TOPICS IN DIFFERENTIAL GEOMETRY

by

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1951

Introduction

These notes are based on lectures which I gave in Professor Veblen's Seminar during the spring of 1949. In these two years some of the material have been further clarified and some problems solved. An attempt is made to include a few of the latest results. As a consequence the presentation given here is hardly the original form, particularly in Chapters III and IV.

It is my pleasure to express here my thanks to Professor Veblen for his interest in this work. I wish also to acknowledge my privilege of having frequent conversations with André Weil. An unpublished manuscript of his has greatly influenced the presentation in Chapter III.

S. S. C., April 24, 1951

Chapter I

General Notions on Differentiable Manifolds

This chapter gives a summary, for later applications, of some notions and results on the topology of differentiable manifolds and the algebra of exterior differential forms. Proofs are only indicated in the simple cases. For lengthy proofs we content ourselves by giving a reference.

1. Homology and cohomology groups of an abstract complex

An <u>abstract complex</u> K is a collection of <u>cells</u> $\{\sigma_i^r\}$, with the following properties:

1) To each cell there is associated a non-negative integer, its <u>dimension</u> (which will be denoted by the superscript), and to two cells of consecutive dimensions σ_i^r, σ_j^{r-1}, there is associated an integer $[\sigma_i^r : \sigma_j^{r-1}]$, their <u>incidence number</u>.

2) To a cell σ_i^r there exists only a finite number of cells σ_j^{r-1}, such that $[\sigma_i^r : \sigma_j^{r-1}] \neq 0$.

3) To two cells σ_i^{r+1}, σ_j^{r-1} whose dimensions differ by two, the incidence numbers satisfy the relation

$$\sum_k [\sigma_i^{r+1} : \sigma_k^r][\sigma_k^r : \sigma_j^{r-1}] = 0. \tag{1}$$

It is of such an abstract complex that we shall define the homology and cohomology groups.

Let G be an abelian group. A finite sum

$$c_r = \sum \lambda_i \sigma_i^r, \quad \lambda_i \in G$$

is called a <u>chain</u>, r being its dimension. If

$$d_r = \sum \mu_i \sigma_i^r, \mu_i \in G$$

is another r-dimensional chain, we define addition by

(2) $$c_r + d_r = \sum (\lambda_i + \mu_i)\sigma_i^r.$$

With this addition all r-dimensional chains form a group $C_r(K, G)$.

To the chains a boundary operation is defined, by

(3) $$\partial c_r = \sum_i \lambda_i \partial \sigma_i^r = \sum_{i,j} \lambda_i [\sigma_i^r : \sigma_j^{r-1}] \sigma_j^{r-1}.$$

By definition the boundary operation commutes with the addition of chains:

(4) $$\partial(c_r + d_r) = \partial c_r + \partial d_r,$$

so that it defines a homomorphism

(5) $$\partial : C_r(K, G) \longrightarrow C_{r-1}(K, G).$$

Elements of the kernel of this homomorphism, that is, chains whose boundaries are zero, are called cycles. The r-dimensional cycles form a subgroup $Z_r(K, G) \subset C_r(K, G)$.

It follows from Property 3) of K that the boundary operation has the property:

(6) $$\partial \partial c_r = 0,$$

so that the boundary of a chain is a cycle, called a bounding cycle. All r-dimensional bounding cycles form a subgroup $B_r(K, G) \subset Z_r(K, G)$.

The difference group

$$H_r(K, G) = Z_r(K, G) - B_r(K, G)$$

is called the r-dimensional homology group of K, with coefficient group G.

Let $C_r(K)$ be the group of r-dimensional chains of K, with integer coefficients, and let G be a topological group. A linear function over $C_r(K)$, with values in G, is called an r-dimensional cochain. An r-dimensional cochain γ^r satisfies therefore the conditions:

$$1) \quad \gamma^r(c_r + d_r) = \gamma^r(c_r) + \gamma^r(d_r).$$

$$2) \quad \gamma^r(-c_r) = -\gamma^r(c_r).$$

If β^r and γ^r are two r-cochains, we define their sum $\beta^r + \gamma^r$ to be the cochain given by

$$(7) \qquad (\beta^r + \gamma^r)(c_r) = \beta^r(c_r) + \gamma^r(c_r).$$

With this addition all the r-cochains form a group $C^r(K, G)$.

Of an r-cochain γ^r an (r+1)-cochain can be defined, called its coboundary, by means of the relation

$$(8) \qquad \delta\gamma^r(c_{r+1}) = \gamma^r(\partial c_{r+1}).$$

The coboundary operation commutes with the addition of cochains:

$$(9) \qquad \delta(\beta^r + \gamma^r) = \delta\beta^r + \delta\gamma^r,$$

and therefore defines a homomorphism

$$(10) \qquad \delta : C^r(K, G) \longrightarrow C^{r+1}(K, G).$$

A cochain whose coboundary is zero is called a cocycle. The r-cocycles form the kernel of the homomorphism δ and hence a subgroup $Z^r(K, G) < C^r(K, G)$.

It follows from (6) and (8) that

$$(11) \qquad \delta\delta\gamma^r = 0,$$

so that the coboundary of a cochain is a cocycle. All r-dimensional coboundaries form a subgroup $B^r(K, G) < Z^r(K, G)$. Their difference group

$$H^r(K, G) = Z^r(K, G) - B^r(K, G)$$

is called the r-dimensional cohomology group of K, with coefficient group G.

An important part of algebraic topology consists in the identification of the homology and cohomology groups of different complexes constructed from a space. For instance, the main theorem in the topology of polyhedra asserts that the complex of its singular cells and the complex of its simplicial decomposition have isomorphic homology and cohomology groups.

Let K and K' be two complexes. A mapping f of the cells of K into the cells of K' is called a <u>chain mapping</u>, if it commutes with the boundary operation:

(12) $$f\partial = \partial f.$$

It follows that a chain mapping induces the homomorphisms

$$f: Z_r(K, G) \longrightarrow Z_r(K', G),$$

$$f: B_r(K, G) \longrightarrow B_r(K', G),$$

and hence the homomorphism

(13) $$f: H_r(K, G) \longrightarrow H_r(K', G).$$

To the chain mapping f we can define a dual mapping

(14) $$f^*: C^r(K', G) \longrightarrow C^r(K, G)$$

as follows: Let $\gamma'^r \in C^r(K', G)$. Then

(15) $$(f^* \gamma'^r)(c_r) = \gamma'^r(f(c_r)).$$

It is easily verified that the dual cochain mapping commutes with the coboundary operation

(16) $$f^* \delta = \delta f^*.$$

Hence there results the <u>dual homomorphism</u>

(17) $$f^*: H^r(K', G) \longrightarrow H^r(K, G).$$

This fact will play an important role in differential geometry. Reference: S. Eilenberg, Singular homology theory, Vol. 45, 407-447 (1944).

2. Product theory

In order to develop a satisfactory product theory for a complex some additional notions and assumptions are necessary.

A cell σ_j^{r-1} is called a face of σ_i^r, if the incidence number $[\sigma_i^r : \sigma_j^{r-1}] \neq 0$. In general, σ^{r-p} is called a face of σ^r, if either $p = 0$ and the two cells are identical or $p > 0$ and there exists a sequence of cells $\sigma^{r-p}, \sigma^{r-p+1}, \ldots, \sigma^r$ such that each is a face of the next one. The closure $\bar{\sigma}_i^r$ of a cell σ_i^r is the subcomplex formed by all its faces. The star st σ_i^r is the subcomplex of all cells having σ_i^r as a face. A cycle is called boundary-like, if it is either of dimension > 0 or is of dimension 0 and has the sum of its coefficients equal to zero.

We shall first use the ring of integers as the coefficient ring for the product theory. Denote by I the 0-dimensional cochain which has the value one for every 0-cell.

Two further conditions will now be imposed on the complex in the establishment of a product theory:

(I) Every boundary-like cycle in $\bar{\sigma}_i^r$ bounds a chain in $\bar{\sigma}_i^r$.

(II) I is a cocycle.

For simplicity the notation σ_i^r will be used to denote at the same time the cell σ_i^r, the chain $1 \cdot \sigma_i^r$, and the cochain which has the value 1 for σ_i^r and 0 for other cells of dimension r. More generally, the notation $\sum \lambda_i \sigma_i^r$ will occasionally be used to denote the cochain having the value λ_i for σ_i^r and the value zero for other cells.

The cup product of two cochains of dimensions r and s is a cochain of dimension $r + s$ which satisfies the following conditions:

($\cup$1) $$(\beta_1^r + \beta_2^r) \cup \gamma^s = \beta_1^r \cup \gamma^s + \beta_2^r \cup \gamma^s.$$

$$\beta^r \cup (\gamma_1^s + \gamma_2^s) = \beta^r \cup \gamma_1^s + \beta^r \cup \gamma_2^s,$$

$$(\lambda\beta^r) \cup \gamma^s = \beta^r \cup (\lambda\gamma^s) = \lambda(\beta^r \cup \gamma^s), \lambda = \text{integer}$$

(∪2) $$\delta(\beta^r \cup \gamma^s) = \delta\beta^r \cup \gamma^s + (-1)^r \beta^r \cup \delta\gamma^s$$

(∪3) $\sigma_i^r \cup \sigma_j^s$ is a cochain which has the value zero for any cell $\bar{\epsilon}(st\,\sigma_i^r)(st\,\sigma_j^s)$.

(∪4) $$I \cup \sigma_i^r = \sigma_i^r \cup I = \sigma_i^r.$$

<u>Theorem 1</u> (Fundamental Existence and Uniqueness Theorem) The cup product of cochains induces a multiplication of cohomology classes. For a complex satisfying the conditions (I), (II) there exists a multiplication of cochains which fulfills the conditions (∪ 1) - (∪ 4). Any two kinds of multiplications with these properties lead to the same multiplication of the cohomology classes.

Let R be a commutative ring and consider the chains and cochains with R as coefficient group. The <u>cup product</u> of the cochains $\beta \in C^r(K, R)$, $\gamma \in C^s(K, R)$ is defined by the conditions:

1. $\beta \cup \gamma$ is bilinear in both variables;
2. If $\lambda\sigma^r \in C^r(K, R)$, $\mu\sigma^s \in C^s(K, R)$, then
   $$(\lambda\sigma^r) \cup (\mu\sigma^s) = \lambda\mu(\sigma^r \cup \sigma^s).$$

Let K, K' be simplicial complexes, and f: K⟶K' a simplicial mapping. If $f^*: H^r(K', R) \longrightarrow H^r(K, R)$ is the dual homomorphism of the cohomology groups, then

(18) $$f^*(\beta') \cup f^*(\gamma') = f^*(\beta' \cup \gamma').$$

This is called <u>Hopf's inverse homomorphism</u>, which can be described by simply saying that the dual homomorphism preserves the cup product.

Theorem 2 (Topological invariance). Let P be a polyhedra and K its or simplicial decomposition. There exists between the cohomology groups of K and the cohomology groups of the singular complex of P an isomorphism which preserves the cup product.

For later applications we shall only be interested in the case that R is either the ring of integers or the real field or the finite field mod 2. We shall therefore assume that R is the ring of integers or a field. Then, if β^r, γ^s are cochains and c^{r+s} a chain (all with coefficients in R), the relation

$$\beta^r \cdot (\gamma^s \wedge c^{r+s}) = (\beta^r \cup \gamma^s)(c^{r+s}), \tag{19}$$

for β^r arbitrary, defines a chain $\gamma^s \wedge c^{r+s}$ of dimension r, called the cap product of γ^s and c^{r+s}. Under our assumption for R a cup product determines a cap product, and vice versa.

Let M be a manifold, oriented if R is the ring of integers and otherwise if R is the field mod 2. In both cases there is a fundamental cycle which we also denote by M. Define

$$\Theta \gamma^r = \gamma^r \wedge M. \tag{20}$$

Then Θ establishes an isomorphism between $H^r(M, R)$ and $H_{n-r}(M, R)$. For $u_r \in H_r(M, R)$, $u_s \in H_s(M, R)$, define

$$u_r \circ u_s = \Theta(\Theta^{-1} u_r \cup \Theta^{-1} u_s). \tag{21}$$

Theorem 3. The product $u_r \circ u_s$ of homology classes on a manifold defined by (21) is identical with the intersection class of u_r and u_s.

This theorem gives the connection between product theory and intersection theory.

Reference: H. Whitney, On products in a complex, Annals of Math. Vol. 39, 397-432 (1938).

For the uniqueness in Theorem 1 we have to assume that the complex K is also star-finite and that the cochains under consideration are finite.

3. An example

As an illustration we consider the n-dimensional real projective space P^n and take as coefficient field the field mod 2. P^n contains a sequence of projective spaces of lower dimensions

$$P^n > P^{n-1} > P^{n-2} > \dots > P' > P^o,$$

and has a cellular subdivision consisting of the cells

$$P^n - P^{n-1},\ P^{n-1} - P^{n-2},\ \dots,\ P' - P^o,\ P^o.$$

It is easy to verify that each of these cells is a cycle and $H_r(P^n, I_2)$, $H^r(P^n, I_2)$, $n \geqq r \geqq 0$, are cyclic groups of order two. Without danger of confusion we can denote the generator of $H_r(P^n, I_2)$ by P^r and the generator of $H^r(P^n, I_2)$ by ζ^r.

We shall prove that the cohomology ring of P^n is

$$H(P^n) = \left\{1,\ \zeta\,(= \zeta'),\ (\zeta)^2,\ \dots,\ (\zeta)^n\right\},$$

when the superscripts outside the parentheses denote powers in the sense of the cup product. We notice that the isomorphism Θ maps ζ^r into P^{n-r}, so that $\zeta^r(P^r) = KI(P^{n-r}, P^r) = 1$. By applying induction on r we suppose $(\zeta)^r(P^r) = 1$ and then find

$$\Theta((\zeta)^{r+1}) = \Theta(\zeta)^r \cup \zeta) = \Theta(\Theta^{-1}(P^{n-r}) \cup \Theta^{-1}(P^{n-1})) = P^{n-r}{}_o P^{n-1} = $$
$$= P^{n-r-1}.$$

This proves that $(\zeta)^{r+1}$ is the generator of $H^{r+1}(P^n, I_2)$ and hence the above form of the cohomology ring of P^n.

A) Let $g: P^{r-1} \longrightarrow P^{n-1}$ be a continuous mapping such that the induced homomorphism g carries a projection line (that is, the homology class of it) into a projective line. Then $n \geqq r$.

Proof. Let ξ and ζ be the generators of the cohomology rings of P^{r-1} and P^{n-1} respectively. Then

$$g^*(\zeta)(P') = \zeta(g(P')) = \zeta(P') = 1.$$

It follows that $g^*(\zeta) = \xi$. Since $\zeta^n = 0$, we have

$$\xi^n = (g^*(\zeta))^n = g^*(\zeta^n) = 0.$$

Hence $n \geq r$.

B) Let $g_1, \ldots, g_n$ be continuous odd functions in $x_1, \ldots, x_r$ defined on the sphere

$$x_1^2 + \ldots + x_r^2 = 1.$$

If $n < r$, the functions $g_1, \ldots, g_n$ have a common zero on the sphere.

Proof. Suppose there be no common zero. We can then assume that

$$g_1^2 + \ldots + g_n^2 = 1.$$

The functions $g_i = g_i(x_1, \ldots, x_r)$, $i = 1, \ldots, n$, then define a mapping of a sphere S^{r-1} into a sphere S^{n-1} and, after identifying the antipodal pairs of both spheres, a mapping $g: P^{r-1} \longrightarrow P^{n-1}$. Moreover, since the functions g_i are odd, the mapping g has the property that it carries the homology class of a projection line into the homology class of a projective line. But this contradicts A).

C) (Borsuk-Ulam) Let an n-sphere S^n be mapped continuously into the n-dimensional Euclidean space E^n. There exists in S^n at least one pair of antipodal points which are mapped into the same point of E^n.

Proof. Let $x_1, \ldots, x_n$ be the coordinates of E^n. Suppose the mapping be

$$x_i = f_i(p),\ p \in S^n,\ i = 1, \ldots, n.$$

Denote by p^* the antipodal point of p. Put

$$g_i(p) = f_i(p) - f_i(p^*).$$

From B) it follows that $g_i(p)$ have a common zero p_o. At this p_o we have

$$f_i(p_o) = f_i(p_o^*),\ i = 1, \ldots, n.$$

4. Algebra of a vector space

The differentiable manifolds which will be our later concern have the property that there is associated at each point a finite dimensional vector space. The study of the algebraic properties of the vector space and of various associated vector spaces will therefore constitute a necessary prerequisite for later developments.

We denote by V^n or V an n-dimensional vector space over the real field. To V there is associated its dual space V*, the space of all linear functions over V, and the relation between V and V* is reciprocal. We shall denote elements of V by small Gothic letters and elements of V* by small Greek letters. Then $\alpha(\mathfrak{v})$ or $\alpha\cdot\mathfrak{v}$, $\mathfrak{v}\in V$, $\alpha\in V^*$, is a real number.

Consider the direct product

$$V(k,\ell) = \underbrace{V\times \dots \times V}_{k} \times \underbrace{V^*\times \dots \times V^*}_{\ell} .$$

A tensor of type (k,ℓ) is a multilinear function in $V(k,\ell)$, with real values, that is, a real-valued function linear in each argument when the other $k+\ell-1$ arguments are kept fixed. k is called the covariant order and ℓ the contravariant order. The tensor is called covariant or contravariant, when $\ell = 0$ or $k = 0$ and is in general called mixed. Covariant (contravariant) tensors of order one are called covariant (contravariant) vectors. Given a tensor f of type (k,ℓ) and a tensor y of type (k',ℓ'), we define $f\times g$ to be the tensor of type $(k+k',\ell+\ell')$ by the relation

$$(22)\quad \begin{aligned}&(f\times g)(\mathfrak{v}_1,\dots,\mathfrak{v}_k,\mathfrak{v}_{k+1},\dots,\mathfrak{v}_{k+k'};\alpha_1,\dots,\alpha_\ell,\alpha_{\ell+1},\dots,\alpha_{\ell+\ell'})\\ &= f(\mathfrak{v}_1,\dots,\mathfrak{v}_k;\alpha_1,\dots,\alpha_\ell)g(\mathfrak{v}_{k+1},\dots,\mathfrak{v}_{k+k'};\alpha_{\ell+1},\dots,\alpha_{\ell+\ell'}).\end{aligned}$$

With a natural addition all tensors of type (k,ℓ) form a vector space of dimension $n^{k+\ell}$. Because of the duality between V and V* the space of all covariant vectors can be identified with V* and the space of all contravariant vectors with V itself.

A tensor may have the property of symmetry or anti-symmetry. For simplicity take a covariant tensor of order two, given by $f(\mathscr{V}, \mathscr{W})$. The tensor is called <u>symmetric</u> or <u>anti-symmetric</u>, according as $f(\mathscr{V}, \mathscr{W}) = f(\mathscr{W}, \mathscr{V})$ or $f(\mathscr{V}, \mathscr{W}) = -f(\mathscr{W}, \mathscr{V})$ holds. From a given covariant (contravariant) tensor f of order k we can construct its <u>symmetrized</u> or <u>alternated</u> tensor respectively by the equations

$$S(f)(\mathscr{V}_1, \ldots, \mathscr{V}_k) = \frac{1}{K!} \sum f(\mathscr{V}_{i_1}, \ldots, \mathscr{V}_{i_k}), \tag{23}$$

$$T(f)(\mathscr{V}_1, \ldots, \mathscr{V}_k) = \frac{1}{K!} \sum \epsilon_{i_1 \ldots i_k} f(\mathscr{V}_{i_1}, \ldots, \mathscr{V}_{i_k}), \tag{24}$$

where the summations are extended over all permutations $i_1, \ldots, i_k$ of $1, \ldots, k$ and $\epsilon_{i_1 \ldots i_k} = +1$ or -1 according as $i_1, \ldots, i_k$ is an even or odd permutation of $1, \ldots, k$.

Of particular importance will be the vector spaces A^r, $r = 1, \ldots, n$, of anti-symmetric or alternating tensors of order (r, 0). Let A^o be the (one-dimensional) vector space isomorphic to the real field, and let

$$A = A^o \dotplus A' \dotplus \ldots \dotplus A^n. \tag{25}$$

Then A is a vector space of dimension 2^n.

We shall convert A into a ring by defining a multiplication which has the properties:

1) It is distributive:

$$f \wedge (g_1 + g_2) = f \wedge g_1 + f \wedge g_2, \tag{26}$$

$$(f_1 + f_2) \wedge g = f_1 \wedge g + f_2 \wedge g.$$

2) If $f \in A^r$, $g \in A^s$, then

(27) $$f \wedge g = T(f \times g).$$

With this multiplication the vector space A becomes a ring (of dimension 2^n), called the Grassmann ring associated to V. An element of the Grassmann ring which belongs to one of the A^r's, that is, whose other components in the direct summand are zero, is called homogeneous dimensional or an alternating form. It follows from definition that if $f \in A^r$, $g \in A^s$, then

(28) $$f \wedge g = (-1)^{rs} g \wedge f.$$

If α_i, $i = 1, \ldots, n$, form a base in V* (which is then identified to A'), a base in the associated Grassmann ring will be formed by the elements

(29) $$1,\ \alpha_i,\ \alpha_i \wedge \alpha_j (i < j),\ \alpha_i \wedge \alpha_j \wedge \alpha_k (i < j < k), \ldots,\ \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n,$$

which are 2^n in number.

We shall give two theorems in the Grassmann ring, which are particularly useful later on.

A) Let $\omega_1, \ldots, \omega_r \in V^*$. Then $\omega_1, \ldots, \omega_r$ are linearly dependent, if and only if

$$\omega_1 \wedge \ldots \wedge \omega_r = 0.$$

This can either be proved by induction on r, or by choosing a base α_i, $i = 1, \ldots, n$ in V*, writing

$$\omega_s = \sum_{i=1}^{n} \ell_{si} \alpha_i, \quad s = 1, \ldots, r,$$

and observing that

(30) $$\omega_1 \wedge \ldots \wedge \omega_r = \sum_{i_1 < \ldots < i_r} \begin{vmatrix} \ell_{1i_1} \ldots \ell_{1i_r} \\ \ell_{ri_1} \ldots \ell_{ri_r} \end{vmatrix} \alpha_{i_1} \ldots \alpha_{i_r}.$$

B) If $\omega_1, \ldots, \omega_r$ are linearly independent and $\pi_s \in V^*$, $s = 1, \ldots, r$, are such that

(31) $$\sum_{s=1}^{r} \pi_s \wedge \omega_s = 0,$$

then

(32) $$\pi_s = \sum_{t=1}^{r} a_{st}\omega_t,$$

where

(33) $$a_{st} = a_{ts}.$$

A tensorial form of order (k, ℓ) and degree r is a multilinear function of $V(k,\ell)$, with values in A^r. Clearly, all tensorial forms of given order and degree form a vector space.

There is an operation, called the tensor product, which pairs two abelian groups (and hence two vector spaces) into an abelian group. Let A and B be two abelian groups, with the elements a_i and b_i respectively. Take the finite sums of the formal products $\sum a_i b_i$. Two such sums are called equivalent, if one can be transformed into the other by a finite number of the following elementary transformations: 1) $(a_i + a_i')b_i = a_i b_i + a_i' b_i$; 2) $a_i(b_i + b_i') = a_i b_i + a_i b_i'$; 3) addition or deletion of $a.0$ or $0.b$. Among the equivalence classes so obtained we can define an addition which, for the representatives, is defined just by adding the terms formally. The equivalence classes with such an addition form a group, called the tensor product of A and B and to be denoted by $A \otimes B$.

5. Differentiable manifolds

A (topological) manifold M is said to have a differentiable structure, if the following conditions are satisfied:

1) There is an open covering $\{U_i\}$ of M such that for each i there exists a homeomorphism θ_i of an n-cell E into U_i.

2) For any two open sets U_i, U_j of the covering the mapping $\theta_j^{-1}\theta_i(S)$, $S = \theta_i^{-1}(U_i \cap U_j)$, of S into E is differentiable of class $r > 0$.

