ANALYSIS SITUS

Lectures by

James W. Alexander

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Notes by Leo Zippin

The Institute for Advanced Study

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Lecture 1.

The Physical Notion of Continuity and its Mathematical Interpretation.

In these lectures we shall develop the subject of Analysis Situs, or Topology, from its very beginnings. We shall lay especial emphasis upon the combinatorial aspect of this subject, for reasons which we shall explain in a moment. Nevertheless, we shall seek to develop our combinatorial apparatus so that it may be applicable to the most general spaces.

The key to Analysis Situs lies in the notion of topological equivalence, or homeomorphism. By definition, two geometrical figures are homeomorphic, or equivalent in the sense of Analysis Situs, if there exists a bi-uniform, bicontinuous correspondence between them. That the correspondence is bi-uniform means, of course, that we are given a pairing off of the points of the two figures, in such a way that with each point of the one figure there is paired one and only one point of the other. Before we define continuity, however, let us first try to describe the underlying physical concept and to give it a sharper mathematical formulation.

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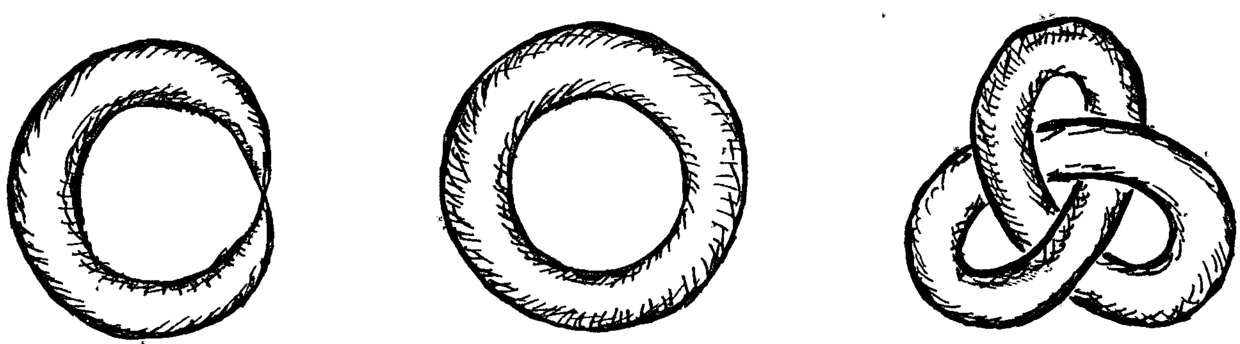
Consider, for example, the surface of a sphere in ordinary 3-space. As a physical model of the sphere, we can take an elastic, spherical membrane -- a toy balloon, if we like. This membrane can be deformed, without tearing it, into an endless variety of shapes, such as ovoids, the surfaces of cubes, tetrahedra, etc. We say that every such deformed surface is homeomorphic with the sphere, because the very deformation determines a bi-uniform, bicontinuous correspondence carrying each particle of the sphere into a corresponding particle of the deformed surface. A further word of explanation is, however, necessary. In a sense, we can also deform our spherical surface into a crescent shaped figure with touching horns, such as the one pictured in Fig. 1a.



1 (a) (b) (c)

Nevertheless, the latter figure is not to be regarded as homeomorphic with the sphere, because when we bring the tips of the horns together we bring into coincidence two particles belonging to different parts of the membrane. Thus the correspondence set up by the deformation fails to be one-one, or bi-uniform.

We must recognise, however, that the possibility of carrying our balloon from one shape to another depends as much on the properties of 3-space as upon the spherical surface. Thus, it is impossible to deform the torus in Fig. 1b into the knotted tube in Fig 1c. On the other hand, it is possible to do this if we permit ourselves to pass over into some containing 4-space. Also, the requisite correspondence between the two surfaces can be obtained in this case by cutting the torus along a generating circle, separating the two lips of the cut so as to obtain a tube open at both ends, twisting the open tube into the knotted shape of the other surface, and healing the lips of the cut again in such a way that initially adjacent particles on opposite lips of the cut become adjacent once more. We need a notion of homeomorphism, therefore, which will not depend on the possibility of deforming one figure into another within the frame of the particular space in which we may happen to find them.

When we try to translate our somewhat vague, empirical notion of physical continuity into more precise mathematical language, there seems to be more than one plausible method of procedure. In other words, several different topological theories are possible. We shall have the occasion to discuss three different theories and to study their relation to one another. We shall call the three theories point-theoretical, linear, and pure combinatorial topology, respectively.

A) Point-theoretical analysis situs is important because it is a natural development of classical function theory. If mathematics had happened to develop along different lines our interest in it might have been much smaller than it is. To formulate a rather easy, but also very narrow approach to this theory, let us consider a cartesian space of n dimensions. A point

$$a = (a_1, a_2, \cdots, a_n)$$

is called a contact point of a set of points

$$\{x\} = \{(x_1, x_2, \cdots, x_n)\}$$

provided that to every positive number ϵ there corresponds a point x of $\{x\}$ such that its coordinates satisfy the relations

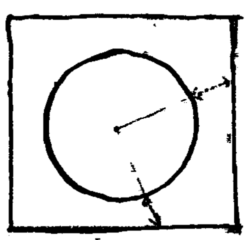
$$|x_i - a_i| < \epsilon \qquad (i = 1, 2, \ldots, n).$$

The notion of a contact point is handier than the related notion of a limit point which is generally used in analysis. A contact point of a set need not be a limit point, in the usual sense, but if it is not, then it must be a point of the set itself. A topological figure is an arbitrary set of points of the n-space. However, we think of it as something more than a mere aggregate of unrelated points, in that we give it a rudimentary internal structure, which may be expressed mathematically by saying that to each subset σ of points of the figure there corresponds a definite set of points $C(\sigma)$ consisting of all the contact points of σ belonging to the figure. Two topological figures F and F' are homeomorphic in the sense of point theoretic analysis situs if, and only if, there exists a one-one correspondance between the points of F and F' such that if a subset σ of F is paired in the correspondance with a set σ' of F' then the set $C(\sigma)$ is always paired with the set $C(\sigma')$, regardless of the choice of σ. For example a square and a circle are homeomorphic in this sense, and also a line and a circle with a missing point. We have indicated in Fig. 2a and Fig.2b simple correspondences which give the desired homeomorphisms.

For simplicity, we began with the case of a topological figure in ~~cartesian n-space~~. An ~~obvious generalisation~~ consists in defining a topological figure as an arbitrary set of quite arbitrary objects, which we call the points of the figure, together with an arbitrary law associating with each subset σ of points of the figure a definite subset $C(\sigma)$, which is then said to consist of the contact points of σ. This definition is too general for practical purposes practical purposes, but we get important classes of figures by restricting the nature of the set function $C(\sigma)$. We shall not consider now the kind of restrictions which lead to interesting theories, but you can form an idea of them by reflecting on the properties which the set function $C(\sigma)$ possesses in the case with which we began, namely that our point set belongs to an n-space and the notion of contact point is used in the sense of that space.

As we stated above, point theoretical analysis situs is important because it fits into the framework of classical analysis. Its great defect is the defect of over refinement which leads to difficulties inherent in the mathematical machinery itself rather than in the underlying problems that we are trying to solve. For example, we can construct surfaces in 3-space with the most fantastic properties, in spite of the fact that these surfaces can be approximated to any degree of accuracy by other surfaces of a perfectly regular type. The presumption is, therefore, that point theoretical analysis situs is, for most purposes, much more general than necessary.

2a.



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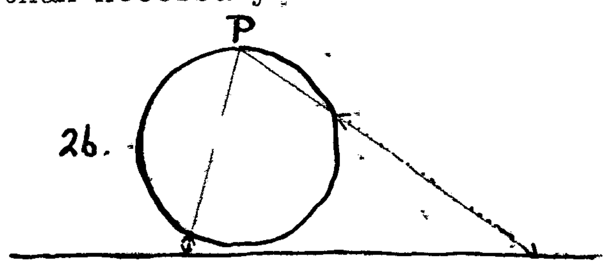


B) Linear Analysis Situs. Let us begin again with a Cartesian n-space F^n, where we use the letter F to suggest that the space is "flat". Here, again, we might have begun with a more general space defined axiomatically and enjoying many but not all of the properties of an F^n. Now we are interested only in point sets which can be broken up into a finite number of pieces (possibly overlapping) each of which satisfies a certain finite number of linear relations. These relations may be in the form of equalities or inequalities.

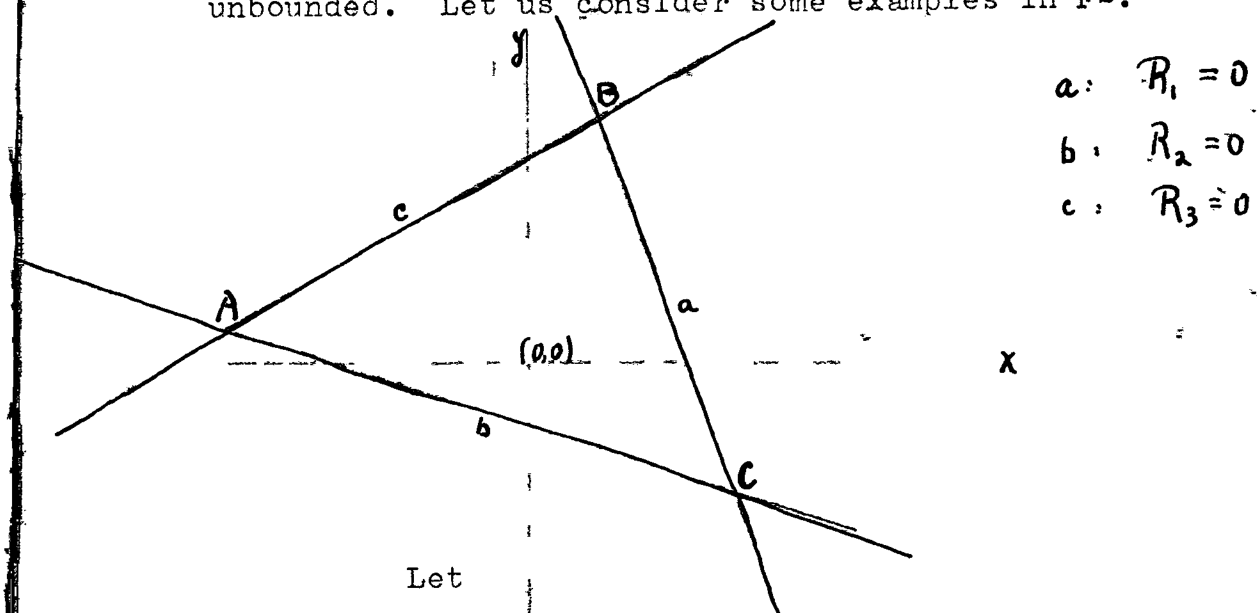
Suppose that we are given such a system of relations:

$$R^{=}) \qquad \sum a_{ij}x_i + b_j = 0 \quad ; \quad j = 1, 2, \cdots, k.$$
$$R \;\;) \qquad \sum c_{ij}x_i + d_j > 0 \quad ; \quad j = 1, 2, \cdots, k'.$$

If these are consistent, that is if there exists any point of F^n whose coordinates

$$x_1, x_2, \cdots, x_n$$

satisfy all of these relations, we shall say that they define a <u>convex</u>. The use of this word is justified by the easily observed fact that if two points belong to such a set, then the straight line interval connecting them will also belong to it. There are many kinds of convexes, depending on the nature of the defining relations. We should emphasize, in particular, that a convex may be finite or infinite, that is bounded or unbounded. Let us consider some examples in F^2.



Let

$$R_i = 0 \quad ; \quad i = 1, 2, 3 ,$$

be the relations determining the three lines, a, b, c. respectively, of our figure.

This entire system is not consistent, and therefore does not determine any convex. The relation $R_2 = 0$ determines the infinite convex consisting of the entire line b. The system of relations

$$R_1 = 0 , \quad R_3 = 0$$

determines the convex consisting of the single point B. Let us suppose that R_2 is positive for x and y both equal to zero. In this case, the relation

$$R_2 > 0 ,$$

determines the infinite convex consisting of all points of the plane which lie on the same side of the line b as the origin. The system

$$R_1 = 0 , \quad R_2 > 0 ,$$

determines that part of the line a which lies above the line b . Let us suppose, further, that in our example R_3 is positive for x and y both equal to zero (if this is not the case, we can obviously make it so by multiplying this relation by -1). Then

$$R_1 = 0, \quad R_i > 0 \; ; \quad i = 2, 3,$$

determines the convex consisting of all points on the line a which lie between the points B and C.

We now define a topologic figure as the totality of points lying on a finite number of convexes. We say that two such figures are L-homeomorphic, or homeomorphic in the sense of linear analysis situs, if they can be cut up so that the correspondence between them is not only continuous but linear as between their pieces, that is, linear in patches. For example, if we consider a triangle and a square (each with its "interior") we can set up such a correspondance between them as follows.

We choose some inner point on one of the ecges of the triangle, and an arbitrary inner point of the triangle itself. Also, we choose an arbitrary point within the square. Now we draw the lines indicated in the figure above. We then have each of our original topological figures cut up into four triangles and it is obvious that we can make a linear correspondance between appropriately chosen triangles from each figure so that the entire correspondance is one-one and, by its construction, linear in patches.

We come now to a very important question. Suppose we are given two L-figures A and B, that is, two topologic figures in the linear sense, above. Suppose further that these figures are P-homeomorphic, so that there exists a one-one bicontinuous correspondance between them, but now in the sense of point-theoretic analysis situs. Can we say that these figures are also L-homeomorphic? This is closely related to another question of which a simple case is the following. Suppose that we are given a surface in 3-space defined by overlapping pieces, each of these homeomorphic to the interior of a triangle, but in the point-theoretic sense. We would like to know whether it is possible to draw a system of wavy lines in this surface, which may be quite crinkly, so that the entire surface is cut up into a system of triangular pieces. If this were possible we could at once construct a <u>linear</u> model of this surface, in the sense of an L-figure. Now for the case of surfaces, both of our questions are rather easily answered in the affirmative. In the general case the question is still open. It must be remarked that Nöbeling has recently published a paper which claims to settle these questions. His proof, however, contains several errors of which he is aware and which he claims to have corrected.

Nöbeling's proof does not contribute any new idea to these questions as much as it seeks to carry through a type of argument which has been tried many times before without success. The fundamental difficulty appears, in all such attempts, to center around the following point. Given two L-figures which are P-homeomorphic, it is very easy to obtain a linear (in patches) transformation which approximates the given homeomorphism. It is possible to do this to any degree of approximation. This linear transformation will not, in general, be one-one. There will be more than one triangle, in the case of surfaces, in the one figure which are mapped upon the same triangle of the other. Roughly speaking, in placing one figure upon the other we may find it necessary to fold one of them at various places. One must iron out these overlappings to complete the proof.

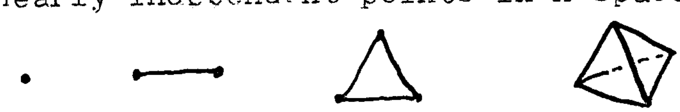
If the questions we have raised are settled in the affirmative, we see that for two L-figures the concepts of P- and L-homeomorphism coincide and we can study all the topologic invariants of these figures with the simpler machinery of transformations linear in patches.

Let us consider now an L-figure which is given to us as a set of convexes with possible overlappings. We shall show how to derive another L-figure from it whose convexes do not overlap. We regard the entire system of equalities and of inequalities which define the given figure, and order the left members of these relations into a simple sequence (finite, of course)

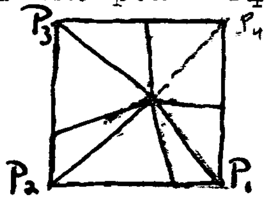
$$R_1, R_2, \cdots, R_N .$$

With each point x of the given figure we associate a signature $\sigma(x)$ with the components $\sigma_i(x)$. Here, $\sigma_i(x)$, $i = 1, \cdots, N$, has one of the three values, 1, -1, or 0 according as the coordinates of the point x give a positive, negative, or zero value to the form R_i. It is now easy to see that the set of points with the same signature determine a convex, and that the convexes which we obtain in this way are non-overlapping and yield the same topological figure.

There is a further way of dividing an L-figure into convexes which do not overlap in which we obtain convexes of a particularly simple sort ~~xxxxxxxxx~~ known as simplexes. We may arrange these simplexes according to their dimension. The first, or 0-simplex is a single point. The next, the 1-simplex, is the set of points on a straight line interval, not including the two endpoints. The 2-simplex is the interior of a plane triangle, the 3-simplex the interior of a space tetrahedron. In general, the n-simplex is the interior of the n-space figure determined by n+1 linearly independent points in n-space.



We can obtain this subdivision as follows. We introduce a point on each edge of our L-figure, but not one which is already present as a convex. Then we introduce a point in each two-dimensional piece of our figure, not on any 1-convex. And we continue this for each dimension for which there are convexes present, of that dimension, in our figure. Now, with each convex, we draw the straight line segments joining the chosen point in that convex to the chosen point in the convexes of lower dimension abutting on it, including the 0-convexes present. We shall find that some of these 1-simplexes which we have introduced determine, when taken in threes, certain 2-simplexes contained in our figure (as point sets). Further, some of the 2-simplexes taken together will determine 3-simplexes of the figure, etc. The entire system of simplexes obtained in this way will give the desired subdivision. Strictly speaking, what we have done is not quite sufficient. For it may happen that our figure does not contain certain convexes which would seem properly to belong to it, and in this case our subdivision may fail to be complete. A simple exam-le will make this clear. Suppose that our L-figure consists of all the convexes contained in the square (Figure below) $P_1P_2P_3P_4$, with the exception of the point P_4. Then when we have carried



through the process indicated above we should want to add the further 1-simplex corresponding to the dotted line, but without adjoining the point P_4 to our final figure.

After we have cut up a given figure into a simplicial one we can give a formal scheme which will serve to identify it to us. Let us call each vertex, that is 0-simplex, by some name P_i. Then we may denote a simplex which is actually present in our figure as a (formal) product of the vertices determining it. Thus an n-simplex present in the figure with vertices $P_{i_1}, P_{i_2}, \cdots, P_{i_n}$, will be symbolized by

$$P_{i_1}P_{i_2}\cdots P_{i_n} .$$

The entire figure will then be expressed as a sum of terms each of them in the form pictured above. Thus a triangle $P_0P_1P_2$ in which the vertex P_2 is not counted as belonging to the figure will be symbolized by

$$\sum_{i=0}^{1} P_i + \sum_{\bmod 3} P_iP_{i+1} + P_0P_1P_2 .$$

If we are now given two L-figures, simplicially divided and symbolized as above, it may happen that we can find a one-one correspondance between their vertices such that every n-simplex present in one figure goes over when we make the corresponding change in notation into a simplex present in the other. In this case we can recognise at once that our two figures are L-homeomorphic. This suggests to us the following secondary definition of L-homeomorphism: Two L-figures are said to be L-homeomorphic if it is possible to subdivide them into simplexes so that there exists a one-one correspondance between their vertices which carries a simplex of the one figure into the corresponding simplex of the other.

To show that this definition is equivalent to our earlier one it remains to prove that if the L-figures A and B are L-homeomorphic in the first sense they are also L-homeomorphic in this new restricted sense. It is quite easy to do this. For the given homeomorphism of A into B carries each convex of A into a set consisting of one or more subsets, each of them a convex which is contained in the set B. Now B can be cut up by this mapping into the convexes determined in this way. The mapping looked at from B, after it is cut, to A is now convex-linear. That is to say, each new convex of B is carried over into a single convex subset of A. If we further subdivide B so that each convex is simplicially divided, each simplex of B will be carried over into a (flat) simplex of A. We may now make the subdivision of A which is induced by this mapping and our homeomorphism is seen to be simplicial. If we call the new figures (subdivided) A' and B' respectively, we see that A' and B' are now congruent and formally identical, that is, to within a change of notation.

C) Pure Combinatorial Analysis Situs.

The considerations above lead us quite naturally to an abstract notion of topologic figure and topologic equivalence, which corresponds to conceiving our earlier figures as made up of chunks of space related to each other after some pattern, like a mosaic. The notion of cutting our convexes into two convex subsets is now replaced by some formal device of cutting one of these pieces into two comparable pieces.

We shall regard a C-figure, then, as a collection of certain abstract entities which we may denote by letters $A_1, A_2, \dots,$ with a certain structural or mosaic relation R which tells us when two of these pieces enjoy a property analogous to that of abutting on each other, or being incident. To determine whether two C-figures are to be regarded as equivalent, or C-homeomorphic, we shall introduce certain operations, to parallel the process of cutting up convexes above, and shall consider whether it is possible to pass from one of these figures to the other by a finite sequence of these operations. It is clear that we shall want to choose our relation R so that for geometrical figures it shall have a well-defined geometric significance and we shall wish our operations to correspond to the various ways of partitioning or cutting up a simplex. It seems to be very difficult to formulate a theory in which the most general division of a simplex can be expressed. However, it is possible to give a single operation which with its inverse is adequate for a geometric theory.

The approach which we shall now make was initiated by Newman and subjected to many simplifications by himself and by others. We begin with a system of marks, or vertices,

$$x_1, x_2, \dots, x_n.$$

Any formal product

$$x_{i_0}x_{i_1}\cdots x_{i_k}$$

in which all of the marks are distinct will be called a k-simplex. A C-figure, or mosaic, will be a formal sum of such terms. We prefer the word mosaic to the more usual word complex because we wish to emphasise that a simplex may be present although some of the simplexes on its boundary (for the moment we may say that these are all simplexes whose symbolic expression is a proper part of the given one) are not. It is clear that we can always make a geometrical model of a C-figure. Let us look into the relation of two C-figures and their geometrical models.

We shall first introduce an operation which may be applied to any C-figure, and in terms of which we shall define C-equivalence, or C-homeomorphism. We shall call this operation the operation of breaking an edge. Let us pick an edge x_ix_j of our figure. In the given mosaic let us consider all the simplexes which contain the edge x_ix_j and let us replace each of them by two formal simplexes. We obtain the first by replacing x_i by a new mark x_k (i.e. a mark not the same as any of those we have already used in our C-figures), and we obtain the second by replacing x_j by this same mark x_k. Thus a simplex

$$\cdots x_ix_j\cdots$$

is to be replaced by a sum of two simplexes

$$\cdots x_kx_j\cdots \quad + \quad \cdots x_ix_k\cdots$$

With this operation we shall, of course, admit the inverse operation of remaking an edge which consists in replacing a sum of terms, of the form given above, by the single one in x_ix_j. We may now define two C-figures, or mosaics, as C-homeomorphic if it is possible to pass from one to the other by means of a finite number of applications of this simple operation and its inverse.

The notion of "breaking an edge" is not applicable, as it stands, to a general L-figure. For all L-figures which are simplicially divided, however, it has a very simple geometrical significance which corresponds to introducing the median of a plane triangle and expressing this triangle as the sum of two others. Two L-figures which are L-homeomorphic can be simplicially divided, as we have seen, so that the resulting L-figures are simplicially equivalent. In this case both of them can be interpreted as C-figures and are then C-equivalent. It is possible also to prove the rather more difficult converse that if two C-figures are C-equivalent, inthe sense above, then the geometrical models which correspond to them are L-equivalent. We have thus a complete agreement between these two theories for all spaces to which they both apply. As we remarked before, it seems very likely that this is the case also for all three of the theories which we have sketched.

Lecture 2.

Linear Analysis Situs.

In our first lecture we considered three distinct topological theories and their interconnection. We emphasized the fact that these theories were not the only possible ones, but that they were essentially typical of the possible approaches to the subject. The linear theory was really initiated by Poincaré in a somewhat different form. He considered figures which were analytic in patches regarding them as equivalent if they were in a one-one correspondance which was analytic in patches. This differs from the linear theory only in its trimmings. What one can do in the one can also be done, for the most part, in the other but with more difficulty.

We begin with a Cartesian space F^n where a point ξ is an ordered set of numbers

$$\xi = (\xi_1, \xi_2, \cdots, \xi_n) .$$

By a coordinate system in this space we shall mean a system of n linear forms

$$x_i = \sum a_{ij}\xi_j + b_i , \qquad \left| a_{ij} \right| \neq 0 .$$

These systems fall into two classes,

$$\left| a_{ij} \right| > 0 \qquad \left| a_{ij} \right| < 0 .$$

Let us fix on an arbitrary one of these and call it the positive, or right hand class, The other we shall call the negative, or left hand class. As soon as we have agreed upon this we shall say that the space is oriented.†

Now let F^k denote a k-flat in F^n. It is determined by the vanishing of n - k linearly independent consistent linear forms:

$$x_{k+1} = x_{k+2} = \cdots = x_n = 0 .$$

† Orientation is treated in an Addendum, at the end of the lectures, following p. 31

How shall we orient F^k? We shall see that there exist two quite distinct ways of doing this. It is worth while insisting upon the fact that they are essentially independent of each other. This is often lost sight of, particularly since in many cases no real confusion arises from overlooking the fact.

1) <u>The absolute, or inner orientation:</u> There exists k linear forms

$$x_1, x_2, \cdots, x_k$$

such that the determinant of the entire system of forms

$$x_1, \cdots, x_k, x_{k+1}, \cdots, x_n$$

is different from zero. The first k forms give a coordinate system for the space F^k. If we now consider any other k forms

$$x'_1, \cdots, x'_k$$

with the same property we find that there is a linear transformation from the one system to the other with non-vanishing determinant. The totality of admissable sets of these forms falls, ~~again,~~ into two distinct classes and we may take either of these to be the positive, the other to be the negative or left hand class.

2) <u>The relative, or outer orientation.</u> This is the one used by Poincaré. The flat-space F^k may also be defined by some system of forms

$$x'_{k+1} = x'_{k+2} = \cdots = x'_n = 0$$

Between this system and the first there will exist a linear transformation with non-vanishing determinant. Again the systems fall into two classes and a particular choice determines an orientation.

However, if the space F^n is definitely oriented, then one of the possible orientations above for F^k can be used to determine the other. For if

$$x_1, \cdots, x_k$$

determines an inner orientation, and

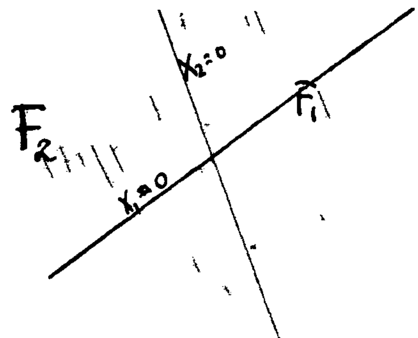
$$x_{k+1}, \cdots, x_n$$

determines an outer orientation, then the ordered system of forms

$$x_1, \cdots, x_2, \cdots, x_n,$$

determines an orientation for F^n. We may say that the inner and outer orientations are consistent for F^k, if the final orientation agrees with that established for F^n.

We can see something of the intuitive basis for the two different orientations above if we consider the example of an F^2 and a sub F^1 given, say, by a relation



$$x_1 = 0 \quad .$$

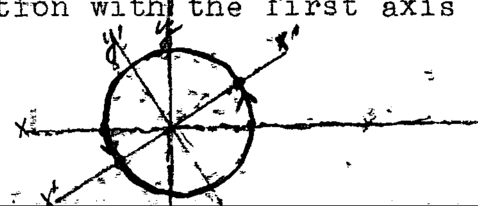
If we now take another line in F^2,

$$x_2 = 0 ,$$

the positive direction on the line x_1 may be taken as that one in which the coordinate x_2 increases. The outer orientation is related to the idea of the two sides of the line $x_1 = 0$, given by

$$x_1 > 0 \quad \text{and } x_1 < 0.$$

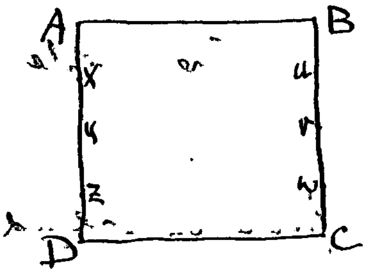
We have seen that in the plane, absolute orientation corresponds to choosing a new pair of axes, in a fixed order. If we now consider any closed curve in the plane we can ascribe to it a positive sense when we traverse its points in one direction and a negative if we run over them in the other. We can do this unambiguously for all closed curves in the plane by moving them continuously without allowing them to collapse until they are brought into coincidence with the unit circle. On this curve, however, we can take as the positive sense the one determined by starting with an intersection with the first axis of our original system and moving at once to the intersection with the first axis of the new system.



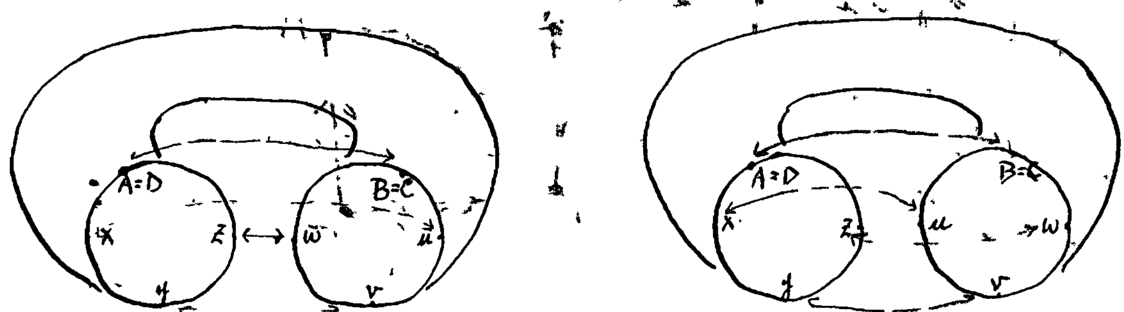
Now for the case of a general surface we can certainly do the same thing locally. We have only to take a piece ~~small enough to be~~ simply connected, to fix on a positive sense for some one curve entirely within this piece, and to transfer this sense from ~~our~~ the fixed curve to any other by deform~~ing it into the other~~ation. Here, however, it may happen that if we allow our curve to move out along ~~any~~ some large path and to return to coincidence with itself we shall find that the original positive sense is now reversed. It may happen, of course, that this is never the case. We are led, then, to a classification of surfaces into two categories, the orientable and non-orientable.

There is another notion which really belongs to the notion of outer orientation and which is sometimes confused with this. Thus orientability, in the sense above, is often confounded with two-sidedness as distinguished from one-sidedness. The confusion arises naturally, perhaps, from the simplest example of a non-orientable surface in three space - the Möbius strip. This is also one-sided. We can take a sensed curve on this surface and move it around so that it returns to its original position with reversed sense; ~~that is, it has~~ non-orientability. It is also true that if we consider a normal to this surface we can move this along the surface and return it to its initial point (keeping it normal, if the surface is regular enough, and never crossing the edge) so that it now points in the opposite direction. We may take this as a definition of one-sidedness.

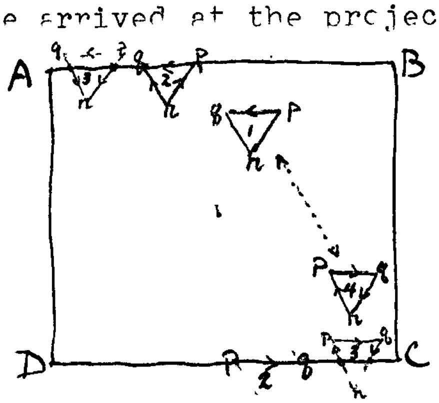
It will be interesting, perhaps, to show by examples that there is no necessary connection whatever between the notions of ~~two~~ vari-sidedness and orientability. Our examples, in each case that of a three-dimensional manifold containing a two-dimensional manifold, will exhibit the four possible combinations of these properties. The two dimensional manifolds which e hall use will be the torus and the projective plane, the three dimensional manifolds will be certain rather direct generalisations of these. We can derive all of these manifolds in the following systematic way.



We begin with a square ABCD. We identify the side AB with the side CD so that A falls on D, B on C. We now have a tube in which we propose to identify the two bases. There are two distinct ways in which we may do this and which we have indicated inthe accompanying figures. We have



the choice whether we shall preserve the original sense AyD and BwC on the base-circles, or whether we shall reverse it. The first identification, which can be carried out in 3-space, leads to the torus which is orientable. The second, which may be done in 4-space, or which we may supposed defined abstractly, leads to the projective plane. This is non-orientable. We observe that we arrived at the projective plane in two steps. If we interchange

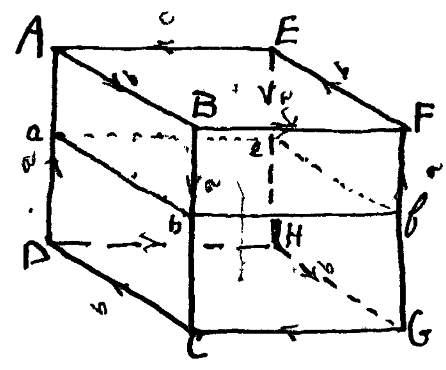


their order, as we may, it becomes clear that the final figure contains in it a surface identical with the Möbius strip on which, we already know, an oriented curve may be "reversed". We have indicated in the accompanying figure how this change of order takes place in the projective plane.

Let ~~us interpolate here the parenthetic remark~~ that

~~one might be tempted to suppose that there is still another possible surface which we could obtain from the square by the same kinds of identification.~~ where all four corners are brought into coincidence! ~~This is not the case and it can be shown that only the two kinds of tubes are possible. We shall return to this later.~~

Let us now extend the same type of construction to the case of a cube. We shall obtain different kinds of three-dimensional manifolds in which we shall be able to immerse our tubes so that they exhibit different "external", or relative properties. Let ABCDEFGH be the cube in the next figure, where we have sketched in, also, a plane section **abfe**. We are goi g to identify the opposite faces of this cube (i.e. in pairs), and we are going to do this in four different ways. We shall indicate the identification by giving the correspondance between the vertices of the faces, the correspondance will then be understood to extend linearly to the rest of the faces in question. The pairing off of the vertices will be exhibited by writing them in corresponding order. Thus



ABCD ←--→ EFGH

means that A is mapped on E, B on F, C on G, and D on H.

