SHAFAREVICH MAPPINGS AND PERIOD MAPPINGS

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To our friend Enrico Arbarello, whose zest for life and enjoyment of mathematics continue to inspire us

ABSTRACT. We shall show that a smooth, quasi-projective variety X has a holomorphically convex universal covering $\widetilde{X}$ when (i) $\pi_1(X)$ is residually nilpotent and (ii) there is an admissable variation of mixed Hodge structure over X whose monodromy representation has a finite kernel, and where in each case a corresponding period mapping is assumed to be proper.

OUTLINE

I. INTRODUCTION AND STATEMENTS OF RESULTS

In [Sh72] Shafarevich posed a beautiful question, a variant of which has become known as Shafarevich's conjecture (cf. [Ko93] and [Ko95]):

I.1: *Let X be a smooth projective variety. Is the universal covering $\widetilde{X}$ of X holomorphically convex?*

This means that there is a Stein analytic variety $\widetilde{S}$ and a proper holomorphic mapping $\widetilde{X} \to \widetilde{S}$ that contracts the connected compact analytic subvarieties of $\widetilde{X}$ to points. This mapping is the Cartan-Remmert reduction of $\widetilde{X}$ ([Car79]).

In practice for a quasi-projective variety X one seeks a *Shafarevich mapping* ([Ko93], [Ko95] and [Cam94])

$$(I.2) \qquad\qquad Sh : X \to S$$

onto an analytic variety with the properties

(i) If $Y \subset X$ is a connected complex analytic subvariety of X that is contracted to a point by Sh, then Y is compact and the image

$$\pi_1(Y) \to \pi_1(X)$$

    is a finite subgroup (we shall say that the mapping $\pi_1(Y) \to \pi_1(X)$ is finite).

(ii) In the diagram

$$(I.3) \qquad\qquad \begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{Sh}} & \widetilde{S} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ X & \xrightarrow{Sh} & S \end{array}$$

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the universal cover $\widetilde{S} = \widetilde{X} \times_X S$ is Stein.

We note that if (i) is satisfied, then the connected fibres of $\widetilde{Sh}$ are the connected components of $\pi^{-1}(Y)$ where $Y \subset X$ is a connected fibre of Sh. Moreover the components of $\pi^{-1}(Y)$ are compact.

The Shafarevich conjecture has been established when

(a) X is projective and $\pi_1(X)$ is residually nilpotent;
(b) X is projective and $\pi_1(X)$ has a faithful linear representation.

The initial proof of (a) is in [Ka97]. Subsequently there have been numerous further works (cf. [Cam95] and [Cl08]). The proof of (b) is in [Ey04] and [EKPR12]; it is the culmination of a series of works drawing on both classical and non-abelian Hodge theory (cf. [CM-SP17] and [Si88], [Si92]). Some of the history and references to the literature are given in [EKPR12]. Further related references are [BdO06], [EF21], [KR98], and [Ey09].

The purpose of this partly expository paper is to extend (a) and a special case of (b) to the case when X is quasi-projective. The essence of what will be proved here may be informally expressed as saying

proper period mappings give Shafarevich mappings.

In order to understand what is meant by proper for the period mappings to be considered we introduce some standard terminology and notations. By a *completion* of X we mean a smooth projective variety X in which $\overline{X}$ is a Zariski open set with complement $Z = \cup Z_i$ a reduced normal crossing divisor with Z_i irreducible.

In case (a) there are two choices for a period mapping. One is the classical Albanese mapping

(I.4a) $$\alpha : X \to \mathrm{Alb}(X)$$

where $\mathrm{Alb}(X) = H^0(\Omega^1_X(\log Z))^* / H_1(X, \mathbb{Z})$ is a semi-abelian variety. The other choice is to use a higher Albanese mapping ([Ha87a] and [Cl08])

(I.4b) $$\alpha_s : X \to \mathrm{Alb}^s(X), \quad s = 1, 2, 3, \ldots.$$

This is a period mapping for a unipotent variation of mixed Hodge structure (UVMHS) ([HZ85]). It reduces to (I.4a) when $s = 1$, and when $\pi_1(X)$ is residually nilpotent the mappings (I.4b) are all essentially the same for $s \gg 0$.

The properness of the Albanese mapping (I.4a) implies that the image of the residue mapping

$$H^0(\Omega^1_{\overline{X}}(\log Z)) \xrightarrow{i} \overset{i}{\oplus} H^0(\mathcal{O}_{Z_i})(-1)$$

projects non-trivially to each summand $H^0(\mathcal{O}_{Z_i})(-1)$. These conditions are independent of the completion of X. Additional for each stratum $Z_I := \cap_{i \in I} Z_i$ with $|I| \geqq 2$ and each ray λ emanating from Z_I^* out into X there should be an $\omega \in H^0(\Omega^1_{\overline{X}}(\log Z)$ having a logarithmic singularity at $\lambda \cap Z_I^*$. If the second of these conditions is not satisfied, then only the closure of the image of the Shafarevich map will be Stein. From the second of the three proofs of Theorem A below it will follow that α is proper if, and only if, the α_s are proper. In fact as a consequence of the proof of Theorem II.4 below we will have the

THEOREM I.5: *The images $\alpha_s(X)$ are algebraic varieties and the mapping $\alpha_s(X) \to \alpha(X)$ is a proper surjective morphism. If $Y \subset X$ is a compact subvariety with $\alpha\big|_Y = \text{constant}$, then $\alpha_s\big|_Y = \text{constant for } s \gg 0$.*

To state the result in case (a) we will use the diagram

(I.6)
$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\;\widetilde{\alpha}\;} & \widetilde{\text{Alb}(X)} \\
{\scriptstyle\pi}\big\downarrow & & \big\downarrow \\
X & \xrightarrow{\;\alpha\;} & \text{Alb}(X)
\end{array}
$$

where $\widetilde{\text{Alb}(X)}$ is the universal cover $\text{Alb}(X)$, together with similar diagrams (I.6$_s$) for the higher Albanese mappings.

THEOREM A: *Assume that $\pi_1(X)$ is residually nilpotent and that α is proper.*

  (i) *The mapping $\widetilde{\alpha}$ is proper;*
  (ii) *the image $\widetilde{\alpha}(\widetilde{X})$ is Stein;*
 (iii) *the connected components of the fibres of $\widetilde{\alpha}$ are the connected components of $\pi^{-1}(Y)$ where $Y \subset X$ is a fibre of α;*
 (iv) *the connected components of the fibres of α are characterized by the condition that the map*
$$
H^1(X, \mathbb{Q}) \to H^1(Y, \mathbb{Q})
$$
be trivial.

The substantive parts of this theorem are (ii), (iii) and (iv). We note that (iii) and (iv) imply

(I.7) $\qquad \pi_1(Y) \to \pi_1(X)$ is finite $\iff H^1(X, \mathbb{Q}) \to H^1(Y, \mathbb{Q})$ is trivial.

The implication $\Rightarrow$ is clear. The proof of the converse will use Hodge theory. One argument will be an extension of that used in [Ka97]. A second argument, one that is more in the conceptual framework of general period mappings, will use that the implication $\Leftarrow$ is a consequence of Theorem I.5. A third argument, different from the first two, will describe a construction of the minimal model of $\widehat{\pi}_1(X)$ as in [Mo78] and [Ha87a]. For use elsewhere this construction will give a bound on the singularities along Z of the differential forms in the minimal model. We will also give an extension to the quasi-projective case of the classical $\partial\bar{\partial}$-lemma.

For case (b) we shall first give a result where we have a standard variation of Hodge structure (VHS) whose underlying local system $\mathbb{V} \to X$ has a monodromy representation
$$
\rho : \pi_1(X) \to \Gamma \subset \text{Aut}(V)
$$
with a finite kernel ([CM-SP17]). We denote by

(I.8) $\qquad\qquad\qquad\qquad \Phi : X \to \Gamma\backslash D$

the corresponding period mapping (loc. cit.). The properness of Φ is equivalent to the logarithm N_i of monodromy around each Z_i being non-zero. Again this condition is independent of the smooth completion of X. In this case it is well known that the image
$$
\Phi(X) := P \subset \Gamma\backslash D
$$

is a closed analytic subvariety of $\Gamma\backslash D$.[1] We then have a diagram

(I.9)
$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\ \widetilde{\Phi}\ } & \widetilde{P} \subset D \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \xrightarrow{\ \Phi\ } & P \subset \Gamma\backslash D \end{array}$$

where $\widetilde{P}$ is the image in D of $\widetilde{X}$.

THEOREM B: *Assume that Φ is proper and that ρ has a finite kernel.*

(i) *$\widetilde{\Phi}$ is a proper holomorphic mapping;*
(ii) *$\widetilde{P}$ is a Stein subvariety of D;*
(iii) *the connected fibres of $\widetilde{\Phi}$ are the connected components of $\pi^{-1}(Y)$ where $Y \subset X$ is a fibre of Φ;*
(iv) *these Y are characterized by the condition that $\rho\big|_{\pi_1(Y)}$ is finite.*

Given our assumptions the main substantive statement in the theorem is (ii). As will be seen in Section III this will be a direct consequence of classical results about the geometry of period domains.

Theorem B may be summarized by saying that Φ is a Shafarevich mapping for X. What is missing is a cohomological, rather than a homotopy-theoretic, description of the fibres of $\widetilde{\Phi}$. One would like some analogue of (iv) in Theorem A. This will be further discussed in Section III.

Our final main result is an amalgam of Theorems A and B. To state it we assume given an admissible variation of mixed Hodge structure (VMHS) with the local system $V_M \to X$ having corresponding monodromy representation $\rho_M : \pi_1(X) \to \Gamma \subset \mathrm{Aut}(V_M)$ (cf. [SZ85] and [Us83]). We denote by

(I.10)
$$\Phi_M : X \to \Gamma_M\backslash D_M$$

the corresponding period mapping where D_M is now a mixed period domain (loc. cit.).