A manifold with a differentiable structure is called a differentiable manifold, r (> 0) being its class. U_i are called coordinate neighborhoods; the coordinates in $E = \theta_i^{-1}(U_i)$ are called local coordinates.

On a differentiable manifold of class r we can define differentiable functions of class r. They are real-valued functions which are differentiable of class r in each coordinate neighborhood, it being sufficient to have the property at every point p relative to one of the coordinate neighborhoods containing p.

Let $p \in M$, and D(p) the family of differentiable functions of class r at p. A tangent vector at p is a mapping $\mathfrak{v} : D(p) \longrightarrow R$ (real field), satisfying the conditions:

1) $\mathfrak{v}$ is linear, that is, for any $f, g \in D(p)$ and any real numbers a, b, we have $\mathfrak{v}(af+bg) = a\,\mathfrak{v}(f) + b\,\mathfrak{v}(g)$.

2) $\mathfrak{v}$ is a differentiation, that is, for any $f, g \in D(p)$,

$$\mathfrak{v}(fg) = f(p)\,\mathfrak{v}(g) + g(p)\,\mathfrak{v}(f).$$

If M is of dimension n, the tangent vectors at p form a vector space T_p of dimension n, called the tangent space at p. To this space the considerations of the last section will apply, so that we can consider its dual space, the spaces of tensors of different types, and the Grassmann ring, etc. Let F_p be the space of tensors of a definite type associated to T_p, and let

$$X = \bigcup_{p \in M} F_p.$$

A natural topology can be defined in X, so that X becomes a topological space. X is then called a tensor bundle over M. There is a natural mapping, called projection,

$$\psi : X \longrightarrow M,$$

defined by

$$\psi(F_p) = p.$$

We shall give the relation of the tensors defined here with those of classical differential geometry. For definiteness consider a tensor Ξ of type (1, 1). Let $x^1, \ldots, x^n$ be a system of local coordinates at p. For $f \in D(p)$ define

$$\mathcal{H}_i(f) = \left(\frac{\partial f}{\partial x^i}\right)_p, \quad i = 1, \ldots, n.$$

Then $\mathcal{H}_i$ are vectors and span the tangent space T_p. In the dual space T_p^* of T_p we choose the base α^i defined by

$$\alpha^i(\mathcal{H}_j) = \delta^i_j.$$

We put

$$\xi^j_i = \Xi(\mathcal{H}_i, \alpha^j),$$

which are called the components of Ξ relative to the local coordinate system x^i. Suppose $\bar{x}^i$ be another system of local coordinates at p. Put

$$a^i_j = \left(\frac{\partial \bar{x}^i}{\partial x^j}\right)_p, \quad b^i_j = \left(\frac{\partial x^i}{\partial \bar{x}^j}\right)_p,$$

so that

$$\sum_j a^i_j b^j_k = \sum_j b^i_j a^j_k = \delta^i_k.$$

Then we have by definition, for the vectors $\bar{\mathcal{H}}_i$, $\bar{\alpha}^i$ relative to the coordinate system $\bar{x}^i$,

$$\bar{\mathcal{H}}_i = \sum_j b^j_i \mathcal{H}_j, \quad \bar{\alpha}^i = \sum_j a^i_j \alpha^j.$$

It follows that the components of Ξ relative to the local coordinate system $\bar{x}^i$ are

(34) $$\bar{\xi}^j_i = \sum_{k,\ell} b^\ell_i a^j_k \xi^k_\ell .$$

These are the well-known equations of transformation in classical differential geometry.

6. Multiple integrals

For the theory of multiple integrals on a differentiable manifold M we have to consider the bundle of Grassmann rings over M, which is the union of the Grassmann rings associated to the tangent spaces T_p, $p \in M$, with a natural topology. We denote by A_p the Grassmann ring associated to p, and let $\mathcal{U} = \bigcup_{p \in M} A_p$. A differential polynomial is a mapping $\omega : M \longrightarrow \mathcal{U}$ such that the projection of $\omega(p)$, $p \in M$, is p itself. The mapping is assumed to be locally differentiable of class $\geqq 2$. If $\omega(p)$, $p \in M$, is a form of degree r, ω is called a differential form of degree r.

We shall define an operation d, called exterior differentiation, which carries differential polynomials into differential polynomials, by the following properties:

1) $d(\omega + \theta) = d\omega + d\theta$,

2) $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^r \omega \wedge d\theta$,

where ω is a differential form of degree r.

3) If f is a scalar (that is, a differential form of degree zero), df is the covariant vector such that

$$df(\mathcal{V}) = \mathcal{V}(f)$$

holds for every contravariant vector $\mathcal{V}$.

4) For every scalar f,

$$d(df) = 0.$$

In terms of a local coordinate system x^i we can take as a base for A' the differentials dx^i, $i = 1, \ldots, n$, which are covariant vectors defined by

(35) $$dx^i(\mathfrak{v}) = \mathfrak{v}(x^i),$$

$\mathfrak{v}$ being any contravariant vector. Then a base for A^r is formed by

$$dx^{i_1}\wedge dx^{i_2}\wedge \dots \wedge dx^{i_r}, \quad i_1 < i_2 < \dots < i_r,$$

so that a differential form of degree r can be written

(36) $$\omega = \sum_{i_1<\dots<i_r} a_{i_1\dots i_r} dx^{i_1}\wedge dx^{i_2}\wedge \dots \wedge dx^{i_r},$$

where the coefficients may be assumed to be anti-symmetric.

It follows from 2) and 4), by induction on the degree r, that

$$d(dx^{i_1}\wedge dx^{i_2}\wedge \dots dx^{i_r}) = 0.$$

Therefore we have the formula

(37) $$d\omega = \sum_{i_1<\dots<i_r} da_{i_1\dots i_r} \wedge dx^{i_1}\wedge \dots \wedge dx^{i_r}.$$

We also have, by 3),

(38) $$df = \sum_i \frac{\partial f}{\partial x^i} dx^i,$$

Concerning the exterior differentiation of differential forms two facts are of importance:

A) For any differential form ω,

(39) $$d(d\omega) = 0.$$

B) Let $f: M \longrightarrow M'$ be a differentiable mapping of a manifold M into a manifold M'. f induces a differential mapping df of the tangent space T_p of M at p into the tangent space $T_{p'}$, $p' = f(p)$. The dual mapping f^* of df is a linear mapping of the Grassmann ring $A_{p'}$ at p' into A_p. Then

(40) $$f^*(d\omega) = d(f^*\omega).$$

In other words, exterior differentiation commutes with the induced dual mapping of a mapping of one differentiable manifold into another.

A differential form ω is called <u>exact</u>, if $d\omega = 0$, and is called <u>derived</u>, if there exists a differential form Θ such that $\omega = d\Theta$. It follows from

(39) that every derived form is exact.

Differential forms can be taken as integrands of multiple integrals in the manifold. The details of a satisfactory integration theory will be too long to be reproduced here. For simplicity we take a cellular decomposition K of M which is so fine that each cell belongs to a coordinate neighborhood. Let ω be a differential form of degree r and c_r an r-chain of K. By the equation

$$\omega(c_r) = \int_{c_r} \omega, \tag{41}$$

ω defines a cochain of dimension r, with real coefficients. An important link of exterior differentiation with cohomology theory is now the generalized Stokes' formula:

$$\int_{c_r} \omega = \int_{\partial c_r} d\omega. \tag{42}$$

In other words, if ω is (or defines) a cochain, $d\omega$ is its coboundary, and is a cocycle, if $d\omega = 0$.

The study of the ring of differential forms and its exterior differentiation is justified by the following fundamental theorem due to de Rham:

Theorem. Let M be a compact differentiable manifold. To every cohomology class of M, with real coefficients, there exists an exact differential form which defines a cocycle belonging to this class. The cohomology class containing the product (in the sense of the Grassmann ring) of two exact differential forms is the cup product of the classes which contain the factors.

Example. The de Rham theorem asserts the existence of a differential form, while in concrete cases it is important to construct the forms explicitly, and the ones with simple properties. Consider the unit n-sphere in an (n+1)-dimensional Euclidean space E^{n+1}. $H^n(S^n, R) \approx R$, so that we wish to construct the exact differential form which defines a generator of $H^n(S^n, R)$.

Let O be the center of S^n, and consider the frames $O\mathfrak{u}_1 \ldots \mathfrak{u}_{n+1}$ formed by O and n+1 mutually perpendicular unit vectors in a definite orientation. Identify a point of S^n with the end-point of $\mathfrak{u}_{n+1}$, and put

$$\omega_{in+1} = (d\mathfrak{u}_{n+1}\mathfrak{u}_i), \quad i = 1, \ldots, n, \tag{43}$$

where the product in the right-hand side is the scalar product in E^{n+1}. The differential form

$$\omega = \frac{1}{O_n}\,\omega_{1n+1}\wedge\cdots\wedge\omega_{nn+1}, \tag{44}$$

where O_n is the area of S^n, has the property that the value of its integral over a fundamental cycle of S^n is ± 1. Hence ω defines a generator of $H^n(S^n, R)$.

Let Σ be a hypersurface in E^{n+1} and let $\mathfrak{v}(p)$, $p \in \Sigma_1$ be a continuous vector field on Σ. Choose $\mathfrak{u}_{n+1}$ to be parallel to $\mathfrak{v}(p)$. Then $f(p) = \mathfrak{u}_{n+1}$ defines a mapping $f: \Sigma \longrightarrow S^n$. Its induced dual mapping f^* maps ω into a differential form $f^*\omega$ in Σ. The integral

$$\int_{\Sigma} f^*\omega = \int_{f(\Sigma)} \omega \tag{45}$$

is equal to the <u>index of the vector field</u>. It is the <u>Kronecker integral</u>.

Reference: Hodge, Theory and Applications of Harmonic Integrals.

Chapter II

Riemannian Manifolds

This chapter is devoted to the theory of Riemannian manifolds, and in particular to the Gauss-Bonnet formula which, for a compact orientable Riemannian manifold, expresses its Euler-Poincaré characteristic as an integral of a scalar invariant over the manifold. We begin with the study of Riemannian manifolds imbedded in an Euclidean space, because this case appeals more to geometrical intuition and usually furnishes a first test for properties in general Riemannian manifolds. To simplify the formulas repeated indices always denote summation.

1. Riemannian manifolds in Euclidean space

Let E^{n+N} be an oriented Euclidean space of dimension n+N. E^{n+N} is transformed transitively by the group of motions. We call a <u>frame</u> the figure $p e_1 \cdots e_{n+N}$ formed by a point p and an ordered set of n+N mutually perpendicular unit vectors through p. The set of frames has the property that there exists one and only one motion which carries one frame to another. It can therefore be made a differentiable manifold isomorphic to the group of motions.

Since the vectors are in Euclidean space, we can write*

$$(1)\qquad \begin{aligned} dp &= \theta_A e_A, \\ de_A &= \theta_{AB} e_B. \end{aligned}$$

---

* We agree in this section to use the following ranges of indices:

$A, B, C = 1, \ldots, n+N;\ \alpha, \beta, \gamma = 1, \ldots, n;\ r, s = n+1, \ldots, n+N.$

The differential forms θ_A, θ_{AB} are in the manifold of frames and satisfy

(2) $$\theta_{AB} + \theta_{BA} = 0.$$

Since

$$d(dp) = d(de_A) = 0,$$

we derive from (1) that

(3) $$d\theta_A = \theta_B \wedge \theta_{BA},$$
$$d\theta_{AB} = \theta_{AC} \wedge \theta_{CB}.$$

Formulas (3) are called the <u>equations of structure</u> of the Euclidean space.

Let M be an n-dimensional manifold, differentially imbedded (of class ≥ 3) in E^{n+N}. M has a Riemannian metric, induced by E^{n+N}. To study M we consider the submanifold of the frames $p e_1 \dots e_{n+N}$, such that $p \in M$, and $e_1, \dots, e_n$ are the tangent vectors of M at p. This submanifold is mapped into the manifold of frames by the inclusion mapping ι. Let ι^* be its dual mapping of differential forms, and let

(4) $$\omega_A = \iota^* \theta_A, \quad \omega_{AB} = \iota^* \theta_{AB}.$$

Since ι^* commutes with both exterior differentiation and multiplication of the Grassmann ring, we have

(5) $$d\omega_r = \omega_\alpha \wedge \omega_{\alpha r} = 0.$$

It follows that

(6) $$\omega_{\alpha r} = \lambda_{r\alpha\beta}\omega_\beta, \quad \lambda_{r\alpha\beta} = \lambda_{r\beta\alpha}.$$

The second equation of (3) gives

(7) $$d\omega_{\alpha\beta} = \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta},$$

where

(8) $$\Omega_{\alpha\beta} = -\omega_{\alpha r} \wedge \omega_{\beta r} = -\lambda_{r\alpha\rho}\lambda_{r\beta\sigma}\omega_\rho \wedge \omega_\sigma.$$

We shall prove that $\Omega_{\alpha\beta}$ depends only on the Riemannian metric of M. The proof depends on the following

Lemma. There exists only one set of $\omega_{\alpha\beta}$, which are anti-symmetric in its indices and which satisfy the equations

$$d\omega_\alpha = \omega_\beta \wedge \omega_{\beta\alpha}.$$

Proof. Suppose a second set $\bar{\omega}_{\beta\alpha}$ also possesses these properties. Then

$$\omega_\beta \wedge (\bar{\omega}_{\beta\alpha} - \omega_{\beta\alpha}) = 0,$$

and we have

$$\bar{\omega}_{\beta\alpha} - \omega_{\beta\alpha} = P_{\beta\alpha\gamma}\,\omega_\gamma ,$$

where $P_{\beta\alpha\gamma}$ is symmetric in β, γ. Since $\bar{\omega}_{\beta\alpha} - \omega_{\beta\alpha}$ is anti-symmetric in β, α and ω_γ are linearly independent, $P_{\beta\alpha\gamma}$ is anti-symmetric in β, α. But a three-indexed symbol symmetric in one pair and anti-symmetric in another is zero. Hence $\bar{\omega}_{\beta\alpha} = \omega_{\beta\alpha}$.

Our statement that $\Omega_{\alpha\beta}$ depends only on the Riemannian metric of M then follows from the lemma. $\Omega_{\alpha\beta}$ will be called the curvature forms. Equations (8) are the Gauss equations. The quadratic differential forms

$$\Phi_r = \omega_\alpha \omega_{\alpha r} = \lambda_{r\alpha\beta}\,\omega_\alpha \omega_\beta \tag{9}$$

are called the second fundamental forms of M.

We shall make two applications of the above formulas.

A) Let N = 1, so that M is a closed hypersurface. The normals n_{n+1} define a vector field over M, whose index is, according to Kronecker's integral formula,

$$I = \frac{1}{O_n}\int_M \omega_{1\,n+1} \cdots \omega_{n\,n+1} = \frac{1}{n!\,O_n}\int_M \epsilon_{i_1\cdots i_n}\,\omega_{i_1 n+1}\cdots\omega_{i_n n+1}.$$

where O_n is the area of the n-dimensional unit hypersphere. If <u>n is even,</u> the integrand can be written

$$\epsilon_{i_1 \dots i_n} \omega_{i_1 n+1} \dots \omega_{i_n n+1} = (-1)^{\frac{n}{2}} \epsilon_{i_1 \dots i_n} \Omega_{i_1 i_2} \dots \Omega_{i_{n-1} i_n},$$

and depends only on the metric in M. On the other hand, it can be proved that $I = \frac{1}{2}\chi$, where χ is the Euler-Poincare characteristic of M. We get therefore the Gauss-Bonnet formula for a hypersurface:

(10) $$\frac{2(-1)^{\frac{n}{2}}}{n! O_n} \int \epsilon_{i_1 \dots i_n} \Omega_{i_1 i_2} \dots \Omega_{i_{n-1} i_n} = \chi.$$

B) Let m(p) be the minimum number of linear differential forms in which the second fundamental forms Φ_r can be expressed. m(p) is obviously equal to the number of linearly independent equations in the system

$$\frac{\partial \Phi_r}{\partial \omega_\alpha} = 0.$$

We put $m = \underset{p \in M}{\operatorname{Max}} m(p)$. Then we have the theorem:

<u>Let M be a closed manifold of dimension n, differentially imbedded in E^{n+N}. Then $m \geqq N$.</u>

Proof. We take a fixed point O and a fixed system of coordinate axes $\mathfrak{n}_A^0$ through O. For $p \in M$ let $s = \overline{Op}^2$ (square of the distance Op). Then s attains a maximum at a point p_o. Writing

$$\overrightarrow{Op} = x_A \mathfrak{n}_A^0,$$

we have

$$\frac{1}{2} ds = x_A dx_A = \overrightarrow{Op}\, dp,$$

$$\frac{1}{2} d^2 s = dpdp + \overrightarrow{Op}\, d^2 p = \omega_\alpha \omega_\alpha + \overrightarrow{Op}\, d^2 p.$$

At p_o we have

$$ds = 0, \quad d^2 s \leqq 0.$$

Since ω_α are linearly independent, the first equation implies

$$\overrightarrow{op}\, \mathfrak{n}_\alpha = 0.$$

Now

$$\overrightarrow{op}\, d^2p = (\ldots)\, (\overrightarrow{op}\, \mathfrak{n}_\alpha) + \Phi_r (\overrightarrow{op}\, \mathfrak{n}_r) ,$$

so that the inequality implies

$$\omega_\alpha \omega_\alpha + \Phi_r (\overrightarrow{op}\, \mathfrak{n}_r) \leqq 0.$$

If $m \leqq n-1$, there exists at least a direction in M for which $\Phi_r = 0$ and along this direction we have

$$\omega_\alpha \omega_\alpha \leqq 0,$$

which contradicts the positive definiteness of the quadratic differential form in the left-hand side.

From this theorem we shall deduce the following theorem which was first proved by Tompkins:

A closed flat Riemannian manifold of dimension n cannot be isometrically imbedded in an Euclidean space of dimension 2n-1.

Proof. We suppose that such an imbedding exists, so that $N = n-1$. By the last theorem it suffices to prove that $m \leqq n-1$.

Suppose that $m = n$. Consider the vectors

$$\gamma_{ij} = (\lambda_{1ij}, \ldots, \lambda_{Nij})$$

in an N-dimensional Euclidean space. By (8) the flatness of the induced metric implies that

$$(*) \qquad (\gamma_{ik} \gamma_{j\ell}) = (\gamma_{i\ell} \gamma_{jk}),$$

where the products in the parentheses are scalar products of vectors (in the auxiliary N-space). Since $m = n$, the matrix of vectors

$$V = (\gamma_{ik})$$

has the property that no column is a linear continuation of the other columns. We also observe that the conditions (*) remain invariant, if we add to a column a linear combination of the other columns and if we permute the columns.

Since N = n-1, the vectors γ_{1k} are linearly dependent. By applying the above elementary transformations on the columns, we can assume $\gamma_{1n} = 0$. The vectors γ_{in} span a linear space of dimension $\geqq 1$, and condition (*) implies that $\gamma_{11}, \ldots, \gamma_{1n-1}$ are perpendicular to this linear space, and hence belong to a linear space of dimension $\leqq n-2$. It follows that $\gamma_{11}, \ldots, \gamma_{1n-1}$ are linearly dependent and we can assume $\gamma_{1n-1} = 0$. Since there is no linear combination of the columns $\gamma_{in-1}, \gamma_{in}$ which is zero, the vectors $\gamma_{in-1}, \gamma_{in}$ span a linear space of dimension $\geqq 2$. The above process can again be applied, and finally we prove that $\gamma_{1k} = 0$ and $\gamma_{ik} = 0$. But this is a contradiction, and the theorem follows.

Remark. The above argument can be applied to establish the following slightly more general theorem:

Let k(p) be the minimum number of linear differential forms in terms of which the curvature forms Ω_{ij} at p can be expressed, and let $k = \max_{p \in M} k(p)$. Then a closed Riemannian manifold of dimension n can not be isometrically imbedded in an Euclidean space of dimension 2n-k-1.

2. Imbedding and rigidity problems in Euclidean space

Let M be an <u>abstract Riemannian manifold</u>, that is, a differentiable manifold with a positive definite symmetric covariant tensor field of the second order or, what is the same, a positive definite (ordinary) quadratic differential form. Two abstract Riemannian manifolds are <u>isometric</u>, if they are differentially homeomorphic and if under the homeomorphism the differential forms are mapped to each other.

Concerning the relations between abstract Riemannian manifolds and submanifolds of an Euclidean space, two problems naturally arise:

1) Imbedding problem: Is an abstract Riemannian manifold isometric to a submanifold in an Euclidean space?

2) Rigidity problem: Are two isometric submanifolds in an Euclidean space necessarily congruent or symmetric?

Our knowledge of the first problem is extremely meagre. It is not known whether the hyperbolic plane (that is, the two-dimensional R. M. with Gaussian curvature = -1) can be imbedded in an Euclidean space of sufficiently high dimension.

The purpose of this section is to study the rigidity problem for a hypersurface in Euclidean space. The most interesting case is that of surfaces in three-dimensional Euclidean space, about which the rigidity theorem was first proved by Cohn-Vossen under the further assumption that the surfaces are convex. We shall present a proof due to G. Herglotz and apply this idea to prove a generalization of a theorem of Christoffel.

We use the notation of the last section, with $N = 1$. Two hypersurfaces are now given in E^{n+1} and we denote the quantities for the second hypersurface by the same symbols with dashes. From the isometry of the hypersurfaces it follows that the homeomorphism h between them can be extended into a homeomorphism h' between the manifolds of their tangent frames such that

$$\omega_\alpha = h'^* \bar{\omega}_\alpha, \tag{11}$$

h'^* being again the dual homomorphism on the differential forms under h'.

Taking the exterior derivative of this equation and applying the lemma of the last section, we get

(12) $$\omega_{\alpha\beta} = h'^{*}\bar{\omega}_{\alpha\beta}$$

Another exterior differentiation gives

(13) $$\Omega_{\alpha\beta} = h'^{*}\bar{\Omega}_{\alpha\beta}$$

Put

(14) $$\lambda_{\alpha\beta} = \lambda_{n+1\alpha\beta}, \quad \lambda'_{\alpha\beta} = h'^{*}\bar{\lambda}_{n+1\alpha\beta},$$

and consider the matrices

(15) $$\Lambda = (\lambda_{\alpha\beta}), \quad \Lambda' = (\lambda'_{\alpha\beta}).$$

Condition (13) signifies in matrix language that corresponding two-rowed determinants of the matrices Λ and Λ' are equal. If $n \geqq 3$ and the ranks of Λ and Λ' are $\geqq 2$, we conclude that $\Lambda' = \pm \Lambda$, which implies that the hypersurfaces are congruent or symmetric.

More interesting is therefore the case $n = 2$, about which we shall prove the following theorem:

Theorem 1. (Cohn-Vossen.) Two closed convex surfaces in E^3 which are isometric are congruent or symmetric.

