NOTES ON MINIMAL SURFACES

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1. Preliminaries on Minimal Surfaces

We introduce the following notations:

Vector \( \vec{a}_1, \vec{a}_2, \vec{a}_3 \);

Scalar product \( a_1 \cdot a_2 = a_1 b_1 + b_1 c_1 + c_1 a_1 \);

Length of \( \vec{a}_1 = |\vec{a}_1| = \sqrt{\sum a_1^2} \);

Vector product: \( [\vec{a}_1, \vec{a}_2] = (b_1 c_2 - b_2 c_1, c_1 a_2 - c_2 a_1, a_1 b_2 - a_2 b_1) \);

Identity of Lagrange: \( [\vec{a}_1, \vec{a}_2]^2 = a_1^2 + a_2^2 - (\vec{a}_1 \cdot \vec{a}_2)^2 \);

Determinant

\[
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3
\end{vmatrix}
\]

In this introductory lesson we shall not worry about assumptions regarding differentiability. We consider the surface \( S \):

\[
x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \text{ or simply}
\]

We designate the vector \( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \) as follows:

\[
\frac{\partial x}{\partial u} = (x_u, y_u, z_u), \quad x_u = \frac{\partial x}{\partial u},
\]

\[
\frac{\partial y}{\partial u} = (x_v, y_v, z_v), \quad y_v = \frac{\partial y}{\partial u},
\]

\[
\frac{\partial z}{\partial u} = (x_w, y_w, z_w), \quad z_w = \frac{\partial z}{\partial u}.
\]

Then we have for the

Linear element or the

\[
\{ \frac{1}{2} ds^2 = dx^2 + dy^2 + dz^2 = E du^2 + 2 F du dv + G dv^2 \}
\]

First fundamental quadratic form

\[
E = \frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial u}, \quad F = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial u}, \quad G = \frac{\partial y}{\partial u} \cdot \frac{\partial y}{\partial u}.
\]
Element of area: \( d\sigma = \sqrt{E\dot{\gamma}^2 - F^2} \, du \, dv \), \( \sqrt{E\dot{\gamma}^2 - F^2} = W \)

Normal vector: \( \vec{s} = (x, y, z) = \frac{[\bar{x}_u, \bar{x}_v]}{\sqrt{\bar{x}_u^2 + \bar{x}_v^2}} = \frac{[\bar{x}_u, \bar{x}_v]}{W} \)

\( W = \sqrt{\bar{x}_u^2 + \bar{x}_v^2} \) follows from the Identity of Lagrange.

\( \vec{s} \) is a unit vector, the point \((x, y, z)\) is always on the sphere \( x^2 + y^2 + z^2 = 1 \). \( \vec{s} \) is the spherical transform of the point \( \bar{x}(u, v) \).

Second fundamental quadratic form: \(-d\bar{x} \cdot d\vec{s} = L \, du^2 + 2M \, du \, dv + N \, dv^2 \)
with
\[
L = -\bar{x}_u \bar{s}_u = \frac{\bar{s}_u}{W} \left( \bar{x}_u \bar{x}_u \bar{x}_v \bar{x}_v \right) , \quad M = -\bar{x}_u \bar{s}_v = -\bar{x}_v \bar{s}_u = \frac{\bar{s}_v}{W} \left( \bar{x}_u \bar{x}_u \bar{x}_v \bar{x}_v \right) ,
\]
\[
N = -\bar{x}_v \bar{s}_v = \frac{\bar{s}_v}{W} \left( \bar{x}_u \bar{x}_u \bar{x}_v \bar{x}_v \right) .
\]

Then, according to the theorem of Euler, the radius of curvature \( s(du:dv) \) of the normal section of \( S \) in \( \bar{x}(u, v) \) in the direction \( du:dv \) is given by
\[
\frac{1}{s(du:dv)} = \frac{L \, du^2 + 2M \, du \, dv + N \, dv^2}{E \, du^2 + 2F \, du \, dv + G \, dv^2}.
\]

The Euler Theorem can be expressed in a little different manner: In the tangential plane of \( S \) at \( \bar{x} \) we lay off the segment \( \sqrt{s(du:dv)} \) in the direction \( du:dv \); then we get a conic section, the so-called Dupin Indicatrix of \( S \) at \( \bar{x} \), either an ellipse or a pair of conjugate hyperbolas. The radii of curvature of \( S \) belonging to the axes of the Dupin Indicatrix are called the principal radii of curvature; we shall designate them by \( R_1 \) and \( R_2 \). The only thing we need here is the relation
\[
\frac{1}{R_1} + \frac{1}{R_2} = \frac{EN - 2FM + GL}{E\dot{\gamma}^2 - F^2}.
\]

Two tangential directions to \( S \) through \( \bar{x} \) are conjugate when they belong to conjugate diameters of the Dupin Indicatrix, i.e., when they are divided
of the Dupin Indicatrix. The asymptotes harmonically by the asymptotes are the directions for which \( f = 0 \); therefore, they are given by:

\[
L \, du^2 + 2M \, du \, dv + N \, dv^2 = 0
\]

and the directions \( du : dv \) and \( du : dv \) are conjugate if, and only if,

\[
L \, du \, dv + M \, (du \, dv + du \, dv) + N \, dv \, dv = 0
\]

If the parameter lines, i.e., the lines

\[
du \text{ arbitrary } \neq 0, \, dv = 0 \text{ and } dv = 0, \, dv \text{ arbitrary } \neq 0
\]

are conjugate, we find

\[
M = 0
\]

This has many applications; for instance, the directions belonging to the principal radii of curvature are conjugate and perpendicular; therefore these directions and this is clearly sufficient for the parameter lines being conjugate at \( x \).

The equation \( M = 0 \) can be expressed in the following form: there exist numbers that must be parallel to their spherical transforms. This is the essential part of the formula of the points of the curve, and the conjugate directions are conjugate to themselves; therefore, they are perpendicular to their spherical transforms.

More geometrically, the conjugate directions can be defined as follows: Take any move \( \gamma \) on \( S \) through \( x \). Its tangent may have the direction \( du : dv \). The spherical representation is as follows:

Consider the intersection of the plane tangent to \( S \) at a point \( \rho \) with the plane tangent to \( S \) in \( x \). If \( \rho \rightarrow x \), this intersection approaches the straight line, which is conjugate to \( du : dv \). This fact gives rise to a nice geometrical consideration:

\( \mathbf{x} \) is the spherical transform of \( \mathbf{x} \). The planes tangential to the sphere in \( \mathbf{x} \) and to \( S \) in \( \mathbf{x} \) are parallel. When the sphere is moved by a translation \( \mathbf{a} \), \( \mathbf{b} \) be two directions, \( \mathbf{x} \) through \( \mathbf{a} \), \( \mathbf{b} \) their spherical transforms, so that the point \( \mathbf{x} \) falls in \( \mathbf{x} \), the tangential planes are identical. Then \( \mathbf{x} = \mathbf{x} \) \( (\alpha, \beta) \). Then from the above configuration follows that

Consider a curve on \( S \) and the corresponding curve on the sphere. Since in corresponding points the tangential planes are parallel, the directions conjugate equal \( \mathbf{x} = \mathbf{x} \) \( (\alpha, \beta) \), \( \alpha \) and \( \beta \) must also be responding points. On the sphere the conjugate to any tangential direction in a
point Q is the perpendicular tangential direction. If we make the above translation, we therefore get the following configuration:

![Diagram of the spherical transform and tangents]

Therefore the spherical representation of $S$ is conformal if, and only if, the angle between the directions is equal to the angle of their conjugates.

The only cases in which this condition is satisfied are the circle and the equidistant hyperbola.

This has many applications; for instance: The directions belonging to the principal radii of curvature are conjugate and perpendicular; therefore these directions must be parallel to their spherical transforms. This is the essential part of the formula of Olindo Rodrigues. Or the asymptotic directions are conjugate to themselves; therefore they are perpendicular to their spherical transforms.

Let $S$ be a spherical surface, $x$ a point of it. We introduce a special coordinate system for which $x = (x, y, z)$, and the $(x, y, z)$-plane is the plane tangent to $S$ at $(x, y, z)$ and we allow the variables $x, y$ to take complex values. Then $S$ has a representation

in the neighborhood of $(x, y, z)$. The asymptotic directions are perpendicular. They are divided harmonically by the straight lines

Let $a, b$ be two directions on $S$ through $x, y, z$ their spherical transforms. When $\mathbf{X}(a, b) = \mathbf{X}(x, y)$, then from the above configuration follows that the angle between the conjugate direction $c_a$ and $c_b$ of $a$ and $b$ must also be equal to $\mathbf{X}(c_a, c_b) = \mathbf{X}(x, y)$. And obviously the converse holds too:
The angle between two directions \( \mathbf{a}, \mathbf{b} \) on \( S \) is equal to the angle of their spherical transforms if, and only if, it is equal to the angle of the direction conjugate to \( \mathbf{a} \) and \( \mathbf{b} \).

Therefore the spherical representation of \( S \) is conformal in \( \mathbb{X} \) if, and only if, the angle between any two directions tangential to \( S \) in \( \mathbb{X} \) is equal to the angle of their conjugates.

The only conic sections for which this condition is satisfied are the circle and the equilateral hyperbola. If the spherical representation is conformal throughout, the Dupin Indicatrix must be either always a circle or always an equilateral hyperbola. In the first case the surface is a sphere, since the sphere is the only surface all of whose points are umbilics; in the second case it is a so-called minimal surface. \( \frac{1}{k_1} + \frac{1}{k_2} = 0 \) therefore is the necessary and sufficient condition for minimal surfaces.

Let the surface \( S \) be a minimal surface, \( \mathbb{X} \) a point of it. We introduce a special coordinate system for which \( \mathbb{X} = (0, 0, 0) \), and the \( (x, y) \)-plane is the plane tangential to \( S \) at \((0, 0, 0)\). We allow the variables \( x, y \) to take complex values. Then \( S \) has a representation

\[
z = f(x, y)
\]

in the neighborhood of \((0, 0, 0)\). The asymptotic directions are perpendicular. They are divided harmonically by the straight lines

\[
y = ix \quad \text{and} \quad y = -ix.
\]

It is well known that the pairs of perpendicular directions, and only these, are divided by the lines \((*)\).

We see, the equilateral hyperbolas with center \((0, 0)\) and only these are conic sections for which the lines \( y = ix \) and \( y = -ix \) are conjugate. Since

\[
2 \sum \bar{x}_n \bar{x}_n = 2 (\bar{x}^2 \Lambda + \bar{x} \bar{y} \bar{B}) = 2 \bar{x} \bar{y} \bar{B} + \bar{B} = 0
\]
Since \( d_z \) is not identically 0 in \( \mathbb{R}^n \), \( d_z \) must be different from 0.

Therefore:

\[
\begin{align*}
z_x(0, 0) &= f_x(0, 0) = 0, \\
z_y(0, 0) &= f_y(0, 0) = 0
\end{align*}
\]

we have in \((0,0)\):

\[
\mathbf{v} = 0
\]

and similarly \( \lambda = 0 \) or \( \alpha \).

\[
d s^2 = d x^2 + d y^2
\]

and for the directions \( y = ix \) and \( y = -ix \):

\[
ge_1(\lambda, \beta) = \frac{z_x}{z} = \left( f_x - f_y \right) (\lambda, \beta)
\]

The last equation is independent of the choice of the coordinate system.

In general coordinates they are determined as the roots of the equation

\[
E = \sum d s^2 = E d u^2 + 2 F d u d v + G d v^2 = 0
\]

and are called the isotropic direction through the point. The lines on the surface which we see immediately that any functions \( x(\alpha, \beta), y(\alpha, \beta), z(\alpha, \beta) \) face whose tangent is isotropic in every point are called the isotropic lines and satisfying the conditions (1) and (3) give a minimal surface. It is \( E = F + G = 0 \).

we see:

and in account of \( \lambda = 0 \) it must vanish too. Therefore \( \lambda = 0 \).

On the minimal surfaces, and only on these, the isotropic lines are conjugate. If (1) is a minimal surface, \( \lambda = 0 \).

Let us take the isotropic lines as parameter lines: when \( \alpha, \beta \) are the new parameters and

\[
\overline{x} = \overline{x}(\alpha, \beta)
\]

is also a minimal surface. It has especially strong connections to (1) and

is the surface, then

\[
E = k_x^2 + y_x^2 + z_x^2 = \overline{x_x^2} = 0 \quad \text{and} \quad \gamma = \overline{x_x^2} = 0
\]

\[
d s^2 = 2 \overline{F} d \alpha d \beta
\]

Further on the lines \( \alpha = \text{const.} \) and \( \beta = \text{const.} \) are conjugate; that means \( \gamma = 0 \).

or the existence of two functions \( A(\alpha, \beta) \) and \( B(\alpha, \beta) \) which satisfy

the equation

\[
\overline{A} \overline{x} - \overline{B} \overline{x} = 0
\]

Further on

\[
\overline{x_x} = \overline{x}_x = 0
\]

and

\[
2 \overline{x_x} = 2 \left( \overline{x_x}^2 A + \overline{x_x} x_B \right) = 2 \overline{x_x} \overline{x_B} - B = 0
\]
Since $d\xi^2$ is not identically 0 in $d\alpha, d\beta$, $F$ must be different from 0, therefore

\[ B = 0 \]

and similarly $A = 0$. We find \( \vec{x}_{\alpha\beta} = 0 \) or

\[
\begin{cases}
  x(\alpha, \beta) = f_1(\alpha) + f_2(\beta) \\
y(\alpha, \beta) = f_2(\alpha) + f_2(\beta) \\
z(\alpha, \beta) = f_3(\alpha) + f_3(\beta)
\end{cases}
\]

with

\[ E = \sum_i \frac{f_i'(\alpha)^2}{f_i''(\beta)} = 0, \quad \gamma = \sum_i \frac{f_i'(\beta)^2}{f_i''(\alpha)} = 0 \]

Since we have

\[ (\vec{x}_{\alpha\beta} = 0) \]

satisfying the conditions (1) and (2) give a minimal surface: it is

and on account of \( \vec{x}_{\alpha\beta} = 0 \) must vanish too. Therefore \( \frac{1}{\vec{f}_1} + \frac{1}{\vec{f}_2} = 0 \).

Hence, if (1) is a minimal surface,

\[
\begin{cases}
  x_\alpha(\alpha, \beta) = \vec{f}_1(\alpha) - f_1(\beta) \\
y_\alpha(\alpha, \beta) = \vec{f}_2(\alpha) - f_2(\beta) \\
z_\alpha(\alpha, \beta) = \vec{f}_3(\alpha) - f_3(\beta)
\end{cases}
\]

is also a minimal surface. It has especially strong connections to (1) and is called the surface adjoint to (1). We represent the surfaces (1) and (1') on each other by making points correspond which belong to the same \( (\alpha, \beta) \).

This representation is an application, for

\[
d \bar{x}^2 = \bar{x}_\alpha \bar{x}_\beta d\alpha d\beta = \sum_i \frac{f_i'(\alpha)^2}{f_i''(\beta)} = \sum_i \frac{f_i'(\beta)^2}{f_i''(\alpha)} = 0
\]

is conformal, and the stereographic projection. Let \( (u, v) \) be rectangular coordinates in the plane for \( \alpha, \beta \).