THEOREM C: *Assume that Φ_M is proper and that ρ_M has a finite kernel. Then Φ_M is a Shafarevich map for X.*

The obvious idea behind the proof is to consider the factorization

$$\begin{array}{ccc} X & \xrightarrow{\ \Phi_M\ } & \Gamma_M\backslash D_M \\ & {\scriptstyle\Phi}\searrow & \downarrow \\ & & \Gamma\backslash D \end{array}$$

where Φ is the period mapping obtained by passing to the associated graded of the mixed Hodge structures parametrized by D_M. For this period mapping we have a diagram (I.9) where the image $\widetilde{\Phi}(\widetilde{X}) \subset D$ will be a Stein analytic variety. Note that we are not saying that $\widetilde{\Phi}$ is proper.

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[1]It is in fact a quasi-projective algebraic subvariety ([BBT06]).

Along the fibres of Φ we will have an admissible VMHS such that the associated graded polarized Hodge structures are locally constant on the irreducible components of a fibre Y.[2] Then a strengthening of (I.5) will give that the image $\Phi_M(Y)$ of the fibre will map finitely to the image under Φ_M of the extension data of levels $\leqq 2$ (this condition will be explained below). The proof of Theorem A can be adapted to this situation to show $\widetilde{\Phi}(\pi^{-1}(Y))$ is Stein. We will then have an analytic variety mapping to a Stein analytic variety with Stein fibres. In the situation at hand the total space can then be shown to be a Stein variety (cf. [St56]).

Following the proof in Section III of Theorems B and C we will discuss the question of describing cohomologically the fibres of the Shafarevich mapping in case B. This leads naturally into the description of the mixed Hodge structure on the completion relative to ρ of $\pi_1(X, x)$ ([Ha98] and [EKPR12]). The cohomology groups that enter here have one interpretation as defining the extension classes of that mixed Hodge structure, and among these are the classes of first order deformations of the local system $\mathbb{V} \to X$ underlying a VHS. Both of these are central ingredients in the proof of the main theorem in [EKPR12]. In [Le19] and [Le21] these results have been extended to the quasi-projective case. Here for expository purposes we will discuss very briefly special cases of the results in these references.

In Section IV we will discuss some questions and conjectures related to different perspectives on the Shafarevich conjecture.

II. THE NILPOTENT CASE

Referring to the diagram (I.6) and statement of Theorem A there are two things to be proved:

(II.1)   if $Y \subset X$ is a connected fibre of α, then the map $\pi_1(Y) \to \pi_1(X)$ is finite;

(II.2)   there is a plurisubharmonic (psh) exhaustion function $\varphi : \widetilde{\alpha}(\widetilde{X}) \to \mathbb{R}$.

For (II.1) we will give two different arguments. The first is basically to extend the proof in [Ka97] to the quasi-projective case (cf. also [EKPR12] and [Cl08]). The steps are

- reduce to the case where $Y \subset X$ has dimension 1;
- further reduce to the case where Y has nodes.

These steps are essentially the same as in [Ka97] and that are reviewed in Section 3 of [EKPR12].

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[2]This is the situation that one encounters in the Satake-Baily-Borel completion of a classical period mapping ([GGLR20], [GGR22] and [Gr22]). In these references a more general version of Theorem 1.5 appears; also given there are further details of some arguments that are only sketched here.

- analyze the case when Y is a cycle; i.e., the dual graph of Y is topologically an S^1.
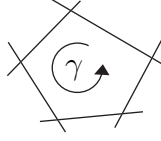
In this case what (II.1) means is that the inverse image $\pi^{-1}(Y) \subset \widetilde{X}$ is not an infinite chain of irreducible curves. Equivalently, some multiple of the circuit γ maps to the identity in $\pi_1(X)$. The general case of a nodal curve can be done by extending the argument in this case.

Our assumption is that the map $H_1(Y, \mathbb{Q}) \to H_1(X, \mathbb{Q})$ is trivial. We want to use this statement about homology to infer one about homotopy. For this we shall use that the unipotent completions

$$\widehat{\mathbb{Z}\pi_1}(X; x) = \lim_{s \to \infty} \pi_1(X, x)/J^{s+1}$$

of $\pi_1(X, x)$ and similarly for Y have mixed Hodge structures ([Mo78] and [Ha87b]). Here $\mathbb{Z}\pi_1(X, x)$ is the group ring of $\pi_1(X, x)$ and J is the augmentation ideal. The weights of the generators coming from $H_1(X)$ and $H_1(Y)$ are

- $\widehat{\pi_1(X)}$ is generated in weights $-1, -2$ (cf. [Mo78]);
- $\widehat{\pi_1(Y)}$ is generated in weights $-1, 0$ (cf. [CG14]).

If $Y = \cup Y_\alpha$, the -1 part comes from the $H_1(Y_\alpha, \mathbb{Q})$. The 0-part, which is only well defined modulo the -1 parts, corresponds to the circuit γ. This may be seen by considering the exact sequence of the pair (Y, D)

$$H_1(Y) \to H_1(Y, D) \xrightarrow{\partial} H_0(D)$$

where D is the set of nodes. Here γ should be considered as a class in $\ker \partial$ modulo the images of the $H_1(Y_\alpha)$. From this weight considerations give that the class of γ maps to zero in $\widehat{\pi_1(X)}$. Since $\pi_1(X)$ is assumed to be residually nilpotent we may conclude that γ is of finite order in $\pi_1(X)$. $\qquad \square$

We will now give the second argument for the proof of (II.1). This argument is more conceptual. It takes place within the general framework of period mappings and will be used again in the proof of Theorem C. We begin by recalling from (I.4b) the higher Albanese mappings constructed in [Ha87a] and [HZ87]. For each $s \geqq 1$ there is a diagram of mappings

(II.3)

$$\begin{array}{ccc}
 & & \mathrm{Alb}^s(X) \\
 & \nearrow \alpha_s & \downarrow \pi_s \\
X & & \\
 & \searrow \alpha & \\
 & & \mathrm{Alb}(X)
\end{array}$$

where $\mathrm{Alb}(X) = \mathrm{Alb}^1(X)$ and $\alpha = \alpha_1$ is the usual Albanese mapping. These maps may be viewed as period mappings associated to unipotent variations of mixed Hodge structure

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whose underlying local systems are the groups

$$\mathbb{V}_x^s = \mathbb{Z}\pi_1(X, x)/J^{s+1}.$$

A basic property of the induced mappings in (II.3) is given by Theorem I.5 above. Here we only need the following part of that result.

THEOREM II.4: *For $Y \subset X$ a compact subvariety and $s \gg 0$*

$$\alpha\big|_Y = constant \iff \alpha_s\big|_Y = constant.$$

The proof of Theorem II.4 will be a consequence of a general result about the variation of the extension data in a VMHS that we will now explain and give a sketch of the proof.

Let

(II.5) $$\Phi : X \to \Gamma\backslash D$$

be any admissible variation of mixed Hodge structures whose associated graded is a constant family $\{H^0, \ldots, H^n\}$ of weight k Hodge structures H^k on a fixed vector space $\mathrm{Gr}_k^W V$. What is then varying is the extension data for the mixed Hodge structures. This extension data comes in various levels (cf. II.8 below), and associated to (II.5) there are maps

(II.6a) $$X \xrightarrow{\Phi_1} \{\text{extension data of level 1}\},$$

(II.6b) $$\{\text{fibre of } \Phi_1\} \xrightarrow{\Phi_{1,2}} \{\text{extension data of level 2}\},$$

(II.6c) $$\{\text{fibre of } \Phi_2\} \xrightarrow{\Phi_{2,3}} \{\text{extension data of level 3}\}.$$
$$\vdots$$

The right-hand sides of the above maps are isomorphic to

$$\text{level 1} \quad \overset{\ell}{\oplus} \mathrm{Ext}^1_{\mathrm{MHS}}(H^{\ell+1}, H^\ell),$$

$$\text{level 2} \quad \overset{\ell}{\oplus} \mathrm{Ext}^1_{\mathrm{MHS}}(H^{\ell+2}, H^\ell),$$

$$\text{level 3} \quad \overset{\ell}{\oplus} \mathrm{Ext}^1_{\mathrm{MHS}}(H^{\ell+3}, H^\ell)$$
$$\vdots$$

This means that if we pick a point in a fibre, then the difference of the extension data for a variable point and the fixed point is in the $\oplus \mathrm{Ext}^1_{\mathrm{MHS}}(\bullet, \bullet)$'s.

