1. Potential Functions of their Generalized Derivatives

The notion of a "potential function $w(x^1, \ldots, x^n)$ of its generalized derivatives" was introduced by G. E. Evans 1). For simplicity of notation we shall assume that $w$ is a continuous function and that $\partial w/\partial x_i$ exists for each $x$ in a square $T = (a, b) \times (c, d)$.


restrict ourselves to two variables $x, y$, most proofs admitting immediate extensions to more variables. It will be convenient to use the following notation:

Let $G$ be an open plane region and $a > 0$. $G_a$ shall consist, then, of all those points $(x_0, y_0)$, for which the points

$$x_0 - a \leq x \leq x_0 + a$$

$$y_0 - a \leq y \leq y_0 + a$$

are in $G$. It is $G_a \subset G_b$ for $b > a$ and $G_a \rightarrow G$ for $a \rightarrow 0$.

Definition. A function $u(x, y)$, defined in an open region $G$, is called a potential function of its generalized derivatives (P. F. G. D.) if for each $\alpha > 0$ it satisfies the following conditions:

1) $u(x, y)$ is summable over $G$

2) two functions $v(x, y)$ and $w(x, y)$, summable over $G_\alpha$, exist such that

$$\int_a^b \int_c^d [u(x, y) - u(x_0, y_0)]dydx = \int_a^b \int_c^d w(x, y)dydx$$

for almost all rectangles $(a, b; c, d; a \leq x \leq b, c \leq y \leq d)$ interior to $G$, where "almost all" means: there exist sets $Z_x$ and $Z_y$ of measure 0 on the $x$-axis and $y$-axis respectively, such that (1) is true for all values $a, b, c, d$ with $a \notin Z_x, b \notin Z_x, c \notin Z_y, d \notin Z_y$.

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We recall the concept of a derivative in the sense of Lebesgue of a set function $P(E)^2$. One says $P(E)$ has a derivative at $P_0$ if for any sequence of measurable sets $E_n$ such that $E_n$ is contained in a square $\cap (P_0, \alpha_n)$ with center $P_0$, diameter $\sqrt{2} \cdot \alpha_n$, $\alpha_n \to 0$, and

$$\lim_{n \to \infty} \frac{m(E_n)}{\mu(P_0, \alpha_n)} \geq \eta > 0$$

(4) $\mu(P_0, \alpha_n)$

$\eta$ must be independent of $n$ but may vary with the sequence $\{E_n\}$, the existence of which is guaranteed by (3). Since the right side of (3) is continuous in $x_0$ for every $y$ from (4) that (3) holds everywhere, (3) is an immediate consequence of (3).

Lemma 2. Let $\nu \in \mathcal{M}_\mathbb{R}$ be summable on the open set $G$. Since in one case $\nu$ is summable of $G$ for each $\alpha > 0$, the set function $\nu \chi_E$ is differentiable almost everywhere in $G$ and, except for a new set of measure 0, its derivative is equal to $\nu$.

If $\left\{ \nu \chi_E \right\}$ is differentiable at $P_0 = (x_0, y_0)$ we call its derivative $D_xu(x_0, y_0)$ the generalized derivative of $u$ with respect to $x$ at $P_0$, and define $D_yu$ correspondingly. We aim at a characterization of the $P$, $F$, $G$, $D$. For this purpose we need the following two lemmas:

**Lemma 1.** If $u(x, y)$ is a $P$, $F$, $G$, $D$, then the function

$$u_h(x, y) = \frac{1}{4h^2} \int \int \frac{u(x, y)}{d} \ dx \ dy$$

is of class $C^1$ on each closed region where it is defined and one has (the subscript $x$ meaning partial differentiation with respect to $x$)

$$u_{h, k} = \frac{1}{4h^2} \int \int \frac{D_x u(x, y)}{d} \ dx \ dy$$

and correspondingly with respect to $y$.

**Proof.** For small $k$ one has

$$u_h(x + k, y) - u_h(x, y) = \frac{1}{4k^2} \int \int \frac{u(x + k, y)}{d} \ dx \ dy - \frac{1}{4k^2} \int \int \frac{u(x, y)}{d} \ dx \ dy$$
hence \[ \lim_{k \to 0} \frac{1}{k} \int_{x=0}^{x_1} \int_{y=0}^{y_4} u(x, y) \, dx \, dy = \int_{x=0}^{x_1} \int_{y=0}^{y_4} u(x, y) \, dx \, dy \]

for almost all \( x \) for which this question has a meaning and therefore
\[
\frac{\partial u}{\partial x} = \frac{1}{4h^2} \left( u(x+h, y) - u(x-h, y) \right) dy \]

where the region arising from \( D \) by the translation with co-ordinates
\[
\frac{\partial u}{\partial x} = \frac{1}{4h^2} \int_{y=0}^{y_4} \left( u(x+h, y) - u(x-h, y) \right) dy \]

The function \( u_h(x, y) \) is absolutely continuous in \( x \) on each closed segment interior to the set, where it is defined; hence

(4) \[ u_h(x, y) = \int_{x_0}^{x} \int_{y_0}^{y_4} \frac{1}{4h^2} \int_{y=0}^{y_4} \left( u(x+h, y) - u(x-h, y) \right) dy \, dx \]

Since the right side of (3) is continuous, we see from (4) that (3) holds everywhere. (2) is an immediate consequence of (3).

**Lemma 2.** Let \( |y(x, y)|^p \) be summable on the open set \( D \), and \( u_h(x, y) \) be defined in the same way as \( u_h \) (compare (2)). \( y_h(x, y) \) exists on \( D_h \). Then one has

(5) \[ \int \int_{D_h} |y_h(x, y)|^p \, dx \, dy \leq \int \int_{D_h} |y_h(x, y)|^p \, dx \, dy \]

(6) \[ \lim_{h \to 0} \int \int_{D_h} |y - y_h| \, dx \, dy = 0 \] for each \( x > 0 \) (\( x > h \))

**Proof.** As a consequence of the Hölder inequality we have:

(7) \[ \int \int_{D_h} \left( \left| \int \int_{D_h} \left| y_h(x, y) \right|^p \, dx \, dy \right|^q \right) \, dx \, dy \]

(8) \[ \int \int_{D_h} \left( \left| \int \int_{D_h} \left| y(x, y) \right|^p \, dx \, dy \right|^q \right) \, dx \, dy \]

Theorem 1. \( u(x, y) \) is in \( P_2, P_2, 0, 0 \) on \( D \) in \( x, y \) and only if, for each \( x > 0 \), there are \( \alpha, \beta > 0 \) such that

\[ \int \int_{D_h} \left| y_h(x, y) \right|^p \, dx \, dy = 0 \]
where \( \Delta ^{5,7} \) is the region arising from \( D_h \) by the translation with components \( \delta, \eta \) is in \( D \) since \( \delta, \eta \) vary between \(-h\) and \(h\).

In order to prove (6) let \( \{ \gamma_n(x, y) \} \) be a sequence of functions continuous on \( D \) and such that

\[
\lim_{n \to \infty} \int_D \| \gamma_n - \gamma_h \|^{-\frac{1}{p}} \, dx \, dy = 0
\]

For \( h < \infty \) one has, \( \gamma_{n, h} \) being defined in the same way as \( u_{n, h} \) and \( \gamma_h \)

\[
\int_D \| \gamma_n - \gamma_{n, h} \|^{-\frac{1}{p}} \, dx \, dy \leq \left( \int_D \| \gamma_n - \gamma_h \|^{-\frac{1}{p}} \, dx \, dy \right)^{\frac{1}{p}} + \left( \int_D \| \gamma_{n, h} - \gamma_h \|^{-\frac{1}{p}} \, dx \, dy \right)^{\frac{1}{p}}
\]

which proves (7). Putting \( y = D_u \) we have, by Lemma 1.

It follows from (5) that the right side of this inequality is at most

\[
\frac{2}{\mu} \left( \int_D \| \gamma_n - \gamma_h \|^{-\frac{1}{p}} \, dx \, dy \right)^{\frac{1}{p}} + \left( \int_D \| \gamma_{n, h} - \gamma_h \|^{-\frac{1}{p}} \, dx \, dy \right)^{\frac{1}{p}}
\]

Let now \( \varepsilon > 0 \) be given. We first choose \( \eta \) so large that the first term is less than \( \varepsilon \). Since \( \gamma_n \) is continuous \( h \) can be chosen so small that the second term becomes less than \( \varepsilon \).

We can give now a characterization of the \( F, E, G, D \) as follows:

**Theorem 1.** \( u(x, y) \) is a \( F, E, G, D \) on \( G \) if, and only if, for each \( \alpha > 0 \) there exists a sequence of functions \( u_n \) of class \( C^\alpha \) on \( \gamma_{n, h} \) such that

\[
\lim_{n \to \infty} \int_D \| u - u_n \|^{-\frac{1}{p}} \, dx \, dy = 0
\]

(7)

\[
\lim_{n \to \infty} \int_D \| u_n - u_{n, h} \|^{-\frac{1}{p}} \, dx \, dy = 0
\]

(8)

\[
\lim_{n \to \infty} \int_D \| u_{n, h} - u_{n, \gamma} \|^{-\frac{1}{p}} \, dx \, dy = 0
\]

for almost all \( \alpha > 0 \) and correspondingly for \( \gamma \). Since for each interval \((a, b, c, d)\) in \( G \) one has for each such interval...
When this is true one has

\[
\lim_{h \to 0} \int \int \left| u_{\alpha} - D_x u_{\alpha} \right| \, dx \, dy = 0
\]

(9)

\[
\lim_{h \to 0} \int \int \left| u_{\alpha} - D_y u_{\alpha} \right| \, dx \, dy = 0
\]

Especially, as \( u_{\alpha} \) one may take a sequence \( u_{h_n}(x) \) with \( h_n \to 0 \) and \( h_n \to 0 \).

**Proof.** A. **Necessary.** Let \( u(x, y) \) be a \( F_i \) \( F_j \) \( G_k \) \( D \). Then, according to Lemma 1, \( u_{h_n} \) is of class \( C_1 \) on \( f_{\alpha} \) for \( h \leq h_0 \), say. By Lemma 2 we have

1) for almost every \( (x, y) \) the function \( u_{h_n}(x, y) \) is absolutely continuous in \( x \) on each closed segment \( \gamma \) such that the points \( (x, y) \) are in \( \gamma \), and cor-

which proves (7). Putting \( y = D_x u \) we have, by Lemma 1,

\[
\frac{\partial u_{h_n}}{\partial x} = \sqrt{h}
\]

2) \( u_{h_n} = D_x u \) and \( u_{h_n} = D_y u \) everywhere.

3) \( \tilde{u}(x, y) \) is measurable to \( x(y) \) uniformly in \( y(x) \) in each closed interval \( (a, b; c, d) \) in \( \gamma \), i.e., the integrals

\[
\lim_{h \to 0} \int \int \left| u_{h_n} - D_x u \right| \, dx \, dy = 0
\]

This proves (9), and from (9) follows (8).

**B. Sufficient.** Suppose that there exists for each \( f_{\alpha} \) a sequence \( \{u_{n}\} \) of the type indicated. It follows from (8) that \( u_{\alpha} \) converges in the mean to a function \( v \). Designate for a fixed \( x \) by \( f_{\alpha} x \) the set of values \( y \) for which \( (x, y) \) lies in \( f_{\alpha} \). Then a subsequence \( \{u_{n}^1\} \) of \( \{u_{n}\} \) exists such that

\[
\lim_{n \to 0} \int \int \left| u - u_{n} \right| \, dy = 0
\]

for almost all \( x \) and correspondingly for \( y \). Since for each interval \( (a, b; c, d) \) in \( \gamma \)

\[
\lim_{n \to 0} \int \left[ u_{n}^1(x, y) - u_{n}^1(c, y) \right] \, dy = \lim_{n \to 0} \int \left[ u_{n}^1(x, y) - u_{n}^1(a, y) \right] \, dy
\]

one has for each such interval
for almost all \( x \). This sequence \( \{ h^1_\gamma \} \) corresponds to \( \gamma \). We then take a subsequence \( \{ h^{1}_{\gamma_n} \} \) of \( \{ h^{1}_{\gamma} \} \) with \( \gamma_n \) instead of \( \gamma \), similarly we define subsequences \( \{ h^{2}_{\gamma_n} \}, \{ h^{3}_{\gamma_n} \}, \ldots \) with respect to \( \gamma \) and the fact that \( u^{1}_{\gamma_n} \) tends to \( v \) in the mean shows that the left side vanishes for almost all intervals in \( \gamma \).

Theorem 2. A P, F, G, D, \( N \) is equivalent to a function \( \overline{u}(x, y) \)

("equivalent" means: \( u(x) = \overline{u}(x) \) except for a set of measure 0) with the following properties:

1) for almost every \( y_0 \), the function \( \overline{u}(x, y_0) \) is absolutely continuous in \( x \) on each closed segment \( a \leq x \leq b \) such that the points \( (x, y_0) \) are in \( G \), and correspondingly with respect to \( y \).

2) \( \overline{u} = D_u \) and \( \overline{u}_y = D_y \) almost everywhere.

3) \( \overline{u}(x, y) \) is summable in \( x (y) \) uniformly in \( y(x) \) in each closed interval \( (a, b, c, d) \) in \( G \), i.e. the integrals

\[
\int_a^b \left[ \int_c^d \overline{u}(x, y) \right] dy
\]

are bounded for \( a \leq x \leq b, c \leq y \leq d \).

Proof. Let \( \alpha \) tend monotonically to 0. The proof of Theorem 1 shows that

\[
\lim_{\alpha \to 0} \int_a^b \left[ \int_c^d \left[ \overline{u}(x, y) - u_h(x, y) \right] + \left[ D_u u - u_{h,x} \right] \right] dy dx = 0.
\]

Designating by \( \gamma \) the set of values \( x (y) \) such that \( (x, y) \) is in \( G \) we can choose a sequence \( \{ h^1_v \}, h_v \to 0 \), such that

\[
\lim_{v \to \infty} \int \left[ \int \left[ \overline{u}(x, y) - u_h(x, y) \right] + \left[ D_u u - u_{h,x} \right] \right] dy dx = 0
\]

for almost all \( y_0 \). We then choose a subsequence \( \{ h^1_v \} \) of \( \{ h_v \} \) such that
for almost all \( x \). This sequence \( \{ h^{1}_{v} \} \) corresponds to \( \alpha_{2} \). We then take a subsequence \( \{ h^{2}_{v} \} \) of \( \{ h^{1}_{v} \} \) satisfying (11) and (12) with \( \alpha_{2} \) instead of \( \alpha_{1} \); similarly we define successive subsequences \( \{ h^{3}_{v} \}, \{ h^{4}_{v} \}, \ldots \) with respect to \( \alpha_{3}, \alpha_{4}, \ldots \). For the sequence \( h^{v}_{v} \) (11) and (12) will hold with any \( \alpha_{v} \) for almost all \( x \) and \( y \).

Take now a fixed value \( y_{0} \) not in the exceptional set and let \( a \leq x \leq b \) be any closed segment such that the points \((x, y_{0})\) are in \( \mathcal{D} \). We then have

\[
\lim_{v \to \infty} \int_{\mathcal{D}} \left| \frac{1}{D_{x} u} d_{x} d_{y} \right| \, d \gamma = 0
\]

and \( D_{x} u(x, y_{0}) \) is summable over \( a \leq x \leq b \). We shall see in Lemma 3, following this proof, that it follows herefrom that the functions \( u(h^{m}_{v}, x, y_{0}) \) tend uniformly towards a function \( u(x, y_{0}) \). \( u(x, y_{0}) \) is absolutely continuous in \( x \), similarly one treats the case where the roles of \( x \) and \( y \) are exchanged.

In order to prove 3 it is sufficient to show that the functions

\[
u(h^{m}_{v}, x, y_{0}) = u_{m}(x, y_{0})
\]

converge in the mean in \( y \) uniformly for all \( x \), i.e. that to a given \( \varepsilon > 0 \) a number \( N(\varepsilon) \) can be found such that

\[
\lim_{v \to \infty} \int_{\mathcal{D}} \left| u_{m}(x, y_{0}) - u_{v}(x, y) \right| d x d y < \varepsilon
\]

for \( u_{m}, u_{v} \geq N(\varepsilon) \) and all \( x \) with \( a \leq x \leq b \). We first choose \( \eta \) so small that

\[
\int_{\mathcal{D}} \left| D_{x} u \right| d x d y < \frac{\varepsilon}{6}
\]
as soon as \( 0 \leq x_{1} - x_{0} \leq \eta \); then a finite set of values \( x_{0} \) for which \( u_{m}(x_{0}, y) \) converges uniformly so densely that each value \( x_{1} \) has at most distance \( \eta \) from a

\[
\int_{\mathcal{D}} \left| f_{m}(x, y) - f_{v}(x, y) \right| d x d y < \frac{\varepsilon}{9}
\]

for \( v > N \).
suitable value $x_0$. We then choose $N(\varepsilon)$ so large that

$$\int \int |D_x u - u_{x_0}| \, dx \, dy < \frac{\varepsilon}{6} \quad \text{for} \quad u > N(\varepsilon)$$

Furthermore,

$$\int \int \int |D_x u - u_{x_0}| \, dx \, dy < \frac{\varepsilon}{6} \quad \text{for} \quad u, u > N(\varepsilon)$$

and each of the finite number of values $x_0$. We then have for $0 \leq x_1 - x_0 < \varepsilon$

and $\varepsilon > N(\varepsilon)$, for instance $x_1 > x_0$.

$$\int \int |u_\varepsilon(x_1, y) - u(x_0, y)| \, dx \, dy < \frac{\varepsilon}{3} \quad \text{for} \quad u, u > N(\varepsilon)$$

(15)

Let now $x_1$ be arbitrary in $(a, b)$ and choose $x_0$ within the above set such that $0 \leq x_1 - x_0 < \varepsilon$. Then

$$\int \int |u_m(x_1, y) - u_m(x_0, y)| \, dx \, dy < \int \int |u_m(x_1, y) - u_m(x_0, y)| \, dx \, dy$$

and the right side is smaller than $\varepsilon$ for $u, u > N(\varepsilon)$ on behalf of (13) and (14).

We finally prove the lemma used in this proof:

**Lemma 3.** Let $f(x)$ and $\varphi(x)$ be summable over $a \leq x \leq b$, and let $f_0(x)$ be of class $C^\infty$. If then

$$\lim_{n \to \infty} \int_a^b |f(x) - f_0(x)| \, dx = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_a^b |\varphi(x) - f_0'(x)| \, dx = 0$$

the functions $f_0(x)$ converge uniformly in $a \leq x \leq b$.

**Proof.** It is

$$\int_{x_0}^{x_1} |f_0'(x)| \, dx - \int_{x_0}^{x_1} |\varphi(x)| \, dx \leq \int_{x_0}^{x_1} |f(x) - f_0(x)| \, dx$$

hence

$$|f_0(x) - f_n(x)| < \frac{\varepsilon}{3} (x_1 - x_0)$$

for $n \geq N$.
where \( \eta \), as indicated, only depends on \( |x_1 - x_0| \) and tends to 0 with \( |x_1 - x_0| \).

We say that \( u(x, y) \) is strictly absolutely continuous in the sense of Cesari if \( \lim_{m \to +\infty} \int \sum_{n=1}^{m} \left| f_n(x) - f_{n+1}(x) \right| \ dx \to 0 \).

Designating by \( S_{m, n, \eta} \) the set of points \( x \) for which

\[
1 \leq \left| f_n(x) - f_{n+1}(x) \right| \geq \eta
\]

one has for each fixed \( \eta > 0 \),

\[
\lim_{m \to +\infty} \int \sum_{n=1}^{m} \left( t_n(u(x, y)) \right) dx = 0
\]

where \( t_n(u(x, y)) \) designates the total variation of \( u(x, y) \) on \( x \in [y, y + \delta] \).

Let now \( \epsilon > 0 \) be given. We choose \( \delta \) such that \( (\delta \eta)(\delta) < \frac{\epsilon}{3} \). We then determine \( N \) such that

\[
\lim_{m \to +\infty} \int \sum_{n=1}^{m} \frac{\epsilon}{3} < 2 \int \sum_{n=1}^{N} \frac{\epsilon}{3} < \epsilon.
\]

To each point \( x \) a point \( x_1 \) of the complement of \( S_{m, n, \frac{\epsilon}{3}} \) can be found with distance less than \( \frac{\delta}{3} \) from \( x \). We then have for \( n, m > N \)

\[
\left| f_n(x) - f_{n+1}(x) \right| \leq \left| f_n(x) - f_n(x_1) \right| + \left| f_n(x_1) - f_{n+1}(x_1) \right| < \frac{\epsilon}{3},
\]

for the first and third members are smaller than \( \frac{\epsilon}{3} \) on account of (15), the second is smaller than \( \frac{\epsilon}{3} \) because \( x_1 \) is not in \( S_{m, n, \frac{\epsilon}{3}} \).

Theorem 5. If \( u(x, y) \) is A.C.T. on an open set \( \Omega \), then \( \frac{\partial u}{\partial x} \) and \( \frac{\partial u}{\partial y} \) exist almost everywhere on \( \Omega \) and are continuous on each closed subdomain of \( \Omega \).

Second is smaller than \( \frac{\epsilon}{3} \) because \( x_1 \) is not in \( S_{m, n, \frac{\epsilon}{3}} \).

\[
\left\{ \int \frac{\partial u}{\partial x} \ dx \right\} = \left\{ \int \frac{\partial u}{\partial y} \ dy \right\}.
\]

\[
\left\{ \int \frac{\partial u}{\partial x} \ dx \right\} = \left\{ \int \frac{\partial u}{\partial y} \ dy \right\}.
\]

Proof: It is sufficient to show that, for each ball \( B \) \( (a, \delta b, \delta d) \)

interior to \( \Omega \), \( u \), and \( u \), exist almost everywhere on \( \Omega \) and are continuous on \( \Omega \). If this is true, the equations (27) and (28) are immediate. We shall consider only \( u \), the proof for \( u \) being similar.
Definition. We first observe that the four real partial derivatives of \( u \) with respect to \( x \) and \( y \) exist in the sense of Tonelli on the rectangle \((a, b; c, d)\) if 1) \( u(x, y) \) is continuous on \((a, b; c, d)\); if 2) the four derivatives coincide and \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \) exist almost everywhere on \( G \). That is, in the existence of the and Abel integral

\[
\int_{a}^{b} \int_{c}^{d} \frac{\partial u}{\partial x}(x, y) \, dy \, dx < \infty
\]

and

\[
\int_{a}^{b} \int_{c}^{d} \frac{\partial u}{\partial y}(x, y) \, dx \, dy < \infty
\]

where \( \frac{\partial u}{\partial x}(x, y) \) designates the total variation of \( u(x, y) \) on \( c \leq y \leq d \) and \( \frac{\partial u}{\partial y}(x, y) \) the total variation of \( u(x, y) \) on \( a \leq x \leq b \). Since \( u(x, y) \) is absolutely continuous in \( x \) for almost all \( y \) on \((a, b)\) and \( u(x, y) \) is absolutely continuous in \( y \) for almost all \( x \) on \((a, d)\).

We call \( u(x, y) \) absolutely continuous in the sense of Tonelli (A.C.T.) in an open set \( G \), if it is strictly so on each closed rectangle \((a, b; c, d)\) in \( G \).

Concerning these functions, we first prove the following theorem:

Theorem 3. If \( u(x, y) \) is A.C.T. on an open set \( G \), then \( \frac{\partial u}{\partial x} \) and \( \frac{\partial u}{\partial y} \) exist almost everywhere on \( G \) and are summable over each bounded closed subset of \( G \). Furthermore the following equations hold for each cell \((a, c; b, d)\)

in \( G \):

\[
\int_{a}^{b} \int_{c}^{d} \frac{\partial u}{\partial x} \, dx \, dy = \int_{c}^{d} \left[ u(b, y) - u(a, y) \right] dy,
\]

(17)

\[
\int_{a}^{b} \int_{c}^{d} \frac{\partial u}{\partial y} \, dx \, dy = \int_{a}^{a} \left[ u(x, b) - u(x, a) \right] dx.
\]

(18)

Proof: It is sufficient to show that, for each cell \( R: (a, c; b, d) \)
in interior to \( G \), \( \frac{\partial u}{\partial x} \) and \( \frac{\partial u}{\partial y} \) exist almost everywhere on \( R \) and are summable on \( R \). If this is true, the equations (17) and (18) are immediate. We shall consider only \( \frac{\partial u}{\partial y} \) the proof for \( \frac{\partial u}{\partial x} \) being similar.
We first observe that the four Dini partial derivatives of \( u \) with respect to \( x \) are measurable on \( R \). Moreover, for almost every \( y_0 \), \( c \leq y_0 \leq d \), all four of these derivatives coincide for all \( x, a \leq x \leq b \), not on a set of measure zero. Hence the four derivatives coincide and \( u_x \) exists almost everywhere on \( R \). That \( u_x \) is summable follows from the existence of the repeated integral

\[
\int_a^b \left\{ \int_c^d \left\{ \frac{\partial u}{\partial x} \right\} \, dy \right\} \, dx = \int_a^b [u(x, y)] \, dy.
\]

Theorem 4. A necessary and sufficient condition that a function \( u(x, y) \) be A.C.T. on \( G \) is that it be a continuous potential function of its generalized derivatives on \( G \). A function \( u(x, y) \) to be of class \( D^1 \) or \( D_0 \) is invariant with respect to a change of variables in \( x \) and \( y \). The necessity follows immediately from Theorem 3. To prove the sufficiency let \( R = (a, b; c, d) \) be any rectangle interior to \( G \). So \( u(x, y) \) converges uniformly to \( u(x, y) \) on \( (a, b; c, d) \) and we have by Theorem 3 and Lemma 2:

\[
\lim_{k \to \infty} \int_R \left( \int u_k(x, y) - D_x u(x, y) \, dy \right) \, dx = 0
\]

As in the proof of Theorem 2 we can find a sequence of values \( \lambda_k > 0 \) tending to \( 0 \) such that

\[
\lim_{k \to \infty} \int_a^b u_k(x, y_0) - D_x u(x, y_0) \, dx = 0
\]

and

\[
\lim_{k \to \infty} \int_c^d u_k(x_0, y) - D_y u(x_0, y) \, dy = 0
\]

for all \( x, y \), and \( y_0 \) not in certain linear sets of measure 0. For each such \( x_0 (y_0) \) the function \( u(x_0, y) \) is absolutely continuous in \( y(x) \) and

\[ \frac{\partial^2 u(x, y)}{\partial x \partial y} = \frac{\partial u(x, y)}{\partial y} \frac{\partial^2 u(x, y)}{\partial x \partial y} = \frac{\partial u(x, y)}{\partial x} \frac{\partial^2 u(x, y)}{\partial y \partial x} \]

for almost all \( y(x) \). The measurability of the Dini partial derivatives of \( u(x, y) \) shows, then, in the usual manner that \( u_x \) and \( u_y \) exist almost everywhere on \( G \) and are equal to the corresponding generalized derivative; \( u_x \) and \( u_y \) are therefore summable over \( G \). This being true for each \( R \) in \( G \), \( u(x, y) \) is A.C.T. in \( G \).