Furthermore

\[
d \bar{x} d\bar{x} = \sum_i \left( f_i' d\alpha + f_i' d\beta \right) \left( i f_i' d\alpha - i f_i' d\beta \right) = i \sum f_i'^2 d\alpha^2 + i \sum f_i'^2 d\beta^2 - i \sum f_i'^2 d\alpha d\beta = 0
\]
We see that the tangents to corresponding curves at corresponding points are perpendicular. Because the representation is an application, it is informal; hence the tangential planes at corresponding points must be parallel (see the figure):

![Diagram]

Hence we have

\[
\overline{\mathbf{s}} \cdot d\overline{x} = 0 \quad \text{and} \quad \overline{\mathbf{s}} \cdot d\overline{x}_0 = 0
\]

The vector \( d\overline{x}_0 \) is perpendicular to both \( \overline{\mathbf{s}} \) and \( d\overline{x} \); it must therefore be parallel to \( \left[ \overline{\mathbf{s}}, \overline{\mathbf{R}} \right] \): tectorial parameters on any surface \( \Sigma(u, v) \), then the equation

\[
d\overline{x}_0 = k \left[ \overline{\mathbf{s}}, d\overline{x} \right],
\]

\( k \) a scalar.

On account of the Lagrange Identity we have

\[
\left[ \overline{\mathbf{s}}, d\overline{x} \right] \cdot \overline{\mathbf{s}} = \overline{\mathbf{s}} \cdot d\overline{x} - \left( \overline{\mathbf{s}} \cdot d\overline{x} \right)^2 = d\overline{x}_0 \cdot d\overline{x}_0
\]

hence \( k = \pm 1 \). It is easy to prove \( k = \pm 1 \), but we do not need that. What is important for us is the fact that the expressions

\[
\eta \, dz - \xi \, dy, \ 
\xi \, dx - \eta \, dz, \ 
\xi \, dy - \eta \, dx
\]

are complete differentials.

This property is independent of the choice of the parameter system.

Until now we have not used the fact that the spherical representation is conformal. The sphere can be represented conformally on the plane by stereographical projection. Let \((u, v)\) be rectangular coordinates in the plane and \( \pi(u, v) \) the corresponding point of the surface. Since the curves \( u = \text{const.} \) and \( v = \text{const.} \) are perpendicular, the corresponding curves on the surface must be perpendicular; therefore

\[
\overline{x}_u \cdot \overline{x}_v = F = 0
\]
Furthermore, linear elements of equal length issuing from one point must correspond to linear elements of equal length on the surface. When we take the linear elements $du = a$, $dv = 0$ and $du = 0$, $dv = a$ in the plane, the corresponding elements are

$$ E_2^2 \text{ and } G_2^2 $$

and we find $E = \frac{\gamma}{\nu} = \frac{1}{\nu(u, v)}$. In the parameters $u$, $v$, the linear element of surface therefore gets the form

$$ ds^2 = l(u, v) \left( du^2 + dv^2 \right) $$

On the other hand, if a surface $x(u, v)$ is given in terms of isothermic parameters $u$, $v$, and the corresponding differential $x(u, v)$, the parameters satisfying the conditions $E = 0$, $F = 0$ are called isothermic or isometric. We see: On a minimal surface there exist isothermic parameters in the neighborhood of each point.

It is easy to verify the following fact:

When $u$, $v$ are isothermic parameters on any surface $x(u, v)$, then the equations hold.

* If the surface is minimal, these expressions must be complete differentials.

$$ E = \frac{1}{x_{uu}^2} \quad F = \frac{1}{x_{uv}} = \frac{1}{W}, \quad F = \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} = 0 $$

$$ W_\gamma = x_{uu} x_v - x_{uv} x_u, \quad W_\delta = x_{uu} y_v - x_{uv} y_u, \quad x_\delta = x_{uu} du + x_{uv} dv $$

$$ W(\gamma dz - \delta dy) = \alpha du + \beta dv $$

$$ \alpha = x_{uu} z_v^2 - x_{uv} z_u z_v - x_{uv} y_v y_u + x_{uv} y_u^2 = x_{uv} \left( x_{uu}^2 + y_{uv}^2 + z_u^2 \right) = x_{uv} W $$

$$ \beta = x_{uv} z_u z_v - x_{uu} z_v^2 - x_{uv} y_v^2 + x_{uv} y_u y_v = x_{uv} \left( x_{uv}^2 - x_{uu}^2 - x_{uv}^2 \right) = -x_{uu} W $$

In our case we therefore get $E = 0$, $F = 0$.

and

$$ E = \frac{1}{x_{uu}^2} \quad F = \frac{1}{x_{uv}}, \quad \frac{E}{F} = \frac{1}{W} $$

and

$$ E = \frac{N + L}{E} $$

and

$$ E \in \mathbb{F}^2 $$

and

$$ E \in \mathbb{F}^2 $$
According to a well-known theorem, if our surfaces have got the same "minimal surfaces", i.e. if the area of the fundamental integral
\[ I = \int (f(u, v) du + g(u, v) dv) \]
is a complete differential if, and only if, by lying off on the normal at the point \( \vec{x} \) the tangent
\[ f_u = g_v \]
In our case we therefore get the relations
\[ x_u = -x_v v, \quad y_u = -y_v v, \quad z_u = -z_v v \]
the coordinates \( x, y, z \) are harmonical functions of \( u \) and \( v \). On the other hand, if a surface \( \Sigma(u, v) \) is given in terms of isothermic parameters \( u, v \), and the co-
ordinates are harmonic functions of \( u, v \), then the surface is minimal. For then
\[ E = G, \quad F = 0, \]
\[ \begin{align*}
\kappa &= \frac{1}{K_1} + \frac{1}{K_2} = \frac{E(N+L)}{EG-F^2} \\
\gamma &= \frac{N}{w} \left( N - 2L \left( \frac{K_1}{K_2} + \frac{K_2}{K_1} \right) + 2 \right) = F - 2u N + i \varphi + i \psi \end{align*} \]
and
\[ N \equiv \frac{1}{w} \left| x_u \right| \bar{x}_u = -\frac{1}{w} \left| x_v \right| \bar{x}_v = -L \]
We have proved:

(1) A surface \( \Sigma = \Sigma(u, v) \), which is given in terms of isothermic parameters
\( u, v \), is minimal if, and only if, the coordinates \( x(u, v), y(u, v), z(u, v) \) are
harmonic functions of \( u \) and \( v \).

When \( x(u, v), y(u, v), z(u, v) \) are harmonic, they are the real parts of
certain analytic functions
\[ x(u, v) = R \bar{F}_1(w), \quad y(u, v) = R \bar{F}_2(w), \quad z(u, v) = R \bar{F}_3(w) \]
we find
\[ \begin{align*}
\bar{F}_1'(w) &= x_u(u, v) - i x_v(u, v) \frac{1}{l} \bar{F}_2' = y_u - i y_v \\
\bar{F}_3' &= z_u - i z_v \end{align*} \]
and
\[ \sum F_i'^2 = E - \gamma - 2i \bar{F} = 0 \]
\[ \sum |F_i'|^2 = E + \gamma. \]
case and if this surface is of the least possible area than it is minimal.

At last we shall discuss why our surfaces have got the name "minimal".

Using the Euler equations and the fact that a neighborhood of surfaces. We take an arbitrary surface \( \overline{X}(u, v) \) and ask when its area is mini-

mum. We vary the surface (in a special way) by laying off on the normal at the
different points the segment

\[ n(u, v) = \varepsilon v(u, v) \]

The Euler equations belonging to the variation problem: to mini-

mize the integral

\[ \int (f_1(u, v) \overline{X}_u \overline{X}_u + f_2(u, v) \overline{X}_v \overline{X}_v + \varepsilon v(u, v) \overline{X}_u \overline{X}_v) \, du \, dv \]

By differentiation we find

\[ \overline{X}_u = \overline{X}_u + \varepsilon \overline{X}_u \overline{X}_v + \varepsilon \overline{X}_u \overline{X}_v \]

\[ \overline{X}_v = \overline{X}_v + \varepsilon \overline{X}_u \overline{X}_v + \varepsilon \overline{X}_v \overline{X}_v \]

\[ E_u = \overline{X}_u^2 = \overline{X}_u^2 + 2 \overline{X}_u \overline{X}_v + \varepsilon \overline{X}_u \overline{X}_v = E - 2 \overline{X}_u \overline{X}_v + \varepsilon \overline{X}_u \overline{X}_v \]

\[ F_v = \overline{X}_v^2 = \overline{X}_v^2 + 2 \overline{X}_u \overline{X}_v + \varepsilon \overline{X}_u \overline{X}_v = F - 2 \overline{X}_u \overline{X}_v + \varepsilon \overline{X}_u \overline{X}_v \]

\[ \gamma = \overline{X}_u \overline{X}_v = \overline{X}_u \overline{X}_v + \varepsilon \overline{X}_u \overline{X}_v + \varepsilon \overline{X}_u \overline{X}_v \]

Therefore

\[ W_u = \frac{\partial}{\partial u} (\gamma - F_v) = \gamma - F_v - 2 \overline{X}_u \overline{X}_v (E - 2 \overline{X}_u \overline{X}_v) + \varepsilon \overline{X}_u \overline{X}_v \]

\[ = (E \gamma - F_v) (1 - 2 \overline{X}_u \overline{X}_v) + \varepsilon \overline{X}_u \overline{X}_v \]

According to the Binomial development

\[ (1 + x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots \]

we find

\[ W = W (1 - \varepsilon \overline{X}_u \overline{X}_v) + \varepsilon \overline{X}_u \overline{X}_v \]

Introducing isothermic parameters, we have \( E = \varepsilon, F = 0 \), and we find

\[ SS \, W \, d u \, d v = SS \, d u - SS \overline{X}_u \overline{X}_v d u + \varepsilon \overline{X}_u \overline{X}_v d u \]

We see: the area of \( S \) is stationary only if the surface is minimal.

And: If through a simple closed curve \( \Gamma \) passes an analytical sur-
face and if this surface is of the least possible area then it is minimal.

Using the Euler equations and the fact that a neighborhood of each point on a surface (of class \( C^2 \)) can be represented in terms of isothermic parameters, the fundamental result (1), page 10, can be established very quickly as follows.

The Euler equations belonging to the variation problem to minimize the integral

\[
\int_D \left( J(x, y, z, x_1, x_2, y, y, z, z) \right) d\sigma \ d\nu
\]

are

\[
\frac{\partial J}{\partial x} - \frac{\partial J}{\partial x} = 0
\]

\[
\frac{\partial J}{\partial y} - \frac{\partial J}{\partial y} = 0
\]

\[
\frac{\partial J}{\partial z} - \frac{\partial J}{\partial z} = 0
\]

If

\[
\frac{\partial J}{\partial x} = \sqrt{E y - F^2} = \sqrt{x_{u}^2 - (x_{u} x_{v})^2}
\]

we have

\[
\frac{\partial J}{\partial x} = 0
\]

and similar equations in \( y \) and \( z \). Therefore the three Euler equations belonging to

\[
\int_{D} J(x, y, z, x_1, x_2, y, y, z, z) d\sigma \ d\nu
\]

are

\[
\frac{\partial}{\partial x} \left[ \frac{x_{u} E - x_{v} F}{W} \right] + \frac{\partial}{\partial y} \left[ \frac{x_{v} E - x_{u} F}{W} \right] = 0
\]

Introducing isothermic parameters, we have \( E = 0, F = 0 \), and we find again the equation

\[
\bar{x}_{u} + \bar{x}_{v} = 0
\]

In his paper with this title J. Douglas \(^*\) gives a simultaneous solution of the following two problems:

Given in the \((x, y, z)\)-space a simple closed Jordan curve \(\Gamma\).

(a) To determine a minimal surface of the type of the circular disk bounded by \(\Gamma\) which satisfies the further conditions: the surface admits a representation \(\mathbf{x} = \mathbf{x}(u, v)\), \(u^2 + v^2 \leq 1\), where \(x(u, v), y, z\) are continuous for \(u^2 + v^2 \leq 1\), harmonic for \(u^2 + v^2 < 1\), and satisfy for \(u^2 + v^2 < 1\) the equation \(E = G, F = 0\). Furthermore the equations \(\mathbf{x} = \mathbf{x}(u, v)\) and \(u^2 + v^2 = 1\) give a topological representation of \(u^2 + v^2 = 1\) on \(\Gamma\).

(b) To lay through \(\Gamma\) a surface of the type of the circular disk whose area is a minimum. By \(s(\Omega)\) we designate the area of \(\Omega\) in the usual sense; we here some remarks are to be made:

Ad (a) * E and G do not vanish identically, for with the notations of page 10 (below) that would mean

\[ \sum |F_i'|^2 = 0 \]

formed for the different \(i\) and \(k\) satisfying the above conditions,

or \(F_i' = \text{constant}\), \(i = 1, 2, 3\). But the possibility that somewhere \(E = G = 0\) is not excluded. The equation

\[ \sum |F_i'|^2 = E + \delta \]

shows that the points where \(E = G = 0\) are isolated. They are singularities of the surface. A closer research shows that they are similar to those branch-points which occur in study of Riemann surfaces. Furthermore (as

\[^*\) The mapping theorem of Koebe and the problem of Plateau. Journal of Mathematics and Physics, vol. 10 (1921), pp. 106-130.\]
follows from this statement) self-crossings of the surface are admitted. *)

\[ E = \sum \left( \frac{\partial x_i}{\partial \mu} \right)^2, \quad F = \sum \frac{\partial x_i}{\partial \mu} \frac{\partial x_i}{\partial \nu}, \quad G = \sum \left( \frac{\partial x_i}{\partial \nu} \right)^2. \]

With this definition of a minimal surface the problems (a), (b) are solved simultaneously for the n-space \( (\Sigma_n) \).

The paper begins with some preparatory considerations: let \( g(v) \).

Ad (b) The area is to be declared according to the definition which Lebesgue gave in his thesis. Let \( \Sigma \) be any continuous surface of the type of the circular disk bounded by \( \oint \) and \( \pi, \pi, \ldots \) a sequence of simply connected polyhedral surfaces tending to \( \Sigma \) whose boundary is in the interior of the outside of \( \pi_n(u, v) \)-plane.

By \( S(\pi) \) we designate the area of \( \pi \) in the usual sense; we put

\[ S(\pi_1, \pi_2, \ldots) = \lim_{\nu \to \infty} \inf \{ S(\pi) \} \]

and understand by \( S(\Sigma) \) the lower bound of the numbers \( S(\pi_1, \pi_2, \ldots) \) formed for the different sequences satisfying the above conditions.

In reality Douglas treats a slightly more general problem than (a, b). He calls minimal surfaces in the euclidean n-space with the cartesian coordinates \( x_1, \ldots, x_n \) surfaces which can be represented in the form

\[ x_i(u, v) = \mathcal{R} f_i(w), \quad l = 1, \ldots, n; \quad w = u + i v \]

where the \( f_i \) are analytic varying in a certain domain of the \((u, v)\)-plane (the same for all \( f_i \)) and satisfy the further condition

\[ \sum \frac{\partial^2 f_i \mathcal{R}}{\partial w^2} = 0. \]

Setting

\[ \mathcal{E} = \sum \left( \frac{\partial x_i}{\partial \mu} \right)^2, \quad \mathcal{F} = \sum \frac{\partial x_i}{\partial \mu} \frac{\partial x_i}{\partial \nu}, \quad \mathcal{G} = \sum \left( \frac{\partial x_i}{\partial \nu} \right)^2. \]
we have (as on p. 10)
\[ \sum F_i^2 = E - \gamma - 2i \cdot F = 0 \]
hence \( E = G, F = 0 \), and
\[ \sum |F_i|^2 = E + \gamma . \]
With this definition of a minimal surface the problems (a), (b) are solved simultaneously for the \( n \)-space \( (n \geq 3) \).

