Notes on Jacquet - Langlands' theory

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1970

Preface

The purpose of these notes would have been better explained if we had chosen another title, namely, "Jacquet - Langlands' theory made easy"; it occurred to us at the last moment that a more pedestrian choice would be more prudent, since after all the author is in a rather bad position to judge...

These notes cover a very large part of §§2, 3, 5, 6, 9, 10 and 11 of Jacquet-Langlands' work, Automorphic Forms on GL(2), VII + 548 pp, 1970, Springer (Lecture Notes in Mathematics, No. 114). Since the volume of our notes is about one fifth of 548 pp., it is not to be expected that we have been able here to explain everything. In fact we have entirely omitted the explicit construction of discrete series from quadratic extensions or quaternion algebra (§4 of J. L.), the connection with zêta functions of matrix algebras (§13), and the most interesting, or at any rate newest, part of their work, namely, the relations between the "spectra" of a quaternion algebra and a 2×2 matrix algebra. The reader who is sufficiently interested by the present notes will of course have to go back to Jacquet and Langlands anyway.

We have given full proofs in §1 and nearly complete ones in §3, but not in §2. For the bibliography, we refer the reader to Jacquet and Langlands, where references will be found.

These notes have been written after lectures on the same subject at The Institute for Advanced Study, where we found from September, 1969, to April, 1970, a very welcome atmosphere of quiet intellectual work. It is for us a great pleasure to express here our deep gratitude not only for the conveniences we were provided with, but also for the fact that we were spared the duty to thank the U. S. Air Force for its main contribution to Culture and Civilization, namely, the highly palatable Napalm-and-Mathematics cocktail that is the mark of the times in the most advanced country of the world.

Princeton, March, 1970

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§1. Representations of the GL_2 group of a $\mathfrak{p}$-adic field

In this section we denote by F a non-archimedean locally compact field, by $\mathcal{O}_F$ and $E_F = \mathcal{O}_F^*$ the ring of integers and group of units of F, and we choose once and for all a non-trivial character τ_F of the additive group, which will be used to define Fourier transforms. The prime ideal of F will be denoted by $\mathfrak{p}$ and a generator of it by ϖ. The largest ideal on which τ_F is trivial will be $\mathfrak{p}^{-d}$ for a certain integer d (of course $\mathfrak{p}^d$ is the different of F if τ_F is chosen as in Weil's <u>Basic Number Theory</u>, which we may assume). There is an absolute value on F defined for instance by the relation $d(ax) = |a|dx$, where dx is an invariant measure on the additive group of F. We shall assume dx chosen in such a way that Fourier inversion formula can be written as

$$\hat{f}(y) = \int f(x)\bar{\tau}_F(xy)dx \Longrightarrow f(x) = \int \hat{f}(y)\tau_F(xy)dy$$

for nice functions (e.g. for $f \in \mathcal{S}(F)$, the space of locally constant functions with compact support on F). The invariant measure d^*x of F^* will be chosen in such a way that

$$\int_{E_F} d^*x = 1 ,$$

so that $d^*x = c|x|^{-1}dx$ with a constant c whose value is unimportant for the time being.

We shall put

$$G_F = GL(2, F), \qquad M_F = GL(2, \mathcal{O}_F)$$

so that M_F is a maximal compact (and open) subgroup of G_F. The set of locally constant functions with compact support on G_F will be denoted by $\mathcal{H}_F$; it is an algebra (the <u>Hecke algebra</u> of G_F) under convolution product

$$f * g(x) = \int_{G_F} f(xy^{-1})g(y)d^*y$$

where d^*y denotes an invariant measure on G_F such that $\int_{M_F} d^*y = 1$.

1. Admissible representations

Let π be a linear representation of G_F on a complex vector space $\mathcal{V}$. For every (finite dimensional) irreducible continuous representation ϑ of the compact group M_F, let $\mathcal{V}(\vartheta)$ be the set of all $\xi \in \mathcal{V}$ which transform under $\pi(M_F)$ according to a finite multiple of ϑ. The representation π will be called admissible if

$$(1) \qquad \mathcal{V} = \oplus \, \mathcal{V}(\vartheta) \quad \text{and} \quad \dim \mathcal{V}(\vartheta) < +\infty .$$

Equivalent conditions: every $\xi \in \mathcal{V}$ is fixed under some open subgroup M of G_F, and the set of all $\xi \in \mathcal{V}$ which are fixed under a given open subgroup M is finite-dimensional. These conditions arise in a natural way from the study of automorphic functions as well as from general representation theory. For such a representation we can define, for every $f \in \mathcal{H}_F$, a linear operator $\pi(f)$ on $\mathcal{V}$ by

$$(2) \qquad \pi(f)\xi = \int_{G_F} f(x)\pi(x)\xi \, d^*x$$

(the "integral" of course reduces to a finite sum--look at the open stabilizer of ξ in G_F). Hence π extends to a representation of the group algebra $\mathcal{H}_F$, with two properties which characterize, as can easily be proved, the representations of $\mathcal{H}_F$ which can be obtained in that way: (i) for every $\xi \in \mathcal{V}$ there is an $f \in \mathcal{H}_F$ such that $\pi(f)\xi = \xi$; (ii) every $\pi(f)$ maps $\mathcal{H}_F$ on a finite-dimensional subspace of $\mathcal{V}$. Such representations of $\mathcal{H}_F$ will still be called admissible.

Let π be an admissible representation of G_F on $\mathcal{V}$. We may consider the representation $g \longmapsto {}^t\pi(g^{-1})$ on the dual space $\mathcal{V}^* = \prod \mathcal{V}(\vartheta)^*$. The subspace of those $\xi^* \in \mathcal{V}^*$ which are invariant under some open subgroup of G_F is evidently

$$(3) \qquad \check{\mathcal{V}} = \oplus \, \mathcal{V}(\vartheta)^* ;$$

we denote by $\check{\pi}(g)$ the restriction of ${}^t\pi(g)^{-1}$ to $\check{\mathcal{V}}$. We thus get an admissible representation of G_F on $\check{\mathcal{V}}$, which we call the contragredient of π.

If a subspace $\mathcal{V}_1$ of $\mathcal{V}$ is invariant under $\pi(G_F)$, i.e. under $\pi(\mathcal{H}_F)$, then the subspace $\mathcal{V}_1^{\perp}$ of all $\xi^* \in \check{\mathcal{V}}$ which are orthogonal to $\mathcal{V}_1$ is invariant under $\check{\pi}$, and we have $(\mathcal{V}_1^{\perp})^{\perp} = \mathcal{V}_1$. Thus we get a one-to-one correspondence between invariant subspaces of $\mathcal{V}$ and of $\check{\mathcal{V}}$. A representation with no non trivial invariant subspaces will be called <u>irreducible</u>. The purpose of this section is to classify these representations, and to associate to each irreducible admissible representation a "local zeta function" which will more or less characterize it.

Note finally that the Schur lemma is valid for irreducible admissible representations: if an operator $T \in \mathcal{L}(\mathcal{V})$ commutes with π, then it operates in every $\mathcal{V}(\mathfrak{r})$, hence must have eigenvectors, so that T is a scalar.

2. <u>The Kirillov model: preliminary construction</u>

Our first goal (number 2 to 4) will be to show that every admissible irreducible representation of G_F can be realized in a very concrete way on a space of functions on F^*, the multiplicative group of non zero elements of F. For finite dimensional representations the problem is not interesting--a finite dimensional irreducible admissible representation π of G_F is one dimensional, and given by $\pi(g) = \chi(\det g)$ for some character* χ of F^*. In fact, the finiteness of $\dim \mathcal{V}$ implies that the kernel of π is an <u>open</u> hence non trivial invariant subgroup of G_F; but any non trivial invariant subgroup of $GL(2, F)$ contains $SL(2, F)$; hence π is trivial on $SL(2, F)$, the space of π is one dimensional by Schur's lemma, and we get the result by taking into account the fact that every $g \in GL(2, F)$ is the product of something in $SL(2, F)$ and the matrix $\begin{pmatrix} \det g & 0 \\ 0 & 1 \end{pmatrix}$. We shall thus consider infinite-dimensional representations only. For such representations the following theorem will be proved:

<u>Theorem 1. Let π be an irreducible admissible representation of G_F on an infinite dimensional vector space $\mathcal{V}$. Then there exists one and only one space $\mathcal{V}'$ of complex valued functions on F^*, and one and only one representation</u>

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* By a character we mean a continuous homomorphism in $\mathbb{C}^*$. Characters such that $|\chi(x)| = 1$ will be called <u>unitary</u>.

π' of G_F on $\mathcal{V}'$, satisfying the following two conditions: π' is equivalent to π, and we have

(4) $$\pi'\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi'(x) = \tau_F(bx)\xi'(ax) \qquad (a,\ x \in F^*,\ b \in F)$$

for every $\xi' \in \mathcal{V}'$, where τ_F is a given non trivial additive character of F. Furthermore each function in $\mathcal{V}'$ is locally constant, and vanishes outside some compact subset of F; each locally constant function which vanishes outside some compact subset of F^* belongs to $\mathcal{V}'$, and the space $\mathcal{S}(F^*)$ of such functions has finite codimension in $\mathcal{V}'$.

Suppose for a moment we have constructed $\mathcal{V}'$ and π', and let $\xi \longmapsto \xi'$ denote an isomorphism of $\mathcal{V}$ on $\mathcal{V}'$ compatible with π and π'; hence

(5) $$\eta = \pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi \Longrightarrow \eta'(x) = \tau_F(bx)\xi'(ax).$$

Consider the linear form L on $\mathcal{V}$ given by $L(\xi) = \xi'(1)$; we evidently have

(6) $$L(\pi\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\xi) = \tau_F(b)L(\xi) \quad \text{for all } \xi \in \mathcal{V} \text{ and } b \in F,$$

and furthermore

(7) $$\xi'(x) = L(\pi\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\xi) \quad \text{for all } \xi \in \mathcal{V} \text{ and } x \in F^*.$$

From (6) it follows that

(8) $$\int_{\mathfrak{p}^{-n}} \overline{\tau_F(x)}\cdot L(\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\xi)dx = L(\xi)\int_{\mathfrak{p}^{-n}} dx$$

for each n; if we consider in $\mathcal{V}$ the subspace $\mathcal{V}_o$ of all vectors ξ such that

(9) $$\int_{\mathfrak{p}^{-n}} \overline{\tau_F(x)}\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\xi\, dx = 0 \quad \text{for all large } n,$$

then it is clear that $\mathcal{V}_o \subset \mathrm{Ker}(L)$. The main step in the proof will be to show that

(10) $$\dim(\mathcal{V}/\mathcal{V}_o) = 1.$$

If we can prove (10), and then denote by L a non zero linear form vanishing on $\mathcal{V}_o$, then we shall get the space $\mathcal{V}'$ by associating to every $\xi \in \mathcal{V}$ the function (7), and the existence and uniqueness of $\mathcal{V}'$ and π' will easily follow as we shall see later.

For the time being we start with the subspace $\mathcal{V}_o$ defined by condition (9) and denote by X the factor space $\mathcal{V}/\mathcal{V}_o$ and by L the canonical map from $\mathcal{V}$ to X. For every $\xi \in \mathcal{V}$ we consider the X-valued function (7) on F^*.

Lemma 1. The relation $\eta = \pi\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix}\xi$ implies $\eta'(t) = \tau_F(bt)\xi'(at)$.

We have to show that $\mathcal{V}_o$ contains $\pi\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}\eta - \tau_F(bt)\pi\begin{pmatrix} at & 0 \\ 0 & 1\end{pmatrix}\xi$ i.e. that

(11) $$\int_{\mathcal{Y}^{-n}} \overline{\tau_F(x)}\pi\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\left[\pi\begin{pmatrix} ta & tb \\ 0 & 1\end{pmatrix} - \tau_F(bt)\pi\begin{pmatrix} ta & 0 \\ 0 & 1\end{pmatrix}\right]\xi\cdot dx = 0$$

for large n, which is clear (take n large enough so that $tb \in \mathcal{Y}^{-n}$, and replace the integration variable x by $x - bt$ in the first term of the difference).

Lemma 2. Each function ξ' is locally constant, and vanishes outside a compact subset of F.

For every $a \in F^*$ sufficiently close to 1 we have $\pi\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\xi = \xi$ and hence $\xi'(xa) = \xi'(x)$ for all x, from which the first assertion follows. Similarly there is in F a non zero ideal $\mathfrak{n}$ such that $\pi\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\xi = \xi$ for all $b \in \mathfrak{n}$, hence $\xi'(x) = \tau_F(bx)\xi'(x)$, which of course implies the second property.

Lemma 3. The map $\xi \longmapsto \xi'$ is injective.

Assume $\xi' = 0$ i.e. $\pi\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}\xi \in \mathcal{V}_o$ for all $t \neq 0$. We see at once that for every $t \neq 0$ we have

(12) $$\int_{\mathcal{Y}^{-n}} \overline{\tau_F(tx)}\pi\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\xi\cdot dx = 0 \quad \text{for all large } n.$$

The first step is to prove that the function $\varphi(x) = \pi\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\xi$ is constant. In fact, there is a non zero ideal $\mathfrak{n} \subset \mathcal{Y}$ such that $\pi\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}\xi = \xi$ for all $u \in 1 + \mathfrak{n}$. Defining

(13) $$\varphi_n(t) = \int_{\mathcal{Y}^{-n}} \tau_F(tx)\varphi(x)dx$$

we then see at once that $\varphi_n(tu) = \pi\begin{pmatrix} u^{-1} & 0 \\ 0 & 1\end{pmatrix}\varphi_n(t)$ for all n, all t, and all u in $1 + \mathfrak{n}$. Since any compact subset K of F^* is a finite union of cosets

$\bmod^* 1 + \mathfrak{m}$, it follows that $\varphi_n(t) = 0$ for all $t \in K$ provided n is large enough, because we assume (12). But let ψ be a Schwartz-Bruhat function on F, and suppose its Fourier transform

(14) $$\hat{\psi}(t) = \int_F \overline{\tau_F(tx)}\psi(x)dx$$

vanishes at $t = 0$, hence outside of a compact subset K of F^*. Since ψ vanishes outside of $\mathfrak{y}^{-n}$ for large n, we get, by making use of Fourier inversion formula,

(15) $$\int_F \psi(x)\varphi(x)dx = \int_{\mathfrak{y}^{-n}} \varphi(x)dx \int_K \hat{\psi}(t)\tau_F(tx)dt = \int_K \varphi_n(t)\hat{\psi}(t)dt = 0$$

for large n. Hence the function $\varphi(x)$, which is translation invariant under an open subgroup of F, is orthogonal to all $\psi \in \mathcal{S}(F)$ which are orthogonal to the function 1. It follows that $\varphi(x)$ is constant, i.e. that

(16) $$\pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\xi = \xi \quad \text{for all } x \in F.$$

The second step in the proof of lemma 3 is to show that (16) implies $\xi = 0$. Let H be the subgroup of all $g \in G_F$ such that $\pi(g)\xi = \xi$. It is open and contains the subgroup U_F of all matrices of the form $\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}$. Since H is open, it is not contained in the subgroup $c = 0$ of G_F, hence H intersects the "big cell" $c \neq 0$, and since H contains U_F, it follows that there is in H a matrix with $a = d = 0$ (Bruhat decomposition). But then H must contain the subgroup generated by U_F and such a matrix, namely $SL(2, F)$. But the set of all $\xi \in \mathcal{V}$ that are fixed under $SL(2, F)$ is an invariant subspace of $\mathcal{V}$ on which $GL(2, F)$ operates as a commutative group. There can be no such $\xi \neq 0$ if $\dim \mathcal{V} > 1$, q.e.d.

Lemma 3 makes it possible to identify each vector $\xi \in \mathcal{V}$ with the corresponding function ξ', and from now on we shall write ξ and $\xi(x)$ instead of ξ' and $\xi'(x)$, so that the elements of $\mathcal{V}$ will be certain X-valued functions on F^* on which G_F operates through π in such a way that $\pi\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\xi(x) = \tau_F(bx)\xi(ax)$. The canonical map $L : \mathcal{V} \longrightarrow X$ can now be identified with $\xi \longmapsto \xi(1)$.

<u>Lemma</u> 4. <u>The space</u> $\mathcal{S}_X(F^*)$ <u>of</u> X-<u>valued locally constant functions with compact support on</u> F^* <u>is contained in</u> $\mathcal{V}$. <u>Furthermore,</u> $\pi\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\xi - \xi \in \mathcal{S}_X(F^*)$ <u>for all</u> $b \in F$ <u>and</u> $\xi \in \mathcal{V}$.

The last assertion is obvious since $\tau_F(bx) - 1$ vanishes in a neighborhood of zero for every $b \in F$. To show that $\mathcal{S}_X(F^*) = \mathcal{S}(F^*) \otimes X$ is contained in $\mathcal{V}$, it will be enough to prove that, for vectors $c \in X$ which generate X, all functions $x \longmapsto \varphi(x)c$, with $\varphi \in \mathcal{S}(F^*)$, belong to $\mathcal{V}$. But the subspace $\mathcal{S}_c(F^*)$ of those $\varphi \in \mathcal{S}(F^*)$ such that $\mathcal{V}$ contains the function $\varphi(x)c$ is of course stable under the operators(*)

$$\{x \longmapsto \varphi(x)\} \longmapsto \{x \longmapsto \tau_F(bx)\varphi(ax)\} \qquad (a \in F^*,\ b \in F). \tag{17}$$

Hence it will be enough to prove that (i) the space $\mathcal{S}(F^*)$ is irreducible under the above operators, and that (ii) one has $\mathcal{S}_c(F^*) \neq 0$ for enough vectors $c \in X$.

To prove (i) let $\mathcal{H}$ be a subspace of $\mathcal{S}(F^*)$ invariant under the operators (17). Since every $\xi \in \mathcal{S}(F^*)$ is invariant under a subgroup of finite index of the group E_F of units of F, it is clear that $\mathcal{H} = \Sigma\mathcal{H}(\chi)$ where we denote, for every character χ of E_F, by $\mathcal{H}(\chi)$ the set of all $\xi \in \mathcal{H}$ such that $\xi(xu) = \xi(x)\chi(u)$. If we define

$$\chi_*(x) = \begin{cases} \chi(x) & \text{if } x \in E_F \\ 0 & \text{if } x \notin E_F, \end{cases} \tag{19}$$

then $\mathcal{S}(F^*)$ is generated by the functions $\chi_*(ax)$ for all χ and all $a \in F^*$. To prove that $\mathcal{H} = \mathcal{S}(F^*)$ if $\mathcal{H} \neq 0$, it will be enough to prove that $\chi_* \in \mathcal{H}$ for every χ.

But there is a $\chi' \neq \chi$ such that $\mathcal{H}(\chi') \neq 0$ [otherwise we would have $\mathcal{H}(\chi) = \mathcal{H}$, which is not compatible with the behaviour of the additive characters of F]. Choose such a $\chi' \neq \chi$ and a non zero $\xi' \in \mathcal{H}(\chi')$. Clearly $\mathcal{H}(\chi)$ contains, for all $a \neq 0$ and b, the function

(*) The use of this notation springs from an attempt by the author to bridge the generation gap. We hope it will have a good reception.

(20) $$\xi(x) = \int_{E_F} \tau_F(bxu)\xi'(axu)\overline{\chi(u)}d^*u = \gamma(bx, \chi'\bar{\chi})\xi'(ax)$$

with a Gaussian sum defined by

(21) $$\gamma(x, \lambda) = \int_{E_F} \tau_F(xu)\lambda(u)d^*u$$

for every character λ of E_F. But it is well known that if λ is non trivial then

(22) $$\gamma(x, \lambda) \neq 0 \Longleftrightarrow v_{\mathfrak{P}}(x) = -d - f(\lambda)$$

where $f(\lambda)$ is the exponent of $\mathfrak{P}$ in the conductor of λ. Since $\chi \neq \chi'$ in (20) we can thus choose $b \neq 0$ in such a way that $\gamma(bx, \chi'\bar{\chi}) \neq 0 \Longleftrightarrow x \in E_F$. We can also choose a such that $\xi'(ax) \neq 0$ if $x \in E_F$. Then (20) is evidently proportional to χ_*; hence $\mathcal{H} = \mathcal{S}(F^*)$ and the irreducibility of $\mathcal{S}(F^*)$ under the operators (17).

The proof of property (11) is similar. To prove that $\mathcal{S}_c(F^*) \neq 0$ for enough vectors $c \in X$ we may of course limit ourselves to vectors c for which there is in $\mathcal{V}$ a function ξ' such that $\xi'(1) = c$, and satisfying a relation $\xi'(xu) = \xi'(x)\chi'(u)$. Consider then such a ξ' and choose any character $\chi \neq \chi'$; clearly $\mathcal{V}$ still contains the function $\gamma(bx, \chi'\bar{\chi})\xi'(ax)$ given by (20). If we choose $a = 1$ and a suitable b, we thus get in $\mathcal{V}$ a non zero function proportional to $\chi_*(x)c$, from which it follows that $\mathcal{S}_c(F^*) \neq 0$.

3. The commutativity lemma

For every character χ of E_F, every $t \in F$ and every $a \in X$ define

(23) $$\chi_{t,a}(x) = \begin{cases} \chi(t^{-1}x)a & \text{if } x \in tE_F \\ 0 & \text{if } x \notin tE_F \end{cases}.$$

These functions generate the vector space $\mathcal{S}_X(F^*)$ and every $\xi \in \mathcal{S}_X(F^*)$ can be written as a series (in fact, a finite sum)

(24) $$\xi = \sum_{tE_F} \sum_{\chi} \chi_{t,a} \quad \text{where } a = \int_{E_F} \xi(xu)\overline{\chi(u)}du$$

if we assume the total mass of the Haar measure of E_F is 1. Consider now the action on $\mathcal{S}_X(F^*) \subset \mathcal{V}$ of the operator $\pi(w)$, where

(25) $$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ;$$

for every $t \in F^*$ and every character χ of E_F, we define in X a linear operator

(26) $$J_\pi(t, \chi) : a \longmapsto \pi(w)\chi_{t,a}(1) = L[\pi(w)\chi_{t,a}] .$$

If we put $\chi_a = \chi_{1,a}$ then we evidently have $\chi_{t,a}(x) = \chi_a(t^{-1}x)$, hence

(27) $$J_\pi(t, \chi)a = L[\pi(w)\pi\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix}\chi_a] = L[\pi\begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix}\pi(w)\chi_a]$$
$$= \omega_\pi(t^{-1})L[\pi\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\pi(w)\chi_a] = \omega_\pi(t^{-1})\pi(w)\chi_a(t),$$

where ω_π is the character of F^* defined by

(28) $$\omega_\pi(t)1 = \pi\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} .$$

(27) and lemma 2 show that each function $J_\pi(t, \chi)a$ is locally constant and vanishes outside a compact subset of F; it is furthermore clear that

(29) $$J_\pi(xu, \chi) = J_\pi(x, \chi)\overline{\chi(u)}$$

for every character χ of E_F. By making use of (24) we get

$$\pi(w)\xi(1) = \sum_{tE_F} \sum_\chi J_\pi(t, \chi)a = \sum J_\pi(t, \chi) \int_{E_F} \xi(tu)\overline{\chi(u)}du$$

hence

(30) $$\pi(w)\xi(1) = \sum_\chi \int_{F^*} J_\pi(y, \chi)\xi(y)d^*y$$

for every $\xi \in \mathcal{S}_X(F^*)$ - a substitute for the more pleasant formula

(31) $$\pi(w)\xi(1) = \int J_\pi(y)\xi(y)d^*y$$

which we cannot write at this point. If we now apply (30) to the function $\pi\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\xi$ instead of ξ, we get at once

(32) $$\pi(w)\xi(x) = \omega_\pi(x)\sum_\chi \int_{F^*} J_\pi(xy, \chi)\xi(y)d^*y ;$$

each integral converges in a trivial way, and the series is actually a finite sum.

Lemma 5. The family of operators $J_\pi(x, \chi)$ is commutative.

To prove this lemma we define $u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for every $t \in F$ and $h(t) = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}$ for every $t \in F^*$ and start from the relation

(33) $$wu(t)w^{-1} = u(-1/t)wh(t)u(-1/t), \qquad t \in F^* .$$

We shall compute the function

(34) $$\eta_t = \pi(wu(t)w^{-1})\xi = \pi[u(-1/t)wh(t)u(-1/t)]\xi$$

for a given $\xi \in \mathcal{S}_\chi(F^*)$ by making use of lemma 1 and relation (32). Using the right hand side of (33) we find at once

(35) $$\begin{aligned}\eta_t(x) &= \tau_F(-x/t)\omega_\pi(x)\sum_\chi \int J_\pi(xy, \chi)\omega_\pi(1/t)\tau_F(-ty)\xi(t^2y)d^*y \\ &= \omega_\pi(x/t)\sum_\chi \int J_\pi(xy/t^2, \chi)\tau_F(-t^{-1}(x+y))\xi(y)d^*y .\end{aligned}$$

To compute the same function from the left hand side of (33) we write

(36) $$\begin{aligned}\eta_t &= \pi(wu(t)w^{-1})\xi = \pi(w)[\pi(u(t))\pi(w^{-1})\xi - \pi(w^{-1})\xi] + \xi \\ &= \omega_\pi(-1)\pi(w)[\pi(u(t))\pi(w)\xi - \pi(w)\xi] + \xi\end{aligned}$$

and observe that $\pi[u(t)]\pi(w)\xi - \pi(w)\xi$ belongs to $\mathcal{S}_\chi(F^*)$ although $\pi(w)\xi$ may not. Using (32) twice we thus get

(37) $$\begin{aligned}\eta_t(x) &= \xi(x) + \omega_\pi(-x)\sum_{\chi'}\int J_\pi(xz, \chi')[\tau_F(tz)-1]\omega_\pi(z)d^*z \times \sum_{\chi''}\int J_\pi(zy, \chi'')\xi(y)d^*y \\ &= \xi(x) + \omega_\pi(-x) \sum_{\chi', \chi''} \int\int J_\pi(xz, \chi')J_\pi(zy, \chi'')\xi(y)[\tau_F(tz)-1]\omega_\pi(z)d^*y d^*z .\end{aligned}$$

If we choose any two $t_1, t_2 \in F^*$ and compute $\eta_{t_1} - \eta_{t_2}$ we thus get

$$\begin{aligned}
(38)\qquad & \omega_\pi(-t_1)^{-1}\sum_\chi \int J_\pi(xy/t_1^2,\ \chi)\tau_F(-t_1^{-1}(x+y))\xi(y)d^*y - \\
& - \omega_\pi(-t_2)^{-1}\sum_\chi \int J_\pi(xy/t_2^2,\ \chi)\tau_F(-t_2^{-1}(x+y))\xi(y)d^*y = \\
& = \sum_{\chi',\chi''} \int\int J_\pi(xz,\ \chi')J_\pi(zy,\ \chi'')\xi(y)[\tau_F(t_1z) - \tau_F(t_2z)]\omega_\pi(z)d^*y d^*z .
\end{aligned}$$

Since the kernels $J_\pi(xy/t^2,\ \chi)\tau_F(t^{-1}(x+y))$ in the left hand side are symmetric functions of x and y, the same must be true in the right hand side, i.e. we must have

$$\begin{aligned}
(39)\qquad & \sum_{\chi',\chi''} \int J_\pi(xz,\ \chi')J_\pi(zy,\ \chi'')[\tau_F(t_1z) - \tau_F(t_2z)]\omega_\pi(z)d^*z = \\
& = \sum_{\chi',\chi''} \int J_\pi(zy,\ \chi'')J_\pi(xz,\ \chi')[\tau_F(t_1z) - \tau_F(t_2z)]\omega_\pi(z)d^*z .
\end{aligned}$$

Looking at the way both sides transform under $x \longmapsto xu$ or $y \longmapsto yu$ for $u \in E_F$, we see that for any two characters χ' and χ'' of E_F the function

(40) $$\varphi(z) = \omega_\pi(z)[J_\pi(xz,\ \chi')J_\pi(yz,\ \chi'') - J_\pi(yz,\ \chi'')J_\pi(xz,\ \chi')]$$

(where x and y are arbitrary elements of F^*) must satisfy condition

(41) $$\int \varphi(z)[\tau_F(t_1z) - \tau_F(t_2z)]d^*z = 0 \quad \text{for all } t_1,\ t_2 \in F^* ;$$

since $\varphi(z)a$, for every $a \in X$, is a locally constant function on F^* which vanishes outside a compact subset of F, it follows at once from (41) that $\varphi = 0$, which concludes the proof of the lemma.

Lemma 6. The space X is one-dimensional.

We first prove that

(42) $$\mathcal{V} = \mathcal{V}_* + \pi(w)\mathcal{V}_* \quad \text{where } \mathcal{V}_* = \mathcal{S}_X(F^*).$$

In fact $\mathcal{V}$ is spanned by the subspaces $\pi(g)\mathcal{V}_*$ since it is irreducible; but $\mathcal{V}_*$ is stable under the operators $\pi\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$; Bruhat's decomposition thus shows that $\mathcal{V}$ is sum of $\mathcal{V}_*$ and the subspaces $\pi(u(t)w)\mathcal{V}_*$. Since we know that $\pi(u(t)w)\xi - \pi(w)\xi$ belongs to $\mathcal{V}_*$ for every $t \in F$ and every $\xi \in \mathcal{V}$, our assertion follows.

We now prove that if a linear operator A on X commutes with the $J_\pi(x, \chi)$ then A is a scalar. To see that we denote by T_A the operator (defined on functions $F^* \longrightarrow X$) given by $\eta(x) = A(\xi(x))$ if $\eta = T_A\xi$; we shall prove first of all that $\mathcal{V}$ is invariant under T_A and that T_A induces in $\mathcal{V}$ an operator which commutes with π (hence a scalar, from which it will evidently follow that A itself is a scalar). In fact any $\xi \in \mathcal{V}$ can be written, as we have just seen, as $\xi = \xi' + \pi(w)\xi''$ with two functions ξ' and ξ'' in $\mathcal{S}_X(F^*)$. We then have

$$\begin{aligned} T_A\pi(w)\xi(x) &= A[\pi(w)\xi'(x) + \omega_\pi(-1)\xi''(x)] \\ &= \omega_\pi(x)A[\Sigma \int J_\pi(xy, \chi)\xi'(y)d^*y] + \omega_\pi(-1)A(\xi''(x)), \end{aligned} \tag{43}$$

and since A commutes with the $J_\pi(x, \chi)$ we see (use the fact that the series and integrals above are actually finite sums) that

$$T_A\pi(w)\xi(x) = \omega_\pi(x)\Sigma \int J_\pi(xy, \chi)A(\xi'(y))d^*y + \omega_\pi(-1)A(\xi''(x)); \tag{44}$$

since the function $x \longmapsto A(\xi(x))$ is still in $\mathcal{S}_X(F^*)$ for every $\xi \in \mathcal{S}_X(F^*)$, this can be written as $T_A\pi(w)\xi = \pi(w)T_A\xi' + \omega_\pi(-1)T_A\xi'' = \pi(w)T_A\xi$, which concludes the second part of the proof.

Finally, the above argument shows that in particular all operators $J_\pi(x, \chi)$ are scalars, hence that every linear operator A on X commutes with the $J_\pi(x, \chi)$, hence is a scalar. This implies $\dim(X) = 1$, q.e.d.

4. The finiteness property

Since X is one dimensional we may identify it (in a non canonical way) with $\underline{C}$, and replace $\mathcal{S}_X(F^*)$ by $\mathcal{S}(F^*)$. We thus get an identification of $\mathcal{V}$ with a space of complex-valued functions on F^* (in fact locally constant and zero outside compact subsets of F) on which π operates in such a way that $\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi(x) = \tau_F(bx)\xi(ax)$; and we know that $\mathcal{S}(F^*) \subset \mathcal{V}$. We are now going to prove that $\dim(\mathcal{V}/\mathcal{S}(F^*))$ is finite. Because of (42) it would of course be enough to prove that

$$\dim[\mathcal{V}_* / \pi(w)\mathcal{V}_* \cap \mathcal{V}_*] < +\infty ; \tag{44}$$

if we denote by $\mathcal{V}_*(\chi) = \mathcal{S}(F^*, \chi)$, for every character χ of E_F, the subspace of all $\xi \in \mathcal{V}_*$ such that $\xi(xu) = \xi(x)\chi^{\cdot}(u)$, then in order to prove (44) it will clearly be enough to prove the following two lemmas:

Lemma 7. The space $\pi(w)\mathcal{V}_* \cap \mathcal{V}_*(\chi)$ has finite codimension in $\mathcal{V}_*(\chi)$ for every character χ of E_F.

Lemma 8. The space $\pi(w)\mathcal{V}_*$ contains $\mathcal{V}_*(\chi)$ for almost all characters χ of E_F.

To prove lemma 7 we observe that it is actually enough to show that the subspace $\mathcal{H} = \pi(w)\mathcal{V}_* \cap \mathcal{V}_*(\chi)$ of $\mathcal{V}_*(\chi) = \mathcal{S}(F^*, \chi)$ is non zero. In fact every linear form λ on $\mathcal{S}(F^*, \chi)$ is given by a formula

(45) $$\lambda(\xi) = \sum_{n \in \mathbb{Z}} \lambda_n \xi(\varpi^n)$$

where ϖ is a generator of $\mathfrak{p}$, and where the λ_n are arbitrary complex coefficients. Since $\mathcal{H}$ is invariant under (multiplicative) translations, it is clear that if $\mathcal{H} \neq 0$ then all λ orthogonal to $\mathcal{H}$ satisfy a non trivial recursive relation

(46) $$\sum_1^p a_i \lambda_{n-i} = 0 \ .$$

But the space of solutions λ of (46) is finite-dimensional, hence the lemma.

Before we start the proof of the fact that

$$\pi(w)\mathcal{V}_* \cap \mathcal{V}_*(\chi) \neq 0$$

we observe that

(47) $$\pi[u(t)w^{-1}]\xi - \pi(w^{-1})\xi - \pi[h(t)u(-1/t)]\xi \in \mathcal{V}_* \cap \pi(w)\mathcal{V}_*$$

for all $t \in F^*$ and $\xi \in \mathcal{V}_*$. First of all it is clear that $\mathcal{V}_*$ contains $\pi[u(t)w^{-1}]\xi - \pi(w^{-1})\xi$ and $\pi[h(t)u(-1/t)]\xi$, hence it remains to prove that the left hand side belongs to $\pi(w)\mathcal{V}_* = \pi(w^{-1})\mathcal{V}_*$, i.e. that

(48) $$\pi[wu(t)w^{-1}] - \xi - \pi[wh(t)u(-1/t)]\xi \in \mathcal{V}_* \ ;$$

but this follows from (33) since we can then write the above expression as $\pi[u(-1/t)]\eta - \eta - \xi$, with $\eta = \pi[wh(t)u(-1/t)]\xi$. Hence (47).

The value of (47) at $x \in F^*$ is

(49) $$\omega_\pi(-1)[\tau_F(tx)-1]\pi(w)\xi(x) - \omega_\pi(t^{-1})\tau_F(-tx)\xi(t^2x);$$

since $\pi(w)\mathcal{V}_* \cap \mathcal{V}_*$ is invariant under the operators $h(t)$, we conclude that for all $t \in F^*$, $\xi \in \mathcal{V}_*$ and characters χ of E_F the space $\pi(w)\mathcal{V}_* \cap \mathcal{V}_*(\chi)$ contains the function

(50) $$x \longmapsto \omega_\pi(-1)\int[\tau_F(txu)-1]\pi(w)\xi(xu)\cdot\bar{\chi}(u)d^*u - \omega_\pi(t^{-1})\int\tau_F(-txu)\xi(t^2xu)\bar{\chi}(u)d^*u.$$

Choosing

(51) $$\xi(x) = \chi'_*(x) = \begin{cases} \chi'(x) & \text{if } x \in E_F \\ 0 & \text{if } x \notin E_F \end{cases}$$

where χ' is another character of E_F, we have

(52) $$\pi(w)\xi(x) = \omega_\pi(x)J_\pi(x, \chi')$$

and thus see that $\pi(w)\mathcal{V}_* \cap \mathcal{V}_*(\chi)$ contains the function

(53) $$\begin{aligned} x \longmapsto \omega_\pi(-x)J_\pi(x, \chi')\int[\tau_F(txu)-1]\omega_\pi(u)\overline{\chi\chi'}(u)d^*u - \\ - \omega_\pi(t^{-1})\chi'_*(t^2x)\int\tau_F(-txu)\chi'\bar{\chi}(u)\cdot d^*u \end{aligned}$$

i.e. the function

(54) $$\begin{aligned} \omega_\pi(-x)J_\pi(x, \chi')[\gamma(tx, \omega_\pi\overline{\chi\chi'}) - \delta(\omega_\pi\overline{\chi\chi'})] - \\ - \omega_\pi(t^{-1})\chi'_*(t^2x)\gamma(-tx, \bar{\chi}\chi') \end{aligned}$$

with Gaussian sums γ given by (20), and the obvious meaning for the Dirac symbol δ. We shall show that (if $\mathcal{V}_* \neq \mathcal{V}$) it is always possible to choose χ' such that (54) does not identically vanish; this will prove lemma 7, as we have seen. Hence the lemma will be proved if we can choose x, χ' and t in such a way that

(55) $$J_\pi(x, \chi') \neq 0, \quad \gamma(tx, \omega\overline{\chi\chi'}) - \delta(\omega\overline{\chi\chi'}) \neq 0, \quad 2v(t) + v(x) \neq 0.$$

Now for every character χ' of E_F there is at least one integer $n(\chi')$ such that

(56) $$v(x) = n(\chi') \Longrightarrow \gamma(tx, \omega\overline{\chi\chi'}) - \delta(\omega\overline{\chi\chi'}) \neq 0$$

[and in fact exactly one if $\chi' \neq \omega\bar{\chi}$]. The problem thus is to choose x, χ' and t such that

(57) $$2v(t) + v(x) \neq 0, \quad v(t) + v(x) = n(\chi'), \quad J_\pi(x, \chi') \neq 0 .$$

But if we have $v(t) + v(x) = n(\chi')$ and $2v(t) + v(x) = 0$ then $v(x) = 2n(\chi')$. Hence the problem is to choose χ' and x such that

(58) $$J_\pi(x, \chi') \neq 0 \quad \text{and} \quad v(x) \neq 2n(\chi') .$$

If it is not possible then all functions $J_\pi(x, \chi')$ belong to $\mathcal{S}(F^*)$, and we evidently have then $\pi(w)\mathcal{V}_* \subset \mathcal{V}_*$ i.e. $\mathcal{V}_* = \mathcal{V}$, which contradicts our assumption [or proves the lemma!].

We still have to prove lemma 8. This is clear if $\pi(w)\mathcal{V}_* \cap \mathcal{V}_*(\chi)$ contains a function whose support reduces to one single class mod E_F. To prove that such is the case for almost every χ, we consider the function (54) with $\chi' = \mathrm{id}$, and assume the conductors of $\omega_\pi\bar{\chi}$ and $\bar{\chi}$ are the same (which is true as soon as the conductor of χ is large enough, hence for almost all χ). Let $\mathfrak{P}^f$ be this conductor. In the expression (54), which now reduces to

(59) $$\omega_\pi(-x)J_\pi(x, \mathrm{id})\gamma(tx, \omega_\pi\bar{\chi}) - \omega_\pi(t^{-1})\mathrm{id}_*(t^2x)\gamma(-tx, \bar{\chi}),$$

the second term is 0 except if

(60) $$2v(t) + v(x) = 0, \quad v(t) + v(x) = -d - f ,$$

i.e. except if $v(t) = d + f$ and $v(x) = -2(d+f)$. But $J(x, \mathrm{id}) = 0$ if $v(x)$ is large negative. If f is large enough and if t is such that $v(t) = d+f$, we thus see that (59) is non zero if and only if $v(x) = -2(d+f)$; this concludes the proof since there are finitely many χ such that $f \leq f_0$.