I) We shall get our first manifold by the following scheme:

1) ABCD ←--→ EFGH
2) BFGC ←--→ AEHD
3) ABFE ←--→ DCGH

In this case, the square abfe has its opposite edges identified according to the following pattern:

ab ←--→ ef
bf ←--→ ae

The square abfe becomes a torus, i.e. orientable. At the same time, if we take a vector normal to the surface of this square interior to our figure and move it about then, no matter what face we carry it through, it will reappear across the opposite face pointing in the same direction. That is, the torus is, in this case, two-sided. Remark that this torus does not separate our manifold. If we permit the vector to <u>leave</u> the surface of the square, we can lift it to the top face and bring it up through the bottom one. But this is an altogether different property from that of two-sidedness, or one-sidedness, with which we are not now concerned. One-, two-sidedness are <u>local</u> properties. We are concerned with whether it is possible to go from one side of a surface to the other without leaving it: in the neighborhood of a point the distinction between one side and the other can be made in all the cases which we shall consider.

II) For our second manifold we take the following scheme:

ABCD ←--→ FEHG
BFGC ←--→ AEHD
ABFE ←--→ DCGH

For the square, we have

ab ←--→ fe
bf ←--→ ae

The square becomes a projective plane, which is non-orientable. However, it is now two-sided.

III) For the third example:

ABCD ←--→ HGFE
BFGC ←--→ DHEA
AEFB ←--→ DHGC

Here we get a one-sided torus.

IV) For the fourth:

ABCD ←--→ HGFE
BFGC ←--→ EADH
AEFB ←--→ DHGC

Here a one-sided projective plane.

We should point out that in all of these examples we really have three-dimensional manifolds, and that the neighborhood of one point is exactly like that of another in spite of the seeming loss of symmetry introduced by our identifications.

That the surfaces in Examples III and IV are really one-sided becomes clear when we observe that in each case the side aA goes over into the side eH.

Let us now return to our investigation of linear subspaces of F^n. Suppose that F^k and F^m are two general subspaces, flat of course and of dimensions k and m. By saying that they are general we shall mean that the entire system of $(n-k) + (n-m)$ equations are linearly independent and consistent. The intersection of these two spaces will be a flat of dimension $(k+m-n)$, which we may call F^h. We shall write this:

$$F^h = F^k \cdot F^m .$$

The exterior orientations for F^k and F^m will now determine uniquely an exterior orientation for F^h. For if

$$x_{k+1} = \cdots = x_n = 0$$

is a system of relations defining the positive exterior orientation for F^k, and if

$$x'_{m+1} = \cdots = x'_n = 0$$

is a system defining that for F^m, then the entire set (which we have supposed independent and consistent) will define an orientation for F^h, and this may be taken to be the positive one.

It is clear that this full system will define an orientation only if we agree upon some definite ordering of the relations. We shall therefore write out first all the relations for F^k in their given order, and then all of those for F^m in their proper order. Once we have agreed upon this, or any other definite scheme, the positive orientation for F^h is determined. We see that we shall not expect the orientation of $F^k . F^m$ to agree with that of $F^m \cdot F^k$. However, there is always a simple relation between their orientations, which depends upon their dimensions:

$$\text{Orient.}(F^k \cdot F^m) = (-1)^{(n-k)(n-m)}\,\text{Orient.}(F^m \cdot F^k) \quad .$$

We do not know how to define an orientation of this intersection which shall depend on the inner orientations of F^k and F^m alone.

Positive and negative incidence.

An F^{k-1} on an F^k divided the F^k into two parts, called the sides of F^{k-1} in F^k. Suppose we choose a coordinate system

$$x_1, x_2, \cdots, x_n$$

in F^n (this orients F^n) such that F^k is determined by the vanishing of the last n-k coordinates

$$x_{k+1} = \cdots = x_n = 0,$$

and F^{k-1} by the vanishing of the last n-k+1,

$$x_k = x_{k+1} = \cdots = x_n = 0.$$

Then if we give F^k the outer orientation determined by the first set of relations, and F^{k-1} the outer orientation defined by the second set, then we shall say that the positive side F^k_+ is the one determined by the condition $x_k > 0$ and the negative side F^k_- the one determined by $x_k < 0$. We shall say, moreover, that F^{k-1} is positively incident to the side F^k_+ and negatively incident to the side F^k_-.

We shall also define the positive and negative sides of F^{k-1} for given inner orientations of F^k and F^{k-1}. Suppose we take k linear forms

$$x_1, x_2, \cdots, x_k$$

so chosen that they determine a right-handed (positive) orientation on F^k and that the first k-1 of them determine a right handed coordinate system on F^{k-1}. Then we shall say that the positive side is again the one determined by $x_k > 0$ and the negative side the one determined by $x_k < 0$.

It is important to notice the following. Suppose we define the positive side of F^{k-1} corresponding to definite inner orientations of F^k and F^{k-1}. Suppose we then orient the containing space F^n and assign to F^k and F^{k-1} the outer orientations associated with the given inner ones. Then, as we can verify immediately, the positive side relative to the given inner orientations is the same as the positive side relative to the associated outer orientations. This result is, of course, independent of the orientation assigned to the containing space F^n in spite of the fact that when we reverse the orientation of F^n we also reverse the associated outer orientations of F^k and F^{k-1}. It depends essentially on the fact that the outer orientations of F^k and F^{k-1} are both reversed simultaneously.

k-Convex

Let

$$p' = (x'_1, x'_2, \cdots, x'_n)$$

and

$$p'' = (x''_1, x''_2, \cdots, x''_n)$$

be any two distinct points of F^n. Then the set of all points p with the coordinates

$$x_i = x'_i + \lambda(x''_i - x'_i) \quad , \quad (\lambda \text{ real})$$

will be called the line $Lp'p''$ determined by p' and p''. The subset consisting of all points such that $\lambda > 0$ will be called the ray $Rp'p''$ from p' through p''. The subset consisting of all points such that $1 > \lambda > 0$ will be called the segment $Sp'p''$.

Lemma 1. If a linear function

$$\phi = \sum_i a_i x_i + b$$

has the same value at two different points p' and p'' then it is constant on the line $Lp'p''$. We see at once by reference to the equation of the line that $x_i = x'_i$ for every point of it. Moreover if it takes on different values at these points it takes on all values along the line and is monotonic (properly) on the line. This is also obvious from the equation.

Lemma 2 Suppose the inequalities

$$\phi_i = \sum_j a_{ij} x_j + b_i > 0 \quad (i = 1, 2, \ldots, k),$$

are all satisfied at the point p' of the line Lp'p". Then there exists an $\epsilon > 0$ such that these inequalities are satisfied at all points of the line such that $|\lambda| < \epsilon$.

This is also obvious.

A set of linear relations, containing equalities and also inequalities

$$\phi_i = 0 \quad (i = 1, 2, \ldots, s)$$
$$\psi_j \phi_j > 0 \quad (j = 1, 2, \ldots, t)$$

is said to be <u>consistent</u> if there is at least one point of the space Fn at which the relations are simultaneously satisfied. If the equations are consistent, then we shall say that one of them is <u>redundant</u> if it is satisfied at all points where all the other relations are satisfied. Finally, the set is called <u>independent</u> if it is consistent and contains no redundant relation. It is clear that if the set is not independent (but consistent) we can suppress at least one redundant relation, and we can repeat this if necessary until we reach an independent system.

Consider a set of linear equations

$$\sum a_{ij} x_j + b_j = \phi_i = 0 \quad (i = 1, 2, \ldots, s).$$

Then by the classical theory of linear dependence and independence the condition that the equations be consistent is that the rank of the matrix $\|a_{ij}\|$ of the equations be the same as the rank of the augmented matrix formed by adding to $\|a_{ij}\|$ a new column composed of the constants b_i. The condition that they be independent (and therefore also consistent) is that the rank of the matrix $\|a_{ij}\|$ be equal to s, the number of the equations of the system. If the equations are consistent and if the rank of the matrix $\|a_{ij}\|$ is t, $t < s$, then exactly s-t of the equations are redundant and may be discarded. The set of all points of F^n at which n-k independent linear equations are satisfied is called a <u>k-flat F^k</u> of F^n. The number k ($n > k \geqslant 0$) is called the <u>dimensionality</u> of F^k. Its ~~values~~ is independent of the particular set of relations used in determining F^k, for the relations in one set will be linearly dependent on those in the other and therefore the two sets will have the same rank n-k.

The set of all points at which a consistent set of linear relations, above, is satisfied will be called a <u>k-convex</u> C^k provided the number of linearly independent <u>equations</u> of the system is n-k. The number k is called the dimensionality of the convex C^k; we shall prove that it is independent of the particular linear system used to determine C^k. To understand the simple nature of a k-convex let us remark that each of the relations defining it determines, by itself, another convex. If this relation is an equation, the convex is an F^{n-1}. If this relation is an inequality the convex is then a whole <u>side</u> in F^n determined by the sign and by the particular F^{n-1} which we get if we change this inequality to an equality. Either of these sets is convex in the usual sense that if it contains two points it contains also the whole segment determined by them. Now we see that a k-convex is the set of points common to a finite number of these larger convexes, and is also convex in the usual sense. Its dimensionality, is simply the dimension of the lowest flat in which it is entirely contained.

Invariance of dimensionality.

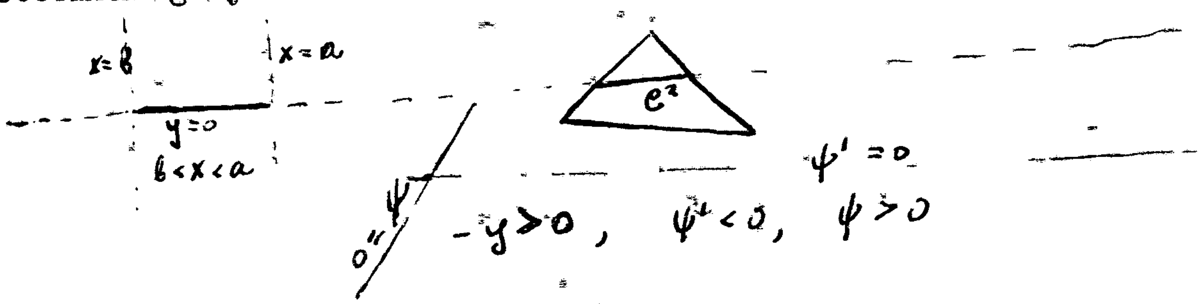
<u>Lemma 3</u>. If a linear function φ vanishes at every point p of a convex C^k then the equation $\varphi = 0$ is linearly dependent on the equations $\varphi_i = 0$ in the determining system of the convex.

<u>Proof</u>. If the equation φ were <u>not</u> linearly

dependent on the equations $\varphi_i = 0$ there would be some point q of F^n at which we would have $\varphi_i = 0$ but $\varphi > 0$ (or $\varphi < 0$). Thus at every point of the segment pq (where p is any point in the convex) we should have $\varphi \neq 0$ ~~(by lemma 1)~~. ~~But from the second~~ lemma we see that the inequalities in the determining relations of C^k are satisfied at p and therefore at all points sufficiently closedto p. Then there must be a point r on the segment pq which belongs to C^k. At such a point we must have $\varphi = 0$, and we see that φ must also be zero at q.

What has been shown here is the fact that a convex C^k determines uniquely its associated F^k. It is perhaps not yet clear what relationship obtains between the linear inequalities of two determining systems for the same C^k. The inequalities will

some typical plane convexes



give us the notion of the boundary of the convex.

Lemma 4. If a linear system is consistent then the function ψ appearing in an inequality cannot be linearly dependent on the set of functions φ_i appearing in the equalities.

For if it were it would vanish wherever these functions vanished. Therefore the relations would not be consistent.

We may observe however that if we change this inequality to an inequality then the new system remains consistent, and this is then true, of course, for any single inequality. Provided, however, that the inequality is not redundant. We shall prove this, shortly. For the moment, let us take the determining relations of the convex C^k and change one or more of the inequalities into equalities. The resulting system need not be consistent, but if it is it determines a certain C^j, necessarily of lower dimensionality than C^k (by lemma 4). We shall say that C^j is on the boundary of C^k and that the boundary of C^k consists of all (See P.46) the convexes C^j that can be obtained by the process just described.

Theorem If C^i is on the boundary of C^j and C^j is on the boundary of C^k then C^i is on the boundary of C^k.

This is obvious from the definition.

We must now show that the boundary of C^k is independent of the choice of the determining relations of C^k. Of course, a Ck may very well not have a boundary. It will not if there are no inequalities present.

Lemma 5. Let C^j be a convex on the boundary of a convex C^k. Then if a linear function vanishes at every point p of C^k it also vanishes at every point q of C^j.

Every point of the segment pq (this does not include the endpoints!) belongs to C^k(by lemma1). Therefore the function vanishes at two points of this segment, and therefore also at q.

Lemma 6. Let C^j be a convex on the boundary of a convex C^k. Then if a linear function ψ is positive at every point p of C^k it is either positive at every point q of C^j or else it is zero at every such point.

The function cannot be negative at q, otherwise it would vanish at a point r of the segment pq (by lemma 1). But every point of this segment belongs to C^k. For both the point p and q satisfy the original equalities determining C^k and both of them satisfy the inequalities which are not disturbed in the changes which define C^j. Then this is true for every point of the segment pq. Now let ψ' be one of the original inequalities which was set equal to zero in defining C^j. If this were also zero at a point r of pq it would be zero at p, since it is zero at q. But this is impossible. Since ψ is positive at every point of C^k we have a contradiction.

Suppose now that ψ vanishes at q. Then it also vanishes at every other point q' of C^j. Otherwise, ψ would be positive at q' by the first part of the argument, in which case it would be negative at every point q" of the line determined by q and q' for which the parameter value was negative. But a point q" sufficiently near to q would be a point of C^j, which would contradict the first part of the argument.

Therefore, finally, if our function does not vanish everywhere on C^j it must be positive everywhere on C^j.

Theorem: The boundary of the convex C^k is independent of the determining set of C^k.

Proof. Call two different determining sets R and R' respectively. Let C^j be a convex on the boundary of C^k in terms of the determining set R. Each function ϕ' of the second determining set R' is either positive everywhere on C^j or zero everywhere on C^j, by lemmas 5 and 6. Therefore C^j is part of ~~the~~ a convex C^i on the boundary of C^k as determined by relations R'. By reversing the argument C^i must be part of Conv C^j. Therefore C^i and C^j must coincide. Therefore the boundary of C^k in terms of R must coincide with the boundary of C^k in terms of R'.

Theorem: Let C^k be a convex determined by a set of independent relations

$$\phi_i = 0 \qquad (i = 1, 2, \ldots, n-k)$$
$$\phi_j > 0 \qquad (j = n-k+1, \ldots, n-k+s).$$

Then the number of k-1 convexes on the boundary of C^k is equal to the number of inequalities in the set.

If we change any one inequality of the set into an equality we obtain a set of relations of the form

$$\phi_i = 0, \quad \phi_{j_0} = 0, \quad \phi_j > 0 \qquad (j \neq j_0).$$

These relations are consistent. For there is surely a point of F^n, call it p, at which the conditions

$$\phi_i = 0, \quad \phi_{j_0} \leqq 0, \quad \phi_j > 0 \qquad (j \neq j_0)$$

are satisfied, otherwise the relation $\phi_{j_0} > 0$ would be redundant originally. If we have $\phi_{j_0} = 0$ at p the consistency is established. If we have $\phi_{j_0} < 0$ we can choose a point q of C^k at which $\phi_{j_0} > 0$ and there will then be a point r on the segment pq at which $\phi_{j_0} = 0$. The other relations will be satisfied at this point, and the consistency of the derived set is established. Since the new relations involve one more independent equation than the old, by an earlier lemma, they determine a C^{k-1} on the boundary of C^k. There is one such corresponding to each inequality. But if we change two inequalities, then by lemma 4, we get a convex of dimensionality k-2. It follows that there are not more than <u>s</u> of these $(k-1)$-convexes on the boundary of C^k, and that no two of them can be identical. Of course, they may overlap on convexes of lower dimensionality.

<u>Corollary</u> Every C^j on the boundary of C^k such that $j < k-1$ is on the boundary of a C^{k-1} belonging to the boundary of C^k.

<u>Corollary</u> If C^j is on the boundary of C^k there exists a sequence of convexes of the form

$$C^j, C^{j+1}, \ldots, C^k$$

beginning and ending with C^j and C^k respectively, such that each convex of the sequence is on the boundary of all the convexes that follow it.

At the expense, possibly, of the reader's patience we should like to expatiate on the subject of "orientation". It is hoped that the following remarks will not make the subject seem more confusing, to those who are not already clear what it is about.

Let us begin in a quite elementary way with an ordinary straight line, given to us in a euclidean plane. Let us mark off on this line two distinct points P and Q. On the segment PQ we are confronted, at once, with a notion of "sense". It is clearly possible and sometimes necessary to distinguish between a positive direction, say PQ, and a negative direction, say QP. It is, perhaps, not so obvious that it is reasonable to distinguish two senses in a <u>cube</u>. On the other hand, it may not seem clear to some why we distinguish only <u>two</u> senses in a cube and not more.

On the line, the segment PQ gives rise to a "coordinate" system. That is, remaining within the framework of euclidean plane geometry we can lay off the segment PQ indefinitely in either direction, and on each segment interpolate rationally a whole infinity of points. We can associate with these points the corresponding rational numbers so that order relations are preserved, and using some axiom of continuity on the line arrive at an identification between the points of the line and the set of all real numbers. Let us call this coordinate system X: to each point the association of a real number, x. This system exhibits, at once, the notion of sense on the line. We may call the positive sense the direction of increasing, and the negative the direction of decreasing reals.

Let us begin with two other points P' and Q'. This gives rise to another coordinate system X', where P' is the number zero, and P'Q' the unit of length, i.e. Q' is the number 1.

Now it is clear, without more ado, that there must exist some sort of functional relation between these two coordinate systems. Each number in the coordination X determines a point of the line, this point determines a number in the coordination X'. This functional relation is easily shown to be of the following quite simple type:

$$x = cx' + b \quad ,$$

for some pair of reals c and b of which c is not zero.

To the intuitive notion that P'Q' has the same "sense" on the line as the original segment PQ, corresponds the slightly more formal notion that in the X coordinate system P' is a smaller real number (its sign being taken into account) than Q'. ~~Thix txxtdanxixxkxwithxthx~~ To the notion that P'Q' is of opposite sense corresponds the fact that P' is a greater real than Q', in X-coordinates. But this relation is exhibited exactly in the sign of the number c, above. If c is greater than zero, P'Q' will have the same intuitive "sense", if c is le ss the opposite sense to PQ.

Let us now compare all the possible coordinate systems on our line, introduced as above. Between any pair of them there obtains a relation which we may call positive or negative, depending on the sign of the corresponding number c. If we start with a fixed, or preferred system, then all coordinate systems are separated into two distinct classes, those positively and those negatively related to it. Suppose we now take X' as preferred, forgetting X. Again we shall have a division of all coordinate systems into two classes. We shall now find that if X' is in the positive class with respect to X, then the positive class with respect to X and to X' will coincide, and their negative .

classes will coincide. On the other hand, if X' is in the negative class with respect to X, then the negative class with respect to X will coincide with the positive class with respect to X' and the positive class with respect to X will coincide with the negative class with respect to X'. Therefore the division into two classes is identical for the two coordinate systems, each as the preferred, but the names we agreed to give these classes may be confounded. What we intended to call positive we may now have called negative. There are two classes, only two, and they are somehow ambiguous. To choose one particular preferred coordinate system is to give a definite ordering to these classes, and is identical with the act of "orienting" or sensing the line.

There is another intuitive notion which is brought out in "orienting" a line. If PQ and P'Q' are positively related, we can slide one of these segments along the line so that P falls on P'. Then, by a linear contraction, we can bring Q into coincidence with Q'. We cannot do this, remaining on the line, if PQ and P'Q' are oppositely sensed. But now, suppose our line is in a plane and we are willing, or obliged, to move our segment about in the plane. Then the distinction between PQ and QP' will be lost, we can put one of these upon the other. Before we allow this to confuse us, let us appreciate that we have defined something which we may call an absolute or inner orientation which relates to our line as the entire space. Let us extend this concept directly to n-spaces. To define these linear n-spaces axiomatically does not here interest us. The point is that they can be defined so that it is possible to introduce coordinate systems in them, and the manner of introducing coordinates can be so described that all admissable coordinate systems (for our geometry) come out linearly related, with non-vanishing determinant. For example, in our n-space abstractly defined, we shall have a whole system of things we recognise as hyperplanes and the admissable coordinate systems will be those in whihh these hyperplanes have linear equations.

A coordinate system in n-space will be a one-one correspondence between the points of our space and the set of all <u>ordered</u> sets of n real numbers.

$$X: \quad P \longleftrightarrow (x_1, x_2, \ldots, x_n), \quad x_i \text{ real.}$$

In another coordinate system, X', we shall have the correspondenc

$$X': \quad P \longleftrightarrow (x'_1, x'_2, \ldots, x'_n) \quad x'_i \text{ real.}$$

Between these systems we shall have the following linear relations:

$$x'_i = \sum c_{ij}x_j + b_i, \quad i = 1, 2, \ldots, n.$$

The determinant of this system of relations will not be zero:

$$|c_{ij}| \neq 0$$

Again we shall find that all coordinate systems fall into two classes, that with respect to a <u>preferred</u> coordinate system it is easy to distinguish a positive from a negative, ~~in-terms-of~~ from the sign of the determinant, and that these classes so defined become interchanged (possibly) when we decide on <u>another</u> preferred system. To orient the space, <u>absolutely</u>, is to pick one of these classes as the "positive" one, or what comes to the same thing to pick one coordinate system as the preferred one. So much for formal definition. What reasonable distinction is thereby made, for example, in the plane or in three-space?

In the plane, we shall take three non-collinear points P, Q, and R and the line segments PQ and PR. This gives us a "2-leg" with P as center, or vertex. (This 2-leg could introduce for us, at once, a coordinate system in the plane.) Consider another 2-leg P'Q'R', with P' as center. It may be possible, moving this leg about in the plane (allowing the angle and the lengths to change continuously)* to put P'Q'R' on PQR so that corresponding ~~vertices~~ points are brought into coincidence. But it may not be possible. ~~Nxxxxxxxxxxxxxxxxxxxxx~~ Now if we cannot do this, and if we take any other 2-leg, P"Q"R" which likewise cannot be moved over continuously into PQR then it is possible to carry P'Q'R' into P"Q"R" remaining in the plane. This is the

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* the lengths must never become zero, of course, and the angle must be kept greater than zero and less than 180°.

---

distinction (in the plane) which is made by our interior, or absolute) orientation. It is clear that the same difference obtains in 3-space, and in n-space. There are two classes of n-legs: two in the same class can be pushed around into coincidence, two in different classes cannot. The reader may verify for himself that there are not more than these two classes which are distinct in the intuitive sense above. (This distinction can be made quite rigorous... it is so made in the formal definition of orientation.)

Let us turn now to exterior or relative orientation. We have seen some necessity for this in considering the line in a plane. Again, the two classes of 2-legs which we have defined in the plane are lost in ~~the~~ a containing 3-space if we permit ourselves to move an element of one of these classes into 3-space and back to the plane.

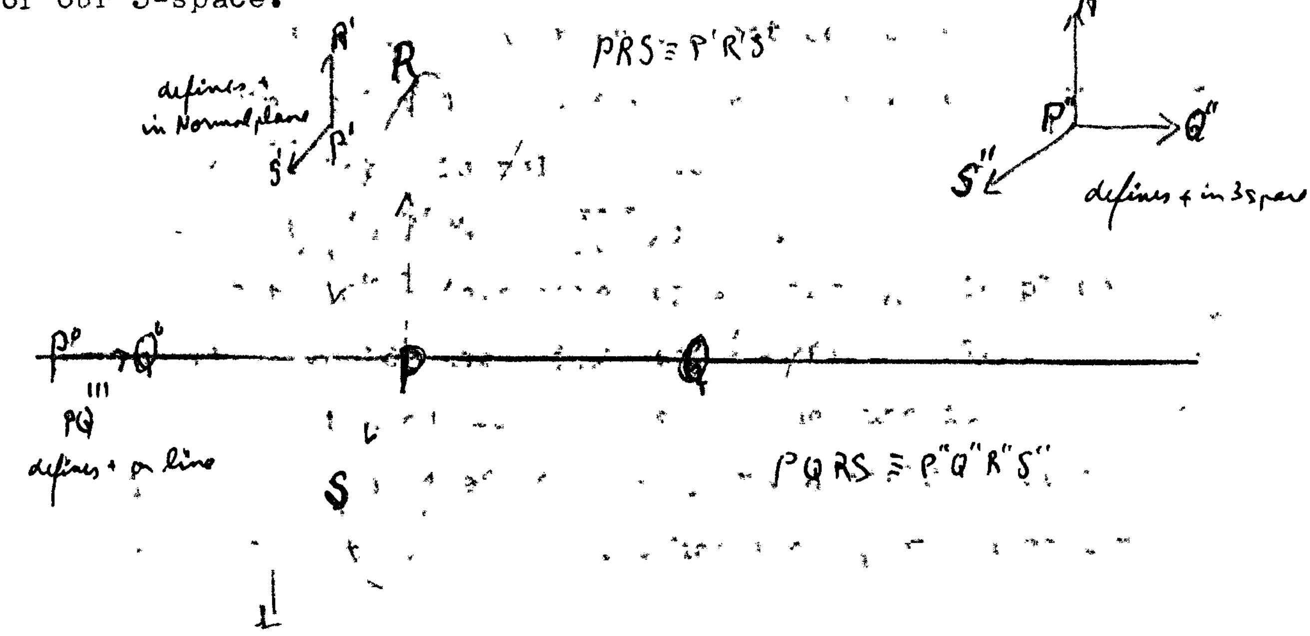
The intuitive notion that corresponds to the exterior orientation of a line in a containing plane is this. Let P be any point on this line, R any point not on it. Let us move the "vector" PR ~~this line~~ about so that P remains on the line and so that R never meets the line. We get, in this way, a certain class of vectors. There is one other class PR' of equivalent vectors, and these classes are distinct, <u>in the plane</u>. Of course, in this case (an n-1 flat in n-space) we get the division more simply by oberving that space is separated into two pieces. If we consider a line in 3-space, all vectors PR become equivalent. But in this case if we consider a 2-leg PRQ, where P is on the line and Q and R are not, we shall find that we have two classes of ~~the~~ such 2-legs, the members of one class being equivalent in 3-space and members of two different classes inequivalent.

For a line in 3-space, we obtain this separation of our 2-legs, i.e. an exterior orientation of the line with respect to the containing 3-space, by observing that in all admissable coordinate systems for the 3-space, the line is always given by two linear (independent) relations. Now all the systems of such pairs of relations defining the line fall into two classes: those in the <u>same</u> class have the property that the determinant of the transformation from one system to the other is positive. To pick on <u>one</u> of these classes and to call <u>it</u> positive is equivalent to deciding on some one representative 2-leg. This would give us a notion of positive and negative area, for example, in all planes normal to the given line. These considerations extend to an arbitrary F^k in an F^n.

There is a hint here of something which will be considered later in the lectures. The primitive notion of separating a space into two distinct pieces, as the line does the plane, is going to be generalised to a notion of linking. Consider a line in 3-space, and a unit circle (the circumference only) which has its center on the line and lies in a normal plane. Any two such circles can be moved over into each other, without our needing to cross the line. <u>But now</u> if we impose a sense on these circles, that is a direction, we shall find that it is not always possible to move them into each other so that the directions come out in agreement, <u>without crossing the line</u>. The fact that the circles are not "free" of the line, that we can't drag them out to arbitrary portions of our space is expressed in the concept of linking. The external orientation corresponds to a distinction between the "positive" and "negative" linking. which we observe above.

It is perhaps clear in what sense an exterior orientation of our line and an interior orientation are <u>independent</u>. Whatever direction we fasten to the line, it is still quite arbitrary which of the two possible "senses" we shall give to the linking circles.

However, if we have a definite orientation of 3-space before us then we can <u>oblige</u> one of these orientations to determine the other uniquely. We can do this, for example, by the convention that the positive one leg, say PQ, on the line and the positive 2-leg, say PRS, in a normal plane shall give us when taken in the definite order PQRS the ~~positive~~ 3-leg which defines the positive orientation of our 3-space.



Lecture 3.

We shall lead up quickly to the very important incidence lemma. This will tell us that if we orient the convex C^k and also orient the convexes of its boundary, then the convexes of dimensionality k-2 and less will, in a certain sense, be cancelled out.

Theorem. If C^{k-2} is on the boundary of C^k it is on the boundaries of exactly two (k-1)-convexes belonging to the boundary of C^k.

Proof. Let

1) $$\varphi_i = 0 \quad , \quad \varphi_j > 0$$

be an independent set of relations determining C^k. Then the relations determining C^{k-2} are of the form

2) $$\varphi_i = 0 \qquad \varphi'_j = 0, \quad \varphi''_j > 0 .$$

Now, C^{k-2} is on the boundary of as many (k-1)-convexes of the boundary of C^k as there are equations of the form $\varphi'_j = 0$. (cf. proof of previous theorem). Furthermore there are at least two such equations since the dimensionality of C^{k-2} is lower by 2 than the dimensionality of C^k. We have left to show that there are not more than two such equations.

Suppose there were three equations (and possibly others)

3) $$\varphi'_{j_1} = 0, \quad \varphi'_{j_2} = 0 \quad , \quad \varphi'_{j_3} = 0 \quad .$$

Then there would be a linear relation between these and the equations defining C^k: otherwise C^{k-2} would have to be of lower dimensionality than k-2. Let this relation be

4) $$\lambda'_1 \varphi'_{j_1} + \lambda'_2 \varphi'_{j_2} + \lambda'_3 \varphi'_{j_3} = \sum \lambda_i \varphi_i \quad .$$

At a point of the C^{k-1} determined by setting $\varphi'_{j_1} = 0$ in place of $\varphi'_{j_1} > 0$ in 1) we would have $\varphi'_{j_2} > 0$ and $\varphi'_{j_3} > 0$ in the left member of 4) while in the right hand ~~hand~~ member all the functions would be zero. Therefore, the coefficients λ_2 and λ_3 would be of opposite sign (they could not both vanish, otherwise φ'_{j_1} would be linearly dependent

dependent on the ϕ_i's, contrary to lemma 4). By a similar argument, λ_1 and λ_3 would be of opposite sign, and λ_1 and λ_2 of opposite sign. We thus arrive at a contradiction, because the first two conditions imply that λ_1 and λ_2 are of the same sign.

We shall introduce the notation

$$[c^j,\ c^{j-1}]$$ p. 24

for the incidence number (which we have defined to be ± 1) of the two convexes c^j and c^{j-1}. Actually, we defined the incidence number for the two oriented j- and (j-1)- flats, respectively, associated with these convexes. However, exactly as in the case of the flat, the orientation of a j-convex is defined by chooding a right-hand j-leg in the convex (for inner orientatio n) or a right hand (n-j)-leg in the space, "normal" to the flat(for exterior orientation. For incidence, as we remarked, we shall use the exterior orientation.

Incidence lemma. Let c^k, c_1^{k-1}, c_2^{k-1}, and c^{k-2} (the last three on the boundary of c^k, and the last on all their boundaries) be given arbitrary exterior orientations. Then

$$[c^k, c_1^{k-1}][c_1^{k-1}, c^{k-2}] + [c^k, c_2^{k-1}][c_2^{k-1}, c^{k-2}] = 0.$$

Proof. In the determining relations of c^k, choose an independent set of the equations

$$\phi_{k+1} = \phi_{k+2} = \cdots = \phi_n = 0$$

such that their order determines the given orientation of c^k. Then we may suppose that the given orientations of c_1^{k-1} and c_2^{k-1} are determined by the ordered sets of equations

$$\psi_1 = \phi_{k+1} = \cdots = \phi_n = 0$$

and respectively.

$$\psi_2 = \phi_{k+1} = \cdots = \phi_n = 0$$

where ψ_1 and ψ_2 vanish on C_1^{k-1} and C_2^{k-1} respectively. (There is no loss in assuming this ordering, since the linear forms may be positive or negative.) Moreover, we may suppose that the subscripts 1 and 2 have been so assigned that the order

$$\psi_1 = \psi_2 = \phi_{k+1} = \quad = \phi_n = 0,$$

determines the given orientation of C^{k-2}. Then we have

$$\begin{aligned} &[C^k, C_1^{k-1}] = \text{sign}\,\psi_1 \qquad &&[C_1^{k-1}, C^{k-2}] = -\text{sign}\,\psi_2 \\ &[C^k, C_2^{k-1}] = \text{sign}\,\psi_2 \qquad &&[C_2^{k-1}, C^{k-2}] = \text{sign}\,\psi_1, \end{aligned}$$

whence our lemma follows at once. (The reader may refer back to page 24).

We have proved the lemma in terms of outer orientations. It is true also in terms of inner orientations, since as we have seen an incidence number determined by outer orientations is the same as the one determined by the corresponding inner orientations (for an arbitrary orientation of the containing F^n).

We are in a position now to define certain quite simple and important invariants of L-figures. To establish their invariance under L-equivalence will be rather tedious. We shall therefore want to break the notion of L-equivalence down into some simple operations, and we shall find that there is one which is sufficient, such that an invariant under these operations is necessarily an L-invariant.

Connectivity Theory. Complex.

A complex is a finite number of convexes (we shall also call them cells, on occasion) such that

1) no two of its convexes have a point in common,

2) if it contains a convex C^k, it contains every C^{k-1} on the boundary of the given C^k.

(See P.46)

A complex is what we have been wont to call a mosaic which contains all the <u>boundary</u> cells associated with its cells. In point-theoretic language, it is closed. The first restriction is more formal, however. Thus two intersecting closed segments do not constitute a complex under our definition. But if we introduce their point of intersection as a vertex (0-convex) and break each of the segments into two then the new figure, which is identical with the old as a point-set, does become a complex.