Proof. We put

$$y_A = (p\mathbb{I}_A),$$

$$\omega'_{\alpha n+1} = h'^{*}\bar{\omega}_{\alpha n+1}.$$

Then we have

$$dy_1 = \omega_1 + y_2\omega_{12} + y_3\omega_{13},$$

$$dy_2 = \omega_2 + y_1\omega_{21} + y_3\omega_{23}.$$

$$d(y_1\omega'_{23} - y_2\omega'_{13}) = \omega_1\wedge\omega'_{23} - \omega_2\wedge\omega'_{13} + y_3(\omega_{13}\wedge\omega'_{23} - \omega_{23}\wedge\omega'_{13})$$

$$= \left\{(\lambda'_{11} + \lambda'_{22}) + y_3(\lambda_{11}\lambda'_{22} + \lambda'_{11}\lambda_{22} - 2\lambda_{12}\lambda'_{12})\right\}\omega_1\wedge\omega_2.$$

Introduce the mean and Gaussian curvatures

$$H = \frac{1}{2}(\lambda_{11} + \lambda_{22}) , \ K = \lambda_{11}\lambda_{22} - \lambda_{12}^2, \text{ etc.}$$

and define

$$J = \lambda_{11}\lambda'_{22} + \lambda_{22}\lambda'_{11} - 2\lambda_{12}\lambda'_{12} = 2K - \begin{vmatrix} \lambda'_{11}-\lambda_{11} & \lambda'_{12}-\lambda_{12} \\ \lambda'_{12}-\lambda_{12} & \lambda'_{22}-\lambda_{22} \end{vmatrix} .$$

From the above formula we have, on integrating over the surface M,

$$2\int H'dS + \int y_3 J dS = 0,$$

where $dS = \omega_1 \wedge \omega_2$ is the element of area of M. In particular, if we identify the surfaces M and $\overline{M}$,

$$2\int HdS + \int 2y_3 KdS = 0.$$

Subtracting, we get

$$2\int HdS - 2\int H'dS = \int y_3(J-2K)\, dS = -\int y_3 \begin{vmatrix} \lambda'_{11}-\lambda_{11} & \lambda'_{12}-\lambda_{12} \\ \lambda'_{12}-\lambda_{12} & \lambda'_{22}-\lambda_{22} \end{vmatrix} dS.$$

We can choose the origin inside M, so that $y_3 < 0$. Since $K \geqq 0$, the integrand in the right-hand side of the equation is $\geqq 0$. It follows that

$$\int H \wedge S - \int H'dS \leqq 0.$$

By symmetry we must also have

$$\int H'dS - \int HdS \leqq 0.$$

Therefore

$$\int H'dS - \int HdS = 0,$$

and

$$\int y_3 \begin{vmatrix} \lambda'_{11}-\lambda_{11} & \lambda'_{12}-\lambda_{12} \\ \lambda'_{12}-\lambda_{12} & \lambda'_{22}-\lambda_{22} \end{vmatrix} dS = 0.$$

But this is possible, only when

$$\lambda'_{11} = \lambda_{11}, \ \lambda'_{12} = \lambda_{12}, \ \lambda'_{22} = \lambda_{22}.$$

Hence the two surfaces are congruent.

Theorem 2. <u>Let two convex closed hypersurfaces M and $\overline{M}$ in E^{n+1} be differentiably homeomorphic such that at corresponding points the third fundamental forms and the sums of the principal radii of curvature are equal. Then M and $\overline{M}$ are congruent or symmetric.</u>

For n = 2 this theorem is due to Christoffel. Its generalization to arbitrary n was made by Kubota. The classical proof, due to A. Hurwitz, makes use of spherical harmonics.

Proof. By definition the third fundamental form of M is

$$\mathrm{III} = \omega_{n+1,1}^{2} + \cdots + \omega_{n+1,n}^{2} .$$

Since the total curvature of M is > 0, we have $(\lambda_{\alpha\beta}) \neq 0$ and we can write

$$\omega_\alpha = \ell_{\alpha\beta}\, \omega_{\beta,n+1},$$

where $(\ell_{\alpha\beta})$ is the inverse matrix of $(\lambda_{\alpha\beta})$. By considering the normal forms of these matrices under orthogonal transformations, it is easy to see that $L = \ell_{\alpha\alpha}$ is the sum of the principal radii of curvature.

Suppose that M and $\overline{M}$ are two convex hypersurfaces satisfying the hypotheses. Using the notations of the proof of Theorem 1, we find

$$d(\epsilon_{\alpha_1\cdots\alpha_n} y_{\alpha_1} \omega'_{\alpha_2} \wedge \omega_{\alpha_3 n+1} \wedge \cdots \wedge \omega_{\alpha_n n+1})$$

$$= \epsilon_{\alpha_1\cdots\alpha_n} (\omega'_{\alpha_1} \wedge \omega_{\alpha_2} \wedge \omega_{\alpha_3 n+1} \wedge \cdots \wedge \omega_{\alpha_n n+1} + y_{n+1} \epsilon_{\alpha_1\cdots\alpha_n} \omega'_{\alpha_1} \wedge \omega_{\alpha_2 n+1} \wedge \cdots \wedge \omega_{\alpha_n n+1}$$

$$= \left\{ \epsilon_{\alpha_1\cdots\alpha_n} (\ell'_{\alpha_1\alpha_1} \ell_{\alpha_2\alpha_2} - \ell'_{\alpha_1\alpha_2} \ell_{\alpha_1\alpha_2}) + y_{n+1} \epsilon_{\alpha_1\cdots\alpha_n} \ell'_{\alpha_1\alpha_1} \right\} \omega_{\alpha_1 n+1} \wedge \cdots \wedge \omega_{\alpha_n n+1},$$

where $\epsilon_{\alpha_1\cdots\alpha_n}$ is +1 or −1 according as $\alpha_1, \ldots, \alpha_n$ form an even or odd permutation of 1, ..., n, and is otherwise zero. We denote by

$$dV = \omega_{1n+1} \wedge \cdots \wedge \omega_{nn+1}$$

the element of volume of the spherical image of M. It follows from the last formula by integration that

$$\int \Big\{ \sum_{\substack{\alpha,\beta \\ \alpha \neq \beta}} (\ell'_{\alpha\alpha}\ell_{\beta\beta} - \ell'_{\alpha\beta}\ell_{\alpha\beta}) + y_{n+1} \sum_{\alpha} \ell'_{\alpha\alpha} \Big\} dV = 0,$$

the integration being taken over M. Taking $\overline{M} = M$, we get

$$\int \Big\{ \sum_{\substack{\alpha,\beta \\ \alpha \neq \beta}} (\ell_{\alpha\alpha}\ell_{\beta\beta} - \ell^2_{\alpha\beta}) + y_{n+1} \sum_{\alpha} \ell_{\alpha\alpha} \Big\} dV = 0.$$

Using the condition L' = L, we get

$$\int \Big\{ \sum_{\substack{\alpha,\beta \\ \alpha \neq \beta}} (\ell'_{\alpha\alpha}\ell_{\beta\beta} - \ell'_{\alpha\beta}\ell_{\alpha\beta}) - \sum_{\substack{\alpha,\beta \\ \alpha \neq \beta}} (\ell_{\alpha\alpha}\ell_{\beta\beta} - \ell^2_{\alpha\beta}) \Big\} dV = 0.$$

By symmetry between M and $\overline{M}$ we have a similar relation with $\ell_{\alpha\beta}$ and $\ell'_{\alpha\beta}$ interchanged. Combining these two relations we can write

$$\int \sum_{(\alpha,\beta)} \begin{vmatrix} \ell'_{\alpha\alpha} - \ell_{\alpha\alpha} & \ell'_{\alpha\beta} - \ell_{\alpha\beta} \\ \ell'_{\alpha\beta} - \ell_{\alpha\beta} & \ell'_{\beta\beta} - \ell_{\beta\beta} \end{vmatrix} dV = 0,$$

where the summation under the integral sign is taken over all combinations of α, β, with $\alpha \neq \beta$. But

$$\sum(\ell'_{\alpha\alpha} - \ell_{\alpha\alpha})(\ell'_{\beta\beta} - \ell_{\beta\beta}) = \tfrac{1}{2}\sum_{\alpha,\beta}(\ell'_{\alpha\alpha} - \ell_{\alpha\alpha})(\ell'_{\beta\beta} - \ell_{\beta\beta}) - \tfrac{1}{2}\sum_{\alpha}(\ell'_{\alpha\alpha} - \ell_{\alpha\alpha})^2$$

$$= -\tfrac{1}{2}\sum_{\alpha}(\ell'_{\alpha\alpha} - \ell_{\alpha\alpha})^2.$$

It follows that

$$\int \Big\{ -\sum_{\substack{(\alpha,\beta) \\ \alpha \neq \beta}} (\ell'_{\alpha\beta} - \ell_{\alpha\beta})^2 - \tfrac{1}{2}\sum_{\alpha}(\ell'_{\alpha\alpha} - \ell_{\alpha\alpha})^2 \Big\} dV = 0.$$

Since the integrand is ≤ 0, this is possible only when it is identically zero, which gives

$$\ell'_{\alpha\beta} = \ell_{\alpha\beta}.$$

From this it follows that M and $\overline{M}$ are congruent.

3. Affine connection and absolute differentiation

In order that differentiation of tensors be defined on a differentiable manifold M, which will be intrinsic, that is, independent of the choice of local coordinates, we shall need an additional structure, an affine connection.

To define the affine connection we consider the space X of all ordered sets of n linearly independent tangent vectors with the same origin (cf. §5, Chapter I). Again denote by $\psi : X \rightarrow M$ the projection. Relative to a base every such set of vectors can be identified with an element of the group G of all $n \times n$ regular matrices. It follows that, corresponding to a coordinate neighborhood U, there is a homeomorphism

$$\varphi_U : U \times G \rightarrow \psi^{-1}(U), \tag{16}$$

called a coordinate function, such that

$$\psi \varphi_U(p, g) = p \quad , \quad p \in U, \; g \in G. \tag{17}$$

If $p \in U \wedge V$, there are matrices $g_{UV}(p)$ defined by

$$\varphi_U(p, g_{UV}(p)g) = \varphi_V(p, g) \quad , \quad g \in G.$$

We suppose the elements of $g_{UV}(p)$ to be differentiable functions, and put

$$\Theta_{UV} = g_{UV}^{-1} dg_{UV}, \tag{18}$$

so that Θ_{UV} is an $n \times n$ matrix of linear differential forms. In $U \wedge V \wedge W$ we have

$$\Theta_{UW} = g_{VW}^{-1} \Theta_{UV} g_{VW} + \Theta_{VW}. \tag{19}$$

An affine connection is defined, by giving in every coordinate neighborhood U, a matrix ω of linear differential forms, such that, in $U \wedge V$,

$$\Theta_{UV} = \omega_V - g_{UV}^{-1} \omega_U g_{UV}. \tag{20}$$

It is easily verified that this definition is compatible.

We put

$$\Omega_U = d\omega_U + \omega_U \wedge \omega_U, \tag{21}$$

so that Ω_U is an $n \times n$ matrix of quadratic differential forms. A little calculation will give

$$\Omega_V = g_{UV}^{-1} \Omega_U g_{UV}. \tag{22}$$

We shall call Ω_U the <u>curvature matrix</u> of the affine connection.

To define absolute differentiation let T be a finite-dimensional vector space and let R be a linear representation of G in T, that is, a homomorphism of G into the group of linear endomorphisms of T. We consider entities of the form (p, U, t), $p \in U$, $t \in T$, where U is a coordinate neighborhood. Two such entities (p, U, t), (p', V, t') are called <u>equivalent</u>, if $p = p'$, $p \in U \cap V$, $t = R(g_{UV})t'$. A natural topology can be defined in the set of such equivalence classes, and the space so obtained is called the <u>tensor bundle</u> of type $R(G)$. A tensor of type $R(G)$ is a <u>cross section</u> of this bundle, and is thus defined in each coordinate neighborhood U by $t = f_U(p) \in T$, and is such that $f_U(p) = R(g_{UV})f_V(p)$ for $p \in U \cap V$. More generally, we can consider the tensor product $T \otimes A^r$ of T and the vector space of exterior differential forms of degree r, operated on by a linear representation of G. A cross section in this bundle is called a tensorial differential form of degree r and type $R(G)$. It is thus defined in each coordinate neighborhood U by $t = f_U(p) \in T \otimes A^r$ and is such that $f_U(p) = R(g_{UV})f_V(p)$, $p \in U \cap V$.

We take a base in $T \otimes A^r$ and consider its linear endomorphisms as matrices. Put

$$\overline{R} = R(g)^{-1} dR(g),$$

which is then a matrix of linear differential forms. It is also left invariant, so that its elements are linear combinations with constant coefficients of the Maurer-Cartan forms of G. We denote the matrix by $\overline{R}(\omega_U)$ when the Maurer-Cartan forms are replaced by the corresponding forms of the connection. (In terms of a base the Maurer-Cartan forms of a linear group can be considered as the elements of an n x n matrix.) Put

$$Df_U = df_U + \overline{R}(\omega_U)f_U. \tag{23}$$

Then we have, in $U \wedge V$,

$$dg_{UV} = g_{UV}\,\omega_V - \omega_U g_{UV},$$

$$dR(g_{UV}) = R(g_{UV})\overline{R}(\omega_V) - \overline{R}(\omega_U)R(g_{UV}),$$

and it follows that

$$Df_U = R(g_{UV})Df_V.$$

Hence Df_U is a tensorial form of type R(G) and degree r + 1. We call Df_U the <u>absolute differential</u> of f_U.

To every differentiable manifold there is naturally defined a tensor of type (1, 1), which defines the identity mapping of the tangent vectors. For geometrical reasons we denote it by dp. The affine connection is called <u>without torsion</u>, if

$$D(dp) = 0.$$

In a coordinate neighborhood U put

$$\omega_U = (\omega_i^j) = (\Gamma_{ik}^j dx^k).$$

Then we have

$$\overline{R}(\omega_U) = \omega_U,$$

and dp has the components dx^i with respect to the coordinate system whose coordinate vectors are tangent vectors to the parametrized coordinate curves. The condition for the affine connection to be without torsion then becomes

$$\Gamma_{ik}^j dx^i \wedge dx^k = 0 \quad \text{or} \quad \Gamma_{ik}^j = \Gamma_{ki}^j.$$

4. Riemannian metric

A differentiable manifold M is called Riemannian, if there is given a quadratic differential form

$$ds^2 = g_{ij}dx^i dx^j.$$

We shall always assume this form to be positive definite. It is clear that the quadratic differential form defines a scalar product in each tangent space, so that we can talk about the length of a vector, the angle between two vectors with the same origin, etc.

The group G in the definition of an affine connection plays an important role. Sometimes the matrices g_{UV} can be so chosen that they belong to a subgroup of G. In particular, if the subgroup is the orthogonal group, then θ_{UV} is anti-symmetric. If ω_U is also anti-symmetric, the affine connection is called a metrical connection.

The fundamental theorem on local Riemannian geometry is the following:

Theorem. On a Riemannian manifold there is exactly one metrical connection without torsion.

To prove this theorem we first notice that in a coordinate neighborhood U, ds^2 can be written as a sum of squares:

$$ds^2 = \omega_1^2 + \dots + \omega_n^2 .$$

It follows that we can choose G to be the orthogonal group. ω_i defines a tensorial form of degree 1. Let (ω_{ij}) be the matrix in U, which defines the metrical connection. The condition that the connection is without torsion gives

$$(24) \qquad D\omega_i = d\omega_i + \omega_{ij} \wedge \omega_j = 0.$$

By the Lemma of §1 a set of $\omega_{ij} = -\omega_{ji}$ satisfying these conditions is unique.

To prove the existence write

(25) $$d\omega_i = A_{ijk}\omega_j \wedge \omega_k \quad , \quad A_{ijk} + A_{ikj} = 0.$$

It is sufficient to put

(26) $$\omega_{ik} = -(A_{kij} + A_{ijk} + A_{jik})\omega_j.$$

This proves the theorem.

We can derive from the above considerations the theory of curvatures of a Riemannian manifold. In fact, exterior differentiation of (24) gives

$$(d\omega_{ik} + \omega_{ij} \wedge \omega_{jk}) \wedge \omega_k = 0.$$

We therefore put

$$d\omega_{ik} + \omega_{ij} \wedge \omega_{jk} = \Omega_{ik},$$

with

$$\Omega_{ik} \wedge \omega_k = 0 \; , \; \Omega_{ik} + \Omega_{ki} = 0.$$

From the last equation it follows that Ω_{ik} is of the form

$$\Omega_{ik} = \Theta_{ikj} \wedge \omega_j + R_{ikj\ell}\,\omega_j \wedge \omega_\ell,$$

when Θ_{ikj} does not contain ω_ℓ. But then we see that Θ_{ikj} is symmetric in k, j and is skew-symmetric in i, k. Therefore $\Theta_{ikj} = 0$ and we have

$$\Omega_{ik} = R_{ikj\ell}\,\omega_j \wedge \omega_\ell.$$

Let us summarize the fundamental equations for local Riemannian geometry in the following form

(27) $$\begin{cases} d\omega_i = -\omega_j \wedge \omega_{ji}, \\ d\omega_{ik} = -\omega_{ij} \wedge \omega_{jk} + \Omega_{ik}, \\ \omega_{ik} + \omega_{ki} = 0 \; , \; \Omega_{ik} + \Omega_{ki} = 0, \\ \Omega_{ik} = R_{ik\ell j}\,\omega_\ell \wedge \omega_j \; , \; R_{ik\ell j} + R_{ikj\ell} = 0, \; R_{ik\ell j} + R_{ki\ell j} = 0. \end{cases}$$

These equations will be called the equations of structure of the Riemann space. Ω_{ik} are called the curvature forms. $R_{ik\ell j}$ gives essentially what is known as the Riemann-Christoffel curvature tensor.

Applying exterior differentiation to the first two equations of (27), we get

$$\omega_j \wedge \Omega_{ji} = 0,$$

$$-\Omega_{ij} \wedge \omega_{jk} + \omega_{ij} \wedge \Omega_{jk} + d\Omega_{ik} = 0.$$

These are called the Bianchi identities.

Over a Riemannian manifold M there are associated different tensor bundles. The more important ones are: 1) the principal bundle, the bundle of all frames $p\mathfrak{n}_1 \cdots \mathfrak{n}_n$, where $\mathfrak{n}_i$ are mutually perpendicular unit vectors at p; 2) the bundle of unit tangent vectors. We denote their total spaces by $X^{(n)}$ and $X^{(1)}$ respectively. There are natural projections

(29) $$X^{(n)} \xrightarrow[\Psi_{n,1}]{} X^{(1)} \xrightarrow[\Psi_{1,0}]{} M,$$

defined by

$$\Psi_{n,1}(p\mathfrak{n}_1 \cdots \mathfrak{n}_n) = p\mathfrak{n}_n,$$

$$\Psi_{1,0}(p\mathfrak{n}_n) = p,$$

and we put

(31) $$\Psi = \Psi_{1,0}\Psi_{n,1}.$$

These projections induce dual mappings of the differential forms in the inverse direction. Our problem is to determine the differential forms in $X^{(n)}$ and $X^{(1)}$, which are dual images of differential forms in $X^{(1)}$ or M, onto which they have been projected. Perhaps the simplest way to do this is to examine the effect of a change of frame. We put

(32) $$\mathfrak{n}_i = u_{ik}\mathfrak{n}^*_k.$$

where (u_{ik}) is an orthogonal matrix, so that $\mathcal{U}_k^*$ are now frames. Denote the differential forms relative to $\mathcal{U}_k^*$ by the same symbols but preceded with asterisks. Then we find

$$\omega_i = u_{ik}\omega_k^* \quad , \quad \omega_k^* = u_{ik}\omega_i ,$$

$$d\omega_i = \omega_j \wedge (-du_{ik}u_{jk} + u_{ik}u_{j\ell}\,\omega_{\ell k}^*),$$

so that

$$\omega_{ji} = du_{ik}u_{jk} - u_{ik}u_{j\ell}\omega_{\ell k}^* .$$

It follows that

$$d\omega_{ik} + \omega_{ij} \wedge \omega_{jk} = u_{ij}u_{k\ell}\,\Omega_{j\ell}^* ,$$

and hence that

(33) $$\Omega_{ik} = u_{ij}u_{k\ell}\,\Omega_{j\ell}^* .$$

From (33) we see that the following are forms in $X^{(n)}$ which are dual images of forms in M:

(34) $$\Delta_4 = \Omega_{ij}\,\Omega_{ji} ,$$
$$\Delta_8 = \Omega_{ij}\,\Omega_{jk}\,\Omega_{k\ell}\,\Omega_{\ell i} ,$$

and more generally, Δ_{4m}, $4m \leqq n$. We shall denote by the same symbols the originals of these forms in M, and we say simply that these forms are <u>in M.</u>

If M is orientable, we can restrict ourselves to the frames $p\mathcal{U}_1 \cdots \mathcal{U}_n$ whose orientations are coherent with an orientation of the manifold. Then two frames at a point p are related by a proper orthogonal transformation, and we can assume (u_{ij}) to be properly orthogonal. In this case the form

(35) $$\Delta_0 = \epsilon_{i_1 \cdots i_n}\,\Omega_{i_1 i_2} \cdots \Omega_{i_{n-1} i_n} ,$$

which exists when the manifold is of even dimensions, is also in M.

By the Bianchi identities it can be verified that Δ_o, Δ_{4m} are closed. A main theorem in differential geometry in the large asserts that their cohomology classes depend only on the differentiable structure of the manifold.

We shall add the following remarks:

A) If M is the Euclidean space with the Euclidean metric, then $\Omega_{ij} = 0$ and ω_i, ω_{ij} are the Maurer-Cartan forms of the group of motions. This can be seen by comparing the discussion of this section to §1.

B) The relation of our presentation to that of classical Riemannian geometry can be given as follows: Let $S_{ikj\ell}$ be the Riemann-Christoffel tensor in the classical sense, and let $\mathfrak{U}_i$ have the components u_i^k (relative to the coordinate system x^j). Then

(36) $$\Omega_{ik} = u_i^m u_k^q S_{mqj\ell}\, dx^j \wedge dx^\ell .$$

4. The Gauss-Bonnet formula

In this section we assume our Riemannian manifold M to be compact and orientable. With Δ_o defined by (35), put

(37) $$\Omega = \begin{cases} (-1)^p \dfrac{1}{2^{2p}\pi^p p!} \Delta_o , & \text{if } n = 2p \text{ is even} \\ 0 , & \text{if } n \text{ is odd.} \end{cases}$$

The Gauss-Bonnet formula states that

(38) $$\int_M \Omega = \chi(M),$$

where $\chi(M)$ is the Euler-Poincare characteristic of M.

We shall make use of the projections as defined in (29) and shall deduce differential forms which are in $X^{(1)}$. Perform a change of frames which leaves $p\mathfrak{N}_n$ unaltered;*

(39) $$\mathfrak{N}_\alpha = u_{\alpha\beta}\, \mathfrak{N}_\beta^* ,$$

where $(u_{\alpha\beta})$ is properly orthogonal. Denoting the quantities formed from the new frames by the same notations preceded with asterisks, we get

(40) $$\omega_{\alpha n} = u_{\alpha\beta}\, \omega^*_{\beta n} ,$$
$$\Omega_{\alpha\beta} = u_{\alpha\gamma} u_{\beta\delta}\, \Omega_{\gamma\delta} .$$

Define

$$\Phi_k = \epsilon_{\alpha_1 \cdots \alpha_{n-1}} \Omega_{\alpha_1\alpha_2} \cdots \Omega_{\alpha_{2k-1}\alpha_{2k}}\, \omega_{\alpha_{2k+1}n} \cdots \omega_{\alpha_{n-1}n},$$

(41)

$$\Psi_k = 2(k+1)\epsilon_{\alpha_1 \cdots \alpha_{n-1}} \Omega_{\alpha_1\alpha_2} \cdots \Omega_{\alpha_{2k-1}\alpha_{2k}} \Omega_{\alpha_{2k+1}n}\, \omega_{\alpha_{2k+2}n} \cdots \omega_{\alpha_{n-1}n},$$

where $k = 0, 1, \ldots, [\frac{n}{2}] - 1$, $[\frac{n}{2}]$ = largest integer $\leq \frac{n}{2}$. If n is odd, $\Phi_{[\frac{n}{2}]}$ is also defined. By convention we set

(42) $$\Psi_{-1} = \Psi_{[\frac{n}{2}]} = 0.$$

Using (40) it is easily verified that Φ_k and Ψ_k, of degrees n-1 and n respectively, are differential forms in $X^{(1)}$.