To conclude the proof of theorem 1 we still have to prove the uniqueness of the space $\mathcal{V}'$ and of the representation π' on $\mathcal{V}'$. If we use again temporarily the notation of theorem 1 then it is clear that all we need to prove is that there is, up to a constant factor, at most one mapping $\xi \longmapsto \xi'$ from $\mathcal{V}$ to a space of complex valued functions on F^* such that

(61) $$\eta = \pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi \Longrightarrow \eta'(x) = \tau_F(bx)\xi'(ax) .$$

But consider, for such a mapping, the linear form $L(\xi) = \xi'(1)$ on $\mathcal{V}$; we clearly have

(62) $$\xi'(x) = L[\pi\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\xi] ,$$

as well as

(63) $$L[\pi\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\xi] = \tau_F(b)L(\xi) .$$

By (62) the map $\xi \longmapsto \xi'$ is uniquely determined by L. Hence it is enough to prove there is on $\mathcal{V}$ (up to a constant factor) at most one linear form L satisfying (63). But as we have seen after the statement of theorem 1 such a linear form vanishes on the subspace $\mathcal{V}_o$. Since $\dim(\mathcal{V}/\mathcal{V}_o) = 1$, the result follows.

5. <u>Whittaker functions</u>

Let π be an irreducible admissible representation of G_F. If the space $\mathcal{V}$ of π is made up of complex valued functions on F^* on which π operates in such a way that $\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi(x) = \tau_F(bx)\xi(ax)$, then π will be called a <u>Kirillov representation</u> of G_F (or the <u>Kirillov model</u> of the corresponding class of irreducible representations), and the space $\mathcal{V}$ of π will be denoted by $\mathcal{K}(\pi)$. Each class of irreducible admissible representations of G_F contains exactly <u>one</u> Kirillov representation.

Let π be a Kirillov representation of G_F. For every $\xi \in \mathcal{K}(\pi)$ consider the function

(64) $$W_\xi(g) = \pi(g)\xi(1) = L[\pi(g)\xi]$$

on G_F; we get a bijection $\xi \longmapsto W_\xi$ of $\mathcal{K}(\pi)$ on a space $\mathcal{W}(\pi)$ of functions on G_F satisfying

(65) $$W[\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}g] = \tau_F(x)W(g)$$

and locally constant; clearly π acts on $\mathcal{W}(\pi)$ through right translations. The elements of $\mathcal{W}(\pi)$ will be called the Whittaker functions of π, and $\mathcal{W}(\pi)$ will be the Whittaker space of π.

If π is an irreducible admissible representation on an "abstract" vector space $\mathcal{V}$, then as we have seen there is on $\mathcal{V}$ essentially one non zero linear form L such that

(66) $$L[\pi\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}\varphi] = \tau_F(x)L(\varphi)$$

for all $x \in F$ and $\varphi \in \mathcal{V}$; and the choice of such an L defines an isomorphism $\varphi \longmapsto \xi_\varphi$ of $\mathcal{V}$ on the Kirillov space $\mathcal{K}(x)$ of π, given by

(67) $$\xi_\varphi(x) = L[\pi\begin{pmatrix}x & 0\\0 & 1\end{pmatrix}\varphi] .$$

The Whittaker function $W_\xi = W_\varphi$ is then given by

(68) $$W(g) = \pi(g)\xi_\varphi(1) = L[\pi(g)\varphi] .$$

In particular, suppose $\mathcal{V}$ is contained in the space of solutions of (65) and that G_F operates on $\mathcal{V}$ through right translations. We may then choose for L the linear form $L(\varphi) = \varphi(e)$; it satisfies (66) because each $\varphi \in \mathcal{V}$ satisfies (65), and it is not zero everywhere on $\mathcal{V}$ because $\varphi(g) = L[\pi(g)\varphi]$ since $\pi(g)$ is the right translation defined by g. We then have $W_\varphi = \varphi$, and thus $\mathcal{V} = \mathcal{W}(\pi)$. In other words we get the following

Corollary of Theorem 1. Let π be an irreducible admissible infinite dimensional representation of G_F. Then there is in the set of solutions of

(69) $$W[\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}g] = \tau_F(x)W(g)$$

one and only one right invariant subspace on which the right translations define a representation isomorphic to π, namely the Whittaker space $\mathcal{W}(\pi)$ of π.

This result will play a fundamental role in the applications to automorphic forms.

6. A theorem on the contragredient of a representation

Let π be an admissible representation of G_F on a vector space $\mathcal{V}$, and suppose we have another admissible representation π' on another vector space $\mathcal{V}'$, as well as a non degenerate bilinear form $\langle\xi, \xi'\rangle$ on $\mathcal{V} \times \mathcal{V}'$ such that

(70) $$\langle\pi(g)\xi, \pi'(g)\xi'\rangle = \langle\xi, \xi'\rangle .$$

Then π' is isomorphic to the contragredient $\check{\pi}$ of π defined in no. 1. In fact we get from (70) an homomorphism from $\mathcal{V}'$ into $\check{\mathcal{V}}$ by associating to every $\xi' \in \mathcal{V}'$ the linear form $\xi \longmapsto \langle\xi, \xi'\rangle$ on $\mathcal{V}$; and this homomorphism transforms π' into $\check{\pi}$. Hence it remains to prove that it is bijective. But we have $\mathcal{V}' = \oplus\ \mathcal{V}'(\vartheta)$ and $\check{\mathcal{V}} = \oplus\ \check{\mathcal{V}}(\vartheta)$ as in no. 1, and the homomorphism of $\mathcal{V}'$ into $\check{\mathcal{V}}$ evidently maps $\mathcal{V}'(\vartheta)$ into $\check{\mathcal{V}}(\vartheta)$ for every ϑ. On the other hand, since the canonical bilinear form on $\mathcal{V} \times \check{\mathcal{V}}$ and the given form on $\mathcal{V} \times \mathcal{V}'$ are invariant and non degenerate, it is clear that we can identify $\check{\mathcal{V}}(\vartheta)$ and $\mathcal{V}'(\vartheta)$ with the dual of the finite dimensional vector space $\mathcal{V}(\check{\vartheta})$, where $\check{\vartheta}$ is the contragredient of ϑ in the usual sense. Hence the homomorphism $\mathcal{V}' \longrightarrow \check{\mathcal{V}}$ under consideration induces a bijection $\mathcal{V}'(\vartheta) \longrightarrow \check{\mathcal{V}}(\vartheta)$ for every ϑ, which shows that it is an isomorphism as was to be proved.

We shall now use these trivial remarks and theorem 1 to prove

Theorem 2. *Let π be an irreducible admissible representation of G_F. Then $\check{\pi}$ is equivalent to the representation*[(*)] $g \longmapsto \omega_\pi(g)^{-1}\pi(g)$, *and the Kirillov space $\mathcal{K}(\check{\pi})$ is the set of functions* $\omega_\pi(x)^{-1}\xi(x)$ *with* $\xi \in \mathcal{K}(\pi)$. *Furthermore the invariant duality between $\mathcal{K}(\pi)$ and $\mathcal{K}(\check{\pi})$ is given by the bilinear form $\langle\xi, \eta\rangle$ such that*

(71) $$\langle\xi, \eta\rangle = \int \xi_1(x)\cdot\eta(-x)d^*x + \int \xi_2(x)\cdot\check{\pi}(w)\eta(-x)d^*x$$

if $\xi = \xi_1 + \pi(w)\xi_2$ *with* $\xi_1, \xi_2 \in \mathcal{S}(F^*)$ *and* $\eta \in \mathcal{K}(\check{\pi})$.

(*) We put $\omega(g) = \omega(\det g)$ for every $g \in G_F$ and every character ω of F^*.

To prove that $\check{\pi}$ is equivalent to the representation

$$\pi'(g) = \omega_\pi(g)^{-1}\pi(g) \tag{72}$$

it is enough (since π is admissible and irreducible) to construct on $\mathcal{K}(\pi)$ a non degenerate bilinear form $\langle \xi, \eta \rangle_\pi$ such that $\langle \pi(g)\xi, \pi'(g)\eta\rangle_\pi = \langle \xi, \eta\rangle_\pi$. The construction and study of this form will be cut into several steps. In what follows we put $\mathcal{V} = \mathcal{K}(\pi)$ and $\mathcal{V}_* = \mathcal{S}(F^*) \subset \mathcal{V}$ as in the proof of lemma 6.

Step 1. We define

$$\langle \xi, \eta\rangle_\pi = \int \xi(x)\eta(-x)\omega_\pi(x)^{-1}d^*x \quad \text{if} \quad \xi \in \mathcal{V}_*, \ \eta \in \mathcal{V}; \tag{73}$$

the integral converges in a trivial way. We first show that

$$\langle \pi(w)\xi, \eta\rangle_\pi = \langle \xi, \pi(w)^{-1}\eta\rangle_\pi \quad \text{if} \quad \xi \in \mathcal{V}_* \cap \pi(w)\mathcal{V}_* \quad \text{and} \quad \eta \in \mathcal{V}_*. \tag{74}$$

In fact we have by no. 3

$$\begin{aligned} \pi(w)^{-1}\eta(x) &= \omega_\pi(-1)\pi(w)\eta(x) = \\ &= \omega_\pi(-x)\sum_\chi \int J_\pi(xy, \chi)\eta(y)d^*y \end{aligned} \tag{75}$$

since $\eta \in \mathcal{S}(F^*)$, and thus

$$\begin{aligned} \langle \xi, \pi(w)^{-1}\eta\rangle_\pi &= \int \xi(x)\omega_\pi^{-1}(x)d^*x \cdot \omega_\pi(x)\Sigma\int J_\pi(-xy, \chi)\eta(y)d^*y \\ &= \Sigma\int\int J_\pi(-xy, \chi)\xi(x)\eta(y)d^*x d^*y \\ &= \int \eta(-y)\omega_\pi(y)^{-1}d^*y \cdot \omega_\pi(y)\Sigma\int J_\pi(xy, \chi)\xi(x)d^*x \\ &= \int \pi(w)\xi(y)\cdot\eta(-y)\omega_\pi(y)^{-1}d^*y = \langle \pi(w)\xi, \eta\rangle_\pi \end{aligned} \tag{76}$$

hence the result. Note that this kind of formal computation is justified as soon as $\xi, \eta \in \mathcal{S}(F^*)$, because the summation over the characters χ of E_F is actually a finite sum.

Step 2. We now observe that $\mathcal{V} = \mathcal{V}_* + \pi(w)\mathcal{V}_*$ and define $\langle \xi, \eta\rangle_\pi$ on the whole of $\mathcal{V} \times \mathcal{V}$ by

$$\langle \xi, \eta\rangle_\pi = \langle \xi_1, \eta\rangle_\pi + \langle \xi_2, \pi(w)^{-1}\eta\rangle_\pi \tag{77}$$

if $\xi = \xi_1 + \pi(w)\xi_2$ with $\xi_1, \xi_2 \in \mathcal{V}_*$, and $\eta \in \mathcal{V}$. This definition makes sense because if $\xi_1 + \pi(w)\xi_2 = 0$ then $\xi_1 \in \mathcal{V}_* \cap \pi(w)\mathcal{V}_*$, so that if we write $\eta = \eta_1 + \pi(w)\eta_2$ with $\eta_1, \eta_2 \in \mathcal{V}_*$ we get

$$
\begin{aligned}
&\langle\xi_1, \eta\rangle_\pi + \langle\xi_2, \pi(w)^{-1}\eta\rangle_\pi = \\
&= \langle\xi_1, \eta_1\rangle_\pi + \langle\xi_1, \pi(w)\eta_2\rangle_\pi + \langle\xi_2, \pi(w)^{-1}\eta_1\rangle_\pi + \langle\xi_2, \eta_2\rangle_\pi \\
&= \langle\xi_1, \eta_1\rangle_\pi + \langle\pi(w)^{-1}\xi_1, \eta_2\rangle_\pi + \langle\pi(w)\xi_2, \eta_1\rangle_\pi + \langle\xi_2, \eta_2\rangle_\pi \\
&= \langle\xi_1, \eta_1\rangle_\pi - \langle\xi_2, \eta_2\rangle_\pi - \langle\xi_1, \eta_1\rangle_\pi + \langle\xi_2, \eta_2\rangle_\pi = 0 ;
\end{aligned}
\tag{78}
$$

we have of course made use of step 1.

Step 3. We prove that

$$\langle\pi(w)\xi, \eta\rangle_\pi = \langle\xi, \pi(w)^{-1}\eta\rangle_\pi \tag{79}$$

for all $\xi, \eta \in \mathcal{V}$. In fact if we write $\xi = \xi_1 + \pi(w)\xi_2$ and apply definition (77), we get

$$
\begin{aligned}
\langle\pi(w)\xi, \eta\rangle_\pi &= \langle\xi_1, \pi(w)^{-1}\eta\rangle_\pi + \omega_\pi(-1)\langle\xi_2, \eta\rangle_\pi \\
\langle\xi, \pi(w)^{-1}\eta\rangle_\pi &= \langle\xi_1, \pi(w)^{-1}\eta\rangle_\pi + \langle\xi_2, \pi(w)^{-2}\eta\rangle_\pi ,
\end{aligned}
\tag{80}
$$

hence the result.

Step 4. Computation (76) shows that

$$\int \pi(w)\xi(x)\cdot\eta(-x)\omega_\pi(x)^{-1}d^*x = \int \xi(x)\cdot\pi(w^{-1})\eta(-x)\cdot\omega_\pi(x)^{-1}d^*x \tag{81}$$

for any two $\xi, \eta \in \mathcal{V}_*$. The right hand side is $\langle\xi, \pi(w)^{-1}\eta\rangle_\pi$ by (73), hence $\langle\pi(w)\xi, \eta\rangle_\pi$ by (77); hence formula (73) is still valid if $\xi \in \pi(w)\mathcal{V}_*$ provided $\eta \in \mathcal{V}_*$, from which we conclude that we still have

$$\langle\xi, \eta\rangle_\pi = \int \xi(x)\eta(-x)\omega_\pi(x)^{-1}d^*x \quad \text{if } \xi \in \mathcal{V},\ \eta \in \mathcal{V}_* . \tag{82}$$

Step 5. We prove that

$$\langle\pi\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\xi, \eta\rangle_\pi = \langle\xi, \pi'\begin{pmatrix} a^{-1} & 0 \\ 0 & 1\end{pmatrix}\eta\rangle_\pi \tag{83}$$

for all $\xi, \eta \in \mathcal{V}$, i.e. that

$$\langle\pi\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\xi, \eta\rangle_\pi = \omega_\pi(a)\langle\xi, \pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1\end{pmatrix}\eta\rangle_\pi . \tag{84}$$

If $\xi \in \mathcal{V}_*$ we can use (73) to compute both sides and then the result is obtained at once by replacing $\xi(x)$ by $\xi(ax)$ in (73) and then x by $a^{-1}x$ in the integral. If $\xi \in \mathcal{V}$ is not in $\mathcal{V}_*$ we are (with obvious notation) reduced to proving that

(85) $$<\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\pi(w)\xi_2,\ \eta>_\pi = \omega_\pi(a)<\pi(w)\xi_2,\ \pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\eta>_\pi ;$$

but

$$<\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\pi(w)\xi_2,\ \eta>_\pi = <\pi(w)\pi\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\xi_2,\ \eta>_\pi =$$

$$= <\pi\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\xi_2,\ \pi(w)^{-1}\eta>_\pi \text{ by Step 3}$$

$$= \omega_\pi(a)<\pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\xi_2,\ \pi(w)^{-1}\eta>_\pi$$

(86) $$= \omega_\pi(a)\cdot\omega_\pi(a^{-1})<\xi_2,\ \pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\pi(w)^{-1}\eta>_\pi \text{ because } \xi_2 \in \mathcal{V}_*$$

$$= <\xi_2,\ \pi(w)^{-1}\pi\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\eta>_\pi$$

$$= <\pi(w)\xi_2,\ \pi\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\eta>_\pi \text{ by Step 3}$$

$$= \omega_\pi(a)<\pi(w)\xi_2,\ \pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\eta>_\pi, \qquad \text{q.e.d.}$$

Step 6. We prove that

(87) $$<\pi(u)\xi,\ \eta>_\pi = <\xi,\ \pi(u)^{-1}\eta>_\pi \quad \text{for all } \xi, \eta \in \mathcal{V}$$

if $u = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Since $\pi(u)\xi(x) = \tau_F(bx)\xi(x)$, this formula is clear if we can compute the scalar products by means of (73), i.e. if ξ or η belongs to $\mathcal{V}_*$ (Step 4). It is thus clear that we are reduced to prove

(88) $$<\pi(u)\pi(w)\xi_2,\ \pi(w)\eta_2>_\pi = <\pi(w)\xi_2,\ \pi(u)^{-1}\pi(w)\eta_2>_\pi$$

in case $\xi_2, \eta_2 \in \mathcal{V}_*$, which by Step 3 reduces to

(89) $$<\pi(w^{-1}uw)\xi_2,\ \eta_2>_\pi = <\xi_2,\ \pi(w^{-1}uw)^{-1}\eta_2>_\pi .$$

Now we have

$$w^{-1}uw = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} = \begin{pmatrix} 1 & -b^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix}w\begin{pmatrix} 1 & -b^{-1} \\ 0 & 1 \end{pmatrix}$$

(90) $$= \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & -b^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b^{-2} & 0 \\ 0 & 1 \end{pmatrix}w\begin{pmatrix} 1 & -b^{-1} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}u'hwu'$$

with obvious notations. Hence

$$\begin{aligned}
<\pi(w^{-1}uw)\xi_2, \eta_2>_\pi &= \omega_\pi(b)<\pi(u'hwu')\xi_2, \eta_2>_\pi \\
&= \omega_\pi(b)<\pi(hwu')\xi_2, \pi(u')^{-1}\eta_2>_\pi \quad \text{because } \eta_2 \in \mathcal{V}_* \\
&= <\pi(wu')\xi_2, \pi(h)^{-1}\pi(u')^{-1}\eta_2>_\pi \quad \text{by Step 5} \qquad (91)\\
&= <\pi(u')\xi_2, \pi(w)^{-1}\pi(h)^{-1}\pi(u')^{-1}\eta_2>_\pi \quad \text{by Step 3} \\
&= <\xi_2, \pi(u')^{-1}\pi(w)^{-1}\pi(h)^{-1}\pi(u')^{-1}\eta_2>_\pi \quad \text{because } \xi_2 \in \mathcal{V}_*,
\end{aligned}$$

hence the result.

To conclude the proof of the theorem, we observe that we have proved identity

$$(92) \qquad <\pi(g)\xi, \eta>_\pi = <\xi, \pi'(g)^{-1}\eta>_\pi \qquad (\xi, \eta \in \mathcal{V})$$

for matrices g which generate G_F, so that it is valid for all $g \in G_F$. On the other hand the bilinear form $<\xi, \eta>_\pi$ on $\mathcal{K}(\pi) \times \mathcal{K}(\pi)$ is non degenerate, because a function orthogonal to $\mathcal{V}_*$ is zero by formula (81).

We have thus shown thus far that $\check{\pi}$ is equivalent to π'. To conclude the proof, we observe that we have

$$(93) \qquad \pi'\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi(x) = \omega_\pi(a)^{-1}\tau_F(bx)\xi(ax)$$

for all $\xi \in \mathcal{K}(\pi)$. To get the Kirillov model of π' or $\check{\pi}$ we must get rid of the factor $\omega_\pi(a)$ in the above formula, which can be done at once by transforming $\mathcal{K}(\pi)$ and π' under the mapping T_π given by $T_\pi\xi(x) = \omega_\pi(x)^{-1}\xi(x)$.

Since a given representation has only one Kirillov realization, it follows that $\mathcal{K}(\check{\pi}) = T_\pi(\mathcal{K}(\pi))$ and that the Kirillov realization of $\check{\pi}$ is given by

$$(94) \qquad \check{\pi}(g) = T_\pi \circ \pi'(g) \circ T_\pi^{-1} = \omega_\pi(g)^{-1} T_\pi \circ \pi(g) \circ T_\pi^{-1}.$$

Since the duality $<\xi, \eta>$ between $\mathcal{K}(\pi)$ and $\mathcal{K}(\pi')$ is given by $<\xi, \eta> = <\xi, T_\pi^{-1}\eta>_\pi$, the proof of the theorem is now complete.

7. Supercuspidal representations

Let π be a given irreducible admissible representation of G_F. We shall say that π is <u>supercuspidal</u> if $\mathcal{K}(\pi) = \mathcal{S}(F^*)$, i.e. if all functions $\xi(x)$ in the Kirillov model $\mathcal{K}(\pi)$ of π vanish around 0.

It is easy to see that an equivalent property is the fact that for every $\xi \in \mathcal{K}(\pi)$ we have

(95) $$\int_{\mathfrak{P}^{-n}} \pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\xi \cdot dx = 0 \quad \text{for } n \text{ large.}$$

In fact it is clear in all cases that the left hand side of (95) is the function

(96) $$y \longmapsto \int_{\mathfrak{P}^{-n}} \tau_F(xy)\xi(y)dx = \xi(y) \int_{\mathfrak{P}^{-n}} \tau_F(xy)dx \ ;$$

if $\mathfrak{P}^{-d}$ is the largest ideal on which τ_F is trivial, then

(97) $$\int_{\mathfrak{P}^{-n}} \tau_F(xy)dx \neq 0 \Longleftrightarrow y \in \mathfrak{P}^{n-d} \ ;$$

for the expression (96) to be identically zero for n large, it is thus necessary and sufficient that $\xi(y) = 0$ in some neighborhood of zero, hence the result. If π is a supercuspidal representation then $\mathcal{K}(\check{\pi}) = \mathcal{S}(F^*)$ since $\mathcal{K}(\check{\pi})$ is obtained by multiplying all functions $\xi \in \mathcal{K}(\pi) = \mathcal{S}(F^*)$ by the locally constant function $\omega_\pi(x)$. Hence $\check{\pi}$ is also supercuspidal, and the invariant duality between $\mathcal{K}(\pi)$ and $\mathcal{K}(\check{\pi})$ reduces here to the bilinear form

(98) $$\langle \xi, \eta \rangle = \int_{F^*} \xi(x)\eta(-x)d^*x$$

on $\mathcal{S}(F^*)$, which thus satisfies

(99) $$\langle \pi(g)\xi, \check{\pi}(g)\eta \rangle = \langle \xi, \eta \rangle \ .$$

Let Z_F be the center of G_F (of course Z_F is isomorphic to F^*). Then for any two $\xi, \eta \in \mathcal{K}(\pi)$ the "coefficient" $\langle \pi(g)\xi, \eta \rangle$ of π is a locally constant function on G_F, whose support is <u>compact mod</u> Z_F. In fact we have $G_F = M_F H_F M_F$ where M_F is compact and H_F is the diagonal subgroup of G_F, which is the product of Z_F and the subgroup $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$; but

(100) $$\langle \pi\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\xi, \eta \rangle = \int_{F^*} \xi(tx)\eta(-x)d^*x$$

belongs to $\mathcal{S}(F^*)$ as a function on t; hence the result.

This property of supercuspidal representations is a characteristic one. In fact let π be an irreducible admissible representation on a vector space $\mathcal{V}$, and assume the function $\langle\pi(g)\xi, \eta\rangle$ has compact support mod Z_F for all $\xi \in \mathcal{V}$ and $\eta \in \check{\mathcal{V}}$. We may assume $\mathcal{V} = \mathcal{K}(\pi)$ and $\check{\mathcal{V}} = \mathcal{K}(\check{\pi})$ and take ξ in $\mathcal{S}(F^*)$; the function

(101) $$\langle\pi\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}\xi, \eta\rangle = \int \xi(tx)\eta(-x)d^*x$$

must then vanish outside a compact subset of F^* for all $\xi \in \mathcal{S}(F^*)$ and all $\eta \in \mathcal{K}(\check{\pi})$. Evidently it follows from this condition that $\mathcal{K}(\check{\pi}) = \mathcal{S}(F^*)$, q.e.d. In short:

Theorem 3. *Let π be an irreducible admissible representation of G_F on a vector space $\mathcal{V}$. Then the following conditions are equivalent:*

(i) *π is supercuspidal, i.e.* $\mathcal{K}(\pi) = \mathcal{S}(F^*)$

(ii) $\int_{\mathfrak{p}^{-n}} \pi\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\xi \cdot dx = 0$ *for* n *large for every* $\xi \in \mathcal{V}$

(iii) *the function* $\langle\pi(g)\xi, \eta\rangle$ *has compact support mod* Z_F *for all* $\xi \in \mathcal{V}$, $\eta \in \check{\mathcal{V}}$.

8. Introduction to the principal series

As we shall see, all irreducible non supercuspidal representations of G_F can be explicitly described in a simple way. The first step is to define, for any two characters μ_1, μ_2 of F^*, a representation ρ_{μ_1,μ_2} of G_F as follows: the space $\mathcal{B}_{\mu_1,\mu_2}$ of ρ_{μ_1,μ_2} is the set of all locally constant functions φ on G_F such that

(102) $$\varphi\left[\begin{pmatrix} t' & * \\ 0 & t''\end{pmatrix}g\right] = \mu_1(t')\mu_2(t'')|t'/t''|^{1/2}\varphi(g),$$

and the group operates on $\mathcal{B}_{\mu_1,\mu_2}$ through right translations. We evidently get admissible representations, which we shall refrain from calling the "principal series" because not all of them are irreducible (see theorem 6 below). The first basic fact is the following:

Theorem 4. *If an irreducible admissible representation π of G_F is not* supercuspidal, then it is a subrepresentation of ρ_{μ_1,μ_2} for some choice of μ_1, μ_2.

In fact consider the Kirillov space $\mathcal{K}(\pi) \supset \mathcal{S}(F^*)$ of π. Then $\mathcal{S}(F^*)$ is invariant under the operators $\pi\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, so that they operate on the finite dimensional space $\mathcal{K}(\pi)/\mathcal{S}(F^*)$; furthermore the matrices $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ operate trivially there because $[\tau_F(bx)-1]\xi(x)$ is always in $\mathcal{S}(F^*)$. We conclude at once that if $\mathcal{S}(F^*) \neq \mathcal{K}(\pi)$ there is on $\mathcal{K}(\pi)$ a non zero linear form B and characters μ_1, μ_2 of F^* such that

(103) $$B[\pi\begin{pmatrix} t' & * \\ 0 & t'' \end{pmatrix}\xi] = \mu_1(t')\mu_2(t'')|t'/t''|^{1/2}B(\xi) .$$

We then get an isomorphism of π into ρ_{μ_1,μ_2} by associating to every $\xi \in \mathcal{K}(\pi)$ the function

(104) $$\varphi_\xi(g) = B[\pi(g)\xi] ,$$

which evidently belongs to $\mathcal{B}_{\mu_1,\mu_2}$, q. e. d.

Theorem 5. The contragredient of ρ_{μ_1,μ_2} is $\rho_{-\mu_1,-\mu_2}$ (where $-\mu = \mu^{-1}$).

We need first of all some remarks on invariant measures on groups. Let P be a closed subgroup of a locally compact unimodular group G. If P is unimodular, there is an invariant measure on $P\backslash G$. In the general case, consider the character β_P of P given by

(105) $$d_\ell(pp_0^{-1}) = d_\ell(p_0pp_0^{-1}) = \beta_P(p_0)d_\ell p ,$$

where $d_\ell p$ is a left invariant measure on P. Let L(G, P) be the space of continuous functions on G such that

(106) $$\varphi(pg) = \beta_P(p)\varphi(g)$$

and whose support is compact mod P. Then there exists on L(G, P) essentially one positive linear form which is invariant under right translations. If we denote it by

(107) $$\varphi \longrightarrow \oint_{P\backslash G} \varphi(g)d\dot{g} ,$$

then we have a decomposition formula

(108) $$\int_G \varphi(g)dg = \oint_{P\backslash G} d\dot{g} \int_P \varphi(pg)d_l p$$

for every continuous function φ with compact support on G. Finally if M is a closed subgroup of G such that $P \cap M$ is compact and $G = MP$ up to a set of measure zero, then

(109) $$\oint_{P\backslash G} \varphi(g)d\dot{g} = \int_M \varphi(mx)d_r m \quad \text{for all } x \in G \text{ and } \varphi \in L(G, P) .$$

The "twisted" invariant measure (107) is useful in particular in the following context. Let F and F' be two topological vector spaces in duality, and suppose we are given two continuous representations μ and μ' of P on F and F'; suppose they are contragredient to each other, i.e. that $\langle\mu(p)a, \mu'(p)a'\rangle = \langle a, a'\rangle$ for all $p \in P$, $a \in F$ and $a' \in F'$. Denote by $L(G, P, \mu)$ the vector space of all continuous mappings $\varphi : G \longrightarrow F$ which satisfy

(110) $$\varphi(pg) = \mu(p)\beta_P^{1/2}(p)\varphi(g)$$

and have compact support mod P; define in a similar way the space $L(G, P, \mu')$; these spaces are stable under right translations by elements of G (and right translations in $L(G, P, \mu)$ more or less define the representation of G "induced" by μ). It is now clear that

(111) $$\langle\varphi(pg), \varphi'(pg)\rangle = \beta_P(p)\langle\varphi(g), \varphi'(g)\rangle$$

for all $\varphi \in L(G, P, \mu)$ and $\varphi' \in L(G, P, \mu')$; hence we can define a duality between these two vector spaces by

(112) $$\langle\varphi, \varphi'\rangle = \oint_{P\backslash G} \langle\varphi(g), \varphi'(g)\rangle d\dot{g} ,$$

and this bilinear form is invariant under right translations; this means that the representations of G induced by μ and μ' are more or less (i.e. depending on your definition of "contragredience") contragredient to each other.

We now prove Theorem 5. If we consider the subgroup $P = P_F$ of all triangular matrices $p = \begin{pmatrix} t' & * \\ 0 & t'' \end{pmatrix}$ in G, then we evidently have $\beta_P(p) = |t'/t''|$.

We can thus define a pairing

(113) $$\langle\varphi,\ \psi\rangle = \oint_{P_F\backslash G_F} \varphi(g)\psi(g)d\dot{g} = \int_{M_F} \varphi(m)\psi(m)dm$$

between the spaces $\mathcal{B}_{\mu_1,\mu_2}$ and $\mathcal{B}_{-\mu_1,-\mu_2}$, which satisfies

(114) $$\langle\rho_{\mu_1,\mu_2}(g)\varphi,\ \rho_{-\mu_1,-\mu_2}(g)\psi\rangle = \langle\varphi,\psi\rangle\ .$$

To conclude the proof it remains to prove that the bilinear form $\langle\varphi,\psi\rangle$ is <u>non degenerate</u>. But the restrictions to M_F of the functions $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$ are the locally constant functions on M_F such that

(115) $$\varphi[\begin{pmatrix} u' & v \\ 0 & u'' \end{pmatrix}m] = \mu_1(u')\mu_2(u'')\varphi(m)$$

for all $u', u'' \in E_F$ and $v \in \mathcal{O}_F$; and the restrictions of the $\psi \in \mathcal{B}_{-\mu_1,-\mu_2}$ are similarly characterized, with $\mu_1(u')^{-1} = \overline{\mu_1(u')}$ and $\mu_2(u'')^{-1} = \overline{\mu_2(u'')}$ instead. Hence the restrictions to M_F of the functions ψ are the conjugate functions of the restrictions of the φ. Thus (113) reduces to the $L^2(M_F)$ scalar product, and the proof is now complete.

9. <u>A lemma on Fourier transforms</u>

We are now going to show that there exists a Kirillov model for the representation ρ_{μ_1,μ_2} on the vector space $\mathcal{B}_{\mu_1,\mu_2}$, even though ρ_{μ_1,μ_2} may not be irreducible; this construction will furthermore lead to complete results as to the decomposition of ρ_{μ_1,μ_2}.

Since the "big cell" is everywhere dense in G_F, it is clear from (102) that every $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$ is uniquely determined by the function $x \longmapsto \varphi[w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}]$ on the additive group F; in fact, the decomposition

(116) $$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c^{-1}.\det g & * \\ 0 & c \end{pmatrix} w^{-1} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix} \quad \text{if } c \neq 0$$

shows that

(117) $$\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu_1(\det g)|\det g|^{1/2}\mu^{-1}(c)|c|^{-1}\phi(d/c) \quad \text{if } c \neq 0\ ,$$

where we put

(118) $$\varphi[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}] = \phi(x) .$$

The function $\phi(x)$ is clearly locally constant (in fact it is translation invariant under an open subgroup of F), and its behaviour at infinity is given by

(119) $$\phi(x) = \varphi(e)\mu^{-1}(x)|x|^{-1} \qquad \text{for } |x| \text{ large enough.}$$

In fact we have $w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & -x^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}x^{-1} & 0\\ 0 & x\end{pmatrix}\begin{pmatrix}1 & 0\\ -x^{-1} & 1\end{pmatrix}$ for $x \neq 0$, hence $\varphi[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}] = \mu(x)^{-1}|x|^{-1}\varphi\begin{pmatrix}1 & 0\\ -x^{-1} & 1\end{pmatrix}$ by (102); but since φ is locally constant we have $\varphi\begin{pmatrix}1 & 0\\ -x^{-1} & 1\end{pmatrix} = \varphi(e)$ for $|x|$ large, whence (119). Conversely it is easy to see that every locally constant function ϕ on F such that $\mu(x)|x|\phi(x)$ is constant for $|x|$ large, is given by (118) with a function $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$.

To get a Kirillov model for the representation ρ_{μ_1,μ_2} we shall associate to every $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$ the function

(120) $$\xi_\varphi(x) = \mu_2(x)|x|^{1/2}\int \varphi[w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}]\overline{\tau_F(xy)}dy ,$$

which is nearly the Fourier transform of (118); it will be seen in a moment that this Fourier transform does make sense if we consider ϕ as a distribution(*) and that this Fourier transform is actually a function on F^* (not always on F). Taking that for granted for the time being it is "of course not difficult" to see first that the mapping $\varphi \longmapsto \xi_\varphi$ is injective, and second that if we look upon ρ_{μ_1,μ_2} as a representation of G_F on the space of functions (120), then the fundamental condition for Kirillov's models, namely

(121) $$\rho_{\mu_1,\mu_2}\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\xi(x) = \tau_F(bx)\xi(ax) ,$$

is satisfied.

(*) It even makes sense in the traditional way if $\int_{|x|\geq 1} |\mu(y)^{-1}| d^*y < +\infty$ i.e. if $|\mu(x)| = |x|^\sigma$ with $\sigma > 0$. The case $\sigma < 0$ could be reduced to the previous one by using theorems 2 and 3. Unfortunately the case $\sigma = 0$ cannot be handled in that simple way.

To replace (120) by something more meaningful, we shall need the following lemma:

Lemma 9. Let μ be a character of F^* and let $\mathcal{F}_\mu$ be the space of locally constant functions ϕ on F such that $\phi(x)\mu(x)|x|$ is constant for $|x|$ large. Define

$$(122) \qquad \hat{\phi}(x) = \sum_{n\in\mathbb{Z}} \int_{v(y)=n} \phi(y)\bar{\tau}_F(xy)dy \quad \text{for } x \in F^*.$$

Then the above series converges uniformly on every compact subset of F^*, and the mapping $\phi \longmapsto \hat{\phi}$ is injective except if $\mu(x) = |x|^{-1}$, in which case its kernel is the set of constant functions in $\mathcal{F}_\mu$. The image $\hat{\mathcal{F}}_\mu$ of $\mathcal{F}_\mu$ under $\phi \longmapsto \hat{\phi}$ is the set of locally constant functions ψ on F^* which vanish outside some compact subset of F, and whose behaviour in some neighborhood of 0 is given by the following formulas:

$$(123) \qquad \psi(x) = \begin{cases} a\mu(x) + b & \text{if } \mu(x) \not\equiv 1, \ |x|^{-1} \\ av(x) + b & \text{if } \mu(x) \equiv 1 \\ b & \text{if } \mu(x) \equiv |x|^{-1}, \end{cases}$$

with arbitrary constants a and b.

It is clear that $\mathcal{F}_\mu$ is the direct sum of $\mathcal{S}(F)$ and the one-dimensional subspace spanned by the function

$$(124) \qquad \phi_\mu(x) = \begin{cases} \mu^{-1}(x)|x|^{-1} & \text{if } |x| \geq 1 \\ 0 & \text{if } |x| < 1. \end{cases}$$

The convergence of (122) and the behaviour of $\hat{\phi}$ near 0 and ∞ are clear if $\phi \in \mathcal{S}(F)$, so that the main part of the proof will be for ϕ_μ.

The corresponding series (122) is clearly (up to an immaterial constant factor due to the choice of Haar measures)

$$(125) \qquad \sum_{n\leq 0} \int_{v(y)=n} \bar{\tau}_F(xy)\mu^{-1}(y)d^*y;$$

assume first μ is ramified, and let $\mathfrak{p}^f$ be its conductor. Then

(126) $$\int_{v(y)=n} \bar{\tau}_F(xy)\mu^{-1}(y)d^*y \neq 0 \Longleftrightarrow v(x) = -d-f-n\,;$$

this makes obvious the fact that (125) converges uniformly on every compact subset of F^*, is locally constant, and vanishes for $|x|$ large. Furthermore we have

(127) $$\hat{\phi}_\mu(x) = \int_{v(xy)=-d-f} \bar{\tau}_F(xy)\mu^{-1}(y)d^*y = a\mu(x) \quad \text{for} \quad v(x) \geq -d-f$$

where

(128) $$a = \int_{v(z)=-d-f} \tau_F(z)\mu^{-1}(z)d^*z \neq 0\,,$$

hence (123) in this case for all $\phi \in \mathcal{F}_\mu$.

If now $\mu(x) = |x|^s$ for some s, then we have

(129) $$\int_{v(y)=n} \bar{\tau}_F(xy)|y|^{-s}d^*y = q^{ns}\int_{\mathfrak{o}^*} \bar{\tau}_F(\varpi^n xu)du = q^{ns}\left[\int_{\mathfrak{o}} - \int_{\mathfrak{p}}\right] =$$
$$= q^{ns}[h(\varpi^n x) - |\varpi|h(\varpi^{n+1}x)]$$

where ϖ is a uniformizing variable, $q = N(\mathfrak{p})$, and

(130) $$h(x) = \int_{\mathfrak{o}} \bar{\tau}_F(xu)du = \begin{cases} 1 & \text{if} \quad v(x) \geq -d \\ 0 & \text{if} \quad v(x) < -d\,. \end{cases}$$

The series (122) thus reduces to

(131) $$\hat{\phi}(x) = F_s(x) - |\varpi|F_s(\varpi x)$$

where

(132) $$F_s(x) = \sum_{n\leq 0} q^{ns}h(\varpi^n x) = \sum_{-d-v(x)\leq n\leq 0} q^{ns}\,;$$

it is thus clear that (122) converges uniformly on compact subsets of F^*, is locally constant on F^*, and vanishes if $v(x) < -d-1$. If $q^s \neq 1$ i.e. if μ is non trivial, then we have for $v(x) \geq -d$ a relation $F_s(x) = a'|x|^s + b'$ with $a' \neq 0$; hence $\hat{\phi}(x) = a|x|^s + b$ with $a = a'(1-|\varpi|^{s+1}) \neq 0$ if μ is not the character $x \longmapsto |x|^{-1}$, and $a = 0$ if $\mu(x) = |x|^{-1}$. If $q^s = 1$ then $F_s(x) = v(x)+d+1$ and $\hat{\phi}_\mu(x) = av(x) + b$ with $a = 1 - |\varpi| \neq 0$, for $v(x) \geq -d$.