**Definition.** We say \( u(x, y) \) is of "class \( D^\alpha \)" (\( \alpha \geq 1 \)) on an open set \( G \) if it is a P.F. of \( G \). \( D \) on \( G \) and if \( (D_x u) \) and \( (D_y u) \) are summable over each bounded closed set interior to \( G \). We say \( u(x, y) \) is of "class \( D^\alpha \)" on \( G \) if it is of class \( D^\alpha \) and furthermore continuous. Theorem 4 and its proof show that \( u(x, y) \) is of class \( D^\alpha \) on \( G \) if, and only if, it is A.C.T. on \( G \) and if \( (D_x u) \) and \( (D_y u) \) are summable over each bounded closed set in \( G \).

The property of a function \( u(x, y) \) to be of class \( D^\alpha \) or \( D_\infty \) is invariant under sufficiently smooth transformations of the coordinates. This is the content of \( \exists \). This theorem is due to C. C. Evans (see reference 1).

**Theorem 5.** If \( u(x, y) \) is of class \( D^\alpha \) (or \( D_\infty \)) on a region \( G \) and

\[ x = x(s, t), \quad y = y(s, t) \]

is a topological transformation of \( G \) into a region \( \Gamma \), and if \( x(s, t) \) and \( y(s, t) \) are of class \( \ell' \) and their Jacobian is different from 0 on \( \Gamma \), then \( u(x(s, t), y(s, t)) \) is of class \( D^\alpha \) on \( \Gamma \) and

\[ D_x u = D_s u \frac{\partial x}{\partial s} + D_t u \frac{\partial x}{\partial t} \]

\[ D_y u = D_s u \frac{\partial y}{\partial s} + D_t u \frac{\partial y}{\partial t} \]

(19)

at each point \((s, t)\) corresponding to a point \((x, y)\) where \( D_x u \) and \( D_y u \) both exist.

(Remark: It ought to be remarked that (19) holds in general only for the generalized derivative \( D_s, D_{ts}, D_x, D_y \) and not for the partial as has been shown by S. Saks 4).

Proof. Let $F$ be a bounded closed subset of $H$ corresponding to a condition that it be of class $D$. Let $u$ be of class $C^1$ on $E$ such that

$$\lim_{n \to \infty} \iint_{E} \left[\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial y} - D_u\right) \frac{\partial u}{\partial y} + \left(\frac{\partial u}{\partial x} - D_u\right) \frac{\partial u}{\partial x}\right] dx dy = 0$$

Putting

$$v = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}, \quad u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$$

we have on $F$ (the Jacobian being bounded and bounded away from zero) that

$$\lim_{n \to \infty} \iint_{F} \left[\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial y} - v\right) + \left(\frac{\partial u}{\partial y} - w\right)\right] ds dt = 0$$

But

$$\iint_{F} v ds dt = \iint_{F} \left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right] \frac{ds}{\partial s} = \iint_{E} \left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right] dx dy$$

If now $F_n$ is a normal sequence closing down to $(s_0, s_0)$, $(x_0, y_0) = (x(s_0, t_0), y(s_0, t_0))$ being a point where $D_u$ and $D_y u$ exist, then the image $E_n$ of $F_n$ is a normal sequence with respect to $(x_0, y_0)$, and

$$\lim_{n \to \infty} \frac{\iint_{F_n} v ds dt}{\iint_{E} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}} = \lim_{n \to \infty} \frac{\iint_{E} \left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right] dx dy}{\iint_{E} \left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right] dx dy}$$

exists, since $\frac{u}{\partial u} E_n$ tends to $\frac{u}{\partial u} E$, therefore the general derivatives of $u(x(s, t)), y(s, t))$ exist almost everywhere in $G$ and satisfy (19); the above formulae show furthermore that $(D_u)\alpha$ and $(D_y u)\alpha$ are summable of each bounded closed set in $H$.

The next theorem which we are going to prove characterizes the functions of class $D_\alpha$ and $D_\alpha$.

For the proof of the next theorem we need the following lemma:

**Lemma 4.** Let $f(x)$ and $\{f_n(x)\} \alpha (n = 1, 2, \ldots)$ be summable on the set $E$ and suppose that

$$\lim_{n \to \infty} \int_{E} |f - f_n| dx = 0, \quad \int_{E} |f_n(x)| dx \leq M, \quad M \text{ being independent of } n.$$ 

Then $\{f\} \alpha$ is summable over $E$ and

$$\int_{E} |f| dx \leq \lim_{n \to \infty} \int_{E} |f_n| dx.$$ 

In particular, if $f$ is summable over the open set $D$, a necessary and sufficient

\[ \text{(Fundamental)} \]
condition that it be of class $L^\alpha$ on $D$ is that

$$\int_{D_h} |f_n^\alpha| \, dx \leq M,$$

$M$ being independent of $n$.

**Proof.** Since a subsequence $\{f_{n_k}(x)\}^\alpha$ tends almost everywhere to $[f(x)]^\alpha$ on $E$ we have, according to Fatou's Lemma,

$$\int |f(x)|^\alpha \, dx \leq \liminf_{k \to \infty} \int |f_{n_k}(x)|^\alpha \, dx \leq \liminf_{k \to \infty} \int |f_{n_k}(x)| \, dx \leq M.$$

Putting

$$f_h^\alpha = f_n^\alpha$$

in $D_h$ we have

$$\lim_{h \to 0} \int |f - f_h^\alpha| \, dx = 0$$

in $D = D_h$, independently of $h$, provided $h$ is so small that $D_h$ contains $E$. From Lemma 1 it follows that $u$ is of class $L^\alpha$ on $F$. Let $h > 0$ one has

$$\int_n |u_n - u_h^\alpha| \, dx \leq M$$

and

$$\int |u_n - u_h^\alpha| \, dx \leq M$$

for all $h$;

hence, according to the first part of the proof, $[f]^\alpha$ is summable over $D$.

The proof of the next theorem being more difficult for more than two than for two variables, we formulate and prove it for $n$ variables:

**Theorem 6.** A necessary and sufficient condition that $u(x^1, \ldots, x^n)$

be of class $D^\alpha (\alpha \geq 1)$ on the bounded open set $G$ is (1) that $u$ be of class 

$L^\alpha$ on each closed subset of $G$, (2) that there exist functions $V_i(x^1, \ldots, x^n)$, 

$i = 1, \ldots, n$ of class $L^\alpha$ on each closed subset of $G$, and (3) that there exists a sequence of functions $\{u_p\}$ of class $C^1$ on the set $G$ such that for each closed subset $F$ of $G$ we have

$$\lim_{p \to \infty} \int_F \left[ |u_p - u|^\alpha + \sum_{i=1}^n |u_p x_i - V_i|^\alpha \right] \, dx = 0$$

A necessary and sufficient condition that $u$ be of class $D^\alpha$ on $G$ is that the above be true, the convergence of $\{u_p\}$ being uniform on each such $F$.

In particular, we may choose $u_p = u_{h_p}$ where $h_p \to 0$. 
Proof of Theorem 6: At Necessary. By Theorem 1 and Lemma 1, we know that
\[ u \cdot x^* = \frac{1}{(2\pi)^n} \int \sum_{x \in G} u(x) \int x \cdot u(x) \, dx = 0 \]
for every closed set in \( G \). It follows from Lemma 2 that
\[ \lim_{k \to 0} \sum_{x \in k} u(x) \cdot x = 0 \]
for every closed \( F \) in \( G \).

If we can prove that \( u \) is of class \( L^\infty \) then \( A \) will follow from Lemma 2. We take \( u = 3 \). Let \( R = \{ a, b, c, d, e, f \} \) be interior to \( G \). For \( h > 0 \) one has
\[ \int \int \int \left| \frac{u_h}{(x, y, z)} \right|^\infty \, dxdydz \leq K \]
independently of \( h \), provided \( h \) is so small that \( G_h \) contains \( R \). From Lemma 4 it follows that \( u \) is of class \( L^\infty \) on \( R \) if
\[ \int \int \int \left| \frac{u_h}{(x, y, z)} \right|^\infty \, dxdydz \leq k \]
independently of \( h \).

To prove (27) we choose \( h > 0 \) and \( x_0 \) with \( a \leq x_0 \leq b \) such that
\[ \int \int \int \left| \frac{u_h}{(x_0, y, z)} \right|^\infty \, dxdydz \leq \frac{M}{(b-a)(c-a)} \]
and then \( y_0 \) such that
\[ \int \left( u_h(x_0, y_0, z) + |u_h(x_0, y_0, z)|^\alpha \right) \, dz \leq \frac{M}{(b-a)(c-a)} \]
and \( z_0 \) such that
\[ \left| u_h(x_0, y_0, z_0) \right| \leq \frac{M}{(b-a)(c-a)} \]
Then we have
\[ \left| u_h(x_0, y_0, z) - u_h(x_0, y_0, z_0) \right| \leq \left| z - z_0 \right|^\alpha \int \left| u_h(x_0, y_0, z) \right|^\alpha \, dz \leq \frac{M}{(b-a)(c-a)} \left( \frac{f-c}{f-c} \right)^{\alpha - 1} \]
Consequently
\[ \left( u_h(x_0, y_0, z) \right) \leq 2^\alpha \left( \left( \frac{b-c}{b-c} \right)^{\alpha - 1} \left( \frac{f-c}{f-c} \right)^{\alpha - 1} \right) \]
for almost all \( y \) in \( C \) and \( v \) and \( w \) which we designate again by \( y \) and \( w \), we designate again by \( v \) and \( w \). Furthermore
Continuing this process one finds (21).

B. Sufficient. If the conditions of Theorem 6 are satisfied, (20) holds with \( \alpha = 1 \), and hence, by Theorem 1, \( u \) is of class \( D^\alpha \) with \( D_i u = v_i \) almost everywhere. Since the \( v_i \) are each of class \( L^\alpha \) on each closed set \( F \) in \( G \), the function \( D_i u \) are also. Therefore \( u \) is of class \( D^\alpha \).

The second part of Theorem 6 follows from this together with Theorem 4.

We now prove:

Theorem 7: Let \( u \) and \( v \) be of class \( D^\alpha \) and \( D^\beta \) respectively on \( G \), with \( \alpha + \beta = 1 \). Then, for almost rectangles \( R = (a, b; e, d) \) we have

\[
\int_R \left( \sum_{i,j} (D_i u D_j v - D_i v D_j u) \right) \, dx \, dy
\]

where \( \sum_{i,j} u_i D_j v \) means the integral of \( u(D_x v dx + D_y v dy) \overline{R^\infty} \). Before proving the \( k \)-th inequalities on \( P, F, G, D \) we establish three lemmas for the proofs:

Proof. Let \( D: (A,C,B,D) \) be a cell interior to \( G \). On account of

Theorem 6 there exist sequences \( \{u_n\} \) and \( \{v_n\} \) of class \( L^\alpha \) on \( D \) such that

\[
\lim_{n \to \infty} \int_D \left( u_n - u \right)^2 + \left( v_n - v \right)^2 + \left( D_u u \right)^2 + \left( D_v v \right)^2 \, dx \, dy = 0
\]

and

\[
\lim_{n \to \infty} \int_D \left( D_u u \right)^2 + \left( D_v v \right)^2 \, dx \, dy = 0.
\]

As several times before we choose subsequences of \( \{u_n\} \) and \( \{v_n\} \), which we designate again by \( \{u_n\} \) and \( \{v_n\} \) such that

\[
\int_D \left( u_n(x,y) - u(x,y) \right)^2 + \left( v_n(x,y) - v(x,y) \right)^2 + \left( D_u u_n(x,y) \right)^2 + \left( D_v v_n(x,y) \right)^2 \, dx \, dy = 0
\]

for almost all \( y \), and correspondingly for \( x \) instead of \( y \) and \( v \) instead of \( u \).
One has, for each \( u \), and each cell \( R \) in \( \mathcal{D} \), that

\[
\int_D u_n \, d\nu_n = \mathcal{L} \frac{\partial (u_n, v_n)}{\partial (x, y)} \, dx \, dy
\]

The Hölder inequality shows that the integral on the right side of (22) exists and that the right side of (24) tends to a limit. (23) shows (after a new application of the Hölder inequality) that the left side of (22) has a meaning for almost all rectangles in \( \mathcal{D} \), and that \( \int u d\nu \) is the limit of the left side of (24) for these rectangles. As \( \mathcal{D} \) may be written as the sum of a denumerable number of such cells \( \mathcal{D} \), our theorem follows.

**Remark.** In case \( u \) and \( v \) are continuous, the theorem can be given a more general form. It is then sufficient to assume that \( D, u, D, u, D, v \), \( D, v \) are of classes \( L^1, L^1, L^1, L^1 \) (on each bounded closed subset of \( \mathcal{D} \)) respectively, with \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

Before proving the final theorem on \( \mathcal{P}, \mathcal{F}, \mathcal{G}, \mathcal{D} \), we establish three lemmas which will be needed for the proof:

**Lemma 5.** Let \( \{ \gamma_n(x) \} \) be of class \( L^1(x \rightarrow 1) \) on a cell \( R \) with

\[
\int_R |\gamma_n| \, dx \, dy = A \quad (A \text{ independent of } n).
\]

Suppose furthermore that for each cell \( D \) in \( \mathcal{R} \), the limit

\[
\lim_{n \to \infty} \int_D \gamma_n \, dx
\]

exists. Then there exists a function \( \gamma \) of class \( L^1 \) on \( \mathcal{R} \), such that

\[
\int_R |\gamma| \, dx = \lim_{n \to \infty} \int_R |\gamma_n| \, dx
\]

Hence, by Lemma 4, \( \gamma \) is of class \( L^1 \) on \( \mathcal{R} \). This, together with (27), proves (26).

(25) follows from Lemma 3.

(26) follows from Lemma 4.

(28) follows from Lemma 5.
Proof: Define a function of intervals, \( \phi(D) \), by
\[
\phi(D) = \lim_{n \to \infty} \int_D \frac{1}{n} \, d\lambda
\]
Designate by \( \sum \) the sums that occur in infinitely many sets.

If \( D = \sum_{i=1}^{N} D_i \), the \( D_i \) being non-overlapping intervals, we have
\[
\phi(D) = \sum_{i=1}^{N} \phi(D_i)
\]

This lemma is due to Porath and its proof is immediate.

Now let \( D_1, \ldots, D_N \) be any set of non-overlapping cells in \( \mathbb{R}^d \). Then
\[
\sum_{i=1}^{N} |\phi(D_i)| = \lim_{n \to \infty} \sum_{i=1}^{N} \int \frac{1}{n} \, d\lambda \leq \lim_{n \to \infty} \int \frac{1}{n} \, d\lambda \leq \int \frac{1}{n} \, d\lambda
\]

This shows in the first place that the function \( \phi(D) \) is of bounded variation and it can therefore be extended to a function \( \phi(E) \) defined on all measurable sets in \( \mathbb{R}^d \). But this same inequality shows in the second place that \( \phi(E) \) is absolutely continuous. Therefore there exists a function \( y \) summable over \( \mathbb{R} \) such that
\[
\phi(D) = \int_D y \, d\lambda = \lim_{n \to \infty} \int_D \frac{1}{n} \, d\lambda
\]

for each cell \( D \) in \( \mathbb{R}^d \).

The functions \( \gamma_{\frac{1}{h}} \) are equicontinuous on \( R_h \) for each fixed \( h > 0 \) and they converge at each point to \( \gamma_1 \). They therefore converge uniformly to \( \gamma_1 \) and we have
\[
\int \frac{1}{h} \, \gamma_{\frac{1}{h}} \, d\lambda = \lim_{n \to \infty} \int \frac{1}{h} \, \gamma_{\frac{1}{h}} \, d\lambda \leq \int \frac{1}{h} \, \gamma_{\frac{1}{h}} \, d\lambda \leq M.
\]

Hence, by Lemma 4, \( \gamma \) is of class \( L_\infty \) on \( \mathbb{R} \). This, together with (27), proves (26). (25) follows from Lemma 2.

Lemma 6: Let the sets \( S_n \) be measurable and all situated in a bounded part of the space. Suppose furthermore
if neither \( f \) nor \( g \) is in a certain set \( Z_f \) of measure zero, \( u(\beta, \gamma, \delta) \)
and \( u(\alpha, \gamma, \delta) \) being of class \( D_p \) for \( \alpha \neq \beta \) and \( \delta \neq \gamma \). Definite by \( S^+ \) the set of those points which occur in infinitely many sets 
\( S_n \). Then

\[
\sum_{n=1}^{\infty} |\sum_{1=1}^{<\infty} S_n|
\]

This lemma is due to E. Borel and its proof is immediate:

\[
S^+ = \lim_{n \to \infty} \left( \sum_{n=1}^{\infty} S_n \right)
\]

The sets \( \sum_{n=1}^{\infty} S_n \) are bounded and monotonically decreasing. Hence

\[
\lim_{n \to \infty} \left( \sum_{n=1}^{\infty} S_n \right) = \sum_{n=1}^{\infty} \left( \sum_{n=1}^{\infty} S_n \right)
\]

Lemma 7: Let \( u(x^1, \ldots, x^n) \) be of class \( D_p \), \( p \geq 1 \), on the open set \( G \). Then, for each \( \ell \), \( 1 \leq \ell \leq n \), there exists a set \( Z \), of values of \( x^\ell \) which is of measure zero such that if \( x^\ell \) is not in \( Z \), then

\[
u(x^1, \ldots, x^\ell - 1, x^\ell, x^\ell + 1, \ldots, x^n) \]

is of class \( D_p \) in \( \sum_{n=1}^{\infty} S_n \).

Proof: It is evidently sufficient to prove this for each cell \( (x, y, z) \) interior to \( G \). For simplicity of notation, we assume \( n = 3 \), and let \( x^1 = x \), \( x^2 = y \), \( x^3 = z \), and \( R \) be the cell \((a, b, c; d, e, f)\). We shall then prove, as an illustration, that \( u(x, y, z) \) is of class \( D_p \) in \((x, y)\) for almost all \( z \).

From the definitions concerning functions of class \( D_p \), it follows that

\[
\int_{x^1}^{x^2} \int_{y^1}^{y^2} \int_{z^1}^{z^2} u(x, y, z) \, dx \, dy \, dz
\]

is non-decreasing in \( x \) and \( y \) and non-increasing in \( z \). Hence the convergence is uniform with respect to \( z \) if \( x \neq \alpha \) or \( y \neq \beta \). Consequently, if neither \( \beta \) nor \( \beta \) is in a certain set \( Z \) of measure zero, and

\[
\int_{x^1}^{x^2} \int_{y^1}^{y^2} D(x, y, z) \, dx \, dy \, dz = \int_{x^1}^{x^2} \int_{y^1}^{y^2} [u(x, y, z) - u(x, 0, z)] \, dx \, dy \, dz
\]
if neither \( f \) nor \( g \) is in a certain set \( Z \) of measure zero, \( u(\beta, y, z) \) and \( u(x, y, z) \) being of class \( L_p \) for \( c \leq y \leq d, e \leq z \leq f \) and \( u(x, \sigma, z) \) and \( u(x, \sigma, z) \) are of class \( L_p \) in \((x, z)\) for \( a \leq x \leq b, e \leq z \leq f \).

Now let \( x_0 \) be not in \( Z \). Define \( v_1(x, y, z) \) equal to \( D_u(x, y, z) \) when this exists and is positive, and equal to zero otherwise. Define \( v_2(x, y, z) \) equal \(-D_u(x, y, z)\) when this exists and is positive, and equal to zero otherwise. Define

\[
\begin{align*}
u_i(x, y, z) &= \int \int \nu_i(\delta, y, z) \, d\delta, \quad i = 1, 2, \quad \delta \in X_0 \end{align*}
\]

letting a \( u_i \) be zero if the integral is not defined. Clearly \( u_i(x, y, z) \) is of class \( L_p \) in \((y, z)\) for each \( x, i = 1, 2 \), and if \( \beta \) is not in \( Z \), \( u(\beta, y, z) \) is equivalent to the function

\[
u(x, y, z) = u_1(x, y, z) + u_2(\beta, y, z) - u_2(\beta, y, z)
\]

as is easily seen, using (27a).

Now if \( z \) is not in a certain set \( \delta \) of measure zero, we have

\[
\lim_{h \to 0} \frac{1}{2h} \int \int \nu_i(\delta, y, z) \, d\delta \, dy \, dz = \int \int \nu_i(\delta, y, z) \, d\delta \, dy \, dz
\]

for every \( x, \sigma, \) and \( \delta, a \leq x \leq b, c \leq \sigma \leq d, v_i(x, y, z, \delta) \) being of class \( L_p \) in \((x, y)\) in \((a, c; b, d)\). Also if \( z \) is not in a set \( \delta \) of measure zero

\[
\lim_{h \to 0} \frac{1}{2h} \int \int u_i(x, y, z) \, dy \, dz = \int u_i(x, y, z) \, dy, \quad i = 1, 2.
\]

Then a subsequence of \( z \) may be chosen which converges in the mean of order for every \( (f, f) \) and \( x_n \) rational \( a \leq x_n \leq b \). But now, each \( u_i \) is monotone non-decreasing and absolutely continuous in \( x \) for almost all \((y, z)\) and \( v_i \geq 0 \) so that the functions on both sides in (27c) and (27d) are all monotone non-decreasing in \( x \). Hence the convergence as \( h \to 0 \) is uniform with respect to \( x \). Combining all of these results we see that if \( z \) is not in a certain...
set $\tilde{Z}$ and $\alpha$ and $\beta$ are not in the set $Z_x$ (independent of $\tilde{Z}$), we have

$$\left(27e\right) \quad \left(\sum \left[\frac{\mu_{0}^{(n)}}{\gamma_{0}^{(n)}} \cdot u_{\gamma_{0}^{(n)}} \right] \right) = \int \int \int D_{x} u_{\gamma_{0}^{(n)}} \, dx \, dy \, dz$$

with respect to $\gamma_{0}^{(n)}$. For $n = 1$ the theorem is known, since the $u_{\gamma_{0}^{(n)}}$ are $\alpha, \beta$, and for every $(\gamma_{0}^{(n)}, \gamma_{0}^{(n)})$ and $D_{x} u_{(x, y, z_{0})}$ is of class $L^{\infty}(x, y)$. A similar proof establishes that, by selecting from $\gamma_{0}^{(n)}$ a further set of measure zero, the limiting function $u$ is bounded. This may be considered as a special case of (29) with $\gamma = 1 = l = 0$ at $\gamma_{0}^{(n)}$, we may also arrange it so that if $\gamma$ and $\gamma_{0}^{(n)}$ are not in $\gamma_{0}^{(n)}$, we have

$$\left(27f\right) \quad \left(\sum \left[\frac{\mu_{0}^{(n)}}{\gamma_{0}^{(n)}} \cdot u_{\gamma_{0}^{(n)}} \right] \right) = \int \int \int D_{y} u_{\gamma_{0}^{(n)}} \, dx \, dy \, dz$$

for every $(\alpha, \beta)$ and $D_{y} u_{(x, y, z_{0})}$ is of class $L^{\infty}(x, y)$. Thus $u(x, y, z_{0})$ is of class $L^{\infty}((a, c), (b, d))$.

**Remark:** We stated this lemma for $n = 2$ is a consequence of Theorem 2.

**Theorem 3.** Let $\left\{u_{p_{k}}\right\}$ be a sequence of functions of class $L^{\infty}$ defined on a cell $R \equiv (a, b)$, $\alpha > 1$. Suppose that there exists an $M$ independent of $p_{k}$ such that

$$\int_{a}^{b} \int_{a}^{b} \left(\sum_{i=1}^{\infty} \left|u_{p_{k}}^{(i)} \right| \right)^{\alpha} \, dx \, dy < \infty$$

Then a subsequence $\left\{u_{p_{k}}\right\}$ may be chosen which converges in the mean of order $\alpha$ to a function $u$ which is $L^{\infty}$ of class $L^{\infty}$ and for which

$$\left(28\right) \quad \int_{R}^{R} \left|u_{p_{k}}^{(i)} \right| \, dx \, dy < \infty$$

Choose the point $x_{0}$ and a subsequence $p_{k}$ of $p_{k}$ such that $x_{0}^{(i)}$ is a point of $R$ and for which

Then the subsequence may be chosen so that for each $\xi_{0}$, $i \leq \xi_{0} \leq \beta$, we have

$$\left(29\right) \quad \int_{a}^{b} \int_{a}^{b} \left(\sum_{i=1}^{\infty} \left|u_{p_{k}}^{(i)} \right| \right)^{\alpha} \, dx \, dy < \infty$$

In fact the subsequence may be chosen so that for each $\xi_{0}$, $i \leq \xi_{0} \leq \beta$, we have

$$\left(30\right) \quad \int_{a}^{b} \int_{a}^{b} \left(\sum_{i=1}^{\infty} \left|u_{p_{k}}^{(i)} \right| \right)^{\alpha} \, dx \, dy < \infty$$
then \( x \rightarrow 0 \) will belong to \( S \) for \( v \geq a \) or \( x^r = x^{r+1} \) will belong to \( S \).

\[ \text{Proof. } \text{We shall prove (29) first. We proceed by induction with respect to } n. \text{ For } n = 1 \text{ the theorem is known, since the } u \text{ are } A_c \text{ and equicontinuous on } (a, b), \text{ as is seen by the Hölder inequality, and uniformly bounded. This may be considered as a special case of (29) with } i = n = 1. \text{ Suppose the whole theorem on convergence has been proved for all values for each } m, \text{ let } (a^r, b^r) \text{ be divided into } 2^m \text{ equal closed intervals } x_{m,i}^r \leq x_{m,i}^r \leq x_{m,i}^r \text{ such that } \]

\[ \ell \sum \int \left| u(x, i \cdots, x, x_{m,i}^r) \right| dx \leq \sum \left| u(x, i \cdots, x, x_{m,i}^r) \right| dx \leq \frac{2}{\varepsilon} \theta + 2. \]

We have \( m^{2m} \geq 2^{m-1} (b^r - a^r) - a^r \). We omit in \( S_{P^2}^m \) those points of which at least one coordinate is in one of the exceptional sets of measure 0 occurring in Lemma 7. We may get in this way the set \( S_{P^2}^m \) having the same measure as \( S_{P^2}^m \). Let \( m \) and \( i \) be fixed and let \( S_{P^2}^m \) be any subsequence of \( S_{P^2}^m \) then we know from Lemma 6 that there is a point which occurs in infinitely many of the \( S_{P^2}^m \). Let \( (m_1, i_1), (m_2, i_2), \ldots \) be any ordering of the admissible pairs \((m, i)\). Put \( S_{P^2}^m \rightarrow S_{P^2}^m \) and let \( x_1^r \) belong to \( S_{P^2}^m \) such that \( x_1^r \) belongs to all \( S_{P^2}^m \). We continue this process choosing generally a point \( x_{m+1}^r \) and a subsequence \( \{ p_{m+1}^r \} \) of \( \{ p_{m}^r \} \) such that \( x_{m+1}^r \) belongs to all \( S_{P^2}^m \).
Then \( x_i^{+1} \) will belong to \( S'_v \) for \( v > s \) or \( x_s^{+1} = x_{m,s}^{+1} \) will belong to \( S'_{\rho, i, s} \) for \( f \geq k(m, s, i, s) \).

