March, 1974

Geometry of Basic Physics: Flat Spacetime

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Foreword

I will be very grateful for any criticism or suggestions concerning the following exposition, which describes special relativity from a point of view somewhere between geometry and physics. The following books have been particularly useful to me:

J. L. Synge, "Relativity: the Special Theory," 2nd ed., North-Holland, 1965.

E. F. Taylor and J. A. Wheeler, "Spacetime Physics," Freeman, 1963.

E. Whittaker, "A History of the Theories of Aether and Electricity," 2 volumes, Philosophical Library, New York, 1951 and 1954.

A few comments on the text, and particularly the notations, may be helpful. Minkowski's 4-dimensional viewpoint is used wherever possible. In order to keep the mathematics on a fairly elementary level, all vectors and tensors are presented very concretely as matrices. As an example, the energy-momentum vector is identified with the 1×4 matrix $\underset{\sim}{p} = [\varepsilon, p, q, r]$, where ε is the energy, and $\vec{p} = [p, q, r]$ is the momentum 3-vector. The word "tensor" is always used for a 4×4 matrix which represents a 2-index tensor, covariant in the first index and contravariant in the second index. Such a tensor $T = [T_i^{\ j}]$ is "symmetric" if it satisfies $T_i^{\ j} = T_j^{\ i}$, but $T_0^{\ j} = -T_j^{\ 0}$, for $1 \le i, j \le 3$. A product such as $\underset{\sim}{u}T$, where $\underset{\sim}{u}$ is a vector and T is a tensor, should always be interpreted as a matrix product.

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December, 1973

§1. Historical and Empirical Prelude

According to Galileo, any particle on which no forces act must continue in a state of uniform linear motion, without acceleration. If we introduce cartesian coordinates, then this principle can be formulated more precisely as follows.

Working Hypothesis 1.1. An omniscent observer can set up a coordinate system for time and space so that each event, that is each precise location in time and space, is specified by four real numbers t, x, y, z, and so that the total history (of the center of mass) of any material object on which no forces act is described by a linear graph of the form

$$x = a_1 + c_1 t, \quad y = a_2 + c_2 t, \quad z = a_3 + c_3 t, \tag{1.2}$$

the a_i and c_i being suitable constants.

We could think of these equations as describing the "position" x, y, z of the object at "time" t. It is better however to place the four coordinates t, x, y, z on an equal footing. The linear equation (1.2) then simply describes a straight line L in the space of all 4-tuples $[t, x, y, z]$. This line L is called the worldline of the object.

The ordinary earthly observer can of course not hope to achieve such a precise description. He can only set up an approximate coordinate system for some large region in time and space. But the sharp form of 1.1 is convenient for mathematical analysis.

This hypothesis contains technical terms such as "force," which will be defined in §7.3, and "center of mass," which will be introduced under a different name in §9.5. For the moment the reader must simply accept these as undefined terms. One has to start somewhere.

Now let us try to incorporate optical phenomena into this mathematical model. It is perhaps easiest to adopt a naive Newtonian viewpoint for the time being, thinking of light simply as a stream of small particles which we call photons.

Experimental evidence, ever since Römer showed in 1675 that light travels at finite speed, has suggested that the speed of light in vacuum is the same all directions, and is independent of the speed of the object from which the light is emitted. This empirical observation can be formulated more precisely as follows.

Working Hypothesis (1.3). The coordinate system (1.1) can be chosen so that the "speed"

$$c = \sqrt{(c_1)^2 + (c_2)^2 + (c_3)^2} = \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} ,$$

associated with the worldline of a photon, is the same for all photons in vacuum.

In practice it is always convenient to choose coordinates so that this constant c (the speed of light with respect to the given coördinate system) equals 1. For example if the observer chooses the coordinate t to measure time in seconds, then he must apply an appropriate scale change to the coordinates x, y, z so that distance $\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$ will also be measured in seconds.

[One second of distance, by definition equal to the distance traveled by light in one second of time, equals 299792.5 kilometers or somewhat less than the distance from Earth to moon. A more practical unit for everyday life would be the nanosecond (10^{-9} seconds) which is approximately one English foot.]

From Newton's time until the late nineteenth century it was believed that this hypothesis, the constancy of the speed of light in all directions, could only be true if one used an "absolute" coordinate system: one such that particles in a state of "absolute rest" are represented by worldlines parallel to the t-axis. But theoretical work by Maxwell, together with experimental work by Michelson, Morley and others, cast considerable

After making this scale change, note that (1.3) can be reformulated as follows. For any two points along the worldline of a photon the difference equation

$$(1.4) \qquad (\Delta t)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

must be satisfied.

doubt on this interpretation, and led Henri Poincaré and H. A. Lorentz to a quite different conclusion. As stated by Poincaré* in 1899:

"I regard it as very probable that optical phenomena depend only on the relative motions of the material bodies, luminous sources, and optical apparatus concerned ... ."

In 1904 he stated this principle in sharper form, and gave it the name by which it is now known:

"According to the Principle of Relativity, the laws of physical phenomena must be the same for a 'fixed' observer as for an observer who has a uniform motion of translation relative to him: so that we have not, and cannot possibly have, any means of discerning whether we are, or are not, carried along in such a motion .... From all these results there must arise an entirely new kind of dynamics, which will be characterized above all by the rule, that no velocity can exceed the velocity of light."

These ideas had been cast in a precise mathematical form by Lorentz in 1903, and were further developed and greatly clarified by Einstein in 1905.

Our object is to give an exposition of the resulting theory.

*Quoted by Whittaker, "A History of the Theories of Aether and Electricity," Philosophical Library, New York, 1951 and 1954.

§2. The Geometry of Light Paths

Let us see how much geometry or physics we can develop using only the two working hypotheses 1.1 and 1.3. In order to make mathematical sense out of these hypotheses, we must suppose that we are given the following four primitive concepts:

(i) The concept of a point $\underset{\sim}{x} = [t, x, y, z]$ in the 4-dimensional coordinate space. It will be convenient to introduce the name spacetime for this coordinate space.

(ii) The usual concept of a line L (also called "straight line") in spacetime. Unless it happens to be parallel to the x, y, z-coordinate hyperplane, such a line can be defined by equations of the form 1.2.

(iii) Suppose also that we are told which lines L satisfy (1.4), or in other words we are told which lines have coordinate speed $\sqrt{(\Delta x/\Delta t)^2 + (\Delta y/\Delta t)^2 + (\Delta z/\Delta t)^2} = \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}$ equal to 1. It is convenient to use the term light path for these distinguished lines with speed equal to 1.

(iv) Finally, in order to make some use of ordinary time oriented language, let us suppose that we are given a preferred direction, namely the direction of increasing coordinate time t, along each light path L. This will enable us to think of a light signal as being "emitted" at one point $\underset{\sim}{x}_0$ of L, and "received" at a "later" point $\underset{\sim}{x}_1$ of L. Briefly we will speak of a signal from $\underset{\sim}{x}_0$ to $\underset{\sim}{x}_1$.

Using only these four primitive concepts, we will use a mixture of geometric and physical arguments to develop various derived concepts such as time, distance, speed, and (hyperbolic) angle.

As a first step, it is useful to consider the totality of all light paths through some given point $\underset{\sim}{x}_0 = [t_0, x_0, y_0, z_0]$ of spacetime. By definition, the union of all such light paths is called the light cone N based at $\underset{\sim}{x}_0$. (Compare Figure 2.1.) Clearly a given point $\underset{\sim}{x} = [t, x, y, z]$ belongs to this light cone if and only if the quadratic equation

$$(t - t_0)^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$

is satisfied.

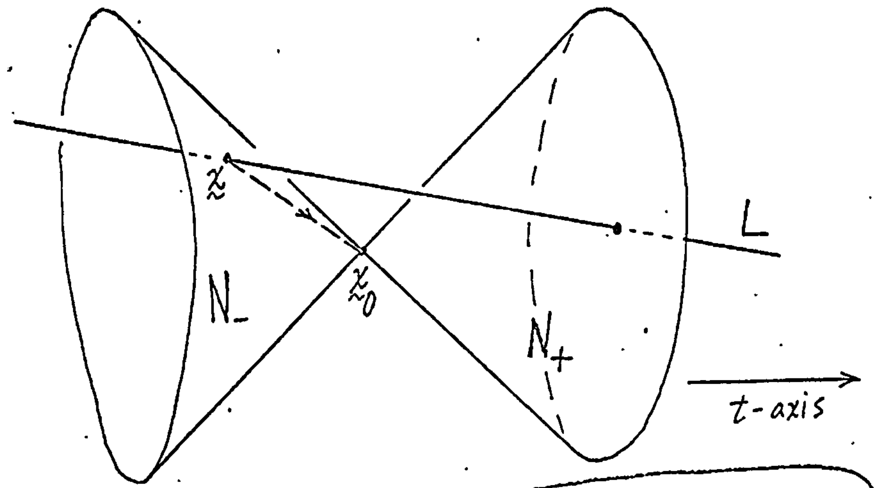

Figure 2.1. A timelike line.

We can further distinguish two different halves of the light cone. A point $\underset{\sim}{x} = [t, x, y, z]$ satisfying this equation belongs to the forward or future light cone N_+ if $t > t_0$, and to the backward light cone N_- if $t < t_0$.

Using the four primitive concepts (i), (ii), (iii), (iv), an important first qualitative observation is that we can distinguish three different kinds of straight lines, corresponding to objects which travel at a uniform velocity slower than light, faster than light, or precisely at the speed of light.

Those straight lines which would correspond to objects traveling slower than light are called timelike. Timelike lines are of primary physical importance, since they represent the possible worldlines of actual material objects.

Those straight lines which would correspond to objects traveling faster than light are called spacelike. In fact no such objects have ever been observed, and most physicists hope and believe* that it will stay that way. So spacelike lines must be thought of as purely mathematical constructions, with no direct physical meaning.

Caution: In referring to the speed of light, we always mean the speed at which light would travel in vacuum. By definition this is +1. In any material medium, light waves travel at some speed less than 1. (The precise speed is a function of frequency.) For example in water the actual speed of light is approximately 3/4. It is perfectly possible for a particle to travel through water at a speed which is between 3/4 and 1. Any such particle generates "Cherenkov radiation" which is analogous to the shock wave generated by an airplane moving faster than the speed of sound.

To distinguish between timelike and spacelike lines using only the primitive concepts (i), (ii), (iii), (iv), we introduce the concept of visibility.

Definition. The line L is visible from a point x_0 in spacetime if L intersects the backwards light cone N_- based at x_0. The line L is uniquely visible from x_0 if there is just one intersection point.

The physical meaning of this definition is clear. If L intersects the backward light cone N_- at some point x, then evidently it is possible for a light signal leaving L at the point x in spacetime to reach the point x_0. (Figure 2.1.)

The distinction between timelike and spacelike lines can now be expressed as follows: A timelike line is uniquely visible from every point x_0 in spacetime. A spacelike line, on the other hand, is either doubly visible or completely invisible from most points in spacetime. [Compare Figure 2.2. To an ordinary timelike observer, an object moving faster than light would appear to spring into existence (with a flash of light analogous to a sonic boom?) and then dash off to infinity in two different directions at once.]

*See however G. Feinberg, "Possibility of faster-than-light particles," Phys. Review 159 (1967), 1089-1105, as well as R. Newton, "Particles that travel faster than light?" Science 167 (1970), 1569-1574.

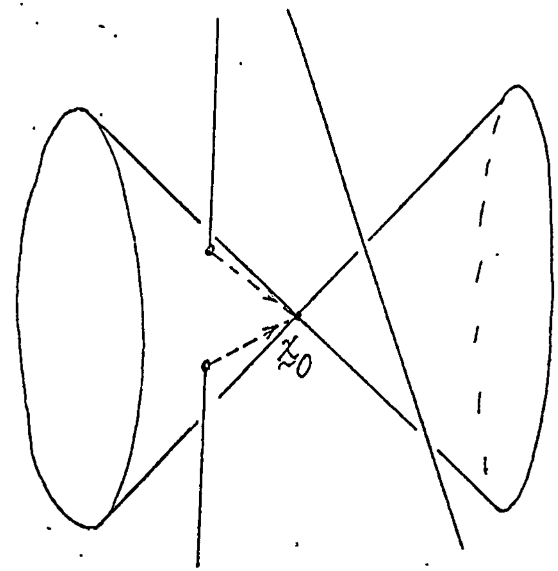

Figure 2.2. Spacelike Lines.

To simplify the discussion (of these two statements), we will only consider the case of a straight line L which pases through the origin and has finite coordinate speed, so that it can be defined by equations of the form

$$x = at, \quad y = bt, \quad z = ct .$$

Thus the coordinate speed $\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}$ is equal to $\sqrt{a^2 + b^2 + c^2}$. In order to find a point $x = [t, at, bt, ct]$ on this line which belongs to the light cone based at x_0 we must solve the quadratic equation

$$(t - t_0)^2 - (at - x_0)^2 - (bt - y_0)^2 - (ct - z_0)^2 = 0$$

for t. Note that the left side of this equation is strictly negative when $t = t_0$

(unless the point x_0 itself lies on L). Expressing the left side as a quadratic polynomial $At^2 + Bt + C$ in t, note that the coefficient of t^2 is given by

$$A = 1 - a^2 - b^2 - c^2 .$$

For a timelike line we have $A > 0$, hence this polynomial is positive for large values of $|t|$. Therefore it has exactly one root with $t < t_0$ (and exactly one root with $t > t_0$). On the other hand for a spacelike line we have $A < 0$, so there are an even number of roots with $t < t_0$. (The possibility of a double root, corresponding to a line which is tangent to the light cone, cannot be excluded, but such a double root can always be eliminated by moving the point x_0 slightly.)

Now let us try to justify the term "timelike" by introducing a procedure for measuring time along any timelike line L. In fact we will describe a kind of "geometrical clock" which might actually be used by an observer whose worldline is L. The first step is to choose another line L_0 which is "parallel" to L.

Geometrically, two lines are called parallel if they are contained in a common plane but do not meet. The definition of the concept of "plane," starting only with the primitive concepts of point and line in 4-space, will be left as an exercise for the reader. (Physically, two objects have parallel worldlines if they are "at rest" relative to each other, so that they are neither approaching nor retreating nor revolving about each other.)

Given two parallel worldlines L and L_0 let us suppose that an observer on L sends a light signal to L_0, which is then reflected back to L. (Compare Figure 2.3.) If the signal is sent from the point x in spacetime, reflected at x_0, and received back on L at the point x', then we will say that one unit of time has elapsed between x and x'. Iterating this procedure, the entire line L can be divided up into equal intervals.

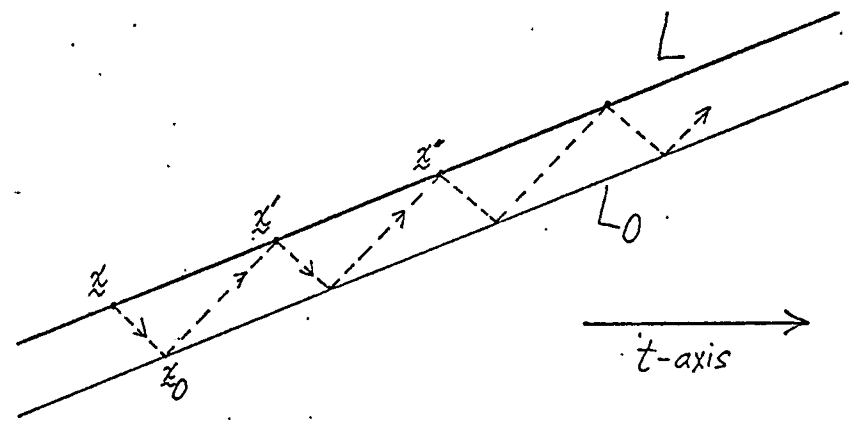

Figure 2.3. A geometrical clock.

Of course this choice of unit is completely arbitrary. A different choice of parallel line would yield a different unit of time along L.

Once chosen, the unit of time can be subdivided into finer intervals by making use of auxilliary parallel lines which are closer to L. In this way the observer on L could theoretically measure time to any required degree of accuracy, and assign a real number τ to any point on L, starting with some arbitrary base point at which $\tau = 0$. The real number τ constructed in this way is called the proper time* along L, to distinguish it from the coordinate time t. Clearly the proper time τ is a monotone linear function of the coordinate time t.

The concept of distance is closely related. Suppose that an observer with worldline L wishes to measure the distance to some object. Using his handy radar set, he can do this by sending out a light signal, say at proper time τ. If this signal is reflected from the object at the point x_1 in spacetime and received back on the worldline L at proper time $\tau' = \tau + 2r$,

* This definition will be sharpened in §3.

This ratio $k > 1$ describes a slowing down process: If the observer on L sends out a periodic signal, say with one pulse every second, then the reflected signal which he receives will be slower, with only one pulse every k seconds. In particular if we think of visible light itself as a periodic signal (with frequency ranging from one cycle every 1.32×10^{-15} seconds at the violet end of the spectrum to one cycle every 2.54×10^{-15} seconds at the red end of the spectrum*), then we see that the reflected visible light signal will be shifted towards the red end of the spectrum.

More generally suppose that the observer on L is tracking an arbitrary moving object on his radar set. Again suppose that the signal is emitted at time τ and returns at time τ'. Then the derivative

$$k = d\tau'/d\tau$$

can again be called the <u>radar red shift ratio</u>.

This ratio k is clearly independent of the choice of a unit of time along L. In other words it is a <u>dimensionless</u> number.

With this more general definition, k can be either greater than 1 (for a receding object), or less than 1 (for an approaching object), or equal to 1. Note that values of k less than 1 correspond to a speeding up process. In the case of visible light they correspond to a shift towards the violet end of the spectrum.

Closely related is the concept of "speed of recession." If the observer on L, using his radar set as described above, finds that the moving object has distance r at proper time σ (as measured along L) then by definition the derivative

$$v = dr/d\sigma$$

is called the <u>speed of recession</u> of the object from L. Clearly v is also a dimensionless number.

This speed of recession v can be computed from the radar red shift

* International Critical Tables **1**, McGraw Hill, 1926, p. 92.

ratio k as follows. Recalling the definitions

$$r = (\tau' - \tau)/2, \quad \sigma = (\tau' + \tau)/2,$$

we have

$$v = \frac{dr}{d\sigma} = \frac{d(\tau' - \tau)}{d(\tau' + \tau)} = \frac{k-1}{k+1}, \tag{2.6}$$

or equivalently

$$k = \frac{1+v}{1-v}.$$

<u>Thus as the radar red shift ratio</u> k <u>varies from</u> 0 <u>to</u> ∞ <u>the corresponding speed</u> v <u>varies from</u> -1 <u>to</u> +1. (In the real world the ratio k is always positive, and equivalently $|v|$ is always less than 1.)

For many purposes it is convenient to express the radar red shift ratio as an exponential $k = e^{2\varphi}$. In other words we work not with the ratio k or $\sqrt{k}$ itself but rather with the logarithm

$$\varphi = \log \sqrt{k}.$$

In particular consider two worldlines L and L_1 which lie in a common plane, so that the ratio $k = d\tau'/d\tau$ is constant.

<u>Definition.</u> In this case the number $\log \sqrt{k}$ will be called the <u>hyperbolic angle</u> $\varphi = \varphi(L, L_1)$ between the two lines L and L_1. (Taylor and Wheeler call φ the <u>velocity parameter</u> associated with L and L_1.)

Our signs are chosen so that φ is positive if the two lines L and L_1 are receding from each other, and negative if they are approaching. Of course φ changes sign at the intersection point.

The hyperbolic angle φ is related to the speed of recession v by the formula

$$v = \frac{k-1}{k+1} = \frac{e^{2\varphi}-1}{e^{2\varphi}+1} \ .$$

(Compare 2.6.) If we recall the definition of the hyperbolic tangent function

$$\tanh \varphi = \frac{\sinh \varphi}{\cosh \varphi} = \frac{(e^{\varphi} - e^{-\varphi})/2}{(e^{\varphi} + e^{-\varphi})/2} \ ,$$

then this formula can be written more compactly as follows.

Theorem 2.7. The speed of recession v is related to the hyperbolic angle φ by the formula $v = \tanh \varphi$.

In this form, there is a strong analogy with the elementary formula of Euclidean trigonometry which asserts that the tangent of an angle in a right triangle is the ratio of opposite side to adjacent side. If we consider the Minkowskian "right triangle" of Figure 2.8 whose base is part of the worldline L and whose hypotenuse is part of L_1, then the argument above shows that the ratio r/σ of opposite side to adjacent side is equal to $v = \tanh(\varphi)$.

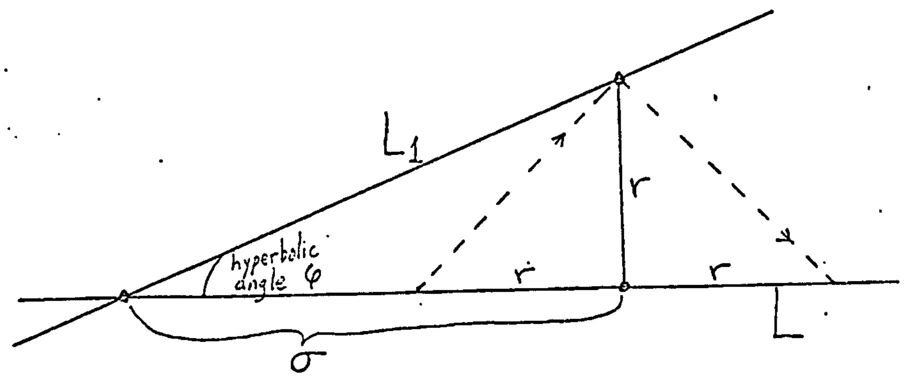

Figure 2.8. Hyperbolic Trigonometry: $r/\sigma = \tanh \varphi$.

For small values of v or φ note that the speed v is approximately equal to the hyperbolic angle φ. In fact one has the power series expansion

$$v = \tanh \varphi = \varphi - \frac{1}{3}\varphi^3 + \frac{2}{15}\varphi^5 - + \dots \ ,$$

which is convergent for $|\varphi| < \pi/2$.

For the worldlines encountered in everyday life $|\varphi|$ is very small, so this approximation is very accurate indeed. For example a crawling pace of one foot per second corresponds to

$$v \approx \varphi \approx 10^{-9} \ .$$

A typical freeway speed of 108 kilometers per hour corresponds to

$$v \approx \varphi \approx 10^{-7} \ .$$

The speed of sound at sea level corresponds to

$$v \approx \varphi \approx 1.2 \times 10^{-6} \ .$$

In other words the hyperbolic angle between a stationary observer and a receding sound wave is roughly one millionth of a "hyperbolic radian." As a final example, the average orbital speed of the Earth around the Sun is approximately 10^{-4}. Much larger speeds can be generated in the laboratory. For example electrons in the Stanford linear accelerator attain an energy* of 21×10^9 electron volts. Working out the corresponding hyperbolic angle, it turns out that $\varphi = 11.3$ hyperbolic radians. The corresponding speed $v = 0.9999999997$ is very close to 1.

To conclude this section, let us show that the number φ really has properties that one would expect of an "angle." One natural requirement is symmetry.

Lemma 2.9. Let L_1 and L_2 be any two intersecting worldlines. Then the hyperbolic angle $\varphi(L_1, L_2)$ between L_1 and L_2 is equal to the

* Compare §7.

hyperbolic angle $\varphi(L_2, L_1)$. Hence the speed of recession of L_2, as measured by an observer on L_1, is equal to the speed of recession of L_1, as measured by an observer on L_2.

Proof. Suppose that a light signal sent from L_1 at proper time τ_1 (using whatever personal choice of units is preferred by the observer on L_1) is received on L_2 at proper time τ_2 (using a presumably different system of units). Conversely, suppose that a signal sent from L_2 at time τ_2 is received on L_1 at time τ_1'. Note that the two derivatives $d\tau_2/d\tau_1$ and $d\tau_1'/d\tau_2$ are both constant. The radar red shift ratio as measured by the observer on L_1 is now given by

$$e^{2\varphi} = d\tau_1'/d\tau_1 = (d\tau_1'/d\tau_2)(d\tau_2/d\tau_1) .$$

Interchanging the two factors on the right, we see that the radar red shift ratio measured by an observer on L_2 would be precisely the same. ∎

Remark. This proof raises an interesting and fundamentally important question. How can two observers on divergent worldlines compare their systems of time measurement? We will return to this question in §3.

Another basic property of the concept of "angle" is additivity.

Lemma 2.10. If three worldlines L_1, L_2, L_3 lie in a common plane, with L_2 in the middle so that every light signal from L_1 to L_3 must pass through L_2, then $\varphi(L_1, L_3) = \varphi(L_1, L_2) + \varphi(L_2, L_3)$.

Proof. Consider a light signal, as in Figure 2.11, which leaves L_1 at proper time τ_1, crosses L_2 at proper time τ_2, is reflected from L_3 at proper time τ_3, crosses back over L_2 at proper time τ_2', and is received on L_1 at proper time τ_1'. Then the radar red shift ratio for L_1, L_3 is equal to

$$d\tau_1'/d\tau_1 = \left(\frac{d\tau_2}{d\tau_1}\frac{d\tau_1'}{d\tau_2'}\right)\left(\frac{d\tau_3}{d\tau_2}\frac{d\tau_2'}{d\tau_3}\right) .$$

Evidently the first expression in parentheses is equal to the radar red shift ratio for L_1, L_2, and the second is equal to the radar red shift ratio for L_2, L_3. Taking half the logarithm of both sides, the conclusion follows. ∎

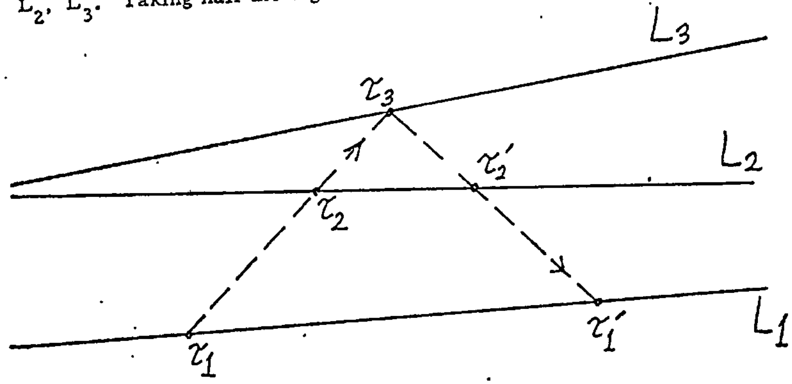

Figure 2.11. Proof of the addition formula for hyperbolic angles.

§3. The Longest Path between two Points

In §2 we managed to develop a surprising number of physical concepts starting with very meager assumptions as to the nature of the real world. In particular we were able to build up a scale of time along any one timelike worldline. To compare time scales along different worldlines we need some further hypotheses about the real world.

Working Hypothesis 3.1. *It is physically meaningful to compare intervals of time on different worldlines, and in fact to choose a common unit of time for all timelike worldlines.*

Actually, there are *many* different procedures which can be used in the real world to define a universal unit of time. One can choose some particular unstable particle or element and use its average lifetime as a unit. (The neutron, with average lifetime of 1013 seconds, would be a natural choice. Of course this lifetime must be measured in proper time along the worldline of the neutron.) Another possibility would be to use the "radius" of a hydrogen atom ($= 1.8 \times 10^{-19}$ seconds) as a unit of distance. One can even combine two of the most fundamental natural constants, namely Planck's constant $\hbar = 1.1734 \times 10^{-43}$ gram seconds* and Newton's gravitational constant $G = 2.4767 \times 10^{-39}$ seconds/gram to obtain the fundamental unit $\sqrt{\hbar G} = 5.3909 \times 10^{-44}$ seconds.

In actual practice, it is most convenient to choose some particular well defined spectral line, defining the unit of time or distance as a specified multiple of its wave length.

Remark. There is a very sharp hypothesis about the real world implicit in this discussion of different possible units of time. If there is to be just one unique concept of time, then we must assume that the ratio of any

*Remember that we are using the second as unit of time and distance.

two of these natural units, for example the dimensionless ratio

$$(\text{mean life of neutron})/(\text{radius of hydrogen atom}) = 5.6 \times 10^{21},$$

is a universal constant, having precisely the same value on any worldline anywhere in spacetime.

Henceforth, whenever we speak of proper time along two different worldlines, we will always assume that both proper times are measured in terms of some common unit of time.

Now let us consider light transmission between two different worldlines. Suppose that a light signal sent from L_1 at proper time τ_1 is received on L_2 at proper time τ_2.

Definition. The derivative $d\tau_2/d\tau_1$ will be called the *red shift ratio* associated with light signals from L_1 to L_2.

The term *red shift*, without any qualification, is used for the difference $\frac{d\tau_2}{d\tau_1} - 1$.

In general this red shift ratio varies with time. But in the special case where L_1 and L_2 lie in a common plane, the ratio is constant (or rather it has one constant value before the lines intersect and the reciprocal constant value afterwards.) Compare Figure 2.5.

In order to actually compute red shifts, we will need some version of the Poincaré-Lorentz Principle of Relativity, which implies that no experiment involving only time measurements and light signals can distinguish in any way between two different worldlines. (Compare §1.) The following special case of this principle will suffice for our purposes.

Symmetry Axiom 3.2. *If two timelike lines L_1 and L_2 lie in a common plane, then the constant red shift ratio associated with light signals from L_1 to L_2 must be precisely equal to the constant red shift ratio associated with light signals from L_2 to L_1.*

(A slightly different axiom, which would also suffice, is the following. Suppose that L_1 and L_2 intersect at the point x_0. Choose a line L through x_0 which bisects the angle between L_1 and L_2, and suppose that a light signal from one point of L hits L_1 at proper time τ_1 after the intersection point and hits L_2 at proper time τ_2 after the intersection point. The alternative axiom states that $\tau_1 = \tau_2$.)

Using 3.2, the red shift ratio can be computed as follows. Consider a light signal emitted from L_1 at proper time τ_1, reflected from L_2 at proper time τ_2, and received back on L_1 at proper time τ'_1. Then

$$d\tau_2/d\tau_1 = d\tau'_1/d\tau_2$$

by Axiom 3.2. But according to §2 the product $(d\tau'_1/d\tau_2)(d\tau_2/d\tau_1)$ is equal to $k = e^{2\varphi}$, where φ is the hyperbolic angle between L_1 and L_2. This proves the following.

Theorem 3.3. If L_1 and L_2 lie in a common plane, then the red shift ratio $d\tau_2/d\tau_1$ associated with light signals from L_1 to L_2 is equal to $\sqrt{k} = e^{\varphi}$, where φ is the hyperbolic angle between L_1 and L_2.

This red shift ratio e^{φ} is related to the speed of recession v by the equation $e^{\varphi} = \sqrt{(1+v)/(1-v)}$. (Compare §2.6.) Using the power series $e^{\varphi} - 1 = \varphi + \frac{1}{2}\varphi^2 + \frac{1}{6}\varphi^3 + \ldots$, we see that the red shift $e^{\varphi} - 1$ is approximately equal to the hyperbolic angle φ, or to the speed of recession v, providing that $|\varphi|$ is small.

To give a numerical example where φ is not so small, the quasi stellar object known as OQ172 has an observed* red shift of

$$\frac{d\tau_2}{d\tau_1} - 1 = 3.53 \ .$$

*Wampler, Robinson, Baldwin, and Burbidge; Nature 243 (1973), 336-337.

This is the largest red shift which has been measured by astronomers to date. If we assume that OQ172 is receding radially from the Earth (so that Theorem 3.3 applies), then it follows that

$$\varphi = \log(4.53) = 1.51 \text{ hyperbolic radians} \ ,$$

corresponding to a speed of recession of more than 9/10.

Caution: If we consider the more general case of an astronomical object whose relative velocity with respect to the Earth has not only a radial component (= speed of recession) v_{rad} but also a transverse component v_{trans}, then the correct formula* is

$$d\tau_2/d\tau_1 = (1+v_{rad})/\sqrt{1 - v_{rad}^2 - v_{trans}^2} \ .$$

Thus for the object OQ172 the most one can say with certainty is that $v_{rad} \le .907 \le \sqrt{v_{rad}^2 + v_{trans}^2}$. (In particular we cannot even be sure that v_{rad} is positive. The object OQ172 could even be approaching us. For example the values $v_{rad} = -.800$, $v_{trans} = .598$ satisfy the required equation. However this possibility seems extremely unlikely in view of the empirical fact that all distant astronomical objects exhibit large red shifts.)

Using 3.3, we will prove a result which plays a central role in relativity theory, analogous to the role of the Theorem of Pythagorus in classical geometry.

Fundamental Theorem 3.4 (Minkowski). The proper time interval $|\tau' - \tau|$ between any two points $x = [t, x, y, z]$ and $x' = [t', x', y', z']$ along any timelike line, anywhere in spacetime, is equal to

$$a\sqrt{(t'-t)^2 - (x'-x)^2 - (y'-y)^2 - (z'-z)^2} \ ,$$

where a is a positive constant.

In practice, we will always assume that the constant a is equal to +1. For the precise value of a can always be altered by applying an

* See for example C. Møller, "The Theory of Relativity", Oxford U. Press, 1952.

appropriate scale change to the coordinates t, x, y, z; so we may as well take the most convenient value for α.

Remarks. Taking $\alpha = 1$, we will write this equation briefly as

$$\Delta\tau = \sqrt{(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2} .$$

The expression under the square root sign is strictly positive, since the two points lie along a timelike line.

If we tried to apply this formula to a spacelike line segment, we would get an imaginary answer. However, changing the sign, it does make sense to define the proper distance between two points along a spacelike line by the formula $\Delta r = \sqrt{-(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$.

If we apply either of these formulas to a light path, we will always get zero. By definition, proper time and distance along any light path must be set equal to zero.

Proof of Theorem 3.4. We must show that the ratio $\Delta\tau/\sqrt{(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2}$ is the same for any two timelike line segments. Clearly it is the same for two segments which lie along the same timelike line.

First consider two lines L and L_1 which intersect, say at the origin. Choose a point $\underset{\sim}{x} = [t, x, y, z]$ on L to the right of the origin $(t > 0)$. Suppose that a light signal from $\underset{\sim}{x}$ hits the line L_1 at the point $\underset{\sim}{x}_1 = [t_1, x_1, y_1, z_1]$, and then is reflected back to hit L at the point $k\underset{\sim}{x}$. By §2 we have $k = e^{2\varphi} > 1$, where φ is the hyperbolic angle between the two lines. Evidently the two equations

$$(t - t_1)^2 - (x - x_1)^2 - (y - y_1)^2 - (z - z_1)^2 = 0$$

and

$$(kt - t_1)^2 - (kx - x_1)^2 - (ky - y_1)^2 - (kz - z_1)^2 = 0$$

must be satisfied. Subtracting k times the first equation from the second,

we obtain

$$(k^2 - k)(t^2 - x^2 - y^2 - z^2) = (k - 1)(t_1^2 - x_1^2 - y_1^2 - z_1^2) .$$

Dividing by $k - 1$ and taking the square root, this yields

$$e^{\varphi}\sqrt{t^2 - x^2 - y^2 - z^2} = \sqrt{t_1^2 - x_1^2 - y_1^2 - z_1^2} .$$

But it follows from 3.3 that

$$e^{\varphi}\tau = \tau_1 ,$$

where τ is the proper time interval from $\underset{\sim}{0}$ to $\underset{\sim}{x}$, and τ_1 the proper time interval from $\underset{\sim}{0}$ to $\underset{\sim}{x}_1$. Dividing these two equations we obtain

$$\tau/\sqrt{t^2 - x^2 - y^2 - z^2} = \tau_1/\sqrt{t_1^2 - x_1^2 - y_1^2 - z_1^2}$$

as required.

The proof for lines which intersect away from the origin is similar. Since any two disjoint timelike lines can easily be joined to each other by a third timelike line, this completes the proof. ∎

More generally consider any curve which can be described by smooth functions

$$x = x(t) , \quad y = y(t) , \quad z = z(t) .$$

(The word smooth will always mean having continuous first derivatives.)

Definition. Such a curve will be called timelike if the inequality

$$(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 < 1$$

is satisfied everywhere. (Compare §6.1.)

We will think of a timelike curve as a possible worldcurve, representing the total history of some material object whose motion may be subject to acceleration. Evidently a timelike line is one special case of a timelike curve.

Now consider a timelike curve segment C joining two points $\underset{\sim}{x}_0$ and $\underset{\sim}{x}_1$.

Definition. The Minkowski arc length of C will mean the integral

$$\int_C \sqrt{1 - (dx/dt)^2 - (dy/dt)^2 - (dz/dt)^2}\, dt \ .$$

We write this briefly as $\int_C \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$. In the special case of a timelike line segment, note that the Minkowski arc length is precisely equal to the expression $\Delta\tau = \sqrt{(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2}$ of §3.4.

Corresponding to the mathematical concept of Minkowski arc length, there is the physical concept of proper time as measured by an accurate clock.

Definition. For any timelike curve segment C we will identify the Minkowski arc length $\int_C \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$ with the length of proper time $\Delta\tau$ which would be measured by an accurate clock whose worldcurve is C.

To justify this definition, consider the following thought-experiment. Take two high quality Swiss watches, and synchronize them at the point $\underset{\sim}{x}_0$ in spacetime. Move the first watch (subjecting it to very high accelerations if necessary) so that its worldcurve is precisely equal to C. Move the second watch so that its worldcurve C' is a polygonal path which is very close to C, and whose first derivatives are very close to those of C. In other words we require the second watch to move in straight line segments, without any acceleration at all, but every now and then we tap it with a sledge hammer, with taps which are precisely judged to keep its position and velocity close to those of the first watch.

Now our experience with watches hopefully tells us that the two

watches will have approximately the same reading when the experiment ends at the point $\underset{\sim}{x}_1$ in spacetime. On the other hand a straightforward mathematical analysis shows that the Minkowski arc length of C' tends to the Minkowski arc length of C as the approximation of C by C' becomes arbitrarily close. ◻

The most fundamental, and perhaps surprising, property of Minkowski arc length (and hence of accurate clocks) is the following.

Theorem 3.5. A timelike straight line segment is the longest possible timelike path between its endpoints.

In other words it has the longest possible Minkowski arc length, or the longest possible proper time interval as measured by an accurate clock.

Using this Theorem, we can restate Galileo's hypothesis 1.1 in the following form.

Principle of Maximum Delay 3.6. Any material object on which no forces act moves in such manner as to take the maximum length of proper time in getting from one point of spacetime to another.

For the moment we will only prove 3.5 in one very special case. Suppose in fact that our timelike line segment joins two points $\underset{\sim}{x}_0 = [t_0, 0, 0, 0]$ and $\underset{\sim}{x}_1 = [t_1, 0, 0, 0]$ along the t-axis. Then for any timelike curve C from $\underset{\sim}{x}_0$ to $\underset{\sim}{x}_1$ we clearly have

$$\int_C \sqrt{1 - (dx/dt)^2 - (dy/dt)^2 - (dz/dt)^2}\, dt \le \int_C dt \ ,$$

where equality holds if and only if

$$dx/dt \equiv dy/dt \equiv dz/dt \equiv 0 \ .$$

This completes the proof in the special case. For the general case we refer to §6.2. (See also §4.5.) ◻

In practice the difference between the Minkowski arc length of such a curve C and of the corresponding straight line segment is very difficult to measure. This is because the speeds which we run into in everyday life tend to be in the range

$$\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} < 10^{-6} .$$

(Compare p. 16.) For such "small" speeds the integral $\int\sqrt{1 - (dx/dt)^2 - (dy/dt)^2 - (dz/dt)^2}\,dt$ will differ from $\int dt$ by less than one part in 10^{12}.

Theorem 3.5 has been paraphrased very vividly as the story of twin brothers: If Peter stays at home (so that his worldline is straight) while Paul travels hither and yon in a fast spaceship, then Paul, still a young man when he returns home, will find that Peter has grown old. For a numerical example, see §6.4.

Perhaps because such a phenomenon is never observed in everyday life, or perhaps because of a misunderstanding of the Principle of Relativity, this statement has often been considered paradoxical. In fact there is no paradox. One must simply learn to accept the fact that, in the actual world, a crooked worldcurve really is shorter than a straight one.

§4. Minkowskian Geometry

Previous discussion (particularly in §3.4) has emphasized the importance of the quadratic expression

$$(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2,$$

where for example $\Delta \mathbf{x} = \mathbf{x} - \mathbf{x}_0$. In order to avoid a plethora of deltas, let us restrict attention (for the moment) to the special case $\mathbf{x}_0 = \mathbf{0}$, and write this expression simply as

$$t^2 - x^2 - y^2 - z^2.$$

Here we must think of the 4-tuple $\mathbf{x} = [t,x,y,z]$ not as a "point" in spacetime, but rather as a "vector" from the origin to $\mathbf{x}$. (Compare §5.2.)

More generally consider two different vectors $\mathbf{x}_1 = [t_1,x_1,y_1,z_1]$ and $\mathbf{x}_2 = [t_2,x_2,y_2,z_2]$.

<u>Definitions</u>. The <u>Minkowski inner product</u> of $\mathbf{x}_1$ and $\mathbf{x}_2$ will mean the real number

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = t_1t_2 - x_1x_2 - y_1y_2 - z_1z_2 .$$

Clearly $\mathbf{x}_1 \cdot \mathbf{x}_2$ is symmetric and bilinear as a function of $\mathbf{x}_1$ and $\mathbf{x}_2$.