We have now proved everything except for the determination of the kernel of $\phi \longmapsto \hat{\phi}$. For every $f \in \mathcal{S}(F^*)$ we have

$$\int_{F^*} f(x)\hat{\phi}(x)dx = \sum_n \int_F f(x)dx \int_{v(y)=n} \bar{\tau}_F(xy)\phi(y)dy \tag{133}$$

$$= \sum \int_{v(y)=n} \hat{f}(y)\phi(y)dy = \int_F \hat{f}(y)\phi(y)dy ,$$

which means that the Fourier transform of the distribution $\phi(x)dx$ induces on F^* the measure $\hat{\phi}(x)dx$. If $\hat{\phi} = 0$ we thus see that the Fourier transform of $\phi(x)dx$ must be proportional to the Dirac measure, which means that ϕ must be constant--and this can happen if and only if $\mu(x) = |x|^{-1}$. This concludes the proof.

It is still useful to observe that if $|\mu(x)| = |x|^{\sigma}$ with $\sigma > 0$, then $\hat{\phi}(x) = \int_F \phi(y)\bar{\tau}_F(xy)dy$ with an absolutely convergent integral. If $\sigma > -1/2$ then ϕ is square integrable on F, and $\hat{\phi}$ is its Fourier transform in the L^2 sense. Finally it is clear by (123) that the functions $\hat{\phi}$ are integrable on F provided $\sigma > -1$, and that in this case we have $\phi(x) = \int \hat{\phi}(y)\tau_F(xy)dy$ for every $\phi \in \mathcal{F}_\mu$.

10. The principal series and the special representations

We can now go back to the representation ρ_{μ_1,μ_2}. It follows from (119) that the representation space $\mathcal{B}_{\mu_1,\mu_2}$ is the same as $\mathcal{F}_\mu$ under the map $\varphi \longmapsto \phi$ given by (118). With the same meaning for $\hat{\phi}$ as in lemma 9, let us associate to every $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$ the function ξ_φ given by

$$\xi_\varphi(x) = \mu_2(x)|x|^{1/2}\hat{\phi}(x) = \mu_2(x)|x|^{1/2}\sum \int_{v(y)=n} \varphi[w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}]\bar{\tau}_F(xy)dy. \tag{134}$$

A trivial computation then shows that

$$\varphi' = \rho_{\mu_1,\mu_2}\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\varphi \Longrightarrow \xi_{\varphi'}(x) = \tau_F(bx)\xi_\varphi(ax) , \tag{135}$$

and since the functions ξ_φ are locally constant and vanish outside compact subsets of F, we see that the mapping $\varphi \longmapsto \xi_\varphi$ yields a Kirillov model for the representation ρ_{μ_1,μ_2}, except for the fact that this mapping may not be one-to-one. This occurs if and only if $\mu(x) = |x|^{-1}$, in which case the kernel of this mapping is generated by the function φ for which $\phi(x) = 1$, i.e. [use (117)] by the function

(136) $$\varphi_o(g) = \mu_1(\det g)|\det g|^{1/2}.$$

Of course the one-dimensional subspace generated by (136) is invariant under ρ_{μ_1,μ_2}, and $\varphi \longmapsto \xi_\varphi$ then induces a bijection of the corresponding factor space on the space of functions ξ_φ.

Note that the image $\mathcal{K}_{\mu_1,\mu_2}$ of $\mathcal{B}_{\mu_1,\mu_2}$ under $\varphi \longmapsto \xi_\varphi$ can be described from lemma 9; it is the space of locally constant functions on F^* which vanish for $|x|$ large and whose behaviour near 0 is given by the following formulas, which follow at once from (123) and (124):

(137) $$\xi(x) = \begin{cases} |x|^{1/2}(a\mu_1(x) + b\mu_2(x)) & \text{if } \mu(x) \neq 1, \ |x|^{-1}, \\ |x|^{1/2}(a\mu_2(x)v(x) + b\mu_2(x)) & \text{if } \mu(x) = 1, \\ b|x|^{1/2}\mu_2(x) & \text{if } \mu(x) = |x|^{-1}. \end{cases}$$

This space contains always $\mathcal{S}(F^*)$ as a subspace of codimension 2, except if μ is the character $x \longmapsto |x|^{-1}$ in which case $\mathcal{S}(F^*)$ has codimension 1. We shall now be able to decide whether ρ_{μ_1,μ_2} is irreducible or not:

<u>Theorem</u> 6. <u>The representation</u> ρ_{μ_1,μ_2} <u>is irreducible except if</u> $\mu(x) = |x|$ <u>or</u> $|x|^{-1}$. <u>If</u> $\mu(x) = |x|^{-1}$ <u>then</u> $\mathcal{B}_{\mu_1,\mu_2}$ <u>contains a one-dimensional invariant subspace, generated by the function</u> $g \longmapsto \mu_1(\det g)|\det g|^{1/2}$, <u>and the representation on the factor space is irreducible. If</u> $\mu(x) = |x|$ <u>then</u> $\mathcal{B}_{\mu_1,\mu_2}$ <u>contains an irreducible subspace of codimension one, namely the set of</u> φ <u>such that</u>

(138) $$\oint_{P_F\backslash G_F} \varphi(g)\mu_1^{-1}(\det g)|\det g|^{1/2}d\dot{g} = \int \varphi\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right]dx = 0.$$

Since the kernel of $\varphi \longmapsto \xi_\varphi$ is invariant in all cases, this mapping transforms ρ_{μ_1,μ_2} into a representation of G_F on the image space $\mathcal{K}_{\mu_1,\mu_2}$, and property (135) shows that if an invariant subspace of $\mathcal{K}_{\mu_1,\mu_2}$ contains a function ξ then it also contains the function $\xi(x) - \tau_F(bx)\xi(x)$ for all b, and thus contains (if it is not zero) non zero functions in $\mathcal{S}(F^*)$. But $\mathcal{S}(F^*)$ is irreducible under the operators (135) as we have already seen; hence every non zero invariant subspace of $\mathcal{K}_{\mu_1,\mu_2}$ contains $\mathcal{S}(F^*)$, hence contains the function $\xi(x) - \tau_F(bx)\xi(x)$ for all $\xi \in \mathcal{K}_{\mu_1,\mu_2}$ and all $b \in F$, and has furthermore codimension at most 2 in $\mathcal{K}_{\mu_1,\mu_2}$ (and at most 1 if μ is the character $x \longmapsto |x|^{-1}$).

If we apply this to the image under $\varphi \longmapsto \xi_\varphi$ of a non zero invariant subspace $\mathcal{V}$ of $\mathcal{B}_{\mu_1,\mu_2}$ we see that unless $\mu(x) = |x|^{-1}$ and $\mathcal{V}$ is the obvious one-dimensional subspace then $\mathcal{V}$ will contain $\varphi - \rho_{\mu_1,\mu_2}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\varphi$ for all $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$ and all $b \in F$. That means that in the contragredient representation $\rho_{-\mu_1,-\mu_2}$ on $\mathcal{B}_{-\mu_1,-\mu_2}$ (Theorem 5) the subspace $\mathcal{V}^\perp$ orthogonal to $\mathcal{V}$ must contain only functions ψ invariant under the operators corresponding to matrices $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$. Such a function must satisfy

$$\psi\left[\begin{pmatrix}t' & *\\ 0 & t''\end{pmatrix} w \begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}\right] = \mu_1^{-1}(t')\mu_2^{-1}(t'')|t'/t''|^{1/2}\psi(w) \tag{139}$$

and furthermore be continuous on G_F. Since the big cell is dense in G_F, this shows that $\operatorname{codim}(\mathcal{V}) \leq 1$, and it is furthermore not difficult to see that a continuous and non zero function satisfying (139) exists if and only if $\mu(x) = |x|$, in which case we may choose $\psi(g) = \mu_1^{-1}(\det g)|\det g|^{1/2}$, which explains why the only non trivial invariant subspace $\mathcal{V}$ of $\mathcal{B}_{\mu_1,\mu_2}$ is then defined by (138). To conclude the proof we still have to look more closely in the case where $\mu(x) = |x|^{-1}$--but by Theorem 5 the invariant subspaces of $\mathcal{B}_{\mu_1,\mu_2}$ are the orthogonal supplements of the invariant subspaces of $\mathcal{B}_{-\mu_1,-\mu_2}$, which shows that the situation for $\mu(x) = |x|^{-1}$ follows at once from the situation for $\mu(x) = |x|$ which we have just cleared off, q. e. d.

Theorem 6 makes it possible to define an irreducible representation π_{μ_1,μ_2} of G_F for every couple of characters of F^*. If $\mu(x) = \mu_1(x)\mu_2(x)^{-1}$ is neither the character $x \longmapsto |x|$ nor $x \longmapsto |x|^{-1}$, we define π_{μ_1,μ_2} to be the representation ρ_{μ_1,μ_2} on the space $\mathcal{B}_{\mu_1,\mu_2}$; we get in this way the so-called principal series of representations. Lemma 9 yields at once the Kirillov model for it: if $\pi = \pi_{\mu_1,\mu_2}$ we denote by $\mathcal{K}(\pi)$ the space of functions $\xi(x)$ on F^* which are locally constant, which vanish outside of a compact set in F, and whose behaviour in some neighborhood of 0 is given by

$$\xi(x) = \begin{cases} |x|^{1/2}(a\mu_1(x) + b\mu_2(x)) & \text{if } \mu \text{ is not trivial} \\ |x|^{1/2}(a\mu_2(x)v(x) + b\mu_2(x)) & \text{if } \mu \text{ is trivial;} \end{cases} \tag{140}$$

then the map $\varphi \longmapsto \xi_\varphi$ given by (56) is a bijection of $\mathcal{B}_{\mu_1,\mu_2}$ on $\mathcal{K}(\pi)$, so that we may assume that $\pi_{\mu_1,\mu_2}(g)$ is a linear operator on $\mathcal{K}(\pi)$, and since we then have

$$\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi(x) = \tau_F(bx)\xi(ax) \tag{141}$$

we have found in this way the Kirillov model for π.

If $\mu(x) = |x|$ we denote by $\pi_{\mu_1,\mu_2}(g)$ the restriction of $\rho_{\mu_1,\mu_2}(g)$ to the invariant hyperplane of $\mathcal{B}_{\mu_1,\mu_2}$ whose existence is asserted by Theorem 6. To get the Kirillov model for $\pi = \pi_{\mu_1,\mu_2}$ we still use lemma 9 and formula (134) but we have to find a characterization of the functions ξ_φ for those $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$ which belong to the hyperplane under consideration, i.e. are such that $\int \varphi\left[w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right]dx = 0$. Now we have $\varphi\left[w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right] = \phi(x)$, see (118), and $\phi(x)$ is proportional to $\mu^{-1}(x)|x|^{-1}$ i.e. to $|x|^{-2}$ for large values of $|x|$, hence is integrable on F; consequently the far-flung Fourier transform $\hat{\phi}$ given by lemma 9 is nothing in this case but the obvious one, and condition (138) means that $\hat{\phi}(0) = 0$; since $\hat{\phi}(x) = a|x| + b$ for $|x|$ small this means that $b = 0$. In other words, and taking care of (137), we see that in this case the Kirillov model is a representation on the space $\mathcal{K}(\pi)$ of functions $\xi(x)$ on F^* which are locally constant, which vanish outside compact subsets of F, and which behave near 0 according to formula

(142) $$\xi(x) = a|x|^{1/2}\mu_1(x) \ ;$$

in other words, $\mathcal{K}(\pi)$ is the space of functions $|x|^{1/2}\mu_1(x)f(x)$ with $f \in \mathcal{S}(F)$.

Finally, if $\mu(x) = |x|^{-1}$, there is in $\mathcal{B}_{\mu_1,\mu_2}$ a one-dimensional invariant subspace; we shall then define π_{μ_1,μ_2} to be the obvious representation on the corresponding factor space. Since this one-dimensional subspace is the kernel of the mapping $\varphi \longmapsto \xi_\varphi$ we see that this mapping will induce an isomorphism between the representation space of $\pi = \pi_{\mu_1,\mu_2}$ and the image $\mathcal{K}(\pi)$ of $\mathcal{B}_{\mu_1,\mu_2}$ under $\varphi \longmapsto \xi_\varphi$, hence the Kirillov model for π. In this case $\mathcal{K}(\pi)$ is therefore the space of all functions $\xi(x)$ on F^* which are locally constant, which vanish for $|x|$ large, and which behave near the origin according to (137), i.e. as

(143) $$\xi(x) = b|x|^{1/2}\mu_2(x) \ ;$$

in other words, $\mathcal{K}(\pi)$ is the set of functions $|x|^{1/2}\mu_2(x)f(x)$ with $f \in \mathcal{S}(F)$.

The representations π_{μ_1,μ_2} for $\mu(x) = |x|$ or $|x|^{-1}$ are the so-called <u>special representations</u>. It follows from theorem 4 that there are no other irreducible admissible representations of G_F than the one we have found: the principal series, the special representations, and the cuspidal ones. Since the argument above shows that the Kirillov space $\mathcal{K}(\pi)$ for an irreducible representation π satisfies

(144) $$\dim[\mathcal{K}(\pi)/\mathcal{S}(F^*)] = \begin{cases} 2 & \text{for the principal series} \\ 1 & \text{for the special representations} \\ 0 & \text{for the cuspidal representations,} \end{cases}$$

we see that these three "series" of representations are mutually disjoint.

The following table describes the space $\mathcal{K}(\pi)$ for the various representations in terms of "arbitrary" functions f, f_1, f_2 in $\mathcal{S}(F)$.

principal series π_{μ_1,μ_2} with $\mu_1 \neq \mu_2$	$\lvert x\rvert^{1/2}\mu_1(x)f_1(x) + \lvert x\rvert^{1/2}\mu_2(x)f_2(x)$
principal series π_{μ_1,μ_2} with $\mu_1 = \mu_2$	$\lvert x\rvert^{1/2}\mu_1(x)f_1(x) + \lvert x\rvert^{1/2}\mu_2(x)v(x)f_2(x)$
special representation π_{μ_1,μ_2} with $\mu_1\mu_2^{-1}(x) = \lvert x\rvert$	$\lvert x\rvert^{1/2}\mu_1(x)f(x)$
special representation π_{μ_1,μ_2} with $\mu_1\mu_2^{-1}(x) = \lvert x\rvert^{-1}$	$\lvert x\rvert^{1/2}\mu_2(x)f(x)$
supercuspidal representations	$f(x)$ with $f \in \mathcal{S}(F)$, $f(0) = 0$

11. <u>The equivalence</u> $\pi_{\mu_1,\mu_2} \sim \pi_{\mu_2,\mu_1}$

Theorems 2 and 5 can be used to give a very simple proof of the following result:

<u>Theorem</u> 7. <u>The representations</u> $\pi_{\lambda_1,\lambda_2}$ <u>and</u> π_{μ_1,μ_2} <u>are equivalent if and only if</u> $(\mu_1, \mu_2) = (\lambda_1, \lambda_2)$ <u>or</u> $= (\lambda_2, \lambda_1)$.

The necessity of the condition is clear if we look at the Kirillov models, so that we only have to show that $\pi_{\mu_1,\mu_2} \sim \pi_{\mu_2,\mu_1}$. But from the fact that $\check{\rho}_{\mu_1,\mu_2} = \rho_{-\mu_1,-\mu_2}$ we see at once that

(145) $$\check{\pi}_{-\mu_1,-\mu_2} = \pi_{\mu_1,\mu_2} ,$$

even for the special representations. By theorem 2 this can be written as

(146) $$\pi_{\mu_1,\mu_2} \sim \omega_\pi^{-1} \otimes \pi_{-\mu_1,-\mu_2}$$

where $\pi = \pi_{-\mu_1,-\mu_2}$; but $\omega_\pi^{-1}(x) = \mu_1\mu_2(x)$, hence the theorem since the mapping

(147) $$\{g \longmapsto \varphi(g)\} \longmapsto \{g \longmapsto \mu_1\mu_2(\det g)\varphi(g)\}$$

is an isomorphism of $\mathcal{B}_{-\mu_1, -\mu_2}$ onto $\mathcal{B}_{\mu_2,\mu_1}$.

Another way of proving the theorem would consist in showing that

(148) $$\pi_{\mu_1,\mu_2}(w) = \pi_{\mu_2,\mu_1}(w) \qquad \text{on } \mathcal{S}(F^*),$$

which would of course be enough to prove that the Kirillov realizations of π_{μ_1,μ_2} and π_{μ_2,μ_1} are identical. To do that one can compute explicitly $\pi_{\mu_1,\mu_2}(w)$; one then easily gets

(149) $$\pi_{\mu_1,\mu_2}(w)\xi(x) = \mu_1\mu_2(x)\int J_{\mu_1,\mu_2}(xy)\xi(y)d^*y$$

with

(150) $$J_{\mu_1,\mu_2}(x) = |x|^{1/2}\mu_2(x)^{-1}\sum_n \int_{v(z)=n} \tau_F(xz+z^{-1})\mu(z)d^*z,$$

an expression that is obviously symmetrical with respect to μ_1 and μ_2. Note that (150) reduces to a finite sum on every compact subset of F^*, and that (149) is valid for $\xi \in \mathcal{S}(F^*)$ only. Similar formulas are to be found in a recent paper by P. J. Sally, (Am. J. of Math., 1968, vol. 90, p.

12. The fundamental functional equation

Let π be an irreducible admissible representation of G_F; we may consider that π is obtained by letting G_F act on the space $\mathcal{W}(\pi)$ of no. 5 through right translations. For every $W \in \mathcal{W}(\pi)$ and every character χ of F^*, we define

(151) $$L_W(g;\chi, s) = \int W[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g]\chi(x)^{-1}|x|^{2s-1}d^*x ,$$

at first formally. We shall now prove

Theorem 8. The integral (151) converges for $\operatorname{Re}(s)$ large, and can be analytically continued to a meromorphic function with at most two poles. Furthermore there exists a meromorphic function $\gamma_\pi(\chi, x)$, which depends neither on $W \in \mathcal{W}(\pi)$ nor on $g \in G_F$, and is such that

(152) $$L_W(wg;\omega_\pi - \chi, 1-s) = \gamma_\pi(\chi, s)L_W(g;\chi, s)$$

for all $W \in \mathcal{W}(\pi)$ and $g \in G_F$. It satisfies

(153) $$\gamma_\pi(\omega_\pi - \chi, 1-s)\gamma_\pi(\chi, s) = \omega_\pi(-1) .$$

To prove the convergence and analytical continuation we may assume that $g = e$ (replace W by its right translate by g); we then have to study the integral

(154) $$M_\xi(\chi, s) = \int \xi(x)\chi(x)^{-1}|x|^{2s-1}d^*x$$

where $\xi(x) = W\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ belongs to the space $\mathcal{K}(\pi)$. If π is supercuspidal we have $\xi \in \mathcal{S}(F^*)$ and then (154) is clearly an entire function of s. If π is not supercuspidal then we have the following possibilities:

(155) $$\xi(x) = \begin{cases} |x|^{1/2}[f_1(x)\mu_1(x) + f_2(x)\mu_2(x)] \\ |x|^{1/2}\mu_1(x)[f_1(x) + f_2(x)v(x)] \\ |x|^{1/2}\mu_1(x)f_1(x) \\ |x|^{1/2}\mu_2(x)f_2(x) \end{cases}$$

where f_1, $f_2 \in \mathcal{S}(F)$, as we have seen at the end of section 10. Hence (154) is the sum of at most two integrals of the following kind:

(156) $$\int f(x)\lambda(x)|x|^{2s}d^*x, \quad \int f(x)\lambda(x)v(x)|x|^{2s}d^*x$$

with a character λ of F^*, hence the results (more detailed information will be found in no. 14).

It thus remains to prove the existence of a functional equation (152). Here too we may assume $g = e$; since $\xi(x) = W\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ implies

(157) $$W\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}g\right] = \pi(g)\xi(x),$$

it is clear that (152) can still be written as

(158) $$M_{\pi(w)\xi}(\omega_\pi - \chi, 1 - s) = \gamma_\pi(\chi, s)M_\xi(\chi, s).$$

We shall prove this in three steps.

<u>Step</u> 1. We prove (158) for all $\xi \in \mathcal{S}(F^*) \subset \mathcal{K}(\pi)$.

It is clear from the definition that if (158) is true for a given ξ it is also true for all its multiplicative translates. To get (158) for all $\xi \in \mathcal{S}(F^*)$ it is thus enough to prove it when

(159) $$\xi(x) = \begin{cases} \lambda(x) & \text{if} \quad x \in E_F \\ 0 & \text{if} \quad x \notin E_F, \end{cases}$$

where λ is a given character of E_F. Both sides of (158) then vanish unless λ is the restriction of χ to E_F. In this case we have $\pi(w)\xi(x) = \omega_\pi(x)J_\pi(x, \chi)$ by (27), where we still denote by χ the restriction of χ to E_F, and thus

(160) $$M_{\pi(w)\xi}(\omega_\pi - \chi, 1 - s) = \int J_\pi(x, \chi)\chi(x)|x|^{1-2s}d^*x$$

(161) $$M_\xi(\chi, s) = \int_{E_F} \chi(x)\chi(x)^{-1}|x|^{2s-1}d^*x = 1$$

if $\int_{E_F} d^*x = 1$. We thus see that (158) is satisfied for <u>all</u> $\xi \in \mathcal{S}(F^*)$ if we choose

(162) $$\gamma_\pi(\chi, s) = \int J_\pi(x, \chi)\chi(x)|x|^{1-2s}d^*x,$$

provided $\int_{E_F} d^*x = 1.$

Step 2. We now prove (153) by choosing a function $\xi \in \mathcal{S}(F^*) \cap \pi(w)\mathcal{S}(F^*)$ and applying (158) twice; we get

$$(163)\qquad \begin{aligned}\omega_\pi(-1)M_\xi(\chi, s) &= M_{\pi(w)\pi(w)\xi}(\chi, s) = \gamma_\pi(\omega_\pi - \chi, 1-s)M_{\pi(w)\xi}(\omega_\pi - \chi, 1-s)\\ &= \gamma_\pi(\omega_\pi - \chi, 1-s)\gamma_\pi(\chi, s)M_\xi(\chi, s),\end{aligned}$$

from which (153) follows provided we can choose ξ in such a way that $M_\xi(\chi, s)$ is not identically 0. But we know by Lemma 7 that there are non-zero ξ in $\mathcal{S}(F^*) \cap \pi(w)\mathcal{S}(F^*)$ which satisfy $\xi(xu) = \xi(x)\chi(u)$ for every $u \in E_F$; for such a ξ we have

$$(164)\qquad M_\xi(\chi, s) = \sum_{-\infty}^{\infty} \xi(\varpi^n)\chi(\varpi^n)^{-1}|\varpi^n|^{2s-1}$$

(a finite sum), and this of course is not identically 0 if $\xi \neq 0$.

Step 3. We eventually prove (158) for an arbitrary $\xi \in \mathcal{K}(\pi)$ by using the fact that $\xi = \xi_1 + \pi(w)\xi_2$ with $\xi_1, \xi_2 \in \mathcal{S}(F^*)$. If we use steps 1 and 2 of the proof we get

$$(165)\qquad \begin{aligned} M_{\pi(w)\xi}(\omega_\pi - \chi, 1-s) &= M_{\pi(w)\xi_1}(\omega_\pi - \chi, 1-s) + \omega_\pi(-1)M_{\xi_2}(\omega_\pi - \chi, 1-s)\\ &= \gamma_\pi(\chi, s)\{M_{\xi_1}(\chi, s) + \gamma_\pi(\omega_\pi - \chi, 1-s)M_{\xi_2}(\omega_\pi - \chi, 1-s)\}\\ &= \gamma_\pi(\chi, s)\{M_{\xi_1}(\chi, s) + M_{\pi(w)\xi_2}(\chi, s)\},\end{aligned}$$

which concludes the proof.

13. Computation of $\gamma_\pi(\chi, s)$ for the principal series and the special representations.

Before we start performing the computation announced in the title of this section, we recall that if we define

$$(166)\qquad L_\varphi(\chi, s) = \int_{F^*} \varphi(x)\chi(x)|x|^s d^*x$$

for every $\varphi \in \mathcal{S}(F)$ and every character χ of F^*, then we have the following properties:

(i) the integral (166) converges for $\mathrm{Re}(s)$ large enough--in fact for $\mathrm{Re}(s) > 0$ if χ is unitary;

(ii) the function (166) is meromorphic in the whole plane for every $\varphi \in \mathcal{S}(F)$ [more about the poles later!];

(iii) there is a factor $\gamma(\chi, s)$ depending only on χ and s and such that

(167) $$L_{\varphi}(-\chi, 1 - s) = \gamma(\chi, s) L_{\hat{\varphi}}(\chi, s)$$

for all $\varphi \in \mathcal{S}(F)$.

We now propose to compute $\gamma_{\pi}(\chi, s)$ in terms of such factors when $\pi = \pi_{\mu_1, \mu_2}$ belongs to the principal series or is a special representation.

To do it we choose a function

(168) $$\xi \in \mathcal{S}(F^*) \cap \pi(w)\mathcal{S}(F^*)$$

such that $M_{\xi}(\chi, s)$ is not identically zero (the existence of such a ξ has been proved in step 2 of the proof of Theorem 8) and we observe that there exists a function $\varphi \in \mathcal{B}_{\mu_1, \mu_2}$ such that

(169) $$\varphi[w^{-1}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}] = \int \mu_2^{-1}(x)|x|^{-1/2}\xi(x)\cdot\tau_F(xy)dx \; ;$$

in fact, the left hand side of (169) is clearly the Fourier transform of a function in $\mathcal{S}(F^*)$, hence is in $\mathcal{S}(F)$, hence in the space $\mathcal{F}_{\mu}$ of Lemma 9, and thus does extend to a function $\varphi \in \mathcal{B}_{\mu_1, \mu_2}$ by making use of (117), p. 1.27. Comparing (169) and (134) we see by making use of Fourier's inversion formula that

(170) $$\xi(x) = \mu_2(x)|x|^{1/2}\int\varphi[w^{-1}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}]\bar{\tau}_F(xy)dy = \xi_{\varphi}(x) \; ,$$

with no need here for the sophisticated Fourier transform of Lemma 9. But since the mapping $\varphi \longmapsto \xi_{\varphi}$ is one-to-one* and compatible with the obvious

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* except if $\mu(x) = |x|^{-1}$ as we have seen at bottom of p. 1.31, but since $\pi_{\mu_1, \mu_2} = \pi_{\mu_2, \mu_1}$ we may assume this is not the case.

actions of G_F, we conclude from (170) that $\pi(w)\xi$ corresponds under this mapping to the function $g \longmapsto \varphi(gw)$. Since $\pi(w)\xi \in \mathcal{S}(F^*)$ by our assumption (168), we can still use (169) with $\pi(w)\xi$ and $g \longmapsto \varphi(gw)$ instead of ξ and $g \longmapsto \varphi(g)$, and we thus get

(171) $$\varphi[w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}w] = \int \mu_2^{-1}(x)|x|^{-1/2}\pi(w)\xi(x)\cdot \tau_F(xy)dx .$$

Since we have

(172) $$w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}w = \begin{pmatrix}1 & 0\\ -y & 1\end{pmatrix} = \begin{pmatrix}1 & -y^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}y^{-1} & 0\\ 0 & y\end{pmatrix}w\begin{pmatrix}1 & -y^{-1}\\ 0 & 1\end{pmatrix}$$

and since $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$ we get [use (102) and $\mu = \mu_1 - \mu_2$]

(173) $$\begin{aligned}\varphi[w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}w] &= \mu(y^{-1})|y^{-1}|\varphi[w\begin{pmatrix}1 & -y^{-1}\\ 0 & 1\end{pmatrix}]\\ &= \omega_\pi(-1)\mu(y^{-1})|y^{-1}|\cdot\varphi[w^{-1}\begin{pmatrix}1 & -y^{-1}\\ 0 & 1\end{pmatrix}];\end{aligned}$$

making use of Fourier inversion formula in (171) we thus get

(174) $$\begin{aligned}&\mu_2^{-1}(x)|x|^{-1/2}\pi(w)\xi(x) = \eta'(x) =\\ &= \omega_\pi(-1)\int \mu(y^{-1})|y^{-1}|\varphi[w^{-1}\begin{pmatrix}1 & -y^{-1}\\ 0 & 1\end{pmatrix}]\cdot\bar{\tau}_F(xy)dy ,\end{aligned}$$

while (170) can also be written as

(175) $$\mu_2^{-1}(x)|x|^{-1/2}\xi(x) = \xi'(x) = \int\varphi[w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}]\bar{\tau}_F(xy)dy .$$

We can now start the computation of $\gamma_\pi(\chi, s)$. Making use of the definition (175) of ξ' we get

(175) $$\begin{aligned}M_\xi(\chi, s) &= \int \xi(x)\chi(x)^{-1}|x|^{2s-1}d^*x =\\ &= \int \xi'(x)\mu_2\chi^{-1}(x)|x|^{2s-\frac{1}{2}}d^*x = L_{\xi'}(\mu_2 - \chi, 2s - \tfrac{1}{2})\end{aligned}$$

where we use definition (166). But by (175) the function ξ' is the Fourier transform of $y \longmapsto \varphi[w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}]$; using (167) we thus get*

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* We put $s' = 2s - 1/2$ in order to simplify the computation.

$$\gamma(\mu_2 - \chi,\ s')M_\xi(\chi,\ s) = \int \varphi[w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}]\chi\mu_2^{-1}(y)|y|^{1-s'}d^*y\ . \tag{176}$$

Consider now

$$\begin{aligned} M_{\pi(w)\xi}(\omega_\pi - \chi,\ 1-s) &= \int \pi(w)\xi(x)\mu_1^{-1}\mu_2^{-1}\chi(x)\cdot|x|^{1-2s}d^*x \\ &= \int \eta'(x)\mu_1^{-1}\chi(x)|x|^{\frac{3}{2}-2s}d^*x = L_{\eta'}(\chi - \mu_1,\ 1-s'); \end{aligned} \tag{177}$$

by (174) and (167) we get in a similar way

$$\begin{aligned} &\gamma(\chi - \mu_1,\ s')M_{\pi(w)\xi}(\omega_\pi - \chi,\ 1-s) = \\ &= \omega_\pi(-1)\int\mu(y^{-1})|y^{-1}|\varphi[w^{-1}\begin{pmatrix}1 & -y^{-1}\\ 0 & 1\end{pmatrix}]\mu_1\chi^{-1}(y)|y|^{s'}d^*y \\ &= \omega_\pi(-1)\int\varphi[w^{-1}\begin{pmatrix}1 & -y^{-1}\\ 0 & 1\end{pmatrix}]\mu_2\chi^{-1}(y)|y|^{s'-1}d^*y \\ &= \mu_1\chi^{-1}(-1)\int\varphi[w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}]\chi\mu_2^{-1}(y)|y|^{1-s'}d^*y\ . \end{aligned} \tag{178}$$

Comparing with (176), and since $M_\xi(\chi,\ s)$ is not identically 0, we get at once

$$\begin{aligned} \gamma_\pi(\chi,\ s) &= M_{\pi(w)\xi}(\omega_\pi - \chi,\ 1-s)/M_\xi(\chi,\ s) \\ &= \mu_1\chi^{-1}(-1)\gamma(\mu_2 - \chi,\ s')/\gamma(\chi - \mu_1,\ 1-s')\,. \end{aligned} \tag{179}$$

But it is clear that by iterating functional equation (167) we get

$$\gamma(\chi,\ s)\gamma(-\chi,\ 1-s) = \chi(-1)\ . \tag{180}$$

Hence

$$\gamma_\pi(\chi,\ s) = \gamma(\mu_1 - \chi,\ s')\gamma(\mu_2 - \chi,\ s')\ , \tag{181}$$

and if we replace s' by $2s - \frac{1}{2}$ we eventually get the result:

<u>Theorem</u> 9. <u>For every character</u> χ <u>of</u> F^* <u>and every function</u> $\varphi \in \mathcal{S}(F)$, <u>define</u>

$$L_\varphi(\chi,\ s) = \int\varphi(x)\chi(x)|x|^s d^*x \tag{182}$$

<u>and let</u> $\gamma(\chi,\ s)$ <u>be the factor such that</u>

$$L_\varphi(-\chi,\ 1-s) = \gamma(\chi,\ s)L_{\hat\varphi}(\chi,\ s) \tag{183}$$

for all $\varphi \in \mathcal{S}(F)$. Then if μ_1 and μ_2 are any two characters of F^* and if $\pi = \pi_{\mu_1,\mu_2}$, then the factor $\gamma_\pi(\chi, s)$ is given by

(184) $$\gamma_\pi(\chi, s) = \gamma(\mu_1 - \chi, 2s - \tfrac{1}{2})\gamma(\mu_2 - \chi, 2s - \tfrac{1}{2}) .$$

14. The local factors $L_\pi(\chi, s)$.

Let's go back to Tate's Ph. D. for a while and consider the Mellin transforms

(185) $$L_\varphi(\chi, s) = \int \varphi(x)\chi(x)|x|^s d^*x$$

of the functions $\varphi \in \mathcal{S}(F)$. We evidently have

(186) $$L_\varphi(\chi, s) = \varphi(0) \int_{|x|\leq 1} \chi(x)|x|^s d^*x + \text{entire function,}$$

so that if we define

(187) $$L(\chi, s) = \begin{cases} 1 & \text{if } \chi \text{ is ramified} \\ \dfrac{1}{1-\chi(\mathfrak{p})N(\mathfrak{p})^{-s}} & \text{if } \chi \text{ is non ramified} \end{cases}$$

we see at once that (for given χ and variable φ) this expression is the highest common divisor of all functions $L_\varphi(\chi, s)$, i.e. that the ratio $L_\varphi(\chi, s)/L(\chi, s)$ is always an entire function which furthermore is 1 for a suitable choice of φ. Strangely enough, the functions $L(\chi, s)$ are the local factors used to define global Hecke's L functions by means of Euler products.

We shall now try to define in a similar way local factors $L_\pi(\chi, s)$ for every irreducible admissible representation π of $GL(2, F)$. We put

(188) $$L_\pi(\chi, s) = \text{g.c.d. of all functions } M_\xi(\chi, s) \text{ for all } \xi \in \mathcal{K}(\pi),$$

and require in addition, to avoid ambiguities, that $L_\pi(\chi, s)$ should be a finite product of Eulerian factors or, which is clearly the same, that

(189) $$L_\pi(\chi, s) = P(q^{-s})^{-1}$$

where $q = N(\mathfrak{p})$ and P is a polynomial such that $P(0) = 1$. The computation will be very easy to perform because of the characterization given on p. 1.36 of the various spaces $\mathcal{K}(\pi)$.

The case of a supercuspidal π. Then the functions

(190) $$M_\xi(\chi, s) = \int \xi(x)\chi(x)^{-1}|x|^{2s-1}d^*x$$

are entire since $\mathcal{K}(\pi) = \mathcal{S}(F^*)$, and there is a $\xi \in \mathcal{K}(\pi)$ such that $M_\xi(x, s) = 1$, e.g. the function equal to χ on E_F and 0 elsewhere. Hence we define

(191) $$L_\pi(\chi, s) = 1$$

in this case.

The case of a special π. Assume

(192) $$\pi = \pi_{\mu_1, \mu_2} \qquad \text{with } \mu(x) = |x|;$$

then $\mathcal{K}(\pi)$ is the set of all functions

(193) $$\xi(x) = |x|^{1/2}\mu_1(x)\varphi(x) \quad \text{with } \varphi \in \mathcal{S}(F);$$

for such a ξ we clearly have

(194) $$M_\xi(\chi, s) = L_\varphi(\mu_1 - \chi, 2s - 1/2);$$

since φ is arbitrary in $\mathcal{S}(F)$, we must therefore choose

(195) $$L_\pi(\chi, s) = L(\mu_1 - \chi, 2s - 1/2)$$

in this case. The case where $\mu(x) = |x|^{-1}$ is the same, with μ_2 instead of μ_1-- use Theorem 7.

The case of a generic member of the principal series. We now assume $\pi = \pi_{\mu_1, \mu_2}$ with

(196) $$\mu(x) \neq |x|, \ |x|^{-1}, \ 1 .$$

Then

(197) $$\xi(x) = |x|^{1/2}[\mu_1(x)\varphi_1(x) + \mu_2(x)\varphi_2(x)]$$

with arbitrary φ_1, $\varphi_2 \in \mathcal{S}(F)$, hence*

(198) $$M_\xi(\chi, s) = L_{\varphi_1}(\mu_1 - \chi, s') + L_{\varphi_2}(\mu_2 - \chi, s'),$$

so that $L_\pi(\chi, s)$ must be the g.c.d. of the two functions $L(\mu_1 - \chi, s')$ and $L(\mu_2 - \chi, s')$. This g.c.d. is also their product. This is clear by (187) unless $\mu_1 - \chi = \lambda_1$ and $\mu_2 - \chi = \lambda_2$ are unramified; but then

(199) $$L(\mu_i - \chi, s') = \frac{1}{1 - \lambda_i(\mathfrak{p})N(\mathfrak{p})^{-s'}}$$

and since $\lambda_1(\mathfrak{p}) \neq \lambda_2(\mathfrak{p})$ [because we assume $\mu_1 \neq \mu_2$ by (196)] our assertion follows. Thus we define in this case

(200) $$L_\pi(\chi, s) = L(\mu_1 - \chi, 2s - \tfrac{1}{2})L(\mu_2 - \chi, 2s - \tfrac{1}{2}).$$

The existence of a $\xi \in \mathcal{K}(\pi)$ such that $M_\xi(\chi, s) = L_\pi(\chi, s)$ follows from the fact that, if λ_1 and λ_2 are distinct unramified characters, there are constants c_1 and c_2 such that

(201) $$L(\lambda_1, s)L(\lambda_2, s) = c_1L(\lambda_1, s) + c_2L(\lambda_2, s).$$

There remains to study the case where

(202) $$\pi = \pi_{\mu_1,\mu_2} \qquad \text{with } \mu_1 = \mu_2.$$

Then $\mathcal{K}(\pi)$ is the set of functions

(203) $$\xi(x) = |x|^{1/2}\mu_1(x)[\varphi_1(x) + \varphi_2(x)v(x)]$$

with arbitrary φ_1, $\varphi_2 \in \mathcal{S}(F)$. We thus have

(204) $$M_\xi(\chi, s) = L_{\varphi_1}(\lambda, s') + \int \varphi_2(x)v(x)\lambda(x)|x|^{s'}d^*x$$

where we set $\mu_1 - \chi = \lambda$. The second integral is an entire function if $\varphi_2(0) = 0$. Hence it will be enough to look at

(205) $$\int_{|x|\leq 1} v(x)\lambda(x)|x|^{s'}d^*x;$$

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* We again set $s' = 2s - \frac{1}{2}$.

this is clearly 0 if λ is ramified; if not, it is proportional to

(206) $$\sum_{0}^{\infty} n\lambda(\mathfrak{p})^n N(\mathfrak{p})^{-ns'} = \frac{\lambda(\mathfrak{p})N(\mathfrak{p})^{-s'}}{[1-\lambda(\mathfrak{p})N(\mathfrak{p})^{-s'}]^2} = \lambda(\mathfrak{p})N(\mathfrak{p})^{-s'}L(\lambda, s')^2 .$$

We thus see that in all cases we have

(207) $$M_\xi(\chi, s) = L(\mu_1 - \chi, s')^2 \times \text{entire function},$$

which suggests formula

(208) $$L_\pi(\chi, s) = L(\mu_1 - \chi, 2s - \tfrac{1}{2})^2 ,$$

the same as for the generic members of the principal series. In fact it will be justified if we prove the existence of a $\xi \in \mathcal{K}(\pi)$ such that

(209) $$M_\xi(\chi, s) = L(\mu_1 - \chi, s')^2 ,$$

which of course is clear if $\lambda = \mu_1 - \chi$ is ramified. If it is not then the computation of (205) at any rate shows there is a ξ such that

(210) $$M_\xi(\chi, s) = L(\mu_1 - \chi, s')^2 N(\mathfrak{p})^{-s'};$$

if we replace $\xi(x)$ by $c\xi(ax)$ with suitable constants $a \neq 0$ and c we evidently get the result.