In more detail we are here using the general fact that if we have a MHS M with graded pieces A, B, C that are pure Hodge structures, then there is a fibration

$$\text{Extension data for } M$$
$$\downarrow$$
$$\mathrm{Ext}^1_{\mathrm{MHS}}(B, A) \times \mathrm{Ext}^1_{\mathrm{MHS}}(C, B)$$

whose fibres have connected components isomorphic to $\mathrm{Ext}^1_{\mathrm{MHS}}(C, A)$. To see this we have

$$0 \longrightarrow B \longrightarrow M/A \longrightarrow C \longrightarrow 0$$

$$0 \longrightarrow A \longrightarrow \mathrm{Ker}(M \to C) \longrightarrow B \longrightarrow 0$$

and a master diagram

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
A & = \!\!= & A \\
\downarrow & & \downarrow \\
0 \to \mathrm{Ker}(M \to C) \longrightarrow & M \longrightarrow & C \to 0 \\
\downarrow & & \downarrow \qquad \| \\
0 \longrightarrow B \longrightarrow & M/A \to & C \to 0 \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}
$$

From [Car80] the extension classes are

$$e_{BC} \in \frac{\mathrm{Hom}_{\mathbb{C}}(C, B)}{F^0 \, \mathrm{Hom}_{\mathbb{C}}(C, B) + \mathrm{Hom}_{\mathbb{Z}}(C, B)}, \quad e_{AB} \in \frac{\mathrm{Hom}_{\mathbb{C}}(B, A)}{F^0 \, \mathrm{Hom}_{\mathbb{C}}(B, A) + \mathrm{Hom}_{\mathbb{Z}}(B, A)}.$$

We also have the extension classes

$$e_1 \in \frac{\mathrm{Hom}_{\mathbb{C}}(C, \mathrm{Ker}(M \to C))}{F^0 + \mathrm{Hom}_{\mathbb{Z}}}, \quad e_2 \in \frac{\mathrm{Hom}_{\mathbb{C}}(M/A, A)}{F^0 + \mathrm{Hom}_{\mathbb{Z}}}$$

with

$$e_1 \to e_{BC}, \quad e_2 \to e_{AB}$$

induced respectively by

$$\mathrm{Hom}(C, \mathrm{Ker}(M \to C)) \to \mathrm{Hom}(C, B), \quad \mathrm{Hom}(M/A, A) \to \mathrm{Hom}(B, A).$$

Knowing e_{AB} and e_1 or e_{BC} and e_2 determines M as a MHS. For example,

• e_{AB} determines $\mathrm{Ker}(M \to C)$ and

• $e_1 \in \{u \in \mathrm{Hom}(C, \mathrm{Ker}(M \to C))/F^0 + H_{\mathbb{Z}} : u \to e_{BC} \text{ under } \ker(M \to A) \to B\}$.

Thus if we know e_{AB} and e_1, then we know the u in the second bullet and from this we know M.

If $e_{BC} = 0$, then the second bullet reduces to

$$\frac{\mathrm{Hom}_{\mathbb{C}}(C, A)}{F^0 \, \mathrm{Hom}_{\mathbb{C}}(C, A) + \mathrm{Hom}_{\mathbb{Z}}(C, A)}.$$

In general we obtain a fibre space.

We note that even though the action of monodromy on the associated graded of the MHS's is trivial, its action on extension data of various levels is abelian and will generally not be trivial.

The key observation is the use of horizontality (transversality) to prove the

PROPOSITION II.7: *For $k \geqq 2$ the differentials of the maps $\Phi_{k,k+1}$ are zero.*

A proof of this result is given in [GGR22] (cf. also [Gr22]). The following is a sketch of the argument. There is a sequence of period mappings

$$(\text{II.8}) \qquad\qquad \Phi_k : X \to \Gamma_k \backslash D_k$$

to the extension data of levels $\leqq k$ in (II.5). Denoting by $X_k \subset X$ a typical fibre of Φ_{k-1}, we have

$$\Phi_{k-1,k} : X_k \to E_k$$

where

$$E_k \cong \overset{\ell}{\oplus} \operatorname{Ext}^1_{\mathrm{MHS}}(H^{k+\ell}, H^\ell)$$

with

$$\operatorname{Ext}^1_{\mathrm{MHS}}(H^{k+\ell}, H^\ell) = \frac{\operatorname{Hom}_{\mathbb{C}}(H^{k+\ell}, H^\ell)}{F^0 \operatorname{Hom}_{\mathbb{C}}(H^{k+\ell}, H^\ell) + \operatorname{Hom}_{\mathbb{Z}}(H^{k+\ell}, H^\ell)}.$$

By horizontality the differential of $\Phi_{k-1,k}$ gives a map

$$TX_k \to H_k \subset TE_k$$

where the following diagram depicts the complex tangent spaces TE_k with the subspaces H_k being the part over the red:

$$k = 1 \quad \underbrace{(\ell-1,-\ell) \oplus \cdots \oplus (0,-1)}_{F^0} \oplus \underbrace{\underbrace{(-1,0)}_{} \oplus \cdots \oplus (-\ell,\ell-1)}_{TE_1}$$

$$k = 2 \quad \underbrace{(\ell-2,-\ell) \oplus \cdots \oplus (0,-2)}_{F^0} \oplus \underbrace{\underbrace{(-1,-1)}_{} \oplus (-2,0) \oplus \cdots \oplus (-\ell,\ell-2)}_{TE_2}$$

$$k = 3 \quad \underbrace{(\ell-3,-\ell) \oplus \cdots \oplus (0,-3)}_{F^0} \oplus \underbrace{\underbrace{(-1,-2)}_{} \oplus (-2,-1) \oplus \cdots \oplus (-\ell,\ell-3)}_{TE_3}.$$

Denoting any of the E_k by E, as a mapping of real manifolds we have

$$T_{\mathbb{R}} E \cong \mathbb{R}^{2m}$$

and the differential of $\Phi_{k-1,k}$ maps to a subspace $H_{\mathbb{R}} \subset \mathbb{R}^{2m}$ that is invariant under the complex structure $J : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$. On the complexification of $\mathbb{R}^{2m}$ given by the action of J the complexification of $H_{\mathbb{R}}$ is $H \oplus \overline{H}$ where H is the term over the $\smile$ above. Denoting by Λ the discrete subgroup of H induced by the $\operatorname{Hom}_{\mathbb{Z}}(H^{k+\ell}, H^\ell)$ terms above we see that

- H/Λ is a compact complex torus for $k = 1$;
- $H/\Lambda \cong (\mathbb{C}^*)^m$ for $k = 2$;[3]
- $H \cong \mathbb{C}^n$ for $k \geqq 3$.

The maps (II.6) correspond to maps arising from a diagram

$$\pi_1(X_k) \longrightarrow \Lambda$$
$$\searrow \qquad \nearrow$$
$$H_1(X_k, \mathbb{Z}).$$

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[3]In this case we have a fibration over a compact, complex torus whose connected components of the fibres are isomorphic to $(\mathbb{C}^*)^m$'s. This does *not* mean that the total space is a semi-abelian variety.

Using a weight argument this implies that TX_k maps to a sub-Hodge structure of TE_k whose complex part lies under the red. It also means that for a desingularization X_k' of X_k and smooth completion $\overline{X}_k'$ with $\overline{X}_k' \backslash X_k' = Z_k'$ a normal crossing divisor

- the pullback by $\Phi_{k-1,k}$ of the dual H^* of H will give 1-forms in $H^0(\Omega^1_{\overline{X}_k'}(\log Z_k'))$, and
- the mapping $\Phi_{k-1,k}$ will be given by integrating these logarithmic 1-forms taken modulo periods from $H_1(X_k', \mathbb{Z})$.

For $k \geqq 3$ there are no periods and so $\Phi_{k-1,k}$ is locally constant. $\qquad\square$

COROLLARY II.9: *In* (II.5) *the induced map*

$$\Phi(X) \to \Phi_2(X)$$

is a finite morphism of algebraic varieties.

Proof. It follows from (II.7) that the connected components of fibres of the above map are points. An analysis of how these maps are defined (they are iterated integrals of logarithmic 1-forms taken modulo periods) shows that both $\Phi(X)$ and $\Phi_2(X)$ may be completed to algebraic varieties and that $\Phi(X) \to \Phi_2(X)$ extends to a rational map on the completions (cf. [GGR22] and [Gr22]). $\qquad\square$

Proof of Theorem II.4. The first basic observation is that the tower of Albanese mappings (II.3) is *not* the tower obtained by considering a higher Albanese mapping as a period mapping and then taking extension data of level $\leqq k$ for each k. For example, the first level of the Albanese tower is

$$\mathrm{Alb}(X) = H^0(X, \Omega_X^1)^* / H_1(X, \mathbb{Z})$$

and in general $H_1(X, \mathbb{Z})$ is a mixed Hodge structure with weights $-2, -1$ and for which there is non-trivial extension data (cf. (II.15) below).

A second point is that the period mappings (II.3) are only defined upon choice of a base point $x_0 \in X$. Thus the relevant Hodge structures are a $\mathbb{Q}$ in weight zero for the point and $H^1(X, \mathbb{Q})$ for the Albanese variety. The mapping α in (II.3) needs both of these (cf. the discussion in 5.32 ff. in [SZ85]).

We now change notation and denote by (II.8) the level of extension data mappings associated to UVMHS given by $\alpha_s : X \to \mathrm{Alb}^s(X)$ for $s \gg 0$. Then $\Phi_2(X)$ contains the information in $\mathrm{Alb}_{X,x_0} : X \to \mathrm{Alb}(X)$. The D_k in the tower of mappings (II.8) are obtained from

(i) the associated graded of the cohomology groups in $\widehat{\pi_1(X, x)}$; and
(ii) the various levels of extension data among these groups, as explained above.

Since we are dealing with a *unipotent* VMHS, the terms in (i) are constant. The basic observation from Corollary II.9 is that the only varying extension data in the period mapping (II.5) in this case is that arising from $\mathrm{Alb}_{X,x}$. In terms of extension data arising from the UVMHS the information in $\mathrm{Alb}_{X,x}$ is contained in the mapping Φ_2 to extension data of level $\leqq 2$. Thus

> *If $Y \subset X$ is a compact subvariety such that $\alpha\big|_Y = constant$,*
> *then $\alpha^s\big|_Y = constant$.*

Since $\pi_1(X)$ is assumed to be residually nilpotent, this statement implies Theorem II.4 and (I.7). $\qquad\square$

Proof of (II.2). Choosing a base point $x_0 \in X$ and a basis ω_λ for $H^0(\Omega^1_{\widetilde{X}}(\log Z))$ the Albanese mapping of X is given by

$$(\text{II.10}) \qquad \alpha(x) = \left(\dots, \int_{x_0}^x \omega_\lambda, \dots\right) \quad \text{modulo periods.}$$

The assumed completion of X means that at every point p of Z there is an ω_λ having a logarithmic singularity at p. In more detail, if in coordinates Z is locally given by

$$z_i = 0, \qquad i \in I$$

then there is an $\omega \in H^0(\Omega^1_{\widetilde{X}}(\log Z))$ such that

$$\omega = \sum_{i \in I} a_i \frac{dz_i}{z_i} + \text{ holomorphic terms}$$

where all $a_i \neq 0$.