The Minkowski <u>length</u> of the vector $\mathbf{x}$ will mean the real number

$$\|\mathbf{x}\| = \sqrt{|\mathbf{x} \cdot \mathbf{x}|} = \sqrt{|t^2 - x^2 - y^2 - z^2|} .$$

In making use of this Minkowski length we must be very careful to distinguish between <u>timelike vectors</u>, with $\mathbf{x} \cdot \mathbf{x} > 0$, and <u>spacelike vectors</u>, with $\mathbf{x} \cdot \mathbf{x} < 0$. The remaining vectors, which point along the light cone based at the origin so that $\mathbf{x} \cdot \mathbf{x} = 0$, will be called <u>lightlike vectors</u>.

In the case of timelike or lightlike vectors, we must further distinguish between vectors which point forward or backward in time.

Definition. A vector $\underset{\sim}{x} = [t,x,y,z]$ is said to be a forward vector if $\underset{\sim}{x} \cdot \underset{\sim}{x} \geq 0$ (so that it is timelike or lightlike) and if $t > 0$. In other words $\underset{\sim}{x}$ is forward if $\underset{\sim}{x} \neq \underset{\sim}{0}$ and $t \geq \sqrt{x^2 + y^2 + z^2}$.

(In the case of a spacelike vector, it turns out that the sign of t has no particular physical meaning, since this sign can always be changed by switching to a different coordinate system. Compare §5.)

In particular we can speak of a forward unit vector, that is a forward vector $\underset{\sim}{u}$ which satisfies

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = 1.$$

We will use the notation $\underset{\sim}{u} = [c,u,v,w]$ for such a vector (where the letter c is supposed to suggest the hyperbolic cosine function, for reasons which will become apparant in a moment).

The set consisting of all forward unit vectors is 3-dimensional. In fact the coordinates u,v,w can be specified arbitrarily, and the initial coordinate c is then determined by the equation

$$c = \sqrt{1 + u^2 + v^2 + w^2}.$$

This set, consisting of all forward unit vectors, can be thought of as forming one sheet of a "hyperboloid of two sheets," consisting of all $\underset{\sim}{u}$ with $\underset{\sim}{u} \cdot \underset{\sim}{u} = 1$. (Similarly the "spacelike unit vectors," with $\underset{\sim}{u} \cdot \underset{\sim}{u} = -1$, form a hyperboloid of one sheet.)

Remark. The unit sphere in classical Euclidean geometry provides a model for the "spherical" non-Euclidean geometry, or for Riemannian geometry of constant positive curvature. Similarly, the set of forward unit vectors in Minkowski space provides a model for the "hyperbolic" non-Euclidean geometry of Lobachevski (and Bolyai), or for Riemannian geometry of constant negative curvature. In the same spirit, the set of all spacelike unit vectors in Minkowski space provides a model for a 3-dimensional "curved spacetime" of constant curvature.*

Lemma 4.1. If $\underset{\sim}{u}$ and $\underset{\sim}{u}_1$ are forward unit vectors, then the Minkowski inner product $\underset{\sim}{u} \cdot \underset{\sim}{u}_1$ is equal to the hyperbolic cosine $\cosh \varphi$ where φ is the hyperbolic angle between the line L joining the origin to $\underset{\sim}{u}$ and the line L_1 joining the origin to $\underset{\sim}{u}_1$.

Proof. According to §3.3 a light signal which leaves the line L at the point $\underset{\sim}{u}$ will hit the line L_1 at the point $e^{\varphi}\underset{\sim}{u}_1$. Therefore, by §1.4, the quadratic expression

$$(e^{\varphi}\underset{\sim}{u}_1 - \underset{\sim}{u}) \cdot (e^{\varphi}\underset{\sim}{u}_1 - \underset{\sim}{u}) = e^{2\varphi}\underset{\sim}{u}_1 \cdot \underset{\sim}{u}_1 - 2e^{\varphi}\underset{\sim}{u} \cdot \underset{\sim}{u}_1 + \underset{\sim}{u} \cdot \underset{\sim}{u}$$

must be zero. Since $\underset{\sim}{u}_1 \cdot \underset{\sim}{u}_1 = \underset{\sim}{u} \cdot \underset{\sim}{u} = 1$, we can solve this equation to obtain

$$\underset{\sim}{u} \cdot \underset{\sim}{u}_1 = (e^{\varphi} + e^{-\varphi})/2.$$

The expression on the right, by definition, is equal to $\cosh \varphi$. ◻

*See de Sitter, Proc. Roy. Acad. Sci. (Amsterdam) 19 (1917), 1217, and 20 (1917), 229 & 1309.

As an example, taking $\underset{\sim}{u}_1 = [1,0,0,0]$, we see that the initial component

$$[c,u,v,w] \cdot [1,0,0,0] = c$$

of the vector $[c,u,v,w]$ is equal to the hyperbolic cosine of an appropriate angle.

The 2-dimensional case is particularly instructive. If $\underset{\sim}{u} = [c,u,v,w]$ is a forward unit vector with $v = w = 0$, then it is easy to check that

$$\underset{\sim}{u} = [\cosh \varphi, \sinh \varphi, 0, 0]$$

for some uniquely defined hyperbolic angle φ. Compare Figure 4.2. (The points

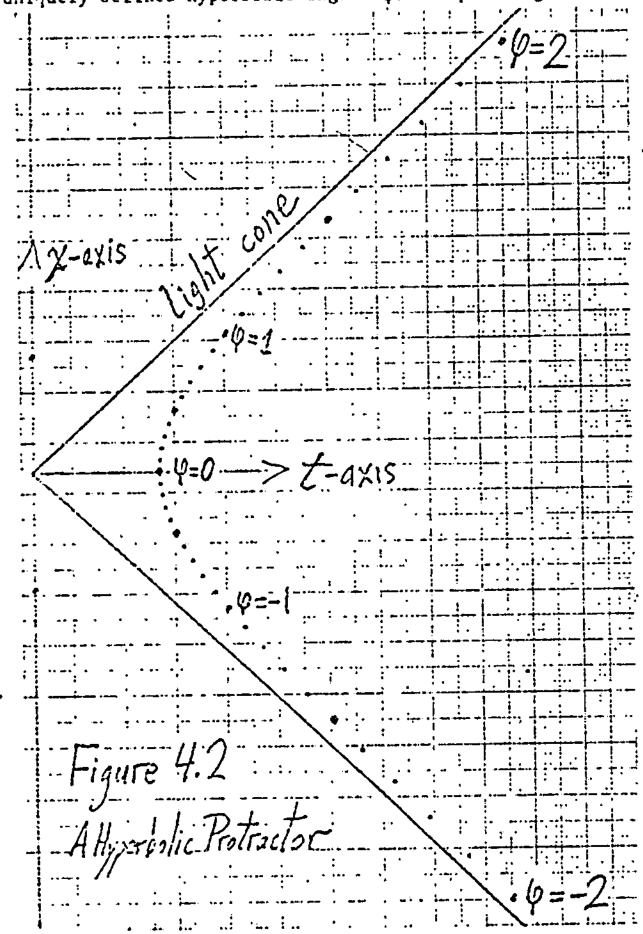

Figure 4.2
A Hyperbolic Protractor

intervals of one tenth of a hyperbolic radian.)

The following basic result is closely related to 4.1.

Backwards Schwarz Inequality 4.3. <u>If $\underset{\sim}{x}$ and $\underset{\sim}{x}_1$ are forward vectors, then</u>

$$\underset{\sim}{x} \cdot \underset{\sim}{x}_1 \geq \|\underset{\sim}{x}\| \ \|\underset{\sim}{x}_1\|$$

<u>where equality holds only if the vector $\underset{\sim}{x}$ is a multiple of $\underset{\sim}{x}_1$.</u>

<u>Proof</u>. If $\underset{\sim}{x}$ and $\underset{\sim}{x}_1$ are timelike, then setting

$$\underset{\sim}{x} = \|\underset{\sim}{x}\|\underset{\sim}{u}, \quad \underset{\sim}{x}_1 = \|\underset{\sim}{x}_1\|\underset{\sim}{u}_1$$

it follows from 4.1 that

$$\underset{\sim}{x} \cdot \underset{\sim}{x}_1 = \|\underset{\sim}{x}\| \ \|\underset{\sim}{x}_1\| \cosh \varphi,$$

where

$$\cosh \varphi = 1 + \frac{1}{2!}\varphi^2 + \frac{1}{4!}\varphi^4 + \cdots \geq 1.$$

Clearly $\cosh \varphi = 1$ only if $\underset{\sim}{u} = \underset{\sim}{u}_1$. This completes the proof if both $\underset{\sim}{x}$ and $\underset{\sim}{x}_1$ are timelike.

Now suppose that $\underset{\sim}{x}$ or $\underset{\sim}{x}_1$ is a lightlike vector. The inequality $x \cdot x_1 \geq \|x\| \|x_1\|$ follows easily by continuity. To analyse the situation when equality holds, ~~Then~~ we will use a different argument, based on the classical Schwarz Inequality*

$$|xx_1 + yy_1 + zz_1| \leq \sqrt{x^2 + y^2 + z^2} \ \sqrt{x_1^2 + y_1^2 + z_1^2}\,.$$

Since $\underset{\sim}{x}$ and $\underset{\sim}{x}_1$ are forward vectors we have $t \geq \sqrt{x^2 + y^2 + z^2}$, $t_1 \geq \sqrt{x_1^2 + y_1^2 + z_1^2}$, hence

$$tt_1 \geq |xx_1 + yy_1 + zz_1| \geq xx_1 + yy_1 + zz_1\,.$$

*See for example, Courant and Hilbert, "Methods of Mathematical Physics $\underset{\sim}{1}$," Interscience, 1953.

Clearly this implies that

$$tt_1 - xx_1 - yy_1 - zz_1 \geq 0 = \|\underset{\sim}{x}\| \ \|\underset{\sim}{x}_1\|,$$

as expected.

If equality holds in this final equation, then we must have equality throughout the argument, therefore

$$t = \sqrt{x^2 + y^2 + z^2} > 0, \qquad t_1 = \sqrt{x_1^2 + y_1^2 + z_1^2} > 0,$$

so both $\underset{\sim}{x}$ and $\underset{\sim}{x}_1$ are lightlike. Furthermore

$$\sqrt{x^2 + y^2 + z^2}\ \sqrt{x_1^2 + y_1^2 + z_1^2} = xx_1 + yy_1 + zz_1,$$

which, according to the classical Schwarz statement*, implies that $[x,y,z]$ is a positive multiple of $[x_1,y_1,z_1]$. Evidently it follows that $\underset{\sim}{x}$ is a positive multiple of $\underset{\sim}{x}_1$. ∎

Corollary 4.4 (Backwards Triangle Inequality). If $\underset{\sim}{x}$ and $\underset{\sim}{x}_1$ are forward vectors, then $\underset{\sim}{x} + \underset{\sim}{x}_1$ is a forward vector with

$$\|\underset{\sim}{x} + \underset{\sim}{x}_1\| \geq \|\underset{\sim}{x}\| + \|\underset{\sim}{x}_1\|,$$

where equality holds only if $\underset{\sim}{x}$ is a multiple of $\underset{\sim}{x}_1$.

In particular, if the sum of two forward vectors is lightlike, then the two vectors must be lightlike and linearly dependent.

Proof. This follows from the computation

$$(\underset{\sim}{x} + \underset{\sim}{x}_1)\cdot(\underset{\sim}{x} + \underset{\sim}{x}_1) = \underset{\sim}{x}\cdot\underset{\sim}{x} + 2\underset{\sim}{x}\cdot\underset{\sim}{x}_1 + \underset{\sim}{x}_1\cdot\underset{\sim}{x}_1$$
$$\geq \underset{\sim}{x}\cdot\underset{\sim}{x} + 2\|\underset{\sim}{x}\|\,\|\underset{\sim}{x}_1\| + \underset{\sim}{x}_1\cdot\underset{\sim}{x}_1$$
$$= (\|\underset{\sim}{x}\| + \|\underset{\sim}{x}_1\|)^2,$$

where equality holds only if $\underset{\sim}{x}$ is a multiple of $\underset{\sim}{x}_1$. ∎

It follows easily that any weighted average $\alpha\underset{\sim}{x} + (1-\alpha)\underset{\sim}{x}_1$ of forward vectors, with $0 \leq \alpha \leq 1$, is again a forward vector. Thus the collection of all forward vectors is convex. This collection is usually referred to as the forward convex cone.

We may think of 4.4 as a polygonal version of Theorem 3.5. The straight line from the origin to the point $\underset{\sim}{x} + \underset{\sim}{x}_1$ is longer than the polygonal path from $\underset{\sim}{0}$ to $\underset{\sim}{x}$ to $\underset{\sim}{x} + \underset{\sim}{x}_1$.

More generally, if $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n$ are forward vectors, then a straightforward induction on n shows that

$$\|\underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_n\| \geq \|\underset{\sim}{x}_1\| + \cdots + \|\underset{\sim}{x}_n\|,$$

where equality holds only if the vectors $\underset{\sim}{x}_i$ are all multiples of $\underset{\sim}{x}_1$. In other words a straight line from $\underset{\sim}{0}$ to the point $\underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_n$ is ~~shorter~~ longer than a polygonal path which proceeds from $\underset{\sim}{0}$ to $\underset{\sim}{x}_1$ to $\underset{\sim}{x}_1 + \underset{\sim}{x}_2$ to $\underset{\sim}{x}_1 + \underset{\sim}{x}_2 + \underset{\sim}{x}_3 \cdots$ and finally to the endpoint $\underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_n$.

Remark 4.5. Since every timelike curve can be approximated arbitrarily closely by such a polygonal path, it would not be difficult to use this result to complete the proof of Theorem 3.5.

*Ibid.

§5. The Poincaré-Lorentz Group

Consider a homogeneous linear transformation which maps each 4-tuple $\underset{\sim}{x} = [t,x,y,z]$ of real numbers to a 4-tuple $\underset{\sim}{x}' = [t',x',y',z']$. If we think of $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ as 1x4 matrices, then any such transformation can be expressed by the matrix equation

$$\underset{\sim}{x}' = \underset{\sim}{x}\Lambda,$$

where $\Lambda = [\Lambda_i{}^j]$ is a uniquely determined 4x4 matrix of real numbers.

Definition 5.1. The linear transformation $\underset{\sim}{x}' = \underset{\sim}{x}\Lambda$ is called a Lorentz transformation, and Λ is called a Lorentz matrix, if the identity

$$t^2 - x^2 - y^2 - z^2 = (t')^2 - (x')^2 - (y')^2 - (z')^2$$

is satisfied for every 4-tuple $\underset{\sim}{x}$.

Note that a Lorentz transformation also preserves Minkowski inner products. For if $\underset{\sim}{v}' = \underset{\sim}{v}\Lambda$ and $\underset{\sim}{w}' = \underset{\sim}{w}\Lambda$, then subtracting the equations $\underset{\sim}{v}\cdot\underset{\sim}{v} = \underset{\sim}{v}'\cdot\underset{\sim}{v}'$ and $\underset{\sim}{w}\cdot\underset{\sim}{w} = \underset{\sim}{w}'\cdot\underset{\sim}{w}'$ from $(\underset{\sim}{v}+\underset{\sim}{w})\cdot(\underset{\sim}{v}+\underset{\sim}{w}) = (\underset{\sim}{v}'+\underset{\sim}{w}')\cdot(\underset{\sim}{v}'+\underset{\sim}{w}')$ and dividing by 2, we obtain

$$\underset{\sim}{v}\cdot\underset{\sim}{w} = \underset{\sim}{v}'\cdot\underset{\sim}{w}'.$$

More generally consider an inhomogeneous linear transformation of the form

$$\underset{\sim}{x}' = \underset{\sim}{x}\Lambda + \underset{\sim}{c},$$

where $\underset{\sim}{c}$ is a constant vector. Such a transformation will be called a Poincaré-Lorentz transformation if Λ is a Lorentz matrix. Poincaré-Lorentz transformations can be characterized among all one-to-one transformations of 4-space, by two properties:

(1) The image of any line L in 4-dimensional coordinate space* is a line L'.

(2) For any pair of points $\underset{\sim}{x}_0$ and $\underset{\sim}{x}_1$ with images $\underset{\sim}{x}'_0$ and $\underset{\sim}{x}'_1$ respectively the identity

$$(t_1-t_0)^2-(x_1-x_0)^2-(y_1-y_0)^2-(z_1-z_0)^2 = (t'_1-t'_0)^2-(x'_1-x'_0)^2-(y'_1-y'_0)^2-(z'_1-z'_0)^2$$

is satisfied.

Thus a Poincaré-Lorentz transformation preserves the straight lines of §1.1 and the light paths of §1.3. Furthermore it preserves the proper time interval (or Minkowski arc length) of §3.4. If we also add the requirement that it preserve time orientation, then it will preserve all physically meaningful concepts:

Definition. A Lorentz matrix preserves time orientation if it transforms forward vectors into forward vectors.

It is not difficult to check that the Lorentz matrix Λ preserves time orientation if and only if its upper left-hand entry $\Lambda_0{}^0$ is positive.

Note the behavior of the difference vector $\Delta\underset{\sim}{x} = \underset{\sim}{x}_1 - \underset{\sim}{x}_0$ when we apply a Poincaré-Lorentz transformation to $\underset{\sim}{x}_0$ and $\underset{\sim}{x}_1$. Setting $\underset{\sim}{x}'_0 = \underset{\sim}{x}_0\Lambda + \underset{\sim}{c}$ and $\underset{\sim}{x}'_1 = \underset{\sim}{x}_1\Lambda + \underset{\sim}{c}$, the difference $\Delta\underset{\sim}{x}' = \underset{\sim}{x}'_1 - \underset{\sim}{x}'_0$ is given by the formula $\Delta\underset{\sim}{x}' = (\Delta\underset{\sim}{x})\Lambda$.

Definition 5.2. A 4-tuple $\underset{\sim}{v}$ is called a vector if it transforms in this manner, so that $\underset{\sim}{v}' = \underset{\sim}{v}\Lambda$.

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*A classical theorem of affine geometry asserts that any one-to-one transformation carrying lines into lines is linear. Compare Coxeter, "The Real Projective Plane," 2nd ed., Cambridge U. Press 1955, §8.101.

In other words a vector is not just a 4-tuple of real numbers, but rather a 4-tuple which is associated with a specific "Lorentz coordinate system" for spacetime, and which transforms in the following precisely specified manner. If we change the names of points in spacetime by introducing new coordinates $\underset{\sim}{x}' = \underset{\sim}{x}\Lambda + \underset{\sim}{c}$, then we must also change the names of vectors by setting $\underset{\sim}{v}' = \underset{\sim}{v}\Lambda$. (A [illegible] 4-tuple such as [altitude, temperature, barometric pressure, wind speed] may be very interesting, but is not a vector.)

For the moment, our only example of a vector is the rather trivial example of the difference $\Delta\underset{\sim}{x} = \underset{\sim}{x}_1 - \underset{\sim}{x}_0$ between two specified points in spacetime. Many other important vectors will be constructed in later sections (velocity and acceleration in §6, energy-momentum and force in §7, angular momentum in §9, gradient in §11, amplitude and frequency in §13, and current density in §15).

Caution. In the case of some of these "vectors," the transformation law 5.2 must be modified by introducing a sign. Thus the angular momentum vector $\underset{\sim}{a}$ transforms as $\underset{\sim}{a}' = \pm\underset{\sim}{a}\Lambda$ where the sign is $+1$ or -1 according as the determinant of Λ is $+1$ or -1; and the velocity vector $\underset{\sim}{u}$ transforms as $\underset{\sim}{u}' = \pm\underset{\sim}{u}\Lambda$ where the sign is $+1$ or -1 according as Λ preserves or reverses time orientation. We will not emphasize these distinctions, but will simply use the word vector in all cases.

Associated with every vector $\underset{\sim}{v} = [v^0, v^1, v^2, v^3]$ is a certain 4×1 matrix

$$v^* = \begin{bmatrix} v^0 \\ -v^1 \\ -v^2 \\ -v^3 \end{bmatrix}$$

which could be called the "associated covector." This notation is chosen so that the Minkowski inner product $\underset{\sim}{v}\cdot\underset{\sim}{w} = \underset{\sim}{w}\cdot\underset{\sim}{v}$ can be written simply as a matrix product $\underset{\sim}{v}\,\underset{\sim}{w}^*$.

Remark. Mathematicians usually prefer to identify the vector $\underset{\sim}{v} = [v^0, v^1, v^2, v^3]$ with the directional derivative operator

$$v^0\frac{\partial}{\partial t} + v^1\frac{\partial}{\partial x} + v^2\frac{\partial}{\partial y} + v^3\frac{\partial}{\partial z}.$$

Similarly they prefer to identify the covector $\underset{\sim}{v}^*$ with the linear differential form

$$\underset{\sim}{v}\cdot d\underset{\sim}{x} = v^0 dt - v^1 dx - v^2 dy - v^3 dz.$$

If these identifications are understood, then the transformation laws appropriate to vectors or to covectors follow automatically. We will not use these identifications, but will rather base our presentation on the formal definition 5.2.

If M is an arbitrary 4×4 matrix, we define the Minkowski adjoint M^* to be the unique 4×4 matrix which satisfies the identity

$$M^*\underset{\sim}{v}^* = (\underset{\sim}{v}M)^*$$

for every 4-tuple $\underset{\sim}{v}$.

Lemma 5.3. The Minkowski adjoint M^* is uniquely determined by this requirement. Furthermore the Minkowski adjoint $(MN)^*$ of a product is equal to the product N^*M^* of the Minkowski adjoints. The determinant of M^* is equal to the determinant of M; and the adjoint of the adjoint, M^{**}, is equal to the original matrix M.

The proof is not difficult. In fact one has the formula

$$M^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} M^{tr} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}^{-1}$$

where M^{tr} is the transpose of M, or more explicitly

$$\begin{bmatrix} a & b & c & d \\ A & B & C & D \\ \alpha & \beta & \gamma & \delta \\ \mathcal{A} & \mathcal{B} & \mathcal{C} & \mathcal{D} \end{bmatrix}^* = \begin{bmatrix} a & -A & -\alpha & -\mathcal{A} \\ -b & B & \beta & \mathcal{B} \\ -c & C & \gamma & \mathcal{C} \\ -d & D & \delta & \mathcal{D} \end{bmatrix}.$$

Further details will be left to the reader. □

Using this Minkowski adjoint operation, we can characterize Lorentz matrices as follows.

Lemma 5.4. <u>The</u> 4x4 <u>matrix</u> Λ <u>is a Lorentz matrix if and only if the product</u> $\Lambda\Lambda^*$ <u>equals the identity matrix</u> I.

For setting $\underset{\sim}{v}' = \underset{\sim}{v}\Lambda$ and $\underset{\sim}{w}' = \underset{\sim}{w}\Lambda$ the required identity $\underset{\sim}{v}\cdot\underset{\sim}{w} = \underset{\sim}{v}'\cdot\underset{\sim}{w}'$ takes the form

$$\underset{\sim}{v}\underset{\sim}{w}^* = (\underset{\sim}{v}\Lambda)(\underset{\sim}{w}\Lambda)^* = \underset{\sim}{v}(\Lambda\Lambda^*)\underset{\sim}{w}^*,$$

or in other words

$$\underset{\sim}{v}(I - \Lambda\Lambda^*)\underset{\sim}{w}^* = 0.$$

Clearly this identity is satisfied for all $\underset{\sim}{v}$ and $\underset{\sim}{w}$ only if $I = \Lambda\Lambda^*$. □

<u>It follows that every Lorentz matrix has determinant</u> ± 1.

We will use the notation $O(1,3)$ for the group consisting of all Lorentz matrices. (This notation indicates that we are working with 1 time dimension and 3 space dimensions.) Those Lorentz matrices which preserve time orientation and have determinant $+1$ form a subgroup which we will call the <u>Lorentz-rotation group</u>† and denote by $SO(1,3)$.

† The terms "proper" or "restricted" Lorentz group are used in the literature.

If we examine the actual formulas for the 16 different components of $\Lambda\Lambda^*$, then Lemma 5.4 can be paraphrased as follows.

Corollary 5.5. <u>The</u> 4x4 <u>matrix</u> Λ <u>is a Lorentz matrix if and only if the top row of</u> Λ <u>is a timelike unit vector, and the remaining three rows are spacelike unit vectors, all four rows being Minkowski orthogonal to each other</u>.

As an example, consider a Lorentz-rotation matrix of the special form

$$\Lambda = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The top row

$$[1,0,0,0]\Lambda = [a,b,0,0]$$

must be a forward unit vector, and therefore must have the form $[\cosh\varphi, \sinh\varphi, 0, 0]$ for some uniquely determined hyperbolic angle φ. The second row must be a Minkowski orthogonal unit vector. Taking account of the requirement that the determinant must be $+1$, the only possible solution is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{bmatrix}.$$

Such a transformation can be thought of as a "rotation" of the Minkowski plane through φ hyperbolic radians. It carries any timelike line L in the (t,x)-plane to a new timelike line L' so that the hyperbolic angle between L and L' is equal to $|\varphi|$. The eigenvalues of this hyperbolic rotation are e^{φ} and $e^{-\varphi}$. The corresponding eigenvectors point along the light cone. (Compare Figure 5.6.)

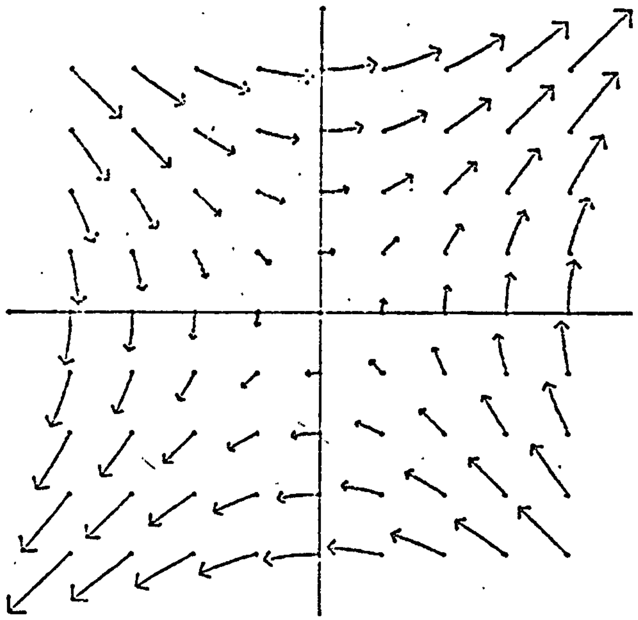

Figure 5.6 Rotation of the Minkowski plane through .22 hyperbolic radians. (Each arrow leads from x to $x\Lambda$.)

The following Lemma tells us that there are "enough" Lorentz matrices.

Lemma 5.7. <u>The top row of a Lorentz-rotation matrix Λ (or in other words the image $\underset{\sim}{u}_0\Lambda$ of the standard basis vector $\underset{\sim}{u}_0 = [1,0,0,0]$) can be a completely arbitrary forward unit vector</u>.

It follows as an easy corollary that <u>the image of a timelike line under a Poincaré-Lorentz transformation can be a completely arbitrary timelike line</u>.

If two different Lorentz-rotation matrices Λ_1 and Λ_2 have the same top row

$$\underset{\sim}{u}_0 \Lambda_1 = \underset{\sim}{u}_0 \Lambda_2,$$

then the quotient $\Lambda_1\Lambda_2^{-1}$ has top row equal to $\underset{\sim}{u}_0$. It follows easily that $\Lambda_1\Lambda_2^{-1}$ is a matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & P & \\ 0 & & & \end{bmatrix}$$

where the 3x3 matrix P belongs to the group $SO(3)$ of rotations of Euclidean 3-space.

<u>Proof of</u> 5.7.. The matrix Λ can easily be constructed inductively, starting with a given top row and then constructing successive rows so as to satisfy the requirements of 5.5. Alternatively, choosing the hyperbolic angle φ and the Euclidean rotation P appropriately, the product matrix

$$\Lambda = \begin{bmatrix} \cosh\varphi & \sinh\varphi & 0 & 0 \\ \sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & P & \\ 0 & & & \end{bmatrix}$$

will have any required forward unit vector as first row. ☐

We can gain a deeper insight into the structure of the Lorentz-rotation group by introducing the exponential power series

$$\exp M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots .$$

Here M can be an arbitrary square matrix; this series always converges*. Typical examples are provided by the hyperbolic rotation

$$\exp\begin{bmatrix} 0 & \varphi \\ \varphi & 0 \end{bmatrix} = \begin{bmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{bmatrix},$$

and the Euclidean rotation

$$\exp\begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$$

---

*<u>Proof</u>. Choose any positive definite "norm" on the algebra of $n \times n$ real matrices, that is any function $\|M\| > 0$ for $M \neq 0$ satisfying the inequalities $\|M + N\| \leq \|M\| + \|N\|$, $\|MN\| \leq \|M\|\ \|N\|$, and $\|aM\| = |a|\ \|M\|$. For example the function $\|M\| = \Sigma\, |M_i^j|$ is a norm in this sense. Then $\|M^k/k!\| \leq \|M\|^k/k!$, so the series is absolutely convergent, with $\|\exp M\| \leq e^{\|M\|}$.

If the matrices M and N commute, $MN = NM$, then by rearranging power series one can check that

$$\exp(M + N) = (\exp M)(\exp N).$$

In particular, taking $N = -M$, it follows that

$$\exp(-M) = (\exp M)^{-1}.$$

Specializing to the case of 4×4 matrices, note that the exponential of a Minkowski adjoint is given by

$$\exp(M^*) = (\exp M)^*.$$

Now suppose that the matrix M is <u>skew self adjoint</u>, in the sense that

$$M^* = -M.$$

<u>Then it follows that</u>

$$(\exp M)(\exp M)^* = I,$$

<u>so that</u> $\exp M$ <u>is a Lorentz matrix</u>.†

---

†If the matrix M is close to 0, then a converse statement is also true. Suppose that $\exp M$ is a Lorentz matrix, where $\|M\| < \log 2$ and $\|M^*\| < \log 2$. Applying the logarithmic power series $\log(I + N) = N - \frac{1}{2}N^2 + - \cdots$ to both sides of the equation $\exp M^* = \exp(-M)$, it follows that $M^* = -M$.

In fact $\exp M$ is a Lorentz-rotation matrix, whenever M is skew self adjoint. For the 1-parameter family of Lorentz matrices $\exp(\alpha M)$, with $0 \le \alpha \le 1$, can be used to prove that $\exp M$ preserves time orientation and has determinant $+1$.

Using this notation, the Euclidean rotation group $SO(3)$ can be described very explicitly as follows. Every matrix in $SO(3)$ can be expressed as an exponential $P = \exp S$ where

$$S = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

is a skew symmetric matrix with $a^2 + b^2 + c^2 \le \pi^2$. Furthermore this expression is unique, except that $\exp S = \exp(-S)$ whenever $a^2 + b^2 + c^2 = \pi^2$.

In more familiar terms, every rotation of Euclidean 3-space can be described as a rotation through some angle $\sqrt{a^2+b^2+c^2} \le \pi$ about some axis $[a,b,c]$. It follows that $SO(3)$ can be described topologically as a disk of radius π with antipodal boundary points identified.

The full Lorentz-rotation group can now be described as follows. Every matrix Λ in $SO(1,3)$ can be expressed as a product

$$\exp\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & S & \\ 0 & & & \end{bmatrix} \exp\begin{bmatrix} 0 & p & q & r \\ p & 0 & 0 & 0 \\ q & 0 & 0 & 0 \\ r & 0 & 0 & 0 \end{bmatrix}$$

with S as above. Again this expression is unique except that $\exp S = \exp(-S)$ when $a^2 + b^2 + c^2 = \pi^2$. The proofs of these statements will be left to the reader.

§6. Velocity and Acceleration

Let us return to the study of a "timelike curve." This was defined in §3 as a curve C in spacetime along which the coordinates x,y,z can be expressed as smooth functions

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

of the coordinate time t, with

$$(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 < 1.$$

This definition is not really satisfactory, since it depends too strongly on one particular choice of coordinate system. We will make use of this definition for the moment, but only in order to motivate a more invariant definition.

Consider the proper time (= Minkowski arc-length)

$$\int \| d\underset{\sim}{x} \| = \int \sqrt{dt^2-dx^2-dy^2-dz^2} = \int \sqrt{1-(dx/dt)^2-(dy/dt)^2-(dz/dt)^2}\, dt,$$

integrated along the curve C. Starting from some arbitrarily chosen base point $\underset{\sim}{x}_0$ along C, we can define the function

$$\tau(\underset{\sim}{x}) = \int_{\underset{\sim}{x}_0}^{x} \| d\underset{\sim}{x} \|$$

integrated along C. If we think of x,y,z as functions of the coordinate time t, then evidently

$$d\tau/dt = \sqrt{1-(dx/dt)^2-(dy/dt)^2-(dz/dt)^2} > 0.$$

Therefore, by the inverse function theorem, we can solve for

$$t = t(\tau)$$

as a smooth function of the proper time τ. Hence we can express all four coordinates t,x,y,z along C as smooth functions of τ. We will write briefly

$$\underset{\sim}{x} = \underset{\sim}{x}(\tau).$$

Note that the derivative $d\underset{\sim}{x}/d\tau$ is a forward unit vector. For the formula

$$\tau = \int \| d\underset{\sim}{x} \| = \int \| d\underset{\sim}{x}/d\tau \| \, d\tau$$

certainly implies that $\| d\underset{\sim}{x}/d\tau \| = 1$; and more explicitly the formula

$$d\underset{\sim}{x}/d\tau = (d\tau/dt)^{-1}[1, dx/dt, dy/dt, dz/dt]$$

implies that $d\underset{\sim}{x}/d\tau$ is a forward unit vector. This suggests the following alternative definition.

Definition 6.1. A subset C of spacetime is called a smooth timelike curve if it can be described parametrically as the image of a smooth function

$$\underset{\sim}{x} = \underset{\sim}{x}(\tau),$$

where τ varies over some interval of real numbers, such that the derivative

$$\underset{\sim}{u} = d\underset{\sim}{x}/d\tau$$

is a forward unit vector for all τ.

This unit vector $\underset{\sim}{u} = \underset{\sim}{u}(\tau)$ is called the unit velocity vector of the curve C. The reader should check that $\underset{\sim}{u}$ really does transform as a vector (compare §5.2), providing that we only allow Lorentz transformations which preserve time orientation.

Since both the proper time τ and the four components of $\underset{\sim}{x}$ must be expressed in units of time, it is clear that $\underset{\sim}{u} = d\underset{\sim}{x}/d\tau$ is a dimensionless vector.

That is, its four components do not depend on any particular choice of a unit of time. By way of contrast, the components of a difference vector $\Delta\underset{\sim}{x} = \underset{\sim}{x}_1 - \underset{\sim}{x}_0$ must be expressed in terms of seconds, years, or some other unit of time or distance. Briefly we say that the vector $\Delta\underset{\sim}{x}$ has the dimensions of time.

Note that the velocity vector $\underset{\sim}{u}$ has only three degrees of freedom. For its initial component can be expressed as a function of the remaining three components.

Given a timelike curve in the sense of Definition 6.1, we can clearly express the coordinates x,y,z along C as smooth functions of t with $(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 < 1$. Thus the new definition is compatible with the old one.

Remark 6.2. It is now easy to complete the proof that a straight line segment is the longest timelike curve between its endpoints. We must simply choose a Poincaré-Lorentz transformation which maps the line segment into the t-axis. This is possible by §5.7. Then any timelike curve will map into a new timelike curve with the same Minkowski arc length. Thus we reduce the general case to the special case for which the argument has already been given in §3.5. ∎

Suppose that the function $\underset{\sim}{x} = \underset{\sim}{x}(\tau)$ is twice continuously differentiable. The second derivative

$$\underset{\sim}{a} = d\underset{\sim}{u}/d\tau = d^2\underset{\sim}{x}/d\tau^2$$

is called the acceleration vector of the curve C. Evidently this vector has the dimensions of time^{-1}.

Differentiating the identity $\underset{\sim}{u} \cdot \underset{\sim}{u} = 1$, we see that

$$\underset{\sim}{u} \cdot \underset{\sim}{a} = 0.$$

Thus the acceleration vector must always be Minkowski orthogonal to the velocity vector. It follows easily that $\underset{\sim}{a}$ is always a spacelike vector (or zero).

Thus the acceleration vector, like the velocity vector, has just three degrees of freedom; for the set of vectors orthogonal to some fixed $\underset{\sim}{u}$ is a 3-dimensional vector space.

The case of motion in just one space dimension (and one time dimension) is particularly instructive. If

$$\underset{\sim}{x} = \underset{\sim}{x}(\tau)$$

is a timelike curve lying in the hyperplane $y = z = 0$, then its velocity vector $\underset{\sim}{u} = d\underset{\sim}{x}/d\tau$ must have the form

$$\underset{\sim}{u} = [\cosh \varphi, \sinh \varphi, 0, 0]$$

for some uniquely defined hyperbolic angle $\varphi = \varphi(\tau)$. Therefore

$$\underset{\sim}{a} = [\sinh \varphi, \cosh \varphi, 0, 0] d\varphi/d\tau.$$

Note that

$$\|\underset{\sim}{a}\| = |d\varphi/d\tau|.$$

The real number $a = d\varphi/d\tau$ can be described as the scalar acceleration of this 1-dimensional motion.

The most familiar acceleration in everyday life is the acceleration of 9.8 meters/second^2 due to Earth's gravity. If we use the year* as unit of both time and distance, then a short computation shows that this acceleration has magnitude

$$\|\underset{\sim}{a}\| = 1.03 \text{ year}^{-1}.$$

*1 equinoctial year = 3.15569 x 10^7 seconds.

In other words, if the x-coordinate measures height above the Earth's surface in years, then the scalar acceleration of a falling body is

$$d\varphi/d\tau = -1.03 \text{ hyperbolic radians/year}$$

(at least for small* values of $|\varphi|$).

The concept of "acceleration" in Minkowskian geometry is completely analogous to the concept of "curvature" for curves in Euclidean space. In particular, a well-known theorem about curves in the Euclidean plane has the following Minkowskian analogue.

Fundamental Theorem 6.2. The scalar acceleration $a = d\varphi/d\tau$ of a curve in the Minkowski plane can be prescribed as an arbitrary continuous function of the proper time τ. The curve

$$\underset{\sim}{x}(\tau) = [t(\tau), x(\tau), 0, 0]$$

is then uniquely determined up to a Poincaré-Lorentz motion.

*According to general relativity theory, the precise formula for the acceleration of a particle falling freely in the (vertical) x-direction in a fixed gravitational field is

$$d\varphi/d\tau = -(\partial V/\partial x)\cosh \varphi,$$

where $V(x)$ is the gravitational potential function. Thus acceleration increases rapidly with speed, for large φ. By way of contrast, if a charged particle falls freely in the direction of a fixed electric field, we will see in §10 that its acceleration is independent of speed.

Proof. The formula $\varphi(\tau) = \int a(\tau)d\tau$ shows that φ is uniquely determined up to an additive constant, and the formulas

$$t(\tau) = \int \cosh\varphi\, d\tau\ , \quad x(\tau) = \int \sinh\varphi\, d\tau$$

then show that t and x are uniquely determined, up to additive constants. □

As an example, consider the case of constant scalar acceleration. If

$$d\varphi/d\tau = a$$

is constant, then (choosing the constant of integration appropriately) we may assume that

$$\varphi(\tau) = a\tau,$$

hence

$$t = \int(\cosh a\tau)d\tau = a^{-1}\sinh a\tau + t_0$$

and

$$x = \int(\sinh a\tau)d\tau = a^{-1}\cosh a\tau + x_0.$$

Taking $t_0 = x_0 = 0$, the solution curve is just one of the two branches of the hyperbola

$$x^2 - t^2 = 1/a^2,$$

consisting of all vectors $x = [t,x,0,0]$ with

$$\underset{\sim}{x}\cdot\underset{\sim}{x} = -1/a^2.$$

We can think of this hyperbola as the Minkowskian analogue of a circle of radius $1/a$.

For a graph of the resulting worldcurve, the reader need only stand Figure 4.2 on its side. Successive dots in this figure represent equal intervals of proper time.

One example of a worldcurve with constant acceleration is provided by a charged particle which is freely falling in a constant electromagnetic field. (Compare §10.5.) Another can be described as follows.

Example 6.4. Suppose that we are given a magical* spaceship which is able to maintain the constant acceleration $a = d\varphi/d\tau$ of one hyperbolic radian per proper year for long periods of time. (Recall that this is approximately one gravity.) If the spaceship starts at the origin, $\underset{\sim}{x} = \underset{\sim}{0}$, with coordinate speed 0 (so that its velocity vector is [1,0,0,0]), then after accelerating for τ years of proper time it will attain the position

$$\underset{\sim}{x}(\tau) = [\sinh\tau, (\cosh\tau)-1,0,0] \text{ years.}$$

As an example, in 2.5 proper years it will attain the position

$$\underset{\sim}{x}(2.5) = [\sinh 2.5, (\cosh 2.5)-1,0,0] \approx [6,5,0,0] \text{ years.}$$

The coordinate speed $dx/dt = \tanh 2.5$ will then be very close to the speed of light. Now reversing engines and decelerating for 2.5 proper years, the ship can bring its coordinate speed back down to zero, attaining the position

$$\underset{\sim}{x} \approx [12,10,0,0] \text{ years.}$$

Thus in a total of 5 years of proper time, this spaceship can make a trip to a star† at a distance of 10 years. Reversing the entire procedure, in 5 more years

*Compare §7.9.