Now choose \( \epsilon > 0 \) and let \( x_0^{+1} \) be any value with
\[
a^{+1} < x_0^{+1} < b^{+1}.
\]
Then for each \( p, [x, a, b, \ldots, x, a] \), \( (b^1, \ldots, b^r) \), we have
\[
\left[ \sum \left[ \int_{\rho} (x, x_i^{+1}) \right] d\rho \right]^{\frac{1}{p}} < \int \left[ \sum \int D \rho \left( x_i^{+1}, x^{+1} \right) d\rho \right]^{\frac{1}{p}} \leq \int \left[ \sum \int D \rho \left( x_i^{+1}, x^{+1} \right) d\rho \right]^{\frac{1}{p}} \leq \left[ \int \left( 2 - \frac{m - m'}{n} \right) \right]^{\frac{1}{2}} \leq \frac{\epsilon}{3}
\]
if \( m \) is large enough and \( x_{m,i}^{+1} \) denote the one of the above points which is nearest to \( x_0^{+1} \). Hence, choose such an \( m \) and \( V(m) \) such that for \( k, \ell > V(m) \)
\[
\left[ \sum \left[ \int_{\rho} (x, x_i^{+1}) \right] d\rho \right]^{\frac{1}{p}} < \frac{\epsilon}{3}, \quad j = 1, \ldots, \ell
\]
By Lemmas 5 there is a function \( v_0 \) so that the formulas (1) hold for every
interval \( 0 \) for \( u \) and \( y \) and such that \( v \) is of class 1 and

This is possible on account of the hypothesis of induction. We then have for
\( k, \ell > V(m) \)
\[
\left[ \sum \left[ \int_{\rho} (x, x_i^{+1}) \right] d\rho \right]^{\frac{1}{p}} < \frac{\epsilon}{3}, \quad j = 1, \ldots, \ell
\]

Thus (29) is proved for \( n = r^{+1}, i = r^{+1} \). By choosing further subsequences of \( \{ p, k \} \) we may arrive at one for which (29) is true for \( i = 1, \ldots, r^{+1} \). Then, the remainder of the statements in the

theorem follow immediately.
It is clear that the convergence in the mean of order $\alpha$ for the whole of $R$ follows from this. Let $u$ be a function which is the limit in the mean of $\{u_{P_k}\}$ and choose $C$. There is a function $u_0$, equivalent to $u$, such that

$$\{u_{P_k} (x_1^1, \ldots, x_1^{\varepsilon-1}, x_0^1, x_0^2, x_0^3, \ldots, x_0^n) \}$$

converges in the mean of order $\alpha$ in $\{x_1^1, \ldots, x_1^{\varepsilon-1}, x_0^1, x_0^2, x_0^3, \ldots, x_0^n\}$ to $u_0 (x_1^1, \ldots, x_1^{\varepsilon-1}, x_0^1, x_0^2, x_0^3, \ldots, x_0^n)$ for each $x_0^1, a \leq x_0^1 \leq b$. Therefore we have for all rectangles $D = (\alpha, \beta) \subset R$

$$\int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \cdots \int_{\alpha}^{\beta} \left[ \sum_{P_k} \left\{ u (x_1^1, \ldots, x_1^{\varepsilon-1}, x_0^1, x_0^2, x_0^3, \ldots, x_0^n) - u_0 (x_1^1, \ldots, x_1^{\varepsilon-1}, x_0^1, x_0^2, x_0^3, \ldots, x_0^n) \right\} \right] d\alpha d\beta \cdots d\alpha.$$

Theorem 1: If $K$ is any aggregate of convex sets, then

$$\int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \cdots \int_{\alpha}^{\beta} \left[ \sum_{P_k} \left\{ u (x_1^1, \ldots, x_1^{\varepsilon-1}, x_0^1, x_0^2, x_0^3, \ldots, x_0^n) - u_0 (x_1^1, \ldots, x_1^{\varepsilon-1}, x_0^1, x_0^2, x_0^3, \ldots, x_0^n) \right\} \right] d\alpha d\beta \cdots d\alpha.$$

Theorem 2: The closure of a convex set is convex.

The next theorem is also about convex.

By Lemma 5 there exists a function $v_0$ so that the formulas (1) hold for every interval $D$ for $u_0$ and $v_0$, and such that $v_0$ is of class $L_\infty$. Separating the $\int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \cdots \int_{\alpha}^{\beta} \left[ \sum_{P_k} \left\{ u (x_1^1, \ldots, x_1^{\varepsilon-1}, x_0^1, x_0^2, x_0^3, \ldots, x_0^n) - u_0 (x_1^1, \ldots, x_1^{\varepsilon-1}, x_0^1, x_0^2, x_0^3, \ldots, x_0^n) \right\} \right] d\alpha d\beta \cdots d\alpha.$

For every $\varepsilon_0 > 0$, there exist two points $A_1 < B_1$ and $A_2 < B_2$ such that

$$\int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \cdots \int_{\alpha}^{\beta} \left[ \sum_{P_k} \left\{ u (x_1^1, \ldots, x_1^{\varepsilon-1}, x_0^1, x_0^2, x_0^3, \ldots, x_0^n) - u_0 (x_1^1, \ldots, x_1^{\varepsilon-1}, x_0^1, x_0^2, x_0^3, \ldots, x_0^n) \right\} \right] d\alpha d\beta \cdots d\alpha.$$

are both continuous in $x_0$. From this, the remainder of the statements in the theorem follow immediately.
Let $F$ be an interior. 

**2. Convex sets and functions**

We shall not attempt a systematic development of the properties of convex functions, but shall give here only those properties which will be needed later on or which are especially fundamental.

**Definition.** A set $S$ of points (in the $n$-dimensional euclidean space) is called convex if, whenever the points $P_1$ and $P_2$ belong to $S$, the straight-line segment $P_1P_2$ belongs to $S$.

The statements of the following theorem are immediate consequences of this definition.

**Theorem 1.** If $\{S\}$ is any aggregate of convex sets, then $\bigcap S$ is either empty or convex. If $s_1 \leq s_2 \leq s_3 \leq \ldots$ and $s_1', s_2', \ldots$ are convex then is the boundary of $\bigcap S$. A hyperplane separating no two points of $\bigcap S$ is called a bounding hyperplane. An equivalent definition is: $\bigcap S$ is a bounding hyperplane if it contains $\bigcap S$.

The closure of a convex set is convex.

The next theorem is also almost trivial:

**Theorem 2.** Let $S_1$ be a closed convex set, $S_2$ a bounded closed convex set, and $S_1S_2 = 0$. Then there exists a hyperplane separating $S_1$ from $S_2$.

We designate by $\epsilon (\mu, \nu)$ the distance of the two sets $\mu$ and $\nu$, i.e.,

$$\epsilon (\mu, \nu) = \sqrt{\langle L \cdot \mu - \epsilon (x, y) \rangle}$$

where $\epsilon (x, y)$ is the euclidean distance of $x$ and $y$. Under the assumptions of the theorem there exist two points $A_1 < S_1$ and $A_2 < S_2$ such that

$$\epsilon (A_1, A_2) = \epsilon (S_1, S_2) > 0$$
Let \( P \) be an interior point of \( A_1A_2 \). Then one has

\[
\varepsilon(P, S_1) = \varepsilon(P, A_1) \quad \varepsilon(P, S_2) = \varepsilon(P, A_2)
\]

wholly contained in a hyperplane. In the first case each point of \( S \) is

for if \( e(P, B) < e(P, A_1) \), one would have an accumulation point of its interior points. More precisely, if \( S \) is an

interior point of \( S \) and if \( e(P, A_1) \), then all points of \( S \) except

Therefore the hyperplane \( \Pi \) perpendicular to \( A_1A_2 \) at \( P \) cannot contain a point

possibly \( P \) are interior points of \( S \). of either \( S_1 \) or \( S_2 \). For \( \varepsilon(P, S_1) \), the segment \( \overline{A_1} \) would belong to \( S_1 \)

and \( S_1 \) if \( e(P, A_1) \) is a right angle, a point \( Q \) if \( e(P, A_1) \) would exist with

If it does not lie in a hyperplane, it contains at points not in it are points.

If \( \varepsilon(P, A_1) \), the segment \( \overline{A_1} \) would belong to \( S_1 \)

The whole simplex with these points as vertices then belongs to \( S \) in amount of

\( \Pi \) therefore separates \( S_1 \) from \( S_2 \). We remark that \( \Pi \) has positive distance

from \( S_2 \).

Let \( S \) contain an interior point \( Q \) and let \( P \) be any point of \( S \). There

Definitions: We generally designate the closure of a set \( \mu \) by \( \overline{\mu} \)

and call the set \( \mu^* = \mu \cup \overline{\mu} \), where \( \overline{\mu} \) is the complement of \( \mu \)

and the points of this sphere to \( P \) form a set \( Q \) belonging to \( S \), and each point of

the boundary of \( \mu \). A hyperplane \( \Pi \) separating two points of \( \mu \) is

\( Q \) if \( e(P, A_1) \) is an interior point of \( P \) and therefore of \( S \), which proves the

called a bounding plane for \( \mu \); an equivalent definition is: \( \Pi \) is a bounding

plane if one of the two closed halfspaces bounded by \( \Pi \) wholly contains \( \mu \).

Obviously any converging sequence of bounding planes for \( \mu \) tends to a

of \( \mu \), \( \Pi \) is supporting plane of \( \mu \) at each point of \( \mu \). This explains

bounding plane for \( \mu \). A bounding plane for \( \mu \) containing a point \( P \) of

why we restrict ourselves in the following theorems to sets with interior

the boundary of \( \mu \) is called a supporting plane of \( \mu \) at \( P \).

As a consequence of the last theorem we have:

\[ \text{Theorem } 4. \quad \text{A closed set } S \text{ with interior points is convex if } \overline{S} \text{ and } S \text{ is closed.} \]

Corollary: If \( S \) is a convex set and the point \( P \) is not in the

\[ \text{closure } S \text{ of } S, \text{ then there exists a } \overline{S} \text{ bounding plane for } S \text{ through } P. \]

For applying Theorem 2 to \( S = S_1 \), \( P = S_2 \), we see that a plane

require that \( S \) be closed. For the sufficiency it is however, essential, the

exists separating \( P \) from \( S_1 \) the plane parallel to \( \Pi \) through \( P \) meets the

requirements of the corollary.

Proof. A line \( \Pi \) is convex and \( P \) a boundary point of \( S \). Let \( Q \)

be an interior point of \( S \). The preceding theorem shows that no point \( X \) on
Theorem 3. A convex set $S$ either contains interior points or is wholly contained in a hyperplane. In the first case each point of $S$ is an accumulation point of its interior points. More precisely: if $Q$ is an interior point of $S$, and $P$ any point of $S$, then all points of $PQ$ except possibly $P$ are interior points of $S$.

If $S$ contains more than one point it contains infinitely many points. If it does not lie in a hyperplane, it contains $n+1$ points not in a hyperplane. The whole simplex with these points as vertices then belongs to $S$ on account of the convexity.

Let $S$ contain an interior point $Q$ and let $P$ be any point of $S$. There exists an open sphere with center $Q$ contained in $S$, the segments connecting the points of this sphere to $P$ form a set $T$ belonging to $S$, and each point of $PQ$ except $P$ is an interior point of $T$ and therefore of $S$, which proves the theorem.

If $P$ is any set in the hyperplane $\Pi$, then $P$ is the boundary of $\mu$ and $\Pi$ is supporting plane of $\mu$ at each point of $P$. This explains why we restrict ourselves in the following theorem to sets with interior points:

Theorem 4. A closed set $S$ with interior points is convex if, and only if, there exists a supporting plane of $S$ at each boundary point of $S$.

Remark. The proof of the necessity of the condition does not require that $S$ be closed. For the sufficiency it is, however, essential, the theorem not being true otherwise.

Proof. A. Assume $S$ is convex and $P$ a boundary point of $S$. Let $Q$ be an interior point of $S$. The preceding theorem shows that no point $R$ on
the prolongation of QP beyond P can belong to S. For R cannot belong to S either. Otherwise P would be an interior point of S, a whole sphere around P would belong to S, and S would have to be everywhere dense in this sphere.

Theorem 2: Let S be a closed convex set with interior points and L a linear subspace of S. Then, for any point C of S, there exists a bounding hyperplane P through L such that S contains no point of P and P is contained in L.

Proof: We first take the case n = m = 2. Then m = 1 and

outside S with P containing S. According to the corollary to Theorem 2, there exists a bounding plane P through P. Choose a subsequence of \{ \Pi_n \} such that \Pi_n \to \Pi through P. As we have remarked, \Pi must also be a bounding plane of S, and since \Pi \supset P, \Pi is a supporting plane of S at P.

B. Assume that there exists a supporting plane of S at each boundary point of S, S being closed. If two points P_1, P_2 in S should exist such that P_1 and P_2 belong to S but an interior point C of P_1P_2 does not, one could connect C to an interior point D of S, which is not on the straight line through P_1 and P_2. Then a boundary point E of S would lie in the interior of CED. According to our assumption, a supporting plane \Pi of S at E would exist. Since D is an interior point of S, the plane \Pi cannot pass through D, but then \Pi would separate two of the points P_1, P_2, D.

That B cannot be proved without the assumption that S is closed is shown by the following example: Take a closed tetrahedron in 3-space and modify the preceding proof can be applied. If k > 2 we consider an interior point, leave out a boundary point which is no vertex. The remaining set would still have the property that a supporting plane exists at each boundary point of L_{k-1} and S considered as a convex set of n-k-2 dimensions. The in-k point, but the set is no longer convex.

In the higher dimensional cases it is clear that for k = 2 the

point, which is a bounding plane for S in-k and therefore
We now prove a theorem which generalizes the necessary condition for convexity of the last theorem:

Theorem 5. Let $S$ be a closed convex set with interior points and $L_{n-k}$ a linear subspace of dimension $0 < n-k \leq n-2$ which contains no interior point of $S$. Then a bounding hyperplane for $S$ through $L_{n-k}$ exists.

Proof. We first take the case $n = 3 \neq 2$. Then $n-k = 1$ and $L_{n-k} = L_1$ is a straight line. We consider the halfplanes bounded by $L_1$.

The subset $\alpha$ of those halfplanes which contain interior points of $S$ has the following properties: (1) With any halfplane the neighboring halfplanes are in $\alpha$; (2) $\alpha$ contains no pair of opposite halfplanes, (3) two different halfplanes in $\alpha$ therefore always determine uniquely a convex angular space bounded by them. This whole space belongs to $\alpha$. (2) and (3) follow from the fact that if $P_1$ and $P_2$ are interior points of $S$, all points of $P_1 P_2$ are interior points of $S$. The set $\alpha$ therefore has two uniquely determined limit-halfplanes $a'$ and $a''$, which according to (1) do not belong to $\alpha$.

On account of (2) and (3) $a'$ and $a''$ are either opposite or $\alpha$ is the interior of the convex angle bounded by $a'$ and $a''$.

In both cases it is clear that the plane containing $a'$ is a bounding one for $S$ through $L_1$.

In the higher dimensional cases it is clear that for $k = 2$ the method of the preceding proof can be applied. If $k \geq 3$ we consider an $L_{n-k+2}$ through the $L_{n-k}$ and an interior point of $S$. This point is an interior point of $L_{n-k+2}$ and an interior point of this convex set, therefore an $L_{n-k+1}$ exists in the $L_{n-k+2}$ which is a bounding plane for $S L_{n-k+2}$ and therefore...
does not contain an interior point of $S$. Thus we have reduced the case of $k$ to the case $k=1$ and it is clear that we can go on until we reach $k = 2$.

**Definition.** One calls convex closure of a set $S$ the product of all closed halfspaces containing $S$ if there are any, otherwise the whole space. The convex closure is characterized by the following

**Theorem 6.** The convex closure $\tilde{c}$ of a set $S$ is identical with the product $c^\infty$ of all closed convex sets containing $S$. Therefore if $S$ is convex, the convex closure coincides with the closure of $S$.

The proof is easy. Clearly we have $c^\infty \subseteq \tilde{c}$. Let $P \in c^\infty$. Then according to Theorem 2, a plane $\pi$ exists separating $P$ from $c$.

Hence $P \notin c$, which proves $\tilde{c} \subseteq c^\infty$.

**Definition:** A function $f_t (p)$ defined on a convex set $S$ is said to be convex on $S$ if for each $P_1$ and $P_2$ in $S$ and each $\lambda$, $0 < \lambda < 1$, we have

$$f_t \left( (1-\lambda) P_1 + \lambda P_2 \right) \leq (1-\lambda) f_t (P_1) + \lambda f_t (P_2)$$

We first characterize the convex functions in terms of convex sets:

**Theorem 7.** A necessary and sufficient condition that the function $f(x^1, \ldots, x^n)$ defined on the convex set $S$ in $(x^1, \ldots, x^n)$-space be convex on $S$ is that the set $Z$ of points $(x^1, \ldots, x^n, z)$ where $(x^1, \ldots, x^n) \in S$ and $z \geq f(x^1, \ldots, x^n)$ be convex in the $(x^1, \ldots, x^n, z)$ space.

**Proof.** As an illustration we take the case $n = 2$, $x^2 \leq x$, $x^1 \leq y$.

A. Suppose $f$ convex on $S$ and let $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ be points of $Z$.

Then $z_1 \geq f(x_1, y_1)$. $i = 1, 2$, and since $S$ is convex the point

$$[(1-\lambda) x_1 + \lambda x_2, (1-\lambda) y_1 + \lambda y_2] \ (0 < \lambda < 1)$$

is in $S$. Therefore by (1)

$$(1-\lambda) z_1 + \lambda z_2 \geq (1-\lambda) f(x_1, y_1) + \lambda f(x_2, y_2) \geq f \left[ (1-\lambda) x_1 + \lambda x_2, (1-\lambda) y_1 + \lambda y_2 \right].$$
Hence the point \((1-\lambda)x_1 + \lambda x_2, (1-\lambda)y_1 + \lambda y_2, (1-\lambda)z_1 + \lambda z_2\) is in \(\Sigma\).

B. Suppose \(\Sigma\) convex and let \((x_1, y_1)\) and \((x_2, y_2)\) be two points of \(\Sigma\) and suppose \(z_1 = f(x_1, y_1), z_2 = f(x_2, y_2)\). Then \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) are in \(\Sigma\). Therefore \([(1-\lambda)x_1 + \lambda x_2, (1-\lambda)y_1 + \lambda y_2, (1-\lambda)z_1 + \lambda z_2]\) is in \(\Sigma\), and hence

\[(1-\lambda)x_1 + \lambda x_2 = (1-\lambda)f(x_1, y_1) + \lambda f(x_2, y_2) = f((1-\lambda)x_1 + \lambda y_2, (1-\lambda)z_1 + \lambda z_2)\]

The next theorem is obvious, but it consists of a fact which will be useful later on:

\[\gamma(x, y) = \frac{\gamma(x_1, y_1) + \gamma(x_2, y_2)}{\gamma(x_1, y_2) + \gamma(x_2, y_1)}\]

**Theorem 8.** A necessary and sufficient condition that

\(f(x^1, ..., x^n)\) be convex on the convex set \(S\) is that, for each pair \(P_1, P_2\) in \((x^1, ..., x^n)\)-space, \(f[(1-\lambda)P_1 + \lambda P_2]\) be convex in \(\lambda\) for those values of \(\lambda\) (an interval, a point, or the empty set) for which \((1-\lambda)P_1 + \lambda P_2\) is in \(S\). If \(f(x^1, ..., x^n)\) is convex, then \(f(x, a)\) is convex in \(x\) on the set \(S_a\) of values \(x\) for which \((x, a)\) lies in \(S_a\); here \(x\) stands for \((x_1, ..., x_n)\) and \(a\) for \((a_1, ..., a_n)\) where \((i_1, ..., i_n, j_1, ..., j_n)\) is a permutation of \((1, 2, ..., n)\) and the \(x\)'s and \(a\)'s are supposed arranged in their natural order; the set \(S_a\) is convex or empty for each such \(a\).

**Theorem 9.** The limit of a convergent sequence of convex functions is convex. The least upper bound of an aggregate of convex functions is convex (or \(-\infty\)).

The first statement follows at once from the definition of convex functions. The second part follows from Theorems 1 and 7 since \(f(\gamma)\) designating the given aggregate,
Theorem 10. Let \( y(x) \) be a convex function of the single variable \( x \) on \((a, b)\). Then

\[
\frac{y(x_2) - y(x_1)}{x_2 - x_1} \leq \frac{y(x_3) - y(x_2)}{x_3 - x_2} \leq \frac{y(x_4) - y(x_3)}{x_4 - x_3}
\]

Furthermore, if \( y(x) \) is \( \geq 0 \) except on a set of points which is at most countable, then \( y(x) \) is non-negative. Moreover, \( y(x) \) satisfies a uniform Lipschitz condition on each closed interval interior to \( (a, b) \) and if \( y(x) \) is \( \geq 0 \) on \((a, b)\), this Lipschitz condition takes the form

\[
\frac{y(x_4) - y(x_3)}{x_4 - x_3} \geq \frac{y(x_2) - y(x_1)}{x_2 - x_1}
\]

where \( \gamma \) denotes a uniform Lipschitz condition.

Proof. Let \( x_2 = (1-\lambda)x_1 + \lambda x_3, 0 < \lambda < 1 \). Then

\[
\gamma(x_2) \leq (1-\lambda)\gamma(x_1) + \lambda\gamma(x_3)
\]

and (2) follows. In (4), if ordering \( a < x < x_1 < x_2 \), for instance, it follows from (2) that

\[
\frac{\gamma(x_2) - \gamma(x_1)}{x_2 - x_1} \leq \frac{\gamma(x_3) - \gamma(x_2)}{x_3 - x_2}
\]

Since \( x - x_1 < 0 \) in this case, (4) follows. The case \( x > x_3 \) is treated similarly.

Theorem 11. Let \( \gamma(x) \) be a convex function of the single variable \( x \) on \((a, b)\). Then at each interior point \( x_0 \), the derivative \( D_+ \gamma(x_0) \) on the right at \( x_0 \) and \( D_- \gamma(x_0) \), the derivative on the left of \( x_0 \), both exist
and are finite. Moreover, each is a monotone non-decreasing function of $x$ on $(a, b)$ and for each interior $x_0$ we have

$$
\lim_{h \to 0^+} D_R \gamma(x_0 + h) \leq D_L \gamma(x_0), \quad \lim_{h \to 0^+} D_L \gamma(x_0 - h) \leq D_R \gamma(x_0).
$$

Furthermore, $D_R \gamma(x_0) = D_L \gamma(x_0)$ except on a set of points which is at most denumerable, and $\gamma''(x)$ exists almost everywhere and is non-negative. $\gamma(x)$ satisfies a uniform Lipschitz condition on each closed interval interior to $(a, b)$, and if $|\gamma| \leq M$ on $(a, b)$, this Lipschitz condition takes the form

$$
|\gamma(x_1) - \gamma(x_2)| \leq \frac{M}{L} |x_1 - x_2|, \quad \alpha \leq x_1, x_2 \leq \beta,
$$

where $l$ is the distance of $(\alpha, \beta)$ from the boundary of $(a, b)$.

**Proof.** Let $a < x_0 < b$. If then $0 < h_1 < h_2$, we have, according to (2), that

$$
\frac{\gamma(x_0 + h_2) - \gamma(x_0)}{h_2} \geq \frac{\gamma(x_0 + h_1) - \gamma(x_0)}{h_1}
$$

hence

$$
\lim_{h \to 0^+} \frac{\gamma(x_0 + h) - \gamma(x_0)}{h} \geq D_R \gamma(x_0)
$$

exists and is $< + \infty$. In a similar way $D_L \gamma(x_0)$ exists and is $> - \infty$. Moreover, for each $h > 0$, we have (using (3)) that

$$
\frac{\gamma(x_0 + h) - \gamma(x_0)}{h} \geq \frac{\gamma(x_0 - h) - \gamma(x_0)}{-h}
$$

so that $D_R \gamma(x_0) \geq D_L \gamma(x_0)$ and both are finite. This proves the first part of (5). If $h \neq 0$ and $x_2 > x_1$, it follows from (3) that

$$
\frac{\gamma(x_2 + h) - \gamma(x_2)}{h} \geq \frac{\gamma(x_1 + h) - \gamma(x_1)}{h}
$$

so that $D_R \gamma$ and $D_L \gamma$ are monotone non-decreasing. Now take $x_0$ and then $x_1 > x_0$. If $0 < h < \frac{x_2 - x_0}{2}$ we see by Theorem 10 that

$$
\frac{\gamma(x_1 - h) - \gamma(x_1)}{-h} \geq \frac{\gamma(x_0 + h) - \gamma(x_0)}{h}
$$
If \( h \to 0 \) one sees that \( D_L \gamma^0 (\mathbf{x}_0) \geq D_R \gamma^0 (\mathbf{x}_0) \). From this the third of the inequalities (8) follows. The second is proved similarly. If \( D_R \gamma^0 (\mathbf{x}) \) or \( D_L \gamma^0 (\mathbf{x}) \) is continuous at \( \mathbf{x}_0 \), it follows from (5) that \( \gamma'(\mathbf{x}_0) \) exists. Therefore \( \gamma'(\mathbf{x}_0) \) exists except for an at most denumerable set of values \( x \) and the monotony of \( \gamma'(x) \) shows that \( \gamma''(x) \) exists almost everywhere and that

\[
\gamma''(x) = a_0 x^2 + c
\]

Now let \( a < x < \beta \leq x_1 < x_2 \leq \beta < \bar{\beta} < b \). Then, by Theorem 10 if \( f \) is of class \( C^1 \) on \( S \), a necessary and sufficient condition that it be convex on the open convex set \( S \) is that for each \( x \) and that

\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(\beta) - f(\beta)}{\beta - \beta}
\]

so that \( \gamma'(x) \) satisfies a uniform Lipschitz condition on \( (a, b) \). If we assume \( |\gamma'(x)| \leq M \) on \( (a, b) \) the inequality (6) follows easily by taking \( x \) arbitrarily close to \( a \) and \( \bar{\beta} \) to \( b \).

(7)

\[
\int_{x_1}^{x_2} f(x) \, dx 
\]

for all \( x \) in \( S \) and all \( f \). In two dimensions, (10) is equivalent to

(11)

\[
\begin{align*}
\sum_{i=1}^{n} f_i(x_i) & = 0, \\
\sum_{i=1}^{n} f(x_i) & = 0, \\
\sum_{i=1}^{n} f(x_i) & = 0
\end{align*}
\]

for all \( (x_1, y_1) \) in \( S \).

Proof. (8) follows immediately by applying Theorem 4 to the set of Theorem 7. Then (9), (10), (11) follow from (8) by applying the mean values theorem.

In a consequence of Theorem 12 we have that a function \( f(x) \) which is convex on an open convex set \( S \) is continuous on \( S \). For if that was not true a sequence of points \( x_1 \) tending to a point \( x_0 \) in \( S \) would exist such that

\[
\lim_{n \to \infty} f(x_1) = f(x_2)
\]

\[
f(x_1) < \lim_{n \to \infty} f(x_1) \]

is impossible since for each plane \( z = a z + b \) with
We now turn to functions of several variables and prove first:

**Theorem 13.** A necessary and sufficient condition that \( f(x^1, \ldots, x^n) \) be convex on the open convex set \( S \) of the \((x^1, \ldots, x^n)\)-plane is that for each point \( x_0 = (x_0^1, \ldots, x_0^n) \) of \( S \) there exists a linear function

\[
\sum_{k=1}^{n} a_k x_k + b = a_0 x^0_k + b
\]

and that

\[
f(x) = a_0 x^0_k + b
\]

If \( f \) is of class \( C^1 \) on \( S \), a necessary and sufficient condition that it be convex on \( S \) is that

\[
E(x, x^0) = E(x^1, \ldots, x^n; x^0, \ldots, x^0) = f(x) - (x^0_k - x^0_k) f_y(x^0) \geq 0
\]

for all \((x, x^0)\) on \( S \). If \( f \) is of class \( C^2 \) on \( S \), a necessary and sufficient condition that \( f(x^1, \ldots, x^n) \) be convex on \( S \) is that

\[
\int_{x} x^k \beta \left( x^1, \ldots, x^n \right) x^k \hat{x} \geq 0
\]

for all \( x \) in \( S \) and all \( \hat{x} \). In two dimensions, (10) is equivalent to

\[
\int_{x} x^k \beta \left( x^1, \ldots, x^n \right) x^k \hat{x} \geq 0
\]

for all \((x, y)\) in \( S \).