The pullback to $\widetilde{X}$ of $\mathrm{Alb}(X)$ is a trivial bundle $\widetilde{X} \times \mathbb{C}^N$. Choosing coordinates t_λ for $\mathbb{C}^N$ the pullback to $\widetilde{X}$ of ω_λ is dt_λ. Then by the above

$$\varphi = \sum |t_\lambda|^2$$

is an exhaustion function on $\widetilde{X}$; the projection to X of the sets $\varphi < C$ stay a fixed distance from Z. Moreover, the Levi form $\left(\frac{i}{2}\right)\partial\overline{\partial}\varphi$ is positive in the Zariski tangent spaces to the analytic subvariety $\widetilde{\alpha}(\widetilde{X}) \subset \mathbb{C}^N$. $\qquad\square$

II.11: *An interpretation of the Albanese map as an extension class.*

The following is intended to provide background for the discussion in the next section of the question raised in the introduction of detecting cohomologically the fibres of the Shafarevich mapping in the VHS case.

We have noted that the analytic group

$$\mathrm{Alb}(X) = H^0(\Omega^1_{\widetilde{X}}(\log Z))^* / H_1(X, \mathbb{Z})$$

is a semi-abelian variety that fits in an exact sequence

$$(\text{II.12}) \qquad\qquad 0 \to T \to \mathrm{Alb}(X) \to \mathrm{Alb}(\overline{X}) \to 0$$

where $T \cong (\mathbb{C}^*)^m$ is an algebraic torus. Completing $\mathrm{Alb}(X)$ by replacing the $\mathbb{C}^*$'s by $\mathbb{P}^1$'s the Albanese mapping on X extends to a rational mapping; the graph of α in $X \times \mathrm{Alb}(X)$ closes up in $\overline{X} \times \overline{\mathrm{Alb}(X)}$.

By way of an interlude we will first describe

II.13: *The extension class of the semi-abelian variety* $\mathrm{Alb}(X)$.

The exact sequence of connected abelian Lie groups (II.12) is constructed from the exact cohomology sequence

$$(\text{II.14}) \qquad\qquad 0 \to H^1(\overline{X}, \mathbb{Q}) \to H^1(X, \mathbb{Q}) \to H^1(T, \mathbb{Q}) \to 0.$$

The term in the middle has a mixed Hodge structure with weight filtration $W_1 \subset W_2$ where
- $\mathrm{Gr}_1^W(H^1(X, \mathbb{Q})) = H^1(\overline{X}, \mathbb{Q})$;
- $\mathrm{Gr}_2^W H^1(X, \mathbb{Q}) \cong H^1(T, \mathbb{Q}) \cong \overset{m}{\oplus} \mathbb{Q}(-1)$.

The extension class of (II.14) is in

$$\text{Ext}^1_{\text{MHS}}(\overset{m}{\oplus} \mathbb{Q}(-1), H^1(\overline{X}, \mathbb{Q})) \cong \overset{m}{\oplus} \left\{ \frac{H^1(\overline{X}, \mathbb{C})}{F^1 H^1(\overline{X}, \mathbb{C}) + H^1(\overline{X}, \mathbb{Z})} \right\}$$

$$\cong \overset{m}{\oplus} H^1(\overline{X}, \mathcal{O}_{\overline{X}})/H^1(\overline{X}, \mathbb{Z})$$

$$\cong \overset{m}{\oplus} \text{Pic}^\circ(\overline{X}).$$

From this we may infer that the extension class of (II.14) gives the image in $\text{Pic}^\circ(\overline{X})$ of the kernel of the Gysin mapping

$$\oplus^i H^0(\mathcal{O}_{Z_i}) \xrightarrow{\text{Gy}} H^2(X, \mathbb{Z}).$$

Next we shall describe

II.15: *The extension class of* $\text{Alb}(X, x_0)$.

Thus far we have made no reference to the choice of a base point $x_0 \in X$. When we include this as in [Ha87b] we obtain a variation of mixed Hodge structure with underlying local system that we denote by $\mathbb{V}_{x_0}$. When $X = \overline{X}$ is projective then for $x \in X$ the fibre $\mathbb{V}_{x_0, x} \cong H^1(X, \{x_0, x\}; \mathbb{Q})$ is a MHS with weight filtration $W_0 \subset W_1$ where

- $\text{Gr}^W_0 \mathbb{V}_{x_0, x} = \mathbb{Q}$ (constant local system);
- $\text{Gr}^W_1 \mathbb{V}_{x_0, x} \cong H^1(X, \mathbb{Q})$.

The extension class is in

$$\text{Ext}^1_{\text{MHS}}(H^1(X, \mathbb{Q}), \mathbb{Q}) \cong \frac{H^1(X, \mathbb{C})^*}{F^1 H^1(X, \mathbb{C})^* + H^1(X, \mathbb{Z})^*}$$

$$\cong H^0(\Omega^1_X)^*/H_1(X, \mathbb{Z}).$$

It may be identified with the linear function on $H^0(X, \Omega^1_X)$ given by the usual Albanese mapping defined for $\omega \in H^0(\Omega^1_X)$ by

$$(\text{II.16}) \qquad \qquad \alpha_{x_0}(x)(\omega) = \int_{x_0}^x \omega \qquad \text{mod periods.}$$

As in [HZ87] this defines a canonical UVMHS associated to (X, x_0).

In the quasi-projective case we again have a VMHS with local system $\mathbb{V}_{x_0} \to X$ where

$$(\text{II.17}) \qquad \qquad 0 \to \mathbb{Q} \to \mathbb{V}_{x_0} \to H^1(X, \mathbb{Q}) \to 0.$$

This time $\mathbb{V}_{x_0, x} \cong H^1(X, \{x_0, x\}; \mathbb{Q})$ is a MHS with weight filtration $W_0 \subset W_1 \subset W_2$ with

- $\text{Gr}^W_0 \mathbb{V}_{x_0, x} = \mathbb{Q}$;
- $W_2(\mathbb{V}_{x_0, x})/W_0(\mathbb{V}_{x_0, x}) \cong H^1(X, \mathbb{Q})$.

PROPOSITION II.18: *The sequence* (II.17) *may be constructed by the same method as in* (II.15) *where the extension class is given for* $\omega \in H^0(\Omega^1_{\overline{X}}(\log Z))$ *by the Albanese mapping* (II.16).

What this means is that the prescription for constructing the Hodge filtration F^1_x on $\mathbb{C} \oplus H^1(X, \mathbb{C})$ may be done using the same formulas as in the projective case above. It is when we put in the variable point $x \in X$ that we get a varying F^p_x in the VMHS.

It is clear that the F^1_x defines a holomorphic sub-bundle of $\mathbb{V} \otimes \mathcal{O}_X$. What must be verified is that horizontality for F^1 and conjugate horizontality for $\overline{F}^1$ are satisfied. If $e_x \in$

$\frac{H_1(X,\mathbb{C})}{F^0 H_1(X,\mathbb{C}) + H_1(X,\mathbb{Z})}$ is the extension class, then the horizontality condition is $\nabla e \in F^{-1} H_1(X,\mathbb{C})$ which is satisfied in this case. Conjugate horizontality is done similarly. $\qquad\square$

We have put this argument in because it will be used verbatim when we replace $\mathbb{Q}$ by $\mathbb{V}$ below.

Third proof of I.7 and the $\partial\overline{\partial}$-lemma.

From rational homotopy theory it is known that the unipotent completion $\widehat{\pi}_1(X) \otimes \mathbb{Q}$ is determined by $H_1(X,\mathbb{Q})$, $H_2(X,\mathbb{Q})$ and the map

(II.19) $$H_2(X,\mathbb{Q}) \to \Lambda^2 H_1(X,\mathbb{Q})$$

that is dual to the cup product

(II.20) $$\Lambda^2 H^1(X,\mathbb{Q}) \to H^2(X,\mathbb{Q})$$

(cf. [Mo78] and the references cited there). Denoting by $\mathcal{L}H_1(X,\mathbb{Q})$ the free Lie algebra generated by $H_1(X,\mathbb{Q})$, there is a surjection

(II.21) $$\mathcal{L}H_1(X,\mathbb{Q}) \to \mathcal{L}(\widehat{\pi_1(X)} \otimes \mathbb{Q})$$

where the right-hand side is the Lie algebra$/\mathbb{Q}$ corresponding to the completion of the group algebra of $\pi_1(X)$. We will use the construction of the Sullivan minimal model to determine the ideal $\mathfrak{I}$ in $\mathcal{L}H_1(X,\mathbb{Q})$ that gives the kernel of the mapping (II.21). Both $\mathcal{L}H_1(X) \otimes \mathbb{Q}$ and $\mathfrak{I}$ are graded by degree and we denote by $(\bullet)_k$ the part up to degree k. The key points will be that in building $\mathfrak{I}$ step-by-step with $\mathfrak{I}_2 \subset \mathcal{L}H_1(X,\mathbb{Q})_2$, $\mathfrak{I}_3 \subset \mathcal{L}H_1(X,\mathbb{Q})_3, \ldots$, at each step we will have a morphism of mixed Hodge structures

$$\mathcal{L}H_1(X,\mathbb{Q})_k / \mathfrak{I}_k \to H^2(X,\mathbb{Q}),$$

and then a weight argument will give that $\mathfrak{I}$ is generated in degrees $2, 3, 4$ ([Mo78]). When $X = \overline{X}$ is projective the filtrations of $\mathcal{L}H_1(X,\mathbb{Q})$ by degree and weight conincide; this is not the case when X is only quasi-projective.

