)d_i$. This gives

$$Z_p(s) = (1-q^{-1})^{-1} q^{-mas} \prod_{i=o}^{m-1} (1-q^{ri-s})^{-1} \int_U \omega_p .$$

In order to compute the last integral, we change from the basis (α_i) to a basis (β_i) of $M_m(\underline{o}'_p)$ considered as an $\underline{o}_p$-module; for $x = \Sigma_i \alpha_i t_i = \Sigma_i \beta_i u_i$, $\beta_i = \Sigma_j \eta_{ij}\alpha_j$, $\eta_{ij} \in k_p$, we get:

$$\int_U \omega_p = \int_U (\prod_i dt_i)_p = |\det(\eta_{ij})|_p \int_U (\prod_i du_i)_p .$$

If we put $d = \det(\mathrm{Tr}(\alpha_i\alpha_j))$, $d' = \det(\mathrm{Tr}(\beta_i\beta_j))$, where Tr is as before the reduced trace in $M_m(D'_p)$ over k_p, we have, by the theory of the different in p-adic algebras, $|d'|_p = q^{n(m-n)}$; since $d^{-1}d' = \det(\eta_{ij})^2$, this gives the value of $|\det(\eta_{ij})|_p$. Finally, $\underline{o}'_p/\pi'\underline{o}'_p$ is the finite field with q^r elements, and a matrix in $M_m(\underline{o}'_p)$ is in U if and only if its reduction modulo $\pi'\underline{o}'_p$ is a matrix of non-zero determinant over that field; the number of such matrices is

$$\nu = (q^{rm}-1)(q^{rm}-q^r)\ldots(q^{rm}-q^{r(m-1)}) ,$$

and $\int_U (\prod_i du_i)_p$ has the value $q^{-rm^2}\nu$. Thus:

$$Z_p(s) = |d|_p^{-\frac{1}{2}} q^{-mas-n(n-m)/2}(1-q^{-1})^{-1} \prod_{i=o}^{m-1} \frac{1-q^{ri-n}}{1-q^{ri-s}} ,$$

For $p \notin S$, we have $a = 0$, $m = n$, $r = 1$. This shows that $\prod_p Z_p(s)$ is absolutely convergent for $\mathrm{Re}(s) > n$.

Now we calculate the $Z_\lambda(s)$: we identify R_λ with $M_m(D_\lambda)$, writing again m, r for m_λ, r_λ; also, put $\rho = [k_\lambda : \underline{\underline{R}}]$. As $R_\lambda - R_\lambda^*$ is of measure 0, we have

$$Z_\lambda(s) = \int_{R_\lambda} \exp(-A_\lambda \mathrm{Tr}({}^t\overline{X} \cdot X)) \, |N(X)|_\lambda^s \omega_\lambda \quad ;$$

we recall that N and Tr mean the reduced norm and trace in R_λ over k_λ, and, if $z \in k_\lambda$, $|z|_\lambda$ means $|z|^\rho$ if $|\ |$ is the "usual" absolute value. Use the "Iwasawa decomposition" $X = UDT$, where ${}^t\overline{U} \cdot U = 1_m$, $D = \mathrm{diag}(\delta_1, \ldots, \delta_m)$ is the diagonal matrix with the elements $\delta_i > 0$, and $T = (t_{ij})$ is triangular ($t_{ij} = 0$ for $i > j$, $t_{ii} = 1$ for $1 \leq i \leq m$). Then, if dU, dD, dT denote Haar measures on the groups $\{U\}$, $\{D\}$, $\{T\}$, one sees at once that a Haar measure on R_λ must be of the form $f(D)dUdDdT$; here we can take for dT the additive Haar measure in the space of the vectors $(t_{ij})_{i<j}$, i.e. $dT = \prod_{i<j} d^+t_{ij}$ if d^+t is the measure in the additive group D'_λ; and we can take $dD = \prod_i (d\delta_i/\delta_i)$. Expressing now that $f(D)dUdDdT$ is invariant under $X \to XD_o$, i.e. under $(U, D, T) \to (U, DD_o, D_o^{-1}TD_o)$, we see that f, up to a constant factor, must be given by

$$f(D) = \prod_i \delta_i^{r^2 \rho(m-2i+1)} \quad .$$

Also, we have

$$|N(X)|_\lambda = |N(D)|_\lambda = \prod_i \delta_i^{r\rho}$$

$$\mathrm{Tr}({}^t\overline{X} \cdot X) = r \sum_{i=1}^{m} \delta_i^2 \left(1 + \sum_{j=i+1}^{m} \bar{t}_{ij} t_{ij}\right) \quad .$$

As ω_λ is a Haar measure on R_λ^*, one finds now immediately

$$Z_\lambda(s) = C_\lambda A_\lambda^{-\rho ns/2} \prod_{i=0}^{m-1} \Gamma(\tfrac{1}{2}r\rho(s-ri))$$

where C_λ is a constant factor, which can be computed by calculating $Z_\lambda(n)$; this is easily done by changing the basis (α_i) for a basis adapted to the identification $R_\lambda = M_m(D'_\lambda)$; one finds

$$Z_\lambda(n) = |d|_\lambda^{-\frac{1}{2}} (A_\lambda/\pi r\rho)^{-\rho n^2/2} .$$

In view of the product formula $\prod_v |d|_v = 1$, the product for $Z(s)$ does not contain d, as was to be expected ($Z(s)$ cannot depend upon the choice of the basis (α_i)).

Now we prove:

<u>Theorem 3.8.1.</u> <u>Let</u> R_k <u>be a central simple algebra over</u> k. <u>If, for every valuation</u> v <u>of</u> k, R_{k_v} <u>is isomorphic to a matrix algebra over</u> k_v, R_k <u>is isomorphic to a matrix algebra over</u> k.

The calculation given above shows that, except for a factor of the form $C_1 C_2^s$, with constant C_1, C_2, "the" zeta-function of R depends only upon the "ramification indices" r_v; in particular, if R "splits everywhere", i.e. if $r_v = 1$ for all v, we have

$$Z(s) = C_1 C_2^s \prod_{i=0}^{n-1} F(s-i)$$

with

$$F(s) = \Gamma(s/2)^{r_1} \Gamma(s)^{r_2} \zeta_k(s) \ ;$$

in particular, for $n = 1$, we see that $F(s)$ is "the" zeta-function of k itself.

In order to prove our theorem, it is clearly enough to show that a central division algebra over k, other than k, cannot "split everywhere", i.e. that this $Z(s)$ cannot be the zeta-function of a division algebra unless $n = 1$. In fact, the additive calculation in 3.1 has shown that, for such an algebra, $Z(s)$ has no other poles than $s = 0$ and $s = n$ on the real axis; in particular, $F(s)$ has the poles $s = 0$, $s = 1$. The above formula shows now that $Z(s)$ has simple poles at $s = 0$, $s = n$, and double poles at $s = 1, \ldots, n - 1$; therefore $n = 1$.

CHAPTER IV
THE OTHER CLASSICAL GROUPS

4.1. Classification and general theorems.

We consider only the algebraic groups, over a groundfield k, which, over the universal domain, are isogenous to products of simple groups of the three "classical" types: "special" linear, orthogonal and symplectic. Excluding the case of characteristic 2 (which has not been fully investigated) and certain "exceptional" forms of the orthogonal group in 8 variables (depending upon the principle of triality), such groups, up to isogeny, can be reduced to the following types, which will be called "classical" (the letter indicates the type over the universal domain, and K denotes any separably algebraic extension of k):

L1. Special linear group (or projective group) over a division algebra D_K over K.

L2. (a) Hermitian (i.e., "special" unitary) group for a hermitian form over a quadratic extension K' of K.

(b) Id. for a non-commutative central division algebra $D_{K'}$ over K', with an involution inducing on K' the non-trivial automorphism of K' over K.

O1. Orthogonal group for a quadratic form over K.

O2. Antihermitian group for an antihermitian (or "skew-hermitian") form over a quaternion algebra over K, with its usual involution.

S1. Symplectic group over K.

S2. Hermitian group for a hermitian form over a quaternion algebra over K, with its usual involution.

Our problem was to solve, if possible, the questions (I), (II), (III) listed in 2.4, for these various groups, their products and the groups isogenous to such products; only a fraction of this program will be fulfilled. In Chapter III, (I) has been solved (affirmatively) for $D^{(1)}$, the special linear group of a division algebra (cf. Remark following Lemma 3.1.1); it will be shown presently that the answer to (I) is negative for $R^{(1)}$ when $R = M_m(D)$, $m \geq 2$; and the problem will be solved for all remaining types except L2(b), for which only a partial answer will be given. Problem (II) has been solved affirmatively, in Chapter III, for the types L1, S1; an affirmative answer will be given to it, in this Chapter, for the types L2(a), O1, S2; perhaps the same method could be applied also to the types L2(b), O2, but this would require computations which have not been carried out. Problem (III) has been solved, in Chapter III, for the types L1, S1; it will be solved in this Chapter, for the types L2(a), O1, S2, by a method depending upon the construction of zeta-functions for quadratic forms (the same idea can be applied to the exceptional group G_2, as shown by Demazure in the Appendix); there is at present no obvious way in which one could hope to extend this method to the types L2(b) and O2. In many special cases, once the problems (I), (II), (III) have been solved for a group G, it is not too difficult to obtain a solution for a group G' isogenous to G; but general theorems by which this could be effected are still lacking. Perhaps the method described in 3.6 could be generalized.

From now on, we consider exclusively the types other than L1; it is enough to consider the case $K = k$ (since the case of an arbitrary K can be reduced to this by the operation $R_{K/k}$). The groups in question can all be described as follows:

Put $k' = k$ in the cases O1, O2, S1, S2; in the case L2, call k' a quadratic extension of k. Let D' be an algebra variety over k', such that $D'_{k'}$ is a division algebra with the center k'; let n^2 be the dimension of D'; put $D = R_{k'/k}(D')$, this being the same as D' for $k' = k$. In $D'_{k'}$, let $x \rightarrow \bar{x}$ be an involution (i.e., an involutory antiautomorphism) inducing the identity on k, and the non-trivial automorphism of k' over k if $k' \neq k$; this can be extended in an obvious manner to an involution in D, defined over k. Take $R' = M_m(D')$, $R = R_{k'/k}(R') = M_m(D)$; then $X \rightarrow {}^t\bar{X}$ is an involution in R, defined over k; so is the mapping $X \rightarrow S^{-1} \cdot {}^t\bar{X} \cdot S$ if S is an element of R^*_k such that ${}^t\bar{S} = \pm S$. The "classical group" defined by these data is the one given by

$$G = \left\{ X \in R^* \mid {}^t\bar{X}SX = S, \; N(X) = 1 \right\} \tag{1}$$

where N is the reduced norm, mapping R^* into its center $Z^* = R_{k'/k}(G_m)$. For the symplectic group (type S1), we have to take $k' = k$, $n = 1$, ${}^tS = -S$, and the relation ${}^t\bar{X}SX = S$ implies $N(X) = 1$ (as shown by the consideration of the pfaffian of S and of ${}^t\bar{X}SX$); therefore the same is true for the type S2, since these two types are one and the same over the universal domain. Similarly, in the case O1, we have $k' = k$, $n = 1$, ${}^tS = S$; and ${}^t\bar{X} \cdot S \cdot X = S$ implies $N(X) = \pm 1$; the additional condition $N(X) = 1$ serves to single out the component of the identity in the group ${}^t\bar{X} \cdot S \cdot X = S$; therefore the same holds for the type O2. On the other hand, in the case L2, the mapping $X \rightarrow N(X)$ maps the group ${}^t\bar{X} \cdot S \cdot X = S$ onto the subgroup U of $R_{k'/k}(G_m)$

determined by $\bar{z}z = 1$, and G is the kernel of that homomorphism.

Now we put $F(x) = {}^t\bar{x}Sx$, with $x \in D^m$; as the characteristic is not 2, S is uniquely determined by the values of F on D_k^m, except in the case S1, when F = 0; F is called "quadratic" in the case O1, "hermitian" in cases L2, S2, "antihermitian" in the case O2; it is a scalar polynomial function in cases L2(a), O1, S2; in the case O2, it takes its values in the (3-dimensional) subspace D^- of odd elements of the quaternion algebra D (the elements such that $\bar{x} = -x$); in the case L2(b), it takes its values in the n^2-dimensional space D^+ of even elements of D (the elements of D such that $\bar{x} = x$).