We gather the various definitions of $L_\pi(\chi, s)$ in the following table:

π	$L_\pi(\chi, s)$
principal series π_{μ_1,μ_2}	$L(\mu_1-\chi, 2s-\frac{1}{2})L(\mu_2-\chi, 2s-\frac{1}{2})$
special representations π_{μ_1,μ_2} with $\mu(x) = \|x\|$	$L(\mu_1-\chi, 2s-\frac{1}{2})$
special representation π_{μ_1,μ_2} with $\mu(x) = \|x\|^{-1}$	$L(\mu_2-\chi, 2s-\frac{1}{2})$
supercuspidal representations	1

The meaning of these local factors is now expressed in the following way:

Theorem 10. *For every irreducible representation* π *of* G_F *and every character* χ *of* F^*, *let* $L_\pi(\chi, s)$ *be the Euler factor defined as above. Then the ratio*

$$L_W(g; \chi, s)/L_\pi(\chi, s) \tag{211}$$

is an entire function for every $W \in \mathcal{W}(\pi)$ *and every* $g \in G_F$, *and there is a* $W \in \mathcal{W}(\pi)$ *such that*

$$L_W(e; \chi, s) = L_\pi(\chi, s) . \tag{212}$$

15. *The factors* $\varepsilon_\pi(\chi, s)$.

The introduction of the factors $L_\pi(\chi, s)$ leads to a different way of writing the functional equation (152) of Theorem 8; instead of

$$L_W(wg; \omega_\pi - \chi, 1 - s) = \gamma_\pi(\chi, s) L_W(g; \chi, s) \tag{213}$$

we can still write it as

$$\frac{L_W(wg; \omega_\pi - \chi, 1 - s)}{L_\pi(\omega_\pi - \chi, 1 - s)} = \varepsilon_\pi(\chi, s) \frac{L_W(g; \chi, s)}{L_\pi(\chi, s)} \tag{214}$$

with new factors $\varepsilon_\pi(\chi, s)$ instead of $\gamma_\pi(\chi, s)$; clearly

$$\gamma_\pi(\chi, s) = \varepsilon_\pi(\chi, s) \frac{L_\pi(\omega_\pi - \chi, 1 - s)}{L_\pi(\chi, s)} . \tag{215}$$

Since we may assume $L_W(g; \chi, s)/L_\pi(\chi, s) = 1$, it is clear that $\varepsilon_\pi(\chi, s)$ is an entire function of s; it never vanishes, because it follows from (215) and

$$\gamma_\pi(\omega_\pi - \chi, 1 - s)\gamma_\pi(\chi, s) = \omega_\pi(-1) \tag{216}$$

that

$$\varepsilon_\pi(\omega_\pi - \chi, 1 - s)\varepsilon_\pi(\chi, s) = \omega_\pi(-1) . \tag{217}$$

In fact, $\varepsilon_\pi(\chi, s)$ is an *exponential* function, because the computations of this number evidently show that all ratios $L_W(g; \chi, s)/L_\pi(\chi, s)$, hence also $\varepsilon_\pi(\chi, s)$, are finite formal series in q^{2s}.

It is easy to compute $\varepsilon_\pi(\chi, s)$ for the principal series and the special representations.

The first thing to do is to replace Tate's functional equation

(218) $$L_\varphi(-\chi, 1-s) = \gamma(\chi, s)L_{\hat\varphi}(\chi, s)$$

of local Mellin transforms, see (167), by something similar to (214), namely

(219) $$\frac{L_\varphi(-\chi, 1-s)}{L(-\chi, 1-s)} = \varepsilon(\chi, s)\frac{L_{\hat\varphi}(\chi, s)}{L(\chi, s)},$$

the factors $\varepsilon(\chi, s)$ are easily computed, and well known, but we don't need now their exact values. Going back to $\gamma_\pi(\chi, s)$ we must distinguish two cases.

If $\pi = \pi_{\mu_1,\mu_2}$ belongs to the <u>principal</u> series, then $L_\pi(\chi, s) = L(\mu_1 - \chi, s')L(\mu_2 - \chi, s')$ as we have seen, and we thus get by (215)

(220) $$\varepsilon_\pi(\chi, s) = \gamma(\mu_1 - \chi, s')\gamma(\mu_2 - \chi, s') \times \frac{L(\mu_1 - \chi, s')L(\mu_2 - \chi, s')}{L(\chi - \mu_2, 1-s')L(\chi - \mu_1, 1-s')}$$

whence

(221) $$\varepsilon_\pi(\chi, s) = \varepsilon(\mu_1 - \chi, 2s - \tfrac{1}{2})\varepsilon(\mu_2 - \chi, 2s - \tfrac{1}{2}).$$

If on the other hand $\pi = \pi_{\mu_1,\mu_2}$ is a special representation with $\mu(x) = |x|$, which we may assume since $\pi_{\mu_1,\mu_2} \sim \pi_{\mu_2,\mu_1}$, then we have

(222) $$L_\pi(\chi, s) = L(\mu_1 - \chi, s')$$

and thus

(223) $$\varepsilon_\pi(\chi, s) = \gamma(\mu_1 - \chi, s')\gamma(\mu_2 - \chi, s') \times \frac{L(\mu_1 - \chi, s')}{L(\chi - \mu_2, 1-s')} ;$$

comparing with (220) we get

(224) $$\varepsilon_\pi(\chi, s) = \varepsilon(\mu_1 - \chi, s')\varepsilon(\mu_2 - \chi, s') \times \frac{L(\chi - \mu_1, 1-s')}{L(\mu_2 - \chi, s')}.$$

Now put $\lambda_2 = \mu_2 - \chi$; then $\chi - \mu_1$ is the character $x \longmapsto |x|^{-1}\lambda_2(x)^{-1}$ since we assume that $\mu_1\mu_2^{-1}(x) = |x|$. Hence we get at once

(225) $$L(\chi - \mu_1, 1 - s') = L(\chi - \mu_2, -s'),$$

so that the fraction in (224) equals

(226) $$\frac{L(\chi - \mu_2, -s')}{L(\mu_2 - \chi, s')} .$$

This is one if $\chi - \mu_2$ is ramified. In the opposite case, putting again $\mu_2 - \chi = \lambda_2$, this equals

(227) $$\frac{1 - \lambda_2(\mathfrak{p})N(\mathfrak{p})^{-s'}}{1 - \lambda_2(\mathfrak{p})^{-1}N(\mathfrak{p})^{s'}} = -\lambda_2(\mathfrak{p})N(\mathfrak{p})^{-s'} ,$$

an exponential factor as we knew in advance since the left hand side of (224) must be one in all cases.

To conclude these computations we recall the computation of the factors $\gamma(\chi, s)$ and $\varepsilon(\chi, s)$ in the functional equation of local Mellin transforms. We start from

(228) $$L_{\varphi}(-\chi, 1 - s) = \gamma(\chi, s)L_{\hat{\varphi}}(\chi, s) ,$$

and assuming first that χ is ramified we choose φ such that

(229) $$\hat{\varphi}(x) = \begin{cases} \chi(x)^{-1} & \text{if } x \in E_F \\ 0 & \text{if } x \notin E_F ; \end{cases}$$

hence

(230) $$\varphi(x) = \int_{E_F} \chi(u)^{-1}\tau_F(xu)du .$$

Since $m^*(E_F) = 1$ we have $L_{\hat{\varphi}}(\chi, s) = 1$, hence

(231) $$\gamma(\chi, s) = \int\int \chi(x)^{-1}\chi(u)^{-1}\tau_F(xu)|x|^{1-s}d^*xdu = m^+(E_F)\int \chi(x)^{-1}\tau_F(x)|x|^{1-s}d^*x .$$

Since we assume that χ is ramified, this reduces to

(232) $$\gamma(\chi, s) = m^{+}(E_F)N(\mathfrak{p})^{(d+f)(1-s)} \int_{v(x)=-d-f} \chi(x)^{-1}\tau_F(x)d^*x .$$

If we put

(233) $$\varepsilon(\chi) = m^{+}(E_F)N(\mathfrak{p})^{\frac{1}{2}(d+f)} \int_{v(x)=-d-f} \chi(x)^{-1}\tau_F(x)d^*x = \gamma(\chi, \tfrac{1}{2}),$$

so that $|\varepsilon(\chi)| = 1$ if χ is unitary, then

(234) $$\gamma(\chi, s) = \varepsilon(\chi)N(\mathfrak{p})^{(d+f)(\frac{1}{2}-s)} .$$

If now χ is unramified, we choose φ such that

(235) $$\hat{\varphi}(x) = \begin{cases} 1 & \text{if } x \in \mathfrak{o}_F \\ 0 & \text{if } x \notin \mathfrak{o}_F \end{cases},$$

hence

(236) $$\varphi(x) = \begin{cases} m^{+}(\mathfrak{o}_F) & \text{if } x \in \mathfrak{p}^{-d} \\ 0 & \text{if } x \notin \mathfrak{p}^{-d} \end{cases}.$$

We thus get

(237) $$L_{\hat{\varphi}}(\chi, s) = \int_{\mathfrak{o}_F} \chi(x)|x|^s d^*x = L(\chi, s)$$

and

(238) $$\begin{aligned} L_{\varphi}(-\chi, 1-s) &= m^{+}(\mathfrak{o}_F) \int_{\mathfrak{p}^{-d}} \chi(x)^{-1}|x|^{1-s}d^*x \\ &= m^{+}(\mathfrak{o}_F)\chi(\mathfrak{p})^{d}N(\mathfrak{p})^{d(1-s)} \int_{\mathfrak{o}_F} \chi(x)^{-1}|x|^{1-s}d^*x \\ &= m^{+}(\mathfrak{o}_F)\chi(\mathfrak{p})^{d}N(\mathfrak{p})^{d(1-s)}L(-\chi, 1-s) . \end{aligned}$$

Hence

$$\begin{aligned} \gamma(\chi, s) &= m^{+}(\mathfrak{o}_F)\chi(\mathfrak{p})^{d}N(\mathfrak{p})^{d(1-s)} \frac{L(-\chi, 1-s)}{L(\chi, s)} \\ &= \chi(\mathfrak{p})^{d}N(\mathfrak{p})^{d(\frac{1}{2}-s)} \frac{L(-\chi, 1-s)}{L(\chi, s)} . \end{aligned}$$

We thus get, in all cases, the following value for the factor $\varepsilon(\chi, s)$ in (219)

(240) $$\varepsilon(\chi, s) = \varepsilon(\chi) N(\varphi)^{(d+f)(\frac{1}{2} - s)}$$

where $\varepsilon(\chi)$ is given by (223) in the ramified case, and by

(241) $$\varepsilon(\chi) = \chi(\varphi)^d$$

in the nonramified case. Formulas (221) and (224) then yield the values of $\varepsilon_\pi(\chi, s)$ in terms of the "root numbers" $\varepsilon(\chi)$.

16. The case of spherical representations.

We shall say that an irreducible admissible representation π of G_F is spherical if its restriction to $M_F = GL(2, \mathcal{O}_F)$ contains the identity representation of M_F. These representations have been well known since a long time (Mautner, Amer. J. Math., LXXX, 1958, with generalizations to semi-simple groups by Satake). The results are as follows:

Theorem 11: An infinite-dimensional irreducible admissible representation π of G_F is spherical if and only if there are unramified characters μ_1 and μ_2 of F^* such that $\pi = \pi_{\mu_1, \mu_2}$ and π is not a special representation. The identity representation of $GL(2, \mathcal{O}_F)$ is then contained exactly once in π. If $\mathcal{O}_F$ is the largest ideal on which τ_F is trivial, then $\mathcal{W}(\pi)$ contains one and only one function W^0 invariant under $GL(2, \mathcal{O}_F)$ and such that $W^0(e) = 1$; it is given by

(242) $$W^0\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} 0 & \text{if } x \notin \mathcal{O}_F \\ |x|^{\frac{1}{2}} \sum\limits_{i+j=v_\varphi(x)} \mu_1(\varphi^i)\,\mu_2(\varphi^j) & \text{if } x \in \mathcal{O}_F . \end{cases}$$

We then have

(242 bis) $$\int W^0\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} |x|^{2s-1}\, d^*x = L_\pi(\mathrm{id}, s) .$$

We first prove that a supercuspidal representation π cannot contain the identity representation of $M_F = GL(2,\mathcal{O}_F)$. Suppose a $\xi \in \mathcal{H}(\pi) = \mathcal{S}(F^*)$ is invariant under M_F, hence under the matrices

(243) $$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \ b \in \mathcal{O}_F \ ; \ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \ a \in \mathcal{O}_F^* \ ; \ w .$$

We already get $\tau_F(bx)\,\xi(x) = \xi(x)$, so that

(244) $$\xi(x) \neq 0 \implies x \in \mathcal{M}$$

where $\mathcal{M}$ is the largest ideal on which τ_F is trivial. Furthermore $\xi(ax) = \xi(x)$ for all $a \in \mathcal{O}_F^*$. Evidently ω_π has to be unramified, and if we now write the functional equation

(245) $$M_{\pi(w)\xi}(\omega_\pi - \chi \ , \ 1-s) = \gamma_\pi(\chi, s)\, M_\xi(\chi \ , \ 1-s)$$

for χ = id, we get, since $\pi(w)\xi = \xi$,

(246) $$\begin{aligned} \int \xi(x^{-1})\,\omega_\pi(x)\,|x|^{2s-1}\,d^*x &= \gamma_\pi(\chi, s) \\ &= \gamma_\pi(\mathrm{id}, s) \int \xi(x)\,|x|^{2s-1}\,d^*x . \end{aligned}$$

But $\gamma_\pi(\mathrm{id}, s) = \varepsilon_\pi(\mathrm{id}, s)$ is an <u>exponential</u> function [see (217) and the argument thereafter] of the form $a \cdot q^{2ns}$ for some integer n. Choosing a $u \in F^*$ such that $|u| = q^{-n}$ we thus get

(247) $$\int \xi(x^{-1})\,\omega_\pi(x)\,|x|^{2s-1}\,d^*x = a \int \xi(ux)\,|x|^{2s-1}\,d^*x ;$$

since $\xi(x)$ depend only on $|x|$ we conclude that

(248) $$\xi(x^{-1}) = a\omega_\pi(x^{-1})\,\xi(ux) .$$

Comparing with (244) we conclude that

(249) $$\xi(x) \neq 0 \implies x, x^{-1} \in \mathcal{M} .$$

But we may choose τ_F in such a way that $\mathcal{U} = \mathcal{Y}$ for instance ; then conditions $x \in \mathcal{U}$ and $x^{-1} \in \mathcal{U}$ are not compatible with each other, and we get $\xi = 0$. [The above argument is essentially Jacquet and Langlands' proof ; see p. 118 of their paper].

If π is spherical we thus have $\pi = \pi_{\mu_1, \mu_2}$, for some choice of μ_1 and μ_2 ; the fact that π_{μ_1, μ_2} contains the identity representation of M_F if and only if μ_1 and μ_2 are unramified with π non special is clear (in the "special" case the identity representation of M_F is that one-dimensional component of ρ_{μ_1, μ_2} we have discarded to define π_{μ_1, μ_2}, so that it is not contained in π_{μ_1, μ_2}). It is no less clear that the space $\mathcal{B}_{\mu_1, \mu_2}$ of π_{μ_1, μ_2} then contains only essentially one vector invariant under M_F, namely, the functions φ such that

$$\varphi\left[\begin{pmatrix} t' & * \\ 0 & t'' \end{pmatrix} m\right] = \mu_1(t')\, \mu_2(t'')\, |t'/t''|^{\frac{1}{2}} \varphi(e) . \tag{250}$$

To compute the corresponding Whittaker function

$$W(g) = \pi(g)\xi_{\varphi}(1) = \sum_{n} \int_{v(y)=n} \varphi\left[w^{-1}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} g\right] \overline{\tau_F}(y)\, dy \tag{251}$$

by (134), it is better to replace brute force computations by a recursion formula expressing that W is an eigenfunction of the Hecke operators, i.e., that W satisfies conditions

$$W\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right] = \tau_F(x)\, W(g) \quad , \quad W\left[\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} g\right] = \omega_{\pi}(t)\, W(g) \tag{252}$$

and, in addition,

$$W * \alpha = \lambda(\alpha) W \tag{253}$$

for every $\alpha \in \mathcal{H}_F$ left- and right-invariant under the compact group M_F, where $\lambda(\alpha)$ is determined by the fact that the function (250) satisfies the same condition as W.

If α is the characteristic function of a coset $M_F\, h\, M_F$, we get

(254)
$$\lambda(h)\ W(x) = \int W(xy^{-1})\,\alpha(y)\,dy = \int_{M_F h M_F} W(xy^{-1})\,dy$$
$$= \sum_{M_F / M_F \cap h^{-1} M_F h} W(xmh^{-1})$$

If in particular we choose $h = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}$ where ϖ is a uniformizing variable, we see at once that $M_F \cap h^{-1} M_F h$ is the subgroup $c \equiv 0 \bmod \mathfrak{p}$ of M_F ; if we denote it by B , and by $U_{\mathfrak{O}}$ the subgroup of matrices $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ in M_F , then the Bruhat decomposition for the $GL(2)$ group over the finite field $\mathfrak{O}/\mathfrak{p}$ shows that*

(255)
$$M_F = B \cup U_{\mathfrak{O}}\, w\, B \quad .$$

Hence (254) can be written as

(256)
$$W\left[x\begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right] + \sum_{\eta \in \mathfrak{O}/\mathfrak{p}} W\left[x\begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} w\begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right] = \lambda W(x) \ .$$

It is of course enough to compute the numbers $W_n = W\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}$. We then get at once

(257)
$$\lambda W_n = W_{n-1} + \omega_\pi(\varpi^{-1}) \sum_{\eta \in \mathfrak{O}/\mathfrak{p}} \tau_F(\eta\varpi^n)\ W_{n+1} \ ,$$
$$= W_{n-1} + \omega_\pi(\varpi^{-1})\ W_{n+1} \sum_{\eta \in \mathfrak{O}/\mathfrak{p}} \tau_F(\eta\varpi^n) \ .$$

It should first of all be observed that if we denote by $\mathfrak{p}^{-d}$ the largest ideal on which τ_F is trivial, then

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* These results, which are trivial for $GL(2)$, can be extended to general reductive groups; see N, Iwahori and H. Matsumoto (I. H. E. S., no. 25, 1965), or R. Steinberg s notes on Chevalley Groups (Yale, Dept. of Mathl, 1968), or forthcoming papers by F. Bruhat and J. Tits.

(258) $$W_n \neq 0 \Longrightarrow \varpi^n \in \mathfrak{y}^{-d} \qquad \text{i.e.,} \quad n \geq -d$$

because of the fact that, if $\eta \in \mathcal{O}$, we have

(259) $$W_n = W\left[\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}\right] = \tau_F(\varpi^n \eta)\, W_n \ .$$

If $n \geq -d$, then (257) reduces to

(260) $$q\omega_\pi(\varpi^{-1})\, W_{n+1} - \lambda W_n + W_{n-1} = 0$$

where $q = N(\mathfrak{y}) = \text{Card}\,(\mathcal{O}/\mathfrak{y})$. The formal series

(261) $$W(X) = \sum W_n X^n = \sum_{-d}^{+\infty} W\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix} X^n$$

thus satisfies

(262) $$\left[X^2 - \lambda X + q\omega_\pi(\varpi^{-1})\right] W(X) = q\omega_\pi(\varpi^{-1})\, W_{-d}\, X^{-d} \ ,$$

whence

(263) $$\sum W\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix} X^n = \frac{q\omega_\pi(\mathfrak{y})^{-1}\, W_{-d}\, X^{-d}}{q\omega_\pi(\mathfrak{y})^{-1} - \lambda X + X^2} \ .$$

The factor λ is computed at once in terms of the characters $\mu_i(x) = |x|^{s_i}$ if we apply (256) to the function (250) for $x = e$; we get

(264) $$\lambda = q^{1/2}(q^{s_1} + q^{s_2}.)$$

while

(265) $$q\omega_\pi(\mathfrak{y})^{-1} = q^{s_1 + s_2 + 1} \ .$$

hence

(266) $$\sum^{\infty} W_n X^n = \frac{q^{s_1+s_2+1}\, W_{-d}\, X^{-d}}{\left(X - q^{s_1+1/2}\right)\left(X - q^{s_2+1/2}\right)}$$
$$= W_{-d}\, X^{-d} \sum_0^{\infty} q^{-i(s_1+\frac{1}{2})} X^i \sum_0^{\infty} q^{-j(s_2+\frac{1}{2})} X^j \ .$$

from which we conclude that

(267) $$W\begin{pmatrix}\varpi^n & 0\\ 0 & 1\end{pmatrix} = W\begin{pmatrix}\varpi^{-d} & 0\\ 0 & 1\end{pmatrix}\sum_{i+j=n+d} q^{-i(s_1+\frac{1}{2})-j(s_2+\frac{1}{2})}$$

(268) $$W\begin{pmatrix}\varpi^n & 0\\ 0 & 1\end{pmatrix} = q^{-\frac{d}{2}}\,|\varpi^n|^{1/2}\, W\begin{pmatrix}\varpi^{-d} & 0\\ 0 & 1\end{pmatrix}\sum_{i+j=n+d} \mu_1(\varpi^i)\,\mu_2(\varpi^j)\ .$$

This formula leads at once to the proof of Theorem 11, including (242 bis).

17. Unitary representations: results

Let π be an admissible representation of G_F on a complex vector space $\mathcal{V}$. We shall say that π is pre-unitary if there exists on $\mathcal{V}$ an invariant positive-definite hermitian form (ξ, η). It is then clear that the operators $\pi(g)$ can be extended to unitary operators on the Hilbert space obtained from $\mathcal{V}$ by completion. We get in this way a unitary representation of G_F in the classical sense ; we call it the completion of π.

Lemma 10: Let π be a pre-unitary admissible representation of G_F on a vector space $\mathcal{V}$. Then the completion of π is topologically irreducible (no invariant closed subspace) if and only if π is algebraically irreducible.

Denote the completion of $\mathcal{V}$ by $\hat{\mathcal{V}}$ and the extension of $\pi(g)$ to $\hat{\mathcal{V}}$ by $\hat{\pi}(g)$. For every irreducible representation ϑ of M_F, let $\hat{\mathcal{V}}(\vartheta) \supset \mathcal{V}(\vartheta)$ be the subspace of vectors $\xi \in \hat{\mathcal{V}}$ which, under $\hat{\pi}(M_F)$, transform according to ϑ. The subspaces $\hat{\mathcal{V}}(\vartheta)$ are mutually orthogonal ; since $\hat{\mathcal{V}}$ is evidently the Hilbert direct sum of the various $\mathcal{V}(\vartheta)$ we thus see that

(269) $$\hat{\mathcal{V}}(\vartheta) = \mathcal{V}(\vartheta)\ ;$$

in other words $\mathcal{V}$ is the set of all M_F-finite vectors of $\hat{\mathcal{V}}$. Since M_F-finite vectors are dense in every closed invariant subspace $\mathcal{M}$ of $\hat{\mathcal{V}}$ (for trivial reasons M_F is compact) we thus see that such a subspace is the closure of its intersection with $\mathcal{V}$. Hence irreducibility of π implies irreducibility of $\hat{\pi}$. On the other hand let $\mathcal{M}$ be an invariant subspace of $\mathcal{V}$; since $\hat{\mathcal{V}} = \hat{\oplus}\, \mathcal{V}(\mathcal{O})$ and since $\mathcal{M} = \oplus \mathcal{M}(\mathcal{O})$ where $\mathcal{M}(\mathcal{O}) = \mathcal{M} \cap \mathcal{V}(\mathcal{O})$ we see that the closure $\hat{\mathcal{M}}$ of $\mathcal{M}$, which is invariant and given by $\hat{\mathcal{M}} = \hat{\oplus} \mathcal{M}(\mathcal{O})$, is nontrivial if $\mathcal{M}$ is. This concludes the proof of the Lemma.

We shall now state the main result:

<u>Theorem 12</u>: <u>The pre-unitary irreducible admissible representations of</u> G_F <u>are the following ones</u>:

(1) <u>the supercuspidal representations</u> π <u>such that</u> $|\omega_\pi(x)| = 1$;

(2) <u>the representations</u> π_{μ_1, μ_2} <u>of the principal series for which</u> μ_1 <u>and</u> μ_2 <u>are unitary</u>;

(3) <u>the representations</u> π_{μ_1, μ_2} <u>of the principal series for which</u> $\mu_2(x) = \overline{\mu_1(x)}^{-1}$ <u>and</u> $\mu(x) = |x|^\sigma$, $0 < \sigma < 1$;

(4) <u>the special representations</u> π <u>for which</u> $|\omega_\pi(x)| = 1$.

<u>In these four cases, the invariant scalar product is furthermore given, on the Kirillov model of</u> π, <u>by</u> (270) $(\xi, \eta) = \int \xi(x)\, \overline{\eta(x)}\, d^*x$. The theorem will be proved in several steps.

18. <u>Unitary representations</u>: <u>the supercuspidal case</u>

Let π be a supercuspidal representation, and assume $|\omega_\pi(x)| = 1$. Let $\mathcal{V}$ be the space of π and denote by $<.\,,\,.>$ the canonical duality between $\mathcal{V}$ and the space $\check{\mathcal{V}}$ of $\check{\pi}$. Choose once and for all a nonzero vector $\zeta_0 \in \check{\mathcal{V}}$ and consider, for any two $\xi, \eta \in \mathcal{V}$, the function

$$g \mapsto <\pi(g)\xi\,,\ \zeta_0><\pi(g)\eta\,,\ \zeta_0>\,;$$

it is invariant mod Z_F and, by Theorem 3, its support is compact mod Z_F. We thus get an invariant scalar product on $\mathcal{V}$ by defining

(271) $$(\xi, \eta) = \int_{G_F/Z_F} \langle\pi(g)\xi, \zeta_0\rangle \overline{\langle\pi(g)\eta, \zeta_0\rangle}\, dg .$$

It is positive definite, because $(\xi, \xi) = 0$ implies that ξ is orthogonal to all $\check{\pi}(g)\zeta_0$, hence vanishes since $\check{\pi}$ is irreducible. Hence π is pre-unitary, and this proves part (1) of Theorem 12.

It should be ovserved that if π is given as a Kirillov representation, so that $\mathcal{V} = \mathcal{K}(\pi) = \mathcal{S}(F^*)$, then the scalar product on $\mathcal{S}(F^*)$ has to be invariant under the irreducible family of operators

(272) $$\{x \mapsto \xi(x)\} \mapsto \{x \mapsto \tau_F(bx)\, \xi(ax)\} ;$$

this leaves no choice, and we get, up to a constant factor,

(273) $$(\xi, \eta) = \int \xi(x)\, \overline{\eta(x)}\, d^*x ,$$

as asserted in Theorem 8.

19. Unitary representations in the principal series

Let π be a pre-unitary irreducible admissible representation on a vector space $\mathcal{V}$. We can then define a semi-linear mapping $J : \mathcal{V} \to \check{\mathcal{V}}$ by

(274) $$(\xi, \eta) = \langle \xi, J\eta \rangle ,$$

and the invariance of the scalar product means that $J \circ \pi(g) = \check{\pi}(g) \circ J$. If we consider the conjugate $\bar{\pi}$ of π [defined by replacing the complex structure of $\mathcal{V}$ by the conjugate one] it is clear that J will now define an isomorphism between $\bar{\pi}$ and $\check{\pi}$. Conversely such an isomorphism defines on $\mathcal{V}$ an invariant nondegenerate hermitian form

(275) $$J(\xi, \eta) = \langle \xi , J\eta \rangle$$

which, however, may not be positive definite.

Now suppose that $\pi = \pi_{\mu_1,\mu_2}$. We know that $\check{\pi} = \pi_{-\mu_1,-\mu_2}$. It is clear on the other hand that $\bar{\pi} = \pi_{\bar{\mu}_1,\bar{\mu}_2}$ [consider the mapping $\varphi \mapsto \bar{\varphi}$ from $\mathcal{B}_{\mu_1,\mu_2}$ to $\mathcal{B}_{\bar{\mu}_1,\bar{\mu}_2}$]. For π to be quasi-unitary we must thus have

(276) $$\pi_{-\mu_1,-\mu_2} \sim \pi_{\bar{\mu}_1,\bar{\mu}_2}$$

i.e., (Theorem 7) either $(-\mu_1,-\mu_2) = (\bar{\mu}_1,\bar{\mu}_2)$ or $(-\mu_1,-\mu_2) = (\bar{\mu}_2,\bar{\mu}_1)$.

The first case means that μ_1 and μ_2 are unitary. In this case there actually is a positive definite invariant scalar product on $\mathcal{B}_{\mu_1,\mu_2}$, namely,

$$(\varphi,\psi) = \oint_{P_F/G_F} \varphi(g)\,\overline{\psi(g)}\,dg = \int_{M_F} \varphi(m)\,\overline{\psi(m)}\,dm$$

(277) $$= \int \varphi\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right] \overline{\psi\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right]}\,dx$$

$$= \int \xi_\varphi(x)\,\overline{\xi_\psi(x)}\,d^*x$$

by Plancherel's formula [use (113) and (120)]. This proves the case (2) of Theorem 12.

Suppose now that $-\mu_1 = \bar{\mu}_2$, $-\mu_2 = \bar{\mu}_1$ i.e., that

(278) $$\mu_1(x)\,\overline{\mu_2(x)} = 1 .$$

If we assume we are not in the case already studied then we have

(279) $$\mu(x) = \mu_1(x)\,\mu_2(x)^{-1} = |\mu_2(x)|^{-2} = |x|^\sigma$$

with a real exponent $\sigma \neq 0$, since otherwise μ_1 and μ_2 would be unitary. Since $\pi_{\mu_1,\mu_2} \sim \pi_{\mu_2,\mu_1}$ we may even assume $\sigma > 0$, and since the special representations are excluded for the time being we have $\sigma \neq 1$. This will allow us to construct an "intertwining operator"

(280) $$A : \mathcal{B}_{\mu_1,\mu_2} \mapsto \mathcal{B}_{\mu_2,\mu_1} = \mathcal{B}_{-\bar{\mu}_1,-\bar{\mu}_2} ,$$

given by

(281) $$A\varphi(g) = \int \varphi\left[w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right] dx .$$

This integral converges since we have

(282) $$\varphi\left[w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right] \asymp \mu^{-1}(x)|x|^{-1} = |x|^{-\sigma-1}$$

for $|x|$ large, by (119) ; and the fact that A defines a mapping of $\mathcal{B}_{\mu_1,\mu_2}$ in $\mathcal{B}_{\mu_2,\mu_1}$ compatible with ρ_{μ_1,μ_2} and ρ_{μ_2,μ_1} can be seen easily. Of course (*) $A \neq 0$.

It is now clear that if there is a positive definite invariant scalar product on the space $\mathcal{B}_{\mu_1,\mu_2}$ of π_{μ_1,μ_2} [we exclude the special representations for the time being] it is given by

(283) $$\begin{aligned}(\varphi_1,\varphi_2) &= c\langle A\varphi_1 , \overline{\varphi_2}\rangle = c \oint_{P_F/G_F} A\varphi_1(g)\cdot\overline{\varphi_2(g)}\, dg\\ &= c\int A\varphi_1\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right]\cdot\overline{\varphi}_2\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right] dx .\end{aligned}$$

with a constant c. But for a $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$ put

(284) $$\phi(x) = \varphi\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right] , \ \phi'(x) = A\varphi\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right] ;$$

we have, by the definition of A,

(*) Use formula (285) below.

$$\phi'(x) = \int\varphi\left[w\begin{pmatrix}1 & -y\\ 0 & 1\end{pmatrix}w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right]dy = \int\varphi\left[\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right]dy$$

(285)
$$= \int\varphi\begin{pmatrix}1 & x\\ y & 1+xy\end{pmatrix}dy = \int\mu^{-1}(y)|y|^{-1}\phi\left(\frac{1+xy}{y}\right)dy \qquad \text{by (117)}$$

$$= \int\phi(x+y)\,\mu(y)\,d^*y \qquad \text{where } d^*y = |y|^{-1}dy\,.$$

Hence if we put

(286)
$$\varphi_i\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right] = \phi_i(x)$$

we get

(287)
$$(\varphi_1, \varphi_2) = c\int\phi_1'(x)\,\overline{\phi_2(x)}\,dx = c\int\int\phi_1(x+y)\,\overline{\phi_2(x)}\,|y|^{\sigma}dx\,d^*y\quad .$$

We thus see that π_{μ_1,μ_2} is unitary if and only if there is a constant $c(\sigma) \neq 0$ such that

(288)
$$c(\sigma)\int\int\phi(x+y)\,\overline{\phi(x)}\,|y|^{\sigma}dx\,d^*y \geq 0$$

for all $\phi \in \mathcal{F}(\mu)$ - cf. Lemma 9. Since this must be the case at least for all $\phi \in \mathcal{S}(F)$ we see that the measure $c(\sigma)|y|^{\sigma}d^*y$ must be positive-definite. If conversely this condition is satisfied, and if we consider, for any $\phi \in \mathcal{F}(\mu)$, the positive-definite function

(289)
$$\Psi(y) = \int\phi(x+y)\,\overline{\phi(x)}\,dx\;,$$

then $c(\sigma)\Psi(y)|y|^{\sigma}d^*y$ is a positive-definite measure with finite total mass, from which it follows that $\int c(\sigma)\,\Psi(y)|y|^{\sigma}d^*y \geq 0$.

We thus see that the representation π_{μ_1,μ_2} under consideration is unitary if and only if $|y|^{\sigma}d^*y$ is proportional to a positive-definite measure, i.e., if the Fourier transform distribution of $|y|^{\sigma}d^*y$ is proportional to a positive measure. But it is clear that this Fourier transform induces on F^* a distribution proportional to $|x|^{1-\sigma}d^*x$. This cannot extend to a measure on F unless $\sigma < 1$.

If conversely we have $0 < \sigma < 1$ then by (167) there is a constant $\gamma(\sigma)$ such that

$$\int \hat{\phi}(y)|y|^{\sigma}d^*y = \frac{1}{\gamma(\sigma)} \int \phi(x)|x|^{1-\sigma}d^*x \tag{290}$$

for all $\phi \in \mathcal{S}(F)$, which means that $|y|^{\sigma}d^*y$ is the Fourier transform of the measure $\frac{1}{\gamma(\sigma)}|x|^{1-\sigma}d^*x$. We then get an invariant scalar product on $\mathcal{B}_{\mu_1,\mu_2}$ by putting

$$\begin{aligned}(\varphi_1, \varphi_2) &= \gamma(\sigma)\int\int \phi_1(x+y)\,\overline{\phi_2(x)}\,|y|^{\sigma}\,dx\,d^*y \\ &= \gamma(\sigma)\int\int \varphi_1\left[w^{-1}\begin{pmatrix}1 & x+y\\ 0 & 1\end{pmatrix}\right]\bar{\varphi}_2\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right]|y|^{\sigma}\,dx\,d^*y\end{aligned} \tag{291}$$

i.e.,

$$(\varphi_1, \varphi_2) = \gamma(\sigma)\int\int \varphi_1\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right]\bar{\varphi}_2\left[w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right]|x-y|^{\sigma-1}\,dx\,dy\,. \tag{292}$$

This concludes the proof of assertion (3) of Theorem 12.

We still have to prove that (φ_1, φ_2) is given by formula (270) in terms of the images ξ_{φ_1} and ξ_{φ_2} of φ_1 and φ_2 in the Kirillov model of π_{μ_1,μ_2}. But we have

$$\begin{aligned}(\varphi_1, \varphi_2) &= \int \bar{\phi}_2(x)\,dx \cdot \gamma(\sigma)\int \phi_1(x+y)|y|^{\sigma}\,d^*y \\ &= \int\int \hat{\phi}_1(z)\,\tau_F(xz)\,\bar{\phi}_2(x)|z|^{1-\sigma}\,dx\,d^*z\end{aligned} \tag{292}$$

since the Fourier transform of $y \mapsto \phi_1(x+y)$ is evidently $z \mapsto \hat{\phi}_1(z)\,\tau_F(xz)$. Hence

$$(\varphi_1, \varphi_2) = \int \hat{\phi}_1(z)\,\overline{\hat{\phi}_2(z)}\,|z|^{1-\sigma}\,d^*z\,, \tag{293}$$

and since we have in a general way

(294) $$\xi_{\varphi}(z) = \mu_2(z)\,|z|^{1/2} \int \varphi\left[w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right] \overline{\tau}_F(yz)\, dy$$

by (134), we get

(295) $$((\varphi_1, \varphi_2)) = \int \xi_{\varphi_1}(z)\,\overline{\xi}_{\varphi_2}(z)\, |\mu_2(z)|^{-2}\, |z|^{-1}\, |z|^{1-\sigma}\, d^*z$$

which, since $|\mu_2(z)|^{-2} = |z|^{\sigma}$ in this case, eventually leads to the formula of Theorem 12, namely

(296) $$((\varphi_1, \varphi_2)) = \int \xi_{\varphi_1}(x)\, \overline{\xi_{\varphi_2}(x)}\, d^*x.$$

The convergence is clear from the table on p. 1.36.

20. Unitary representations : the special case

Let π_{μ_1,μ_2} be a special representation ; we may assume that $\mu(x) = |x|$. If π is pre-unitary then $\overline{\pi} \sim \check{\pi}$, which, as we have seen for the principal series, implies that either μ_1 and μ_2 are unitary, which is clearly impossible here, or $\mu_1(x)\,\overline{\mu}_2(x) = 1$. We thus see that if π is to be unitary then we must have

(297) $$\mu_1(x) = |x|^{1/2}\,\chi(x) \quad , \quad \mu_2(x) = |x|^{-1/2}\,\chi(x)$$

with a unitary character χ.