†There are nine known stars within ten years of the sun, the closest being the three Alpha Centauri stars at a distance of 4.3 years.

of proper time, the spaceship can return to Earth at the point

$$\underset{\sim}{x} \approx [24,0,0,0]$$

of spacetime. Thus the round trip journey would take only 10 years according to the ship's chronometer, but would last for 24 years according to an observer on Earth.

This formulation puts the "twin paradox" of §3 into particularly sharp perspective, since an astronaut who travels out and back on our spaceship for 10 years, and his twin who watches the trip from Earth for 24 years, will both experience the same apparent acceleration of one gravity throughout the entire period.

The distance which can be covered in such a trip grows very rapidly with proper time, according to the formula

$$\text{distance} = 2a^{-1}(\cosh(\tfrac{1}{2}a\Delta\tau) - 1),$$

where a is the acceleration and $\Delta\tau$ the proper time for the one way trip. As a dramatic example, suppose that we wish to visit the Andromeda galaxy, at a distance of 1.5 million years. Computation shows that such a trip would require $\Delta\tau = 28.5$ years at an acceleration of 1 year^{-1}. The astronaut who made such a journey to the Andromeda galaxy would return, after 57 proper years, to an Earth which had aged by 3 million years.

§7. The Energy-Momentum Vector

It is time to infuse another basic idea from the real world into our mathematical model. The concept of the energy-momentum vector was implicit in the work of Planck and Minkowski in 1908. Using this vector we will begin to formulate the law of conservation of energy-momentum, surely the most fundamental and pervasive law in all of physics. More complete versions of this law will be given in later sections.

The basic empirical fact is the following.

Axiom 7.1. To every particle, and every point $\underset{\sim}{x}$ on the worldcurve of this particle, there is associated a forward vector

$$\underset{\sim}{p} = [\varepsilon, p, q, r]$$

called the energy-momentum of the particle at the point $\underset{\sim}{x}$ in spacetime. This vector $\underset{\sim}{p}$ is always tangent to the worldcurve C at the point $\underset{\sim}{x}$.

The initial component ε of $\underset{\sim}{p}$ is called the energy of the particle with respect to the given coordinate system. Note that ε is always strictly positive. The remaining three components p,q,r describe the momentum* of the particle with respect to this coordinate system.

The norm $\|\underset{\sim}{p}\| = \sqrt{\varepsilon^2-p^2-q^2-r^2}$ is an invariant called the mass m of the particle. (In the older literature, m was called the "rest-mass," and ε was called the "mass.") We can distinguish two essentially different cases.

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*The word "momentum" seems difficult to translate into other languages. In German it comes out as "Impuls" or "Bewegungsgrösse"; but no two dictionaries agree as to the correct translation into French.

Case 1. If the worldcurve C is timelike, with velocity vector $\underset{\sim}{u} = d\underset{\sim}{x}/d\tau$, then evidently

$$\underset{\sim}{p} = m\underset{\sim}{u}.$$

Thus the mass must be strictly positive. This equation is often taken as the definition of the energy-momentum vector for a particle of positive mass.

The difference $\varepsilon - m$ is called the kinetic energy of the particle with respect to the given coordinate system. Note the inequalities

$$\varepsilon > \sqrt{p^2+q^2+r^2} \geq \varepsilon - m \geq 0,$$

or in other words

$$\text{energy} > \|\overrightarrow{\text{momentum}}\| \geq \text{kinetic energy} \geq 0.$$

(Proofs will be left as an exercise.) If we express the velocity vector as $\underset{\sim}{u} = [\cosh\varphi, u, v, w]$, where φ is the hyperbolic angle between $\underset{\sim}{u}$ and the coordinate vector $[1,0,0,0]$, then evidently $\varepsilon = m\cosh\varphi$. Expanding $\cosh\varphi$ as a power series, it follows that the kinetic energy is given by

$$\varepsilon - m = \frac{1}{2}m\varphi^2 + \frac{1}{24}m\varphi^4 + \cdots .$$

The first term on the right is essentially the pre-relativistic formula for kinetic energy.

Here is a numerical example. Suppose that we use the gram as unit of mass, and hence also of momentum and energy. Consider a ten ton truck ($m = 10^7$ grams) traveling at freeway speed ($\varphi = 10^{-7}$ hyperbolic radians). The momentum of this truck is approximately 1 gram. This follows from the computation

$$\|\overrightarrow{\text{momentum}}\| = m\sinh\varphi \approx m\varphi.$$

The kinetic energy is approximately 0.5×10^{-7} grams, since

$$\varepsilon - m = m(\cosh\varphi - 1) \approx \frac{1}{2}m\varphi^2.$$

Thus it would take twenty million such trucks, all traveling at freeway speed, to achieve a total kinetic energy of just one gram. The gram is a small unit of mass, but a large unit of momentum, and an enormous unit of kinetic energy. To give another numerical example, the total electric power output of the United States (roughly 200,000 megawatts) amounts to just .002 grams/second.

Case 2. Now consider a particle with mass $m = 0$. Then $\varepsilon = \sqrt{p^2+q^2+r^2}$ or briefly

$$\text{energy} = \text{kinetic energy} = \|\overrightarrow{\text{momentum}}\| > 0.$$

The worldcurve C in this case must have coordinate speed $\sqrt{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2}$ equal to 1. In fact we will always assume that the worldcurve of a massless particle is a straight line or line segment (i.e., a "light path").

Only two kinds of particles of mass zero have actually been observed, namely photons and neutrinos*. A third particle of mass zero, the "graviton," has been postulated but never observed.

Let us begin to formulate the Conservation Law. We will assume that all of our worldcurves are piecewise smooth, so that the velocity vector $\underset{\sim}{u}$ along any worldcurve is allowed to have simple jump discontinuities only. Our world-curves are allowed to have beginning points (particle creation) or endpoints (particle annihilation).

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*One can distinguish between four different kinds of neutrino; but we will ignore these finer distinctions.

Definition. A collision point in spacetime will mean any point at which a worldcurve begins, or ends, or changes direction.

Axiom 7.2 (Conservation Law). For any collision point $\underset{\sim}{x}$ in spacetime, the sum $p_1 + \cdots + p_k$ of the energy-momentum vectors of all particles whose worldcurves converge at $\underset{\sim}{x}$ is equal to the sum $p_1' + \cdots + p_\ell'$ of the energy-momentum vectors of all particles which emerge from this collision.

Thus energy-momentum can neither be created nor destroyed in a collision.*

Note that the sum $p_1 + \cdots + p_k$ cannot be zero, since it is a sum of forward vectors. Hence k particles coming together $(k \geq 1)$ cannot simply annihilate each other. Something must emerge from the collision point. Similarly, particles cannot spontaneously appear at a point of spacetime without violating the Conservation Law. There must be at least one worldline which goes into the collision point.

Note also that a particle cannot gratuitously change direction. If the energy-momentum vector of a particle has an instantaneous jump at some point of spacetime, we can be sure that the particle has either collided with something or ejected something.

To illustrate this Conservation Law, let us look at collisions of "elementary particles." A multitude of different elementary particles are known to physicists. Each one has a distinctive fixed mass, which helps to identify it. The following table lists a few of the better known elementary particles. For convenience we have taken the mass of the electron (= 9.10956×10^{-28} grams or .511004 MeV) as the unit of mass.

*The law of conservation of energy was first formulated by Helmholtz, while the law of conservation of momentum goes back essentially to Newton. This 4-dimensional formulation is implicit in the work of Planck and Minkowski.

particle	mass	mean lifetime
electron	1	∞
muon	206.77*	2.2×10^{-6} seconds
neutral pion	264.13*	1.9×10^{-16} seconds
proton	1836.10	∞
neutron	1838.63*	1013 seconds

These elementary particles can be used as building blocks for the formation of more complicated particles. The simplest example is the deuteron, built up out of one proton and one neutron, with mass equal to 3669.37 electron masses.

One particularly interesting form of "collision" is spontaneous decay or fission. This is the case $k = 1$, where just one worldcurve leads into the collision point, but two or more worldcurves emerge from it. A basic observation is the following.

Law of Non-Conservation of Mass. If a single particle of mass m undergoes fission, decaying into a collection of ℓ particles with masses $m_1', \ldots, m_\ell'$, then $m > m_1' + \cdots + m_\ell'$.

This is essentially just the backwards triangle inequality

$$\|p_1' + \cdots + p_\ell'\| \geq \|p_1'\| + \cdots + \|p_\ell'\|$$

of §4.4. Equality could hold here only if the worldcurves which emerged from the collision point were all tangent to each other. In fact this never happens.

One important consequence is the following statement: A particle of mass zero can never spontaneously decay. This follows since particles of negative mass are not permitted.

* In the case of an unstable particle with mean lifetime τ, the mass is only well defined to within an uncertainty of the order of $\hbar/\tau$ where $\hbar \approx 10^{-48}$ gram-seconds.

Here are three numerical examples.

(i) The neutron, with a mass of 1838.63, decays spontaneously into a proton, an electron, and a neutrino. The total mass of these decay products is

$$1836.10 + 1 + 0 = 1837.10.$$

Thus the total mass decreases by 1.53 electron masses. According to the conservation law, this difference must go into kinetic energy. More precisely, if the neutron is at rest ($u = [1,0,0,0]$) before fission, then the sum of the kinetic energies of the resulting proton, neutron, and neutrino must be precisely 1.53 electron masses.

(ii) The neutral pion decays spontaneously into two photons. In this case there is no mass at all but only kinetic energy after the decay.

(iii) The deuteron is stable. It cannot decay into a neutron and a proton since its mass is smaller than the total mass of a neutron and a proton. The difference

$$m_{neutron} + m_{proton} - m_{deuteron} = 1838.63 + 1836.10 - 3669.37$$
$$= 5.36 \text{ electron masses}$$

represents the additional energy which would have to be supplied from outside in order to split up the deuteron. (From another point of view this "binding energy" of 5.36 electron masses represents additional energy which must be ejected in some manner if we can manage to fuse a neutron and a proton together.)

The opposite of the process of fission is of course fusion, in which two or more particles collide and fuse together. Note that the total mass $m_1 + \cdots + m_k$ before fusion, must be strictly less than the mass m' after fusion. Again this follows from §4.4.

Here is a macroscopic example. If two ten ton trucks ($m_1 = m_2 = 10^7$ grams) collide head on and fuse together, then the total mass m' after the collision must be strictly greater than $m_1 + m_2$. In fact we can compute m' as follows. Recall that the kinetic energy of each truck (assuming a freeway speed of 108 kilometers per hour) is

$$\varepsilon_i - m_i = 0.5 \times 10^{-7} \text{ grams.}$$

Hence the total mass of the wreckage will be

$$m' = \varepsilon_1 + \varepsilon_2 = m_1 + m_2 + 10^{-7} \text{ grams.}$$

This extra 10^{-7} grams will consist mostly of heat energy*. It will disappear when the trucks have cooled off.

Next let us introduce the concept of force. Consider a particle of positive mass whose worldcurve C is twice continuously differentiable.

Definition 7.3. The derivative of the vector $p = mu = mdx/d\tau$ with respect to the proper time τ is called the total force f which is acting on the particle.

Evidently this force vector f has the dimensions of mass/time. To give a numerical example, the gravitational field of the Earth exerts a force on any object which is proportional to its mass. In the case of a ten ton truck, this

*The mass of a macroscopic object (such as a truck) can be defined as the length of the "total energy-momentum vector," formed by adding the energy-momenta of all of the constituent particles. (Compare 7.6.) Hence this mass is equal to the sum of the masses of the constituent particles plus a sum of kinetic energy terms which describe the heat energy of the object. (In §17 we will correct this definition of "total energy-momentum" by adding an electromagnetic term.)

gravitational force works out as 0.33 grams/second. Thus, if our truck falls off a vertical cliff, it will take just 3 seconds to reach the freeway momentum of 1 gram.

In most cases of interest, the mass of a particle or object is independent of time, so that the total force vector is given by

$$\underset{\sim}{f} = m\underset{\sim}{a} = m d\underset{\sim}{u}/d\tau = m d^2\underset{\sim}{x}/d\tau^2.$$

For example this will certainly be the case if the force is acting on an elementary particle, whose mass cannot change. One must be careful however with macroscopic objects such as an automobile (which propels itself by burning fuel) or a flashlight (which ejects photons and thereby loses mass).

If no force at all acts on a particle, $\underset{\sim}{f} = \underset{\sim}{0}$, then

$$(dm/d\tau)\underset{\sim}{u} + m(d\underset{\sim}{u}/d\tau) = \underset{\sim}{0}.$$

Since velocity is orthogonal to acceleration, this implies that

$$m = \text{constant}, \qquad \underset{\sim}{u} = \text{constant},$$

so the worldcurve must be a straight line. Thus we have recovered Galileo's principle 1.1.

We can now state a more global version of the Conservation Law 7.2.

Lemma 7.4. Suppose that no forces act on our particles, so that they interact only by collision. Then for any bounded region R in spacetime the sum of the energy-momentum vectors of all particles entering R is equal to the sum of the energy-momentum vectors of all particles leaving R.

Proof. (Compare Figure 7.5.) Adding up the conservation laws 7.2 corresponding to all collision points in R, and then canceling out those terms corresponding to worldlines which both begin and end in R, we obtain the required formula. □

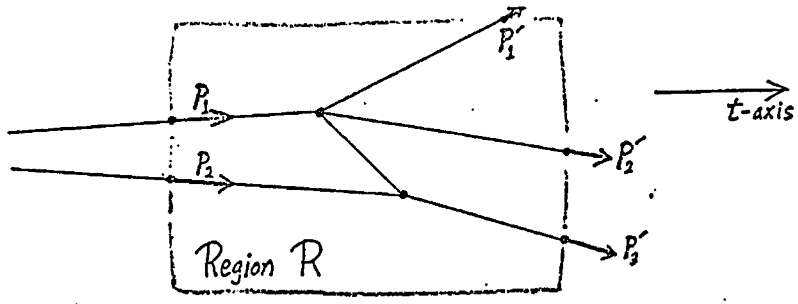

Figure 7.5 The Conservation Law: $\underset{\sim}{p}_1 + \underset{\sim}{p}_2 = \underset{\sim}{p}_1' + \underset{\sim}{p}_2' + p_3'$

An even more global formulation can be given as follows. We will need the following concept.

Definition. A subset H of spacetime is called a spacelike hyperplane if there exists a forward unit vector $\underset{\sim}{u}$ and a constant c so that H consists of all points $\underset{\sim}{x}$ in spacetime with $\underset{\sim}{x} \cdot \underset{\sim}{u} = c$.

Clearly this concept is Lorentz invariant. As an example, the coordinate hyperplane $t = t_0$ is a spacelike hyperplane in this sense.

Now consider an isolated system consisting of finitely many particles which interact only with each other. We continue to assume that no forces act, so that these particles interact only by collision.

Definition 7.6. The total energy-momentum vector p_{total} for this system of particles is the sum of the energy-momentum vectors p_α of all of the particles in the system whose worldlines intersect some fixed spacelike hyperplane H.

In other words we specifically exclude all particles which are annihilated before reaching the hyperplane H, and all particles which are created "after" the hyperplane H. To avoid confusion, it is best to assume that H does not contain any collision points.

To justify this definition we must prove the following.

Lemma 7.7. This sum (of energy-momentum vectors of particles whose worldlines intersect H) is independent of the particular choice of spacelike hyperplane H.

In particular, the total energy-momentum of the system at coordinate time $t = t_0$ is independent of t_0.

Outline of Proof. (Compare Figure 7.8.) If we move the hyperplane H around by means of a 1-parameter family of Poincaré-Lorentz transformations, then evidently this total energy-momentum vector does not change at all. It cannot change as H sweeps through ordinary points on the various worldlines, since we have assumed that the vector p_α is constant along the α-th worldcurve. It does not change as H sweeps through a collision point, since the conservation law 7.2 is satisfied. Finally, H can never become parallel to one of our worldlines, since H was assumed to be spacelike. Thus the total energy-momentum of our system is uniquely defined. □

Clearly this sum p_{total} is a vector in the sense of §5.2.

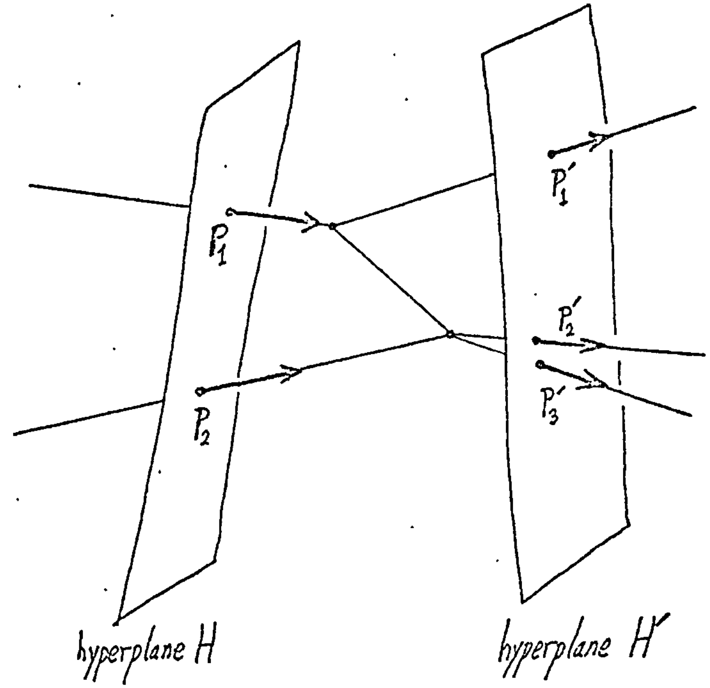

Figure 7.8. The Total Energy-Momentum $p_1 + p_2$ is equal to $p_1' + p_2' + p_3'$.

Remark 7.9: It is interesting to re-examine the "spaceship" of §6.4 in light of the conservation law. Let $p = mu$ be the energy-momentum vector of the spaceship. If we assume that this ship propels itself by emitting a continuous stream of particles, then both m and u will be functions of the proper time τ. Hence the derivative vector $dp/d\tau$ will be the sum of a component $(dm/d\tau)u$ which is tangent to the worldcurve of the spaceship and a component $m du/d\tau = m\underset{\sim}{a}$ which is orthogonal to this worldcurve. Evidently the relative speed v of the ejected particles is just the ratio of the magnitudes of these two components:

$$v = \|m\underset{\sim}{a}\| \big/ \|(dm/d\tau)\underset{\sim}{u}\| = -m\|\underset{\sim}{a}\| / (dm/d\tau).$$

Therefore

$$m^{-1}dm/d\tau = -v^{-1}\|\underset{\sim}{a}\|,$$

where $v \leq 1$. Integrating this equation with respect to $d\tau$, we obtain a fundamental equation of rocket theory:

$$\Delta \log m = -\int_{\tau_0}^{\tau_1} v^{-1}\|\underset{\sim}{a}\|\, d\tau.$$

Changing sign, and taking the exponential of both sides, it follows that the mass ratio

(initial mass)/(final mass)

must be equal to $\exp(\int v^{-1}\|\underset{\sim}{a}\|\, d\tau)$.

Here is a numerical example. If we assume the most optimistic value $v = 1$ for the exhaust speed*, then a voyage of 5 proper years at constant acceleration $\|\underset{\sim}{a}\| = 1\ \text{year}^{-1}$, as assumed in 6.4, would require a mass ratio of $e^5 \approx 148$.

*Present day rocket technology is based on exhaust speeds of $v \approx 10^{-5}$ (i.e., 3 kilometers/second).

§8. Tensors

The concept of a two-index tensor in Minkowski space can be introduced as follows. First some matrix formalism.

Recall from §5 that to every vector $\underset{\sim}{v} = [v^0, v^1, v^2, v^3]$ there is assigned an "associated covector"

$$\underset{\sim}{v}^* = \begin{bmatrix} v^0 \\ -v^1 \\ -v^2 \\ -v^3 \end{bmatrix}.$$

If we change coordinates by a Lorentz transformation, replacing $\underset{\sim}{v}$ by $\underset{\sim}{v}\Lambda$, then this associated covector must be replaced by

$$(\underset{\sim}{v}\Lambda)^* = \Lambda^*\underset{\sim}{v}^* = \Lambda^{-1}\underset{\sim}{v}^*.$$

(See §5.3.) If $\underset{\sim}{w}$ is another vector, it follows of course that the 1×1 product matrix $\underset{\sim}{w}\underset{\sim}{v}^*$ is invariant under Lorentz transformations. For if we replace $\underset{\sim}{w}$ by $\underset{\sim}{w}\Lambda$ and $\underset{\sim}{v}^*$ by $\Lambda^{-1}\underset{\sim}{v}^*$, then the product

$$\underset{\sim}{w}\underset{\sim}{v}^* = (\underset{\sim}{w}\Lambda)(\Lambda^{-1}\underset{\sim}{v}^*)$$

will remain unchanged. In fact (as we noted in §5) this matrix product $\underset{\sim}{w}\underset{\sim}{v}^*$ is precisely equal to the Minkowski inner product $\underset{\sim}{w} \cdot \underset{\sim}{v}$.

Now consider the matrix product

$$\underset{\sim}{v}^*\underset{\sim}{w} = \begin{bmatrix} v^0 \\ -v^1 \\ -v^2 \\ -v^3 \end{bmatrix} [w^0, w^1, w^2, w^3]$$

in the other order. Evidently this product is a 4×4 matrix M, which vanishes only if $\underset{\sim}{v}$ or $\underset{\sim}{w}$ is $\underset{\sim}{0}$. If we apply a Lorentz transformation, replacing $\underset{\sim}{v}$ by $\underset{\sim}{v}\Lambda$ and $\underset{\sim}{w}$ by $\underset{\sim}{w}\Lambda$, then this matrix $M = \underset{\sim}{v}^* \underset{\sim}{w}$ will be replaced by

$$(\Lambda^{-1}\underset{\sim}{v}^*)(\underset{\sim}{w}\Lambda) = \Lambda^{-1}M\Lambda.$$

Definition 8.1. A 4×4 matrix M, associated to a given Lorentz coordinate system for spacetime, is called a tensor if it obeys this transformation law, so that, whenever we change the names of points in spacetime by a Poincaré-Lorentz transformation $\underset{\sim}{x}' = \underset{\sim}{x}\Lambda + \underset{\sim}{c}$, we must replace M by the matrix $\Lambda^{-1}M\Lambda$.

The word "tensor", with something like this meaning, was first introduced by Gibbs.

For the moment, our only example of a tensor is the somewhat artificial one of the matrix $\underset{\sim}{v}^* \underset{\sim}{w}$ associated with two vectors. Another example, rather dull but very important, is the identity matrix I. The equation $I = \Lambda^{-1}I\Lambda$ shows that I is a tensor which is represented by the same matrix in every coordinate system.

Physically interesting tensors will be constructed in later sections: the angular momentum tensor in §9, the electromagnetic tensor in §10, and the energy (density) tensor in §16, 17, and 20.

In more conventional terminology, our tensors would be called "two-index tensors, covariant in the first index and contravariant in the second." Such mixed tensors are particularly easy to work with, as is shown by the following.

Lemma 8.2. If M and N are tensors, then the matrix product MN is also a tensor, and the Minkowski adjoint M^* is also a tensor. To any tensor M there is associated a real number

$$\text{trace}(M) = \text{trace}(M^*)$$

which is invariant under change of coordinates, and satisfies the equation trace(MN) = trace(NM). Finally, if M is a tensor and $\underset{\sim}{v}$ is a vector, the matrix product $\underset{\sim}{v}M$ is a vector.

The proofs are all straightforward. For example if we replace M by $\Lambda^{-1}M\Lambda$ and N by $\Lambda^{-1}N\Lambda$, then the product MN will be replaced by $\Lambda^{-1}MN\Lambda$, and the adjoint M^* by $(\Lambda^{-1}M\Lambda)^* = \Lambda^* M^* (\Lambda^{-1})^* = \Lambda^{-1}M^*\Lambda$. (Compare §5.3.) The identity trace(ΛP) = trace$(P\Lambda)$ can be verified by inspection. Substituting $\Lambda^{-1}M$ for P it follows that trace(M) = trace$(\Lambda^{-1}M\Lambda)$. Further details will be left to the reader. □

The identity $\det(\Lambda^{-1}M\Lambda) = \det(M)$ shows that the determinant of a tensor is also a well defined real number. Using this determinant function, we can define the eigenvalues of a tensor M as the four real or complex roots of the equation

$$\det(M - \lambda I) = 0.$$

Given any real eigenvalue λ we can find a corresponding eigenvector $\underset{\sim}{v}$, satisfying the equation $\underset{\sim}{v}M = \lambda \underset{\sim}{v}$. Clearly these constructions are invariant under Lorentz transformations.

The tensor $\underset{\sim}{v}^* \underset{\sim}{w}$ which is associated with two vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$ will play an important role in our development, so we give it a special name.

Definition 8.3. If $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are vectors, then the tensor $\underset{\sim}{v}^* \underset{\sim}{w}$ will be denoted by the symbol $\underset{\sim}{v} \otimes \underset{\sim}{w}$ and called the tensor product of $\underset{\sim}{v}$ and $\underset{\sim}{w}$.

Remark. This notation is unconventional, since it is not customary to consider a "tensor product" which is a mixed tensor, half covariant and half contravariant. However it will be extremely convenient for our purposes. Here is a list of three basic properties.

Lemma 8.4. The trace of the tensor product $\underset{\sim}{v} \otimes \underset{\sim}{w}$ is equal to the Minkowski inner product $\underset{\sim}{v} \cdot \underset{\sim}{w}$; and the Minkowski adjoint $(\underset{\sim}{v} \otimes \underset{\sim}{w})^*$ is equal to $\underset{\sim}{w} \otimes \underset{\sim}{v}$. If $\underset{\sim}{p}$ is another vector, then the matrix product $\underset{\sim}{p}(\underset{\sim}{v} \otimes \underset{\sim}{w})$ is equal to $(\underset{\sim}{p} \cdot \underset{\sim}{v})\underset{\sim}{w}$.

The proofs are immediate. □

A tensor M is called symmetric if the matrix M is self adjoint[†]

$$M^* = M,$$

and is called skew if M is skew self adjoint

$$M^* = -M.$$

One basic example of a symmetric tensor is provided by the self tensor product

$$\underset{\sim}{v} \otimes \underset{\sim}{v}.$$

Correspondingly, one basic example of a skew tensor is provided by the difference

$$\underset{\sim}{v} \otimes \underset{\sim}{w} - \underset{\sim}{w} \otimes \underset{\sim}{v}.$$

This difference is called the skew product of $\underset{\sim}{v}$ and $\underset{\sim}{w}$, denoted by the symbol $\underset{\sim}{v} \wedge \underset{\sim}{w}$.

Lemma 8.5. The skew product $\underset{\sim}{v} \wedge \underset{\sim}{w}$ is zero if and only if the vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are linearly dependent.

[†]Thus symmetry does not mean that M is equal to its transpose. In fact the transpose of a tensor is not a tensor, according to our definitions. The equation $M = M^{tr}$ would not make any real sense, since it would not be invariant under a change of coordinates.

Proof. Note that any non-zero row of the matrix $\underset{\sim}{v} \otimes \underset{\sim}{w}$ is a multiple of the vector w. So if $\underset{\sim}{v} \otimes \underset{\sim}{w} = \underset{\sim}{w} \otimes \underset{\sim}{v}$ then it follows that the two vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$ must be linearly dependent. Conversely, if say $\underset{\sim}{v} = \lambda \underset{\sim}{w}$, then $\underset{\sim}{v} \otimes \underset{\sim}{w} = \lambda(\underset{\sim}{w} \otimes \underset{\sim}{w}) = \underset{\sim}{w} \otimes \underset{\sim}{v}$. □

Since a 4 x 4 matrix has 16 distinct entries, the collection consisting of all tensors forms a 16 dimensional vector space over the real numbers. Choosing 4 linearly independent vectors $\underset{\sim}{u}_0, \underset{\sim}{u}_1, \underset{\sim}{u}_2, \underset{\sim}{u}_3$, it is not difficult to check that the 16 tensor products $\underset{\sim}{u}_i \otimes \underset{\sim}{u}_j$ form a basis for this 16 dimensional vector space.

Evidently the skew tensors form a 6 dimensional subspace, with a basis consisting of all products $\underset{\sim}{u}_i \wedge \underset{\sim}{u}_j$, $i < j$. Similarly the symmetric tensors form a 10 dimensional subspace. Note that every tensor M can be expressed uniquely as the sum of a symmetric tensor

$$\frac{1}{2}(M + M^*)$$

and a skew tensor

$$\frac{1}{2}(M - M^*).$$

Hence the vector space consisting of all tensors splits canonically as the direct sum of the 10 dimensional subspace consisting of symmetric tensors and the 6 dimensional subspace consisting of skew tensors.

There is a further canonical splitting which plays an important role in relativity theory. Any symmetric tensor M can be expressed uniquely as the sum of a symmetric tensor

$$M - \frac{1}{4}(\text{trace } M)I$$

which has trace zero, and a tensor

$$\frac{1}{4}\,(\text{trace } M) I$$

which is a multiple of the identity. Thus the space of symmetric tensors splits canonically as the direct sum of a 9 dimensional subspace and a 1 dimensional subspace.

No further decomposition is possible. The space of symmetric tensors with trace zero is irreducible, in the sense that no non-trivial subspace is invariant under the action of the Lorentz group. Similarly the space of skew tensors is irreducible. Details will be omitted.

§9. Angular Momentum

As in §7.6, we consider a finite system of particles which interact only by collision. We will construct:

(1) a preferred line L called the center of energy worldline of the system of particles, and

(2) a skew tensor A called the angular momentum of the system about this line L.

The construction will be based on the energy-momentum conservation law of §7.2. Results in this section will not be needed in subsequent sections.

First a preliminary version of the definition. Consider a particle with straight worldline L_a and with energy-momentum vector p_a. Let b be some arbitrary fixed base point in spacetime.

Definition. The angular momentum A_a of this particle about the base point b is defined to be the skew product†

$$A_a = (x - b) \wedge p_a$$

where x is any point along the worldline L_a.

Clearly A_a does transform as a tensor. For if we replace x by $x\Lambda + c$, replace b by $b\Lambda + c$, and replace p_a by $p_a\Lambda$, then we must replace A_a by $\Lambda^{-1} A_a \Lambda$. Evidently the 16 components of the skew tensor A_a have the dimensions of mass × time.

This tensor A_a does not depend on the particular choice of x. For if x' is another point along the same worldline L_a then the difference

$$x' - x = (x' - b) - (x - b)$$

must be a multiple of p_a. Hence $(x' - x) \wedge p_a = 0$ by 8.5, and it follows that $(x' - b) \wedge p_a = (x - b) \wedge p_a$.

---

†In other words A_a is the skew self adjoint 4×4 matrix $(x - b)^* p_a - p_a^*(x - b)$. Compare §8.5.

The tensor A_a does depend very strongly on the particular choice of base point b. We will analyze this dependence later.

Now consider several particles whose worldlines $L_1, \ldots, L_k$ converge at a collision point x in spacetime. Let $L'_1, \ldots, L'_\ell$ be the worldlines which emerge from this collision point.

Lemma 9.1 (Conservation of Angular Momentum). Using any fixed base point b, the sum $A_1 + \ldots + A_k$ of the angular momenta of the particles which collide at x is equal to the sum $A'_1 + \ldots + A'_\ell$ of the angular momenta of the particles which emerge from the collision.

Proof. This equation is just the skew product of the vector $x - b$ with the conservation equation

$$p_1 + \cdots + p_k = p'_1 + \cdots + p'_\ell$$

of §7.2. □

Another closely related formulation is the following. We continue to assume that no forces act on our particles.

Lemma 9.2. For any bounded region R in spacetime, the sum of the angular momenta (with respect to some fixed base point b) of all particles entering the region R is equal to the sum of the angular momenta of all particles leaving the region R.

Proof. This follows immediately from 9.1. (Compare the proof of 7.4.) □

We can illustrate the physical content of this new conservation law by thinking of the region R as a "black box" in spacetime, within which we are not permitted to carry out any observations. Consider the situation illustrated in Figure 9.3 where a particle is observed entering the black box R, and a particle with the identical energy-momentum is observed leaving R. This picture conforms perfectly with the energy-momentum conservation law, as formulated in §7.4. But the illustrated behavior cannot actually occur, since it violates 9.2.

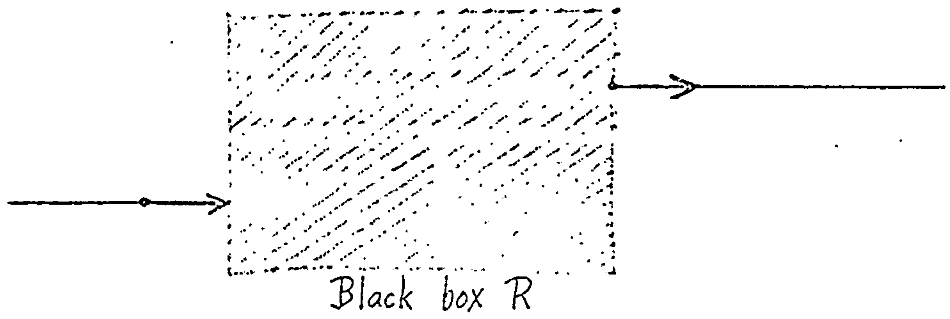

Figure 9.3. A Forbidden History

Let us formulate a global version of this conservation law. As in §7.6 we consider a finite system of particles which interact only with each other and only by collision. Let H be a spacelike hyperplane which does not contain any collision point. We continue to work with a fixed base point b.

Lemma 9.4. The sum of the angular momenta of all particles in our system whose worldlines intersect H is independent of the particular choice of spacelike hyperplane H.

In particular, the sum of the angular momenta of all particles which exist at the coordinate time t_0 is independent of t_0.

As in §7.7, this can be proved by deforming H by means of a 1-parameter family of Poincaré-Lorentz motions, and noting that the sum described in 9.4 can not change during this deformation. □

This sum is called the total angular momentum of the system of particles about the base point b.

We must analyze the dependence of this total angular momentum tensor A on the choice of base point b. Writing the defining equation in the form

$$A = \sum_a x_a \wedge p_a - \sum_a b \wedge p_a ,$$

to be summed over all worldlines hitting the hyperplane H, clearly we can simplify to

$$A = A_0 - b \wedge p$$

where $p = \sum p_a$ is the total energy-momentum vector of our system of particles and $A_0 = \sum x_a \wedge p_a$. If we used a different base point b', then the corresponding tensor $A' = A_0 - b' \wedge p$ would differ by

$$A' - A = (b - b') \wedge p .$$

Thus, if the two base points b and b' both lie along a line parallel to p, then this skew product will be zero by 8.5, so A' will equal A. In order to determine the tensor A it is not necessary to specify a base point. It would suffice to specify a base line parallel to p.

But we can do better than this: we can actually pick out one preferred base line parallel to p.

Definition 9.5. The base point b belongs to the center of energy worldline* L for our system of particles if the matrix product

$$pA = p(A_0 - b \wedge p)$$

equals 0.

The motivation for this definition will become clear if we use a coordinate system for which the vector p points along the 0-th coordinate axis, say

$$p = [\varepsilon, 0, 0, 0]$$

*We must assume for the rest of this section that the total energy-momentum vector p is timelike. This is a very mild assumption. It would fail to be satisfied only if all of the particles involved had parallel light paths as worldlines.

where $\varepsilon = \|p\|$ is the "mass" of the system (i.e., the total energy of the system using these preferred coordinates). Setting $b = [a, b, c, d]$, computation shows that

$$A = A_0 - b \wedge p = A_0 + \varepsilon \begin{bmatrix} 0 & b & c & d \\ b & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ d & 0 & 0 & 0 \end{bmatrix} .$$

Thus, by choosing the base point b appropriately, we can choose the top row of the skew tensor A as we please (with a corresponding choice of initial column). But we cannot touch the rest of the tensor. The simplest choice is clearly to choose b so that the top row $[1, 0, 0, 0]A = \varepsilon^{-1}pA$ equals 0.

This argument essentially proves the following statement, which helps to justify our definition 9.5.

Lemma 9.6. Suppose that the total energy-momentum vector p is timelike. Then the matrix product $pA = p(A_0 - b \wedge p)$ will equal 0 if and only if the base point b lies on a certain line L parallel to the vector p. The resulting total angular momentum tensor A is independent of the choice of b along L.

Proof. Let $\overline{H}$ be the hyperplane consisting of all points x with $x \cdot p = 0$, and let $\overline{b}$ be the orthogonal projection of b on this hyperplane $\overline{H}$. Thus

$$b = \overline{b} + \beta p$$

for some real number β. Then

$$A = A_0 - b \wedge p = A_0 - \overline{b} \wedge p$$

by §8.5. It follows using 8.4 that

$$pA = pA_0 + \|p\|^2 \overline{b} .$$

Hence the required equation $\underset{\sim}{p}A = \underset{\sim}{0}$ will be satisfied if and only if

$$\bar{\underset{\sim}{b}} = -\|\underset{\sim}{p}\|^{-2}\underset{\sim}{p}A_0 .$$

Further details will be left to the reader. □

Switching back to a coordinate system for which $\underset{\sim}{p} = [\varepsilon, 0, 0, 0]$, the construction of this central worldline L can be made more explicit as follows. Suppose that the basepoint $\underset{\sim}{b}$ lies in the hyperplane $t = t_0$. For each worldline which intersects this hyperplane, let $\underset{\sim}{x}_\alpha$ be the intersection point and $\underset{\sim}{p}_\alpha = [\varepsilon_\alpha, p_\alpha, q_\alpha, r_\alpha]$ the energy-momentum vector. Using 8.4 we see that the matrix product $\underset{\sim}{p}A = \underset{\sim}{p}\sum(\underset{\sim}{x}_\alpha - \underset{\sim}{b}) \wedge \underset{\sim}{p}_\alpha$ is equal to $\varepsilon\sum\varepsilon_\alpha(\underset{\sim}{b} - \underset{\sim}{x}_\alpha)$, where $\varepsilon = \sum\varepsilon_\alpha$. Setting this equal to $\underset{\sim}{0}$, we obtain the basic formula

$$\underset{\sim}{b} = \varepsilon^{-1}\sum\varepsilon_\alpha \underset{\sim}{x}_\alpha \tag{9.7}$$

which shows that $\underset{\sim}{b}$ is a weighted average of the intersection points $\underset{\sim}{x}_\alpha$. This formula finally justifies the phrase "center of energy."

In this special coordinate system, the skew tensor A will have the form

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a^3 & a^2 \\ 0 & a^3 & 0 & -a^1 \\ 0 & -a^2 & a^1 & 0 \end{bmatrix} .$$

The 3-dimensional vector $[a^1, a^2, a^3]$ is just what was called "angular momentum" in pre-relativistic physics.

In fact this angular momentum vector can be interpreted, using strictly 4-dimensional language, as follows. Let A be the total angular momentum of our system about a completely arbitrary base point $\underset{\sim}{b}$. According to the Appendix, the skew tensor A has a well defined "dual" skew tensor $\hat{A}$. The angular momentum vector or spin vector is now defined to be the matrix product

$$\|\underset{\sim}{p}\|^{-1}\underset{\sim}{p}\hat{A} .$$

Since $\hat{A}$ is skew, this angular momentum vector is always orthogonal to $\underset{\sim}{p}$. Hence it is a spacelike vector (or zero). If we use a special coordinate system so that $\underset{\sim}{p} = [\varepsilon, 0, 0, 0]$, then

$$A = \begin{bmatrix} 0 & * & * & * \\ * & 0 & -a^3 & a^2 \\ * & a^3 & 0 & -a^1 \\ * & -a^2 & a^1 & 0 \end{bmatrix} \quad \text{hence } \hat{A} = \begin{bmatrix} 0 & a^1 & a^2 & a^3 \\ a^1 & 0 & * & * \\ a^2 & * & 0 & * \\ a^3 & * & * & 0 \end{bmatrix} .$$

(Compare the explicit formula for the dual skew tensor given in the Appendix.) Here the *'s stand for numbers which depend on the particular choice of base point $\underset{\sim}{b}$. Therefore the angular momentum vector

$$\varepsilon^{-1}[\varepsilon, 0, 0, 0]\hat{A} = [0, a^1, a^2, a^3]$$

does not depend on the choice of base point.

An important invariant of any tensor A is the trace of the product $A^2 = AA$. (The trace of the skew tensor A itself is of course zero.) Using the special coordinate system above (and assuming that $\underset{\sim}{p}A = \underset{\sim}{0}$), a short computation shows that

$$-\tfrac{1}{2}\operatorname{trace}(A^2) = (a^1)^2 + (a^2)^2 + (a^3)^2 .$$

Thus the quantity

$$\sqrt{-\operatorname{trace}(A^2)/2} = \sqrt{(a^1)^2 + (a^2)^2 + (a^3)^2} ,$$

with dimensions of mass × time, is an appropriate measure of the magnitude of the angular momentum tensor A.

Remark 9.8. All of our discussion of angular momentum has been based on the hypothesis that each individual particle is a pointlike object, with no internal structure. But in actual fact, many elementary particles have an inherent angular momentum or "spin." To build this into our mathematical model, we must assume, as a new axiom, that each particle has an intrinsic angular momentum A_a^{int} about its own worldline. This intrinsic angular momentum must be a skew tensor satisfying the condition $p_a A_c^{int} = 0$. The angular momentum of the particle about an arbitrary base point b is then defined as the sum

$$A_a^{int} + (x - b) \wedge p_a ,$$

where x is any point along the worldline. The conservation law 9.1 must now be assumed as an axiom; and the remaining constructions of this section go through without any change.