Writing T for any one of the symbols L2(a), L2(b), O1, O2, S1, S2, we say that the group G defined by (1) is of type T_m. With this notation, we have the lemma:

<u>Lemma</u> 4.1.1. <u>Let</u> G <u>be the group of type</u> T_m <u>defined by</u> (1); <u>let</u> ξ <u>be a vector in</u> D_k^m, <u>other than</u> 0; <u>let</u> g <u>be the subgroup of</u> G <u>leaving</u> ξ <u>fixed.</u> <u>Then</u>: (a) <u>if</u> m = 1, $g = \{e\}$; (b) <u>if</u> $m \geq 2$ <u>and</u> $F(\xi) \neq 0$, g <u>is isomorphic to a group of type</u> T_{m-1}; (c) <u>if</u> m = 2 <u>and</u> $F(\xi) = 0$, g <u>is</u> $\{e\}$ <u>or isomorphic to a group</u> $(G_a)^r$; (d) <u>if</u> $m \geq 3$ <u>and</u> $F(\xi) = 0$, g <u>is the semi-direct product of a group</u> g' <u>and of a group</u> G'' <u>of type</u> T_{m-2}, <u>where</u> g' <u>is either isomorphic to a group</u> $(G_a)^r$ <u>or to the semidirect product of two groups of that type.</u>

For m = 1, we must have $F(\xi) \neq 0$; and (a) is trivial. In case (b), g is isomorphic to the group of type T, acting on the vector-space ${}^t\bar{\xi}Sy = 0$ and leaving invariant the form induced on it by F. In the cases (c), (d), one can, by a suitable change of coordinates over k, transform S and ξ into matrices

$$S = \begin{pmatrix} 0 & 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & S'' \end{pmatrix} , \qquad \xi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} .$$

Then g consists of the matrices

$$\begin{pmatrix} 1 & x & -v \\ 0 & 1 & 0 \\ 0 & u & X'' \end{pmatrix}$$

with

$$\bar{x} \pm x + {}^t\bar{u}S''u = 0, \ v = \pm {}^t\bar{u}S''X'', \ {}^t\bar{X}''S''X'' = S'' , \quad N(X'') = 1 .$$

Call G'' the group consisting of the X'', which is of type T_{m-2}; call g' the subgroup of g for which $X'' = 1_{m-2}$; call g'' the subgroup of g' for which $u = 0$; if $m = 2$, $g = g' = g''$. Then g is the semidirect product of g' and $G'' = g/g'$; g' is the semidirect product of g'' and g'/g''; $g'' = \{e\}$ in the case Ol; otherwise g'' is isomorphic to G_a, or to D^-, as the case may be; g'/g'' is isomorphic to D^{m-2}. This proves the lemma.

Now we apply the lemma, and Witt's theorem, to the consideration of "spheres"; by <u>the sphere of radius</u> ρ, we understand the variety $\Sigma = \Sigma(\rho)$ defined in D^m by the equation $F(x) = \rho$ (in the cases Ol, L2(a), S2, this is actually, in the classical terminology, the sphere of radius $\sqrt{\rho}$). To begin with, Witt's theorem says that, if ξ, ξ' are two vectors, other than 0, in Σ_k, there is $M \in R_k^*$ such that ${}^t\bar{M}SM = S$ and $\xi' = M\xi$; for our purposes, however, we need more. Let H, as in 3.4, be the orbit of the vector e (and therefore also of every vector ξ, other than 0, in D_k^m) under R^*;

put $\Sigma^* = \Sigma \cap H$; this is a Zariski-open subset of Σ. With these notations:

Lemma 4.1.2. *If* $\rho \in k$ (*case* L2(a), O1, S2), $\rho \in D_k^+$ (*case* L2(b)), $\rho \in D_k^-$ (*case* O2), *and* $\rho \neq 0$, *and* Σ *is the sphere of radius* ρ, *we have* $\Sigma \subset H$, *and consequently* $\Sigma^* = \Sigma$.

Let K be a field containing k; then $D_K = D_k \otimes K$ is either isomorphic to a matrix algebra $M_r(D'')$ over a central division algebra D'' over K (cases O, S, and L2 for $K \not\supset k'$) or to a direct sum $M_r(D'') \oplus M_r(D'')$ of two such algebras (case L2 for $K \supset k'$). In the former case, H is as described in 3.4; call σ an isomorphism of D_K onto $M_r(D'')$; we have to show that, if $x \in D_K^m$ and $F(x) = \rho \neq 0$, the (mr, r)-matrix $\sigma(x)$ is of rank r; this follows at once from the relation

$$\sigma({}^t\bar{x}) \cdot \sigma(S) \cdot \sigma(x) = \sigma(\rho)$$

since ρ is invertible in D_k, so that $\sigma(\rho)$ must be of rank r. Now consider the case L2, with $K \supset k'$; if σ is any isomorphism of D_K onto $M_r(D'') \oplus M_r(D'')$, the involution $x \rightarrow \bar{x}$ of D, transported to the latter algebra by means of σ, must exchange the two components, since it induces the non-trivial automorphism on the center $K \oplus K$; therefore we can choose σ so that this involution is $(Y, Z) \rightarrow ({}^tZ, {}^tY)$. Put $\sigma = (\sigma_1, \sigma_2)$, where σ_1, σ_2 are two homomorphism of D_K onto $M_r(D'')$, and extend σ, σ_1, σ_2 in the obvious manner to D_K^m and $R_K = M_m(D_K)$. Then H_K consists of the elements x of D_K^m such that the two (mr, r)-matrices $\sigma_1(x)$, $\sigma_2(x)$ over D'' are both of rank r; and one sees, just as above, that $F(x) = \rho \neq 0$ implies $x \in H_K$.

We can now generalize Witt's theorem as follows:

Lemma 4.1.3. (i) Let a, a' be in Σ_K^*; then there is $X \in R_K^*$ such that $a' = Xa$, ${}^t\bar{X} \cdot S \cdot X = S$. (ii) Let G be of any type except L2(b); also, assume, if G is of type O1 or L2(a), that $m \geq 3$, or that $\rho \neq 0$, $m \geq 2$; and let a, a' be in Σ_K^*; then there is $X \in G_K$ such that $a' = Xa$.

In the cases O1, S1, (i) is nothing else than Witt's theorem; in the case S1, (ii) is the same as (i); in case O1, take $X_1 \in R_K^*$ such that $a' = X_1 a$, ${}^tX_1 S X_1 = S$; if $N(X_1) = 1$, take $X = X_1$; if $N(X_1) = -1$, take $X_2 \in R_K^*$ such that $X_2 a = a$, ${}^tX_2 S X_2 = S$, $N(X_2) = -1$ (this can be constructed by reasoning just as in the proof of Lemma 4.1.1), and $X = X_1 X_2$. In the case S2, (ii) is the same as (i); (i) is Witt's theorem if D_K is a quaternion algebra over K; if not, there is an isomorphism σ of D_K onto $M_2(K)$; $\sigma(a)$ is a $(2m, 2)$-matrix, of rank 2 since $a \in H_K$, which can be written as $(b_1\ b_2)$, where b_1, b_2 are in K^{2m}, and we can similarly write $\sigma(a') = (b_1'\ b_2')$. The extension of σ to $M_m(D_K)$ maps $M_m(D_K)$ onto $M_{2m}(K)$, and G_K onto the symplectic group of an alternating bilinear form $\Phi(y_1, y_2)$ on $K^{2m} \times K^{2m}$; and it is easily seen that the assumption $F(a) = F(a')$ is then equivalent to $\Phi(b_1, b_2) = \Phi(b_1', b_2')$; our assumption is then a special case of Witt's theorem, applied to Φ and to the subspaces of K^{2m} respectively spanned by b_1, b_2 and by b_1', b_2'. The proof of (i) for the case O2 is quite similar; in order to deduce (ii) from this, we observe, if D_K is a quaternion algebra over K, that $N(X) = 1$ for all $X \in G_K$ (in fact, this is so for $m = 1$, as one finds by direct calculation; in the general case, it follows then immediately from the fact that G_K is generated by the "quasi-symmetries", i.e. by the elements which leave invariant a nonisotropic hyperplane; cf. Dieudonné, Géom. des gr. class. p. 24 and 41); if there is an isomorphism σ of D_K onto $M_2(K)$, this transforms G_K into a group of type $(O1)_{2m}$, and one has merely to apply what has been said above for the type O1, observing that the exceptional cases for that type cannot occur here since $2m \geq 2$, and since ρ cannot be 0 if $m = 1$ (for all types, Σ^* is empty if $m = 1$, $\rho = 0$). In the case L2(a), (i) is Witt's theorem if $K \not\supset k'$, and (ii) can be deduced from (i) just as in the case O1; if $K \supset k'$, we have $D_K = D_k \otimes K = k' \otimes K \cong K \oplus K$; let σ be an isomorphism of D_K onto $K \oplus K$; this transforms the involution $x \to \bar{x}$ of D_K into $(y, z) \to (z, y)$; if it maps S into (A, B), where A, B are in $M_m(K)$, we have $B = {}^tA$; it if maps a, a' into (b, c), (b', c'), b, c, b', c' must be vectors, other than 0, in K^m; and the assumption $F(a) = F(a')$ becomes ${}^tc \cdot A \cdot b = {}^tc' \cdot A \cdot b'$; the statement (i) amounts to saying that there is Y in $M_m(K)^*$ such that $b' = Yb$ and ${}^tA \cdot c' = {}^tY^{-1} \cdot {}^tA \cdot c$; (ii) says that we can take Y such that $\det(Y) = 1$; these statements are easily verified. It seems to be an open question whether (ii) is still valid for the type L2(b).

Combining the above lemmas with Theorem 2.4.2, we have now:

Theorem 4.1.1. *Let* ξ *be a vector other than* 0 *in* D_k^m; *let* g *be the subgroup of* G *which leaves* ξ *fixed; put* $\rho = F(\xi)$, *and let* Σ *be the sphere of radius* ρ *in* D^m. *Then (except for the case* $m = 2$, $\rho = 0$ *and the case* $m = 1$ *of types* O1, L2(a), *and possibly for the type* L2(b)), *we have* $\Sigma^* = G/g$, $\Sigma_K^* = G_K/g_K$ *for every field* $K \supset k$, *and* $\Sigma_A^* = G_A/g_A$; *the same is true in the case* L2(b) *if* G *is replaced by the group* $G^* = \{X \in R^* \mid {}^t\bar{X}SX = S\}$, *and* g *by the subgroup of that group leaving* ξ *fixed.*

We now consider problems (I), (II) for the groups in question. All available evidence goes to show that (II) is to be answered affirmatively (i.e., that G_A/G_k has finite measure) for all semisimple groups, and that the answer to (I) is given by "Godement's conjecture": if G is semisimple, G_A/G_k is compact if and only if G_k contains no unipotent element.

(Added in December 1960: these statements have now been proved by Borel and Harishchandra for all semisimple groups over number-fields.)

In the direction of Godement's conjecture, we prove:

Lemma 4.1.4. Let Γ be any locally compact group; let γ be a discrete subgroup of Γ, such that Γ/γ is compact. Then the orbit of any $\xi \in \gamma$ under the group of inner automorphisms of Γ is closed.

In fact, this orbit is the union of the compact sets $\{k^{-1}\xi' k \mid k \in K\}$, where K is a compact set such that $\Gamma = \gamma K$, and where one takes for ξ' all the distinct transforms of ξ under inner automorphisms of γ; and the family consisting of these compact sets is locally finite (i.e., only finitely many of them can meet a given compact subset of Γ). (This lemma is, in substance, due to Selberg; cf. Bombay Colloquium on Function-Theory, 1960, p. 148-149.)

In view of this, the necessity of the condition in Godement's criterion will be proved if one shows the following: if G is a semisimple algebraic group, and ξ is a unipotent element of G_k, the closure of the orbit of ξ under the inner automorphisms of G_A contains the neutral element of G. We do not discuss this in general. For the classical groups, it can be verified easily:

Type L1: Consider the group $R^{(1)}$, $R = M_m(D)$, $m \geq 2$; as $M_2(D)^{(1)}$

is a subgroup of $M_m(D)^{(1)}$ for $m > 2$, it is enough to discuss the case $m = 2$. Consider the orbit of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ under the inner automorphisms induced by elements $\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}$, $x \in I_k$, for a sequence of values of x tending to 0 in A_k.

<u>Type</u> $(O1)_m$, $m \geq 3$, F <u>not of index</u> 0: Take coordinates so that S is as in Lemma 4.1.1; consider the elements $X^{-1}AX$, with

$$A = \begin{pmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & a & 1_{m-2} \end{pmatrix}, \qquad X = \begin{pmatrix} x^{-1} & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1_{m-2} \end{pmatrix}$$

where $a \in k^{m-2}$, $a \neq 0$, $b = -\frac{1}{2}\,{}^t aS''a$, $c = -\,{}^t aS''$, $x \in I_k$; and, as above, take a sequence of values of x tending to 0 in A_k.