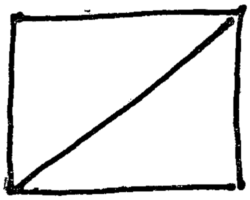
There is one minor point which should be got out of the way here. Some of the convexes of our complex may well be infinite: eg., rays. In this case our definition of L-equivalence is not strictly accurate since a ray may be linearly equivalent to a line which consists of infinitely many successive linear pieces, and we wish to rule this out in our development. We shall suppose that when two figures are L-homeomorphic, there are a finite number of linear patches (~~possibly overlapping~~) which cover the figures and are respectively linearly equivalent.

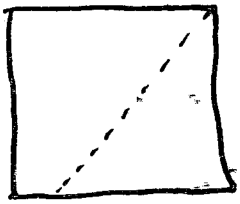
<u>Definition of an elementary operation on a convex:</u>

Suppose we are given a convex C^k and a hyperplane (i.e. (n-1)-flat) F^{n-1}, which we ~~may suppose~~ assume is determined by some single relation <u>independent</u> of the relations determining C^k.

$$\phi^0 = 0$$

The set of points of ~~C^k~~ satisfying this relation may be vacuous. If it is not vacuous, however, then it is a convex D^{k-1}, of dimensionality k-1, contained in C^k and separating C^k into two convexes: one consisting of all points at which ϕ^0 is positive, the other of points at which it is negative. Now it may happen that the boundary convexes of D^{k-1} are already contained among the boundary convexes present in C^k. <u>In this case</u> we shall permit ourselves to introduce the convex D^{k-1} and to write the original C^k as the sum of three convexes; namely, D^{k-1} itself, and the two distinct k-convexes into which C^k is split by it. We shall call this an elementary operation. In the case of the square below, the four vertices being present, adding a diagonal segment is an elementary operation, but adding the dotted segment is <u>not</u>.





There is one trivial case not covered by our definition, the situation which arises when we want to introduce a vertex into a segment. Here we may take the convention that the boundary of a vertex is empty (which seems plausible enough) and we can call the introduction of a vertex into a segment an elementary operation. We see now that it is possible to bring in the dotted segment above if we will first make the elementary divisions of the corresponding edges and <u>then</u> introduce the desired segment. See Note on reverse side 52

Now this is quite general. If the boundary convex, say D^{k-2}, of D^{k-1} above is not contained in the boundary convexes C^{k-2}_s of C^k then there is some C^{k-1} on the boundary of C^k such that C^{k-1} contains D^{k-2} as subset. If we can introduce D^{k-2} by an elementary operation (for all the convexes of this dimension in question) we can then introduce D^{k-1}. The idea here is very simple if we begin with convexes of dimension 1. These are cut by F^{n-1}, if at all, in single points, which we may introduce at once. If we now consider the 2-convexes, we shall find that these are cut along 1-convexes whose boundary vertices were already introduced. So we may proceed inductively. We see further that if we are given any finite set of hyperplanes cutting C^k into a number of flat pieces, then we can by a series of elementary operations obtain a subdivision of C^k into convexes with this property:

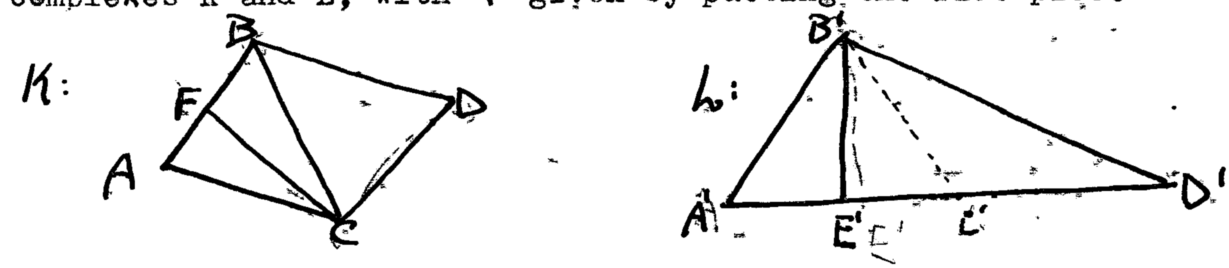
if one of our hyperplanes has any point in common with a convex of the subdivision, then it contains the convex.

What we have done is merely to reproduce the cutting up of C^k given by the hyperplanes, by elementary operations only. Obviously we can achieve the same result for an arbitrary complex by performing these operations on its successive convexes, in some order. It may be worth remarking that if we are given any figure made up of a finite number of flat pieces, not necessarily convex, we can by elementary operations convert it into a complex.

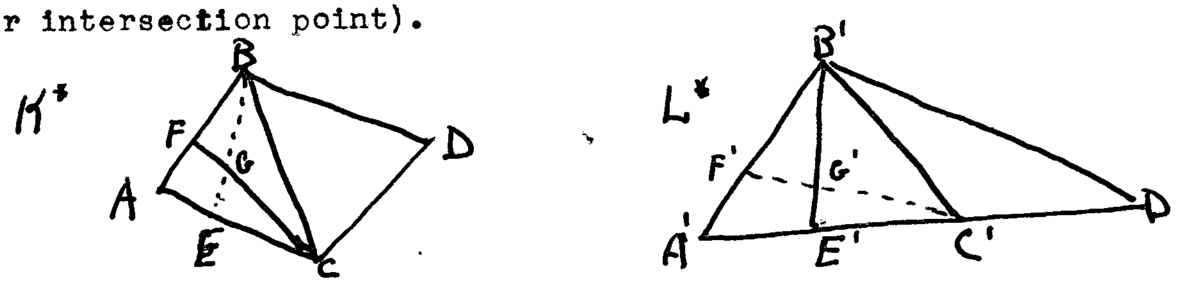
THEOREM. If we are given two complexes K and L, and a homeomorphism T which is linear in patches, then we can perform a series of elementary operations on these figures and obtain two complexes K' and L' such that the mapping T becomes a congruence; i.e. in the sense that each convex of K' is paired off by T with one of L' and the incidence relations are preserved.

Now when we say that T is linear in patches we imply that for each convex of K there exists a finite set of linear relations (hyperplanes) such that any piece of ~~K~~ this convex all of whose points have the same signature with respect to the hyperplanes, is carried by T into a single flat piece of L. These hyperplanes determine a certain subdivision of the convex in question into convex subsets. We can reproduce this division by a series of elementary operations on the convex. Suppose we do this for all the convexes present in K, obtaining a new complex K^*. Suppose that we have also performed a similar construction upon L with respect to K, obtaining a new complex L^*. We shall now have this result that each convex of K^* will be mapped by T upon a single flat piece of L^*: this piece willbe a convex point set but not necessarily a <u>convex</u> of L^*. A similar situation will obtain for L^* and its mapping by τ upon K^*.

In the accompanying figure we have indicated in bold lines two complexes K and L, with τ given by putting the flat piece



ABC on the flat piece A'B'C' (the segment B'C' and the point C' not being present as <u>convexes</u> of the complex L) and the flat piece BCD on the flat piece B'C'D'. Here the flat piece B'E'D' is not carried into a flat piece of K. The effect of our first step, in this case, is to introduce the segment B'C', and the vertex C', obtaining the complex L^*. The complex K^* is the same as the complex K. It is clear that these are not congruent, in our sense, and we have yet to get in the lines dotted below (and their intersection point).



Let us now take an arbitrary convex of K^*. This is carried over <u>linearly</u>, by τ, to L^*. Let us add to it all the cells (convexes) of K^* such that the resulting figure is still flat (in K^*) ~~and convex (as a point set)~~ and such that it is carried <u>linearly</u> over into L^*. Let us call this complex ~~convex set~~ C. Our set C is carried by τ into a complex ~~convex point set~~ H in L^*, which is the sum of a finite number of convexes of L^*. The correspondance imposes a certain subdivision upon H. We can reproduce this subdivision by a set of elementary operations.

Let us repeat this construction for every convex present in K^*. Then we shall have cut L^* into a complex L', such that each convex of K^* is carried by T into the sum of a finite number of convexes of L'. Now if we consider any convex of K^*, the set of convexes of L' which are mapped into it will determine a certain subdivision of it, and if we achieve this division by elementary operations, for every convex of K^*, we shall get the desired complex K'.

As we remarked before, this theorem enables us to tell in a very simple way whether a property of our complexes is linearly-in-patches invariant.

Lecture 4.

We come now to the connectivity theory of complexes. Let K be an arbitrary complex. It contains a certain finite number, α^0, of zero-cells (vertices), $C^0{}_i$, a certain finite number, α^1, of 1-cells (segments), $C^1{}_i$, and so forth up to a certain finite number n such that K contains n-cells but not cells of higher dimensionality. We may suppose these cells represented in a table:

$$\begin{array}{lll} C^0{}_i & , & i = 1, 2, \ldots, \alpha^0, \\ C^1{}_i & , & i = 1, 2, \ldots, \alpha^1, \\ & - - - - - & \\ C^n{}_i & , & i = 1, 2, \ldots, \alpha^n. \end{array}$$

It is not important, but we may as well remark that if K contains an n-cell it must also contain cells of all lower dimensionality since it contains the boundary of the cell.

Now let us assign to each of the cells, above, a certain definite orientation. There are many ways of doing this, of course, but we shall see that for the theory we are about to develop they will be equivalent.

We shall define a k-chain as a linear form in the symbols for the k-cells where the coefficients are arbitrary integers. The restriction to integers is not important and, in fact, we shall want to abandon it presently. It is simply easier, perhaps, to think of the coefficients as integers. Thus a k-cell

$$C^k = \sum_{i=1}^{\alpha^k} a_i C^k{}_i \quad , \quad a_i \text{ integral} ,$$

is not merely a geometrical figure but a certain set of cells with a definite "multiplicity". We can think of it as a geometrical figure where different parts are differently "weighted", and where the "densities" may cancel out when we add two chains.

We can add two chains by the simple convention of adding their coefficients:

$$C^k + \overline{C}^k = \sum (a_i + \overline{a}_i) C^k_i \quad .$$

With this definition of sum, the chains give rise to a group — we shall call it the group of k-chains and denote it by Γ^k. It is easy to see that each of these groups, for different k, is a free group and the direct sum (usually called direct product, for general non-abelian groups) of α^k infinite cyclic groups. An infinite cyclic group, of course, is any group simply isomorphic to the group of integers. The group Γ^k is not L-invariant, for we have only to subdivide our complex K to increase the number of its generators, i.e. the number of its k-cells.

We shall now introduce a linear operator β which maps one of these groups (homomorphically, in the group sense) upon the group of ~~next~~ lower dimensionality. This operator β will correspond to the geometric notion of an oriented boundary. We recall that our cells have been oriented. Therefore with each pair of cells, C^k_i and C^{k-1}_j there is associated, by our earlier definitions, an incidence number

$$\left[C^k_i,\ C^{k-1}_j\right]$$

which is ± 1 according as the two cells are positively or negatively incident. We shall make the further convention that this number is zero in case the cells are not incident at all.

Since β is, by assumption, linear it is sufficient to define it for each k-cell, C^k_i:

$$\beta C^k_i = \sum_j \left[C^k_i,\ C^{k-1}_j\right] C^{k-1}_j \quad .$$

Then we shall have to have,

$$\beta C^k = \sum_{i,j} a_i \left[C^k_i,\ C^{k-1}_j\right] C^{k-1}_j \quad .$$

By definition, a chain C^k is closed if $\beta C^k = 0$.

It is clear, from the linearity of β, that the closed chains of K of dimensionality k, form a subgroup of Γ^k. We denote this group, the group of closed k-chains, by Γ_C^k.

We shall call a chain C a <u>bounding</u> chain if there exists a chain D (necessarily of one higher dimensionality) such that

$$\beta D = C .$$

The bounding k-chains form a subgroup which we denote by Γ_B^k. It is a consequence, most important for us, of our incidence lemma that a <u>bounding chain is closed</u>, i.e. its boundary vanishes.

To prove this we have merely to compute the "boundary" of such a chain, that is to evaluate

$$\beta^2 D = \beta(\beta D) \quad , \text{ for a chain D of our complex.}$$

Let C^k denote an arbitrary k-chain of our complex. Then

$$\beta^2 C^k = \sum_{i,j,m} a^k{}_i \left[C^k{}_i,\ C^{k-1}{}_j\right]\left[C^{k-1}{}_j,\ C^{k-2}{}_m\right] C^{k-2}{}_m ,$$

as we see by direct substitution. But our incidence lemma tells us that for each $C^{k-2}{}_m$ appearing on the boundary of C^k, there are exactly two $C^{k-1}{}_j$'s incident with it, and these incidences are of opposite sign. Therefore every $C^{k-2}{}_m$ is cancelled, and the expression on the right is zero. We can, in fact, express our incidence lemma in this form:

$$\sum_{j} \left[C^k{}_i,\ C^{k-1}{}_j\right]\left[C^{k-1}{}_j,\ C^{k-2}{}_m\right] = 0$$

We now form the difference group Γ_C^k mod Γ_B^k, which we call <u>the k-th connectivity group</u> (or k-th Betti group). We shall show, later, that this is an invariant of our complex.

We shall first introduce another invariant group associated with our complex. We shall get it by defining a different linear operator δ which will map the group of k-chains on the group of <u>k+1</u>-chains. Although this operator is simpler than our β in some important respects, it does not have a simple geometric signficance. It is tied up with the fact that a chain may be regarded as a subset of our space, with parts differently weighted, <u>or</u> as a function defined over the entire space, giving certain values to the different pieces of this space. To emphasize this difference we shall want to use a new set of symbols which we shall obtain by putting bars over the old ones. Thus a dual k-chain will be a

$$\overline{C}^k = \sum_i b_i \overline{C}^k{}_i ,$$

where $\overline{C}^k{}_i$ is a new symbol associated with the old k-cell $C^k{}_i$. This will not seem so confusing, after a while. We now define the linear operator δ by defining it for k-cells (k = 0, 1, ..., n-1),

$$\delta \overline{C}^k{}_i = \sum_j \left[\overline{C}^{k+1}{}_j, \overline{C}^k{}_i\right] \overline{C}^{k+1}{}_j .$$

Now $$\delta \overline{C}^k = \sum_{i,j} b_i \left[\overline{C}^{k+1}{}_j, \overline{C}^k{}_i\right] \overline{C}^{k+1}{}_j .$$

We shall say that $\overline{C}^k$ is exact, if $\delta \overline{C}^k = 0$. This ~~is~~ suggested by multiple integrals, the k-chains playing the role of domains of integration and dual-k-chains the role of integrands. It is clear, from the linearity of δ, that the exact chains form a subgroup $\overline{\Gamma}^k{}_E$ of the group of dual k-chains $\overline{\Gamma}^k$.

We shall say that a dual chain $\overline{C}$ is derived if ~~there exists~~ a dual chain $\overline{D}$ (necessarily of one lower dimensionality) such ~~that~~

$$\delta \overline{D} = \overline{C} .$$

We see that the derived (k-) chains also form a group, $\overline{\Gamma}^k{}_D$. In consequence of our incidence lemma, the group of derived k-chains is a subgroup of the group of exact k-chains. To show this we have merely to compute $\delta^2 \overline{C}^{k-2}$ for an arbitrary (k-2)-chain of our complex. This amounts to interchanging the roles of k and k-2 in our computation, above, of the operator β^2.

Now we can form the difference group $\overline{\Gamma}^k{}_E$ mod. $\overline{\Gamma}^k{}_D$, i.e. we regard two exact chains as equivalent if their difference is a derived chain and all derived chains as equivalent to zero. This group is also a topologic invariant, which we call the k-th dual connectivity group. It is different, in general, from the k-th connectivity group, although related to it in a very definite way. To exhibit this relationship we shall have to abandon our integral coefficients and, in fact, to use different types of coefficients for ordinary chains from those which we use for dual chains. The two kinds of coefficients will want to be related to each other in that each of them will be the character group of the other. Then, very generally, our connectivity groups will also be character groups one of the other. For the moment, however, we shall not define this concept of a character group and shall confine ourselves to integers as before.

Now let

$$C^k = \sum_i a_i C^k_i \qquad \text{be an ordinary } k\text{-chain}$$

and

$$\overline{C}^k = \sum_i b_i \overline{C}^k_i \qquad \text{be a \underline{dual} } k\text{-chain.}$$

Then, by <u>the integral of $\overline{C}^k$ over C^k</u>, we shall mean the number

$$(C^k, \overline{C}^k) = \sum_i a_i b_i \ .$$

<u>STOKES' theorem for complexes.</u>

Let C be an arbitrary k-chain, and $\overline{C}$ an arbitrary dual $(k-1)$-chain. Then:

$$(C, \delta \overline{C}) = (\beta C, \overline{C}) \ .$$

Proof. We can verify this at once by direct computation.

$$\begin{array}{llcl}
1) & \overline{C} \equiv \overline{C}^{k-1} & = & \sum_j b_j \overline{C}^{k-1}_j \ , \\
2) & C \equiv C^k & = & \sum_i a_i C^k_i \ ; \\
3) & \delta \overline{C}^{k-1} & = & \sum_{j,i} b_j \left[\overline{C}^k_i, \overline{C}^{k-1}_j\right] \overline{C}^k_i \ . \\
4) & \beta C^k & = & \sum_i a_i \left[C^k_i, C^{k-1}_j\right] C^{k-1}_j
\end{array}$$

finally, $(C, \delta \overline{C}) = \sum_{i,j} a_i b_j \left[C^k_i, C^{k-1}_j\right] = (\beta C, \overline{C})$,

since the barred and unbarred incidence relations are, by definition, the same.

Now if we read the relation which we have just established , it tells us that the integral of a derived function (using this word for dual chain) over a domain (for ordinary chain) is equal to the integral of the function over the boundary of the domain. We shall call this Stokes' theorem for a complex. The resemblance is not accidental and it is possible to derive Stokes' theorem from the corresponding theorem for complexes by a passage to a limit.

Lecture Five.

We shall have to make a digression on groups before we can proceed with our development of connectivity theory. We take the opportunity to interpolate here a correction to our definition of a complex, which is required if we want to perform elementary operations on them so that they remain complexes (as formally defined). Thus a plane triangle with its interior is a complex on our old definition. Its boundary is the sum of the three edges. We got this boundary by changing (in succession) ~~khm~~ one of the inequalities defining the interior -- i.e. the 2-cell -- to an equality. Suppose now we introduce ~~xm~~ a vertex into one of the edges -- this was the simplest elementary operation. The 2-cell is not altered, its defining relations are the same, therefore its boundary. But now we ~~wmmxxthexboundaryxim~~ don't get the new boundary, but the old one; since one of the edges is broken up, what we should like to call a new complex seems, formally, not to contain the boundary of one ~~m~~ of its cells.

We can get around this quite simply as follows. Given a convex, we shall say that a point is a boundary point, or a point-boundary if it satisfies the relations defining the convex where one of these relations at least has been changed from an inequality to an equality. Thus the point which we stuck in above becomes a point-boundary. Further any cell of points each of which is a ~~mmimx~~ boundary point of our given cell (convex) will be called a boundary cell. Now we shall take as our new definition of a complex this that it is the sum of a finite number of cells, no two of them having a point in common, and such that if any cell C is present every point of the boundary of C belongs to some boundary cell C' which is also present in the complex.

With this correction out of the way we shall proceed to a discussion of abelian groups.

We shall confine our attention, for the present, to countable abelian groups. We shall say that a set of elements

$a_1, a_2, \ldots, a_n, \ldots$ (possibly infinite)

is a set of generators of an abelian group A if every element a of A can be expressed as a finite linear form in these elements:

$$a = \sum n_i a_i \quad , \quad \sum \text{ finite.}$$

It is convenient to make the convention that we regard this form as an infinite form in which almost all (that is with a finite number of exceptions) the coefficients are zero. Every group has at least one set of generators, namely the set of all elements of the group. In general it will have much less inclusive systems of generators. We shall sometimes speak of such a set as a basis for the group without implying that it has any special properties of linear independence, etc.

We shall not go into the theory of the sort of relations which must exist between two arbitrary distinct sets of generators. We shall indicate, however, a few elementary transformations which lead from one basis to another:

i) the permutation of two generators: if we interchange a_i and a_j in our sequence above, it is clear that every element a is still a finite linear form in this new set.

ii) replacing an element a_i by its negative ("inverse") $-a_i$.

iii) replacing a_i by $a_i + a_j$, $j \neq i$.

In each case, above, all the other elements remain unaltered. From these three operations (transformations) we can derive the following:

iv) replacing a_i by the element $a_i + ka_j$, where k is any integer and, as before, $i \neq j$.

To get the fourth operation from the others in case k is positive, we merely repeat the third operation k times. In case k is negative, we first change a_j to $-a_j$ (by ii), perform iii) ~~xxx~~ -k times (i.e. k times, numerically), and then change $-a_j$ back to a_j.

By our convention that we are to think of the coefficients in

$$a = \sum n_i a_i$$

as always present (almost all zero), we regard this form as a fundamental form defining the general element of the group. Let us see how the coefficients of this form change as we perform the operations above (we shall verify at the same time that we do get a new basis in all these cases).

i) the coefficients n_i and n_j are permuted.

ii) n_i is replaced by $-n_i$

iii) the coefficient n_j is replaced by $n_j - n_i$, all others including n_i are unchanged.

(thus: $n_i a_i + n_j a_j = n_i(a'_i - a'_j) + n_j a'_j = n_i a'_i + (n_j - n_i)a'_j$, in the new variables.)

iv) n_i is unchanged, n_j is replaced by $n_j - kn_i$.

We may summarize all these changes by the remark that the coefficients transform contragrediently to the elements.

Theorem. By means of elementary transformations we can bring the fundamental form down to na_1, where n is the highest common factor of the coefficients $n_1, n_2, \ldots,$ (mostly zero).

It is obviously not possible ~~for~~ in a general group to find a basis consisting of one generator such that every element of the group is an integral multiple n of that generator and the theorem is to be understood, of course, in the sense that if we take any fixed element a of the group we can perform elementary transformations on the generators so that this element becomes the multiple n of the first generator.

If the element a is expressed as a form with at most one non-vanishing coefficient, the theorem is proved for it. Let us define by the <u>complexity</u> of a form, the sum of the absolute values of its coefficients. If a form has two non-vanishing coefficients, at least, say n_i and n_j then either $n_j - n_i$ or $n_j + n_i$ has a smaller absolute value than n_j. Now we can replace n_j by the numerically smaller of these by making the change in ~~xxxxx~~ iv) taking k as +1 or -1. The complexity of our form has now been <u>reduced</u>, it is clear that the highest common factor of the coefficients is invariant. Since this factor represents a lower bound to the <u>complexity</u> we must after a finite number of steps reduce our form to $n' a'_m$ (in new generators), and by applying i) we get it as $n' a^*_1$. It is clear that n' is $\pm n$, and we can make it positive by using ii).

As a simple corollary we have the simple theorem:

<u>Corollary</u>: if $n_1, n_2, \ldots, n_k$, denote integers whose highest common factor is n, there exist integral multipliers $a_1, a_2, \ldots, a_k$ such that

$$a_1 n_1 + \cdots + a_k n_k = n .$$

We regard the a's as indeterminates, and by the theorem above convert this form to the form $n a^*_1$. This is achieved by a change of basis, each a_i being a finite linear integral form in the new variables a^*_i. The equation

$$a_1 n_1 + \cdots + a_k n_k \equiv n a^*_1$$

becomes an identity in the new variables a^*. If we now set $a^*_1 = 1$, and $a^*_i = 0$, $i \neq 1$, we may compute the values of the a_i.

Definition. A set of generators is said to be <u>free</u> if there exists no non-trivial relation between them: i.e. the only linear form in these elements which can represent the zero element is the one all of whose coefficients are zero.

A group is said to be a free group if it possesses at least one free basis, i.e. one set of free generators. In this case the set of all elements of the group in particular, and also other bases, will not be free.

In general, for a group A, there will exist a whole set of relations for a given basis:

$$\varphi_i = \sum k_{ij} a_j = 0 \quad , i = 1, 2, \ldots,$$
$$k_{ij} \text{ integral}, \sum \text{ finite.}$$

This means that there will exist forms which do not vanish identically, but do represent the zero-element of the group. Thus for a finite cyclic group of order p, say, with generator a we have $pa = 0$. This is a "typical" relation. We shall say that a set of **relations** is a fundamental set (of relations) provided every relation among the elements of the group is expressible as a finite linear form in them. That is, if φ is any relation and the φ_i above are a fundamental set, then

$$\varphi = \sum m_i \varphi_i \quad \text{for some finite set of}$$

integers m_i.

Theorem: Given a group A (countable) with generators $a_1, a_2, \ldots, a_n, \ldots,$ there always exists a countable fundamental set of relations.

Let us classify all relations on these generators according to their last non-vanishing coefficient. Within each class (there are at most a countable number of classes) let us pick one such that the numerical value of this "last non-vanishing" coefficient is a minimum. We may suppose it positive, since zero is its own inverse and therefore the negative of a relation is also a relation. Let us call these particular relations, one from each non-vacuous class, $\varphi_1, \varphi_2, \ldots, \varphi_n, \cdots$, arranged in order of "lateness" — the last non-zero term in φ_n preceding that in φ_m if $n < m$.

It is easy to see that these φ_i form a fundamental set. To show this we consider two arbitrary relations ϕ and ϕ' with ~~the same~~ last non-zero coefficient of the same index. By our previous corollary we can form a linear combination of them (which is <u>also</u> a relation) whose last non-zero term has as coefficient the highest common factor of the corresponding coefficients. Then it follows from our choice of the ϕ_i that if a relation ϕ has the same last-term a_k, with a coefficient n_k, as ϕ_i then the coefficient n'_k of a_k in ϕ_i must <u>divide</u> n_k. Therefore by subtracting from ϕ a proper integral multiple λ of ϕ_i we get a new relation $\phi'' = \phi - \lambda\phi_i$ whose last non-zero term <u>precedes</u> a_k. Now to every ϕ such a ϕ_i, as we require, exists by construction. In a finite number of steps, then, we can find an expression for ϕ in terms of our ϕ_i.

The connection between the generators and the fundamental set of relations can be exhibited conveniently in the form of a matrix:

$$
\begin{array}{c|cccccc}
 & a_1 & a_2 & a_3 & \cdots\cdots & a_n & \cdots\cdots \\
\phi_1 & x & 0 & 0 & 0 & 0 & 0 \\
\phi_2 & - & - & x & 0 & 0 & 0 \\
\phi_m & - & - & - & ------ & x & 0 \\
 & ---- & ------- & & -\ -- & & \\
\vdots & & & & & &
\end{array}
$$

In this matrix the x's denote non-zero coefficients. To the right of them stand zeros, always. I.e., the x's are the <u>last</u> non-zero coefficients. Left of them the hyphens denote coefficients which may or may not be zero. There are only a finite number, of course, in any relation: each row representing the coefficients of such a relation, each column a generator.

The problem of when two groups are isomorphic can be converted into the problem of when two matrices of this sort are equivalent under a set of matrix operations whose nature we indicated earlier in the lecture. For infinite matrices they do not exhaust the transformations which must be admitted. We call these matrices finite by rows since in each row there are only a finite number of non-zero coordinates. We shall have occasion, later, to consider matrices that are finite by columns, and correspond to ~~infinite~~ topologic groups. We shall prove now the very important

Theorem. Every subgroup B of a free group A is a free group.

The proof has already been given, in essentials. We let a_1, a_2, ..., denote a free basis of the given group A. By definition there are no relations on these generators. But we can certainly treat the elements of B, the forms, as we did the relations and obtain a fundamental set of elements of B. This is precisely the aggument we gave before, if we choose to regard the group B as the new zero, i.e. to consider A mod B. Now we can reinterpret the elements of this fundamental set as elements of the subgroup B. And we see at once that they constitute a set of free generators of the group B. First, they are a set of generators by the construction which made them a fundamental set. Secondly, there cant be any relation on them because no two of them have the same length": thus any finite sum of these fundamental elements of B will become a non-vanishing form in the a's, therefore it cannot be a relation because the a's are free by hypothesis.

In group theory the question is not whether two groups are the same group, but whether they are isomorphic, and although isomorphism defines sameness, in a sense, it will avoid confusion if we write

$$A \ (=) \ B$$

with a circle around the equality sign to indicate this kind of sameness. We may read it A is of the type of B. They may be given to us as very different looking groups.

We shall say that a group A is of the type of a direct sum of certain other groups, A_1, A_2, ..., possibly infinitely many, and shall write it

$$A \oplus A_1 \oplus A_2 \oplus \cdots$$

provided there is a (1-1) correspondance between the elements of A and the composite symbols

$$(a_1 , a_2 , \cdots, a_n , \cdots)$$

where each a_i is an element of the group A_i, almost all of them however denoting the zero of the corresponding group. Further to the element $a + a'$ of the group A there corresponds the symbol

$$(a_1+a_1' , a_2+a_2', \cdots, a_n+a_n', \cdots).$$

Of course, a_n+a_n' denotes an element of A_n.

Given a group A and a subgroup B we shall define the residue group of A with respect to B, writing it A mod. B , as the group of the cosets. Thus each element a of A determines a unique set of elements of A which we may represent by $(a + b^*)$, b^* an arbitrary element of B, such that the difference of any two of the elements of this set is an element of B. This set is called the B-coset of the element a. It is clear that if we take any other element a' of this coset, $a' - a = b'$ for some choice of b' in B. Therefore $a = a' - b'$, and a is in the B-coset of a'. The coset is independent of the element we use to define it. Now we can add two cosets by adding any two representative elements out of these cosets, and determining the coset which contains their sum. This addition is independent, as one verifies easily, of the choice of representatives. Further it is a group-addition, the coset defined by an element of B, that is to say the entire group B, playing the role of the zero or identity element.

<u>Theorem</u> Let B be a subgroup of A, and suppose that A mod B is a free group. Then,

$$A \;\textcircled{=}\; B \oplus A \bmod B \quad .$$

Proof. Let us choose some ~~free~~ basis for A mod. B ;

(a_1+b^*) , (a_2+b^*), ..., (a_n+b^*), ...

In each of these <u>cosets</u> let us pick an arbitrary element, say a_n from the coset (a_n+b^*). Then the set of elements: a_1, a_2, ..., generate a free subgroup of A which is isomorphic to A mod.B. The isomorphism is given by the correspondence, and our definition of the coset-group, A mod. B. From the fact that the coset group is free it follows that no linear <u>form</u> in the elements a_i which we have selected can be equal to any element of B, unless it is the <u>trivial form</u> with coefficients zero. For otherwise the same form in the corresponding <u>cosets</u> would equal the B-coset, which is impossible. Therefore the zero-element of A has the <u>unique</u> expression as the zero-form in the a_i 's and the zero of the group B. Now every element of A is obviously expressible in terms of the generators a_i plus some element of B. Our theorem is established.

It [redacted] is quite important [redacted] to [redacted] observe that even for very simple groups A the theorem may fail if B is arbitrary, i.e. if A mod B is not free. For example, consider the cyclic group A of order four with a single generator which we may denote by <u>a</u>: it contains the four elements, (a), (2a) , (3a) and (4a) = (0). Let B be the group generated by (2a). This has the two elements (2a), (0). A mod. B is not free and the group A [redacted] does not have B as a "direct summand".

<u>Theorem</u> Any abelian group A can always be expressed as the residue group of a free group C by a free group B, i.e. there

To prove this let us construct an arbitrary basis of the group A: $a_1, a_2, \ldots, \ldots$ (at least one exists) and let us construct a fundamental set of relations for this basis. We have shown that such a set exists, in a previous theorem. Denote the forms in this fundamental set by $F_1, F_2, \ldots$. Let C be a group which is of the type of a direct sum of the infinite cyclic groups with generators $a_1, a_2, \ldots$. That is to say, ignore the relations upon the generators. The group C defined in this way is a free group. Now each F_n represents an element, i.e. a form, of this group C. Therefore the group generated by the elements F_n is also a free group which we may denote by B. Then it is clear that

$$A \ominus C \text{ mod.} B$$

and the theorem is proved.

Group Invariants. Our concept of group is defined to within an isomorphism, isomorphic groups are equivalent. A group invariant will be any "function" (a number, a subgroup, etc.) which is defined for a group and has the same value for isomorphic groups. Let us consider some very simple invariants. An element a of a group A is said to be divisible by an integer n if there exists an element b of the group such that $a = nb$. The set of all elements divisible by n form a subgroup of A, which we may denote by nA. This is an invariant, for if $A \ominus B$ then $nA \ominus nB$. This is quite obvious, for if a and b are paired under the isomorphism of the original groups, and if a is of the form nc then b is of the form nc' for that element c' of B which is paired with c.

Even the very simple invariant defined above can be of great assistance in distinguishing between two groups. Thus, let φ_n denote the free group on n generators, and φ_m the free group on m generators. Let us consider $\varphi_n \text{ mod}(2\varphi_n)$, and $\varphi_m \text{ mod}(2\varphi_m)$. In both groups each element is of order two. This means that if we consider the forms in the generators of ϕ_n, for example, we may reduced all coefficients modulo 2. The group $\phi_n \bmod (2\phi_n)$ has 2^n distinct elements! Since 2^n and 2^m are equal when and only m and n are equal we see that ϕ_n and ϕ_m cannot be isomorphic unless m = n. Implicitly, we have used another group invariant -- the number of elements in the group. Of course we could not have applied this directly, as a criterion, to ϕ_n and ϕ_m since both are infinite.

We turn now to the classification of all abelian groups which have a finite basis .. i.e. a finite set of generators. We shall prove a preliminary

Theorem. A necessary and sufficient condition that an abelian group A with a finite basis be free is that it contain no element of finite order (the zero-element, of course, excepted).