By exterior differentiation we have

$$d\Phi_k = k\epsilon_{\alpha_1 \cdots \alpha_{n-1}} d\Omega_{\alpha_1\alpha_2} \Omega_{\alpha_3\alpha_4} \cdots \Omega_{\alpha_{2k-1}\alpha_{2k}}\, \omega_{\alpha_{2k+1}n} \cdots \omega_{\alpha_{n-1}n}$$
$$+ (n-2k-1)\epsilon_{\alpha_1 \cdots \alpha_{n-1}} \Omega_{\alpha_1\alpha_2} \cdots \Omega_{\alpha_{2k-1}\alpha_{2k}}\, d\omega_{\alpha_{2k+1}n}\, \omega_{\alpha_{2k+2}n} \cdots \omega_{\alpha_{n-1}n}.$$

This exterior differentiation is carried out in $X^{(n)}$. But as the resulting forms are in $X^{(1)}$, the terms involving $\omega_{\alpha\beta}$ must cancel with each other. Hence we immediately get

(43) $$d\Phi_k = \Psi_{k-1} + \frac{n-2k-1}{2(k+1)}\Psi_k.$$

* Greek indices in this section run from 1 to n-1.

Solving for Ψ_k, we get

$$\Psi_k = d\,\text{Ⓗ}_k \quad , \quad k = 0, 1, \ldots, [\tfrac{n}{2}] - 1,$$

where

$$\text{Ⓗ}_k = \sum_{\lambda=0}^{k} (-1)^{k-\lambda} \frac{(2k+2)\ldots(2\lambda+2)}{(n-2\lambda-1)\ldots(n-2k-1)} \Phi_\lambda \quad , k = 0, 1, \ldots, [\tfrac{n}{2}] - 1.$$

If n is even, say = 2p, then we have

$$d\,\text{Ⓗ}_{p-1} = \Psi_{p-1},$$

where

$$\Psi_{p-1} = n\,\epsilon_{\alpha_1 \ldots \alpha_{n-1}} \Omega_{\alpha_1 \alpha_2} \ldots \Omega_{\alpha_{n-1} n} = \epsilon_{i_1 \ldots i_n} \Omega_{i_1 i_2} \ldots \Omega_{i_{n-1} i_n},$$

If n is odd, say = 2q+1, then

$$d\,\text{Ⓗ}_{q-1} = \Psi_{q-1}.$$

But in this case we have also

$$d\,\Phi_q = \Psi_{q-1},$$

so that

$$d(\text{Ⓗ}_{q-1} - \Phi_q) = 0.$$

By defining

$$(44) \quad \Pi = \begin{cases} \dfrac{1}{\pi^p} \displaystyle\sum_{\lambda=0}^{p-1} (-1)^\lambda \dfrac{1}{1.3\ldots(2p-2\lambda-1)2^{p+\lambda}\lambda!} \Phi_\lambda & , n = 2p, \\ \dfrac{1}{2^{2q+1}\pi^q q!} \displaystyle\sum_{\lambda=0}^{q} (-1)^{\lambda+1} \binom{q}{\lambda} \Phi_\lambda & , n = 2q+1, \end{cases}$$

we therefore get

$$(45) \qquad -d\,\Pi = \Omega .$$

Π is a differential form in $X^{(1)}$ whose exterior derivative is in M, Let K^{n-1} be the (n-1)-dimensional skeleton of a cellular decomposition of M. It is well-known that a continuous non-zero vector field can be defined in K^{n-1}. We extend this field over M, with the possibility of introducing a number of isolated singularities where the vector is zero. Draw about each singularity a small sphere with radius ϵ, and call M_ϵ the domain of these spheres. The vector field defines a mapping $f: M - M_\epsilon \longrightarrow X^{(1)}$, with $\psi_{1,0} f$ = identity. Applying the Theorem of Stokes, we get

$$\int_{M-M_\epsilon} \Omega = \int_{f(M-M_\epsilon)} \Omega = -\int_{f(M-M_\epsilon)} d\Pi = -\int_{\partial f(M-M_\epsilon)} \Pi = I + \eta,$$

where I is the sum of indices of the vector field (Cf. §6, Chapter I), and $\eta \longrightarrow 0$ as $\epsilon \longrightarrow 0$. It follows that, as $\epsilon \longrightarrow 0$,

$$\int_M \Omega = I.$$

This shows that I is independent of the choice of the vector field. By considering a particular field, we verify that it is equal to the Euler-Poincare characteristic $\chi(M)$ of M. Hence the formula is proved.

Chapter III

Theory of Connections

We shall develop in this chapter the theory of connections in a fiber bundle in the sense of Elie Cartan. Its modern treatment was first carried out by Ehresmann and Weil. Our main theorem consists in giving a relationship between a characteristic homomorphism defined by topological properties of the fiber bundle and a homomorphism defined by the local properties of the connection. As we shall see, the Gauss-Bonnet formula for a compact orientable Riemann manifold is a corollary of this theorem.

1. Resume on fiber bundles

There are now available several accounts of the general theory of fiber bundles, in particular, N. E. Steenrod's forthcoming book. We can therefore restrict ourselves to a resume of the notions and results which are necessary for our purpose. To conform with some notations currently in use (at least in Princeton), we change our previous notation by interchanging B and X, so that B will be the bundle and X the base space.

Let F be a space acted on by a topological group G of homeomorphisms. A <u>fiber bundle</u> with the <u>director space</u> F and <u>structural group</u> G consists of topological spaces B, X and a mapping ψ of B onto X, together with the following:

1) X is covered by a family of neighborhoods $\{U_\alpha\}$, called the <u>coordinate neighborhoods</u>, and to each U_α there is a homeomorphism (a coordinate function), $\varphi_\alpha : U_\alpha \times F \longrightarrow \psi^{-1}(U_\alpha)$, with $\psi\varphi_\alpha(x, y) = x$, $x \in U_\alpha$, $y \in F$.

2) As a consequence of 1), a point of $\psi^{-1}(U_\alpha)$ has the coordinates (x, y), and a point of $\psi^{-1}(U_\alpha \cap U_\beta)$ has two sets of coordinates (x, y) and (x, y'), satisfying $\varphi_\alpha(x, y) = \varphi_\beta(x, y')$. It is required that $g_{\alpha\beta}(x)\colon y' \longrightarrow y$ is a continuous mapping $g_{\alpha\beta}$ of $U_\alpha \cap U_\beta$ into G.

The spaces X and B are called the <u>base space</u> and the <u>bundle</u> respectively. Each subset $\psi^{-1}(x) \subset B$ is called a <u>fiber</u>.

Since we wish to allow changes of coordinate neighborhoods and coordinate functions in a bundle, an equivalence relation is introduced. Two bundles (B, X), (B', X) with the same base space and the same F, G are called <u>equivalent</u>, if, $\{U_\alpha, \varphi_\alpha\}$, $\{V_{\alpha'}, \Theta_{\alpha'}\}$ being respectively their coordinate neighborhoods and coordinate functions, there is a fiber-preserving homeomorphism $T\colon B \longrightarrow B'$ such that the mapping $h_{\alpha\alpha'}(x)\colon y \longrightarrow y'$ defined by $\Theta_{\alpha'}(x, y) = T\,\phi_\alpha(x, y')$ is a continuous mapping of $U_\alpha \cap V_{\alpha'}$ into G.

An important operation on fiber bundles is the construction from a given bundle of other bundles with the same structural group, in particular, the principal fiber bundle which has G as director space acted upon by G itself as the group of left translations. It can be defined as follows: For $x \in X$ let G_x be the totality of all maps $\varphi_{\alpha, g}(x)\colon F \longrightarrow \psi^{-1}(x)$ defined by $y \longrightarrow \varphi_\alpha(x, g(y))$, $y \in F$, $g \in G$, relative to a coordinate neighborhood U_α containing x. G_x depends only on x. Let $B^* = \cup_{x \in X} G_x$ and define the mapping $\psi^*\colon B^* \longrightarrow X$ by $\psi^*(G_x) = x$ and the coordinate functions $\varphi^*_\alpha(x, g) = \varphi_{\alpha, g}(x)$. Topologize B^* such that the φ^*_α's define homeomorphisms of $U_\alpha \times G$ into B^*. The bundle (B^*, X) so obtained is called a <u>principal fiber bundle</u>. This construction is an operation on the equivalence classes of bundles in the sense that two fiber bundles are equivalent if and only if their principal fiber bundles are equivalent. Similarly, an inverse operation can be defined, which will permit us to construct bundles with a given principal bundle and having as director space a given space acted upon by the structural group G. Such bundles are called associate bundles. An important property of the principal fiber bundle is that B^* is acted upon by G as right translations.

Let G' be a subgroup of G. If the mappings $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ have their images in G' for every pair of coordinate neighborhoods U_α and U_β with a non-empty intersection, we say that the bundle has the structural group G'. A bundle is called trivial, if it is equivalent to a bundle with a structural group which consists of the unit element only.

A cross section of a bundle is a mapping $f : X \to B$ such that ψf is the identity. Using this notion, it is easy to establish the following statements, of which the second is a consequence of the first:

1) A bundle with the group G is equivalent to a bundle with the group $G' \subset G$, if and only if the associate bundle having G/G' as director space has a cross section.

2) A bundle is trivial if and only if its principal bundle has a cross section.

Suppose a bundle be given, with the above notations. Let f be a mapping of a space Y into X. The neighborhoods $\{f^{-1}(U_\alpha)\}$ then form a covering of Y and coordinate functions $\varphi_\alpha' : f^{-1}(U_\alpha) \times F \to f^{-1}(U_\alpha) \times \psi^{-1}(U_\alpha)$ can be defined by $\varphi_\alpha'(\eta, y) = \eta \times \varphi_\alpha(f(\eta), y)$. This defines a fiber bundle $Y \times \psi^{-1}(f(Y))$ over Y, with the same director space F and the same group G. The new bundle is said to be induced by the mapping f.

This method of generating new fiber bundles from a given bundle is very useful, particularly in the case when a so-called universal bundle exists. Let the director space and the group G be given and fixed for our present considerations. A bundle with the base space X_0 is called universal relative to a space X, if every bundle over X is equivalent to a bundle induced by a mapping $X \to X_0$ and if two such induced bundles are equivalent when and only when the mappings are homotopic. If, for a space X, there exists a universal bundle with the base space X_0, then the classes of bundles over X are in one-one correspondence with the homotopy classes of mappings $X \to X_0$, so that the enumeration of the bundles over X reduces to a homotopy classification problem.

For our purpose the existence of a universal bundle has another consequence. Let $H(X_0, R)$, $H(X, R)$ denote the cohomology rings of the spaces X_0, X respectively, with the coefficient ring R. It follows from the above that the induced dual homomorphism

$$h': \quad H(X_0, R) \longrightarrow H(X, R)$$

is completely determined by the bundle. h' will be called the characteristic homomorphism, its image $h'(H(X_0, R)) \subset H(X, R)$ the characteristic ring, and an element of the characteristic ring a characteristic (cohomology) class.

A necessary and sufficient condition that a bundle is universal relative to all polyhedra of dimension n is that its principal bundle B satisfies the conditions: $\pi_i(B) = 0$, $0 \leqq i \leqq n$, where $\pi_0(B) = 0$ means that B is connected. The proof of this theorem can be found in Steenrod's book or in: S. S. Chern and Y. Sun, The imbedding theorem for fiber bundles, Trans.Amer.Math.Soc. 67, 286-303 (1949).

When a universal bundle exists, it may not be unique. However, we shall show that the characteristic homomorphism is independent of the choice of the universal bundle by proving the theorem: Let $\psi_o: B_o \rightarrow X_o$, $\psi'_o: B'_o \rightarrow X'_o$ be two universal principal bundles relative to complexes of dimension n. There are one-one isomorphisms

$$H^{r}(X_o, G) \cong H^{r}(X'_o, G), \quad r \leq n.$$

For simplicity we assume X_o and X'_o to be cellular complexes. Denote by X_o^n and $X_o'^n$ their n-dimensional skeletons. Since every continuous mapping is homotopic to a cellular mapping, the sub-bundles over X_o^n, $X_o'^n$ can be induced respectively by mappings

$$f: X_o^n \longrightarrow X_o'^n, \quad g: X_o'^n \longrightarrow X_o^n.$$

It follows that these sub-bundles are equivalent to bundles induced by the mappings gf and fg. Since the given bundles are universal, we conclude that gf and fg are homotopic to the identity mapping. This proves the theorem. The theorem is valid under more general assumptions of the base space of the universal bundle. The proof will then make use of the singular complex and is more complicated.

We shall show that a universal bundle exists whenever the base space X is compact and the structural group is a connected Lie group. According to a theorem due to E. Cartan, Malcev, Iwasawa, and Mostow, all the maximal compact subgroups of a connected Lie group G are conjugate to each other and the homogeneous space G/G_1 is homeomorphic to an Euclidean space, where G_1 is a maximal compact subgroup. (Cf. in particular, K. Iwasawa, On some types of topological groups, Annals of Math. 50, p. 530 (1949)). Using this theorem, it follows that every bundle with the group G is equivalent to a bundle with the group G_1 and that two bundles with the group G are equivalent when and only when their equivalent bundles with the group G_1 are equivalent relative to G_1. In other words, so long as the equivalence classes of bundles are concerned, we can replace G by its maximal compact subgroup G_1.

G_1 being a compact Lie group, it can be considered as a subgroup of the rotation group R(m) in m variables. Imbed R(m) as a subgroup of R(m+n+1) which operates on the first m variables, while R(n+1) operates on the last n+1 variables. Then we have

$$R(m+n+1) \supset G \times R(n+1) \supset I_m \times R(n+1),$$

where I_m denotes the group of the identity. By the natural projection

$$\psi : R(m+n+1) / I_m \times R(n+1) \longrightarrow R(m+n+1) / G \times R(n+1),$$

we get a principal fiber bundle with the group G. It is universal relative to complexes of dimension n, since, by the covering homotopy theorem, all homotopy groups of the bundle up to the dimension n inclusive are zero.

2. Connections

We consider a fiber bundle and adopt the notation of the last section. For the purpose of differential geometry the following assumptions will be made: 1) B, X, F are differentiable manifolds; 2) G is a Lie group which acts differentiably on F; 3) the projection of B onto X is differentiable.

Let L(G) be the Lie algebra of G. L(G) is invariant under the left translations of G, while the right translations and the inner automorphisms of G induce on L(G) a group of linear endomorphisms ad(G), called the <u>adjoint group</u> of G. Relative to a base of L(G) there are the left-invariant linear differential forms ω^i and the right-invariant linear differential forms π^i, each set consisting of linearly independent forms whose number is equal to the dimension of G. A fundamental theorem on Lie groups asserts that their exterior derivatives are given by

(1)
$$d\omega^i = \tfrac{1}{2}\sum_{j,k} c^i_{jk}\,\omega^j \wedge \omega^k ,$$
$$d\pi^i = -\tfrac{1}{2}\sum_{j,k} c^i_{jk}\,\pi^j \wedge \pi^k , \quad 1 \leq i, j, k \leq \dim G.$$

where c^i_{jk} are the so-called constants of structure. They are anti-symmetric in the lower indices and satisfy the Jacobi relations obtained by expressing that the exterior derivative of the right-hand side of (1) is equal to zero:

(2)
$$\sum_m \left(c^i_{mj}\, c^m_{k\lambda} + c^i_{mk}\, c^m_{\lambda j} + c^i_{m\lambda}\, c^m_{jk}\right) = 0.$$

This being said about the structural group G, the dual mapping of the mapping $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$ carries ω^i and π^i into linear differential forms in $U_\alpha \cap U_\beta$, which we shall denote by $\omega^i_{\alpha\beta}$ and $\pi^i_{\alpha\beta}$ respectively. Since $g_{\alpha\gamma} = g_{\alpha\beta}\, g_{\beta\gamma}$ in $U_\alpha \cap U_\beta \cap U_\gamma$, we have

(3)
$$\omega^i_{\alpha\gamma} = \sum_j ad\,(g_{\beta\gamma})^i_j\, \omega^j_{\alpha\beta} + \omega^i_{\beta\gamma},$$
$$\pi^i_{\alpha\gamma} = \pi^i_{\alpha\beta} + \sum ad\,(g^{-1}_{\alpha\beta})^i_j\, \pi^j_{\beta\gamma}.$$

We can also interpret $\omega^i_{\alpha\beta}$ as a vector-valued linear differential form in $U_\alpha \cap U_\beta$, with values in L(G), and shall denote it simply by $\omega_{\alpha\beta}$, when so interpreted.

The generalization of the notion of a tensor field in classical differential geometry leads to the following situation: Let E be a vector space acted on by a representation M(G) of G. A <u>tensorial</u>

differential form of degree r and type M(G) is an exterior differential form u_α of degree r in each coordinate neighborhood U_α, with values in E, such that, in $U_\alpha \cap U_\beta$, $u_\alpha = M(g_{\alpha\beta}) u_\beta$. The exterior derivative du_α of u_α is in general not a tensorial differential form. To recover its tensorial character, a connection is introduced into the fiber bundle.

A connection in the fiber bundle is a set of linear differential forms Θ_α in U_α, with values in L(G), such that

(4) $$\omega_{\alpha\beta} = -ad(g_{\alpha\beta})\,\Theta_\alpha + \Theta_\beta.$$

It follows from (3) that such relations are consistent, in $U_\alpha \cap U_\beta \cap U_\gamma$. With the help of a connection absolute or covariant differentiation can be defined as follows: Let $\bar{M}(X)$, $X \in L(G)$, be the representation of the Lie algebra L(G) induced by the representation M(G) of G. M(G) being a linear group, the elements of $\bar{M}(X)$ can be identified with linear endomorphisms in E. If a base is chosen in the vector space E, both M(G) and $\bar{M}(X)$ can be represented by matrices. Moreover, we can take as their elements differential forms with values in L(G). With this understanding equation (4) goes under the homomorphic mapping $G \longrightarrow M(G)$ into the equation

$$dM(g_{\alpha\beta}) = M(g_{\alpha\beta})\,\bar{M}(\Theta_\beta) - \bar{M}(\Theta_\alpha)\,M(g_{\alpha\beta}).$$

This, together with the equation obtained by exterior differentiation of $u_\alpha = M(g_{\alpha\beta})u_\beta$, shows that

(5) $$Du_\alpha = du_\alpha + \bar{M}(\Theta_\alpha) \wedge u_\alpha$$

is a tensorial differential form of degree r + 1 and the same type M(G).

It is easy to prove that a connection can always be defined in a fiber bundle. In fact, let $\{U_\alpha\}$ be a finite or countable covering of X by coordinate neighborhoods. There exists an open covering $\{V_\alpha\}$ of

X, such that $V_\alpha \subset U_\alpha$. We define the connection in $V_1 + \dots + V_\alpha$ by induction on α. If the connection has been defined in $V_1 + \dots + V_{\alpha-1}$, Θ_α is given in $V_\alpha \cap (V_1 + \dots + V_{\alpha-1})$ and is consistent there because of (4). By an elementary extension theorem, Θ_α exists in V_α such that it becomes the pre-assigned form in $V_\alpha \cap (V_1 + \dots + V_{\alpha-1})$ and is zero in $X-U_\alpha$. Thus we define a connection relative to the covering $\{V_\alpha\}$. We shall give a second proof of the existence of a connection at the end of this section.

The above definition of a connection makes use of the coordinate neighborhoods and is entirely analytic. We shall give equivalent but intrinsic definitions and at the same time interpret the definition geometrically.

For this purpose we consider the principal bundle, whose bundle space we again denote by B. If (x, s), (x, t), s, $t \in G$, are the coordinates of a point of B relative to the coordinate neighborhoods U_α and U_β respectively, we have $g_{\alpha\beta} t = s$. Since the left-invariant differential forms of G can be regarded as the components of such a form ω with values in L(G), the form $ad(s)\Theta_\alpha + \omega$ is a linear differential form in $U_\alpha \times G$, with values in L(G). From (4) we can verify that in $U_\alpha \times U_\beta$ this differential form is equal to the same form constructed from U_β. The set of these differential forms defines therefore an L(G)-valued linear differential form in B, which we shall denote by $\varphi(b)$, $b \in B$. Since B is acted on by G as right translations, we can study the effect of such a translation on $\varphi(b)$, and we find $\varphi(bg) = ad(g^{-1})\varphi(b)$. This leads to the following definition of a connection:

A connection in a fiber bundle is defined by an L(G)-valued linear differential form in the principal fiber bundle having the properties:

1) Under right translations it is transformed according to the adjoint group, $\varphi(bg) = ad(g^{-1})\,\varphi(b)$; 2) It is transformed by the dual mapping of the identity mapping of a fiber into the bundle into the left-invariant form of the fiber. The second property has a sense, because a fiber in the principal bundle has a group structure defined up to a left-translation.

Since an $L(G)$-valued linear differential form can be interpreted as a linear mapping of the tangent space of B into $L(G)$, it follows from the above definition that a connection is a linear mapping at every point of B such that: 1) every tangent vector is mapped into a tangent vector to the fiber; 2) every tangent vector to the fiber remains invariant; 3) the mapping is invariant under right translations. It is an elementary theorem of linear algebra that under such a mapping the set of tangent vectors which are mapped into zero form a linear space complementary to the tangent space of the fiber. A connection therefore gives rise to a family of tangent subspaces in the principal fiber bundle which are transversal to the fibers (that is, which span with the tangent space of the fiber the tangent space of the bundle) and are invariant under right translations.

From these geometrical considerations it follows easily that we can define a connection in a homogeneous space whose group is a semi-simple Lie group. In fact, let R be a semi-simple Lie group and H a closed subgroup of R. Then R is a principal fiber bundle over R/H with the director space H. It is known that a positive definite Riemannian metric can be defined in R, which is both left and right invariant. Using this metric, we can define a mapping which maps a tangent vector of R to its orthogonal projection in the fiber through the origin of the vector. This mapping satisfies the three conditions above and therefore defines a connection in the bundle.

In particular, it follows that we can define a connection in the particular universal bundle chosen above. Let $f_1\ X' \longrightarrow X$ be a mapping which induces a bundle over X'. If the original bundle has a connection given by the differential form Θ_α in U_α, the dual mapping f^* of f carries Θ_α into $f^* \Theta_\alpha$ in $f^{-1}(U_\alpha)$ for which the relation corresponding to (4) is valid. The forms $f^* \Theta_\alpha$ therefore define an induced connection in the induced bundle. Since every bundle is equivalent to one induced by mapping its base space into the base space of a universal bundle and since the latter has a connection, it follows that a connection can be defined in any bundle. This gives a second proof of the statement that a connection can always be defined in a bundle.

3. Local theory of connections; the curvature tensor

To study the local properties of the connection we again make use of a base of the Lie algebra, relative to which the form Θ_α has the components Θ^i_α. We put

$$(6) \qquad \textcircled{H}^i_\alpha = d\Theta^i_\alpha - \tfrac{1}{2}\sum_{j,k} c^i_{jk}\,\Theta^j_\alpha \wedge \Theta^k_\alpha \qquad 1 \leqq i, j, k \leqq \dim G.$$

The form $\textcircled{H}_\alpha$, whose components relative to the base are $\textcircled{H}^i_\alpha$, is then an exterior quadratic differential form of degree 2, with values in L(G). It is easy to verify that $\textcircled{H}_\alpha = ad(g_{\alpha\beta})\,\textcircled{H}_\beta$ in $U_\alpha \cap U_\beta$. The $\textcircled{H}$'s therefore define a tensorial differential form of degree 2 and type ad(G), called the <u>curvature tensor</u> of the connection.

The following formulas for absolute differentiation can easily be verified:

$$\overline{M}(\text{Ⓗ}_\alpha) = d\overline{M}(\Theta_\alpha) + \overline{M}(\Theta_\alpha)^2,$$

(7) $$D\,\text{Ⓗ}_\alpha = 0,$$

$$D^2 u_\alpha = \overline{M}(\text{Ⓗ}_\alpha) \wedge u_\alpha .$$

The second relation is known as the Bianchi identity. It shows that absolute differentiation of the curvature tensor does not give further invariants.

It follows from our intrinsic definition of connection that a connection in a bundle gives rise to a connection in every bundle of its equivalence class, so that we can speak of a connection in an equivalence class of bundles. The connections in two equivalent bundles are called equivalent, if they define the same connection in the class of bundles.