For $\mathcal{G}_{\mathbb{Q}} := \mathcal{L}H_1(X,\mathbb{Q})/\mathfrak{I}$ we set

$$\mathcal{G} = \mathcal{G}_{\mathbb{Q}} \otimes \mathbb{C}.$$

We next denote by $A^\bullet(\overline{X}, \log Z)$ the C^∞ log complex generated by the smooth forms on $\overline{X}$ adjoined by the dz_i/z_i's along $z_1 \cdots z_k = 0$. From [GS75] or [Mo78] the inclusion $A^\bullet(\overline{X}, \log Z) \hookrightarrow A^*(X)$ is a quasi-isomorphism so that we have

$$H^*(X; \mathbb{C}) \cong H^*(A^\bullet(\overline{X}, \log Z)).$$

Assuming that $\pi_1(X)$ is residually nilpotent and with terms to be explained in the proof we have

PROPOSITION II.22 ([Ha87b]): *There exists a form $\omega \in \mathcal{G} \otimes A^1(\overline{X}, \log Z)$ that satisfies*

(i) $d\omega + \frac{1}{2}[\omega, \omega] = 0$;
(ii) *integrating ω defines the higher Albanese mapping;*
(iii) *the entries of ω give the minimal model of $\widehat{\pi_1(X)} \otimes \mathbb{C}$;*
(iv) *ω_{-k} has $(\log z)^k$ singularities along Z; and*
(v) *the ideal $\mathfrak{I} \subset \mathcal{L}H_1(X)$ is generated in degrees $(2, 3, 4)$.*

LEMMA II.23 ($\partial\bar{\partial}$ Lemma): *Let $\eta \in A^2(\overline{X}, \log Z)$ be $\partial, \bar{\partial}$ closed and with cohomology class $[\eta] = 0$ in $H^2(X, \mathbb{C})$. Then there exists a function $f \in A^0(X)$ that satisfies*

$$\partial\bar{\partial}f = \eta.$$

We may choose this function to have logarithmic singularities along Z. Any such function is then unique up to $\mathrm{Gr}_2^W H^1(X, \mathbb{C})$.

The proof of the lemma will be given following that of Proposition II.22. Before giving that argument we will make some preliminary observations. The first is that we shall use the standard identifications

$$
(\text{II.24}) \quad
\begin{cases}
(\text{a}) \quad H^1(X, \mathbb{C}) \cong \mathbb{H}^1(\Omega_X^\bullet(\log Z)) \cong \overbrace{H^0(\Omega_{\overline{X}}^1(\log Z))}^{F^1} \oplus H^1(\overline{X}, \mathcal{O}_{\overline{X}}) \\[2em]
(\text{b}) \quad H^2(X, \mathbb{C}) \cong \mathbb{H}^2(\Omega_{\overline{X}}^\bullet(\log Z)) \cong \overbrace{\underbrace{H^0(\Omega_{\tilde{X}}^2(\log Z))}_{F^2} \oplus H^1(\Omega_{\overline{X}}^1(\log Z))}^{F^1} \oplus H^2(\mathcal{O}_{\overline{X}}).
\end{cases}
$$

In both cases

$$H^q(\mathcal{O}_{\overline{X}}) \cong \overline{H^0(\Omega_{\overline{X}}^q)}.$$

A result of Deligne is that

$$H^0(\Omega_{\overline{X}}^q(\log Z)) \text{ is a direct summand of } H^1(\overline{X}, \mathbb{C});$$

i.e., forms in $H^0(\Omega_{\overline{X}}^q(\log Z))$ are closed and never exact. A proof of this using $A^\bullet(\overline{X}, \log Z)$ is in [GS75].

The Lie algebra $\mathcal{G}$ is nilpotent and may be realized as a Lie algebra of lower triangular matrices. We then have

$$\omega = \omega_{-1} + \omega_{-2} + \omega_{-3} + \cdots$$

where ω_{-k} are 1-forms that on a diagonal that is k-steps below the principal diagonal. The equation (i) in Proposition II.22 is then a sequence of equations

$$(\text{II.25}_k) \qquad d\omega_{-k} = -\left(\frac{1}{2}\right) \sum_{i+j=k} \omega_{-i} \wedge \omega_{-j}, \quad k \geqq 0 \ ^4$$

We will argue that since
 (i) $\widehat{\pi_1}(X) \oplus \mathbb{Q}$ is determined by $H_2(X, \mathbb{Q}) \to \wedge^2 H_1(X, \mathbb{Q})$;
 (ii) the minimal model is isomorphic to the dual of $\widehat{\pi_1}(X) \otimes \mathbb{Q}$; and
 (iii) constructing the minimal model entails solving equations $\alpha = d\beta$
we will be able to inductively solve the equations (II.25_k). Using the mixed Hodge structures on $H^1(X, \mathbb{Q})$ and $\ker\{\wedge^2 H^1(X, \mathbb{Q}) \to H^2(X, \mathbb{Q})\}$, from a weight argument we will see that the only "new" equations (II.25_k) arise for $k = 2, 3, 4$ ([Mo78]).

Proof of Proposition II.22. We take the entries of ω_{-1} to be bases $\varphi_i \in H^0(\Omega_{\overline{X}}^1(\log Z))$, $\overline{\psi}_\alpha \in \overline{H^0(\Omega_{\overline{X}}^1)}$. The (II.25_k) is

$$(\text{II.25}_2) \qquad d\omega_{-2} = -\left(\frac{1}{2}\right)\left\{\sum a_{ij}\varphi_i \wedge \varphi_j + \sum b_{i\alpha}\varphi_i \wedge \overline{\psi}_\alpha + \sum c_{\alpha\beta}\overline{\psi}_\alpha \wedge \overline{\psi}_\beta\right\}.$$

---

^{4}For $k = 1$ the equation (II.25_k) is $d\omega_{-1} = 0$.

The entries of ω_{-2} will correspond to a basis of the kernel of (II.20)$\otimes\mathbb{C}$. From (II.24$_b$) the first and third entries in the right-hand side of (II.25$_2$) are zero as differential forms. This leaves the middle term and by (II.23) we may take

$$\omega_{-2} = -\overline{\partial} f_{-2}\ ^5$$

and we have defined the part $(\mathcal{L}H_1(X)_2/\mathcal{I}_2)^*$ of the minimal model up through degree 2.

For the next step (II.25$_3$) is

$$d\omega_{-3} = -[\omega_{-1}, \omega_{-2}].$$

We think of the right-hand side as giving a map

$$(\mathcal{L}H_1(X))_3/\mathcal{I}_2 \cdot H_1(X) \to H^2(X).$$

Since $\widehat{\pi_1(X)}\otimes\mathbb{Q}$ is accounted for by the kernel of (II.21), this map must be zero which means we can solve for ω_{-3}. Using Lemma II.23 we may take

$$\omega_{-3} = \overline{\partial} f_{-3}.$$

This process continues inductively giving us (i) and (iii).

The proof of (ii) follows from the construction of the higher Albanese varieties in [Ha87a].

The proof of (iv) will follow from the proof of Lemma II.23.

Finally, for (v) the maps

$$\mathcal{L}H_1(X)_k/\mathcal{I}_k \to H^2(X)$$

that are used to inductively to define ω_{-k} are morphisms of MHS's. In (II.25$_k$) the weights of ω_{-k} are the weights on the right-hand side. In (II.25$_k$) the correspondence on weights between the left- and right-hand sides is

$$
\begin{array}{ll}
k = 2 & 2, 3, 4 \longleftrightarrow 2, 3, 4 \\
k = 3 & 3, 4, 5, 6 \longleftrightarrow 2, 3, 4 \\
k = 4 & 4, 5, 6, 7, 8 \longleftrightarrow 2, 3, 4 \\
k = 5 & 5, 6, 7, 8, 9, 10 \longleftrightarrow 2, 3, 4.
\end{array}
$$

It follows that these maps are zero for $k \geqq 5$. Thus for $k \geqq 5$

$$\mathcal{I}_k = \mathcal{I}_5 \{\overset{k-5}{\otimes} H_1(X)\}. \qquad \square$$

Proof of Lemma II.23. We will use (II.24)(b). Writing $\eta = \eta^{2,0} + \eta^{1,1} + \eta^{0,2}$ we have $\overline{\partial}\eta^{2,0} = 0$. Thus

$$\eta^{2,0} \in H^0\left(\Omega^2_{\overline{X}}(\log Z)\right).$$

Since the cohomology class $[\eta] = 0$ in $H^2(X, \mathbb{C})$, it follows from the result of Deligne that the differential form $\eta^{2,0} = 0$. Similarly, $\eta^{0,2} = 0$ and

$$\eta = \eta^{1,1} \in A^{1,1}(\overline{X}, \log Z)$$

is $\partial, \overline{\partial}$ closed and with cohomology class

$$[\eta] = 0 \ \text{ in } \ H^1\left(\Omega^1_{\overline{X}}(\log Z)\right).$$

From the exact sequence

$$\oplus H^0(Z_i, \mathbb{C})(-1) \xrightarrow{\mathrm{Gy}} H^2(\overline{X}, \mathbb{C}) \to H^2(X, \mathbb{C})$$

---

[5]We could also take $\omega_{-2} = \partial f_{-2}$. Then $[\omega_{-1}, \omega_{-2}]$ is represented by closed forms with two different (p, q) types. Since it is supposed to have intrinsic Hodge-theoretic meaning this suggests (but does not prove) that we can solve the equations (II.25$_k$) for $k = 3$.

since $[\eta]$ maps to zero in $H^2(X, \mathbb{C})$, it follows that in $H^2(\overline{X}, \mathbb{C})$

$$[\eta] = \sum c_i \operatorname{Gy}(1_{Z_i}), \quad c_i \in \mathbb{C}.$$

It is well known, and a proof will be recalled below, that there is $g_i \in A^0(X)$ such that $\partial\overline{\partial}g_i$ gives a de Rham representative of $\operatorname{Gy}(1_{Z_i})$. Then

$$\alpha := \eta - \sum c_i \partial\overline{\partial}g_i \text{ is a } \partial, \overline{\partial} \text{ closed (1,1) form whose cohomology class } [\alpha] = 0 \text{ in } H^2(\overline{X}, \mathbb{C}).$$

A standard Kähler fact is that then $\alpha = \partial\overline{\partial}h$ for some $h \in A^0(\overline{X})$. This follows from the result that the inclusion

(II.26) $$A^\bullet(\overline{X}) \cap \ker \partial \hookrightarrow A^\bullet(\overline{X})$$

is a quasi-isomorphism (Section 5 in [DGMS75]).