**Proof.** (8) follows immediately by applying Theorem 4 to the set of Theorem 7. Then (9), (10), (11) follow from (8) by applying the mean values theorem.

As a consequence of Theorem 12 we have that a function \( f(x) \) which is convex on an open convex set \( S \) is continuous on \( S \). For if that was not true a sequence of points \( x_i \) tending to a point \( x_0 \) in \( S \) would exist such that

\[
\lim_{i \to \infty} f(x_i) = f(x_0)
\]

\( f(x_0) < \lim f(x_i) \) is impossible since for each plane \( z = a_\omega x^\omega + b \) with
If \( x_0^\infty = f(x_0) \) one has for large \( i \)

\[
\left( f(x_i) \right) < \left( f(x_0) \right) = a_\infty x_0^\infty + b
\]

which contradicts Theorem 12. If \( f(x_0) < \lim f(x_i) \) we could construct

planes \( \Pi_i : \mathbf{z} = a_i x_i^\infty + b_i \) with

\[
\left( f(x) \right) > a_i x_i^\infty + b_i \quad \text{and} \quad \left( f(x_i) \right) = \sum a_x x_i^\infty + b_i
\]

therefore in particular

\[
a_x^i x_i^\infty + b_i \leq f(x_i)
\]

(12) and (13) show that the slope of \( \Pi_i \) tends to infinity with \( i \), the limit plane \( \Pi \) of a convergent subsequence \( \left\{ \Pi_i \right\} \) of \( \left\{ \Pi_i \right\} \) contains the parallel to the \( z \)-axis through \( x_0 \). But as limit of bounding plane of the set \( \Sigma \) (see Theorem 3), \( \Pi \) would have to be a bounding plane for \( \Sigma \) which is not true.

The condition that \( S \) be open is essential; for the function

\[
\begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{pmatrix} \rightarrow \begin{pmatrix} x_1^2 + y_1^2 \leq 1 \\ \vdots \\ x_n^2 + y_n^2 \leq 1 \end{pmatrix}
\]

is convex on \( x^2 + y^2 \leq 1 \) without admitting a plane of the type (8) at the points of \( x^2 + y^2 = 1 \) and without being continuous at these points.

We prove now:

**Theorem 14.** Let \( f(x^1, \ldots, x^n) \) be convex on the convex set \( S \).

Then \( f \) satisfies a uniform Lipschitz condition on each interior closed bounded subset \( E \) of \( S \). If \( |E| \leq M \) on \( S \), the Lipschitz condition takes the form

\[
|f(x_i) - f(x_L)| \leq \frac{2^n \delta(x_i, x_L)}{\gamma}
\]

where \( \gamma \) is the distance of \( E \) from the boundary \( S^* \) of \( S \). If the functions \( f_e(x^1, \ldots, x^n) \) are convex on \( S \) and converge to \( f(x^1, \ldots, x^n) \) at each point of \( S \), the convergence is uniform in each interior bounded closed subset \( E \) of \( S \).
Proof. (14) is an immediate consequence of (6), p. 36. If one does not know that \( f \) is bounded on \( S \), let \( S_1 \) be any bounded open convex set in \( S \) containing \( E \) and such that \( S_1 \) is interior to \( S \). Since \( f \) is continuous on \( S \), it is bounded on \( S_1 \), and we can apply (14) to \( S_1 \) and \( E \).

From this, the last statement follows if we know that the functions are uniformly bounded on each bounded closed set \( S_1 \) interior to \( S \). If this were not true, there would exist a subsequence \( \{ p_k \} \) and a sequence \( x_k \) of points of \( S_1 \) tending to a point \( x_0 \) of \( S_1 \) such that

\[
\lim_{k \to \infty} f(p_k(x_k)) = +\infty
\]

Case I. The above limit is +\( \infty \). Let

\[
z = a_k^* x + f^* x
\]

be the supporting plane to \( z = f(p_k(x)) \) at \( x = x_k \). Then \( f(p_k(x)) = a_k^* x + b^k \) for every \( x \) and \( k \) and

\[
\lim_{k \to \infty} f(p_k(x_0)) = f(x_0)
\]

which is some finite number. The plane (16) cannot coincide with the plane \( z = f(p_k(x_0)) \) (a constant) for infinitely many \( k \) as (17) and the definition of (16) would imply that \( f(x_0) = +\infty \). Hence, we assume that the two planes never coincide. Then if \( x \) is interior to the half space (of \( x \)-space)

\[
a_k^* x + f^* x > 0
\]

the boundary of which contains \( x_0 \), we see that

\[
f(p_k(x_k)) > a_k^* x + f^* x \geq f(p_k(x_0))
\]

Evidently (since \( x_k \to x_0 \)) we may choose a further subsequence (still called \( p_k \)) such that the half planes (18) converge to a limit. Then \( x_0 \) is on the boundary of this limiting half plane, and if \( x_1 \) is interior to this half plane, we have by (15) and (18), that

\[
\lim_{k \to \infty} f(p_k(x_1)) = +\infty.
\]
which is impossible as \( x_0 \) is interior to \( S \).

**Case II.** The limit in (15) is \(-\infty\). Let \( S_2 \) be the closed bounded set of points at a distance \( \leq f \) from \( S_1 \); if \( f \) is sufficiently small, \( S_2 \) is interior to \( S \). It is clear from (15) and (17) that \( x_k \) cannot coincide with \( x_0 \) for infinitely many \( k \). Hence, for each \( k \), let \( x_k \) be the point on the line joining \( x_k \) and \( x_0 \) which is at a distance \( f(x_k) \) from \( x_0 \) and is such that \( x_0 \) is between \( x_k \) and \( x_{k+1} \). Then, by Theorems (6) and (6) it follows that

\[
\int p_k(x_n) = (1 - \lambda_k) \int p_k(x_k) + \lambda_k \int p_k(x_0)
\]

where

\[
\lambda_k = (1 - \lambda_k) x_k + \lambda_k x_0, \quad \lambda_k \to \infty, \quad \lambda_k \to \infty.
\]

Hence

\[
\lim_{k \to \infty} \int p_k(x_n) = +\infty
\]

and we have Case I as all the \( x_k \) are in \( S_2 \) and a further subsequence may be chosen so that they converge to a point \( x_0 \) of \( S_2 \). One has

Choose now a subsequence of \( p_n \) such that converges to a half-plane \( x_0 \) is on the boundary of and \( f(x) = \) if \( x \) is in the interior of but this is impossible, \( x_0 \) being an interior point of \( S \).

The next theorem is a consequence of the following

**Lemma 1.** Let \( \sum_{i=1}^{N} \lambda_i = 1, \lambda_i \geq 0, i = 1, \ldots, N \), and let \( p_i, \]

\( i = 1, \ldots, N \) be points of a convex set \( S \) on which the convex function \( f \) is defined. Then \( \lambda \cdot \sum p_i \) belongs to \( S \) and

\[
\int (\lambda \cdot p_i) \leq \lambda \cdot \int (p_i)
\]

The proof is by induction on \( N \). If \( N = 1 \), the theorem is trivial.

Hence, suppose the theorem has been proved for \( N = k \) and consider \( k+1 \) points

Proof. Let \( x_0 \) be any point and \( x_0 = f(x_0) \). By Theorems 1 and 2,
of $S$ and $k+1$ non-negative numbers $\lambda_i$ whose sum is 1. Then
\[
\sum_{i=1}^{k+1} \lambda_i \cdot \rho_i = \sum_{i=1}^{k} \lambda_i \cdot \rho_i + \lambda_{k+1} \cdot \rho_{k+1} = (1 - \lambda_{k+1}) \sum_{i=1}^{k} \lambda_i \cdot \rho_i + \lambda_{k+1} \cdot \rho_{k+1},
\]
But the point
\[
\sum_{i=1}^{k+1} \frac{\lambda_i}{1 - \lambda_{k+1}} \cdot \rho_i
\]
is in $S$ according to the hypothesis of induction, and $\rho_{k+1}$ is too, such that
\[
\sum_{i=1}^{k+1} \lambda_i \cdot \rho_i
\]
is in $S$. We then have as a consequence of (1) and (15)
\[
+ \left( \sum_{i=1}^{k+1} \lambda_i \cdot \rho_i \right) \leq (1 - \lambda_{k+1}) + \left( \sum_{i=1}^{k} \lambda_i \cdot \rho_i \right) + \lambda_{k+1} + \left( \rho_{k+1} \right) \leq \sum_{i=1}^{k+1} \lambda_i \cdot \rho_i
\]

**Theorem 15.** Let $f(x^1, \ldots, x^N)$ be defined and convex for all $(x^1, \ldots, x^N)$, and let $\phi(E)$ be a non-negative completely additive set function defined as a set $E$ for which $\phi(E)$ exists and is finite. Let $r^1, \ldots, r^n$ be functions which are summable (in the Lebesgue-Stieltjes sense) over $E$ with respect to $\phi$. Finally, let $M_{\phi, E}$ be defined by
\[
M_{\phi, E}(r) = \frac{1}{\phi(E)} \int_E r \, d\phi
\]
Then
\[
+ \left[ M_{\phi, E}(r^1), \ldots, M_{\phi, E}(r^n) \right] \leq M_{\phi, E} \left[ f(r^1, \ldots, r^n) \right]
\]

**Proof.** This follows immediately from Lemma 1 and the usual processes of approximation by step functions.

**Theorem 16.** Let $f(x^1, \ldots, x^N)$ be a convex function such that
\[
\lim_{(x^1, \ldots, x^N) \to \infty} f(x^1, \ldots, x^N) = \infty
\]
Then $f(x)$ takes on its minimum, and if $a^1, \ldots, a^h$ are any given numbers there exists a unique number $d$ such that $z = a^1 x^1 + \ldots + a^h x^h + d$ is a supporting plane to $f(x) = f(a^1, \ldots, a^h)$.

**Proof.** Let $x_0$ be any point and $z_0 = f(x_0)$. By Theorems 1 and 7,
the set \( Z_0 \) of \( x_1, ..., x_n \) for which \( f(x) \leq z_0 \) is a closed convex set, which
is bounded on account of (16). Since \( f \) is continuous on \( Z_0 \), it takes its
minimum value \( c \) on \( Z_0 \), and \( c \leq z_0 \). For \( x \not\in Z_0 \) one has \( f > z_0 \geq c \).
To prove the second part of the theorem we notice that \( f(x) - a^\infty x^\infty \) also
satisfies (16). Let \( d \) be the minimum of this function. Then we have that
\( z = a^\infty x^\infty + d \) is a supporting plane for \( z = f \). It is obviously unique.

From this theorem we conclude as a

Corollary: Let \( f(x_1, ..., x_n) \) and \( g(x_1, ..., x_n) \) be convex and sat-
isfy (16), and let \( f \geq g \) everywhere. Let \( a_1, ..., a_n \) be given numbers.
Then, if \( a^\infty x^\infty + c \) and \( a^\infty x^\infty + d \) are supporting planes to \( f \) and \( g \), \( c \geq d \).

This follows from the proof of Theorem 16, since \( c \) and \( d \) are the
minima of \( f(x) - a^\infty x^\infty \) and \( g(x) - a^\infty x^\infty \) respectively.

Theorem 17. Suppose that the functions \( f \) and \( g \) are
all convex and satisfy (16). Suppose also that \( f^\rho (x) \to f(x) \) at each
point. Let \( c \) be the minimum of \( f^\rho \) and \( c \) that of \( f \). Then \( c \to c^\rho \).

Proof: Let \( x_0 \) be a point where \( f(x_0) = c \). Then \( f(x_0) \to c^\rho \)
so that \( c \geq \lim c^\rho \). Now choose \( \varepsilon > 0 \) and let \( Z_{\varepsilon} \)
be the closed convex set where \( f(x) \leq c + 3 \varepsilon \) and let \( R_{\varepsilon} \)
be the smallest sphere with center \( x_0 \)
which contains \( Z_{\varepsilon} \); clearly \( f \geq c + 3 \varepsilon \) on \( R_{\varepsilon} \). Let \( r_\varepsilon \) be the radius
of \( R_{\varepsilon} \) and choose \( N \) so large that for \( p > N \) \( |f^\rho(x) - f(x)| < \varepsilon \) for all \( x \)
in \( R_{\varepsilon} \). Now let \( x_0 + \frac{\varepsilon}{2} \) be any point with \( f^\rho > f_{\varepsilon} \). Put \( \gamma \frac{\varepsilon}{2} = \frac{\varepsilon}{2} \)
Then \( x_0 + \frac{\varepsilon}{2} \) is on \( R_{\varepsilon} \) and by Theorems 8 and 10 we have for \( p > N \)

\[
\frac{f^p(x_0 + \frac{\varepsilon}{2})}{\gamma} \leq \frac{f^p(x_0) + \frac{\varepsilon}{2} f^p(x_0 + \frac{\varepsilon}{2}) - f^p(x_0)}{\gamma} \leq \frac{f^p(x_0)}{\gamma} + \frac{\varepsilon}{\gamma} \leq c + \frac{\varepsilon}{\gamma} \]

Hence for \( p > N \) we have \( f^p(x) > c + \frac{\varepsilon}{2} \) for all \( x \) so that \( c \leq \lim c^\rho \).

The definition of bounded variation, differentiation, and absolute con-
tinuity of a normal cell function \( \gamma (H) \) over cells \( D \) in \( G \) and over \( G \) are the
Theorem 18. Let \( f(x_1, \ldots, x_n) \) be convex and satisfy (2). Let \( a_0 x + c_0 \) be a supporting plane to \( z = f \) at \((x_0^1, \ldots, x_0^n)\). Let \( (a_p^\kappa, c_p^\kappa) \) be a sequence such that \( a_p^\kappa \to a^\kappa, \ k = 1, \ldots, n, \) and suppose \( c_p \) is chosen so that \( a_p^\kappa x^\kappa + c_p \) is supporting to \( f \). Then \( c_p \to c_0 \) so that

\[
\lim_{p \to \infty} (a_p^\kappa x^\kappa + c_p) = a^\kappa x^\kappa + c_0
\]

Proof. Let \( \gamma_p = f - a_p^\kappa x^\kappa \). Then \( \{\gamma_p\} \) and \( \gamma_0 = f - a^\kappa x^\kappa \) satisfy the hypotheses of Theorem 17 so that if \( c_p \) and \( c_0 \) denote the minima of \( \gamma_p \) and \( \gamma_0 \) respectively, \( c_p \to c_0 \). But \( c_0 = c_0 \) and, for each \( p \), \( a_p^\kappa x^\kappa + c_p \) is a supporting plane to \( f \), which proves the theorem.

3. Lower Semicontinuity Theorem.

Definition 1. Let \( \gamma(R) \) be a function of cells defined on an open set \( G \) with the property that, if \( R \) is divided into the non-overlapping cells \( R_1, \ldots, R_n \), we have

\[
\gamma(R) \leq \sum_{i=1}^{n} \gamma(R_i)
\]

Such a cell function will be called normal on \( G^* \).


The following function of linear intervals is an example for a normal cell function: Let \( x(t) \) and \( y(t) \) be defined for \( 0 \leq t \leq 1 \), let \([t_1, t_2]\) be an arbitrary subinterval of \([0,1]\), and put

\[
\gamma([t_1, t_2]) = \left[ (x(t_1) - x(t))^2 + (y(t_1) - y(t_2))^2 \right]^{\frac{1}{2}}
\]

The definition of bounded variation, variation, and absolute continuity of a normal cell function \( \gamma(R) \) over cells \( D \) in \( G \) and over \( G \) are the
usual ones. If \( \psi(y) \) is of bounded variation over \( G \), the variation \( V_y(y) \) of \( \psi \) over \( R \) is an additive cell function whose variation over \( G \) is the same as that of \( \psi \).\(^*)\)

\(^*)\) Compare S. Saks, Théorie de l'intégrale, Warszawa 1933, Chapter I and Chapter VI, §5.

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**Definition 2:** Let \( \psi(y) \) be a cell function. Then the derivative \( D_y \psi \) of \( \psi \) at \( y_0 \) is defined by

\[
D_y \psi = \lim_{R \to y_0} \frac{V_y(R)}{u(R)}
\]

if it exists, \( R \) being a square containing \( y_0 \). If this limit does not exist and we may assume that each \( y \) is entirely interior to the cell \( R \) in which we let \( \overline{D_y} \psi \) be the upper limit and \( \underline{D_y} \psi \) be the lower limit.

**Lemma 1:** Let \( \mu(y) \) be a non-negative normal set function of bounded variation on \( G \). Then its derivative \( D_y \mu \) exists and is equal to \( D \mu \), almost everywhere and is therefore summable, and we have

\[
V_y(y) = \int \overline{D_y} \mu \, dx \, dy
\]

If \( \psi \) is absolutely continuous, the equality holds.

**Proof.** We know that \( D \mu \) exists almost everywhere and is summable, and also that

\[
\overline{D \psi} = \overline{D \mu} \geq \underline{D \mu} \geq \underline{D \psi} \geq 0
\]

which contradicts \( \langle y \rangle \) at each point. Hence, let \( \tilde{S} \) be the set of points \( p \) where \( D \mu \) exists and is \( > \frac{1}{n} \) and where at the same time

\[
\psi(y) < \left(1 - \frac{1}{u} \right) D \psi
\]
It is then sufficient to prove that \( m(\tilde{S}_{m,n}) = 0 \) for each \( m \) and \( n, m \) designating the exterior measure.

Suppose \( m(S_{m,n}) > k > 0 \). Let \( 0 < \varepsilon < \frac{k}{m} \) and choose non-overlapping cells \( H_1, \ldots, H_k \) so that

\[
\sum_{i=1}^{k} \gamma(H_i) > V(\gamma) - \varepsilon, \quad \mu_\gamma(S_{m,n}) > k
\]

where \( S_{m,n} \) consists of those points of \( \tilde{S}_{m,n} \) which are interior to some \( H_i \).

Each point \( P_0 \) of \( \tilde{S}_{m,n} \) is interior to a sequence \( \{ \gamma_i \} \) of squares of arbitrary small diameter such that

\[
V_\gamma(\gamma_i) > \frac{1}{m} \mu_\gamma(\gamma_i), \quad V_\gamma(\gamma_i) < (1 - \frac{i}{N}) V_\gamma(\gamma_i)
\]

and we may assume that each \( \gamma_i \) is entirely interior to the cell \( H_i \) which contains \( P_0 \). By the Vitali covering theorem, we can find a finite number \( R_1, \ldots, R_N \) of these squares which are non-overlapping and such that

\[
\sum m(R_i) > k. \quad \text{Then we have}
\]

\[
\sum_{i=1}^{N} \gamma(R_i) < \sum_{i=1}^{N} V_\gamma(R_i) - \frac{1}{m} \sum_{i=1}^{N} V_\gamma(R_i) < \sum_{i=1}^{N} V_\gamma(R_i) - \frac{k}{m}. \quad \text{(1)}
\]

We now divide \( H_1, \ldots, H_k \) into further squares \( R_1, \ldots, R_\iota, R_{\iota+1}, \ldots, R_\rho \),

where \( R_1, \ldots, R_\rho \) are the squares previously used. From the normality of \( \gamma(R) \) it follows that

\[
\sum_{i=1}^{\rho} \gamma(R_i) > \gamma(H_1) > V_\gamma(\gamma) - \varepsilon > \sum_{i=1}^{\rho} V_\gamma(R_i) - \varepsilon. \quad \text{(4)}
\]

But from (3) it follows that

\[
\sum_{i=1}^{\rho} \gamma(R_i) < \sum_{i=1}^{N} \gamma(R_i) + \sum_{i=1}^{\rho} V_\gamma(R_i) < \sum_{i=1}^{N} V_\gamma(R_i) - \frac{k}{m}.
\]

which contradicts (4).

**Definition 3:** Let \( z^{(v)} (v = 1, 2, \ldots) \) and \( z \) be of class \( D^v \) (see p. 15), on an open set \( G \). We say that \( z^{(v)} \rightarrow z \) if for every cell \( R \) in-
terior to \( G \) we have
\[
\lim_{r \to \infty} \iint_{\mathbb{R}} \left( u^2 \right) d\rho d\gamma = \lim_{r \to \infty} \iint_{\mathbb{R}} \left( \rho \left( t^2 \right) \right) d\rho d\gamma = \lim_{r \to \infty} \iint_{\mathbb{R}} \left( y^2 \right) d\rho d\gamma = 0
\]

where \( p^{(t)} = \partial_x \), \( q^{(t)} = \partial_y \), \( \rho = \partial_x z \), \( \gamma = \partial_y z \).

**Theorem 1.** Let \( f(p, q) \) be a convex function which is bounded below. Let \( \{ z^{(t)} \} \) and \( z \) be functions of class \( W_1^2 \) on the bounded open set \( G \). Then the integrals
\[
\int (z^{(t)}, \gamma) = \iint f(p^{(t)}, q^{(t)}) \ dx \ dy
\]
\[
\int (z, \gamma) = \iint f(p, q) \ dx \ dy
\]
are all finite or \( +\infty \) and if \( z^{(t)} \to z \), we have
\[
\int (z, \gamma) = \lim_{t \to \infty} \int (z^{(t)}, \gamma)
\]

**Proof.** Define the rectangle functions
\[
y(R, z^{(t)}) = \frac{1}{m(R)} \iint_R p \ dx \ dy, \quad \gamma(R, z) = \frac{1}{m(R)} \iint_R q \ dx \ dy
\]
where
\[
p^{(t)} = \frac{1}{m(R)} \iint_R p \ dx \ dy, \quad q^{(t)} = \frac{1}{m(R)} \iint_R q \ dx \ dy
\]
and \( p^{(t)}, q^{(t)} \) are defined correspondingly. Since \( G \) is bounded we may assume \( f \geq 0 \). Now let \( R \) be a cell which is divided into the non-overlapping cells \( R_1, \ldots, R_N \). Then by Lemma 1 on convex function:
\[
\frac{1}{m(R)} y(R, z) = \int \frac{1}{m(R)} \iint_R p \ dx \ dy, \quad \frac{1}{m(R)} \iint_R q \ dx \ dy
\]
\[
= \sum_{i=1}^N \lambda_i \iint_{R_i} p \ dx \ dy, \quad \sum_{i=1}^N \lambda_i \iint_{R_i} q \ dx \ dy
\]
\[
= \sum_{i=1}^N \lambda_i \int \frac{1}{m(R_i)} \iint_{R_i} p \ dx \ dy, \quad \frac{1}{m(R_i)} \iint_{R_i} q \ dx \ dy
\]
\[
= \frac{1}{m(R)} \sum_{i=1}^N y(R_i, z), \quad \lambda_i = \frac{m(R_i)}{m(R)}.
\]
Thus $\varphi (R, z)$ is normal and so is $\varphi (R, z^{(\nu)})$.

From the convexity of $f$ it follows furthermore, that if $R_1, \ldots, R_N$ are non-overlapping rectangles in $G$,

\begin{align*}
\sum_{i=1}^{N} \varphi (R_i, z) &\leq \mathcal{I} (z, g), \\
\sum_{i=1}^{N} \varphi (R_i, z^{(\nu)}) &\leq \mathcal{I} (z^{(\nu)}, g)
\end{align*}

\begin{align*}
\sum_{i=1}^{N} \varphi (R_i, z) &\leq \mathcal{I} (z, R), \\
\sum_{i=1}^{N} \varphi (R_i, z^{(\nu)}) &\leq \mathcal{I} (z^{(\nu)}, R)
\end{align*}

\begin{align*}
\mathcal{D}_y = f (\mathbf{P}_1, \mathbf{g}), \\
\mathcal{D}_{\nu} = f (\mathbf{P}^{(\nu)}, \mathbf{g}^{(\nu)})
\end{align*}

Thus it follows

\begin{align*}
\mathcal{V}_y (\mathbf{g}) &= \mathcal{I} (z, \mathbf{g}), \\
\mathcal{V}_{\nu} (\mathbf{g}) &= \mathcal{I} (z^{(\nu)}, \mathbf{g})
\end{align*}

from which, using Lemma 1 once more, we infer our theorem.

\begin{align*}
\mathcal{S}_y (x) &= \int \int f (x, y, z) \, dx \, dy + \int \int f (x, y, z) \, dx \, dy
\end{align*}

\begin{align*}
\mathcal{S}_{\nu} (x) &= \int \int f (x, y, z) \, dx \, dy + \int \int f (x, y, z) \, dx \, dy
\end{align*}

The $+$ denotes the other two terms corresponding to the last two in (8).
Theorem 2: Let \( u(x), x = (x^1, \ldots, x^n) \), be of class \( D^p_p \) \((p \geq 1)\) on a hypercube \( R: (a; b) \) of side \( h \), then

\[
(i) \quad \int_0^h \left[ \int_{x^1}^{x^1+h} \left( \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \right)^2 \, dx_1 \right] \, dx = \int_{(a_1, a_1+h)} \left( \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \right)^2 \, dx
\]

\[
(ii) \quad \int_0^h \left[ \int_{x^1}^{x^1+h} \left( \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \right)^2 \, dx_1 \right] \, dx = \int_{(a_1, a_1+h)} \left( \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \right)^2 \, dx
\]

where \( u_R \) denotes the average of \( u \) over \( R \).

Proof: From previous theorems, it is sufficient to prove this for functions \( u(x) \) of class \( C^1 \) on \( R \). For simplicity, we shall assume \( n = 3 \), \( x_1 = x, x_2 = y, x_3 = z \). Let

\[
M = \int_a^b \left[ \sum_{i=1}^3 \left( \frac{\partial u}{\partial x_i} \right)^2 \right] \, dx dy dz
\]

There exist values \( \tilde{y}, \tilde{z} \) such that

\[
\int_a^b \left[ \sum_{i=1}^3 \left( \frac{\partial u}{\partial x_i} \right)^2 \right] \, dx dy dz 
\]

such that

\[
\int_a^b \left[ \sum_{i=1}^3 \left( \frac{\partial u}{\partial x_i} \right)^2 \right] \, dx dy dz 
\]

Then

\[
\int_a^b \left[ \sum_{i=1}^3 \left( \frac{\partial u}{\partial x_i} \right)^2 \right] \, dx dy dz 
\]

Hence

\[
\int_a^b \left[ \sum_{i=1}^3 \left( \frac{\partial u}{\partial x_i} \right)^2 \right] \, dx dy dz 
\]

where the * denotes the other two terms corresponding to the last two in (8).
Using the Hölder inequality on the interior integrals and the integrating, using (7), etc., (i) follows immediately. (ii) follows from (i) using the Hölder inequality and the definition of \( u_n \).

Lemma 2: Let \( f(x, y, z, p, q) \) be defined and satisfy a uniform Lipschitz condition with constant \( K \) over the whole \((x, y, z, p, q)\) space. Suppose further that \( f(x, y, z, p, q) \) is convex in \((p, q)\) for each fixed \((x, y, z)\) and that there exist numbers \( k > 0 \) and \( N \) such that

\[
\int f(x, y, z, p, q) \, dx \, dy \geq k \left( \frac{p^2 + q^2}{2} \right)^{\frac{\varepsilon}{2}} + N
\]

for all \((x, y, z, p, q)\). Then, if \( z_n(x, y) \to z(x, y) \), all being of class \( D^1 \) on the bounded open set \( G \), we have

\[
\left\{ \int f(x, y, z, p, q) \, dx \, dy \right\}_{n \to \infty} = \frac{C}{\ln I(z_n, G)}
\]

In this theorem, \( z \) may be a vector function on \( N \) variables \((x_1, \ldots, x_N)\) instead of \((x, y)\).