To prove this we shall show ~~that if~~ we can pick ~~an arbitrary~~ a basis ~~$a_1, a_2, \ldots, a_k$~~ for the group A such that there will exist no relations on these except trivial ones. Let $a_1, a_2, \ldots a_k$ be an arbitrary finite basis. Suppose that

$\sum n_i a_i = 0$ is a relation, and let n be the highest common factor of the coefficients n_i. Then we can write this as: $na = 0$ where $a = \sum n'_i a_i$ and $n_i = n'_i \cdot n$. Therefore the element a must be the identity since it is of finite order n. But now the h.c.f. of the coefficients n'_i is 1. ~~This means that we can~~ express at least one of the a_i as a linear ~~form in the remaining ones so that this generator~~ is ~~redundant~~; i.e. if we discard it the remaining generators still define a basis for the group. It is obvious that in a finite number of steps, when we have eliminated all the redundant generators we shall have no relations left.

Theorem: If a group A has a finite basis, then every subgroup B has a finite basis.

Proof. Consider the group C = A mod. B. Here the elements of B play the role of relations. Therefore we can pick a fundamental set of these relations, by an earlier theorem. In this fundamental set, no two "relations" have the same last term, i.e. each one contains one generator with non-zero coefficient, which does not appear in all any of the others. Therefore the number of these is finite, and at most the number of generators in a basis for A. These "relations" give us a set of generators for B.

~~We turn now to the classification of all groups with a finite basis. We shall achieve this by the use of the matrix associated with a given group. Let A be the group, $a_1, \ldots, a_n$ a set of generators, and $\phi_1, \ldots, \phi_k$ $(k \leq n)$ a set of fundamental relations.~~

<u>Theorem</u>: if A is a group with a finite basis, then

$$A \cong [\alpha_1] \oplus [\alpha_2] \oplus \cdots \oplus [\alpha_k]$$

where each $[\alpha_i]$ is a free cyclic group with a single generator, <u>or</u> a finite cyclic group of order p^{s_i}, p a prime.

We shall give at least two proofs of this, of which the second will bring out the details of the argument in the form of elementary operations on a matrix.

Let $a_1, \ldots, a_n$ be a basis for A. In the group of relations on these generators, let us pick any one whose highest common factor is a <u>minimum</u>. Say this is $\sum k_i a_i = 0$, and that k_n is the h.c.f. Then by our first theorem, we can make a change of basis on the a_i and get a new basis $a'_1, \ldots, a'_n$ in which this relation has the form: $k_n a'_n = 0$. It is clear, by the choice of k_n, that if there are any other relations on the a'_i which involve a'_n with a non-zero coefficient k'_n, the number k'_n must be divisible by k_n. Therefore we can eliminate a'_n from all other relations on the generators $a'_1, \ldots, a'_n$. If there is any relation

~~[illegible]~~

~~generators) that this is [illegible] again we pick a relation~~

left on the first n-1 generators, we pick one whose h.c.f. is a minimum, say k_{n-1}. By a change of basis on the first n-1 generators we reduce this to a form $k_{n-1}a''_{n-1} = 0$. We can continue this, at most a finite number of times, until we get a final basis: $a^*_1, a^*_2, \ldots, a^*_n$ $(a^*_n = a'_n, a^*_{n-1} = a''_{n-1}$, etc.) such that there are no relations whatever involving the first j of these generators, and such that there are n - j relations of the form $k_i a^*_i = 0$, $j+1 \leqq i \leqq n$.

We have already proved a part of the theorem. We have shown that $A \cong [a^*_1] \oplus \cdots \oplus [a^*_j] \oplus [a^*_{j+1}] + \cdots + [a^*_n]$ where the first j groups are free cyclic groups, and the last n-j are finite cyclic groups. To complete the theorem we have to show that each of these finite cyclic groups is of the type of a direct sum of cyclic groups of <u>prime-power order</u>.

Let B denote any cyclic group of order k, k not a power of a prime. Then $k = s \cdot t$ where s and t are relatively prime integers. It is clear that if $\underline{a}$ denotes the generator of B, $s\underline{a}$ is of finite order t and $t\underline{a}$ is of order s. We shall show that

$$B \cong [s\underline{a}] \oplus [t\underline{a}].$$

There exist integers, λ and μ , such that $\lambda s + \mu t = 1$. Therefore $\underline{a} = \lambda(s\underline{a}) + \mu(t\underline{a})$. Further, there cannot exist any non-trivial relation on the elements $s\underline{a}$ and $t\underline{a}$. For, if integers q and r exist such that

$$q(s\underline{a}) + r(t\underline{a}) = 0,$$

it must be true that $qs + rt$ is divisible by k, the order of $\underline{a}$. Therefore qs must be divisible by t and rt must be divisible by s. This implies that q is divisible by t and that r is divisible by s. But then $qs\underline{a} = 0 = rt\underline{a}$, and the relation <u>is</u> trivial.

We have decomposed the group B into a direct sum of cyclic groups of lower order. It is obvious that we can continue this breaking down as long as any groups remain not of prime power order. Then our theorem is proved.

There is one final remark. If we start with an arbitrary basis, the numbers k_i need not be different from 1. This means, merely, that the corresponding generators are redundant, and may be discarded. Further, we may observe that from the manner in which the k_i 's were derived, each k_i divided the following one; i.e. k_{n-1} divides k_n, .., k_{j+1} divides k_{j+2}.

<u>Theorem:</u> The decomposition in the theorem above is unique (~~to within order of the generators~~).

We shall prove this by giving the invariantive significance of the theorem. Let B be the subgroup of A whose elements are of finite order. This is certainly an invariant. A mod B is a group all of whose elements are of <u>infinite</u> order. For, if any element of A mod B were of finite order, say k, then k times the corresponding coset would be the coset B. That would mean that for any element in this coset, the kth multiple of it would be an element of B and therefore of finite order. But then the element in question must itself be of finite order, and belong to B. The only element in A mod B of finite order is the element B, that is the zero. The cosets $(a_i + B)$, not necessarily distinct, certainly give a basis for A mod. B so that this group has a <u>finite</u> basis. <u>Therefore</u> it is a free group. Therefore A $\ominus$ B $\oplus$ A mod. B. It is clear from our theorem that

$$A \text{ mod. } B \ominus \{a^*_1\} \oplus \cdots \oplus [a^*_j].$$ p 55, 56

We have seen, before, that this number j is invariant. It is clear, also, that $$B \ominus [\alpha_{m+1}] \oplus \cdots \oplus [\alpha_k]$$

where each of these is cyclic and of prime power order. We have to prove the invariance of this decomposition. Consider now, for an arbitrary prime p, the invariant subgroup pB, and with it the invariant residue group, B mod. pB. Every element b' of the group B whose order is a power of a prime q different from p is of the form pb for some element b. For, if $q^k b' = 0$, find λ and μ such that $\lambda q^k + \mu p = 1$. Then $b' = p(\mu b')$. The number of independent generators in B mod. pB is precisely the number of groups in the decomposition of B whose orders are powers of the prime p. Therefore this number is invariant for every prime. We may remark that the effect of reducing B mod pB is simply to reduce all coefficients modulo p, from which what we have said above, follows by inspection.

Now the number of groups whose order is a power of the same prime includes the numbers whose orders are the prime, the square of the prime, etc. Then we must finally sort these out. Let us suppose now that the group B is given as a direct sum of a finite number of cyclic groups $[b_i]$ each of order p^{s_i}, $i = 1, \ldots, n$. The group pB consists of all elements of the form $p(\sum n_i b_i)$. The generators of B mod. pB may be taken as the cosets $(b_i + pB)$. It is clear that they are independent, each of order p (mod. pB), and that there are n of them. The number of elements in the group B mod. pB is p^n. Let us now consider the invariant subgroup p^2B. The elements of this are of the form $p^2(\sum n_i b_i)$. If we form B mod p^2B we shall, in general, get a larger group than B mod pB. It will have the same number of generators, but some of them will now be of order p^2 and not of order p. The number, say m_1, of the generators $(b_i + p^2B)$ which are of order p is the same as the number of generators b_i of order p. The number, say m_2, of those of order p^2 is the same as the number of generators of the original decomposition given of order p^2 or greater.

It is clear, by inspection, that the group B mod. p^2B contains $p^{(n_1 + 2n_2)} = p^{(n + n_2)}$ elements. Therefore this number is an invariant of the group. Accordingly n_2 and n_1 are both invariant. The first, we have remarked, is the number of generators of order p, the second of order p^2 or <u>more</u>. By considering B mod p^3B, etc.. we find that the number n_k of generators, in a given decomposition, of order p^k is an invariant.

This concludes our first proof of the decomposition theorem for abelian groups with a finite basis, and gives a complete classification of them.

Lecture Six.

We shall repeat our discussion of groups with a finite basis from a somewhat different point of view.

Let us consider a free group Φ. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, be two free sets of generators. We know that the number of free generators must be the same, by our consideration of $\Phi \bmod 2\Phi$. This is true, also, for free groups with an infinite set of generators: in that case the cardinal number of two distinct sets must be the same.

Now, since the a_i are, by assumption, a basis for Φ, there must exist equations of the form:

$$1) \qquad b_i = \sum_j k_{ij} a_j , \qquad i = 1, 2, \ldots, n .$$

For a similar reason there must exist equations:

$$2) \qquad a_r = \sum_s m_{rs} b_s , \qquad r = 1, 2, \ldots, n.$$

The k_{ij} and m_{rs} are integers. If we replace the a_j 's (in 1) by their values in terms of the b_i's, we shall have

$$b_i = \sum_{j,s} k_{ij} m_{js} b_s , \qquad i = 1, 2, \ldots, n.$$

Because the b_i 's are a free basis there can be none but identical relations on them. This implies that the product of the two matrices (k_{ij}) and (m_{rs}) is the identity matrix of order n. Therefore the product of the two determinants $|k_{ij}|$ and $|m_{rs}|$ is 1, and each of them is ± 1, since both are <u>integral</u>. We see, also, by replacing the b_s 's in 2) by their value in 1), that the product of the matrices in opposite order: $(m_{rs})(k_{ij})$ is also the identity. If we call these matrices M and K we have this: the matrices of transformation from one base to another satisfy

$$M \cdot K = I = K \cdot M .$$

Here I is the identity matrix of corresponding order.

Now if the a_i's form a basis, and if the determinant in 1) is equal to ± 1, then the b_i 's also form a basis. For if we regard the a_i's as indeterminates and the b 's as constants and solve for the a 's by Cramer's rule we shall get for them expressions analagous to 2) where each coefficient m_{rs} is equal to the quotient of two integral determinants. The numerator is obtained from the determinant K by replacing one of the columns by a column of 1's, and the denominator is the determinant K. Therefore the numbers m_{rs} will be integers, and each a_r will be a linear integral form in the b 's. This implies that the b 's form a basis. For, if we can express everything in the a 's and the a 's in the b 's, we can express every element of the group directly in through the b 's. Similarly if the b 's are a basis, and the determinant M is ± 1, then the a 's are a basis.

For infinite groups, the matter is a little complicated by the fact that we cannot speak of the value of the determinant of a matrix A of a transformation analagous to 1). In this case we have to say that if the a 's (which we now suppose infinite in number) form a basis, then the b 's also form a basis provided there exists a transformation like 2) expressing the a 's in terms of the b 's <u>such that</u>

$$A \cdot B = I = B \cdot A .$$

The matrix I is the identity matrix with infinitely many rows and columns, and the matrix multiplication is exactly the same as for finite matrices. There is no question of convergence here, because both matrices are finite by rows so that in the product of the elements of a row of one by a column of the other only a finite number of terms can be different from zero.

What is most interesting here is the fact that one part of this identity is not sufficient. That is, if ~~$A \cdot B = I$~~ $B \cdot A = I$ it may none the less happen (in contrast to the finite case) that $A \cdot B \neq I$ ~~B.A = I~~. Consider the following example:

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & & \\ 1 & 0 & 0 & \cdots & & \\ 0 & 1 & 0 & 0 & \cdots & \\ 0 & 0 & 1 & 0 & \cdots & \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & & & 0 & 1 & 0 \\ \vdots & \cdots & & & & \end{pmatrix} \qquad B = \begin{pmatrix} 0, & 1, & 0, & 0, & & \\ 0, & 0, & 1, & 0, & & \\ 0, & 0, & 0, & 1, & 0, & \\ 0, & 0, & 0, & 0, & 1, & 0 \\ \ddots & & & & & \end{pmatrix}$$

$$AB = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \qquad BA = \begin{pmatrix} 1 & 0 & 0 & \cdots & \\ 0 & 1 & & \ddots & \\ 0 & & 1 & & \ddots \\ \vdots & \ddots & & 1 & \\ \vdots & & \ddots & & 1 \\ \vdots & & & \ddots & & \ddots \end{pmatrix}$$

To interpret this phenomenon in the case of our groups, we may suppose that each column in B is headed by a mark representing a generator: the n-th column by the free generator y_n. Further, that each row in B is prefixed with a mark representing an element of the group: the nth row by x_n. The rows themselves give the expression of the x 's in terms of the y 's. If the y 's form a basis by assumption, we may ask whether the x 's do. (It is clear that they do not, since we cannot express the element y_1 in terms of them). Nevertheless there does exist a transformation of the sort pictured in A (where now the y 's stand to the left, and the x 's above) in which each y is given as a form in the x 's such that B.A = I. Our condition here is this: if a set of marks y_i form a basis for a group, and if a set of elements x_i is given as forms in the y's with a matrix B, then a necessary and sufficient condition that the x_i 's form a basis is that there exist a matrix A finite by rows such that $A \cdot B = I = B \cdot A$. In the example above, the x 's are not a basis and the desired matrix does not exist.

The considerations above are closely allied to the definition of an infinite set according to which it may be mapped (1-1) upon a proper subset of itself. Thus we may think of A as a linear mapping of the group A upon a subgroup of itself, each form in the y 's being mapped upon a form in the x' s. The mapping B is now the identity on this subgroup. That is, in this example, we regard x_{i+1} as another name for y_{i+1} . We shall dismiss these groups for the present, and return to the consideration of groups with a finite basis.

Let A be a group with a finite basis $a_1, a_2, \ldots, a_n$ and let $\varphi_1, \ldots, \varphi_m$ represent a fundamental set of relations on them. We construct the matrix whose columns correspond to the generators a_i, and whose rows to the relations φ_i: each row will give the coefficients in the form φ_i:

$$\begin{array}{c|cccc} & a_1 & a_2 & \cdots & a_n \\ \varphi_1 & k_{11} & k_{12} & & k_{1n} \\ \varphi_2 & k_{21} & k_{22} & \cdots\cdots & \\ \cdot & & & & \\ \varphi_m & k_{m1} & & & k_{mn} \end{array}$$

Let us reduce this matrix to normal form. In the normal form which we shall choose it will be convenient to have all the terms zero except those along the sinister diagonal (upper right-lower left). Later we shall have a number of matrices to reduce simultaneously, and it will be a little easier to follow the argument with this prescription of a normal form.

The operations which we shall use on the rows and columns of the matrix correspond exactly to the changes of basis, of the type indicated in an earlier lecture, both on the generators a_i and on the relations δ_i. There are many ways of achieving this normal form and we shall want to verify later that it is independent of our procedure. It will save words to make it clear now that when we speak of the numbers in the matrix we mean the non-zero coordinates. If all coordinates are zero, the matrix is in its normal form. Since we are entitled to permute rows and columns we shall bring into the upper right hand corner the smallest number (in absolute value, to be understood always -- we can make it positive by changing the sign of a generator, or of a relation) appearing in the matrix. Now if there is any number in its row or column not divisible by it, we can subtract a suitable multiple of it from that number and obtain a smaller one than either. We can bring this one into the upper right hand corner. We can obviously continue until the number which we have brought into this cardinal position divides all the numbers in its row and column. At this point we shall subtract multiples of it from them, so that they all become zero. We shall now have a matrix in which all numbers in the first row and last column are zero, except the one in the right corner. However, in the process, we may have introduced still smaller numbers into the body of the matrix. In this case we shall take the smallest one, put it into the corner and begin all over again. We may suppose, now, that we have a matrix in which the right-corner number k is the only one in its row and column not zero, and that this number is not larger than any appearing inthe matrix. Suppose, however, that there is some number k' in the matrix not divisible by the corner one. By adding k''s row to the first we get a new first row whose last term is still k (since each row except the first has zeros in the last column), and we can now get a smaller number than k

by subtracting some multiple of k from k'. We can bring this newest number into the right-corner position and begin reducing again. It is clear that what must finally happen is this: that we reduce our matrix to one whose first row and last column have zero's everywhere except in the right-corner, and the number k_1 which stands there divides every number in the matrix. We shall have a quite new basis, both for our generators and relations, at this stage.

$$
A' \left\{ \begin{array}{c|cccc}
 & a'_1 & a'_2 & \cdots & a'_n \\
\hline
\phi'_1 & 0 & 0 & \cdot & k_1 \\
\phi'_2 & \cdot & \cdot & \cdots & 0 \\
 & \vdots & & & \vdots \\
\phi'_m & \cdot & \cdots & - & 0
\end{array} \right.
$$

We shall now be able to treat the lower left matrix in precisely the same way, without altering the first row or last column. That is, we shall make a change of basis on the last $m - 1$ relations and the first $n - 1$ generators. Finally, we shall have a matrix such that it has zero's everywhere except along the upper right diagonal. Here each number, beginning at the top, will divide all the succeeding ones.

$$
\begin{array}{c|cccc}
 & a^*_1 & a^*_2 & \cdots & a^*_n \\
\hline
\phi^*_1 & 0 & 0 & \cdots & k_1 \\
\phi^*_2 & 0 & 0 & \cdots k_2 & \\
\vdots & & \cdots & & \\
\phi^*_m & 0 & 0 & k_m &
\end{array}
$$

The relation of this matrix to the decomposition theorem is clear. It asserts that our group is the direct sum of $n - m$ free groups (supposing the relations to have been irredundant) and of m finite cyclic groups, of orders k_i above. Further, these k_i are quite the numbers we got before (somewhat more existentially). Their invariant significance will be clearer, perhaps, from our new approach. We assert, and shall show in a moment, that $d_i = \prod_1^{i} k_i$ is the highest common factor of all i-rowed minors in our original matrix. It is, of course, quite clear that the d_i are the highest common factors of all i-rowed minors in the final matrix merely from the form of this matrix and the fact that k_i divides every k_j $j \geqq i$. To show that this is true for the original matrix we need merely prove that the h.c.f. of all i-rowed minors of a matrix is not altered by a single elementary operation. In the case of permuting rows or columns or of changing the sign of a row or column this is altogether trivial. The only operation we must consider is the one of adding one row (or column) to another. Now, the h.c.f. of all i-rowed minors of an integral matrix is simply the highest common factor of the determinants of these minors, supposing that at least one is not zero. That the h.c.f. is not altered by adding one row to another means that the value of a determinant is not altered by adding one row to another. This is, of course, ~~the case~~ true.

It is clear then that the numbers k_i which we obtain in this way are invariant in this sense at least that if we start with a given basis and a given set of relations they are uniquely defined. But it does not follow, without argument, that if we are given another basis and set of relations for our group, we may not obtain other numbers. We shall prove that we do not, by forming a larger matrix to embrace both sets of bases and relations, as follows: Let $A = B$; let $a_1, \ldots, a_n$ be a basis for A, $\phi_1, \ldots, \phi_k$, a set of fundamental relations on them; let $b_1, \ldots, b_{n'}$ be a basis for B and $\phi'_1, \ldots, \phi'_{k'}$ a set of fundamental relations on them. We form the following composite matrix, and shall consider its reduction to normal form:

$$\begin{array}{c} \begin{array}{cc} a_1, a_2, \dots, a_n \; ; & b_1, b_2, \dots, b_n \end{array} \\ \left(\begin{array}{c|c} \begin{pmatrix} \\ A \\ \\ \end{pmatrix} & \begin{pmatrix} \\ O \\ \\ \end{pmatrix} \\ \hline \begin{pmatrix} \\ R \\ \\ \end{pmatrix} & \begin{pmatrix} 1 & & O \\ & \ddots & \\ O & & 1 \end{pmatrix} \\ \hline \begin{pmatrix} \\ O \\ \\ \end{pmatrix} & \begin{pmatrix} \\ B \\ \\ \end{pmatrix} \\ \hline \begin{pmatrix} 1 & & O \\ & \ddots & \\ O & & 1 \end{pmatrix} & \begin{pmatrix} \\ L \\ \\ \end{pmatrix} \end{array}\right) \end{array}$$

The columns represent the combined generators of the two groups. The first set of rows gives the relations on the a 's alone, the third set on the b's alone. The second set of rows gives the expressions of the b 's in terms of the a 's, since the a 's form a basis, and the final set gives the expression for the a 's in terms of the b 's since these form a basis.

Now we know that however we reduce this comprehensive matrix to its normal form we shall get the same result. We shall do this, therefore, in two distinct ways. By the first, we shall get the numbers k_i appropriate to the original matrix A, and by the second way we shall get the numbers for the matrix B.

First, using the 1 's in the second block of rows, we can cancel all the numbers in the second block of columns (below it) and we can then cancel the matrix R to the left of it. When we are through we shall find that the second bank of columns has only a unit matrix in it, and the first bank has three parts not empty. The matrix A is unchanged, the R is cancelled, the matrices

below are both modified and will have numbers scattered around in them. However the new rows now read as relations upon the original generators a_i alone. Therefore all the new rows in these matrices are consequences of the rows in the matrix A, since this comprises a fundamental set of relations.
Now it is clear, that we can cancel all of these rows, using the matrix A and we are left with a matrix which contains A, and a diagonal of 1 's. It is clear from inspection that this matrix has the same highest common factors as the matrix A.

Now we shall use the 1 's in the last bank of rows, under the a 's. With these we shall cancel all the rows above it, and cancel the matrix L to the right of it. In the second bank of columns, the matrix B will be unaltered, L (of course) cancelled, and the two top sections of rows will be modified in one way or another. But now we see that all the rows which remain are expressions for relations upon the generators b_i (except those corresponding to the 1 's, which merely tell us that certain new generators which have been defined by our process are now redundant). These are all linear consequences of the relations in B, by the assumption that the latte[r] form a fundamental set. Therefore the rows of B will permit us to cancel all of these others, and we shall be left with a matrix composed of B and a unit matrix. This will have the same h.c.f.s as the matrix B.

Therefore, finally, the matrix A and the matrix B will have the same h.c.f.s, and the numbers we defined are group invariants.

not complete
LZ

We are ready to apply our analysis of groups to a study of the connectivity, and dual connectivity group of a complex. The group of i-chains (dual i-chains) is a free group on a finite set (α^i = number of distinct i-cells) of generators. The group of closed i-chains (where the β-operator vanishes) is a subgroup of this, therefore a free group on a finite number of generators. This group has as subgroup the group Γ^i_B of bounding i-chains: those which result from an application of β to $(i+1)$-chains. Therefore the bounding i-chains, also, form a free group on a finite basis. The connectivity group Δ^i was defined as the residue group Γ^i_C mod Γ^i_B. In the group Δ^i, the set of elements of finite order forms a subgroup which we may call F^i, known as the Torsion group. By what we have proved, Δ^i mod F^i is a free group, and

$$\Delta^i \ominus F^i \oplus \Delta^i \bmod F^i.$$

The number of generators of Δ^i mod F^i is denoted by P^i and is called the i-th connectivity number. The group F^i, when reduced to canonical form as direct sum of cyclic groups of prime power order, defines the torsion numbers. In this respect, we may remark that usage has changed and that torsion numbers, as originally defined by Poincaré denoted the divisors of the matrix, which we encountered in our earlier analysis.

The dual theory runs along the same lines exactly. We have the group of dual chains $\bar{\Gamma}^i$, free with a finite basis; the subgroup $\bar{\Gamma}^i_E$ of exact chains (where the δ-operator vanishes), also free, with finite basis; and the subgroup of this: $\bar{\Gamma}^i_D$, of derived chains (resulting from a δ-operation on an $(i-1)$-chain), also free with a finite basis. Finally we have the dual connectivity group $\bar{\Delta}^i = \bar{\Gamma}^i_E \bmod \bar{\Gamma}^i_D$. This, again, is the direct sum of two groups: the dual Torsion group (elements of finite order) and a free group (the residue group) the number of whose generators defines the dual connectivity numbers. We shall see that the connectivity and dual connectivity numbers agree. The torsion numbers will not. None the less there will be a relation between the torsion numbers for chains of one dimensionality and the dual torsion numbers for the dual chains of another dimensionality, so that strictly we do not have two sets of independent invariants. The coincidence of these invariants will exhibit the duality relations of topology.

We define first an <u>integration</u> table, which gives us the value of $(\bar{C}^i_i, C^j_j)$, the integral of $\bar{C}^i$ over C^j. By definition, this was the ordinary Kronecker delta:

$$\begin{array}{c|cccc} & \bar{C}^i_1, & \bar{C}^i_2, & \ldots, & \bar{C}^i_{\alpha^i} \\ \hline C^i_1 & 1 & 0 & \cdots & \\ C^i_2 & 0 & 1 & & \vdots \\ \vdots & & & & \\ C^i_{\alpha^i} & 0 & \cdots & & 1 \end{array}$$

The matrix has the form of a unit matrix. By the bilinearity of the $(\bar{C}, C)$ function, we can read the value of any function (dual-chain) integrated over any domain (ordinary chain) by expanding the bracket and substituting from the table.

We introduce next the Poincaré matrices. These give the <u>incidence</u> relations between the (i-1)- and i-cells.

$$\begin{array}{c|cccc} & C_1^{i-1}, & C_2^{i-1}, & \ldots, & C_{\alpha^{i-1}}^{i-1} \\ \hline C^i_1 & [C^i_1, C^{i-1}_1] & [C^i_1, C^{i-1}_2] & \cdots & [C^i_1, C^{i-1}_{\alpha^{i-1}}] \\ C^i_2 & & \cdots & & \\ \vdots & & & & \\ C^i_{\alpha^i} & [C^i_{\alpha^i}, C^{i-1}_1] & \cdots & \cdots & [C^i_{\alpha^i}, C^{i-1}_{\alpha^{i-1}}] \end{array}$$

The rows of this matrix give the coefficients of the <u>boundary</u> relations, so that each row represents a bounding (i-1)-chain.

We shall set down also the dual matrices. These will concern our dual chains, and will be the transposed matrices to the Poincaré.

$$\begin{array}{c|ccc}
 & \bar{C}^i_1 \;,\; \bar{C}^i_2 & , \ldots , & \bar{C}^i_{\alpha^i} \\
\bar{C}^{i-1}_1 & [C^i_1, C^{i-1}_1] & & [C^i_{\alpha^i}, C^{i-1}_1] \\
\bar{C}^{i-1}_2 & & & \\
\vdots & & & \\
\bar{C}^{i-1}_{\alpha^{i-1}} & & & [C^i_{\alpha^i}, C^{i-1}_{\alpha^{i-1}}]
\end{array}$$

In the expression for the incidence number it is, of course, a matter of indifference whether we choose to bar the C's, or not. We shall, in general, omit them. By definition, the incidence relations were identical for the barred and unbarred cells. Here the rows give the coefficients of the δ-operation. Each row represents a derived chain.

We shall now want to reduce all of these matrices to normal form, and to do this simultaneously.

Theorem: By a change of basis on all the i-cells, for every i, it is possible to bring all the matrices above into normal form, simultaneously.

We have to make this change on the ordinary chains only. For, each transformation of the chains induces a contragredient transformation on the dual chains keeping the form of the integration table unaltered. If the first set of matrices is brought into normal form, then the second set will be in normal form automatically.

Proof: We start with the first of these matrices, expressing the boundaries of the 1-cells. By suitable changes we can bring this into the following normal form:

$$\begin{array}{ll} & C^0_1, \;\dots\; C^0_{\alpha^0} \\ \text{Independent}\left\{\begin{array}{l} C^1_1 \\ \vdots \\ C^1_j\end{array}\right. & \left.\begin{array}{c} 0 \quad \ddots \\ \qquad 0 \end{array}\right\}\rho^0 \\ \text{Closed}\left\{\begin{array}{l} C^1_{j+1}\\ \vdots \\ C^1_{\alpha^1}\end{array}\right. & 0 \end{array}$$

It is to be understood here that the variables are new ones: we have had to make changes of basis which we should have indicated by priming the various C 's, perhaps. It is important, at any rate, to bear in mind that the new variables represent chains, i.e. linear combinations of cells, and not merely cells themselves.

We observe that the C 's whose corresponding row has zeros only in it, represent closed 1-chains. The C 's in the first j rows (above) represent 1-chains that are not closed and such that no linear combination of them can be closed. For, each of them has in its expression a 0-chain which appears in none of the others. Let us suppose that the rank of this matrix is ρ^0. We see that Γ^1_c is a free group with $\alpha^1 - \rho^0$ generators. For, every closed chain must be a linear combination of the closed chains in the matrix above, and they are independent (we know that we started with an independent set, namely the separate 1-cells). In general, we shall see that the group Γ^i_c has $\alpha^i - \rho^{i-1}$ free generators.

Now we have made a change of basis on the C^0 's, and these we shall never have to disturb. The change of basis which we made on the C^1 's we shall repeat for the second matrix, giving the incidence relations between the two-cells and 1-cells. After the change of basis on the 1-cells, it expresses the incidence between the two-cells and the new 1-chains. Let us have a look at it.

$$\begin{array}{c|c}
 & C^1_1, \dots, C^1_{j+1}, \dots, C^1 \\
C^2_1 & \\
C^2_2 & 0 \\
\vdots & \\
C^2_{\alpha^2} &
\end{array}$$

We have indicated that the first columns are zero. (We take $j = \alpha^1 - \rho^0$, in the figure above) This is certainly the case. Because we have just observed that the first j of our 1-chains were <u>not</u> closed, and were linearly independent with respect to closedness. But the boundary of a cell (or chain) <u>is</u> closed. Therefore the expression of this boundary cannot contain any of the 1-chains in question.

Now this is exceedingly fortunate. Because, to bring this new matrix to normal form, we have to make changes of basis on the rows, which dont affect the C^1's at all, and also upon the <u>last</u> columns, but not on the first j. We see that in the first matrix, which is already in normal form, the form will not be disturbed by any change which we may make upon the last C^1's because these have nothing but zero's beside them. Therefore, although our first change of basis upon the C^1's was not final, the normal form is preserved.

The remainder of the proof follows exactly the same lines. We shall return to it the next hour.

Lecture Seven

We considered, last hour, the problem of reducing the Poincaré matrices of a complex to normal form. We should like to emphasize, again, the fact that this reduction is accomplished by a series of elementary changes of basis. These changes reflect the simple matrix operations of changing the sign of all the numbers in a row (or column), ~~and~~ adding one row (column) to another, and permuting rows (columns).

Let A be a free group with a finite basis, and let $a_1, \dots, a_n$ be a free basis. Then an arbitrary change of basis is given by

$$b_i = \sum \lambda_{ij} a_j ,$$

where the determinant of the λ's is ± 1. Here, the elements $b_1, \dots, b_n$ give the new basis. We have seen, in fact, that it is possible to solve for the a's, as integral linear forms in the b's, and that the b's are linearly independent. Let us see how this change of basis is carried through, in terms of elementary changes.

$$\begin{array}{c|cccc}
 & a_1, & a_2, & \dots, & a_n \\
\hline
b_1 & \lambda_{11} & & & \lambda_{1n} \\
b_2 & \vdots & & & \\
\cdot & \vdots & & & \\
b_n & \lambda_{n1} & & & \lambda_{nn}
\end{array}$$

Let k denote the highest common factor of the numbers in the first column. By a series of elementary operations, performed on the rows alone, we obtain a new matrix in which the number k appears in the upper left corner and in which all other numbers in its column are zero. We see at once, simply by expanding, ~~thxxxdetxx~~

the determinant of this matrix in terms of the first column, that the number k must be ± 1. That, remember, is the value of the entire determinant. By a change of sign of the first row, if necessary, we can make $k = 1$. Now, leaving the first row intact we can perform a series of elementary operations on the remaining rows so as to obtain a new matrix in which the number 1 appears in the diagonal term of the second row, with zero's below it. We can obviously continue this until we have a matrix with 1 's on the main (upper right-lower left) diagonal and zero s below. The operations on the rows induce a change of basis on the b 's so that we have a new set of elements $b'_1, \ldots, b'_n \ldots$

$$
\begin{array}{c|cccc}
 & a_1, & \ldots\ldots, & & a_n \\
\hline
b'_1 & 1 & \mathrm{xxxx} & & \mathrm{xx} \\
b'_2 & & 1 & \mathrm{xxxx} & \\
\cdot & & 0 & \ddots & \\
b'_n & & & & 1 .
\end{array}
$$

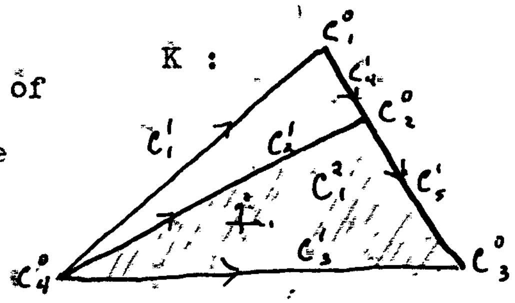
x's mean, maybe not zero

The elements a are unchanged, since we didnt touch the columns. Finally, by subtracting a suitable multiple of the second row from the first, we get zeros in all of the second column. By subtracting a suitable multiple of the third from the first and second row we get zeros in all the third column, etc. At the end we have the unit matrix:

$$
\begin{array}{c|cccc}
 & a_1, & a_2, & \ldots\ldots, & a_n \\
\hline
b^*_1 & 1 & 0 & \ldots\ldots & 0 \\
b^*_2 & 0 & 1 & \ldots & 0 \\
\cdot & \ldots & \ldots & \ddots & \\
b^*_n & 0 & 0 & \ldots & 1
\end{array} ;
$$

this tells us that the new generators b^*, are merely the old generators a. If we reverse all of the operations, we have the passage from the generators a to the generators b, with which we began above, achieved by elementary changes.