More interesting is the notion of local equivalence of two connections. Given two bundles with the same structural group and with a connection defined in each bundle. The structures pertaining to the second bundle we denote by the same notation with dashes. We shall define the notion that the connections are equivalent at a point $x \in X$ and a point $x' \in X'$. In fact, let U_α and U'_α be coordinate neighborhoods containing x and x' respectively. The two connections are said to be equivalent at x and x' if: 1) there exist open sets V, V' satisfying $x \in V \subset U_\alpha$, $x' \in V' \subset U'_\alpha$; 2) there exist a differentiable homeomorphism $f: V \longrightarrow V'$ and a differentiable mapping $g: V \longrightarrow G$ such that

$$\Theta_\alpha = g^* \omega + ad(g) \; (f^* \Theta'_{\alpha'}).$$

We can verify that this condition is independent of the choice of U_α and $U'_{\alpha'}$. Instead of making this verification we can also formulate the definition in an intrinsic form. We shall say that a mapping l: $\psi^{-1}(V) \longrightarrow \psi'^{-1}(V')$ is admissible if there is a mapping l_*: $V \longrightarrow V'$ such that $l\, \psi^{-1}(x) = \psi'^{-1}(l_*(x))$ and that the mapping $g \longrightarrow g'$ defined by $l\, \varphi_\alpha(x, g) = \varphi'_{\alpha'}(\lambda_*(x), g')$ is a left translation of G. Clearly the last condition is independent of the choice of the coördinate neighborhoods U_α and $U'_{\alpha'}$ in the definition. Remembering that a connection gives rise to an L(G)-valued linear differential form $\varphi(b)$, $b \in B$ in the principal bundle B, we can formulate the definition of local equivalence of two connections as follows: The two connections are equivalent at x and x' if there are neighborhoods V and V' of x and x' respectively such that there is an admissible homeomorphism between $\psi^{-1}(V)$ and $\psi'^{-1}(V')$ under which $\varphi(b)$ and $\varphi'(b')$ are equal. The equivalence of the two definitions can be easily verified by making use of (3).

A connection is called locally flat at $x \in X$ if $\Theta_\alpha = 0$ in a neighborhood of x (relative to a coordinate neighborhood U_α containing x) or is equivalent to one with this property. It follows that a necessary and sufficient condition for a connection to be locally flat at x is that there exist an open set V containing x and contained in a coordinate neighborhood U_α and a mapping f of V into G such that $\Theta_\alpha = f^*\omega$ in V. When a connection is locally flat at x, the curvature tensor vanishes in a neighborhood of x. Conversely, when the curvature tensor is zero, it follows from the theorem of Frobenius that the system of differential equations

$$\Theta_\alpha - f^*\omega = 0,$$

where the mapping f is the unknown function, is completely integrable. Hence f exists in a neighborhood of x and the connection is flat at x.

As a first instance of the problem concerning the relationship between properties of a bundle with those of the connections which can be defined on it, we consider the case that the connection is everywhere flat. Then, by Frobenius's Theorem, there exists to every coordinate neighborhood U_α a mapping $f_\alpha : U_\alpha \longrightarrow G$ such that $f_\alpha^* \omega = \theta_\alpha$. In $U_\alpha \cap U_\beta$ we define $g'_{\alpha\beta} = f_\alpha \; g_{\alpha\beta} \; f_\beta^{-1}$. Then we have $g'^*_{\alpha\beta} \omega = 0$, which in turn implies that the mapping $g'_{\alpha\beta}$ is constant in every connected component of $U_\alpha \cap U_\beta$. These mappings $g'_{\alpha\beta}$ define a bundle over X equivalent to B. It follows that the bundle will not be affected if we replace the topology in G by the discrete topology. Such a fiber bundle is a covering space. If X is simply connected, B is a topological product of G and X. In general, B is a product of the connected component of G and a covering of X. Thus the flatness of a connection implies topological properties of the bundle.

4. The homomorphism h and its independence of connection

We consider a fiber bundle (B,X), with a structural group G which is a Lie group, and we assume that a connection is given in the bundle.

A real-valued multilinear function $P(Y_1, \cdots, Y_k)$, with arguments $Y_1, \cdots, Y_k \in L(G)$, is called <u>invariant</u>, if

$$P(ad(a)Y_1, \cdots, ad(a)Y_k) = P(Y_1, \cdots, Y_k) \text{ for all } a \in G.$$

Replacing a by an infinitesimal transformation, the condition of invariance implies

$$\sum_{i=1}^{k} P(Y_1, \cdots, Y_{i-1}, [Z,Y_i], Y_{i+1}, \cdots, Y_k) = 0, \tag{8}$$

where Z is any element of L(G). If G is connected, condition (8) is equivalent to the definition of invariance. Instead of taking Y_i to be in L(G), we can take them to be L(G)-valued exterior differential forms of degree p_i. If, therefore, Y_i are tensorial differential forms of degree p_i and type ad(G), P is a differential form of degree $\sum_{i=1}^{k} p_i$ defined in the whole manifold X.

We can define the bracket operation [Z,W], where Z,W are L(G)-valued exterior differential forms. In fact, let Z^i, W^i be the components of Z,W relative to a base of L(G). The components $\sum_{j,k} c^i_{jk} Z^j \wedge W^k$ define an L(G)-valued differential form which is independent of the choice of the base and will be denoted by [Z,W]. The degree of [Z,W] is the sum of the degrees of Z and W.

The formula (8) can be generalized to the case that Z,Y_i are differential forms. In particular, when Z is a linear L(G)-valued differential form and p_i = dim Y_i, we have

$$(9) \qquad \sum_{i=1}^{k} (-1)^{p_1+\cdots+p_{i-1}} P(Y_1, \cdots, Y_{i-1}, [Z,Y_i], Y_{i+1}, \cdots, Y_k) = 0.$$

We now suppose our invariant functions P to be symmetric in their arguments and call them for simplicity invariant polynomials. We shall make them into a ring. By the definition of addition

$$(10) \qquad (P + Q)(Y_1, \cdots, Y_k) = P(Y_1, \cdots, Y_k) + Q(Y_1, \cdots, Y_k),$$

all invariant polynomials of degree k form an abelian group. Let I(G) be the direct sum of these abelian groups for all $k \geq 0$. If P and Q are invariant polynomials of degrees k and l respectively, we define their product PQ to be an invariant polynomial of degree k + l given by

$$(11) \qquad (PQ)(Y_1, \cdots, Y_{k+\ell}) = \frac{1}{N} \sum P(Y_{i_1}, \cdots, Y_{i_k}) Q(Y_{i_{k+1}}, \cdots, Y_{i_{k+\ell}}),$$

where the summation is extended over all permutations of the vectors Y_i and

N is the number of such permutations. This definition of multiplication, together with the distributive law, makes I(G) into a commutative ring, the ring of invariant polynomials of G.

Let P be an invariant polynomial of degree k and let us substitute for its arguments the curvature tensor $\textcircled{H}$ of the connection. Then $P(\textcircled{H}) = P(\textcircled{H}, \cdots, \textcircled{H})$ is a differential form in X, of degree 2k. Its exterior derivative is, by the Bianchi identity (7_2) and by the observation that the absolute derivative of a product follows the same rule as the exterior derivative,

$$dP(\textcircled{H}) = D\,P\,(\textcircled{H}) = 0.$$

Hence $P(\textcircled{H})$ is a closed differential form and defines, according to the de Rham theory, an element of the cohomology ring H(X) of X having as coefficient ring the field of real numbers. We shall denote the resulting mapping by

$$h: \quad I(G) \dashrightarrow H(X). \tag{12}$$

It is clear that the ring operations in I(G) are so defined that h is a ring homomorphism. Notice that I(G) depends only on G and that h is defined with the help of a connection in the bundle.

Concerning the homomorphism h the following <u>theorem of Weil</u> is fundamental: h is independent of the choice of the connection. In other words, two different connections in the bundle give rise to the same homomorphism h.

We proceed to give Weil's proof of this theorem. Let two connections be given in the same bundle, defined by the differential forms Θ_α and Θ'_α respectively. Then $u_\alpha = \Theta_\alpha - \Theta'_\alpha$ is a linear differential form of the type ad(G), and their curvature tensors are related by the formula

$$\textcircled{H}^1 = \textcircled{H} - Du - \tfrac{1}{2}[u,u]. \tag{13}$$

The theorem will be proved, if we express $P(\textcircled{H}') - P(\textcircled{H})$ as an exterior derivative. For this purpose we introduce the notations

$$(14) \qquad P(Y) = P(Y, \cdots, Y), \quad Q(Z,Y) = P(Z,Y, \cdots, Y),$$

where Y,Z are L(G)-valued differential forms. With an auxiliary variable t we put

$$F(t) = P(W-tY-t^2Z)-P(W).$$

Then

$$F'(t) = -k\, Q(Y+2tZ, W-tY-t^2Z),$$

and we have

$$(15) \qquad P(W-Y-Z)-P(W) = -k \int_0^1 Q(Y+2tZ,\ W-tY-t^2Z)dt.$$

Making use of (9) and the relations

$$(16) \qquad D^2u = -[\textcircled{H},\ u], \quad [[u,u],u] = 0,$$

we get the formula

$$(17) \qquad dQ(u, \textcircled{H}-tDu-\frac{t^2}{2}[u,u]) = Q(Du + t\,[u,u],\ \textcircled{H}-tDu-\frac{t^2}{2}[u,u]).$$

Integrating with respect to t from 0 to 1, we find

$$(18) \qquad dR(u, \textcircled{H}, Du, \tfrac{1}{2}[u,u]) = P(\textcircled{H}')-P(\textcircled{H}),$$

where R is defined by

$$(19) \qquad R(V,W,Y,Z) = -k\int_0^1 Q(V,W-tY-t^2Z)dt.$$

This proves the theorem.

Another consequence can be derived from the above considerations. We consider the principal bundle B and the L(G)-valued linear differential form φ which defines the connection. Recalling the definition of φ in terms of a coordinate neighborhood, we immediately get

$$(20) \qquad d\varphi - \frac{1}{2}[\varphi,\varphi] = \Phi,$$

where Φ is an L(G)-valued quadratic differential form in B, which, in a coordinate neighborhood U_α, has the representation $ad(s)\textcircled{H}_\alpha$. Manipulations similar to those in the above proof then give the formula:

$$dR(\varphi, \Phi, d\varphi, -\frac{1}{2}[\varphi,\varphi]) = -P(\Phi). \tag{21}$$

Since $P(\Phi)$ is clearly the dual image of the form $P(\textcircled{H})$ in X, it follows that $\psi{*}P(\textcircled{H})$ is cohomologous to zero in B. We identify G with a fiber of B and denote the inclusion mapping by $i: G \dashrightarrow B$. Then $i{*}R(\varphi, \Phi, d\varphi, -\frac{1}{2}[\varphi,\varphi])$ is closed and defines an element of H(G), which is defined up to an element of $i{*}H(B)$. The result is a group homomorphism

$$t:\ I(G) \dashrightarrow H(G)/\ i{*}H(B) \tag{22}$$

which maps the polynomial P into an element of the quotient group on the right-hand side having as representative the differential form $c_k\, Q(\omega, d\omega)$, where

$$c_k = -k\int_0^1 (-t + t^2)^{k-1}\, dt.$$

We notice that $i{*}\varphi = \omega$.

5. The homomorphism h for the universal bundle

The results of the last section can in particular be applied to the universal bundle

$$R(m+n+1)/\ I_m \times R(n+1) \dashrightarrow R(m+n+1)/\ G \times R(n+1)$$

considered in §1. In this case we denote the base space by X_o and the homomorphism h by

$$h_o:\ I(G) \dashrightarrow H(X_o).$$

The purpose of this section is to prove the theorem: In the dimensions $\leqq n$, h_o is a one-one isomorphism.

X_0 being a homogeneous space, the structure of the ring $H(X_0)$ can be studied by the method of integral invariants of Elie Cartan (Cf. E. Cartan, Sur les invariants intégraux de certains espaces homogènes clos et les propriétés topologiques de ces espaces, Annales de la Soc. Pol. de Math. t. 8, pp. 181-225, 1929). The main ideas and results are as follows: Let G/K be the homogeneous space, G a compact Lie group of dimension r and K a closed subgroup of dimension s of G. The Lie algebra L(K) is then a subalgebra of L(G). In the dual vector space L*(G) of L(G) there is determined a subspace M*(K) consisting of all the elements of L*(G) which are perpendicular to L(K). The adjoint group ad(G) acts on both vector spaces L(G) and L*(G). Its subgroup ad(K), consisting of the linear endomorphisms ad(a), $a \in K$, leaves L(K) and M*(K) invariant. If we identify the space L*(G) with the space of left-invariant linear differential forms (Maurer-Cartan forms), a base $\omega_1, \ldots, \omega_r$ in L*(G) can be so chosen that $\omega_{s+1}, \cdots, \omega_r$ span M*(K) and that the system of differential equations

$$\omega_{s+1} = \cdots = \omega_r = 0$$

is completely integrable. Since ad(K) leaves M*(K) invariant, it induces a group of linear endomorphisms on M*(K) and on the exterior algebra $\bigwedge$ (M*(K)) of M*(K). Denote by R(G/K) the subring of $\bigwedge$ (M*(K)) consisting of all elements invariant under the action of ad(K). Cartan proved that R(K) is stable under exterior differentiation and that its cohomology ring (that is, the quotient ring of the subring of closed forms over the ideal of derived forms) is isomorphic to the cohomology ring of the space G/K.

Returning to our problem, consider the rotation group R(m) in which G is imbedded as a subgroup. A positive definite Riemann metric can be defined in R(m), which is invariant under both left and right translations. If we choose the Maurer-Cartan forms of R(m) such that the ds^2 of the Riemann metric is equal to the sum of their squares, the constants of structure will be anti-symmetric in all three indices.

To the base space of our universal bundle we now apply the method of Cartan. We agree on the following ranges of indices:

$$1 \leqq \alpha, \beta, \gamma \leqq m, \quad m+1 \leqq r,s,t \leqq m+n+1, \quad 1 \leqq A,B,C \leqq m+n+1.$$

$$1 \leqq \lambda, \mu, \nu \leqq \dim R(m) - \dim G, \quad 1 \leqq i,j,k \leqq \dim G.$$

Setting $K = G \times R(n+1)$ and remembering that our underlying group is here $R(m+n+1)$, we see that the vector space $M^*(K)$ is spanned by τ_λ, $\omega_{\alpha r}$, where τ_λ span the space $M^*(G)$ in $L^*(R(m))$. For $a \in I_m \times R(n+1)$ the induced endomorphism of ad(a) on $\omega_{\alpha r}$ is given by

$$\omega'_{\alpha r} = \sum_s a_{rs}\, \omega_{\alpha s},$$

where (a_{rs}) is a proper orthogonal matrix. Now the so-called first main theorem on vector invariants asserts that any integral rational invariant of a system of vectors under the rotation group is an integral rational function of their scalar products and their determinants. It follows that in order that a form of degree $\leqq n$ generated by $\tau_\lambda, \omega_{\alpha r}$ be invariant under ad(K), it must contain $\omega_{\alpha r}$ in the combinations

$$\Omega_{\alpha\beta} = -\sum_r \omega_{\alpha r} \wedge \omega_{\beta r}. \tag{23}$$

By the equations of Maurer-Cartan for $R(m+n+1)$ these forms satisfy the relations

$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}.$$

We take in $L^*(R(m))$ a base (θ_i, τ_λ) such that: 1) τ_λ span $M^*(G)$; the invariant Riemann metric ds^2 is equal to the sum of the squares of θ_i, τ_λ, so that the constants of structure are anti-symmetric in all three indices. We then have

$$d\theta_i = \frac{1}{2}\sum_{j,k} c_{ijk}\,\theta_j \wedge \theta_k + \frac{1}{2}\sum_{\mu,\nu} c_{i\mu\nu}\tau_\mu \wedge \tau_\nu + \textcircled{H}_i' ,$$

$$d\tau_\lambda = \frac{1}{2}\sum_{\mu,i} c_{\lambda\mu i}\tau_\mu \wedge \theta_i + \frac{1}{2}\sum_{\mu,\nu} c_{\lambda\mu\nu}\tau_\mu \wedge \tau_\nu + T_\lambda' ,$$

where (θ_i, τ_λ) are related to the set $(\omega_{\alpha\beta})$ by a non-singular linear transformation. By introducing the forms

$$(24)\quad \textcircled{H}_i = \frac{1}{2}\sum_{\mu,\nu} c_{i\mu\nu}\tau_\mu \wedge \tau_\nu + \textcircled{H}_i' ,\quad T_\lambda = \frac{1}{2}\sum_{\mu,\nu} c_{\lambda\mu\nu}\tau_\mu \wedge \tau_\nu + T_\lambda' ,$$

we can write

$$(25)\quad \begin{aligned} d\theta_i &= \frac{1}{2}\sum_{j,k} c_{ijk}\,\theta_j \wedge \theta_k + \textcircled{H}_i , \\ d\tau_\lambda &= \frac{1}{2}\sum_{\mu,i} c_{\lambda\mu i}\tau_\mu \wedge \theta_i + T_\lambda . \end{aligned}$$

The ring of invariant forms is generated by $1, \tau_\lambda, \textcircled{H}_i, T_\lambda$.

The forms θ_i are in $R(m + n + 1)$, but can be regarded to be in the bundle $R(m + n + 1)/ I_m \times R(n + 1)$, because there are uniquely determined forms in the latter of which they are the dual images under the natural projection. Regarded as in the bundle, these forms θ_i define a connection, of which $\textcircled{H}_i$ is the curvature tensor. Relative to this connection an absolute differentiation is defined in $M^*(G)$ in which $ad(G)$ acts. Since τ_λ can be regarded as an $M^*(G)$-valued linear differential form in X_0, it has an absolute derivative which is a quadratic differential form of the same type. We find

$$(26)\quad D\tau_\lambda = T_\lambda .$$

To describe the ring of invariant forms generated by $1, \tau_\lambda, \textcircled{H}_i$, T_λ, we consider real-valued multilinear functions

$$P(Y_1, \cdots, Y_k; Z_1, \cdots, Z_p; W_1, \cdots, W_q), Y_a, Z_b \in M^*(K), \ W_c \in L^*(G),$$

which are anti-symmetric in the Y's, and symmetric in the Z's and in the W's, and are invariant under the action of ad(G):

$$P(\mathrm{ad}(a)Y_1, \cdots, \mathrm{ad}(a)Y_k; \ \mathrm{ad}(a)Z_1, \cdots, \mathrm{ad}(a)Z_p; \ \mathrm{ad}\ (a)W_1, \cdots, \mathrm{ad}(a)W_q)$$

(27)

$$= P(Y_1, \cdots, Y_k; Z_1, \cdots, Z_p; W_1, \cdots, W_q).$$

An invariant form $P(\tau, T, \textcircled{H})$ is obtained, by substituting τ for each Y, T for each Z, and $\textcircled{H}$ for each W. The degree of the form is $k + 2(p+q)$ and the form itself is said to be of type (k,p,q). As in the case of the ring of invariant polynomials these functions P can be made into a ring by the definition of an addition and a multiplication in such a way that one gets a ring homomorphism under the substitution described above. Denote by R the ring of differential forms $P(\tau, T, \textcircled{H})$. If D denotes the absolute differentiation relative to the connection defined above, we have, for $p \in R$,

(28) $$dP(\tau, T, \textcircled{H}) = DP(\tau, T, \textcircled{H}) .$$

It follows that R is stable under D and that the derived ring relative to D is isomorphic in the dimensions $\leqq n$ to the cohomology ring $H(Y_0)$.

Under D we have

(29) $$D\,\textcircled{H} = 0, \quad DT = 0,$$

the first being the Bianchi identity and the second following from a similar calculation. We shall prove: For $P \in R$ with $DP = 0$ there exist $Q, P_1 \in R$ such that

$$P(\tau, T, \textcircled{H}) = DQ + P_1(\textcircled{H}),$$

where P_1 is of type $(0,0,q')$.

The proof follows an idea of algebraic topology in the construction of a homotopy operator. We define an operator f in R which is an anti-derivation (that is, a differential operator with $f(ab) = f(a)b + (-1)^r af(b)$, $r = \dim(a)$ and is such that

$$f\tau = 0,\ f\,T = \tau\,,\ f\,Ⓗ = 0.$$

Then we have

$$(Df+fD)T = \frac{1}{k+p}\,T,\ (Df+fD)\tau = \frac{\tau}{k+p}\,,(Df+fD)\,Ⓗ = 0.$$

Now P is a sum of terms of types (k,p,q). We can assume P to be homogeneous in the sense that $k + 2(p+q) = \text{const}$. Among the terms of P let m be the largest value for k+p. Then the value of k+p in each term of

$$P-\frac{1}{k+p}\,(fD+Df)P$$

is smaller than m. Since DP = 0, we prove the above statement by induction on m.

It follows that every class of $H(X_0)$ contains as a representative a differential form P(Ⓗ) of the type (0,0,q'). Since the latter is clearly never a derived form in R relative to D, the mapping so established is a one-one isomorphism. Thus the theorem stated in the beginning of this section is completely proved.

6. The fundamental theorem

We consider a fiber bundle with a Lie group as structural group and having as base space a differentiable manifold. A connection is supposed to be defined in the bundle. The problem on the relationship between properties of the connection and topological properties of the bundle can be described as follows:

Let G_1 be a maximal compact subgroup of the structural group G. Since an invariant polynomial under G is an invariant polynomial under

G_1, there is a natural homomorphism

$$\sigma : I(G) \longrightarrow I(G_1). \tag{30}$$

We have also defined a homomorphism

$$h: I(G) \longrightarrow H(X),$$

which, according to the theorem of Weil, is independent of the choice of the connection. On the other hand, the bundle defines a characteristic homomorphism

$$h': H(X_o) \longrightarrow H(X).$$

By the theorem of the last section $H(X_o)$ is isomorphic to $I(G_1)$ in the dimensions $\leqq n$. The characteristic homomorphism can therefore be written

$$h': I(G_1) \longrightarrow H(X).$$

Our fundamental theorem asserts that

$$h = h'\sigma. \tag{31}$$

Let us notice that h is defined by the connection, h' by the topological properties of the bundle, and σ by the relation between the groups G and G_1.

To prove this theorem we observe that the bundle is equivalent to one with the structural group G_1. Define a connection in the bundle with the group G_1. Since $L(G_1)$ is a subalgebra of $L(G)$, the connection can be regarded as relative to the group G. Using this connection, we see that the two sides of (31) are identical.

An important particular case of the theorem is one for which G is itself compact. Then h is identical with the characteristic homomorphism. In particular, it follows that if a connection can be defined in the bundle which is locally flat, then the characteristic ring is zero.

Another consequence of the theorem is that the kernel of σ in I(G) is mapped into zero by h. On the other hand, as we shall see later from examples, σ is not necessarily onto.

The description of the homomorphism σ depends on the study of the relation between the Lie algebras L(G) and $L(G_1)$, and the cohomology structure of these Lie algebras. Their study has recently been successfully carried out by H. Cartan, Chevalley, Koszul, and Weil. (Cf. Koszul, J. L., Homologie et cohomologie des algebres de Lie, Bull. Soc. Math. de France 78, 65-127 (1950))

Chapter IV

Bundles with the Classical Groups as Structural Groups

Fiber bundles which have as structural groups the classical groups, namely the orthogonal group, the rotation group, the unitary group, and the symplectic group, play an important role in problems of geometry. In fact, such bundles include those naturally associated to differentiable manifolds and complex manifolds. When a Riemann metric is given on a differentiable manifold or an Hermitian metric on a complex manifold, the metrics will define intrinsically connections in the tangent bundles, and the determination of the characteristic homomorphism by the connection gives rise to a relationship between the tangent bundle and the metric. Moreover, at least at the present stage, we have a better knowledge of the characteristic homomorphisms of such bundles. This chapter will be devoted to the study of these particular cases.

1. Homology groups of Grassmann manifolds

Let $R(n), O(n), U(n)$ denote respectively the rotation group, the orthogonal group, and the unitary group in n variables. For bundles with these groups as structural groups and with base spaces of dimension $\leq k$, universal bundles are respectively given by

$$R(n+N)/R(N) \dashrightarrow R(n+N)/\, R(n) \times R(N), \qquad k \leq N-1,$$

$$(1) \qquad R(n+N)/R(N) \dashrightarrow R(n+N)/\, R(n+N) \cap (R(n) \times R(N)), \quad k \leq N-1,$$

$$U(n+N)/U(N) \dashrightarrow U(n+N)/\, U(n) \times U(N), \quad k \leq 2N.$$

That these are universal bundles follows simply from the vanishing of homotopy groups of dimensions r, $0 \leq r \leq k$, of the bundle.