Proof: First, let \( R \) be a square of side \( h \) interior to \( G \). Then, from our hypotheses and Theorem 2, it follows that

\[
\int \left| \left\{ f(x, y, z, p, q) \right\}_{R} - f(x, y, z, p, q) \right| \, dx \, dy \
\]

\[
\leq K \int \left| \left[ x - x_R \right] + \left| y - y_R \right| + \left| z - z_R \right| \right| \, dx \, dy \
\]

\[
\leq K \int \left( \frac{h^2}{2} + 3h \int (p^2 + q^2) \, dx \, dy \right)
\]

and the same holds for \( z \).

Next, let \( D \) be any square interior to \( G \). Then \( I(z, D) \) and \( I(z_n, D) \) are all finite. If \( \lim_{n \to \infty} I(z_n, D) = + \infty \) it is clear that

\[
I(z, D) \leq \lim_{n \to \infty} I(z_n, D)
\]

If \( I(z, D) \) is finite, we may consider a subsequence (still called \( z_n \)) such that \( I(z_n, D) \) tends to the above lower limit. From our hypotheses, it then follows that \( D^1(z_n, D) \leq M \), for some

\[
^{*} D_n \text{ is defined in } p \, 53, \text{ below.}
\]
If \( \varepsilon \) is large enough, independently of \( n \), \( d \) being the side of \( D \). But now, for each \( \varepsilon \) and \( n \)

\[
\sum_{i=1}^{\varepsilon} \left| \int_{R_i} \sum_{j=1}^{m_n} (\mathbf{z}_l;_i) \mathbf{z}_l;_i \right| d\mathbf{x} d\mathbf{y} \leq \varepsilon \sum_{i=1}^{\varepsilon} \left( \sum_{j=1}^{m_n} (\mathbf{z}_l;_i) \mathbf{z}_l;_i \right)
\]

which tends to zero as \( n \to \infty \) since \( \mathbf{z}_n \to \mathbf{z} \). Hence, for each \( \varepsilon \)

\[
\sum_{i=1}^{\varepsilon} \left( \int_{R_i} \sum_{j=1}^{m_n} (\mathbf{z}_l;_i) \mathbf{z}_l;_i \right) d\mathbf{x} d\mathbf{y} \leq \lim_{n \to \infty} \sum_{i=1}^{\varepsilon} \left( \int_{R_i} \sum_{j=1}^{m_n} (\mathbf{z}_l;_i) \mathbf{z}_l;_i \right) d\mathbf{x} d\mathbf{y}
\]

Lemma 3: Let \( f(x,y,z,p,q) \) be continuous all over \( S \)-space, convex in \( (p,q) \) for each fixed \( (x,y,z) \), and suppose \( f(x,y,z,p,q) \geq \gamma(p,q) \) for all \( (x,y,z,p,q) \) where \( \gamma(p,q) \) is a convex function satisfying the condition

\[
\lim_{p^2+q^2 \to \infty} \left( \frac{p^2+q^2}{2} \gamma(p,q) \right) = +\infty.
\]

Then there exists a monotone non-decreasing sequence \( f_n(x,y,z,p,q) \) of functions of the type described in Lemma 2 which converge at each point (and hence uniformly over each closed bounded set) to \( f(x,y,z,p,q) \). It is clearly sufficient to assume \( f(x,y,z,p,q) \) lower semicontinuous.

Proof: Let \( a \) and \( b \) be any numbers. Let the function \( c(x,y,z,a,b) \)

be chosen so that \( w = \gamma(x,y,z,p,q; a,b) = ap + bq + c(x,y,z; a,b) \) is the unique
supporting plane to \( w = f(x, y, z, p, q) \) determined by \((a, b)\) we regard \((a, b, x, y, z)\) as fixed. By Theorems 16 and 17, \(\mathcal{C}(x, y, z; a, b)\) is continuous in \((x, y, z)\) for each fixed \((a, b)\) and by Theorem 16, \(\mathcal{C}(x, y, z; a, b) \geq \mathcal{C}(a, b)\) for all \((x, y, z)\), \(\mathcal{C}(a, b)\) being the corresponding function if \(f = \mathcal{C}(p, q)\). Clearly, for each \((a, b)\), we can find a non-decreasing sequence \(\mathcal{C}_n(x, y, z; a, b)\) which converges uniformly on each bounded region of \((x, y, z)\) space to \(\mathcal{C}(x, y, z; a, b)\), each \(\mathcal{C}_n\) satisfying a uniform Lipschitz condition condition over the whole \((x, y, z)\) space. We then define

\[
\mathcal{C}_n(x, y, z; p, q; \lambda) = a \mathcal{P} + b \mathcal{Q} + c(x, y, z; \lambda) = a \mathcal{P} + b \mathcal{Q} + c(x, y, z; \lambda)
\]

and \(\mathcal{C}_{n+1}(x, y, z; p, q; a, b) \geq \mathcal{C}_n(x, y, z; p, q; a, b)\) for each \(n, a, b\), and each \(\mathcal{C}_n\) satisfies a uniform Lipschitz condition condition over the whole \((x, y, z; p, q)\) space.

Now, we define \(f_1(x, y, z, p, q)\) at each point as the largest of

\(\mathcal{C}_1(x, y, z; p, q; a, b)\) where \(a\) and \(b\) may independently take on the values 

-1, 0, 1.

We then define \(f_2(x, y, z, p, q)\) as the largest of the numbers

\(\mathcal{C}_2(x, y, z; p, q; a, b)\) where \(a\) and \(b\) may independently take on the values 

-2, -1, -\(\frac{1}{2}\), 0, \(\frac{1}{2}\), 1, 2.

In general, we define \(f_n(x, y, z, p, q)\) as the largest of the numbers \(\mathcal{C}_n(x, y, z; p, q; a, b)\) where \(a\) and \(b\) may take on the numbers

0, 1, -1, \(\frac{1}{2}\), 2, -\(\frac{1}{2}\), -2, 1/3, 3, 2/3, 3/2, -1/3, -3, -2/3, -3/2, ...

1/n, n/2, n/4, ..., n-1/n, n/n-1, 1/n, -n, -2/n, -n/2, ..., n-1/n, -n/n-1.

Clearly \(f_n(x, y, z, p, q) \leq f_{n+1}(x, y, z, p, q)\) for each \(n\), each \(f_n(x, y, z, p, q)\) is convex in \((y, q)\) for each fixed \((x, y, z)\), and satisfies a uniform Lipschitz condition condition over \((x, y, z, p, q)\) space, and \(f_n(x, y, z, p, q) \geq f_1(x, y, z, p, q) \geq n + k(p^2 + q^2)^{\frac{1}{2}}\).

Now, let \((x_0, y_0, z_0, p_0, q_0)\) be a point and let \((a_0, b_0)\) be chosen so that

\[
\mathcal{C}(x_0, y_0, z_0; p_0, q_0; a_0, b_0) = \mathcal{C}_n(x_0, y_0, z_0; p_0, q_0; a_0, b_0).
\]

\[
\mathcal{C}(x_0, y_0, z_0; p_0, q_0; a_0, b_0) \leq \mathcal{C}(x_0, y_0, z_0; p_0, q_0) + c(x_0, y_0, z_0; a_0, b_0).\]
for all \((p, q)\). Choose \(\varepsilon > 0\). By Theorem 16 on convex functions, there exists a rational \(\tilde{a}, \tilde{b}\) so that

\[
\gamma(x_0, y_0, z_0, p_0, q_0; \tilde{a}, \tilde{b}) + \frac{\varepsilon}{2} > \gamma(x_0, y_0, z_0)
\]

Now we may choose \(N(\varepsilon)\) so large that the denominators of \(\tilde{a}\) and \(\tilde{b}\) or their reciprocals are \(\geq N(\varepsilon)\) and so that, if \(n > N(\varepsilon)\)

\[
f(x_0, y_0, z_0, p_0, q_0) \geq f_n(x_0, y_0, z_0, p_0, q_0) \geq \gamma_n(x_0, y_0, z_0, p_0, q_0; \tilde{a}, \tilde{b}) > \gamma(x_0, y_0, z_0, p_0, q_0) - \varepsilon.
\]

Thus \(f_n \to f\) at each point and hence uniformly on each bounded portion of \((x, y, z, p, q)\) space.

**Theorem 3.** Let \(f(x, y, z, p, q)\) be a function of the type described in

**Lemma 3.** Then if \(z_n(x, y) \to z(x, y)\), all being of class \(D^1\) on the region \(G\), we have

\[
I(z, y) = \lim_{n \to \infty} I(z_n, y)
\]

**Proof:** We shall assume \(I(z, y)\) finite. If it is \(+\infty\) the proof given below takes care of this case also if interpreted in the obvious way.

Choose \(\varepsilon > 0\). Approximate to \(f(x, y, z, p, q)\) as in Lemma 3. Then there exists an \(N\) such that

\[
I(z, y) - \varepsilon < I_N(z, y) \leq \lim_{n \to \infty} I_N(z_n, y) \leq \lim_{n \to \infty} I(z_n, y)
\]

where

\[
I_N(z, y) = \sum_{y} f_N(x, y, z, p, q) \, dx \, dy.
\]

§4. Boundary value questions

**Definition 1:** Let \(z\) be of class \(D^1\) on a region \(G\). We shall define \(D_+(z, G)\) and \(D_-(z, G)\) by

\[
D_+(z, y) = \sum_{i=1}^n \left( z(x_i) + \frac{1}{i} \right) \, dx, \quad D_-(z, y) = \sum_{i=1}^n \frac{1}{|x_i - x|} \, dx
\]

when these integrals are finite; otherwise we define them to be \(+\infty\).
Clearly $D_\alpha(z, \mathbb{H})$ and $\overline{D}_\alpha(z, \mathbb{H})$ are both finite for each bounded region $\mathbb{H}$ with $\mathbb{H} \subset \mathbb{G}$ if $z$ is of class $D^\alpha$ on $\mathbb{G}$.

The third statement is also immediate; for $z$ may be extended as above to $\mathbb{G}$ and then to the whole space by successive reflections. The fourth statement follows immediately from Theorem 3.5 (b).

Lemma 1. Let $z$ be of class $D_\alpha$ on the open cell $R_1$ $(a^1 < x^i < b^1)$, with $D_\alpha(z, R)$ finite. Then $D_\alpha(z, R)$ is finite and there exist functions $f^1(x^1, \ldots, x^{i-1}, x^i, \ldots, x^n)$ and $f^1(x^1, \ldots, x^{i-1}, x^i, \ldots, x^n)$, $i = 1, \ldots, n$ of class $L_\alpha$ such that

$$\int_{a^1}^{b^1} \int_{a^1}^{b^1} \int_{a^1}^{b^1} \cdots \int_{a^1}^{b^1} \left| \frac{\partial^i f}{\partial x^i} \right| dx^1 \cdots dx^n = 0$$

Furthermore $z$ can be extended so as to be of class $D^\alpha$ in a region which

contains $R$ in its interior. If $\alpha > n$, $z$ is continuous on $R$ as extended.

Proof: If $k$ is small enough but $> 0$, we know that $D_\alpha(z, R_k)$ is finite, where $R_k$: $a^1 + k \leq x^i \leq b^1 - k$. Now if $a^1 < x^i < a^1 + k$ or $b^1 - k \leq x^i < b^1$, and $a^1 + k \leq x^i \leq a^1 - k$, we have

$$f^1(x, x^i) - f^1(x, \beta)^i \left| dx^i \right| = \frac{1}{x^i - \beta^i} \cdot \overline{D}_\alpha(z, R)$$

so that $|z|^2$ is summable over $R_{1k}$: $(a^1, a^2 + k, \ldots, a^n + k; b^1, \ldots, b^n - k)$.

By successive steps, we see that $z$ is of class $L_\alpha$ over $R$. This proves the first statement.

The second statement is now immediate for

$$\left| f^i(x^i) - f^i(x^i) \right| \leq \left( \beta^i - \alpha^i \right)^j \int_{a^i}^{b^i} \int_{a^i}^{b^i} \cdots \int_{a^i}^{b^i} \left| \frac{\partial^j f^i}{\partial x^i} \right| dx^1 \cdots dx^n$$

and this tends to zero as $\alpha^i \to 0$ and $\beta^i \to 0$ independently of $a^i$ or $b^i$. 

(3)
The third statement is also immediate, for \( z \) may be extended as above to \( R \) and then to the whole space by successive reflections. The fourth statement follows immediately from Theorem 2, §3.

Lemma 2: Let \( z(x^1, \ldots, x^n) \) be of class \( D^\infty \) on a cell \( a^i \leq x^i \leq b^i \), \( i = 1, \ldots, n-1 \), \( 0 < x^n \leq b^n \) with \( D^\infty (x^n, R) \) finite (\( R \) being the above cell).

Suppose that

\[
\lim_{(x^n)^0 \to 0} \int \left| z \left( x^n, x^n_0 \right) - y \left( x^n \right) \right|^2 \, dx^n = 0.
\]

as in Lemma 1. Let \( x^i = x^i(y^1, \ldots, y^n) \) be a transformation, of class \( C^\infty \) with non-vanishing Jacobian, of the cell \( T \) \( a^i \leq y^i \leq d^i \) \( (i = 1, \ldots, n, c^n = 0) \) into a subset of \( R \) in such a way that \( z^n(y^1, \ldots, y^n, 0) = 0 \). Let \( w(y) \) be the transformed function (also of class \( D^\infty \) on \( T \) with \( D^\infty w \) is finite) of \( z(x) \) and let

\[
\Phi \left( y^1, y^n \right) = y \left[ x^i(y^1, \ldots, y^n, 0), \ldots, x^{n-1}(y^1, \ldots, y^n, 0) \right].
\]

Then

\[
\lim_{y^n(0) \to 0} \int \left( \Phi \left( y^1, y^n_0 \right) - \Phi \left( y^n \right) \right)^2 \, dy^n = 0.
\]

Proof: Let \( z \) be extended to be of class \( D^\infty \) in a region containing \( R \). Then there exists a sequence \( \{ z_p \} \) of class \( C^\infty \) on \( R \) such that

\[
\lim_{p \to \infty} \int_{a^n} \left( z_p - z \right)^2 + \sum_i \left| D_{y_i} (z_p - z) \right|^2 \, dy = 0.
\]

Clearly, we have \( w_p \) of class \( C^\infty \) on the closed cell \( (c, d) \) and

\[
\lim_{p \to \infty} \int_c \left( w_p - w \right) + \sum_i \left| D_{y_i} (w_p - w) \right|^2 \, dy = 0.
\]

As in the proof of Theorem 2, §1 we can choose a subsequence (still called \( z_p \)) so that

\[
\int_{a^n} \left| z_p \left( x^n, x^n_0 \right) - z \left( x^n, x^n_0 \right) \right|^2 \, dx^n \leq \varepsilon_p
\]

\[
\int_{c^n} \left| w_p \left( y^n, y^n_0 \right) - w \left( y^n, y^n_0 \right) \right|^2 \, dy^n \leq \varepsilon_p
\]

\[
\lim_{p \to \infty} \varepsilon_p = 0.
\]
ξ being independent of \(x^o_0\) and \(y^o_0\). We include the cases \(x^o_0 = 0\), \(y^o_0 = 0\).

It being understood that

\[
\bar{Z}(x^o_0, 0) = \varphi(x^o_0), \quad \bar{Z}(y^o_0, 0) = \varphi(y^o_0)
\]

Now, there exists a function \(\psi(f) \to 0\) with \(f\) such that

\[
\int \left| \frac{\partial^2 \psi}{\partial y^o_0} \right| dy^o_0 \leq h(y^o_0)
\]

\[
\int \left| \frac{\partial^2 \psi}{\partial x^o_0} \right| dx^o_0 \leq h(x^o_0)
\]

independently of \(p\). The lemma follows by combining (1) to (5).

**Lemma 5:** Let \(z(x)\) be of class \(D^1\) on an open set \(G\). Let \(R\) be a subregion of \(G\) of the form \(a^< x^< b \rightarrow f_1(x^<)\), \(f_1\) and \(f_2\) being continuous. Then

\[
\bar{Z}(x^o_0, 0) \leq f_1(x^o_0), \quad i = 1, 2
\]

are summable with respect to \(x^c\) and

\[
\int \left( \bar{Z}[x^c, f_1(x^c)] - \bar{Z}[x^c, f_2(x^c)] \right) dx^c = \int \frac{\partial \bar{Z}}{\partial x^c} dx^c
\]

Proof: Evidently we may split such a region up into a finite number of similar regions in each of which either \(f_1\) or \(f_2\) is a constant. It is therefore sufficient to prove the theorem for such regions.

Let us suppose, for instance, that \(f_1 = 0\). Then we know that

\[
\bar{z}(x^c, 0)\]

is summable. Now we can approximate \(f_2(x^c)\) uniformly by means of step functions \(\gamma_p(x^c)\) constant over each of a finite number of cells of \((a^c, b^c)\). Evidently \(\bar{z}[x^c, \gamma_p(x^c)] \to \bar{z}[x^c, f_2(x^c)]\) almost everywhere, and \(\bar{z}[x^c, \gamma_p(x^c)]\) is measurable for each \(p\). Thus \(\bar{z}[x^c, f_2(x^c)]\) is measurable. Now...
so that \( f[x^e, f_2(x^e)] \) is summable. From this, the theorem follows easily.

**Definition 2:** A transformation \( x = x(y) \) of a set \( S \) consisting of an open region \( S_1 \) plus some of its boundary points into a set \( S_2 \) of the same type is said to be a regular transformation of class \( C^* \) if it is 1-1 and continuous, if the functions \( x(y) \) are of class \( C^* \) on the whole set \( S_1 \) and if the inverse has the same properties. It is said to be a regular transformation of class \( D^* \) if all the above is true except that the functions involved are of class \( D^* \). It is said to be a regular transformation of class \( L \) if the above hold except that the functions involved satisfy uniform Lipschitz conditions.

**Definition 3:** A region \( G \) is said to be of class \( C^* \) if there exists a \( \varepsilon > 0 \) such that every point \( x \) of \( G \) at a distance \( \leq \varepsilon \) from \( G^* \) can be covered by a finite number of regions \( R_j \) into which the cells \( T_j \) (\( a_j^i < y_j^i < b_j^i \), \( i = 1, \ldots, n-1 \), \( 0 \leq y_j^N < b_j^N \), \( j = 1, \ldots, N \)) are carried by the regular transformations \( x = x_j(y_j) \). \( G \) is said to be of class \( D^* \) if the above holds except that the transformations are regular transformations of class \( D^* \). \( G \) is said to be of class \( L \) if the above holds except that the transformations are regular of class \( L \). Clearly regions of class \( D^* \) are of class \( L \), and regions of class \( C^* \) are of class \( D^* \).

**Theorem 1:** Let \( G \) be a region of class \( L \) and let \( z(x) \) be of class \( D^* \) in \( G \) with \( D^*_z(x,0) \) finite. Then \( D_z(z,0) \) is finite and there exists a function \( \gamma(P) \) of class \( L^* \) on \( G^* \) such that, if \( x = x(y) \) is a regular transformation of class \( D^* \) of a closed cell \( (a, b) \) \( (a^n = 0) \) onto a sufficiently small portion of \( G \), the points \( y_n \) corresponding to points of \( G^* \), then...
where \( w \) is the transform of \( z \) and \( \mathcal{V}^* (y^\mu) = \mathcal{V} (P) \) where \( P \) is the point of \( G^* \) corresponding to \((y^\mu, 0)\). Moreover, if \( x^\ell = x^\ell(x) \) is a regular transformation of class \( D^\ell \) of \( G \) onto the region \( \mathcal{H} \) (of the same type), then the transformed function takes on the transformed boundary values in the same sense.

Furthermore, if \( G \) is of class \( D^\ell \), we have for almost every point \( x_0 \) of \( G^* \) at which the tangent hyperplane to \( G^* \) exists and is not parallel to the \( x^\ell \) axis, the function \( z(x_0^\ell, x^\ell) \) is A.C. in \( x^\ell \) and tends to \( \mathcal{V} (x_0) \) as \( x^\ell \to x_0^\ell \) (\( x_0^\ell, x^\ell \) being in \( G (\ell = 1, \ldots, n) \)). Finally, if \( \alpha > n \), \( z \) is continuous on \( G \) of class \( L \).

**Proof:** We may evidently cover \( G^* \) and the points of \( G \) at a distance \( \leq p (p > 0) \) from \( G^* \) by a finite number of regular transformations \( T_j \) of class \( L \), where

\[
T_j: x = x_j (y_1^j, \ldots, y^\mu_j) \quad y_j \subset [y_j^0, y_j^\mu] \quad 0 \leq j \leq \ell^\mu
\]

We let \( w_j(y_j) = z(x_j) \), and we see that \( w_j \) is of class \( D^\ell \) on this cell \( R_j \) and so that \( D^\ell \) of the form is finite. Hence it follows immediately that

\( D^\ell (z, G) \) is finite. Next we define \( \mathcal{V}_j (P) \) on the points of \( G^* \) corresponding to \( y_j^\mu = 0 \) as the transform of the function \( \mathcal{V}_j (y_j^\mu) \) to which \( w_j(y_j^\mu, y_j^\mu) \) converge in the mean of order \( \alpha \) as \( y_j^\mu \to 0 \). Now let \( x = x(y) \) be a regular transformation of class \( L \) of a closed cell \( (a, b) (a^\mu = 0) \) into a portion of \( G \) of diameter \( \phi \) and the points of \( y^\mu = 0 \) corresponding to points of \( G^* \). If \( \phi \) is small enough, this portion of \( G \) will be entirely covered by one of the representations \( \mathcal{T}_j \), and if \( w(y) = z(x(y)) \), then (by Lemma 2) \( w(y^\mu, y^\mu) \) will tend in the mean of order \( x \) to the transform of \( \mathcal{V}_j (y_j^\mu) \), i.e., of \( \mathcal{V}_j (P) \).
Of our portion of \( \tilde{G} \) is covered by two different \( T_j \), it follows from this that the two different \( \gamma_j(p) \) agree almost everywhere on the part of \( \tilde{G} \) in our uniformly compacted, provided that there exists a cell \( K \) interior to \( \tilde{G} \) on which portion. Thus the \( \gamma_j(p) \) define a single function \( \gamma(p) \) which is of class \( L^\infty \) over \( \tilde{G} \) and \( z \) takes on these boundary values in the sense of class \( L^1 \) for each bounded region \( \tilde{H} \) with \( \tilde{H} \subset \tilde{G} \). If \( \tilde{G} \) is of class \( L^1 \), \( D(x,p) \) is uniformly bounded for each stated theorem. This proves the first statement, and the second statement is now obvious.

Proof. It is sufficient to prove that each point \( \tilde{G} \) is interior to a cell \( K \) in which \( \tilde{G} \) is uniformly bounded. Now let \( \tilde{G} \) be in \( C-H \) and let \( \tilde{P} \) be a tangent hyperplane to \( \tilde{G} \) exists and is not parallel to the \( \tilde{x}^1 \) axis for interior to \( \tilde{G} \) and is open and bounded, we can find a finite sequence of small cells \( (a^1, b^1) \) with center at \( \tilde{x}^1 \) such that \( \tilde{G} \) contains \( \tilde{G} \) in its interior and is contained in \( \tilde{x}^1, \tilde{x}^2 \) such that \( \tilde{G} \) is of class \( C \). interior to \( \tilde{G} \) on \( \tilde{x}^1 \) contains \( \tilde{G} \) in its interior and \( \tilde{G} \) is uniformly bounded for some where \( k \) is \( \pm 1 \) according as \( (\tilde{x}^1, \tilde{x}^2) \) belongs to \( \tilde{G} \) for \( \tilde{x}^1 > f(x^1) \) or \( \tilde{x}^1 < f(x^1) \), the difference being sufficiently small. Let \( W(y) \) be the transformed function of \( z \), and we see that \( W(\tilde{y}_0, y) \) is absolutely continuous in \( y \) for each \( y \), for which \( z(y) \) is absolutely continuous and the integral of \( z \) is uniformly bounded on some faces of \( \tilde{G} \) on \( \tilde{x}^1, \tilde{x}^2 \). Thus

\[
(\tilde{y}_0, y) = \int (x^1, x^2, x^3)
\]

if \( y^1 \) is not in a set of measure 0. From this the third statement follows at once. The fourth statement follows immediately from Lemma 1 as follows. If we choose a small closed cell \( \tilde{K} \) interior to the cell all the \( \gamma_j(p) \) must be continuous with \( \tilde{W}_j(p) \) continuous on \( \tilde{K}_j \), and \( z \) is continuous (by Theorem 2, §3) on each closed region interior to \( \tilde{G} \).

\[
(\tilde{x}) \text{ is uniformly bounded.}
\]

Since all the region is at a distance \( \tilde{p}_j \) from \( \tilde{G} \), can be covered by a finite number of the \( \tilde{K}_j \), the final statement follows from the above.
Theorem 2. Let $z_p(x)$ be of class $D^r$ on a region $G$ with $D(z_p, G)$ uniformly bounded. Suppose that there exists a cell $R$ interior to $G$ on which $D(z_p, R)$ is uniformly bounded. Then $D(z_p, H)$ is uniformly bounded for each bounded region $H$ with $H \subset G$. If $G$ is of class $L$, $D(z_p, G)$ is uniformly bounded.

Proof: It is sufficient to prove that each point $Q$ is interior to a cell $R_Q$ in which $D(z_p, R_Q)$ is uniformly bounded. Now let $Q$ be in $G$ and let $P$ be interior to $R$. Since $G$ is open and connected we can find a finite sequence of hypercubes $R_1, \ldots, R_N$ of the same dimensions and all parallel to and interior to $G$ so that $R_1$ contains $P$ in its interior and is contained in $R$, $R_N$ contains $Q$ in its interior, and $R_{i-1}$ and $R_i$ have a face in common, $i = 2, \ldots, N$.

Now, we have seen that, if $D(z_p, R)$ is uniformly bounded for some cell $R$ interior to $G$, we have that $z_p$ is of class $L_\infty$ on $R$ with
\[
\int_{R} |z_p| d\lambda \leq C,
\]
uniformly bounded. It is also immediate that if $D(z_p, R)$ is uniformly bounded and the integral of $|z_p|$ is uniformly bounded on some face of $R$, then $D(z_p, R_{i-1})$ is uniformly bounded. Using these two principles, we see that $D(z_p, R_{i-1})$ is uniformly bounded, $i = 2, \ldots, N$.

Now suppose $G$ is of class $L$, and consider a representation $T_j$ (of definition 3). If we choose a small closed cell $r$ interior to the cell $T_j$, $a_j < y_j < b_j, i = 1, \ldots, n-1, 0 \leq y_j^2 < b_j$, we see that $D(w_{jp}, r)$ is uniformly bounded. From Lemma 1 and the above, it follows that $D(w_{jp}, r_j)$ is uniformly bounded so that $D(z_p, R_j)$ is uniformly bounded. Since all the points of $G$ at a distance $< \delta$ from $G$ can be covered by a finite number of the $R_j$, the final statement follows from the above.
Theorem 3: Let \( z_p(x) \) be of class \( D^\alpha \) on a region \( G \), of class \( L_\alpha \), with 
\( D_\alpha(z_p, G) \) uniformly bounded. Suppose that there exists a set \( E \), open on \( G \)

such that 
\[
\sum_{E} |y_p| \ dx \Sigma
\]
is uniformly bounded, \( y_p \) being the boundary values of \( z_p \) and \( d \Sigma \) being 


the element of hyper-area on \( G^* \). Then \( D_\alpha(z_p, G) \) is uniformly bounded. If 

\( \alpha > n \), the \( z_p \) are uniformly bounded and equicontinuous on \( G \).