<u>Other types</u>, F <u>not of index</u> 0: As the matrix S must be invertible, the latter condition implies $m \geq 2$; take coordinates so that S is as in Lemma 4.1.1; then, as for the type L1, it is enough to discuss the case $m = 2$. Consider the unipotent element $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ in G_k, where $a \pm \bar{a} = 0$, $a \in D_k$, $a \neq 0$ (notations of Lemma 4.1.1; take $a \in k'$, $a + \bar{a} = 0$, $a \neq 0$, in the cases L2, S2, and $a \in k$, $a \neq 0$, in the cases O2, S1); take its transforms under the inner automorphisms induced by $\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}$, with $x \in I_k$, x tending to 0 in A_k.

We now seek to prove that G_A/G_k is compact, for the types other than L1, whenever F is of index 0. This will be done for all types except L2(b); for the type L2(b), we only do it for the group G^* defined in Theorem 4.1.1.

We begin by considerations which are valid, whether F is of index

0 or not. For the time being, however, we exclude the type L2(b), and also, for the types O1, L2(a), the cases $m = 2$, F not of index 0, and $m = 1$. Let C be a compact subset of D_A^m, of measure > 1; as in the proof of Lemma 3.1.1, put $C' = C + (-C)$. For $X \in G_A$, the automorphism $x \to X^{-1}x$ of D_A^m has the module 1; therefore it maps C onto a set $X^{-1}C$ of measure > 1, which cannot be mapped in a one-to-one manner onto its image in D_A^m/D_k^m; this means that $X^{-1}C' \cap D_k^m$ must contain an element $\xi \neq 0$, so that $\xi = X^{-1}c$ with $c \in C'$. Then $F(c) = F(\xi)$, and, if we put $\rho = F(\xi)$, ρ is in $F(C') \cap D_k$, which is a finite set since $F(C')$ is compact and D_k discrete in D_A; write that set as $\{\rho_o = 0, \rho_1, \ldots, \rho_h\}$. For each i, let Σ_i be the sphere of radius ρ_i (the variety $F(x) = \rho_i$); choose a vector ξ_i in $(\Sigma_i^*)_k$, if there is one (i.e. a vector $\neq 0$ in D_k^m, such that $F(\xi_i) = \rho_i$; for $i = 0$, there is such a vector if and only if F is not of index 0); let g_i be the subgroup of G leaving ξ_i fixed, and call ϕ_i the mapping $X \to X\xi_i$ of G into Σ_i^*; by Theorem 4.1.1, we can identify Σ_i^* with G/g_i, and then ϕ_i becomes the canonical mapping of G onto G/g_i. Now put $E_i = (\Sigma_i^*)_A \cap C'$, and $B_i = \phi_i^{-1}(E_i)$. Our proof shows that, if X is any element of G_A, there is an i, and an element c of C', such that $\xi = X^{-1}c$ is in $(\Sigma_i^*)_k$, i.e., by Lemma 4.1.3, of the form $M^{-1}\xi_i$ with $M \in G_k$; then $X\xi = XM^{-1}\xi_i$ is in E_i, so that $XM^{-1} \in B_i$, $X \in B_iG_k$. We formulate this as a lemma:

<u>Lemma</u> 4.1.5. <u>There are finitely many fectors</u> $\xi_i \neq 0$ <u>in</u> D_k^m, <u>and a compact subset</u> C' <u>of</u> D_A^m, <u>with the following properties</u>: (a) <u>the</u> $\rho_i = F(\xi_i)$ <u>are distinct elements of</u> D_k; (b) <u>let</u> Σ_i <u>be the sphere of radius</u> ρ_i, g_i <u>the subgroup of</u> G <u>leaving</u> ξ_i <u>fixed</u>, ϕ_i <u>the mapping</u> $X \to X\xi_i$ <u>of</u> G <u>into</u> Σ_i^*;

put $E_i = (\Sigma_i^*)_A \cap C'$, $B_i = \phi_i^{-1}(E_i)$; then $G_A = \bigcup_i B_i G_k$.

As indicated above, we choose notations so that $\rho_i \neq 0$ for $i \neq 0$; in particular, if F is of index 0, we have $\rho_i \neq 0$, i.e., in this notation, $i \neq 0$, for all i. By Lemma 4.1.2, for $i \neq 0$, we have $\Sigma_i^* = \Sigma_i$; as $(\Sigma_i)_A$ is a closed subset of D_A^m, $(\Sigma_i)_A \cap C'$, is a compact subset of $(\Sigma_i)_A$. Therefore, for every $i \neq 0$, there is a compact subset K_i of G_A such that $\phi_i(K_i) \supset E_i$; then $B_i \subset K_i(g_i)_A$.

Now assume that F is of index 0, and that $(g_i)_A/(g_i)_k$ is compact for every i, so that there is, for every i, a compact subset K_i' of $(g_i)_A$ such that $(g_i)_A = K_i'(g_i)_k$; then $B_i \subset K_i K_i'(g_i)_k$, and, in view of Lemma 4.1.5, we have $G_A = (\bigcup_i K_i K_i')G_k$, so that G_A/G_k is compact. But for each i, by Lemma 4.1.1, g_i is the group, of the same type as G but with $m-1$ substituted for m, acting on the space ${}^t\bar{\xi}_i Sx = 0$ and leaving invariant the form induced on that space by F; obviously, the latter form is of index 0 if F is of index 0. Now, by induction on m, we can prove:

Theorem 4.1.2. For all types other than L2(b), the group G determined by (1) is such that G_A/G_k is compact whenever ${}^t\bar{x}Sx \neq 0$ for $x \neq 0$ in D_k^m.

The theorem is trivially true for $m = 1$ in the cases L2(a), O1 (for then G is reduced to the neutral element), and vacuously true for $m = 0$ in the cases O2, S2; the induction proof is valid for $m \geq 2$ for L2(a), O1, and for $m \geq 1$ for O2, S2 (one could also deduce the cases $m = 2$ of L2(a), O1, and $m = 1$ of O2, S2, directly from Lemma 3.1.1).

In the case L2(b), we consider, instead of G, the group G^* defined in Theorem 4.1.1. One proves then, exactly in the same manner, that G_A^*/G_k^* is compact if F is of index 0. We observe that G^* is isogenous to $G \times U$, where U is, as before, the commutative subgroup of $R_{k'/k}(G_m)$ determined by $z\bar{z} = 1$. We shall not proceed further with the investigation of the type L2(b), which, in all respects, is the most difficult of all.

Now we apply Lemma 4.1.5 to proving that, for all types except possibly L2(b) and O2, G_A/G_k is of finite measure. Apply Lemma 2.4.1 to G_A, $(g_i)_A$, $(g_i)_k$, and to the characteristic function $f_i(w)$ of E_i on $(\Sigma_i^*)_A$; this shows that the image of B_i in $G_A/(g_i)_k$ is of finite measure if and only if $(g_i)_A/(g_i)_k$ and E_i are so (of course we are using invariant measures on G_A, $(g_i)_A$ and $(\Sigma_i^*)_A$). For $i \neq 0$, we have seen that E_i is compact, hence of finite measure. Proceeding by induction on m, and using Lemma 4.1.1, we may assume that $(g_i)_A/(g_i)_k$ is of finite measure; so the image of B_i in $G_A/(g_i)_k$ is of finite measure; as the obvious mapping from $G_A/(g_i)_k$ onto G_A/G_k is locally a measure-preserving isomorphism, this implies that the image of B_i, which is also the image of B_iG_k, in G_A/G_k, is of finite measure. In view of Lemma 4.1.5, the induction part of our proof will be complete if we show that E_o and $(g_o)_A/(g_o)_k$ are of finite measure. The latter fact, in view of Lemma 4.1.1, is also a consequence of the induction assumption. Thus it only remains to show that $E_o = \Sigma_A^* \cap C'$ is of finite measure (for the invariant measure on Σ_A^*) when C' is compact and Σ is the sphere of radius 0; this will be done in 4.2 for the types L2(a) ($m \geq 3$), O1 ($m \geq 5$), S2 ($m \geq 2$); the case S1 has been treated in 3.5. The case L2(a), $m = 1$, is trivial, and the case L2(a), $m = 2$, has been treated in 3.7; the cases O1, $m = 3$ and 4, have been treated in 3.7; the case S2, $m = 1$, is included in Theorem 3.3.1; therefore this will prove:

Theorem 4.1.3. If G is defined by (1), G_A/G_k is of finite measure for the types L2(a), O1, S1, S2, except only for the case O1, $m = 2$, F not of index 0.

The same would be proved for the type O2 if we could show, also in that case, that E_o is of finite measure. As to the type L2(b), our method could be applied to the group G^*, and would show that G_A^*/G_k^* is of finite measure, again under the assumption that E_o is so. These cases will not be considered any further.

4.2. End of proof of Theorem 4.1.3 (types O1, L2(a), S2).

In the remainder of this chapter, we shall consider only the cases O1 (quadratic case), L2(a) (hermitian case) and S2 (quaternionic case); we put $\delta = [D_k:k]$; in the quadratic case, $D_k = k$ and $\delta = 1$; in the hermitian case, $D_k = k'$ and $\delta = 2$; in the quaternionic case, D_k is a field of quaternions with the center k, and $\delta = 4$. In all cases, D_k^m is a vector-space of dimension δm over k, $F(x) = {}^t\bar{x}Sx$ is a k-valued quadratic form in that space, D^m is an affine space of dimension δm in the sense of algebraic geometry, and the sphere of radius ρ is the hypersurface defined by $F(x) = \rho$.

In this section, we assume that F is not of index 0, and Σ will denote the sphere of radius 0, i.e. the hypersurface $F(x) = 0$ in D^m; as before, we put $\Sigma^* = \Sigma \cap H$, where H is the orbit of the vector $e = (1, 0, \ldots, 0)$ in D^m under the group $R^* = M_m(D)^*$.

Lemma 4.2.1. For the variety Σ^*, (1) is a set of convergence factors, provided $\delta m > 4$.

This is done by computing the number of points of Σ^* modulo p for almost all p (the formulas for this are well known) and applying Theorem 2.2.5. We exclude all p for which S is not in $M_m(D_{o_p})$, all p which divide 2N(S), and all p which are ramified in k' (resp. in D_k) in the hermitian (resp. quaternionic) case. Then, in the quadratic case, the number of solutions, other than 0, of F(x) = 0 in the field F_q with q = N(p) elements is $q^{m-1} - 1$ if $m \equiv 1$ mod. 2, and $(q^{m'} - \epsilon)(q^{m'-1} + \epsilon)$ with $m' = m/2$ and $\epsilon = \pm 1$ if $m \equiv 0$ mod. 2 ($\epsilon = +1$ or -1 according as $(-1)^{m'}$ det (S) is or is not a square in F_q). In view of Theorem 2.2.5, this proves the lemma in that case. In the hermitian case, consider first the case when p does not split in k', i.e. when it can be extended in only one way to k', so that k'_p is a quadratic extension of k_p, and o'_p/p is a quadratic extension of $F_q = o_p/p$; then F(x) determines a quadratic form in $(o'_p/p)^m$ considered as a vector-space of dimension 2m over F_q, so that the number of solutions, other than 0, of F(x) = 0 modulo p is given by $(q^m - \epsilon)(q^{m-1} + \epsilon)$ with a suitable $\epsilon = \pm 1$. If p "splits" in k', i.e. if it can be extended to two distinct valuations p', p'' of k', then, reasoning as in the latter part of the proof of Lemma 4.1.3, we see that the number of points of $\Sigma^*_{o_p}$ modulo p is the number of pairs of vectors x, y in F_q^m, other than 0, satisfying a relation ${}^t yS'x = 0$, where S' is an invertible matrix in $M_m(F_q)$; this is equal to $(q^m - 1)(q^{m-1} - 1)$. The conclusion is the same as before. In the quaternionic case, reasoning as in the first part of the proof of Lemma 4.1.3, we see that the number of points of $\Sigma^*_{o_p}$ is the number of (2m, 2)-matrices $(x_1\ x_2)$ of rank 2 over F_q such that $\Phi(x_1, x_2) = 0$, where Φ is a non-degenerate alternating bilinear form on $F_q^{2m} \times F_q^{2m}$; this has the value $(q^{2m}-1)(q^{2m-1}-q)$. The conclusion is again the same.