In order to describe the characteristic homomorphism, it is therefore necessary to study the homology properties of the base spaces of these bundles. While this gives rise to a new treatment in the case of real coefficients, it is wider in scope in the sense that more general coefficients can be taken into consideration. Geometrically these base spaces are the so-called Grassmann manifolds and are respectively: 1) the manifold $\widetilde{G}(n,N,R)$ of oriented n-dimensional linear spaces through a point in a real Euclidean space $E^{n+N}(R)$ of dimension n+N; 2) the manifold G(n,N,R) of non-oriented n-dimensional linear spaces through a point in a real Euclidean space $E^{n+N}(R)$ of dimension n+N; 3) the manifold G(n,N,C) of n-dimensional linear spaces through a point in a complex Euclidean space $E^{n+N}(C)$ of dimension n+N. The homology groups of the Grassmann manifolds have been studied by Ehresmann who defined cellular decompositions which we proceed to describe.

As the three cases admit a common treatment, we shall adopt the convention to denote the Grassmann manifold by G(n,N) and the Euclidean space by E^{n+N}, when the results in question are valid for all three cases.

Let O be a point of E^{n+N}, the n-dimensional linear spaces through which constitute the Grassmann manifold G(n,N) under consideration. Take through O a sequence of linear spaces

$$(2) \qquad L^{a_1+1} \subset L^{a_2+2} \subset \cdots \subset L^{a_n+n}, \quad 0 \leq a_1 \leq \cdots \leq a_n \leq N,$$

whose superscripts are the dimensions. The set of all elements X of G(n,N), i.e., n-dimensional linear spaces through O, satisfying the conditions

$$\dim (X \cap L_i^{a_i+i}) \geq i, \quad i = 1, \cdots, n,$$

is called a Schubert variety, to be denoted by $(a_1 \cdots a_n)$. The Schubert varieties have the following properties:

1) $\dim (a_1 \cdots a_n) = \sum_{i=1}^{n} a_i$.

2) $(a_1 \cdots a_n)$ depends on the choice of the sequence (2).

However, relative to the same set of integers $a_1, \cdots, a_n$, with $0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n \leqq N$, the Schubert varieties defined by different sequences (2) are equivalent under the group of motions about 0 in E^{n+N}.

3) If $a_i = a_{i+1}$, condition (C_i) is a consequence of (C_{i+1}).

4) Define the open Schubert varieties:

$$(3) \qquad (a_1 \cdots a_n)^* = (a_1 \cdots a_n) - \sum_{\substack{i=1,\cdots,n \\ a_{i-1} < a_i}} (a_1 \cdots a_{i-1} a_i - 1 \cdots a_n).$$

If the sequences used in the definition of the Schubert varieties are chosen from a fixed sequence of linear spaces

$$(4) \qquad 0 \subset L^1 \subset L^2 \subset \cdots \subset L^{n+N-1} \subset L^{n+N} (= E^{n+N}),$$

the open Schubert varieties form a family of disjoint subsets of $G(n,N)$, which cover $G(n,N)$.

5) An open Schubert variety is an open cell in the case of $G(n,N,R)$ and $G(n,N,C)$ and is the union of two open cells in the case of $\tilde{G}(n,N,R)$.

The properties 1),2),3),4) are easily verified. We shall give a proof of 5). Let $(a_1 \cdots a_n)^*$ be the open Schubert variety, defined by means of the sequence (4). The statement being clear for $n = 1$, we assume its truth for $n-1$ and give the proof by induction on n. If $a_1 = 0$, we take a hyperplane K through 0 and not containing L^1. For $X \in (a_1 \cdots a_n)^*$ we have $\dim(X \cap K) = n-1$. Moreover, $X \in (a_1 \cdots a_n)^*$ if and only if $X \cap K \in (a_2 \cdots a_n)'^*$, the latter being defined by means of the sequence in which K intersects the sequence (4). Hence the theorem follows by our induction hypothesis.

Suppose now that $a_1 > 0$. We take a hyperplane M satisfying the conditions: 1) $M \cap L^{a_1+1} = L^{a_1}$; 2) M intersects $L^{a_1+2}, \cdots, L^{n+N-1}$ respectively in the linear spaces

$$(5) \qquad L'^{a_1+1}, \cdots, L'^{n+N-2} .$$

Put $X' = X \cap M$, $Y = X \cap L^{a_1+1}$. If $X \in (a_1 \cdots a_n)^*$, then

$$\dim X' = n-1, \ \dim Y = 1,$$

and we have

$$Y \subset L^{a_1+1} - L^{a_1}, \ X' \in (a_2 \cdots a_n)'^* ,$$

the latter being defined by the sequence (5). Conversely, the Y and X' satisfying the last conditions span an $X \in (a_1 \cdots a_n)^*$. Since the locus of Y is an a_1-cell, the theorem follows by induction.

Since a Grassmann manifold is an algebraic variety, it follows from general theorems on the covering of algebraic varieties by complexes that it has a simplicial decomposition of which our cellular decomposition in 4) is a consolidation. The additive homology structure can therefore be determined by the boundary relations of our cellular decomposition. To determine the boundary relations we must first orient the open cells. For G(n,N,C) the open cells have a complex structure which determines a natural orientation. Let $(a_1 \cdots a_n)$ denote the chain carried by the oriented open cell and also the cochain taking the value one for this chain and the value zero for other chains of the same dimension. Then we have:

6) The boundary relations of the cellular decomposition of $G(n,N,C)$ defined in 4) are

$$\partial(a_1 \cdots a_n) = 0, \quad \delta(a_1 \cdots a_n) = 0. \tag{6}$$

It follows that the $(a_1 \cdots a_n)$, $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq N$, form a homology or cohomology base of $G(n,N,C)$. In particular, all Betti numbers of odd dimension are zero.

Concerning the real Grassmann manifolds we first observe that $\widetilde{G}(n,N,R)$ is a covering space of $G(n,N,R)$. An element $X \in (a_1 \cdots a_n)^*$ is spanned by the vectors

$$\xi_i = (x_{iA}) \in L^{a_i+i} - L^{a_i+i-1}, \quad i = 1, \cdots, n; \; A = 1, \cdots, n+N,$$

where x_{iA} are the components of ξ_i in a coordinate system having the linear spaces in the sequence (4) as the coordinate linear spaces. Moreover, we have

$$\Delta = \prod_{i=1}^{n} x_{i\, a_i+i} \neq 0 .$$

In the case of $\widetilde{G}(n,N,R)$, $(a_1 \cdots a_n)^*$ consists of two open cells, to be denoted by $(a_1 \cdots a_n)^+$ and $(a_1 \cdots a_n)^-$ and defined respectively by $\Delta > 0$ and $\Delta < 0$. In order to determine the vectors ξ_i uniquely, we also assume

$$x_{i a_i+i} = 1, \quad x_{i\, a_j+j} = 0, \; j < i, \; i = 1, \cdots, n, \; j = 1, \cdots, n-1 \qquad \text{for } (a_1 \cdots a_n)^+,$$

(7)

$$x_{k\, a_k+k} = 1, \; x_{n\, a_n+n} = -1, \; x_{i\, a_j+j} = 0, \; j < i,$$
$$i = 1, \cdots, n, \; j, k = 1, \cdots, n-1 \qquad \text{for } (a_1 \cdots a_n)^- .$$

The remaining x's, namely,

$$x_{11},\dots,x_{1a_1},x_{21},\ ,x_{2a_2},x_{2a_2+2},\dots,x_{2a_2+1},x_{31},\dots,x_{3a_3+2},\dots x_{n1},\dots,x_{na_n+n-1},$$

then form a coordinate system on each of the open cells. We orient the open cell by the above lexicographic order of the coordinates. In this way the cells of $\tilde{G}(n,N,R)$ are oriented. By projecting into $G(n,N,R)$ by the covering mapping, we orient the cells of $G(n,N,R)$. This orientation of the cells has the property that under a covering transformation in $\tilde{G}(n,N,R)$ which transforms an $X \in \tilde{G}(n,N,R)$ into the same linear space with the opposite orientation, the oriented cell $(a_1 \dots a_n)^+$ goes into the oriented cell $(a_1 \cdots a_n)^-$, and vice versa.

It is clear that the incidence number of two cells is zero, unless they are of the forms $(a_1 \dots a_i+1 \dots a_n)^{\pm}$ and $(a_1 \cdots a_n)^{\pm}$. Denoting these cells by E^{m+1} and E^m respectively, we shall determine their incidence number by the following process: Let an element X of E^m be determined by the vectors ξ_i normalized by the conditions (7). Put $\xi_i' = \xi_i + t\, e_{a_i+i}$, where e_{a_i+i} is the (a_i+i)th coordinate vector. For $t > 0$, t and the coordinates in E^m provide a system of coordinates in E^{m+1}. When E^{m+1} is oriented by this ordered system of coordinates, it has $-E^m$ as its boundary. Since the relation between the two systems of coordinates is easily determined, we find the incidence number between E^m and E^{m+1}. The result may be summarized by the following theorem:

7) The boundary relations of $\tilde{G}(n,N,R)$ are

$$\partial(a_1 \dots a_n)^+ = \sum (-1)^{a_1+\dots+a_i}[(-1)^{n-i+1}(a_1 \dots a_i-1 \dots a_n)^+ + (-1)^{a_i+1}(a_1 \dots a_i-1 \dots a_n)^-],$$

(8)

$$\partial(a_1 \dots a_n)^- = \sum (-1)^{a_1+\dots+a_i}[(-1)^{a_i+1}(a_1 \dots a_i-1 \dots a_n)^+ + (-1)^{n-i+1}(a_1 \dots a_i-1 \dots a_n)^-],$$

$$\delta(a_1..a_n)^+ = \sum(-1)^{a_1+\cdots+a_i}[(-1)^{n-i}(a_1..a_i+1..a_n)^+ + (-1)^{a_i+1}(a_1..a_i+1..a_n)^-],$$

$$\delta(a_1..a_n)^- = \sum(-1)^{a_1+\cdots+a_i}[(-1)^{a_i+1}(a_1..a_i+1..a_n)^+ + (-1)^{n-i}(a_1..a_i+1..a_n)^-].$$

The boundary relations of $G(n,N,R)$ are

$$\partial(a_1\cdots a_n) = \sum(-1)^{a_1+\cdots+a_i}[(-1)^{n-i+1}+(-1)^{a_i+1}](a_1..a_i-1...a_n),$$

(9)

$$\delta(a_1\cdots a_n) = \sum(-1)^{a_1+\cdots+a_i}[(-1)^{n-i}+(-1)^{a_i+1}](a_1..a_i+1...a_n).$$

In all these formulas the sums in the right-hand sides are taken for $i = 1, \cdots, n$ such that the symbols have a sense, that is, that the inequalities in (2) are satisfied.

Having the boundary relations, the determination of the additive homology structure is a purely combinatorial problem. For definiteness we shall carry this out for the cohomology groups of dimensions $< N$ of $G(n,N,R)$. We shall call cochains of the type $(a_1 \cdots a_n)$ <u>elementary cochains</u>. For such an elementary cochain define two sets of integers b_k, i_k, by the conditions

$$a_1 = \cdots = a_{i_1} < a_{i_1+1} = \cdots = a_{i_1+i_2} < \cdots < a_{i_1+\cdots+i_{s-1}+1} = \cdots = a_{i_1+\cdots+i_s},$$

(10)

$$i_1 + \cdots + i_s = n,$$

$$b_1 = a_{i_1}, \cdots, b_s = a_{i_1+\cdots+i_s},$$

so that

$$0 \leqq b_1 < b_2 < \cdots < b_s \leqq N,$$

(11)

$$1 \leqq i_1, \cdots, i_s, \quad i_1+\cdots+i_s = n.$$

Every elementary cochain determines therefore two sequences of non-negative integers $(b_1, \cdot\cdot, b_s)$, $(i_1, \cdot\cdot, i_s)$, satisfying the relations (11), and is determined by them. According to the nature of these sequences the elementary cochains are classified into three kinds: 1) It is of the first kind if all the b's and i's are even or if $b_2, \cdot\cdot, b_s, i_2, \cdot\cdot, i_s$ are even and $b_1 = 0$; 2) It is of the second kind if $b_{k+1}, \cdot\cdot, b_s, i_{k+1}, \cdots, i_s$ are even and b_k is odd; 3) It is of the third kind if $b_k, \cdots, b_s, i_{k+1}, \cdot\cdot, i_s$ are even, $b_k \neq 0$, and i_k is odd. It follows from (9) that an elementary cochain of the first kind is an integral cocycle. Moreover, if y is an elementary cochain of the second kind, with $b_k < N$, we have

$$\frac{1}{2}\delta y = \pm z + \sum \text{ elem. cochains of second kind,}$$

where z is an elementary cochain of the third kind obtained from y by replacing b_k by $b_k + 1$. This correspondence between y and z is one-one.

As stated above, we restrict ourselves to the study of cochains of dimension $r < N$. From the above remark we have the following consequences: 1) A cochain of dimension r is a linear combination of cochains x^{γ}, y^{γ}, $\frac{1}{2}\delta y^{\gamma-1}$, where x^{γ} is a linear combination of cochains of dimension r of the first kind and y^{γ}, $y^{\gamma-1}$ those of cochains of dimensions r, r-1 respectively of the second kind. A sum $\lambda x^{\gamma} + \mu y^{\gamma} + \nu\frac{1}{2}\delta y^{\gamma-1} = 0$ only if $\lambda = \mu = \nu = 0$; 2) y^{γ} is a cocycle only if it is zero.

It follows that an integral cocycle of dimension r is of the form $x + \frac{1}{2}\delta y^{\gamma-1}$ and that the latter is a coboundary only when $x^{\gamma} = 0$, $y^{\gamma-1} = 0$. We have therefore the theorem:

8) An integral cohomology base of dimension $r < N$ is formed by x^{γ}, $\frac{1}{2}\delta y^{\gamma-1}$, where x^{γ} runs over the elementary cochains of dimension r of the first kind and $y^{\gamma-1}$ those of dimension r-1 of the second kind.

Weak cohomology and cohomology with coefficients in a finite field have an even simpler structure. They are given by the theorem:

9) All elementary cochains $(a_1 \cdots a_n)$ of dimension r are cocycles and form a cohomology base mod 2. All elementary cocycles of the first kind of dimension $r < N$ form a cohomology base with rational coefficients and with coefficients mod $p \geqq 3$.

Since the dimension of an elementary cocycle of the first kind is a multiple of 4, a cocycle of dimension r such that $r \not\equiv 0 \ (4)$, $r < N$, is cohomologous to 0 in rational coefficients or in coefficients mod $p \geqq 3$.

Before concluding this section, it may be of interest to show how a formula of Whitney can be derived from the boundary relations (8). On $\widetilde{G}(n,N,R)$ we consider the cochain

$$w^r = (0\cdots 0 \underbrace{1\cdots 1}_{r})^+ - (0\cdots 0 \underbrace{1\cdots 1}_{r})^-$$

From (8) we get

$$\delta w^r = \left\{(-1)^r + 1\right\} w^{r+1}$$

Hence w^r is a cocycle if r is odd and is a cocycle mod 2 if r is even. Moreover, we have

$$w^{2r+1} = \frac{1}{2}\,\delta w^{2r} \,. \tag{12}$$

When a bundle over X is induced by a mapping $f: X \dashrightarrow \widetilde{G}(n,N,R)$, the cohomology classes of $f^* w^r$ are called the Stiefel-Whitney classes. In that context the above formula was first given by Whitney.

If X is an orientable differentiable manifold of dimension n and B is the tangent bundle over X, it can be verified that

$$(f^* w^r)\, X = \chi(X) \tag{13}$$

is equal to the Euler-Poincaré characteristic of X.

2. Differential forms in Grassmann manifolds

The Grassmann manifolds are compact homogeneous spaces acted on transitively by a compact Lie group. It follows that the method of integral invariants described in §5, Chapter III, is applicable and that it is possible to describe the cohomology classes with real coefficients by invariant differential forms. We shall set up this relationship in this section. Because of the different features of the results we divide the discussions into cases:

Case A. Complex Grassmann manifolds $G(n,N,C)$.

The Maurer-Cartan forms ω_{AB}, $A, B = 1, \cdots, n+N$, of the unitary group $U(n+N)$ are the elements of an Hermitian matrix. We put

$$(14)\quad \Omega_{ik} = \sum_r \omega_{ir}\omega_{rk}\ , \quad i, k = 1, \cdots, n,\ r = n+1, \cdots, n+N$$

From these differential forms we construct the following forms:

$$(15)\quad \begin{aligned} \Psi_m &= \sum \delta(p_1 \cdots p_m;\ q_1 \cdots q_m)\Omega_{p_1 q_1} \cdots \Omega_{p_m q_m}, \\ \Phi_m &= \sum \delta^2(p_1 \cdots p_m;\ q_1 \cdots q_m)\Omega_{p_1 q_1} \cdots \Omega_{p_m q_m}, \\ \Delta_m &= \sum \Omega_{p_1 p_2}\Omega_{p_2 p_3} \cdots \Omega_{p_m p_1}, \end{aligned}$$

where $p_1, \cdots, p_m$ is a permutation of $q_1, \cdots, q_m$, the summation is over all such permutations and all $p_1, \cdots, p_m = 1, \cdots, n$, and $\delta(p_1, \cdots, p_m; q_1, \cdots, q_m)$ is the Kronecker index, equal to +1 or -1 according as the permutation is even or odd. These can be regarded as differential forms in $G(n,N,C)$ in the sense that there are uniquely determined forms in $G(n,N,C)$ of which they are the dual images under the natural projection $U(n+N) \longrightarrow G(n,N,C)$. It is clear that each of the three sets of forms in (15) can be expressed as polynomials of the forms of another, with numerical coefficients. A simple direct computation shows that the forms Δ_m are closed, so that the same is true of the forms Φ_m and Ψ_m. Moreover, by the first main theorem on vector invariants for the unitary group, it follows that every invariant form in G(n,N,C) is a polynomial in the forms of one of the three sets, with numerical coefficients.

An invariant differential form Ω of degree r in G(n,N,C) defines a cochain γ according to the equation $\gamma \cdot z = \int_z \Omega$, z = any r-dim cycle. γ is a cocycle if Ω is closed. We say that Ω belongs to the cohomology class of γ. Of interest is therefore the question of deciding the cohomology class to which a given closed form belongs. Concerning this we have the following theorem:

Theorem 1. The form $\frac{1}{(2\pi\sqrt{-1})^m m!}\Psi_m$ belongs to the class $(0\cdots0\ 1\cdots1)$. The form $\frac{1}{(2\pi\sqrt{-1})^m m!}\Phi_m$ belongs to the class $(0\cdots0\ m)$.

To prove this theorem we shall integrate the forms over the Schubert varieties. We introduce, for a Schubert variety S, the integers b_k, i_k, defined in (10), (11), so that its dimension is $2m = 2(b_1 i_1 + \cdots + b_s i_s)$. If X belongs to the corresponding open Schubert variety, we define a linear space of dimension $i_1+i_2+\cdots+i_k$ by

$$Y^{i_1+\cdots+i_k} = X \cap E^{i_1+\cdots+i_k+b_k}, \quad k = 1, \cdots, s.$$

Then we have $Y^{i_1+\cdots+i_k} \subset E^{i_1+\cdots+i_k+b_k}$, $k = 1, \cdots, s$, and

$$Y^{i_1} \subset Y^{i_1+i_2} \subset \cdots \subset Y^{i_1+\cdots+i_s} = X.$$

Define the vectors $e_1, \cdots, e_{i_1+\cdots+i_s}$, $f_1, \cdots, f_{b_1+\cdots+b_s}$, which form an orthonormal system and which are such that: 1) $e_1, \cdots, e_{i_1+\cdots+i_k}$ span $Y^{i_1+\cdots+i_k}$; 2) $e_1, \cdots, e_{i_1+\cdots+i_k}$, $f_{b_1+\cdots+b_{k-1}+1}, \cdots, f_{b_1+\cdots+b_k}$ span $E^{i_1+\cdots+i_k+b_k}$, $k = 1, \cdots, s$.

We now notice that the group $U(n+N)$ can be identified with the space of orthonormal systems of vectors or frames, each element to the frame to which it carries the coordinate frame. If we denote the frame by $e_1, \cdots, e_n, f_1, \cdots, f_N$, we have

$$(16) \qquad \omega_{ir} = de_i \cdot \overline{f}_{r-n}, \quad i = 1, \cdots, n, \quad v = n+1, \cdots, n+N$$

where the product in the right-hand side is the scalar product.

To integrate a differential form over S we shall find what it reduces on S, under the above choice of vectors. Every polynomial of degree m in Ω_{ij} reduces on S to a multiple of the form

$$(17) \qquad \prod_{k=1,\cdots,s} \Big(\prod_{\substack{\rho = i_1+\cdots+i_{k-1}+1,\cdots,i_1+\cdots+i_k \\ \sigma = b_1+\cdots+b_{k-1}+1,\cdots,b_1+\cdots+b_k}} (de_\rho \cdot \overline{f}_\sigma)(d\overline{e}_\rho \cdot f_\sigma) \Big)$$

It follows that the form

$$\Omega_{p_1q_1} \Omega_{p_2q_2} \cdots \Omega_{p_mq_m}$$

reduces to a non-zero form on S only when the set $p_1, \cdots, p_m$ contains $b_k i_k$ indices among $i_1+i_2+\cdots+i_{k-1}+1, \cdots, i_1+i_2+\cdots+i_k$, $k = 1, \cdots, s$.

We apply this criterion to the forms Ψ_m, Φ_m defined in (15). In the case of Ψ_m, it is clear that the summation can be restricted to the indices $p_1, \cdots, p_m$, which are mutually distinct. Hence its integral over S will be non-zero, only when S has the symbol $(0 \cdots 0\ 1 \cdots 1)$.

As for the forms Φ_m, we first observe that $\Phi_m = 0$, if $m > N$. In fact, in this case, Φ_m is a sum of terms as

$$\sum_{\substack{(p_1\cdots p_m)=(q_1\cdots q_m)\\ p_1,\cdots,p_m=1,\cdots,n}} \omega_{p_1r}\,\omega_{rq_1}\,\omega_{p_2r}\,\omega_{rq_2}\cdots\cdots\omega_{p_ms}\,\omega_{sq_m},$$

where the index r occurs at least twice and is not summed. The sum changes its sign, when ω_{rq_1} and ω_{rq_2} are interchanged. Since the summation is taken over all $p_1, \cdots, p_m = 1, \cdots, n$, we have $\Phi_m = 0$, for $m > N$.

We shall next prove that

$$\int_{(a_1\cdots a_n)} \Phi_m = 0, \quad \sum_{i=1}^{n} a_i = m,$$

if $a_{n-1} \neq 0$. This will be done by induction on N. For $N = 1$ it is easily verified. Suppose the statement be true for N-1. If $a_n < N$, S lies in a complex Euclidean space of dimension $n+N-1$ and the result follows from our induction hypothesis. If $a_n = N$, we have $m > N$ and the result follows from the lemma of the last paragraph. Thus it follows that the integral of Φ_m over S is non-zero only when S has the symbol $(0 \cdots 0\ m)$.

It remains to evaluate these integrals. The Schubert variety $(0 \cdots 0\ \underbrace{1 \cdots 1}_{m})$ consists by definition of all n-dimensional planes through a fixed E^{n-m} and belonging to E^{n+1}. Let $e_1, \cdots, e_{n+1}$ be a frame in E^{n+1} such that $e_1, \cdots, e_{n-m}$ span E^{n-m} and that $e_1, \cdots, e_n$ span X, a general element of S. Then Ψ_m reduces on S to the form

$$(-1)^m (m!)^2 \prod_{r=n-m+1,\cdots,n} \omega_{n+1,r}\,\omega_{r,n+1}.$$

The latter may be considered as the measure of all lines lying in the space E^{m+1} in E^{n+1} orthogonal to E^{n-m}. Similarly, the Schubert variety $(0\cdots0\ m)$ consists of all X containing a fixed E^{n-1} and contained in a fixed E^{n+m}.