Here is a numerical example. Suppose that a neutral pion with energy-momentum

$$p = [264.13,\ 0,\ 0,\ 0] \text{ electron masses}$$

decays into two photons (i.e., γ-rays) with energy-momenta $p_1 = [\varepsilon,\ \varepsilon,\ 0,\ 0]$ and $p_2 = [\varepsilon,\ -\varepsilon,\ 0,\ 0]$ respectively, where $2\varepsilon = 264.13$. Since the intrinsic angular momentum of the pion is zero, the equations

$$0 = A_1 + A_2, \quad p_1 A_1 = p_2 A_2 = 0 ,$$

imply that $A_1 = -A_2$ is a tensor of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \end{bmatrix} .$$

In fact the entry a must be Planck's constant $\hbar \approx 1.17 \times 10^{-48}$ gram-seconds, up to sign, since the photon is a spin 1 particle. Thus we have computed the angular momenta A_1 and A_2 up to sign.

§10. The Electromagnetic Field

In classical electromagnetic theory we learn that every particle can be assigned a real number e called its charge. This charge can be measured by immersing the particle in an "electromagnetic field" and observing that it then experiences a force $\underset{\sim}{f}$ which is directly proportional to e. Since this force also depends on the velocity $\underset{\sim}{u}$ of the particle and on the position $\underset{\sim}{x}$ in spacetime, we can write this force law provisionally as

$$\underset{\sim}{f} = e\underset{\sim}{g}(\underset{\sim}{x}, \underset{\sim}{u})$$

where $\underset{\sim}{g}$ is some unknown vector valued function of spacetime position and velocity which describes the electromagnetic field.

The unit of charge can be specified, for example, by the convention that the charge of a proton is equal to $+1$. With this choice of units, it is an empirical fact that the charge of every particle is an integer (i.e., a whole number). Particles of fractional charge have sometimes been postulated, but have never been actually observed.

The most fundamental empirical fact about charge is the following statement, associated with the names of William Watson and Benjamin Franklin.

Conservation Law 10.1. The sum of the charges of all particles entering any bounded region of spacetime is equal to the sum of the charges of all of the particles leaving this region.

A different, but completely equivalent, formulation is the following.

Conservation Law 10.2. If the worldlines of one* or more particles come together at a collision point in spacetime, then the sum $e_1 + \ldots + e_k$ of the charges of the colliding particles is equal to the sum $e'_1 + \ldots + e'_\ell$ of the charges of the particles which emerge from the collision.

*As in §7, we consider the spontaneous decay of a particle as one form of a "collision."

As a numerical example, consider the decay

$$\text{neutron} \longmapsto \text{proton} + \text{electron} + \text{neutrino}$$

mentioned in §7. The neutron has charge 0, while the three particles which emerge from the collision have charges $+1$, -1, and 0 respectively.

One final important property of charge is the following

Empirical Fact 10.3. Every charged particle has positive mass.

In fact there are only two known types of particles with zero mass, namely photons and neutrinos, and both of these have zero charge.

Let us examine the nature of the electromagnetic force, described by the vector equation

$$\underset{\sim}{f} = e\underset{\sim}{g}(\underset{\sim}{x}, \underset{\sim}{u}) \ .$$

We know that this force is always orthogonal to velocity

$$\underset{\sim}{f} \cdot \underset{\sim}{u} = 0 \ .$$

(This is just the statement that an electromagnetic force acting on a particle cannot change its mass. Compare §7.3.) Therefore the function $\underset{\sim}{g}(\underset{\sim}{x}, \underset{\sim}{u})$ must really depend on the velocity $\underset{\sim}{u}$; unless $\underset{\sim}{f}$ is simultaneously orthogonal to all forward unit vectors, which would clearly imply that $\underset{\sim}{f} = \underset{\sim}{0}$. It turns out that the dependence is the simplest possible one:

The electromagnetic force exerted on any particle depends linearly on the velocity $\underset{\sim}{u}$ of the particle.

In matrix notation we can write this force law as

$$\underset{\sim}{f} = e\underset{\sim}{u}F \ , \tag{10.4}$$

where F is a 4×4 matrix which is a function only of the position $\underset{\sim}{x}$ in spacetime.

Definition 10.5. This matrix $F = F(\underset{\sim}{x})$ is called the electromagnetic field at the point $\underset{\sim}{x}$ of spacetime.

Clearly F transforms as a tensor. For if we apply a Lorentz transformation, replacing $\underset{\sim}{u}$ by $\underset{\sim}{u}\Lambda$ and $\underset{\sim}{f}$ by $\underset{\sim}{f}\Lambda$, then we must replace F by $\Lambda^{-1}F\Lambda$ in order to preserve the force law 10.4.

Note that this tensor field F has the dimensions of force/charge, or in other words of mass/(time × charge).

The requirement that $\underset{\sim}{f}\cdot\underset{\sim}{u} = 0$ now corresponds to the matrix equation

$$\underset{\sim}{u}F\underset{\sim}{u}^{*} = 0 .$$

It is not difficult to check that this equation is satisfied for all vectors $\underset{\sim}{u}$ (or for all forward unit vectors), if and only if the tensor F is skew:

$$F^{*} = -F .$$

Thus F has six independent components. It is convenient to label these by setting

$$(10.6) \qquad F = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix} .$$

As a first example, consider a particle with zero coordinate speed, so that the velocity vector is $\underset{\sim}{u}_0 = [1, 0, 0, 0]$. Then the force acting on this particle is

$$\underset{\sim}{f} = e\underset{\sim}{u}_0F = e[0, E^1, E^2, E^3] .$$

By definition these three numbers E^1, E^2, E^3 describe the electric field with respect to the given coordinate system. We will sometimes use the 3-dimensional vector notation

$$\vec{E} = [E^1, E^2, E^3]$$

for this electric field.

The remaining three numbers B^1, B^2, B^3 describe a force which acts on a particle only if it has non-zero coordinate speed. By definition, these are the components of the magnetic field $\vec{B} = [B^1, B^2, B^3]$.

Of course this distinction is meaningful only for the given coordinate system. If we apply a Lorentz transformation, then the electric and magnetic components will be inextricably mixed.

It is interesting to look for invariants of F, that is real numbers which are unchanged by a Lorentz transformation $F \longmapsto \Lambda^{-1}F\Lambda$. One such invariant, which we encountered already in §9, is the trace of F^2. Inspection shows that

$$\tfrac{1}{2}\mathrm{trace}(F^2) = \vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B} ,$$

where the heavy dot stands for the positive definite Euclidean inner product. Evidently this invariant measures the relative preponderance of the electric and magnetic portions of the field F. In particular it follows from this formula that no Lorentz transformation can transform a purely electric field into a purely magnetic field.

Another invariant is the real number

$$\sqrt{\det F} = \vec{E}\cdot\vec{B}$$

which measures the Euclidean inner product of the electric and magnetic fields. (Actually this number is invariant in a somewhat weaker sense, since $\vec{E}\cdot\vec{B}$ changes sign if we apply a Lorentz transformation with determinant -1 to F.) This invariant is described more adequately in the Appendix.

To illustrate the force law 10.4, let us study the motion of a particle which is freely falling in a constant electromagnetic field.

Lemma 10.7. If only electromagnetic forces act on a particle, and if the field F is independent of position in spacetime, then the acceleration vector $a = d^2x/d\tau^2$ of the particle will have constant length.

Proof. We will use the notation $a = \dot{u} = \ddot{x}$, where the dot stands for the derivative with respect to proper time τ. The equation of motion can be written as

$$m\dot{u} = euF \ .$$

Differentiating, we obtain $m\ddot{u} = e\dot{u}F$, or in other words

$$\dot{a} = m^{-1}eaF \ .$$

The derivative of $\|a\|^2 = -a \cdot a$ is evidently equal to

$$-2\dot{a} \cdot a = -2\dot{a}a^* = -2m^{-1}eaFa^* \ .$$

But $aFa^* = 0$ since F is skew. Therefore $\|a\|$ is constant, as asserted. □

The precise shape of this worldcurve of constant acceleration depends of course on the matrix F and on the initial velocity. In the case of a pure electric field, say

$$\vec{E} = [E^1, 0, 0], \quad \vec{B} = \vec{0} \ ,$$

one possible solution is the hyperbola

$$x(\tau) = k^{-1}[\sinh k\tau, \cosh k\tau, 0, 0]$$

of "radius" k^{-1}, where $k = eE^1/m$. Thus the particle accelerates uniformly. In the case of a pure magnetic field

$$\vec{E} = \vec{0}, \quad \vec{B} = [B^1, 0, 0] \ ,$$

a typical solution is provided by the helix

$$x(\tau) = [\tau\sqrt{1 + a^2}, 0, k^{-1}a \sin k\tau, k^{-1}a \cos k\tau] \ ,$$

where $k = eB^1/m$ and where a is an arbitrary constant. In 3-dimensional language, the particle travels in a circle of radius a/k with coordinate speed equal to $a/\sqrt{1 + a^2}$, returning to its initial position after a proper time interval $\Delta\tau = 2\pi/k$ which is independent of the radius.

We conclude this section with a confession.

Warning 10.8. All of the above statements about electromagnetic force have been somewhat simplified, since we have ignored the fact that any (infinitely small) charged particle creates its own electromagnetic field which becomes infinitely large in the immediate neighborhood of the particle itself. This creates very serious problems. For example the exact solutions discussed above must be modified by higher order correction terms. For the moment, the reader must simply be tolerant. We will discuss these problems in detail in §19.

§11. Gradient, Divergence, Curl, and the Wave Equation

Let $\psi(x) = \psi(t, x, y, z)$ be a smooth real valued function, defined throughout some region R of spacetime.* Then the gradient $\nabla\psi$, by definition, is the continuous vector field

$$[\partial\psi/\partial t, \ -\partial\psi/\partial x, \ -\partial\psi/\partial y, \ -\partial\psi/\partial z] \ .$$

The minus signs in this definition may appear strange. They can be justified by any one of the following arguments.

(a) Consider a smooth timelike curve C in the region R, with position $x(\tau)$ and velocity vector $u(\tau) = dx/d\tau$ at proper time τ. Then with this notation the derivative

$$\frac{d\psi}{d\tau} = \frac{\partial\psi}{\partial t}\frac{dt}{d\tau} + \frac{\partial\psi}{\partial x}\frac{dx}{d\tau} + \frac{\partial\psi}{\partial y}\frac{dy}{d\tau} + \frac{\partial\psi}{\partial z}\frac{dz}{d\tau}$$

of the function $\psi(x(\tau))$ can be written simply as a Minkowski inner product $(\nabla\psi)\cdot u$.

(b) With this notation the total differential $d\psi = (\partial\psi/\partial t)dt + (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy + (\partial\psi/\partial z)dz$ can be written as an inner product $(\nabla\psi)\cdot dx$.

(c) With this notation the linear differential operator

$$v^0\frac{\partial}{\partial t} + v^1\frac{\partial}{\partial x} + v^2\frac{\partial}{\partial y} + v^3\frac{\partial}{\partial z}$$

associated with any vector field v can be written as an inner product $v\cdot\nabla$. In other words the directional derivative $v^0\partial\psi/\partial t + \ldots + v^3\partial\psi/\partial z$ of the function ψ in the direction v is equal to $v\cdot(\nabla\psi)$.

Here the symbol ∇ by itself stands for the 1×4 matrix

$$\nabla = [\partial/\partial t, \ -\partial/\partial x, \ -\partial/\partial y, \ -\partial/\partial z]$$

*As a typical example, if R is a domain in the Earth's atmosphere throughout some time interval, then ψ might be the temperature or pressure function.

of differential operators. It is understood that these operators are to operate on whatever smooth function is placed to the right of ∇. (Thus if we place the function ψ to the right of ∇ we obtain the gradient vector field $\nabla\psi$ described above. But if we placed ψ to the left we would obtain a new 1×4 matrix

$$\psi\nabla = \left[\psi\frac{\partial}{\partial t}, \ -\psi\frac{\partial}{\partial x}, \ -\psi\frac{\partial}{\partial y}, \ -\psi\frac{\partial}{\partial z}\right]$$

of differential operators.)

Lemma 11.1. This matrix ∇ of differential operators transforms, under a Poincaré-Lorentz change of coordinates, just as if it were a vector. That is, if we introduce new coordinates $x' = [t', x', y', z']$ by the formula $x' = x\Lambda + c$, then the new $\nabla' = [\partial/\partial t', \ -\partial/\partial x', \ -\partial/\partial y', \ -\partial/\partial z']$ of operators will be equal to the matrix product $\nabla\Lambda$. More precisely, for any smooth function ψ the new gradient vector $\nabla'\psi = [\partial\psi/\partial t', \ -\partial\psi/\partial x', \ -\partial\psi/\partial y', \ -\partial\psi/\partial z']$, evaluated at a point with new coordinates $x' = x\Lambda + c$, will be equal to the matrix product $\nabla\psi\Lambda$ evaluated at x.

Thus the gradient $\nabla\psi$ is a vector in the sense of §5.2. (The symbol ∇ by itself is not a vector, since its four components are not numbers.)

Proof of 11.1. Applying the * operation of §5 to both sides of the alleged equation $\nabla' = \nabla\Lambda$, we obtain a completely equivalent equation

$$(\nabla')^* = \Lambda^*\nabla^* = \Lambda^{-1}\nabla^* \ ,$$

or in other words $\Lambda(\nabla')^* = \nabla^*$. Writing this as

$$\Lambda\begin{bmatrix}\partial/\partial t' \\ \partial/\partial x' \\ \partial/\partial y' \\ \partial/\partial z'\end{bmatrix} = \begin{bmatrix}\partial/\partial t \\ \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z\end{bmatrix} ,$$

and substituting in an explicit description of the 16 entries of Λ as partial derivatives, this alleged equation becomes

$$\begin{bmatrix} \partial t'/\partial t & \partial x'/\partial t & \partial y'/\partial t & \partial z'/\partial t \\ \partial t'/\partial x & \partial x'/\partial x & \partial y'/\partial x & \partial z'/\partial x \\ \partial t'/\partial y & \partial x'/\partial y & \partial y'/\partial y & \partial z'/\partial y \\ \partial t'/\partial z & \partial x'/\partial z & \partial y'/\partial z & \partial z'/\partial z \end{bmatrix} \begin{bmatrix} \partial/\partial t' \\ \partial/\partial x' \\ \partial/\partial y' \\ \partial/\partial z' \end{bmatrix} = \begin{bmatrix} \partial/\partial t \\ \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} ,$$

which we easily recognize as a standard formulation of the chain rule for functions of 4 variables. □

Now suppose that we want to differentiate a smooth vector field $\underset{\sim}{v}$. Letting the 4×1 matrix $\underset{\sim}{\nabla}^*$ of differential operators act on the 1×4 matrix $\underset{\sim}{v}$ of smooth functions, we obtain a 4×4 matrix

$$\underset{\sim}{\nabla} \otimes \underset{\sim}{v} = \underset{\sim}{\nabla}^* \underset{\sim}{v} = \begin{bmatrix} \partial v^0/\partial t & \partial v^1/\partial t & \partial v^2/\partial t & \partial v^3/\partial t \\ \partial v^0/\partial x & \partial v^1/\partial x & \partial v^2/\partial x & \partial v^3/\partial x \\ \partial v^0/\partial y & \partial v^1/\partial y & \partial v^2/\partial y & \partial v^3/\partial y \\ \partial v^0/\partial z & \partial v^1/\partial z & \partial v^2/\partial z & \partial v^3/\partial z \end{bmatrix}$$

of continuous functions. It follows from 11.1 that $\underset{\sim}{\nabla} \otimes \underset{\sim}{v}$ transforms as a tensor. That is, if $\underset{\sim}{v}' = \underset{\sim}{v}\Lambda$ and $\underset{\sim}{\nabla}' = \underset{\sim}{\nabla}\Lambda$, then

$$\underset{\sim}{\nabla}' \otimes \underset{\sim}{v}' = \Lambda^{-1}(\underset{\sim}{\nabla} \otimes \underset{\sim}{v})\Lambda .$$

The proof is straightforward.

Definition 11.2. This continuous tensor field $\underset{\sim}{\nabla} \otimes \underset{\sim}{v}$ is called the derivative tensor of the smooth vector field $\underset{\sim}{v}$.

Evidently $\underset{\sim}{\nabla} \otimes \underset{\sim}{v} = 0$ if and only if $\underset{\sim}{v}$ is a constant vector field.

As an example, if the function ψ is twice continuously differentiable, then we can form the derivative tensor $\underset{\sim}{\nabla} \otimes (\underset{\sim}{\nabla}\psi)$ of the gradient $\underset{\sim}{\nabla}\psi$. It is easy to check that this second derivative tensor field is symmetric, that is, $\underset{\sim}{\nabla} \otimes \underset{\sim}{\nabla}\psi = (\underset{\sim}{\nabla} \otimes \underset{\sim}{\nabla}\psi)^*$.

For any smooth vector field $\underset{\sim}{v}$ the trace of the derivative tensor

$$\text{trace}(\underset{\sim}{\nabla} \otimes \underset{\sim}{v}) = \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = \frac{\partial v^0}{\partial t} + \frac{\partial v^1}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial v^3}{\partial z}$$

is an important invariant called the divergence of the vector field $\underset{\sim}{v}$.

As an example, taking the trace of the second derivative tensor $\underset{\sim}{\nabla} \otimes \underset{\sim}{\nabla}\psi$ (or in other words taking the divergence of the gradient of ψ) we obtain the invariant

$$\text{trace}(\underset{\sim}{\nabla} \otimes \underset{\sim}{\nabla}\psi) = \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla}\psi = \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial z^2} .$$

Definition. The second order differential operator

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

is called the d'Alembertian operator. The corresponding homogeneous linear differential equation

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla}\psi = 0$$

is called the wave equation.

Since solutions of the wave equation play a very important role in physics, let us describe their basic properties. The following result is due to Poisson.

Theorem 11.3. Given a three times continuously differentiable function $\psi_0(x, y, z)$ and a twice continuously differentiable function $\psi_0'(x, y, z)$, there is one and only one solution $\psi(t, x, y, z)$ to the wave equation

$$\partial^2\psi/\partial t^2 - \partial^2\psi/\partial x^2 - \partial^2\psi/\partial y^2 - \partial^2\psi/\partial z^2 = 0$$

which satisfies the initial conditions

$$\psi(t_0, x, y, z) = \psi_0(x, y, z), \ \frac{\partial\psi}{\partial t}(t_0, x, y, z) = \psi_0'(x, y, z) .$$

Furthermore this solution ψ is twice* continuously differentiable. Its value $\psi(x)$ at any point x of spacetime is equal to the average of the expression

$$\psi(\bar{x}) + (t - \bar{t})\frac{\partial\psi}{\partial t}(\bar{x}) - (x - \bar{x})\frac{\partial\psi}{\partial x}(\bar{x}) - (y - \bar{y})\frac{\partial\psi}{\partial y}(\bar{x}) - (z - \bar{z})\frac{\partial\psi}{\partial z}(\bar{x})$$

as $\bar{x} = [\bar{t}, \bar{x}, \bar{y}, \bar{z}]$ varies over the 2-dimensional sphere obtained by intersecting the hyperplane $t = t_0$ with the light cone based at x.

Thus solutions of the $(1+3)$-dimensional wave equation travel cleanly, at precisely the speed of light.

For the proof of a statement equivalent to 11.3, we refer to Duff and Naylor, "Differential Equations of Applied Mathematics," Wiley, 1966, p. 383.

Let us again consider a smooth vector field v.

Definition. The curl $\nabla \wedge v$ is the continuous skew tensor field

$$(\nabla \otimes v) - (\nabla \otimes v)^*$$

which is obtained by subtracting the Minkowski adjoint from the derivative tensor field $\nabla \otimes v$.

(Caution. $\nabla \wedge v$ can not be identified with the difference $\nabla \otimes v - v \otimes \nabla$ since, according to our conventions, the operator ∇ placed to the right of v would not operate on v.)

A basic property of the curl operator can be described as follows.

Lemma 11.4. A given smooth vector field v, defined on a convex region of spacetime, can be expressed as the gradient,

$$v = \nabla\psi \ ;$$

of some function ψ, if and only if the curl $\nabla \wedge v$ is identically zero.

Proof. If $v = \nabla\psi$, then the derivative tensor $\nabla \otimes v$, being equal to a second derivative $\nabla \otimes \nabla\psi$, is certainly a symmetric tensor.

*We must put in somewhat more differentiability than we get out.

To prove the converse it is convenient to temporarily introduce the notation

$$x_0 = t,\ x_1 = -x,\ x_2 = -y,\ x_3 = -z$$

so that the differential form

$$v \cdot dx = v^0 dt - v^1 dx - v^2 dy - v^3 dz$$

can be written as $\sum v^i dx_i$. Then inspection shows that the matrix $\nabla \otimes v$ is self adjoint if and only if

$$\partial v^i/\partial x_j = \partial v^j/\partial x_i \ .$$

If these conditions are satisfied, then a well known theorem asserts that

$$v^i = \partial\psi/\partial x_i$$

for some smooth function ψ.

[For completeness here is a proof, by induction on the number of variables. Suppose inductively that we are given a function $\psi(x_0, \ldots, x_{n-1}, 0)$ which satisfies

$$\partial\psi(x_0, \ldots, x_{n-1}, 0)/\partial x_i = v^i(x_0, \ldots, x_{n-1}, 0)$$

for $i < n$. It will be convenient to use the abbreviation $\vec{x}$ for the variables $x_0, \ldots, x_{n-1}$. Setting

$$\psi(\vec{x}, x_n) = \psi(\vec{x}, 0) + \int_0^{x_n} v^n(\vec{x}, \xi)d\xi \ ,$$

the equation $\partial\psi/\partial x_n = v^n$ is certainly satisfied. For $i < n$, differentiating under the integral sign and replacing $\partial v^n/\partial x_i$ by $\partial v^i/\partial x_n$, we obtain

$$\frac{\partial\psi}{\partial x_i}(\vec{x}, x_n) = v^i(\vec{x}, 0) + \int_0^{x_n} \frac{\partial v^i}{\partial x_n}(\vec{x}, \xi)d\xi \ ,$$

which clearly equals $v^j(\vec{x}, x_n)$ as required. } □

To conclude this section, let us look at the problem of differentiating a smooth tensor field. Since our notation does not allow for 3-index tensor fields (i.e., $4\times 4\times 4$ arrays of real valued functions), we can not introduce the full derivative of a (2-index) tensor field. However we can introduce an important subsidiary construction.

Definition 11.5. The divergence of a smooth tensor field T will mean the continuous vector field ∇T obtained by multiplying (and operating) on the left by the matrix of differential operators ∇. In other words ∇T is the vector whose j-th component is

$$\frac{\partial}{\partial t}T_0{}^j - \frac{\partial}{\partial x}T_1{}^j - \frac{\partial}{\partial y}T_2{}^j - \frac{\partial}{\partial z}T_3{}^j \ .$$

This divergence transforms as a vector since, if we replace ∇ by $\nabla\Lambda$ and T by $\Lambda^{-1}T\Lambda$, the divergence ∇T will be replaced by $(\nabla T)\Lambda$.

Here is an explicit example which will be useful in the next section.

Lemma 11.6. Suppose that the skew tensor field S is equal to the curl

$$S = \nabla \wedge v$$

of a twice continuously differentiable vector field. Then the divergence ∇S can be expressed as the difference

$$(\nabla\cdot\nabla)v - \nabla(\nabla\cdot v)$$

where $(\nabla\cdot\nabla)v$ is the (componentwise) d'Alembertian of v and $\nabla(\nabla\cdot v)$ is the gradient of the divergence of v.

This is just the analogue for the ∇ operator of the vector identity

$$p(p\wedge v) = (p\cdot p)v - p(p\cdot v) \ ,$$

which follows easily from 8.4.

Proof. If T denotes the derivative tensor field $\nabla \otimes v$, then

$$\nabla T = \nabla\nabla^* v = (\nabla\cdot\nabla)v \ .$$

The computation of ∇T^* is more confusing. Perhaps it is easiest to write T^* as $v\otimes\nabla = v^*\nabla$ with the temporary understanding that ∇ is supposed to operate on v^* even though, to keep the matrix notation straight, it appears to the right of v^*. With this understanding we can write

$$\nabla T^* = \nabla v^* \nabla = (\nabla\cdot v)\nabla \ .$$

Moving the function $\psi = \nabla\cdot v$ back to the right of ∇ where it belongs, we have $\nabla T^* = \nabla\psi$ hence $\nabla S = (\nabla\cdot\nabla)v - \nabla\psi$ as required. □

§12. The Maxwell Equations in Vacuum

The electromagnetic tensor field F satisfies a beautiful set of differential equations which were formulated by Maxwell in 1864. This section will try to motivate these equations.

As a starting point let us take Coulomb's law* which can be stated in 3-dimensional language as follows. Any two stationary charged particles experience a mutual force, directed along the line joining the two particles, whose magnitude is proportional to the product ee' of the charges divided by the square of their distance r. This force is repulsive or attractive according as the charges have the same sign or opposite sign. Thus

$$\text{force} = \kappa ee'/r^2 \ ,$$

where κ is a certain universal constant. If we use the second as unit of time and distance, and the gram and proton charge as units of mass and charge, then this Coulomb constant is given by

$$\kappa = 8.56265 \times 10^{-51} \text{ gram seconds/(proton charge)}^2 \ .$$

At this point it is convenient to eliminate this constant κ by switching to a different unit of charge, chosen precisely so that Coulomb's law will take the simpler form**

(12.1) $$\text{force} = ee'/r^2 \ .$$

The appropriate unit of charge, known as the $\sqrt{\text{gram second}}$, is equal to 1.08068×10^{25} proton charges. (This is an enormous unit, equal to

*This law was first proposed by Joseph Priestley in 1767, in order to explain an experiment by Benjamin Franklin. It was confirmed by direct measurement by Charles Coulomb in 1785.

**Thus we are using "Gaussian" units. Many authors prefer to work with the alternative "rational" units, defined by the requirement that force = $ee'/4\pi r^2$.

$1.731\!45 \times 10^6$ coulombs, or to the total charge on all of the protons in 6 grams of matter.) The corresponding unit of electric and magnetic field strength is known as the $\sqrt{\text{gram/second}^3}$. It is equal to 0.173145 volts/meter, or to 5.77550×10^{-6} gauss.

We will also need the following empirical statement.

12.2. A stationary charged particle does not generate any magnetic field.*

Let us restate these two facts, using the 4-dimensional language of §10. Consider two particles with the same velocity vector $\underset{\sim}{u} = [1, 0, 0, 0]$. If the worldline of the first particle with charge e is the t-axis, $x = y = z = 0$, then the force exerted on a second particle with charge e' at the position $\underset{\sim}{x} = [t, x, y, z]$ in spacetime will be

$$\underset{\sim}{f} = ee'[0, x/r^3, y/r^3, z/r^3]$$

according to 12.1, where $r = \sqrt{x^2 + y^2 + z^2}$. Comparing this with the force law

$$\underset{\sim}{f} = e'\underset{\sim}{u}F$$

of §10.4, we see that the top row $\underset{\sim}{u}F$ of the matrix F is given by

$$\underset{\sim}{u}F = e[0, x/r^3, y/r^3, z/r^3] \ .$$

Using 12.2 and the fact that F is a skew tensor, this determines F completely. In fact:

(12.3) $$F(\underset{\sim}{x}) = \frac{e}{r^3}\begin{bmatrix} 0 & x & y & z \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ z & 0 & 0 & 0 \end{bmatrix} .$$

* In the real world many elementary particles do actually have an intrinsic magnetic moment. Our presentation does not allow for this fact.

= 5.19076 × 10^7 volts*.

We will call this tensor field 12.3 the Coulomb field generated by a particle with charge e whose worldline is the t-axis.

Remark. In computing the force acting on the second particle, we made use of the field generated by the first particle only. This is essential, since the field generated by a given charged particle always has a bad infinite singularity along the worldcurve of the particle itself. We will give a precise explanation of this point in §19.

Next consider a charged particle whose worldcurve is an arbitrary timelike line L. Choosing a Poincaré-Lorentz transformation which carries the t-axis to L, and transforming the Coulomb field 12.3 correspondingly, we obtain a new electromagnetic field which we may call the Coulomb field* generated by the particle with worldline L.

Now consider a system of many moving charged particles. The basic empirical fact in this case is that the total electromagnetic field generated by all of the particles together is just the sum of the electromagnetic fields generated by the individual particles. Using this fact, we can compute the field F generated by any system of finitely many particles with straight (but not necessarily parallel) worldlines. We will make use of this construction presently.

Let us return to a study of the Coulomb field 12.3, generated by a single particle at $x = y = z = 0$. We will write this in 3-dimensional language as

$$E^1 = ex/r^3, \quad E^2 = ey/r^3, \quad E^3 = ez/r^3 .$$

As noted by Poisson in 1812, this Coulomb field (and hence any sum of Coulomb fields arising from finitely many distinct stationary particles) has two important mathematical properties.

*Actually, the first correct description of the electromagnetic field generated by a charged particle in a state of uniform linear motion was perhaps given by Liénard in 1898?

Observation 12.4. The Coulomb field $\vec{E}$ can be expressed as the negative gradient of a suitable "potential function" V. That is

$$E^1 = -\partial V/\partial x, \quad E^2 = -\partial V/\partial y, \quad E^3 = -\partial V/\partial z ,$$

where $V(\underset{\sim}{x}) = e/r + \text{constant}$.

(A completely equivalent statement would be that the 3-dimensional curl of the vector field $\vec{E}$ is zero.)

Furthermore this potential function satisfies Laplace's equation $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = 0$, or in other words:

Observation 12.5. The divergence $\partial E^1/\partial x + \partial E^2/\partial y + \partial E^3/\partial z$ is identically zero.

The reader can easily check both of these properties of the Coulomb field 12.3 by direct computation. □

Of course both observations break down along the worldline $x = y = z = 0$ itself, since $\vec{E}$ becomes infinite. We will need new mathematical techniques to give an adequate formulation of 12.4 and 12.5 along this singular worldline. These techniques will be developed in §14. Note that the potential function V has the dimensions of mass/charge =

We would like to formulate these two properties of the Coulomb field in an invariant 4-dimensional language, so that they will transform properly under the Poincaré-Lorentz group.

The most obvious differential operation to apply to the tensor F is the divergence operation $F \mapsto \underset{\sim}{\nabla} F$ of §11.5. The divergence $\underset{\sim}{\nabla} F$ is a well defined 4-dimensional vector field. If we take the divergence of the Coulomb field 12.3, then inspection, together with Observation 12.5, shows that $\underset{\sim}{\nabla} F = \underset{\sim}{0}$. This statement is clearly invariant under Poincaré-Lorentz transformations. Since the divergence of a sum of tensors is equal to the sum of the divergences, this proves the following.

Lemma 12.6. The electromagnetic field F produced by any finite collection of charged particles with straight (but not necessarily parallel)

* To complete the comparison with classical electrical units, our unit of current is $1\sqrt{\text{gram}/\text{second}} = 1.73145 \times 10^6$ amperes, and our unit of resistance is the dimensionless quantity 29.97925 ohms.

worldlines has divergence ∇F identically zero wherever this divergence is defined, i.e., except on the worldlines themselves.

This argument cannot tell us anything about electromagnetic fields produced by accelerating or colliding particles. Nevertheless the special case considered in 12.6 should be sufficiently general to motivate the following.

Maxwell Axiom 12.7. For any electromagnetic field, in any region of spacetime which contains no worldcurves (i.e., in vacuum), the divergence ∇F is identically zero.

Of course Maxwell stated this law in 3-dimensional language. The 4-dimensional formulation is due to Minkowski. We can recover Maxwell's version as follows.

The vector equation $\nabla F = 0$ is equivalent to a system of four scalar equations. Recalling the definition of ∇ and the notation 10.6 for the components of F, these four equations can be written explicitly as

$$(12.8)\quad \begin{cases} \quad - \partial E^1/\partial x - \partial E^2/\partial y - \partial E^3/\partial z = 0 \\ \partial E^1/\partial t - \partial B^3/\partial y + \partial B^2/\partial z = 0 \\ \partial E^2/\partial t + \partial B^3/\partial x - \partial B^1/\partial z = 0 \\ \partial E^3/\partial t - \partial B^2/\partial x + \partial B^1/\partial y = 0 \; ; \end{cases}$$

or in 3-dimensional vector language:

$$(12.9)\qquad \mathrm{div}(\vec{E}) = 0, \; \partial\vec{E}/\partial t = \vec{\mathrm{curl}}(\vec{B}) \,.$$

Now let us construct a corresponding 4-dimensional formulation of the property 12.4 of the Coulomb field, that is the property

$$E^1 = -\partial V/\partial x, \quad E^2 = -\partial V/\partial y, \quad E^3 = -\partial V/\partial z \,,$$

where $V = e/r$. As a first attempt, consider the derivative tensor $\nabla \otimes v$ of the vector field

$$v = [V, 0, 0, 0] \,.$$

By definition

$$\nabla \otimes v = \begin{bmatrix} -\partial V/\partial t & 0 & 0 & 0 \\ -\partial V/\partial x & 0 & 0 & 0 \\ -\partial V/\partial y & 0 & 0 & 0 \\ -\partial V/\partial z & 0 & 0 & 0 \end{bmatrix} .$$

Clearly, in order to obtain the required Coulomb field 12.3, it is only necessary to subtract the Minkowski adjoint $(\nabla \otimes v)^*$ of this matrix. Thus we have proved the following statement.

If v is the vector field $[V, 0, 0, 0] = [-e/r, 0, 0, 0]$, then the curl

$$\nabla \wedge v = (\nabla \otimes v) - (\nabla \otimes v)^*$$

is equal to the Coulomb field 12.3.

Again we can obtain a corresponding statement for any particle with straight worldline by applying an appropriate Poincaré-Lorentz transformation. Since the curl operation is linear, this proves the following.

Lemma 12.10. The electromagnetic field generated by any finite system of charged particles with straight worldlines, can be expressed as a curl

$$F = \nabla \wedge v$$

for some appropriately chosen "potential vector field" v.

Again we generalize this statement by assuming that it holds for any electromagnetic field.

Maxwell Axiom 12.11. Any electromagnetic field F can be expressed as a curl, $F = \nabla \wedge v$, for some suitably chosen vector field v.

More explicitly, using 10.6 and the precise definition of the curl operator, this formula for F is equivalent to the system of six equations

$$(12.12)\quad \begin{cases} E^1 = \dfrac{\partial v^1}{\partial t} + \dfrac{\partial v^0}{\partial x}, & E^2 = \dfrac{\partial v^2}{\partial t} + \dfrac{\partial v^0}{\partial y}, \quad E^3 = \dfrac{\partial v^3}{\partial t} + \dfrac{\partial v^0}{\partial z}, \\[2ex] B^1 = \dfrac{\partial v^2}{\partial z} - \dfrac{\partial v^3}{\partial y}, & B^2 = \dfrac{\partial v^3}{\partial x} - \dfrac{\partial v^1}{\partial z}, \quad B^3 = \dfrac{\partial v^1}{\partial y} - \dfrac{\partial v^2}{\partial x} \ . \end{cases}$$

Note that the potential vector field $\underset{\sim}{v}$ has the dimensions of mass/charge = $\sqrt{\text{mass}}$/time. Our unit of potential is $\sqrt{\text{gram}}$/second = [illegible] volts.

Of course the potential vector field $\underset{\sim}{v}$ is not uniquely determined. For according to §11.4 we can add a completely arbitrary gradient vector field $\underset{\sim}{\nabla}\psi$ to $\underset{\sim}{v}$ without changing the curl $\underset{\sim}{\nabla} \wedge \underset{\sim}{v}$. For this reason, the formulation 12.11 is not completely satisfying. Nevertheless it is often very convenient to use. Following is an illustration of this. We will assume that the potential field $\underset{\sim}{v}$ is three times continuously differentiable.

Lemma 12.13. Every one of the 16 components of the electromagnetic tensor in vacuum satisfies the wave equation. In other words $(\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla})F = 0$.

Proof. According to §11.6 one has the identity

$$\underset{\sim}{\nabla}(\underset{\sim}{\nabla} \wedge \underset{\sim}{v}) = (\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla})\underset{\sim}{v} - \underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{v}) \ .$$

If $\underset{\sim}{v}$ is the vector potential of 12.11, then the left side of this equation equals $\underset{\sim}{\nabla}F = \underset{\sim}{0}$ by 12.7. Setting $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \psi$, it follows that

$$(\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla})\underset{\sim}{v} = \underset{\sim}{\nabla}\psi \ .$$

Now taking the curl of both sides of this equation, it is not difficult to check that the left side becomes

$$\underset{\sim}{\nabla} \wedge (\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla})\underset{\sim}{v} = (\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla})\underset{\sim}{\nabla} \wedge \underset{\sim}{v} = (\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla})F \ ,$$

[illegible]

while the right side becomes $\underset{\sim}{\nabla} \wedge \underset{\sim}{\nabla}\psi = 0$ by 11.4. Therefore $(\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla})F = 0$. □

Using well known properties of the wave equation, as described in §11.3, it follows that any electromagnetic disturbance in vacuum must be propagated at precisely unit speed. We will return to this point in §13.

The formulation 12.11 has the defect of involving a moderately arbitrary vector field $\underset{\sim}{v}$. It is useful to have an alternative formulation which does not involve any arbitrary quantities. This can be constructed as follows.

By inspection of the explicit equations 12.12, the reader can easily check that the following four equations are satisfied.

$$(12.14)\quad \begin{cases} \quad - \partial B^1/\partial x - \partial B^2/\partial y - \partial B^3/\partial z = 0 \\ \partial B^1/\partial t + \partial E^3/\partial y - \partial E^2/\partial z = 0 \\ \partial B^2/\partial t - \partial E^3/\partial x + \partial E^1/\partial z = 0 \\ \partial B^3/\partial t + \partial E^2/\partial x - \partial E^1/\partial y = 0 \ . \end{cases}$$

These are the required additional Maxwell equations. In 3-dimensional vector language they can be written as

$$(12.15)\qquad \operatorname{div}(\vec{B}) = 0, \quad \partial\vec{B}/\partial t = -\operatorname{curl}(\vec{E}) \ .$$

[Remark. The best mathematical tool for calculations of this type is provided by the exterior differential calculus. (See Flanders, "Differential Forms with Applications to the Physical Sciences," Academic Press, 1963; or Spivak, "Calculus on Manifolds," Benjamin, 1965.) Using this calculus, the electromagnetic field is described by a differential 2-form

$$\Phi = \tfrac{1}{2}(d\underset{\sim}{x})F(d\underset{\sim}{x})^* = E^1 dxdt + E^2 dydt + E^3 dzdt + B^1 dydz + B^2 dzdx + B^3 dxdy \ .$$

The axiom 12.11 is now expressed by the equation $\Phi = d\varphi$ where

$$\varphi = \underset{\sim}{v}\cdot d\underset{\sim}{x} = v^0 dt - v^1 dx - v^2 dy - v^3 dz \ .$$

It follows from this equation that $d\Phi = d(d\varphi) = 0$, which is just the set of equations 12.14.]

The reader who notes the strong similarity between equations 12.8 and 12.14 will suspect that there must be a more symmetrical 4-dimensional formulation. This can be provided as follows. According to the Appendix, we can associate to the skew tensor F a dual tensor*

(12.16)
$$\hat{F} = \begin{bmatrix} 0 & B^1 & B^2 & B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix}.$$

In terms of this dual tensor, the Maxwell equations 12.14 can clearly be written in the more elegant form

$$\nabla \hat{F} = 0 .$$

Thus the full set of Maxwell equations in vacuum can now be written as follows:

(12.17)
$$\nabla F = \nabla \hat{F} = 0 .$$

We will examine the implications of this system of differential equations in the next section.

For later use, we note the following fact, which follows from the above discussion.

Lemma 12.18. *If $F = \nabla \wedge v$ where v is a twice continuously differentiable vector field, then $\nabla \hat{F} = 0$.*

*More precisely, $\hat{F}$ transforms as a tensor provided that we only make use of Lorentz transformations with determinant +1.

Conversely, if F is defined and smooth throughout a convex region of spacetime, with $\nabla \hat{F} = 0$, then using "Poincaré's Lemma" in the theory of exterior differential forms one can check that $F = \nabla \wedge v$ for some smooth vector field v.

Here is an exercise for the reader.

Problem. Consider a particle freely falling in a steady state electromagnetic field, so that $\vec{E} = -\overrightarrow{grad}(V)$ with $\partial V/\partial t = 0$. Show that its energy ε and potential V are related by the equation

$$\varepsilon + eV = \text{constant}.$$

§13. Electromagnetic Waves

We continue to study the vacuum Maxwell equations

$$\nabla F = \nabla \hat{F} = 0 \ ;$$

or in 3-dimensional vector language

$$\mathrm{div}(\vec{E}) = 0, \quad \partial\vec{E}/\partial t = \overrightarrow{\mathrm{curl}}(\vec{B}) \ ,$$
$$\mathrm{div}(\vec{B}) = 0, \quad \partial\vec{B}/\partial t = -\,\overrightarrow{\mathrm{curl}}(\vec{E}) \ .$$

The mathematical theory of these equations can be summarized as follows. (Compare §§11.3 and 12.13.)