Now let (ϵ_ν), for $0 \leq \nu \leq \delta - 1$, be a basis of D_k over k, with $\epsilon_0 = 1$; call E_ν the mapping $x \rightarrow x\epsilon_\nu$ of D^m, considered as an automorphism of the (δm)-dimensional affine space. Let L be the group of all such automorphisms, which is isomorphic, over k, to $M^*_{\delta m}$, the full linear group in δm variables; then $R^* = M_m(D)^*$ is the subgroup of the elements of L which commute with E_ν for $1 \leq \nu \leq \delta - 1$ (in particular, $R^* = L$ for $\delta = 1$). Call H_o the orbit of the vector $e = (1, 0, \ldots, 0)$ of D^m under the group L; this is a Zariski-open subset of D^m, and the orbit H of e under R^* is a Zariski-open subset of H_o. Let G_o be the group of type Ol, consisting of the elements of L of determinant 1, leaving the quadratic form F invariant; the group G defined by 4.1 (1) is then a subgroup of G_o. As before, put $\Sigma^* = \Sigma \cap H$; and put $\Sigma^*_o = \Sigma \cap H_o$; then Σ^* is a Zariski-open subset of Σ^*_o. Assume, from now on, that $\delta m > 4$; in view of Lemma 4.2.1, and of Lemma 3.4.1, the Tamagawa measure on Σ^*_o, derived from any gauge-form dw, induces on Σ^* the Tamagawa measure derived from dw on Σ^*, and $\Sigma^*_o - \Sigma^*$ is of measure 0. Now take any $\xi \in \Sigma^*_k$ (F is not of index 0), and call g, g_o the subgroups of G and G_o leaving ξ fixed; by Theorem 4.1.1, we can identify Σ^*_o with G_o/g_o, and Σ^* with G/g. Choose the gauge-form dw on Σ^*_o so that it matches algebraically with invariant gauge-forms on G_o and g_o (in the sense of 2.4); then the Tamagawa measure dw_A derived from it is invariant under $(G_o)_A$, hence also under G_A; therefore it induces on Σ^*_A the (uniquely determined) Tamagawa measure on Σ^*_A which is invariant under G_A. In order to complete the proof of Theorem 4.1.3, we have to show that $\Sigma_A^* \cap C'$ is of finite measure, for that measure, whenever C' is compact in D_A^m; it is now clear that it is enough to prove this for Σ_o^* instead of Σ^*; in other words, it is enough to consider the quadratic case (with $m > 4$).

Actually, we shall prove a stronger result (needed in the next sections):

Theorem 4.2.1. Assume that $\delta m > 4$, and call dw_A the Tamagawa measure on Σ_A^*, invariant under G_A. Then $\int_{\Sigma_A^*} \Phi(w)dw_A < +\infty$ for every function Φ in D_A^m, defined for $x = (x_v) \in D_A^m$ by $\Phi(x) = \prod_v \phi_v(x_v)$, where the ϕ_v are as follows: (i) for almost all p, ϕ_p is the characteristic function of $D_{o_p}^m$; (ii) for all p, ϕ_p is continuous with compact support; (iii) for any infinite place v_λ, $|\phi_\lambda(x)| \leq C_\lambda \exp(-Q_\lambda(x))$, where C_λ is a constant and Q_λ is a positive-definite quadratic form on $D_{k_\lambda}^m$ considered as a vector-space over $\mathbb{R}$.

We first show how this implies the finiteness of the measure of $\Sigma_A^* \cap C'$ for compact C'. Take for ϕ_p^o the characteristic function of $D_{o_p}^m$ for every p; take for ϕ_λ^o the characteristic function of $Q_\lambda(x) < 1$, where Q_λ is as in (iii); then $\Phi_o(x) = \prod_v \phi_v(x_v)$ is the characteristic function of an open neighborhood of 0 in D_A^m, so that C' can be covered by finitely many translates of this neighborhood; therefore the characteristic function of C' is majorized by a finite sum $\Sigma_i \Phi_o(x+a_i)$, with $a_i \in D_A^m$; as every term of that sum satisfies the conditions for Φ in Theorem 4.2.1, this proves our assertion.

Now we prove our theorem. It is clearly enough to consider the quadratic case (with $m > 4$); then $\Sigma^* = \Sigma - \{0\}$. Proceeding as in similar calculations in Chapter III, we see that it is enough to show that all the factors in the product

$$\prod_v \int_{\Sigma^*_{k_v}} \phi_v(w)(dw)_v$$

are finite and that the product is absolutely convergent. As to the first point, let Q be the quadric defined in the projective space of dimension $m - 1$ by the homogeneous equation $F(x) = 0$; let f be the obvious mapping of Σ^* onto Q; for each v, this determines a mapping of $\Sigma^*_{k_v}$ onto the compact space Q_{k_v}; and $\Sigma^*_{k_v}$ is fibered over Q_{k_v} by that mapping, with the fibre k^*_v. As Q_{k_v} is compact, the finiteness of our integral will be proved provided we show that each point of Q_{k_v} has a neighborhood Ω such that the same integral, taken over $f^{-1}(\Omega)$, is finite. Assume (as we may, since the characteristic is not 2) that coordinates have been taken so that $F(x) = \Sigma_i a_i x_i^2$; if Ω is a suitable neighborhood of a point of Q where $w_1 \neq 0$, we can write, in $f^{-1}(\Omega)$, $w_1 = t$, $w_i/w_1 = u_i$ for $2 \leq i \leq m$, so that $\Sigma_{i \neq 1} a_i u_i^2 = - a_1$. It is easily seen that the invariant gauge-form on Σ^* can be taken to be $dw = dw_2 \ldots dw_m / a_1 w_1$; this is equal to $t^{m-3}\omega(u)dt$, where $\omega(u)$ is a gauge-form on Ω. This gives:

$$\int_{f^{-1}(\Omega)} \phi_v(w) \cdot (dw)_v = \int_{\Omega} \omega_v \int_{k^*_v} \phi_v(w) \cdot |t|_v^{m-3} (dt)_v \ .$$

Because of the conditions on ϕ_v, the last integral is absolutely convergent and is a continuous function of u in Ω, provided $m \geq 3$.

Now take any p such that all the a_i and 2 are units in $\underline{o}_p$ and that ϕ_p is the characteristic function of $(\underline{o}_p)^m$; let π be a generator of the maximal ideal in $\underline{o}_p$. For every point w of $\Sigma^*_{k_p} \cap (\underline{o}_p)^m$, we can write $w = \pi^\nu w'$, where $\nu \geq 0$, $w' \in (\underline{o}_p)^m$, $w' \not\equiv 0 \pmod{p}$, $F(w') = 0$; the latter conditions amount to $w' \in \Sigma^*_{\underline{o}_p}$, since $\Sigma^* = \Sigma \cap H$ and we have seen in 3.4 that $H_{\underline{o}_p}$ consists of the points $x \in (\underline{o}_p)^m$ such that $x \not\equiv 0 \pmod{p}$. As the formula given above for dw shows that the mapping $w \to \pi^\nu w$ changes $(dw)_p$ into $q^{\nu(2-m)}(dw)_p$, with $q = N(p)$, this gives

$$\int_{\Sigma^*_{k_p}} \phi_p(w)\cdot(dw)_p = \sum_{\nu=0}^{\infty} q^{\nu(2-m)}\mu_p = (1-q^{2-m})^{-1}\mu_p$$

where μ_p is the $(dw)_p$-measure of $\Sigma^*_{\underline{o}_p}$. By Lemma 4.2.1, $\prod_p \mu_p$ is absolutely convergent (for $m > 4$). As the same is true of $\prod_p (1-q^{2-m})$, this completes our proof.

4.3. The local zeta-functions for a quadratic form.

Notations remain the same as above.

Lemma 4.3.1. *Let V be the Zariski-open set defined by $F(x) \neq 0$ in the affine space D^m of dimension $\delta m \geq 4$. Then $(1-q^{-1})$, $q = N(p)$, is a set of convergence factors for V.*

In fact, by the same formula which was used in the proof of Lemma 4.2.1, the number of points of V modulo p, for almost all p, is $q^{\delta m-1}(q-1)$ if δm is odd, and $q^{m'-1}(q^{m'}-\epsilon)(q-1)$ if $\delta m = 2m'$. In view of Theorem 2.2.5, this proves the lemma.

As V is isomorphic to the variety, in the affine space of dimension $\delta m + 1$, with the generic point $(x, 1/F(x))$, V_A is the set of the points $x \in D^m_A$

such that $F(x) \in I_k$.

We now wish to calculate, for almost all p, the following "local zeta-function":

(1) $$Z_p(s) = \int_{V_{k_p}} |F(x)|_p^s \, \phi_p(x) d'_p x \quad ;$$

here $\phi_p(x)$ is the characteristic function of $D^m_{\underline{o}_p}$; $d'_p x$ is the local measure for V_{k_p}, derived from the gauge-form $dx_1 \ldots dx_{\delta m}$ for V (the x_i are the coordinates, for any basis of D^m_k over k) and from the convergence factor $1 - q^{-1}$. Clearly this can also be written:

(2) $$Z_p(s) = (1-q^{-1})^{-1} \int_{(\underline{o}_p)^{\delta m}} |F(x)|_p^s \, (dx)_p$$

where $(dx)_p$ is the additive measure in $(\underline{o}_p)^{\delta m}$, normalized so that the group has the measure 1; we may assume that the quadratic form $F(x)$ has been written as $F(x) = \Sigma_i a_i x_i^2$. Then:

<u>Lemma</u> 4.3.2. <u>For almost all</u> p, <u>we have</u>

$$Z_p(s) = (1-q^{-s-m})(1-q^{-s-1})^{-1}(1-q^{-2s-m})^{-1} \quad \underline{\text{for}} \ \delta = 1, \ m \ \underline{\text{odd}};$$

$$Z_p(s) = (1-\epsilon q^{-m'})(1-q^{-s-1})^{-1}(1-\epsilon q^{-s-m'})^{-1} \quad \underline{\text{for}} \ \delta m = 2m' \ ,$$

<u>where in the latter case</u> ϵ <u>is</u> +1 <u>or</u> -1 <u>according as</u> $(-1)^{m'} a_1 a_2 \ldots a_{\delta m}$ <u>is or is not a quadratic residue in</u> $\underline{o}_p$ <u>modulo</u> p.

In fact, it will be shown that this is so whenever all a_i and 2 are units in $\underline{o}_p$. Let π be a prime element in $\underline{o}_p$; for $x \in (\underline{o}_p)^{\delta m}$, we can write $x = \pi^\nu x'$ with $\nu \geq 0$, $x' \in (\underline{o}_p)^{\delta m}$, $x' \not\equiv 0$ mod. p, and get

$$Z_p(s) = (1-q^{-1})^{-1}(1-q^{-2s-\delta m})^{-1}\, Z'_p(s) \; ,$$

with $Z'_p(s) = \int |F(x)|_p^s \cdot (dx)_p$, the integral being taken over $x \in (o_p)^{\delta m}$, $x \not\equiv 0$ mod. p. Now let N_ν, for $\nu \geq 1$, be the number of those solutions in $(o_p)^{\delta m}$ mod. p^ν for the congruence $F(x) \not\equiv 0$ mod. p^ν which are $\not\equiv 0$ mod. p; we know that N_1 is $q^{\delta m-1} - 1$ if δm is odd, and $(q^{m'} - \epsilon)(q^{m'-1} + \epsilon)$ if $\delta m = 2m'$; and it is easily seen that $N_{\nu+1} = q^{\delta m-1} N_\nu$ for $\nu \geq 1$. In the set $x \in (o_p)^{\delta m}$, $x \not\equiv 0$ mod. p, the measure of the subset where $F(x) \equiv 0$ mod. p^ν, i.e. where $|F(x)|_p \leq q^{-\nu}$, is equal to $q^{-\delta m\nu} N_\nu$ for $\nu \geq 1$; and the measure of the subset where $F(x) \not\equiv 0$ mod. p, i.e. where $|F(x)|_p = 1$, is $q^{-\delta m}(q^{\delta m}-1-N_1)$. This gives:

$$Z'_p(s) = q^{-\delta m}(q^{\delta m}-1-N_1) + \sum_{\nu=1}^{\infty} q^{-\nu s}(q^{-\delta m\nu} N_\nu - q^{-\delta m(\nu+1)} N_{\nu+1}) \; ,$$

A trivial calculation gives the result in the lemma.

As to the value of ϵ, we remark the following:

(a) <u>Quadratic case</u> ($\delta = 1$), m <u>even</u>: then $\epsilon = (\Delta/p)$ (quadratic residue character of Δ mod. p), where $\Delta = (-1)^{m/2} \det(S)$ is the discriminant of F;

(b) <u>Hermitian case</u> ($\delta = 2$): then, write $F = \Sigma_1^m \bar{x}_i a_i x_i$, $k' = k(\alpha)$ with $\alpha^2 = a \in k$, $x_i = y_i + \alpha z_i$, $\bar{x}_i = y_i - \alpha z_i$; then, in terms of the δm variables y_i, z_i, F has the coefficients a_i, $-a_i a$; therefore $\epsilon = (a^m/p)$, i.e. $\epsilon = 1$ for m <u>even</u>, and $\epsilon = (a/p)$ for m <u>odd</u>.