The space of $\pi = \pi_{\mu_1,\mu_2}$ is the hyperplane $\mathcal{B}^0_{\mu_1,\mu_2}$ of $\mathcal{B}_{\mu_1,\mu_2}$ defined by the (invariant) condition

(298) $$\int \varphi\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right] dx = 0\ ,$$

and the space of $\check{\pi} = \pi_{-\mu_1,-\mu_2}$ is the quotient of $\mathcal{B}_{-\mu_1,-\mu_2}$ by the one-dimensional subspace orthogonal to $\mathcal{B}^0_{\mu_1,\mu_2}$. One should observe that since $\mathcal{B}^0_{\mu_1,\mu_2}$ is invariant under ρ_{μ_1,μ_2} we have

$$\int \varphi\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right] dx = 0 \tag{299}$$

for all $\varphi \in \mathcal{B}^0_{\mu_1,\mu_2}$ and $g \in G_F$, which means that the intertwining operator $A: \mathcal{B}_{\mu_1,\mu_2} \to \mathcal{B}_{\mu_2,\mu_1}$ here vanishes on $\mathcal{B}^0_{\mu_1,\mu_2}$. We thus cannot use it to define a nonzero invariant scalar product on $\mathcal{B}^0_{\mu_1,\mu_2}$. However, since we have here $\sigma = 1$ in the notation of the previous n^0, it would be natural to define the scalar product by a limiting process, i.e., by

$$\begin{aligned}(\varphi_1, \varphi_2) &= \lim_{\sigma=1-0} \gamma(\sigma) \int\int \varphi_1\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right] \overline{\varphi}_2\left[w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right] |x-y|^{\sigma-1}\, dx dy \\ &= \lim_{\sigma=1-0} \int \hat{\phi}_1(z) \overline{\hat{\phi}_2(z)}\, |z|^{1-\sigma}\, d^* z\,;\end{aligned} \tag{300}$$

since the condition for a $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$ to be in $\mathcal{B}^0_{\mu_1,\mu_2}$ means that the corresponding function

$$\phi(x) = \varphi\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right] \tag{301}$$

satisfies $\hat{\phi}(0) = 0$, we see that the limit under consideration does exist, and that

$$(\varphi_1, \varphi_2) = \int \hat{\phi}_1(x)\, \overline{\hat{\phi}_2(x)}\, d^* x = \int \xi_{\varphi_1}(x)\, \overline{\xi_{\varphi_2}(x)}\, d^* x. \tag{302}$$

To conclude the proof, we need only to check that the above scalar product is invariant under π_{μ_1,μ_2}. The invariance under triangular matrices being clear, we need only to check the invariance under w. To do that we first compute, for a given $\varphi \in \mathcal{B}^0_{\mu_1,\mu_2}$, the number

$$\lim_{\sigma\to 1} \gamma(\sigma) \int \phi(x)\, |x-y|^{\sigma-1}\, dx = \lim \gamma(\sigma) \int \phi(x+y)\, |x|^{\sigma}\, d^* x\,. \tag{303}$$

It can be seen at once [use (240) and (241)] that

(304) $$\gamma(\sigma) = \frac{1-q^{-\sigma}}{1-q^{\sigma-1}}\, q^{d(\frac{1}{2}-\sigma)} \qquad \text{where } q = N(\mathfrak{P}) .$$

Assuming $y = 0$ we thus have to compute

(305) $$\lim \frac{\int \phi(x)|x|^{\sigma}\, d^*x}{1-q^{\sigma-1}} = \lim \frac{\int \phi(x)\,(|x|^{\sigma}-|x|)\, d^*x}{1-q^{\sigma-1}}$$

since $\int \phi(x)\, dx = 0$ in $\mathcal{B}^{0}_{\mu_1,\mu_2}$. Taking derivatives with respect to σ (L'Hospital's rule) and observing that

(306) $$\frac{d}{d\sigma}|x|^{\sigma} = |x|^{\sigma}\log|x| = -v(x)|x|^{\sigma}\log q$$

we eventually see that

(307) $$\lim_{\sigma\to 1-0} \gamma(\sigma)\int \phi(x+y)|x|^{\sigma}\, d^*x = q^{-d/2}(1-\frac{1}{q})\int \phi(x+y)\, v(x)\, dx ,$$

provided of course that $\Phi \in \mathcal{B}^{0}_{\mu_1,\mu_2}$; the derivation under the $\int$ sign is justified because of the fact that the integral $\int \phi(x+y)\, v(x)\, dx$ is absolutely convergent.

We now get

(308) $$\begin{aligned}(\varphi_1, \varphi_2) &= \lim \gamma(\sigma)\int\int \phi_1(x)\, \overline{\phi}_2(y)\,|x-y|^{\sigma-1}\, dx\, dy \\ &= q^{-d/2}(1-\frac{1}{q})\int\int \phi_1(x)\, \overline{\phi}_2(y)\, v(x-y)\, dx\, dy .\end{aligned}$$

We can now prove the invariance under $\pi(w)$. We have [we neglect the constant $q^{-d/2}(1-\frac{1}{q})$ in the following computation]

$$(\pi(w)\varphi_1\,,\ \pi(w)\varphi_2) = \int\int \varphi_1\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}w\right]\,\overline{\varphi}_2\left[w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}w\right]\,v(x-y)\,dx\,dy$$

$$= \int\int \varphi_1\begin{pmatrix}1 & 0\\ -x & 1\end{pmatrix}\overline{\varphi}_2\begin{pmatrix}1 & 0\\ -y & 1\end{pmatrix}v(x-y)\,dx\,dy$$

(309)
$$= \int\int \phi_1(-x^{-1})\,|x|^{-2}\,\overline{\phi}_2(-y^{-1})\,|y|^{-2}\,v(x-y)\,dx\,dy$$

$$= \int\int \phi_1(x)\,\overline{\phi}_2(y)\,v_1(x^{-1}-y^{-1})\,dx\,dy$$

$$= \int\int \phi_1(x)\,\overline{\phi}_2(y)\,[v_1(x-y) - v(xy)]\,dx\,dy\,,$$

so that it remains to check that

(310)
$$\int\int \phi_1(x)\,\overline{\phi}_2(y)\,v(xy)\,dx\,dy = 0\,.$$

But this is clear since we assume that $\int \phi_1(x)\,dx = \int \phi_2(y)\,dy = 0$.

§2. The archimedean case

In this section we assume the ground field F to be $\mathbb{R}$ or $\mathbb{C}$, and we intend to show how the results of §1 can be "extended" to this case (a not too surprising fact, since the archimedean case was studied twenty years before the $\mathfrak{p}$-adic one...). We still put $G_F = GL(2, F)$ and let M_F denote the obvious (i.e. orthogonal or unitary) maximal compact subgroup of G_F.

We shall not here give full proofs of the results because of two reasons. First of all we see no way of substantially improving Jacquet and Langlands' account of these results (§§5 and 6 of their monster). Secondly, the theory of unitary representations of $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})$ has been well known to many people for quite some time now, and extending the results to (non necessarily unitary) representations of $GL(2, \mathbb{R})$ or $GL(2, \mathbb{C})$ is a comparatively routine matter.

1. Admissible representations

Let π be a representation of G_F on a reasonable* topological vector space $\mathcal{H}$; for each $x \in G_F$ we have a continuous automorphism $\pi(x)$ of $\mathcal{H}$, and the mapping $x \longmapsto \pi(x)\xi$ is continuous for every $\xi \in \mathcal{H}$. We can then define an operator $\pi(f)$ on $\mathcal{H}$ for every continuous function (or even measure) with compact support, by means of formula

(1) $$\pi(f)\xi = \int \pi(x)\xi \cdot f(x)dx ,$$

and the mapping $f \longmapsto \pi(f)$ transforms convolution products into products of operators. For closed subspaces of $\mathcal{H}$, invariance under $\pi(G_F)$ is equivalent to invariance under $\pi(f)$ for "sufficiently many" functions f.

Let ϑ be an irreducible finite-dimensional representation of M_F; define

(2) $$\chi_{\vartheta}(m) = \dim(\vartheta) \cdot \mathrm{Tr}[\vartheta(m)]$$

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* We shall assume that $\mathcal{H}$ is locally convex and that the closed convex envelope of a compact subset of $\mathcal{H}$ is still compact. This makes it possible to integrate continuous functions with values in $\mathcal{H}$ and compact support.

for all $m \in M_F$, so that $\chi_{\vartheta} * \chi_{\vartheta} = \chi_{\vartheta}$. Considering χ_{ϑ} as a measure on G_F (with support M_F), we get on $\mathcal{H}$ a projection operator

$$E(\vartheta) = \pi(\bar{\chi}_{\vartheta}) = \int \pi(m)\cdot\overline{\chi_{\vartheta}(m)}\,dm \tag{3}$$

on a closed subspace $\mathcal{H}(\vartheta)$ of $\mathcal{H}$ --the space of vectors $\xi \in \mathcal{H}$ which, under $\pi(M_F)$, transform according to a finite multiple of ϑ.

The subspace

$$\mathcal{H}_0 = \bigoplus_{\vartheta} \mathcal{H}(\vartheta) \tag{4}$$

(the sum is direct) is dense in $\mathcal{H}$; it is the set of all M_F-finite vectors in $\mathcal{H}$ [vectors the transforms of which under $\pi(M_F)$ generate a finite dimensional subspace of $\mathcal{H}$]. Though $\mathcal{H}_0$ is stable under $\pi(M_F)$, it is generally not stable under $\pi(G_F)$; this is the main technical difference between the archimedean and the $\mathfrak{p}$-adic cases. However $\mathcal{H}_0$ is stable under many operators $\pi(f)$, e.g. those for which the function or measure f is left M_F-finite (i.e. such that the left translates $f(mx)$ generate a finite dimensional vector space), or is such that the transforms $f(mxm^{-1})$ of f under the elements of M_F stay in a finite-dimensional space, etc.

We shall be interested mainly in those representations for which $\mathcal{H}(\vartheta)$ is finite-dimensional; this is true (under mild assumptions of a topological nature*) if π is irreducible (no invariant closed subspace) - in this case we always have $\dim \mathcal{H}(\vartheta) \leq \dim(\vartheta)^2$ and even**

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* These assumptions are automatically satisfied if π is unitary, or finite-dimensional. See R. Godement, A Theory of Spherical Functions. I (Trans. Am. Math. Soc., 73 (1953).

** Formula (5) can be proved without looking at the classification of irreducible representations by first checking it is satisfied for finite-dimensional representations (which, if $F = \mathbb{C}$, rests upon the Clebsch-Gordan formula for decomposing the tensor product of two representations of $SU(2, \mathbb{C})$), and then making use of the general principle explained in the paper quoted in footnote (*), where a complete and direct proof of (5) will be found.

(5) $$\dim \mathcal{H}(\mathfrak{s}) \leq \dim \mathfrak{s},$$

which means that every irreducible representation of M_F occurs at most <u>once</u> in every (reasonable, e.g. unitary) irreducible representation of G_F. In any case, as soon as a representation π satisfies

(6) $$\dim \mathcal{H}(\mathfrak{s}) < +\infty$$

for all $\mathfrak{s}$, the vectors $\xi \in \mathcal{H}_0$ are "analytic", which means among other things that the "coefficient" $\langle\pi(x)\xi, \eta\rangle$ is an ordinary analytic function on the real Lie group G_F for every continuous linear form η on $\mathcal{H}$; and for every $\xi \in \mathcal{H}_0$ and every <u>distribution</u> μ with compact support on G_F there is in $\mathcal{H}$ a vector $\pi(\mu)\xi$ such that

(7) $$\langle\pi(\mu)\xi, \eta\rangle = \int \langle\pi(x)\xi, \eta\rangle d\mu(x)$$

for all η.

There are two obvious cases where we have $\pi(\mu)\xi \in \mathcal{H}_0$ for all $\xi \in \mathcal{H}_0$: if μ is a Dirac measure at a point m of M_F -- then $\pi(\mu) = \pi(m)$, evidently -- or if the transforms $d\mu(mxm^{-1})$ of μ under the elements of M_F generate a finite dimensional vector space, e.g. if $\mu \in \mathcal{U}(\mathfrak{g})$, the algebra (under convolution product) of distributions with support reduced at e; it is well known that $\mathcal{U}(\mathfrak{g})$ is canonically isomorphic to the envoloping algebra of the complex Lie algebra $\mathfrak{g}$ of the real Lie group G_F; and its elements can be identified with left invariant differential operators on G_F: the operator defined by μ is $f \longmapsto f * \check{\mu}$, where $\check{\mu}(x) = \mu(x^{-1})$.

We can now define a "Hecke algebra" $\mathcal{H}_F$ as follows*. We choose

(8) $$\mathcal{H}_F = \mathcal{U}(\mathfrak{g}) \qquad \text{if } F = \mathbb{C}$$

because, since $G_{\mathbb{C}}$ is connected, there is a good chance that the map (representation of $G_F \longmapsto$ representation of $\mathfrak{g}$) will be one-to-one. If $F = \mathbb{R}$ we cannot entertain such hopes, and something must be added to $\mathcal{U}(\mathfrak{g})$ in

* Jacquet and Langlands choose a much bigger one. Note also that Hecke never developed a taste for Lie algebras...

order to take care of the two connected components of $G_{\mathbb{R}}$. We add to $\mathcal{U}(\mathfrak{g})$ the Dirac measure ε_- at the point $\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$ of $G_{\mathbb{R}}$. In other words we define

(9) $$\mathcal{H}_F = \mathcal{U}(\mathfrak{g}) \oplus \varepsilon_- * \mathcal{U}(\mathfrak{g}) \text{ if } F = \mathbb{R},$$

the algebra (under convolution product) of distributions whose support is contained in the subgroup $\begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}$ of G_F.

If π is a representation of G_F on a topological vector space $\mathcal{H}$ and is such that $\dim \mathcal{H}(\mathcal{J}) < +\infty$ for all $\mathcal{J}$, then we get in a natural way a representation (still denoted by π) of $\mathcal{H}_F$ on the subspace $\mathcal{H}_0$; and if we associate with every closed subspace $\mathcal{L}$ of $\mathcal{H}$, invariant under $\pi(G_F)$, the subspace $\mathcal{L}_0 = \mathcal{L} \cap \mathcal{H}_0$ of $\mathcal{H}_0$, we get in this way a bijection $\mathcal{L} \longmapsto \mathcal{L}_0$ on the set of subspaces of $\mathcal{H}_0$ invariant under $\pi(\mathcal{H}_F)$. For unitary representations on Hilbert spaces the correspondence between representations of G_F and representations of $\mathcal{H}_F$ is furthermore one-to-one.

These well-known facts explain the following definition of admissible representations. Let π be a representation of the algebra $\mathcal{H}_F$ on a complex vector space $\mathcal{V}$. We shall say that π is admissible if the following conditions are satisfied:

(i) the restriction of π to the Lie algebra of M_F decomposes into finite-dimensional irreducible representations, with finite multiplicities;

(ii) for every $\xi \in \mathcal{V}$ and every $\eta \in \check{\mathcal{V}}$ (see below) there exists on G_F a function, which we denote by $\langle\pi(x)\xi, \eta\rangle$, such that

(10) $$\langle\pi(\mu)\xi, \eta\rangle = \int \langle\pi(x)\xi, \eta\rangle d\mu(x)$$

for all $\mu \in \mathcal{H}_F$.

What we denote by $\check{\mathcal{V}}$ is of course the set of all linear forms on $\mathcal{V}$ which, under the transposed of the operators $\pi(\mu)$, transform according to a finite dimensional representation of the Lie algebra of M_F. If we denote, for a given irreducible representation $\mathcal{J}$ of M_F, by $\mathcal{V}(\mathcal{J})$ the set of all $\xi \in \mathcal{V}$ which transform under a multiple of $\mathcal{J}$ (we consider $\mathcal{J}$ as a representation of the obvious subalgebra of $\mathcal{H}_F$ as well), then condition (i) insures that

(11) $$\mathcal{V} = \oplus \mathcal{V}(\mathcal{S}), \quad \check{\mathcal{V}} = \oplus \mathcal{V}(\mathcal{S})^* \subset \mathcal{V}^*,$$

and its purpose is of course to avoid considering representations of $\mathcal{H}_F$ which do not correspond to representations of G_F (note that G_F is not simply connected, even if $F = \mathbb{C}$, and still less if $F = \mathbb{R}$).

For an admissible representation π of $\mathcal{H}_F$ on a vector space $\mathcal{V}$, irreducibility will mean the usual and purely algebraic concept--no invariant subspace whatsoever. Condition (ii) implies that π can then (in many different ways) be realized on a space of functions on G_F. More accurately there is always an isomorphism $\xi \longmapsto \varphi_\xi$ of $\mathcal{V}$ on a space of functions on G_F with the following properties:

(a) the functions φ_ξ are analytic and right M_F-finite;

(b) for every $\xi \in \mathcal{V}$ and $\mu \in \mathcal{H}_F$ we have

(12) $$\varphi_{\pi(\mu)\xi} = \varphi_\xi * \check{\mu}.$$

Such an isomorphism can be obtained for instance by putting

(13) $$\varphi_\xi(x) = \langle \pi(x)\xi, \eta \rangle$$

with a given non zero $\eta \in \check{\mathcal{V}}$; in this case the functions φ_ξ are also left M_F-finite. But as we shall see there are other ways of constructing irreducible admissible representations by letting $\mathcal{H}_F$ operate through right convolutions on function spaces.

2. <u>The representations</u> ρ_{μ_1,μ_2}

Let μ_1 and μ_2 be two characters of F^*. As in §1 we shall denote by $\mathcal{B}_{\mu_1,\mu_2}$ the space of functions $\varphi(g)$ satisfying

(14) $$\varphi\left[\begin{pmatrix} t' & * \\ 0 & t'' \end{pmatrix} g\right] = \mu_1(t')\mu_2(t'')\,|t'/t''|_F^{1/2}\,\varphi(g)$$

and which are right M_F-finite. It is clear that $\mathcal{B}_{\mu_1,\mu_2} * \mathcal{H}_F \subset \mathcal{B}_{\mu_1,\mu_2}$, hence a representation ρ_{μ_1,μ_2} of $\mathcal{H}_F$ on $\mathcal{B}_{\mu_1,\mu_2}$ given by

$$\rho_{\mu_1,\mu_2}(X) = \varphi * \check{X} \tag{15}$$

for any $X \in \mathcal{H}_F$. This representation is clearly admissible: the functions φ are determined by their restrictions to M_F, which are "trigonometric polynomials" on M_F. The first fundamental result is

Theorem 1. Every irreducible admissible representation of $\mathcal{H}_F$ is contained in a representation ρ_{μ_1,μ_2}.

This is in fact an already old result of Harish-Chandra's*, valid with minor modifications for all reductive real Lie groups. It means that the super-cuspidal representations of §1 do not exist here--there are not many analytic functions with compact support mod Z_F, either...

To get an elementary proof of Theorem 1, one first needs to select a basis of $\mathcal{B}_{\mu_1,\mu_2}$ adapted to the action of M_F; we shall explain it in the case where $F = \mathbb{R}$, the other case being similar.

We can write

$$\mu_i(t) = |t|^{s_i}\operatorname{sgn}(t)^{m_i} \tag{16}$$

with $m_i = 0$ or 1; the character $\mu = \mu_1 - \mu_2$ then is given by

$$\mu(t) = |t|^s\operatorname{sgn}(t)^m \qquad s = s_1 - s_2, \quad m = |m_1 - m_2| \,. \tag{17}$$

For every integer $n \equiv m \pmod 2$ there is in $\mathcal{B}_{\mu_1,\mu_2}$ a function φ_n such that

$$\varphi_n\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = e^{ni\theta} \tag{18}$$

and only one since $G_F = P_F M_F$ where P_F is the subgroup $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. The set of functions φ_n is a basis of $\mathcal{B}_{\mu_1,\mu_2}$. Now the complex Lie algebra $\mathfrak{g} = M_2(\mathbb{C})$ of $GL(2, \mathbb{R})$ has a basis whose elements are the matrices

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* Harish-Chandra's theorem actually states that every irreducible admissible representation of $\mathcal{U}(\mathfrak{g})$ can be realized on $\mathcal{B}'/\mathcal{B}''$ where $\mathcal{B}'$ and $\mathcal{B}''$ are two invariant subspaces of a suitable $\mathcal{B}_{\mu_1,\mu_2}$, with $\mathcal{B}'' \subset \mathcal{B}'$. But for $GL(2)$ it turns out that we can always choose $\mathcal{B}'' = 0$.

$$(19)\qquad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$V_+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad V_- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$

and it is easy to see that

$$(20)\qquad \rho(U)\varphi_n = in\varphi_n, \qquad \rho(Z)\varphi_n = (s_1 + s_2)\varphi_n,$$
$$\rho(V_+)\varphi_n = (s+1+n)\varphi_{n+2}, \qquad \rho(V_-)\varphi_n = (s+1-n)\varphi_{n-2},$$

where we write ρ instead of ρ_{μ_1,μ_2}. Furthermore the Dirac measure ε_- at $\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$ operates as

$$(21)\qquad \rho(\varepsilon_-)\varphi_n = (-1)^{m_1}\varphi_{-n}.$$

Since it is clear in advance that every irreducible admissible representation of $\mathcal{H}_F$ has a basis whose elements are eigenvectors of U, it is easy, by making use of the classical computations of Bargman for the unitary representations of $SL(2, \mathbb{R})$, to see that every such representation can be imbedded in some ρ_{μ_1,μ_2}.

3. <u>Irreducible components of ρ_{μ_1,μ_2}</u> (case $F = \mathbb{R}$)

Assume $F = \mathbb{R}$. The relations above show that

$$(22)\qquad \rho(V_+)^p\varphi_n = (s+1+n)\dots(s+2p-1+n)\varphi_{n+2p}$$

$$(23)\qquad \rho(V_-)^p\varphi_n = (s+1-n)\dots(s+2p-1-n)\varphi_{n-2p}.$$

It follows that ρ is irreducible if there is no integer $n \equiv m \bmod 2$ such that $s \equiv n+1 \bmod 2$; in other words ρ is irreducible unless

$$(24)\qquad s \equiv m+1 \bmod 2,$$

which means that

$$(25)\qquad \mu(t) = |t|^{m+1+2q}\operatorname{sgn}(t)^m = t^p\operatorname{sgn}(t)$$

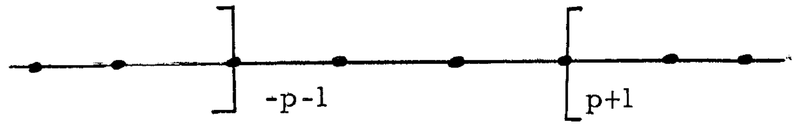
for some integer p [writing $\mu(t) = |t|^s\operatorname{sgn}(t)^m$ we then have $s = p$ and $m \equiv p+1 \bmod 2$].

If $\mu(t) = t^p \operatorname{sgn}(t)$ for some integer p, so that $s = p$ and $m \equiv p+1 \bmod 2$, we must distinguish two cases depending on the sign of p.

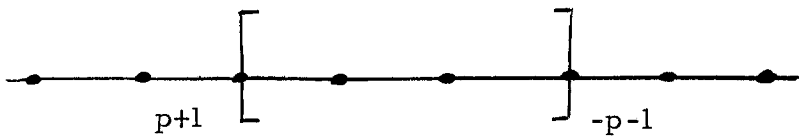
In all cases we have, since $s = p \equiv m+1 \bmod 2$,

(26) $$\rho(V_-)\varphi_{p+1} = \rho(V_+)\varphi_{-p-1} = 0 \ ,$$

and $\rho(V_-)\varphi_n \neq 0$ for $n \neq p+1$, as well as $\rho(V_+)\varphi_n \neq 0$ for $n \neq -p-1$. We find at once the non-trivial subspaces of $\mathcal{B}_{\mu_1,\mu_2}$ invariant by $\mathcal{U}(\mathfrak{g})$; there are three of



$p \geq 0$



$p < 0$

them, namely the two subspaces below:

(27) $$\{\ldots, \varphi_{-p-5}, \varphi_{-p-3}, \varphi_{-p-1}\} \ ,$$
$$\{\varphi_{p+1}, \varphi_{p+3}, \varphi_{p+5}, \ldots\},$$

and either their intersection (if $p < 0$) or their sum (if $p \geq 0$). If we take into account the operator $\rho(\varepsilon_-)$, which maps φ_n on $\pm\varphi_{-n}$, then we see that $\mathcal{B}_{\mu_1,\mu_2}$ contains only one non trivial subspace invariant under $\rho(\mathcal{H}_F)$ if $p \neq 0$, and none if $p = 0$. We eventually get the following result:

Theorem 2. The representation ρ_{μ_1,μ_2} of $\mathcal{H}_{\mathbb{R}}$ is irreducible except if

(28) $$\mu(t) = t^{\varphi} \operatorname{sgn}(t)$$

for some integer $p \neq 0$.

If $p > 0$ then $\mathcal{B}_{\mu_1,\mu_2}$ contains exactly one invariant subspace

(29) $$\mathcal{B}^s_{\mu_1,\mu_2} = \{\ldots, \varphi_{-p-3}, \varphi_{-p-1}, \varphi_{p+1}, \varphi_{p+3}, \ldots\}$$

and the quotient $\mathcal{B}^f_{\mu_1,\mu_2} = \mathcal{B}_{\mu_1,\mu_2} / \mathcal{B}^s_{\mu_1,\mu_2}$ is finite dimensional.

If $p < 0$ *then the only invariant subspace of* $\mathcal{B}_{\mu_1,\mu_2}$ *is*

(30) $$\mathcal{B}^f_{\mu_1,\mu_2} = \{\varphi_{p+1}, \varphi_{p+3}, \ldots, \varphi_{-p-3}, \varphi_{-p-1}\};$$

it is finite-dimensional, and the quotient

(31) $$\mathcal{B}^s_{\mu_1,\mu_2} = \mathcal{B}_{\mu_1,\mu_2}/\mathcal{B}^f_{\mu_1,\mu_2}$$

is infinite dimensional.

We shall denote by π_{μ_1,μ_2} the representation ρ_{μ_1,μ_2} if it is irreducible, or, in case it is not, the obvious representation on the finite dimensional space $\mathcal{B}^f_{\mu_1,\mu_2}$. The representation on the infinite dimensional subspace or quotient $\mathcal{B}^s_{\mu_1,\mu_2}$ will be denoted by σ_{μ_1,μ_2}; it is defined only if $\mu(t) = t^p \operatorname{sgn}(t)$ with a non zero integer p. With these notations we get a complete classification of the representations:

(a) *every irreducible admissible representation is a* π_{μ_1,μ_2} *or a* σ_{μ_1,μ_2};

(b) *we have the following equivalences between these representations:*

(32) $$\pi_{\mu_1,\mu_2} \sim \pi_{\mu_2,\mu_1}$$

(33) $$\sigma_{\mu_1,\mu_2} \sim \sigma_{\mu_2,\mu_1} \sim \sigma_{\mu_1+\eta,\,\mu_2+\eta} \sim \sigma_{\mu_2+\eta,\,\mu_1+\eta}$$

where $\eta(t) = \operatorname{sgn}(t)$;

(c) *there are no other relations between the irreducible representations than the ones listed above.*

The *infinite* dimensional π_{μ_1,μ_2} make up the *principal series*; the set of representations σ_{μ_1,μ_2} will be called the *discrete series.*

It should be observed that if $\mu(t) = t^p \operatorname{sgn}(t)$ with an integer $p < 0$, then the subspace

(34) $$\mathcal{B}^f_{\mu_1,\mu_2} = \{\varphi_{p+1}, \ldots, \varphi_{-p-1}\}$$

of $\mathcal{B}_{\mu_1,\mu_2}$ can be described quite simply. In fact we have $\mu_1\mu_2^{-1}(t) = t^p\operatorname{sgn}(t)$, hence every $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$ satisfies

(35) $$\begin{aligned}\varphi[\begin{pmatrix} t' & * \\ 0 & t'' \end{pmatrix} g] &= \mu_1(t')\mu_2(t'')|t'/t''|^{1/2}_{\mathbb{R}}\varphi(g) \\ &= \mu_1(t't'')t''^{-p}\operatorname{sgn}(t'')|t'/t''|^{1/2}_{\mathbb{R}}\varphi(g) \\ &= \mu_1(t't'')|t't''|^{1/2}_{\mathbb{R}}t''^{-p-1}\varphi(g).\end{aligned}$$

Hence $\mathcal{B}_{\mu_1,\mu_2}$ contains all functions

(36) $$\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu_1(\det g)|\det g|^{1/2}_{\mathbb{R}}f(c, d)$$

where f is an homogeneous polynomial of degree $-p-1 \geq 0$. The set of these functions is clearly $\mathcal{B}^f_{\mu_1,\mu_2}$.

The equivalences (32) and (33) are easily explained. If we consider the (not always irreducible) representation ρ_{μ_1,μ_2}, which is described by formulas (20) and (21), then ρ_{μ_2,μ_1} is obtained from these formulas by replacing s by $-s$ and m_1 by m_2. If we denote by (φ_n) and (φ'_n) the canonical basis for $\mathcal{B}_{\mu_1,\mu_2}$ and $\mathcal{B}_{\mu_2,\mu_1}$, and if we assume first that $\mu(t)$ is not $t^p\operatorname{sgn}(t)$, then we get an isomorphism T of $\mathcal{B}_{\mu_1,\mu_2}$ on $\mathcal{B}_{\mu_2,\mu_1}$, compatible with the action of $\mathcal{H}_{\mathbb{R}}$ by

(37) $$T\varphi_n = \frac{\Gamma(\frac{-s+1+n}{2})}{\Gamma(\frac{s+1+n}{2})}\varphi'_n$$

[Observe that if either $\frac{s+1+n}{2}$ or $\frac{-s+1+n}{2}$ is a negative integer then $s \equiv m+1 \mod 2$, which is impossible if we are <u>not</u> in the discrete series ; hence T is defined and bijective] ; (32) follows from this construction. To prove the $\sigma_{\mu_1,\mu_2} \sim \sigma_{\mu_2,\mu_1}$

part of (33) we may assume that $s = -p \equiv m + 1 \pmod 2$ with a positive integer p. The number

$$(38) \qquad a_n = \lim_{s = -p} \frac{\Gamma\left(\frac{-s+1+n}{2}\right)}{\Gamma\left(\frac{s+1+n}{2}\right)}$$

is then defined for every $n \equiv m \pmod 2$ and the mapping $T : \mathcal{B}_{\mu_1, \mu_2} \to \mathcal{B}_{\mu_2, \mu_1}$ given by $T\varphi_n = a_n\varphi'_n$ still commutes with the actions of $\mathcal{H}_{\mathbb{R}}$. It is easily seen that $\mathrm{Ker}(T)$ is the invariant subspace $\mathcal{B}^f_{\mu_1, \mu_2}$ of $\mathcal{B}_{\mu_1, \mu_2}$, and that T induces an isomorphism from $\mathcal{B}^s_{\mu_1, \mu_2} = \mathcal{B}_{\mu_1, \mu_2} / \mathcal{B}^f_{\mu_1, \mu_2}$ onto the subspace $\mathcal{B}^s_{\mu_2, \mu_1}$ of $\mathcal{B}_{\mu_2, \mu_1}$, from which $\sigma_{\mu_1, \mu_2} \sim \sigma_{\mu_2, \mu_1}$ follows.

Finally, to prove that $\sigma_{\mu_1, \mu_2} \sim \sigma_{\mu_1 + \eta,\, \mu_2 + \eta}$ we may assume that $\mu(t) = t^p \operatorname{sgn}(t)$ with $p > 0$. The representation σ_{μ_1, μ_2} restricted to the subalgebra $\mathcal{U}(\mathfrak{g})$ of $\mathcal{H}_{\mathbb{R}}$ then decomposes into a direct sum of two invariant subspaces, namely, $\{\varphi_{p+1}, \varphi_{p+3}, \ldots, \}$ and $\{\ldots, \varphi_{-p-3}, \varphi_{-p-1}\}$ and we have a similar decomposition into $\{\varphi'_{p+1}, \varphi'_{p+3}, \ldots\}$ and $\{\ldots, \varphi'_{-p-3}, \varphi'_{-p-1}\}$ for $\sigma_{\mu_1 + \eta,\, \mu_2 + \eta}$. We then get at once an intertwining operator T by requiring that

$$(39) \qquad T\varphi_n = \begin{cases} \varphi'_n & \text{if } n > 0 \\ -\varphi'_n & \text{if } n < 0 . \end{cases}$$

4. <u>Irreducible components</u> of ρ_{μ_1, μ_2} (case $F = \mathbb{C}$). If $F = \mathbb{C}$ the situation is similar to what we have just seen except that there is no discrete series. The fundamental result then is

<u>Theorem</u> 3. <u>The representation</u> ρ_{μ_1, μ_2} <u>of</u> $\mathcal{H}_{\mathbb{C}}$ <u>is irreducible except if</u>

(40) $$\mu(t) = t^p \bar{t}^q \quad \text{with } p, q \in \mathbb{Z} \text{ and } pq > 0.$$

If $\mu(t) = t^p \bar{t}^q$ with $p, q < 0$ then $\mathcal{B}_{\mu_1, \mu_2}$ contains one invariant subspace $\mathcal{B}^f_{\mu_1, \mu_2}$; it is finite-dimensional and spanned by the functions

(41) $$\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu_1(\det g)\,|\det g|_{\mathbb{C}}^{-1/2}\,\phi_1(c, d)\,\bar{\phi}_2(c, d),$$

where ϕ_1 and ϕ_2 are homogeneous polynomials of degree $-p-1$ and $-q-1$ respectively. If $\mu(t) = t^p \bar{t}^q$ with $p, q > 0$ then $\mathcal{B}_{\mu_1, \mu_2}$ contains one invariant subspace $\mathcal{B}^s_{\mu_1, \mu_2}$, namely the orthogonal supplement to $\mathcal{B}^f_{-\mu_1, -\mu_2}$.

If we denote by ρ_n the $\mathbb{C}$-rational irreducible representation of $GL(2, \mathbb{C})$ [or $\mathcal{H}_{\mathbb{C}}$] on the space of homogeneous polynomials of degree $n-1$ in two variables, then for $\mu(t) = t^p \bar{t}^q$ we see that the restriction of ρ_{μ_1, μ_2} to $\mathcal{B}^f_{\mu_1, \mu_2}$ yields a representation π_{μ_1, μ_2} equivalent to

(42) $$\mu_1(\det g)\,|\det g|_{\mathbb{C}}^{-1/2}\,\rho_{-p}(g) \otimes \bar{\rho}_{-q}(g),$$

which is the most general irreducible finite-dimensional representation of the real Lie group $G_{\mathbb{C}}$. In other words ρ_{μ_1, μ_2} fails to be irreducible if and only if either ρ_{μ_1, μ_2} or its contragredient $\rho_{-\mu_1, -\mu_2}$ contains an irreducible finite dimensional representation of $G_{\mathbb{C}}$; this result is actually valid for all fields F.

For every couple of characters μ_1, μ_2 of $\mathbb{C}^*$ we shall now define an irreducible representation π_{μ_1, μ_2} as follows. We shall take

(43) $$\pi_{\mu_1, \mu_2} = \rho_{\mu_1, \mu_2} \quad \text{if it is irreducible;}$$

and if it is not then π_{μ_1, μ_2} will denote the finite dimensional representation contained (as a subspace or as a quotient space, depending on the situation) in ρ_{μ_1, μ_2}.

If $\mu(t) = t^p \bar{t}^q$ one can also define a representation σ_{μ_1,μ_2} on the infinite dimensional subspace $\mathcal{B}^s_{\mu_1,\mu_2}$ if $pq > 0$, or quotient space $\mathcal{B}^s_{\mu_1,\mu_2} = \mathcal{B}_{\mu_1,\mu_2}/\mathcal{B}^f_{\mu_1,\mu_2}$ if $pq < 0$. However, it can be proved that

$$\sigma_{\mu_1 \mu_2} = \pi_{\nu_1 \nu_2} \tag{44}$$

where ν_1 and ν_2 are defined by

$$\nu_1(t) = \bar{t}^{-q}\mu_1(t) \quad , \quad \nu_2(t) = \bar{t}^{q}\mu_2(t) \tag{45}$$

if $\mu(t) = t^p \bar{t}^q$. Jacquet and Langlands' proof of (44) [bottom of p. 231 of their paper] rests upon a general theorem of Harish-Chandra for semi-simple groups, and is not very illuminating. Somebody ought to improve it by explicitly constructing an isomorphism between the representation spaces of $\sigma_{\mu_1 \mu_2}$ and $\pi_{\nu_1 \nu_2}$.

We finally observe that the only nontrivial equivalences between irreducible representations of $G_{\mathbb{C}}$ are obtained from

$$\pi_{\mu_1,\mu_2} \sim \pi_{\mu_2,\mu_1}. \tag{46}$$

5. Kirillov model for an irreducible representation

We again let F denote $\mathbb{R}$ or $\mathbb{C}$, and we define

$$\tau_F(x) = \begin{cases} e^{2\pi i x} & \text{if} \quad F = \mathbb{R} \\ e^{2\pi i (x+\bar{x})} & \text{if} \quad F = \mathbb{C}. \end{cases} \tag{47}$$

Theorem 6. Let π be an irreducible infinite dimensional admissible representation of $\mathcal{H}_F$ on a vector space $\mathcal{V}$. Then there exists an isomorphism $\xi \mapsto W_\xi$ of $\mathcal{V}$ on a space $\mathcal{W}(\pi)$ of functions on G_F with the following properties:

(i) we have

$$W_\xi\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right] = \tau_F(x)W_\xi(g) \tag{48}$$

for all $x \in F$, $g \in G_F$ *and* $\xi \in \mathcal{V}$;

(ii) *each* W_ξ *is* C^∞ *and*

(49) $$W_{\pi(X)\xi} = W_\xi * \check{X}$$

for all $\xi \in \mathcal{V}$ *and* $X \in \mathcal{H}_F$;

(iii) *for each* $\xi \in \mathcal{V}$ *there is an integer* N *such that*

(50) $$W_\xi \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = O(|t|^N) \quad \text{as} \quad |t| \to +\infty .$$

The space $\mathcal{W}(\pi)$ *is unique, and the mapping* $\xi \mapsto W_\xi$ *is unique up to a constant factor.*

The method of construction of $\mathcal{W}(\pi)$ which we are going to explain uses heavily the classical "integral formulas" for Whittaker functions; we could have used it for the principal series over a $\mathfrak{p}$-adic field as well (as Jacquet and Langlands actually do).

We may assume that π is contained in a representation ρ_{μ_1,μ_2}. Since $\rho_{\mu_1,\mu_2} \sim \rho_{\mu_2,\mu_1}$ we may assume that $|\mu(t)| = |t|_F^\sigma$ with $\sigma \geq 0$. We now consider on the plane F^2 the space $\mathcal{S}_0(F^2)$ of all M_F-finite functions ϕ in the Schwartz space $\mathcal{S}(F^2)$, and for such a ϕ we consider the function

(51) $$\varphi(g) = \mu_2(\det g)|\det g|_F^{-1/2} \int \phi\left[g^{-1}\begin{pmatrix} t \\ 0 \end{pmatrix}\right] \mu(t)\, dt ;$$

the integral converges always at infinity, and it converges around $t = 0$ as soon as $\sigma > -1$. It is more or less obvious that $\phi \mapsto \varphi$ maps $\mathcal{S}_0(F^2)$ *onto* $\mathcal{B}_{\mu_1,\mu_2}$. In fact, the map is surjective even if we replace $\mathcal{S}_0(F^2)$ by the much smaller space of functions of the form (*)

(*) In the following argument x, y, t, a, b should be assumed to be real if $F = \mathbb{R}$.