Theorem 13.1. Let $F(t_0, x, y, z)$ be a twice differentiable skew tensor valued function on the hyperplane $t = t_0$ in spacetime. Suppose that the two equations*

$$\mathrm{div}(\vec{E}) = \mathrm{div}(\vec{B}) = 0$$

are satisfied throughout this hyperplane. Then this tensor field on $t = t_0$ can be extended uniquely to a tensor field which is defined and smooth throughout spacetime, and satisfies the vacuum Maxwell equations. In fact the electric field $\vec{E}(\mathbf{x})$ at a given point of spacetime is equal to the average of the expression

$$\vec{E}(\bar{\mathbf{x}}) + (t-\bar{t})\,\overrightarrow{\mathrm{curl}}(\vec{B}(\bar{\mathbf{x}})) - (x-\bar{x})\frac{\partial\vec{E}}{\partial x}(\bar{\mathbf{x}}) - (y-\bar{y})\frac{\partial\vec{E}}{\partial y}(\bar{\mathbf{x}}) - (z-\bar{z})\frac{\partial\vec{E}}{\partial z}(\bar{\mathbf{x}})$$

as $\bar{\mathbf{x}} = [\bar{t}, \bar{x}, \bar{y}, \bar{z}]$ varies over the 2-dimensional sphere obtained by intersecting the hyperplane $t = t_0$ with the light cone based at $\mathbf{x}$. Similarly $\vec{B}(\mathbf{x})$ is the average of the expression

$$\vec{B}(\bar{\mathbf{x}}) - (t-\bar{t})\,\overrightarrow{\mathrm{curl}}(\vec{E}(\bar{\mathbf{x}})) - (x-\bar{x})\frac{\partial\vec{B}}{\partial x}(\bar{\mathbf{x}}) - (y-\bar{y})\frac{\partial\vec{B}}{\partial y}(\bar{\mathbf{x}}) - (z-\bar{z})\frac{\partial\vec{B}}{\partial z}(\bar{\mathbf{x}})$$

as $\bar{\mathbf{x}}$ varies over this 2-dimensional sphere.

*Here the 3-vectors $\vec{E}$ and $\vec{B}$ are to be defined as in §10.6.

Thus electromagnetic disturbances travel "cleanly" with speed 1. For the proof, the reader is referred for example to Duff and Naylor, "Differential Equations of Applied Mathematics," Wiley, 1966, p. 391. ∎

As an example, suppose that the electromagnetic field at coordinate time $t = 0$ is zero, except in a small neighborhood of the origin. Then at coordinate time $t = t_1 > 0$ the field F will be zero except near the 2-dimensional sphere

$$x^2 + y^2 + z^2 = t_1^2$$

of radius t_1. We can describe this situation intuitively by saying that an electromagnetic disturbance near the origin generates an expanding spherical wave which travels with speed 1.

Consider this wave from the point of view of a "stationary" observer with worldline

$$x = x_1, \ y = 0, \ z = 0$$

where x_1 is large. Since the surface of a large sphere looks very much like a plane, what he will actually observe (to a fair order of approximation) will be a "plane wave" which sweeps past him in the direction of increasing x at coordinate time $t = x_1$.

Before defining the concept of a plane wave, it will be convenient to consider a somewhat more general type of "wave." Let $\boldsymbol{\ell}$ be a fixed non-zero vector.

Definition 13.2. A smooth electromagnetic field F satisfying the vacuum Maxwell equations will be called a directed electromagnetic wave with direction $\boldsymbol{\ell}$ if the matrix equations $\boldsymbol{\ell}F = \boldsymbol{\ell}\hat{F} = 0$ are satisfied everywhere.

Of course the vector $\boldsymbol{\ell}$ is not uniquely defined. Any multiple of $\boldsymbol{\ell}$ would do just as well.

To motivate this definition we prove the following.

Lemma 13.3. Let F be a solution of the vacuum Maxwell equations which is constant along each hyperplane orthogonal to a fixed vector $\underset{\sim}{\ell}$, so that $F(\underset{\sim}{x})$ can be expressed as a function of the real number $\underset{\sim}{x}\cdot\underset{\sim}{\ell}$. Then F is equal to the sum of a directed electromagnetic wave with direction $\underset{\sim}{\ell}$ and a constant tensor.

Definition. Any directed electromagnetic wave which is constant along hyperplanes perpendicular to $\underset{\sim}{\ell}$ is called a plane wave with direction $\underset{\sim}{\ell}$.

Proof of 13.3. After a Lorentz transformation we may assume that the vector $\underset{\sim}{\ell}$ is a multiple of either $[1, 0, 0, 0]$ or $[0, 1, 0, 0]$ or $[1, 1, 0, 0]$. In the first case, our hypothesis is that F is a function of t only, so that $\partial F/\partial x = \partial F/\partial y = \partial F/\partial z = 0$. The Maxwell equations then imply that $\partial F/\partial t = 0$ also, so that F is constant.

If $\underset{\sim}{\ell}$ is spacelike, it follows similarly that F is constant. These two cases are completely uninteresting.

Suppose then that $\underset{\sim}{\ell} = [1, 1, 0, 0]$, so that F is a function of $\underset{\sim}{x}\cdot\underset{\sim}{\ell} = t - x$. Then

$$\partial F/\partial t + \partial F/\partial x = 0, \quad \partial F/\partial y = \partial F/\partial z = 0 .$$

Combining these equations with the Maxwell equations 12.8 and 12.14, we see easily that

$$E^1 = \text{constant},\ B^1 = \text{constant},\ E^2 - B^3 = \text{constant},\ E^3 + B^2 = \text{constant} .$$

Therefore F is the sum of a constant electromagnetic field and an electromagnetic field satisfying the identities

$$E^1 = 0,\ B^1 = 0,\ E^2 = B^3,\ E^3 = -B^2 . \qquad (13.4)$$

In matrix notation, F is the sum of a constant matrix and a matrix of the form

$$\begin{bmatrix} 0 & 0 & E^2 & E^3 \\ 0 & 0 & -E^2 & -E^3 \\ E^2 & E^2 & 0 & 0 \\ E^3 & E^3 & 0 & 0 \end{bmatrix} . \qquad (13.5)$$

Multiplying this matrix 13.5 on the left by $\underset{\sim}{\ell} = [1, 1, 0, 0]$ we certainly get $\underset{\sim}{0}$. The reader can easily verify that the dual skew tensor (§12.16) has the same property. ☐

Lemma 13.6. The most general plane wave with direction $\underset{\sim}{\ell} = [1, 1, 0, 0]$ is given by a matrix F of the form 13.5. Here $E^2 = B^3$ and $E^3 = -B^2$ can be arbitrary smooth functions of $t - x$.

The proof is straightforward. ☐

It is interesting to note that any such plane wave can be expressed as the sum

$$\begin{bmatrix} 0 & 0 & E^2 & 0 \\ 0 & 0 & -E^2 & 0 \\ E^2 & E^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & E^3 \\ 0 & 0 & 0 & -E^3 \\ 0 & 0 & 0 & 0 \\ E^3 & E^3 & 0 & 0 \end{bmatrix}$$

of two "plane polarized" plane waves.

Let us return to the study of the somewhat more general "directed electromagnetic waves."

Lemma 13.7. If F is a non-zero directed electromagnetic wave with direction $\underset{\sim}{\ell}$, then the vector $\underset{\sim}{\ell}$ is lightlike, and the field F is constant along every line parallel to $\underset{\sim}{\ell}$. Furthermore the two Lorentz invariants*

$$\tfrac{1}{2}\mathrm{trace}(F^2) = \|\vec{E}\|^2 - \|\vec{B}\|^2$$

*Compare §10 and the Appendix.

and

$$\frac{1}{2}\mathrm{trace}(F\hat{F}) = 2\vec{E}\cdot\vec{B}$$

of the field F are identically zero.

(The geometry is mildly confusing since every line parallel to $\underset{\sim}{\ell}$ is contained in a hyperplane perpendicular to $\underset{\sim}{\ell}$.)

Proof. As in §13.3, we may assume that $\underset{\sim}{\ell}$ is a multiple of [1, 0, 0, 0] or [0, 1, 0, 0] or [1, 1, 0, 0]. In the first two cases it is easy to check that F must be identically zero.

Suppose then that $\underset{\sim}{\ell} = [1, 1, 0, 0]$. If F satisfies the equations $\underset{\sim}{\ell}F = \underset{\sim}{\ell}\hat{F} = \underset{\sim}{0}$, inspection shows that F must be a tensor of the form 13.5. In other words:

(i) The electric and magnetic components E^1 and B^1 associated with the direction of the wave are zero. This fact is usually expressed by saying that electromagnetic waves are transverse.

(ii) The remaining electric and magnetic components determine each other according to the formula $E^2 = B^3$, $E^3 = -B^2$. Therefore the 3-dimensional vectors $\vec{E}$ and $\vec{B}$ have the same length, and are mutually orthogonal. This last statement has only been proved for one Lorentz coordinate system. But, as noted in §10, it follows that it is true in any Lorentz coordinate system.

Consider the Maxwell equation $\nabla F = \underset{\sim}{0}$ for a tensor F of the form 13.5. Evidently the last two components of this vector equation assert that

$$\partial E^2/\partial t + \partial E^2/\partial x = 0 = \partial E^3/\partial t + \partial E^3/\partial x .$$

Therefore the directional derivative of F in the direction $\underset{\sim}{\ell} = [1, 1, 0, 0]$ is zero, hence F is constant along lines parallel to $\underset{\sim}{\ell}$. ◻

[Remark. It is interesting to note that the remaining Maxwell equations for an electromagnetic field of the form 13.5 reduce to the Cauchy-Riemann equations

$$\partial E^2/\partial y + \partial E^3/\partial z = 0, \quad \partial E^3/\partial y = \partial E^2/\partial z .$$

In other words the complex number $E^3 + iE^2$ is a complex analytic function of $y + iz$ for each fixed value of $t - x$. If these complex analytic functions are bounded, then of course they are constant, and it follows that F is a plane wave.]

Let us measure the size of a wave.

Definition. For any directed electromagnetic wave, the real number

$$\|\vec{E}\| = \|\vec{B}\| \geq 0$$

will be called the amplitude α^0 of the wave F at the given point of spacetime with respect to the given Lorentz coordinate system.

The amplitude α^0 is not invariant under Lorentz transformations. To construct an invariant description of amplitude we proceed as follows.

Lemma 13.8. If F is a directed electromagnetic wave with direction $\underset{\sim}{\ell}$, then the matrix product FF is a scalar multiple of the matrix $\underset{\sim}{\ell} \otimes \underset{\sim}{\ell} = \underset{\sim}{\ell}^*\underset{\sim}{\ell}$.

(Compare §17.4.)

Proof. Since the statement is Lorentz invariant, it suffices to check the special case $\underset{\sim}{\ell} = [1, 1, 0, 0]$. But for this special case, inspection shows that the square of the matrix 13.5 is given by

$$FF = (\alpha^0)^2 \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = (\alpha^0)^2 \underset{\sim}{\ell} \otimes \underset{\sim}{\ell} . \;\square$$

(It is interesting to note that the third power FFF is zero. Hence all eigenvalues of the matrix F are zero.)

This computation motivates the following.

Definition 13.9. The amplitude vector $\underset{\sim}{a}$ of the directed wave F at a given point of spacetime is the unique forward (or zero) vector with the property that $FF = \underset{\sim}{a} \otimes \underset{\sim}{a}$.

Clearly $\underset{\sim}{a}$ is well defined, and transforms as a vector. In fact $\underset{\sim}{a}$ can be characterized as the unique multiple of the direction vector $\underset{\sim}{\ell}$ whose initial component $\underset{\sim}{a}\cdot[1, 0, 0, 0]$ is equal to the amplitude $a^0 = \|\vec{E}\| = \|\vec{B}\|$. Like the electromagnetic field, this vector $\underset{\sim}{a}$ has the dimensions of force/charge $= \sqrt{\text{mass}/\text{time}^3}$.

Let us specialize once more to the case of a plane wave with direction $\underset{\sim}{\ell} = [1, 1, 0, 0]$. Using Fourier analysis we can try to express the arbitrary smooth functions $E^2(t-x)$ and $E^3(t-x)$ as sums of sinusoidal functions of the form

$$a \cos(\omega^0(t-x)) + b \sin(\omega^0(t-x)) \ .$$

If we introduce the vector

$$\underset{\sim}{\omega} = \omega^0[1, 1, 0, 0] = [\omega^0, \omega^0, 0, 0] \ ,$$

then the resulting field F can be written as

$$F = F_1 \cos(\underset{\sim}{\omega}\cdot\underset{\sim}{x}) + F_2 \sin(\underset{\sim}{\omega}\cdot\underset{\sim}{x})$$

where F_1 and F_2 are constant tensors of the form 13.5. This suggests the following.

Definition 13.10. Let $\underset{\sim}{\omega} = [\omega^0, \omega^1, \omega^2, \omega^3]$ be a forward lightlike vector. Any electromagnetic wave of the form

$$F = F_1 \cos(\underset{\sim}{\omega}\cdot\underset{\sim}{x}) + F_2 \sin(\underset{\sim}{\omega}\cdot\underset{\sim}{x}) \ ,$$

where F_1 and F_2 are constant tensors which must satisfy the conditions

$\underset{\sim}{\omega}F_i = \underset{\sim}{\omega}\hat{F}_i = \underset{\sim}{0}$, is called a sinusoidal plane wave with angular frequency vector $\underset{\sim}{\omega}$.

Clearly $\underset{\sim}{\omega}$ is well defined, and transforms as a vector. Note that $\underset{\sim}{\omega}$ has the dimensions of time^{-1}. The initial component ω^0 of this vector is called the angular frequency of the wave with respect to the given coordinate system, and the number $2\pi/\omega^0$ is called the wave length with respect to the given coordinate system.

(13.11) Remark. There is a close connection between this vector $\underset{\sim}{\omega}$ and the energy-momentum vector $\underset{\sim}{p}$. For according to Einstein any sinusoidal wave with angular frequency vector $\underset{\sim}{\omega}$ must be made up out of photons, each of which has energy-momentum vector

$$\underset{\sim}{p} = \hbar\underset{\sim}{\omega} \ .$$

Here $\hbar = 1.173391 \times 10^{-48}$ gram seconds is Planck's constant. In particular the energy ϵ of each such photon is related to the angular frequency ω^0 by the equation $\epsilon = \hbar\omega^0$. [More generally, according to de Broglie, any particle at all is associated with a sinusoidal wave with frequency vector $\underset{\sim}{\omega}$ equal to $\underset{\sim}{p}/\hbar$.]

Now let us compare the angular frequency vector $\underset{\sim}{\omega}$ with the amplitude vector field $\underset{\sim}{a}$ associated with a sinusoidal wave. Both point in the same direction, so the equation

$$\underset{\sim}{a} = (a^0/\omega^0)\underset{\sim}{\omega}$$

must be satisfied. Therefore the ratio a^0/ω^0 of amplitude to angular frequency at any point of spacetime is invariant under the Lorentz group.

This ratio a^0/ω^0 is evidently a periodic function of $\underset{\sim}{\omega}\cdot\underset{\sim}{x}$. This function is constant only in the special case of a "circularly polarized" wave.

Let us illustrate these statements by a concrete example. Consider the sinusoidal wave

$$\vec{E} = \vec{E}_1 \cos(\underset{\sim}{\omega}\cdot\underset{\sim}{x})$$

where $\vec{E}_1$ and $\underset{\sim}{\omega} = [\omega^0, \omega^0, 0, 0]$ are constant. (The magnetic field $\vec{B}$ is then uniquely determined.) The corresponding amplitude vector field is given by

$$\underset{\sim}{a} = \|\vec{E}_1 \cos(\underset{\sim}{\omega} \cdot \underset{\sim}{x})\| [1, 1, 0, 0] .$$

Thus the ratio

$$a^0/\omega^0 = \|\vec{E}_1 \cos(\underset{\sim}{\omega} \cdot \underset{\sim}{x})\| / \omega^0$$

is a Lorentz invariant.

Consider an observer with velocity vector $\underset{\sim}{u} = [\cosh \varphi, \sinh \varphi, 0, 0]$, where φ is very large so that he is traveling "almost" as fast as the wave. To this observer, the wave will appear to have an amplitude

$$\underset{\sim}{a} \cdot \underset{\sim}{u} = a^0 e^{-\varphi},$$

which is very small. Similarly its apparent angular frequency

$$\underset{\sim}{\omega} \cdot \underset{\sim}{u} = \omega^0 e^{-\varphi}$$

will be very low. Correspondingly its wavelength

$$2\pi / \underset{\sim}{\omega} \cdot \underset{\sim}{u} = 2\pi e^{\varphi} / \omega^0$$

will be very long. (Compare the discussion of "red shift" in §3.3.)

§14. Generalized Functions and Smoothing

In this section we interrupt the physical exposition to describe some useful mathematical tools.

P. A. M. Dirac, in 1926, introduced a function $\delta(t)$, zero for $t \neq 0$ and infinitely large for $t = 0$, with the marvelous property that

$$\int_{-\infty}^{\infty} \delta(t)\psi(t)dt = \psi(0)$$

for any smooth ψ. A mathematically coherent theory which makes sense of such "functions" and their derivatives was first provided by Laurent Schwartz in 1946. This section will outline some of Schwartz's ideas. To fix our ideas, we will always work in the space of four real variables $\underset{\sim}{x} = [t, x, y, z]$. Four-fold integrals such as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\underset{\sim}{x})dtdxdydz$$

will be written briefly as $\int f(\underset{\sim}{x})d^4\underset{\sim}{x}$. Thus $d^4\underset{\sim}{x}$ denotes the 4-dimensional volume element.

Let $\psi(\underset{\sim}{x})$ be a real valued function of four variables. We say that ψ is <u>infinitely differentiable</u> if it has continuous partial derivatives of all orders, and that ψ has <u>compact support</u> if it vanishes outside of some bounded set (say $\psi(\underset{\sim}{x}) = 0$ whenever $|t| + |x| + |y| + |z| \geq$ constant). If ψ satisfies both of these conditions, then it will be called briefly a <u>test function</u>. As the name indicates, test functions are introduced, not because we really want to study them in their own right, but rather because they provide an important tool for analyzing or describing other mathematical objects.

We will also need (something like a) topology on the space consisting of all test functions. (More precisely, we will define a concept of "convergence" for sequences of test functions.) The definition will be given presently.

<u>Definition</u> 14.1. A <u>generalized function</u> or <u>distribution</u>* f of the

*Schwartz's term "distribution is usually used in the literature for this particular type of generalized function. We will prefer the generic term since it lends itself more readily to combinations such as "generalized vector field," and since the word distribution is useful for other purposes.

four real variables $\underset{\sim}{x} = [t, x, y, z]$ will mean a continuous linear operator which assigns to each test function ψ a real number. We will use the notation

$$\int f\psi = \int \psi f$$

for the value of a generalized function f on a test function ψ. This notation is supposed to indicate that we think of $\int f\psi$ intuitively as the integral of the product $f\psi$ over 4-space (even though the value of f at any particular point of 4-space is not defined).*

The word linear in the above definition means of course that

$$\int (a_1\psi_1 + a_2\psi_2)f = a_1\int\psi_1 f + a_2\int\psi_2 f$$

for any test functions ψ_1, ψ_2 and real numbers a_1, a_2. The word continuous is to be interpreted as follows.

Definition 14.2. We will say that a sequence of test functions $\psi_1, \psi_2, \ldots$ converges to the test function ψ if the following extremely stringent requirements are satisfied:

(a) the ψ_i all vanish outside of some fixed bounded set,

(b) the sequence ψ_i converges uniformly to ψ, and furthermore

(c) any partial derivative of ψ_i, of first order or of higher order, converges uniformly to the corresponding partial derivative of ψ.

The requirement of continuity now means that $\int f\psi_i$ converges to $\int f\psi$ whenever the sequence ψ_i converges to ψ.

As an example, given any fixed point $\underset{\sim}{a}$ of 4-space, the Dirac generalized function $\delta_{\underset{\sim}{a}}$, concentrated at the point $\underset{\sim}{a}$, is defined by the equation

$$\int \psi\delta_{\underset{\sim}{a}} = \psi(\underset{\sim}{a})$$

*Such fake integrals can be distinguished from honest integrals, in our notation, by the fact that the symbol $d^4\underset{\sim}{x}$ is omitted.

for any test function ψ. This is a well defined generalized function since, for any fixed $\underset{\sim}{a}$, the value $\psi(\underset{\sim}{a})$ clearly depends continuously and linearly on ψ.

More prosaically, to any continuous function g of four real variables there is associated a generalized function $\tilde{g}$ defined by the equation

$$\int \tilde{g}\psi = \int g(\underset{\sim}{x})\psi(\underset{\sim}{x})d^4\underset{\sim}{x} ,$$

where the expression on the right stands for an honest 4-fold integral extended over the entire 4-dimensional coordinate space. This 4-fold integral is a well defined real number since the product $g(\underset{\sim}{x})\psi(\underset{\sim}{x})$ is continuous with compact support. Clearly $\int g(\underset{\sim}{x})\psi(\underset{\sim}{x})d^4\underset{\sim}{x}$ is continuous and linear as a function of ψ.

Note that this generalized function $\tilde{g}$ determines the continuous function g uniquely. In fact to compute the value $g(\underset{\sim}{a})$ at any point of 4-space we need only choose a test function ψ_i which

(1) satisfies $\int \psi_i(\underset{\sim}{x})d^4\underset{\sim}{x} = 1$,

(2) satisfies $\psi_i(\underset{\sim}{x}) \geq 0$ for all $\underset{\sim}{x}$, and

(3) vanishes outside of a small neighborhood N_i of $\underset{\sim}{a}$.

Then the value

$$\int \tilde{g}\psi_i = \int g(\underset{\sim}{x})\psi_i(\underset{\sim}{x})d^4\underset{\sim}{x}$$

is a weighted average of values $g(\underset{\sim}{x})$ as $\underset{\sim}{x}$ varies over the neighborhood N_i of $\underset{\sim}{a}$. Since g is continuous, this weighted average must be close to $g(\underset{\sim}{a})$. Passing to the limit for a sequence of neighborhoods $N_1, N_2, \ldots$ which shrink down to the point $\underset{\sim}{a}$, it follows easily that the numbers $\int \tilde{g}\psi_i$ converge to $g(\underset{\sim}{a})$. □

In practice we will identify the generalized function $\tilde{g}$ with the continuous function g whenever there is no danger of confusion.

Now suppose that g happens to be a smooth function (i.e., with continuous first partial derivatives). Integrating by parts we obtain the formula

$$\int \frac{\partial g}{\partial t}(\underset{\sim}{x})\psi(\underset{\sim}{x})d^4\underset{\sim}{x} = - \int g(\underset{\sim}{x})\frac{\partial \psi}{\partial t}(\underset{\sim}{x})d^4\underset{\sim}{x} ,$$

with corresponding formulas for the other partial derivatives. This computation motivates the following.

Definition 14.3. For any generalized function f, the partial derivatives $\partial f/\partial t, \ldots, \partial f/\partial z$ are the generalized functions defined by the formulas

$$\int \frac{\partial f}{\partial t}\psi = - \int f\frac{\partial \psi}{\partial t}, \ \ldots, \ \int \frac{\partial f}{\partial z}\psi = - \int f\frac{\partial \psi}{\partial z}$$

for any test function ψ.

Iterating this procedure, the higher partial derivatives of f are also well defined generalized functions. Note that the various partial differentiation operators commute with each other. For example $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$.

[One word of caution is necessary. If we start with a continuous function g whose derivatives are not continuous, then the derivatives of the associated generalized function $\tilde{g}$ may not correspond to the derivatives of g. As an example, if $g(\underset{\sim}{x}) = t^{1/3}$ then the second derivative $\partial^2 g/\partial t^2 = -\frac{2}{9}t^{-5/3}$ is well defined but discontinuous. The generalized function $\tilde{g}$ associated with g (of course) has a well defined second derivative $\partial^2\tilde{g}/\partial t^2$; but the value

$$\int \frac{\partial^2 \tilde{g}}{\partial t^2}\psi = \int \tilde{g}\,\frac{\partial^2 \psi}{\partial t^2} = \int g(\underset{\sim}{x})\frac{\partial^2 \psi}{\partial t^2}(\underset{\sim}{x})d^4\underset{\sim}{x}$$

on a test function ψ cannot be identified with the integral

$$\int \frac{\partial^2 g}{\partial t^2}(\underset{\sim}{x})\psi(\underset{\sim}{x})d^4\underset{\sim}{x} = \int -\frac{2}{9}t^{-5/3}\psi(\underset{\sim}{x})d^4\underset{\sim}{x}$$

unless ψ vanishes for $t = 0$. In general this last integral is divergent, and does not make any reasonable sense.]

Now consider a 4-tuple $\underset{\sim}{v} = [v^0, v^1, v^2, v^3]$ where v^0, v^1, v^2, v^3 are generalized functions. We will call $\underset{\sim}{v}$ a vector valued generalized

function, or briefly a generalized vector field. Extending our notation slightly, for any test function ψ, we will use the abbreviation $\int \underset{\sim}{v}\psi$ for the vector $[\int v^0\psi, \int v^1\psi, \int v^2\psi, \int v^3\psi]$.

One important example of a generalized vector field is provided by the gradient

$$\underset{\sim}{\nabla} f = [\partial f/\partial t, -\partial f/\partial x, -\partial f/\partial y, -\partial f/\partial z]$$

of a generalized function f. Note the vector identity

$$\int (\underset{\sim}{\nabla} f)\psi = - \int f(\underset{\sim}{\nabla}\psi)$$

for any test function ψ.

The divergence of a generalized vector field $\underset{\sim}{v}$ is defined to be the generalized function

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \frac{\partial v^0}{\partial t} + \frac{\partial v^1}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial v^3}{\partial z} .$$

Extending our notation still further, this divergence can be defined by the identity

$$\int (\underset{\sim}{\nabla}\cdot\underset{\sim}{v})\psi = - \int \underset{\sim}{v}\cdot(\underset{\sim}{\nabla}\psi) = - \int \left(v^0\frac{\partial \psi}{\partial t} + v^1\frac{\partial \psi}{\partial x} + v^2\frac{\partial \psi}{\partial y} + v^3\frac{\partial \psi}{\partial z}\right)$$

for any test function ψ.

Next let us take up the problem of "smoothing." Suppose that we are given a continuous function $g(\underset{\sim}{x})$, which is perhaps not differentiable, and want to approximate it by a nearby function $s(\underset{\sim}{x})$ which is infinitely differentiable. This can be done as follows. Choose some fixed test function ψ and let s be the convolution $\psi * g = g * \psi$, defined by the formula

$$s(\underset{\sim}{x}) = \int \psi(\underset{\sim}{a})g(\underset{\sim}{x} - \underset{\sim}{a})d^4\underset{\sim}{a}$$
$$= \int \psi(\underset{\sim}{x} - \underset{\sim}{a})g(\underset{\sim}{a})d^4\underset{\sim}{a} .$$

Then $s(x)$ is continuous as a function of x. Furthermore, differentiating under the integral sign, we see for example that

$$\partial(\psi * g)/\partial t = (\partial\psi/\partial t) * g \ .$$

Hence $s = \psi * g$ is continuously differentiable. Continuing inductively, it follows that s has continuous partial derivatives of all orders.

(If g itself happens to be continuously differentiable, then a similar argument shows that $\partial(\psi * g)/\partial t = \psi * (\partial g/\partial t)$.)

So far we have not put any restrictions on the test function ψ.

Definition 14.4. Suppose that the test function ψ satisfies

(1) $\int \psi(x) d^4x = 1$, and

(2) $\psi(x) \geq 0$ for all x.

Then the correspondence

$$g \longmapsto g * \psi$$

will be called a smoothing operator. We will say that this smoothing operator is close to the identity if ψ vanishes outside of a small neighborhood N of the origin.

If these conditions are satisfied, then clearly $(g * \psi)(x)$ is a weighted average of values $g(x - a)$ as a varies over the small neighborhood N. Therefore, in some sense, $g * \psi$ must be close to g.

Let us try to write down a formula which defines the convolution $s = g * \psi$ in terms of the generalized function $\tilde{g}$ associated with g. This can be done as follows. Given any fixed point x in 4-space, let ψ_x denote the reflected test function whose value at a is defined to be

$$\psi_x(a) = \psi(x - a) \ .$$

Then clearly

$$\int \tilde{g}\psi_x = \int g(a)\psi_x(a) d^4a$$

is equal to

$$\int g(a)\psi(x - a) d^4a = s(x) \ .$$

This motivates the following.

Definition 14.5. For any generalized function f and any test function ψ the convolution

$$s = f * \psi = \psi * f$$

is a real valued function defined by the formula

$$s(x) = \int f\psi_x$$

where ψ_x is the reflected function $\psi_x(a) = \psi(x - a)$.

As an example, for the Dirac generalized function δ concentrated at the origin (so that $\int \delta\psi = \psi(0)$), the computation

$$(\delta * \psi)(x) = \int \delta\psi_x = \psi_x(0) = \psi(x)$$

shows that $\delta * \psi = \psi$.

Lemma 14.6. For any generalized function f and any test function ψ the function $s = f * \psi$ defined in this way is infinitely differentiable. ~~function~~ Furthermore

$$\partial(f * \psi)/\partial t = (\partial f/\partial t) * \psi = f * (\partial\psi/\partial t) \ ,$$

with analogous formulas for all other partial derivatives.

Now if ψ satisfies conditions (1) and (2) of 14.4 we will say that the infinitely differentiable function $f * \psi$ is a smoothing of the generalized function f. The smoothing $v * \psi = [v^0 * \psi, v^1 * \psi, v^2 * \psi, v^3 * \psi]$ of a generalized vector field is defined similarly.

Proof of 14.6. We must first show that the function $s(\underset{\sim}{x}) = \int f\psi_{\underset{\sim}{x}}$ is continuous. But if $\Delta\underset{\sim}{x}_1, \Delta\underset{\sim}{x}_2, \ldots$ is any sequence of vectors converging to zero, then it is not difficult to show that the corresponding sequence of test functions $\psi_{\underset{\sim}{x}+\Delta\underset{\sim}{x}_1}, \psi_{\underset{\sim}{x}+\Delta\underset{\sim}{x}_2}, \ldots$ converges (in the sense of 14.2) to the test function $\psi_{\underset{\sim}{x}}$. Therefore the sequence of numbers $s(\underset{\sim}{x}+\Delta\underset{\sim}{x}_1), s(\underset{\sim}{x}+\Delta\underset{\sim}{x}_2), \ldots$ converges to $s(\underset{\sim}{x})$.

Next we must compute the derivative $\partial s/\partial t$. To this end, choose any sequence $\Delta t_1, \Delta t_2, \ldots$ of non-zero real numbers converging to zero, and let

$$\Delta\underset{\sim}{x}_i = [\Delta t_i, 0, 0, 0] .$$

Then each difference quotient

$$(14.7) \qquad (s(\underset{\sim}{x}+\Delta\underset{\sim}{x}_i) - s(\underset{\sim}{x}))/\Delta t_i$$

is equal to

$$\int f(\psi_{\underset{\sim}{x}+\Delta\underset{\sim}{x}_i} - \psi_{\underset{\sim}{x}})/\Delta t_i .$$

Setting $\psi' = \partial\psi/\partial t$, the identity

$$(14.8) \qquad (\psi_{\underset{\sim}{x}+\Delta\underset{\sim}{x}_i} - \psi_{\underset{\sim}{x}})/\Delta t_i = \int_0^1 \psi'_{\underset{\sim}{x}+u\Delta\underset{\sim}{x}_i} du$$

can easily be used to show that the sequence of test functions (14.8) converges to the test function $\psi'_{\underset{\sim}{x}}$. Therefore the sequence of difference quotients (14.7) converges to $\int f\psi'_{\underset{\sim}{x}} = (f * (\partial\psi/\partial t))(\underset{\sim}{x})$. This proves that the derivative $\partial s/\partial t$ is defined and equal to $f * (\partial\psi/\partial t)$, hence continuous. Since similar arguments work for other partial derivatives, it follows inductively that s is infinitely differentiable.

Finally, to compute $(\partial f/\partial t) * \psi$, we need only recall the definitions 14.3 and 14.5 which give us

$$(14.9) \qquad ((\partial f/\partial t) * \psi)(\underset{\sim}{a}) = \int(\partial f/\partial t)\psi_{\underset{\sim}{a}} = -\int(f)(\partial\psi_{\underset{\sim}{a}}/\partial t) .$$

Setting $\psi' = \partial\psi/\partial t$, and recalling the definition $\psi_{\underset{\sim}{a}}(\underset{\sim}{x}) = \psi(\underset{\sim}{a} - \underset{\sim}{x})$ where $\underset{\sim}{x} = [t, x, y, z]$ we see that

$$\partial\psi_{\underset{\sim}{a}}(\underset{\sim}{x})/\partial t = -\psi'(a - x) = -\psi'_{\underset{\sim}{a}}(\underset{\sim}{x}) .$$

Therefore the expression 14.9 is equal to

$$\int f\psi'_{\underset{\sim}{a}} = (f * \psi')(\underset{\sim}{a}) .$$

This proves that $(\partial f/\partial t) * \psi = f * (\partial\psi/\partial t)$, as required. ◻

§15. The Current Density Vector

In this section we will show that the distribution of charge throughout spacetime can be described by a (generalized) vector field $\underset{\sim}{j}$. The law of conservation of charge will then be expressed by the equation $\underset{\sim}{\nabla}\cdot\underset{\sim}{j} = 0$, and Maxwell's equations will be formulated throughout spacetime in the form

$$\underset{\sim}{\nabla}F = -4\pi\underset{\sim}{j}\,, \quad \underset{\sim}{\nabla}\hat{F} = \underset{\sim}{0}\,.$$

To begin the exposition, let us try to answer the following question in a physically meaningful way. Consider a fixed bounded region R in spacetime.

What is the average density of charge throughout the 4-dimensional region R?

First consider a region of the special form $R = (t_0, t_1) \times D$, consisting of all points $\underset{\sim}{x} = [t, x, y, z]$ in spacetime with $t_0 < t < t_1$ and with $[x, y, z]$ in the bounded 3-dimensional region D.

We can proceed as follows. For any fixed coordinate time $\bar{t}$ let $e(\bar{t}\times D)$ denote the total quantity of charge located in the 3-dimensional region D at time $\bar{t}$. To compute this total charge we must look for all worldcurves C_α of charged particles which cross the hyperplane $t = \bar{t}$ at a point of $\bar{t} \times D$, and form the sum

(15.1) $$e(\bar{t}\times D) = \sum_{C_\alpha \text{ intersecting } \bar{t}\times D} e_\alpha$$

of the associated charges. Clearly the ratio

$$e(t\times D) / \iiint_D dxdydz$$

of total charge to 3-dimensional volume can be described as the "average density of charge" throughout the 3-dimensional region $t \times D$. Now averaging over the interval $t_0 < t < t_1$ of coordinate time, we obtain the

expression

(15.2) $$\frac{1}{t_1 - t_0}\int_{t_0}^{t_1} e(t\times D)dt \Big/ \iiint_D dxdydz = \int_{t_0}^{t_1} e(t\times D)dt \Big/ \text{volume (R)}\,,$$

where

$$\text{volume (R)} = \iiiint_R dtdxdydz\,.$$

By definition, we will call this expression 15.2 the average charge density

$$\rho = \rho_{av}(R)$$

throughout the 4-dimensional region $R = (t_0, t_1) \times D$.

Remark. In order to make sense of 15.1 and 15.2 we must assume that only finitely many worldcurves intersect the bounded region R.

Let us compute this average charge density more explicitly, so that we can decide whether or not it is Lorentz invariant. Combining the formulas 15.1 and 15.2, we easily obtain the formula

(15.3) $$\rho_{av} = \sum_\alpha \int_{R\cap C_\alpha} e_\alpha dt/\text{volume (R)}\,,$$

to be integrated over that portion of the worldcurve C_α which lies in the region R, and then summed over all worldcurves. From this formula, we see clearly that ρ_{av} by itself is not a Lorentz invariant; for the time coordinate t plays a very special role in its definition. To obtain a Lorentz invariant object, we must place the four coordinates t, x, y, z on an equal footing. Instead of the single integral $\int e_\alpha dt$ we must look at the four integrals

$$\int e_\alpha dt, \quad \int e_\alpha dx, \quad \int e_\alpha dy, \quad \int e_\alpha dz\,,$$

each of which is to be integrated over the curve segment $R \cap C_a$ and then summed over a. Clearly these four integrals form the components of a vector, which we denote briefly by $\int e_a d\underset{\sim}{x}$.

Definition 15.4. For any bounded region R in spacetime, the vector

$$\sum_a \int_{R\cap C_a} e_a d\underset{\sim}{x} = \sum_a \int_{R\cap C_a} e_a \underset{\sim}{u}_a d\tau ,$$

where $\underset{\sim}{u}_a = d\underset{\sim}{x}/d\tau$ is the velocity vector along C_a, will be called the total charge displacement across the region R. The quotient

$$\underset{\sim}{j}_{av} = (\text{charge displacement})/\text{volume}(R)$$

will be called the average current density throughout the region R.

Thus we can set $\underset{\sim}{j}_{av} = [\rho, i, k, \ell]_{av}$ where the initial component ρ_{av} is just the average charge density 15.3. This 4-tuple $\underset{\sim}{j}_{av}$ transforms as a vector, since, if we replace $\underset{\sim}{u}_a$ by $\underset{\sim}{u}_a\Lambda$, each integral $\int e_a \underset{\sim}{u}_a d\tau$ will be replaced by $\int e_a \underset{\sim}{u}_a d\tau\Lambda$. Note that $\underset{\sim}{j}_{av}$ has the dimensions of charge/time^3 = $\sqrt{\text{mass/time}^5}$. Correspondingly the "charge displacement vector" has the dimensions of charge × time = $\sqrt{\text{mass} \times \text{time}^3}$.

The integral $\int e_a d\underset{\sim}{x}$ is of course trivial to evaluate. For example if the curve C_a enters the region R at the point $\underset{\sim}{x}'_a$ and leaves R at the point $\underset{\sim}{x}''_a$, never to return, then evidently

$$\int_{R\cap C_a} e_a d\underset{\sim}{x} = e_a(\underset{\sim}{x}''_a - \underset{\sim}{x}'_a) .$$

(Compare Figure 15.5.) If all of the C_a behave in this way, then we can write simply

$$\text{charge displacement} = \sum_a e_a(\underset{\sim}{x}''_a - \underset{\sim}{x}'_a) .$$

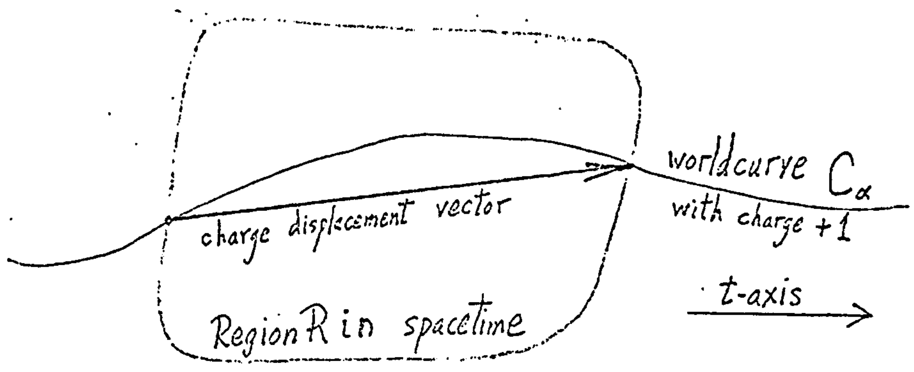

Figure 15.5. Charge displacement across the region R

Here is a concrete example to illustrate the meaning of the statement that the average current density transforms as a vector.

Example. Consider an electrical conductor, part of an electric circuit, composed of equally many protons and electrons (together with neutrons which we ignore). Suppose, for the sake of the argument, that the protons are stationary and the electrons are moving at the (ridiculously large) speed of $\sqrt{2}/2$ in the negative x direction. Then the average current density vector will be the sum of a vector $[\rho, 0, 0, 0]$ describing the protons and a vector $[-\rho, \rho/\sqrt{2}, 0, 0]$ describing the electrons. Combining these two terms we obtain

$$\underset{\sim}{j}_{av} = [0, \rho/\sqrt{2}, 0, 0] ,$$

with average charge density zero. Now look at this same conductor from the point of view of an observer who is traveling with the electrons. A short computation shows that he will describe the protons by the vector $[\rho\sqrt{2}, \rho, 0, 0]$ and the electrons by the vector $[-\rho/\sqrt{2}, 0, 0, 0]$, with sum equal to

$$\underset{\sim}{j}_{av}\Lambda = [\rho/\sqrt{2}, \rho, 0, 0] .$$

To this moving observer, the moving protons will seem to be densely packed together and the stationary electrons will seem to be spread out, so that there is a net positive charge.

We would like to define a current density vector field $j(x)$ with the property that the integral

$$\int_R j(x) d^4 x$$

over any bounded region R is equal to the charge displacement across R. This is not literally possible (as long as we assume that charge is concentrated along 1-dimensional worldcurves). But it is easy to define a generalized vector field j with this property. (See §14.)

Definition. Let j be the generalized vector field whose value

$$\int j\psi$$

on any test function ψ is equal to

$$(15.6) \qquad \sum_{a} \int_{C_a} e_a \psi(x) dx = \sum_{a} \int_{C_a} e_a \psi(x) u_a d\tau ,$$

to be summed over all worldcurves of charged particles. Again we must assume that only finitely many worldcurves intersect any bounded region of spacetime.