On this the form Φ_m reduces to

$$(m!)^2 \prod_{r=n+1,\cdots,m+n} \omega_{nr}\omega_{rn}$$

In each case the evaluation of the integral in question reduces to the determination of the volume of $G(1,m,C)$.

Let E^{m+1} be the complex Euclidean space such that $G(1,m,C)$ is the manifold of all lines through the origin O of E^{m+1}. A vector in E^{m+1} can be written as $v = v' + \sqrt{-1}\, v''$, where v' and v'' are real vectors. To v we can therefore associate in a real Euclidean space R of dimension 2m+2 the vectors

$$f(v) = (v', v''), \quad g(v) = (-v'', v').$$

The relation between the scalar products in E^{m+1} and R is given by

$$v\bar{w} = f(v)f(w) - \sqrt{-1}\, f(w)\, g(v)$$
$$= g(v)g(w) + \sqrt{-1}\, g(w)\, f(v).$$

It follows that the vectors $f(e_i), g(e_i)$, $i = 1, \cdots, m+1$, associated to the vectors e_i of a frame in E^{m+1} form a frame in the real Euclidean space R.

As the volume element in $G(1,m,C)$ we can take

$$\Lambda = \prod_{i=1,\cdots,m} \omega_{m+1,i}\,\omega_{i,m+1}\,.$$

Since

$$\omega_{m+1,i}\,\omega_{i,m+1} = 2\sqrt{-1}\,(df(e_{m+1})\cdot f(e_i))(df(e_{m+1})\cdot g(e_i)),$$

we find by substitution

$$\Lambda = (2\sqrt{-1})^m \prod_{i=1,\cdots,m} (df(e_{m+1})\cdot f(e_i))(df(e_{m+1})\cdot g(e_i)).$$

To integrate this over $G(1,m,C)$ we consider the unit hypersphere S^{2m+1} described by the end-points of the vectors v satisfying $v\bar{v} = 1$. $G(1,m,C)$ is obtained by identifying all v which differ from each other by a scalar factor of absolute value 1. In other words, S^{2m+1} can be fibered by circles with $G(1,m,C)$ as base space. It was shown in Chapter I that the volume element of S^{2m+1} is

$$\prod_{i=1,\cdots,m} (df(e_{m+1})\cdot f(e_i))(df(e_{m+1})\cdot g(e_i))\cdot(df(e_{m+1})\cdot g(e_{m+1})),$$ if we identify v with e_{m+1}.

To evaluate its integral over S^{2m+1} we fix a point of $G(1,m,C)$ and integrate over the fiber, giving

$$V(S^{2m+1}) = \frac{2\pi}{(2\sqrt{-1})^m} \int_{G(1,m,C)} \bigwedge .$$

Since the volume of S^{2m+1} is known to be

$$V(S^{2m+1}) = \frac{2\pi^{m+1}}{m!} ,$$

we get

$$\int_{G(1,m,C)} \bigwedge = \frac{(2\pi\sqrt{-1})^m}{m!} .$$

This completely proves our Theorem 1.

Case B. Real Grassmann manifolds $G(n,N,R)$.

As we shall show in the next section, a multiplicative base of the cohomology ring up to dimensions $\leq N$, will be formed by each of the sets of cohomology classes having the symbols

$$(18)\qquad \begin{aligned} P^{4k} &= (0 \cdots 0\ 2 \cdots 2), & k &= 1, \cdots, [\tfrac{1}{2}n], \\ \bar{P}^{4k} &= (0 \cdots 0\ 2k\ 2k), & k &= 1, \cdots, [\tfrac{1}{2}N]. \end{aligned}$$

The P^{4k} will be called the Pontrjagin classes and their images in the base space of a sphere bundle the Pontrjagin characteristic classes. As in the complex case we take the Maurer-Cartan forms of the orthogonal group $O(n + N)$, which are elements of a skew-symmetric matrix: $(\widetilde{\omega}_{AB})$. We put

$$\widetilde{\Omega}_{ij} = \sum_A \widetilde{\omega}_{iA}\widetilde{\omega}_{Aj}, \quad i, j = 1, \cdots, n, \tag{19}$$

and from these we construct the forms

$$\begin{aligned}\widetilde{\Psi}_{2m} &= \sum \delta(p_1 \cdots p_m; q_1 \cdots q_m)\widetilde{\Omega}_{p_1q_1} \cdots \widetilde{\Omega}_{p_mq_m}, \\ \widetilde{\Phi}_{2m} &= \sum \delta^2(p_1 \cdots p_m; q_1 \cdots q_m)\widetilde{\Omega}_{p_1q_1} \cdots \widetilde{\Omega}_{p_mq_m},\end{aligned} \tag{20}$$

where the summation convention is as above. These forms can be regarded as differential forms in $G(n,N,R)$ and are invariant under the action of $G(n,N,R)$ by the orthogonal group $O(n + N)$. Because of the skew-symmetry of $\widetilde{\Omega}_{ij}$ in its two indices, it can be seen that $\widetilde{\Psi}_{2m}$, $\widetilde{\Phi}_{2m}$ are zero, when m is odd. The relation between these forms and the cohomology classes of $G(n,N,R)$ is given by the following theorem:

Theorem 2. *The form* $\frac{1}{(2\pi)^{2k}(2k)!}\widetilde{\Psi}_{4k}$ *belongs to the class* P^{4k} *and the form* $\frac{1}{(2\pi)^{2k}(2k)!}\widetilde{\Phi}_{4k}$ *to the class* $\bar{P}^{4k}$.

It is of course possible to prove this theorem by a process similar to the one used in the complex case. However, a simpler and more conceptual procedure would be to reduce the proof to the complex case.

We assume the real Euclidean space of dimension n+N to be imbedded in the complex Euclidean space of the same (complex) dimension:

$$i : \widetilde{E}^{n+N} \longrightarrow E^{n+N}.$$

This induces a mapping

$$f : G(n,N,R) \longrightarrow G(n,N,C).$$

Denote by C^{2m} and $\bar{C}^{2m}$ respectively the classes having the symbols $(0 \cdots 0 \underbrace{1 \cdots 1}_{m})$ and $(0 \cdots 0\ m)$. Then we have the lemma:

Lemma. Under the mapping f,

(21) $$f^*C^{2m} = \begin{cases} 0 & , m \text{ odd}, \\ (-1)^m P^{2m}, & m \text{ even}, \end{cases}$$

$$f^*\bar{C}^{2m} = \begin{cases} 0 & , m \text{ odd} \\ (-1)^m \bar{P}^{2m}, & m \text{ even} \end{cases}$$

Before proceeding to the proof, we remark that the Schubert varieties C^{2m} and $\bar{C}^{2m}$, being complex manifolds, have orientations defined by their complex structure, so that the real cohomology classes with the same symbols are well defined. The statements for m odd are trivial, because the non-zero classes of G(n,N,R) have as dimensions multiples of 4. We shall give the proof for $f^* C^{2m}$, m even, the proof concerning the expression for $f^* \bar{C}^{2m}$, m even, being similar.

Between two Schubert varieties of complementary dimensions on G(n,N,C) we can define their intersection number or Kronecker index. It can be proved that

$$KI\ ((b_1 \cdots b_n), (N-a_n, \cdots, N-a_1)) = 0 \text{ or } 1,$$

according as $(b_1 \cdots b_n)$ is distinct from $(a_1 \cdots a_n)$ or not. (Cf. next section.) Denote by S the Schubert variety having the symbol

$$(\underbrace{N-1 \cdots N-1}_{2k}\ N \cdots\cdots N).$$

It reduces to show that

$$KI\ \ (S, f(a_1 \cdots a_n)) = 0, \ \sum_{i=1}^{n} a_i = 4k,$$

unless $(a_1 \cdots a_n) = P^{4k}$, and that in the latter case the Kronecker index is $(-1)^k$.

The first statement is easy to prove. In fact, we have, by definition,

$$S = \left\{ X \;\middle|\; \text{comp. dim. } (X \cap E^{N+2k-1}) \geq 2k \right\}.$$

If $(a_1 \cdots a_n) \neq P^{4k}$, the fact that the a's are even implies that $a_{n-2k+1} = 0$. If Y is an element of G(n,N,R) belonging to the corresponding Schubert variety, Y must contain a fixed linear space of dimension n-2k+1, and f(Y) contains a fixed linear space of complex dimension n-2k+1. When the latter is chosen to be in general position with E^{N+2k-1}, they will have only the origin in common, by dimension considerations. It follows that S and $f((a_1 \cdots a_n))$ are then set-theoretically disjoint and hence that their intersection number is zero.

To prove the second statement we shall determine the set-theoretical intersection of S and $f(P^{4k})$. The latter consists of all n-dimensional linear spaces Y satisfying

$$\tilde{E}^{n-2k} \subset Y \subset \tilde{E}^{n+2} \qquad ,$$

where $\tilde{E}^{n-2k}$ and $\tilde{E}^{n+2}$ are fixed linear spaces of real dimensions n-2k and n + 2 respectively. Let $\tilde{E}^{2k+2}$ be the linear space orthogonal to $\tilde{E}^{n-2k}$ in $\tilde{E}^{n+2}$. Y is then determined by a linear space ξ of real dimension 2k in $\tilde{E}^{2k+2}$. Denote by E^{n-2k}, E^{n+2}, E^{2k+2} the complex linear spaces determined by $\tilde{E}^{n-2k}$, $\tilde{E}^{n+2}$, $\tilde{E}^{2k+2}$ respectively. Since E^{N+2k-1} is in general position with them, we have

$$E^{n-2k} \cap E^{N+2k-1} = 0,$$

$$E^{n+2} \cap E^{N+2k-1} = L^{2k+1}, \text{ say.}$$

In order that f(Y) belongs to S, ξ must belong to $L^{2k+1} \cap \tilde{E}^{2k+2}$. But L^{2k+1} and $\tilde{E}^{2k+2}$, both belonging to E^{2k+2}, have exactly one linear space of real dimension 2k in common. It follows that S and $f(P^{4k})$ have exactly one element in common.

By using a local coordinate system, whose details we shall not give here, we verify that the intersection number is actually equal to $(-1)^k$. This completes the proof of the lemma.

We now observe that under the mapping f we have

$$f^*\Omega_{ij} = \tilde{\Omega}_{ij}.$$

Hence Theorem 2 is an immediate consequence of Theorem 1 and the above lemma.

Remark. The mapping f induces a unitary bundle over G(n,N,R). It is equivalent to the Whitney product of the universal bundle over G(n,N,R) with itself. For if $e_1, \cdots, e_n$ are n vectors in an element X of G(n,N,R), then f(X) will be spanned by $e_i, \sqrt{-1}\, e_i$, $i = 1, \cdots, n$. Thus f(X) can be considered as the vector space spanned by two copies of X, with the orthogonal group acting coherently.

Since a sphere bundle can always be induced by mapping its base space into G(n,N,R), this relationship is valid for a general sphere bundle. We express it by saying that the Whitney square of a sphere bundle has an almost complex structure. The above lemma then gives the relationship between the real characteristic ring of a sphere bundle with that of the almost complex structure of its Whitney square. This result is useful in differential geometry, where the primary concern is the real characteristic ring.

3. Multiplicative properties of the cohomology ring of a Grassmann manifold

Since G(n,N,C) has no torsion, it is sufficient to determine the multiplication in the real cohomology ring. For both G(n,N,C) and G(n,N,R), when the coefficient ring is the real field, the cohomology classes can be described by differential forms and their multiplication by the exterior multiplication of the latter. According to the general theory described in §5, Chapter III, these differential forms can be supposed to be invariant under the actions of the unitary group and the rotation group respectively. For definiteness of description take the case of G(n,N,C). Applying the so-called first main theorem on vector invariants under the unitary group (Weyl, Classical Groups, p. 45) and using an argument of §5, Chap. III, it follows that an invariant differential form of degree $\leqq 2N$ is a polynomial of Δ_m, $m = 1, \cdots, N$, and hence a polynomial of Ψ_m, $m = 1, \cdots, N$, with constant coefficients. Since all these differential forms are of even degree, it follows from Cartan's theorem that they are closed and are not cohomologous to zero unless identically equal to zero.

The case of G(n,N,R) can be described in a similar manner. Here we restrict ourselves to classes of dimension < N. Every invariant differential form of degree < N is a polynomial of $\tilde{\Psi}_{2m}$ or of $\tilde{\Phi}_{2m}$ with constant coefficients. Such a form has as degree a multiple of 4. For $\tilde{G}(n,N,R)$ there exists for even n the further invariant form

$$\sum_{i} \epsilon_{i_1 \cdots i_n} \tilde{\Omega}_{i_1 i_2} \cdots \tilde{\Omega}_{i_{n-1} i_n}, \tag{22}$$

corresponding to the determinant in vector invariants.

For coefficient rings other than the real field, more topological methods have to be employed to describe the multiplicative structure of the cohomology rings. We shall illustrate this method by determining the cohomology ring of G(n,N,R) mod 2.

In this case every cochain $(a_1 \cdots a_n)$ is a cocycle and all these symbols form an independent cohomology base. We shall use the convention of omitting the zeros of such a symbol, thus defining

(23a) $$(a_{i+1} \cdots a_n) = (0 \cdots 0\, a_{i+1} \cdots a_n)$$

and

(23b) $$(0) = 1,\ (c) = 0,\ \text{if } c < 0.$$

Using the same symbols to denote also the cohomology classes, the main multiplication formula to be proved is

(24) $$(a_1 \cdots a_n) \cup (h) = \sum (b_1 \cdots b_n),$$

where the summation is extended over all combinations $b_1, \cdots, b_n$, such that

$$0 \leqq b_1 \leqq b_2 \leqq \cdots \leqq b_n \leqq N,$$

(25) $$a_i \leqq b_i \leqq a_{i+1} \quad (a_{n+1} = N),\ i = 1, \cdots, n,$$

$$\sum_{i=1}^{n} a_i + h = \sum_{i=1}^{n} b_i \ .$$

Formula (24) implies the following multiplication formula:

(26) $$(a_1 \cdots a_n) = \begin{vmatrix} (a_1) & (a_1 - 1) & \cdots\cdot & (a_1 - \overline{n-1}) \\ (a_2 + 1) & (a_2) & \cdots\cdot & (a_2 - \overline{n-2}) \\ & \cdots\cdots & & \\ (a_n + \overline{n-1}) & (a_n + \overline{n-2}) & \cdots & (a_n) \end{vmatrix}$$

where the right-hand side is to be expanded by the Laplace development with cup product as multiplication. This is easily proved by induction on n. In fact, assuming the truth of the formula for n-1 and expanding the determinant according to the first column, we see that the determinant is equal to

$$\sum_{i=1}^{n} (a_i + i - 1) \cup (a_1 - 1 \cdots a_{i-1} - 1 \ \ a_{i+1} \cdots a_n).$$

Using (24), it can be seen that the sum is equal to the left-hand side of (26).

Since the multiplication by cup product is associative, formulas (24) and (26) together give the multiplication of any two cohomology classes.

We shall prove the formula (24) by establishing a corresponding formula on intersection numbers. Consider first the cycles

(27) $$(b_1 \cdots b_n),\ (N-a_n, \cdots, N-a_1).$$

Suppose that they are defined by two sequences of linear spaces in general position:

(28) $$E_1 \subset E_2 \subset \cdots \subset E_n,$$
$$F_1 \subset F_2 \subset \cdots \subset F_n,$$

whose dimensions are given by

$$\dim E_i = b_i + i,$$

$$\dim F_{n-i+1} = N - a_i + n - i + 1,\ i = 1, \cdots, n.$$

In order that the two cycles have an element X in common, we must have

$$\dim (X \cap E_i) \geqq i,$$

$$\dim (X \cap F_{n-i+1}) \geqq n-i+1.$$

Since these two intersections both belong to X, they intersect in a linear space of dimension $\geqq 1$. It follows that the same is true of E_i and F_{n-i+1}. Hence we have

$$(b_i + i) + (N - a_i + n - i + 1) \geqq N + n + 1,$$

or $b_i \geqq a_i$.

On the other hand, by making use of a coordinate system, we can arrange that $(a_1 \cdots a_n)$ and $(N-a_n \cdots N-a_1)$ intersect in exactly one element and have there the intersection number 1. It follows that when

$(b_1 \cdots b_n)$ and $(N-a_n \cdots N-a_1)$ are of complementary dimensions, their intersection number is 1 or 0 according as $(a_1 \cdots a_n)$ is equal to $(b_1 \cdots b_n)$ or not.

Using this result, it is seen that formula (24) is equivalent to the statement that the intersection number of the three cycles

$$(29) \qquad (b_1 \cdots b_n),\ (N-a_n \cdots N-a_1),\ (N-h\ N \cdots N)$$

is one if the conditions (25) are satisfied and is otherwise zero. We have shown in the above that in order the intersection number be non-zero it is necessary that $a_i \leq b_i$.

Suppose that these necessary conditions are satisfied. We put

$$M_i = E_i \cap F_{n-i+1},\ P_i = E_i \cap F_{n-i},\ i = 1, \cdots, n,$$

and let M be the space spanned by $M_1, \cdots, M_n$. Since dim $M = b_i - a_i + 1$, we have dim $M \leq h + n$, and this dimension is equal to $h + n$, if any two distinct M_i have only the point 0 in common. An element X belonging to the intersection of the first two cycles of (29) meets each M_i in a linear space of dimension ≥ 1. Such an X must therefore belong to M. It follows that in order that the three cycles in (29) have a non-empty intersection we must have dim $M = h + n$. A necessary condition for this is $M_i \cap M_{i+1} = 0$. Since both contain P_i, this condition implies $P_i = 0$. But dim $P_i = b_i - a_{i+1}$, so that we have $b_i \leq a_{i+1}$.

It remains to show that when the conditions (25) are satisfied, the three cycles in (29) have the intersection number one. In fact, it can be arranged that they have exactly one element in common and we verify the intersection to be simple. For details, compare S. Chern, On the multiplication in the characteristic ring of a sphere bundle, Annals of Math. 49, 362-372 (1948). We remark that the proof given above is a slight simplication of the one given in this paper.

From the above multiplication formulas we can draw a number of consequences. The coefficient ring is always the field mod 2.

1) We introduce the classes

$$(30)\qquad \begin{aligned} &w^i = (0 \cdots 0\underbrace{1 \cdots 1}_{i}),\ 1 \leq i \leq n, \\ &\bar{w}^k = (k), \qquad 1 \leq k \leq N, \\ &w^0 = \bar{w}^0 = 1. \end{aligned}$$

Applying (24) we get the formula

$$(31)\qquad \sum_{0,\gamma - N \leq i \leq n} w^i \cup \bar{w}^{\gamma - i} = 0 \ , \quad \gamma > 0.$$

This permits us to express the w's in terms of the $\bar{w}$'s, and vice versa.

2) Every cohomology class of G(n,N,R) is a polynomial in $\bar{w}^k$, $k = 0, \cdots, N$, and is a polynomial in w^i, $i = 0, \cdots, n$.

This follows from (26) and (30).

3) There is a natural homeomorphism

$$f:\ G(n,N,R) \dashrightarrow G(N,n,R),$$

under which an n-dimensional linear space is mapped into its N-dimensional orthogonal space. Denote the classes of G(N,n,R) by the same symbols with dashes. The dual homomorphism induced by f is given by

$$(32)\qquad \begin{aligned} &f^* \bar{w}'^i = w^i,\ i = 1, \cdots, n, \\ &f^* w'^k = \bar{w}^k,\ k = 1, \cdots, N. \end{aligned}$$

To prove this, we show that both sides have the same value over any homology class, an argument which has been applied several times before. We shall omit the details here.

4) There is no non-trivial relation between the w^i, $i = 0, \cdots, n$ or the $\bar{w}^k$, $k = 0, \cdots, N$. When the classes $\bar{w}^k$ are concerned, we see this from (26). In fact, let $F(\bar{w})$ be a polynomial in $\bar{w}^k$, which is a sum of terms of the form

$$\bar{w}^{i_1}\,\bar{w}^{i_2} \cdots \bar{w}^{i_r}\,, \quad i_1+\cdots+i_r = d, \text{say}.$$

We introduce an ordering of such terms by defining

$$\bar{w}^{i_1} \cdots \bar{w}^{i_r} < \bar{w}^{k_1} \cdots \bar{w}^{k_s},$$

if $r < s$ or $r = s$, $i_1 = k_1, \cdots, i_t = k_t$, $i_{t+1} < k_{t+1}$. Relative to this ordering let $\bar{w}^{m_1} \cdots \bar{w}^{m_r}$ be the largest term with non-zero coefficient in $F(\bar{w})$. We now carry out the multiplication by writing $F(\bar{w})$ as a sum of Schubert symbols. From the form of (26) it is observed that the expansion contains the term $(m_1 \cdots m_r)$ with non-zero coefficient. Since the Schubert symbols are homologically independent, it follows that the class $F(\bar{w})$ is not zero, unless the polynomial is identically zero.

The statement concerning w^i follows from a consideration of $G(N,n,R)$.

4. Some applications

It goes without saying that our interest in Grassmann manifolds lies in the fact that their study leads to a description of the characteristic homomorphism of a general sphere bundle or of a general differentiable manifold. A sphere bundle (B,X), whose base space is of dimension $< N$, can always be induced by a mapping

$$f: \quad X \dashrightarrow G(n,N,R),$$

which is determined up to a homotopy. For any cohomology class γ of $G(n,N,R)$ the class $f^*\gamma$ is therefore an invariant of the bundle. The geometric interpretation of such invariants was given by Pontrjagin and later by the author (Pontrjagin, C. R. Doklady), 35, 34-37 (1942); Mat. Sbornik N.S., 24(66), 129-162 (1949); Chern, Proc.Nat.Acad.Sci., USA, 33, 78-82 (1947)). It consists in their identification with cohomology classes defined in obstruction theory. We shall carry this out for the classes $W^i = f^* w^i$.

To the bundle (B,X) of (n-1)-spheres over X with the orthogonal group as the structure group we consider its associate bundle $(E_{n,p}, X)$, whose fibers are the Stiefel manifolds $V_{n,p}$, being the manifolds of ordered sets of p mutually perpendicular unit vectors in an n-space. It is known that $V_{n,p}$ is connected and that its first non-vanishing homotopy group is of dimension n-p, which is infinite cyclic if n-p is even or p=1 and is otherwise cyclic of order 2. Suppose X be a simplicial complex, and X^k its k-dimensional skeleton. According to a well-known procedure due to Stiefel, Whitney, and Steenrod, a cross-section can be defined over X^{n-p}. To such a cross-section h we define an (n-p+1)-dimensional cochain c^{n-p+1} as follows: To an (n-p+1)-cell σ, $\psi^{-1}(\sigma)$ is homeomorphic to $\sigma \times V_{n,p}$. By taking its projection into $V_{n,p}$, the cross-section $h/\partial\sigma$ defines a mapping of an (n-p)-sphere into $V_{n,p}$, and hence an element of $\pi_{n-p}(V_{n,p})$. This we take to be the value of c^{n-p+1} for σ. The cochain c^{n-p+1} so defined is to be understood with local coefficients in the sense that the homotopy groups $\pi_{n-p}(V_{n,p})$ related to different cells are connected by isomorphisms. This cochain is a cocycle and its cohomology class is independent of the choice of the cross-section h over X^{n-p}. (For details, cf. Steenrod, Fibre Bundles, 155-183.) We call it the primary obstruction class of the bundle.

Each of the groups $\pi_{n-p}(V_{n,p})$, whether infinite cyclic or cyclic of order 2, can be mapped homomorphically into the cyclic group of order

2, I_2, such that a generator of the former goes into a generator of the latter. Applying this homomorphism to the primary obstruction c^{n-p+1}, we get a cohomology class $\bar{c}^{n-p+1}$, with coefficients in I_2. Since the homomorphism commutes with the isomorphisms of the local groups, the class $\bar{c}^{n-p+1}$ is an ordinary cohomology class. It will be called the reduced primary obstruction. A theorem due to Pontrjagin can be stated as follows:

Theorem 1. $W^i = f^* \bar{c}^i$, $i = 1, \cdots, n$.