Proof: Evidently there exists a regular transformation \( x = x(y) \)
of class \( L \) of a closed cell \( \mathcal{T}: (a, b)(a^N = 0) \) into a portion of \( G \) such that 
all the points \( (y^{N-1}, 0) \) correspond to points of \( E \). Then if \( \varphi(y^{N-1}) \) is the 
transform of \( \varphi(x) \) and \( w_p(y) \) is that of \( z_p \), we see that 
\[
\int_{a}^{b} \varphi(y^{N-1}) |w_p(y)| \ dy
\]
is uniformly bounded so that \( D_\alpha(w_p, \mathcal{T}) \) is uniformly bounded. We may let 
\( R \) be any cell interior to the transform of \( \mathcal{T} \) and \( D_\alpha(z_p, R) \) and hence 
\( D_\alpha(z_p, G) \) is uniformly bounded.

If \( \alpha > n \), it is clear that the \( z_p \) are equicontinuous and uni-
formly bounded on any region \( H \) with \( H < G \). If we have a representation 
\( x = x(y) \) as above, it is clear that the \( z_p \) are equicontinuous and uniformly 
bounded on the transform of \( \mathcal{T} \). Hence the second statement follows.

Theorem 4: Let \( z_p(x) \) be of class \( D^\alpha \) on a region \( G \), of class \( L_\alpha \), 
with \( D_\alpha(z, G) \) uniformly bounded. Then 
\[
(6) \quad \int_{y_0}^{y} |y_p| \ dx \Sigma
\]
is uniformly bounded. If only \( D_\alpha(z_p, G) \) is assumed uniformly bounded and
If \( z \to z \) on \( G \), then \( D(z, G) \) is uniformly bounded and \( x_p \to x \) in the mean of order \( \alpha \). If \( \alpha > n, D(z, G) \) is uniformly bounded, and \( \left| \frac{z_p(x_p)}{z_p} \right| \) is uniformly bounded (\( x_p \) in \( G \)), then \( z_p \) is equicontinuous and uniformly bounded on \( G \); if \( z \to z \) on \( G \), the convergence of \( z_p \) to \( z \) is uniform on \( G \).

**Proof:** Let \( x = x(y) \) be a regular transformation of class 1 as above of \( T \) into a portion \( R \) of \( G \). Then \( D(z, G) \) is uniformly bounded so that the integral of \( \left| \psi_p(y^{n_p}) \right|^\alpha \) over \( (a^{n_p}, b^{n_p}) \) is uniformly bounded. Thus (6) is uniformly bounded.

If \( z \to z \) on \( G \), then it is clear that \( D(z, R) \) is uniformly bounded for some \( R \) in \( G \) and hence \( D(z, G) \) is uniformly bounded. Now, let \( \{ z_j \} \) be any subsequence of \( \{ z_p \} \), and let \( T_j \) \((j = 1, \ldots, N) \) be a finite number of representations covering all the points of \( G \) at a distance \( \leq b_j > 0 \) from \( G \), let \( \psi_j(y_j) \) be the transforms of \( z \) and \( \psi_j(y_j) \) those of \( z_p \). Then by Theorem 8, \( \psi_j \) can be chosen a subsequence \( \psi \) of \( \psi_j \) so that \( \psi_j \) is uniformly bounded. Then if \( \psi_j \) is uniformly bounded, we can choose a subsequence \( \psi_j \) of \( \psi_j \) which is also of class \( \alpha \) on \( G \). By Theorem 4, the boundary values of \( \psi_j \) tend to the boundary values of \( \psi \). From this, it is clear that \( \left| \psi_j \right|^\alpha \) is uniformly bounded. The remainder of the theorem is immediate.

**Theorem 5 (a general existence theorem):** Let \( f(x, z, p) \)

\[
\begin{bmatrix}
  \begin{array}{c}
    x = (x^1, \ldots, x^n), \\
    z = (z^1, \ldots, z^m), \\
    p = (p_1^1, \ldots, p_n^m, \ldots, p_1^m, \ldots, p_n^m)
  \end{array}
\end{bmatrix}
\]

be defined and continuous in \( (x, z, p) \), be convex in \( p \) for each \( (x, z) \), and satisfy

\[
\int (x, z, p) \geq 0, \quad \sum_{i=1}^n \int (x, z, p) \int_{ - |P| \alpha^1}^{ |P| \alpha^1} \leq 0, \quad f > 0
\]
for all \((x, z, \rho)\). Let \(G\) be a region of class \(L\), let \(E\) be a set which is open on \(G^c\), and let \(\{\gamma\}\) be a closed (with respect to the metric of \(L\)) family of functions of \(L\), such that
\[
\int_{\gamma} |f(z, z, \rho)| \, d\gamma
\]
is uniformly bounded. Suppose that there exists a function \(z\) of class \(D_1\) on \(G\) which takes on boundary values in \(\{\gamma\}\) for which
\[
\Gamma(z, \frac{\pi}{2}) = \int_{\gamma} f(x, z, \rho) \, dx
\]
is finite.

Then there exists a function \(z\) which minimizes \(\Gamma(z, \frac{\pi}{2})\) among all such functions. If \(\alpha > n\), \(z\) is continuous on \(G\).

Proof: It is clear that if a function \(z\) is of class \(D_1\) with \(\Gamma(z, \frac{\pi}{2})\) finite, then it is of class \(D_{1-}\) on \(G\) with \(D_{1-}^\alpha(z, G)\) finite. Since such a function \(z_1\) exists which takes on boundary values in \(\{\gamma\}\), we may therefore consider only those functions \(z\) with \(\Gamma(z, \frac{\pi}{2}) \leq \Gamma(z, \frac{\pi}{2})\), so that \(D_{1-}^\alpha(z, G)\) is uniformly bounded, where \(z\) takes on boundary values in \(\{\gamma\}\). Let \(\{z_p\}\) be a sequence from this class such that \(\Gamma(z_p, \frac{\pi}{2})\) tends to its greatest lower bound \(k\). Then, by Theorem 3, \(D_{1-}^\alpha(z_p, G)\) is uniformly bounded. We may then choose a subsequence \(\{z_q\}\) tending to a function \(z_0\) which is also of class \(D_{1-}^\alpha\) on \(G\). By Theorem 4, the boundary values of \(z_q\) tend in the mean of order \(\alpha\) to those of \(z_0\). By Theorem 1 or Theorem 5 of \(\S5\), it follows that
\[
k \leq \Gamma(z_0, \frac{\pi}{2}) \leq \Gamma(z_q, \frac{\pi}{2}) \to k
\]
which proves the first statement. If \(\alpha > n\), it follows by Theorem 1 that \(z_0\) is continuous on \(G\).

Lemma 4: A necessary and sufficient condition that \(z(x)\) be of class \(D_{1-}^\alpha\) on an open set \(G\) is that for each cell \(\overline{D}_s\) \((A, B)\) interior to \(G\), \(z\) be of class \(L_{\omega}\) on \(D_s\), and there exist functions \(f_s(x)\) of class \(L_{\omega}\) on \(D_s\).
such that
\[
\int_{a^c} Z(x^c, \ell^c) - Z(x^a, a^c) \, d\ell^c = \int_{\alpha} Z(\alpha) \, d\alpha
\]
provided that \(a^c\) and \(b^c\) do not belong to a certain set \(\gamma^c\) of measure zero.

A necessary and sufficient condition that \(z(x)\) be of class \(D^\alpha\) on \(G\) is that each point \(z_o\) of \(G\) be interior to some cell \(D\) of the type described above.

Proof: This lemma is obvious.

**Theorem 6:** Let \(z\) be of class \(D^\alpha\) on an open set \(G\), and let \(D\) and \(R\) be subregions of \(G\) which are of class \(C^\alpha\), and suppose that \(\bar{R}\) is interior to \(D\). Then the boundary values taken on by \(z\) considered as a function defined only in \(R\) coincide on \(R^*\) with those taken on by \(z\) considered only as a function defined in \(D\).\(\bar{R}\).

Proof: For, let \(z_o\) be any point on \(R^*\). A cell \((a, b)\) with center at \(z_o\) may be found such that for some \(\varepsilon^o\), the portion of \(R^*\) which is in \((a, b)\) may be represented in the form

\[
x^o = f(x^o), \quad a^o < x^o < b^o, \quad a^o < f(x^o) < b^o
\]

where \(f\) is of class \(C^\alpha\). By Theorem 2, we see that the boundary values of \(z\) from both sides of \(R^*\) coincide, for almost every \(x^o\), with the value which the function \(z(x^o, \varepsilon^o)\), which is \(L^C\), in \(x^o\), takes on for \(x = f(x^o)\), \(z\) considered as being defined in \(G\). This proves the theorem.

**Theorem 7:** Let \(z\) be of class \(D^\alpha\) on a region \(G\), and suppose that \(D\) is a region of class \(C^\alpha\) such that \(D^\alpha \subset G\). Let \(u\) be of class \(D^\alpha\) on \(D\) and take on the same boundary values as \(z\) on \(D^*\), and suppose that \(D_{\alpha}(u, D)\)
\[(and hence \(D_{\alpha}(u, D)\) is finite). Then the function \(w\), which coincides with \(u\) in \(D^\alpha\), is of class \(D^\alpha\) on \(G\).
Proof: Let \( P_0 \) be any point of \( D^* \). For some \( e \) there exists a cell \( (a^e, b^e) \) and a number \( \epsilon > 0 \) such that a portion of \( D^* \) containing \( P_0 \) has the representation \( x^e = f(x^e), a^e \leq x^e \leq b^e \), where \( f \) is of class \( C^e \), and all the points \( (x^e, x^e) \) with \( a^e \leq x^e \leq b^e \), \( f(x^e) - \epsilon \leq x^e < f(x^e) \) are in \( D \) say, and all the points \( a^e \leq x^e \leq b^e \), \( f(x^e) < x^e \leq f(x^e) + \epsilon \) are in \( G - \overline{D} \). Then, if we make the transformation \( y^i = x^i, i \neq e, y^e = x^e - f(x^e) \) which is of class \( C^e \), we see that \( \tilde{w}(y) \) (the transform of \( w(x) \)) is of class \( D^e \) for \( a^e \leq y^e \leq b^e \), \( -\epsilon \leq y^e < 0 \) with \( \overline{D} \cup (w, R_1) \) finite, and is also of class \( D^e \), for \( a^e \leq y^e \leq b^e \), \( 0 < y^e \leq \epsilon \) with \( \overline{D} \cup (w, R_2) \) finite (\( R_1 \) and \( R_2 \) being the respective cells involved), and the two parts of \( \tilde{w} \) have common boundary values of class \( L_x \) for \( y = 0 \). It follows easily that \( \tilde{w} \) is of class \( D^e \), for \( a^e \leq y^e \leq b^e \), \( -\epsilon \leq y^e \leq \epsilon \).

Hence each point \( P_0 \) of \( D^* \) is interior to a cell in which \( w(x) \) is of class \( D^e \). Obviously each \( P_0 \) of \( G - D^* \) is interior to a cell of \( G \) which contains no point of \( D^* \) and in which \( w(x) \) is of class \( D^e \). Hence \( w(x) \) is of class \( D^e \) in \( G \).
§5. The borderline case \( n = 2, \alpha = 2 \)

In this section we consider continuity theorems for the case \( \alpha = n \) when \( n = 2 \). For \( n = 2 \) it is clear that any region bounded by a finite number of simple closed curves is of class \( C_2 \) and conversely. We now prove

**Lemma 1:** Let \( z(x, y) \) be of class \( D^2 \) on a circle \( C(P, R) \) with \( D_2[z, C(P, R)] \) finite. Then if we represent \( z(x, y) \) on this circle by

\[
(1) \quad z(r, \theta) = \frac{a_0(r)}{2} + \sum_{n=1}^{\infty} \left[ a_n(r) \cos n\theta + b_n(r) \sin n\theta \right], \quad 0 \leq r \leq R
\]

we find that the \( a_n(r) \) and \( b_n(r) \) are absolutely continuous for \( 0 \leq r \leq R \), and

\[
(2) \quad D_2[z, C(P, R)] = \pi \int_{0}^{2\pi} \left[ \frac{a_0'^2}{2} + \sum_{n=1}^{\infty} \left( a_n'^2 + b_n'^2 + \frac{a_n^2 b_n^2}{r^2} \right) \right] r \, dr
\]

Proof: Let \( z(r, \theta) \) be the function coinciding with \( z \) almost everywhere and absolutely continuous in \( \theta \) for almost all \( r \) and in \( r \) for almost every \( \theta \) with

\[
\bar{z}(r, \theta) = \bar{z}(r_0, \theta) + \int_{r_0}^{r} \bar{z}(r', \theta) \, dr'
\]

\[
\bar{z}(r, \theta) = \bar{z}(r_0, \theta) + \int_{r_0}^{r} \bar{z}(r', \theta) \, dr'
\]

from this.

**Lemma 2:** Let \( z(x, y) \) be harmonic in the circle \( C(P, R) \) and let the boundary values \( H(r, \theta) \) (polar coordinates with pole at \( P \)) be of class \( D^2 \) and be given by

\[
H(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left( \cos n\theta + b_n \sin n\theta \right)
\]

Thus

\[
D_2 \left[ H, C(P, R) \right] = \pi \sum_{n=1}^{\infty} \frac{a_n^2}{r^2}
\]

both sides being simultaneously \( \pi \) times.
for every \( r_0 \), \( r_0 \) being a value for which \( \mathfrak{z}(r_0, \mathfrak{v}) \) is A.C. Then, for almost every \( r \), \( 0 < r \leq R \), \( \overline{z_1}^2 \) and \( \overline{z_2}^2 \) are enumerable with respect to \( \mathfrak{v} \). Using a well known \(^1\) theorem on trigonometric series, it follows that the series

\[ \mathfrak{z}_0 \]

is obtained by termwise differentiation of (1) with respect to \( \mathfrak{v} \) for those values of \( r \). Let \( r_0 > 0 \) be the above value; then

\[
a_n(r) = a_n(r_0) + \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{n=1}^{\infty} Z_n(f(s)) \cos(\mathfrak{v} f s) d\mathfrak{v} \right) d\mathfrak{v}, \quad b_n(r) = b_n(r_0) + \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{n=1}^{\infty} Z_n(f(s)) \sin(\mathfrak{v} f s) d\mathfrak{v} \right) d\mathfrak{v}.
\]

and \( z_1^2 \) and \( z_2^2 \) are summable on such a rectangle if \( r > 0 \). Hence we see that \( a_n \) and \( b_n \) are absolutely continuous on any interval \( (\varepsilon, R) \) with \( \varepsilon > 0 \).

We see also that \( a_n^1(r) \) and \( b_n^1(r) \) are the Fourier coefficients for \( \overline{z_1} \) for almost every \( r \). By the Riesz-Fischer theorem \(^2\) and its converse, it follows that

\[ (3) \quad \int_0^{2\pi} \left( a_n^1 + \frac{1}{\gamma \varepsilon} a_n^2 \right) d\mathfrak{v} = \pi \int_0^{2\pi} \left( \frac{a_n^2}{\varepsilon} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right) d\mathfrak{v} + \gamma^{-2} \sum_{n=1}^{\infty} \frac{1}{\varepsilon^2}(a_n^2 + b_n^2)
\]

for almost every \( r \), the right side being finite. The lemma follows immediately from this.

**Lemma 2:** Let \( H(x,y) \) be harmonic in the circle \( C(P, R) \) and let its boundary values \( H(R, \mathfrak{v}) \) (polar coordinates with pole at \( P \)) be of class \( L_2 \) and be given by

\[
H(R, \mathfrak{v}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos(n \mathfrak{v}) + b_n \sin(n \mathfrak{v}) \right)
\]

Then

\[ (4) \quad D_2 \left[ H, R, \rho, \mathfrak{v}, (P, R) \right] = \pi \sum_{n=1}^{\infty} \frac{1}{\varepsilon^2} (a_n^2 + b_n^2)
\]

both sides being simultaneously finite.
Proof: This is well known and follows from Lemma 1 and the fact that

\[ H(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{1}{r^n} \left( a_n \cos n\theta + b_n \sin n\theta \right) \]

Suppose also that there is a smaller \( R \) such that we have

\[ D_2 \left[ z, C(P, R) \right] \leq K D_2 \left[ H(r, \theta), C(P, R) \right], \quad \forall r < R, \]

where \( H(r, \theta) \) denotes the harmonic function extending with \( z \) on \( C^0(P, P) \).

Then

\[ D_2 \left[ z, C(P, R) \right] = K \cdot \pi \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{1}{r^n} \right)^n, \quad \forall \ r < R. \]

If, instead of (1), we have

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{1}{r^n} \right)^n, \quad \forall \ r < R, \]

then we have

\[ D_2 \left[ z, C(P, R) \right] = K \cdot \pi \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{1}{r^n} \right)^n, \quad \forall \ r < R, \]

and the right side tends to zero with \( r \).

Proofs: We shall prove (8) first. Define

\[ \Psi(r) = \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{1}{r^n} \right)^n, \quad \forall \ r < R. \]

Then, for almost every \( r \), we see using (8), (4), (5), that

\[ \psi(r) \sim P(\Psi(r)), \quad \forall \ r < R. \]

In other words, we see that

\[ \frac{d}{dr} \Psi(r) \leq 0, \quad \forall \ r < R, \]

so that it follows that

\[ \Psi(r) \leq M \cdot R \quad \forall \ r < R, \]

and hence

\[ \psi(\theta) \leq M \cdot \Theta(\theta) \quad \forall \ \theta. \]

But now, from (4) and (5) we see that

\[ D_2 \left[ z, C(P, R) \right] \leq \text{Pr} \Psi(\theta) \quad \forall \ \theta. \]
Theorem 1: Let \( z(x,y) = z^i(x,y), i = 1, \ldots, N \) be of class \( D_2 \) on the circle \( C(P, r) \) with

\[
(i) \quad D_2 \left[ z, C(P, r) \right] \leq \sum_{i=1}^{N} D_2 \left[ z^i, C(P, r) \right] = M < \infty.
\]

Suppose also that there is a number \( K \geq 1 \) such that we have

\[
(ii) \quad D_2 \left[ z, C(P, r) \right] \leq K D_2 \left[ \mathcal{H} \left\{ z; C(P, r) \right\}, C(P, r) \right], \quad 0 < r \leq R,
\]

where \( \mathcal{H}(z, C(P, r)) \) denotes the harmonic function coinciding with \( z \) on \( C^*(P, r) \).

Then

\[
D_2 \left[ z, C(P, r) \right] \leq K \cdot M \left( \frac{r}{R} \right)^{\frac{1}{K}}, \quad 0 < r \leq R
\]

If, instead of (ii), we have

\[
(iii) \quad D_2 \left[ z, C(P, r) \right] \leq K \cdot D_2 \left[ \mathcal{H} \left\{ z; C(P, r) \right\}, C(P, r) \right] + y(\omega), \quad \int_0^R y(\omega) d\omega < \infty,
\]

then we have

\[
\frac{1}{2} D_2 \left[ z, C(P, r) \right] \leq K \cdot M \left( \frac{r}{R} \right) + r \int_0^R y(s) ds + \frac{1}{2} \int_0^R y(p) ds
\]

and the right side tends to zero with \( R \).

Proof: We shall prove (6) first. Define

\[
\Psi(\omega) = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{C_k}{2^k} \cos k \omega + \frac{C_k}{2^k} \sin k \omega.
\]

Then, for almost every \( r \), we see, using (2), (4), (5), that

\[
\Psi(\omega) = C(\omega), \quad \Psi(\omega) \in D_2 \left[ z, C(P, r) \right] = M
\]

In other words, we see that

\[
\frac{1}{r} \int_0^r \frac{d}{d\omega} \Psi(\omega) d\omega \geq 0, \quad 0 < r \leq R
\]

so that it follows that

\[
\frac{1}{r} \Psi(\omega) \leq M, \quad 0 < r \leq R
\]

so that

\[
\Psi(\omega) \leq M \left( \frac{r}{R} \right)^{\frac{1}{K}}
\]

But now, from (4) and (5) we see that \( D_2(z, C(P, r)) \leq \frac{1}{r} \Psi(\omega) \) so that we have

\[
\int_0^R D_2 \left[ z, C(P, s) \right] ds \leq P \cdot M \left( \frac{r}{R} \right)^{\frac{1}{K}}
\]
Since $D_2[z, C(P, r)]$ is monotone now decreasing in $r$, we see that
\[
\frac{1}{2} D_2[z, C(P, \frac{r}{2})] = \frac{1}{2} \int \int_{D_2[z, C(P, \frac{r}{2})]} dS \leq \int \int_{D_2[z, C(P, r)]} dS \leq \int \int_{D_2[z, C(P, \frac{r}{2})]} dS \leq \int \int_{D_2[z, C(P, r)]} dS.
\]

From this, (6) follows immediately.

To prove (8), we see from (2), (4), and (7), that
\[
\psi(1) \leq P \cdot \frac{\psi(1)}{\psi(r)} + \psi(r), \quad \psi(\mathcal{R}) \leq \mathcal{C}.
\]
In other words, we see that
\[
\frac{d}{dr} \left( \frac{r}{\psi} \right) \geq -\frac{1}{r} \frac{\psi}{\psi(r)}.
\]

Hence if we define $\chi(r)$ by the equations
\[
\frac{d}{dr} \left( \frac{r}{\chi} \right) = -\frac{1}{r} \frac{\psi}{\psi(r)},
\]
we see that $\psi(1) \leq \chi(r)$ for $0 < r \leq \mathcal{R}$ and $\chi(r)$ is given by
\[
\chi(r) = \frac{1}{r} \int \int_{D_2[z, C(P, \frac{r}{2})]} dS.
\]
or, by simplifying, we see that
\[
\psi(1) \leq \mathcal{R} \left( \frac{r}{\psi} \right) \frac{\psi}{\psi(r)} \int \int_{D_2[z, C(P, \frac{r}{2})]} dS.
\]

But by (7), $D_2[z, C(P, r)] \leq P \cdot \frac{\psi(1)}{\psi(r)} + \frac{\psi(1)}{\psi(r)} \int \int_{D_2[z, C(P, \frac{r}{2})]} dS$ so that
\[
\int \int_{D_2[z, C(P, r)]} dS \leq P \cdot \int \int_{D_2[z, C(P, \frac{r}{2})]} dS + \frac{\psi(1)}{\psi(r)} \int \int_{D_2[z, C(P, \frac{r}{2})]} dS.
\]
To see that the right side tends to zero with $r$, we need to confine ourselves to a discussion of the middle term and to small values of $r$. Then we see that
\[
\frac{1}{r^2} \int \int_{D_2[z, C(P, \frac{r}{2})]} dS \leq \frac{1}{r^2} \int \int_{D_2[z, C(P, \frac{r}{2})]} dS + \frac{1}{r^2} \int \int_{D_2[z, C(P, \frac{r}{2})]} dS,
\]
each term of which clearly tends to zero. Clearly (8) follows from (10) as (6) followed from (9) above.

**Definition 1:** We say that $z(x, y)$ satisfies a condition $A[\lambda; \mathcal{N}(a, d)]$ on the region $G$ if it is of class $D_2$ on $G$ and if
\[
D[z, C(P, r)] \leq \mathcal{N}(a, d) \left( \frac{r}{a} \right) \lambda, \quad 0 \leq r \leq a, \quad P = (x, y) \leq \mathcal{B}, \quad \lambda > 0.
\]
where \( a > 0 \), \( d > 0 \), and \( a + d \) is the distance of \( P \) from \( G^* \); \( M(a, d) \) is supposed to depend only on \( a \) and \( d \) and not on \((x, y)\).

**Definition 2:** We say that \( z(x, y) \) satisfies a condition \( B[ \lambda, M(a, d)] \) on \( G \) if

\[
| z(x_i, y_i) - z(x_j, y_j) | \leq N(a, d) \left( \frac{r}{a} \right)^{\lambda}, \quad D \leq r \leq a, \quad r = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \]

provided that every point on the segment joining \((x_1, y_1)\) to \((x_2, y_2)\) is at a distance \( \geq a/2 + d \) from \( G^* \).

**Theorem 2:** If \( z(x, y) \) satisfies a condition \( A[ \lambda, M(a, d)] \) on \( G \), it satisfies a condition \( B[ \lambda/2, M(a, d)] \) where

\[
N(a, d) = \frac{8}{\lambda \sqrt{3}} N \left( \frac{a}{2}, d \right) \]


**Theorem 3:** Let \( G \) be a region which can be mapped conformally on a region bounded by a finite number of circles. Let \( z(x, y) \) be a vector function of class \( \text{H}^2 \) on \( G \) with \( D_2(z, G) = \sum_{i=1}^n D_2(z, G) \) finite, and suppose that its boundary values are continuous. Suppose also that there exists a number \( \rho \geq 1 \) such that

\[
D(z, R) \leq \rho \cdot D \left[ H(z, R), R \right]
\]

for every region \( R \) of class \( C^* \), \( H(z, R) \) being the harmonic function on \( R \) with the same boundary values as \( z \) on \( R \). Then \( z \) is continuous on \( G \).

**Proof:** From our definition of boundary values, it follows that it is sufficient to prove our theorem for the case that \( G \) is a region bounded by a finite number of non-intersecting circles as \( G \) may be mapped onto such a region preserving condition (9) and then \( z \) will take on the corresponding continuous boundary values.
From Theorems 1 and 2 it follows that $z$ satisfies a condition $A[1/P, M(a, d)]$ and a condition $B[(1/P), N(a, d)]$ on $G$ and is therefore continuous at each interior point of $G$.

Now let $P_0$ be a point of $G^*$ and take polar coordinates with pole at $P_0$ and initial line pointing in the positive direction of the tangent to the boundary circle on which $P_0$ lies. If $r_0$ is sufficiently small, each circle $C(P_0, r)$ intersects $G^*$ in exactly two points $[r, \varphi(r)]$ and $[r, \pi - \varphi(r)]$, both on the boundary circle on which $P_0$ lies. We have

$$\lim_{r \to 0^+} \varphi(r) = 0$$

where $\varphi(r) > 0$ for $r > 0$ if $P_0$ is on the outer boundary of $G^*$ and $\varphi(r) < 0$ if $P_0$ is on an inner boundary.

The region $0 < r < r_0$, $\varphi(r) < \varphi < \pi - \varphi(r)$ lies in $G$ and $z(r, \varphi)$ is of class $D_2$ in $(r, \varphi)$ there with

$$\int_0^{r_0} \int_{\varphi(r)}^{\pi - \varphi(r)} \frac{1}{r} (Z_{xy} + \frac{i}{r} Z_{y}) \, dx \, dy = \gamma(R), \quad \text{as} \quad R \to 0$$

Thus, for almost every $r$, $z(r, \varphi)$ is $A.C.$ in $\varphi$ with $Z^2$ summable and converges to the continuous boundary values as $\varphi$ tends to $\varphi(r)$ or $\pi - \varphi(r)$ (using Theorem 2).