Clearly the vector $\int j\psi$ is continuous and linear as a function of ψ. This generalized vector field j is called the current density vector field, and its initial component ρ is called the charge density function.

(This generalized vector field j is related to the charge displacement vector in the following sense. If we were willing to allow, in place of the infinitely differentiable test function, a discontinuous function ψ which is identically 1 on the region R and zero elsewhere, then $\int j\psi$, as defined in (15.6), would be equal to the charge displacement across R.)

Using the current vector field j, the law of conservation of charge can be expressed very neatly as follows.

Theorem 15.7. The divergence $\nabla \cdot j$ of the generalized vector field j is identically zero.

For by definition the value of the divergence

$$\nabla \cdot j = \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} + \frac{\partial k}{\partial y} + \frac{\partial \ell}{\partial z}$$

on a test function ψ is equal to

$$\int (\nabla \cdot j)\psi = -\int j \cdot (\nabla \psi) = -\int \left(\rho \frac{\partial \psi}{\partial t} + j \frac{\partial \psi}{\partial x} + k \frac{\partial \psi}{\partial y} + \ell \frac{\partial \psi}{\partial z} \right) .$$

Substituting in the definition (15.6) it follows that

$$\int (\nabla \cdot j)\psi = -\sum_{a} \int_{C_a} e_a \left(\frac{\partial \psi}{\partial t} dt + \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial z} dz \right) .$$

Evidently the expression is equal to

$$-\sum_{a} e_a \int_{C_a} d\psi .$$

If there are no collisions, so that the curve C_a extends from coordinate time $-\infty$ to coordinate time $+\infty$, then $\int_{C_a} d\psi = 0$. In general, let x'_a be the "beginning point" of C_a. This can either be a collision point, or the formal symbol $-\infty$ if C_a extends infinitely far to the left. Similarly let x''_a be the "end point" of C_a, which can be either a collision point or the symbol $+\infty$. Then

$$\int_{C_a} d\psi = \psi(x''_a) - \psi(x'_a) .$$

Multiplying by $-e_\alpha$ and summing over α, the total contribution from each collision point is zero by the Conservation Law 10.2. Therefore $\nabla \cdot j = 0$. □

Remark. There are other physical objects whose density can be measured by a similar vector field. For example in the study of elementary particles there is a law of "conservation of baryon number" which is completely analogous to the law of conservation of charge. Hence it is possible to describe the distribution of baryons by a "baryon current density vector" with divergence zero. Similarly, in studying thermodynamics, one can introduce a "molecule density vector" to describe the density of molecules throughout spacetime. But in this case the divergence may well be non-zero, since the number of molecules is usually not conserved in a chemical reaction.

It is often more convenient in practice to replace the generalized vector field j by a smooth vector field. There are two possible points of view.

(a) If one believes that charge really is distributed along strictly 1-dimensional worldcurves, then the true current density j is a generalized vector field as described above. But one can approximate j arbitrarily closely by a smooth (and in fact infinitely differentiable) vector field $j * \psi$, by §14.5. Furthermore the divergence

$$\nabla \cdot (j * \psi) = (\nabla \cdot j) * \psi$$

will still be zero. (Compare §14.6.) If the smoothing operator $*\psi$ is sufficiently close to the identity, then it is hard to see how any physical experiment could distinguish between j and $j * \psi$. Of course in practice it may very well be convenient to choose $*\psi$ not too close to the identity, so as to blur out any unnecessary fine detail.

(b) Alternatively one may believe that charge really is distributed smoothly, rather than being concentrated on 1-dimensional curves. In this case we must simply postulate that there exists a smooth current density vector field j satisfying the conservation law $\nabla \cdot j = 0$. From this

point of view it is the generalized vector field which is the simplification, introduced for mathematical convenience.

[In the case of a smooth current density vector field, it is customary to interpret the conservation law $\nabla \cdot j = 0$ as follows. Consider the integral

(15.8) $$\int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} \int_{t_0}^{t_1} \left(\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} + \frac{\partial k}{\partial y} + \frac{\partial \ell}{\partial z} \right) dt\,dx\,dy\,dz$$

of $\nabla \cdot j$ over a rectangular box B in spacetime. Each of the four summands can easily be integrated directly, so that 15.8 can be expressed as a sum of integrals over the eight faces of B. Inspection shows that the resulting integral over the boundary simply measures the charge budget:

(total charge leaving B) - (total charge entering B) .

This difference is zero by 10.1. Since this statement must be true for all boxes in spacetime, it follows again that $\nabla \cdot j = 0$.]

Next let us develop a form of the Maxwell equations appropriate to a region of spacetime which contains charged particles. As in §12, we begin by considering the Coulomb tensor field F generated by a particle with charge e and with worldline $x = y = z = 0$. But now we want to consider F as a generalized tensor field, so that we will be free to differentiate without worrying about singularities. Hence we replace formula 12.3 by the following. For any test function ψ define the value of the generalized tensor field F on ψ to be

$$\int \psi F = \int_{r \neq 0} \psi(x) \frac{e}{r^3} \begin{bmatrix} 0 & x & y & z \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ z & 0 & 0 & 0 \end{bmatrix} d^4 x$$

where $r^2 = x^2 + y^2 + z^2$.

Lemma 15.9. <u>The integral on the right is convergent, so that F is a well defined generalized tensor field. Furthermore the divergence ∇F is</u>

equal to $-4\pi j$ where j is the (generalized) current density field associated with a particle with charge e on the t-axis; and the divergence $\nabla\hat{F}$ of the dual tensor is zero.

The proof will be given at the end of this section.

Now consider a system composed of finitely many charged particles, each with straight worldline. We claim that the equations

$$\nabla F = -4\pi j \,, \quad \nabla\hat{F} = 0$$

are valid for this system also. Since both sides are additive, it is sufficient to consider the case of a single charged particle. But the equations in this case follow from 15.9 by applying an appropriate Poincaré-Lorentz transformation.

This should help to motivate the following.

Revised Maxwell Axiom 15.10. For any electromagnetic field F the divergence ∇F is equal to $-4\pi j$ where j is the current density vector field; and the divergence $\nabla\hat{F}$ of the dual tensor is 0.

Writing out the four components of the vector equation $\nabla F = -4\pi j$, we obtain

$$\begin{cases} \quad - \partial E^1/\partial x - \partial E^2/\partial y - \partial E^3/\partial z = -4\pi\rho \\ \partial E^1/\partial t \quad - \partial B^3/\partial y + \partial B^2/\partial z = -4\pi j \\ \partial E^2/\partial t + \partial B^3/\partial x \quad - \partial B^1/\partial z = -4\pi k \\ \partial E^3/\partial t - \partial B^2/\partial x + \partial B^1/\partial y \quad = -4\pi \ell \end{cases}$$

or in 3-dimensional vector language

$$(15.11) \quad \begin{cases} \mathrm{div}(\vec{E}) = 4\pi\rho, \text{ and} \\ \mathrm{curl}(\vec{B}) = \partial\vec{E}/\partial t + 4\pi\vec{j} \,. \end{cases}$$

These equations can be interpreted in two different ways. If we use the mathematical model in which charge is concentrated on 1-dimensional worldcurves, then F must be a generalized tensor field and j must be

a generalized vector field. On the other hand if we assume that charge is distributed smoothly then we may assume that F is a smooth tensor field and j a smooth vector field. One can of course pass from the first interpretation to the second by applying a smoothing operator. For if $\nabla F = -4\pi j$ then

$$\nabla(F * \psi) = -4\pi(j * \psi)$$

by §14.6.

Remark 1. There is a very direct connection between the Maxwell equation $\nabla F = -4\pi j$ and the conservation law $\nabla\cdot j = 0$. In fact for a completely arbitrary skew tensor field S it is not difficult to check that $\nabla\cdot(\nabla S)$, the divergence of the divergence of S, is identically zero.* Thus 15.7 follows from 15.10. This may be taken as additional evidence suggesting the truth of 15.10.

Remark 2. The current density vector field in 15.10 must take into account the distribution of all charged particles. Since ordinary matter is made up of large numbers of charged particles (about 10^{24} protons and electrons to the gram) with motions which are not easy to understand, this is a real difficulty. As a typical example, in a bar of iron any magnetic field will tend to line up the orbits of electrons thus creating a circulating current which in turn tends to create a stronger magnetic field. To really understand the innocuous looking equations 15.10 one would need a complete catalogue of all of the forces which act on all of the constituent charged particles.

Remark 3. A complete theory would also tell how to incorporate the

*This observation points up the difficulty in giving a satisfactory theory of gravitation within the framework of Special Relativity. Although the gravitational field generated by a stationary point mass seems to resemble the Coulomb field generated by a stationary point charge, we must not try to set up a strictly analogous theory of gravitation. For such a theory would lead to a mathematical "proof" of the law of conservation of mass; which is known to be false.

intrinsic magnetic moment of an elementary particle into the vector field $\underset{\sim}{j}$. We will not attempt this.

Proof of Lemma 15.9. To show that the integral

$$(15.12) \qquad \int_{r\neq 0} \psi(\underset{\sim}{x})\frac{e}{r^3}\begin{bmatrix} 0 & x & y & z \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ z & 0 & 0 & 0 \end{bmatrix} d^4\underset{\sim}{x}$$

is well behaved, we introduce spherical coordinates

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta$$

for each fixed coordinate time t, where

$$0 \le r, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi < 2\pi .$$

Since

$$d^4\underset{\sim}{x} = r^2 \sin\theta \, dr d\theta d\varphi dt ,$$

a typical entry in the matrix 15.12 (namely the (0, 1)-st entry) is given by

$$\int_{r\neq 0} \psi(\underset{\sim}{x}) e r^{-3} x d^4\underset{\sim}{x} = \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \psi(\underset{\sim}{x}) e \sin^2\theta \cos\varphi \, dr d\theta d\varphi dt .$$

Clearly this is well defined, linear, and continuous (§14.2) as a function of ψ, so F is a well defined generalized tensor field.

The divergence $\underset{\sim}{\nabla}F$ can be defined by the identity

$$(15.13) \qquad \int \psi \underset{\sim}{\nabla} F = -\int (\underset{\sim}{\nabla}\psi)F = -\int_{r\neq 0} \frac{e}{r^3}(\underset{\sim}{\nabla}\psi(\underset{\sim}{x}))\begin{bmatrix} 0 & x & y & z \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ z & 0 & 0 & 0 \end{bmatrix} d^4\underset{\sim}{x}$$

for any test function ψ. Evidently the initial component of this vector (15.13) is equal to

$$e\int_{r\neq 0} r^{-2}\left(\frac{\partial\psi}{\partial x}\frac{x}{r} + \frac{\partial\psi}{\partial y}\frac{y}{r} + \frac{\partial\psi}{\partial z}\frac{z}{r}\right) d^4\underset{\sim}{x} .$$

Switching to polar coordinates, and noting that the expression in parentheses is just $\partial\psi/\partial r$, this can be written as

$$e\int\int\int\int(\partial\psi/\partial r)\sin\theta \, dr d\theta d\varphi dt .$$

Using the formulas

$$\int_0^{\infty}(\partial\psi/\partial r)dr = -\psi(t, 0, 0, 0), \quad \int_0^{\pi}\sin\theta \, d\theta = 2, \quad \int_0^{2\pi} d\varphi = 2\pi ,$$

this reduces to

$$-4\pi e\int_{-\infty}^{\infty}\psi(t, 0, 0, 0)dt .$$

Thus we have computed the initial component of the vector 15.13. The next component

$$-\int_{r\neq 0} e r^{-3}(\partial\psi/\partial t)x \, d^4\underset{\sim}{x}$$

is zero since

$$\int_{-\infty}^{\infty}(\partial\psi/\partial t)dt = 0 ,$$

and the remaining components are similarly zero. Thus, all together, we have shown that

$$\int\psi\underset{\sim}{\nabla}F = -4\pi\int_{-\infty}^{\infty} e\psi(t, 0, 0, 0)\underset{\sim}{u}\,dt$$

where $\underset{\sim}{u} = [1, 0, 0, 0]$. But the current density vector $\underset{\sim}{j}$ is defined (§15.6) by the formula

$$\int \phi j = \int_{-\infty}^{\infty} c\phi(t, 0, 0, 0) u \, dt \; .$$

Therefore $\nabla F = -4\pi j$, as asserted.

The proof that $\nabla \hat{F} = 0$ is completely analogous. The details (involving integration by parts) will be left to the reader. ◻

§16. The Energy Density Tensor

In §15 we saw that the distribution of charge throughout spacetime can be described by a (generalized) vector field j. This section will make a similar analysis of the distribution of energy-momentum throughout spacetime.

Suppose that an observer tries to answer the following question in a physically meaningful way. What is the average density of energy throughout some bounded region R of spacetime? Since energy, by definition, is just the initial component of the energy-momentum vector, this question only makes sense if we work with some fixed Lorentz coordinate system. To simplify the discussion, let us first assume that the region R has the special form

$$R = (t_0, t_1) \times D$$

where D is a bounded region in 3-space.

To begin, we can choose some fixed coordinate time, say $t = \bar{t}$, and consider all worldcurves C_a which intersect the hyperplane $t = \bar{t}$ at a point of $\bar{t} \times D$. Adding the corresponding energy-momentum vectors, we obtain the total energy-momentum

$$(16.1) \qquad p(\bar{t} \times D) = \sum_{\text{all } C_a \text{ intersecting } \bar{t} \times D} p_a$$

in this 3-dimensional region (at time $\bar{t}$). (Here p_a denotes the energy-momentum associated with the curve C_a at the time of intersection $t = \bar{t}$.)

Now, dividing by the 3-dimensional volume of D, we obtain a vector

$$p(\bar{t} \times D) \Big/ \iiint_D dx\,dy\,dz$$

which can legitimately be described as the average energy-momentum density throughout the 3-dimensional set $\bar{t} \times D$.

Averging over the time interval $t_0 < \bar{t} < t_1$ we obtain a vector

(16.2) $\frac{1}{t_1 - t_0}\int_{t_0}^{t_1} \underset{\sim}{p}(t\times D)dt \Big/ \iiint_D dxdydz = \int_{t_0}^{t_1} \underset{\sim}{p}(t\times D)dt \Big/ \text{volume }(R)$

which can be described as the average energy-momentum density throughout the 4-dimensional region R.

We will see however, that this vector (16.2) does not behave properly with respect to Lorentz transformations. To examine its dependence on the particular Lorentz coordinate system, let us integrate 16.1 to obtain the formula

(16.3) $$\int_{t_0}^{t_1} \underset{\sim}{p}(t\times D)dt = \sum_a \int_{R\cap C_a} \underset{\sim}{p}_a\, dt ,$$

where the expression on the right is to be integrated over that portion of the worldcurve C_a which lies in R, and then summed over all worldcurves C_a. From this last formula we see that the time coordinate t plays a very special role. Instead of integrating with respect to dt, we could equally well integrate with respect to dx or dy or dz. In this way we could obtain 4 different vectors associated with the region R, or altogether a 4×4 matrix.

Definition. For any bounded region R in spacetime, the 4×4 matrix

$$P = \sum_a \int_{R\cap C_a} \underset{\sim}{p}_a \otimes d\underset{\sim}{x}$$

will be called the total energy-momentum displacement across the 4-dimensional region R. We will be particularly interested in the quotient P/volume(R) which will be denoted by $T = T_{\text{average}}(R)$.

Evidently the matrix P has the dimensions of mass × time, while the matrix T = P/volume(R) has the dimensions of mass/time^3.

At this point we have finally obtained objects P and T which are Lorentz invariant. For if we replace the vector $\underset{\sim}{p}_a$ by $\underset{\sim}{p}_a\Lambda$ and the vector $d\underset{\sim}{x} = [dt, dx, dy, dz]$ by $d\underset{\sim}{x}\Lambda$ where Λ is a Lorentz matrix, then the integrand $\underset{\sim}{p}_a \otimes d\underset{\sim}{x} = \underset{\sim}{p}_a^* d\underset{\sim}{x}$ will be replaced by $\Lambda^{-1}\underset{\sim}{p}_a^* d\underset{\sim}{x}\Lambda$. Therefore the energy-momentum displacement P will be replaced by $\Lambda^{-1}P\Lambda$, and similarly T will be replaced by $\Lambda^{-1}T\Lambda$. Thus P and T transform as tensors.

Lemma 16.4. These tensors P and T are symmetric: $P = P^*$ and $T = T^*$.

For if we describe the worldcurve C_a parametrically by specifying the position along C_a as a smooth function $\underset{\sim}{x} = \underset{\sim}{x}_a(s)$ of some parameter s, then the vector $d\underset{\sim}{x}_a/ds$ is necessarily a multiple of the energy-momentum vector $\underset{\sim}{p}_a$. (See §7.1.) Hence the matrix $\underset{\sim}{p}_a \otimes (d\underset{\sim}{x}_a/ds)$ is self adjoint:

$$\underset{\sim}{p}_a \otimes d\underset{\sim}{x}_a/ds = (d\underset{\sim}{x}_a/ds)\otimes \underset{\sim}{p}_a = (\underset{\sim}{p}_a \otimes d\underset{\sim}{x}_a/ds)^* ,$$

by 8.4 and 8.5. Integrating and summing, it follows that $P = P^*$ hence $T = T^*$. ∎

Consider the matrix product $\underset{\sim}{u}T$, where $\underset{\sim}{u}$ is some fixed forward unit vector. In the special case where $\underset{\sim}{u}$ is the basis vector

$$\underset{\sim}{u}_0 = [1, 0, 0, 0]$$

we can evaluate $\underset{\sim}{u}_0 T$ as follows. Using 8.4 we have

$$\underset{\sim}{u}_0(d\underset{\sim}{x} \otimes \underset{\sim}{p}_a) = (\underset{\sim}{u}_0 \cdot d\underset{\sim}{x})\underset{\sim}{p}_a = (dt)\underset{\sim}{p}_a .$$

Integrating over the curve segment $R \cap C_a$ and then summing over a, we see that the matrix product $\underset{\sim}{u}_0 P$ is precisely equal to the expression 16.3. Therefore, dividing this expression by the 4-dimensional volume of R, we see that the vector $\underset{\sim}{u}_0 T$ is equal to the average density of energy-momentum throughout the region R of spacetime as measured in the given Lorentz coordinate system, or in other words as measured by an observer with velocity $\underset{\sim}{u}_0$.

The initial component $p \cdot u_0 = p u_0^*$ of any energy-momentum vector p can of course be described as the associated energy, measured by an observer with velocity equal to u_0. Hence the real number

$$(u_0 T) \cdot u_0 = u_0 T u_0^*$$

can be described as the average energy density throughout the region R, as measured by an observer with velocity u_0.

Since the real number uTu^* is invariant under Lorentz transformations, and since the expression uT transforms as a vector, we have proved the following statement.

Lemma 16.5. The average density of energy-momentum throughout the region R of spacetime, as measured by an observer with velocity u, is equal to uT. Hence the average density of energy as measured by this observer is equal to uTu^*.

For this reason, the symmetric tensor

$$T = T_{\text{average}}(R) = \sum_a \int_{R \cap C_a} p_a \otimes dx/\text{volume}(R)$$

is called the average energy density tensor for the region R of spacetime.

In fact the average energy density uTu^* is strictly positive, except in vacuum:

Lemma 16.6. This average energy density tensor T is positive semi-definite in the sense that

$$vTv^* \geq 0$$

for any vector v. In fact

$$uTu^* > 0$$

for every timelike vector u unless no worldcurve at all intersects the region R.

Proof. We must show that each integral

$$v \int_{R \cap C_a} (p_a \otimes dx) v^*$$

is ≥ 0 (respectively > 0). Using the identity

$$v(p_a \otimes dx)v^* = (v \cdot p_a)(v \cdot dx) \; ,$$

this can be written as

$$\int (v \cdot p_a)(v \cdot dx/ds) ds \; .$$

But p_a is a positive multiple of dx/ds, so

$$(v \cdot p_a)(v \cdot dx/ds) \geq 0 \; ,$$

hence the integral is ≥ 0. In the case of a timelike vector u, the energy $u \cdot p_a$ is non-zero, hence the integrand is strictly positive. ∎

In any reasonable case, the tensor T will actually be positive definite;* that is:

$$vTv^* > 0 \quad \text{for} \quad v \neq 0 \; .$$

For the equality $vTv^* = 0$ can hold only if v is orthogonal to every energy-momentum vector p_a associated with a worldcurve C_a at a point of R. If there exist four different particles with linearly independent energy-momentum vectors, then this cannot happen.

If T is positive definite, then it is not difficult to construct a Lorentz change of coordinates reducing T to diagonal form

* Caution: Since we are working in Minkowski space, a "positive definite" tensor will have some negative eigenvalues, and may even have trace zero.

$$\Lambda^{-1}T\Lambda = \begin{bmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} .$$

(Compare Synge, "Relativity: the Special Theory," 2nd ed., p. 291.) We will call the special Lorentz coordinate system selected in this way an eigen coordinate system.

Using such an eigen coordinate system, clearly the eigenvalue λ_0 corresponding to a timelike eigenvector is strictly positive. But the eigenvalues λ_1, λ_2, λ_3 corresponding to spacelike eigenvectors must be strictly negative, in order to guarantee that $\underset{\sim}{v}T\underset{\sim}{v}^* > 0$ for $\underset{\sim}{v} \neq \underset{\sim}{0}$.

In particular this description will be true for the average energy density tensor associated with a [illegible] gas; for the particles making up the gas will certainly have a wide variety of velocity vectors. In the case of a gas, one expects to find spacial isotropy

$$\lambda_1 \approx \lambda_2 \approx \lambda_3 < 0 .$$

In order to get one specific number, we will take the average.

Definition. The negative of the average, $-(\lambda_1 + \lambda_2 + \lambda_3)/3$, of these spacelike eigenvalues is called the pressure of the gas.

Evidently pressure has the dimensions of mass/time^3 = force/area. The following discussion may help to motivate this definition. Consider a 2-dimensional rectangle

$$x = \text{constant}, \quad y_0 < y < y_1, \quad z_0 < z < z_1 ,$$

throughout some time interval $t_0 < t < t_1$. Let $\underset{\sim}{u}_1 = [0, 1, 0, 0]$ be the spacelike unit vector orthogonal to the resulting 3-dimensional box. Then evidently

$$\underset{\sim}{u}_1 T \underset{\sim}{u}_1^* = -\lambda_1 .$$

For each worldcurve C_a which crosses this box in either direction, consider the absolute value of the normal component

$$|\underset{\sim}{u}_1 \cdot \underset{\sim}{p}_a|$$

of energy-momentum. Assertion: If we sum this normal component of momentum over all worldcurves which intersect the 3-box, then divide by 3-dimensional volume, and average over $x_0 < x < x_1$, we get precisely $\underset{\sim}{u}_1 T \underset{\sim}{u}_1^*$; where T is the average energy density tensor for the resulting 4-dimensional box. Thus $-\lambda_1$ measures the average of momentum per unit area per unit time in the $\underset{\sim}{u}_1$ direction. The proof is similar to the proof of 16.3, and will be left to the reader.

Remark. The opposite of pressure is of course tension. Since a gas cannot support any tension, this concept does not yet make sense in our mathematical model. However in §17 we will introduce a more complete model which takes certain forces into account and hence allows for the possibility of tension. In other words we will introduce an energy density tensor which is not necessarily positive semi-definite.

The lower-right hand 3×3 submatrix of T, which describes these properties of pressure and tension, is sometimes called the (3-dimensional) stress tensor.

A Digression into Thermodynamics

One important quantity associated with any gas is the temperature. We will give a brief description of this concept, based on Synge, "The Relativistic Gas," North-Holland and Interscience, 1957.

Definition. Using an eigen coordinate system, as described above, the temperature θ of a gas is defined to be the quotient

$$\theta = \text{pressure}/(\text{number of molecules per unit 3-volume}) .$$

This real number θ has the dimensions of mass. To convert the temperature expressed in grams into the temperature in degrees Kelvin, it is only necessary to divide by Boltzmann's constant

$$1.53615 \times 10^{-37} \text{ grams per degree Kelvin} .$$

Here is an alternative description.

Lemma 16.7. Using such an eigen coordinate system, the temperature θ can be defined as the average over all molecules of the quantity

$$\frac{1}{3}\,\text{momentum} \times \text{speed} = \frac{1}{3}\,\text{momentum}^2/\text{energy} .$$

Here the momentum of a molecule with total energy-momentum vector $\underline{p} = [\varepsilon;\ p,\ q,\ r]$ in the eigen coordinate system is defined to be the real number

$$\sqrt{p^2 + q^2 + r^2} ,$$

and the speed (or "coordinate speed) is defined to be the number

$$\text{momentum}/\text{energy} = \sqrt{p^2 + q^2 + r^2}/\varepsilon .$$

Proof of 16.7. Inspecting the definition of the tensor T we see that

$$-T_1{}^1 = \sum_{\alpha} \int_{R \cap C_\alpha} p\,dx/\text{volume}(R) .$$

Substituting $dx = \varepsilon^{-1} p\,dt$, and then averaging with the corresponding expressions in the y and z directions, we see that the pressure $-(T_1{}^1 + T_2{}^2 + T_3{}^3)/3$ is given by

$$\text{pressure} = \frac{1}{3} \sum_{\alpha} \int_{R \cap C_\alpha} \varepsilon^{-1}(p^2 + q^2 + r^2)\,dt/\text{volume}(R) .$$

Now dividing by the quantity

$$(16.8) \qquad \sum_{\alpha} \int_{R \cap C_\alpha} dt/\text{volume}(R) ,$$

which measures the average number of molecules per unit 3-volume, we obtain the required expression 16.7.

This computation becomes clearer if we use a region R of the form $(t_0, t_1) \times D$ and then pass to the limit as t_1 tends to t_0. Evidently the pressure tends to the limit

$$\frac{1}{3} \sum_{C_\alpha \text{ intersecting } t_0 \times D} \varepsilon^{-1}(p^2 + q^2 + r^2) \Big/ \iiint_D dx\,dy\,dz ,$$

as $t_1 \longrightarrow t_0$, and the expression 16.8 tends to the limit

$$\sum_{C_\alpha \text{ intersecting } t_0 \times D} 1 \Big/ \iiint_D dx\,dy\,dz = (\text{number of molecules})/(3\text{-volume of } D)$$

Thus the ratio θ is equal to the average of the expression $p^2 + q^2 + r^2/3\varepsilon$ over all molecules. ◻

The most fundamental and important property of this concept of temperature is the following.

If a gas is composed of molecules having several different masses, $m_1, m_2, \ldots, m_k$, then the temperature, computed using only those molecules of mass m_i, is independent of i.

More precisely this statement holds (in an appropriate statistical sense) for a "perfect" gas, i.e., a gas composed of "molecules" with 1-dimensional worldlines, which is in a state of statistical equilibrium. For the (precise statement and) proof we refer to [The Relativistic Gas, p. 44]. ◻

As a limiting case, we could even apply this statement to a gas which is partly composed of mass zero particles, i.e., photons. The conclusion in this special case would be the following.

The energy of the average photon in a gas at temperature θ should be equal to 3θ.

Although this statement is very roughly true, it does need some modification. The mathematical model of a "perfect gas" assumes that all of the constituent particles behave like ping-pong balls, able to rebound from each other, but not to be emitted or absorbed. In the case of mass zero particles, this mathematical model leads to an energy distribution in which the probability that a given particle will have energy between $a\theta$ and $b\theta$ is equal to $\int_a^b \frac{1}{2} u^2 e^{-u} du$. (Compare Synge, p. 50.) However photons do not quite behave in this way. Using the correct energy distribution, constructed by Planck in 1900, the probability that a given photon will have energy between $a\theta$ and $b\theta$ is rather equal to $\int_a^b u^2(e^u - 1)^{-1} du \Big/ \int_0^\infty u^2(e^u - 1)^{-1} du$.

In order to be visible to the human eye, a photon must have energy somewhere between 2.91×10^{-33} grams, for a red photon, and 5.57×10^{-33} grams, for a violet photon. (Compare p. 13 as well as §13.11.) Consider, for example, the warm room temperature of 300°K. At this temperature, using either of the above energy distributions, a short computation shows that only one photon in 10^{24} attains the minimum energy of 2.91×10^{-33} grams associated with a photon of visible light. Hence objects at room temperature do not glow in the dark. However, if an object is heated to 1700°K, then about one photon in a thousand will attain this minimum energy, so that we perceive a dim red glow, which increases rapidly with increasing temperature. (This increase is the more rapid since the total energy emitted per unit time, according to Stefan and Boltzmann, is proportional to θ^4.)

The above estimate 3θ for average photon energy is misleading for another reason, since it lumps all photons together on an equal basis. It would be more reasonable to consider a weighted average in which each photon is weighted according to its energy. In this case, using Planck's

energy distribution and carrying out an appropriate integration, we obtain the following corrected formula.

Statement 16.9. The weighted average of the energies of the photons emitted by a "black body" at temperature θ is equal to 3.83 times θ.

As an example, for the visible surface of the Sun, with temperature $\theta = 6000°K = 9.2 \times 10^{-34}$ grams, the weighted average energy is equal to

$$3.83\ \theta = 3.5 \times 10^{-33} \text{ grams per photon.}$$

This lies right in the middle of the visible spectrum.

For any reasonable temperature, the concept of temperature can be defined in terms of average kinetic energy.

Assertion 16.10. If the ratio θ/m of temperature to molecular mass is close to zero, then the average kinetic energy per molecule is approximately equal to $3\theta/2$. In particular, this average kinetic energy is independent of the molecular mass.

Intuitive Proof. Let $\tanh\varphi$ be the coordinate speed of a typical molecule, so that

$$\text{energy} = m\cosh\varphi, \qquad \text{momentum} = m\sinh\varphi .$$

Then by 16.7 the number $3\theta/2$ is equal to the average of

$$\tfrac{1}{2}m\sinh\varphi\tanh\varphi = \tfrac{1}{2}m\left(\varphi^2 - \tfrac{1}{6}\varphi^4 + \ldots\right) .$$

Comparing this with the kinetic energy

$$m(\cosh\varphi - 1) = \tfrac{1}{2}m\left(\varphi^2 + \tfrac{1}{12}\varphi^4 + \ldots\right) ,$$

we see that the two expressions are approximately equal as long as φ^2 is close to zero; or in other words as long as θ/m is close to zero. ☒

Remark. This intuitive argument can be made rigorous by including some precise estimate as to the probability that a molecule will have $\tfrac{1}{2}m\sinh\varphi\cosh\varphi$ very much larger than the average value $3\theta/2$. (Synge, p. 48.)

Since the lightest possible particle of positive mass is the electron, with $m = 9\times 10^{-28}$ grams, it follows that the approximation 16.10 is accurate whenever

$$\theta << 9\times 10^{-28}\text{ grams} = 6\times 10^{9}\ {}^{\circ}K .$$

In particular, it is quite adequate for most stellar interiors, which typically have temperatures of the order of 10^7 °K.

Synge (pp. 63, 88) gives a precise formula for the average kinetic energy of particles of mass m at temperature θ in terms of Bessel functions. Expanded as an asymptotic series, this formula becomes:

$$\text{average kinetic energy} \sim \frac{3}{2}\theta\left(1 + \frac{5}{4}\frac{\theta}{m} - \frac{5}{4}\left(\frac{\theta}{m}\right)^2 + \ldots\right) .$$

Thus, at ultra high temperatures, particles of small mass get a little extra kinetic energy. In other words they behave more like photons.

Reduced-Mass Density

An important invariant of any symmetric tensor is the trace (§8.2). The trace of the energy-momentum displacement tensor P can be explicitly computed as follows. Note that the trace of the integrand

$$\underset{\sim}{p}_a \otimes d\underset{\sim}{x}/ds$$

is equal to

$$\underset{\sim}{p}_a\cdot d\underset{\sim}{x}/ds \geq 0 .$$

In the case of a lightlike particle, with mass zero, this integrand is clearly zero. But for a timelike particle, with positive mass, we can set

$$\underset{\sim}{p}_a = m_a\underset{\sim}{u}_a,\quad \underset{\sim}{u}_a = d\underset{\sim}{x}/d\tau ,$$

hence

$$\underset{\sim}{p}_a\cdot d\underset{\sim}{x} = m_a\underset{\sim}{u}_a\cdot\underset{\sim}{u}_a\, d\tau = m_a\, d\tau .$$

Therefore the integral

$$\int_{R\cap C_a} \underset{\sim}{p}_a\cdot d\underset{\sim}{x} = \int_{R\cap C_a} m_a\, d\tau$$

is equal to the product of the mass m_α of the α-th particle and the length of proper time

$$\int_{R\cap C_\alpha} d\tau$$

that this particle remains in the region R.

Summing over α we obtain a real number $\text{trace}(P) \geq 0$ which can be described as the sum of mass $\times$ duration for all particles in the region R. Evidently

$$\text{trace}(P) > 0$$

unless there are no particles at all of positive mass in the 4-dimensional region R. Hence the quotient

$$\text{trace}(T) = \text{trace}(P)/\text{volume}(R)$$

is also positive unless there are no particles of positive mass whose world-curves intersect R.

If we use an eigen coordinate system, note that the trace of T is equal to the average energy density minus three times the pressure. Hence, in any gas composed of particles of positive mass, we have

$$\text{energy density} > 3 \times \text{pressure} \;.$$

In the limiting case of a gas composed completely of photons, equality would hold.

In order to interpret the real number trace(T) more explicitly, let us introduce the following rather strange concept. Consider a particle with mass m, coordinate speed $\tanh\varphi$, and energy $\varepsilon = m\cosh\varphi$.

Definition. The reduced-mass of this particle, in the given coordinate system, will mean the number

$$m^2/\varepsilon = m/\cosh\varphi \quad = m\sqrt{1-\tanh^2\varphi} \;.$$

Thus

$$\text{reduced mass} \leq \text{mass} \;,$$

where equality holds only if the particle is at rest (or has mass zero).

Lemma 16.11. The trace of the average energy density tensor T is equal to the average density of reduced-mass throughout the 4-dimensional region R.

In other words, if we take the total reduced-mass in a 3-dimensional region D at time t = constant, then divide by the 3-volume of D and average over $t_0 < t < t_1$, we will obtain trace(T); where T is the average energy density tensor for the region $R = (t_0, t_1) \times D$. The proof is not difficult, and will be left to the reader. ∎

It follows of course that the concept of reduced-mass density is Lorentz invariant.

The Energy Density Tensor Field

So far we have only talked about the average energy density tensor associated with a region in 4-space. Classically, there is an energy density tensor $T(\underset{\sim}{x})$ associated with each individual point $\underset{\sim}{x}$ in 4-space. This does not make sense in our mathematical model, since we have assumed that energy-momentum is concentrated along strictly 1-dimensional worldcurves. But we can still define the energy density tensor as a generalized matrix valued function of four variables.

Definition 16.12. The energy density tensor field $T = T_{\text{matter}}$ associated with the distribution of matter in spacetime is the matrix valued generalized function whose value on a test function ψ is given by

$$\int \psi T = \sum_\alpha \int_{C_\alpha} \psi(\underset{\sim}{x}) \underset{\sim}{p}_\alpha \otimes d\underset{\sim}{x} \quad = \sum_\alpha \int_{C_\alpha} \psi(\underset{\sim}{x})\, d\underset{\sim}{x} \otimes \underset{\sim}{p}_\alpha \;,$$

to be summed over all worldcurves C_a.

Notice that this definition would reduce to the definition of the energy-momentum displacement tensor P if we were willing to allow ψ to be the (discontinuous) function which is identically 1 on the region R and zero elsewhere.

Conservation Law 16.13. If no forces act on our particles, so that they interact with each other only by collision, then the (generalized) energy density tensor field T has divergence $\underset{\sim}{\nabla}T$ identically equal to $\underset{\sim}{0}$.

Proof. The divergence $\underset{\sim}{\nabla}T$ of a generalized tensor field can be defined by its values on test functions by the rule

$$\int \psi(\underset{\sim}{\nabla}T) = -\int(\underset{\sim}{\nabla}\psi)T = -\int\left[\frac{\partial\psi}{\partial t}, -\frac{\partial\psi}{\partial x}, -\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial z}\right]T .$$

Substituting in the definition 16.12 this becomes

$$-\sum_a \int_{C_a}\left(\frac{\partial\psi}{\partial t}dt + \dots + \frac{\partial\psi}{\partial z}dz\right)\underset{\sim}{p}_a = -\sum_a \int_{C_a}\underset{\sim}{p}_a d\psi .$$

But the hypothesis that no forces act means mathematically that the energy-momentum vector $\underset{\sim}{p}_a$ is constant along each worldcurve C_a. Hence this last expression equals

$$-\sum_a \underset{\sim}{p}_a(\psi(\underset{\sim}{x}''_a) - \psi(\underset{\sim}{x}'_a)) ,$$

where $\underset{\sim}{x}'_a$ and $\underset{\sim}{x}''_a$ are the two endpoints (possibly infinite) of the curve C_a. (Compare the proof of 15.7.) Thus $\int\psi(\underset{\sim}{\nabla}T)$ can be expressed as a sum over collision points. Using the conservation law §7.2, it follows that the total contribution from each collision point is zero, hence $\underset{\sim}{\nabla}T = \underset{\sim}{0}$. ∎

Remark 16.14. If we allow forces to act on our particles, then the divergence $\underset{\sim}{\nabla}T$ would definitely not be $\underset{\sim}{0}$. In fact the argument above can easily be adapted to give the precise formula

$$\int\psi\underset{\sim}{\nabla}T = -\sum_a \int_{C_a}\underset{\sim}{p}_a d\psi$$

$$= \sum_a \int_{C_a}\psi d\underset{\sim}{p}_a = \sum_a \int_{C_a}\psi\underset{\sim}{f}_a d\tau$$

for the divergence $\underset{\sim}{\nabla}T$, where $\underset{\sim}{f}_\alpha$ is the total force which acts on the α-th particle.

If we wish to pass to the classical point of view in which T is a continuous (and in fact infinitely differentiable) tensor field, then it is only necessary to apply a smoothing operator, as in §14 and §15, replacing T by $T * \psi$. The divergence

$$\underset{\sim}{\nabla}(T * \psi) = (\underset{\sim}{\nabla}T) * \psi$$

will be zero whenever $\underset{\sim}{\nabla}T = \underset{\sim}{0}$. (Compare §14.6.) If the smoothing operator $*\psi$ is sufficiently close to the identity, it is difficult to see how any experiment could distinguish the smooth tensor field $T * \psi$ from T.

§17. Electromagnetic Energy

In the preceding section we saw that the distribution of the energy-momentum of material particles throughout spacetime can be described by a symmetric (generalized) tensor field $T = T_{matter}$. If these particles interact only by collision, then the divergence ∇T is 0. But if forces act on the particles, then the divergence ∇T is non-zero. In order to restore the conservation law $\nabla T = 0$ it is necessary to add supplementary terms to the tensor T by ascribing suitable energy and momentum to whatever exerts such forces on the particles. In the case of electromagnetic forces, the computation of electromagnetic energy, stress, and momentum was carried out by Thomson, Maxwell, Poynting, and Heaviside, and formulated in an invariant 4-dimensional language by Minkowski. This section will give a preliminary account of this material, which will be further developed in §20.

In fact nearly all of the forces which we encounter in everyday life are ultimately electromagnetic in character. The only non-electromagnetic forces known to physics are the extremely short ranged nuclear and weak forces; and gravitational forces which are properly the subject of general relativity theory.

If we recall that the electromagnetic tensor F has the dimensions of force/charge $= \sqrt{\text{mass/time}^3}$ while the energy density tensor T has the dimensions of mass/time^3, then it is tempting to try to identify the "electromagnetic energy tensor" T_F with the square of the matrix F. This idea becomes particularly attractive if we notice that the square of a skew tensor

$$F^* = -F$$

is automatically symmetric

$$(FF)^* = FF \ .$$

Taking

$$F = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix}$$

as usual, clearly the (0, 0)-th entry

$$\underset{\sim}{u}_0 F F \underset{\sim}{u}_0^*$$

of the square FF is equal to

$$\sum (E^i)^2 = \|\vec{E}\|^2 \ ,$$

which is ≥ 0 as an energy density should be. However the invariant

$$\text{trace}(FF) = 2\|\vec{E}\|^2 - 2\|\vec{B}\|^2$$

may well be negative. This is awkward since we do not expect any "negative reduced-mass" associated with the electromagnetic field. Furthermore we would expect not only the E^i but also the B^i to contribute to the energy density. In fact, according to Thomson and Maxwell:

The energy density associated with an electromagnetic field is equal to

(17.1) $$(\|\vec{E}\|^2 + \|\vec{B}\|^2)/8\pi \ .$$

We will take 17.1 as an axiom, which will be justified by its success in restoring the conservation law. (See §17.8.) The usual motivation for this axiom involves a computation of the energy needed to build up an electric field in a condenser, or a magnetic field in a Rowland ring. (See for example Sears, "Principles of Physics," Addison-Wesley, 1947.)

The problem of finding a symmetric tensor T_F whose (0, 0)-th entry $\underset{\sim}{u}_0 T_F \underset{\sim}{u}_0^*$ is equal to (17.1) is easily solved. We can take

$$T_F = \frac{1}{8\pi}\left(F^2 + \hat{F}^2\right) \quad (17.2)$$

or alternatively

$$T_F = \frac{1}{4\pi}\left(F^2 - \tfrac{1}{4} I \,\mathrm{trace}(F^2)\right) \quad (17.3)$$

where I is the identity matrix and $F^2 = FF$. The solution is unique. For if we have two symmetric tensors T and T' which have the same (0, 0)-th component in every Lorentz coordinate system, then it is not difficult to show that $T = T'$. In particular, the expression (17.2) is equal to (17.3); as one can check by direct computation.