(c) <u>Quaternionic case</u> ($\delta = 4$): we can take a base $1, i, j, ij$ for D_k over k, with $i^2 = a \in k$, $j^2 = b \in k$, $ij = -ji$; if we put $F = \Sigma_i \bar{x}_i a_i x_i$, $x_i = t_i + iu_i + jv_i + ijw_i$, then, in terms of the δm variables t_i, u_i, v_i, w_i, F has the coefficients a_i, $-a_i a$, $-a_i b$, $a_i ab$, so that $\epsilon = 1$.

4, 4. The Tamagawa number (hermitian and quaternionic cases).

From now on (in this section) we assume that $\delta = 2$ (hermitian case) or $\delta = 4$ (quaternionic case); we use "resp." to refer to these two cases (in that order). In both cases, we shall denote by $z \rightarrow \nu(z) = \bar{z}z$ the norm-mapping of D^* into G_m; its kernel is U (in the notation of 3, 7) resp. $D^{(1)}$. In both cases, ν maps D_A^* onto an open subgroup of I_k. In the hermitian case, by class-field theory, $\nu(D_A^*)\cdot k^*$ is an open subgroup of I_k of index 2; in the quaternionic case (cf. Lemma 3. 3. 2), $\nu(D_A^*)$ contains all elements of I_k whose components at the infinite places of k are all > 0, so that $\nu(D_A^*)\cdot k^* = I_k$. In both cases, we define a character λ of I_k by putting $\lambda = 1$ on $\nu(D_A^*)\cdot k^*$ and $\lambda = -1$ on the complement of that group in I_k. In the hermitian case, λ is the character of I_k of order 2 belonging to the quadratic extension k'/k in the sense of class-field theory; in the quaternionic case, λ is the trivial character of I_k. By $|\;|$, we always denote the idele-module taken in I_k.

In the quaternionic case, we shall construct Fourier transforms of functions in D_A^m by means of the character $\chi_D({}^t\bar{x}Sy)$, where χ_D is the character of D_A introduced in 3.1. In the hermitian case, we have $D_A = A_{k'}$, and we do the same by means of $\chi'({}^t\bar{x}Sy)$, where χ' is the character of $A_{k'}$ defined by Theorem 2.1.1; in both cases, we simplify notations by writing χ instead of χ' resp. χ_D. If $\Psi(y)$ is the Fourier transform of $\Phi(x)$ in D_A^m, defined by

$$\Psi(y) = \int_{D_A^m} \Phi(x)\chi({}^t\bar{x}Sy)dx$$

(dx = Tamagawa measure in D_A^m), and if $X \in M_m(D_A)$ is such that ${}^t\bar{X}SX = S$, then the Fourier transform of $\Phi(Xx)$ is $\Psi(X'y)$ with $X' = S^{-1} \cdot {}^t\bar{X}^{-1} \cdot S$, as one sees by replacing x, y by Xx, $X'y$ and observing that, for ${}^t\bar{X}SX = S$, the module of the automorphism $x \to Xx$ of D_A^m is 1. Similarly, if $z \in D_A^*$, the Fourier transform of $\Phi(xz)$ is $|\bar{z}z|^{-\delta m/2}\,\Psi(y\bar{z}^{-1})$.

Our method will depend upon the construction of a zeta-function (whose residue, as usual, gives the Tamagawa number) by means of a function $\Phi(x)$ in D_A^m of which we assume that it is "of standard type" in a sense similar to that defined in 3.1, and also that Theorem 4.2.1 is valid both for Φ and for its Fourier transform Ψ; such functions can be obtained by the procedure described in 3.1 (following the definition of the "standard type").

We are concerned with the group

$$G = \left\{ X \in M_m(D) \,\middle|\, {}^t\bar{X}SX = S;\ N(X) = 1 \right\}$$

(we know that $N(X) = 1$ is a consequence of ${}^t\bar{X}SX = S$ in the quaternionic case, but not in the hermitian case). By 3.7(b) resp. Theorem 3.3.1, we know that $\tau(G) = 1$ for $\delta m = 4$. From now on, we assume $\delta m > 4$.

If V is as in Lemma 4.3.1, we denote by V_A^+ the open subset of V_A given by $\lambda(F(x)) = 1$. With this notation, we introduce the function

$$Z^\Phi(s) = \int_{V_A^+} |F(x)|^s \Phi(x)\,d'x \ , \qquad (1)$$

where $d'x$ is the Tamagawa measure on V_A derived from the gauge-form $dx_1 \ldots dx_{\delta m}$ (if the x_i are the coordinates of x for any choice of a basis of D_k^m over k) and from the convergence factors $(1-q^{-1})$, $q = N(p)$.

We put, for $\nu = 0, 1$ (hermitian case) and for $\nu = 0$ (quaternionic case):

$$I_\nu = \int_{V_A} |F(x)|^s \lambda(F(x))^\nu \Phi(x)d'x \quad ;$$

then we have $Z^\Phi(s) = \frac{1}{2}(I_o+I_1)$ resp. $= I_o$. We give now a multiplicative calculation for I_o, I_1 resp. for I_o; this is similar to the corresponding calculations in Chapter III; I_ν is the product of a "finite part" (i.e., of an integral over a finite product $\prod_{v\in S} V_{k_v}$) and of a product of "local zeta-functions"

$$\int_{V_{k_p}} |F(x)|_p^s \lambda_p(F(x))^\nu \phi_p(x)d'_p x \ .$$

For $\nu = 0$, this is given by Lemma 4.3.2; for $\nu = 1$, λ_p is the local character induced by λ on k_p^* considered (in the obvious manner) as a subgroup of I_k; if p is not ramified in k', this is given by $\lambda_p(t) = \lambda(p)^r$ if $|t|_p = q^r$, with $\lambda(p) = +1$ or -1 according as p "splits" or not in k' (i.e, according as there are two valuations p', p'' of k' extending p, with $k'_{p'} \cong k'_{p''} \cong k_p$, or there is only one such valuation p', with $k'_{p'}$ quadratic and non-ramified over k_p). But then we have

$$|F(x)|_p^s \lambda_p(F(x)) = |F(x)|_p^{s'} \ , \quad s' = s + \frac{\log \lambda(p)}{\log q} \quad ,$$

so that Lemma 4.3.2 gives the value of the local zeta-function also in this case. In the quaternionic case, we find (in view of the remarks following Lemma 4.3.2) that the infinite product for I_o coincides, for almost all p, with that for

$$\zeta_k(s+1)\zeta_k(s+2m)\zeta_k(2m)^{-1} \quad ,$$

which shows that it converges absolutely for $Re(s) > 0$, and that

$$[sI_o]_{s=o} = \rho_k \int_{D_A^m} \Phi(x)dx \quad .$$

Similarly, in the hermitian case, the infinite product for I_o is the same (for almost all p) as that for

$$\zeta_k(s+1)\zeta_k(s+m)\zeta_k(m)^{-1} \qquad \text{(m even)},$$

$$\zeta_k(s+1)L_{k'/k}(s+m)L_{k'/k}(m)^{-1} \qquad \text{(m odd)},$$

where $L_{k'/k} = \zeta_{k'}/\zeta_k$ is the L-function belonging to the quadratic extension k' of k, i.e. to the character λ. The infinite product for I_1 is the same as that for

$$L_{k'/k}(s+1)L_{k'/k}(s+m)\zeta_k(m)^{-1} \qquad \text{(m even)},$$

$$L_{k'/k}(s+1)\zeta_k(s+m)L_{k'/k}(m)^{-1} \qquad \text{(m odd)}.$$

As $m > 2$ in this case, this proves the absolute convergence for $Re(s) > 0$. Furthermore, we find that

$$[sI_o]_{s=o} = \rho_k \int_{D_A^m} \Phi(x)dx \quad , \quad [sI_1]_{s=o} = 0 \quad .$$

Thus, in all cases, the integral for $Z^\Phi(s)$ is absolutely convergent for $Re(s) > 0$, and

$$(2) \qquad [s\cdot Z^\Phi(s)]_{s=o} = \frac{1}{4}\delta\rho_k \int_{D_A^m} \Phi(x)dx \quad .$$

Now we give the additive calculation. Take any $x \in V_A^+$; this means that $x \in D_A^m$ and that $F(x)$ is of the form $\bar{z}\rho z$ with $\rho \in k^*$, $z \in D_A^*$, i.e. that $F(xz^{-1}) = \rho$. By Hasse's fundamental theorem on quadratic forms, the fact that the equation $F(x') = \rho$ has a solution x' in D_A^m implies that it has a solution ξ in D_k^m; then we have $F(x) = F(\xi z)$. On V_k, which is the set of the vectors $\xi \in D_k^m$ such that $F(\xi) \neq 0$, consider the equivalence relation $F(\xi')/F(\xi) = \bar{\zeta}\zeta$ with $\zeta \in D_k^*$; by Lemma 4.1.3, two vectors ξ, ξ' are in the same equivalence class for this if and only if $\xi' = M\xi\zeta$ with $M \in G_k$, $\zeta \in D_k^*$. Let (ξ_μ) be a complete set of representatives for the equivalence classes on V_k under this relation; put $\rho_\mu = F(\xi_\mu)$; for $\mu \neq \nu$, ρ_ν/ρ_μ cannot be of the form $\bar{\zeta}\zeta$ with $\zeta \in D_k^*$; therefore (by the norm theorem for cyclic extensions, applied to k'/k in the hermitian case, and by Eichler's norm theorem in the quaternionic case) ρ_ν/ρ_μ cannot be of the form $\bar{z}z$ with $z \in D_A^*$. From this, one concludes at once that, for every $x \in V_A^+$, there is one and only one μ such that $F(x) = \bar{z}\rho_\mu z$ with $z \in D_A^*$; let Ω_μ be the subset of V_A^+ where this is so for a given μ; this is an open subset of V_A, and we get:

$$(3) \qquad Z^\Phi(s) = \sum_\mu \int_{\Omega_\mu} |F(x)|^s \Phi(s) d'x \quad .$$

Put $\Gamma = G \times D^*$, and make it act on D^m by $((X, z), x) \to Xxz$. By Lemma 4.1.3, Ω_μ is the same as the orbit of ξ_μ under Γ_A; call $\Gamma^{(\mu)}$ the subgroup of Γ leaving ξ_μ fixed; this consists of the (X, z) such that $X\xi_\mu = \xi_\mu z^{-1}$; when that is so, we have $F(\xi_\mu) = F(\xi_\mu z^{-1})$, and therefore $\bar{z}z = 1$. In order to determine the structure of $\Gamma^{(\mu)}$, change coordinates so that S, ξ_μ appear as matrices

$$S = \begin{pmatrix} a & 0 \\ 0 & S' \end{pmatrix} \quad , \quad \xi_\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ;$$

then, for $(X, z) \in \xi_\mu$, X must be a matrix $\begin{pmatrix} z^{-1} & 0 \\ 0 & X' \end{pmatrix}$, with ${}^t\bar{X}'S'X' = S'$, and, in the hermitian case, $\det(X') = z$. This shows that, in the quaternionic case, $\Gamma^{(\mu)}$ is isomorphic to $G' \times D^{(1)}$, where G' is the group of type $(S2)_{m-1}$ belonging to the matrix S'; in the hermitian case, $\Gamma^{(\mu)}$ is isomorphic to the group $G'^* = \left\{ X' \mid {}^t\bar{X}'S'X' = S' \right\}$. Algebraically, the orbit of ξ_μ under Γ is V, so that we can identify V, algebraically (i.e., over the universal domain) with $\Gamma/\Gamma^{(\mu)}$; moreover, it is easily seen that the gauge-form $F(x)^{-\delta m/2}dx$ on V is invariant under Γ, so that we can find invariant gauge-forms on Γ and $\Gamma^{(\mu)}$ which match algebraically with that form on V.