For the statement about $\operatorname{Gy}(1_{Z_i})$, in the line bundle $L_i := [Z_i]$ we choose a metric and a section $\sigma_i \in H^0(\overline{X}, L_i)$ with divisor $(\sigma_i) = Z_i$. If ρ_i is a C^∞ bump function with $\rho_i\big|_{Z_i} = 1$ and that is compactly supported in a neighborhood of Z_i, then

$$\left(\frac{\sqrt{-1}}{2\pi}\right) \partial\overline{\partial} \log \|\rho_i \sigma_i\|^2$$

represents $\operatorname{Gy}(1_{Z_i})$ in $H^2(X, \mathbb{C})$. $\qquad\square$

The statement about the singularities of f is then clear.

If we have f with $\partial\overline{\partial}f = 0$, then $\partial f \in A^1(\overline{X}, \log Z)$ gives a cohomology class in $H^1(X)$ such that $\operatorname{Res}[\partial f] \in \ker\{\overset{i}{\oplus} H^0(C_{Z_i})(-1) \to H^2(\overline{X})\} \cong \operatorname{Gr}_2^W H^1(X, \mathbb{C})$, which implies the uniqueness statement.

Finally to prove Theorem A as in the first argument we may reduce to showing that for a nodal curve $Y = \cup Y_\alpha$

$$H^1(X, \mathbb{Q}) \to H^1(Y, \mathbb{Q}) \text{ is trivial} \implies \widehat{\pi}_1(Y) \to \widehat{\pi}_1(X) \text{ is trivial}.$$

The assumption gives that all

$$\omega_{-1}\big|_{Y_\alpha} = 0.$$

From (II.23$_k$) it follows that

$$d\omega_{-2}\big|_{Y_\alpha} = 0.$$

From Lemma II.23 we have $\omega_{-2} = df_{-2}$ where $\overline{\partial}f_{-2} = 0$. Then

$$\partial\overline{\partial}f_{-2}\big|_{Y_\alpha} = 0 \implies f_{-2}\big|_{Y_\alpha} = \text{constant} \implies \omega_{-2}\big|_{Y_\alpha} = 0.$$

Proceeding inductively on k we obtain

$$\omega\big|_{Y_\alpha} = 0,$$

which implies the result that all maps $\widehat{\pi}_1(Y_2) \to \widehat{\pi}_1(X)$ are trivial. From [Ha87b] the higher Albanese mappings

$$X \to \operatorname{Alb}^s(X)$$

contract all Y_α and hence send Y to a point. $\qquad\square$

Remark: If $\widetilde{\omega}$ is the pullback of ω to the universal cover $\widetilde{X} \xrightarrow{\pi} X$, then it is well known that there is a linear complex Lie group G with Lie algebra $\mathcal{G}$ and a mapping

(II.27)
$$\widetilde{f} : \widetilde{X} \to G$$

with

$$\widetilde{f}^*(g^{-1}dg) = \widetilde{\omega}.$$

If $\Gamma \subset G$ is the monodromy group of the flat connection ω, then (II.27) induces a map

$$f : X \to \Gamma\backslash G.$$

From [Ha87b] there is a subgroup $F^0 G$ such that the induced map

$$X \to \Gamma\backslash G/F^0 G$$

is the Albanese map for $s \gg 0$. This gives another argument that

$$\omega\big|_{Y_\alpha} = 0 \implies \alpha_s(Y) = \text{point}.$$

Finally one may ask about the singularities of the locally defined matrix-valued function g along Z. Consider the first non-trivial case where locally on X

$$g = \begin{pmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ b & a_2 & 1 \end{pmatrix}$$

$$\omega = g^{-1}dg = \omega_{-1} \oplus \omega_{-2}.$$

From our construction, ω_{-1} has dz_i/z_i and $d\bar{z}_i/\bar{z}_i$ terms so that a_1, a_2 can have linear terms in $\log z_i, \overline{\log z_j}$. From

$$\omega_{-2} = db - a_2 da_1$$

and the above construction of ω_{-2}, it follows that b can have quadratic terms in $\log z_i, \log \bar{z}_j$. In general

$$\omega_{-k} \text{ is a polynomial of degree } k \text{ in the } \log z_i\text{'s and their conjugates.}$$

This illustrates a general result ([Gr22]) about the period matrices of the extension data in a VMHS.

III. The variation of Hodge structure and mixed Hodge structure cases

Proof of Theorem B. The proof is basically an observation using known results.

The assumption that the monodromy representation

$$\rho : \pi_1(X) \to \text{Aut}(V)$$

has a finite kernel is used in two ways. One is that the covering of X corresponding to $\pi_1(X)/\ker\rho$ is a finite quotient of the universal covering. The other is that for a connected compact subvariety $Y \subset X$

(III.1)
$$\pi_1(Y) \to \pi_1(X) \text{ is finite} \iff \rho\big|_{\pi_1(Y)} \text{ is finite}.$$

It is a classical result from Hodge theory that

$$\rho\big|_{\pi_1(Y)} \text{ is finite} \iff \Phi\big|_Y \text{ is constant.}$$

Thus in (I.9) the connected fibres of $\widetilde{\Phi} : \widetilde{X} \to \widetilde{P}$ are the connected components of $\pi^{-1}(Y)$ where (III.1) is satisfied.

It remains to prove that in (I.9) the image $\widetilde{\Phi}(\widetilde{P}) \subset D$ is Stein. We denote by $\check{D}$ the compact dual of D ([CM-SP17] and [GGK13]). Then

$$D = G_{\mathbb{R}}/H$$
$$\cap$$
$$\check{D} = G_{\mathbb{C}}/B = M/H$$

where $G_{\mathbb{R}}$ is a real semi-simple Lie group, $G_{\mathbb{C}}$ is its complexification, H is a compact subgroup of $G_{\mathbb{R}}$ and M is the maximal compact subgroup of $G_{\mathbb{C}}$. On D there is a $G_{\mathbb{R}}$-invariant volume form Ω_D, and on $\check{D}$ there is an M-invariant volume form $\Omega_{\check{D}}$. From [GGK13, Lecture 6] the ratio

$$\varphi = \Omega_D/\Omega_{\check{D}}$$

is then a function on D with the property

$\varphi : D \to \mathbb{R}$ is an exhaustion function and the Levi form $(i/2)\partial\overline{\partial}\varphi$ is positive in the horizontal sub-bundle $I \subset TD$.

From the assumed completeness of the period mapping Φ on X it follows that the image $\widetilde{\Phi}(\widetilde{X}) = \widetilde{P} \subset D$ is a closed analytic subvariety of D whose Zariski tangent spaces lie in I. Consequently

$\varphi|_{\widetilde{P}}$ is a psh exhaustion function

and then by [Na62] we may conclude that $\widetilde{P}$ is a Stein variety. $\square$

Proof of Theorem C. We have outlined the proof in Section I; here we will give some further details. To give a rigorous argument involves technical issues in the theory of complex analytic geometry that we shall not deal with here. Ones that are similar arise in the projective $X = \overline{X}$ case and are treated in Section 5 in [EKPR12].

By the argument just given the image $\widetilde{\Phi}(\widetilde{X}) = \widetilde{P}$ is a Stein variety. This uses the assumption that Φ_M is proper, which implies that the monodromies of Φ around the Z_i are of infinite order.

Using the terminology and notations from Section II, we denote by $\Phi_{M,2}$ the mapping given by taking the associated graded together with the extension data of levels $\leqq 2$ in the MHS's described by (I.10). The arguments given in the proof of (II.4) apply to the VMHS along the fibres of $\Phi_M(X) \to \Phi(X)$. Thus the map

$$\Phi_M(X) \to \Phi_{M,2}(X)$$

has finite fibres. Moreover, the fibres of

$$\Phi_{M,2}(X) \to \Phi(X)$$

are images of the mappings of the fibres to extension data of levels $\leqq 2$. Due to the assumed completeness of Φ_M their universal coverings are Stein. From this we see that

(III.2) $$\widetilde{\Phi}_M(\widetilde{X}) \to \widetilde{\Phi}(\widetilde{X})$$

is holomorphic mapping onto a Stein variety and with Stein fibres. As in Section 5 of [EKPR12] one may use [Car79] and [Se53] to conclude that the total space is Stein. $\square$

Comments regarding possible cohomological descriptions of the fibres of Φ; deformations of the monodromy representation ρ.

The connected components $Y \subset X$ of the fibres of a Shafarevich map are defined homotopy-theoretically by the finiteness of the map $\pi_1(Y) \to \pi_1(X)$. One may ask if they can be characterized cohomologically as is possible in the nilpotent case by (iv) in Theorem A. This is an interesting question that we shall now discuss in the case of a VHS.

In order to be able to cohomologically describe the fibres some cohomology groups must be non-zero, and we note that

(III.3) (i) *Given the local system $\mathbb{V} \to X$ underlying a VHS, the group $H^1(X, \operatorname{End}(\mathbb{V}))$ is zero if, and only if, the monodromy representation*

$$\rho : \pi_1(X) \to \operatorname{Aut}(V_{\mathbb{C}})$$

is to first order rigid.