To make the Poincaré matrices, and our operations with them, somewhat more concrete we shall analyze a simple example. In the accompanying figure, only the shaded portions are regarded as part of the complex K. (essentially, we have dropped one of the two-cells which seem to go with the diagram). The



complex K consists of the four 0-cells, vertices, labelled $C^0_1, \ldots, C^0_4$; five 1-cells, $C^1_1, \ldots, C^1_5$, and one two-cell C^2_1. The orientation of these cells is indicated by arrows, for the one-cells, and by a two-leg for the two-cell. There are several remarks to make. We are not interested, at the moment, in a containing space and have therefore chosen inner orientations. If the figure had been supposed in a 3-space, we would have specified outer-orientations, had we wanted them, by constructing two-legs at each of the 1-cells (normal to them) and a one-leg normal to the two-cell. Further, we do not need to orient the 0-cells, in fact we have not defined an orientation for them. The orientation is an arbitrary matter, and all of the cells are oriented independently (we shall see later that the connectivity groups are not affected by this arbitrariness). Finally, there is the question of <u>incidence</u>. This depends, of course, on the precise orientations which we have decided upon. The effect of our earlier conventions on incidence are these: an oriented 1-cell is positively incident with the zero-cell from which it originates and negatively incident with the 0-cell to which it leads.

To determine the incidence of a two-cell and bounding 1-cell, we take the two-leg and move it about so that the vector marked 1 is brought into coincidence with the orienting vector on the 1-cell. Now, if the vector marked 2 points inside, the incidence is positive; if it points outside the incidence is negative.

Let us give the Poincaré matrices for the complex K.

	c^0_1	c^0_2	c^0_3	c^0_4
c^1_1	-1	0	0	1
c^1_2	0	-1	0	1
c^1_3	0	0	-1	1
c^1_4	1	-1	0	0
c^1_5	0	1	-1	0

----

	c^1_1	c^1_2	c^1_3	c^1_4	c^1_5
c^2_1	0	-1	1	0	-1

The dual matrices are determined by these. We merely spin them across the (sinister) diagonal. why? The incidence numbers are, by convention, invariant with respect to the bars.

	$\bar{c}^2_1$
$\bar{c}^1_5$	-1
$\bar{c}^1_4$	0
	1
	-1
$\bar{c}^1_1$	0

Now, by adding the second, third and fourth columns to the first (in the top matrix) we get a column of zeros. If we then add the second and third columns to the last we get this matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

This induces a change of basis on the 0-cells, which we do not bother to indicate. Now, by adding the first row to the fourth, changing the sign of the fourth, adding the second to the fourth and fifth, changing the sign of the fifth, adding the third to the fifth, then changing the sign of the second and third we get this matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We bring the matrix to normal form, finally, by permuting the second and third rows.

	$D^0{}_1$,	$D^0{}_2$,	$D^0{}_3$,	$D^0{}_4$
$D^1{}_1$	0	0	0	1
$D^1{}_2$	0	0	1	0
$D^1{}_3$	0	1	0	0
$D^1{}_4$	0	0	0	0
$D^1{}_5$	0	0	0	0 .

To understand this, let us consider what these D 's represent. They are, of course, 0- and 1-chains. If we have made no mistake in retracing the operations on our basis, they are these:

$D^0{}_1 = C^0{}_1$; $D^0{}_2 = C^0{}_2 - C^0{}_1 - C^0{}_4$; $D^0{}_3 = C^0{}_3 - C^0{}_1 - C^0{}_4$; $D^0{}_4 = C^0{}_4 - C^0{}_1$. We have indicated in an earlier lecture how this change of basis takes place. When, as our first step, we added the fourth column to the first we had to make the change of basis on the 0-cells which consisted in replacing $C^0{}_4$ by $D^0{}_4$: if the matrix is to be interpreted as representing the bounding relations between the original 1-chains and a new set of 0-chains.

When we manipulate the rows of our matrix, this induces a change of basis on the 1-chains which is <u>covariant</u>: i.e. if we add the first row to the fourth, the new basis consists in replacing $\underline{C^1_4}$ by $\underline{C^1_4 + C^1_1}$. This, again, is forced on us by the meaning of the new matrix. The D^1's have the following significance.
$D^1_1 = C^1_1$; $D^1_2 = -C^1_3$; $D^1_3 = -C^1_2$; $D^1_4 = C^1_2 - (C^1_1 + C^1_4)$; $D^1_5 = C^1_3 - (C^1_2 + C^1_5)$.

The matrix tells us that D^1_4 and D^1_5 are closed. Further, the other D^1's are not closed, and no linear combination of them is closed. Again, there is one 0-chain which is not bounding, and all other non-bounding 0-chains are linear combinations of it (that is, multiples) modulo the group of bounding 0-chains generated by D^0_2, D^0_3, and D^0_4. The zero-th connectivity number is $1 = P^0$.

Now the change of basis which we have made upon the 1-chains, alters the second matrix: giving the relations between the two-cells and 1-chains. We can determine the new matrix by recalling that the columns behave contragrediently to the change in the basis. The new matrix has the appearance:

$$\begin{array}{c|ccccc} & D^1_1 & D^1_2 & D^1_3 & D^1_4 & D^1_5 \\ \hline C^2_1 & 0 & 0 & 0 & 0 & 1 \end{array}$$

and it is already in normal form. We get this form by writing down the explicit change of basis on the C^1's which leads to the D^1's , spinning this on its diagonal to get the contragredient change of basis on the coefficients (the incidence numbers) and then evaluating the new coefficients from the old in terms of <u>this</u> change of basis.

The matrix tells us that there are no closed two-chains, and that there is one bounding 1-chain of which all others must be integral multiples. Only the last two 1-chains are closed. Therefore the first connectivity number P^1 is 1, since one of these is unbounding.

According to our formulas:

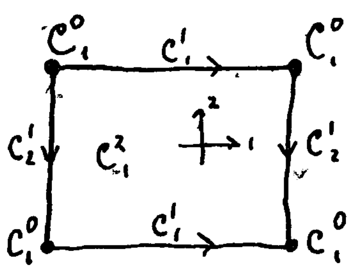
$$p^1 = \alpha^1 - \rho^{0,1} - \rho^{1,2} .$$

In the case above, this becomes

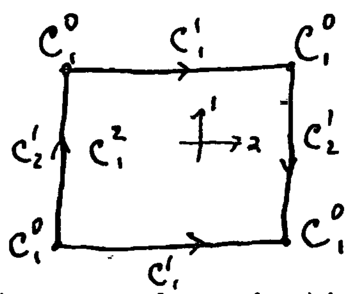
$$p^1 = 5 - 3 - 1 = 1 . \quad \text{Check.}$$

We may interpolate here the remark that the computation of connectivity numbers need not be as tedious a business as the example above would intimate. Observe that although our incidence numbers were originally defined for cells, they extend immediately to chains. In fact, our final matrix is still an incidence matrix giving now the incidence relations between certain chains of dimension two and others of dimension one. The incidence function $\left[C_j^{i-1} , C_k^i\right]$ is bilinear, and we can replace the individual cells by combinations of them. The connectivity numbers, as we shall show later, are invariants of the geometric figure. We shall prove this, for the present, within the frame of linear geometry. This will show that however we represent the complex K above as a sum of suitable convexes, the new matrices will give the same groups. If we consent to be a little lax about the linearity of our spaces we can compute at once the connectivity numbers of a torus and ~~projective plane~~ the Klein's bottle, as follows.

T



P



In the representations above, both the torus and ~~projective plane~~ Klein bottle are given by a single two-cell, two 1-cells, and one vertex. The matrices read as follows:

T

$$\begin{array}{c|c} & c^0_1 \\ \hline c^1_1 & 0 \\ c^1_2 & 0 \end{array}$$

P

$$\begin{array}{c|c} & c^0_1 \\ \hline c^1_1 & 0 \\ c^1_2 & 0 \end{array}$$

T

$$\begin{array}{c|cc} & c^1_1 , & c^1_2 \\ \hline c^2_1 & 0 & 0 \end{array}$$

P

$$\begin{array}{c|cc} & c^1_1 , & c^1_2 \\ \hline c^2_1 & 0 & 2 \end{array}$$

All of the matrices are in normal form.

We see that $p^0 = 1$, for both T and P , since in each there is one 0-chain which does not bound.

In the complex T, $p^1 = 2$; there are two non-bounding 1-chains, linearly independent with respect to bounding.

In the complex P, $p^1 = 1$ (this agrees with the formula we have used above: $p^1 = 2 - 0 - 1$) since there is one closed 1-chain not bounding and linearly independent with respect to bounding. There is a second non-bounding 1-chain, namely C^1_2. However, $2C^1_2$ does bound. We have here the simplest instance of <u>torsion</u>. This emphasizes the fact that it is not possible, in general, to divide the coefficients of a relation of bounding. Thus $2C^1_2$ bounds, but C^1_2 does <u>not</u>.

We may remark in this connection that it is not necessary to bring the matrices to a normal form in order to compute the connectivity numbers, since these are given by formulas which involve the numbers of basic chains, and the <u>ranks</u> only of the matrices in question: two of these for each connectivity number. Some sort of normal form is needed to exhibit the torsion coefficients.

Let us verify the formula

$$p^i = \alpha^i - \rho^{i-1,i} - \rho^{i,i+1} ,$$

for a general i.

Consider, first, the Poincaré matrix mapping the i-chains on the i-1-chains (in virtue of the boundary-operator). We can bring this to the following form

(i-1)-chains

i-chains			
	0	X	not closed ≡ open linearly independent & with respect to being closed.
	0	0	closed. generating all closed i-chains
	non-bounding (i-1)chains.	bounding (i-1)-chains.	

We have not yet bothered to reduce the matrix X to a normal form: we understand that it is a square matrix, with non-zero determinant. The rank of this matrix is the number of its rows (or columns).

Let us now consider the matrix expressing the dependence of the (i+1)-chains on the i-chains of our new basis, i.e. the basis with which we are left when we bring the matrix above to the form given. We shall write the i-chains from left to right, in the order in which they occur in the first matrix from up to down.

i-chains

(i+1)-chains		
	0	
	0	

We can be certain that the first columns, corresponding to the open i-chains, have zeros only. They cannot occur in the boundary of an (i+1)-chain, since this boundary is closed and they are linearly independent with respect to "closedness", as we have seen above.

We now make a change of basis on the closed i-chains so that the second matrix has the form:

	i-chains		
	(open)	(closed): (non-)	(closed): (bounding)
(i+1)-chains	0	0	Y
	0	0	0

Here the matrix Y, not necessarily in normal form, is square: its rank is the number of its rows (or columns) and is the rank of the entire matrix. The i-th connectivity number is the number of columns in the middle section. This is obviously the number of i-chains (which is the number of i-cells we began with) minus the rank of the first matrix, $\rho^{i-1,i}$ (giving the number of columns in the first part of the matrix) minus the rank of the matrix above, $\rho^{i,i+1}$ (giving the number of columns in the last part).

$$p^i = \alpha^i - \rho^{i-1,i} - \rho^{i,i+1} . \quad \text{q.e.d.}$$

We may remark, in passing, that the change of basis which we made upon the second matrix considered, affected the basis of the first, since it affected the i-chains, but did not affect its form because the corresponding i-chains had zeros only to the right of them. I.e. the rows which we have to add to each other, are rows of zeros.

We notice that in the expression for P^i, the terms $\alpha^i - \rho^{i-1,i}$ give the number of independent closed i-chains, i.e. the number of generators (free) in the free group Γ^i_0. Let us now consider how the whole thing looks from the point of view of the dual chains.

	dual (i)-s
dual (i-1)-s	

	dual (i+1)'s
dual (i)-s	

In these matrices the rows correspond to the linear operator δ. Since the roles of rows and columns are interchanged here, the changes of basis on the original C 's which bring the first two matrices to the forms above, are here paralleled by contragredient changes of basis bringing these matrices into a similar form: the difference (which could obviously be compensated for by permutations among the rows and columns) consists in this that the non-zero matrix occupies the lower left corner instead of the upper right one. Now the dual connectivity number has a slightly different significance. It is, by definition, the number of exact chains which are linearly independent with respect to being derived. The number of independent exact chains is the number of elements in the basis (free) of the free group Γ^i_E. This number we get from the ~~second matrix,~~ in ~~this case, and it is~~ the rank of this matrix, ~~$\rho^{i,i+1}$~~.

$$\begin{array}{c|cc|l} & \multicolumn{2}{c}{\text{dual } i+1\text{-chains}} & \\ & 0 & 0 & \text{exact} \\ \text{dual } i\text{-chains} & 0 & 0 & \\ & Y & 0 & \text{"in"-exact.} \end{array}$$

We see from this matrix that the number of generators in Γ^i_E is $\alpha^i - \rho^{i,i+1}$, the last number being the rank of the matrix Y. Dualizing the first matrix, we get

$$\begin{array}{c|cc} & \multicolumn{2}{c}{\text{dual } i\text{-chains}} \\ \text{dual } (i-1) \text{ chains} & 0 & 0 \\ & X & 0 \end{array}$$

The <u>derived</u> i-chains (we know they are also exact) stand above the matrix X. The number of linearly independent ones is the rank of X, and the original matrix, i.e. $\rho^{i-1,i}$. Finally,

$$\bar{p}^i = \alpha^i - \rho^{i,i+1} - \rho^{i-1,i} = p^i .$$

As for the torsion numbers, it is clear from the argument above that the i-th dual torsion numbers are determined by reducing to normal form the matrix X. These correspond to the (i-1)-th torsion numbers of the ordinary chains. We do not, therefore, get any new invariants by our consideration of the dual chains.

It is convenient, at this point, to introduce one of the simplest of the combinatorial relations expressing the dependence of the connectivity numbers and the number of i-cells of all dimensionalities. It is known as the Euler-Poincaré formula.

We have shown that :

$$
\begin{aligned}
p^0 &= \alpha^0 - \rho^{0,1} \\
p^1 &= \alpha^1 - \rho^{0,1} - \rho^{1,2} \\
&\cdots \\
p^i &= \alpha^i - \rho^{i-1,i} - \rho^{i,i+1} \\
&\cdots
\end{aligned}
$$

By adding these equalities, with alternating signs, we get:

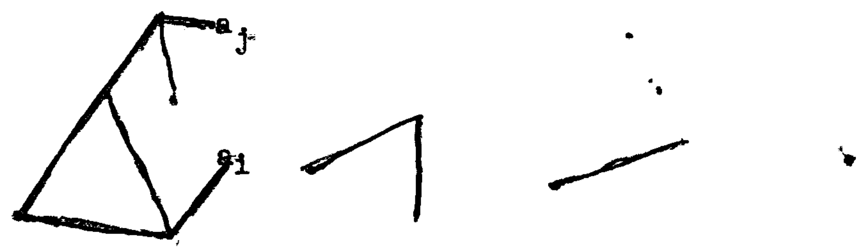
$$\sum (-1)^k p^k = \sum (-1)^k \alpha^k ,$$

the "ranks" of the various matrices having ~~got~~ themselves eliminated.

It will bring out some of the difference in meaning between ordinary and dual chains to consider the numbers p^0 and $\overline{p}^0$, which we know to be equal.

An ordinary 0-chain is a set of weighted vertices. The weights are integers, positive, negative, or zero. We assert: in order that a 0-chain bound, it is necessary ~~and sufficient~~ that the sum of the weights be zero. For, if a 0-chain bounds a 1-chain it must be the sum of the boundaries of the 1-cells (with their proper multiplicities, or weights). Now each 1-cell is positively incident with one vertex (its tail-vertex) and negatively incident with another (its head). Therefore it contributes zero to this sum. It is clear that the total sum of weights must be zero. Conversely, suppose that we have a 0-chain $\sum \lambda_i c^0{}_i$, where $\sum \lambda_i = 0$.

K :



Suppose, further, that this O-chain consists of vertices drawn from a single connected piece of the complex. Such a piece has the characteristic property that every pair of vertices can be connected by a broken line of consecutive 1-cells , and that it includes as much of the complex as it can. Above, we exhibit a complex K, with its pieces. Now, take any λ_i which is positive, non-zero. There must be one such, if the chain is not identically zero (in which case it bounds a O-chain, by definition). There must be another vertex at which a λ_j is negative. Let a_i denote the vertex where λ_i is positive, a_j the one where it is negative. We join these vertices by any broken line of the figure. It is clear that this line, expressed as a chain with all of its 1-cells oriented from a_i to a_j (with suitable coefficients, then, whatever the original orientation may have been) is bounded by $a_i - a_j$. If we now subtract this O-chain from our original one we get a new O-chain the sum of the absolute values of whose coefficients has been reduced. The new chain is equivalent to the old, modulo bounding chains. In a finite number of steps, we can reduce it to zero. That is, the original chain bounds. Suppose now, we consider any O-chain in a piece of our complex, and an arbitrary vertex. By subtracting a suitable multiple of this vertex from the chain, we can get the sum of the coefficients to be zero. That means that the original chain is linearly dependent on the group of chains generated by the single vertex. If we now pick one vertex in each piece, we get a maximum linearly independent set of non-bounding O-chains. The number P^0 is the number of distinct pieces.

No dual 0-chain ~~can~~ be derived, by convention, since we have no -1 chains to derive them from. (Similarly, ~~no~~ every n-chain ~~can be~~ is exact, if n is the highest dimensionality.) For this reason, we have them linearly independent with respect to "derivation", and the number $\overline{P}^0$ is the number of linearly independent ~~exact~~ ones. If a 0-chain (dual) is exact, then the δ-operation gives us the zero 1-chain. Now the dual 0-chain is a function which assigns to each vertex a_i a certain integral number (until, as presently, we introduce other coefficients) k_i. The linear operator δ converts this into a function defined on the 1-cells. We can see at once what the value of this function must be on a particular 1-cell of the complex. For if the original function gives its positive vertex the value k, and its negative vertex the value m, it must give the 1-cell the value k-m (one has merely to recall the definition of δ). Therefore, if it is to give all 1-cells the value zero, k must equal m. Therefor whenever two vertices can be joined by a succession of 1-cells, the original 0-function (dual 0-chain) must give them both the same value. We get a basis for all such functions at once, by taking the set of functions which have the value 1 at all the vertices of one piece and the value zero at all other vertices. The number of these functions is, of course, the number of pieces.

The difference in meaning between the connectivity and dual connectivity numbers will appear in even sharper relief when we pass to disparate fields of coefficients.

Lecture Eight.

In any free group, whether it has a finite or infinite basis, there is an invariant associated with each element which we may call its divisibility. We shall define it, first, in terms of a free basis for the group. Let A be the group, $a_1, a_2, \dots,$ a free basis. Each element a is a finite linear integral form in these base marks. The highest common factor of all the coefficients is a finite number $n = n(a)$, and will be called the divisibility of a (we suppose a <u>not</u> zero: the zero-element is exceptional). This means, first, that there exists an element b such that $a = nb$, and secondly that there is no element b' such that $a = n'b'$ holds for an n' greater than n. The first part of our remark is obvious. The second part is nearly so, because if we expand the relation $a = n'b'$ into the generators a_i (i.e. replace b' by its form in the a_i) we shall have a <u>new</u> expression for a in terms of the base marks. The formal difference between these expressions will represent the zero element. Since we are supposing that the a's form a free basis, this formal difference must be identically zero. The coefficients in the expansion of n'b' are all divisible by n'. Therefore n' must divide n. From the meaning of this divisibility-number, we see that it must be invariant with respect to changes of basis. Had we not used a basis, it would not have been clear that such a number existed, in the sense that it was finite. There are discrete groups, countable, where no <u>finite</u> number marks the divisibility of any element.

We shall turn now to a more intimate consideration of chains and dual chains and their relation to Stokes's Theorem for a complex. We shall bring out the limitations of the integral coefficients which we have been using.

Stokes's Theorem:

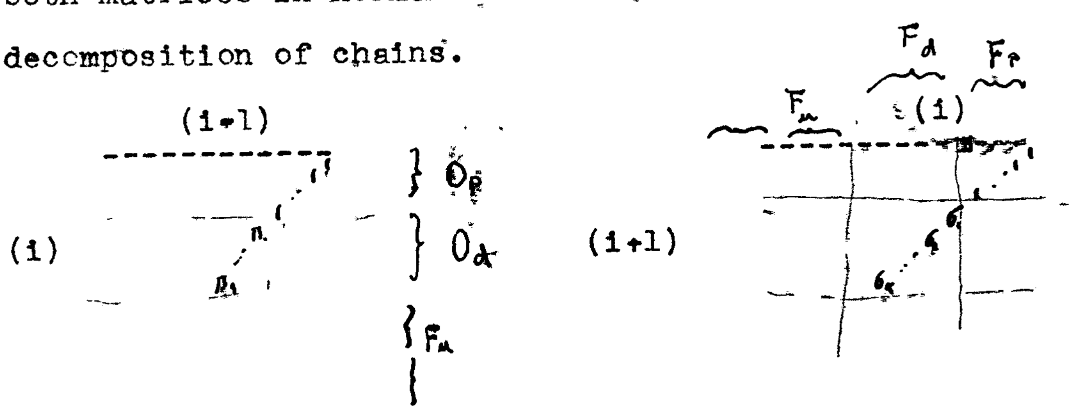
$$(\delta \overline{C},\ C) = (\overline{C},\ \beta C)$$

This theorem tells us, at once, that

1) <u>The integral of an exact $\overline{C}$ over a bounding chain vanishes.</u> For, βC in the right bracket is certainly a bounding chain. If $\overline{C}$ is exact, $\delta\overline{C} = 0$, by definition, and therefore the left hand bracket is zero.

2) <u>The integral of a derived function (dual-chain) over a closed C vanishes.</u>
For, in the left bracket, $\delta\overline{C}$ represents an arbitrary derived dual chain. If C is closed, $\beta C = 0$, and the right bracket is zero.

We are led to the question, what is the value of the integral of an <u>exact</u> function over a <u>closed</u> chain. This will correspond to the notion of the period of an integral over a cycle. We shall want to look at the matrices and integration tables in somewhat sharper focus. We shall set down the i-th and (i-1)-th matrices, where the i-chains are given in the same order, and both matrices in normal form. We shall obtain a rather fine decomposition of chains.



We have merely brought into relief the precise form of the non-vanishing matrices, showing the numbers equal to and the ones different from <u>one</u>. We see that the i-chains fall into five distinct categories: we have indicated them by the symbols, above,

O_p , O_d , F_u , F_d , F_p.

Decomposition of the i-chains:

O_p	O_d	F_u	F_d	F_p
--- all open	-----	--- all	<u>closed</u>	------
Open i-chains, i.e. with boundary, <u>but</u> boundary of divisibility 1.	Open, bound. of divis. greater 1.	linearly indep. with respect to bounding	bounding <u>if</u> (<u>only</u>) counted more than " "	<u>bounding</u> counted just once.

The middle group, F_u, we recognise as the one which gives the connectivity number. The fourth group give the torsion numbers: they are sometimes called the zero-divisors. The last three groups, together, determine the Γ^i_C (group of closed i-chains). The last two classes determine the Γ^i_B (group of bounding chains). The difference group $\Delta^i = \Gamma^i_C$ mod. Γ^i_B is of the type of a free group with as many generators as are in the section F_u.

We obtain an analagous division of the dual i-chains into the classes:

Dual u-chains:

---- The exact ones	-----	-----+-- :	--- the "in"-exact ones	
<u>the derived</u>:	some <u>multiple</u> is derived.	the merely exact.	the derived of these chains have the divisibility greater than	the derived just <u>one</u>

We can now put all these classes together into a rather imposing integration table:

Integration table

i- Chains

	Open		closed			
exact	$\begin{smallmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{smallmatrix}$	0	0	0	0	derived counted once
exact	0	$\begin{smallmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{smallmatrix}$	0	0	0	derived *if*, and only if, counted multiply.
exact	0	0	$\begin{smallmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{smallmatrix}$	0	0	linearly independent with respect to being derived
"in"-exact	0	0	0	$\begin{smallmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{smallmatrix}$	0	"derivatives" have divisibility one
"in"-exact	0	0	0	0	$\begin{smallmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{smallmatrix}$	"derivatives" have multiple divisibility
	boundaries have divisibility one	boundaries have *multiple* divisibility	linearly independent with respect to bounding	these *bound* as they stand	these bound *if* (and only if) counted multiply	

the arrangement of the symbols above is a little different from the one in the Poincaré Normal form for [illegible]al matrices

From this table we can read off the integral of an arbitrary function (dual-chain) over an arbitrary domain (ordinary chain). That's what the table is for.

Now: any exact function must be a combination of the functions corresponding to the first three batches of rows. Any bounding chain must be a combination of the last two batches of columns. The table shows at once that the integral of an exact function over a bounding chain <u>vanishes</u>.

Any derived function must be a combination of the functions corresponding to the first two batches of rows, and any closed chain must correspond to the last three batches of columns. Again, the integral of a derived function over a closed chain <u>vanishes</u>.

We come, once more, to the question which prompted this analysis. What is the integral of an exact function over a closed chain? We see from the table that this integral need not vanish: the two types of chains overlap, in our table, on the small underscored matrix in the middle. Thus if $\overline{C}$ is exact, we can write it as

$$\overline{C} = \overline{C}_1 + \overline{C}_2 + \overline{C}_3 ,$$

corresponding to the first, second, and third batches of rows. If C is closed, we may write it as

$$C = C_1 + C_2 + C_3 ,$$

corresponding to the third, fourth, and fifth batches of columns.

Then $(\overline{C}, C) = (\sum \overline{C}_i, \sum C_j)$; $i, j, = 1, 2, 3$.

Expanding this bracket, and dropping the terms which we know to be zero (from the table) we have

$$(\overline{C}, C) = (\overline{C}_3, C_1).$$

We see from this that the value of an exact chain over a closed chain is given by what we may call the period table: the integration table considered only for the exact chains which are linearly independent with respect to derivation taken over the closed chains which are linearly independent with respect to bounding.

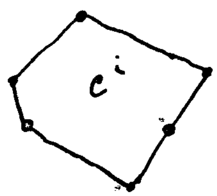
We should like to have the result that a necessary and sufficient condition that an exact chain be derived is that its periods be zero. The necessity of this condition is clear, in our case, from what we have shown before. But it is not sufficient. For, if an exact chain is derived then it must be ~~a~~ an arbitrary combination of the functions corresponding to the first batch of rows. Suppose, however, that we have an exact chain which contains at least one function with arbitrary coefficient from the second batch and none from the third. Its periods will be zero. On the other hand it will not be a derived chain.

If our field of coefficients for the dual chains had been more general than the integers, if it had been the field of reals or of rational numbers for example, these k-derived functions would not have occurred. Thus if $1/k$ is a coefficient, a function which is derived if taken k times becomes a derived function when multiplied by $1/k$. There are other, related reasons for extending the generality of our coefficient domain for the connectivity groups.

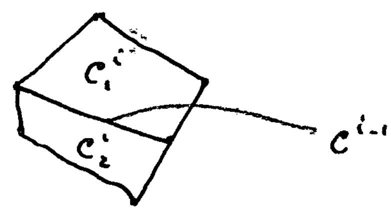
We shall dispose now of the unfinished business of proving the connectivity groups are actual invariants. To do this it is sufficient to show that they are invariant with respect to a single elementary division, as we defined this previously. For, we showed whenever two figures were equivalent in the sense of the linear topology which we are now studying, it was always possible to pass from one of these to the other by a sequence of elementary divisions. We shall show the invariance, first, for ordinary chains.

An elementary division consists in inserting into an i-convex C^i, an (i-1)-convex C^{i-1} whose boundary cells are contained in the boundary cells of C^i. Let us call the original complex K, the "new" one K': in the figure we indicate the essential change.

K



K'



We have introduced a new (i-1)-chain, and two new i-chains. The connectivity groups for dimensionalities <u>less</u> than (i-1) or <u>greater</u> than (i+1) <u>cannot</u> be affected. The (i-1)'th groups might be, and the i-th because we have new base elements. The (i+1) might be, and the i-th, because it is possible that some chain of these dimensionalities might now have a boundary, which did not before.

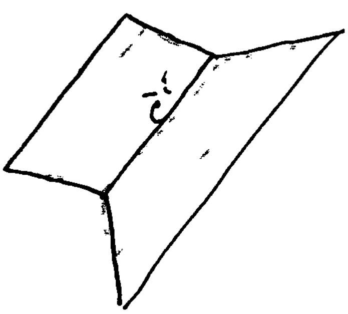
The i-th groups: To each chain in K there corresponds a uniquely determined chain in K' (i-chains, always), as follows. - If the chain does not involve the divided C^i, then it corresponds to the same formal chain in K'. If it does involve this C^i then we identify it with the chain in K' in which this C^i is replaced by the <u>sum</u> $C^i{}_1 + C^i{}_2$. This mapping preserves the β-operator, because $\beta(C^i{}_1 + C^i{}_2) = \beta(C^i)$ since the (i-1) cell which we introduced is positively incident with one of the convexes C^i_a and negatively incident with the other C^i_b $(a \neq b, = 1,2)$, <u>no matter what</u> orientation we may decide on for C^{i-1}, and the C^i_a 's. But conversely, if we have a chain in K' which involves one of the part i-cells but not the other, we can add to it a proper multiple of the other so that the new chain contains the original C^i to that multiplicity. The free groups of i-chains are not in (1-1) correspondence, but the closed i-chains are: for, if a <u>closed</u> i-chain in K' contains C^i_1 with multiplicity k, say, it must contain C^i_2 with the same multiplicity. Otherwise, we see at once, that the boundary of this chain must contain C^{i-1} with a coefficient equal to the difference in these multiplicities. Therefore there is a (1-1) correspondence between the groups of closed chains in the two complexes. This (1-1) correspondence is an isomorphism. Further, the groups of bounding i-chains are isomorphic. This is trivial, because there are no <u>new</u> (i+1)-chains and if C^i is on the boundary of some (i+1)-chain of K, then $(C^i_1 + C^i_2)$ is on the boundary of the <u>same</u> chain, and conversely. Therefore the corresponding residue groups, which define the connectivity groups, are also isomorphic.

The (i-1) th groups: Here we get a new (i-1) chain, and also (what is more significant for us) a new <u>closed</u> (i-1)-chain; the boundary, for example, of C^i_1. This is, essentially, the only new one (in the sense of linear independence) for if we consider a closed chain which involves the cell C^{i-1} we can add to it a proper multiple of the boundary of C^i_1 and eliminate the cell (except as it appear in that boundary). <u>But</u> this new closed chain is identically zero in the connectivity groups: for it is also a bounding chain, and we reduce modulo the bounding chains. Therefore, the connectivity groups, here also, are unaltered.

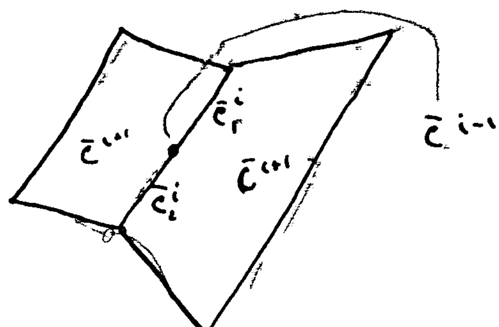
The i+1-th group: Here we have no new (i+1)-chains, and therefore no new closed ones. The closed chains of K and K' are simply isomorphic; we have only to identify chains which are formally identical. For this, however, it is important to observe that the β-operator is unaltered. If we have an (i+1)-chain in K' whose boundary involves $C^i{}_1$ with multiplicity k, say, then this boundary must contain $C^i{}_2$ k times. This follows from the fact that its boundary must be a closed i-chain. We have already seen that a closed i-chain must contain both pieces of C^i with equal multiplicity. It is quite clear, here, that there can be no new bounding relations, and the connectivity groups are isomorphic.

The argument above proves, of course, the invariance of the dual connectivity groups. We have seen exactly how these are determined from the former ones. It is of some interest, however, to see the argument directly. We shall use a different picture.

K



K'



We cannot now identify chains in K containing $\bar{C}^i$ with chains in K' containing the sum $\bar{C}^i{}_1 + \bar{C}^i{}_2$ because we must preserve the δ-operator, and the δ's of these two chains will not be the same. Given $\bar{C}^i$, its δ will be zero on all (i+1) dual chains which do not have it on their boundary, + 1 on positively incident, and -1 on negatively incident (i+1) dual cells. Now, if an (i+1) cell is incident with $\bar{C}^i$ it will be incident with both $\bar{C}^i{}_1$ and $\bar{C}^i{}_2$ and the value which the δ-operator will ascribe to it will be the sum of the separate numbers (among others) given to it by $\delta\bar{C}^i{}_1$ and $\delta\bar{C}^i{}_2$.

Therefore, to preserve this δ-operation, we must identify $\overline{C}^i$ with $k\overline{C}^i_1 + m\overline{C}^i_2$, where k and m are any two integers whose sum is one. This ambiguity seems a bit of a nuisance, but it corresponds precisely to the fact that the difference between any two such dual chains is derived : in the dual connectivity groups, the derived chains are set equal to zero. That these are derived chains is clear if we notice that they are of the form $n\overline{C}^i_1 - n\overline{C}^i_2$; they are derived from $n\overline{C}^{i-1}$ with which one of them is positively, the other negatively, incident.

It is not much of an exercise, now, to carry out the details of an invariance proof of the dual groups. We shall leave this and turn to the subject of character groups.

Lecture Nine:

Abelian, countable, groups and their character groups.

Let A denote a countable abelian group,

$a_1, \ldots, a_n, \ldots$ an arbitrary basis.