We now proceed with the proof in the quaternionic case, and will then indicate the changes required to adapt it to the hermitian case. By the induction assumption, (1) is a set of convergence factors for G', and $\tau(G') = 1$; therefore (1) is a set of convergence factors for $\Gamma^{(\mu)}$, and $\tau(\Gamma^{(\mu)}) = 1$ (since $\tau(D^{(1)}) = 1$); reasoning as in Theorems 2.4.3 and 2.4.4, we conclude, since the orbit Ω_μ of ξ_μ under Γ_A is open in V_A, and since $(1-q^{-1})$ is a set of convergence factors for V, that $(1-q^{-1})$ is a set of convergence factors for Γ (and therefore, since it is such a set for D^*, that (1) is a set of convergence factors for G), and also that the Tamagawa measure on Γ_A, $\Gamma_A^{(\mu)}$ and V_A, derived from matching gauge-forms and from these convergence factors, match together topologically when one identifies Ω_μ with $\Gamma_A/\Gamma_A^{(\mu)}$. We can now apply Lemma 2.4.2 to Γ_A, $\Gamma_A^{(\mu)}$, Ω_μ and to the discrete groups Γ_k, $\Gamma_k^{(\mu)}$; this gives

$$\int_{\Omega_\mu} |F(x)|^{s+\frac{1}{2}\delta m}\Phi(x)\cdot |F(x)|^{-\frac{1}{2}\delta m} d'x$$

$$= \int_{\Gamma_A/\Gamma_k} \left(\sum_{\xi} |F(X\xi z)|^{s+\frac{1}{2}\delta m}\Phi(X\xi z)\right) d'(X, z) ,$$

where the sum in the right-hand side is extended to all $\xi \in \Gamma_k/\Gamma_k^{(\mu)}$, i.e. to the orbit of ξ_μ under Γ_k; this is nothing else than the equivalence class of ξ_μ in V_k for the equivalence relation defined above; therefore, when we take the sum of both sides over all μ, the sum in the right-hand side gets extended to all $\xi \in V_k$. At the same time, we have $F(X\xi z) = \bar{z}F(\xi)z$, with $F(\xi) \in k^*$, hence $|F(\xi)| = 1$; also, we can write $d'(X, z) = dX \cdot d'z$, where dX, $d'z$ are the Tamagawa measures, on G and on D^*, derived from the convergence factors (1) and $(1-q^{-1})$, respectively. Therefore:

$$(4) \qquad Z^{\Phi}(s) = \int_{\Gamma_A/\Gamma_k} |\bar{z}z|^{s+\frac{1}{2}\delta m}\cdot\left(\sum_{\xi\in V_k} \Phi(X\xi z)\right) dX\cdot d'z ,$$

and the multiplicative calculation shows that this is absolutely convergent for $\mathrm{Re}(s) > 0$. Just as in the similar calculation in 3.1, we split this into two parts, $Z_+^{\Phi}(s)$ and $Z_-^{\Phi}(s)$, by introducing into the integral a factor $1 = f_+(z) + f_-(z)$, where f_+ is 0, $\frac{1}{2}$ or 1 according as $|\bar{z}z|$ is < 1, $= 1$ or > 1, and $f_-(z) = f_+(z^{-1})$; then $Z_+^{\Phi}(s)$ is an entire function. In $Z_-^{\Phi}(s)$, we apply Poisson summation, observing that $V_k = D_k^m - \Sigma_k$, where Σ is the sphere of radius 0, i.e. the variety $F(x) = 0$ in D^m. If Ψ is the Fourier transform of Φ, the Fourier transform of $\Phi(Xxz)$, considered as a function of $x \in D_A^m$ for a given $(X, z) \in \Gamma_A$, is $|\bar{z}z|^{-\delta m/2}\Psi(X'y\bar{z}^{-1})$ with $X' = S^{-1}\cdot {}^t\bar{X}^{-1}\cdot S$; this gives

$$\sum_{\xi\in D_k^m} \Phi(X\xi z) = |\bar{z}z|^{-\frac{1}{2}\delta m} \sum_{\eta\in D_k^m} \Psi(X'\eta\,\bar{z}^{-1}) .$$

On the other hand, $(X, z) \to (X', \bar{z}^{-1})$ is a measure-preserving automorphism of Γ_A, which maps Γ_k onto itself; if we make that substitution in the integral for $Z_+^{\Psi}(-s-\frac{1}{2}\delta m)$, and compare it with the expression for $Z_-^{\Phi}(s)$ obtained by Poisson summation as we have just said, we get:

$$Z_-^{\Phi}(s) - Z_+^{\Psi}(s+\tfrac{1}{2}\delta m)$$

$$= \int_{\Gamma_A/\Gamma_k} \left(|\bar{z}z|^{s} \sum_{\eta\in\Sigma_k} \Psi(X'\eta\,\bar{z}^{-1}) - |\bar{z}z|^{s+\frac{1}{2}\delta m} \sum_{\xi\in\Sigma_k} \Phi(X\xi z)\right) f_-(z)dXd'z \,.$$

If F is of index 0, then $\Sigma_k = \{0\}$, and the calculation can be completed immediately by applying Theorem 3.1.1 (iii) to D^*; it shows that $\tau(G)$ is finite, since the right-hand side must be absolutely convergent for $\mathrm{Re}(s) > 0$, and it gives the value of the right-hand side, showing in particular that it has the residue $\tau(G)\Psi(0)\rho_k$ for $s = 0$; comparing this with (2), we get $\tau(G) = 1$. In the general case, we apply Lemma 2.4.2 to Σ_A^*, which, by Lemma 4.1.4, we may identify with G_A/g_A, where g is the subgroup of G leaving some vector $\xi_o \in \Sigma_k^*$ fixed; by Lemma 4.1.1 and the induction assumption, we have $\tau(g) = 1$. This gives:

$$\int_{\Sigma_A^*} \Phi(s)dw_A = \int_{G_A/G_k} \left(\sum_{\xi\in\Sigma_k^*} \Phi(X\xi)\right) dX \quad ;$$

replacing here $\Phi(w)$ by $\Phi(wz)$, and applying the same formula to $\Psi(w\bar{z}^{-1})$, we get (since the automorphism $w \to wz$ of Σ_A^*, for $z \in D_A^*$, changes dw_A into $|\bar{z}z|^{\frac{1}{2}\delta m-1}dw_A$):

$$(5) \qquad Z^{\Phi}(s) = Z_+^{\Phi}(s) + Z_+^{\Psi}(-s-\tfrac{1}{2}\delta m) + \rho_k\tau(G)\left(\frac{\Psi(0)}{s} - \frac{\Phi(0)}{s+\frac{1}{2}\delta m}\right)$$

$$+ \rho_k\left(\frac{1}{s+\frac{1}{2}\delta m-1} \int_{\Sigma_A^*} \Psi(w)dw_A - \frac{1}{s+1} \int_{\Sigma_A^*} \Phi(w)dw_A \right)$$

in the number-field case, and a similar formula, which we omit, in the function-field case. Since (2) shows that the residue at $s = 0$ must be $\rho_k \Psi(0)$, we get $\tau(G) = 1$. One may observe that, when G_A/G_k is compact, i.e. when F is of index 0, the zeta-function has no other residue than $s = 0$, $s = \frac{1}{2}\delta m$ (and these residues give the value of the Tamagawa number) while otherwise it also has poles at $s = -1$, $s = 1 - \frac{1}{2}\delta m$; this should be compared with similar results in 3.8 for the zeta-functions of simple algebras.

In the hermitian case, let again G^* be the group $\left\{ {}^t\overline{X}SX = S \right\}$; if we take coordinates so that S appears as $\begin{pmatrix} a & 0 \\ 0 & S' \end{pmatrix}$, with $a \in k^*$ and $S' \in M_{m-1}(k')$, we see that G^* is the semidirect product of G and of the group $\left\{ \begin{pmatrix} z & 0 \\ 0 & 1_{m-1} \end{pmatrix} \middle| \bar{z}z = 1 \right\}$, which is isomorphic to U. Now we have seen that $\Gamma^{(\mu)}$ is isomorphic to a group $G'^* = \left\{ {}^t\overline{X}'S'X' = S' \right\}$; if G' is the subgroup of G'^* determined by $\det(X') = 1$, we see that $\Gamma^{(\mu)}$ is isomorphic to a semidirect product of G' and U; and, by the induction assumption, (1) is a set of convergence factors for G', and $\tau(G') = 1$. As we have seen in 3.7(c) that $(1-\lambda(p)q^{-1})$ is a set of convergence factors for U, it is also such for $\Gamma^{(\mu)}$; and the measure of $\Gamma_A^{(\mu)}/\Gamma_k^{(\mu)}$, for the Tamagawa measure derived from those convergence factors, is $\tau' = 2\rho_{k'}/\rho_k$, since it has been shown in 3.7(c) that this is so for U. As we have seen that $(1-q^{-1})$ is a set of convergence factors for V, we conclude from this, as above, that the factors

$$(1-q^{-1})(1-\lambda(p)q^{-1})$$

are convergence factors for $\Gamma = G \times D^*$; as they are such for D^*, this shows that (1) is a set of convergence factors for G. Using these sets of convergence factors, we can again apply Lemma 2.4.2 to Γ_A, $\Gamma_A^{(\mu)}$, Ω_μ, Γ_k, $\Gamma_k^{(\mu)}$, and get a formula similar to (4), except that $Z^{\Phi}(s)$ has now to be replaced by $\tau' Z^{\Phi}(s)$. The continuation of the calculation is just as before, except that the application of Theorem 3.1.1 (iii) to D^* introduces now the constant $\rho_{k'}$ instead of ρ_k. Thus (5), or the corresponding formula in the function-field case, will be valid provided we replace Z^{Φ}, Z^{Φ}_+, Z^{Φ}_- by $\tau' Z^{\Phi}$, $\tau' Z^{\Phi}_+$, $\tau' Z^{\Phi}_-$, and ρ_k by $\rho_{k'}$; since $\tau' = 2\rho_{k'}/\rho_k$, this means that (5) is valid in the hermitian case if ρ_k is replaced by $\frac{1}{2}\rho_k$. Comparing this with (2), we get $\tau(G) = 1$ as before; and the result is the same in the function-field case.

Theorem 4.4.1. We have $\tau(G) = 1$ for the group G defined by (1) in 4.1, in the cases L2(a) (hermitian case) and S2 (quaternionic case).

Remark. The group Γ, acting on D^m by $x \to Xxz$, is not effective; in fact, (X, z) induces the identity on D^m if z is in Z^*, where Z is the center of D, and $X = z^{-1} \cdot 1_m$; the condition $N(X) = 1$ gives then $z^{\delta m/2} = 1$. We have found it more convenient to use Γ, rather than the effective group which could be derived from it. On the other hand, for even $m \geq 4$ in the hermitian case, our methods can also be applied to the following group:

$$\Gamma' = \left\{ (X, \mu) \in M_m(D)^* \times G_m \mid {}^t\bar{X}SX = \mu S,\ \det(X) = \mu^{m/2} \right\} ;$$

if $(X, \mu) \in \Gamma'$, X is called a similitude of multiplicator μ; if M_k, M_A are the sets of the multiplicators belonging respectively to the elements of Γ'_k and Γ'_A, it follows from Dieudonné's theorems on similitudes that $M_k = M_A \cap k^*$, and that M_A consists of the $\mu \in I_k$ such that $\mu_v > 0$ for every real infinite place v of k for which (i) $k_v = \mathbb{R}$, $k'_w = \mathbb{C}$ for $w|v$, and (ii) F and $-F$ are not equivalent as hermitian forms over k_v. Using this, it can be shown that, when we write $2Z^{\Phi}(s) = I_o + I_1$ as above, I_1 is an entire function and consequently I_o is a meromorphic function, and that both of them satisfy "functional equations" similar to (5). The method fails for odd m.

4.5. The Tamagawa number of the orthogonal group.

From now on, we consider exclusively the quadratic case $\delta = 1$. Our purpose is to prove, by induction on m, the Siegel-Tamagawa theorem $\tau(G) = 2$ (this can be shown, by purely formal calculations, to be equivalent to Siegel's main theorem on the number of representations of a quadratic form by a genus of quadratic forms). The reduction from m to m-1 can be effected, for even m, by the consideration of the following zeta-function

$$\int_{V_A^+} |F(x)|^s \Phi(x) d'x ,$$

where V, Φ, d'x are as explained in 4.4, and V_A^+ is defined, as in 4.4, by $\lambda(F(x)) = 1$, except that here λ is the character of the quadratic extension k' of k given by $k' = k(\Delta^{1/2})$, $\Delta = (-1)^{m/2} \det(S)$. The method used in 4.4 can be applied with small changes. This fails, however, for odd m (because the group of similarity transformations has not the same structure for odd m as for even m). The following treatment is valid for all values of m.

We change our notation by writing F for the matrix S, so that we have now $F(x) = {}^t xFx$. Whenever convenient, we may assume that F has been put into diagonal form, $F(x) = \Sigma_i a_i x_i^2$. For m even, we denote by Δ the discriminant of F, $\Delta = (-1)^{m/2} \det(F)$; for m odd, we write $D = (-1)^{(m-1)/2} \det(F)$.

By Hasse's theorem, if $\rho \in k^*$, and Σ is the sphere of radius ρ,

Σ_k is empty if and only if Σ_A is empty, or also if and only if there is v such that Σ_{k_v} is empty.

Lemma 4.5.1. If $\rho \in k^*$ and Σ is the sphere of radius ρ, and if Σ_A is not empty, (1) is a set of convergence factors for Σ, provided $m \geq 4$.