To prove this theorem, notice that it is sufficient to establish it for the universal bundle. In E^{n+N} we take a system of p linearly independent vectors, say the first p coordinate vectors $e_1, \cdots, e_p$. To an element X of G(n,N,R) let x_i be the orthogonal projection of e_i in X. If R^{n+N-p} is the linear space orthogonal to e_i, $i = 1, \cdots, p$, it is seen that the vectors x_i, $i = 1, \cdots, p$, are linearly dependent if and only if X satisfies the condition $\dim (X \cap R^{n+N-p}) \geq n-p+1$. The latter form the Schubert variety $(\underbrace{N-1 \cdots N-1}_{n-p+1} N \cdots N)$ of dimension $nN-(n-p+1)$, whose dual cohomology class is W^{n-p+1}. It follows that if X does not belong to this Schubert variety, a field of p linearly independent vectors can be defined in X. By a well-known orthogonalization process they can be taken to be mutually perpendicular. The remaining part of the proof can be achieved by taking a simplicial decomposition of G(n,N,R) such that the Schubert variety in question is a sub-complex and calculating the reduced primary obstruction relative to this cross-section.

As a second application we shall discuss the question of relations between the characteristic classes. Since the characteristic homomorphism f^* preserves multiplication, relations between the cohomology classes on G(n,N,R) remain valid for the characteristic classes. As an illustration we shall give a proof of a theorem of Pontrjagin on a relation between the characteristic classes of a four-dimensional manifold (that is, relative to its tangent bundle).

Related to this question is the question whether there are relations between the characteristic classes. With coefficients mod 2 it follows from 4), §3 that there is no nontrivial relation between the W's. However, the question assumes a different aspect if the bundle is not a general one, in particular, if it is the tangent bundle of a differentiable manifold or the normal bundle of an imbedded differentiable manifold in an Euclidean space. The following theorem is due to Whitney, but proved in a different way:

Theorem 2. For the tangent bundle of a compact orientable n-dimensional differentiable manifold, the characteristic class $\overline{W}^n = 0$, the coefficients being mod 2.

We imbed X differentiably in an Euclidean space of dimension $n + N$: $X \subset E^{n+N}$. Consider the normal bundle of X and denote its characteristic classes by dashes. By dimension considerations we have $W'^{r} = 0$, $r \geqq n + 1$. It also follows from dimension considerations that an N-n field of normal vectors can be defined over X, because the primary obstruction, being a cohomology class of dimension $n + 1$, is zero. We take such a field and consider the bundle A of its normal spaces of dimension n over X. Let A' be the bundle of (n-1)-spheres over X obtained from A by taking the unit sphere in each fiber. We assert that the bundle (A',X) has a cross-section. In fact, A' can be realized as a small tube of unit vectors about X. Suppose its fiber F be ~ 0 in A'. Let D be a disc of unit vectors with F as its boundary. Then D + C, where C is a chain in A' with F as boundary, is a cycle in the Euclidean space having with X the intersection number 1. But this contradictd the fact that such a cycle is ~ 0 in E^{n+N}. It follows that no fiber of A' is ~ 0 in A', and that this is true with any coefficient group. By a theorem of Gysin (Comm.Math.Helv. 14, 61-122 (1942); cf. also Chern and Spanier, Proc.Nat.Acad.Sci. 36, 248-255 (1950)), the primary obstruction class of the bundle is zero. Hence the bundle (A',X) has a cross-sedtion.

On the other hand, it is clear that the characteristic classes of (A',X) are the same as those of the normal bundle over X. Hence $W'^n = 0$ and, by 3), §3, we have $\overline{W}^n = 0$.

Let X be now a compact orientable four-dimensional manifold, and P^4 its characteristic class which has the symbol $f^*(22)$. The following theorem was stated by Pontrjagin:

Theorem 3. With coefficients mod 2,

$$P^4 + W^4 = 0. \tag{33}$$

Since the manifold is orientable, we have $W^1 = 0$. Equations (31) then give

$$\overline{W}^1 = 0,$$

$$\overline{W}^2 + W^2 = 0,$$

$$\overline{W}^4 + W^4 + (W^2)^2 = 0.$$

On the other hand, (26) gives

$$P^4 + (W^2)^2 = 0.$$

Combining these equations and using Theorem 2, we get (33).

The problem of showing that the characteristic classes of a differentiable manifold are non-trivial is undoubtedly of interest. In particular, the question has been raised whether there exists an orientable differentiable manifold with a non-zero Stiefel-Whitney class (Steenrod, Fibre Bundles, p. 212). We shall give such an example (Wu Wen-tsun, C.R.Acad.Paris,230, 508-511(1950)).

Let M^4 be the complex projective plane, and $M^4 \times I$ its topological product with the unit interval, so that its points can be represented as (p,t), $p \in M^4$, $t \in I$. Identify $(p,0)$ and $(\bar{p},1)$, where $\bar{p}$ is the conjugate complex point of p. The resulting space N is a 5-dimensional orientable manifold. To see this, let L be a complex projective line in M^4. The conjugation $\tau : p \dashrightarrow \bar{p}$, which maps a point p of M^4 to its conjugate complex point $\bar{p}$, preserves the orientation of M^4 and reverses the orientation of L. The space M^4 has a cellular decomposition consisting of the 4-cell $\sigma^4 = M^4 - L$, the 2-cell $\sigma^2 = L - \sigma^0$, and the 0-cell σ^0, where σ^0 is a point of L. Denoting by I' the open unit interval, we see that N has a cellular decomposition with the cells σ^0, σ^2, σ^4, $\sigma^0 \times I'$, $\sigma^2 \times I'$, $\sigma^4 \times I'$. The above remark om the conjugation gives the following incidence relations

(34)
$$\partial(\sigma^4 \times I') = 0,$$
$$\partial(\sigma^2 \times I') = \pm 2\sigma^2.$$

In particular, the first relation implies that N is an orientable manifold. If, as usual, we use the same notation to denote both a chain and a cochain, then $C^2 = \sigma^2$ is a cocycle mod 2 and $C^3 = \sigma^2 \times I'$ is an integral cocycle.

We shall show that the class W^3 of N is $\neq 0$. We first consider the space M^4. Its characteristic classes can be determined directly by suitably chosen fields of vectors. Just to show the usefulness of some recent results of Wu (C.R.Paris, loc.cit.), we shall find them as follows: Our coefficients are mod 2. Since the Steenrod squares define homomorphisms and since M^4 is a manifold, there exist uniquely determined cohomology classes U^1, U^2, of dimensions 1 and 2 respectively, satisfying

(35) $$Sq^1 Y^3 = U^1 \cup Y^3, \quad Sq^2 Y^2 = U^2 \cup Y^2,$$

where Y^2 or Y^3 is an arbitrary class whose dimension is given by the superscript. Then we have the formula

(36) $$W^i = \sum_{p=0}^{i} Sq^p U^{i-p}$$

Since M^4 is orientable, we have $W^1 = 0$. Notice that $H^2(M^4)$ and $H^4(M^4)$ are both cyclic of order two. The above formula shows that W^2 and W^4 are the non-zero elements.

We can therefore define a 3-field on M^4 with a single singularity in L. From this a 4-field can be constructed in N by taking as the fourth vector the one in the direction of I. Hence the class W^2 of N is $\neq 0$. Using Whitney's formula

(37) $$W^3 = \frac{1}{2}\, \delta \omega W^2.$$

we find $W^3 \neq 0$ for N.

The manifold N is 5-dimensional. But $N \times S^1$ is 6-dimensional, and its characteristic class W^3 is also $\neq 0$. The latter cannot therefore have an almost complex structure.

As a third application of our results, we mention the fact that the characteristic classes give rise to necessary conditions that a differentiable manifold can be differentiably imbedded in a certain Euclidean space. Clearly we have the theorem;

Theorem 4. In order that a compact differentiable manifold of dimension n can be differentiably imbedded in an Euclidean space of dimension n + N, it is necessary that

(38) $$\overline{W}^r = 0, \quad r \geqq N+1,$$

(39) $$\overline{P}^{4k} = 0, \; 2k \geqq N+1.$$

By means of the multiplication formulas these conditions can be transformed into a different form. In particular, (38) gives, for N = 1, the conditions

(40) $$W^r = (W^1)^r \quad , r = 1, 2, \cdots, n,$$

and, for N = 2, the onditions

(41) $$W^r + W^{r-1}W^1 + W^{r-2}(W^2 + (W^1)^2) = 0, \; 3 \leqq r \leqq n.$$

It is of course possible to have similar conditions for larger values of N. Notice also that (40) permits us to express all W^r as polynomials of W^1 and (41) permits us to express all W^r as polynomials of W^1 and W^2. The conditions (41) have some easy consequences which would perhaps be simpler to apply to practical problems. In fact, if in addition $W^1 = 0$, then (41) implies

(41a) $$W^{2r} = (W^2)^r \quad , \quad 2r \leqq n.$$

If in addition $W^2 = 0$ (without necessarily having $W^1 = 0$), (41) implies

(41b) $$W^r = \begin{cases} (W^1)^r, & r \not\equiv 2 \; (3), \\ 0, & r \equiv 2 \; (3). \end{cases}$$

We proceed to apply these conditions to the real projective space Π^n of n dimensions. We have shown before that its cohomology groups mod 2 are

$$H^r(\Pi^n) = I_2, \quad 0 \leqq r \leqq n.$$

Moreover, if ζ is a generator of $H^1(\Pi^n)$, then $(\zeta)^r$ is a generator of $H^r(\Pi^n)$, the power being in the sense of the cup product. The Stiefel-Whitney classes of Π^n have been determined by Stiefel (Comm.Math.Helv.

13, 201-218). They are

$$W^r = \binom{n+1}{r} (\zeta)^r.$$

We now suppose that these classes satisfy the conditions (41). Since $\binom{n+1}{3} = \binom{n+1}{2}\frac{n-1}{3}$, the condition that $\binom{n+1}{2}$ is even implies that $\binom{n+1}{3}$ is also even. This means that $W^2 = 0$ implies $W^3 = 0$, and by (41b), that $W^1 = 0$. Then all the W's are 0. Hence $W^2 = 0$ implies all $W^2 = 0$. Suppose next $W^2 \neq 0$. If $W^1 = 0$, we derive from (41a) that $W^r = 0$ or not according as r is odd or even. If $W^1 \neq 0$, then $W^2+(W^1)^2 = 0$ and from (41) we see that no W^r is = 0 for $0 \leqq r \leqq n$. It follows that if Π^n can be differentiably imbedded in E^{n+2}, n must be a positive integer satisfying one of the following conditions

$$(1+t)^{n+1} \equiv 1 + t + t^2 + \cdots + t^{n+1}, \ (2)$$

$$\equiv 1 + t^{n+1} \qquad , \ (2)$$

$$\equiv 1 + t^2 + t^4 + \cdots + t^{n+1}, (2).$$

An elementary argument shows that these conditions are respectively equivalent to the conditions that n be of one of the forms: 1) $n = 2^k - 2$, $k \geqq 2$; 2) $n = 2^k - 1$, $k \geqq 2$; 3) $n = 2^k - 3$, $k \geqq 3$.

We get therefore the theorem:

Theorem 5. A real projective space of dimension n cannot be differentiably imbedded in E^{n+2}, if n is not of one of the forms: 1) $n = 2^k - 2$, $k \geqq 2$; 2) $n = 2^k - 1$, $k \geqq 2$; 3) $n = 2^k - 3$, $k \geqq 3$.

Further applications of our criteria can be made, in particular, the application of conditions (39) to the imbedding of the complex projective space in an Euclidean space. It may be remarked that conditions (39) are expressed in terms of cohomology classes with real coefficients.

As characteristic classes with real coefficients they can be represented by differential forms obtained from a Riemann metric. There is no difficulty in finding the actual expressions. Thus we get in this way criteria that a Riemann manifold cannot be imbedded in a certaih Euclidean space in terms of curvature properties of the manifold. These criteria are particularly useful, when the Riemann manifold admits a transitive group of transformations.

5. Duality Theorems

An operation on sphere bundles which has various geometrical applications was introduced by Whitney. Let (B_1,X) and (B_2,X) be two principal bundles over the same base space X, whose structural groups are orthogonal groups $O(n_1)$, $O(n_2)$ in n_1 and n_2 variables respectively. Then the group $O(n_1) \times O(n_2)$ can be imbedded in $O(n_1+n_2)$, and we get a bundle over X with $O(n_1+n_2)$ as structural group, called the product of the given bundles. Denote by W^r, P^{4k} the Whitney and Pontrjagin characteristic classes of the product bundle and by W_1^r, P_1^{4k}, W_2^r, P_2^{4k} those of the given bundles. We recall here that the W's are with coefficients mod 2 and the P's with real coefficients. The so-called duality theorems deal with relations between the characteristic classes of the three bundles. More precisely, they express the classes of one of these bundles in terms of those of the other two. To express this relationship we introduce the polynomials

$$(42)\quad \begin{aligned} W(t) &= \sum_{r=0}^{n} W^r t^r \qquad W^0 = 1, \\ P(t) &= \sum_{0 \leqq 4k \leqq n} (-1)^k P^{4k}, \quad P^0 = 1, \end{aligned}$$

with the independent variable t. Then we have the following formulas

$$W(t) = W_1(t)W_2(t),$$
(43)
$$P(t) = P_1(t)P_2(t),$$

of which the first one is due to Whitney.

The proof of both formulas can be reduced to the case of universal bundles, following an idea of Wu Wen-tsun. Take an Euclidean space $E^{n_1+n_2+N_1+N_2}$, spanned by two Euclidean spaces $E^{n_1+N_1}$ and $E^{n_2+N_2}$, with N_1 and N_2 sufficiently large. Let $G = G(n_1, N_1)$, $G_2 = G(n_2, N_2)$ be the Grassmann manifolds in $E^{n_1+N_1}$ and $E^{n_2+N_2}$ respectively, and $G = G(n_1+n_2, N_1+N_2)$ that in $E^{n_1+n_2+N_1+N_2}$. An element in G_1 and an element in G_2 span an element in G, thus giving rise to a mapping

$$f : G_1 \times G_2 \dashrightarrow G.$$

Denote the projections of $G_1 \times G_2$ into its two factors by

$$p_1 : G_1 \times G_2 \dashrightarrow G_1,$$

$$p_2 : G_1 \times G_2 \dashrightarrow G_2.$$

Suppose the two given bundles be induced by the mappings

$$h_1 : X \dashrightarrow G_1, \; h_2 : X \dashrightarrow G_2.$$

We compose these two mappings into a mapping

$$h : X \dashrightarrow G_1 \times G_2,$$

defined by $h(x) = (h_1(x), h_2(x))$, $x \in X$, so that the two bundles are induced by the mappings p_1h and p_2h. Then the bundle over X induced by the mapping fh is their product. It is therefore sufficient to prove the formulas (43) for the characteristic classes in $G_1 \times G_2$ relative to the bundles induced by the mappings p_1, p_2, and f.

Let us restrict ourselves to the proof of the first formula of (43). Denote the classes in G, G_1, G_2 by w^r, w_1^r, w_2^r respectively. It will be sufficient to prove the following formula

$$f^* w^r = \sum_{i=0}^{r} p_1^* w_1^i \otimes p_2^* w_2^{r-i} . \tag{44}$$

To prove this, we show that both sides of this equation have the same value for any homology class of $G_1 \times G_2$ of dimension r. Such a class is of the form $z_1^k \otimes z_2^{r-k}$, where z_1^k and z_2^{r-k} are homology classes of G_1 and G_2 respectively. Suppose, for instance, that $z \neq (\underbrace{1 \cdots 1}_{k})$. A representative Schubert variety of this class will consist of linear spaces which contain a fixed linear space of dimension n_1-k+1. The linear spaces of a Schubert variety of the class z_2^{r-k} contain a fixed linear space of dimension $n_2-(r-k)$. A linear space spanned by them will then contain a linear space L_0 of dimension n_1+n_2-r+1. This condition is therefore satisfied by the linear spaces of a representative cycle of the class $f(z_1^k \times z_2^{r-k})$. To prove that w^r has the value 0 for $f(z_1^k \times z_2^{r-k})$, we notice that the value is also the intersection number of the latter with the cycle $(\underbrace{N_1+N_2-1 \cdots N_1+N_2-1}_{r} \underbrace{N_1+N_2 \cdots N_1+N_2}_{n_1+n_2-r})$, which consists of all linear spaces of dimension n_1+n_2 in a fixed linear space of dimension N_1+N_2+r-1. When this is in general position with L_0, they have only the origin 0 in common. Hence the intersection number is actually zero.

It remains to show that w^r has the value 1 when z_1^k and z_2^{r-k} are of the forms $z_1 = (1 \cdots 1)$, $z_2 = (1 \cdots 1)$. This will be reduced to the calculation of an intersection number. We shall omit the details here.

So far we have studied the additive and multiplicative homology structures of Grassmann manifolds in order to get conclusions on the characteristic classes of sphere bundles. Recently it has been found useful to study other topological operations in the Grassmann manifold (Cf., for instance, Wu Wen-tsun, C.R. Paris 230, 918-920 (1950)), in particular, the Steenrod squaring operations. It has been found possible to express the Steenrod squares of the Stiefel-Whitney classes as their quadratic polynomials. The formulas are

$$Sq^r W^s = \sum_{t=0}^{r} \binom{s-r+t-1}{t} W^{r-t} W^{s+t}, \quad s \geqq r > 0, \tag{45}$$

where $\binom{p}{q}$ is the binomial coefficient reduced mod 2, with the following conventions:

$$\begin{aligned} \binom{p}{q} &= 0, \text{ if } p < q,\ q > 0, \\ &= 1, \text{ if } q = 0. \end{aligned} \tag{46}$$

Let n-1 be the dimension of the spheres. We prove (45) by induction on n. For n = 1 it is trivial. Suppose therefore that (45) is true for bundles of spheres of dimension < n-1. It is sufficient to prove this on the universal bundle over $G = G(n,N) = G(n_1+n_2,\ N_1+N_2)$. We use the above notation and take $n_1 = n-1$, $n_2 = 1$, so that there is a mapping

$$f: \ G_1 \times G_2 \dashrightarrow G.$$

We put

$$F^{rs} = Sq^r W^s + \sum_{t=0}^{r} \binom{s-r+t-1}{t} W^{r-t} W^{s+t},$$

and denote the corresponding expressions for the bundles over G_1, G_2 by F_1^{rs}, F_2^{rs} respectively. Then we find

$$\begin{aligned} f^*F^{rs} &= Sq^r f^*W^s + \sum_{t=0}^{r} \binom{s-r+t-1}{t} f^*W^{r-t} f^*W^{s+t} \\ &= Sq^r (W_1^s \otimes 1 + W_1^{s-1} \otimes W_2^1) + \sum_{t=0}^{r} \binom{s-r+t-1}{t} (W_1^{r-t} \otimes 1 + W_1^{r-t-1} \otimes W_2^1) \\ &\qquad (W_1^{s+t} \otimes 1 + W_1^{s+t-1} \otimes W_2^1, \\ &= F_1^{rs} \otimes 1 + F_1^{r,s-1} \otimes W_2^1 + F_1^{r-s,s-1} \otimes (W_2^1)^2, \end{aligned}$$

which is zero by our induction hypothesis. But by (44) f^* is an isomorphism in the dimensions $\leq N_1, N_2$. Hence $F^{rs} = 0$, and (45) is proved.

The interest in the formula (45) lies in the fact that it enables us to formulate necessary conditions that a sphere bundle is a tangent bundle. In fact, we introduce the cohomology classes U^r of Wu by the equations

$$W^i = \sum_{p \geq 0} Sq^{i-p} U^p, \quad i \geq 0 \tag{47}$$

which completely determine U^p. Comparing with the equation (36), we have the following theorem:

A necessary condition for a sphere bundle to be a tangent bundle is the vanishing of the following classes of Wu:

$$U^p = 0, \quad p > \frac{n}{2}, \tag{48}$$

n-1 being the dimension of the spheres.

6. An application to projective differential geometry

We shall conclude these notes by giving an application of a different kind, namely, to a problem of projection differential geometry.

Let P be the three-dimensional real projective space, and E the four-dimensional space of its lines. We define a ruled surface in P to be a differentiable mapping $f: S^1 \dashrightarrow E$ of a circle into E such that no two lines intersect. In our early notation E was written as G(2,2). The mapping f therefore induces a bundle of circles over S^1 and the ruled surface is a realization of the bundle space in P. Since $\pi_1(E) \approx I_2$, there are two such bundles according as the mapping f defines the zero or non-zero element of $\pi_1(E)$, the ruled surface being homeomorphic to a torus and a Klein bottle respectively. We shall prove the following theorem due to Wu:

A ruled surface in P is homeomorphic to a torus.

In other words, the non-orientable bundle cannot be realized as a ruled surface in P.

To prove this theorem let us recall that if x_i, $i = 1, \cdots, 4$, are the homogeneous coordinates in P, the Plücker coordinates p_{ij} of a line joining the points x_i, y_i are defined by

$$p_{ij} = x_i y_j - x_j y_i \quad , \; i, j = 1, \cdots, 4. \tag{49}$$

These coordinates p_{ij} are homogeneous and satisfy the identity

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0. \tag{50}$$

Instead of these we introduce another set of coordinates by

$$\begin{aligned} p_{12} &= \xi_1 + \eta_1, \; p_{13} = \xi_2 + \eta_2, \; p_{14} = \xi_3 + \eta_3, \\ p_{34} &= \xi_1 - \eta_1, \; p_{42} = \xi_2 - \eta_2, \; p_{23} = \xi_3 - \eta_3. \end{aligned} \tag{51}$$

Then the identity (50) becomes

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = \eta_1^2 + \eta_2^2 + \eta_3^2 \tag{52}$$

The coordinates ξ_i, η_i being still homogeneous, we can normalize them so that

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = 1, \quad \eta_1^2 + \eta_2^2 + \eta_3^2 = 1. \tag{53}$$

The normalized coordinates are determined up to a sign. We can therefore take two spheres (two-dimensional) S_1, S_2 and represent the lines of P as pairs of points of these spheres such that the pairs (ξ, η), (ξ^*, η^*), where ξ^*, η^* are the antipodal points of ξ, η, determine the same line. If we take the lines to be oriented, then the oriented lines are in one-one correspondence with the pairs of points of the two spheres.

The mapping f will be represented by a pair of curves $\xi(t)$, $\eta(t)$, $0 \leq t \leq 1$, such that

$$(\xi(0), \eta(0)) = (\xi(1), \eta(1)) \text{ or } (\xi(0), \eta(0)) = (\xi^*(1), \eta^*(1)).$$

The bundle is orientable or non-orientable, according as the first or second case happens. Suppose now that the second is the case. Denote by d_1, d_2 the spherical distances on the two spheres, and write

$$d_1(t,t') = d_1(\xi(t), \xi(t')),$$

$$d_2(t,t') = d_2(\eta(t), \eta(t')).$$

Then we have

$$d_1(0,t)+d_1(t,1) = d_1(0,1),$$

$$d_2(0,t)+d_2(t,1) = d_2(0,1).$$

An elementary argument will then give the following lemma: There exist two values $0 \leq t \leq 1$, $0 \leq t' \leq 1$, with $t \neq t'$, $(t,t') \neq (0,1)$, such that

$$d_1(t,t') = d_2(t,t').$$

In terms of the coordinates this can be written

$$\xi_1(t)\xi_1(t')+\xi_2(t)\xi_2(t')+\xi_3(t)\xi_3(t') = \eta_1(t)\eta_1(t')+\eta_2(t)\eta_2(t')+\eta_3(t)\eta_3(t'),$$

which is the condition that the lines corresponding to the parameters t,t' intersect. But this contradicts our assumption that no two distinct lines intersect.