Thus, for each $R$, $0 < R < r_0$, there is an $r_*$, $R < r < 2R$ such that

$$r_*^{-1} \int_{\varphi}^{\pi - \varphi} \left| \frac{Z_{y}}{r} \right| \, d\varphi \leq \frac{\gamma(R)}{r_*} \int_{\varphi}^{\pi - \varphi} \left| \frac{Z_{y}}{r} \right| \, d\varphi \leq \gamma(R)$$

For this $r_*$, we see that

$$\int_{\varphi}^{\pi - \varphi} \left| \frac{Z_{y}}{r} \right| \, d\varphi \leq \left( \epsilon \gamma(R) \right)^{\frac{1}{2}}$$

Now, choose $\epsilon > 0$. From (10) and the fact that the boundary values are continuous, it follows that we can choose $R$ so small that the oscillation of $z(r_*, \varphi)$ on the boundary of the region $D$: $0 < r < r_*$, $\varphi(r) < \varphi < \pi - \varphi(r)$,
is less than $\varepsilon/2$. Now map this region conformally onto the unit circle and let $w(z, y)$ be the transform of $z$. Condition (9) is preserved and $w$ is of class $D_2$ in $\Sigma$ and takes on the corresponding continuous boundary values.

Let $(r, \theta)$ be polar coordinates in $\Sigma$. Define

$$
\psi(r) = \int_0^{2\pi} \left| W_2(r, \theta) \right| \, d\theta,
$$

$$
\chi(r) = \int_0^{r} \psi(s) \, ds.
$$

In case the integrals exist. Now, by Theorem 1, we find that

$$
\int_0^r \frac{\partial^2}{\partial r^2} D[w, \Sigma(0, r)] \, dr \leq P \cdot \psi(R) r^{1-\varepsilon}.
$$

Using (11) and Schwarz's inequality, we find that

$$
\psi(r) = \left( \int_0^r \psi(s) \, ds \right)^2 \leq \frac{1}{\pi} \int_0^r \psi(s)^2 \, ds \leq 2\pi \varepsilon \int_0^r D[w, \Sigma(0, r)] \, ds \leq 2\pi \varepsilon \frac{\partial^2}{\partial r^2} D[w, \Sigma(0, r)]
$$

so that $\psi(r)$ is certainly defined. Now, for each $\varepsilon > 0$

$$
\left[ \int_0^r \psi(s) \, ds \right]^2 \leq (r-\varepsilon) \int_0^{r-\varepsilon} \psi(s)^2 \, ds \leq 2\pi \varepsilon \int_0^r D[w, \Sigma(0, r)] \, ds \leq 2\pi \varepsilon P \psi(R) r^{1-\varepsilon},
$$

so that $\chi(r)$ is defined and

$$
\chi(r) \leq \left[ 2\pi P \psi(R) \right] \frac{r^{1-\varepsilon}}{r^{1-\varepsilon} + \frac{\varepsilon}{2}} \psi(1) \leq \left[ 2\pi P \psi(R) \right] \frac{1}{2}.
$$

So $P$ and $P' \geq 0$ and $\psi(1) \leq \left[ 2\pi P \psi(R) \right] \frac{1}{2}$.

Suppose also that (9) is satisfied for some $\varepsilon$ independent of $z$, and that there exist numbers $k$ and $\varepsilon > 0$ and a function $\psi(r) \to 0$ with $r$, both independent of $z$, such that

$$
D[w, \Sigma(0, r)] \geq \psi(r) \quad \text{for } r > 0.
$$

Then the family $\Sigma$ is equicontinuous.

Theorem 6. Let $f(z, y, c) = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4$ be
Hence, for each \( \varepsilon > 0 \),

\[
\int_{\partial D} (w(1, \gamma) - \bar{w}(\varepsilon, \rho)) d\gamma \leq \int_{\partial D} \psi(1, \gamma) d\gamma = \int_{\partial D} \frac{1}{2} \psi(1, \gamma) d\gamma = \frac{3}{2} \int_{\partial D} \psi(1, \gamma) d\gamma = \frac{3}{2} \int_{\partial D} \psi(1, \gamma) d\gamma
\]

Since \( w \) is continuous on the interior, the inequality holds for \( \varepsilon = 0 \). From the fact that the oscillation of \( \bar{w}(1, \gamma) \) is \( < \frac{\varepsilon}{2} \), it follows that

\[
\max_{|x_0| < R} \bar{w}(x_0, y_0) \leq \frac{3}{2} \int_{\partial D} \psi(1, \gamma) d\gamma 
\]

The above proof has demonstrated the following theorem:

**Theorem 4:** Let \( \{z\} \) be a family of vector functions, each defined and of class \( D' \) on a region \( G \) bounded by \( N \) circles, \( N \) being independent of \( z \). Suppose that the regions \( G \) are all within some large circle, that two distinct circles of any \( G^* \) are at a distance \( > \varepsilon > 0 \) apart (\( \varepsilon \) independent of \( z \)), and that the boundary values of the \( z \) are equicontinuous. Suppose also that (7) is satisfied for some \( \rho \) independent of \( z \) and that there exist numbers \( \rho_0 \) and \( \varepsilon \) such that

\[
D_{\rho} [z, \gamma] \leq \frac{1}{2} \int_{\partial D} \frac{1}{2} \psi(1, \gamma) d\gamma
\]

Then the family \( z \) is equicontinuous.

**Theorem 5:** Let \( f(x_0, y_0, z_1, \ldots, z_n, p_1, \ldots, p_n, q_1, \ldots, q_n) \) be
of the type described in Theorem 5, §4, with \( \alpha = 2 \) and \( f \) satisfying the additional condition that there exist an \( M \) such that

\[
\left( 13 \right) \quad \int \leq M \cdot \sum_{i=1}^{n} \left( \rho_i^2 + \phi_i^2 \right)
\]

Let \( \mathcal{G} \) be a region bounded by a finite number of simple closed curves and suppose that \( \mathbf{z} \) is a continuous vector function on \( \mathcal{G} \) which is such that a \( \mathbf{z} \) of class \( D' \) exists which takes on these boundary values and \( f \) or with \( I(z, G) \) is finite. Then the solution \( \mathbf{z} \) of \( I(z, G) = \text{minimum} \), which takes on these boundary values, is continuous on \( G \), and satisfies conditions \( A[1, M(a, d)] \) and \( B[N_2, N(a, d)] \) on \( G \) for \( \lambda = M/m, N(a, d) = M(a) \), and \( N(a, d) = N(a) \).

If \( f \) satisfies (13), we see that \( \mathbf{z} \) must satisfy (9) with \( P = M/m \).

For if there should exist an open set \( \mathcal{R} \) of class \( C^2 \) on which (9) did not hold, we could define a function \( \mathbf{z}' = z \) on and outside of \( \mathcal{R} \) and equal in \( \mathcal{R} \) to the harmonic function with the same boundary values as \( z \) on \( \mathcal{R} \). Then \( \mathbf{z}' \) is of class \( D' \) on \( G \), has the same boundary values as \( z \) on \( \mathcal{G} \), and

\[
I(z', \mathbf{z}) = I(z', \mathbf{z} - \mathbf{z}') + I(z', R) = I(z, \mathbf{z} - \mathbf{z}') + I(z, R) 
\]

This contradicts the fact that \( z \) minimized \( I(z, G) \) among all functions with the given boundary values. Thus \( z \) is continuous on \( \mathcal{G} \) and satisfies conditions \( A[1, M(a, d)] \), \( B[1/2, N(a, d)] \) on \( G \), where \( \lambda = m/M \), and

\[
\frac{M(q, d)}{\mu} \cdot D_2[z, \mathcal{C}(p, \alpha)] 
\]

Thus, by Theorem 3, it follows that \( z \) is continuous on \( \bar{G} \).

and we see that \( \mathbf{z} \) is of class \( C^2 \) in all four variables and that the function

\[

\mathbf{z}(x, y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^2} \mathbf{z}(x, y) e^{-\frac{(x^2 + y^2)}{2}} \, dx \, dy
\]

is of class \( C^2 \) everywhere with
§6. Lemmas on potential theory

**Theorem 1**: Let \( \Phi(e) \) be a completely additive set function defined on a bounded region \( \mathcal{F} \). Then the potential function

\[
V(x, y) = \frac{1}{2\pi} \int \int_\mathcal{F} \phi(\epsilon, \gamma) (\gamma - x)^2 + (\gamma - y)^2 \, d\epsilon d\gamma \alpha \Phi(e, \gamma)
\]

is of class \( C^1 \) for each \( 1 \leq \alpha < 2 \), with \( D < V, F \) finite for each bounded region \( F \). Moreover, for almost every rectangle \( R \) in the plane

\[
\int \int_V V_x \, dy - \int \int_V V_y \, dx = \Phi(\epsilon, \gamma)
\]

Finally, \( V_x \) and \( V_y \) are defined almost everywhere by the formulas

\[
V_x = \frac{1}{2\pi} \int \int_\mathcal{F} \frac{\phi(\epsilon, \gamma)}{(\gamma - x)^2 + (\gamma - y)^2} \, d\epsilon d\gamma
\]

\[
V_y = \frac{1}{2\pi} \int \int_\mathcal{F} \frac{\phi(\epsilon, \gamma)}{(\gamma - x)^2 + (\gamma - y)^2} \, d\epsilon d\gamma
\]

**Proof**: It is evidently sufficient to prove this lemma under the assumption that \( \Phi(e) \geq 0 \) on \( G \). For convenience, we extend the definition of \( \Phi(e) \) to the whole plane by \( \Phi(e) = \Phi(e + G) \).

We now define

\[
h_t(x, y) = \frac{1}{2\pi} \int \int_\mathcal{F} \phi(\epsilon, \gamma) (\gamma - x)^2 + (\gamma - y)^2 \leq t^2
\]

\[
= \frac{1}{2\pi} \int \int_\mathcal{F} \phi(\epsilon, \gamma) (\gamma - x)^2 + (\gamma - y)^2 \leq t^2, \quad t > 0,
\]

and we see that \( h_t \) is of class \( C^1 \) in all four variables and that the function

\[
V_t(x, y) = \frac{1}{2\pi} \int \int_\mathcal{F} h_t(x, y, \gamma, \gamma) \alpha \phi(\epsilon, \gamma)
\]

is of class \( C^1 \) everywhere with

where \( R \) denotes the set of all points at a distance \( \geq t \) from \( \mathcal{F} \). Evidently
Next, let the set functions \( \Phi_n(x) \) tend weakly to \( \Phi(x) \), each \( \Phi_n(x) \) being of the form

\[
\phi_n(x) = \sum \phi_{n_i}(x_i), \quad \phi_{n_i}(x_i) = \lambda_{n_i}^{(i)} > 0, \quad \lambda_{n_i}^{(i)} \in \Lambda, \quad x_i \in \mathbb{R}, \quad i = 1, \ldots, N_n.
\]

where \( \phi_{n_i}(x) = 0 \) if \( x \) contains none of the \( P_i^n \). Then we see that the functions (also of class \( C^1 \))

\[
V_n(x, y) = \frac{1}{2\pi} \sum_n \int \phi(t, y) \cdot \Phi_n(x) \, dt
\]

converge uniformly together with their derivatives to \( V(x, y) \) on each bounded region \( \Gamma \) if \( t > 0 \). Next, we see immediately that

\[
\int_{\Gamma} |V_n(x, y)|^p \, dx \, dy \leq \sum_n \lambda_n^{(n)} \int_{\Gamma} \left| \int_{\Gamma} V_n(x, y) \, dx \, dy \right|^p \leq \sum_n \lambda_n^{(n)} \int_{\Gamma} \phi(t, y) \, dt \, dy
\]

independently of \( t \) and \( n \). Now, for each \( \epsilon \),

\[
V_{n_1}(x, y) \leq V_{n_2}(x, y), \quad V_{n_1}(x, y) \leq V_{n_2}(x, y), \quad 0 < \epsilon < \delta
\]

so that we see that \( V \) and \( V_n \) are summable to any power on any segment parallel to either axis and \( V_n \) and \( V \) converge to \( V \) and \( V_n \) respectively in the mean of any order on each such segment and any rectangle. It follows also that \( V_n \) converges to \( V \) in the mean of any order on each such segment and any rectangle.

Next, let \( R \) be any cell and let \( \alpha \) be any number, \( 1 \leq \alpha \leq 2 \). Then

\[
\alpha (\Omega) \leq M \text{ and hence } (\Omega, \Omega) = \rho
\]

for each cell \( \Omega \) in a region \( \mathbb{G} \) containing the point \( R \). This means that \( \alpha (\Omega) \) is of class \( C^2 \) in the region \( \mathbb{G} \).

Finally, let \( R \) be any cell and let \( \alpha \) be any number, \( 1 \leq \alpha \leq 2 \). Then

\[
\alpha (\Omega) \leq M \text{ and hence } (\Omega, \Omega) = \rho
\]

for each cell \( \Omega \) in a region \( \mathbb{G} \) containing the point \( R \). This means that \( \alpha (\Omega) \) is of class \( C^2 \) in the region \( \mathbb{G} \).

Evidently
similar estimates hold for $V_{t^v}$, $V_{ntx}$ and $V_{nty}$ independently of $n$. Thus we see that $V$ and $V_n$ are of class $D^\nu$, with $D^\nu_n(V, \gamma)$ and $D^\nu_n(V, \Gamma)$ uniformly bounded for each fixed bounded $\gamma$ and each fixed $\alpha$, $1 \leq \alpha < 2$. Moreover, the above also shows that $\overline{D}^\nu_n(V - V, \Gamma)$ tends to zero for each fixed bounded $\Gamma$ and each $\alpha$, $1 \leq \alpha < 2$, and that formulas (3) hold.

Since $\overline{D}^\nu_n(V - V, \Gamma)$ tends to zero as above, it follows that, for a subsequence $V_{n_k}$ we have

$$\int_{R^2} V_n \partial_x^\alpha y - V_n^\alpha \partial_x y \, dx = C_{\alpha, x} \int_{R^2} V_\alpha \partial_x^\alpha y - V_\alpha^\alpha \partial_x y \, dx$$

for almost every rectangle $R$. Furthermore $\phi_{n_k}(R)$ tends to $\phi(R)$ for almost every $R$ (as extended) and it is clear that

$$\int_{R^2} V_n \partial_x^\alpha y - V_n^\alpha \partial_x y \, dx = \phi_{n_k}(R)$$

for almost all $R$ and every $k$. From this (8) follows.

Remark: In this section, if $U(x, y)$ is a function of class $D^\nu$, $\alpha \geq 1$, then $\overline{U}(x, y)$ will denote the function

$$\overline{U}(x, y) = \lim_{\rho \to 0} \frac{1}{\pi} \int_{(x, y) \rho} U(x, y) \, dx \, dy$$

wherever it is defined. It is clear that $\overline{U}$ is $A.C.$ along all lines parallel to each axis and is of class $L_\infty$ on each segment parallel to an axis within its region of definition.

Lemma 1: Let $U(x, y)$ be of class $D^\nu$ in a region $G$ containing the point $P_0$, and suppose that

$$\int_{\gamma} \int G(x, y, \gamma) \, dx \, dy = 0$$

for all $r$. Then $\overline{U}(x, y)$ is defined at $P_0$. In fact, there exists a summable function $\gamma(\gamma)$ such that

$$\lim_{\gamma \to 0} \int_{|x + r| \leq r_0, \gamma_0 + r \sin \gamma = 0} U(x, y) \, d\gamma = 0$$
if \( r \) is not allowed to assume values in a certain set of measure zero. Then

\[
U(x_0, y_0) = \frac{1}{2\pi} \int_0^1 \int_0^1 \frac{U(x, y) \, dx \, dy}{r} \leq \frac{1}{2\pi} \int_0^1 \int_0^1 \frac{1}{(1 + x^2)}} \, dx \, dy \leq \frac{1}{2\pi} \int_0^1 \int_0^1 \frac{1}{(1 + x^2)}} \, dx \, dy.
\]

Proof: Let \( r \) and \( J \) be polar coordinates with pole at \( P_o \) and let \( W(r, J) \) be the transform of \( U(x, y) \) and let \( \tilde{W}(r, J) \) be formed as above from \( W(r, J) \) in the \((r, J)\) plane. Then \( \tilde{W}(r, J) \) is of class \( B_2 \) in \((r, J)\) for \( r \geq r_o > 0 \), no matter how small \( r_o \) is, and \( \tilde{W} \) is A.C. in \( J \) with \( \frac{\tilde{W}}{r^2} \) summable for almost every \( r > 0 \), and is A.C. in \( r \) with \( \frac{\tilde{W}}{r^2} \) summable in a region

\[
0 \leq J \leq 2\pi, \quad 0 < r_o \leq r \leq a, \quad \text{if} \; C(P_o, a) \; \text{lies in} \; G, \quad \text{and finally}
\]

\[
0 \leq J \leq 2\pi, \quad 0 \leq r \leq a.
\]

Define

\[
k(r) = \int_0^{2\pi} \left( \frac{1}{r} \int_0^r \frac{\tilde{W}(r, J)}{r} \, dJ \right) \, dJ.
\]

Then

\[
k(r) = \left( \int_0^{2\pi} \left( \frac{1}{r} \int_0^r \frac{\tilde{W}(r, J)}{r} \, dJ \right)^2 \, dJ \right)^{1/2} \leq \left( \int_0^{2\pi} \left( \int_0^r \frac{\tilde{W}(r, J)}{r} \, dJ \right)^2 \, dJ \right)^{1/2} + \epsilon
\]

for almost all \( r \) and \( k(r) \) is A.C. for \( 0 < \epsilon \leq r \leq a \) for each \( \epsilon > 0 \) and

\[
k^2(r) \leq \left[ \int_0^{2\pi} \left( \frac{1}{r} \int_0^r \frac{\tilde{W}(r, J)}{r} \, dJ \right)^2 \, dJ \right] \leq \epsilon \pi r \left[ \int_0^{2\pi} \left( \frac{1}{r} \int_0^r \frac{\tilde{W}(r, J)}{r} \, dJ \right)^2 \, dJ \right] \leq \epsilon \pi r.
\]

Also, for each \( \epsilon > 0 \)

\[
\left( \frac{1}{r} \int_{r-o}^{r+o} \tilde{W}(r, J) \, dJ \right) \leq \int_0^{2\pi} \left( \frac{1}{r} \int_0^r \frac{\tilde{W}(r, J)}{r} \, dJ \right)^2 \, dJ \leq \left( \frac{1}{r} \int_0^r \frac{\tilde{W}(r, J)}{r} \, dJ \right)^2 \, dJ = \left( \frac{1}{r} \int_0^r \frac{\tilde{W}(r, J)}{r} \, dJ \right)^2 \, dJ.
\]

Hence the existence of \( \gamma(J) \) to satisfy (10) is evident as \( W(r, J) \) is equivalent in \((r, J)\) to \( U(x_0 + r \cos J, y_o + r \sin J) = W(r, J) \). The existence of \( \tilde{U}(x_0, y_o) \) and equation (5) follow easily.

Lemma 2: Let \( d \) and \( e \) be of class \( L_2 \) on a bounded region \( G \) and satisfy

\[
\int_G (d^2 + e^2) \, dx \, dy \leq \frac{1}{\pi r^2}, \quad 0 < \lambda < 1
\]

\( C(P_o, \gamma) \).
for all circles \( C(p_0, r) \) with center at a fixed point \( p_0 \). Then there exist sequences \( d_n \) and \( e_n \) of functions of class \( C^\infty \) all over the plane \( \mathbb{R}^2 \), zero outside a bounded region \( U \supset G \), satisfying (7) uniformly, and such that, if \( d = e = 0 \) outside \( G \), then

\[
\lim_{n \to \infty} \iint \left[ (d_n - d)^2 + (e_n - e)^2 \right] \, dx \, dy = 0
\]

Proof: We first observe that, if \( d_n \) and \( e_n \) are given by

\[
d_n \left( x, y \right) = \frac{1}{\pi h^2} \iint 2 \iint \left\{ \alpha(x + \xi, y + \eta) \alpha(x, y) + \alpha(x, y) \alpha(x + \xi, y + \eta) \right\} \, dx \, dy \, d\xi \, d\eta
\]

we have

\[
\iint \left[ \frac{d_n^2 + e_n^2}{\pi h^2} \right] \, dx \, dy \leq \frac{1}{\pi h^2} \iint \left\{ \iint \left[ \alpha(x + \xi, y + \eta) + \alpha(x, y + \eta) \right] \, dx \, dy \, d\xi \, d\eta \right\} \, dx \, dy
\]

Next we choose a sequence \( r_n \to 0 \) and define

\[
d_n \left( x, y \right) = \alpha \left( x, y \right), \quad e_n \left( x, y \right) = \epsilon \left( x, y \right), \quad (x-x_0)^2 + (y-y_0)^2 \leq r_n^2
\]

For each \( n \), we define a positive number \( h_n \) so small that

\[
\left( \frac{r_n}{h_n} \right)^4 \leq \frac{1}{\pi h_n} \iint \left[ \left( d_n - d_{n, h_n} \right)^2 + \left( e_n - e_{n, h_n} \right)^2 \right] \, dx \, dy \leq \frac{1}{u}
\]

\( d_{n, h_n} \) and \( e_{n, h_n} \) being the circular averages of \( d_n \) and \( e_n \). We next let

\[
d_n \left( x, y \right) = \left( 1 - \frac{r_n}{r} \right) d_{n, h_n}, \quad e_n \left( x, y \right) = \left( 1 - \frac{r_n}{r} \right) e_{n, h_n}
\]

and we see that \( d_{2n} \) and \( e_{2n} \) are continuous, zero near infinity, and satisfy

\[
\iint \left[ \frac{d_{2n}^2 + e_{2n}^2}{\pi h_{2n}^2} \right] \, dx \, dy \leq \frac{1}{u}
\]
\[ \int \frac{1}{\pi} \left( \int (\phi_{n} - \phi) \left( \psi_{n} - \psi \right) \, dx \, dy \right)^{2} \leq \int \frac{1}{\pi} \left( \int \left( \phi_{n} - \left( 1 - \frac{L}{n} \right) \phi_{n} \right) \, dx \, dy \right)^{2} \left( \psi_{n} - \left( 1 - \frac{L}{n} \right) \psi_{n} \right) \, dx \, dy \]

which evidently tends to zero as \( n \to \infty \). The determination of the \( d_{n} \) and \( e_{n} \)
only at \( P \), then an approximation is possible in \( L^{1} \) which preserves (7) at \( P \).

Theorem 2: Let \( d \) and \( e \) be of class \( L^{2} \) on the bounded region \( G \). Then

the function

\[ U(x, y) = \frac{1}{2\pi} \int \frac{(x - \xi)(x' - \xi) + (y - \eta)(y' - \eta)}{(x - \xi)^{2} + (y - \eta)^{2}} \, d\xi \, d\eta \]

is defined almost everywhere (the right side existing as a Lebesgue integral)
and is of class \( D_{2} \) all over the plane \( \mathbb{R}^{2} \) satisfying

\[ D_{2}(U, \eta) \leq 4 \int (d^{2} + e^{2}) \, dx \, dy. \]

If \( d \) and \( e \) satisfy (7) for all circles with center at some point \( P \),
then \( U(P) \) is defined by (8) and

\[ |U(P)| \leq 2 \sqrt{1 + 4^{-1}} \left( 2\pi \right)^{\frac{1}{2}} R \]

where \( C(P, R) \) is the circle of smallest radius about \( P \) which contains \( G \).

If (7) holds for every \( P \) and \( r \), then \( U \) is continuous over \( \mathbb{R}^{2} \) and

satisfies conditions \( A[\lambda, M(a)] \) and \( B[\lambda/2, N(a)] \) everywhere where

\[ M(a) = 4 \int \left( \lambda^{-1} + (1 - \lambda)^{-1} \right) \alpha^{2} \, d\alpha, \quad N(a) = \frac{1}{2} \left( \lambda^{-1} + (1 - \lambda)^{-1} \right) \alpha^{1/2} \]

Proof: It is clear that we can find sequences \( \{d_{n}\} \) and \( \{e_{n}\} \) of functions of class \( C^{2} \) all over \( \mathbb{R}^{2} \) and zero outside a bounded region \( H \) containing \( G \)
in its interior such that

\[ \lim_{n \to \infty} \int_{H} \left( (d_{n} - d)^{2} + (e_{n} - e)^{2} \right) \, dx \, dy = 0. \]
If \( d \) and \( e \) satisfy (7) for every \( P \) and \( r \), it follows from Lemma 1, §3 of D.E.

*) The author's paper, "On the Solutions of Quasi-Linear Elliptic Partial Differential Equations", Transactions of the American Mathematical Society, vol. 43 (1938), pp. 126-166. We shall hereafter refer to this paper as D.E.

that we may require (7) uniformly with \( H \) replacing \( G \). If (7) is satisfied only at \( P \), then an approximation is possible in \( L_2 \) which preserves (7) at \( P \).

Now let \( R \) be any bounded region. Then

\[
\int_R \left| \frac{1}{\lambda_n} \sum_{j=1}^n \left( d_k - d \right)^2 \right| dx \, dy \leq \frac{2}{\pi} \int_R \left| \sum_{j=1}^n \left( d_k - d \right)^2 \right| dx \, dy
\]

which obviously tends to zero as \( n \to \infty \), being chosen large enough so that \( C(\xi, \eta, \zeta) \) contains \( R \) for each \( (\xi, \eta) \) in \( H \). Thus, it is easily seen that \( U_n \to U \) in the sense defined in §3 all over \( \Omega \) so that

\[
D_2(U_n R) \leq \lim_{n \to \infty} D_2(U_n R)
\]

for every region \( R \). If \( d, d_n, e, \) and \( e_n \) satisfy (7) uniformly on \( H \), it follows from Theorem 1, §3 of D.E., that the corresponding \( U_n \) and \( U \) satisfy the condition B of the theorem uniformly so that \( U_n \) tends uniformly to \( U \) over \( \Omega \).