(52) $$\phi\begin{pmatrix}x\\y\end{pmatrix} = e^{-\pi(x\bar{x}+y\bar{y})}\, P(x,y,\bar{x},\bar{y})$$

where P is any polynomial in $x, y, \bar{x}$ and $\bar{y}$. To prove this, it is enough to compute the function (52) on $SO(2,\mathbb{R})$ or $SU(2,\mathbb{C})$, and to show one gets in this way all polynomial functions on these compact subgroups. But clearly if $a\bar{a}+b\bar{b}=1$ then

(53) $$\varphi\begin{pmatrix}a & b\\ -\bar{b} & \bar{a}\end{pmatrix} = \int \phi\begin{pmatrix}\bar{a}t\\ \bar{b}t\end{pmatrix}\mu(t)\,dt = \int e^{-\pi t\bar{t}}\, P(\bar{a}t, \bar{b}t, a\bar{t}, b\bar{t})\,\mu(t)\,dt$$

and the result follows at once.

The representation ρ_{μ_1,μ_2} is thus a quotient of the obvious representation of G_F (or rather $\mathcal{H}_F$) on $\mathcal{S}_0(F^2)$ or the subspace of it we have just described. To construct $\mathcal{W}(\pi)$, we put

(54) $$W\varphi(g) = \int \varphi\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right]\bar{\tau}_F(x)\,dx$$

for every $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$. Since we have, instead of relation (119) of §1, the less precise but equally useful estimate

(55) $$\varphi\left[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right] \prec \mu^{-1}(x)|x|_F^{-1} \qquad \text{for } |x|_F \text{ large}$$

(same proof as in §1) the convergence of the above integral is clear at any rate if $\sigma > 0$. Assuming for the time being $\sigma > 0$ and making use of (52) we get

(56) $$W\varphi(g) = \mu_2(\det g)|\det g|_F^{-1/2}\int\int \phi\left[g^{-1}\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix} w\begin{pmatrix}t\\0\end{pmatrix}\right]\bar{\tau}_F(x)\mu(t)\,dx\,dt\ ,$$

from which it follows by standard computations that

(57) $$W\varphi(g) = \mu_2(\det g)|\det g|_F^{-1/2}\int \phi_g\begin{pmatrix}t^{-1}\\ -t\end{pmatrix}\mu(t)|t|_F^{-1}\,dt = W_\phi^{\mu_1,\mu_2}(g)\ ,$$

where

(58) $$\phi_g\begin{pmatrix}x\\y\end{pmatrix} = \int \phi\left[g^{-1}\begin{pmatrix}z\\y\end{pmatrix}\right]\bar{\tau}_F(xz)\,dz$$

belongs, for every $g \in G_F$, to the space $\mathcal{S}(F^2)$, so that the integral for $W_\varphi(g)$ makes sense for all values of μ since the function $t \mapsto \phi_g\begin{pmatrix} t^{-1} \\ -t \end{pmatrix}$ is rapidly decreasing at 0 and at ∞.

If $\sigma > -1$ then we can, for every $\phi \in \mathcal{S}_0(F^2)$, define $W_\phi^{\mu_1,\mu_2}$ and the function (52); we shall see that although $W_\phi^{\mu_1,\mu_2}$ may not be given by (54), which makes sense only if $\sigma > 0$, we can, however, compute φ in terms of $W_\phi^{\mu_1,\mu_2}$ by means of a Fourier transformation. In fact, we have, if $\sigma > -1$,

(59)
$$\begin{aligned}\varphi(w) &= \int \phi\left[w^{-1}\begin{pmatrix} t \\ 0 \end{pmatrix}\right] \mu(t)\, dt = \int \phi\begin{pmatrix} 0 \\ t \end{pmatrix} \mu(t)\, dt \\ &= \int \mathcal{F}_1 \phi\begin{pmatrix} x \\ t \end{pmatrix} \mu(t)\, dx\, dt = \int \mathcal{F}_1 \phi\begin{pmatrix} xt^{-1} \\ t \end{pmatrix} \mu(t)\, |t|_F^{-1}\, dx\, dt\end{aligned}$$

where

(60)
$$\mathcal{F}_1 \phi\begin{pmatrix} x \\ y \end{pmatrix} = \int \phi\begin{pmatrix} z \\ y \end{pmatrix} \bar{\tau}_F(xz)\, dz = \phi_e\begin{pmatrix} x \\ y \end{pmatrix}.$$

But

(61)
$$g = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \Longrightarrow \phi_g\begin{pmatrix} u \\ v \end{pmatrix} = |x|_F\, \mathcal{F}_1 \phi\begin{pmatrix} xu \\ v \end{pmatrix}$$

so that

(62)
$$\begin{aligned}W_\phi^{\mu_1,\mu_2}\begin{pmatrix} -x & 0 \\ 0 & 1 \end{pmatrix} &= \mu_2(-x)|x|_F^{+1/2} \int \mathcal{F}_1 \phi\begin{pmatrix} -xt^{-1} \\ -t \end{pmatrix} \mu(t)|t|_F^{-1}\, dt \\ &= \mu_1(-1)\,\mu_2(x)|x|_F^{1/2} \int \mathcal{F}_1 \phi\begin{pmatrix} xt^{-1} \\ t \end{pmatrix} \mu(t)|t|_F^{-1}\, dt\,.\end{aligned}$$

Comparing with (59) we thus get

$$\begin{aligned}\varphi(w) &= \int W_\phi^{\mu_1\,\mu_2}\begin{pmatrix} -x & 0 \\ 0 & 1 \end{pmatrix} \mu_1(-1)\,\mu_2(x)^{-1}\, |x|_F^{-1/2}\, dx \\ &= \omega_\pi(-1) \int W_\phi^{\mu_1,\mu_2}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \mu_2^{-1}(x)|x|_F^{-1/2}\, dx\,;\end{aligned}$$

if we replace ϕ by its transform under $w^{-1}g$ we get more generally

$$(64) \qquad \varphi(g) = \omega_\pi(-1)\int W_\phi^{\mu_1\mu_2}\left[\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix} w^{-1}g\right] \mu_2^{-1}(x)|x|_F^{-1/2}\, dx \ ;$$

this formula is valid for $\sigma > -1$. It still leads at once to

$$(65) \qquad \varphi\left[w^{-1}\begin{pmatrix} 1 & y \\ 0 & 1\end{pmatrix} g\right] = \int W_\phi^{\mu_1\mu_2}\left[\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix} g\right] \mu_2^{-1}(x)|x|_F^{-1/2}\, \tau_F(xy)\, dx \ ,$$

a result that would be equivalent to (52) if we could make use of Fourier's inversion formula - but we cannot if $-1 < \sigma \leq 0$.

The above formula at any rate shows that $\varphi = 0$ implies $W_\phi^{\mu_1\mu_2} = 0$. We may thus define W_φ by (65) ; in other words, we now denote by W_φ the function on G_F such that

$$(66) \qquad \varphi\left[w^{-1}\begin{pmatrix} 1 & y \\ 0 & 1\end{pmatrix} g\right] = \int W_\varphi\left[\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix} g\right] \mu_2^{-1}(x)|x|_F^{-1/2}\, \tau_F(xy)\, dx$$

for all g and y. We shall denote by $\mathcal{W}_{\mu_1,\mu_2}$ the set of functions W_φ for all $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$, and by $\mathcal{W}(\pi)$ the subset of W_φ for those elements φ of $\mathcal{B}_{\mu_1,\mu_2}$ which belong to the space of π.

The space $\mathcal{W}(\pi)$ satisfies the conditions of Theorem 4. If we replace g by $\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} g$ we get $y + b$ instead of y in the left hand side of (66), hence

$$(67) \qquad \tau_F(bx)\, W_\varphi\left[\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix} g\right]$$

instead of $W_\varphi\left[\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix} g\right]$, from which condition (48) follows (choose $x = 1$). Condition (49) can be verified at once by differentiating with respect to g. As to the growth condition (50), it is enough to verify it for the function

$$(68) \qquad W\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix} = \mu_2(x)|x|_F^{1/2}\int \mathcal{F}_1\phi\begin{pmatrix} xt^{-1} \\ -t\end{pmatrix} \mu(t)|t|_F^{-1}\, dt$$

where $\phi\begin{pmatrix}x\\y\end{pmatrix} = e^{-\pi(x\bar{x} + y\bar{y})} P(x, y, \bar{x}, \bar{y})$ with a polynomial P. Then $\mathcal{F}_1\phi$ is given by a similar formula, so that (68) is a sum of functions of the form

$$f(x) = \lambda_1(x) \int e^{-\pi(x\bar{x}t^{-1}\bar{t}^{-1} + t\bar{t})} \lambda_2(t)\, d^*t \tag{69}$$

where λ_1 and λ_2 are characters of F^*, i.e., of functions

$$\lambda_1(x) \int_0^{+\infty} e^{-\pi(t^2 + t^{-2}x\bar{x})} t^\alpha\, d^*t\,. \tag{70}$$

But if we set $x\bar{x} = r^2$ with $r > 0$ then we have

$$\begin{aligned}
\int_0^\infty e^{-\pi(t^2 + t^{-2}r^2)} t^\alpha\, d^*t &= e^{-2\pi r} \int_0^\infty e^{-\pi/ t-t^{-1}r)^2} t^\alpha d^*t \\
&= r^{\alpha/2} e^{-2\pi r} \int_0^\infty e^{-\pi r(t-t^{-1})^2} t^\alpha\, d^*t \\
&\leq r^{\alpha/2} e^{-2\pi r} \int_0^\infty e^{-\pi(t-t^{-1})^2} t^\alpha\, d^*t \qquad \text{if } r \geq 1\,,
\end{aligned} \tag{71}$$

from which we get for each $W \in \mathcal{W}(\pi)$ a majoration at infinity of the following kind:

$$\left| W\begin{pmatrix}x & 0\\0 & 1\end{pmatrix}\right| \leq c|x|_F^\gamma\, e^{-2\pi(x\bar{x})^{1/2}}\;; \tag{72}$$

this is much better than (50)!

The above computation proves the existence of a space of functions $\mathcal{W}(\pi)$ satisfying the conditions of Theorem 4. We still have to prove $\mathcal{W}(\pi)$ is unique; we shall explain it for $F = \mathbb{R}$, the complex case being similarly treated. Let $\mathcal{W}$ be a space of functions on $G_{\mathbb{R}}$ satisfying the conditions of Theorem 4. We then have a basis W_n of $\mathcal{W}$ satisfying conditions (20) and (21); more accurately, if $\pi = \pi_{\mu_1,\mu_2}$ (principal series) with μ_1, μ_2 given by (16) then the index n runs over the set of all integers $n \equiv m \pmod 2$, while if $\pi = \sigma_{\mu_1,\mu_2}$ (discrete series) with $\mu(t) = t^p\,\mathrm{sgn}(t)$

and $p > 0$ then

(73) $$n \in \{ ,\ldots, -p-3, -p-1, p+1, p+3, \ldots \} .$$

In all cases, formulas (20) can be written as

(74)
$$W_n * \check{U} = in\, W_n, \quad W_n * \check{Z} = (s_1 + s_2)\, W_n ,$$
$$W_n * \check{V}_+ = (s+1+n)\, W_{n+2} , \quad W_n * \check{V}_- = (s+1-n)\, W_{n-2} ;$$

we identify the elements of the Lie algebra of $G_{\mathbb{R}}$ to distributions at e. Now the trick is to express $W_n * \check{U}$, etc... in terms of the functions

(75) $$F_n(t) = W_n \begin{pmatrix} |t|^{1/2}\operatorname{sgn}(t) & 0 \\ 0 & |t|^{-1/2} \end{pmatrix} = \omega_\pi \left(|t|^{-1/2}\right) W_n \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} ;$$

by expressing U, V_+ and V_- in terms of the elements

(76) $$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of the Lie algebra, it is easily found that the F_n must satisfy the following relations:(*)

(77)
$$(s+1+n)\, F_{n+2}(t) = 2tF_n'(t) - (4\pi t - n)\, F_n(t) ,$$
$$(s+1-n)\, F_{n-2}(t) = 2tF_n'(t) + (4\pi t - n)\, F_n(t) ,$$

from which one gets

(*) The formulas on p. 188 of Jacquet-Langlands are wrong; the factor u in these formulas should be replaced by $2\pi u$, because $\frac{d}{dt} e^{2\pi ut} \neq ue^{2\pi ut}$. Here we choose $u = 1$.

$$(78) \qquad F''_n(t) + \left(\frac{1 - s^2}{4t^2} + \frac{2rn}{t} - 1 \right) F_n(t) = 0 \quad ;$$

this second order equation evidently corresponds to the action of the Casimir operator $D = \frac{1}{2}V_- V_+ - iU - \frac{U^2}{2}$ of G.

We already know that these equations have a set of solutions F_n which, as $|t| \to +\infty$, tend to 0 exponentially. Let G_n be a set of solutions which, at infinity, grow at most as a power of $|t|$. Because of (78) the functions $F_n(t)\, G'_n(t) - F'_n(t)\, G_n(t)$ are constant for $t > 0$ and for $t < 0$; but they must evidently tend to 0 as $|t| \to \infty$. Hence, the G_n are proportional to the F_n, and this concludes the proof of Theorem 4.

One cannot explicitly compute the F_n (they are classical Whittaker functions) except in the case of the discrete series; and we shall need the result a little later. Then n belongs to the set (73), and if $n = p + 1$ then (77) shows that

$$(79) \qquad 2tF'_{p+1}(t) - (4\pi t - p - 1)\, F_{p+1}(t) = 0 \quad ;$$

since F_{p+1} must not blow up at infinity we get

$$(80) \qquad F_{p+1}(t) = \begin{cases} t^{\frac{1}{2}(p+1)} e^{-2\pi t} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t < 0 \ . \end{cases}$$

Using (77) we conclude at once that for every positive integer k we have

$$(81) \qquad F_{p+1+2k}(t) = \begin{cases} t^{\frac{1}{2}(p+1)} P_k(t) e^{-2\pi t} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

with a certain polynomial P_k of degree k. Finally formula (21) shows that

(82) $$F_{-n}(t) = \pm F_n(-t) ,$$

from which the other half of the picture follows. Taking care of (75) we thus see that for

(83) $$\pi = \sigma_{\mu_1, \mu_2} \quad , \quad \mu(t) = t^p \operatorname{sgn}(t) \quad , \quad p > 0 ,$$

the functions $W\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ where $W \epsilon \mathcal{W}(\pi)$ are given by formula

(84) $$W\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} \omega_\pi(|t|^{\frac{1}{2}})|t|^{\frac{1}{2}(p+1)} P_+(t)e^{-2\pi|t|} & \text{if } t > 0 \\ \omega_\pi(|t|^{\frac{1}{2}})|t|^{\frac{1}{2}(p+1)} P_-(t)e^{-2\pi|t|} & \text{if } t < 0 \end{cases}$$

with arbitrary polynomials P_+ and P_-.

6. <u>The functions</u> $L_W(g\,;\,\chi, s)$.

We can now proceed as in the $\mathcal{G}$ - adic case and prove a result similar to Theorem 8 of §1. Let $\mathcal{W}(\pi)$ be the Whittaker space of an irreducible infinite-dimensional representation π of $\mathcal{H}_F$, and let χ be a character of F^*. For any $W \epsilon \mathcal{W}(\pi)$ define

(85) $$L_W(g\,;\,\chi, s) = \int W\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right] \chi(x)^{-1} |x|^{2s-1} d^*x ,$$

at first formally. We may assume π is contained in a representation ρ_{μ_1, μ_2} with $|\mu(t)| = |t|_F^\sigma$ and $\sigma \geq 0$ as in the previous section, so that

(86) $$W\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right] = \mu_2(\det g)|\det g|_F^{-\frac{1}{2}} \mu_2(x)|x|_F^{\frac{1}{2}} \int \phi_g\begin{pmatrix} xt^{-1} \\ -t \end{pmatrix} \mu(t) d^*t$$

for some $\phi \epsilon \mathcal{S}_0(F^2)$; this follows at once from (57).

If we define as in §1

$$(87) \qquad \xi(x) = W\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \quad M_\xi(\chi, s) = L_W(e\,;\,\chi, s)$$

we then get formally

$$(88) \qquad M_\xi(\chi, s) = \int\int \mathcal{F}_1\phi\begin{pmatrix} xt^{-1} \\ -t \end{pmatrix} \mu_2\chi^{-1}(x)|x|_F^{s'} \mu(t) d^*x\, d^*t$$

with $d^*x = |x|_F^{-1}dx$ and $s' = 2s - \frac{1}{2}$. Defining

$$(89) \qquad L_\phi(\lambda_1, s_1, \lambda_2, s_2) = \int\int \phi\begin{pmatrix} x \\ y \end{pmatrix} \lambda_1(x)|x|_F^{s_1} \lambda_2(y)|y|_F^{s_2}\, d^*x\, d^*y$$

for every $\phi \in \mathcal{S}(F^2)$ and any two characters λ_1 and λ_2 of F^*, the change of variable $x \mapsto tx$ in (88) leads at once to

$$(90) \qquad M_\xi(\chi, s) = \mu_1\chi^{-1}(-1)\, L_{\mathcal{F}_1\phi}(\mu_2 - \chi, s', \mu_1 - \chi, s').$$

Now it is clear that (89) converges as soon as $\mathrm{Re}(s_1)$ and $\mathrm{Re}(s_2)$ are large enough (> 0 if λ_1 and λ_2 are unitary) ; hence (88) is absolutely convergent for $\mathrm{Re}(s)$ large, and Lebesque-Fubini's theory then shows the same is true of (85). And since we may assume ϕ, hence also $\mathcal{F}_1\phi$, is given by (52) it is clear that (89), hence (85), is a meromorphic function ; we shall look more closely at its poles a little later.

Now the functional equation! We want to prove there exists a meromorphic function $\gamma_\pi(\chi, s)$ such that

$$(91) \qquad L_W(wg\,;\,\omega_\pi - \chi, 1 - s) = \gamma_\pi(\chi, s)\, L_W(g\,;\,\chi, s)$$

for all $g \in G_F$ and $W \in \mathcal{W}(\pi)$; we shall even prove it for all $W \in \mathcal{W}_{\mu_1, \mu_2}$. As in §1 it will be enough to prove that

$$(92) \qquad M_{\pi(w)\xi}(\omega_\pi - \chi, 1 - s) = \gamma_\pi(\chi, s)\, M_\xi(\chi, s)$$

where $\xi(x) = W\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ and $\pi(w)\xi(x) = W\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} w\right]$.

First of all let us define

(93) $$L\varphi(\lambda, s) = \int \varphi(x)\, \lambda(x) |x|_F^s \, d^*x$$

for every $\varphi \in \mathcal{S}(F)$. We then obviously have a functional equation

(94) $$L\varphi(-\lambda, 1-s) = \gamma(\lambda, s)\, L\hat{\varphi}(\lambda, s)$$

with a very simple meromorphic factor $\gamma(\lambda, s)$ that will be explicitly computed later. Similarly the "double" Mellin transforms (89) can be analytically continued [this is especially clear for functions (52)], with two functional equations

(95) $$\begin{aligned} L\phi(-\lambda_1, 1-s_1, \lambda_2, s_2) &= \gamma(\lambda_1, s_1)\, L_{\mathcal{F}_1\phi}(\lambda_1, s_1, \lambda_2, s_2) \\ L\phi(\lambda_1, s_1, -\lambda_2, 1-s_2) &= \gamma(\lambda_2, s_2)\, L_{\mathcal{F}_2\phi}(\lambda_1, s_1, \lambda_2, s_2) ; \end{aligned}$$

hence

(96) $$L\phi(-\lambda_1, 1-s_1, -\lambda_2, 1-s_2) = \gamma(\lambda_1, s_1)\, \gamma(\lambda_2, s_2)\, L\hat{\phi}(\lambda_1, s_1, \lambda_2, s_2) .$$

Now we start from (90) ; this of course leads to

(97) $$M_\xi(\omega_\pi - \chi, 1-s) = \mu_2\chi^{-1}(-1)\, L_{\mathcal{F}_1\phi}(\chi-\mu_1, 1-s', \chi-\mu_2, 1-s') ;$$

we need to write (97) for $\pi(w)\xi$ instead of ξ ; but $\pi(w)\xi$ is given by

(98) $$\pi(w)\xi(x) = W\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} w\right] = W'\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

and since W corresponds to ϕ by (86) the function W' corresponds to

(99) $$\phi'\begin{pmatrix} x \\ y \end{pmatrix} = \phi\left[w^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right] = \phi\begin{pmatrix} -y \\ x \end{pmatrix} .$$

To compute

(100) $$M_{\pi(w)\xi}(\omega_\pi - \chi, 1-s) = \mu_2\chi^{-1}(-1)\, L_{\mathcal{F}_1\phi'}(\chi-\mu_1, 1-s', \chi-\mu_2, 1-s')$$

we thus need to compute

(101) $$\mathcal{F}_1\phi'\begin{pmatrix}x\\y\end{pmatrix} = \int \phi'\begin{pmatrix}z\\y\end{pmatrix}\overline{\tau}_F(xz)\,dz = \int \phi\begin{pmatrix}-y\\z\end{pmatrix}\overline{\tau}_F(xz)\,dz$$
$$= \int \overline{\tau}_F(xz)\,dz \int \mathcal{F}_1\phi\begin{pmatrix}u\\z\end{pmatrix}\cdot\overline{\tau}_F(uy)\,du = \widehat{\mathcal{F}_1\phi}\begin{pmatrix}y\\x\end{pmatrix}$$

where

(102) $$\hat{\phi}\begin{pmatrix}x\\y\end{pmatrix} = \iint \phi\begin{pmatrix}u\\v\end{pmatrix}\overline{\tau}_F(ux+vy)\,du\,dv$$

(This is not quite the same definition as used by Jacquet and Langlands). Comparing with (100) we evidently get

(103) $$M_{\pi(w)\xi}(\omega_\pi-\chi, 1-s) = \mu_2\chi^{-1}(-1)\,L_{\widehat{\mathcal{F}_1\phi}}(\chi-\mu_2, 1-s', \chi-\mu_1, 1-s')\,.$$

Comparing with (90) and using the general relation (96) with $\mathcal{F}_1\phi$ instead of ϕ we get at once

(104) $$M_{\pi(w)\xi}(\omega_\pi-\chi, 1-s) = \gamma_\pi(\chi, s)\,M_\xi(\chi, s)$$

with

(105) $$\gamma_\pi(\chi, s) = \gamma(\mu_1-\chi, s')\,\gamma(\mu_2-\chi, s')$$

as in Theorem 9 of §1.

7. <u>Factors</u> $L_\pi(\chi, s)$

If $\varphi \in \mathcal{S}(F)$, the poles of

(106) $$L_\varphi(\lambda, s) = \int \varphi(x)\,\lambda(x)\,|x|_F^s\,d^*x$$

depend only on the Taylor expansion (sic) of φ around 0 - in fact, if λ is unitary then (106) is holomorphic in $\mathrm{Re}(s) > -n$ as soon as $\varphi(x) = 0(|x|_F^n)$ at the origin. The poles of (106) are always simple, and are at most those of the g.c.d. $L(\lambda, s)$ of these functions. It is easy to compute it. The results are as follows.

If $F = \mathbb{R}$ write $\lambda(x) = |x|_F^r\,\mathrm{sgn}(x)^m$ with $m = 0$ or 1. Then

(107) $$L(\lambda, s) = \pi^{-\frac{1}{2}(s+r+m)} \Gamma\left(\frac{s+r+m}{2}\right) = \int_0^\infty e^{-\pi t^2} t^{s+r+m} d^*t .$$

The functional equation of Mellin transforms can then be written as

(108) $$\frac{L_\varphi(-\lambda, 1-s)}{L(-\lambda, 1-s)} = \varepsilon(\lambda, s) \frac{L_{\hat{\varphi}}(\lambda, s)}{L(\lambda, s)}$$

with

(109) $$\varepsilon(\lambda, s) = i^m .$$

If $F = \mathbb{C}$, write $\lambda(x) = |x|_F^r x^m \bar{x}^n$ where m and n are integers such that $\inf(m, n) = 0$. Then we may choose

(110) $$L(\lambda, s) = 2(2\pi)^{-(s+r+m+n)} \Gamma(s+r+m+n)$$

and we have (108) with

(111) $$\varepsilon(\lambda, s) = i^{m+n} .$$

observe that _in all cases there is a_ φ _such that_ $L_\varphi(\lambda, s) = L(\lambda, s)$ _for all_ s.

We shall now define, for each irreducible representation π of G_F and each character χ of F^*, a g. c. d. $L_\pi(\chi, s)$ for the functions $L_W(g; \chi, s)$ $(g \in G_F\ ,\ W \in \mathcal{W}(\pi))$.

Assume first that $\pi = \pi_{\mu_1, \mu_2}$ belongs to the principal series; hence $\pi = \rho_{\mu_1, \mu_2}$ and the space of π is $\mathcal{B}_{\mu_1, \mu_2}$, so that $\mathcal{W}(\pi)$ is the space $\mathcal{W}_{\mu_1, \mu_2}$ of _all_ functions (57). In other words the functions ξ are all the functions

(112) $$\xi\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \mu_2(x)|x|_F^{\frac{1}{2}} \int \mathcal{F}_1\phi\begin{pmatrix} xt^{-1} \\ -t \end{pmatrix} \mu(t)\, d^*t ,$$

if (86) , and by (90) we get _all_ functions

(113) $$L_{\mathcal{F}_1\phi}(\mu_2-\chi\ ,\ s'\ ,\ \mu_1-\chi\ ,\ s')$$

with all $\phi \in \mathcal{S}_0(F^2)$ - or, which is clearly the same with all functions ϕ of the form (52). It is then rather clear that we can choose

(114) $$L_\pi(\chi, s) = L(\mu_1-\chi\ ,\ s')\ L(\mu_2-\chi,\ s')$$

as a g.c.d. for the functions $L_W(g\ ;\ \chi\ ,\ s)$.

If π belongs to the discrete series $(F = \mathbb{R})$ then $\mathcal{W}(\pi)$ is strictly contained in $\mathcal{B}_{\mu_1,\mu_2}$; this may, and does, result in a different formula for the g.c.d. The simplest thing to do is to observe that in this case we know explicitly the space $\mathcal{W}(\pi)$: it is the set of all functions (84). Hence the functions $M_\xi(\chi, s)$ are nothing but the Mellin transforms

(115) $$M_\xi(\chi, s) = \int_0^{+\infty} e^{-2\pi t}\, P(t)\, t^{\frac{1}{2}(p+1)}\, \omega_\pi(t^{\frac{1}{2}})\, \chi(t)^{-1}\, t^{s'-\frac{1}{2}}\, d^*t$$

with an arbitrary polynomial $P(t)$. It is then clear that

(116) $$L_\pi(\chi, s) = \int_0^{+\infty} e^{-2\pi t}\, t^{p/2}\, \omega_\pi(t^{\frac{1}{2}})\, \chi^{-1}(t)\, t^{s'}\, d^*t$$

is a g.c.d. for the set of functions (115). If

(117) $$\pi = \sigma_{\mu_1,\mu_2} \qquad \text{with} \quad \mu(t) = |t|^p\, \mathrm{sgn}(t)\ ,\ p > 0$$

and if

(118) $$\chi(t) = |t|^r \mathrm{sgn}(t)^m$$

we thus have

$$L_\pi(\chi, s) = \int_0^{+\infty} e^{-2\pi t}\, t^{\frac{1}{2}(p+s_1+s_2)+s'-r}\, d^*t$$

(119) $$= (2\pi)^{r-s'-\frac{1}{2}(p+s_1+s_2)}\, \Gamma(s'-r+\tfrac{1}{2}(p+s_1+s_2))$$

$$= (2\pi)^{r-s'-s_1}\, \Gamma(s'+s_1-r)$$

since $s_1 - s_2 = p$.

These computations show that for every π and every χ there is a $W \in \mathcal{W}(\pi)$ such that $L_W(e\,;\chi\,,\,s) = L_\pi(\chi, s)$ for all s.

8. Factors $\varepsilon_\pi(\chi, s)$

We can now, as we did in the $\mathcal{G}$-adic case, write the functional equation (91) in the form

$$\frac{L_W(wg\,;\,\omega_\pi - \chi\,,\,1-s)}{L_\pi(\omega_\pi - \chi\,,\,1-s)} = \varepsilon_\pi(\chi, s)\,\frac{L_W(g\,;\,\chi\,,\,s)}{L_\pi(\chi, s)} \tag{120}$$

with new factors

$$\varepsilon_\pi(\chi, s) = \gamma_\pi(\chi, s)\,\frac{L_\pi(\chi, s)}{L_\pi(\omega_\pi - \chi\,,\,1-s)}\,. \tag{121}$$

If $\pi = \pi_{\mu_1, \mu_2}$ belongs to the principal series, then formulas (105), (108) and (121) lead at once to

$$\varepsilon_\pi(\chi, s) = \varepsilon(\mu_1 - \chi\,,\,s')\,\varepsilon(\mu_2 - \chi\,,\,s') \tag{122}$$

as in the non-archimedean case.

If $F = \mathbb{R}$ and $\pi = \sigma_{\mu_1, \mu_2}$ belongs to the discrete series, then by our choice (119) of the g. c. d. we have

$$\begin{aligned} \varepsilon_\pi(\chi, s) &= \gamma(\mu_1 - \chi, s')\,\gamma(\mu_2 - \chi, s')\,\frac{(2\pi)^{r-s'-s_1}\,\Gamma(s'-r+s_1)}{(2\pi)^{s_1+s_2-r-1+s'-s_1}\,\Gamma(1-s'+r-s_1-s_2+s_1)} \\ &= (2\pi)^{2r-2s'-s_1-s_2+1}\,\gamma(\mu_1 - \chi\,,\,s')\,\gamma(\mu_2 - \chi\,,\,s')\,\frac{\Gamma(s'-r+s_1)}{\Gamma(1-s'+r-s_2)}\,. \end{aligned} \tag{123}$$

If we write that

(124) $$\chi(x) = |x|^{r}\operatorname{sgn}(x)^{m} \quad , \quad \mu_i\chi^{-1}(x) = |x_i|^{s_i - r}\operatorname{sgn}(x)^{n_i}$$

with

(125) $$n_i = (m_i - m) \qquad (i = 1, 2) \ ,$$

we get by (107), (108) and (109) :

$$\gamma(\mu_i - \chi, s') = \varepsilon(\mu_i - \chi, s')\frac{L(\chi - \mu_i \ , \ 1-s')}{L(\mu_i - \chi \ , \ s')}$$

(126) $$= i^{n_i}(\text{sic}) \frac{\pi^{-\frac{1}{2}(1-s'+r-s_i+n_i)}\,\Gamma\left(\frac{1-s'+r-s_i+n_i}{2}\right)}{\pi^{-\frac{1}{2}(s'+s_i-r+n_i)}\,\Gamma\left(\frac{s'+s_i-r+n_i}{2}\right)} =$$

$$= i^{n_i}\,\pi^{s'-r+s_i-\frac{1}{2}}\,\frac{\Gamma\left(\frac{1-s'+r-s_i-n_i}{2}\right)}{\Gamma\left(\frac{s'+s_i-r+n_i}{2}\right)} \ .$$

Hence

$$\varepsilon_{\pi}(\chi, s) = (2\pi)^{2r-2s'-s_1-s_2+1}\, i^{n_1+n_2}\, \pi^{2s'-2r+s_1+s_2-1} \times$$

(127) $$\times \frac{\Gamma\left(\frac{1-s'+r-s_1+n_1}{2}\right)\Gamma\left(\frac{1-s'+r-s_2+n_2}{2}\right)\Gamma(s'-r+s_1)}{\Gamma\left(\frac{s'+s_1-r+n_1}{2}\right)\Gamma\left(\frac{s'+s_2-r+n_2}{2}\right)\Gamma(1-s'+r-s_2)}$$

$$= i^{n_1+n_2}\, 2^{2r-2s'-s_1-s_2+1} \times \frac{\Gamma(x_1)\,\Gamma\left(\frac{1-x_1+n_1}{2}\right)}{\Gamma\left(\frac{x_1+n_1}{2}\right)} : \frac{\Gamma(x_2)\,\Gamma\left(\frac{1-x_2+n_2}{2}\right)}{\Gamma\left(\frac{x_2+n_2}{2}\right)}$$

with

(128) $$x_1 = s'-r+s_1 \quad , \quad x_2 = 1-s'+r-s_2 \ .$$

By using classical formulas, namely

(129) $$\Gamma(x)\Gamma(x+\tfrac{1}{2}) = \pi^{\frac{1}{2}} 2^{1-2x}\,\Gamma(2x) \quad , \quad \Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$$

and the fact that $n_i = 0$ or 1, it is easy to see that

(130) $$\frac{\Gamma(x)\Gamma\left(\frac{1-x+n}{2}\right)}{\Gamma\left(\frac{x+n}{2}\right)} = \pi^{\frac{1}{2}} 2^{x-1}/\sin\left(\pi\frac{x+1-n}{2}\right)$$

if $n = 0$ or 1. We thus get

$$\varepsilon_\pi(\chi, s) = \iota^{n_1+n_2}\, 2^{2r-2s'-s_1-s_2+1} \times$$

(131) $$\times \frac{\pi^{\frac{1}{2}} 2^{s'-r+s_1-1}}{\sin\left(\pi\frac{s'-r+s_1+1-n_1}{2}\right)} : \frac{\pi^{\frac{1}{2}} 2^{-s'+r-s_2}}{\sin\left(\pi\frac{r-s'-s_2+n_2+2}{2}\right)}$$

$$= i^{n_1+n_2} \frac{\sin\left(\pi\frac{s'-r+s_2-n_2}{2}\right)}{\sin\left(\pi\frac{s'-r+s_1+1-n_1}{2}\right)} .$$

Making use of the fact that

(132) $$s_1 - s_2 = p \quad , \quad n_1 - n_2 \equiv p+1 \pmod 2$$

we get at once

(133) $$\varepsilon_\pi(\chi, s) = (-1)^{\frac{1}{2}(p+1-n_1-n_2)}\, i^{n_1+n_2} \ .$$

This result is not to be found in Jacquet-Langlands, where they compute $\varepsilon_\pi(\chi, s)$ from the Weil construction of the discrete series. It is to be expected that their formula (top of p. 195) agrees with (133).

§3 - The global theory

1. Parabolic forms.

In this section we denote by k an arithmetic field (i.e., a number field, or a field of algebra functions of one variable over a finite field), and by A the ring of adeles of k; we have a natural locally compact topology on A, and k is imbedded in A as a discrete subgroup with compact quotient. If $\mathfrak{p}$ is a (finite or archimedean) place of k we denote by $k_{\mathfrak{p}}$ the corresponding completion of k, by $|x|_{\mathfrak{p}}$ the absolute value on $k_{\mathfrak{p}}$, and by $\mathcal{O}_{\mathfrak{p}}$ the ring of integers of $k_{\mathfrak{p}}$ if $\mathfrak{p}$ is non-archimedean. In Weil's Basic Number Theory and in Jacquet and Langlands they write $\mathfrak{v}$ instead of $\mathfrak{p}$. We prefer our $\mathfrak{p}$'s; they remind us of the Great Dynasty - Gauss, Jacobi, Dirichlet, Riemann, Dedekind, Kronecker, Hilbert, Minkowski, Hecke, Artin, Hasse, etc. . . . - Salvation through Zahlentheorie.

We shall write

$$G_k = GL(2,k) \quad , \quad G_A = GL(2,A) \tag{1}$$

so that G_k is a discrete subgroup of G_A. For every place $\mathfrak{p}$ let $M_{\mathfrak{p}}$ be the obvious maximal compact subgroup of $G_{\mathfrak{p}} = GL(2,k_{\mathfrak{p}})$; then G_A contains the compact subgroup

$$M = \prod_{\mathfrak{p}} M_{\mathfrak{p}} \tag{2}$$

and we have $G_A = U_A H_A M$, where U is the subgroup of matrices $\begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix}$ and H the diagonal subgroup; we denote by U_A, H_A, U_k, H_k the subgroups of matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $x \in A$, $\begin{pmatrix} t' & 0 \\ 0 & t'' \end{pmatrix}$, $t', t'' \in A^*$, $\begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$, $\eta \in k$ and $\begin{pmatrix} \xi' & 0 \\ 0 & \xi'' \end{pmatrix}$, $\xi', \xi'' \in k^*$. With the help of this "Iwasawa decomposition" of G_A we can construct a <u>fundamental</u> <u>open</u> <u>set</u> for G_k in G_A, i.e., an open set $\mathfrak{S} \subset G_A$ such that $G_A = G_k \mathfrak{S}$ and that $\gamma\mathfrak{S}$ intersects $\mathfrak{S}$ for a finite number of $\gamma \in G_k$ only ; to do that we can choose $\mathfrak{S} = \Omega_U \cdot \Omega_H \cdot H_\infty(c) \cdot M$, where Ω_U and Ω_H are suitably chosen open relatively compact subsets of U_A and H_A, and where $H_\infty(c)$ is, in the number-field case, the set of matrices $\begin{pmatrix} t' & 0 \\ 0 & t'' \end{pmatrix} \in H_A$ such that $t'_{\mathfrak{p}} = t''_{\mathfrak{p}} = 1$ for all non-archimedean primes and

$$|t' / t''| > c \tag{3}$$

for a given $c > 0$; in the function field case* one should choose once and for all a place $\mathfrak{p}_0$ and require that $t'_{\mathfrak{p}} = t''_{\mathfrak{p}} = 1$ for all $\mathfrak{p} \neq \mathfrak{p}_0$.

The absolute value in (3) is of course the global absolute value of an idele of k.

Let dx be an invariant measure on G_A. The group G_A operates, through right translations, on the Hilbert space $L^2(G_k \backslash G_A)$ of measurable functions φ such that

$$\varphi(\gamma g) = \varphi(g) \quad , \quad \int_{G_k \backslash G_A} |\varphi(g)|^2 dg < +\infty . \tag{4}$$

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* This section is written with the number field case in mind. The reader will have to use it with care in the function field case.

Let Z_A (resp. Z_k) be the center of G_A (resp. G_k), i.e., the group of matrices $\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$ with $t \in A^*$ (resp. $t \in k^*$). The translations by elements of Z_A commute with the obvious representation of G_A. We can thus, by means of a Fourier transform on Z_A / Z_k, decompose $L^2(G_k | G_A)$ into the "continuous sums" of the spaces $L^2(G_k \backslash G_A, \omega)$, where, for any unitary character ω of Z_A / Z_k (i.e., of A^* / k^*), we denote by $L^2(G_k \backslash G_A, \omega)$ the space of functions such that

$$\varphi(\gamma g z) = \varphi(g)\,\omega(z) \quad , \quad \int_{G_k \backslash G_A / Z_A} |\varphi(g)|^2 dg < +\infty . \tag{5}$$

Of course $G_k \backslash G_A / Z_A = Z_A G_k \backslash G_A$. We still have a unitary representation $g \mapsto T_\omega(g)$ of G_A on $L^2(G_k \backslash G_A, \omega)$.