The paper begins with some preparatory considerations: Let \( g(\varphi) \),
\[ 0 \leq \varphi < 2\pi \]
be a bounded real function \( |g(\varphi)| \leq M \) which is continuous in all but at most a countable number of points in which \( g(\varphi) \) may have discontinuities of the first kind, i.e. the limits \( g(\varphi^+) \) and \( g(\varphi^-) \) exist everywhere and are equal to \( g(\varphi) \) except for an enumerable set of values of \( \varphi \). Then in the interior of the unit circle of the \( (u, v) \)-plane we consider the function \( (u + iv = w) \)
\[ H(w) = H(u, v) = R F(u) \]
with
\[ F(u) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\varphi} + w}{e^{i\varphi} - w} g(\varphi) d\varphi . \]

In the paper the fact is used that
\[ H(w) \to g(\varphi_0) \]
if \( w \to e^{i\varphi_0}, \|w\| < 1 \) and \( g(\varphi) \) is continuous in \( \varphi_0 \). To prove that, let \( w \) be equal to \( re^{i\varphi} \). Then
\[ \frac{e^{i\varphi} + w}{e^{i\varphi} - w} = \frac{(\cos \varphi + r \cos \varphi) + i(\sin \varphi + r \sin \varphi)}{(\cos \varphi - r \cos \varphi) + i(\sin \varphi - r \sin \varphi)} \]
and
\[ \frac{e^{i\varphi} + w}{e^{i\varphi} - w} = \frac{(\cos \varphi + r^2 \cos \varphi - \frac{2}{r} \cos \varphi) + (\sin \varphi)^2 - r^2 \sin \varphi}{(\cos \varphi - r \cos \varphi)^2 + (\sin \varphi - r \sin \varphi)^2} \]
and
\[ \Delta = \left| g(\varphi) - g(\varphi_0) \right| \]
Hence \( R F(\omega) \) is the Poisson integral of the segment \( \gamma \) of the unit circle bounded by \( \delta \) and \( \omega \), and the straight-line segment connecting the end points of \( \gamma \).

we have

\[
R F(\omega) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - \gamma^2}{1 + \gamma^2 - 2\cos(\varphi - \omega)} g(\varphi) \, d\varphi.
\]

The figure shows:

\[
\begin{align*}
\gamma & = 0 \\
\omega & = \omega_0
\end{align*}
\]

If \( e^{i\omega} \) designates the point where the straight line connecting \( e^{i\omega} \) and \( \omega \) crosses the unit circle the second time, and if we put

\[
\phi = |\omega - e^{i\omega}|, \quad \phi^* = |\omega - e^{i\omega}|
\]

we have the relations

\[
\phi^2 = 1 + \omega^2 - 2\omega \cos(\varphi - \omega)
\]

From this the inequality (1) follows immediately, and from (1),

\[
\phi^2 \leq (1 + \omega^2)(1 - \omega^2)
\]

for \( \omega \neq 0 \), \( |\omega| < 1 \).

if \( g(\omega) \) is continuous on \( \gamma \),

\[
\frac{d\phi}{d\omega} = \frac{\phi^*}{\phi}
\]

We see quite generally (if \( g(\omega) \) is summable): the limiting values of \( F(\omega) \) depend linearly on the direction in which \( \omega \) approaches \( \omega_0 \). This can be proved in the following way.

Let \( D \) be the interior of the unit circle, \( \partial D \) its boundary, and \( J_0 \) any point on \( \partial D \); take on \( \partial D \) an interval \( i \) of length \( 2\delta \) with center \( J_0 \) and put

\[
\Delta(J_0, \delta) = \text{Least Upper Bound} \left| g(\omega) - g(J_0) \right|_{\omega \in i}
\]
Then for all points $\omega$ in the interior of the segment $\zeta$ of the unit circle bounded by $1$ and the straight line segment connecting the end points of $i$, we have

$$\left| H(\omega) - g(\zeta_0) \right| \leq \Delta (\zeta_0, \zeta) + \frac{2\pi}{\pi} \zeta.$$

To show this, we set

$$f(\zeta) = g(\zeta) - g(\zeta_0).$$

Then we have

$$H(\omega) = g(\zeta_0) + \frac{1}{2\pi} \int_{0}^{2\pi} f(\theta) d\theta = g(\zeta_0) + \frac{1}{2\pi} \int_{\pi}^{\pi} f(\theta) d\theta + \frac{1}{2\pi} \int_{\pi}^{\pi} f(\theta) d\theta.$$  

Therefore, continuous approaches for $\omega$ converges to $\omega$ in such a manner that according to our previous result, the value

$$\left| \int_{\pi}^{\pi} f(\theta) d\theta \right| \leq \Delta (\zeta_0, \zeta) (2\pi - 2\zeta)$$

and on account of

$$\left| g(\zeta_0) \right| \leq 2\pi,$$

$$\left| \int_{\pi}^{\pi} f(\theta) d\theta \right| \leq 4\pi \zeta.$$

From this, the inequality (1) follows immediately, and from (1),

$$H(\omega) \to g(\zeta_0)$$

for $\omega \to \zeta_0$, $|\omega_1| < 1$

if $g(\zeta)$ is continuous in $\zeta_0$.

Douglas states (without using the statement later) that if $g(\zeta_0)$ has at $\zeta_0$ a discontinuity of the first kind, the limiting values of $H(\omega)$ for $\omega \to \zeta_0$, $|\omega_1| < 1$, depend linearly on the direction in which $\omega$ approaches $\zeta_0$. This can be proved in the following way.

Let $\zeta_0$ be the point $(1, 0)$ of the $(u, v)$-plane. Then $\frac{v}{u-1}$ is defined in $D$. One verifies easily that $\arctg \frac{v}{u-1}$ is a harmonic function in $D$. 

\[ H(\omega) \to g(\zeta_0) \]
The values \( \lambda(\theta) \) of \( \arctg \frac{\nu}{u-1} \) on \( C \) have in \( \nu=0 \) a discontinuity of the first kind:

\[
\lambda(0^+) = \frac{\pi}{2}, \quad \lambda(0^-) = -\frac{\pi}{2}
\]

The equation

\[
\int_{\varphi}^{\varphi+2\pi} \frac{\nu}{u-1} \, d\theta = \lambda(\varphi)
\]

If for our function \( g(\varphi) \)

\[
g(0^+) - g(0^-) = \alpha \neq 0
\]

the values \( \frac{\alpha}{\pi} \lambda(\varphi) \) of

\[
\Delta(\omega) = \frac{\alpha}{\pi} \arctg \frac{\nu}{u-1}
\]

don \( C \) have the same discontinuity as \( g(\varphi) \):

\[
\frac{\alpha}{\pi} \lambda(0^+) - \frac{\alpha}{\pi} \lambda(0^-) = \alpha.
\]

Therefore \( g(\varphi) - \frac{\alpha}{\pi} \Delta(\omega) \) now is continuous at \( \nu = 0 \) and set

\[
R \left( \frac{\alpha}{2\pi} \int_{\varphi}^{\varphi+2\pi} g(\varphi) - \frac{\alpha}{\pi} \Delta(\omega) \, d\varphi \right)
\]

approaches for \( \nu \to 1, |\nu| < 1 \), according to our previous result, the value

\[
g(0^+) - \frac{\alpha}{2} = g(0^-) + \frac{\alpha}{2} \Delta(\omega)
\]

if and only if

\[
\sum \nabla^{(\omega)} \psi(\omega) = 0
\]

If we set

\[
\omega_r = \frac{1}{2} \nu
\]

and let \( \omega_r \) converge to 1 in such manner that

\[ \psi(\nu) \to \psi, \]

then measurable functions \( \psi(\nu) \) with \( |\psi(\nu)| < 1 \). If \( g(0^+) \leq \lambda \) and

\[
\psi(0^+) \to g(0^+), \quad \psi(0^-) \to g(0^-)
\]

(not necessarily uniformly), then \( g(0^+) < \lambda \). This hence

\[
H(\omega_r) \to g(0^+) - \frac{\alpha}{2} + \frac{\alpha}{\pi} \psi
\]

almost everywhere on \( \omega \in J \neq 2\pi \), i.e., except for sets \( \nu \neq \nu \), respectively of measure 0. Then

After these remarks Douglas considers a set \( g_1(\varphi), \ldots, g_n(\varphi) \)

of functions of the same kind as our \( g(\varphi) \) and forms the corresponding functions

\[
H_i(\omega) = H_i(\omega, \nu) = R \int_{\varphi}^{\varphi+2\pi} g_i(\varphi)
\]
We assume that
\[ F_i(u) = \frac{1}{2\pi} \int \frac{e^{iu} + e^{-iu}}{e^{iu} - e^{-iu}} \gamma_i(\theta) \, d\theta. \]
Be enlarge the class \( \mathcal{L} \) by admitting all measurable functions \( \gamma(\theta) \) which define a surface harmonic for \( u^2 + v^2 < 1 \). We introduce the functionals
\[ A_s(\gamma) = \frac{1}{2} \int_{D_s} (E + g) \, d\nu \quad \text{and} \quad S_s(\gamma) = \int_{D_s} \sqrt{E + g} \, d\nu, \]
where \( D_s \) designates the domain \( u^2 + v^2 < 1 \). We set
\[ A(\gamma) = \lim_{s \to 1} A_s(\gamma) \quad \text{and} \quad S(\gamma) = \lim_{s \to 1} S_s(\gamma). \]

Then some properties of the functionals are established.

I. \( S(\gamma) \geq A(\gamma) \) and \( S(\gamma) = A(\gamma) \) if and only if \( A_s(\gamma) \) is continuous. Since
\[ \sum_{i=1}^{n} F_i^2(u) = 0 \]
the functions which interest us here are uniformly bounded.

\[ A_s(\gamma) = A(\gamma) \quad \text{for} \quad |\gamma| \leq M. \]

If \( \gamma(\theta) \leq L \) and \( \gamma(\theta) \to \gamma(\theta) \) (not necessarily uniformly) then \( \gamma(\theta) \leq L \). This can be slightly generalized: Let \( \gamma(\theta) \) and \( \gamma(\theta) \) be defined only almost everywhere on \( 0 \leq \theta < 2\pi \), i.e. except for sets \( N_0 \cup N \cup N_2 \) respectively of measure 0. Then
\[ N = N_0 + N_1 + N_2 \]
and we understand the relation
\[ g(\theta) \to g(\theta). \]
as meaning that
\[ g \nu (\vartheta_0) \to g (\vartheta_0) \quad \text{if} \quad \vartheta_0 \text{ is not in } N. \]
We enlarge the class L by admitting all measurable functions \( f(\vartheta) \) which are defined on \( 0 < \vartheta < 2\pi \) except for a set \( N(\vartheta) \) of measure 0 and for which \( f(\vartheta) \to g \) as \( \vartheta \to \infty \). Then we still have: if \( g \nu < L \) and \( g \nu \to g \) then \( g < L \).

That \( A(g) \) is continuous on \( L \) means: if \( g \nu \to g \) \( (g \nu, g < L) \) then \( A(g) = \frac{\text{lim inf}}{r \to \infty} A(g \nu) \).

On the other hand,
\[ A(g) \leq \lim \inf_{r \to \infty} A(g \nu) \]
The proof of Theorem II consists of two steps: the first is to prove that \( A(g) \) can be represented as limit of a non-decreasing sequence of continuous functions \( A_f(g) \). Douglas proves this by showing that \( A_f(g) \) is continuous. Since
\[ A_f(g) = A_{f_{s_0}}(g) \quad \text{for} \quad 1 > f_2 > f_1 > 0, \]
the functions
\[ A_{g_{s}}(g) = A_{g_{s-n}}(g) \]
form a non-decreasing sequence of functions converging to \( A(g) \).

The second step is the proof of the following general fact:

Let \( L \) be any set of the following type: the relation of convergence is defined and satisfies the conditions: (a) if \( g \nu \to g \) \( (g \nu, g < L) \) then each subsequence \( \{ g_{\nu'} \} \) of \( \{ g_{\nu} \} \) converges to \( g \); (b) if \( g \nu = g \) \( \nu = 1, 2, \ldots \), then \( g \nu \) converges to \( g \). (Our class L satisfies these conditions and is even compact.) Let the continuity and semicontinuity of a function \( A(g) \) on \( L \) be defined as above, then the theorem holds:
If $A_v(g)$ is lower semicontinuous on $L$, $v = 1, 2, \ldots$, and $A_v(g) \geq A_{v-1}(g)$, $v = 2, 3, \ldots$, for each $g$ on $L$, then $A_v(g)$ converges to lower semicontinuous function $A(g)$.

Proof. $A_v(g)$ converges, since $A_v(g) \geq A_{v-1}(g)$.

Let $A(g)$ be the limit. If $A(g)$ were not lower semicontinuous, a sequence $g_n \to g^0$ would exist with

$$\lim_{n \to \infty} \inf A(g_n) < A(g^0)$$

or for sufficiently great $v$ we should have

$$\lim_{n \to \infty} \inf A(g_n) < A_v(g^0).$$

On the other hand

$$A_v(g_n) \leq A(g_n)$$

and therefore

$$\lim_{n \to \infty} \inf A_v(g_n) \leq \lim_{n \to \infty} \inf A(g_n).$$

This, together with $(\ast)$, would give

$$\lim_{n \to \infty} \inf A_v(g_n) < A_v(g^0).$$

for great $v$ in contradiction to the lower semicontinuity of $A_v(g)$. Hence, if $\xi = x + iy$ one of the above equations, say

$$\frac{D(x, z)}{\partial u, \xi}$$

does not vanish at $(x_0, y_0)$ and the equations

$$x^2 + y^2 = a^2, \quad \psi = 0,$$

$$x = H(u, v),$$

$$v = L(u, v)$$
Now let \( g_i(z) \) be again a bounded function with an almost im-
numerable set of discontinuities of the first kind and no others. We shall
obtain later on by a limiting process functions \( g_i(z) \), \( i = 1, \ldots, n \),
which clearly have these properties, and it will be important to exclude the
possibility that \( g_i(z) \) has discontinuities or is constant on certain arcs
of \( \gamma \). To prepare this conclusion Douglas proves the following facts:

If one of the functions \( g_i(z) \) (at least) has a discontinuity,

then

\[
\text{III. } A(g) = +\infty \quad \text{and}
\]

\[
\text{IV. } \sum_i F_i^2(\omega) \neq 0.
\]

V. If all \( g_i(z) \) are constant on an arc \( \gamma \) of \( \gamma \) but not all \( g_i(z) \)
are constant on the whole circle \( \Gamma \), then

\[
\sum_i F_i^2(\omega) \neq 0.
\]

Since \( \beta(u, v) \) is the potential function defined by the Poisson integral with
the boundary values \( \beta(z) \), (see pp. 4-54), we have

\[
\beta(u, v) = \kappa.
\]

The proof of V given in the paper uses a theorem of Fatou. Professor Morse gave the following proof recurring to more elementary means.