From the formula (cf. proof of Lemma 4.2.1) for the number of solutions of a homogeneous quadratic equation mod. p, one deduces the number of solutions of $F(x) \equiv \rho$ mod. p in the field F_q with $q = N(p)$ elements; assuming p to be such that $F \in M_m(\underline{o}_p)$, $\rho \in \underline{o}_p$, and that $2\rho\, \det(F)$ is a unit in $\underline{o}_p$, this number is $q^{m'-1}(q^{m'}-\epsilon)$ for $m = 2m'$, with $\epsilon = (\Delta/p)$ (quadratic residue character of the discriminant Δ modulo p); it is $q^{m'}(q^{m'}+\eta)$ if $m = 2m'+1$, with $\eta = (D\rho/p)$, $D = (-1)^{m'} \det(F)$. The conclusion follows now from Theorem 2.2.5.

For $\rho \in k^*$, we consider the variety $F(x) = \rho y^2$ in the affine space of dimension $m+1$, and, on that variety, the Zariski-open subset $T(\rho)$ defined by $y \neq 0$; this is isomorphic to $\Sigma \times G_m$, if Σ is the sphere of radius ρ (the mapping $(x, y) \rightarrow (xy, y)$ is an isomorphism of $\Sigma \times G_m$ onto $T(\rho)$); in view of Lemma 4.5.1, $(1-q^{-1})$ is therefore a set of convergence factors for $T(\rho)$, for $m \geq 4$. As $dx = dx_1 dx_2 \ldots dx_m$ is a gauge-form on the (non-singular) variety $T(\rho)$, we can take on $T(\rho)$ the Tamagawa measure $d'x$ derived from this and the factors $(1-q^{-1})$. We introduce, for each $\rho \in k^*$, the "spherical zeta-function"

$$(1) \qquad Z(s, \rho) = \int_{T(\rho)_A} |F(x)|^s \Phi(x) d'x = \int_{T(\rho)_A} |y|^{2s} \Phi(x) d'x \quad ,$$

where Φ is a function in A_k^m, of the type described in 4.4. It should be understood that this is 0 if Σ_A, and consequently $T(\rho)_A$, are empty. For every $\lambda \in k^*$, we have $Z(s, \rho\lambda^2) = Z(s, \rho)$, as we see by making the change of variables $(x, y) \to (x, \lambda^{-1}y)$, which maps $T(\rho)$ onto $T(\rho\lambda^2)$ and leaves the integrand in $Z(s, \rho)$ invariant. The zeta-function for the quadratic form F will be defined as

$$(2) \qquad Z(s) = \sum_{\rho \in k^*/k^{*2}} Z(s, \rho) ,$$

where the sum is taken over a full set of representatives of k^* modulo $(k^*)^2$. As usual, we start with a multiplicative calculation (which will give the proof for convergence, and the principal residue), and this depends upon the calculation of the local zeta-functions for almost all p.

Take a finite set S of valuations of k, containing as usual all the infinite places, and such that, for $p \notin S$: (i) all coefficients of F are in $\underline{o}_p$; (ii) 2 det(F) is a unit in $\underline{o}_p$; (iii) the factor ϕ_p occurring in the definition of Φ is the characteristic function of $(\underline{o}_p)^m$ (there will be further conditions on S later on). For any $u \in k_p^*$, consider the integral

$$(3) \qquad Z_p(s, u) = (1-q^{-1})^{-1} \int_{T(u)} |F(x)|_p^s \phi_p(x) \cdot (dx)_p \quad .$$

As before, we see that $Z_p(s, uv^2) = Z_p(s, u)$ for $v \in k_p^*$; therefore, if we write $u = \pi^\alpha v$, where v is a unit in $\underline{o}_p$, $Z_p(s, u)$ can depend only upon α mod. 2 and upon the quadratic residue character (v/p) of v mod. p (in fact, since 2 is a unit in $\underline{o}_p$, every element of $\underline{o}_p$ which is $\equiv 1$ mod. p is in $(k_p^*)^2$). The value of $Z_p(s, u)$ is given by:

Lemma 4.5.2. Put $u = \pi^\alpha v$, where v is a unit in $\underline{o}_p$. Then:

(i) <u>for odd</u> m, $m = 2m' + 1$, <u>we have</u>

$$(1-q^{-2s-2})(1-q^{-2s-m})Z_p(s,u) = \begin{cases} q^{-s-1}(1-q^{1-m}) & \text{for } \alpha \text{ odd,} \\ (1+\eta q^{-m'})(1-\eta q^{-2s-2-m'}) & \text{for } \alpha \text{ even,} \end{cases}$$

<u>where</u> $\eta = (Dv/p)$, $D = (-1)^{m'} \det(F)$.

(ii) <u>for even</u> m, $m = 2m'$, <u>we have</u>

$$(1-q^{-2s-2})(1-q^{-2s-m})Z_p(s,u) = \begin{cases} q^{-s-1}(1-\epsilon q^{-m'})(1+\epsilon q^{1-m'}) & \text{for } \alpha \text{ odd,} \\ (1-\epsilon q^{-m'})(1+\epsilon q^{-2s-m'-1}) & \text{for } \alpha \text{ even,} \end{cases}$$

<u>where</u> $\epsilon = (\Delta/p)$, $\Delta = (-1)^{m'} \det(F)$.

Assume that we have put F into diagonal form, $F(x) = \Sigma_i a_i x_i^2$. We have to take the integral (1) over the set of points (x, y) of k_p^{m+1} for which $F(x) = uy^2$, $x \in (\underline{o}_p)^m$, $y \neq 0$; for each such point, we can write, in one and only one way, $y^{-1}x$ and y in the form $y^{-1}x = \pi^{\lambda} z$, $y = \pi^{\mu} t$, where $z \in (\underline{o}_p)^m$, $t \in \underline{o}_p$, $z \not\equiv 0$ mod. p and $t \not\equiv 0$ mod. p; then we have $\lambda + \mu \geq 0$ (since x must be in $(\underline{o}_p)^m$), and $F(z) = \pi^{\alpha-2\lambda} v$ and therefore $2\lambda \leq \alpha$. We split up our domain of integration into the open subsets such that, on each of these sets, λ, μ have given values, and z, t have given values $\bar{z}, \bar{t}$ modulo p; the latter must then be such that $F(\bar{z}) \equiv v$ mod. p if $2\lambda = \alpha$, and $F(\bar{z}) \equiv 0$ mod. p if $2\lambda < \alpha$. Conversely, assume that $\lambda, \mu, \bar{z}, \bar{t}$ are so given, and e. g. $\bar{z}_1 \not\equiv 0$ mod. p; then, for every choice of $z_2, \ldots, z_m$ such that $z_i \equiv \bar{z}_i$ mod. p for $2 \leq i \leq m$, the equation $F(z) = \pi^{\alpha-2\lambda} v$ has exactly one solution z_1 such that $z_1 \equiv \bar{z}_1$ mod. p, and this is an analytic function of $z_2, \ldots, z_m$; therefore, on the subset of the domain of integration determined by those values of $\lambda, \mu, \bar{z}, \bar{t}$, we can use $z_2, \ldots, z_m$, t as local coordinates, and the mapping $(x, y) \to (z_2, \ldots, z_m, t)$

is an analytic homeomorphism of that set onto the subset of $(\underline{o}_p)^m$ given by $z_i \equiv \bar{z}_i$, $t \equiv \bar{t}$ mod. p. On that set, and in terms of those local coordinates, the gauge-form $dx_1 \ldots dx_m$ is given by

$$dx_1 \ldots dx_m = \pi^{m(\lambda+\mu)} F(z) dt dz_2 \ldots dz_m / a_1 z_1 ,$$

which gives, on that set, since $a_1 z_1$ on that set is a unit in $\underline{o}_p$:

$$(dx)_p = q^{-m(\lambda+\mu)} \cdot q^{2\lambda-\alpha} (dt dz_2 \ldots dz_m)_p ;$$

therefore, on that set, our integral has the value q^N, with

$$N = (s+1)(2\lambda-\alpha) - m(\lambda+\mu+1) .$$

For given values of λ, μ, this gives to our integral a contribution $(q-1)\nu(v)q^N$ if $2\lambda = \alpha$, where $\nu(v)$ is the number of solutions of $F(\bar{z}) \equiv v$ mod. p, and $(q-1)\nu(0)q^N$ if $2\lambda < \alpha$, where $\nu(0)$ is the number of solutions, other than 0, of $F(\bar{z}) \equiv 0$ mod. p; we have to take the sum of these contributions over all values of λ, μ satisfying $2\lambda \leq \alpha$, $\lambda + \mu \geq 0$. Using the formulas given above (in the proofs of Lemmas 4.2.1 and 4.5.1) for $\nu(v)$, $\nu(0)$, we get our result.

Now, for $p \notin S$, denote by U_p the multiplicative group of the units in $\underline{o}_p$, and put $H_S = (I_k)^2 \prod_{p \notin S} U_p$; clearly H_S is an open subgroup of I_k. Also, write $|S|$ for the number of elements of S.

Lemma 4.5.3. *The group $H_S k^*$ is of finite index in I_k. Moreover, if S is large enough, this index is $[I_k : H_S k^*] = 2^{|S|}$, and we have $k^* \cap H_S = k^{*2}$.*

The characters of $I_k / H_S k^*$ are those characters of order 2 of I_k which are 1 on k^*, and on U_p for every $p \in S$; by class-field theory, they are the characters attached to those quadratic extensions $k(d^{\frac{1}{2}})$ of k which are unramified outside S, In the number-field case, take S such that it contains all the primes dividing 2, and that the primes in S generate the group C of ideal-classes of k. If $k(d^{\frac{1}{2}})$ is unramified outside S, every prime $p \notin S$ must occur in d with an even exponent, so that we can write the principal ideal (d) as a product of primes $p \in S$ and of the square m^2 of an ideal m; by our assumption on S, we can write m as $\lambda m'$ with $\lambda \in k^*$, where m' has no prime factor $p \notin S$; therefore every quadratic extension of k, unramified outside S, can be written as $k(d^{\frac{1}{2}})$ with $d \in E_S$, where E_S is the group of the S-units of k (elements of k which are p-units for every $p \notin S$). The index we have to compute is then $[E_S:E_S^2]$. It is well-known (Dirichlet-Chevalley) that E_S is the product of a finite cyclic group of even order, and of a free abelian group with $|S| - 1$ generators. The proof is even simpler in the function-field case. This proves the first part (note that, if the index is finite for large S, it must be finite for all S). Now take $d \in k^* \cap H_S$; then, in $k' = k(d^{\frac{1}{2}})$, each $p \in S$ splits (i.e. can be extended to two distinct valuations of k') and each $p \notin S$ is unramified. As there are only finitely many non-ramified quadratic extensions of k, this implies $d^{\frac{1}{2}} \in k^*$ provided S has been so chosen that, to every non-ramified quadratic extension k' of k, there is at least one $p \in S$ which does not split in k'.

Now take any S satisfying the conditions in Lemma 4.5.3 as well as the earlier ones. Also, let V be, as in 4.3, the open set $F(x) \neq 0$ in the affine m-space; and call Ω_S the open subset of V_A determined by the condition $F(x) \in H_S k^*$; if $\gamma(S)$ is the group of order $2^{|S|}$ consisting of the characters of I_k which are 1 on $H_S k^*$, Ω_S can also be defined as the subset of V_A where $\lambda(F(x)) = 1$ for every $\lambda \in \gamma(S)$. Now we introduce the function

$$Z_S(s) = \int_{\Omega_S} |F(x)|^s \Phi(x) d'x = 2^{-|S|} \sum_{\lambda \in \gamma(S)} \int_{V_A} |F(x)|^s \lambda(F(x)) \Phi(x) d'x \ .$$

The multiplicative calculation for the $2^{|S|}$ integrals in the last sum is the same as the one given in 4.4 for I_o, I_1 (this depends only upon Lemma 4. 3, 2); it shows that these integrals are absolutely convergent for $Re(s) > 0$, and that only the one corresponding to $\lambda = 1$ gives a residue for $s = 0$, this residue being $\rho_k \int \Phi(x) dx$. Therefore the integral for Z_S is absolutely convergent for $Re(s) > 0$, and we have:

$$(4) \qquad [s Z_S(s)]_{s=o} = 2^{-|S|} \rho_k \int_{A_k^m} \Phi(x) dx \ .$$

Now we can also write $H_S k^*$ as the disjoint union of the sets $H_S \rho$ when we take for ρ a complete set of representatives of k^* modulo $k^* \cap H_S$, i.e., under our assumptions on S, modulo k^{*2}; therefore, if we put

$$(5) \qquad Z_S(s, \rho) = \int_{F(x) \in H_S \rho} |F(x)|^s \Phi(x) d'x \quad ,$$

these integrals are absolutely convergent, and we have

$$(6) \qquad Z_S(s) = \sum_{\rho \in k^* / k^{*2}} Z_S(s, \rho) \ ,$$

this series being also absolutely convergent for $Re(s) > 0$.