For every $\varphi \in L^2(G_k \backslash G_A, \omega)$ and almost every $g \in G_A$, the function $x \mapsto \varphi\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right]$ on A is square integrable and periodic, i.e., invariant under k. If we denote by τ a nontrivial character of A/k chosen once and for all, we thus have a Fourier series

$$\varphi\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right] = \sum_{\xi \in k} \varphi_\xi(g)\,\tau(\xi x) \tag{6}$$

with

$$\varphi_\xi(g) = \int_{A/k} \varphi\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right] \overline{\tau}(\xi x)\, dx \; ; \tag{7}$$

We assume of course $\int_{A/k} dx = 1$. By making use of the invariance of φ under $g \mapsto \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g$ and of dx under $x \mapsto \xi x$ if $\xi \neq 0$, it is seen at once that if $\xi \neq 0$ we have

(8) $$\varphi_\xi(g) = W_\varphi\left[\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}g\right] \quad\text{where}\quad W_\varphi(g) = \varphi_1(g) = \int_{A/k}\varphi\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right]\bar{\tau}(x)\,dx$$

Hence

(9) $$\varphi(g) = \varphi_0(g) + \sum_{\xi\neq 0} W_\varphi\left[\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}g\right].$$

We say that φ is <u>parabolic</u> if

(10) $$\varphi_0(g) = \int_{A/k}\varphi\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right]dx = 0 \quad\text{almost everywhere}.$$

It is easily seen that the set of parabolic φ is a <u>closed invariant</u> subspace $L_0^2(G_k\backslash G_A, \omega)$ of $L^2(G_k\backslash G_A, \omega)$. If f is a continuous function with compact support on G_A, then the continuous operator

(11) $$T_\omega(f) = \int_{G_A} T_\omega(x)\, f(x)\, dx,$$

more explicitly given by

(12) $$T_\omega(f)\varphi(x) = \int_{G_A}\varphi(xy)\, f(y)\, dy = \varphi * \check{f}(x),$$

maps into itself every closed invariant subspace of $L^2(G_k\backslash G_A, \omega)$, for trivial reasons (an integral is a limit of finite sums . . .). The first main result is

<u>Theorem</u> 1. <u>For every continuous function</u> f <u>with compact support on</u> G_A, <u>the operator</u> $T_\omega(f)$ <u>is compact on</u> $L_0^2(G_k\backslash G_A, \omega)$.

Proofs of this very simple but clever result are to be found almost everywhere*, at any rate for $SL(2)$, but the extension to $GL(2)$ is easy.

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* See for instance References 3, 11 and 13 in Jacquet and Langlands.

The authors of the Jacquet and Langlands paper are not interested in L^2 theory, and they replace Theorem 1 by "purely algebraic" results (Prop. 10.5, p. 334, for instance) which are more closely related to the classical point of view.

The proof consists in writing (12) on $L^2(G_k \backslash G_A, \omega)$ as

(13) $$T_\omega(f)\varphi(x) = \int_{U_k \backslash G_A} \varphi(y)\, K'_f(x,y)\, dy$$

where U_k is the subgroup $\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$, $\xi \in k$, and where

(14) $$K'_f(x,y) = \sum_{U_k} f(x^{-1}\eta y) - \int_{U_A} f(x^{-1}uy)\, du\ ;$$

if f is good enough this can be estimated by means of Poisson's summation formula, and it is found that, for $x = uh_x m \in \mathfrak{S}$ and $\varphi \in L_0^2(G_k \backslash G_A, \omega)$,

we have estimates

(15) $$|T_\omega(f)\varphi(x)| \leq c_N |\beta(h_x)|^{-N} \cdot ||\varphi||_2$$

for every N, with $\beta(h) = t'/t''$ if $h = \begin{pmatrix} t' & 0 \\ 0 & t'' \end{pmatrix}$; a similar result (without the $||\varphi||_2$ term, and with a constant $c_N(\varphi)$ instead of c_N) would be obtained if the parabolic function φ, instead of being assumed to be L^2, was assumed to be slowly increasing, i.e., dominated in $\mathfrak{S}$ by some power of $|\beta(h_x)|$.

A corollary of Theorem 1 is that <u>the unitary representation of G_A on $L_0^2(G_k \backslash G_A, \omega)$ is a</u> (Hilbert) <u>discrete direct sum of topologically irreducible representations, each occurring with a finite multiplicity</u>, i.e., we can write

$$L^2_0(G_k \backslash G_A, \omega) = \hat{\oplus}\, \mathcal{H}_n \tag{16}$$

where each $\mathcal{H}_n$ is a minimal closed invariant subspace of $L^2_0(G_k \backslash G_A, \omega)$, and with only finitely many p such that the representations on $\mathcal{H}_p$ and $\mathcal{H}_n$ are equivalent. This corollary follows from a general and elementary lemma in the theory of representations*. It will be proved (Theorem 3) that here the multiplicities are one; and that furthermore those irreducible unitary representations of G_A which do occur in $L^2_0(G_k \backslash G_A, \omega)$ can be characterized by conditions concerning an infinite eulerian product (Theorem 5). These are - among many others to be found in Jacquet and Langlands' work - the two results we shall prove in this section.

We shall need later another consequence of Theorem 1. Suppose a function $\varphi \in L^2_0(G_k \backslash G_A, \omega)$ belongs to $\mathcal{H}_n(\vartheta)$ for some n - we use (16) - and some irreducible representation ϑ of M. Since $\dim \mathcal{H}_n(\vartheta) < +\infty$ it is immediately seen there are continuous functions f with compact support on

* Let $x \mapsto T(x)$ be a unitary representation of a locally compact group G on a Hilbert space $\mathcal{H}$, and assume the operator

$$T(f) = \int T(x)\, f(x)\, dx$$

is compact and not zero for a given integrable function f. We may assume $T(f) = T(f)^*$, so that there is a $\lambda \neq 0$ such that the subspace $\mathcal{V}$ of vectors a such that $T(f)a = \lambda a$ is nonzero and finite dimensional. It is clear that $\mathcal{V}$ is invariant under every operator commuting to the representation T. It follows at once from this property that if we consider the closed invariant subspace $\mathcal{H}'$ of $\mathcal{H}$ generated by the orbit of a $a \in \mathcal{V}$, then $\mathcal{H}'' \mapsto \mathcal{H}'' \cap \mathcal{V}$ is an injective mapping of the set of closed invariant subspaces $\mathcal{H}'' \subset \mathcal{H}'$ into the set of subspaces of $\mathcal{V}$. Since $\mathcal{V}$ is finite dimensional, the existence of minimal closed invariant subspaces follows.

G_A such that $T_\omega(f)\varphi = \varphi$; we may even assume f as nice as we want (e.g., C^∞ with respect to the archimedean components). But then $\varphi = T_\omega(f)\varphi = \varphi * \check{f}$ is itself very nice : its right translates under M remain in a finite dimensional space, and φ is C^∞ with respect to the archimedean components of G_A. Furthermore (15) shows that φ is rapidly decreasing in $\mathcal{G}$, i.e., satisfies for every N an inequality

$$|\varphi(x)| \leq c_N(\varphi) \cdot |\beta(h_x)|^{-N} \quad \text{in } \mathcal{G}. \tag{17}$$

The same properties apply to the functions in the dense subspace

$$\bigoplus_{n,\vartheta} \mathcal{H}_n(\vartheta) \tag{18}$$

of $L_0^2(G_k \backslash G_A, \omega)$; they will here be called cusp-forms.

2. Local decomposition of an irreducible component.

For every $\mathcal{Y}$ let $\rho_{\mathcal{Y}}$ be an irreducible unitary representation of $G_{\mathcal{Y}}$ on a Hilbert space $\mathcal{H}_{\mathcal{Y}}$; we shall assume ** that every $\pi_{\mathcal{Y}}$ is admissible, i.e., that the multiplicity of any irreducible representation of $M_{\mathcal{Y}}$ in $\rho_{\mathcal{Y}}$ is finite; for such a representation $\vartheta_{\mathcal{Y}}$ we denote by $\mathcal{H}_{\mathcal{Y}}(\vartheta_{\mathcal{Y}})$ the corresponding isotypical component of $\rho_{\mathcal{Y}} | M_{\mathcal{Y}}$. Finally we assume that $\mathcal{H}_{\mathcal{Y}}(\text{id}) \neq 0$ for almost all $\mathcal{Y}$; we have then $\dim \mathcal{H}_{\mathcal{Y}}(\text{id}) = 1$; we choose once and for all a unit vector $\xi^0_{\mathcal{Y}}$ in $\mathcal{H}_{\mathcal{Y}}(\text{id})$; we denote by S_0 a finite set of places containing all places at infinity as well as those $\mathfrak{p}$ where $\xi_{\mathfrak{p}}^0$ is not defined.

For all finite sets $S \supset S_0$ define

(19) $$\mathcal{H}_S = \bigotimes_{\mathfrak{p} \in S} \mathcal{H}_{\mathfrak{p}} ;$$

this is a pre-Hilbert space, with the scalar product

(20) $$\left(\bigotimes_{\mathfrak{p} \in S} \xi_{\mathfrak{p}} , \bigotimes_{\mathfrak{p} \in S} \eta_{\mathfrak{p}} \right) = \prod_{\mathfrak{p} \in S} \left(\xi_{\mathfrak{p}} , \eta_{\mathfrak{p}} \right) .$$

For $S \subset S'$ we have an isometric injection $\mathcal{H}_S \to \mathcal{H}_{S'}$ namely

(21) $$\xi \mapsto \xi \otimes \bigotimes_{\mathfrak{p} \in S' - S} \xi_{\mathfrak{p}}^0 .$$

This enables us to define the pre-Hilbert space

(22) $$\varinjlim \mathcal{H}_S ,$$

and by completion a Hilbert space

(23) $$\mathcal{H} = \widehat{\bigotimes_{\mathfrak{p}}} \mathcal{H}_{\mathfrak{p}} .$$

Since we have a unitary representation $\rho_{\mathfrak{p}}$ of $G_{\mathfrak{p}}$ on $\mathcal{H}_{\mathfrak{p}}$ for every $\mathfrak{p}$, we get at once a unitary representation

(24) $$\rho = \widehat{\bigotimes_{\mathfrak{p}}} \rho_{\mathfrak{p}}$$

of G_A on $\mathcal{H}$; if $\xi = \bigotimes_{\mathfrak{p}} \xi_{\mathfrak{p}}$ is a "decomposable" element of $\mathcal{H}$, with $\xi_{\mathfrak{p}} = \xi_{\mathfrak{p}}^0$ for almost all $\mathfrak{p}$, we have

(25) $$\rho(g)\xi = \bigotimes_{\mathfrak{p}} \rho_{\mathfrak{p}} (g_{\mathfrak{p}}) \xi_{\mathfrak{p}}$$

for every $g \in G_A$. Since such vectors generate topologically $\mathcal{H}$, this is enough to define ρ.

Let ϑ be an irreducible representation of M. Since M is compact ϑ is finite-dimensional, and since $M = \prod M_{\mathfrak{y}}$ we have $\vartheta = \otimes \vartheta_{\mathfrak{y}}$, with irreducible representations $\vartheta_{\mathfrak{y}}$ of the various $M_{\mathfrak{y}}$, and $\vartheta_{\mathfrak{y}}$ = id for almost all $\mathfrak{y}$. Evidently

(26) $$\mathcal{H}(\vartheta) = \bigotimes_{\mathfrak{y}} \mathcal{H}_{\mathfrak{y}}(\vartheta_{\mathfrak{y}}),$$

and since $\dim \mathcal{H}_{\mathfrak{y}}(\vartheta_{\mathfrak{y}}) = 1$ if $\mathfrak{y} \notin S$ it is clear that $\mathcal{H}(\vartheta)$ is finite dimensional. It is furthermore a routine business to prove that if the $\rho_{\mathfrak{y}}$ are (topologically) irreducible, so is ρ. An irreducible unitary representation ρ of G_A will be said <u>admissible</u> if $\dim \mathcal{H}(\vartheta) < +\infty$ for all ϑ, and <u>decomposable</u> if there are admissible irreducible unitary representations $\rho_{\mathfrak{y}}$ of the $G_{\mathfrak{y}}$ such that $\rho = \hat{\otimes} \rho_{\mathfrak{y}}$.

<u>Theorem</u> 2. <u>Every admissible irreducible unitary representation</u> ρ <u>of</u> G_A <u>is decomposable, and its irreducible factors</u> $\rho_{\mathfrak{y}}$ <u>are uniquely defined</u>.

We first prove that for every $\mathfrak{y}$ the restriction of ρ to $G_{\mathfrak{y}}$ is a direct sum of mutually equivalent irreducible representations, i.e., can be written as the (Hilbert) tensor product of an admissible irreducible unitary representation $\rho_{\mathfrak{y}}$ of $G_{\mathfrak{y}}$ and a multiple of the identity representation of $G_{\mathfrak{y}}$.

Consider the compact subgroup

(27) $$M^{(\mathfrak{y})} = \prod_{\mathfrak{q} \neq \mathfrak{y}} M_{\mathfrak{q}};$$

if ϑ is an irreducible representation of $M^{(\mathfrak{y})}$, let $\mathcal{H}(\vartheta)$ be the corresponding subspace of $\mathcal{H}$. Of course $\mathcal{H} = \widehat{\oplus}\,\mathcal{H}(\vartheta)$, and every $\mathcal{H}(\vartheta)$ is stable under $\rho(G_{\mathcal{Y}})$. Every irreducible representation $\vartheta_{\mathcal{Y}}$ of $M_{\mathcal{Y}}$ occurs in $\mathcal{H}(\vartheta)$ with finite multiplicity, because we assume ρ is admissible. Hence $\mathcal{H}(\vartheta)$ is a Hilbert direct sum of minimal closed subspaces invariant under $G_{\mathcal{Y}}$. The same is therefore true for $\mathcal{H}$. But on the other hand the restriction of ρ to $G_{\mathcal{Y}}$ is a "factor representation" of $G_{\mathcal{Y}}$ [i.e., every continuous operator T on $\mathcal{H}$ which commutes with everything that commutes with $\rho(G_{\mathcal{Y}})$ is a scalar-proof: T commutes with $\rho(G_A)$, and we assume ρ is irreducible]. It follows at once that the irreducible components of $\rho(G_{\mathcal{Y}})$ are mutually equivalent.

Let $\rho_{\mathcal{Y}}$ be the admissible irreducible unitary representation of $G_{\mathcal{Y}}$ that can be imbedded into $\mathcal{H}$. Let $\mathcal{H}_{\mathcal{Y}}$ be the space of $\rho_{\mathcal{Y}}$, so that $\mathcal{H} \sim \mathcal{H}_{\mathcal{Y}} \widehat{\otimes} \mathcal{H}'_{\mathcal{Y}}$ for some Hilbert space $\mathcal{H}'_{\mathcal{Y}}$. This $\mathcal{H}'_{\mathcal{Y}}$ can be canonically defined as the space of all linear mappings $u : \mathcal{H}_{\mathcal{Y}} \to \mathcal{H}$ that are continuous and compatible with the actions of $G_{\mathcal{Y}}$; the scalar product of two such mappings u and v is the number (u,v) such that

(28) $$(u(\xi)\ ,\ v(\eta)) = (u,v)(\xi,\eta)$$

for any two $\xi, \eta \in \mathcal{H}_{\mathcal{Y}}$; and the isomorphism $\mathcal{H}_{\mathcal{Y}} \widehat{\otimes} \mathcal{H}'_{\mathcal{Y}} \to \mathcal{H}$ is the unique isometry which, for any $\xi \in \mathcal{H}_{\mathcal{Y}}$ and $u \in \mathcal{H}'_{\mathcal{Y}}$, maps $\xi \otimes u$ onto $u(\xi)$.

Now it is clear that continuous operators T on $\mathcal{H}$ which commute with $\rho(G_{\mathcal{Y}})$ are in one-to-one correspondence with continuous operators T' on $\mathcal{H}'_{\mathcal{Y}}$, in such a way that $T = 1 \otimes T'$ if we identify $\mathcal{H}$ and $\mathcal{H}_{\mathcal{Y}} \otimes \mathcal{H}'_{\mathcal{Y}}$. In particular ρ defines a representation on $\mathcal{H}'_{\mathcal{Y}}$ of $G_{\mathcal{N}}$ for every $\mathcal{N} \neq \mathcal{Y}$, which is of course the Hilbert tensor product of $\rho_{\mathfrak{M}}$ and an identity representation of $G_{\mathfrak{M}}$ on a Hilbert space $\mathcal{H}'_{\mathfrak{p},\mathfrak{M}}$. We thus see that, for every finite set S of places, we can write $\mathcal{H} = \mathcal{H}_S \hat{\otimes} \mathcal{H}'_S$, where

(29) $$\mathcal{H}_S = \hat{\bigotimes}_{\mathfrak{p} \in S} \mathcal{H}_{\mathfrak{p}},$$

where $\mathcal{H}'_S$ is the Hilbert space of continuous homomorphisms $\mathcal{H}_S \to \mathcal{H}$ which are compatible with the actions of

(30) $$G_S = \prod_{\mathfrak{p} \in S} G_{\mathfrak{p}},$$

and where G_S operates on $\mathcal{H}_S$ through the representations $\rho_{\mathfrak{p}}$, and on $\mathcal{H}'_S$ in a trivial way.

To conclude the proof we first observe that $\rho_{\mathfrak{p}}$ contains the identity representation of $M_{\mathfrak{p}}$ for almost all $\mathfrak{p}$, because the restriction of ρ to $G_{\mathfrak{p}}$ already satisfies this condition for almost all $\mathfrak{p}$ [choose a representation ϑ of M such that $\mathcal{H}(\vartheta) \neq 0$, and consider those $\mathfrak{p}$ for which $\vartheta_{\mathfrak{p}} = \mathrm{id}$]. For all $\mathfrak{p} \notin S_0$, let us choose once and for all a unit vector $\xi^0_{\mathfrak{p}}$ in $\mathcal{H}_{\mathfrak{p}}$ invariant under $M_{\mathfrak{p}}$; we can then define a representation $\hat{\otimes}\rho_{\mathfrak{p}}$ of G_A on $\hat{\otimes}\mathcal{H}_{\mathfrak{p}}$, as we have seen at the beginning of this section. We want to show that $\mathcal{H}$ is isomorphic to $\hat{\otimes}\mathcal{H}_{\mathfrak{p}}$ and ρ to $\hat{\otimes}\rho_{\mathfrak{p}}$; this will conclude the proof.

Since ρ and $\hat{\otimes}\rho_{\mathfrak{p}}$ are irreducible it will be enough to construct a nonzero isometric linear map $\hat{\otimes}\mathcal{H}_{\mathfrak{p}} \to \mathcal{H}$ compatible with the actions of G_A. To do that we observe that, if $S \supset S_0$, there are in $\mathcal{H}$ and hence in $\mathcal{H}'_S$ nonzero vectors invariant by all the $M_{\mathfrak{p}}$, $\mathfrak{p} \notin S$. Choose such a vector

ξ_S^0 for every $S \supset S_0$. On the other hand, if $\mathfrak{p} \in S$ and $S' = S - \{\mathfrak{p}\}$, we clearly have $\mathcal{H}'_S = \mathcal{H}_{\mathfrak{p}} \hat{\otimes} \mathcal{H}'_{S'}$. Since $\xi_{\mathfrak{p}}^0$ is, up to a scalar, the <u>only</u> vector in $\mathcal{H}_{\mathfrak{p}}$ invariant under $M_{\mathfrak{p}}$, we have $\xi_S^0 = \lambda\, \xi_{\mathfrak{p}}^0 \otimes \xi_{S'}^0$, with a scalar λ depending on S and $\mathfrak{p}$. If we index by integers the set of $\mathfrak{p} \notin S_0$, we see at once that we can choose the vectors ξ_S^0 in such a way that $\lambda = 1$ in all cases. Then for every $S \supset S_0$ we get an imbedding $\xi \to \xi \otimes \xi_S^0$ of $\mathcal{H}_S$ into $\mathcal{H}$, with the obvious compatibility conditions satisfied. The remaining proof then follows by standard arguments.

Theorem 2 of course applies to the irreducible components of the representation of G_A on a space $L_0^2(G_k \backslash G_A, \omega)$.

3. <u>Global Kirillov models</u>.

Let $\rho = \hat{\otimes} \rho_{\mathfrak{p}}$ be an admissible irreducible unitary representation of G_A on a Hilbert space $\mathcal{H} = \hat{\otimes} \mathcal{H}_{\mathfrak{p}}$, where $\mathcal{H}_{\mathfrak{p}}$ is the space of $\rho_{\mathfrak{p}}$. For each $\mathfrak{p}$ let $\mathcal{H}_{\mathfrak{p}}^f$ be the everywhere dense subspace of $M_{\mathfrak{p}}$-finite vectors in $\mathcal{H}_{\mathfrak{p}}$; then the subspace of M-finite vectors in $\mathcal{H}$ is of course

(31) $$\mathcal{H}^f = \bigcup_S \bigotimes_{\mathfrak{p} \in S} \mathcal{H}_{\mathfrak{p}}^f \otimes \bigotimes_{\mathfrak{p} \notin S} \mathcal{H}_{\mathfrak{p}}^0 ,$$

where we denote by $\mathcal{H}_{\mathfrak{p}}^0$ the (zero- or one-dimensional) subspace of vectors fixed under $M_{\mathfrak{p}}$.

Denote by $\mathscr{H}_{\mathfrak{p}}$ the Hecke algebra of $G_{\mathfrak{p}}$ for every $\mathfrak{p}$, as defined in Sections 1 and 2. For every $\mathfrak{p}$ we then get a representation (which we shall denote by $\rho_{\mathfrak{p}}^f$) of $\mathscr{H}_{\mathfrak{p}}$ on $\mathcal{H}_{\mathfrak{p}}^f$ - this is clear if $\mathfrak{p}$ is non-archimedean since for every $F \in \mathscr{H}_{\mathfrak{p}}$ the operator $\rho_{\mathfrak{p}}(F) = \int \rho_{\mathfrak{p}}(x) F(x)\, dx$ is defined and continuous on the whole of $\mathcal{H}_{\mathfrak{p}}$, and maps $\mathcal{H}_{\mathfrak{p}}^f$ into itself because F is left $M_{\mathfrak{p}}$-finite. If $\mathfrak{p}$ is archimedean, in which case $\mathcal{H}_{\mathfrak{p}}$ consists of distributions with support $\{e\}$ if $\mathfrak{p}$ is complex, and $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\right\}$ at most if $\mathfrak{p}$ is real, then we have to make use of the results quoted in §2, No. 1, which rest upon the fact that, for every $\xi \in \mathcal{H}_{\mathfrak{p}}^f$, the map $x \mapsto \rho_{\mathfrak{p}}(x)\xi$ of $G_{\mathfrak{p}}$ into $\mathcal{H}_{\mathfrak{p}}$ is real-analytic.

The representation $\rho_{\mathfrak{p}}^f$ of $\mathcal{H}_{\mathfrak{p}}$ on $\mathcal{H}_{\mathfrak{p}}^f$ is of course admissible for every $\mathfrak{p}$; it is furthermore irreducible, and its class determines the unitary representation $\pi_{\mathfrak{p}}$ of $G_{\mathfrak{p}}$ on $\mathcal{H}_{\mathfrak{p}}$ up to unitary equivalence. The irreducibility is due to the fact that if a subspace $\mathcal{M}$ of $\mathcal{H}_{\mathfrak{p}}^f$ is invariant under $\mathcal{H}_{\mathfrak{p}}$, then its closure $\overline{\mathcal{M}}$ in $\mathcal{H}_{\mathfrak{p}}$ is invariant under $G_{\mathfrak{p}}$, and we have $\mathcal{M} = \overline{\mathcal{M}} \cap \mathcal{H}_{\mathfrak{p}}^f$ because of the decomposition of $\mathcal{M}$ into mutually orthogonal subspaces corresponding to the various representations of $M_{\mathfrak{p}}$. The fact that the representation of $\mathcal{H}_{\mathfrak{p}}$ determines that of $G_{\mathfrak{p}}$ is proved as follows. The given scalar product (ξ, η) on $\mathcal{H}_{\mathfrak{p}}$ defines canonically a semi-linear isomorphism $\eta \mapsto \bar{\eta}$ between the representation $\rho_{\mathfrak{p}}^f$ of $\mathcal{H}_{\mathfrak{p}}$ on $\mathcal{H}_{\mathfrak{p}}^f$ and the contragredient representation $\check{\rho}_{\mathfrak{p}}^f$ on $\check{\mathcal{H}}_{\mathfrak{p}}^f$ (where $\check{\pi}_{\mathfrak{p}}$ is defined on page 1.2 for finite primes, and in the obvious way, see page 2.4, for archimedean places). This already shows there is on $\mathcal{H}_{\mathfrak{p}}^f$ only one positive definite scalar product compatible with the representation of $\mathcal{H}_{\mathfrak{p}}$. Furthermore if we know the action of $\mathcal{H}_{\mathfrak{p}}$ on $\mathcal{H}_{\mathfrak{p}}^f$ then we know the number

(32) $$(\rho_{\mathfrak{p}}(F)\xi, \eta) = \langle \rho_{\mathfrak{p}}^f(F)\xi, \bar{\eta}\rangle = \int \langle \rho_{\mathfrak{p}}^f(x)\xi, \bar{\eta}\rangle F(x)\,dx = \int (\rho_{\mathfrak{p}}(x)\xi, \eta)F(x)\,dx$$

for any two $\xi, \eta \in \mathcal{H}_{\mathfrak{y}}^f$ and every $F \in \mathcal{H}_{\mathfrak{y}}$, from which it follows that we know all the "coefficients" $x \mapsto (\rho_{\mathfrak{y}}(x)\xi, \eta)$ for all $\xi, \eta \in \mathcal{H}_{\mathfrak{y}}^f$; but it is well known that if two irreducible unitary representations have a common coefficient, then they are unitarily equivalent.

We thus conclude that an admissible irreducible unitary representation ρ of G_A defines, for every $\mathfrak{y}$, an irreducible admissible representation $\rho_{\mathfrak{y}}^f$ of $\mathcal{H}_{\mathfrak{y}}$ in the sense of Sections 1 and 2, and ρ is determined by the knowledge of the $\rho_{\mathfrak{y}}^f$. Observe that $\rho_{\mathfrak{y}}^f$ contains the unit representation of $M_{\mathfrak{y}}$ for almost all $\mathfrak{y}$.

These remarks lead us to define a global Hecke algebra $\mathcal{H}_A$ as follows. Choose a $\mu_{\mathfrak{y}} \in \mathcal{H}_{\mathfrak{y}}$ for each $\mathfrak{y}$, with the condition that $\mu_{\mathfrak{y}}$ is the characteristic function of $M_{\mathfrak{y}}$ for almost every $\mathfrak{y}$. Let G_∞ (resp. G_f) be the subgroup of all $x \in G_A$ such that $x_{\mathfrak{y}} = 1$ for all non-archimedean (resp. archimedean) places, so that $G_A = G_\infty \times G_f$. We can define on G_f a measure (even a function)

(33) $$\mu_f = \bigotimes_{\mathfrak{y} \text{ non arch.}} \mu_{\mathfrak{y}}$$

and on G_∞ a distribution

(34) $$\mu_\infty = \bigotimes_{\mathfrak{y} \text{ arch.}} \mu_{\mathfrak{y}} .$$

If we have on $G_A = G_\infty \times G_f$ a function $\varphi(x) = \varphi(x_\infty, x_f)$ which, in every sufficiently small open subset of G_A, is the product of a C_∞ function of x_∞ and a constant function of x_f, then we can define the number

(35) $$\mu(\varphi) = \int\int \varphi(x_\infty, x_f)\, d\mu_\infty(x_\infty)\, d\mu_f(x_f) .$$

Denoting by $C^\infty(G_A)$ the set of functions φ just defined, we thus get a linear form μ on $C^\infty(G_A)$; we write $\mu = \otimes\, \mu_{\mathfrak{y}}$, and we define $\mathcal{H}_A$ as the vector space generated in the dual of $C^\infty(G_A)$ by these linear forms. The structure of algebra of $\mathcal{H}_A$ is clear - if $\mu = \otimes \mu_{\mathfrak{y}}$ and $\nu = \otimes \nu_{\mathfrak{y}}$ we define

(36)
$$\mu * \nu = \otimes\, \mu_{\mathfrak{y}} * \nu_{\mathfrak{y}} ,$$

where the $*$ is the convolution product (i.e., the product in $\mathcal{H}_{\mathfrak{y}}$ for every $\mathfrak{y}$).

Now if we are given, for every $\mathfrak{y}$, an irreducible admissible representation $\pi_{\mathfrak{y}}$ of $\mathcal{H}_{\mathfrak{y}}$ on a vector space $\mathcal{V}_{\mathfrak{y}}$, which for almost every $\mathfrak{y}$ contains the identity representation of $M_{\mathfrak{y}}$, and if we choose for every such $\mathfrak{y}$ a nonzero vector $\xi^0_{\mathfrak{y}} \in \mathcal{V}_{\mathfrak{y}}$ invariant under $M_{\mathfrak{y}}$, then we can define a tensor product

(37)
$$\mathcal{V} = \underset{\mathfrak{y}}{\otimes}\, \mathcal{V}_{\mathfrak{y}} = \varinjlim \mathcal{V}_S$$

in the same way as we defined (22), and a representation $\pi = \otimes\, \pi_{\mathfrak{y}}$ of $\mathcal{H}_A$ on $\mathcal{V}$ in the obvious way. It is easy to see that π is irreducible (no invariant subspaces), and that all operators $\pi(\mu)$, $\mu \in \mathcal{H}_A$, have finite rank.

In particular, let us go back to the situation in No. 1, and consider a minimal closed invariant subspace $\mathscr{H}$ of $L^2_0(G_k\backslash G_A, \omega)$. The representation ρ of G_A on $\mathscr{H}$ satisfies the conditions of Theorem 2. Hence $\rho = \hat{\otimes}\rho_{\mathfrak{y}}$ and $\mathscr{H} = \hat{\otimes}\, \mathscr{H}_{\mathfrak{y}}$ with irreducible unitary representations $\rho_{\mathfrak{y}}$ of the $G_{\mathfrak{y}}$ on Hilbert spaces $\mathscr{H}_{\mathfrak{y}}$. Define

(38) $$\mathcal{V}_{\mathfrak{y}} = \mathcal{H}^{f}_{\mathfrak{y}} = \bigoplus_{\vartheta_{\mathfrak{y}}} \mathcal{H}_{\mathfrak{y}}(\vartheta_{\mathfrak{y}})$$

for each $\mathfrak{y}$, and let $\pi_{\mathfrak{y}} = \rho^{f}_{\mathfrak{y}}$ be the corresponding representation of $\mathcal{H}_{\mathfrak{y}}$ on $\mathcal{V}_{\mathfrak{y}}$; it is irreducible and admissible. Denoting by $\xi^{0}_{\mathfrak{y}}$ the unit vector in $\mathcal{H}_{\mathfrak{y}}$ we have chosen for almost every $\mathfrak{y}$ to define the isomorphism between $\mathcal{H}$ and $\hat{\otimes}\,\mathcal{H}_{\mathfrak{y}}$, we can also use the $\xi^{0}_{\mathfrak{y}}$ to define the representation $\pi = \otimes \pi_{\mathfrak{y}} = \otimes \rho^{f}_{\mathfrak{y}}$ of $\mathcal{H}_{A}$ on $\mathcal{V} = \otimes \mathcal{V}_{\mathfrak{y}}$. It is then clear that

(i) the isomorphism $\hat{\otimes}\,\mathcal{H}_{\mathfrak{y}} = \mathcal{H}$ induces an isomorphism between $\mathcal{V}$ and the space $\mathcal{H}^{f}$ of M-finite vectors in $\mathcal{H}$;

(ii) all $\varphi \in \mathcal{H}^{f}$ are C^{∞} and rapidly decreasing in $\mathcal{G}$;

(iii) the irreducible representation π of $\mathcal{H}_{A}$ on $\mathcal{V} = \mathcal{H}^{f}$ is given by

(39) $$\pi(\mu)\varphi = \varphi * \check{\mu} .$$

The assertion (ii) has been proved at the end of No. 1 of this section.

We thus conclude that from a (topologically) irreducible component ρ of the unitary representation of G_{A} on $L^{2}_{0}(G_{k} \backslash G_{A}, \omega)$, we can get an (algebraically irreducible) representation π of $\mathcal{H}_{A}$ on a space of C^{∞} and M-finite functions (in fact, cusp-forms) on $G_{k} \backslash G_{A}$. By making use of the standard arguments of the theory of spherical functions it is seen at once that every $\varphi \in \mathcal{H}^{f}$ satisfies an <u>elliptical</u> differential equation on G_{∞}, hence that $\varphi(g)$ is an <u>analytical</u> function of g_{∞}. This conclusion applies more generally to all <u>cusp-forms</u> since such a function belongs to the sum of a <u>finite</u> set of irreducible subspaces of $L^{2}_{0}(G_{k} \backslash G_{A}, \omega)$.

4. The multiplicity one theorem.

Suppose we are given for each $\mathfrak{p}$ an admissible irreducible representation $\pi_{\mathfrak{p}}$ of $\mathcal{H}_{\mathfrak{p}}$ on a space $\mathcal{V}_{\mathfrak{p}}$ in such a way that, for almost every $\mathfrak{p}$, there is in $\mathcal{V}_{\mathfrak{p}}$ a vector $\xi^0_{\mathfrak{p}} \neq 0$ invariant under $M_{\mathfrak{p}}$. Let $\pi = \otimes \pi_{\mathfrak{p}}$ be the corresponding representation of $\mathcal{H}_A$ on $\mathcal{V} = \otimes \mathcal{V}_{\mathfrak{p}}$, the tensor product being defined by means of the vectors $\xi^0_{\mathfrak{p}}$.

Consider for each $\mathfrak{p}$ the Whittaker space $\mathcal{W}(\pi_{\mathfrak{p}})$ of $\pi_{\mathfrak{p}}$: its elements are functions $W_{\mathfrak{p}}$ on $G_{\mathfrak{p}}$ satisfying *

(40) $$W_{\mathfrak{p}}\left[\begin{pmatrix}1 & x_{\mathfrak{p}}\\ 0 & 1\end{pmatrix} g_{\mathfrak{p}}\right] = \tau_{\mathfrak{p}}(x_{\mathfrak{p}})\, W_{\mathfrak{p}}(g_{\mathfrak{p}})\,;$$

they are right $M_{\mathfrak{p}}$-finite, and C^{∞} (even analytical) if $\mathfrak{p}$ is archimedean ; the representation $\pi_{\mathfrak{p}}$ is equivalent to the representation on $\mathcal{W}(\pi_{\mathfrak{p}})$ given by

(41) $$\pi_{\mathfrak{p}}(\mu)\, W_{\mathfrak{p}} = W_{\mathfrak{p}} * \check{\mu}\,;$$

and finally we have

(42) $$W_{\mathfrak{p}}\begin{pmatrix}x_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix} = \begin{cases} 0 & \text{if } \mathfrak{p} \text{ is non-archimedean}\\ O\left(|x_{\mathfrak{p}}|^{-N}\right) & \text{for all } N \text{ if } \mathfrak{p} \text{ is archimedean}\end{cases}$$

for $|x_{\mathfrak{p}}|$ large enough, as well of course as

(43) $$W_{\mathfrak{p}}\left[\begin{pmatrix}x_{\mathfrak{p}} & 0\\ 0 & x_{\mathfrak{p}}\end{pmatrix} g_{\mathfrak{p}}\right] = \omega_{\mathfrak{p}}(x_{\mathfrak{p}})\, W_{\mathfrak{p}}(g_{\mathfrak{p}})$$

where $\omega_{\mathfrak{p}} = \omega_{\pi_{\mathfrak{p}}}$.

For almost every $\mathfrak{p}$ there is in $\mathcal{W}(\pi_{\mathfrak{p}})$ a function invariant under

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* We denote by $\tau_{\mathfrak{p}}$ the restriction to $k_{\mathfrak{p}}$ of the additive character τ of A.

$M_{\mathfrak{p}}$. If the largest ideal on which $\tau_{\mathfrak{p}}$ is trivial is $\mathcal{O}_{\mathfrak{p}}$ which will be the case for almost all $\mathfrak{p}$ - then it follows from Theorem 11 of Section 1 that $\mathcal{W}(\pi_{\mathfrak{p}})$ contains a unique function $W_{\mathfrak{p}}^{0}$ such that

(44) $$W_{\mathfrak{p}}^{0}(g) = 1 \quad \text{if} \quad g \in M_{\mathfrak{p}}.$$

If we use the spaces $\mathcal{W}(\pi_{\mathfrak{p}})$ instead of $\mathcal{V}_{\mathfrak{p}}$, and the $W_{\mathfrak{p}}^{0}$ instead of $\xi_{\mathfrak{p}}^{0}$, then the tensor product $\pi = \otimes\pi_{\mathfrak{p}}$ can be considered as a representation on the function space $\mathcal{W}(\pi)$ generated (as a vector space) by the functions

(45) $$W(g) = \prod W_{\mathfrak{p}}(g_{\mathfrak{p}})$$

on G_A, where $W_{\mathfrak{p}} \in \mathcal{W}(\pi_{\mathfrak{p}})$ for each $\mathfrak{p}$, and where $W_{\mathfrak{p}} = W_{\mathfrak{p}}^{0}$ for almost all $\mathfrak{p}$. Evidently we have

(46) $$W\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right] = \tau(x)\, W(g)$$

for all $x \in A$, $g \in G_A$; the $W \in \mathcal{W}(\pi)$ are right M-finite and C^{∞} with respect to the archimedean components of G_A; we have

(47) $$W\left[\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} g\right] = \omega_{\pi}(x)\, W(g)$$

for all $x \in A^{*}$ and $g \in G_A$, with

(48) $$\omega_{\pi}(x) = \prod \omega_{\mathfrak{p}}(x_{\mathfrak{p}});$$

and finally we have

(49) $$W\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = O\left(|x|^{-N}\right) \quad \text{for all } N$$

if $x \in A^{*}$ goes to infinity in $\mathcal{G}$ (i.e., if $|x| \to +\infty$). <u>These properties characterize the space</u> $\mathcal{W}(\pi)$. In fact, let $\mathcal{W}'$ be a space of functions on G_A; assume its elements satisfy the above condtions from (46) on; assume finally that $\mathcal{W}' * \mathcal{H}_A = \mathcal{W}'$, and that the obvious representation π' of $\mathcal{H}_A$ on $\mathcal{W}'$ is equivalent to π. We want to prove that $\mathcal{W}' = \mathcal{W}(\pi)$. We shall of course use the uniqueness theorem of Whittaker models in the local case.- (Section 1, No. 5 and Section 2, Theorem 4).

Denote by $W \mapsto W'$ an isomorphism of $\mathcal{W}(\pi)$ on $\mathcal{W}'$ compatible with $\mathcal{H}_A$. Choose a place $\mathfrak{y}$ and, for every $\mathfrak{v} \neq \mathfrak{y}$, a function $W_{\mathfrak{v}} \neq 0$ in $\mathcal{W}(\pi_{\mathfrak{v}})$, with $W_{\mathfrak{v}} = W_{\mathfrak{v}}^0$ almost everywhere. For every $W_{\mathfrak{y}} \in \mathcal{W}(\pi_{\mathfrak{y}})$ consider the function

(50) $$W(g) = W_{\mathfrak{y}}(g_{\mathfrak{y}}) \cdot \prod_{\mathfrak{v} \neq \mathfrak{y}} W_{\mathfrak{v}}(g_{\mathfrak{v}})$$

in $\mathcal{W}(\pi)$, and its image W' in $\mathcal{W}'$. If we give fixed values to the $g_{\mathfrak{v}}$ for each $\mathfrak{v} \neq \mathfrak{y}$ then W' reduces to a function $W'_{\mathfrak{y}}$ on $G_{\mathfrak{y}}$; it is more or less clear (rather more than less) that $W_{\mathfrak{y}} \mapsto W'_{\mathfrak{y}}$ is a nonzero homomorphism between two Whittaker models of $\pi_{\mathfrak{y}}$. Hence $W'_{\mathfrak{y}}$ is proportional to $W_{\mathfrak{y}}$ - in other words we have

(51) $$W'(g) = W_{\mathfrak{y}}(g_{\mathfrak{y}})\, c(g)$$

where $c(g)$ depends only on the $W_{\mathfrak{v}}$ and $g_{\mathfrak{v}}$ for $\mathfrak{v} \neq \mathfrak{y}$.