We shall not suppose that the group is free, so that there may exist relations upon these base elements. We let

$$R_j: \qquad \sum k_{ij} a_i = 0, \quad j = 1, 2, \ldots$$

denote a fundamental set of relations. Let us emphasise that in this expression the numbers k_{ij} are <u>integers</u> and that, for a fixed j, there are at most a finite number of them different from zero. We shall be led, in a moment, to groups of a very different type. The "general element" of such a group is :

$$a = \sum \lambda_i a_i \ ; \qquad \lambda_i \text{ integer}, \qquad \sum \text{ finite.}$$

A <u>character</u> will be, by definition, a function b(a) of the elements of A such that

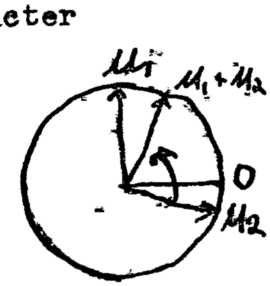
i) <u>b is single valued</u>

ii) the <u>values</u> of b, i.e. b(a) are <u>real</u> numbers reduced <u>modulo</u> <u>one</u>. We are interested only in the fractional part, then, of these values and when we add them we shall reduce mod 1.

iii) b is <u>linear</u> : (mod.1)

$$b(a) \equiv \sum \lambda_i b(a_i) \qquad \text{modulo } 1.$$

We can represent the effect of a single character pictorially, as follows. On a unit circle mark off some point to denote an origin. To each point on the circle assign a real number which measures the number of radians subtended by the corresponding arc measured in a fixed direction from the origin. Then, if the value of b at an element a is μ, we may say that b maps a on the point μ of our circle. If it maps a on μ_1 and a' on μ_2, it will have to



m

map (a + a') on $\mu_1 + \mu_2$. The conditions on b imply, as is easily seen, that the mapping above is a homomorphic one in the group sense, of the group A on the rotation group of a circle. (This is sometimes called a multiple isomorphism). If we think of our circle as the unit circle in the complex plane, then b assigns to each element a a complex number of modulus one and gives a multiple isomorphism of A upon the multiplicative group of these numbers. In this sense, it is spoken of as a representation of the group A. We shall find it more convenient to think of b as a function, since we shall be concerned with the totality of these character-functions and their relation to each other.

We observe, first, that these characters themselves constitute a group with respect to addition, if we define addition as follows:

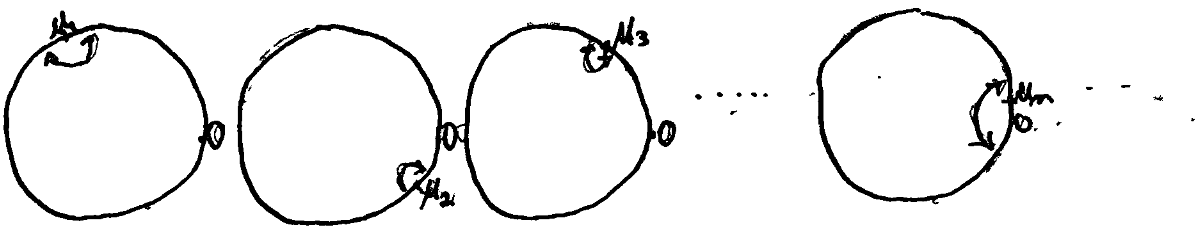
$$b + b' = b'' \text{ if and only if, for every } a \text{ of } A$$
$$b''(a) \equiv b(a) + b'(a), \bmod. 1 \quad .$$

It is clear that the function $b''(a)$ whose values are given by the relation above is a character of A. Obviously, each b has an inverse -b which is unique. The group of these functions b we shall call the character group B of A. We shall find that this group B is of a very different structure from that of A. We shall show that it has a very natural topology. We shall then define characters for it, adding the requirement of continuity (such a condition would have been meaningless on characters of A since we had no topology in A and continuity is not defined). Finally, we shall show that the group of continuous characters of B is the group A with which we began. We shall have to develop a few of the preliminary considerations.

Let us make a picture of the group B. We know that each character b is determined as soon as we have its value at all the base elements of the group A. For then its value at an arbitrary element is given by the condition of linearity (mod 1). We shall therefore set down a circle to correspond to each base element, and we shall have as many circles as base elements. On each of these circles, as above, we shall pick an origin and lay off the values of the angles. Now each character b will correspond to marking one point on each of these circles. We may think of b as given by the coordinates corresponding to the values on these circles. Thus if we have only one base element we shall have the case we illustrated before. If there are two base elements then the set of all coordinates will correspond exactly to the points on a torus, or anchor ring. In the general case where we may have infinitely many circles we shall think of the figure as a generalised torus and shall call it a Toral group. Now we should emphasise at once that not every set of these coordinates will correspond to a character of our group. It is instructive to see why this is so.

An arbitrary set of coordinates in our Toral space will give us a function defined on A. It will be a linear function, if we want it to be. We have only to define its value at a general element to be the linear sum, reduced mod 1. But it will not in general be single valued. For if we extend it linearly we may find that it ascribes to a certain form in the base elements a value different from zero (mod 1) even though the element which this form expresses is really the zero element of the group. If this never happens, then the function will be a character by definition, because it will be single valued. We see, then, that a necessary and sufficient condition that an arbitrary set of coordinates in our Toral space define a character of A is that these coordinates satisfy (mod 1) all the fundamental relations defining the group A. It is perhaps worth laboring this point to make it quite clear. If a function is to be single valued then it must have the same value for two different forms (in the base elements of A) whenever these forms represent the same element of the group. The difference of two such forms is a form representing the zero. Now a character must have the value zero (mod 1) at the zero element. Therefore, if it is to be single valued it must have the value zero for every form which represents the zero element, and if it has this value then it is certainly single valued. But every form representing zero is a finite linear combination of the fundamental forms. It is therefore sufficient (because of the linearity) to verify this condition for the fundamental relations. And, of course, necessary.

T:



If b corresponding to $(\mu_1, \mu_2, \mu_3, \ldots, \mu_n, \ldots)$ is a character then

$$\sum k_{ij}\mu_j \equiv 0 \quad (\text{mod. } 1) \quad .$$

We are now going to topologize the group B. We shall do this by first topologizing the group T. Let b, above, represent an arbitrary element of T. Let us pick a finite set of the circles and on each of them lay off a small arc containing the corresponding coordinate of b. The set of all elements b' of T such that their corresponding coordinates lie on these arcs quite regardless of what their other coordinates do, will be called a neighborhood of the point b. Each determination of an arbitrary finite set of these circles, and each choice of an arc (one on each circle, containing that coordinate of b) will determine a neighborhood of b. We get quite a lot of neighborhoods in this way, and we shall want to fasten on a somewhat smaller set in order to have a smoother apparatus for arguments. We shall show later that the space T topologized by these neighborhoods is not only a Hausdorff space, but actually a compact metric space.

Let us observe, for the moment, that if T consists of a finite set of circles only, then it is adequate if we restrict ourselves to neighborhoods such that there is actually one arc on each of the circles. We mean by this, that the topologic nature of the space is unaltered. It will be recalled that two systems of neighborhoods are regarded as equivalent (and are said to define the same topology) if whenever a point belongs to some neighborhood of one system it belongs to a neighborhood of the other system contained in that one. Now if b is a point of T, and if we are given a neighborhood of b determined by marking off arcs on some only of these circles, then we can get a smaller neighborhood (one contained in this, and containing b) by adding restricting arcs on the circles not already marked. Therefore, in this case, we may consider neighborhoods such that they mark each circle. A moment's reflection will suggest that this is precisely the topology one would associate with the n-dimensional Torus.

In case T contains infinitely many circles, i.e. when T is the infinite product of circle-spaces, we can restrict ourselves to neighborhoods which mark the first n circles, where n is arbitrary and finite. This, as in the case above, merely amounts to replacing a neighborhood of a point by a smaller one.

The topology which we have assigned to T induces a topology upon the subspace B. Observe that T depends very directly upon the choice of basis in A. For example, if A is a free cyclic group and if we take its generator for basis then T is a single circle. On the other hand if we take all the elements of A to form a basis, then T is an infinite toral group. In each case, however, B will be a subspace equivalent to a circle. In the first case B will coincide with T, in the second it will be a circle wrapped around T, not identical with any of the "defining" circles. We shall prove that the topology of B is invariant (it hardly needs to be remarked that B is invariant with respect to change of basis,—its definition was independent of a basis).

Suppose, then, that $a'_1, a'_2, \ldots, a'_n \ldots$ (finite or infinite) define a new basis for A, and suppose that T' is the toral group associated with this new basis. We can make an obvious correspondence between the subgroup B' of T and subgroup B of T' merely by pairing off the same functions. They are the same group. We must show that to each neighborhood in T' of an element of b there corresponds a neighborhood of ~~T~~ lying within it, where we take into account, of course, only the elements of B. Let b be any element of B, having on T the coordinates $(\mu_1, \mu_2, \ldots)$ and on T' the coordinates $(\mu'_1, \mu'_2, \ldots)$. We get a neighborhood of b in T' by considering those points of B whose first n (n arbitrary) coordinates differ from the first n above by arbitrarily small amounts. Now each

$$a'_i = \sum p_{ij} a_j, \quad i = 1, \ldots, n.$$

In all of these relations, there are at most a finite number of the generators a_i appearing explicitly (i.e. with non-zero coefficients). There is a number N so that no later a_k, $k > N$, appears above.

Now, in the relations defining the a'_i in terms of the a_i, we shall replace the a_i by the values which <u>b</u> takes on them; i.e. by the coordinates of <u>b</u>. This will give us the value of <u>b</u> at a'_i, i.e. the coordinate of <u>b</u> in T'. We therefore have,

$$\mu'_i = \sum p_{ij}\mu_j, \qquad i = 1, 2, \ldots, n.$$

Now the μ'_i are obviously continuous functions of the μ_j, where $j = 1, 2, \ldots, N$. By making the variation in the μ_j sufficiently small we can make the variation in the μ'_i arbitrarily small. But this was all we had to show.

Just as we defined the topology of B, which was invariant, through the space T which was not, so we shall find it convenient to establish the topologic properties of B by reference to the properties of T: which depends on a fixed basis.

Theorem. The space T is compact.

This means that if we are given an arbitrary infinite sequence of points of T, we can always find a subsequence of them which converges to some point of T.

Proof. Let S_0 be the given sequence: $p_0, p_1, \ldots, p_n, \ldots$ Each of these points has a certain coordinate on the first circle of T. We can certainly find a subsequence $S_1 : p_0, p_{n_1}, p_{n_2}, \ldots$ such that the first coordinates of these points converges to a definite coordinate on the first circle. In this subsequence we can find another subsequence, S_2, such that the second coordinates (i.e. on the second circle) converge to a definite value. Continuing in this way, and extracting the "diagonal" sequence we get a final sequence such that the values of the coordinates on an arbitrary circle converge to a single definite value. Suppose that $(u_1, u_2, \ldots, u_n, \ldots\quad)$ are the corresponding values determined above. Then the point of T with these coordinates is the sequential limit point of the "diagonal" sequence.

The proof is obvious. We may remark that the advantage of considering T, instead of B, is that we have not had to consider at this moment the question whether if the points of our sequence had belonged to B the final limit point would also have belonged to B. We shall now prove, however, that this is the case.

Theorem. The set B is a closed subset of T.

That is, B contains all of its limit points. Now, each point of T is a linear function on A. It will be a point of B, as we have seen before, if it satisfies all of the congruences imposed upon it by the group A :

$$\sum k_{ij}\mu_j \equiv 0 \mod 1 ,$$ where the μ_j denote the coordinates of the point t, say, in question. Let us show that if the point t is a limit point of B, then it satisfies all of these congruences. Suppose, on the contrary, that there is some one congruence which it does not satisfy. This congruence is certainly a continuous function of the finite set of coordinates (u_j) which it involves explicitly. Therefore for a corresponding set of coordinates (u'_j) which differ from these by very little, the form will suffer an arbitrarily small change; therefore, it will still fail to be congruent to zero, mod. 1. But this restriction of the coordinates defines a neighborhood of t. This neighborhood must contain points of B, by the assumption that t is a limit point of B, and the congruence does hold for all points of B. The contradiction shows that the congruence holds at t also, and that t is an element of B.

Corollary. The space B is compact and closed.

In this form, we have freed ourselves entirely of a reference space T.

Theorem. Every closed subgroup of T is characterized by the solution of certain suitable congruences.

We shall be able to interpret these congruences as the fundamental relations of a certain countable group which we shall know to identify with the character group of B. Before proving this theorem, we shall prove the

Theorem: Every continuous character of a closed subgroup B of T is expressible in the form

$\sum k_i u_i$, where the u_i denote the coordinates of a point of T, the k_i are integers of which, at most a finite number, are different from zero.

By a character of B, we mean a function a(b) of the elements b of B which is single valued, linear and <u>continuous</u> in the topology of B (all mod 1 of course, as before).

We shall prove this theorem inductively, on the dimension of the space T. There will be one final case where we shall have to jump from finite to infinite dimension.

Case 1: Dimension of T is 1, i.e. T is the group of reals mod 1.

Now in this case, the group B is of one of two simple types. Either B coincides with T, or else B is a finite cyclic group of some order k. For, if there are elements of B arbitrarily close to the zero element, there will be elements of B everywhere dense on the group T and, since B is closed, B will coincide with T. If there is a neighborhood of the zero which contains no other element of B, then going in a fixed direction on the circle T we shall encounter a first point b of B, since B is closed. If we consider the elements b, 2b, 3b, .. . there must be some integer k such that kb = 0. Otherwise as we follow these points around on the circle there will be one just before we get to zero, and the next one will go beyond zero. But then it will certainly fall short of b, and accordingly b was not the first element, as it should have been. In this case, then, the group B is a finite cyclic group of order k. In this case, the coordinate~~s~~ of the point b is $1/k$, and each point of B has the coordinate m/k for some value of $m < k$. Now suppose we have a character of B, and suppose that its value at b is v. Its value at kb is kv. But since $kb = 0$, $kv \equiv 0$, mod. 1. That is, kv is an integer, say k'. It is clear that if u designates the coordinate of an arbitrary point of B, the value of our character at this point is given by k'u. Now k'u is a function of the kind required by our theorem, and in this case the theorem is proved.

Suppose now that B coincides with T, and suppose that a is a continuous character of T. Then, since it is continuous, we can certainly find a neighborhood of the ~~identity~~ zero such that the value of a at all points of this neighborhood is less than $1/3$, mod. 1. Now let v be the value of a at a point u of this neighborhood. Let v' be its value at the point $u/2$. It is clear, from the linearity of a, that $2v' \equiv v$ mod. 1. Then v' is $v/2$ or it is $v/2 + 1/2$. Only the first of these lies in the given neighborhood. It is clear that if we restrict ourselves to this neighborhood, the character a is unambiguous, and that at the points u, $u/2$, ..., $u/2^n$, .. it has the values, v, $v/2$, ..., $v/2^n$, ... But now by linear interpolation, and by continuity we see that the value of a at any point of this neighborhood must be of the form k'u', where u' is the coordinate of the point and k' is v/u constant. Then we can extend this function linearly to the whole group, and the character must be of that form at all points of B. In this case, also, our theorem is proved.

Lecture 10

We are considering a countable abelian group A, whose elements we may denote by a_n, n = 1, 2, ... We defined a character of A as any single-valued, linear function b(a) whose values were real numbers, mod. 1, where addition is to be understood modulo 1. We introduce now the notation

(b, a)

to represent the value which the function (character) b takes at the element a. We regard this as a function of two variables, a variable element of A and a variable character b: we have seen that the characters b form a group B which we called the character group B of A. This function (b, a) is linear in a if we keep b fixed: in fact, when we keep b fixed and let a vary over the whole group A we get a set of values which define the function b. But now, suppose we hold a fixed and let b vary over the group B. Our function (b, a) of two variables becomes now a function of the single variable b of B. It is quite easy to see that it is a linear function and single valued. The linearity expresses the fact that (b + b' , a) = (b, a) + (b', a). This is the case, by our definition of b + b' in the preceding lecture: that function, namely, which had at an element a the sum (mod. 1) of the values b(a) and b'(a). The single-valuedness is obvious. We see, then, that each element a of A defines (or we say, is) a character a(b) of the group B. Its values are precisely those which are given by the bilinear expression (b, a) keeping a fixed. Now two questions arise. First, are all the characters a(b) distinct? Can two different elements a of A have the same value at every element b of B? The answer to this will be that the characters a(b) defined in this way are distinct. The second question is, are these all the characters of B? Here we shall find that the answer is definitely negative. The group B must have more characters than are contained in the group A. However, we shall find that if we topologize B in a very natural manner, and we have already given this topology in our previous lecture, then the group A will coincide with the totality (necessarily a group) of the continuous characters of B -- in this topology. There are two parts to such a proof, of course. We shall have to show that each character a (of B) is continuous in this topology (which is not obvious) and that there are no other continuous characters (which is less obvious).

To take our first question, we want to show that $a_i(b)$ and $a_j(b)$ cannot coincide if a_i and a_j are different elements of A. This means that there exists at least one element b of B at which they have different values. We must show, therefore, that $a_i(b) - a_j(b)$ is not identically zero as a function of b. By the linearity, this is equivalent to showing that a(b) is not identically zero, if $a = a_i - a_j$ is not the zero element. But we ~~shall see that~~ given any element of A not the zero element, there exists a character b whose value at that element is not zero. Therefore a(b) is not zero for all elements b of B, and our assertion is proved. The characters a(b) are all distinct.

To prove the underscored statement, we consider first two trivial lemmas.

1) for any integer n, $nx \equiv 0 \pmod{1}$ has precisely the n distinct solutions: $x \equiv 0, 1/n, 2/n, \ldots, n-1/n$.

2) for any integer n, and any real number k, not zero, the congruence $nx \equiv k \pmod{1}$ has precisely n solutions,

$$k/n,\ k+1/n,\ k+2/n,\ \ldots,\ k+n-1/n \quad , \bmod.\ 1.$$

The lemmas may be verified by inspection.

We shall now prove the following

Theorem: Let A_0 be any subgroup of A and a any element of A which does not belong to A_0. Suppose further that we have defined a character of A_0. Then we can extend this to a character of A in such a way that its value at the element a is not zero.

A few remarks: By extending a character from A_0 to A, we understand constructing a character defined over all of A such that its values on the subset A_0 agree with the values already defined there. We suppose nothing whatever about A_0, except that it is a group: it may very well consist of the zero element alone, in which case the character which we suppose given ~~might~~ would be the function whose value at the zero element is zero, and which is not anywhere else defined. For later purposes, we need the theorem above in the generality in which we have stated it.

Proof. Let us enumarate all the elements of A which do not belong to A_0, but in such an order that the given element a is the first. It is essential that a is the first element which we encounter, it is really immaterial whether we include elements of A_0 or not. Let $b(a_0)$ be the character defined on A_0. There are two possibilities. It may happen that no integral multiple of a belongs to A_0. In this case we shall define $b(a)$ by choosing for it an arbitrary value (real, of course) not zero (mod. 1). We shall then extend this function linearly over all elements of the form $a_0 + na$, where a_0 is any element of A_0 and n is any integer. We now have our function b extended to a character of the group (A_0, a), i.e. the group generated by the elements of A_0 and the element a. There is the second possibility, namely that for some integral multiple n, na belongs to the group A_0 (this may happen, of course, even when the element a is of infinite order). In this case there will be a least multiple, which we may suppose to be the integer n above, such that na belongs to A_0. Now the integer n (which we may always take as positive) cannot be 1, because a does not belong to A_0 by assumption

Our character b is already defined for na, since this is in A_0. We shall define it at a by giving to it any one of the n-1 non-zero values which verify the congruence

$$nx \equiv (b, na), \quad \text{mod. } 1 .$$

It is easy to verify that if we extend this new function over the group generated by A_0 and a, together, by defining it linearly for all elements of this group, then it becomes a character. We have only to remark that it is single valued, and this follows from our choice of b(a) and the fact that every element of the larger group can be represented uniquely as a form $a_0 + ka$, where a_0 is a suitable element of A_0, and where $0 \leqq k < n$.

Therefore we have been able to extend our character, defined originally over A_0, to a character defined over a larger group, call it A_1, which includes the element $a = a_1$ (in our enumeration) and is not zero at a_1. To complete the argument, we must extend this function over all of A. We do it by an obvious induction, taking the elements in sequential order. If we suppose that we have extended this function to a character over a subgroup A_n of A which includes all the first n elements, then denoting a_{n+1} by a (for the fun of it) we extend this character to include a_{n+1} by precisely the argument we gave above, replacing A_0 in that argument by A_n.

Then we have, finally, proved our original assertion (and more) that to each non-zero element of A there corresponds at least one character which does not vanish on it. The proof above is essentially arithmetical because we had no topology whatever on A. The question whether the characters we have defined are continuous has, therefore, no meaning. Characters constructed as above for topologic groups will not, in general, be continuous in the given topology so that this process is not reversible when we consider, for example, our groups B with their topology.

Now if we take an arbitrary subgroup A_o of A it will determine for us a well-defined subgroup B_o of B: the subgroup namely which consists of all elements of B vanishing at every element of A_o. Each such element will, in general, vanish at other elements of A. We shall find, however, that there is no element of A, not in A_o, such that every element of B_o vanishes at it. We shall call B_o the annihilator of A_o. What we have intimated above is that A_o will turn out to be the annihilator of B_o. We shall prove that the character group of A mod A_o can be identified with B_o and that the character group of A_o can be identified with the group B mod B_o.

Theorem. Given an arbitrary subgroup A_o of A, any character b(a) of A determines uniquely a character $b(a_o)$ of A_o, and each character $b(a_o)$ may be derived from at least one b(a) of A.

This is now quite obvious. For, a character b(a) determines a character $b(a_o)$ of A_o at once, if we merely regard it as being defined over A_o and ignore its definition at other elements of A. I.e., it will continue to be single valued and linear. On the other hand, we have proved in an earlier theorem that if we take any character $b(a_o)$ we can extend it to a character b(a) of A: it is clearly the case that $b(a_o)$ may now be derived from b(a) by restrictin its range to A_o.

Theorem. Let A_o be an arbitrary subgroup of A, and let B_o be its annihilator. Then B_o is the character group of A mod A_o.

This is to be understood in the sense that B_o is isomorphic to the character group of A mod A_o. The group A mod A_o is a group of cosets, and a character is defined by assigning suitable values to these cosets: i.e. a character is constant on each coset. In particular, a character must vanish at the zero element, i.e. it must have the value zero at every element of A_o. What we are saying is essentially this. Given any character of the group A mod A_o we can at once identify this with a character of A: that character, namely, which has at every element of any coset a + A_o of A the same value which the character of A mod A_o has on that "coset-element". But in this way we have certainly identified it with an element not only of B but also of B_o. Because the character must vanish at A_o, and B_o includes all of these. Conversely, any character belonging to the subgroup B_o of B, because it vanishes on A_o, will be constant on all cosets a + A_o. Clearly it defines a character of A mod A_o. Therefore, each character of A mod A_o can be identified with an element of B_o, and each element of B_o gives rise to one such character. We want to be certain that two distinct elements of B_o give rise to distinct characters of A mod A_o. This is also clear. For, since the elements in question are distinct as characters of A there will exist an element a of A at which they have different values. Then, as characters of A mod A_o, they will have different values on the coset-element a + A_o. (It hardly needs to be remarked, anyway it is immaterial, that this element a will not belong to A_o since there all elements of B_o agree in having the value zero.) Our theorem is proved.

Theorem. If B_o is the annihilator of A_o, then A_o is the annihilator of B_o.

This means that every element of A_o has the value zero for every element of B_o (which is trivial) and that any element of A which vanishes at every point of B_o must belong to A_o (which is not trivial). However, we have already proved the last part of this statement. For, if a is any element of A not belonging to A_o then we know that there exists a character b which vanishes at Ao and does not vanish at a. This is a particular case of our first theorem. But now this character b, since it vanishes at every elemen of A_o, must belong to B_o by definition. Therefore, given <u>a</u> not in A_o we <u>have found</u> at least one element of b_o at which it does not vanish. It follows that A_o is exactly the annihilator of B_o.

We cannot prove the converse theorem without taking into account considerations of topology; in fact without ascribing to B a properly chosen topology. We shall get this topology so that it will turn out that each group B_o arising as annihilator of a subgroup A_o of A is <u>closed</u> and so that, further, every closed subgroup B_o of B will be the annihilator of some subgroup A_o of A. (It is clear that then we shall have the converse theorem, at least for closed subgroups of B). The topology will be such, as we have remarked before, that the elements <u>a</u> will be the only continuous characters of B.

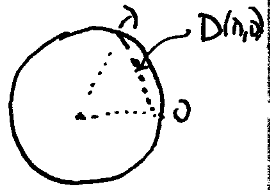
The kind of topologic groups which we shall have to consider will be very restricted. We may call them the uniform-metric groups (we shall be concerned only with the compact ones, among these). We shall say that a group B is a uniform metric group, provided there exists a function $D(x, y)$ where x and y denote arbitrary elements of B such that i) this function is defined for every pair of elements (not necessarily distinct) and its value is a positive real number, or zero; ii) $D(x, y) = 0$ if and only if $x = y$; iii) $D(x, y) = D(y, x)$, <u>symmetry</u>; iv) triangle inequality:

$$D(x, z) \leqq D(x, y) + D(y, z)$$ and finally

v) <u>translation invariant</u>:

$$D(x, y) = D(x+z, y+z).$$

As examples of such groups we may consider the general Toral groups which we defined previously where we introduce such a metric as follows. Consider the one dimensional toral group: realising it as the group of rotations of a circle. We can picture this group as an ordinary unit circle in the plane. As distance between two points we can take the ordinary euclidean distance. This will obviously have the first four properties we enumerated: these are the usual properties of a metric. Each point of the circle denotes a rotation through the corresponding angle: if we think of the plane as rotating with the circle (about the fixed center) then we see that this distance is preserved. In the group, this "rotation" is a "translation", of course, every point of the one-torus being displaced by the same amount. For the case of the finite dimensional torus we can think of the group as the product of a set of unit circles and take as distance between two points of the group the sum of the distance, in the euclidean sense above, between their corresponding coordinates. For the infinite dimensional torus we cannot do the same thing, because we want the distance between two points to be finite. We can therefore choose a fixed set of multipliers whose sum converges, e.g. 1, $\frac{1}{2}$, $\frac{1}{4}$, ... and add the distance between corresponding coordinates with these multipliers. It is to be understood that the metric is always a convenient tool, so that it is quite immaterial exactly how it is constructed .. so it is a metric. We must be sure of course that different choices of multipliers, above, lead to the same topology. It is the topology with which we are concerned. Now it is rather easy to see that if we define limit points in terms of the given metric (for any set of multipliers), then a necessary and sufficient condition that a sequence of points x_n, say, converge to a point x in that sense that corresponding distances converge to zero is that their first N coordinates converge to the corresponding coordinates of x, for every integer N.



The assertion above is quite easily verified: if

$D^n = d_1{}^n + d_2{}^n + \cdots + d_m{}^n + \cdots\cdot ; n = 1, 2, \ldots,$

is a sequence of sums of positive (or zero) terms $d_m{}^n\ldots$ then this sequence can converge to zero if and only if $d_m{}^n$, for fixed m, converges to zero. Of course, by our choice of multipliers the individual sums are finite and the partial remainders are uniformly small. On the other hand the convergence of the partial sums need not be uniform. We are not concerned with that. Our metric merely exhibits conveniently the topology which we have in mind for our groups.

We may remark at this point that the group operations are continuous in the topology we have chosen. Thus if a_n is a sequence of elements converging to an element a, then $-a_n$ converges to $-a$. Expressed in our metric, the first statment means that $D(a_n, a)$ goes to zero with n to infinity. Translating these distances by $-a$, we have that $D(a_n-a, 0)$ goes to zero. Therefore, translating by $-a_n$, $D(-a, -a_n)$ goes to zero so that, finally, $-a_n$ converges to $-a$. Further, if a_n converges to a and b_n converges to b we can show that a_n+b_n converges to $a+b$. For $D(a_n+b_n, a+b)$ is equal to $D(a_n-a+b_n, b) = D(a_n-a, b-b_n)$, translating first by $-a$ and then by $-b_n$. By the triangle inequality, this last number is less (or at most equal) $D(a_n-a, 0) + D(0, b - b_n)$. But each of these numbers goes separately to zero so that their sum goes to zero, and a_n+b_n must converge to $a+b$. (It is also quite easy to verify this directly in terms of the coordinates of all of the points).

We have already shown that the Toral groups T are compact, i.e. every infinite sequence of points has a subsequence converging to some definite element of the group. We have shown, further, that the subgroups B (which arise as character groups of discrete groups A) are closed subgroups of T. As subsets of T, they are also metric, and it is clear that if we have a uniform metric in T then the subgroup B becomes a uniform metric group (merely observe that when we translate by an element of B, the group B goes into itself).

Topologic residue groups: Let B be any uniform metric group, for simplicity we shall suppose it compact. Let B_0 be any closed subgroup of B. By B mod B_0 we shall mean the group of cosets $(b + B_0)$, as b varies in B. It is easy to see that this is a group. We want to show that it is also a uniform metric group. We shall "metrize" this coset group, i.e. we shall define a distance between pairs of its elements. This comes to the same thing as defining a distance, in the group B, between two distinct cosets. Observe, first, that if B_0 is closed then every coset $b + B_0$ is closed. This is quite trivial: if an element b^* is a limit point of such a coset, then $b^* - b$ is seen to be a limit element of B_0, therefore an element of B_0 (which is closed by assumption). But then b^* belongs to $b + B_0$, and this is closed. Now if $b + B_0$ and $b' + B_0$ are two cosets, let the distance between them be defined as the greatest lower bound of all distances $D(x, y)$ where x ranges in the first coset and y in the second and $D(x, y)$ is their distance in the metric group B.

The distance which we have defined above is certainly real and non-negative. It is zero, obviously, when the two cosets coincide. But it is not zero, otherwise: this depends on the fact that our cosets are closed. To prove this most simply we remark that in our definition of the distance of cosets we may keep one of the variable elements fixed. Thus if x' belongs to the coset $x + B_0$, $x' = x + x_0$ where x_0 is a suitable element of B_0. Therefore $D(x', y') = D(x, y' - x_0)$, by translation, and $y' - x_0$ is in the same coset as y'. Now if the distance of two cosets is zero, we see that there must exist a sequence of elements in one of them whose distance to a fixed element in the other converges to zero: therefore this element is a limit point of the sequence. Therefore, the cosets being closed, the limit element belongs to the other coset. The two cosets coincide, since they have an element in common. It may be remarked that our argument has not used the compactness of B. It is trivial that the metric which we have defined is translation invariant: there remains the triangle inequality to verify. Let P, Q, and R denote three cosets, and let $\underline{e}$ denote any positive number. Take any point p in P. Then there exists a point q in Q such that the distance of the cosets P and Q is greater than $D(p, q) - \underline{e}$. There exists a point r in R such that the distance of the cosets Q and R is greater than $D(q, r) - \underline{e}$. By the triangle inequality, $D(p, q) + D(q, r) \geqq D(p, r)$. Now $D(p, r)$ is not less than the distance between the cosets P and R. We see that

$$D(P, Q) + D(Q, R) \geqq D(P, R) - 2\underline{e}$$

, where these symbols denote distances between corresponding cosets. Since $2\underline{e}$ is arbitrarily small, the triangle inequality for this new metric is verified. Here again we have not used the compactness of B: we might have used it by proving (it is easy to) that there existed a point q in Q such that $D(P, Q) = D(p, q)$, and avoiding the $\underline{e}$'s.

The most important application, for us, of the metric which we have introduced in our Toral spaces is that it has facilitated the topologizing of the residue groups. We could have done this more directly, without a metric but it would have been less easy to follow.

Lecture 11

We shall begin a study of Toral groups, their subgroups, and character-groups. Let T denote a Toral group of infinitely many dimensions. Each element b is given by the values of its coordinates

$$b \quad : \quad (\beta_1, \beta_2, \ldots \beta_n, \ldots \quad) .$$

Let T^n denote the subgroup of T such that all coordinates, after the nth are zero. T^n is then an n-dimensional Toral group, and it is obviously a closed subgroup of T. Now if b is any element of T, we shall define its projection on T^n as that element b^n of T^n whose first n coordinates agree with the first n coordinates of b. If B is an arbitrary group of T, then by the projection of B we shall mean the set of all elements of T^n which are projections of an element of B. It is quite easy to see that the projection of a group is itself a group. We must guard ourselves against supposing that the projection is necessarily a subgroup of the original group B (it is, of course, a subgroup of T and of T^n). (A simple example will make this clear. Let T^2 be a toral group, the coordinate on one circle being denoted by x and the coordinate on the other by y. The set of elements of the form $\lambda(x + y)$, where λ takes on real values, mod 1., is a closed subgroup of T^2. Its projection on the x-circle covers the entire circle. But no element of the form λx, except zero of course, belongs to the group in question).

Theorem: If B is a closed subgroup of T, then its projection into T^n is a closed subgroup B^n of T^n.

Let us suppose that some element b^n ~~of T^n~~ is a limit element of B^n. Let b^n_s, s = 1, 2, ..., denote a sequence of elements of B^n converging to b^n. With each of these we may associate one element b_s of B, such that b^n_s is the projection of b_s. We can choose a subsequence of the new sequence b_s such that this subsequence converges to an element b of T. Since the first n coordinates of the elements b_s (and therefore of our subsequence) converge to the first n coordinates of the given element b^n it is clear that the first n coordinates of the limit element b, above, coincide with those of b^n. Therefore b^n is a projection of the element b. But b certainly belongs to B, since this is closed and contains all b_s. Therefore, since b^n was an arbitrary limit element of B^n, B^n is proved closed.

Suppose now that we have any function defined on B, which is single valued. We cannot, in general, define this function on the projection B^n of B so that it shall be single valued there. It may happen, however, that the function has the same value at all points of B which project into the same one point of B^n. In this case we can project the given function by taking its value at a projection point to be its value at any one of the points which project into this point. Let us now prove that

Corollary: if a function defined on B is continuous, and if the function can be projected into B^n, then it is continuous in B^n.

For if b^n is a point of B^n at which the projected function is not continuous, we can find a sequence of elements $b^n{}_s$ of B^n converging to b^n such that there is a sequence b_s of elements of B which project into them and which converge to an element b of B projecting into b^n. But now the contradiction is immediate since our projected function has the same value at b_s and b as it has at $b^n{}_s$ and b^n, respectively, and is continuous in B.