Using either expression for T_F, we find that

$$\mathrm{trace}(T_F) = 0 .$$

Thus there is no "reduced-mass" associated with the field T_F.

The full matrix T_F can be described as follows. The upper left hand entry is of course equal to $(\|\vec{E}\|^2 + \|\vec{B}\|^2)/8\pi$. The remaining three entries on the top row form the Poynting vector $\vec{E} \times \vec{B}$ divided by 4π. According to §16.5, we must interpret this vector as a measure of momentum density. (Classically it is described as the flux of energy transfer.) Finally, for $i, j \geq 1$ the (i, j)-th component of T_F equals

$$\begin{cases} E^i E^j + B^i B^j & \text{if } i \neq j \\ E^i E^j + B^i B^j - \tfrac{1}{2}\|\vec{E}\|^2 - \tfrac{1}{2}\|\vec{B}\|^2 & \text{if } i = j , \end{cases}$$

all divided by 4π. The resulting 3×3 matrix describes "electromagnetic stress." It is the sum of an electric stress term and a magnetic stress term.

Remark. For any forward unit vector $\underset{\sim}{u}$ the number $\underset{\sim}{u} T_F \underset{\sim}{u}^*$ can of course be interpreted as the energy density as measured by an observer with velocity $\underset{\sim}{u}$. Hence $\underset{\sim}{u} T_F \underset{\sim}{u}^* > 0$ except at points where $F = 0$. However the tensor T_F is not positive definite. As an example, suppose that there

is an electric field of unit strength pointing in the x-direction, but no magnetic field. Then computation shows that

$$T_F = \frac{1}{8\pi}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} .$$

Thus there is "pressure" in the y and z directions, but "tension" in the x direction. In terms of this tension, the attractive force between two particles of opposite charge lying along the x-axis can be explained as follows. In the region between the two particles, the two Coulomb fields reinforce each other. Hence there is tension in the x-direction which draws the two particles together. On the other hand, for two particles of identical charge, the Coulomb fields tend to reinforce each other only outside of the region between the two particles. So in this case the electromagnetic tension pulls the particles away from each other.

One example is of particular interest.

Lemma 17.4. Suppose that F is a directed electromagnetic wave with amplitude vector $\underset{\sim}{a}$. Then $T_F = \underset{\sim}{a} \otimes \underset{\sim}{a}/4\pi$.

(See §13.2 and §13.9 for the definitions.) In other words T_F is equal to the energy density tensor which would be associated with a suitably dense cloud of photons, all with worldlines parallel to the direction $\underset{\sim}{\ell}$ of the electromagnetic wave.

Proof. The square FF is equal to $\underset{\sim}{a} \otimes \underset{\sim}{a}$ by the definition of $\underset{\sim}{a}$, and therefore

$$\mathrm{trace}(FF) = \underset{\sim}{a} \cdot \underset{\sim}{a} = 0$$

since $\underset{\sim}{a}$ points along the light cone. Hence $T_F = FF/4\pi = \underset{\sim}{a} \otimes \underset{\sim}{a}/4\pi$ by 17.3. ∎

Let us compute the divergence ∇T_F of the energy density tensor associated with an electromagnetic field. We will assume that F is smooth, and satisfies the Maxwell equations $\nabla F = -4\pi j$ and $\nabla \hat{F} = 0$.

Lemma 17.5. The divergence ∇T_F is equal to $-jF$.

In particular, if F satisfies the vacuum Maxwell equations so that $j = 0$, then $\nabla T_F = 0$.

Proof. Since the alleged vector equation $\nabla T_F = -jF$ is clearly Lorentz invariant, it suffices to check that the initial components of the two sides are equal. For if the vector $\nabla T_F + jF$ has initial component zero with respect to every Lorentz coordinate system, then it is not difficult to check that the entire vector is zero.

Since our notation is not well adapted to this computation, we will cheat and use 3-dimensional vector language. Using the definition 17.3, a brief computation shows that the initial component of the vector ∇T_F is equal to $1/4\pi$ times the expression

$$(17.6)\quad \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{E}+\vec{B}\cdot\vec{E}) + \frac{\partial}{\partial x}(E^2B^3 - E^3B^2) + \frac{\partial}{\partial y}(E^3B^1 - E^1B^3) + \frac{\partial}{\partial z}(E^1B^2 - E^2B^1)$$

or briefly $1/4\pi$ times

$$\frac{\partial \vec{E}}{\partial t}\cdot\vec{E} + \frac{\partial \vec{B}}{\partial t}\cdot\vec{B} + \mathrm{div}(\vec{E}\times\vec{B}) .$$

Now using the identity

$$\mathrm{div}(\vec{E}\times\vec{B}) = \vec{\mathrm{curl}}(\vec{E})\cdot\vec{B} - \vec{E}\cdot\vec{\mathrm{curl}}(\vec{B}) ,$$

which can easily be verified by explicit calculation, together with the Maxwell equations in the 3-dimensional form (§15.11)

$$\vec{\mathrm{curl}}(\vec{B}) = \partial\vec{E}/\partial t + 4\pi\vec{j}$$

and

$$\vec{\mathrm{curl}}(\vec{E}) = -\partial\vec{B}/\partial t ,$$

we see that the expression 17.6 is equal to $-4\pi\vec{j}\cdot\vec{E}$. Therefore the initial component of the 4-vector ∇T_F equals $-\vec{j}\cdot\vec{E}$. But this is clearly equal to the initial component of the 4-vector $-jF$ where $j = [\rho, \vec{j}]$. ☐

Now consider the energy density tensor of §16.12. We will use the notation T_{matter} for this (generalized) tensor field to indicate that it describes the energy-momentum distribution of material particles only.

We want to compare this tensor T_{matter} with the electromagnetic energy density tensor T_F. Unfortunately T_F has only been defined if F is a smooth tensor field. (The more general case will be considered in §20.) So we must assume for the moment that matter is also distributed smoothly through spacetime. In other words we must use a mathematical model in which T_{matter} is assumed to be a smooth tensor field. But this raises a further problem since our definition of "force" acting on a particle only makes sense if the particle is concentrated along a 1-dimensional worldcurve. For the moment the best we can do is to give the following definition, supported by an intuitive argument.

Definition. We will say that a smooth energy density tensor field T_{matter}, a smooth electromagnetic field F, and a smooth current density vector field j are compatible with the hypothesis that only electromagnetic forces act on the matter if the equation

$$(17.7)\qquad \nabla T_{matter} = jF$$

is satisfied everywhere.

If this equation 17.7 is satisfied, then adding 17.5 we obtain a new and improved formulation

$$(17.8)\qquad \nabla(T_{matter} + T_F) = 0$$

of the law of conservation of energy-momentum.

The two portions T_{matter} and T_F of the total energy density tensor play a quite symmetrical role in this equation. If the field F bends the worldcurves of material particles, then symmetrically these charged particles must bend the field F via Maxwell's equations.

To justify 17.7, let us consider a mixed mathematical model in which F is supposed to be a fixed smooth tensor field, while matter is concentrated along 1-dimensional worldcurves satisfying the differential equations

$$dp_a/d\tau = f_a = e_a u_a F ,$$

which assert that only electromagnetic forces act. (Compare §10.4.) This is not really a satisfactory model, since it is not compatible with the Maxwell equations; but it is easy to work with.

Using this model, we can form the current density j, where

$$\int \psi j = \sum_a \int_{C_a} \psi(x) e_a u_a d\tau \ ;$$

and we can define the matrix product jF to be the generalized vector field

$$\int \psi jF = \sum_a \int_{C_a} \psi(x) e_a u_a F d\tau \ .$$

Setting this equal to

$$\sum_a \int_{C_a} \psi(x) f_a d\tau = \int \psi \nabla T_{matter} \ ,$$

by §16.14, it follows that

$$jF = \nabla T_{matter} \ .$$

Thus the equation 17.7 is true in this mixed model.

Hopefully this fact is sufficient to induce belief in 17.7 in the smooth mathematical model also. A more legitimate formulation and proof of the conservation law $\nabla(T_{matter} + T_F) = 0$ will be presented in §20.

§18. Retarded Potential

This section will solve the following problem. Suppose that we are given some well behaved current density vector field $\underset{\sim}{j}$ and want to find a skew tensor field F satisfying the Maxwell equations

$$\underset{\sim}{\nabla} F = -4\pi \underset{\sim}{j}, \quad \underset{\sim}{\nabla} \hat{F} = \underset{\sim}{0} .$$

In the case of a smooth current (density) vector field, the appropriate electromagnetic field F was constructed by Ludwig Lorenz in 1867. For the case in which charge is concentrated along 1-dimensional worldcurves, the appropriate formulas were given by A. Liénard and E. Wiechert in 1898 and 1900.

Of course the solution to these equations is not unique. If F is one particular solution to the equations $\underset{\sim}{\nabla} F = -4\pi \underset{\sim}{j}$, $\underset{\sim}{\nabla}\hat{F} = \underset{\sim}{0}$, then the most general solution can clearly be expressed as a sum F + F' where F' is any solution to the vacuum Maxwell equations $\underset{\sim}{\nabla} F' = \nabla \hat{F}' = \underset{\sim}{0}$. (Compare §13.)

A particular solution F can be constructed in several steps as follows.

Let N_- denote the backwards light cone based at the origin, consisting of all $\underset{\sim}{x} = [t, x, y, z]$ with $t = -\sqrt{x^2 + y^2 + z^2}$. As first step we notice the following statement.

Lemma 18.1. The 3-dimensional volume element

$$dx\, dy\, dz/|t|$$

on the backward light cone N_- is invariant under Lorentz transformations.

For this volume element is certainly invariant under rotation of the three space coordinates. If we apply a Lorentz transformation of the form

$$t' = at + bx, \quad x' = bt + ax, \quad y' = y, \quad z' = z$$

(where $a = \cosh \varphi$, $b = \sinh \varphi$), then the Jacobian determinant $\partial(x', y', z')/\partial(x, y, z)$ is equal to

$$\partial x'/\partial x = a + b(\partial t/\partial x) .$$

Since $t = -\sqrt{x^2 + y^2 + z^2}$, this equals $a + bx/t = t'/t$. Hence $dx'dy'dz' = (t'/t)dxdydz$, and therefore

$$dx'dy'dz'/t' = dx\, dy\, dz/t \;;$$

as asserted. Together with the discussion in §5.7, this completes the proof. ◻

As next step we will invert the d'Alembertian operator $\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla} = \partial^2/\partial t^2 - \partial^2/\partial x^2 - \partial^2/\partial y^2 - \partial^2/\partial z^2$. Let $\eta(\underset{\sim}{x}) = \eta(t, x, y, z)$ be a twice differentiable real valued function on spacetime. We will need some restriction on η in order to guarantee that integrals over the light cone exist and behave well. The following will be convenient for our purposes. For any bounded set S in spacetime let $S + N_-$ be the collection of all sums $\underset{\sim}{x} + \underset{\sim}{\bar{x}}$ with $\underset{\sim}{x}$ in S and $\underset{\sim}{\bar{x}}$ in the backward light cone.

Hypothesis 18.2. For any bounded set S in spacetime, the restriction of η to the set $S + N_-$ vanishes outside of a compact set.

Equivalently, given a bounded set S there must exist a number c so that $\eta(\underset{\sim}{x}+\underset{\sim}{\bar{x}}) = 0$ for all $\underset{\sim}{x}$ in S and all $\underset{\sim}{\bar{x}} = [\bar{t}, \bar{x}, \bar{y}, \bar{z}]$ in the backward light cone with $|\bar{t}| \geq c$. As an example, this hypothesis would certainly be satisfied if $\eta(t, x, y, z) = 0$ for $x^2 + y^2 + z^2 \geq$ constant.

If this hypothesis is satisfied, then we can define a new function ψ by the equation

$$\psi(\underset{\sim}{x}) = \iiint_{N_-} \eta(\underset{\sim}{x}+\underset{\sim}{\bar{x}})d\bar{x}d\bar{y}d\bar{z}/|\bar{t}|$$

to be integrated over all $\underset{\sim}{\bar{x}}$ in the backwards light cone N_-. The following result has been ascribed to Riemann. (See Whittaker's History, $\underset{\sim}{1}$, p. 395.)

Lemma 18.3. With η as above, the function ψ defined by this integration is twice continuously differentiable and satisfies the equation $\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\psi = 4\pi\eta$.

A proof will be given at the end of this section.

Now consider a twice differentiable current density vector field $\underset{\sim}{j}$. We will assume that $\underset{\sim}{j}$ satisfies the conservation law $\underset{\sim}{\nabla}\cdot\underset{\sim}{j} = 0$, and that each component of $\underset{\sim}{j}$ satisfies the hypothesis 18.2. Following Ludwig Lorenz we introduce the <u>retarded potential</u> vector field

$$\underset{\sim}{v}(\underset{\sim}{x}) = -\iiint_{N_-} \underset{\sim}{j}(\underset{\sim}{x}+\bar{\underset{\sim}{x}})d\bar{x}d\bar{y}d\bar{z}/|\bar{t}| \ ,$$

to be integrated over all $\bar{\underset{\sim}{x}} = [\bar{t}, \bar{x}, \bar{y}, \bar{z}]$ on the backward light cone N_-. Here the word <u>retarded</u> indicates that we are integrating over the <u>backward</u> light cone, so that $\underset{\sim}{v}(\underset{\sim}{x})$ depends only on values of $\underset{\sim}{j}$ at earlier points of spacetime.

Theorem 18.4. <u>With $\underset{\sim}{j}$ and $\underset{\sim}{v}$ as above, the curl $\underset{\sim}{\nabla}\wedge\underset{\sim}{v} = F$ satisfies the Maxwell equations $\underset{\sim}{\nabla}F = -4\pi\underset{\sim}{j}$ and $\underset{\sim}{\nabla}\hat{F} = \underset{\sim}{0}$.</u>

<u>Proof</u>. First notice that the divergence $\underset{\sim}{\nabla}\cdot\underset{\sim}{v}$ is zero. In fact, differentiating under the integral sign we see that $\underset{\sim}{\nabla}\cdot\underset{\sim}{v}(\underset{\sim}{x})$ is equal to

$$\underset{\sim}{\nabla}\cdot\iiint \underset{\sim}{j}(\underset{\sim}{x}+\bar{\underset{\sim}{x}})d\bar{x}d\bar{y}d\bar{z}/\bar{t} = \iiint \underset{\sim}{\nabla}\cdot\underset{\sim}{j}(\underset{\sim}{x}+\bar{\underset{\sim}{x}})d\bar{x}d\bar{y}d\bar{z}/\bar{t} = 0 \ .$$

(Here the components of $\underset{\sim}{\nabla}$ are partial derivatives with respect to the $\underset{\sim}{x}$ variables, keeping $\bar{\underset{\sim}{x}}$ fixed.) Therefore, by Lemma 11.6

$$\underset{\sim}{\nabla}F = (\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla})\underset{\sim}{v} - \underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{v})$$

is equal to the d'Alembertian $(\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla})\underset{\sim}{v}$. Applying Lemma 18.3 to each component of $\underset{\sim}{v}$ individually, it follows that

$$\underset{\sim}{\nabla}F = -4\pi\underset{\sim}{j} \ .$$

Together with 12.18, this completes the proof that the Maxwell equations are satisfied. □

Now let us construct a corresponding potential vector field associated

with a single charged particle. Suppose that the worldcurve $C = C_a$ is three times continuously differentiable. In analogy with 18.2 we will need the following. (Compare §2.1.)

<u>Hypothesis of Visibility</u> 18.5. For any point $\underset{\sim}{x}$ not on the worldcurve C, the backward light cone $\underset{\sim}{x} + N_-$ must intersect C.

Using §4.4 it is easy to check that there is just one intersection point. This hypothesis is a significant restriction on C. As an example, the hyperbola

$$\underset{\sim}{x}(\tau) = [\sinh\tau, \ \cosh\tau, \ 0, \ 0]$$

does not satisfy 18.5.

<u>Definition</u>. The <u>Liénard-Wiechert retarded potential field</u> $\underset{\sim}{v}(\underset{\sim}{x})$ associated with the curve C and charge e is defined by the equation

$$\underset{\sim}{v} = -e\underset{\sim}{u}/r$$

where $\underset{\sim}{u}$ is the tangent vector of C at the unique visible point $\underset{\sim}{x}+\bar{\underset{\sim}{x}}$, and where $r = |\bar{\underset{\sim}{x}}\cdot\underset{\sim}{u}|$. (Compare Figure 18.6.)

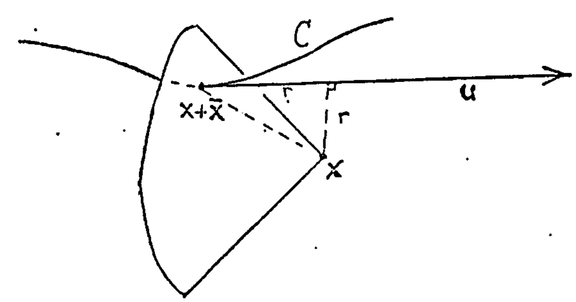

Figure 18.6. The Liénard-Wiechert potential $-e\underset{\sim}{u}/r$

Example. Suppose that the worldcurve C is the t-axis $x = y = z = 0$. Then clearly $r = \sqrt{x^2 + y^2 + z^2}$, so that

$$v = [-e/r,\ 0,\ 0,\ 0]$$

is just the potential associated with the Coulomb field. (See §12.10.)

In general, it is clear that $v(x)$ is a well defined twice (differentiable) vector field, except along the worldcurve C itself. Hence the curl $\nabla \wedge v$ is defined and smooth except on C.

In fact it is clear that the value of $\nabla \wedge v$ at x depends only on the derivatives of C at the unique visible point $x + \bar{x}$. [More precisely, computation shows that

$$\nabla \wedge v(x) = e\bar{x} \wedge a/r^2 + e(\bar{x} \cdot a + 1)\bar{x} \wedge u/r^3$$

where $a = du/d\tau$ is the acceleration vector at $x + \bar{x}$ and $r = -\bar{x} \cdot u$.]

Let F' denote the associated generalized tensor field, defined by the equation

$$\int \psi F = \int \psi(x) \nabla \wedge v(x) d^4x$$

to be integrated over all points x in the complement of C; and let j be the current density vector field associated with the charge e along C.

Theorem 18.7. This generalized tensor field F, coming from the Liénard-Wiechert retarded potential field v, is well defined and satisfies the Maxwell equations $\nabla F = -4\pi j$ and $\nabla \hat{F} = 0$.

Proof. Let us first replace the generalized current density vector field j by a smooth vector field

$$j' = j * \psi .$$

(Compare §14.5.) Here ψ can be any test function. According to 18.4 the Maxwell equations $\nabla F' = -4\pi j'$, $\nabla \hat{F}' = 0$ have a smooth solution $F' = \nabla \wedge v'$ where

$$v'(a) = \iiint_N j(a + \bar{x}) d\bar{x}\, d\bar{y}\, d\bar{z}/\bar{t}$$

is the Lorenz potential field. We will first prove the following statement.

This Lorenz potential v' associated with the smooth current density $j' = j * \psi$, and the Liénard-Wiechert potential v associated with j, are related by the equation $v' = v * \psi$.

In fact if we substitute the definition

$$j'(x) = \int j\psi_x = \int_C e\psi(x - x_\alpha(\tau)) u_\alpha(\tau) d\tau$$

(where $C = C_\alpha$) into the definition of v', we obtain

$$(18.8) \qquad v'(a) = \iiint_N \int_C e\psi(a + \bar{x} - x_\alpha(\tau)) u_\alpha\, d\tau\, d\bar{x}\, d\bar{y}\, d\bar{z}/\bar{t} .$$

Substituting $x = x_\alpha(\tau) - \bar{x}$, note that x varies over all of spacetime as $x_\alpha(\tau)$ varies over C and $\bar{x}$ varies over the backward light cone. Computation shows that the Jacobian determinant

$$|\partial(t, x, y, z)/\partial(\tau, \bar{x}, \bar{y}, \bar{z})|$$

$$= -\det \begin{bmatrix} \partial t_\alpha/\partial\tau & -\bar{x}/\bar{t} & -\bar{y}/\bar{t} & -\bar{z}/\bar{t} \\ \partial x_\alpha/\partial\tau & -1 & 0 & 0 \\ \partial y_\alpha/\partial\tau & 0 & -1 & 0 \\ \partial z_\alpha/\partial\tau & 0 & 0 & -1 \end{bmatrix}$$

is equal to $(\bar{x} \cdot u_\alpha)/\bar{t}$. Hence 18.8 can be written as

$$v'(a) = \iiiint \psi(a - x) e u_\alpha (\bar{x} \cdot u_\alpha)^{-1} d^4x .$$

Inserting the definition

$$\underset{\sim}{v}(\underset{\sim}{x}) = e\underset{\sim}{u}_a/(\bar{\underset{\sim}{x}}\cdot \underset{\sim}{u}_a)$$

of the Liénard-Wiechert potential, this equation becomes $\underset{\sim}{v}' = \psi * \underset{\sim}{v}$, as asserted.

Now applying the curl operator $\underset{\sim}{\nabla}\wedge$ to both sides we obtain

$$F' = \underset{\sim}{\nabla}\wedge \underset{\sim}{v}' = \underset{\sim}{\nabla}\wedge \underset{\sim}{v} * \psi = F * \psi .$$

Therefore the divergence $\underset{\sim}{\nabla}F' = -4\pi\underset{\sim}{j}' = -4\pi\underset{\sim}{j} * \psi$ is equal to $(\underset{\sim}{\nabla}F) * \psi$. Since this is true for every test function ψ, it follows easily that $-4\pi\underset{\sim}{j} = \underset{\sim}{\nabla}F$.

Furthermore the equation $F' = F * \psi$ implies that $\hat{F}' = \hat{F} * \psi$, and hence that $\underset{\sim}{\nabla}\hat{F}' = \underset{\sim}{0}$ is equal to $(\underset{\sim}{\nabla}\hat{F}) * \psi$. Since this is true for every test function ψ, it follows that $\underset{\sim}{\nabla}\hat{F} = \underset{\sim}{0}$. ◻

Remark. Since the Liénard-Wiechert field F at a point $\underset{\sim}{x}$ of spacetime depends on the second derivative of the curve C at the visible point $\underset{\sim}{x} \cdot \bar{\underset{\sim}{x}}$, it is clear that C really must be three times continuously differentiable in order to guarantee that F is smooth throughout the complement of C. If the curve C changes direction at some point $\underset{\sim}{x}$ in spacetime (i.e., if the particle bounces off of something at $\underset{\sim}{x}$), then it is interesting to note that the potential field $\underset{\sim}{v}$ has a discontinuity, an "electromagnetic shock wave," which is propagated along the forward light cone from $\underset{\sim}{x}$.

To complete all of these arguments, we must prove Lemma 18.3. The proof will be based on the following result, which is essentially due to Poisson.

Lemma 18.9. If $V(x, y, z)$ is twice continuously differentiable with compact support, then the integral

$$\iiint_{r\neq 0} (\partial^2V/\partial x^2 + \partial^2V/\partial y^2 + \partial^2V/\partial z^2)dx\,dy\,dz/r ,$$

where $r = \sqrt{x^2+y^2+z^2}$, is well defined and equal to $-4\pi V(0, 0, 0)$.

Proof. Integrating each of the three terms by parts we get

$$\iiint_{r\neq 0} \left(\frac{\partial V}{\partial x}\frac{x}{r} + \frac{\partial V}{\partial y}\frac{y}{r} + \frac{\partial V}{\partial z}\frac{z}{r}\right) r^{-2}dx\,dy\,dz .$$

Now, proceeding exactly as in the proof of 15.9, we can switch to spherical polar coordinates and express this integral as

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}(\partial V/\partial r)\sin\theta\, dr\, d\theta\, d\varphi = -4\pi V(0, 0, 0) . ◻$$

Proof of Lemma 18.3. Given a twice continuously differentiable function $\rho(t, x, y, z)$ satisfying the hypothesis 18.2, we introduce a new function $V(s, x, y, z)$ of four variables by setting

$$V(s, x, y, z) = \rho(s-r, x, y, z) ,$$

where r is the square root of $x^2+y^2+z^2$. Differentiating the equivalent equation

$$\rho(t, x, y, z) = V(t+r, x, y, z)$$

with respect to x, we obtain

$$\rho_x = V_x + V_s x/r ,$$

and therefore

$$\rho_{xx} = V_{xx} + 2V_{sx}x/r + V_s(1/r - x^2/r^3) + V_{ss}x^2/r^2 ,$$

where the subscripts indicate partial derivatives. Adding corresponding expressions for ρ_{yy} and ρ_{zz}, and subtracting $\rho_{tt} = V_{ss}$, we see that

$$-\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\rho = -\rho_{tt} + \rho_{xx} + \rho_{yy} + \rho_{zz}$$

is equal to the sum of the Laplacian

$$V_{xx} + V_{yy} + V_{zz} ,$$

and the expression

$$2(V_{sx}x/r + V_{sy}y/r + V_{sz}z/r + V_s/r) .$$

Therefore the integral

$$\iiint_{N_-} \underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\rho(t, x, y, z)dx\,dy\,dz/t ,$$

integrated over the backward light cone, is equal to the sum of the integrals

(18.10) $$\iiint (V_{xx} + V_{yy} + V_{zz})dx\,dy\,dz/r$$

and

(18.11) $$2\iiint (V_{sx}x/r + V_{sy}y/r + V_{sz}z/r + V_s/r)dx\,dy\,dz/r ,$$

integrated over the hyperplane $s = 0$.

According to 18.9 the integral 18.10 is equal to $-4\pi V(0, 0, 0, 0) = -4\pi\rho(0, 0, 0, 0)$. (The reader should check that the mild singularity of $V(0, x, y, z)$ at the origin does not affect the proof of 18.9.) Switching to spherical coordinates, and setting $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\varphi$, the integral 18.11 takes the form

$$2\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}(r\partial V_s/\partial r + V_s)\sin\theta\,dr\,d\theta\,d\varphi .$$

This is zero since

$$\int_0^\infty \left(r\frac{\partial V_s}{\partial r} + V_s\right)dr = \int_0^\infty d(rV_s) = 0 .$$

Thus (changing the names of the variables) we have shown that

$$\iiint_{N_-} \underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\rho(\bar{t}, \bar{x}, \bar{y}, \bar{z})d\bar{x}\,d\bar{y}\,d\bar{z}/\bar{t} = -4\pi\rho(0, 0, 0, 0) .$$

Now suppose that we are given a twice differentiable function η satisfying the hypothesis 18.2. Choosing some fixed $\underset{\sim}{x}$ and setting

$$\rho(\bar{\underset{\sim}{x}}) = \eta(\underset{\sim}{x} + \bar{\underset{\sim}{x}}) ,$$

we have proved that

$$\iiint_{N_-} \underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\eta(\underset{\sim}{x} + \bar{\underset{\sim}{x}})d\bar{x}\,d\bar{y}\,d\bar{z}/\bar{t}$$

is equal to $-4\pi\rho(\underset{\sim}{0}) = -4\pi\eta(\underset{\sim}{x})$. Therefore, differentiating under the integral sign, it follows that

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\iiint_{N_-} \eta(\underset{\sim}{x} + \bar{\underset{\sim}{x}})d\bar{x}\,d\bar{y}\,d\bar{z}/\bar{t} = -4\pi\eta(\underset{\sim}{x})$$

which completes the proof of 18.3. ∎

§19. The Foresightful Electron

So far we have slurred over two very fundamental problems which arise in making sense out of point charges (i.e., 1-dimensional worldcurves). This section will discuss the first of these two problems:

How can one compute the electromagnetic force $e\underset{\sim}{u}F$ on a charged particle with worldcurve C, in view of the fact that the electromagnetic field F has a bad singularity along the worldcurve C itself?

Our discussion of this question will be based on Dirac's paper, "Classical theory of radiating electrons," Proceedings Royal Society London $\underset{\sim}{167}$ (1938), 148-168.

Let us first suppose that spacetime contains just one charged particle, with three times continuously differentiable worldcurve $C = C_\alpha$ extending from coordinate time $-\infty$ to coordinate time $+\infty$. If we once understand this special case, there will be no particular difficulty in passing to a more reasonable model with many charged particles.

Let $e = e_\alpha$ be the charge, and $\underset{\sim}{j}$ the associated current density vector field. Then the actual electromagnetic field F must satisfy the Maxwell equations $\underset{\sim}{\nabla}F = -4\pi\underset{\sim}{j}$ and $\underset{\sim}{\nabla}\hat{F} = \underset{\sim}{0}$. We will assume that F is smooth except on C.

Let us also assume, to simplify the discussion, that this worldcurve C satisfies the visibility hypothesis 18.2, so that the Liénard-Wiechert retarded potential field is defined everywhere (except on C itself). We will denote this field by

$$\underset{\sim}{v}_{ret} = -e\underset{\sim}{u}_{ret}/r_{ret} .$$

(Compare §18.6.) Let F_{ret} denote the generalized tensor field associated with $\underset{\sim}{\nabla}\wedge\underset{\sim}{v}_{ret}$. Then F_{ret} satisfies the same Maxwell equations $\underset{\sim}{\nabla}F_{ret} = -4\pi\underset{\sim}{j}$ and $\underset{\sim}{\nabla}\hat{F}_{ret} = \underset{\sim}{0}$ by 18.7. Hence the difference

$$F_{in} = F - F_{ret}$$

satisfies the vacuum Maxwell equations

$$\underset{\sim}{\nabla}F_{in} = \underset{\sim}{0}, \quad \underset{\sim}{\nabla}\hat{F}_{in} = \underset{\sim}{0} .$$

Dirac calls this difference the incident radiation field. Since F_{in} is known to be a smooth tensor field except along C, it follows easily that F_{in} extends to a tensor field which is defined and continuous throughout spacetime. (Compare §13.1.) In particular, the field $F_{in}(\underset{\sim}{x})$ is well defined even for points $\underset{\sim}{x}$ on the worldcurve C itself.

It is usual to think of the retarded field F_{ret} as the portion of the total electromagnetic field F which is caused by the particle, and to think of F_{in} as the "given" electromagnetic field which exists independently of the particle. Hence it is reasonable to assume that this field F_{in} is actually smooth throughout spacetime.

It is widely believed, in conformity with much experimental evidence, that the basic laws of electrodynamics must be invariant under time reversal. That is, they must be preserved under a Poincaré-Lorentz transformation which reverses time orientation.* Of course there certainly exist electrodynamic phenomena which are not invariant under time reversal. An example is provided by resistance in an electric circuit. The presumption is that such irreversible phenomena are statistical in nature. As far as the basic underlying laws are concerned, a pocket flashlight could equally well work backwards, sucking in light and using the resulting energy to recharge its batteries. Such behavior is unlikely, but not impossible.

Let us see what happens to the above constructions if we reverse the time orientation. In place of the visibility hypothesis 18.2 we impose a "surveillance hypothesis": Every point $\underset{\sim}{x}$ of spacetime must be visible from some point $\underset{\sim}{x} + \underset{\sim}{\bar{x}}$ on the worldcurve C. Then we can construct the advanced potential

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*The usual convention is that both charge and velocity change sign under time reversal, but that current density and force transform as vectors.

$$\underset{\sim}{v}_{adv}(\underset{\sim}{x}) = -e\underset{\sim}{u}_{adv}/r_{adv} ,$$

where $\underset{\sim}{u}_{adv}$ is the velocity of C at $\underset{\sim}{x} + \bar{\underset{\sim}{x}}$ and $r_{adv} = \bar{\underset{\sim}{x}}\cdot\underset{\sim}{u}_{adv}$. The generalized tensor field F_{adv} associated with $\underset{\sim}{\nabla}\wedge\underset{\sim}{v}_{adv}$ satisfies the same Maxwell equations $\underset{\sim}{\nabla}F_{adv} = -4\pi\underset{\sim}{j}$ and $\underset{\sim}{\nabla}\hat{F}_{adv} = \underset{\sim}{0}$. Therefore the difference

$$F_{out} = F - F_{adv}$$

satisfies the vacuum Maxwell equations, and hence is continuous everywhere. Dirac calls F_{out} the outgoing radiation field.

In computing the electromagnetic force on the particle with worldcurve C, we cannot give a predominant role either to the incoming or to the outgoing radiation field. For that would violate the principle of symmetry under time reversal. The best we can do is to average these two fields, obtaining another electromagnetic field

$$\tfrac{1}{2}(F_{in} + F_{out}) = F - \tfrac{1}{2}(F_{ret} + F_{adv})$$

which is continuous throughout spacetime.

Dirac Axiom 19.1. The electromagnetic force exerted on the particle at a given point on its worldcurve is equal to

$$e\underset{\sim}{u}(F_{in} + F_{out})/2 = e\underset{\sim}{u}(F - \tfrac{1}{2}(F_{ret} + F_{adv})) ,$$

where e is the charge and $\underset{\sim}{u}$ the velocity.

Thus in order to compute electromagnetic force we must know the actual field F near the worldcurve, subtract a correction term $(F_{ret} + F_{adv})/2$ which is determined completely by the local shape of the curve C, and then extend this difference over the singular curve C itself and multiply on the left by the vector $e\underset{\sim}{u}$.

Dirac shows that this force law is the only reasonable one compatible with energy-momentum conservation. I will not try to duplicate his rather

intricate argument, but want to point it out since 19.1 needs all the justification it can get. We will see that this force law has rather strange consequences.

Remark. The visibility hypothesis 18.2, and the symmetrical "surveillance hypothesis," were needed to define F_{ret} and F_{adv} throughout the complement of C. However they are not needed for 19.1. To compute the force at a given point $\underset{\sim}{x}$ on C we need only know the actual field F and the curve C in an arbitrarily small neighborhood of $\underset{\sim}{x}$. The behavior of C in the remote past and in the future is totally irrelevant. For that matter the existence of other particles elsewhere in spacetime would also be irrelevant.

To develop this theory further, we need some precise information about the difference

$$F_{ret} - F_{adv} = F_{out} - F_{in} .$$

This difference is a continuous tensor field, satisfying the vacuum Maxwell equations.* The value of this difference at a point $\underset{\sim}{x}$ of C depends only on the shape of C in an arbitrarily small neighborhood of $\underset{\sim}{x}$. In fact Dirac computes it as follows.

Lemma 19.2. The difference $F_{ret} - F_{adv}$ at a point of C is equal to $\frac{4}{3}e\underset{\sim}{u}\wedge d^2\underset{\sim}{u}/d\tau^2$.

For the proof we refer to Dirac's paper, cited above. (Caution: Dirac's F is our -F). ◻

The appearance of $d^2\underset{\sim}{u}/d\tau^2$, the third derivative of position with respect to proper time, is unfortunate, and leads to much awkwardness. It will be convenient to denote this vector by $\ddot{\underset{\sim}{u}}$, where each dot stands for $d/d\tau$. Note the identities

*By definition, continuous functions are said to satisfy a partial differential equation if the associated generalized functions satisfy the equation. (Compare §14.)

$$\mathbf{u}\cdot\mathbf{u} = 1, \quad \mathbf{u}\cdot\dot{\mathbf{u}} = 0, \quad \text{and} \quad \dot{\mathbf{u}}\cdot\dot{\mathbf{u}} + \mathbf{u}\cdot\ddot{\mathbf{u}} = 0 \ .$$

Combihing 19.1 and 19.2 we obtain the following.

Lemma 19.3. Given the incident radiation field $F_{in} = F - F_{ret}$, the electromagnetic force on a particle is given by

$$\mathbf{f} = e\mathbf{u}F_{in} + \frac{2}{3}e^2\ddot{\mathbf{u}} + \frac{2}{3}e^2(\dot{\mathbf{u}}\cdot\dot{\mathbf{u}})\mathbf{u} \ .$$

Hence, if only electromagnetic forces act on the particle, its equation of motion is

$$(19.4) \qquad m\dot{\mathbf{u}} = e\mathbf{u}F_{in} + \frac{2}{3}e^2\ddot{\mathbf{u}} + \frac{2}{3}e^2(\dot{\mathbf{u}}\cdot\dot{\mathbf{u}})\mathbf{u} \ .$$

Proof. This follows easily from 19.1, 19.2, and 8.4. ∎

Thus in addition to the expected term $e\mathbf{u}F_{in}$ there is a self-force proportional to the normal component of $\ddot{\mathbf{u}}$. (The last term $\frac{2}{3}e^2(\dot{\mathbf{u}}\cdot\dot{\mathbf{u}})\mathbf{u} = -\frac{2}{3}e^2(\mathbf{u}\cdot\ddot{\mathbf{u}})\mathbf{u}$ can be thought of merely as a correction, to guarantee that $\mathbf{f}\cdot\mathbf{u} = 0$.) This is awkward for two reasons.

(1) To specify a solution to 19.4 we must prescribe not only the initial position $\mathbf{x}(0)$ and velocity $\mathbf{u}(0)$, but also the initial acceleration $\dot{\mathbf{u}}(0)$. This is unheard of in mechanics! Furthermore:

(2) For most choices of initial acceleration the behavior of the particle will be ridiculous. Consider for example the special case where the incident field F_{in} is identically zero. If we assume that the last two components of $\mathbf{u}(0)$ are zero, then clearly $\mathbf{u}$ will have the form

$$\mathbf{u}(\tau) = [\cosh\varphi, \ \sinh\varphi, \ 0, \ 0]$$

for every proper time τ. The equation 19.4 reduces to

$$m\dot{\varphi} = 2e^2\ddot{\varphi}/3$$

with general solution

$$\dot{\varphi} = \dot{\varphi}(0)\exp(3m\tau/2e^2) \ .$$

These solutions accelerate wildly, chasing themselves off exponentially towards the speed of light, unless the initial acceleration $\dot{\varphi}(0)$ happens to be precisely zero. In the real world, no such behavior is observed.

Both of these problems can be solved, in some sense, by the following.

Dirac Hypothesis. A charged particle (i.e., an electron) always takes great care to choose its initial acceleration so that no runaway acceleration will take place.

Thus electrons display a combination of precognition (in anticipating future incident radiation) and prudence (in avoiding wild behavior) which cannot help but arouse our admiration.

To make a precise mathematical statement out of this hypothesis, one would have to prove something like the following.

Conjecture. If the field F_{in} is smooth and bounded throughout space-time, then given the initial position $\mathbf{x}(0)$ and the initial velocity $\mathbf{u}(0)$, there is one and only one value for the initial acceleration $\dot{\mathbf{u}}(0)$ so that the expression $\tau^{-1}\log(1 + \|\dot{\mathbf{u}}(\tau)\|)$ will tend to zero as $\tau \longrightarrow \infty$.

For a thorough discussion of the present status of Dirac's hypothesis the reader is referred to Rohrlich, "Classical Charged Particles, Foundations of their Theory," Addison-Wesley, 1965, as well as Plass, "Classical electrodynamic equations of motion with radiative reaction", Reviews of Modern Physics 33 (1961), 57-62.

To illustrate this Dirac hypothesis, consider the case of a constant electromagnetic field, discussed in §10.7. In the case of a constant electric field, with no magnetic field, the exact solution

$$\mathbf{x}(\tau) = k^{-1}[\sinh k\tau, \ \cosh k\tau, \ 0, \ 0] \ ,$$

mentioned on p. 84, happens to satisfy equation 19.4, since the normal component of $\ddot{\mathbf{u}}$ is zero. Thus we have one explicit solution with bounded acceleration. But this is only an accident. In general, it seems extremely difficult to describe the solution curves, or to single out the more reasonable ones.

Here is a numerical example. Consider an electron, with charge to mass ratio

$$e/m \approx -102 \sqrt{\text{seconds/gram}} ,$$

in the Earth's magnetic field, with typical field strength

$$\|\vec{B}\| = B^1 = 0.58 \text{ gauss} \approx 10^5 \sqrt{\text{grams/second}^3} .$$

Then

$$-B^1 e/m \approx 10^7 \text{ second}^{-1}$$

and

$$2e^2/3m \approx 6 \times 10^{-24} \text{ seconds} .$$

Thus the equations of motion become

$$\dot{\underset{\sim}{u}} = 10^7 \underset{\sim}{u} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} + 6 \times 10^{-24} (\ddot{\underset{\sim}{u}} + (\dot{\underset{\sim}{u}} \cdot \dot{\underset{\sim}{u}})\underset{\sim}{u}) .$$

The coefficient of the last expression is extremely small. Unless $\ddot{\underset{\sim}{u}}$ is very large, we can (presumably) ignore the last term and obtain solutions as in §10.7 which are typically helixes. The electron will complete one turn of the helix in about $2\pi \times 10^{-7}$ seconds.

But this procedure of finding an approximate solution to a differential equation by ignoring the highest order term is clearly dangerous and unreliable.

Let us study the motion of a charged particle by posing a somewhat different form of initial value problem. Again we assume for convenience that there is just one charged particle, with worldcurve C, in all of space-time.

Given the actual electromagnetic field F at coordinate time $t = t_0$, suppose that only electromagnetic forces act for $t \geq t_0$. Does it follow that the field F and the worldcurve C are uniquely determined for $t \geq t_0$? (In particular, the position, velocity, and acceleration of C at time t_0 are presumably determined by F at time t_0.) The answer is presumably yes. For technical reasons, it will be much easier to carry out the proof if we assume just a little bit more. In fact we will suppose that the field F and the worldcurve C are specified throughout a thin region of the form $t_0 - \varepsilon \leq t \leq t_0$. We will assume that F is twice continuously differentiable except on C, that C is four times continuously differentiable, and that the Maxwell equations are satisfied throughout this thin slab.