A similar argument [or induction on Card (S)] shows more generally the following. Let S be a finite set of places such that $W_{\mathfrak{y}}^0$ is defined for all $\mathfrak{y} \notin S$. For every

(52) $$W_S \in \bigotimes_{\mathfrak{y} \in S} \mathcal{W}(\pi_{\mathfrak{y}})$$

consider in $\mathcal{W}(\pi)$ the function

(53) $$W(g) = W_S(g_S) \cdot \prod_{\mathfrak{y} \notin S} W_{\mathfrak{y}}^0(g_{\mathfrak{y}})$$

(obvious definition for g_S) and its image W' in $\mathcal{W}'$. Then we have

(54) $$W'(g) = W_S(g_S)\, c_S(g)$$

where $c_S(g)$ depends only on the $g_{\mathfrak{v}}$, $\mathfrak{v} \notin S$. If $S' = S \cup \{\mathfrak{v}\}$ with $\mathfrak{v} \notin S$ we must of course have

(55) $$W_S(g_S)\, c_S(g) = W_S(g_S)\, W^0_{\mathfrak{v}}(g_{\mathfrak{v}})\, c_{S'}(g)$$

for all $W_S \in \bigotimes_{\mathfrak{y} \in S} \mathcal{W}(\pi_{\mathfrak{y}})$, hence

(56) $$c_S(g) = W^0_{\mathfrak{v}}(g_{\mathfrak{v}})\, c_{S'}(g) .$$

We conclude of course that

(57) $$c_S(g) = c \prod_{\mathfrak{v} \notin S} W^0_{\mathfrak{v}}(g_{\mathfrak{v}})$$

with a constant c that depends only upon the isomorphism $W \to W'$. Hence

(58) $$W'(g) = c\, W(g)$$

for all g and all $W \in \mathcal{W}(\pi)$, which concludes the proof of the uniqueness of $\mathcal{W}(\pi)$.

Theorem 3. The multiplicity of an irreducible component of the representation of G_A on $L^2_0(G_k \backslash G_A, \omega)$ is one.

Let $\mathcal{H}$ be a minimal closed invariant subspace of $L^2_0(G_k \backslash G_A, \omega)$; let $\mathcal{H}^f$ be the space of M-finite vectors in $\mathcal{H}$, and consider the representation π of $\mathcal{H}_A$ on $\mathcal{H}^f$, which we define at the end of No. 3. The functions $\varphi \in \mathcal{H}^f$ being parabolic and nice, they have Fourier series expansions

(59) $$\varphi\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right] = \sum_{\xi \neq 0} W_\varphi\left[\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g\right] \tau(\xi x)$$

with functions W_φ which are M-finite, C^∞ with respect to the archimedean variables, satisfy

(60) $$W_\varphi\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right] = \tau(x)\, W_\varphi(g),\; W_\varphi\left[\begin{pmatrix}x & 0\\ 0 & x\end{pmatrix} g\right] = \omega(x)\, W_\varphi(g),$$

and finally are rapidly decreasing at infinity - see (49) - because every $\varphi \in \mathcal{H}^f$ satisfies (17). If $\mu \in \mathcal{H}_A$ then

$$\psi = \pi(\mu)\varphi \Longrightarrow \psi = \varphi * \check{\mu} \Longrightarrow W_\psi = W_\varphi * \check{\mu}.$$

Hence the mapping $\varphi \mapsto W_\varphi$ transforms $\mathcal{H}^f$ into a , i.e., into <u>the</u>, Whittaker space of π, which means that $\mathcal{H}^f$ is the space of all functions

(61) $$\varphi(g) = \sum_{\xi \neq 0} W\left[\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} g\right] \quad \text{where } W \in \mathcal{W}(\pi).$$

Since equivalent irreducible components of the unitary representation T_ω of G_A on $L_0^2(G_k \backslash G_A, \omega)$ define equivalent irreducible representations of $\mathcal{H}_A$, it is clear that the proof of Theorem 3 is now complete.

<u>Exercise.</u> Prove directly that if two continuous operators on $L_0^2(G_k \backslash G_A, \omega)$ commute with the representation of G_A, then they commute with each other. One might even try to prove it for $L^2(G_k \backslash G_A, \omega)$ as well, since the remaining part of the spectrum (continuous spectrum and the obvious one-dimensional representation) is known to be simple.

5. <u>Euler product attached to an irreducible component.</u>

Assume we are given for each $\mathfrak{p}$ an irreducible admissible representation $\pi_\mathfrak{p}$ of $\mathcal{H}_\mathfrak{p}$, with the usual condition that $\pi_\mathfrak{p}$ contains the identity representation of $M_\mathfrak{p}$ for almost all $\mathfrak{p}$. Let χ be a character of A^*/k^* ; denote by $\chi_\mathfrak{p}$ its restriction to $k_\mathfrak{p}^*$ (which is unramified for almost all $\mathfrak{p}$). We set, at least formally,

(62) $$L_\pi(\chi, s) = \prod_{\mathfrak{p}} L_{\pi_{\mathfrak{p}}}(\chi_{\mathfrak{p}}, s)$$

where the factors $L_{\pi_{\mathfrak{p}}}(\chi_{\mathfrak{p}}, s)$ have been defined in Sections 1 and 2.

Let S_0 be a finite set of places containing all archimedean primes and such that, for every $\mathfrak{p} \notin S_0$, the following conditions are satisfied: $\pi_{\mathfrak{p}}$ contains the identity representation of $M_{\mathfrak{p}}$, the largest ideal of $k_{\mathfrak{p}}$ on which $\tau_{\mathfrak{p}}$ is trivial is $\mathfrak{d}_{\mathfrak{p}}$, and $\chi_{\mathfrak{p}}$ is unramified. By Theorem 11 of Section 1 there are for every $\mathfrak{p} \notin S_0$ two characters $\mu_{\mathfrak{p}}$ and $\nu_{\mathfrak{p}}$ of $k^*_{\mathfrak{p}}$, with $\mu_{\mathfrak{p}} \nu^{-1}_{\mathfrak{p}}$ being neither $x \mapsto |x|_{\mathfrak{p}}$ nor $x \mapsto |x|^{-1}_{\mathfrak{p}}$, and such that $\pi_{\mathfrak{p}} = \pi_{\mu_{\mathfrak{p}}, \nu_{\mathfrak{p}}}$) a member of the principal series for $G_{\mathfrak{p}}$. We thus have [see page 1. 48]

(63) $$\begin{aligned} L_\pi(\chi_{\mathfrak{p}}, s) &= L(\mu_{\mathfrak{p}} \chi_{\mathfrak{p}}, s')L(\nu_{\mathfrak{p}} \chi_{\mathfrak{p}}, s') \\ &= \left[1-\mu_{\mathfrak{p}} \chi^{-1}_{\mathfrak{p}}(\mathfrak{p}) N(\mathfrak{p})^{-s}\right]^{-1}\left[1-\nu_{\mathfrak{p}} \chi^{-1}_{\mathfrak{p}}(\mathfrak{p}) N(\mathfrak{p})^{-s}\right]^{-1} \end{aligned}$$

by formula (187) of Section 1. If we assume every $\pi_{\mathfrak{p}}$ is pre-unitary then we know by Theorem 12 of Section 1 that

(64) $$|\mu_{\mathfrak{p}}(x)| = |x|_{\mathfrak{p}}^{\sigma_{\mathfrak{p}}/2}, \ |\nu_{\mathfrak{p}}(x)| = |x|_{\mathfrak{p}}^{-\sigma_{\mathfrak{p}}/2} \quad \text{with} \quad 0 \leq \sigma_{\mathfrak{p}} \leq 1$$

so that

(65) $$|\mu_{\mathfrak{p}}(\mathfrak{p})| = N(\mathfrak{p})^{-\sigma_{\mathfrak{p}}/2}, \ |\nu_{\mathfrak{p}}(\mathfrak{p})| = N(\mathfrak{p})^{\sigma_{\mathfrak{p}}/2}.$$

Hence the product (62) converges at least like

(66) $$\prod \left[1-\chi^{-1}_{\mathfrak{p}}(\mathfrak{p}) N(\mathfrak{p})^{-s \pm \sigma_{\mathfrak{p}}/2}\right]^{-1},$$

i.e., for Re(s) large enough.*

We also observe that, under the above assumptions, we have

(67) $$\varepsilon_{\pi_{\mathfrak{p}}}(\chi_{\mathfrak{p}}, s) = 1 \qquad \text{for all } \mathfrak{p} \notin S_0 .$$

In fact, let $W^0_{\mathfrak{p}}$ be the Whittaker function of $\pi_{\mathfrak{p}}$ such that $W^0_{\mathfrak{p}}(m) = 1$ on $M_{\mathfrak{p}}$. We know by Theorem 11 of Section 1 that

(68) $$L_{\pi_{\mathfrak{p}}}(\chi_{\mathfrak{p}}, s) = L_{W^0_{\mathfrak{p}}}(e\,;\chi_{\mathfrak{p}}, s) .$$

Hence

(69) $$\frac{L_{W^0_{\mathfrak{p}}}(w, \omega_{\mathfrak{p}} - \chi_{\mathfrak{p}}, 1-s)}{L'_{\pi_{\mathfrak{p}}}(\omega_{\mathfrak{p}} - \chi_{\mathfrak{p}}, 1-s)} = \varepsilon_{\pi_{\mathfrak{p}}}(\chi_{\mathfrak{p}}, s) ,$$

where $\omega_{\mathfrak{p}}$ is given by $\pi_{\mathfrak{p}}\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} = \omega_{\mathfrak{p}}(t)\,1$. Of course $\omega_{\mathfrak{p}} - \chi_{\mathfrak{p}}$ is unramified. But (68) is valid for all unramified characters. We also have in a trivial way

(70) $$L_{W^0_{\mathfrak{p}}}(w;\omega_{\mathfrak{p}} - \chi_{\mathfrak{p}}, 1-s) = L_{W^0_{\mathfrak{p}}}(e;\omega_{\mathfrak{p}} - \chi_{\mathfrak{p}}, 1-s)$$

because $W^0_{\mathfrak{p}}$ is right invariant under $M_{\mathfrak{p}}$; this of course proves (67).

We may thus define the <u>finite</u> product

(71) $$\varepsilon_{\pi}(\chi, s) = \prod_{\mathfrak{p}} \varepsilon_{\pi_{\mathfrak{p}}}(\chi_{\mathfrak{p}}, s)$$

for every character χ of A^*/k^* .

We now go back to the irreducible components of the unitary

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* The assumption that the $\pi_{\mathfrak{p}}$ are unitary can of course be weakened. See Jacquet - Langlands, Section 11.

representation T_ω of G_A on $L_0^2(G_k \backslash G_A, \omega)$. Let π be such a component, so that π corresponds to a minimal closed invariant subspace $\mathcal{H}$ of $L_0^2(G_k \backslash G_A, \omega)$. For the sake of simplicity (and confusion) we shall now denote by $\pi_{\mathfrak{y}}$ the corresponding irreducible admissible representation of $\mathcal{H}_{\mathfrak{y}}$ instead of the corresponding irreducible unitary representation of $G_{\mathfrak{y}}$, as we did before. Since the $\pi_{\mathfrak{y}}$ are obviously pre-unitary, the Euler product $L_\pi(\chi, s)$ is defined for $\mathrm{Re}(s)$ large if χ is any character of A^*/k^*.

Theorem 4. Let π be an irreducible component of the representation of G_A on $L_0^2(G_k \backslash G_A, \omega)$. Then for every character χ of A^*/k^* the function $L_\pi(\chi, s)$ is entire, bounded in every vertical strip, and satisfies

(72) $$L_\pi(\chi, s) = \varepsilon_\pi(\chi, s)\, L_\pi(\omega - \chi, 1-s)$$

The proof is simple enough. Choose in $\mathcal{W}(\pi_{\mathfrak{y}})$, for each $\mathfrak{y}$, a function $W_{\mathfrak{y}}$, and assume $W_{\mathfrak{y}} = W^0_{\mathfrak{y}}$ almost everywhere. The function

(73) $$W(g) = \prod_{\mathfrak{y}} W_{\mathfrak{y}}(g_{\mathfrak{y}})$$

then belongs to the Whittaker space of the representation $\otimes\, \pi_{\mathfrak{y}}$ of the global Hecke algebra $\mathcal{H}_A$, as we have seen in No. 4. Hence there is in $\mathcal{H}$ a M-finite function φ such that

(74) $$\varphi(g) = \sum_{\xi \neq 0} W\left[\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g\right],$$

see (61). Consider the integral

(75) $$L_\varphi(g; \chi, s) = \int_{k^* \backslash A^*} \varphi\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right] \chi(x)^{-1} |x|^{2s-1}\, d^*x .$$

Since we know that $\varphi\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right]$ is rapidly decreasing as $|x| \to +\infty$, the part $|x| \geq 1$ of the integral converges for all values of s. But

$$\begin{aligned}\varphi\left[\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} g\right] &= \omega(x)^{-1}\varphi\left[\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} g\right] \\ &= \omega(x)^{-1}\varphi\left[w^{-1}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} wg\right] = \omega(x)^{-1}\varphi\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} wg\right].\end{aligned} \tag{76}$$

Hence $\varphi\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right]$ is rapidly decreasing as $|x| \to 0$ as well. Hence (75) is an entire function, bounded in every vertical strip for trivial reasons.

From (74) we get

$$L_\varphi(g;\chi, s) = \int_{A^*} W\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right] \chi(x)^{-1}|x|^{2s-1}d^*x\,, \tag{77}$$

provided the right-hand side is convergent. We know that $W\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right]$ is rapidly decreasing as $|x| \to +\infty$. On the other hand φ, being a cusp-form (see end of No. 1), is bounded in $\mathcal{G}$, hence on G_A, and since W is a Fourier coefficient of φ the same is true for W. It follows that (77) is justified for Re(s) large enough. Writing that

$$A^* = \bigcup_S \prod_{\mathfrak{p}\in S} k_{\mathfrak{p}}^* \times \prod_{\mathfrak{p}\notin S} \vartheta_{\mathfrak{p}}^* = \bigcup A_S^* \tag{78}$$

we have

$$\begin{aligned} L_\varphi(g;\chi, s) &= \lim_S \int_{A_S^*} W\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right] \chi(x)^{-1}|x|^{2s-1} d^*x = \\ &= \lim \prod_{\mathfrak{p}\in S} \int_{k_{\mathfrak{p}}^*} W_{\mathfrak{p}}\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g_{\mathfrak{p}}\right] \chi_{\mathfrak{p}}(x)^{-1}|x|_{\mathfrak{p}}^{2s-1} d^*x \times \\ &\times \prod_{\mathfrak{p}\notin S} \int_{\vartheta_{\mathfrak{p}}^*} W_{\mathfrak{p}}\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g_{\mathfrak{p}}\right] \chi_{\mathfrak{p}}(x)^{-1}|x|_{\mathfrak{p}}^{2s-1} d^*x\,. \end{aligned} \tag{79}$$

If S is large enough we have

$$W_{\mathfrak{p}}\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g_{\mathfrak{p}}\right] = \chi_{\mathfrak{p}}(x) = |x|_{\mathfrak{p}} = 1 \tag{80}$$

for all $x \in \mathfrak{O}_{\mathfrak{p}}^{*}$ and $\mathfrak{p} \notin S$. Thus

$$L_{\varphi}(g;\chi, s) = \lim \prod_{\mathfrak{p} \in S} L_{W_{\mathfrak{p}}}(g_{\mathfrak{p}};\chi_{\mathfrak{p}}, s), \tag{81}$$

where we use the notation (151) of Section 1. But we know by (68) that, for a given g,

$$L_{W_{\mathfrak{p}}}(g_{\mathfrak{p}};\chi_{\mathfrak{p}}, s) = L_{\pi_{\mathfrak{p}}}(\chi_{\mathfrak{p}}, s) \tag{82}$$

for almost all $\mathfrak{p}$. Since the infinite product of the $L_{\pi_{\mathfrak{p}}}(\chi_{\mathfrak{p}}, s)$ converges for for $\mathrm{Re}(s)$ large, we conclude that

$$L_{\varphi}(g;\chi, s) = \prod_{\mathfrak{p}} L_{W_{\mathfrak{p}}}(g_{\mathfrak{p}};\chi_{\mathfrak{p}}, s) \tag{83}$$

if $\mathrm{Re}(s)$ is large enough. In particular we may choose the $W_{\mathfrak{p}}$ such that $L_{W_{\mathfrak{p}}}(e;\chi_{\mathfrak{p}}, s) = L_{\pi_{\mathfrak{p}}}(\chi_{\mathfrak{p}}, s)$ for <u>all</u> $\mathfrak{p}$; then $L_{\varphi}(e;\chi, s) = L_{\pi}(\chi, s)$ for $\mathrm{Re}(s)$ large, which already shows that $L_{\pi}(\chi, s)$ is entire and bounded in every vertical strip.

In all cases we have

$$\frac{L_{\varphi}(g;\chi, s)}{L_{\pi}(\chi, s)} = \prod_{\mathfrak{p}} \frac{L_{W_{\mathfrak{p}}}(g_{\mathfrak{p}};\chi_{\mathfrak{p}}, s)}{L_{\pi_{\mathfrak{p}}}(\chi_{\mathfrak{p}}, s)}, \tag{84}$$

a <u>finite</u> product. Using the local functional equations we thus get

$$\frac{L_{\varphi}(wg;\omega-\chi, 1-s)}{L_{\pi}(\omega-\chi, 1-s)} = \varepsilon_{\pi}(\chi, s)\frac{L_{\varphi}(g;\chi, s)}{L_{\pi}(\chi, s)}. \tag{85}$$

But since $\varphi(wg) = \varphi(g)$ we have

$$L_{\varphi}(wg;\omega-\chi,1-s) = \int_{k^*\backslash A^*} \varphi\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} wg\right] \omega^{-1}\chi(x)\cdot |x|^{1-2s} d^*x$$

(86)

$$= \int \varphi\left[\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} g\right] \omega^{-1}\chi(x)|x|^{1-2s} d^*x = \int \varphi\left[\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} g\right] \chi(x)|x|^{1-2s} d^*x = L_{\varphi}(g;\chi,s),$$

and this concludes the proof.

6. The converse of Theorem 4.

We now assume we are given, for every $\mathfrak{y}$, an irreducible unitary representation $\pi_{\mathfrak{y}}$ of $G_{\mathfrak{y}}$ and hence an irreducible admissible representation (we still denote it by $\pi_{\mathfrak{y}}$) of $\mathcal{H}_{\mathfrak{y}}$. The problem is to state sufficient conditions for

(87) $$\pi = \hat{\otimes}\, \pi_{\mathfrak{y}}$$

to be contained in the representation T_ω of G_A on $L_0^2(G_k\backslash G_A,\omega)$. We assume of course that

(88) $$\pi_{\mathfrak{y}}\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} = \omega_{\mathfrak{y}}(t)\, 1 \qquad \text{for all } \mathfrak{y},$$

where $\omega_{\mathfrak{y}}$ is the restriction of ω to $k^*_{\mathfrak{y}}$, and that $\pi_{\mathfrak{y}}$ contains the identity representation of $M_{\mathfrak{y}}$ for almost all $\mathfrak{y}$ - we could not even define (87) otherwise.

<u>Theorem</u> 5. π <u>is contained in</u> $L_0^2(G_k\backslash G_A,\omega)$ <u>if and only if the following conditions are satisfied for every character</u> χ <u>of</u> A^*/k^* :

(i) <u>the Euler product</u> $L_\pi(\chi,s)$ <u>extends to an entire function bounded in every vertical strip</u>;

(2) it satisfies

(89) $$L_\pi(\chi, s) = \varepsilon_\pi(\chi, s)\, L_\pi(\omega-\chi, 1-s) .$$

We shall denote by $\mathcal{W}$ the Whittaker space of the representation $\otimes \pi_{\mathfrak{y}}$ of $\mathcal{H}_A$, which was defined in No. 4 and is spanned by the functions (45).

Step 1. We first prove that in the series

(90) $$\varphi_W(g) = \sum_{\xi \neq 0} W\left[\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} g\right],$$

the summation over k^* can be replaced by a summation over an ideal of k.

If we write the Iwasawa decomposition $g = uhm$ of g, then

(91) $$\begin{aligned} W\left[\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} g\right] &= W\left[\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} hm\right] \\ &= \tau(\xi u)\, W\left[\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} hm\right] \end{aligned}$$

and since $|\tau(\xi u)| = 1$ we may as well assume that $u = e$. The right translates of W under the elements of M evidently stay in a finite dimensional subspace of $\mathcal{W}$. Hence we may assume $m = e$, i.e., $g = h$. We may even assume that $g = \begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}$ since the behavior of W under the center of G_A is known in advance.

Assume that $W(g) = \prod W_{\mathfrak{y}}(g_{\mathfrak{y}})$ - we may of course restrict ourselves to such functions. Denote by S a finite set of places such that, for every $\mathfrak{y} \notin S$, we have $W_{\mathfrak{y}} = W_{\mathfrak{y}}^0$ and the largest ideal on which $\tau_{\mathfrak{y}}$ is trivial is $\mathfrak{d}_{\mathfrak{y}}$. We know that for every non-archimedean $\mathfrak{y}$ the function

$W_{\mathfrak{p}}\begin{pmatrix} x_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}$ vanishes outside an ideal $\mathfrak{M}_{\mathfrak{p}}$ of $k_{\mathfrak{p}}$ (Section 1, Lemma 2), and that $\mathfrak{M}_{\mathfrak{p}} = \mathfrak{O}_{\mathfrak{p}}$ if $\mathfrak{p} \notin S$ (Section 1, Theorem 11). If we denote by $\mathfrak{M}$ the ideal of k with $\mathfrak{p}$-adic completion $\mathfrak{M}_{\mathfrak{p}}$ for every finite prime $\mathfrak{p}$, and by $x^{-1}\mathfrak{M}$ (for $x \in A^*$) the ideal of k such that $(x^{-1}\mathfrak{M})_{\mathfrak{p}} = x_{\mathfrak{p}}^{-1}\mathfrak{M}_{\mathfrak{p}}$ for every such $\mathfrak{p}$, then it is clear that

$$W\begin{pmatrix} \xi x & 0 \\ 0 & 1 \end{pmatrix} \neq 0 \implies \xi \in x^{-1}\mathfrak{M}. \tag{92}$$

We thus have, formally at least,

$$\varphi_W\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \sum_{\substack{\xi \in x^{-1}\mathfrak{M} \\ \xi \neq 0}} W\begin{pmatrix} \xi x & 0 \\ 0 & 1 \end{pmatrix}. \tag{93}$$

<u>Step</u> 2. To prove that (93) and hence (90) converges absolutely, we need an estimate for $W\begin{pmatrix} \xi x & 0 \\ 0 & 1 \end{pmatrix}$.

If $W_{\mathfrak{p}} = W_{\mathfrak{p}}^0$ we know (Section 1, Theorem 11) that $\pi_{\mathfrak{p}} = \pi_{\mu_{\mathfrak{p}}, \nu_{\mathfrak{p}}}$ belongs to the principal series, and that

$$W_{\mathfrak{p}}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = |x|_{\mathfrak{p}}^{1/2} \sum_{\substack{i,j \geq 0 \\ i+j = v_{\mathfrak{p}}(x)}} \mu_{\mathfrak{p}}(\mathfrak{p}^i)\, \nu_{\mathfrak{p}}(\mathfrak{p}^j). \tag{94}$$

Since $\pi_{\mathfrak{p}}$ is unitary we have relations of the form (64), hence there is a number $\sigma > 0$ <u>independent</u> <u>from</u> $\mathfrak{p}$ such that

$$\left| W_{\mathfrak{p}}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right| \leq |x|_{\mathfrak{p}}^{1/2} \sum_{\substack{i,j \geq 0 \\ i+j = v_{\mathfrak{p}}(x)}} N(\mathfrak{p})^{(i+j)\sigma} = [v_{\mathfrak{p}}(x)+1]\, |x|_{\mathfrak{p}}^{1/2-\sigma} \tag{95}$$

with furthermore $W_{\mathfrak{p}}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = 0$ if $x \notin \mathfrak{M}_{\mathfrak{p}}$. Since $v_{\mathfrak{p}}(x) + 1 \leq |x|_{\mathfrak{p}}^{-1}$ we get

(96) $$|W_{\mathfrak{p}}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}| \leq |x|_{\mathfrak{p}}^{-\sigma-1} .$$

If $\mathfrak{p}$ is another prime, and if $\mathfrak{p}$ is non-archimedean, then the table, page 1.36, shows that

(97) $$|W_{\mathfrak{p}}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}| \leq M_{\mathfrak{p}} |x|_{\mathfrak{p}}^{-\sigma_{\mathfrak{p}}-1}$$

for a suitable choice of $M_{\mathfrak{p}}$ and $\sigma_{\mathfrak{p}}$. If $\mathfrak{p}$ is archimedean the integral representation of Whittaker functions shows at once we have an estimate (97) near $x = 0$; but we also know, by equation (72) of Section 2, that

(98) $$W_{\mathfrak{p}}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \prec \exp(-c_{\mathfrak{p}} (x\bar{x})^{1/2}) \qquad \text{as} \quad |x|_{\mathfrak{p}} \to +\infty .$$

Hence there are constants $M_{\mathfrak{p}}$, $\sigma_{\mathfrak{p}}$, $c_{\mathfrak{p}} > 0$ such that

(99) $$|W_{\mathfrak{p}}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \leq M_{\mathfrak{p}} |x|_{\mathfrak{p}}^{-\sigma_{\mathfrak{p}}-1} \exp(-c_{\mathfrak{p}} (x\bar{x})^{1/2})$$

for <u>all</u> $x \in k_{\mathfrak{p}}^{*}$. By modifying σ and the (finitely many) $\sigma_{\mathfrak{p}}$ we may assume that $\sigma_{\mathfrak{p}} = \sigma$ for <u>all</u> $\mathfrak{p}$. Then we get

(100) $$|W\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \leq M |x|^{-\sigma-1} \prod_{\mathfrak{p}\ \text{arch.}} e^{-c_{\mathfrak{p}} (x_{\mathfrak{p}} \bar{x}_{\mathfrak{p}})^{1/2}} .$$

If we consider the real vector space (or algebra)

(101) $$k_{\infty} = \prod_{\mathfrak{p}\ \text{arch.}} k_{\mathfrak{p}}$$

and denote by x_{∞} (for an $x \in A$) the vector with coordinates $x_{\mathfrak{p}}$ in k_{∞}, we thus get for W a majoration

(102) $$|W\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}| \leq M \cdot |x|^{-\sigma-1} \exp(-c \| x_{\infty} \|) \quad \text{for all } x \in A^{*} ,$$

where $\| \ \|$ is an Euclidean norm on k_∞, and where M, σ, c are constants, with $c>0$. Since $W\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ vanishes, as we have seen, unless $x_{\mathfrak{p}} \in \mathfrak{M}_{\mathfrak{p}}$ for all finite $\mathfrak{p}$, we conclude from (102) that there is on A a Schwartz-Bruhat function f such that

(103) $$\left| W\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right| \leq |x|^{-\sigma-1} f(x)$$

for all $x \in A^*$. Since $|x\xi| = |x|$, the absolute convergence of the series φ_W is now clear, and we get

(104) $$|\varphi_W(x)| \leq |x|^{-\sigma-1} \sum_{\xi \in k^*} f(\xi x) \ ;$$

but if $f \in \mathcal{S}(A)$ it is clear that $\sum f(\xi x)$ is rapidly decreasing as $|x| \to +\infty$, and (use Poisson's summation formula) is dominated by a power of $|x|$ as $|x| \to 0$. We thus get for $\varphi_W(x)$ estimates of the following kind :

(105) $$\varphi_W(x) = O(|x|^{-N}) \quad \text{for all } N \text{ as } |x| \to +\infty ,$$

(106) $$\varphi_W(x) = O(|x|^{-q}) \quad \text{for some } q \text{ as } |x| \to 0 .$$

Step 3. We consider now, for every $W \in \mathcal{W}$, the Mellin transform

(107) $$\begin{aligned} L_W(g;\chi,s) &= \int_{A^*} W\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right] \chi(x)^{-1} |x|^{2s-1} d^*x \\ &= \int_{k^*\backslash A^*} \varphi_W\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right] \chi(x)^{-1} |x|^{2s-1} d^*x , \end{aligned}$$

where χ is a character of A^*/k^*. Both integrals converge and are equal for $\mathrm{Re}(s)$ large because of the estimates (103), (105) and (106).

We shall prove that L_W can be analytically continued to the whole plane and that

$$(108)\qquad L_W(wg;\omega-\chi, 1-s) = L_W(g;\chi, s) .$$

In fact, if we assume, as we may do, that W is a product $W(g) = \prod W_{\mathfrak{p}}(g_{\mathfrak{p}})$, then the arguments we explained in the previous section show that we have

$$(109)\qquad L_W(g;\chi, s) = \prod_{\mathfrak{p}} L_{W_{\mathfrak{p}}}(g_{\mathfrak{p}};\chi_{\mathfrak{p}}, s)$$

for $\mathrm{Re}(s)$ large, with a convergent infinite product since the local representations $\pi_{\mathfrak{p}}$ are unitary. But then

$$(110)\qquad L_W(g;\chi, s) = L_\pi(\chi, s) \prod_{\mathfrak{p}} \frac{L_{W_{\mathfrak{p}}}(g_{\mathfrak{p}};\chi_{\mathfrak{p}}, s)}{L_{\pi_{\mathfrak{p}}}(\chi_{\mathfrak{p}}, s)} ,$$

with a <u>finite</u> products of <u>entire</u> functions. Since $L_\pi(\chi, s)$ is assumed to be entire, the same is true for $L_W(g;\chi, s)$.

If we compare (110) with

$$(111)\qquad L_W(wg\, ;\, \omega-\chi, 1-s) = L_\pi(\omega-\chi, 1-s) \prod \frac{L_{W_{\mathfrak{p}}}(wg_{\mathfrak{p}}, \omega_{\mathfrak{p}}-\chi_{\mathfrak{p}}, 1-s)}{L_{\pi_{\mathfrak{p}}}(\omega_{\mathfrak{p}}-\chi_{\mathfrak{p}}, 1-s)}$$

and if we take care of the <u>local</u> functional equations and the <u>global</u> functional equation for $L_\pi(\chi, s)$ which we assume is satisfied, then we get at once (108) by a trivial computation.

<u>Step</u> 4. For a given $W \in \mathcal{W}$ and a given $g \in G_A$, consider the functions

$$(112)\qquad F'(x) = \varphi_W\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right] ,$$

$$(113)\qquad F''(x) = \varphi_W\left[w\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right] = \omega(x)\, \varphi_W\left[\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} wg\right]$$

on A^*/k^*. We have formally

(114) $$\int F'(x)\chi(x)^{-1}|x|^{2s-1}d^*x = L_W(g;\chi,s)$$

(115) $$\int F''(x)\chi(x)^{-1}|x|^{2s-1}d^*x = L_W(wg;\omega\text{-}\chi,1-s) .$$

Hence the Mellin transforms of F' and F'' are identical - but for the fact that the Mellin transform of F' is defined for Re(s) large positive, while that of F'' is defined for Re(s) large negative. However, we know that the Mellin transforms of F' and F'' extend to the same entire function (Step 3 of the proof) ; we shall now prove that this entire function is bounded in every vertical strip.

We start from (110) and observe that we already assumed that $L_\pi(\chi,s)$ is bounded in vertical strips. For a finite $\mathfrak{p}$, the ratio $L_{W_\mathfrak{p}}(g_\mathfrak{p};\chi_\mathfrak{p},s)/L_{\pi_\mathfrak{p}}(\chi_\mathfrak{p},s)$ is evidently a polynomial in q^{2s} and q^{-2s}, where $q = N(\mathfrak{p})$, hence is bounded in every vertical strip. If $\mathfrak{p}$ is archimedean, the situation is a little more complicated, but can be handled by making use of the formulas of Section 2.

In fact, we know (Section 2, Theorem 1) there are characters $\mu_\mathfrak{p}$ and $\nu_\mathfrak{p}$ of $k_\mathfrak{p}^*$ such that $\pi_\mathfrak{p}$ is contained in $\rho_{\mu_\mathfrak{p};\nu_\mathfrak{p}}$. We then have [cf. Section 2, e.g., (90)]

(116) $$L_{W_\mathfrak{p}}(g_\mathfrak{p};\chi_\mathfrak{p},s) = \pm L_{\phi_{g_\mathfrak{p}}}(\nu_\mathfrak{p}-\chi,s',\mu_\mathfrak{p}-\chi,s') ,$$

where

(117) $$\phi_{g_\mathfrak{p}}\begin{pmatrix}x\\y\end{pmatrix} = \int_{k_\mathfrak{p}}\phi\left[g_\mathfrak{p}^{-1}\begin{pmatrix}z\\y\end{pmatrix}\right]\overline{\tau}_\mathfrak{p}(xz)\,dz$$

for some function $\phi \in \mathcal{S}_0(k_{\mathfrak{y}} \times k_{\mathfrak{y}})$. Since $\Phi_{g_{\mathfrak{y}}} \in \mathcal{S}(k_{\mathfrak{y}} \times k_{\mathfrak{y}})$ it is clear that $L_{W_{\mathfrak{y}}}(g_{\mathfrak{y}}\,;\chi_{\mathfrak{y}}\,, s)$ is bounded by a power of s in every vertical strip. On the other hand we know that $L_{\pi_{\mathfrak{y}}}(\chi_{\mathfrak{y}}\,, s)$ is, up to an exponential function, the product of at most two functions of the form $\Gamma(as + b)$ with $a > 0$ and complex b. Since

$$\Gamma(\sigma+it) \sim (2\pi)^{1/2}|t|^{\sigma-1/2}e^{-\frac{\pi}{2}|t|} \tag{118}$$

in $a \leq \sigma \leq b$ as $|t| \to +\infty$, we see that

$$\left|\frac{L_{W_{\mathfrak{y}}}(g_{\mathfrak{y}}\,;\chi_{\mathfrak{y}}\,, s)}{L_{\pi_{\mathfrak{y}}}(\chi_{\mathfrak{y}}\,, s)}\right| \prec |\operatorname{Im}(s)|^q e^{c\cdot|\operatorname{Im}(s)|} \tag{119}$$

at infinity in every vertical strip. We thus get the same kind of estimate for $L_W(g;\chi, s)$. Furthermore we know that if that strip is far away on the right or on the left, then $L_W(g;\chi, s)$ is bounded in this strip because of the integral representation (107) and the functional equation (108). An entire function satisfying these properties is bounded in every vertical strip, by a "well known result" of the Phragmen-Lindelöf type.

<u>Step</u> 5. We now prove that φ_W is <u>left-invariant under</u> G_k <u>for every</u> $W \in \mathcal{W}$. The invariance under the triangular group is more or less clear, so that we still have to check the invariance under w, i.e., that $F' = F''$ if we use notations (112) and (113). Consider on A^* any Schwartz-Bruhat function F with compact support, and consider the Mellin transforms of $F * F'$ and $F * F''$ (convolution products on A^*).

Denoting by

$$(120) \qquad \hat{F}(\chi, s) = \int F(x)\, \chi(x)^{-1} |x|^{2s-1}\, d^*x$$

the Mellin transform of F, which for a given χ is an entire function rapidly decreasing in every vertical strip, it is clear that the Mellin transforms of $F * F'$ and $F * F''$ are $\hat{F}(\chi, s)\, L_W(g; \chi, s)$ and $\hat{F}(\chi, s)\, L_W(wg; \omega-\chi, 1-s)$; they are entire and, by Step 4, rapidly decreasing in every vertical strip. The inversion formulas

$$(121) \qquad F * F'(x) = \frac{1}{2\pi i} \sum_{(\chi)} \int_{Re(s) = \sigma} \hat{F}(\chi, s)\, L_W(g; \chi, s) |x|^{1-2s}\, ds$$

$$(122) \quad F * F''(x) = \frac{1}{2\pi i} \sum_{(\chi)} \int_{Re(s) = 1-\sigma} \hat{F}(\chi, s)\, L_W(wg; \omega-\chi, 1-s) |x|^{1-2s}\, ds$$

are valid if σ is large enough (the summation over χ is over all unitary characters, with χ' and χ'' considered as identical if $\chi' - \chi''$ is of the form $x \to |x|^{\alpha}$). But since we integrate entire functions which decrease fast enough at infinity we can shift the integration to $\sigma = 1/2$. We thus get $F * F' = F * F''$ for all $F \in \mathcal{S}(A^*)$ with compact support, which proves that $F' = F''$.

<u>Step 6</u>. By (90), (105) and Step 5 we have $\varphi_W \in L_0^2(G_k \backslash G_A, \omega)$ for all $W \in \mathcal{W}$; furthermore it is more or less obvious that if $W' = W * \check{\mu}$ for some $\mu \in \mathcal{H}_A$ then $\varphi_{W'} = \varphi_W * \check{\mu}$; note that φ_W is C^∞ because the series (90) which defines φ_W remains convergent if we apply an invariant differential operator to its terms; the standard argument using elliptical operators then even shows that every φ_W is actually analytic

with respect to G_∞.

Now let $\mathcal{H}$ be the closure in $L_0^2(G_k \backslash G_A, \omega)$ of the space $\mathcal{V}$ of φ_W. Then $\mathcal{H}$ is invariant under the representation $g \mapsto T_\omega(g)$ of G_A on $L_0^2(G_k \backslash G_A, \omega)$.

This can be proved in about the same way as similar properties of irreducible representations of Lie groups; see reference * on page 2.2. The idea is always the same: to express that the closure of a subspace $\mathcal{V}$ is invariant under G_A, we must express that if a $\psi \in L_0^2(G_k \backslash G_A, \omega)$ is orthogonal to $\mathcal{V}$ then $(T_\omega(g)\varphi, \psi) = 0$ for all $\varphi \in \mathcal{V}$ and $g \in G_A$. But since φ is analytical the same is true of $(T_\omega(g)\varphi, \psi)$, which satisfies the same elliptical equations as φ ; thus it is enough to express that the "derivatives at the origin" of $(T_\omega(g)\varphi, \psi)$ vanish, i.e., that $(\varphi * \check{\mu}, \psi) = 0$ for all $\mu \in \mathcal{H}_A$; but this is clear if $\mathcal{V} * \mathcal{H}_A = \mathcal{V}$!

This argument more generally shows that the closed subspaces of $\mathcal{H}$ invariant under G_A are the closures of the subspaces of $\mathcal{V}$ invariant under $\mathcal{H}_A$. Since $W \mapsto \varphi_W$ is compatible with the actions of $\mathcal{H}_A$, it is clear that the representation of G_A on $\mathcal{H}$ is irreducible. To conclude the proof we should still prove that this representation is equivalent to $\hat{\otimes}\, \pi_\mathfrak{y}$, which is more or less obvious since the representation of $\mathcal{H}_A$ on $\mathcal{H}^f = \mathcal{V}$ is $\otimes\, \pi_\mathfrak{y}$.