Theorem. Let B be an arbitrary closed subgroup of a toral group T. The i-th coordinate β_i of a point b of B is a continuous character of B.

This is quite trivial. It is clear that this coordinate is single-valued and linear, by the definition of a toral group. That it is continuous in our topology, we have already remarked when we noticed that a necessary and sufficient condition that a set of elements converge to a given element is that the correspondin coordinates converge.

In the application which we shall make of this theorem the i-th coordinate β_i will be the value which the given element has at the element a_i of the group A of which B will be the character group. This will tell us at once that each of the elements a_i is a continuous character of the character group B.

Principal Theorem: If B is a closed subgroup of a toral group T, then every continuous character of B can be expressed as a finite linear integral form in the coordinates.

i.e. $c(b) = \sum \lambda_i \beta_i$, λ_i integers, at most a finite number different from zero.

Proof. We shall consider, first, the case that T is of finite dimensionality and we shall make an induction on the dimension. Suppose, to begin the induction, that T is of dimension 1, i.e. a circle group. We have already seen, in an earlier lecture, that ✓ p108 in this case every continuous linear function of B must be of the form: $\lambda\beta$, where β denotes the coordinate on T and where λ is an integer. (This is, essentially (except for the slight complication of the mod. 1), the theorem that a single valued additive continuous function of the straight line is necessarily linear.) We suppose now that the theorem is established for all toral groups of dimension less than n. Consider the subgroup, let us call it B', of B such that the first n-1 coordinates of all points of B' are zero. (There is at least one such element, namely the zero). It is clear that B' is closed, and it belongs to the one dimensional toral group which corresponds to the nth coordinate. Suppose that $c(b)$ is any continuous character of B. Clearly $c(b)$ determines a continuous character of the subgroup B'. Therefore by the one-dimensional case of our theorem $c(b')$ can be expressed <u>on B'</u> as some function $\lambda_n\beta_n$ of the n-th coordinate, where λ_n is a suitable integer. But we have already seen that β_n and ~~in this case~~ $\lambda_n\beta_n$, also, is a continuous character of B. Therefore

$$c(b) - \lambda_n\beta_n = d(b) \text{ is a continuous character}$$

of B. Now we are going to project this function $d(b)$ into the projected group B^{n-1} of B into T^{n-1}, determined by the first n-1 coordinates. The projected function <u>is</u> single-valued, for if we have two points b and b' whose first n-1 coordinates agree then $d(b - b') = d(b) - d(b')$ has the value zero. This follows simply from the fact that, in this case, $b - b'$ is an element of B' so that $c(b - b') = \lambda_n(\beta_n - \beta'_n)$. This is <u>precisely</u> the quantity which we must then subtract from $c(b-b')$ to get $d(b-b')$. It is quite clear that the projected function is linear in B^{n-1} and we have already observed, as a corollary, that it is necessarily continuous in B^{n-1}. Therefore, finally, $d(b)$ is a continuous character of B^{n-1}. By our induction, it follows that $d(b)$ can be expressed as a finite integral form in the first n-1 coordinates

$$d(b) = \sum_1^{n-1} \lambda_i\beta_i \quad \text{, so that}$$

$$c(b) = \sum_1^{n} \lambda_i\beta_i \quad \text{, proving our theorem, for every } \underline{n}$$

We have the final case to consider that T is of infinite dimensionality. In this case we shall prove that if $c(b)$ is continuous character of B there exists an integer <u>n</u> sufficiently large so that $c(b)$ vanishes for every point whose first n coordinates vanish. If this were not so, we could find a point b_n for every n = 1, 2, ..., whose first n coordinates vanished and such that $c(b_n)$ was different from zero. In this case we can certainly find real multipliers t_n

such that $t_n c(b_n) > \frac{1}{4}$ mod 1. Now $t_n c(b_n) = c(t_n b_n)$ (mod. 1) by the linearity of a character, so that $c(t_n b_n)$ cannot converge to zero. But the first n coordinates of b_n and therefore of $t_n b_n$ are zero, by construction. Therefore the elements $t_n b_n$ converge to zero. This violates the ~~assumed~~ continuity of the character c(b), and the contradiction establishes our contention.

This means that there exists an n such that the value of c(b) coincides with the value of $c(b^n)$ where b^n is the projection of b into T^n: n is a fixed integer. This is not quite accurate, since c is not necessarily defined for the element b^n of T: this may not belong to B. We shall define c at b^n by projection, but then our assertion becomes trivial. What we mean is that there exists an n such that the function so projected becomes single-valued. This is not true for every n, but it is true for any n such that the character c vanishes on every element whose first n coordinates vanish. We have shown that such an n exists.

Now the character $c(b^n)$ is defined on a subgroup B^n of the finite dimensional toral group T^n and we may apply the case of our theorem which is already proved: we see that c(b) is an integral linear form in the first n coordinates.

Now let us apply this theorem to the case that the group B is the character group of a given discrete group A, and the group T in which B is imbedded is determined by taking as a basis for A the set of all elements of A enumerated in single sequence: $a_1, a_2, \ldots$ Then we have the result that each continuous character of B is a finite integral form in the coordinates. We have already seen that the i-th coordinate of an element b is the value of the character $a_i(b)$. Therefore the general continuous character of B is of the form

$$\sum \lambda_i a_i(b) = a(b)$$

where a is that element of A which is given by the finite integral expansion

$$\sum \lambda_i a_i$$

We have finally proved, as corollary, that if B is the character group of a discrete group A then A is the continuous character group of B (it being understood that B is topologized, as we have indicated: it is not asserted that this is the only topology which could be assigned to B so that the theorem should still hold, although that is the case).

Now, preparatory to establishing the converse theorem to our theorem on the annihilator of a subgroup A_0 of a discrete group A, let us prove the theorem:

<u>Given a closed subgroup B_0 of B and an element b not in B_0 there exists a character vanishing on B_0 and not vanishing at b.</u>

Let the coordinates of b be $(\eta_1, \eta_2, \ldots \eta_n, \ldots\ )$. Suppose that for each n there is an element b_n in B_0 whose first n coordinates agree with the coordinates of b. It is clear that this sequence converges to b. Since B_0 is closed, by assumption, and b does not belong to it, this is impossible. Therefore there exists an n such that there is no element of B_0 whose first n coordinates are $(\eta_1, \eta_2, \ldots, \eta_n)$ (and there exists a first such n). For <u>this</u> n, let B'_0 be the subgroup of B_0 consisting of those elements whose first n coordinates are zero. Let us consider the coordinates $(0, 0, \ldots, 0, \beta'_{n+1}, \ldots)$ of a general element b' of B'_0. It is clear that β'_{n+1} cannot take ~~different from β_{n+1} otherwise~~ on all real values (mod 1). <u>For</u>, suppose that it could. We know, from our particular choice of n, that there exists an element b" of B_0 whose first n coordinates agree with those of the initial element b. Then if β''_{n+1} is the n+1 st coordinate of b", we should be able to add to b" an element b^0 of B'_0 whose first n coordinates were zero and whose n+1 coordinate was $\eta_{n+1} - \beta''_{n+1}$. This would give us an element of B_0 (since B'_0 is a subgroup of this) whose first n+1 coordinates agreed with b, and such an element does not exist.

In case it happens that there is no element in B_0 whose first coordinate agrees with that of the element b, the group B'_0 will coincide with B_0.

From the fact that the n+1 st coordinates of elements of B'_0 have a restricted range we know that this range must be finite and that there must exist an integer k such that each coordinate is of the form: $1/k, 2/k, \ldots, k/k \equiv 0$, mod.1 . It is clear that $k\beta_{n+1}$ is a single valued function of the first n coordinates of elements of B_0. I.e. if we have any two elements of B_0 whose first n coordinates coincide, then k times the difference in their n+1 st coordinates will be zero (this difference being of the form m/k, mod 1.). Therefore $k\beta_{n+1}$, as a character of B_0, will have the form $k\beta_{n+1} = \sum_1^m \lambda_i\beta_i$, and $k\beta_{n+1} - \sum_1^m \lambda_i\beta_i$ as a character of B will vanish on B_0.

To conclude our argument we have only to observe that this character of B does not vanish at our original element b. Suppose it did. Let b^0 be any element of B_0 whose first n coordinates agree with those of b (we know at least one such element exists by our choice of n). Therefore this character must vanish at their difference (since it vanishes at both). It is clear from the form of the character that the n+1 st coordinate of the difference will have to be a solution of the congruence: $kx \equiv 0$ (mod. 1). By our choice of k there is at least one element of B'_0 whose first n coordinates, of course, are zero and whose n+1 st coordinate is an arbitrary integer multiple of 1/k. It should be clear now that we could find an element of B'_0 which added to b^0 would have the first n+1 coordinates of b. The contradiction establishes our theorem.

Theorem. Let B be the character group of a discrete group A, let Bo be an arbitrary closed subgroup of B, and let A_0 denote the annihilator of B_0: i.e. the subgroup of A each element of which has the value zero at all elements of B_0. Then B_0 is the annihilator of A_o. *q 10 6*

Proof. It is clear that each element of B_o has the value zero at every element of A_o, so that B_o is contained in the annihilator of A_o. On the other hand, if b^o is any element of B <u>not</u> belonging to B_o then, by the theorem above, there exists at least one character, call it a(b) which vanishes on B_o and does not vanish at b^o. Now <u>a</u> belongs to A_o, the annihilator of B_o (since it <u>does</u> vanish there). Since it does not vanish at b^o, it follows that b^o cannot belong to the annihilator of A_o: since b^o was any element not in B_o it follows that B_o <u>is</u> the annihilator of A_o.

We shall continue these converse theorems by showing, later, that if B_o is an arbitrary closed subgroup of the character group B of a discrete group A, and if A_o is the annihilator of B_o, then A_o is the character group (continuous) of B mod B_o.

In the application of this theory to combinatorial topology we shall have to consider simultaneously not merely these two distinct types of groups but certain sets of groups of each type which are given to us with definite mappings. These mappings will correspond to the boundary operator, for example, in our connectivity theory.

Suppose then that we have two arbitrary discrete groups A and A', and their associated character groups B and B' (topologized, always). Suppose, further, that we have a mapping of A upon A', in the sense of a group homomorphism: i.e. if <u>a</u> corresponds to a', then -a to -a', and a+b to a'+b' whenever <u>a</u> to a' and <u>b</u> to b'. The mapping is not supposed to be an isomorphism, nor to cover all of A'; of course, it <u>may</u>. Let us call this mapping: μ. We shall show how to associate with it a definite mapping γ of B' upon B.

Definition of ν p 11.9

Let a, b' be arbitrary elements of A, B' resp.

Define the function $b(a)$ as follows:

$$b(a) = b'(\mu a).$$

Then $\quad b(a_1 + a_2) = b'(\mu(a_1 + a_2)) = b'(\mu a_1 + \mu a_2)$

$$= b'(\mu a_1) + b'(\mu a_2) = b(a_1) + b(a_2).$$

Hence $b(a)$ is a character of A, and b is by definition an element of B.

We then define $\nu b' = b$, where b is obtained from b' as above, and show easily that

$$\nu(b_1' + b_2') = \nu b_1' + \nu b_2'.$$

This mapping ν (by the way, it will go the "other way", i.e. it will map B' on B) will also be a group homomorphism and will be continuous.

The original mapping μ distinguishes two subgroups, one in A the other in A'. There will be a subgroup A_0 of A such that $\mu(A_0) = 0$, i.e. such that each element of A_0 is mapped on the zero-element of A'. On the other hand there will be a subgroup A'_0 of A' which is covered by this mapping, i.e. every element of A'_0 is associated with at least one element of A, and every element of A' which is "covered" belongs to A'_0. Now we shall construct the mapping ν so that it will have the following property: The closed subgroup of B' which is mapped into the zero-element of B will be the annihilator B'_0 of A'_0. The closed subgroup B_0 of B which is covered by this mapping will be the annihilator of A_0.

To determine the mapping ν let us consider an arbitrary element b' of B'. This element is a character b'(a') of the group A'. Now we shall map b' upon that element b of B which regarded as a character of A has the same value at an arbitrary element a of A which b' has upon $a' = \mu(a)$. It is not trivial that we can do this, for it is not trivial that such a character exists: (there won't be more than one).

Observe first that not all elements of A' affect our choice of the function b. For if a' is an element of A' not covered by any a of A, the value of b' at this element is a matter of indifference to us. We are interested, then, in the character b' as a character of the subgroup A'_0: the group of elements that are actually covered. Now A is mapped homomorphically into all of A'_0. Each element a'_0 of A'_0 is covered by a coset of A mod A_0: i.e. our mapping is really an isomorphism between the residue group A mod A_0 and the subgroup A'_0. Therefore the function which has that value at each element of a coset $a + A_0$ which b' has at

$\mu(a) = a'$ is a character of A: this is the desired character b(a). Now we see that if b' belongs to B'_o it will have to have the value zero at every element of A'_o and the function which we are looking for will have the value zero at every element of A: that is to say it will be the zero character. This justifies our statement that $\nu(B'_o) = 0$. But we see, also, that in any event since each b' vanishes at the zero element of A' the character b which we are looking for must vanish at every element of A_o, since these are mapped into zero. Therefore the character b will necessarily belong to B_o the annihilator of A_o.

We shall review this discussion the next hour and complete the demonstration that the mapping ν which we have defined is a continuous homomorphism.

Lecture 12.

Let us review, briefly, the results of the last few lectures. You recall that we began with an arbitrary (countable) discrete abelian group A and defined its character group B. We then topologised the group B. The topology which we chose for B was such that a sequence of elements b_n of B converged to an element b of B if and only if the values $b_n(a)$ converged to the value b(a) for every element a of the original group A.

We saw that if we considered an arbitrary subgroup A* of A there was associated with it a definite closed subgroup B* of B which we called the annihilator of A* and which was defined as the totality of elements of B whose values at every element of A* were zero. It was clear that every element of A* had the value zero at every element of B* and we proved, further, that A* contained every element of A which had this property. That is, we showed that if B* is the annihilator of A*, then conversely A* is the annihilator of B*. We proved that if we started with an arbitrary closed subgroup B* of B, and any element b of B not in B*, there always existed an element a of A which vanished on B* without vanishing on b. From this we were able to conclude that if we began with a closed subgroup B* of B and constructed its annihilator A* in A then B* was the annihilator of A* (this would not be true for an arbitrary (not closed) subgroup B' of B: in this case we should get as its annihilator group in A a group A' whose annihilator was not the original group but its closure $\overline{B}'$).

We then considered a group A, an arbitrary subgroup A*, the character group B of A, and the annihilator B* of A*. We showed that the group B* was the character group of A mod A*, and the group B mod B* the character group of A*. We want now to conclude that A* is the continuous character group of B mod B* and that A modA* is the continuous character group of B*. This can be done in several ways of which the neatest perhaps is this. We already know that if we take an arbitrary discrete group and form its character group ~~than~~ the discrete group becomes the continuous character group of its character group provided we topologize this one in the proper way. That means that we know that A* is the continuous character group of B mod B* and that A mod A* is the continuous character group of B* provided we pick the right topology. We also know how to pick this topology. But we already have a topology for B*, as closed subgroup of the group B which was topologized to begin with. Similarly we already gave a topology to B mod B*. What we have to do now is to verify that these are the proper topologies for our theorem. This is quite easy to do.

Theorem: A mod A* is the continuous character group of B*, on the hypotheses of the preceding paragraph.

We have two topologies for B*, which we must now try to identify. In the first, regarding B* as subgroup of B, we know that a sequence of elements b^*_n of B* converges to an element b* if and only if $b^*_n(a)$ converges to $b^*(a)$ for every element a of the group A. In the second, the one in which our theorem is certainly true, b^*_n converges to b* if and only if $b^*_n(a+A^*)$ converges to $b^*(a+A^*)$ for every coset-element a + A*. But the value of b* at the coset-element a+A* is the same as its value at the element a, since the value of b* at the coset element A* is zero. Therefore

$$(a + A^*, b^*) = (a, b^*) \quad \text{and similarly}$$

$$(a + A^*, b^*_n) = (a, b^*_n) \quad \text{so that convergence in}$$

the one topology is identical with convergence in the other. To keep what has been said from becoming confusing it is important to bear in mind that while the values of the pairs of brackets which we have written above coincide, their interpretations are quite distinct: in one case the element b^* is an element of a group B, character-group of A, in the other it is an element of a group B^*, character group of a coset-group (residue-group) A mod A^*.

Theorem: A^* is the continuous character group of B mod B^*, our hypotheses being as above.

Here we shall have to be a bit more careful since the topology of B mod B^*, as residue group of B, is a little trickier. The topology which verifies our theorem is the one in which a sequence of (coset)-elements $b_n + B^*$ converges to the coset $b + B^*$ if and only $(a^*, b_n + B^*)$ converges to $(a^*, b + B^*)$ for every element a^* of the group A^*. The values of these brackets coincide, respectively, with the values (a^*, b_n) and (a^*, b) where we now regard b_n and b as definite elements of the group B, chosen from the corresponding cosets above. It is immaterial which elements b_n and b we choose from these cosets. Now if a sequence of cosets converge to a coset in the topology which we previously associated with the residue group B mod B^*, there necessarily exists a choice of an element from each coset such that these elements converge to the proper limit. Therefore the values (a^*, b_n) will converge to (a^*, b) (since we know that a^* is continuous over B) and therefore the cosets will converge in the topology which we want them to have. We must now show that if we have a sequence of cosets $(b_n + B^*)$ converging to $(b + B^*)$ in the sense that $(a^*, b_n + B^*)$ converges to $(a^*, b + B^*)$ for every element a^* of A^* then these cosets converge in the sense that there exists an element b'_n in $b_n + B^*$ and an element b' in $b + B^*$ such that b'_n converges to b'. For if this is the case, we shall know that the cosets also converge in their topology as a residue group (i.e. in this case, the corresponding distances between the cosets will go to zero).

To this end, let b_n and b represent arbitrary definite choices of elements out of these cosets. Consider the elements $b_n - b$. Since our group is compact we can suppose that a subsequence of these converges to a definite element $\bar{b}$ of B. Since the values $(a^*, b_n - b)$ goes to zero for every a^* of A^* we may conclude that $(a^*, \bar{b})$ is zero. But in this case we know that $\bar{b}$ must belong to B^*, for we have seen that if it did not there would be an element of A vanishing at B^*, therefore necessarily in A^*, but not vanishing at $\bar{b}$. Now we may take b'_n to be $b_n - \bar{b}$(for that subsequence of the n 's for which $b_n - b$ converges) and we see that (for this sequence) $b_n - \bar{b}$ converges to b. The new elements and the old belong to the same coset, since $\bar{b}$ was shown to be an element of B^*, and our theorem is proved.

We return now to the considerations with which we closed last hour's lecture. We have two groups A and A', discrete, and a definite group homomorphic mapping μ of A into A'. We do not suppose that all of A' is covered. In the application which we shall make the next hour the group A will be the group of k-chains and A' the group of (k-1)-chains and the mapping will correspond to the boundary operator. The mapping μ distinguishes for us a subgroup A_o of A: the subgroup of elements which are mapped into the identity element of A', and a subgroup A'_o of A': the elements a' such that for some element a, $\mu(a) = a'$. In this event, $\mu(a + a_o) = a'$, for every a_o in A_o. We associate with these groups their character groups B and B', properly topologised. We construct a mapping μ' of B' into B as follows: $\mu'(b') = b$ <u>if and only if</u>

$$(b, a) = (b', a')$$

where a is any element of A and $a' = \mu(a)$. We have shown last hour that this does define a mapping of all of B' into a part of B: i.e.

given any element b' of B' there exists one and only one element b of B satisfying the relation above. We see that there is a subgroup B_0 of B which this mapping emphasises: the set of elements, namely, such that there are ~~they have~~ elements of B' ~~which are~~ mapped into them. Similarly there is in B' the subgroup B'_0 of elements which are mapped into the zero element of B. Now we shall show that B_0 is the annihilator of A_0 and B'_0 the annihilator of A'_0.

Theorem: B'_0 is the annihilator of A'_0.

Since each element of B'_0 maps into the zero-element of B, each such element must have the value zero at all elements of A'_0: for the zero of B has the value zero at every element of A, and this must be the value of an element of B'_0 at every point of A' which comes from some a of A. That defines A'_0. On the other hand, every element of B'_0 vanishing on A'_0 defines a character b vanishing on all of A, namely the zero element of B. We see, by our first argument, that B'_0 belongs to the annihilator of A'_0, and by our second that it contains this annihilator. The theorem is proved.

Theorem: B_0 is the annihilator of A_0.

An element a_0 of A_0 is mapped into the zero of A'. The characters b' of B' have the value zero at zero, of course. Therefore the elements b into which they map must have that value at a_0. That is to say, B_0 must belong to the annihilator of A_0, since a_0 is any element of A_0. Conversely, suppose that b is any element of B vanishing on A_0. The element b gives rise to a character of A mod A_0, and therefore to a character of the group A'_0 which is isomorphic. We can extend this character from A'_0 to all of the group A': this defines for us at least one character b' of A' which we easily recognise as an element of B' which is mapped into our original element b. Therefore B_0 contains the annihilator of A_0, and by our first argument it must coincide with this.

We should perhaps have proven before this, in any event we shall do it now, <u>that the mapping μ' of B' into B is a continuous one.</u> I. e. if a sequence of elements b'_n of B' converge to an element b' then the corresponding elements b_n converge to an element <u>b</u> which is the map of b'. It is sufficient, since we are dealing with groups, to show that if ~~the~~ a sequence b'_n converges to zero, then the corresponding sequence b_n converges to zero (we use the word zero rather freely -- sometimes it is the value zero, as element of the group of reals mod 1, sometimes the zero of B, of B', of A, or of A'). This is quite trivial. For if b'_n converges to zero then for every element a' of A', (a', b'_n) converges to zero. But (a', b'_n) = (a, b_n) for that element b_n which is μ' (b'_n) and for every <u>a</u>, since every <u>a</u> maps into some <u>a</u>'. This means that b_n converges to zero, and our theorem is proved.

We can reverse the entire process which we outlined above. That is, we can start with the two groups B and B' and a continuous group homomorphism μ'(B') into B. ~~Thisxinducesxaxmapping ofxAxintoxA'xxwhichxwexmayxdenotexbyxµ*xxxThexmappingxisxobtained asxfollowsxxxletxa(b)xbexanyxelementxofxAxxLetxbxbexanyxelement ofxBxsuchxthatxa(b)x~~. We can show without difficulty that this determines a mapping of A into A', which we may designate by μ^*. We shall want to know that if the mapping μ' is the one which we had up above, then the mapping μ^* will coincide with the mapping μ with which we began the whole discussion.

First, a word about these continuous mappings. We have, as before, a subgroup B'_o of B' consisting of elements mapped into the zero of B and a subgroup B_o of B consisting of elements which are covered, in this mapping. We assert that both groups are closed. This is quite clear. For if we have a sequence of elements

b'_n of B', mapped into zero, and any limit element b' of this sequence, then b' must be mapped into zero because of the continuity. That proves that B'_o is closed. On the other hand if b_n is a sequence of elements of B each of them covered by some element of B' and if b is any limit of this sequence then we can find a sequence of elements $b'n$ such that $\mu'(b'_n) = b_n$. The sequence b'_n by the compactness of the group B' has at least one limit element b'. Now it is an immediate consequence of the continuity that $\mu'(b') = b$. Therefore both of these groups are closed (it is perhaps interesting to note that this fact was known to us in the previous situation from the fact that these groups could be identified with certain annihilators which we already knew to be closed, necessarily). The mapping μ' is really a continuous isomorphic mapping of B' mod B'_o into B_o. The linearity of the mapping, and the fact that B'_o is the group of all elements mapping into zero, assures us of the isomorphism. The continuity of this mapping is an easy consequence of the fact that the group B'_o (and therefore all of its cosets) is closed.

We determine the mapping μ^* of A into part of A' as follows. Let a be any element of A: it is a character $a(b)$ of the group B. It defines for us a character of the group B_o, where we merely restrict its range of definition. Therefore it is equivalent to a character of B' mod B'_o, which is isomorphic to B_o. This, in turn, defines a character of B': that character a' which is constant on cosets of B'_o in B' and has on each such coset the same value that it has on the corresponding coset-element of the residue group. Then μ^* is defined by $\mu^*(a) = a'$. We can express this, also, as follows: if a is given, and b denotes a variable element of B_o, then a' is determined by the relations

$(a, b) = (a', b')$, where b' is a variable element of B' but so chosen that $\mu'(b') = b$.

We have already seen that there is always one a', and only one, which satisfies this relation.

Now the mapping μ^* picks out a subgroup A^*_o of A and a subgroup A'^*_o of A': the group A^*_o is mapped into zero, the group A'^*_o is covered by the mapping.

Theorem: A^*_o is the annihilator of B_o.

If the element a of A is mapped into the zero of A' it must vanish on B_o. Otherwise it would determine for us a character of B_o not vanishing everywhere on B_o, and therefore a character of B' mod B'_o whose value was different from zero for at least one coset of this residue group. This would give rise to a character of B' not identically zero, contradicting our assumption on a. Therefore A^*_o belongs to the annihilator of B_o. On the other hand if an element a of A vanishes on B_o, it determines for us the zero-character of B_o, therefore of B' mod B'_o, therefore of B', and is mapped into zero. Therefore, finally, A^*_o coincides with the annihilator of B_o.

Theorem: A'^*_o is the annihilator of B'_o.

If we begin with any element a'_o of the annihilator of B'_o this must be constant on all the cosets of B'_o in B' (being zero on B'_o) and may be regarded as a character of B' mod B'_o. Then it determines a character of the isomorphic group B_o. This character may be extended to a character a of the entire group B (in general, in many different ways), and it is quite clear from our definition of the mapping μ^* that $a'_o = \mu^*(a)$. Therefore A'^*_o contains ~~the~~ annihilator of B'_o, since the element a'_o was shown to be "covered". On the other hand, it is clear from our definition of the mapping that the only elements a' which can have images are those which are constant on cosets of B'_o in B'. Since every character must have the value zero at zero, we see that all these characters must have the value zero at every element of B'_o. That is to say they must belong to the annihilator group of B'_o.

We see, finally, that if we begin with two discrete groups A and A' and a group mapping μ, and if we determine from μ the continuous mapping μ' of B' (character group of A') into B (the character group of A) and now from this mapping μ' determine a mapping μ^* of A into A', the mapping μ must coincide with the mapping μ^*.

Proof: Using the notation which we have followed throughout this lecture, we see that the mapping μ determines two groups A_o and A'_o respectively and the mapping μ' (induced by it) determines two groups B_o and B'_o which are the respective annihilators of these. Now the mapping μ^* induced by μ' determines two groups A^*_o and $A^{*\prime}_o$ which are the annihilators of B_o and B'_o respectively. Therefore, as we showed in an earlier lecture, A_o coincides with A^*_o and A'_o with A'^*_o. We observed, before, that the mapping μ was an isomorphic mapping of A mod A_o into A'_o. It is quite easy to see that the same holds for μ^*, so that the mappings μ and μ^* coincide.

In the application which we shall make of this theory, the next hour, the groups A and A' will be the groups of i- and of i-1 -chains and the mapping μ will correspond to the β-operator. The character groups B' and B will be the groups of dual i-1 - and dual i-chains, and the mapping μ' will correspond to the δ-operator.

Lecture thirteen.

We return to the connectivity theory of complexes. We shall find it convenient to consider more general complexes than those which we have defined, and used before. We shall define an abstract complex by the conditions that it

1) contains a countable number of elements which we call cells, we think of them as undefined terms;

2) each cell has ascribed to it a finite dimensionality, n. there need be no cell of highest dimensionality in a given complex.

3) given two cells C^n and C^{n-1}, i.e. of dimensions which are immediately adjacent, we suppose that an incidence number is associated with this pair:

$[C^n, C^{n-1}]$, which has one of the three values 0, ± 1

When the incidence number is different from zero we say that C^{n-1} is on the boundary of C^n.

4) Given any C^n there exists a finite number at most of C^{n-1}'s which are on its boundary.

5) the incidence numbers satisfy the relation

$$\sum_i [C^{n+1}, C^n_i] \cdot [C^n_i, C^{n-1}] = 0 , \quad \text{for every}$$

fixed pair C^{n+1}, C^{n-1}, the summation being taken over all the distinct n-cells. This is a weaker form of the incidence lemma which we proved for our old complexes: there we saw, that as a matter of fact there were only two n-cells for which both of the incidence numbers involved could fail to be zero together and this sum really was zero.

These arbitrary complexes will do for the moment, we shall really want to work with others which we shall define presently.

Given an abstract complex K and a definite abelian group A, we construct a connectivity theory for the complex and the group A. It is important to realize that the connectivity groups of a complex depend very essentially on the group of coefficients. In the theory which we shall develop we shall use the elements of the group A to form the group of chains. We shall use the character group B of A to form the groups of dual chains.

An n-chain: is defined by assigning a definite element of A to each n-cell of the complex. However, for each n-chain, all but a finite number of the cells are associated with the zero-element of A. Essentially, then, we are concerned with the groups of finite chains. We represent a chain K^n by the symbol:

$K^n = \sum_i a_i C_i{}^n$, the a_i are elements of A all but a finite number of which are zero, and the $C^n{}_i$ are the distinct n-cells of the complex K. We add two such chains by adding the corresponding coefficients. It is quite clear that these chains determine a group which we may denote by Γ^n. Now Γ^n is obviously the direct sum of a set of groups each of them isomorphic to the group A. There are as many summands as there are distinct cells of the complex K.

A dual n-chain: is defined as a function which has at each n-cell of the complex a definite value, this value being an element of the character group B of A. A dual n-chain may be represented as an infinite form:

$\bar{K}^n = \sum_i b_i \bar{C}^n{}_i$, this being regarded as a formal sum taken over all the distinct n-cells of the complex (we now think of them as dual n-cells, using the barred symbol) the coefficients being elements of B. These coefficients are unrestricted: i.e. all of them, for example, may be distinct from the zero element of B.

the other: it can be expressed as a duality between all <u>continuous</u> characters in each case, when proper topologies are assigned

In order to define <u>all</u> characters of an "unrestricted" direct sum it is necessary to find a "discrete" basis — i.e. a basis (<u>all</u> the elements, e.g.) such that each element of the group is a <u>finite</u> sum of base elements

Of course, each element of A is a character of the unrestricted B, and A itself a subgroup, dense in a suitable topology, of <u>all</u> characters of unrestricted B.

p 13.3

Let us say that a direct sum $A = A_1 \oplus A_2 \oplus \cdots$ is "restricted" if we allow only finite sums $a_1 + a_2 + \cdots$ and "unrestricted" if we allow infinite (formal) sums $a_1 + a_2 + a_3 + \cdots$.

If B_i is the character group of A_i, then the character group B of the restricted direct sum A is the unrestricted direct sum $B_1 \oplus B_2 \oplus \cdots$. If A is unrestricted ... These do not get into difficulties of convergence. ... arise when we consider the continuous characters of the unrestricted direct sum B because of the fact that, if $X(b)$ is such, there exists n such that the value of $X(b)$ depends only on the first n terms in the infinite sum $b = b_1 + b_2 + b_3 + \cdots$

Any comments?

essentially correct - i.e. for countable ... otherwise some n terms.

The duality ... between all ... on the one hand and all continuous ...

We add dual n-chains by forming the sum whose coefficients are the sums of corresponding coefficients (as elements of B). In this way we have defined a group $\overline{\Gamma}^n$ of dual n-chains. Itis clear that the group $\overline{\Gamma}^n$ is the direct sum of groups isomorphic to B, and there are as many summands as there distinct cells C^n.

It is easy to prove that the groups Γ^n and $\overline{\Gamma}^n$ are character groups of each other. As for the matter of the topology of $\overline{\Gamma}^n$ we may remark for the moment that it is to be topologized as a direct sum of the topologized groups B in much the same way as we topologized the infinite toral group as a direct sum of the constituent "circle"-groups. We shall return to this later. That $\overline{\Gamma}^n$ is the character group of Γ^n is clear when we consider that a character of this group is defined as soon as we know its value at every coordinate of an element of the group. Now the coordinates of the elements of Γ^n are simply the elements of A, the mth one corresponding to the definite n-cell C^n_m. The value which a character of Γ^n has at the m-th coordinate of one of its elements is, of course, the m-th coordinate of that character. It is clear that this coordinate must be a character of the first one, i.e.it must be an element of B.

We may therefore write a dual n-chain as a form

$$\overline{K}^n = \sum_i b_i(a)\overline{C}^n_i .$$

Here we have merely emphasized the fact that the elements b_i are actually to be regarded as characters of A, i.e. as ~~certain kinds of~~ linear, etc. functions of the variable element a of A. Now as far as the groups Γ^n and $\overline{\Gamma}^n$ are concerned the complex K has figured only through the number of distinct n-cells which it contains, and we havent yet any "connectivity theory" whatever. This appears only when we consider the set of these groups for all n and what is more to the point the mappings of these groups upon each other.

For these mappings are determined by the incidence relations of the n- and (n-1)-cells which, of course, exhibit the <u>structure</u> of the complex K.

So far as the groups Γ^n and Γ^n are concerned we can wash out the complex K, and think of an n-chain as an element α with coordinates : $(a_1, a_2, \ldots, a_m, \ldots)$ where these are elements of A, all but a finite number of them zero. The number of coordinates is the number of n-cells of K. A dual n-chain is then an element with coordinates $(b_1, b_2, \ldots, \ldots)$ where each coordinate (none of them <u>necessarily</u> zero) may be interpreted as a character of the group Γ^n with the property that

$$(b_m, \alpha) = (b_m, a_m).$$

I.e. b_m is a character whose value depends on the m-th coordinate alone, and has there the value which it should have as a character of the group A. We can now define a Stokes' product for chains and dual-chains, as before,

$$(K^n, K^n) = \sum_m (b_m, a_m).$$

Notice that this is well-defined because it is essentially a <u>finite</u> sum since at most a finite number of the a_m's can be different from zero (the zero element of A).