According to the identity of Lagrange we have

\[
E^g - F^2 = \frac{1}{\pi} \sum_{i,j=1}^{n} \left( \frac{\partial x_i}{\partial u} \frac{\partial x_j}{\partial v} - \frac{\partial x_i}{\partial v} \frac{\partial x_j}{\partial u} \right)^2
\]

(cf. pp. 1, 2, where this is proved for \( n = 3 \)). Hence, if \( E^g - F^2 \neq 0 \)
at \((u_0, v_0)\) one of the above Jacobians, say

\[
\frac{\partial (x_i, x_j)}{\partial (u, v)} (i \neq j)
\]

and for \(|u| > r\)

\[
\frac{\partial (x_i, x_j)}{\partial (u, v)} (i \neq j)
\]
does not vanish at \((u_0, v_0)\) and the equations

\[
x_i = H_i(u, v),
\]

\[
x_j = H_j(u, v)
\]
do not vanish. To prove that \( H_i(u, v) \) is harmonic for all \( u \) not on the closed part of \( \gamma \), consider a circle \( \gamma \) perpendicular to \( \gamma \) intersecting the arc between \( u \) and \( w \) in \( n \) interior points of \( \gamma \). Under the transformation \( (u, v) \) inside \( \gamma \), the part of \( k \) in the interior \( \gamma \) of \( \gamma \) is transformed into the part of \( k \) outside \( \gamma \). The values of \( H_i(\omega) \) on \( k \) are continuous and define (by

means of the Poisson integral) a function \( H_i(\omega) \) harmonic in the in-
set forth a topological mapping of a neighborhood of \( u = u_0 \) of \((u_0, v_0)\) on a neighborhood of the point \( (u_i, v_i, v_0 = u_0) \) in the \((x_i, x_j)\)-plane. Therefore the equations 

\[ x_i = H_i(u) \]

give a one-to-one correspondence between the points 

\[ x_i = H_i(u) \quad |u - u_0| < \eta \]

of the surface and the circle \( |u - u_0| < \eta \) of the \( u \)-plane. Let now 

\[ g_i(u) = k_i = \text{const. on } \Gamma \]

but not all \( g_i(u) \) be constant on the whole of \( \mathcal{C} \). If \( V \) were false we should have 

\[ \sum F_i \cdot (u) = 0 \]

hence (see p. 15) 

\[ E = \mathbb{F}, \quad F = 0 \]

Since \( H_i(u, v) \) is the harmonic function defined by the Poisson integral with the boundary values \( g_i(u) \) (cf. pp. 15-19), we have 

\[ H_i(u) = \mathcal{H}_i \quad \text{everywhere in this domain (we use the fact that a function harmonic in a domain bounded by a piezodric if } \eta \text{ approaches an interior point of } \Gamma \text{. Therefore } H_i(u) \text{ can be continued across } \Gamma \text{ by the method of reflection.} \]

The mapping of our surface must be uniquely determined by these boundary values. Considering

As is well known, this process is defined as follows: We set 

\[ W = \mathbb{F} : \quad \mathbb{F}(u) = \frac{1}{2} \quad (|W|^2 = \infty) \]

and for \( \eta \rightarrow 1 \)

\[ H_i(u) = -H_i(u^*) + \frac{2k_i}{\Gamma} \]

to prove that \( H_i(u) \) is harmonic for all \( u \) not on the closed arc \( \mathcal{C} - \gamma \), consider a circle \( k \) perpendicular to \( \mathcal{C} \) intersecting \( \mathcal{C} \) at two interior points of \( \mathcal{C} \). Under the transformation \( W \rightarrow W^* \), the part of \( k \) in the interior \( D \) of \( C \) is transformed into the part of \( k \) outside \( C \). The values of \( H_i(u) \) on \( k \) are continuous and define (by means of the Poisson integral) a function \( H_i(u) \) harmonic in the in-
of the interior of \( k \) with the boundary of \( \mathcal{S} \) and \( \mathcal{S} \)

\[
\overline{H_i}(u) = H_i(u) \quad \text{for} \quad \omega \in k
\]

The transformation

\[
\omega \rightarrow \omega' (u)
\]

transforms \( \overline{H_i}(u) \) into the function

\[
\overline{H_i}(u) = \overline{H_i}(u) + 2k_i
\]

Since on \( k \) we have

\[
\overline{H_i}(u) = -\overline{H_i}(u) + 2k_i
\]

this equation must also hold in the interior of \( k \), especially for \( u \in \mathcal{S} \).

The last preparatory consideration concerns the set of all topological

Hence on the whole boundary of the domain inside of both \( k \) and \( \mathcal{S} \), we

have as coordinates \( \overline{H_i}(u) \) on take again the angle \( \omega \) with \( \mathcal{S} \) at

and therefore \( \overline{H_i}(u) = H_i(u) \) everywhere in this domain (we use the

fact that a function harmonic in a domain bounded by a piecewise

analytic simple closed Jordan curve with continuous values on this

curve is uniquely determined by these boundary values). Concerning

the method of reflection, see Hurwitz-Courant, Funktionentheorie, 3d


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S can also be continued over \( \mathcal{S} \). Since \( \mathcal{S}; \omega^2 = k \), on \( \mathcal{S} \) this mapping

for \( \omega' \); \( \omega^2 = k \) is impossible since the function

cannot be one-to-one in the neighborhood of any point of \( \mathcal{S} \). Our prelim-

inary remark shows that

\[
E \xi - F^2 = 0 \quad \text{and} \quad \mathcal{S}.
\]

And on account of \( (1) \)

\[
E = \xi = 0, \quad F^2 = 0 \quad \text{and} \quad \mathcal{S}.
\]

would vanish for some values \( t \) between \( 0 \) and \( 1 \).
or (see the definitions of E and G)

\[
\frac{\partial x_i}{\partial u} = \frac{\partial x_i}{\partial v} = 0, \quad i = 1, 2, \ldots, n \quad \text{on } \gamma
\]

This equation is known as the condition on the arguments of \( f_i \) on the right and on the left sides, but

This would involve

\[
F_i(\theta) = \frac{\partial x_i}{\partial u} - \frac{\partial x_i}{\partial v} = 0 \quad \text{on } \delta
\]

and since \( F_i(\theta) \) is analytic in a neighborhood of \( \delta \) of \( \Omega \) we should have

\[
F_i(\theta) = 0, \quad \theta = 0, 1, \ldots, n
\]

or a \( F_i(\theta) \) is constant, and \( (g_i(\theta)) = \) constant on the whole of \( C \) contrary to our hypothesis.

The last preparatory consideration concerns the set of all topological mappings of the unit circle \( \gamma \) onto itself, which leave three points of \( \gamma \) fixed. As coordinate on \( \gamma \) we take again the angle \( \gamma = \gamma \) of \( f_i(\theta) \).

Let \( \delta \) be a sequence \( \gamma \rightarrow f(\theta) \), converging to a function \( f(\theta) \) non-decreasing in \( \gamma \).

If then there may be more on \( \gamma \) on which \( f(\theta) \) is constant, and there

\[
\gamma_{\delta} \leq \gamma' \leq \gamma'' \leq \gamma_{\delta}, \quad (0 \leq \gamma_{\delta} < \gamma_{\delta} < \gamma_{\delta} < 2\pi)
\]

Then we must have at least of the two following

If then there may be more on \( \gamma \) on which \( f(\theta) \) is constant, and there

\[
\gamma_{\delta} \leq \gamma' < \gamma'' \leq \gamma_{\delta}
\]

we must have

\[
\gamma_{\delta} \leq f(\gamma') \leq f(\gamma'') \leq \gamma_{\delta}
\]

for if \( f(\gamma') > f(\gamma'') \) so \( (f(\gamma') = f(\gamma'') \) is impossible since \( \gamma' \neq \gamma'' \)

the function discontinuity belongs a positive number

\[
y(\varphi) = f(\varphi(\gamma) + e\gamma) - f(\gamma'' + (1-e)\gamma_{\delta})
\]

continuous for \( 0 \leq e \leq 1 \) and satisfying the conditions

\[
y(0) = f(\gamma_{\delta}) - f(\gamma_{\delta}) = \gamma_{\delta} - \gamma_{\delta} < 0 \quad \text{and}
\]

\[
y(1) = f(\gamma'') - f(\gamma'') > 0
\]

would vanish for some value \( \bar{e} \) between 0 and 1.
we have
\[ f((1-\varepsilon)\bar{J}_1 + \varepsilon \bar{J}^*) = f((\varepsilon \bar{J}^* + (1-\varepsilon)\bar{J}_2) \]

This equation would involve necessary equality of the arguments of \( f(\cdot) \) on the right and on the left sides, but
\[ (1-\varepsilon)\bar{J}_1 + \varepsilon \bar{J}^* \leq \bar{J}' = \bar{J}'' \leq \varepsilon \bar{J}^* + (1-\varepsilon)\bar{J}_2. \]

We see that the function \( f(\cdot) \) is increasing in each of the intervals
\[ J_2 \leq \bar{J} \leq J_1, \quad J_2 \leq \bar{J} \leq J_3, \quad J_3 \leq \bar{J} \leq J_1 + 2\pi. \]

If \( \bar{J} \) is not consecutive, one of the sets \( J_i \) must be even non-countable.

Now let \( \bar{J} \to f(\bar{J}) \) be any sequence of topological mappings.

Using this fact one easily that any function which has only discontinuities of \( \bar{J} \) onto itself with the three fixed points \( J_1, J_2, J_3 \), has only countably many points of discontinuity, according to a well-known theorem (which we shall prove later) and a subsequence \( \{ f(\bar{J}) \} \) can be chosen which converges for \( \bar{J}_1 \leq \bar{J} \leq \bar{J}_2 \)
to a monotonic (non-decreasing) function, according to the same theorem in the sequence \( \{ f(\bar{J}) \} \). A subsequence \( \{ f(\bar{J}) \} \) exists converging in \( \bar{J}_2 \leq \bar{J} \leq \bar{J}_3 \) to a monotonic function, and finally (choosing a subsequence of \( f(\bar{J}) \)) we find a sequence \( \{ f(\bar{J}) \} \) converging to a function \( f(\cdot) \) non-decreasing in each of the intervals \( J_i \) with

\[ \{ \bar{J}_i \} \text{ is a set of points on } \bar{J} \text{ and whole space } \bar{J}. \]

In general \( f(\cdot) \) will have singularities of the two following kinds: there may be arcs on \( \bar{J} \) on which \( f(\cdot) \) is constant, and there may be discontinuities. These are of the first kind since the monotonicity of \( f(\cdot) \) involves the existence of the limits \( f(\bar{J}^+) \text{ and } f(\bar{J}^-) \) at each point \( \bar{J} \). The set \( \bar{J} \) of all these discontinuities is countable.

For to every discontinuity belongs a positive number
\[ \alpha = f(\bar{J}^+) - f(\bar{J}^-) \text{ in the points of } \bar{J}. \]

Denoting by \( \bar{J}_v \) the subset of those points for which
\[ f(\bar{J}_v) \text{ exists and } f(\bar{J}_v) \text{ is bounded, a converging subsequence } f(\bar{J}_v) \text{ of } f(\bar{J}_v) \text{ exists.}

Observe that in the cases...
we have
\[ \sigma = \sum \delta_v \]

and if \( \sigma \) were not countable at least one of the sets \( \sigma_v \), say \( \sigma_{v_0} \), must be infinite. Then one of the intervals \( (\sigma_{v_0}) \) must contain an infinite subset of \( \sigma_{v_0} \) and \( f(0) \) would not be bounded.\(^1\) The set of (greatest)

\(^1\) If \( \sigma \) is not countable, one of the sets \( \sigma_v \) must be even non-countable; using this fact one sees easily that any function which has only discontinuities of the first kind has only countably many points of discontinuity.

arcs, where \( f(0) \) is constant, is countable (this can be deduced from the preceding investigation, since these arcs are the discontinuities of the function inverse to \( f(0) \)). We see the mapping

\[ \mathcal{J} \to f(0) \]

has in general the following singularities: countably many subarcs of \( C \) are transformed into points to a countable set of points on \( C \). Corresponding whole arcs.

Let \( J \) be the rational values of \( f(0) \) contained in the interval \( J \).

To prove the compactness of the monotonic functions, we start with the sequence

\[ h_v(P), \quad v = 1, 2, \ldots \]

be any sequence of uniformly bounded functions defined on a point set \( \mu \). To any arbitrary countable subset \( \hat{\mu} \) of \( \mu \), a subsequence of \( \{ h_v(P) \} \) can be found converging at the points of \( \hat{\mu} \).

Proof. Let \( A_1, A_2, \ldots \) be the points of \( \hat{\mu} \). The sequence \( h_v(A_1) \) is bounded; therefore a converging subsequence \( h_{1v}(A_1) \) of \( h_v(A_1) \) exists. Since \( h_v(A_2) \) is bounded, a converging subsequence \( h_{2v}(A_2) \) of \( h_{1v}(A_2) \) exists. Going on in the same
manner we get a sequence

\[ h_{n_1}(p), h_{n_2}(p), \ldots \]

and

\[ h_{n_1}(p), h_{n_2}(p), \ldots \]

of subsequences of \( \{ h_n(p) \} \), where \( h_{n_1}(p), h_{n_2}(p) \ldots \)

converges at the points \( A, \ldots, A \) and is a subsequence of the preceding sequences. Then

\[ h_{n_1}(p), h_{n_2}(p), \ldots, h_{n_n}(p), \ldots \]

converges at each point \( A_n \) of \( S \), because

\[ h_{n_{n_1}}(p), h_{n_{n_2}}(p), \ldots \]

is a subsequence of \( h_{n_1}(p), h_{n_2}(p), \cdots \) since there are at the same time those of \( f(v) \) as proved on \( D \) by way of using a countable set.

We now prove the theorem we used above.

Let the functions \( f_n(v) \) be defined and non-decreasing for

\( f_{n_1} \leq f_{n_2} \leq f_{n_3} \ldots \) and uniformly bounded: \( |f(v)| \leq M \). Then a sub-
sequence \( \{ f_{n_k}(v) \} \) of \( f(v) \) exists converging to a non-decreasing func-
tion \( f(v) \).

Let \( \bar{V}, \bar{V}^2, \ldots \) be the rational values of \( V \) contained in the
interval \( \bar{V}_1 \leq \bar{V} \leq \bar{V}_2 \).

According to our Lemma we can find a subse-
quence \( \{ f_{n_k}(\bar{V}) \} \) converging at the points

\( \bar{V}_1, \bar{V}_2, \ldots \). We set

\[ \lim_{v \to \infty} f_{n_k}(v) = f(\bar{V}), k = 1, 2, \ldots \]

Since

\[ f_{n_k}(v) \leq f_{n_k}(v') \]

we have

\[ f(v) \leq f(v') \]

for \( v \leq v' \). Then our Lemma shows that in \( \{ f_{n_k}(\bar{V}) \} \) a
subsequence \( \{ f_{n_k}(\bar{V}) \} \) exists converging at all points where \( f(v) \) and \( f(v') \)
are discontinuous. The sequence \( \{ f_{n_k}(\bar{V}) \} \) has all properties required in
the assertion.
Hence the limits exist for \( V_n \to V \) (for \( f = f_1, f_2 \) respectively, only one of these limits has a sense). The functions are monotonic and we have for all \( V \) onto the whole of itself. That is what we mean by a function mapping any correspondence to a single point.

The discontinuities of \( \overline{f}(V) \) therefore are at the same time those of \( f(V) \); as proved on p. 36 they form at most a countable set and except for these points we have \( f(V) = \overline{f}(V) \). Let \( V_0 \) be a point where \( f(V) \) and \( \overline{f}(V) \) are continuous and let the rational numbers be chosen in such manner that \( V_n \to V \) as \( n \to \infty \). Then we have

\[
\lim_{n \to \infty} f_n(V) = f_n(V_0) \leq \overline{f}(V_0) \leq \overline{f}(V) \leq f(V) = \overline{f}(V),
\]

since \( V_0 \) is a point of continuity for \( f(V) \) as well as \( \overline{f}(V) \) we have

\[
\lim f_n(V) = f(V_0) = \overline{f}(V_0) = \lim \overline{f}(V).
\]

Consequently

\[
\lim \sup f_n(V) = \lim \sup \overline{f}(V_0).
\]

and define as distance of \( f(V) \) and \( \overline{f}(V) \) the number or \( f_n(V) \) converges at \( V_0 \). Then our Lemma shows that in \( \{f_n(V)\} \) a subsequence \( \{f_{n_k}(V)\} \) exists converging at all points where \( \overline{f}(V) \) and \( f(V) \) are discontinuous. The sequence \( \{f_{n_k}(V)\} \) has all properties required in the assertion.
Returning a moment to our previous discussion where we applied this theorem, we see: if \( f(\theta^-) < f(\theta^+) \) by fixing the value \( f(\theta) \), we determine the point \( f(\theta_0) \) as image of \( \vartheta_0 \). The mapping \( x \to f(x) \) then does not cover any point between \( f(\theta^-) \) and \( f(\theta^+) \) except \( f(\theta_0) \). Instead of doing this we can agree to consider all points between \( f(\theta^-) \) and \( f(\theta^+) \) (eventually including one or both of these points) as images of \( \vartheta_0 \). Then the function \( f(\vartheta) \) is not one-valued, but \( x \to f(x) \) gives a mapping of \( F \) onto the whole of itself. That is what we meant by saying on p. 27 that whole arcs may correspond to single points.