Lemma 19.5. With F and C given as above for $t_0 - \varepsilon \leq t \leq t_0$, there is one and only one extension of F and of C to the region $t \geq t_0$ so that F satisfies the Maxwell equations and so that C satisfies the differential equation

$$m\dot{\underset{\sim}{u}} = e\underset{\sim}{u}(F - \tfrac{1}{2}(F_{ret} + F_{adv})) \tag{19.6}$$

of §19.1 for $t \geq t_0$, and has the given position, velocity, and acceleration at coordinate time t_0.

Proof. We will make use of a procedure which is very popular in practical politics: the method of fictitious history. However, in contrast to the political applications of the method, this fictitious history will be used only to predict the future – not to influence it.

To begin this method let us extend the curve C, which is given only for $t_0 - \varepsilon \leq t \leq t_0$, to a curve C^- which lies in the half-space $t \leq t_0$ and extends from coordinate time $-\infty$ to coordinate time t_0. The choice of C^-

is completely arbitrary except that it must be four times continuously differentiable, satisfy the visibility hypothesis 18.2, and coincide with the given curve for $t_0 - \varepsilon \leq t \leq t_0$.

Using this fictitious history C^-, the retarded field F_{ret} is defined, and is twice continuously differentiable, throughout the half-space $t \leq t_0$, except on C^- itself. Hence the field $F_{in} = F - F_{ret}$ is defined, and satisfies the vacuum Maxwell equations, throughout $t_0 - \varepsilon \leq t \leq t_0$.

Now using §13.1, it follows that F_{in} extends to a field which is defined and smooth throughout spacetime. Using this extended field F_{in}, we can formulate the differential equation 19.4. Using the position, velocity, and acceleration of C^- at coordinate time $t = t_0$ as initial conditions, there is clearly a unique solution curve C^+ for $t \geq t_0$.

Finally, using this solution curve C^+ together with the fictitious history C^-, the field F_{ret} is defined throughout spacetime (except on these curves) and hence $F = F_{in} + F_{ret}$ is defined throughout spacetime. Evidently the field F constructed in this way satisfies the Maxwell equations, where the current density j is determined by C^- together with C^+. Furthermore F and C^+ satisfy the differential equation 19.4, which is equivalent to 19.6.

Conversely, given C^+ and given F for $t \geq 0$ so that 19.6 and the Maxwell equations are satisfied, we can paste on C^- so that F_{ret} and F_{in} are determined everywhere, and conclude that 19.4 must be satisfied. Since the solution to 19.4, with given initial conditions, is unique, it follows that C^+ coincides with the solution constructed above. ∎

Remark. There is something rather startling about this procedure. The differential equation 19.4 depends on F_{in}, which in turn depends on the past history C^-. Nevertheless the argument shows that the solution of 19.4 does not depend on C^-.

§20. Infinite Coulomb Energy and the Conservation Law

There is another fundamental problem in making sense out of a charged particle with strictly 1-dimensional worldcurve C: The electromagnetic field grows so rapidly, near the curve C, that there appears to be infinitely much electromagnetic energy in an arbitrarily small neighborhood of the particle. Consider for example the Coulomb field of §12.3, with

$$\|\vec{E}\| = e/r^2, \quad \|\vec{B}\| = 0 \ .$$

Using the formula 17.1,

$$T_0^{\ 0} = (\|\vec{E}\|^2 + \|\vec{B}\|^2)/8\pi \ ,$$

for electromagnetic energy density, we obtain $T_0^{\ 0} = e^2/8\pi r^4$. Hence the total energy $\iiint_{r \leq \varepsilon} T_0^{\ 0}\, dx\, dy\, dz$ in a ball of radius ε, at coordinate time $t =$ constant, appears to be

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\varepsilon} (e^2/8\pi r^2) \sin\theta \, dr\, d\theta\, d\varphi = +\infty \ .$$

To correct this situation, in Dirac's words:

> "If we want a model of the electron, we must suppose that there is an infinite negative mass* at its centre such that, when subtracted from the infinite positive mass of the surrounding Coulomb field, the difference is well defined and is just equal to m."

The appropriate mathematical language for making such a statement precise is of course provided by the theory of generalized functions.

Definition 20.1. By the electromagnetic energy density tensor associated with the electromagnetic field F, singular along the worldcurve C, will be meant the generalized tensor field T'_F defined by the equation

*For "mass" read "energy," to be compatible with our terminology.

$$\int \psi T'_F = \lim_{b\to 0} \left(\iiiint_{r \geq b} \psi(\mathbf{x}) T_F(\mathbf{x}) d^4\mathbf{x} - \frac{2e^2}{3b} \int_C \psi(\mathbf{x})(\mathbf{u} \otimes \mathbf{u} - \tfrac{1}{4} I) d\tau \right)$$

for any test function ψ, where T_F is defined by 17.2 or 17.3, where $r = r(\mathbf{x})$ denotes the length of a perpendicular line segment from $\mathbf{x}$ to C, and where e is the charge. Thus we integrate the product ψT_F only over the complement of a tube of radius b about the singular worldcurve C and then subtract, as a correction term, b^{-1} times the integral along C of a certain symmetric tensor. This correction term is precisely chosen so that the difference will tend to a limit as $b \to 0$.

Lemma 20.2. <u>This limit exists, and</u> T'_F <u>is a well defined generalized tensor field.</u>

Details of the proof will be omitted. We merely remark that it depends on the explicit estimate

$$F(\mathbf{x}) = \frac{e}{\sqrt{1 - \mathbf{a}\cdot\mathbf{r}}} \left(\frac{1}{r^3} (\mathbf{r} \wedge \mathbf{u}) + \frac{1}{2r} (\mathbf{u} \wedge \mathbf{a}) \right) + O(1) ,$$

where $\mathbf{r}$ is a perpendicular vector from C to $\mathbf{x}$ with length $\|\mathbf{r}\| = r$, where $\mathbf{u}$ and $\mathbf{a}$ are the velocity and acceleration at the foot $\mathbf{x} - \mathbf{r}$ of this perpendicular, and where the symbol $O(1)$ stands for a remainder term which remains bounded as $r \to 0$. Using this, we obtain the estimate

$$4\pi r^2 T_F(\mathbf{x}) = r^{-2} \frac{e^2}{1 - \mathbf{a}\cdot\mathbf{r}} \left((1 + \mathbf{a}\cdot\mathbf{r})(\mathbf{u} \otimes \mathbf{u} - \tfrac{1}{4} I) - r^{-2}(\mathbf{r} \otimes \mathbf{r}) + \tfrac{1}{2}(\mathbf{r} \otimes \mathbf{a} + \mathbf{a} \otimes \mathbf{r}) \right) + O(1) ,$$

which can be used to prove 20.2. (Compare the proof of 20.3 below.) Both estimates follow from page 166 of Dirac's paper, cited in §19. ∎

The correction term in 20.1 incorporates not only the infinite negative energy required by Dirac, but also <u>infinite positive tension</u>. Intuitively we may think of this tension as being required to keep the particle from flying apart under its own self repulsion.

Note that the symmetric tensor T'_F has trace zero. This electromagnetic energy density has nothing to do with the mass of the particle.

Theorem 20.3. <u>The generalized tensor field</u> T'_F <u>defined in this way satisfies the conservation law</u>

$$\nabla(T'_F + T_{matter}) = 0 ,$$

<u>assuming that only electromagnetic forces act, so that equation</u> 19.6 <u>is satisfied.</u>

The proof can be outlined as follows. In order to compute the divergence $\nabla T'_F$ we must compute the value

$$\int \psi(\nabla T'_F) = - \int (\nabla\psi) T'_F = \lim_{b\to 0} \left(- \iiiint_{r \geq b} (\nabla\psi) T_F d^4\mathbf{x} + \frac{2e^2}{3b} \int_C (\nabla\psi)(\mathbf{u} \otimes \mathbf{u} - \tfrac{1}{4} I) d\tau \right)$$

for an arbitrary test function ψ. Using 8.4, this last integral over C is clearly equal to

(20.4)
$$\frac{2e^2}{3b} \int_C \mathbf{u} d\psi - \frac{e^2}{6b} \int_C (\nabla\psi) d\tau .$$

To evaluate the 4-fold integral we integrate by parts, using the fact that $\nabla T_F = 0$. After some work we see that the integral equals

(20.5)
$$- \frac{1}{b} \iiint_{r=b} \psi \mathbf{r} T_F dV ,$$

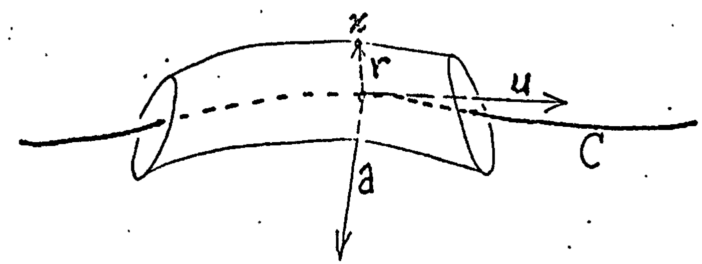


Figure 20.6. The Tubular Neighborhood of Radius b.

to be integrated over the "surface" of the tube of radius b, with respect to the 3-dimensional volume element dV. Here $\underset{\sim}{r}$ again denotes the orthogonal vector from C to $\underset{\sim}{x}$, so that $r = \|\underset{\sim}{r}\|$ is the distance to C and $\underset{\sim}{r}/r = \underset{\sim}{r}/b$ is the outward unit normal vector.

According to Dirac, p. 167, one has the following explicit estimate :

$$(20.7)\quad 4\pi \underset{\sim}{r} T_F = \frac{e^2}{1 - \underset{\sim}{r}\cdot\underset{\sim}{a}}\left(\left(\frac{1}{2r^4} + \frac{\underset{\sim}{a}\cdot\underset{\sim}{a}}{2r^2}\right)\underset{\sim}{r} - \frac{1}{2r^2}\left(1 + \frac{3}{2}\underset{\sim}{r}\cdot\underset{\sim}{a}\right)\underset{\sim}{a}\right) + \underset{\sim}{f}/r + O(1) .$$

Here $\underset{\sim}{a}$ denotes the acceleration vector, while

$$\underset{\sim}{f} = e\underset{\sim}{u}(F - \tfrac{1}{2}(F_{ret} + F_{adv}))$$

denotes the electromagnetic force acting on the particle, and O(1) again stands for a remainder term which remains bounded as $r \to 0$. Dirac notes that the 3-dimensional volume element on the tube boundary can be written as

$$dV = (1 - \underset{\sim}{r}\cdot\underset{\sim}{a})\, dA\, d\tau$$

where dA is the 2-dimensional area element on the 2-sphere obtained by intersecting the tube boundary with a hyperplane orthogonal to C. (Compare Figure 20.6. The factor $(1 - \underset{\sim}{r}\cdot\underset{\sim}{a})$ compensates for the fact that the 2-spheres are squeezed together away from the direction of the acceleration vector $\underset{\sim}{a}$.) Thus, using 20.7, we see that 20.5 is equal to the integral

$$-\frac{1}{b}\iiint_{r=b}(1 - \underset{\sim}{r}\cdot\underset{\sim}{a})\psi\underset{\sim}{r}T_F\, dA\, d\tau$$

$$= \frac{e^2}{8\pi b^3}\iiint_{r=b}\left[-(b^{-2} + \underset{\sim}{a}\cdot\underset{\sim}{a})\psi\underset{\sim}{r} + \left(1 + \frac{3}{2}r\cdot\underset{\sim}{a}\right)\psi\underset{\sim}{a}\right] dA\, d\tau$$

$$- \int_C \psi\underset{\sim}{f}d\tau + bO(1) .$$

Setting

$$\psi(\underset{\sim}{x}(\tau) + \underset{\sim}{r}) = \psi(\underset{\sim}{x}(\tau)) + (\underset{\sim}{\nabla}\psi)\cdot\underset{\sim}{r} + r^2 O(1)$$

for $\underset{\sim}{x}(\tau)$ on C, and noting that odd functions of $\underset{\sim}{r}$ integrate to zero over each 2-sphere, while

$$\iint_{r=b}(\underset{\sim}{r}\cdot\underset{\sim}{v})\underset{\sim}{r}dA = \frac{4\pi b^4}{3}((\underset{\sim}{u}\cdot\underset{\sim}{v})\underset{\sim}{u} - \underset{\sim}{v})$$

for any fixed τ and any fixed vector $\underset{\sim}{v}$, this becomes

$$\int_C\left[-\frac{e^2}{6b}((\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\psi)\underset{\sim}{u} - \underset{\sim}{\nabla}\psi) + \frac{e^2}{2b}\psi\underset{\sim}{a} - \psi\underset{\sim}{f}\right] d\tau + bO(1) .$$

Using the equations $\underset{\sim}{u}\cdot(\underset{\sim}{\nabla}\psi)d\tau = d\psi$ and $\int\psi\underset{\sim}{a}d\tau = -\int\underset{\sim}{u}d\psi$, it becomes

$$\int_C\left[-\frac{2e^2}{3b}\underset{\sim}{u}d\psi + \frac{e^2}{6b}(\underset{\sim}{\nabla}\psi)d\tau - \psi\underset{\sim}{f}d\tau\right] + bO(1) .$$

Now adding 20.4 and passing to the limit as $b \to 0$ we obtain the formula

$$\int\psi\underset{\sim}{\nabla}T'_F = -\int_C \psi\underset{\sim}{f}d\tau .$$

But

$$\int\psi\underset{\sim}{\nabla}T_{matter} = +\int_C \psi\underset{\sim}{f}d\tau$$

by §16.4. Hence

$$\underset{\sim}{\nabla}(T'_F + T_{matter}) = \underset{\sim}{0} ,$$

and we have finally proved the conservation law. ∎

Appendix: The Dual of a Skew Tensor

This Appendix will describe a duality which assigns to each skew tensor in the 4-dimensional Minkowski space a "dual" skew tensor.

It may be helpful to compare this 4-dimensional situation with what would happen in lower dimensions. A skew tensor

$$\begin{bmatrix} 0 & s \\ s & 0 \end{bmatrix}$$

in the Minkowski plane is "dual" to a well defined real number s, which describes the tensor completely. Similarly, a skew tensor in 3-space is dual to a unique vector. In general a skew (2-index) tensor in n-dimensional space is dual to a skew $(n-2)$-index tensor in n-space. Thus it is only in the 4-dimensional case that the dual tensor is exactly the same kind of object as the original tensor.

To construct this dual tensor we proceed as follows. Let $u_0 = [1, 0, 0, 0], \ldots, u_3 = [0, 0, 0, 1]$ be the standard basis for the 4-dimensional Minkowski coordinate space. Then, as noted in §8, the six skew products

$$u_i \wedge u_j \ ,$$

with $i < j$, form a basis for the 6-dimensional vector space consisting of all skew tensors. Let us define a real valued symmetric bilinear inner product on this 6-dimensional vector space as follows.

Definition. The inner product of (two skew tensors) S and T, denoted by $S \wedge T$, is defined by the requirement that the inner product

$$(u_i \wedge u_j) \wedge (u_k \wedge u_\ell)$$

of two basis tensors should be:

$+1$ if i, j, k, ℓ is an even permutation of $0, 1, 2, 3$,

-1 if i, j, k, ℓ is an odd permutation of $0, 1, 2, 3$,

0 otherwise (i.e., if the four indices are not distinct).

Clearly there is one and only one bilinear inner product which satisfies this condition.

Lemma A.1. *If S and T are any two skew tensors of the form $S = a \wedge b$ and $T = c \wedge d$, then the inner product $S \wedge T = (a \wedge b) \wedge (c \wedge d)$ is equal to the determinant of the matrix whose rows are a, b, c, and d respectively.*

Proof. Expressing a, b, c, d as linear combinations $a = \sum a^i u_i, \ldots, d = \sum d^\ell u_\ell$ we obtain

$$(a \wedge b) \wedge (c \wedge d) = \sum a^i b^j c^k d^\ell (u_i \wedge u_j) \wedge (u_k \wedge u_\ell)$$
$$= \sum_{\text{permutations}} \pm a^i b^j c^k d^\ell \ ,$$

which is just the classical definition of the determinant. ∎

Now suppose that we apply a Lorentz transformation, replacing the vectors $a, \ldots, d$ by $a\Lambda, \ldots, d\Lambda$. Then evidently the determinant $(a \wedge b) \wedge (c \wedge d)$ will be multiplied by $\det(\Lambda) = \pm 1$. Since any skew tensors S and T can be expressed as sums of such skew products $a \wedge b$ and $c \wedge d$, we easily conclude the following.

Lemma A.2. *The inner product $(\Lambda^{-1}S\Lambda) \wedge (\Lambda^{-1}T\Lambda)$ is equal to $\pm S \wedge T$, where the sign ± 1 is equal to the determinant of the Lorentz matrix Λ.*

As an example to illustrate this construction, let us compute the inner product of the electromagnetic field F with itself. Using the notation 10.6, it is not difficult to check that

$$F = u_0 \wedge u_1 E^1 + u_0 \wedge u_2 E^2 + u_0 \wedge u_3 E^3$$
$$+ u_2 \wedge u_3 B^1 + u_3 \wedge u_1 B^2 + u_1 \wedge u_2 B^3 \ ,$$

hence

$$F \wedge F = 2\vec{E} \cdot \vec{B} \ .$$

This proves the following. (Compare §10.)

Corollary A.3. If we apply a Lorentz transformation, replacing the electromagnetic field F by $\Lambda^{-1}F\Lambda$, then the Euclidean inner product $\vec{E} \cdot \vec{B}$ of the electric and magnetic components of this field will be multiplied by the sign $\pm 1 = \det(\Lambda)$.

Now let us introduce a second and quite different inner product on the space of skew tensors by setting the inner product of S and T equal to

$$\tfrac{1}{2}\mathrm{trace}(ST) = \tfrac{1}{2}\mathrm{trace}(TS) = \tfrac{1}{2}\mathrm{trace}(\Lambda^{-1}ST\Lambda) \ .$$

It is not difficult to check that the six basis tensors $u_i \wedge u_j$ are mutually orthogonal under this new inner product, with

$$\tfrac{1}{2}\mathrm{trace}((u_i \wedge u_j)(u_i \wedge u_j)) = \pm 1 \ .$$

To compare these two inner products, consider the following construction. Fixing S, consider the linear transformation

$$T \longmapsto S \wedge T$$

from skew tensors to real numbers. Since the trace inner product is non-degenerate, there exists one and only one skew tensor $\hat{S}$ satisfying the identity

$$S \wedge T = \tfrac{1}{2}\mathrm{trace}(\hat{S}T)$$

for all T.

Definition A.4. We will call the skew tensor S constructed in this way the dual of S.

As an illustration of this notation, for any electromagnetic field F we note that the real number $\vec{E} \cdot \vec{B} = \tfrac{1}{2}F \wedge F$ can also be described by the formula

$$\vec{E} \cdot \vec{B} = \tfrac{1}{4}\mathrm{trace}(F\hat{F}) \ .$$

Using the explicit basis $u_i \wedge u_j$, it is not difficult to compute $\hat{S}$ explicitly. In fact if

$$S = \begin{bmatrix} 0 & a & b & c \\ a & 0 & -z & y \\ b & z & 0 & -x \\ c & -y & x & 0 \end{bmatrix} \text{ then } \hat{S} = \begin{bmatrix} 0 & x & y & z \\ x & 0 & c & -b \\ y & -c & 0 & a \\ z & b & -a & 0 \end{bmatrix} .$$

It follows from this formula that the dual $\hat{\hat{S}}$ of $\hat{S}$ is equal to $-S$.

This matrix $\hat{S}$ transforms as a tensor, up to sign:

Lemma A.5. The dual $(\Lambda^{-1}S\Lambda)^\wedge$ of a Lorentz transformed matrix is equal to $\pm\Lambda^{-1}\hat{S}\Lambda$, where the sign ± 1 is equal to the determinant of Λ.

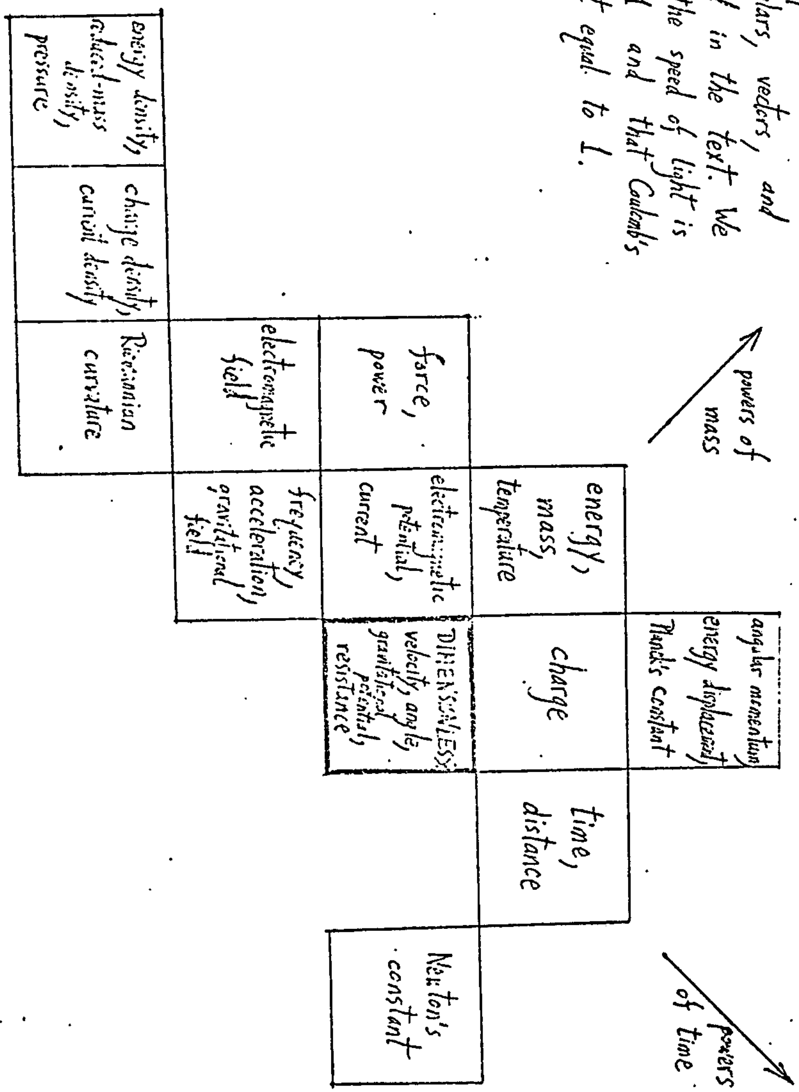
Proof. This follows easily from A.2 and the definition of $\hat{S}$. □

Thus we can think of $\hat{S}$ as a tensor, providing that we only allow Lorentz transformations of determinant $+1$.

For applications of this dual tensor $\hat{S}$ the reader is referred to §9, §12, and later sections.

Dimensional Chart

This chart compares the dimensions of various scalars, vectors, and tensors mentioned in the text. We assume that the speed of light is set equal to 1 and that Coulomb's constant is set equal to 1.



Chronological Table

Galileo Galilei	1564-1642
René Descartes	1596-1650
Isaac Newton	1642-1726
Olaus Römer	1644-1710
Benjamin Franklin	1706-1790
William Watson	1715-1787
Jean-le-Rond d'Alembert	1717-1783
Joseph Priestley	1733-1804
Charles Augustin Coulomb	1736-1806
Pierre Simon Laplace	1749-1827
Karl Friedrich Gauss	1777-1855
Siméon Denis Poisson	1781-1840
Nicolai Ivanovitch Lobachevsky	1793-1856
János Bolyai	1802-1860
Hermann von Helmholtz	1821-1894
William Thomson (Lord Kelvin)	1824-1907
Georg Friedrich Bernhard Riemann	1826-1866
Ludwig Lorenz	1829-1891
Elwin Bruno Christoffel	1829-1900
James Clerk Maxwell	1831-1879
Josef Stefan	1835-1893
Edward Williams Morley	1838-1923
Ludwig Eduard Boltzmann	1844-1906
Oliver Heaviside	1850-1925
George Francis Fitzgerald	1851-1901
John Henry Poynting	1852-1914
Albert Abraham Michelson	1852-1931
Gregorio Ricci-Curbastro	1853-1925

Hendrik Antoon Lorentz	1853-1928
Henri Poincaré	1854-1912
Max Planck	1858-1947
Emil Wiechert	1861-1928
Hermann Minkowski	1864-1909
Alfred Marie Liénard	1869- *
Willem de Sitter	1872-1934
Tullio Levi-Civita	1873-1941
Karl Schwarzschild	1873-1916
Albert Einstein	1879-1955
Aleksandr Aleksandrovich Friedman	1888-1925
Edwin Powell Hubble	1889-1953
Louis Victor Pierre Raymond de Broglie	1892-
Paul Adrien Maurice Dirac	1902-
Laurent Schwartz	1915-
Joseph Weber	1919-

* Liénard retired in 1936. I do not know his date of death.

28. The Schwarzschild Metric and Black Holes

As a first illustration of the vacuum Einstein equations $R_{ij} = 0$, let us study the geometry of spacetime about an isolated, non-rotating, spherically symmetric planet or star. If we assume that the gravitational potential at proper height h above the surface is a smooth function $\Phi(h)$ which depends only on h, and if we assume that the set of all points at height h forms a perfect sphere whose circumference is also a function $2\pi r(h)$, then we are led to a metric of the form

$$e^{2\Phi(h)}dt^2 - dh^2 - r(h)^2(d\varphi^2 + \cos^2\varphi\, d\theta^2) ,$$

where φ and θ measure latitude and longitude on the surface. Our problem is to determine the two unknown functions $\Phi(h)$ and $r(h)$.

Following K. Schwarzschild, this metric can be studied more conveniently by using r, rather than the proper height h, as independent variable. This <u>Schwarzschild distance coordinate</u> r is by definition equal to $1/2\pi$ times the circumference of the spherical surface consisting of all points at some fixed distance from the planet or star. Setting

$$dr/dh = e^{-\Lambda} ,$$

the metric can be written as

(28.1) $$e^{2\Phi}dt^2 - e^{2\Lambda}dr^2 - r^2(d\varphi^2 + \cos^2\varphi\, d\theta^2) .$$

Here Φ and Λ are to be expressed as functions of the Schwarzschild coordinate r. A straightforward but tedious computation shows that the Ricci tensor R_{ij} associated with this metric 28.1 is given by $R_{ij} = 0$ for $i \neq j$, and

(28.2) $$\begin{cases} R_{00} = (\Phi_{rr} + \Phi_r^2 - \Phi_r\Lambda_r + 2r^{-1}\Phi_r)e^{2(\Phi-\Lambda)} \\ R_{11} = -\Phi_{rr} - \Phi_r^2 + \Phi_r\Lambda_r + 2r^{-1}\Lambda_r \\ R_{22} = 1 + (r\Lambda_r - r\Phi_r - 1)e^{-2\Lambda} , \end{cases}$$

with $R_{33} = R_{22}\cos^2\varphi$. Here the subscript r stands for d/dr, and the four variables t, r, φ, θ are indexed as 0, 1, 2, 3 respectively.

We must choose the functions $\Phi(r)$ and $\Lambda(r)$ to satisfy the differential equations $R_{ij} = 0$. Adding $e^{2(\Lambda-\Phi)}R_{00}$ to R_{11} we obtain the equation

$$2r^{-1}(\Phi_r + \Lambda_r) = 0$$

or in other words

$$\Phi + \Lambda = \text{constant} .$$

But the gravitational potential Φ is only well defined up to an additive constant. We may as well choose this constant so that

(28.3) $$\Lambda = -\Phi .$$

Then the equation $R_{22} = 0$ reduces easily to

$$2re^{2\Phi}\Phi_r = 1 - e^{2\Phi} ,$$

or in other words

$$(e^{2\Phi})_r = (1 - e^{2\Phi})/r .$$

It is not difficult to check that the most general solution of this differential equation has the form

$$e^{2\Phi} = 1 - 2M/r$$

where 2M is an arbitrary constant. Thus we obtain the <u>Schwarzschild metric</u>

(28.4) $$(1 - 2M/r)dt^2 - (1 - 2M/r)^{-1}dr^2 - r^2(d\varphi^2 + \cos^2\varphi\, d\theta^2) ,$$

where M can be any constant. In deriving this metric, we have not made full use of the equations $R_{00} = R_{11} = 0$. However, it is easy to check that these equations are actually satisfied for any M.

For the special case $M = 0$ note that 28.4 reduces to the flat Minkowskian metric

$$dt^2 - dr^2 - r^2(d\varphi^2 + \cos^2\varphi \, d\theta^2)$$

in spherical coordinates. For any value of M, it is interesting to note that the Schwarzschild metric approximates this flat Minkowskian metric very closely for large values of r. This near flatness at infinity is not a boundary condition which we must impose, but rather is a consequence of our system of differential equations.

In order to interpret the constant M, let us compare this theory with the Newtonian gravitational theory. In the Newtonian theory, the gravitational potential at distance r from the center of a planet or star is given by

$$\Phi = -Gm/r + \text{constant} \quad ,$$

where m is the mass and

$$G = 7.425 \times 10^{-29} \text{ centimeters/gram} = 2.4767 \times 10^{-39} \text{ seconds/gram} \quad .$$

On the other hand, in the Schwarzschild metric, the relativistic gravitational potential is given by

(28.5) $$\Phi = \tfrac{1}{2}\log(1 - 2M/r) = -(M/r) - (M/r)^2 - \tfrac{4}{3}(M/r)^3 - \ldots \quad .$$

The normalization is such that Φ tends to zero at infinity.

Thus, if this potential Φ is to approximate the Newtonian gravitational potential for large values of r, then the unknown constant M must be precisely equal to Gm.

Here is another comparison. In the Newtonian theory, the gravitational field strength at any point is equal to

$$\partial\Phi/\partial r = Gm/r^2 \quad .$$

On the other hand, using Schwarzschild coordinates, the relativistic expression for gravitational field strength is

$$\partial\Phi/\partial(\text{proper height}) = M/(r^2\sqrt{1 - 2M/r}) \quad .$$

If we assume that $r \gg M$, so that the factor $\sqrt{1 - 2M/r}$ can be ignored, then again we obtain a precise correspondence between the two theories by assuming that $M = Gm$.

Definition. The product $M = Gm$ will be called the gravitational mass of the planet or star. It is directly proportional to mass, but is measured in units of distance (or of time).

Note that the Schwarzschild metric only makes sense for $r > 2M$. The number $2M$ is sometimes called the "Schwarzschild radius." It represents the lower limit of applicability of Schwarzschild coordinates.

Of course, the Schwarzschild metric is only to be used for $r \geq r_0$ where r_0 is the Schwarzschild coordinate at the surface of our planet or star. (Briefly, r_0 is called the radius.) In practice it turns out that r_0 is always much larger than $2M$.

Here are some examples. For the Earth, with mass $m = 5.98 \times 10^{27}$ grams, we obtain

$$M = 0.444 \text{ centimeters} \quad .$$

Therefore the "Schwarzschild radius" $2M$ for the Earth is a little less than one centimeter. Of course the actual radius r_0 of the Earth is much larger. In fact

$$r_0 = 6378 \text{ kilometers} \quad ;$$

so the surface gravitational potential $\Phi \approx -M/r_0$ equals 6.95×10^{-10}.

(Compare the discussion in §24.) Thus the Schwarzschild metric about the Earth coincides with the flat metric $dt^2 - dr^2 - r^2(d\varphi^2 + \cos^2\varphi d\theta^2)$ to an accuracy of one part in 10^9. However, this seemingly minor discrepancy of one part in 10^9 is completely responsible for the powerful force of gravitation on the Earth.

The gravitational mass of the sun is 1.475 kilometers, or almost a mile. Since the radius r_0 is 696,000 kilometers, we obtain a surface gravitational potential of

$$\Phi \approx -M/r_0 = -2.12 \times 10^{-6}$$

and a surface gravitational field strength of

$$\partial\Phi/\partial h \approx M/r_0^2 = 28.8 \text{ year}^{-1}$$

or about 28 Earth gravities.

There exist stars which are much denser than the Sun. Thus a white dwarf star might have a gravitational mass*

$$M = 1.8 \text{ kilometers}$$

even larger than that of the sun, packed into a ball of radius

$$r_0 = 4000 \text{ kilometers} ,$$

even smaller than the radius of the Earth. This would correspond to a surface potential of

$$\Phi \approx -M/r_0 = -.00045$$

and a surface gravity of 10^6 year^{-1}.

*This figure is based on a maximally massive white dwarf. Compare Weinberg, p. 317.

Even denser objects are believed to exist, in which the matter is so completely squeezed together that most protons and electrons will be forced to fuse together into neutrons. In such a "neutron star," the gravitational mass might be as large as* 1 kilometer and the radius r_0 as small as 10 kilometers. This would correspond to a surface gravitational potential of $\Phi = -0.11$ and a surface field strength of $\partial\Phi/\partial h$ = 3350 seconds^{-1} $\approx 10^{11}$ years. Thus light from such a neutron star would be red shifted by $e^{-\Phi} - 1 = .12$.

It is believed that objects denser than this cannot exist in the real world. In particular, gravitational red shifts of more than 12 per cent cannot be observed. For if any more massive object were crowded into such a small volume then it could not sustain its own weight. Rather, its gravitational field strength would be larger than its ability to sustain compression, so that it would collapse, shrinking rapidly and inexorably towards the Schwarzschild radius. The result, to an outside observer, would be that this star would almost instantly** disappear. However, the Schwarzschild metric would remain, or in other words the gravitational field of this star would remain. The resulting gravitational field, with essentially nothing at all at its center, would constitute what is called a black hole.***

There are currently several candidates for possible black holes in the real world, but the existence of black holes has not definitely been established.

It is necessary to qualify the statement that there is "nothing" at the center of a black hole. The surface of a collapsing star certainly cannot travel inward at more than the speed of light. But even a photon, traveling directly towards the collapsing star, so that $d\varphi = d\theta = 0$ and

*Compare Weinberg, p. 321.

**Compare Misner, Thorne, and Wheeler, p. 850.

***More generally one can consider a rotating black hole, and also a black hole with non-zero charge. The mathematical properties are similar to those of the simpler case considered here.

$$(1 - 2M/r)dt^2 - (1 - 2M/r)^{-1}dr^2 = 0 ,$$

would require infinitely much Schwarzschild coordinate time in order to actually reach the Schwarzschild radius $r = 2M$. In fact, integrating the associated differential equation

$$dr/dt = -(1 - 2M/r)$$

for the worldcurve of an incoming photon, we find that

(28.6) $$t + r + 2M \log(r - 2M) = \text{constant} ,$$

so that the coordinate time t tends to $+\infty$ as $r \longrightarrow 2M$.

Hence, in some sense, the star can never actually complete its collapse to the Schwarzschild radius. However, for all practical purposes the star would rapidly cease to exist. The light emitted by this star would effectively disappear in microseconds, as the gravitational red shift became enormously large. Even a radar signal sent to the collapsing star would never reach its surface, much less return.

Concluding Remark. The apparent singularity of the Schwarzschild metric at $r = 2M$ can be removed by a carefully chosen change of coordinates. Consider for example an observer moving towards a black hole on the radial line φ = constant, θ = constant. Then his position in spacetime is specified by the two Schwarzschild coordinates t and r, and his proper time τ is determined by the equation

$$d\tau^2 = (1 - 2M/r)dt^2 - (1 - 2M/r)^{-1}dr^2 .$$

Instead of using the Schwarzschild coordinate time t, suppose that this observer continually resets his watch according to coordinate-time signals sent from a master clock located at some fixed point with Schwarzschild coordinate $r_1 > r$. Using 28.6, we see that the time T on this watch will

be related to the coordinate time t by the equation

(28.7) $$T = t + r + 2M \log(r - 2M) + \text{constant} .$$

Using T and r, in place of t and r, as coordinates, a brief computation shows that the metric takes the form

$$(1 - 2M/r)dT^2 - 2dTdr .$$

Reintroducing the latitude and longitude coordinates, we see that the full Schwarzschild metric is completely equivalent to the metric

(28.8) $$(1 - 2M/r)dT^2 - 2dTdr - r^2(d\varphi^2 + \cos^2\varphi d\theta^2) .$$

In fact the two metrics are related by the smooth change of coordinates 28.7, throughout the region $r > 2M$.

But this new metric is non-singular, and makes perfect mathematical sense, for all values of $r > 0$. The Schwarzschild radius $r = 2M$ no longer seems to play any special role. If our observer firmly believes in this new coordinate system, then he may perfectly well approach the black hole and actually cross the Schwarzschild radius $r = 2M$. However, if he does this, his future history will be of no interest to the outside world. For as he approaches the Schwarzschild radius $r = 2M$ his Schwarzschild coordinate time t will tend to $+\infty$. Thus his history after passing through the Schwarzschild radius will take place at coordinate time $t \geq \infty$, and can never be reported back to the outside world.

The locus $r = 2M$ is called the Schwarzschild horizon, since it represents the limit of what can be observed by the outside world.

It is interesting to note that the sub-locus

$$r = 2M, \quad \varphi = \text{constant}, \quad \theta = \text{constant}$$

is a null geodesic. Thus, if our observer radios for help as he crosses the

Schwarzschild horizon, the signal will simply sit on this horizon forever.

In any case, the history of our intrepid observer, once he has crossed the Schwarzschild horizon, will hardly be worth writing home about. For using the identity

$$(1 - 2M/r)\dot{T}^2 - 2\dot{T}\dot{r} - r^2(\dot{\varphi}^2 + \cos^2\varphi\,\dot{\theta}^2) = 1 ,$$

where the dot stands for the derivative with respect to proper time τ along his worldcurve, it is not difficult to check that

$$\dot{r} \le -\sqrt{(2M/r) - 1}$$

for $r \le 2M$. Thus r must decrease monotonically, converging to zero in a proper time interval

$$\Delta\tau \le \int_0^{2M} \frac{dr}{\sqrt{(2M/r) - 1}} = \left[M \arcsin\left(\frac{r}{M} - 1\right) - \sqrt{r(2M - r)}\right]_0^{2M} = \pi M .$$

As an example, for a black hole of solar mass we have $\pi M = 1.5 \times 10^{-5}$ seconds. Therefore, once the observer crosses the Schwarzschild horizon of such a black hole, he will inevitably be crushed to zero volume within 15 microseconds.

Here are some problems for the reader. The first two will play an important role in subsequent sections.

Problem 28-A (Kepler's Second Law). Consider a freely falling particle or satelite in the equatorial plane $\varphi = 0$. Assuming that the metric in this equatorial plane has the form

$$e^{2\Phi(r)}dt^2 - e^{2\Lambda(r)}dr^2 - r^2 d\theta^2 ,$$

show that the orbital angular momentum

$$A = r^2 d\theta/d\mu$$

is constant along the worldcurve of this particle. Here μ denotes the natural (time/energy)-parameter along the worldcurve. (Compare §23. In the case of a particle of mass $m > 0$ we can write $A = mr^2 d\theta/d\tau$.) The proof depends on a computation of the Christoffel symbols of the form Γ_{ij}^2 for this metric, where we are indexing the variables t, r, θ by $0, 1, 2$ respectively.

Problem 28-B (Kepler's Third Law for circular orbits). If the particle is moving in a circular orbit r = constant, show that

$$d\theta/dt = \pm\sqrt{-\Gamma_{00}^1/\Gamma_{22}^1} = \pm\sqrt{(\partial g_{00}/\partial r)/2r} .$$

Specializing to the Schwarzschild metric, with $g_{00} = 1 - 2M/r$, this reduces to

$$d\theta/dt = \pm\sqrt{M/r^3} .$$

Thus the coordinate time period of revolution is given by

$$\Delta t = 2\pi\sqrt{r^3/M} .$$

If the particle has mass $m > 0$, show that the radius r of this orbit must satisfy* $r > 3M$. For a particle of mass zero in circular orbit, show that the radius r must be precisely equal to $3M$.

Problem 28-C (Geodetic Lag). Consider a gyroscope in circular orbit, with its axis in the plane of revolution. Suppose that the angular momentum vector of this gyroscope points along the axis. Assuming that this angular momentum vector forms a parallel vector field along the worldcurve, show that it lags behind by an angle of $2\pi(1 - \sqrt{1 - 3M/r}) \approx 3\pi M/r$ in each full revolution. (For a satelite in near Earth orbit, this amounts to 8 seconds of arc per year.)

Problem 28-D (Comparison of different measures of height). Show that the proper height h (as measured by a measuring tape) above the

*The orbit is stable only if $r > 6M$. Compare Problem 29-B.

surface of a spherically symmetric planet is related to the Schwarzschild coordinate r by the formula

$$h = \int_{r_0}^{r} dr/\sqrt{1 - 2M/r} = r\sqrt{1 - 2M/r} + M\log(r - M + r\sqrt{1 - 2M/r}) + \text{constant} .$$

Show that the height as measured by a radar set on the surface (using coordinate time) is given by

$$h' = \int_{r_0}^{r} dr/(1 - 2M/r) = r + 2M\log(r - 2M) + \text{constant} .$$

In the case of a black hole, integrating from some arbitrary fixed $r_0 > 2M$, note that the first integral tends to a finite limit as $r \longrightarrow 2M$, but that the second integral tends to $-\infty$.

28-11.