Now consider $Z(s, \rho) Z_S(s, \rho)^{-1}$; the multiplicative calculation shows that this is the product, extended over all valuations v of k, of the factors

(7) $$\left(\int_{T(\rho)} |F(x)|_v^s \phi_v(x)dx_v\right)\cdot\left(\int_{F(x)\in H_v\rho} |F(x)|_v^s \phi_v(x)dx_v\right)^{-1}$$

with $H_v = k_v^{*2}$ if $v \in S$, $H_p = k_p^{*2}U_p$ if $p \notin S$. For $v \in S$, $T(\rho)$, which is the subset of the variety $F(x) = \rho y^2$ determined by $y \neq 0$, covers twice the subset of A_k^m determined by $F(x) \in \rho k_v^{*2}$; therefore the above factor has then the value 2. For $p \notin S$, take $u_p \in U_p - U_p^2$, so that $U_p = U_p^2 \cup U_p^2 u_p$ and $H_p = k_p^{*2} \cup k_p^{*2} u_p$; then the second integral in (7) is the sum of the same integrals taken under the restrictions $F(x) \in k_p^2\rho$, $F(x) \in k_p^2 u_p \rho$, respectively; and these, for the reasons just explained, differ from the same integrals, taken over $T(\rho)$ and $T(u_p\rho)$, only by the factor $\frac{1}{2}$. Thus the factor (7), for $p \notin S$, has the value

$$\frac{2Z_p(s,\rho)}{Z_p(s,\rho) + Z_p(s,u_p\rho)} .$$

By Lemma 4.5.2, <u>this is equal to</u> 1 <u>for</u> m <u>even</u>; in this case, therefore, the integral in (1) and the series in (2) are absolutely convergent for $Re(s) > 0$; and we have:

$$Z(s,\rho) = 2^{|S|}Z_S(s,\rho)\ ,\quad Z(s) = 2^{|S|}Z_S(s)\ ,\quad [sZ(s)]_{s=0} = \rho_k \int_{A_k^m} \Phi(x)dx\ .$$

<u>For</u> m <u>odd,</u> Lemma 4.5.2 shows that the factor (7), for $p \in S$, has the value 1 whenever ρ contains an odd power of p; when that is not so, this same factor has the value

$$\theta(p) = \frac{(1+\eta q^{-m'})(1-\eta q^{-2s-2-m'})}{1-q^{-2s-m-1}}$$

with $\eta = \pm 1$; in all cases, therefore, we have

$$1 - q^{-m'} < \theta(p) < 1 + q^{-m'} \quad .$$

As $m > 4$, we have here $m' \geq 2$. This shows that the infinite product for $Z(s, \rho)$ is absolutely convergent together with that for $Z_S(s, \rho)$; also, if we put

$$\mu(S) = \prod_{p \notin S} (1-q^{-m'}) \quad , \quad \mu'(S) = \prod_{p \notin S} (1+q^{-m'}) \quad ,$$

we see that $2^{-|S|} Z(s, \rho) Z_S(s, \rho)^{-1}$ is $> \mu(S)$ and $< \mu'(S)$ for $Re(s) > 0$; this implies that the series (2) for $Z(s)$ is absolutely convergent, and that $2^{-|S|} Z(s) Z_S(s)^{-1}$ remains between the same constants. In view of (4), this shows that lim. inf. $sZ(s)$ and lim. sup. $sZ(s)$, for $s = 0$, are between $\mu(S)$ and $\mu'(S)$; since we may here take S as large as we please, we have proved:

$$(8) \qquad [sZ(s)]_{s=o} = \rho_k \int_{A_k^m} \Phi(x) dx \quad ,$$

so that this formula holds for all $m > 4$. This completes the "multiplicative calculation".

Now we take up the additive calculation. Take any ρ such that $T(\rho)$ is not empty, i.e. (by Hasse's theorem) such that there is $\xi_o \in k^m$ for which $F(\xi_o) = \rho$. Put $\Gamma = G \times G_m$, and let Γ act on $T(\rho)$ by

$$((X, t), (x, y)) \to (Xxt, yt) \quad ;$$

by Witt's theorem (Lemma 4.1.3), Γ acts transitively on $T(\rho)$, and the subgroup leaving $(\xi_o, 1)$ fixed has a cross-section; this subgroup is $\Gamma' = G' \times \{1\}$, where G' is the subgroup of G leaving ξ_o fixed. By Lemma 4.1.1 and the induction assumption, (1) is a set of convergence factors for G', and $\tau(G') = 2$. Let dX, dX' be invariant gauge-forms for G, G'; then $dX \cdot (dt/t)$

is such a form for Γ. Clearly $y^{-m}dx_1 \dots dx_m$ is a gauge-form on $T(\rho)$, invariant under Γ. Therefore, by Theorem 2.4.3, Γ has the same set of convergence factors as $T(\rho)$, viz. $(1-q^{-1})$, so that G has (1) as a set of convergence factors; also, the Tamagawa measures for $\Gamma, \Gamma', T(\rho)$, derived from these convergence factors and the gauge-forms $dX\cdot(dt/t)$, dX', $y^{-m}dx$, match together topologically. This gives, by Lemma 2.4.2:

$$Z(s,\rho) = \frac{1}{2}\int_{T(\rho)_A} |y|^{2s+m}\Phi(x)|y|^{-m}d'x =$$
$$= \frac{1}{2}\int_{\Gamma_A/\Gamma_k}\left(\sum_{(M,\tau)} |\tau t|^{2s+m}\Phi(XM\xi_o\tau t)\right) dX(dt/t)' ,$$

where the sum is taken over all $(M,\tau)\in\Gamma_k/\Gamma'_k$, i.e. over all $M\in G_k/G'_k$ and all $\tau\in k^*$; but then, by Witt's theorem, the vector $\xi = M\xi_o\tau$ runs twice through the set of all vectors in k^m such that $F(\xi)\in\rho k^{*2}$; if then we let ρ run through a full set of representatives of k^* modulo k^{*2}, ξ runs twice through the set of vectors ξ in k^m such that $F(\xi)\neq 0$. This gives:

$$Z(s) = \int_{\Gamma_A/\Gamma_k} |t|^{2s+m}\left(\sum_{F(\xi)\neq 0}\Phi(X\xi t)\right) dX\cdot(dt/t)' .$$

From here on, the calculation is exactly the same as in 4.4, beginning with formula (4) of that section. The conclusions are the same; in particular, the residue of $Z(s)$ at $s=0$ turns out to be $\rho_k\tau(G)/2$; comparing this with (8), we get:

Theorem 4.5.1. The Tamagawa number of the orthogonal group in $m\geq 3$ variables is 2.

Remark. For indefinite quadratic forms, Siegel has defined zeta-functions for individual classes of such forms. This can perhaps be explained by the fact that, for indefinite forms, classes and "spinor-genera" are the same. (Eichler-Kneser). If G is the orthogonal group, and $\mathcal{G}$ is the corresponding spin-group, the spinor-norm, for an element of G_K (K = any field) is an element of K^*/K^{*2} which gives the obstruction against lifting that element from G_K to $\mathcal{G}_K$. In particular, this defines a homomorphism of G_A into $I_k/I_k^2k^*$, with the value 1 on G_k; therefore, to every character of $I_k/I_k^2k^*$, i.e. to every character of I_k belonging to a quadratic extension of k, one can assign a character of order 2 of G_A, with the value 1 on G_k; it is not unlikely that, by introducing such characters into our zeta-functions, one might get Siegel's zeta-functions for indefinite forms.

Erratum to page 17: Omit lines 12, 13, 14 on that page (from "The end of this paragraph, .." to "... directly to 2, 3,").

APPENDIX
THE CASE OF THE GROUP G_2

(by M. Demazure)

The method used in the case of orthogonal groups can also give the Tamagawa number of the groups of type G_2, which turns out to be 1 as expected.

We first recall some results on Cayley Algebras, after JACOBSON, Composition Algebras and Their Automorphisms, Rend. Palermo, 1958. For the time being, k is any field of characteristic not 2.

A Cayley algebra over k is a vector space Ω_k of dimension 8 over k, together with a k-linear map $\Omega_k \times \Omega_k \to \Omega_k$ denoted $(x, y) \to x \cdot y$ and a non-degenerate quadratic form called the norm $N: \Omega_k \to k$ subject to the following axioms:

(i) there exists in Ω_k a unit-element, i.e. $1 \in \Omega_k$ with $x \cdot 1 = 1 \cdot x = x$;

(ii) for any $x, y \in \Omega_k$, $N(x \cdot y) = N(x)N(y)$.

One can easily show that the form N is uniquely determined by the structure of (non-associative) algebra of Ω_k.

Let Ω be the algebra-variety defined by Ω_k. We denote by Ω_o the orthogonal space, for N, of the one-dimensional line $k \cdot 1$. For $x \in \Omega_o$, $N(x)1 = - x \cdot x$. An automorphism of Ω is a linear mapping $g: \Omega \to \Omega$ such that $g(x \cdot y) = g(x) \cdot g(y)$. Then $g(1) = 1$ and $g(\Omega_o) = \Omega_o$. If $x \in \Omega_o$, then $N(g(x)) = N(x)$. Moreover, one proves (Jacobson, Theorem 2) that g is a rotation, i.e. of determinant 1. Hence the group G of all automorphisms of Ω is imbedded in $SO(\Omega_o, N)$. It is a semi-simple algebraic group defined over k which becomes isomorphic over $\bar{k}$ to the group G_2 of the Cartan-Killing classification.

A Witt-type theorem is true for G (Jacobson § 3):
Let (Ω_k, N) and (Ω'_k, N') be two Cayley algebras with equivalent norms (in the sense of quadratic forms). Let B (resp. B') be a non-isotropic subalgebra of Ω_k (resp. Ω'_k). Let there be given an algebra-isomorphism $f: B \to B'$. Then f can be extended to an isomorphism of Ω_k onto Ω'_k.

Corollary. If $x, y \in \Omega_k$, $x, y \neq 0$, $N(x) = N(y)$, then there exists $g \in G_K$ mapping x on y.

Proof: 1) If $N(x) = N(y) \neq 0$, then K(x) and K(y) are two isomorphic quadratic fields.

2) If $N(x) = N(y) = 0$, then x and y can be imbedded in two quaternion algebras isomorphic under a map carrying x into y.

We finally have the two following results:

(i) Let $a \in \Omega_o$, $a \cdot a = b \cdot 1 \neq 0$. Let $K = k(a)$. Then the orthogonal L of K is a 3-dimensional vector-space over K. The subgroup of G leaving K point-wise fixed is isomorphic to the unimodular unitary group of L as a vector space over K relative to the form $(x, y) + b^{-1} a(ax, y)$. (Jacobson, Theorem 3.)

(ii) Let B be a quaternion subalgebra of Ω. The subgroup of G leaving B point-wise fixed is isomorphic to the multiplicative group of elements of norm 1 in B.

From now on, k is a number field. (In the case of a function field of characteristic not 2, everything is valid, provided that we prove that G_k

is Zariski-dense in G.)

A careful analysis of 4.5 shows that what was proved there amounts to the following:

Let Ω_o be a finite-dimensional vector-space (of dimension ≥ 5) defined over k, with a quadratic form N. Let G be an algebraic group of rotations of N defined over k. Suppose that G verifies a Witt-type theorem (i.e. if x, $y \in \Omega_o$, $N(x) = N(y)$, there exists $g \in G$ carrying x into y; if x and y are rational over k, then g may be taken rational over k). For $a \in \Omega_o$, let G(a) be the isotropy group of a in G. Assume the two following properties:

(i) There exists a finite number τ such that for non-isotropic $a \in \Omega_k$, the Tamagawa number of G(a) is finite and equal to τ, independently of a.

(ii) For any isotropic $a \in \Omega_k$, the Tamagawa number of G(a) is finite. Then the Tamagawa number of G is finite and equal to τ.

In our case, G satisfies a Witt-type theorem. We have only to verify properties (i) and (ii).

(i) By property (i) recalled above, for non-isotropic $a \in \Omega_k$, G(a) is a unimodular unitary group and $\tau(G(a)) = 1$.

(ii) Let now a be isotropic. The subgroup of G leaving the line $k \cdot a$ fixed has at most rank two. It contains a multiplicative factor not contained in G(a). Hence G(a) has at most rank one. On the other hand, let B be a quaternion algebra containing a. Then the subgroup of G leaving B pointwise fixed is semi-simple of rank one and contained in G(a); its Tamagawa number is one (by property (ii) recalled above). By the general properties of algebraic groups, it must be normal in G(a). The quotient being of rank zero is unipotent, hence has Tamagawa number one. On the other hand, the quotient being unipotent, the fibration admits local cross-sections (Rosenlicht). By Theorem 2.4.4, the Tamagawa number of G(a) is finite and equal to 1. This proves:

Theorem. The Tamagawa number of a group of type G_2 is 1.