We now come to the solution of the problem of Plateau. The area of a continuous surface \( S \) of the type of the circular disk has been defined on p. 14. Thereby sequences of polyhedrons \( \mathcal{P}_1, \mathcal{P}_2, \ldots \) occurred whose boundaries \( \mathcal{P}_1, \mathcal{P}_2, \ldots \) tend to the given curve \( \Gamma \). We did not perceive then the sense in which \( \mathcal{P}_1, \mathcal{P}_2, \ldots \) has to approach \( \Gamma \). We mean "approaching" in the sense of Fréchet: \( \Gamma \) and \( \mathcal{P}_n \) are topological images of the unit circle \( G \). Let \( \varphi \) be given by

\[
x_i = \psi_i(t), \quad 0 \leq t \leq 2\pi, \quad i = 1, 2, \ldots, n
\]

and \( \mathcal{P}_n \) by

\[
x_i = \psi_i^{(n)}(t)
\]

If \( t \to \varphi(t) \) is a topological representation of \( G \) onto itself,

\[
x_i = \psi_i^{(n)}(t) = \psi_i(t)
\]

still represents \( \mathcal{P}_n \). We set

\[
\alpha_{\chi}(\Gamma, \mathcal{P}_n) = \frac{\mu}{\sqrt{\sum_{i=1}^{m} (f_i^{(n)}(t) - \psi_i(t))^2}}
\]

and define as distance of \( \Gamma \) and \( \mathcal{P}_n \) the number

\[
\alpha(\Gamma, \mathcal{P}_n) = \sup_{x \in \Omega} \alpha_{\chi}(\Gamma, \mathcal{P}_n)
\]
where \( \tau(t) \) traverses all topological mappings \( t \to \tau(t) \) of \( C \) into itself. We say that \( P_m \) tends to \( \Gamma \) if
\[
0 \left( \Gamma, P_m \right) \to 0.
\]

At first sight this definition seems to be complicated, but it is only an exact (and for more complicated cases than Jordan curves and only useful) formalulation of the requirement: there must be parametric representations of \( P_m \), for which "corresponding" points of \( P_m \) and \( \Gamma \) are near to each other uniformly along \( \Gamma \). Indeed we have:
\[
P_m \to \Gamma \quad \text{holds if and only if such parametrizations } x_i = \gamma_i^{(m)}(t)
\]
and \( x_i = \gamma_i^{(t)}(t) \) of \( P_m \) and \( \Gamma \) can be found that \( \gamma_i^{(m)}(t) \) tends to \( \gamma_i^{(t)}(t) \) uniformly in \( t \).

The proof is obvious since the uniform convergence is equivalent to
\[
\lim_{m \to \infty} \left| \gamma_i^{(m)}(t) - \gamma_i^{(t)}(t) \right| = 0
\]
for all \( i \).

From the uniform convergence of \( \gamma_i^{(m)}(t) \) follows: For \( P_m \to \Gamma \), we have
\[
\lim_{m \to \infty} \left| \gamma_i^{(m)}(t) - \gamma_i^{(t)}(t) \right| = 0
\]
for all \( i \).

Now let \( \Pi_1, \Pi_2, \ldots \) be any sequence of simply connected polyhedrons whose boundaries tend to \( \Gamma \) and put
\[
S(\Pi_1, \Pi_2, \ldots) = \lim_{n \to \infty} S(\Pi_n),
\]
where \( S(\Pi_n) \) is the area of \( \Pi_n \). The greatest lower bound \( \text{in} \{\Gamma\} \) of all simply-connected surfaces spanned by \( \Gamma \) is then equal to the greatest lower bound of all the numbers \( S(\Pi_1, \Pi_2, \ldots) \) formed for the different se-
quences $\overline{\pi_1}, \overline{\pi_2}, \ldots$. Then there exists a sequence $\overline{\pi_n}, \overline{\pi_2}, \ldots$ of simply-connected polyhedra whose boundaries $F_1, F_2, \ldots$ tend to $\Gamma$.

The expression "simply connected" requires an interpretation. For if, for instance, $\Gamma$ is a knotted curve in the 3-space, the curves $F_n$ (for large $n$) will be knotted too, and there exists no polyhedron of the type of the circular disk spanned by $F_n$. We mean that $F_n$ is of the type of the circular disk in the combinatorial sense: Take a finite set of triangles in the plane with certain incidental relations for the sides. These relations must be such that they define a complex $\overline{f_n}$ which is topologically equivalent to the circular disk. (We may suppose that identified sides have equal lengths. Then in a space of sufficiently large dimensions a realization of $\overline{f_n}$ exists whose faces are congruent to the given triangle and which is homeomorphic to the circular disk. In the three-space in general we only get a polyhedron $F_n$ whose faces are congruent to the given triangle and cross each other in certain straight-line segments. $F_n$ is only a continuous, not a one-to-one continuous, image of the abstract complex $\overline{f_n}$.

We now apply the so-called mapping theorem of Koebe:

Let $\overline{\pi^*}$ be a simply-connected polyhedron bounded by the polygon $\Gamma$. Then there exists a topological map of $\overline{\pi^*} \sim \Gamma$ on the closed unit circle $D + C$, which is conformal at every interior point not being a

\[ S(\overline{\pi_n}) \sim \mathbb{C}(\Gamma) \]
vertex. *) (1) define a topological mapping of $G$ onto itself (considered at parameter $u$).

*) In this form the theorem has already been proved by H. A. Schwarz. (Koebe treated much more general cases.) For a proof see C. Caratheodory, Conformal representation (No. 28 of the Cambridge Tracts), Chap. VII.

According to our statement on p. 27, we will change a subsequent.

Regarding the interior points of a face of $H^k$ the meaning of the expression "conformal" is clear. For the points on an edge $\gamma$ of $H^k$ which are not corners the word "conformal" means: we move the two faces touching at $\gamma$ so that they fall into the same plane. As metric on $H^k$ in the neighborhood of an interior point $P$ of $\gamma$ we take the metric of the plane neighborhood of $P$ after the movement. The mapping in the theorem of Koebe is conformal with respect to this metric.

We apply the theorem to our polyhedron $\Pi_m$ (see (**), p. 32).

The boundary $P_m$ of $\Pi_{m-1}$ may be given by

$$x_i = \tilde{f}_i^m(t), \quad 0 \leq t \leq 2\pi$$

where the functions $\tilde{f}_i^m(t), \tilde{f}_i^m(t)$ may be chosen in such manner that $\tilde{f}_i^m(t)$ approaches $f_i(t)$ of uniformly in $t$, where

$$x_i = f_i(t)$$

(see p. 29)

is a representation of $\Gamma$. (The mapping of $\Pi_m$ on $D + C$ induces a topological mapping of $P_m$ on $C$, which may be given by

(1) presentation $\tau = \tilde{J}_m(J)$.

Fixing on $P_m$ the three points $t_1, t_2, t_3$ $(0 \leq t_1 < t_2 < t_3 < 2\pi)$ and taking on $C$ the three fixed points $\bar{\gamma}_1 = t_1, \bar{\gamma}_2 = t_2, \bar{\gamma}_3 = t_3$ we may suppose that

$$\tau_i = \bar{\gamma}_i = \tilde{J}_m(J_i), \quad i = 1, 2, 3; \quad m = 1, 2, \ldots$$

For we can compose our conformal mapping with a linear mapping of $D + C$ onto itself which transforms the original images of $t_1, t_2, t_3$ into the points $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3$.
(1) defines a topological mapping of \( C \) onto itself (considered as parameter on \( C \)):

\[ J \rightarrow \overline{J} = \mathcal{T} \]

According to our statement on p. 27, we can choose a subsequence \( \{ \omega_m(\mathcal{T}) \} \) of \( \{ \overline{J} \} \) converging to function \( \omega(\mathcal{T}) \). We cannot exclude a priori that the mapping

\[ \mathcal{T} \rightarrow \omega(\mathcal{T}) \]

might have singularities of the types mentioned on p. 27 and 30. 

The equations

\[ x_i = f_i^{\omega_m}(\omega_m(\mathcal{T})) = g_i^{\omega_m}(\mathcal{T}) \]

and that this series is uniformly convergent in \( \mathcal{T} \) for \( r < 1 \), we have still represent \( P_m \), but by setting

\[ x_i = f_i(\omega(\mathcal{T})) = g_i(\mathcal{T}) \]

we are not sure to have a proper parametric representation of \( P \). However, from the uniform convergence of the \( f_i^{\omega_m}(\mathcal{T}) \) it follows that

(2)

\[ \lim_{m \to \infty} g_i^{\omega_m}(\mathcal{T}) = g_i(\mathcal{T}) \]

if \( \mathcal{T}_0 \) is point of continuity for \(\omega(\mathcal{T})\). For then

\[ \omega_m(\mathcal{T}_0) \rightarrow \omega(\mathcal{T}_0) \]

(af. p. 29)

and (2) is a consequence of (3), p. 37.

The mapping of \( \Pi_n \) on \( u^2 + v^2 \leq 1 \) sets forth a parametric representation

\[ x_i = x_i^{\omega_m}(u, v) \quad \text{for} \quad u^2 + v^2 \leq 1 \]

and of \( \Pi_n \) since this mapping is conformal for \( u^2 + v^2 < 1 \) except for a finite number of points we have

\[ E = \sum_i \left( \frac{\partial x_i}{\partial u} \right)^2 = \sum_i \left( \frac{\partial x_i}{\partial u} \right)^2 = 0 \]

\[ F = \sum_i \frac{\partial x_i}{\partial u} \frac{\partial x_i}{\partial v} = 0. \]
Therefore

\[ D(x^m) = \frac{1}{2} \iiint_D (E + F) \, dxdydz = \frac{1}{2} \iiint_D \sqrt{E^2 - F^2} \, dxdydz = S(\mathcal{H}_m) \]

Now let \( H_m(u, v) \) be the function harmonic in \( D \) with the boundary values \( g_i^n \). Then

\[ D(H_m) = \iiint_D \left( \frac{2H_m^2}{\partial u} \right) + \left( \frac{2H_m^2}{\partial v} \right) \, dxdydz \leq \iiint_D \left[ \left( \frac{2\lambda^2}{\partial u} \right) + \left( \frac{2\lambda^2}{\partial v} \right) \right] \, dxdydz. \]

From \( D \) and \( (a) \) follows

For, omitting the subscript \( i \) and the superscript \( m \), and considering that

\[ \frac{1}{1 + r^2 - 2r \cos \theta} = 1 + \sum_{n=1}^{\infty} r^n \cos n(\theta - \phi) \]

and that this series is uniformly convergent in \( \mathcal{F} \) for \( r < 1 \), we have

\[ H(u, v) = H(\tau, \rho) = \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos \theta} \, d\mathcal{F} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos n\theta + b_n \sin n\theta \right) \]

with

\[ a_n = \frac{1}{\pi} \int_0^{2\pi} g(\rho) \cos n\theta \, d\mathcal{F}, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} g(\rho) \sin n\theta \, d\mathcal{F} \]

Putting

\[ H_n = H_0 + \sum_{k=1}^{\infty} \left( a_k \cos k\mathcal{F} + b_k \sin k\mathcal{F} \right) \]

we have for \( r = 1 \)

\[ \frac{\partial H_n(u, v)}{\partial r} = \sum_{k=1}^{\infty} k \left( a_k \cos k\mathcal{F} + b_k \sin k\mathcal{F} \right) \]

and we find on account of \((3)\), p. 35

\[ \int_0^{2\pi} \left( x - H_n \right) \cos k\mathcal{F} \, d\mathcal{F} = 0, \quad \int_0^{2\pi} \left( x - H_n \right) \sin k\mathcal{F} \, d\mathcal{F} = 0 \quad k = 1, \ldots, n \]

hence for \( r = 1 \)

\[ A(\rho) = \int_0^{2\pi} \left( x - H_n \right) \frac{\partial H_n}{\partial \mathcal{F}} \, d\mathcal{F} = 0. \]

We then consider the harmonic surface \( H \) defined by the boundary values \( g_i^n \) on \( \partial D \). In a sense defined by Douglas, \( H \) may be
Now

\[ D(x) = D(\overline{H}_n + (x - \overline{H}_n)) = D(\overline{H}_n) + D(x - \overline{H}_n) + \]

\[ + \int_\Omega \left( \frac{\partial H_n}{\partial u} \frac{\partial (x - \overline{H}_n)}{\partial u} + \frac{\partial H_n}{\partial v} \frac{\partial (x - \overline{H}_n)}{\partial v} \right) du dv \]

According to Green's formula the last integral on the right side is equal to

\[- \int_\Omega \frac{\partial H_n}{\partial x} \frac{\partial u}{\partial u} + \frac{\partial H_n}{\partial y} \frac{\partial u}{\partial v} \]

From \( \Delta \overline{H}_n = 0 \) and (4) follows

\[ D(x) = D(\overline{H}_n) + D(x - \overline{H}_n) \]

and

\[ \int_{D_s} \left[ \left( \frac{\partial H_n}{\partial u} \right)^2 + \left( \frac{\partial H_n}{\partial v} \right)^2 \right] du dv \leq D(\overline{H}_n) = D(x) \]

where \( D_s \) designates the domain \( 0 \leq u^2 + v^2 < s < r \). In \( D_s \) the functions \( H_n \) and their derivatives converge uniformly to \( \overline{H}_n \); therefore for \( n \to \infty \) the integral on the left side tends to the Dirichlet integral of \( H \) over \( D_s \) and we have

\[ \int_{D_s} \left[ \left( \frac{\partial H}{\partial u} \right)^2 + \left( \frac{\partial H}{\partial v} \right)^2 \right] du dv \leq D(x) \]

Now, returning to our previous notation

\[ A(g^{\mu}) = \sum_{\gamma=1}^{n} D(H_i^{\mu}) \]

we find on account of (3), p. 35

\[ A(g^{\mu}) \leq S(T_\mu) \]

Since \( A(g^{\mu}) \) is lower semi-continuous we have

\[ A(g^{\mu}) \leq \sup \inf \ A(g^{\mu}) \leq \sup S(T_\mu) = \mu(R) \]

If \( m(R) < \infty \), theorem III, p. 22, shows that the functions \( g^{(r)} \) are continuous. We then consider the harmonic surface \( H \) defined by the boundary values \( g^{(r)} \) on \( C \). In a sense defined by Douglas, \( H \) may be
considered as bounded by $\Gamma$ (though the $f_i$ might be constant on certain arcs of $C$). Therefore

$$\mu_1(\Gamma) \leq \delta(C) = \delta(\Gamma)$$

and the theorems I, IV, V easily lead to the conclusion that $H$ is a minimal surface with the area $\mu_1(\Gamma)$ bounded by $\Gamma$ and so that the equations

$$x_i = g_i(y)$$

set forth a topological mapping of $C$ on $\Gamma$.

Finally Douglas treats the case $\mu_1(\Gamma) = \infty$ and proves that a minimal surface through $\Gamma$ exists.