(ii) *The map $H^1(X, \operatorname{End}(\mathbb{V})) \to H^1(Y, \operatorname{End}(\mathbb{V}))$ is trivial if, and only if, the corresponding first order deformation of ρ induces a trivial deformation of $\rho\big|_{\pi_1(Y)}$.*

As a first step we will reformulate the cohomological criterion in the nilpotent case. Keeping our assumption that $X = \overline{X} \backslash Z$ we denote by $\mathcal{O}(\widehat{\pi_1(X, x)}) := \lim_s \mathcal{O}(\pi_1(X, x)/J^{s+1})$ the Hopf algebra of functions on the unipotent completion of $\pi_1(X, x)$. This algebra has a mixed Hodge structure that is generated by $H^1(X, \mathbb{Q})$ viewed as linear functions on $\pi_1(X, x)$ ([Ha87b]). For Y a normal crossing variety the induced mapping

$$\widehat{\pi_1(Y, y)} \to \widehat{\pi_1(X, x)}, \qquad f(y) = x$$

of a map $f : Y \to X$ is trivial if, and only if, the pullback map

$$f^* : \mathcal{O}(\widehat{\pi_1(X, x)}) \to \mathcal{O}(\widehat{\pi_1(Y, y)})$$

is zero. This happens when this map is trival on generators; i.e., when

$$H^1(X, \mathbb{Q}) \xrightarrow{f^*} H^1(Y, \mathbb{Q})$$

is zero. We have used this in the proof of Theorem A.

In the VHS case we let $G \subset \operatorname{Aut}(V)$ be the $\mathbb{Q}$-Zariski closure of the image of the monodromy representation. Then G is a semi-simple $\mathbb{Q}$-algebraic group. For the purposes of this discussion we shall assume that it is simple and that the members of the set $\{V_\lambda\}$ of irreducible G-modules all arise as sub-quotients of tensor products of V. Then each V_λ gives a local system $\mathbb{V}_\lambda \to X$ that underlies a VHS.

One now uses the completion of $\pi_1(X, x)$ relative to ρ as in [Ha98]. Following the notations there, except that here we use G instead of S, the relative completion is a pro-algebraic group $\mathcal{G}$ that fits in a diagram

(III.4)

$$1 \longrightarrow \mathcal{U} \longrightarrow \mathcal{G} \longrightarrow G \longrightarrow 1$$

with maps $\widetilde{\rho}$ and ρ from $\pi_1(X, x)$.

where $\mathcal{U}$ is the unipotent radical of $\mathcal{G}$. The Hopf algebra $\mathcal{O}(\mathcal{G})$ of regular functions on $\mathcal{G}$ has a functorial mixed Hodge structure with non-negative weights and relative to which the Hopf algebra operations are morphisms.

The regular functions $\mathcal{O}(G)$ may be realized as the matrix coefficients of the above representations.[6] We then have

$$\mathcal{O}(G) \cong \overset{\lambda}{\oplus} V_\lambda^* \otimes V_\lambda$$

where G acts on both the left and right. Referring to (III.4) we have an inclusion $\mathcal{O}(G) \hookrightarrow \mathcal{O}(\mathcal{G})$ and

(III.5) $$\mathcal{O}(G) = \mathrm{Gr}_0^W(\mathcal{O}(\mathcal{G})).$$

From [Ha98] it may be inferred that

(III.6) $$\mathrm{Gr}_1^W(\mathcal{O}(\mathcal{G})) \cong \overset{\lambda}{\oplus} H^1(X, \mathbb{V}_\lambda^*) \otimes V_\lambda.$$

The reason that $\mathbb{V}_\lambda^*$ is inside the parenthesis and V_λ is outside reflects the left and right actions of G on $\mathcal{O}(G)$ (cf. loc. cit.).

Given an inclusion $j : Y \hookrightarrow X$ from (III.4) and (III.5) the condition that

(III.7) $$\widetilde{\rho} \circ j : \pi_1(Y) \to \mathcal{G}$$

be trivial is

$$(\widetilde{\rho} \circ j)^* \mathcal{O}(\mathcal{G}) = \text{ constant functions.}$$

From (III.6) this implies that all restriction mappings

(III.8$_\lambda$) $$H^1(X, \mathbb{V}_\lambda) \to H^1(Y, \mathbb{V}_\lambda)$$

are zero.

QUESTION III.9: *If* (III.8$_\lambda$) *holds for all non-trivial* V_λ, *then does* (III.7) *hold?*

A positive answer would necessitate that

(III.10) $$H^1(X, \mathbb{V}) \neq 0$$

for some $V = V_\lambda$. This group has a MHS with the part not in $H^1(\overline{X}, j_* \mathbb{V})$ having a description involving the part of $\mathbb{V}$ that is not invariant under monodromy around the Z_i's (cf. [Ha98] and [Le21]). We will not get into this here. *For the remainder of Section III we will assume that* $X = \overline{X}$ *is projective.* Then $H^1(\overline{X}, \mathbb{V})$ has a Hodge structure with

(III.11) $$F^1 H^1(X, \mathbb{V}) = H^0(\Omega^1_{\overline{X}}).$$

The non-zero vector space (III.11) gives rise to an interesting construction that we now shall discuss (cf. loc. cit. for a related construction of locally constant iterated integrals).

III.12: *Twisted Albanese map.*

We first note that for any local system $\mathbb{V} \to X$ and $\omega \in H^0(\Omega^1_{\overline{X}} \otimes \mathbb{V})$ the exterior derivative $d\omega$ is well defined. If $\mathbb{V} \to X$ underlies a VHS, then it may be shown that $d\omega = 0$.

Now let $x_0 \in X$ and $\widetilde{x}_0 \in \widetilde{X}$ be base points with $(\widetilde{X}, \widetilde{x}_0) \overset{\pi}{\to} (X, x_0)$. Then canonically

$$\pi^* \mathbb{V} \cong \widetilde{X} \times V.$$

If ω is as above with $\pi^* \omega = \widetilde{\omega}$, then $\widetilde{\omega} \in H^0(\Omega^1_{\widetilde{X}} \otimes \mathbb{V}) \cong H^0(\Omega^1_{\widetilde{X}}) \otimes V$ and since $d\widetilde{\omega} = 0$,

(III.13) $$\int_{\widetilde{x}_0}^{\widetilde{x}} \widetilde{\omega} \in V$$

---

[6] From here on in this section we will work over $\mathbb{C}$ and will set $G_{\mathbb{C}} = G$, $\mathbb{V}_{\lambda}, \mathbb{C} = \mathbb{V}_{\lambda}$ etc.

is well defined. Taking into account the action of $\pi_1(X, x)$ as deck transformations we may descend (III.13) to X to have a well-defined mapping

(III.14) $$\mathrm{Alb}_{X,\mathbb{V}} : X \to H^0(\Omega^1_X \mathbb{V})^* \otimes V_{x_0}/\widetilde{\rho}(\pi_1(X, x_0)).$$

This is the *twisted Albanese mapping* referred to above.

We note that

(III.15) $$(\mathrm{III.8}_\lambda) \text{ is equivalent to } \mathrm{Alb}_{X,\mathbb{V}_\lambda}\big|_Y = constant.$$

Consequently an affirmative answer to (III.9) would mean that we have

$$\rho\big|_{\pi_1(Y)} \text{ is finite } \iff \mathrm{Alb}_{X,\mathbb{V}_\lambda}\big|_Y = \text{constant for all } V_\lambda.$$

(III.16) *Extensions of mixed Hodge structures and the first step in the construction of the mixed Hodge structure on a relative completion.*

This is an extension to the relative completion of a representation of the fundamental group of the discussion in (II.13), (II.15) and of Proposition II.18.

PROPOSITION III.17: *Let* $\mathbb{V} \to X$ *be a local system underlying a VHS where* $\mathbb{V}_\lambda$ *has weight zero. Choosing a base point* $x_0 \in X$ *there is for each* $x \in X$ *a canonical mixed Hodge structure* $\mathbb{V}_{1,x}$ *with*

• $\mathrm{Gr}_0^W(\mathbb{V}_{1,x}) = \mathbb{C}$;
• $\mathbb{V}_{1,x}/W_0(\mathbb{V}_{1,x}) \cong V^*_{x_0} \otimes H^1(X, \mathbb{V})$

and that is constructed using $\mathrm{Alb}_{X,\mathbb{V}}(x)$ *as extension class. Letting* x *vary we obtain an admissible VMHS.*

Proof. As was the case for Proposition II.18, the construction of a mixed Hodge structure from a class in $H^0(\Omega^1_{\overline{X}}\mathbb{V})^* \otimes V_{x_0}/\widetilde{\rho}(\pi_1(X, x_0))$ may be done following the prescription in [Car80]. Using the same argument as in the $\mathbb{V} = \mathbb{C}$ we obtain a local system (flat vector bundle) and horzontality (transversality) of the family of mixed Hodge structures parametrized by $x \in X$ as consequences of

$$H^1(X, \mathbb{V}_\mathbb{C}) \cong H^0(\Omega^1_{\overline{X}} \otimes \mathbb{V}) \oplus \overline{H^0(\Omega^1_{\overline{X}} \otimes \mathbb{V})}$$

where the differential forms appearing in both summands on the right-hand side are closed.