Now let us consider, in the complex K, the linear operator β. Before, when we had integer coefficients, we expressed the boundary of an n-~~chain~~ cell C^n as follows:

$$\beta C^n_i = \sum_j \left[C^n_i, C^{n-1}_j\right] C^{n-1}_j, \text{ or in the equivalent}$$

form

$$C^n_i \rightarrow \sum_j \left[C^n_i, C^{n-1}_j\right] C^{n-1}_j .$$

Now in our theory a chain is by definition a <u>weighted</u> symbol (a formal sum of such symbols, really) and these "weights" are elements of A. Unless the group A happens to contain a unit element correspond-ing to the number 1, we cannot give any meaning to the formula above.

We can, however, interpret it formally and understand by it that for any arbitrary coefficient $\underline{a}$

$$\underline{a}C^n_i \longrightarrow \sum_j \underline{a}\left[C^n_i,\ C^{n-1}_j\right]C^{n-1}_j\ .$$

Now it is of importance to appreciate the following application of our earlier lecture on the mappings of discrete groups and the induced mappings on their character groups.

Let Γ^n and Γ^{n-1} denote the n-th and (n-1)-th groups of chains, $\bar{\Gamma}^n$ and $\bar{\Gamma}^{n-1}$ their character groups (i.e. the groups of dual chains). We shall want to show that if we map the group Γ^n upon the group Γ^{n-1} by means of the operator β(boundary), then the mapping which this induces between $\bar{\Gamma}^{n-1}$ and $\bar{\Gamma}^n$ is precisely the δ-operator.

For, in order to determine the mapping induced by β, we have to show how given a dual-chain $\bar{C}^{n-1}$ (this is a character of Γ^{n-1}) we can determine the dual n-chain $\bar{C}^n$ (this is a character of Γ^n) into which it is to be mapped. But this dual n-chain is determined as the one and only X satisfying the relation:

$$(\bar{C}^{n-1}\ ,\ \beta C^n)\ =\ (\ X,\ C^n).$$

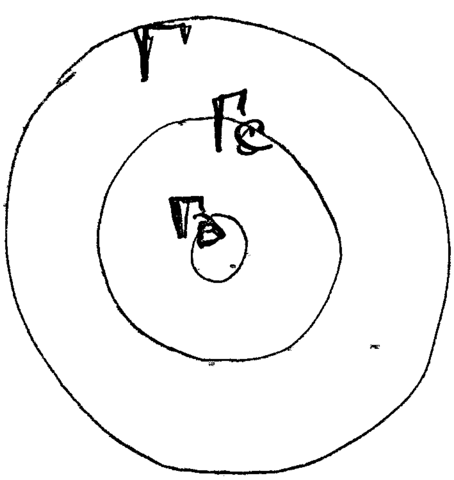
Here X represents for us the desired dual n-chain which we may write as $\delta\bar{C}^{n-1}$, for the moment, allowing that it is perhaps not our original δ-operator. But we know that the mapping operator is uniquely defined by these relations (we proved all of that in our previous lecture). We also know that the original δ-operator of our connectivity theory satisfies this relation: this is simply the Stokes' theorem which we proved some time ago. Therefore the mapping operator is the δ-operator of our theory. The proof of this Stokes' theorem did not depend on the fact that we seemed to be using integral coefficients.

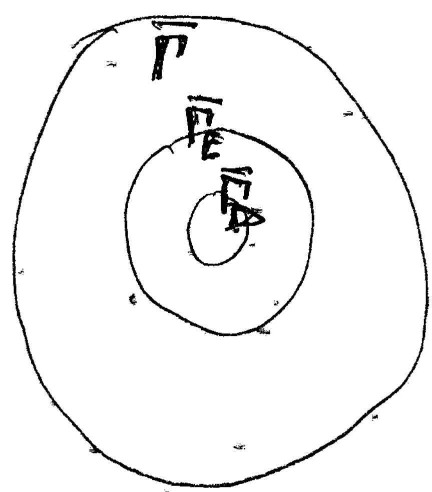
We should remark that the identity

$$(\bar{C}^{n-1}\ ,\ \beta C^n)\ =\ (\delta\bar{C}^{n-1}\ ,\ C^n)$$

is to be regarded as a formal one in the following sense. We may replace the symbols C^n, etc., by any chains (dual chains,..) of our complex K; but a single cell, C^n_i, for example, will not be chain in general and we shall have to understand the formula to hold when this cell is taken with any coefficient which belongs to the group A. With such a coefficient it is a chain.

We now want to prove that the n-th connectivity group and the dual n-th connectivity group are character groups of each other. It will be convenient to drop the superscript n, which is to be understood throughout: thus we write Γ for Γ^n, the group of n-chains. The group Γ contains the subgroup Γ_C of closed n-chains, and this contains the subgroup Γ_B of bounding n-chains. The n-th connectivity group $\Delta \ominus \Gamma_C$ mod. Γ_B. The n-th dual group $\bar{\Gamma}$ contains the subgroup $\bar{\Gamma}_E$ of exact chains, and this the subgroup $\bar{\Gamma}_D$ of derived chains. The n-th dual connectivity group $\bar{\Delta} \ominus \bar{\Gamma}_E$ mod. $\bar{\Gamma}_D$. We have to prove, then, that Δ and $\bar{\Delta}$ are each other's character groups. We shall show this, neglecting for the moment the consideration of the topology of $\bar{\Delta}$ (although it is clear what this topology will be when the dual n-chain group $\bar{\Gamma}$ has its appropriate topology).





We might remark at this point a difference between the theorem we are about to prove and the considerations of our previous lecture. There we had two groups A and A' and their character groups. Here we really have <u>three</u> groups and their character groups. We have the group of n-chains, the subgroup of closed n-chains (those which are mapped into the zero element of the group of n-1 chains, i.e. their boundary is the zero n-1 chain) and the group of bounding n-chains (these are the ones <u>into</u> which are mapped the group of n+1 -chains).

We prove, first, that Γ_C is the annihilator of $\overline{\Gamma}_D$. This is simply the part of our Stokes' theorem which tells us that the integral of a derived chain over a closed chain is zero. Or, we can derive this relation by observing that if we think of the group of n-chains as the group A of our earlier lecture, and the group of (n-1)-chains as the group A', then Γ_C corresponds to the group A_o. On this interpretation, the group of dual n-chains is the group B, and of dual (n-1)-chains the group B' and, finally, the group $\overline{\Gamma}_D$ is the group B_o. We have already proved that Ao and B_o are each other's annihilators.

We notice, secondly, that Γ_B is the annihilator of $\overline{\Gamma}_E$. This is the part of our Stokes' theorem which tells us that the integral of an exact chain over a bounding chain vanishes. Or, we remark that we have already proved it if we interpret the group A of earlier story to be the group of <u>(n+1)-chains</u> and the group A' to be the group of <u>n-chains.</u> In this case the group Γ_B becomes the group A'_o and the group $\overline{\Gamma}_E$ becomes the group B'_o.

We are now ready to prove that $\overline{\Delta}$ is the character group of Δ. Consider an arbitrary character on $\Delta \ominus \Gamma_C$ mod. Γ_B. This is a linear function constant on the cosets of Γ_B in Γ_C, and vanishing of course on Γ_B. We may interpret it, at once, as a character of the group Γ_C (i.e. consider it as defined not for the entire coset-element merely, but for each element of the coset, therefore for each element of the group). Now we can extend this character from the group Γ_C to the group Γ (in many ways). The resulting character is therefore an element of $\overline{\Gamma}$. On the other hand, since it vanishes on Γ_B it must belong to $\overline{\Gamma}_{\cancel{D}E}$, the annihilator of Γ_B. We have the result then, that every character of Δ may be interpreted as an element of $\overline{\Gamma}_{\cancel{D}E}$. We see, at once, that the elements of $\overline{\Gamma}_{\cancel{D}E}$ cannot be regarded as all of them distinct characters of Δ. For two elements of $\overline{\Gamma}_{\cancel{D}E}$ belonging to the same coset of $\overline{\Gamma}_{\cancel{E}D}$ in $\overline{\Gamma}_{\cancel{D}E}$ determine ? same character of Γ_C and therefore of Δ and conversely. That is, two elements of $\overline{\Gamma}_{\cancel{D}E}$ define the same character of Δ if and only if their difference vanishes on Γ_C, in other words if their difference belongs to the annihilator $\overline{\Gamma}_{\cancel{E}D}$ of Γ_C. But this concludes the proof of our theorem.

For the further development of our theory we shall want to restrict ourselves to complexes made up of simplexes. We shall therefore make a fresh start and define a Symbolic Complex of Simplexes. We define this by giving ourselves an arbitrary countable collection of "symbolic" vertices (after a while we shall forget to call them symbolic and no harm will be done) which we may denote by the "marks" $x_1, x_2, \ldots, x_n, \ldots\ldots \ldots$.
We shall say that an n-simplex (symbolic) is an arbitrary set of n + 1 distinct vertices of this set: the order of these vertices is immaterial, the set of them defining an absolute n-simplex

$$|x_{i_0} x_{i_1} \cdots x_{i_n}|.$$

(If you want to have a picture of an abstract or symbolic n-simplex you just take a picture of a real one).

We shall say that a k-simplex S^k is a k-component of an n-simplex S^n if the vertices which define S^k are chosen from among the vertices defining S^n (in a geometric situation, S^k is a face of S^n).

Now we shall define a symbolic complex as an arbitrary collection of simplexes from among our "symbolic simplexes" such that if an n-simplex is present in the collection then every one of its k-components is present, for all k.

Now if we arrange the vertices of a simplex in a definite order we shall find that the possible orderings fall into two distinct classes: the members of the same class are related to each other by even permutations, and the members of opposing classes are related to each other by odd permutation. If we choose one of these classes as the positive one we have an orientation of the given simplex. We may represent the positive class, indifferently, by any one of its members and we then represent the negative class by prefixing a minus sign to one of the positive elements.

Let us suppose now that we have associated some definite orientation with each of the n-simplexes of our complex. An n-simplex will now be denoted by an ordered set of n+1 vertices; two different orderings equivalent modulo even permutations will be regarded as the same simplex, or better the same oriented simplex. Now, again, to define chains we choose a coefficient group A and regard a chain (without restriction to being composed of simplexes all of the same dimension) as an assignment of elements of A to the positively oriented simplexes, where a finite number of these simplexes at most have an element different from zero assigned to them. We shall then understand that we assign to the negatively oriented simplexes the negative value to the one given their positive orientations, so that $\underline{a}(-C) = -\underline{a}C$, where C is a simplex.

Lecture 14

Complex of Simplexes:

We suppose that we have a set of distinct marks, $x_0, x_1, \ldots,$ which we shall designate as vertices (symbolic vertices). It is not necessary ~~to suppose~~ that they ~~are~~ be countable.

We define an i-simplex as an arbitrary set of i+1 distinct vertices of our system:

important. with a [illegible]

$$C^i \equiv \left| x_{s_0} x_{s_1} \cdots x_{s_i} \right| .$$

For the purposes of our formalism it is convenient to represent this simplex as a "product" of the vertices. By the bars we mean to imply that the simplex is regarded, here, as unoriented (absolute), ~~and~~ so that the order of ~~terms~~ vertices in the product is not essential. [It is to be understood that in distinction to the case of geometric complexes a set of distinct vertices determine ~~precisely~~ one absolute simplex.]

By a component of a simplex C^i we shall understand a simplex C^j, $j < i$ necessarily, whose vertices are contained among the vertices of the simplex C^i. Thus if $C^2 = |x_1 x_2 x_3|$, its components are $|x_3 x_1|$, $|x_2|$, etc.

The set of symbolic vertices, above, is supposed given for the entire theory. So also are the innumerable simplexes which are determined by these vertices, ~~in distinct finite sets~~.

We may now define a Complex as an arbitrary collection of simplexes (from this fixed system) such that if a simplex C^i appears in our complex then all of its components appear in the complex. (This is like the condition of closedness which we required of our geometric complexes.)

Let K be a fixed complex, and x_α one of its vertices. By the Star of x_α, which we may denote by S_α, we shall understand the set of all simplexes of K which contain the vertex x_α.

Observe that the star S_α is <u>not</u> a complex, in the sense of our definition, because it contains simplexes without containing all the components of these simplexes. Thus if $|x_\alpha x_\beta x_\gamma|$ is one of the simplexes of K, S_α contains this simplex as well as $|x_\alpha x_\beta|$, $|x_\alpha x_\gamma|$, and (of course) $|x_\alpha|$ itself (the absolute sign around a vertex is fairly meaningless). However, S_α does <u>not</u> contain x_β, x_γ, or $|x_\beta x_\gamma|$.

By the <u>closure</u> $\bar{S}_\alpha$ of the star S_α we shall understand the set of all simplexes of this star <u>and</u> of all simplexes which are components of these. By the <u>Shell</u> S^*_α we understand the set of simplexes belonging to the closure of the star but not to the star itself. Thus

$$\bar{S}_\alpha = S_\alpha + S^*_\alpha .$$

It is understood that <u>star</u>, <u>closure</u>, <u>shell</u> are defined for <u>all</u> vertices of the complex K. It is to be borne in mind, always, that the complex K is a subsystem of the set of all simplexes made up of our symbolic vertices, so that from the fact that K contains two vertices it does not follow that it contains the 1-simplex determined by these (symbolically). regarded, later, as <u>zero</u> in K.

We shall find it necessary to orient the simplexes of K in order to study the linear operators β and δ. However this orientation is not essential to the definition of <u>chain</u>. To define a <u>chain</u> of the complex K we merely need a group of coefficients. Let us choose a quite arbitrary group of coefficients A, abelian, and let us attach (for convenience) a definite symbol C^i_α to each i-simplex of K, for all i. (This is merely a shorthand for the product of the vertices corresponding to the simplex.) Now an A chain is ~~an~~ a ~~arbitrary~~ linear form

$$\sum a_\alpha C^i_\alpha, \text{ with } \underline{\text{coefficients } a_\alpha \text{ in } A}.$$

It will be important for us to distinguish four distinct types of i-chains.

1) Finite i-chains: these are chains in which at most a finite number of the coefficients a_α are different from zero (zero, the identity element of A).

2) Locally finite chains: those where infinitely many coefficients may be different from zero but for every star the number of simplexes of that star which appear in the chain with non-zero coefficient is finite.

one vertex,

3) Semi-infinite chains: here the defining condition is that there shall exist a finite set of vertices such that every i-simplex of the i-chain meets at least one of these vertices. The number of these vertices is not restricted, except to be finite, so that the sum of two such i-chains will be another one.

4) Infinite, or general, chains: these are quite arbitrary linear forms, and all coefficients may be different from zero.

Now in order to study these chains under our linear operators we shall want to introduce a conventionof orientation on the simplexes. If we consider any fixed simplex and the set of all possible orderings of the vertices we find that these fall into two classes, the members of one class being related by an even permutation. We fix on one of these classes, calling it the positive orientation of the simplex. The other class becomes the negative one. We can represent the oriented simplex by an element of the positive class or by minus any element of the negative class. Thus if $|x_1x_2x_3|$ is a simplex C^2 of K, and $x_1x_2x_3$ is a "positive" ordering, we may represent $+C^2$ (that is, the positively oriented simplex) by $x_1x_2x_3$ or $x_2x_3x_1$, etc., or $-x_2x_1x_3$, etc.

~~We recall that a single simplex need not be a chain, is not a chain in fact until we associate with it a coefficient taken from the group A of coefficients.~~ Therefore, if the group A contains no element corresponding to a ~~unit-element a simplex is not a chain and minus a simplex, i.e. $-C^i$ is not a chain.~~ We shall make a convention for chains of oriented simplexes which is formally equivalent to identifying the symbolic minus of orientation with the symbol minus for the inverse of a group element of A. Thus ~~If~~ $\underline{a}C^i$ is a chain which consists in associating the element $\underline{a}$ with the positively oriented simplex C^i, $-\underline{a}C^i$ consists in associating the element $-\underline{a}$ with this simplex. We shall understand that $-\underline{a}C^i$ is to be identified with the chain $\underline{a}(-C^i)$ which associates the coefficient $\underline{a}$ with the negatively oriented simplex. Or, what is equivalent, we can redefine chains for oriented simplexes by the restriction that a chain associates an element of A with each oriented simplex but always so ~~thatxinversexelementxarexassociatedxwithxoppositelyxorientedxxx simplexes~~ that the element associated with a positively oriented simplex is the inverse (in the group, i.e. the minus or negative) of the element associated with the negatively oriented one. (It is because of this convention, in fact, that it is quite immaterial to the connectivity theory precisely what orientations are decided on for each simplex: the resulting groups, rings (we shall have them later) etc., are all isomorphic.)

We can now define the incidence relation between two oriented simplexes. The incidence number $[C^i, C^j]$ will be zero, plus or minus 1, always. It will be zero unless one of these simplexes is a component of the other and of dimension one lower. In this case we may write the number as $[C^i, C^{i+1}]$. If this is not to be zero the vertices of C^i must be contained among the vertices of C^{i+1}, in fact must consist of all but one of them.

To define the incidence, let us suppose for convenience that we are considering here the positively oriented ~~simplexes.~~ Suppose that $x_{s_0}x_{s_1}\cdots x_{s_i}$ is the positively ordered C^i, and suppose that $x_{s_{i+1}}$ is the remaining vertex of C^{i+1}. Then if

$$x_{s_{i+1}}x_{s_0}x_{s_1}\cdots x_{s_i}$$

is the positively ordered C^{i+1}, we shall say that the incidence number is + 1. If, however, this is $-C^{i+1}$ we shall say that the incidence is -1. [We can describe this orientation business a little more systematically, perhaps, as follows. Suppose $x_{s_0}x_{s_1}\cdots x_{s_i}$ is a positive C^i. If we drop the first vertex, the resulting C^{i-1} is positively incident; if we drop the second negatively incident, and in general if we drop the jth vertex the resulting C^{i-1} is $(-1)^{j+1}$ -incident.]

Let us consider our different types of chains and our two operators. The β-operator converts an i-chain into an (i-1)-chain. It associates to each (i-1)-cell C^{i-1} the sum of all the coefficients of the i-cells, each of them multiplied by the corresponding incidence number. Now a given (i-1)-simplex may be ~~on the boundary of infinitely many i-cells; more correctly it may be~~ positively or negatively incident with infinitely many such i-cells. If the i-chain were infinite, or even semi-infinite, the (i-1)-chain could not ~~, in general,~~ be defined since some coefficients would not be defined. These coefficients would be infinite sums of elements of A and, in general, without meaning. We see, then, that the range of application of the β-operator is limited to the locally finite chains at best. Of course, these include the finite chains.

The β-operator does apply to locally finite chains. Suppose that K^i is a locally finite i-chain and C^{i-1} an arbitrary (i-1)-simplex. There are at most a finite number of i-simplexes actually occurring in K^i (i.e. with non-zero coefficient) which are incident with C^{i-1}. Otherwise, since the number of vertices in

C^{i-1} is finite (i, in fact) there would have to be infinitely many i-simplexes occurring in K^i which contained the same one vertex of C^{i-1}. But then all of these would have to belong to the star of that vertex, contradicting the definition of locally finite chain. We see, in fact, that the β-operator converts a locally finite i-chain into a locally finite (i-1)-chain. For if S_α is any star of the complex K (associated, then, with a definite vertex x_α depending on the star) there are at most a finite number of i-cells actually occurring in the i-chain which belong to this star. Therefore there will be an at most finite set of (i-1)-~~cells~~ simplexes of this star which are relevantly incident (that is, plus or minus incident) with these ~~xxxxx~~ i-simplexes and these will be the only ones containing the vertex x_α which can possibly have non-zero coefficients in the associated (i-1)-chain. It is clear, further, that β converts a finite i-chain into a finite (i-1)-chain. For if you consider any finite collection of i-simplexes there are at most (i+1) times as many (i-1)-simplexes which are relevantly incident with them, and these are the only ones which will have non-zero coefficients in the associated (i-1)-chain.

Now the δ-operator applies to the locally finite and to the finite chains, also. However it converts them, in general into infinite (i+1)-chains, or semi-infinite ones. Therefore we shall not consider the δ-operator as ~~applying to~~ acting on locally finite chains. That the δ-operator does apply to general chains is quite obvious. For if we have an i-chain K^i and want the value of the δ-chain at a simplex C^{i+1} we merely have to consider the coefficients associated with the finite number (i+2) of i-simplexes which are incident with it, and to add these ~~xxxxxxx~~ coefficients (some of them may be zero) after we have multiplied each of them by the appropriate incidence number. Therefore, each C^{i+1} has associated with it a definite element of A and this association defines the desired (i+1)-chain.

If we consider a semi-infinite i-chain and apply the δ-operator we shall get a semi-infinite (i+1)-chain. For the only (i+1)-simplexes which will actually occur in the new chain will be such as were incident ~~(really incident)~~ with one of the i-simplexes of the original i-chain. Therefore ~~it~~ each of these (i+1)-simplexes will contain at least one of the defining set of vertices, since it will contain all the vertices of each i-~~cell~~ simplex incident with it. Therefore each of our two last types of chains have the δ-operator as their natural "manipulator".

We shall find that there are natural topologies associated with these different types of chains such that the finite i-chains and the general i-chains are dual and such that the finite i-chains are dense in the locally finite i-chains, and the semi-infinite i-chains are dense in the infinite i-chains.

We shall begin a systematic study of the general i-chains under the δ-operator.

We consider our complex K, all of its ~~cells~~ simplexes oriented (here as elsewhere the word cell is being used, by the recorder, where the word simplex should be used: there are no "cells"). We shall represent each simplex by its ordered vertices and we shall represent a chain, an i-chain, as

$$\sum a_{s_0 s_1 \ldots s_i} x_{s_0} x_{s_1} \cdots x_{s_i} \quad .$$

The coefficients are elements of a ring A (we shall have more to say about this in a moment). We must emphasize the following convention. We suppose that each absolute i-simplex ~~occurs~~ is represented exactly once in Σ ~~, in the chain,~~ (possibly with zero coefficient). Now it is clear ~~that~~ the same set of vertices ~~might~~ may occur many times in different orderings. In this case we shall suppose one definite ordering is fixed on for this simplex and shall order all of ~~them~~ these "products" the same way. When this ordering is equivalent to an odd permutation we shall change the sign of the coefficient, as

...write ...a Σ ...om, or ...of ...'s both ...ying our ...tions we are ~~entitled~~ obliged to. Then we shall add up all of the corresponding terms, that is we shall add up their coefficients as elements of A. In this way our chain Σ is brought into conformity with our conventions. Thus if our chain contains a sum:

$$\underline{a}x_1x_2x_3 + \underline{b}x_1x_3x_2 - \underline{c}x_2x_3x_1$$

we shall have replaced it by

$$(\underline{a} - \underline{b} - \underline{c})x_1x_2x_3.$$

We understand that all chains have been contracted in this way. This agreement to collect terms makes it unnecessary for us to require skew-symmetry in the coefficients $a_{s_0 \ldots s_i}$ which would have amounted to the same thing except for certain numerical coefficients to allow for duplications. It will also be convenient to allow such a chain to contain a formal product of vertices to which no simplex pertains in K: there is always a symbolic simplex if the vertices are distinct. We shall even allow such a product to have non-zero coefficients. Only we shall agree to regard the whole term, coefficient and all, as zero. That is, it is purely a dummy term which we can strike out if it suits us. The point in introducing it is that when we come to multiply chains formally such terms will actually appear: but they will be zero by definition.

We know how to add i-chains and how to add arbitrary chains so as to form an abelian group of them. We shall find it important to multiply them: we shall find that they form a ring which is, in general, a stronger invariant than the connectivity groups. We shall define two distinct types of products of chains: in particular the product of an i-chain by a j-chain. The first type will be the easier to formulate, but the less interesting. However, we shall need it for the second type and it will be instructive. For this purpose we need not a group of coefficients, but a ring of coefficients. We do not require that the ring be commutative (a ring, it will be recalled is a system of elements forming an abelian group under an operation called additinn and in which another operation called multiplication is defined, not in general commutative: the ring need not contain a unit, etc.). We do not need to lose generality, if we wish. We can make any abelian group ~~to~~ a ring by defining the product of two elements as zero. In this case the correspn nding connectivity rings which we shall define presently will reduce to the underlying connectivity groups. The theory becomes interesting, of course, when the ring is not trivial in the sense above.

Now, let

$$\varphi = \sum a_{s_0 s_1 \ldots s_i} x_{s_0} \ldots x_{s_i}$$

be an arbitrary i-chain and

$$\psi = \sum b_{t_0 t_1 \ldots t_j} x_{t_0} \ldots x_{t_j}$$

an arbitrary j-chain.

We shall define the product as

$$\varphi\psi = \sum a_{s_0 \ldots s_i} b_{t_0 \ldots t_j} x_{s_0} \ldots x_{s_i} x_{t_0} \ldots x_{t_j}$$

where we collect terms under one coefficient, which differ by a permutation of the vertices. We must justify this definition. (Observe, ~~first~~, that the product as we have defined it is not commutative). We get ~~it~~ the product by writing down the vertices of a simplex in our i-chain and following this by the vertices in a simplex of our j-chain. The coefficient of this term in the product is the corresponding ordered product of the coefficients in their ring. There is no difficulty about collecting terms because an (i+j+1)-simplex can be represented as the formal product of an i-simplex and a j-simplex in a finite number of distinct ways, so that the coefficient which such a simplex acquires is necessarily a finite sum, and therefore defined.

Take the example that our complex K is a single

3-simplex, with vertices x_1, x_2, x_3, x_4, and all the simplexes determined by them. Let a, b, etc. denote the elements of some ring of coefficients. Let A be the 1-chain $ax_1x_2 + bx_2x_3$ and let B the the 1-chain $cx_3x_4 + dx_1x_4$. Then the product chain A.B, when we collect terms, will be

$$(ac + bd)x_1x_2x_3x_4 \ .$$

The other terms will be zero by the convention that any product of vertices in which a vertex is repeated is ~~identically~~ zero. This convention is highly important and must be borne in mind, always. It is to be understood, and should have been remarked, in the definition of i-chain: there too a term is to be zero if it contains a repeated vertex, and also if there is no simplex in the complex K which corresponds to the given set of vertices.

We notice that in this definition of multiplication the product of an i-chain by a j-chain is always an i+j+1 -chain: it may, of course, vanish ~~identically~~ (in view of our conventions above) even when the original chains do not. (We have a ring with zero-divisors) This ring will not be the signifcant one for us. We can make an algebra of chains as follows. We define a chain without reference to its dimension as a sum of chains of arbitrary dimensions: it is clear that they will form a group. We can always write such a chain as

$$\varphi = \varphi_0 + \varphi_1 + \cdots + \varphi_m + \cdots$$

where each φ_i is an i-chain. Now we define the product of two such chains in a natural manner, distributively, and get another chain of the same kind. While this algebra has the advantages of permitting us to formulate our theorems concisely it is, in a sense, fraudulent because it shuffles up the various dimensions and loses the significance of these dimensions. These are highly important to our linear operators.

We should have remarked in connection with out definition of a product that it satisfies the associative law. It is quite easy to verify this because the formal product of three simplexes, i-, j-, and k-, respectively will certainly be a symbolic (i+j+k+2)-simplex in whatever "association" we take them. The coefficients of our ring multiply associatively by definition. Whether this formal simplex vanishes or not is also independent of any "assocativity" of our multiplication.

<u>The δ-operator as symbolic chain.</u>

We can see, easily enough, that the δ-operator may be "identified" with the symbolic chain $\sum x_\alpha$, summed over the vertices of the complex:

$$\delta \equiv \sum x_\alpha .$$

We have to prove that

$$\delta\varphi = \sum x_\alpha \sum a_{s_0 \ldots s_i} x_{s_0} \cdots x_{s_i} ,$$ where this product is defined exactly as we defined the product of chains above. Observe that although δ corresponds to a symbolic chain, the product will be an actual chain since all the terms of this product will have appropriate coefficients out of our ring A. To prove this theorem we have merely to recall what $\delta\varphi$ comes to: It is an (i+1)-chain whose value on a given (i+1)-simplex $x_{s_0} x_{s_1} \cdots x_{s_{i+1}}$ is obtained by summing the values which φ has on the <u>i-simplexes incident</u> with it multiplied by the corresponding incidence numbers. We can write this as an expanded sum $\sum x_{s_j} a_{s_0 \ldots s_{j-1} s_{j+1} \ldots s_{i+1}} x_{s_0} \cdots x_{s_{j-1}} x_{s_{j+1}} \cdots x_{s_{i+1}}$

where j has one of the values 0 to (i+1). In this clumsy enough looking expression we have simply lifted one of the vertices x_{s_j} out of the (i+1)-simplex, writing it out front, and associated with the remaining i-simplex the coefficient $a_{s_0 \cdots s_{j-1} s_{j+1} \cdots s_{i+1}}$ which it has in the i-chain φ. It is clear that these are exactly the terms in the formal product $\delta\varphi$ which contract to give the desired (i+1)-simplex and its proper coefficient.

The important property of our δ-operator that $\delta^2 = 0$, is very easily verified in terms of this formal product. For δ as a formal chain (symbolic) is $\sum x_\alpha \sum x_\beta$, the summation in each case being over all the vertices of the complex K. Now a term in which a vertex is repeated vanishes. On the other hand, if x_α and x_β are distinct vertices, this formal product will contain a term $x_\alpha x_\beta$ and a term $x_\beta x_\alpha$. These two, when we multiply them by any other collection of vertices, will cancel each other:

symbolically, $x_\alpha x_\beta + x_\beta x_\alpha = x_\alpha x_\beta - x_\alpha x_\beta = 0$.

As before, we define an <u>exact</u> chain φ by the condition

$$\delta\varphi = 0.$$

We define a <u>derived</u> chain φ by the condition that there <u>exist</u> a chain ψ such that $\varphi = \delta\psi$.

Our earlier theorem that <u>derived chains are exact</u> is now automatic: we multiply the equation above, head on, and the right hand side vanishes because of the term δ^2.

Now if we consider the ring of all chains (these being sums of chains of all dimensions) it is easy to show that the exact chains form an ideal in this ring and that the derived chains form a subideal. None the less, the ring is not of great interest because it reduces to the underlying connectivity groups as we shall see presently. We shall want a quite different sort of product.

Theorem: If φ is an <u>exact</u> chain and ψ an arbitrary one, both products $\varphi\psi$ and $\psi\varphi$ are exact.

We have to prove that

$$\delta(\varphi\psi) = 0 = \delta(\psi\varphi).$$

If we write each of these chains as the sum of its constituent i-chain for all dimensions, we can use the distribute law of multiplication to bring our argument down to the case that φ is an i-chain and

ψ a j-chain.

Using the associative law we see, even without breaking the chains down, that

$$\delta(\phi\psi) = (\delta\phi)\psi = 0 ,$$ since ϕ is given to us as exact. But in order to verify the second half of our theorem, we shall have to shift the term δ to the other side of the term ψ. Now if ψ is a j-chain, $\delta\psi = (-1)^{j+1}\psi\delta$. This is trivial: putting the δ on the other side merely means that each term in the expanded product will have a vertex as last element of a product where before it had this vertex as first term. We have to shoot the vertex down j+1 places to the head of the line, because the terms are products representing j-simplexes so ~~there~~ that many vertices (j+1).

Therefore, if we apply the associative law to the right hand side of the relation to prove, permute the first two terms, then couple the δ to the ϕ we shall get zero because ϕ is exact.

Unfortunately, we shall be able to prove much more than this. We shall show later, it requires considerably more analysis, that the product of an arbitrary chain and an exact one is actually <u>derived</u>. What we see now is that the exact chains form a two-sided ideal in the ring of all chains.

Theorem: If ϕ is exact and ψ is derived, then

$$\phi\psi = 0 = \psi\phi.$$

Using the distributive law of multiplication, and breaking our chains down to sums of i-chains, we can suppose that ϕ is an i-chain and ψ a j-chain. Since ψ is derived we can write it as $\delta\chi$.

Now $\phi\delta\chi = (-1)^{i+1}\delta\phi\chi = 0$, since ϕ is exact. Similarly $\delta\chi\phi = (-1)^{j}\chi\delta\phi$, since χ is a (j-1)-chain, and this is zero since ϕ is exact.

We shall show, the next hour, that the product of two exact chains is zero. This will tell us that the ring of exact chains reduces to the group of exact chains and that the connectivity ring of the exact chains mod. the derived chains is ~~not any different from~~ the connectivity group ~~where we~~ ignoring ~~the~~ multiplication. We shall then introduce a new product, a bracket product, which is not trivial.

As a preliminary to this we shall have to analyze a chain into three constituent parts with respect to a fixed, but arbitrary vertex. Let φ be an arbitrary chain and let x_α be an arbitrary vertex. We shall write φ in the following form:

$$\varphi = x_\alpha \frac{\partial\varphi}{\partial x_\alpha} + \varphi_\alpha^* + \varphi_\alpha^\circ .$$

The first batch of terms will consist in those which contain the given vertex x_α. We factor the term x_α out of them, writing it up front, and we have denoted the chain which remains by $\frac{\partial\varphi}{\partial x_\alpha}$ because that is what it is, very nearly, in a formal sense. (The only difference, in fact, between our partial derivative with respect to x_α and what one would expect if one treated φ as a polynomial in distinct variables is that we have to take account of the order of x_α in a product and adjust the sign when we perform the permutations which bring it out front.) The second batch of terms will consist in those terms of our chain which belong to the shell of the given ~~chain~~ vertex. The third batch of terms are the remaining ones: each simplex of this set has at least one vertex not in the closure of the star of the vertex x_α.

It is to be understood that the vertex x_α is a "general" vertex, and that we have a similar decomposition of our chain for each vertex of the complex K. In particular cases one or two of the terms above may be missing, i.e. zero. Not all of them unless the chain is zero!