$\alpha_x$ has a positive distance from the boundary of $\alpha_x$. We have

$$\mu_1(\Gamma) = \infty$$

3. Examples

Let $\Gamma$ be defined as follows: $\Gamma$ consists of the curve $x = \exp(\lambda)$.

The case $\mu_1(\Gamma) = \infty$ can really occur for a suitable simple closed Jordan curve in the three-space. The example which we shall discuss here has been indicated by R. Rado.* Professor S.S. Cairns gave the details of the proof.


Consider the spiral

$$r = \frac{1}{\gamma}, \quad \frac{r}{\gamma} = \gamma \leq \infty$$

in the plane $z = 0$ of a 3-space with the cylindrical coordinates $\varphi, \gamma, z$.

This spiral, together with the straight line segment connecting the endpoints of the spiral, forms a curve $\Gamma$ which divides the plane $z = 0$ into an infinite and enumerably many simply-connected regions.
connected open regions. We designate these latter regions by \( a_1, a_2, \ldots \), where \( a_\nu \) is the region which contains the point

\[
\gamma = \frac{1}{\sqrt{\pi (2\nu - 1) + 1}} \quad \beta = \frac{1}{\pi
\]

Denoting by \( |a_\nu| \) the content of \( a_\nu \), one readily verifies that

\[
|a_1| + 2|a_2| + 3|a_3| + \ldots = \infty
\]

We determine in \( a_\nu \) a simple closed polygon \( \beta_\nu \) bounding a (simply connected) region \( \nu \) with

\[
|\beta_\nu| \geq \frac{|a_\nu|}{2}.
\]

\( \beta_\nu \) has a positive distance from the boundary of \( a_\nu \). We have

\[
|\beta_1| + 2|\beta_2| + 3|\beta_3| + \ldots = \infty
\]

(1)

Let \( \Gamma \) be defined as follows: \( \Gamma \) consists of the curves \( \gamma = \frac{1}{\sqrt{\pi}}, \quad z = \frac{1}{\pi}, \quad \sqrt{z} = \infty \), arguments of \( u(w) - u_0 \), and \( u_0 - u_{00} \) so that their difference is less than \( \pi \) and continuing and the straight line segment connecting the end-points of the curve. \( \Gamma \) is a continuous curve, we see that will not span any simply-connected surface of finite area. In order to prove this we need an auxiliary consideration:

In the \( \nu \)-plane let

\[
\nu = \frac{\nu}{2} + \pi, \quad 0 \leq \nu \leq 2\pi
\]

be any continuous closed curve \( \nu \) \( \omega(\nu) = u(2\pi) \) \( \nu_0 \) a point not on \( \nu \). Then the integral

\[
2\pi i \int_{\nu_0}^{\nu} \omega = \int_{\nu_0}^{\nu} (u - u_0) d\omega
\]

is equal to the variation of the argument of \( u - u_0 \) \( \omega \) (i.e. the angle which the vector \( u - u_0 \) forms with a fixed direction), when \( \nu \) traverses \( \nu \) in the sense of increasing \( \nu \), and this angle is continued in a continuous manner along \( \nu \). From a consideration familiar in the theory of complex
functions it follows that
\[ \omega^k = \omega, \]
if \( \omega \) can be connected with \( \omega_0 \) by an arc which has no common points with \( k \).

Let \( k^1 \) be another continuous curve so near \( k \) that their distance in the sense of Frechet is less than the (usual) distance of either curve from \( \omega_0 \). Then there exist such parametrizations \( \omega(\zeta) \) and \( \omega'(\zeta) \) of \( k \) and \( k^1 \) that
\[ |\omega(\zeta) - \omega'(\zeta)| < \min \{ |\omega(\zeta) - \omega_0|, |\omega'(\zeta) - \omega_0| \}, \quad 0 \leq \zeta \leq 2\pi. \]
We shall prove that
\[ J_{\omega} = J_{\omega^1}. \]
The angle (in the common sense) between the vectors \( \omega(\zeta) - \omega_0 \) and \( \omega'(\zeta) - \omega_0 \) is less than \( \frac{\pi}{2} \). Fixing the arguments of \( \omega(\zeta) - \omega_0 \) and \( \omega'(\zeta) - \omega_0 \) so that their difference is less than \( \frac{\pi}{2} \) and continuing these arguments for \( 0 \leq \zeta \leq 2\pi \) in a continuous manner, we see that
\[ |\arg (\omega(\zeta) - \omega_0) - \arg (\omega'(\zeta) - \omega_0)| < \frac{\pi}{2}, \quad \text{for} \quad 0 \leq \zeta \leq 2\pi, \]
therefore
\[ |\arg (\omega(2\pi) - \omega_0) - \arg (\omega(0) - \omega_0)| - |\arg (\omega'(2\pi) - \omega_0) - \arg (\omega'(0) - \omega_0)| \leq \pi. \]
from this follows \((2)\) immediately.

Let now \( B_0 \) be a point in \( B \) and \( m, m_1, \ldots \) any sequence of simply-connected polyhedrons (in the combinatorial sense, cf. p. 32) whose boundaries \( P_1, P_2, \ldots \) tend to \( \Gamma \) (in the sense of Frechet). We have to show that the areas \( S(\Pi) \) of \( \Pi \) tend to \( \infty \). Let \( P_{\frac{\pi}{2}} \), \( P_{\frac{\pi}{3}} \), \( \ldots \) be the projections of \( P_1, P_2, \ldots \) on the plane \( z = 0 \). Since \( P_{\frac{\pi}{2}} \) tends to \( \Gamma \) in the sense of Frechet, one easily sees that \( P_{\frac{\pi}{2}} \) tends to \( \Gamma' \) in the sense of Frechet (using the fact that the ratio of the lengths of any arc on \( \Gamma' \) and the corresponding arc of \( \Gamma \) is bounded). Therefore
on account of (2), to given $N$ a number $v(N)$ can be found such that for $n > v(N)$ and $m = 1, \ldots, N$, suppose that the faces of $F_m^+$ are triangles and 

(a) $F_m^+$ does not intersect the boundary $B_m$ of $F_m^+$. A face of $F_m^+$ is perpendicular to $P_m^+$. 

(b) $J_{F_m^+} = J_{B_m}$.

From (1) it follows that we have $B_m$ as a continuous image of $F_m^+$. This induces an obvious orientation of the triangles forming $F_m^+$. The oriented polygon $P_m^+$ is the boundary of $F_m^+$. Let $A$ be any point of the line where $P_m^+$ crosses itself. $A$ is covered by certain triangles of $F_m^+$; can easily be determined. Let $J_a$ be the sum of 

\[ J_a = \sum_{m=1}^{N} J_{B_m} \]

for each $A$ point, $A \subset B_m$. We then have at

\[ J_{F_m^+} - J_{B_m} \]

Dence each point of $A$ is not under an edge or a self-crossing of $F_m^+$ is a projection of at least $m-1$ different points of $F_m^+$. Since the content of a plane, we see that the area $\pi (F_m^+)$ of $F_m^+$ for $n > v(N)$ satisfies the relation 

and on account of (b) and (c)

\[ J_{P_m^+} = m \]

for $Q \in B_m$, $m = 1, \ldots, N, n > v(N)$.

We orient the curve $\Gamma$ so that the part over the spiral is traversed positively if $\gamma$ increases. The orientation of $\Gamma$ induces orientations
of \( P^* \) and \( P_1, P_2, \ldots \) (at least for great subscripts) and herewith of \( P_{1}, P_{2}, \ldots \). We may suppose that the faces of \( \pi_n \) are triangles and we can assume without loss of generality that the plane of no face of \( \pi_n \) is perpendicular to \( z = 0 \). The orientation of \( P_n \) induces in the well-known manner a ("coherent") orientation of the triangles forming \( \pi_n \). We consider the projection \( \pi_n^* \) of \( \pi_n \) as a continuous image of \( \pi_n \). This induces as above positive orientations of the triangles forming \( \pi_n^* \); the oriented polygon \( P_n^* \) is the boundary of \( \pi_n^* \). Let \( A \) be any point of the plane \( z = 0 \) which is not the projection of a point on an edge of \( \pi_n \) or a line where \( \pi_n \) crosses itself. \( A \) is covered by certain triangles of \( \pi_n^* \); let \( \tilde{r} \) be the number of those among them which are oriented in the same way as the plane \( z = 0 \) itself (positively if \( \varphi \) increases), and \( \tilde{n} \) the number of those oriented in the opposite manner. We then apply the theorem that

\[
\int_{A} P_{n}^* = \tilde{r} - \tilde{n}.
\]

From our formula (d) it follows that

\[
\int_{A} P_{n}^* = \tilde{r} - \tilde{n} = m \quad \text{for} \quad Q < P_m, \quad u = 1, \ldots, N, \quad n > \sqrt{N}.
\]

Hence each point of \( P_n \) not under an edge or a self-crossing of \( \pi_n \) is a projection of at least \( m \) different points of \( \pi_n \). Since the content of a triangle does not increase by perpendicular projection of the triangle on a plane, we see that the area \( S(\pi_n) \) of \( \pi_n \) for \( n > \sqrt{N} \) satisfies the relation

\[
S(\pi_n) = |\tilde{r}| + 2 \left( |\tilde{r}| + \cdots + N |	ilde{r}| \right)
\]

hence

\[
S(\pi_n) \to \infty.
\]
We shall now consider examples concerning the local behavior of minimal surfaces, and examples showing that the solution of Plateau's problem generally is not unique. We start with a new and more general definition of a minimal surface.

Let \( D^* \) be any domain in the \((u^*, v^*)\)-plane bounded by a simple closed Jordan curve \( C^* \) and \( D^t \) a domain in the \((u^t, v^t)\)-plane bounded by the simple closed curve \( C^t \). Then it is well known that \( D^* + C^* \) can be mapped topologically on \( C^* + D^t \):

\[
\begin{align*}
u^* = \varphi(u^t, v^t) & \quad (u^*, v^*) = \varphi(u^t, v^t) \\
u^t = \psi(u^*, v^*) & \quad (u^t, v^t) = \psi(u^*, v^*)
\end{align*}
\]

Let \( \overline{x} = \overline{x}_0(u^*, v^*) \) be any continuous surface defined on \( D^* + C^* \). Then the equations

\[
\overline{x} = \overline{x}_0(u^*, v^*) = \overline{x}_0(\varphi(u^t, v^t)) = \overline{x}_0(\psi(u^t, v^t)) \quad (u^t, v^t) \subset D^t
\]

represent the same point set \( S \) and we regard the two systems of equations

\[
\overline{x} = \overline{x}_0(u^*, v^*) \quad \text{and} \quad \overline{x} = \overline{x}_0(u^t, v^t)
\]

merely as two different parametrizations of the same surface. Since \( D^* + C^* \) can be mapped topologically on the unit-circle \( D \) plus its boundary \( C \) of a \((u, v)\)-plane, in a suitable parametrization the domain of the parameter space is given by \( u^2 + v^2 < 1 \).

We generalize this process a little introducing local parameters.

Let \( \Delta \) be any neighborhood of a point \((u_0, v_0)\), \( u_0^2 + v_0^2 < 1 \), and \( W \) the points of \( \Delta \) corresponding to \( \Delta \). We transform \( \Delta \) topologically into a vicinity \( \Delta_0 \) of the point \((\alpha_0, \beta_0)\) in a \((\alpha, \beta)\)-plane

\[
\begin{align*}
\alpha = \alpha(u, v) & \quad (u, v) \subset \Delta \\
\beta = \beta(u, v) & \quad (u, v) \subset \Delta \\
u = u(\alpha, \beta) & \quad (\alpha, \beta) \subset \Delta_0 \\
v = \Psi(\alpha, \beta) & \quad (\alpha, \beta) \subset \Delta_0
\end{align*}
\]

and introduce in \( W \) the parameters \( \alpha, \beta \) by setting

\[
\begin{align*}
\overline{x} = \overline{x}(u, v) = \overline{x}(u(\alpha, \beta), \psi(\alpha, \beta)) = \overline{x}_0(\alpha, \beta) \quad (\alpha, \beta) \subset \Delta_0
\end{align*}
\]

We call \( \alpha, \beta \) local parameters for the neighborhood \( W \) of the point \( \overline{x}(u_0, v_0) \).
a branch). By means of this we define again what we mean by a minimal surface $S$ in the three-space bounded by the simple closed Jordan curve $\Gamma$. We require that $S$ can be represented by a continuous vector function $\overline{X}(u, v)$:

$$\overline{X}(u, v) = \overline{X}(\alpha, \beta), \quad u^2 + v^2 \leq 1$$

such that the equations

$$\overline{X} = \overline{X}(\alpha, \beta), \quad u^2 + v^2 = 1$$

yield a topological mapping of $\Gamma$ on the circle $C$ and that a suitable neighborhood $W$ of each point $\overline{X}(u, v)$ of $S$ with $u^2 + v^2 < 1$ can be parametrized by equations of the form (1) in such manner that the functions $x_0(\alpha, \beta), y_0(\alpha, \beta), z_0(\alpha, \beta)$ are harmonic in $V_0$ and satisfy the conditions

$$E = \left(\overline{x}_0\right)_{\alpha}^2 + \left(\overline{y}_0\right)_{\beta}^2 = 0, \quad F = \left(\overline{y}_0\right)_{\alpha} \overline{z}_0 = 0$$

The representation of $W$ on $V_0$ by (1) therefore is conformal. The parameters $\alpha, \beta$ are called local typical parameters for $W$. Putting $y_1 = x_0 - \alpha x_0, y_2 = y_0 - \alpha y_0, y_3 = z_0 - \alpha z_0$, $y_1$ corresponds to our previous $\overline{y}$ (see ref. p. 14). The conditions (2) are equivalent to

$$y_1^2 + y_2^2 + y_3^2 = 0$$

and the surface is given by $x(\xi, \eta) = \left(x_0(\alpha, \beta) + \eta y_1(\alpha, \beta)\right)$ of $S$ and $\alpha, \beta$ local typical parameters.

In order to eliminate one of the functions $x_0, y_0, z_0$, therefore either

(4)

$$y_0(\xi, \eta) \neq 0 \quad \text{or} \quad z_0(\xi, \eta) \neq 0$$

(b) The point $\overline{X}(\alpha_0, \beta_0)$ is a regular point of $S$ in the sense of differential geometry if

$$\sum_i |y_i(\alpha, \beta)|^2 = E \xi^2 - F \eta^2 = 0$$

If $\overline{X}(\alpha_0, \beta_0)$ is not regular and all partial derivatives of $x, y, z$ up to the order $\kappa$ vanish, but not all those of order $\kappa + 1$, we call $\overline{X}(\alpha_0, \beta_0)$
a branch point of order \( n \). \( \bar{\alpha}(\alpha_0, \beta_0) \) is a branch point of order \( n \) if, and only if, at 
\[
\varphi_i = \varphi''_i = \cdots = \varphi^{(n-1)}_i = 0, \quad i = 1, 2, 3
\]
and one of the functions \( \varphi^{(n)}_i \) does not vanish at \( \alpha_0 + i\beta_0 \). Introducing other local typical parameters \( \alpha', \beta' \) for \( \mathcal{W} \), which may be defined by
\[
\alpha' = \alpha'(u, v), \quad \beta' = \beta'(u, v), \quad (u, v) < \nu
\]
or
\[
u = u'(\alpha', \beta'), \quad v = v'(\alpha', \beta'), \quad (\alpha', \beta') < \nu '
\]
the equations
\[
(5) \quad \alpha' = \alpha'(u, \alpha, \beta, \nu(\alpha, \beta)), \quad \beta' = \beta'(u(\alpha, \beta), \nu(\alpha, \beta)), \quad (\alpha, \beta) < \nu
\]
set forth a topological mapping of \( \nu_0 \) on \( \nu_0' \). The representation of \( \mathcal{W} \) onto \( \nu_0 \) (resp. \( \nu_0' \)) is conformal on account of \( E = G, F = 0 \) except for the points where \( EG - F^2 = E^2 = 0 \). Hence the mapping \( (5) \) of \( \nu_0 \) on \( \nu_0' \) is conformal except for an isolated set of points, and since it is topological it must be conformal throughout. From this it follows that the definition of regularity and the order of a branch point does not depend on the choice of the local typical parameters.