Alternatively we may proceed as follows. In general suppose we have over X

$$0 \to A \to B \to C \to 0$$

where A, C are Hodge structures and B is a family of mixed Hodge structures B_x defined by

$$e_x \in \frac{\mathrm{Hom}_\mathbb{C}(C, A)}{F^0 \mathrm{Hom}_C(C, A) + \mathrm{Hom}_\mathbb{Z}(C, A)}.$$

The connection on B is

$$\nabla_B = \begin{pmatrix} \nabla_A & \omega \\ 0 & \nabla_C \end{pmatrix}$$

where

$$\omega \in \mathrm{Hom}_\mathbb{C}(C, A) \otimes \Omega^1_X.$$

From

$$\nabla_B^2 = \begin{pmatrix} \nabla_A & \nabla_A\omega + \omega\nabla_C \\ 0 & \nabla_C^2 \end{pmatrix}$$

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to have integrability we need $\nabla_A \omega + \omega \nabla_B = 0$, which is

(i) $$\nabla_{\mathrm{Hom}(C,A)} \omega = 0.$$

For horizontality a calculation shows that we need

(ii) $$\nabla_{\mathrm{Hom}(C,A)} e \in \frac{F^{-1} \mathrm{Hom}(C,A) \otimes \Omega_X^1}{\nabla F^0 \mathrm{Hom}(C,A)}.$$

This is well defined since $\nabla F^0 \subseteq F^{-1}$. Taking

$$\omega = \frac{\nabla_{\mathrm{Hom}(C,A)} e}{\nabla_{\mathrm{Hom}(C,A)} F^0 \mathrm{Hom}(C,A)}$$

we see that both (i) and (ii) are satisfied. $\qquad \square$

For $\mathbb{V} \to X$ as in Proposition III.17 we have the associated local system $\mathrm{End}(\mathbb{V})$. We note that here there are two interpretations of $H^1(X, \mathrm{End}(\mathbb{V}))$. One is as was just discussed in classifying extensions

(III.18a) $$0 \to \mathrm{End}(\mathbb{V}) \to \mathbb{M} \to \mathrm{End}(\mathbb{V}) \to 0.$$

The other is as first order deformations of $\mathbb{V} \to X$. One may think of the latter as local systems over a scheme $(X, \mathcal{O}_{X,1})$ where $\mathcal{O}_{X,1}$ is an extension of $\mathcal{O}_X$ by nilpotents of order 2. In the case at hand we may take the universal first order deformations to be the middle term in an exact sequence

(III.18b) $$0 \to \mathbb{V} \otimes H^1(X, \mathrm{End}(\mathbb{V}))^* \to \mathbb{V}_1 \to \mathbb{V} \to 0.$$

The extension class defining this sequence is the tautological class given by the identity in

$$H^1(X, \mathrm{Hom}(\mathbb{V}, \mathbb{V} \otimes H^1(X, \mathrm{End}(\mathbb{V}))^*)) \cong H^1(X, \mathrm{End}(\mathbb{V})) \otimes H^1(X, \mathrm{End}(\mathbb{V}))^*$$

and $\mathcal{O}_{X,1} = \mathcal{O}_X[H^1(X, \mathrm{End}(\mathbb{V}))^*]$.

PROPOSITION III.19: *There is a VMHS over the scheme $(X, \mathcal{O}_{X,1})$ whose restriction to $(X, \mathcal{O}_X)$ is the VMHS defined by the class*

$$\mathrm{Alb}_{X, \mathrm{End}(\mathbb{V})} \subset \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{V}, \mathbb{V} \otimes H^1(X, \mathrm{End}(\mathbb{V}))^*).$$

Proof. The proof is the same as that for Proposition III.17 above. We note that the connection form for $\mathbb{V}_1$ as a local system over $\mathcal{O}_X$ is

$$\nabla_{\mathbb{V}} = \begin{pmatrix} \nabla_{\mathbb{V} \otimes H^1(\mathrm{End}(\mathbb{V}))^*} & \Omega \\ 0 & \nabla_{\mathbb{V}} \end{pmatrix}$$

where $\Omega \in H^0(X, \Omega_X^1 \otimes \mathrm{Hom}(\mathbb{V}, \mathbb{V} \otimes H^1(\mathrm{End}(\mathbb{V}))^*))$. Then

$$\nabla_V^2 = 0 \iff \nabla_{\mathrm{Hom}} \Omega + \frac{1}{2}[\Omega, \Omega] = 0$$

where the Hom is $\mathrm{Hom}(\mathbb{V}, \mathbb{V} \otimes H^1(\mathrm{End}(\mathbb{V}))^*)$.

As a local system over $\mathcal{O}_{X,1}$ where $H^1(X, \mathrm{End}(\mathbb{V}))^*$ is treated as deformation parameters since $[\Omega, \Omega]$ is quadratic we have $\nabla_{\mathbb{V}_1}^2 = 0$. $\qquad \square$

For a subvariety $Y \subset X$ the mapping

$$H^1(X, \mathrm{End}(\mathbb{V})) \to H^1(Y, \mathrm{End}(\mathbb{V}))$$

is interpreted as restricting first order deformations of $\rho : \pi_1(X) \to \mathrm{Aut}(V)$ to $\pi_1(Y)$. With this interpretation we have completed the proof of Proposition III.3 and its corollary.

The above may be summarized as follows:

(III.20)   *Given a local system* $\mathbb{V} \to X$ *supporting a VHS, denote by* $\mathcal{O}_{X,1}$ *the scheme* $\mathcal{O}_X[H^1(X, \mathrm{End}(E))^*]$ *obtained by adjoining to* $\mathcal{O}_X$ *nilpotents of order 2. Then*

(i) $\mathcal{O}_{X,1}$ *has a mixed Hodge structure with*

$$\begin{cases} \mathrm{Gr}_0^W(\mathcal{O}_{X,1}) = \mathbb{C}, \\ \mathrm{Gr}_1^W(\mathcal{O}_{X,1}) = H^1(X, \mathrm{End}(\mathbb{V}))^* \end{cases}$$

and whose extension class is $\mathrm{Alb}_{X, \mathrm{End}(\mathbb{V})}$, *and*

(ii) *over* $(X, \mathcal{O}_{X_1})$ *there is a VMHS as described in Proposition III.17.*

In the projective case the extension of the above to all orders of deformation of ρ (Goldman-Millson theory) is carried out in [EKPR12]. Thus there are

- a MHS on the completed local rings $\widehat{\mathcal{O}}_{\mathrm{Def}(\rho)}$ of the Kuranishi space $\mathrm{Def}(\rho)$; and
- a formal VMHS over $\mathrm{Def}(\rho)$.

For further discussion of these results cf. [Pr17], [Pr19], [Le19] and [Le21].

IV. Future directions

In this section we will very briefly outline some future research directions. We have made an attempt to formulate a categorical version of Shafarevich conjecture in [HKL17], [KL14]. In those papers we outline new techniques of perverse sheaves of categories and functors between them. The categorical information is recorded by the skeleton and the Lagrangian sheaves over it. We give an interpretation of the infinite chain conjecture.
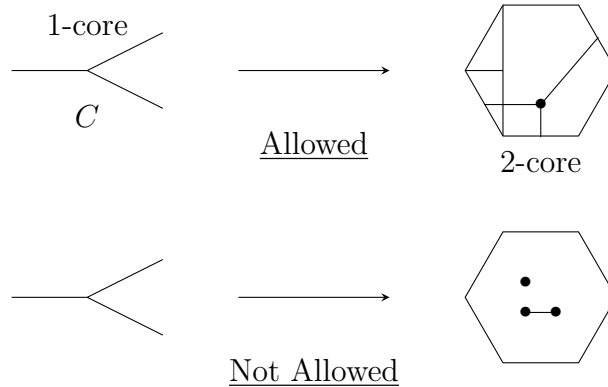
CONJECTURE IV.1 (Categorical Shafarevich): *Let X be a quasi-projective variety with infinite (nilpotent) fundamental group. Assume that for any curve C the functor*

$$F_{wrapped}(C) \to F_{wrapped}(X)$$

is allowed. Then X is holomorphically convex.

Here allowed maps are maps those which map the core of the category of an open Riemann surface in the connected 1 dimensional subcore of the quasipojective variety. Details can be found in the above references.

The above conjecture suggests

CONJECTURE IV.2: *Let X be quasi-projective. Assume that every proper morphism f from a smooth curve C to X has a meridian of infinity of infinite order in $\pi_1(X)$. Then is X holomorphically convex?*

Successful steps in this direction were taken by P. Eyssidieux and R. Aguilar.
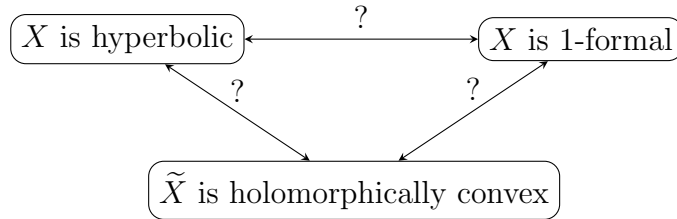
Other considerations naturally lead to.

QUESTION IV.3: *Assume that X is Picard hyperbolic and $\pi_1(X)$ is nilpotent. Does this imply that X is holomorphically convex?*

More plausible may be a positive answer to the following:

QUESTION IV.4: *Assume that X is Picard hyperbolic and $\pi_1(X)$ is a big fundamental group in the sense of* [Br22]. *Does this imply that X holomorphically convex (Stein).*

This question extends the work of loc. cit. who considered the case of a projective $X = \overline{X}$. In fact it will be interesting to investigate how the following three notions are related.

$$
\begin{array}{ccc}
\boxed{X \text{ is hyperbolic}} & \xleftrightarrow{\quad ? \quad} & \boxed{X \text{ is 1-formal}} \\
& \searrow\ ? \qquad ?\ \swarrow & \\
& \boxed{\widetilde{X} \text{ is holomorphically convex}} &
\end{array}
$$

Recently several outstanding results were obtained in direction of algebraicity of the period map in the case of variations of mixed Hodge structures; see [BBT06]. It seems likely that one can drop the condition about infinite monodromy around 2. We would like to pose the following.

QUESTION IV.5: *Let X be a quasi-projective variety such that there is an inclusion*

$$\pi_1(X) \hookrightarrow \mathrm{GL}(n, \mathbb{C}).$$

Then is $\widetilde{X}$ holomorphically convex?

Several results in this direction were obtained by B. Brunebarbe and Y. Deng. Similarly in the case when the fundamental group $\pi_1(X)$ is subgroup of a nilpotent group of $\mathrm{Nil}\, p$ we have the following:

QUESTION IV.6: *Let X be a quasi-projective variety such that*

$$\pi_1(X) \hookrightarrow \mathrm{Nil}\, p.$$

Then is $\widetilde{X}$ holomorphically convex?

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