Let \( \bar{\alpha}(\alpha_1, \beta_0) \) be a regular point of \( \mathcal{S} \) and \( \alpha', \beta' \) local typical parameters. It is natural to use the equation \( (3) \) in order to eliminate one of the functions \( \varphi_i \). We have either \( \varphi_1(\xi_0) \neq 0 \) \( (\xi_0 = \alpha_0 + i\beta_0) \) or \( \varphi_2(\xi_0) \neq 0 \). Therefore either

(a) \( \varphi_1(\xi_0) + i\varphi_2(\xi_0) \neq 0 \) or

(b) \( \varphi_1(\xi_0) - i\varphi_2(\xi_0) \neq 0 \)

In case (a) we set
\[
\psi = \sqrt{-\frac{\varphi_1 + i\varphi_2}{2}}
\]
where \( \sqrt{\cdot} \) means any one of the two distinctions of \( \varphi_1 \), and \( \phi = \sqrt{\cdot} \).
a surface given in the form (4). If \( \mathbf{r}(x, \beta) \) is regular, one of the

In case (b) we put

\[
\phi = \sqrt{\frac{\gamma_1 - i \gamma_2}{2}}
\]

\[
\psi = \frac{\gamma_3}{2 \phi}
\]

In case (b) we have

In both cases we have

\[
\gamma_1 = \phi^2 - \psi^2
\]

\[
\gamma_2 = i (\phi^2 + \psi^2)
\]

\[
\gamma_3 = 2 \phi \psi
\]

Conversely for any two functions \( \phi, \psi \) regular in a neighborhood of \( S_0 \), the functions \( \gamma_i \) defined by (8) satisfy (5); therefore the equations

\[
x = R \int_0^\infty \left( \phi^2 - \psi^2 \right) d S
\]

\[
y = R \int_0^\infty i (\phi^2 + \psi^2) d S
\]

\[
z = R \int_0^\infty 2 \phi \psi d S
\]

always yield a minimal surface and the point \( \mathbf{r}(x_0, \beta_0) \) is regular, unless

\[
\phi(S_0) = \psi(S_0) = 0
\]

We have

\[
\gamma_1 = x_{xx} - i x_{xy} = 2 \left( \phi \phi' - \psi \psi' \right)
\]

\[
\gamma_2 = y_{yy} - i y_{yx} = 2 \left( \phi \phi' + \psi \psi' \right)
\]

\[
\gamma_3 = z_{zz} - i z_{z\bar{z}} = 2 \left( \phi \psi' + \phi' \psi \right)
\]

If \( \mathbf{r}(x_0, \beta_0) \) is singular (= not regular) therefore all partial derivatives of \( x, y, z \) of the first and second order (at least) vanish, and we see:

A branch point of a minimal surface represented in the form (7) is at least of order \( 2 \).

There exists another local representation of minimal surfaces which is especially well suited for the treatment of branch points. We start with
a surface given in the form (4). If \( \bar{x}(x_0, \beta_0) \) is regular, one of the inequalities (a), (b), p. 44 holds; in the case (a) we put \( \beta = \mu \). If not,

\[
\mu = \frac{\varphi_1 - i \varphi_2}{2}, \quad \lambda = \frac{\varphi_3}{2 \mu}, \quad \text{in case (b)}
\]

the assertion is obvious, for if, for instance,

\[
\mu = \frac{\varphi_1 + i \varphi_2}{2}, \quad \lambda = \frac{\varphi_3}{2 \mu}.
\]

we simply make the transformation in both cases we have

\[
\begin{align*}
\varphi_2 &= (1 - \lambda^2) \mu, \\
\varphi_3 &= (1 + \lambda^2) \mu, \\
\end{align*}
\]

and

\[
\begin{align*}
\lambda &= \int (1 - \lambda^2) \mu \, d \bar{s}, \\
\gamma &= \int (1 + \lambda^2) \mu \, d \bar{s}, \\

\end{align*}
\]

Conversely, if \( \lambda, \mu \) are regular analytic in a neighborhood of \( \bar{s}_0 \), the functions \( \varphi_i \) defined by (8) satisfy (3), and (9) represents a minimal surface. If \( \mu(\bar{s}_0) = 0 \) the point \( \bar{x}(x_0, \beta_0) \) is singular. But, as Professor Morse remarked, admitting (usual orthogonal) transformations of the coordinates \( x, y, z \), we can represent the neighborhood of any (not only a regular) point \( \bar{x}(x_0, \beta_0) \) of a minimal surface in the form (9), and even so, that \( \lambda(\bar{s}_0) = 0 \) (The advantage of this last condition will appear later.)

By a change of the local typical parameters we can transform into the origin of the \( \bar{s} \)-plane. Let

\[
\begin{align*}
\varphi_i &= \bar{\varphi}(i, \bar{s}) \sum \zeta_{i \alpha}(i) \bar{s}^\alpha + \cdots, \quad i = 1, 2, 3, \\
\end{align*}
\]

the transformation
be the Taylor expansions of \( y_1, y_2, y_3 \). The relation \( \sum y_i^2 = 0 \) implies that the two least among the numbers \( m, m', m'' \) must be equal. If not, \( m = m' = m'' \) (\( \mu, \omega^{10} = \omega \)), the assertion is obvious, for if, for instance, we simply make the transformation

\[
\begin{align*}
x' &= x, \\
y' &= y, \\
z' &= z;
\end{align*}
\]

then the surface gets the form

\[
\begin{align*}
x' &= R \int y_2 \, dS \\
y' &= R \int y_3 \, dS \\
z' &= R \int y_3 \, dS;
\end{align*}
\]

where

\[
\begin{align*}
a_{nm}^2 - i a_{nm}^3 = 0 \quad \text{or} \quad a_{nm}^2 + i a_{nm}^3 \neq 0. 
\end{align*}
\]

We have either responding either

\[
\begin{align*}
\mu &= \frac{y_2 - i y_3}{2}, \\
\lambda &= \frac{y_2}{2\mu},
\end{align*}
\]

Set (10) where

\[
\begin{align*}
\mu &= \frac{y_2 + i y_3}{2}, \\
\lambda &= \frac{y_2}{2\mu}
\end{align*}
\]

Therefore the functions we get a representation of the form (9) with \( \lambda(0) = 0 \). If \( m = m' = m'' \) we must have

\[
\sum (a_n^i)^2 = 0
\]
on account of (3). Setting as the \( \gamma_i \) in (9), we then get the representation (9) with the vectors \( \gamma_i \) and the \( y_i \) are regular in a neighborhood of \( y_i \) we have \( \gamma_i = 0 \) and the \( \gamma_i \) we are the equations (11) and (12) show that the plane \( a_k = 0 \) is the uniquely determined plane in the surface : \( k = 0 \). Choosing \( a_k \) so that

\[
\begin{align*}
\sum a_i^2 &= k^2, \\
\sum \gamma_i^2 &= k^2, \\
\sum \gamma_i \cdot c_i &= 0
\end{align*}
\]

Choosing \( a_k \) so that

\[
\sum a_i^2 = k^2, \quad \sum \gamma_i \cdot c_i = \sum \gamma_i \cdot c_i = 0
\]

the transformation
The chief advantage of the representation (11) is that one recognizes immediately
\[ x_1 = \frac{1}{k} \left( a_1 x + b_1 y + c_1 z \right) \]
\[ y_1 = \frac{1}{k} \left( a_2 x + b_2 y + c_2 z \right) \]
\[ z_1 = \frac{1}{k} \left( a_3 x + b_3 y + c_3 z \right) \]
is a rigid motion. In this coordinate system the surface has the equations
\[
\begin{align*}
R \int \overline{y}_1 \, dS = 0 \\
R \int \overline{y}_2 \, dS = 0 \\
R \int \overline{y}_3 \, dS = 0
\end{align*}
\]
where
\[
\overline{y}_i = \frac{1}{k} \sum \ell_i x_i y_i = \frac{1}{k} \sum \ell_i x_i y_i + \cdots
\]
and \( x(0, 0) \), i.e., a branch point of order \( m_2 \geq 0 \), is a regular point if
\[
R \int \overline{y}_i \, dS = 0
\]
(11) makes \( m_2 \geq 0 \) and \( k \) a product of constant divisors. Then the functions \( \alpha \) represent the analogous equations with the functions \( \beta \), where
\[
\begin{align*}
\overline{y}_1 &= \frac{1}{k} \sum \ell_i x_i y_i = \frac{1}{k} \sum (a_i b_i + c_i d_i) y_i \\
\overline{y}_2 &= \frac{1}{k} \sum \ell_i x_i y_i = \frac{1}{k} \sum (a_i b_i + c_i d_i) y_i \\
\overline{y}_3 &= \frac{1}{k} \sum \ell_i x_i y_i = \frac{1}{k} \sum (a_i b_i + c_i d_i) y_i
\end{align*}
\]
The point \( x(0, 0) \) of the surface \( x \) is the representation (8) with
But (10) shows that
\[
\sum (a_i b_i + c_i d_i) = 0
\]
therefore the functions
\[
\mu = \frac{\overline{y}_1 - \overline{y}_2}{\overline{y}_2}, \quad \lambda = \frac{\overline{y}_1}{\overline{y}_2}
\]
are regular in a neighborhood of \( 0 \), we have \( \lambda(0) = 0 \), and the \( \overline{y}_i \) are expressed in terms of \( \lambda \) and \( \mu \) as the \( y_i \) in (8). We then get the representation (8) with the properties desired.

In which The equations (11) and (12) show that the plane \( z_1 = 0 \) is the uniquely determined tangential plane of the surface at \( x(0, 0) \), specially.

At a branch point a minimal surface has a uniquely determined tangential plane. The problem with the side condition that suitable parameters \( u, v \) may be found which are local typical parameters for each interior point of \( x \), i.e., \( x(0, 0) \), \( y(0, 0) \), \( z(0, 0) \) have to be independent for \( u^2 + v^2 < 1 \) and to satisfy the condition
\[
R \cdot 0, \quad P = 0.
\]
The chief advantage of the representation (11) is that one recognizes immediately if \( \overline{Z}(0, 0) \) is regular or a branch point of a certain order: since \( A(0) = 0 \) the point \( \overline{Z}(0, 0) \) can only be regular if \( \mu(0) \neq 0 \).

If

\[
\mathcal{M}(\xi) = \sum_{\nu} a_{\nu} \xi^{\nu}, \quad A = a_{\nu} S^{\nu}, \quad a_{\nu} \neq 0, \quad \nu < 0
\]

then \( \overline{Z}(0, 0) \) is a branch point of order \( m_1 \). But it must be remarked: if the numbers \( m_1, m_2, \ldots \) have the greatest common divisor \( r \), then the equations (9) represent the same point set as the analogous equations with the functions \( \mathcal{T}_1, \mathcal{T}_2 \), where

\[
\overline{Z} = A, \quad \overline{E} = \sum_{\nu} a_{\nu} S^{\nu}, \quad a_{\nu} \neq 0, \quad \nu < 0
\]

and it has been shown that \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) have not been solved, when they are modified by requiring that all equations (9) (instead of one) shall be found. But it has been shown that \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) in this modified form are and \( \overline{Z}(0, 0) \) is a branch point of order \( \frac{m}{n} \) for this minimal surface. Our equivalent, namely \( X \) and \( Z \), have always been found which for any minimal surface is this one covered \( r \) times. We therefore see:

The point \( \overline{Z}(0, 0) \) of the surface \( \overline{Z}(0, 0) \) is a (proper) branch point of order \( m_1 \geq 0 \), if

\[
\mathcal{M}(\xi) = \sum_{\nu} a_{\nu} \xi^{\nu}, \quad A = a_{\nu} S^{\nu}, \quad a_{\nu} \neq 0, \quad \nu < 0
\]

and the numbers \( m_1, m_2, \ldots \) are relatively prime.

That the solutions of \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) can be shown by the following simple example due to K. Menger:

It follows that for

\[
A = \sum_{\nu} a_{\nu} S^{\nu}, \quad A = a_{\nu} S^{\nu}, \quad a_{\nu} \neq 0, \quad \nu < 0
\]

we have a branch point of order \( m_1 \).

\[
X = \sum_{\nu} X^{\nu}, \quad Y = \sum_{\nu} Y^{\nu}
\]

We now state (following T. Rado, loc. cit., p. 32) the different forms for a surface of revolution \( x = f(y) \) the parallel circle \( y = \text{const} \), and the in which the problem of Plateau has been treated:

- **P.** Given \( y(\tau), y(\tau) \), how is the limit of curvature? The center of curvature \( f(y) \) is the point where the normal to the surface passes through the point of revolution.

- **P.** The same problem with the side-condition that suitable parameters \( u, v \) may lead to the condition that

\[
E = 0, \quad F = 0.
\]
The same as \( P_2 \), but requiring furthermore that functions \( \varphi, \gamma \) can be found such that equations of the form (7) hold for \( u^2 + v^2 < 1 \).

The same as \( P_3 \), but requiring that the surface has no singularities, i.e., that never \( \varphi = \gamma = 0 \) for \( u^2 + v^2 < 1 \).

When \( \mathcal{F} \) projects in a one-to-one fashion onto an \((x, y)\)-plane to obtain a minimal surface in the form \( z = f(x, y) \), the function \( z(x, y) \) being analytic inside the projection of \( \mathcal{F} \) and one-valued and continuous in and on the projection of \( \mathcal{F} \).

In the paper of Douglas dealt with in §2, problem \( P_2 \) (and therefore \( P_1 \)) is solved under the side condition, that the area of the surface is a minimum. Until problems \( P_1 \) and \( P_2 \) have not been solved, when they are modified by requiring that all minimal surfaces through \( \mathcal{F} \) (instead of one) shall be found. But it has been shown that \( P_1 \) and \( P_2 \) also in this modified form are equivalent, namely Rado and Beckenbach have proved that for any minimal surface through the intersection of the normal with \( x = 0 \) has length parameters \( u, v \) can be found which are locally typical in the neighborhood of each interior point.\(^{a)}\)


That the solution of \( P_1 \) and \( P_2 \) can be shown by the following simple example due to N. Wiener:

\[ (15) \]

We introduce cylinder coordinates \( r, \varphi, z \) related to \( x, y, z \) by

\[ x = r \cos \varphi, \quad y = r \sin \varphi, \quad z = z \]

For a surface of revolution \( r = f(z) \) the parallel circle \( z = \text{const.} \) and the meridians \( y = \text{const.} \) are the lines of curvature. The center of curvature \( \left( r_0, \varphi_0, z_0 \right) \) for a line of curvature being a parallel circle at a point \( \left( r_0 = f(z_0), \varphi_0, z_0 \right) \) is the point where the normal to the plane curve

\[ (13) \]

\[ \sigma = \sigma(z), \quad \gamma = \gamma(z) \]

For large \( z \) the distance of this center is less than (13). Taking \( \sigma \) and \( \gamma \) from
at \((x_0, y_0, z_0)\) intersects the \(z - ax + is\) (hence \(z = 0\)). Hence the two principal radii of curvature of \(r = f(z)\) at \((x_0, y_0, z_0)\) are equal if, and only if, the curvature of (15) at the point \((x_0, y_0, z_0)\) is equal to the distance of this point to \((0, 0, z = z_0)\). The only plane curves for which this relation holds everywhere are the circle and the catenary; the only minimal surface of revolution therefore is the catenary. Who does not know this geometrical property of the catenary verifies easily that the radius of curvature of

\[
\gamma(z) = \frac{a}{q} \left( e^{\frac{2z}{a}} + e^{-\frac{2z}{a}} \right)
\]

for \(z = z_0\) is equal to

\[
\frac{q}{4 \left( e^{\frac{2z}{a}} + e^{-\frac{2z}{a}} \right)} = \frac{\tau^2(z_0)}{a}
\]

and that the segment of the normal to (15) at \((0, \gamma(z_0), z_0)\) from this point to the intersection of the normal with \(z = 0\) has length

\[
\tau(z_0) \sqrt{1 + \gamma^2(z_0)} = \frac{\tau^2(z_0)}{a}
\]

The area of the part \(C_q\) of the catenoid (15) between the planes \(z = \pm q\) is equal to

\[
2 \pi \int_{-q}^{q} \sqrt{1 + \gamma^2(z_0)} \, dz = 2 \pi \int_{-q}^{q} \frac{a}{q} \left( e^{\frac{2z}{a}} + e^{-\frac{2z}{a}} \right) \, dz =
\]

\[
\frac{a^2 \pi}{4} \left( e^{\frac{2q}{a}} - e^{-\frac{2q}{a}} + \frac{2q^2}{a^2} \right)
\]

Each of the boundary circles of \(C_q\) bounds a plane domain \(k_1^q\), resp. \(k_2^q\), with the area

\[
\frac{a^2 \pi}{4} \left( e^{\frac{2q}{a}} + e^{-\frac{2q}{a}} + 2 \right)
\]

For large \(q\) the double of this number is less than (15). Taking out from
the piece $1 \leq r < \bar{r}$ is homeomorphic to the closed circular
disk and bounded by a piecewise analytic Jordan curve $\Gamma$. The two circular
disks $k_\bar{r}, k_r$ together with the part $1 < r < \bar{r}$ of the catenoid form a
surface homeomorphic to the circular disk and equally bounded by $\Gamma$. For
large $\bar{r}$ and small $\varepsilon$ this surface has a smaller area $\text{Area} \bar{r}$. Since we know
that there exists a minimal surface (of the type of the circular disk) bounded
by $\Gamma$ whose area is a minimum, we see that two different minimal surfaces
of the type of the circular disk pass through $\Gamma$. By a closer research one
can show that if $\gamma$ and $\varepsilon$ are properly chosen these two minimal surfaces
are different and have the same area, so that the solution of $P_1$ and $P_2$. even
under the restriction of minimizing area, is not unique. (*)

(*) If one admits as geometrically evident that one of the minimal surfaces for
small $\varepsilon$ must be very near to the second surface through $\Gamma$ which we con-
structed above, one can prove this easily, using the fact that up to a
certain $\bar{r}$ the first minimal surface through $\Gamma$ actually minimizes the
area.

The solution of problem $P_3$ would imply that there always exist mini-
mal surfaces through a given curve $\Gamma$ which have only branch points of order
greater than 1. Professor Morse stated that there are examples which show
that $P_3$ is not always solvable. ($P_4$ is then naturally not solvable either.)

It has long been known that $P_5$ is not always solvable, but different suffi-
cient conditions can be given under which $P_5$ has an (even unique) solution;
for instance, if the projection of $\Gamma$ on the $(x, y)$-plane is convex.

If we pass a point $(x, y)$ where $h(x, y)$ has a zero of type $\gamma > 1$, $h_\gamma$ has (as
we have just proved) a tangent different from the tangents of the other branches
of $h(x, y) = 0$ at $(x, y)$. If we require $h_\gamma$ to have a tangent everywhere
the continuation is uniquely determined, containing $h_\gamma$ in this manner.
Some of these last results can easily be derived from the following
in \( u^2 + v^2 < 1 \) as far as possible we must envelop the unit circle \( C \). We either

**Lemma:** If \( h(u, v) \) is continuous for \( u^2 + v^2 \leq 1 \) and harmonic for
\( u^2 + v^2 < 1 \); and if at \( (u_0, v_0) \) \( (u_0^2 + v_0^2 < 1) \) \( h \) and all its partial deriva-
tives up to the order \( n-1 \) but not all those of order \( n \) vanish (we then call
\( h \) `locally' \( C \)) if each of the branches \( b_1, \ldots, b_n \) is different from those of \( h \) it is \( n \) different.

**Proof.** We consider the curve \( h(u, v) = 0 \) at first in the neighbor-
hood of \( (u_0, v_0) \). We represent \( h \) in polar coordinates \( r, \gamma \)

\[
h = \frac{a_0}{r} + \sum_{n=1}^{\infty} \left( a_n \cos n \gamma + b_n \sin n \gamma \right).
\]

Since the expansion of \( h(u, v) \) in terms of \( u - u_0 \) and \( v - v_0 \) begins with term
of order \( u \), we must have

\[
a_0 = a_1 = \ldots = a_{n-1} = 0, \quad b_0 = \ldots = b_{n-1} = 0,
\]

and \( a_1 \neq 0 \) or \( b_1 \neq 0 \). Therefore \( h(u, v) = 0 \) has a representation of the form

\[
a_n \cos n \gamma + b_n \sin n \gamma + r \tilde{F}(r, \gamma) = 0
\]

where \( \tilde{F}(r, \gamma) \) is a Fourier series converging for small \( r \). Hence the tan-
gents to \( h(u, v) = 0 \) at \( (u_0, v_0) \) are given by

\[
a_n \cos n \gamma + b_n \sin n \gamma = 0.
\]

and we see that there are \( n \) different tangents, the angle between two conse-
cutive tangents being equal to \( \frac{\pi}{n} \). The curve \( h(u, v) = 0 \) has at \( (u_0, v_0) \) \( n \) different branches \( b_1 \ldots b_n \) with \( n \) different tangents. Let \( b_1^+ \) and \( b_1^- \) be the
the two parts of \( b_1 \) issuing from \( (u_0, v_0) \). We traverse \( b_1^+ \) starting at \( (u_0, v_0) \).

If we pass a point \( (\bar{u}, \bar{v}) \) where \( h(u, v) \) has a zero of type \( \gamma > 1 \), \( b_1^+ \) has (as
we have just proved) a tangent different from the tangents of the other branches
of \( h(u, v) = 0 \) at \( (\bar{u}, \bar{v}) \). If we require \( b_1 \) to have a tan-
gent everywhere the
continuation is therefore uniquely determined. Continuing \( b_1 \) in this manner
in \( u^2 + v^2 < 1 \) as far as possible we must approach the unit circle \( G \). We either
approach a definite point \( P \) on \( G \) or all points of a whole subarc \( G^1 \) of \( G \). Since

h is continuous in \( u^2 + v^2 \leq 1 \) we have in this latter case \( h = 0 \) on \( G^1 \) and the
harmonicity of \( h(u, v) \) in the harmonic sense \( f \) passes to \( h(u, v) \) we have \( h(u, v) = 0 \) and the
Lemma is proved. If each of the branches \( b_j^1 \), \( b_j^2 \) belongs a definite point
the very definition of the order of a branch point (cf. p. 43-44) forces that
all these \( 2n \) points must be different. Otherwise, if for instance \( P^1 = P \) the
all partial derivatives of \( h \) up to the order a vanish at \( (u, v) \), it is
for \( u^2 + v^2 \leq 1 \) and bounded. Moreover we have shown that we should have \( h = 0 \). Hence \( h \neq 0 \). Since

\[ h(r_j) = 0, \] the lemma is proved.

We now consider minimal surfaces of the type of the circular disk in
the three-space bounded by a simple closed Jordan curve \( \Gamma \) as defined on p. 45.
From (1) we deduce.

We suppose that \( s \) is represented in the form

\[ x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \]

where \( x(u, v), y(u, v), z(u, v) \) are harmonic for \( u^2 + v^2 < 1 \), and continuous
for \( u^2 + v^2 \leq 1 \) and satisfy the condition

\[ E = G, \quad F = 0, \]

according to (1) \( \Gamma \) must intersect \( \Gamma \) at least twice.

We know on account of the paper of Beckembach and Rado, quoted on p. 50,
that this assumption does not mean a loss of generality. Besides, the follow-
ing considerations be not using only local typical parameters. An
immediate consequence of our lemma is the theorem

\( \Gamma \) in at

(1) If \( \Gamma \) is plane through a branch point \( P = \mathcal{F}(u_0, v_0) \) of order \( u \)
on \( s \) (the regular case included as \( u = 0 \), then \( \Gamma \) intersects \( \Gamma \) in at least 4 distinct points.
The assumption of (2) is obviously satisfied if there exists a simply
\( 2(n+1) \) different points; if \( \Gamma \) is the tangent plane of \( s \) at \( P \) (this plane
exists according to our statement on p. 45), then \( \Gamma \) has at least \( 2(n+2) \) dif-
ferent intersections with \( \Gamma \).

Let \( \Gamma \) be the plane

\[ ax + by + cz + d = 0. \]
The intersections of $\Gamma$ with $\Gamma'$ are given by

$$h(u, v) = ax(u, v) + by(u, v) + cz(u, v) + d = 0,$$

where $a, b, c,$ and $d$ are constants determined by the orthogonality condition $u^2 + v^2 = 1$.

$h(u, v)$ is harmonic. Since $\Pi$ passes $\overline{xy}(u_o, v_o)$, we have $h(u_o, v_o) = 0$ and from the very definition of the order of a branch point (cf. pp. 43-44) follows that all partial derivatives of $h$ up to the order $n$ vanish at $(u_o, v_o)$. If $\Pi$ is the tangent plane of $s$ at $P$ in the sense used on p. 43 (which is easily seen to be equivalent to the definition used in algebraic geometry), we see that all partial derivatives of $h(u, v)$ up to the $(n+1)^{st}$ order vanish at $(u_o, v_o)$.

Hence in both cases our lemma proves the theorem (1).

From (1) we conclude:

(2) If a convex region $K$ contains $\Gamma$ the minimal surface $s$ is contained in $K$.

For if there were a point $P$ of $s$ exterior to $K$ a plane $\Pi$ through $P$ would exist having no corners or points with $K$ and therefore with $\Gamma$, whereas according to (1) $\Pi$ must intersect $\Gamma$ at least twice.

A further consequence of (1) is

(3) If there exists a straight line $g$ such that no plane through $g$ intersects $\Gamma$ in more than two different points, $s$ has no branch points.

For a plane through $g$ and a branch point would intersect $\Gamma$ in at least 4 distinct points.

The assumption of (3) is obviously satisfied if there exists a simply covered bounded and star-shaped central or parallel projection $\Pi'$ of $\Gamma$ upon some plane. For if $P'$ is the center of the star (or one such center if $P'$ can be regarded as star in different manners) we chose as the straight line $g$ of (13) the line connecting $P'$ with the center of projection (everywhere on the plane at infinity).
If \( \Gamma \) has a simply covered parallel projection \( \Gamma' \) on some plane which is convex, then the orthogonal projection \( \Gamma'' \) of \( \Gamma \) upon a plane \( \Pi \) orthogonal to the direction of projection is identical to the orthogonal projection of \( \Gamma' \) on \( \Pi'' \). Therefore \( \Gamma'' \) is convex to a one-to-one image of \( \Gamma \).

As proved before, \( S \) has no branch points. Furthermore there cannot exist any tangent plane of \( S \) perpendicular to \( \Pi'' \) for such a plane would intersect \( \Gamma'' \) and therefore \( \Gamma'' \) in at least 4 distinct points, in contradiction to the convexity of \( \Gamma'' \). Hence the orthogonal projection of \( S \) on \( \Pi'' \) sets forth a one-to-one continuous mapping of a neighborhood of each interior point \( (u_0, v_0) \) of \( S \) on a neighborhood of a point of \( \Pi'' \). Choosing \( \Pi'' \) as \( (x, y) \)-plane, we see that \( S \) can be represented locally in the form \( s = f(x, y) \).

The cylinder consisting of the straight lines perpendicular to \( \Pi'' \) at the points of \( \Gamma'' \) bounds a convex region, whose closure contains \( \Gamma \). From (2) it follows that \( S \) does not contain any point outside this cylinder. We can even say that this cylinder does not contain other points of \( S \) than those of \( \Gamma'' \). For a supporting plane of the cylinder at such a point would be a supporting plane of \( S \) at that point and therefore a tangent plane of \( S \) since

\[ s \text{ is analytic, whereas } s \text{ has no tangent planes perpendicular to } \Pi'' . \]

We designate the convex region bounded by \( \Gamma'' \) with \( S'' \). Now we can show that the orthogonal projection of \( S \) on \( S'' \) gives a one-to-one mapping of \( S \) on \( S'' \).

\[ \left( 1 + \left( \frac{\partial f}{\partial x} \right)^2 \right) \left( 1 + \left( \frac{\partial f}{\partial y} \right)^2 \right) \frac{\partial^2 s}{\partial x \partial y} = 0. \]

From the theory of partial differential equations one knows that the solution of this equation, if it exists, is unique for given values of \( f \) on the boundary of \( \Gamma'' \). We see
Let \( z = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad u^2 + v^2 \leq 1, \)
be the equation of \( S. \) The fact that \( S \) can be represented locally in the
form \( z = f(x, y) \) together with our last statement, means that by

\[
(*) \quad (u, v) \rightarrow (x(u, v), y(u, v)) \quad \text{in the } (u, v)-\text{plane}
\]

a neighborhood of each point \( x(u_0, v_0), y(u_0, v_0), u_0^2 + v_0^2 < 1 \) of \( S \) is mapped topologically
on a neighborhood of the point \( x(u_0, v_0), y(u_0, v_0) \) of \( S \). Furthermore by (*)
\( u^2 + v^2 = 1 \) is mapped topologically on \( S \). From a topological theorem \( *) \) it
follows that (*') sets forth a topological mapping of \( u^2 + v^2 \leq 1 \) on \( S + r \),
i.e., \( u, v \) are single-valued functions of \( x \) and \( y \). Putting these values of \( u, v \)
in the above equations of \( S \) we get a representation of \( S \) as a whole in the form

\[
z = f(x, y).
\]

Since \( S \) has no branch point and no tangent plane perpendicular to the \( (x, y)\)-
plane, \( z \) is analytic in the interior of \( S \). We have (cf. pp. 1-2)

\[
E = 1 + f_x^2, \quad F = f_x f_y, \quad G = 1 + f_y^2
\]

\[
W = EG - F^2 = 1 + f_x^2 + f_y^2
\]

\[
L = \frac{f_{xx}}{W}, \quad M = \frac{f_{xy}}{W}, \quad N = \frac{f_{yy}}{W}
\]

The condition

\[
\frac{1}{E_1} + \frac{1}{E_2} = \frac{EH - 2MF + GL}{W^2} = 0
\]

for minimal surfaces gives us

\[
(1 + f_x^2) f_{yy} - 2 f_x f_y f_{xy} + (1 + f_y^2) f_{xx} = 0.
\]

From the theory of partial differential equations one knows that the solution of
this equation, if it exists, is unique for given values of \( f \) on the boundary
of \( S \). We see:
Problem $P_5$ (p. 50) is solvable in a unique manner if the projection of $\Gamma$ on the $(x, y)$-plane occurring in the formulation of $P_5$ is convex.

4. The area of a surface $z = f(x, y)$

The next subject discussed in the seminar was the theory of the area of surfaces, which can be represented in the form

$$z = f(x, y).$$

The lectures followed closely the 6th chapter of the book: *Théorie de l'Intégrale* by Stanislaw Saks (Warszawa, 1933).