Now, consider such sequences \( a_n, e_n \) and \( U_n \). Then \( U_n \) is of class \( C^a \) and satisfies

\[
\Delta U_n = \alpha_n x + \epsilon_n y \quad (\Delta y = \gamma_x x + \gamma_y y)
\]

all over \( \Omega \). Hence it is easily seen that \( U_n \) minimizes

\[
\int_R \left[ \left( \psi_{x x} - a_n \right)^2 + \left( \psi_{y y} - e_n \right)^2 \right] dx \, dy
\]

among all functions \( \psi \) of class \( D_2^2 \) on \( R \) and coinciding with \( U_n \) on \( R \), \( R \) being
any region of class $C^*$. Thus if we let $H_{nr}$ be the harmonic function coinciding with $U_n$ on $R^*$ and let $U_{nr} = U_n - H_{nr}$, we see that (since $H_{nr}$ minimizes $D_2(\gamma, R)$ among all $\gamma = U_n$ on $R$'s of class $D^*_2$ on $R$)

$$D_2(U_n, R) = D_2(H_{nr}, R) + D_2(U_{nr}, R)$$

and $U_{nr}$ satisfies (12) and hence minimizes (13) among all $\gamma$ of class $D^*_2$ on $R$ and zero on $R^*$. From the remarks at the bottom of page 149 of D.E., it follows that

$$D_2(U_{nr}, R) \leq 2 \iint_{R} \left( (U_{nr} - \alpha_n^2)^2 + (U_{nr} - \epsilon_n)^2 \right) dx dy + 2 \iint_{R} (\alpha_n^2 + \epsilon_n^2) dx dy \leq 4 \iint_{R} (\alpha_n^2 + \epsilon_n^2) dx dy$$

since $U_{nr}$ minimizes (13). Thus for each $R$ of class $C^*$ in $\pi$ we see that

$$D_2(U_n, R) \leq D_2(H_{nr}, R) + 4 \iint_{R} (\alpha_n^2 + \epsilon_n^2) dx dy.$$

Now let $(x, y)$ be a point not in $\bar{H}$. Then

$$U_{nr} = \frac{1}{2\pi} \iint \left[ \frac{(x-x)^2 + (y-y)^2}{\bar{H}^2} \right] dx dy, \quad U_{nr} = -\frac{1}{2\pi} \iint \left[ \frac{(x-x)^2 + (y-y)^2}{\bar{H}^2} \right] dx dy.$$

Now let $P_0$ be any point of $\bar{H}$ and let $\rho$ be so large that $H \subset C(P_0, \rho)$ and the distance of $H$ from $C^*(P_0, \rho)$ is $> \sqrt{2}$. Then, on $C^*(P_0, \rho)$, we see that

$$\left| \frac{\partial}{\partial T} U_n(x_0 + \rho \cos T, y_0 + \rho \sin T) \right| \leq \frac{4 \kappa_0}{\sqrt{2}}, \quad \kappa_0 = \frac{1}{2\pi} \iint (\alpha_n^2 + \epsilon_n^2) dx dy$$

and hence

$$\sum_{\rho = 1}^{\infty} \rho^2 (\alpha_n^2 + \epsilon_n^2) \leq \sum_{\rho = 1}^{\infty} \rho^2 (\alpha_n^2 + \epsilon_n^2) = \frac{8\pi \kappa_0}{\sqrt{2}}$$

which tends to zero as $\rho \to \infty$. Here, we have placed

$$U_n(x_0 + \rho \cos T, y_0 + \rho \sin T) = \rho \frac{\epsilon_n}{2} + \sum_{\rho = 1}^{\infty} \rho^2 \left( \alpha_n^2 \sin \rho T + \epsilon_n \rho \sin \rho T \right)$$

Thus, from Lemma 2, §5, it follows that

$$\lim_{\rho \to 0} D_2 \left[ H \{ U_n, C(P_0, \rho) \} \right] = 0$$

and hence (using this and (16))

$$D_2(U_n, \pi) \leq 4 \iint_{H} (\alpha_n^2 + \epsilon_n^2) dx dy$$
Now, by §4, Theorem 4, the boundary values for $U_n$ tend in $L^2$ on $R^*$ to those of $U$ and $D_2(U_n, R)$ is uniformly bounded, $R$ being any region of class $C^*$. Thus $H_{nR} \to H_R$, $U_{nR} \to U_R$ also. Now, by referring to (12) and (13) and the argument at that point, we see that

$$
E_m \to \infty \quad D_n(U_n - U_{nR}, \pi) \leq E_m^{\infty} 4 \int \int \left[ (d_n(x, y) - e_n(x, y))^2 \right] \, dx \, dy = 0
$$

so that

$$
E_m \to \infty \quad D_n\left( H_{nR} - H_{nR}, \pi \right) = \infty \quad D_n(U_{nR} - U_{nR}, \pi) = 0
$$

so that (by letting $m \to \infty$ first and using the lower semicontinuity of $D_2$)

we have

$$
E_m \to \infty \quad D_n(U_n - U, \pi) = \infty \quad D_n\left( H_{nR} - H_{nR}, \pi \right) = 0 \quad D_n(U_{nR} - U_{nR}, \pi) = 0
$$

so that finally (9) holds, and (16) holds with $U$, $H_R$, and $U_R$ replacing $U_n$, $H_{nR}$, and $U_{nR}$ respectively, and $d$ and $e$ replacing $d_n$ and $e_n$. Thus if (9) holds for a single point $P$ and every $r > 0$, we find that (10) holds if we define $\bar{U}(P)$ by the integral **), and also we see, using Theorem 1, §5, that

** For the proof of this, see the proof of Theorem 1, §3, of $D_2E$.

Thus, by Lemma 1,

$$
\left| \frac{1}{n} \int \int \frac{1}{U_n(x, y)} \, dx \, dy \right| \leq \left( \frac{2\pi}{n} \right)^{\frac{1}{2}} \frac{4}{\sqrt{\lambda + (1-\lambda)^2}} \lambda \left( \frac{\lambda}{\lambda + \lambda} \right)^{\frac{1}{2}} \left( \frac{1 + \lambda}{1 + \lambda} \right)
$$

$$
\left| \frac{1}{n} \int \int \frac{1}{U_n(x, y)} \, dx \, dy \right| \leq \left( \frac{2\pi}{n} \right)^{\frac{1}{2}} \frac{4}{\sqrt{\lambda + (1-\lambda)^2}} \lambda \left( \frac{\lambda}{\lambda + \lambda} \right)^{\frac{1}{2}} \left( \frac{1 + \lambda}{1 + \lambda} \right)
$$

from which it follows immediately that $U_n(x, y) \to \bar{U}(x, y)$.
Now let

$$h_n(r) = \int_0^r \left( \int_0^1 \left( (\sigma_{n,d} - d)^2 + (\epsilon_{n,e} - e)^2 \right) d\sigma d\epsilon \right)^{1/2} d\tau$$

Then

$$h_n(r) \leq \pi^{1/2} \int_0^r \left( \int_0^1 \left( (\sigma_{n,d} - d)^2 + (\epsilon_{n,e} - e)^2 \right) d\sigma d\epsilon \right)^{1/2} \leq \frac{2 \pi}{n} \frac{1}{2} + \frac{4}{r}.$$

Furthermore, \( h_n(r) \) tends uniformly to zero for all \( r \geq 0 \). Thus, if we let

$$2(N M)^{1/2} \frac{1}{r_n}$$

be the maximum of \( h_n(r) \), we see that \( r_n \to 0 \). Now

$$\int_0^R \left( \int_0^1 \left( (\sigma_{n,d} - d)^2 + (\epsilon_{n,e} - e)^2 \right) d\sigma d\epsilon \right)^{1/2} d\tau = \int_0^R h_n(\tau) d\tau = \int_0^1 \frac{1}{2} \pi \frac{1}{2} + \frac{4}{r_n} d\tau + 2(2\pi)^{1/2} \int_0^R \frac{1}{r_n} d\tau +$$

where \( R \) was chosen large enough so that \( C(P_0, R) \) contained \( \overline{H} \). From this, it follows that the corresponding integrals for \( U_n(x_0, y_0) \) converge to the right side of (8) so that in this case \( \overline{U(P_0)} \) is defined by (8), since

$$U_n(x_0, y_0) \to \overline{U(x_0, y_0)}^*.$$

Now if condition (7) is satisfied for every point \( P \) and every \( r \), it follows from (17) that \( U \) satisfies a condition \( A(\wedge, M(a)) \) where \( M(a) \) is given by (11). The condition \( B(\wedge, N(a)) \) where \( N(a) \) is given in (11) has been proved in D.E., §5, Theorem 1. This completes the proof of the theorem.

Lemma 3: Let \( G \) be a region of class \( C^2 \), and let \( u(x, y) \) be of class \( D^2 \) on \( G \) with \( D^2(u, G) \) finite, and suppose \( u \) vanishes on \( G^* \). Then we may find a sequence of functions \( u_n(x, y) \), each of class \( C^\infty \) on \( G \) and zero near \( G^* \), such that \( \overline{D_n(u - u_n, G)} \to 0 \).

where \( r_1 \) is the radius of the \( n \)th bounding circle in \( \mathbb{Z} \). From (15) and (19) it follows that \( D_n(u - w, \mathbb{Z}) \to 0 \) and it is also clear that \( D_n(u - w, \mathbb{Z}) \to 0 \).
Proof: Let \( u \) be of class \( D_2 \) on \( G \) with \( D_2(u, G) \) finite. Let \( G \) be mapped conformally on the region \( \Omega \) bounded by a finite number of circles and let \( w(s, t) \) be the transform of \( u(x, y) \); \( w \) is of class \( D_2 \) on \( \Omega \) with \( D_2(w, \Omega) \) finite and vanishes on \( \Omega^* \). Let \( f(s, t) \) denote the distance of a point \( (s, t) \) of \( \Omega \) from \( \Omega^* \). \( f \) satisfies a Lipschitz condition with constant unity and is analytic near \( \Omega^* \). Now, for each \( n > N \) (sufficiently large), define functions \( k_n(s, t) \) and \( u_n(s, t) \) as follows:

\[
k_n(s, t) = \begin{cases} 1 & \text{if } f(s, t) \leq \frac{1}{2n} \\ 0 & \text{if } f(s, t) > \frac{1}{2n} \end{cases}
\]

\[
k_n (s, t) = 2n \int_{f(s, t) = \frac{1}{2n}} \frac{\partial}{\partial f} \left( \int_0^f \frac{1}{2} \right) ds = \frac{1}{2n} \leq f(s, t) \leq \frac{1}{n}
\]

\[
k_n (s, t) = 0 \quad \text{if } f(s, t) > \frac{1}{2n}
\]

\[
u_n(s, t) = u(s, t) \cdot k_n(s, t)
\]

Let \( \Omega_n \) denote the part of \( \Omega \) for which \( \frac{1}{2n} \leq f(s, t) \leq \frac{1}{n} \) and \( u_n \) that were \( f(s, t) \leq \frac{1}{2n} \). Then

\[
D_2(u_n - u, \Omega) = D_2(u_n - u, \Omega_n) + 2 D_2(u_n, \Omega_n) + 2 D_2(u_n, \Omega^*).
\]

Now

\[
\int \int \left[ \int_0^{1/2n} \frac{k_n}{2} \text{ds} \text{dt} \right] \left[ \int_0^{1/2n} \frac{k_n}{2} \text{ds} \text{dt} \right] = \sum_{i=1}^{r_1} \int_{y_i - \frac{1}{2n}}^{y_i + \frac{1}{2n}} \int_{y_i - \frac{1}{2n}}^{y_i + \frac{1}{2n}} \left[ \int_0^{1/2n} \frac{k_n}{2} \text{ds} \text{dt} \right] \text{dv} \text{dt}
\]

where \( r_1 \) is the radius of the \( i \)th bounding circle in \( \Omega^* \). From (18) and (19) it follows that \( D_2(w_n - w, \Omega) \to 0 \) and it is also clear that \( D_2(w_n - w, \Omega) \to 0 \).
Now for each \( n \geq N \), we may choose \( h_n \) so small that \( w_{n+1}^{3} \) is defined on \( \Sigma \) and zero near \( \Sigma^* \); here \( w_{n}^{3} \) denotes the third iterated average of \( w_n \) and is of class \( C^m \) on \( \Sigma \); \( h \) may also be chosen so that

\[
D_2 \left( \frac{u_n - u_{n+1}^{3}}{h^{3/2}}, \Sigma \right) < \frac{1}{n}
\]

If we let \( u_n \) denote the transform of \( w_{n}^{3} \), we see that we have the desired approximation to \( u \). It is clear that \( u \) is of class \( D^2 \) over \( \mathbb{R}^2 \) and \( u_n \) of class \( C^m \) over \( \mathbb{R}^2 \), and hence it is evident that \( D_2 (u_n, \mathbb{R}^2) \to 0 \) as well as

\[
D_2 (u - u_n, \mathbb{R}^2) \to 0.
\]

**Theorem 3:** Let \( u(x,y) \) be of class \( D^2 \) on a region \( G \) of class \( C_2 \) with \( D_2 (u, G) \) finite, and suppose that \( u \) vanishes on \( G^* \). Then, if we define \( u = 0 \) for \( (x,y) \) not in \( G \), \( u \), as extended, is of class \( D^2 \) over \( \mathbb{R}^2 \) and is given almost everywhere by

\[
\bar{u}(x,y) = \frac{1}{2\pi} \iint \frac{\bar{u}_x(\xi,\eta) + \bar{u}_y(\xi,\eta)}{(\xi-x)^2 + (\eta-y)^2} \, d\xi \, d\eta
\]

In fact (20) holds at each point \( (x,y) \) where the integrand on the right is summable.

If \( u(x,y) \) also satisfies

\[
D_2 \left[ \frac{u}{f}, \mathbb{R} \right] \leq h^{-1}, \quad 0 < h < 1
\]

for every circle with center at some point \( x_0 \), then \( \bar{u}(x_0, y_0) \) and (20) exist, \( \bar{u} \) is given by (20), and

\[
|\bar{u}(x_0, y_0)| \leq \frac{2\pi}{h} \left( \frac{1}{h} \right)^{1/2} \left( 1 + h^{-1} \right) \leq \frac{2\pi}{h}
\]

where \( \bar{c} \) is the diameter of \( \Sigma \). If \( u(x,y) \) satisfies (21) for every \( P \) and \( r \), then \( u \), as extended, satisfies (22) and a condition \( B[1/2, N(a)] \) everywhere, where

\[
N(a) = \frac{8\pi^{1/2}}{a^{1/2} \lambda^{1/2}}
\]

with \( \lambda = \sqrt{3} \).
Proof: According to Lemma 3, we may find a sequence \( \{u_n(x, y)\} \), each of class \( C^m \) and zero near \( G \) such that \( D_2(u_n - u, G) \to 0 \). If we define \( u_n = 0 \) for all \((x, y)\) outside of \( G \) we see that \( u_n \) is of class \( C^m \) all over \( \bar{G} \) and \( D_2(u_n - u, G) \to 0 \). Also, for each \( u_n(x, y) \), we have

\[
- \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{R}{(x - x_0)^2 + (y - y_0)^2} \right) d\theta - \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{R}{(x - x_0)^2 + (y - y_0)^2} \right) d\theta = - \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} u_n(x + \rho \cos \theta, y + \rho \sin \theta) \rho d\rho d\theta.
\]

Let \( R \) be chosen large enough so that \( G(\rho, \theta) \) contains \( G \) in its interior. From the above, \( u_n \to u \) in the mean of order 2 over \( \bar{G} \) and from the proof of Theorem 2 it follows that the corresponding integrals tend in the mean of order 1 to the right side of (20). Thus (20) holds almost everywhere. From this the existence of \( u(x_0, y_0) \) follows at each point where (21) holds for every \( r \).

Now suppose (21) holds at \((x_0, y_0)\). Let \((r, \theta)\) be polar coordinates with pole at \((x_0, y_0)\) and let \( w(r, \theta) = u(x + r \cos \theta, y + r \sin \theta) \). From Lemma 1 and its proof, we see that there exists a summable function \( \phi(\theta) \) such that

\[
\lim_{r \to 0} \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1}{r} - \phi(\theta) \right) d\theta = 0.
\]

Then \( w(r, \theta) \) exists and \( w(r, \theta) \) is continuous, hence \( u(x, y) \) is continuous everywhere by Theorem 2 in case (21) holds for every \( P \) and \( r \).

The condition \( B[\lambda/2, N(a)] \) as given follows from Theorem 2, §5, if (21) holds everywhere.
To show in general that \( u(x_0, y_0) \) exists at each point where (20) exists, we merely observe that

\[
\frac{1}{2\pi} \iint \left( \frac{(x-x_0) \overline{w}(x, y) + (y-y_0) \overline{w}(x, y)}{(x-x_0)^2 + (y-y_0)^2} \right) \, dx \, dy = \frac{1}{2\pi} \iint \overline{u}(x, y) \, dx \, dy
\]

if \( R \) is chosen large enough. Since \( \overline{w}(r, \theta) \) is of class \( D_2 \) in \((r, \theta)\) for \( \varepsilon \leq r \leq R \), \( 0 \leq \theta \leq 2\pi \) and A.C. in \( r \) for almost all \( \theta \), and in \( \theta \) for almost all \( r \) in any such region, \( \varepsilon > 0 \), and since \( \overline{w}(r, \theta) \) is summable for \( 0 \leq r \leq R \), \( 0 \leq \theta \leq 2\pi \), we see immediately that a summable function \( \gamma(\theta) \) exists such that

\[
\left. \left. \frac{1}{2\pi} \int_0^{2\pi} \overline{u}(r, \theta) - \gamma(\theta) \, d\theta \right|_{r \to 0} = 0, \quad \int_0^{2\pi} \left| \overline{w}(r, \theta) - \gamma(\theta) \right| \, d\theta = \int_0^{2\pi} \left| \overline{u}(r, \theta) - \gamma(\theta) \right| \, d\theta
\]

Thus if we let \( u_0 = \) the mean value of \( \gamma(\theta) \), we see that

\[
\left| \int \frac{1}{2\pi} \int_0^{2\pi} \overline{u}(r, \theta) \, d\theta \, dx - u_0 \right| \leq \frac{2}{\pi} \int_0^{2\pi} \left| \int \left| \overline{w}(r, \theta) - \gamma(\theta) \right| \, d\theta \right| \, dx \leq \frac{2}{\pi} \int \left| \int \left| \overline{u}(r, \theta) - \gamma(\theta) \right| \, d\theta \right| \, dx
\]

which tends to zero with \( r \). Thus \( u(x_0, y_0) \) exists and

\[
\overline{u}(x_0, y_0) = u_0 = \frac{1}{2\pi} \int_0^{2\pi} \gamma(\theta) \, d\theta = -\frac{1}{2\pi} \int_0^{2\pi} \left[ \overline{w}(r, \theta) - \gamma(\theta) \right] \, d\theta = -\frac{1}{2\pi} \int_0^{2\pi} \left| \overline{u}(r, \theta) - \gamma(\theta) \right| \, d\theta
\]

as we observed above.

**Theorem 4**: Let \( G \) be a bounded region, let \( \Phi(c) \) be a completely additive set function on \( G \), and let \( \xi \) be of class \( D_2 \) on \( G \), be zero on \( \partial G \), and satisfy (14). Then

\[
\iint_{(c \in G)} \xi \, d\Phi(c) = \iint \left( \xi_x V_x + \xi_y V_y \right) \, dx \, dy
\]

as observed above.
where the right side exists as a Lebesgue integral, \( V \) being the potential of \( \phi (x) \) as defined by (1).

Proof: We first observe that, for any such \( \phi \) and \( \xi \), we have

\[
\int_\mathbb{R}^4 \left( \frac{(x-\xi)^2 + (y-\xi)^2}{(x-\xi)^2 + (y-\xi)^2} \right) dx dy \cdot \phi(\xi, \eta) d\xi d\eta
\]

exists and is finite by (10) so that the four dimensional Lebesgue-Stieltjes integral without the absolute value signs exists and may be evaluated by means of the iterated double integrals in either order. Now let \( V \) be defined by (1) and then, by using formulas (3) and (20), we see that

\[
- \int_\mathbb{R}^2 \left( \frac{(x-x') + (y-y')}{(x-x')^2 + (y-y')^2} \right) dx dy \cdot \phi(x', y') d\xi d\eta = \int_\mathbb{R}^2 \left( \frac{(x-x') + (y-y')}{(x-x')^2 + (y-y')^2} \right) dx dy \cdot \phi(x', y') d\xi d\eta
\]

which proves the theorem.

Lemma 4: Let \( P_1 \) and \( P_2 \) be two points of the plane with \( |P_1 P_2| = \xi > 0 \).

Let \( r_1 \) and \( r_2 \) be the distances of \( P_1 \) and \( P_2 \) from a variable point \((x, y)\). Then

\[
0 \leq \gamma (r_1, r_2) = \min_{x, y, \xi, \eta} \int_{P_1 \leq \eta} \frac{dx dy}{\gamma_{x, y}} \leq \left\{ \begin{array}{ll} \frac{4\pi r}{\alpha} & 0 \leq r \leq \frac{1}{4} \\ \frac{4\pi r \sqrt{\gamma}}{\alpha} & \frac{1}{4} \leq r \leq \alpha \\ \frac{4\pi \sqrt{\gamma}}{\alpha} + \frac{8\pi \sqrt{\gamma}}{3} & \gamma \geq \alpha \end{array} \right.
\]

Proof: First of all, if new coordinates \((x', y')\) are taken in with \( x' = kx, y' = ky \), we see that \( r_1' = kr_1, r_2' = kr_2, k' = k \xi \) so that

\[
\gamma (kr, k\xi) = \gamma (r, \xi), k > 0. \quad \therefore \quad \gamma = h(\frac{\gamma}{\xi}). \quad \text{Hence we choose } \xi = 1.
\]

Thus \( P_1 = (\frac{1}{2}, 0), P_2 = (\frac{1}{2}, 0), P_0 = (x_0, y_0) \), and try to dominate the maximum of the integral. Clearly we may take \( y_0 = 0 \). Now

\[
\gamma_1^2 - \gamma_2^2 = [ (x + \frac{1}{2})^2 + y^2 ] [ (x - \frac{1}{2})^2 + y^2 ] = \gamma (x, y)
\]

\[
\gamma_1 = 4 \sqrt{4x^2 + y^2 - \frac{1}{4}}
\]
Now let \( r \) be fixed and \( \leq \frac{1}{2} \) and let
\[
 f(x_0) = \int \frac{1}{(x + \frac{1}{2})^2 + \gamma^2} \, dx \, d\gamma
\]
Then
\[
 f'(x_0) = \int \frac{1}{(x + \frac{1}{2})^2 + \gamma^2} \, d\gamma
\]
It is seen that if \( x_0 < -\frac{1}{2} - r \), then \( f'(x_0) > 0 \), and if \( -\frac{1}{2} + r < x_0 < 0 \), then \( f'(x_0) < 0 \) (by referring to (27)), and as \( x_0 \to -\frac{1}{2} + r \) from either side
\[
 f'(x_0) \to +\infty \quad \text{and as } x_0 \to -\frac{1}{2} + r \quad \text{from either side } f'(x_0) \to -\infty; \quad \text{also } f'(0) = 0.
\]
Thus \( f(x_0) \) has symmetrically placed maxima at \( x_0 = \pm \bar{x}_0 \), \( \frac{1}{2} - r < \bar{x}_0 < \frac{1}{2} + r \), and a relative minimum at \( x_0 = 0 \). In fact it is also easy to see that \( f'(x_0) > 0 \)
for \( -\frac{1}{2} - r < x_0 \leq -\frac{1}{2} \), so \( \frac{1}{2} - r < \bar{x}_0 < \frac{1}{2} \).

If \( r > \frac{1}{2} \), we see that \( f'(x_0) \to +\infty \) as \( x_0 \to -\frac{1}{2} - r \) or \( \frac{1}{2} - r \), and it is obvious that \( f(\frac{1}{2} - r) > f(-\frac{1}{2} - r) \). Also \( f'(0) = 0 \) and \( f'(x_0) > 0 \) for
\( \frac{1}{2} - r < x_0 < 0 \). Thus \( f(x_0) \) has its max for \( x_0 = 0 \). This last is evidently also true for \( r = \frac{1}{2} \). Clearly if there is only one point \( P_1 \): \((0, 0)\), the maximum of
\[
 \int \frac{1}{\sqrt{x^2 + \gamma^2}} \, dx \, d\gamma
\]
\( C(P_1, r) \) where \( r \) is the diameter of \( C_0 \). Moreover if \( r \) is of class \( C_0 \) and \( f \) is any

Thus, for \( r > \frac{1}{2} \) \( h(r) < 4\pi r \). Now let \( \frac{1}{4} \leq r \leq 1 \), and let \( P_1 : (\bar{x}_0, 0) \).
Then the line \( x = 0 \) divides \( C(P_0, r) \) into two parts of measure \( x_1^2 \) and \( x_2^2 \) respectively, which are at a distance \( \geq \frac{1}{2} \) from \( P_2 \) and \( P_1 \) respectively. Since it is obvious that
\[
 \int \frac{\sqrt{r^2} \, d\gamma}{(x^2 + \gamma^2)^{\frac{1}{2}}}
\]
is a maximum among all \( E \) with \( m(E) \) constant if \( E \) is a circle with center at \((0, 0)\), and we see that
we see that
\[
 h(r) \leq 4\pi \sqrt{r}, \quad \frac{1}{4} \leq r \leq 1.
\]
If \( r > 1 \), then \( \bar{x}_0 = 0 \) and \( (r_1 r_2)^{-1} \leq \frac{\pi}{2} (x^2 + y^2)^{-1} \)

\[ h(r) \leq h(1) + \frac{\mu}{3} \int_0^r \left( \int_0^{2\pi} d\theta \int_0^\pi d\phi \right) = 4\pi V_2 + \frac{8\pi}{3} C_0 f r. \]

From this, the lemma follows.

**Theorem 5:** Let \( \Phi \) be a completely additive set function, defined

on a bounded region \( G \), satisfying

\[ (28) \quad V_\Phi \left[ C(P, r) \cdot S \right] \leq N r^a, \quad 0 < a < 1 \]

for every circle \( C(P, r) \), \( V_\Phi(E) \) denoting the variation of \( \Phi \) over \( E \). Let \( V \) be the potential of \( \Phi \). Then \( V \) is of class \( D_2 \) everywhere, and satisfies conditions \( A[\lambda, M(a)] \) and \( B[\lambda/2, N(a)] \) where

\[ (29) \quad H(a) = \frac{M V_\Phi(S)}{\pi} \left( \frac{2 + \frac{8\pi}{3}\alpha}{r^2 + \frac{8\pi}{3}\alpha} \right) \lambda^\alpha, \quad N(a) = \frac{3}{\alpha(3\pi)^{\frac{3}{2}}} \left[ H V_\Phi(S) \right]^2 \left( \frac{2}{r^2 + \frac{8\pi}{3}\alpha} \right)^2 \]

Moreover if \( h_t(x, y; S, \gamma) \) and \( V_t(x, y) \) are defined as in the proof of Theorem 1, then

\[ (30) \quad |V_t(x, y) - V_t(x, \gamma)| \leq \frac{M}{2\pi} r^\lambda \]

On \( G \) we have

\[ (31) \quad |V(x, \gamma)| \leq \frac{1}{2\pi} \left[ H \int_0^\pi (C_0 + \lambda^{-1}) \right] \]

where \( \delta \) is the diameter of \( G \). Moreover if \( G \) is of class \( C_2 \) and \( S \) is any function of class \( D_2 \) on \( G \) and zero on \( G^c \), then

\[ (32) \quad \int_S \delta d\Phi = - \int_S \left( \bar{\Phi}_x V_x + \bar{\Phi}_y V_y \right) d\alpha d\gamma, \]

where the integral on the left exists as a Lebesgue-Stieltjes integral.

**Proof:** We define (as before \( V_\Phi(E) \) is the variation of \( \Phi \) over \( E \))

\[ H(s, t; \gamma) = V_\Phi \left[ C(s, t; \gamma) \cdot S \right], \quad h(r; \Phi, \gamma; s, r) = \left\{ \begin{array}{ll} \frac{4\pi V_2 + \frac{8\pi}{3} C_0 \gamma}{\alpha} & d \leq r \\ 4\pi V_2 \left( \frac{\gamma}{\alpha} \right)^2 & d > r \end{array} \right. \]

and we see that

Thus (31) follows.
for every circle $C(P, r)$. Then it follows that

$$
\frac{1}{4\pi^2} \sum \sum \left\{ \sum \sum \left[ \frac{(x-s)(y-t)(\xi-s)(\eta-t)}{(x-s)^2 + (y-t)^2} \right] \right\} \frac{1}{(x-s)^2 + (y-t)^2} \frac{d\phi(x,y)}{d\theta(x,y)} \frac{d\phi(s,t)}{d\theta(s,t)} \\
\leq \frac{1}{4\pi^2} \sum \sum \left\{ \sum \sum \left[ \frac{(x-s)(y-t)(\xi-s)(\eta-t)}{(x-s)^2 + (y-t)^2} \right] \right\} \frac{1}{(x-s)^2 + (y-t)^2} \frac{d\phi(x,y)}{d\theta(x,y)} \frac{d\phi(s,t)}{d\theta(s,t)} \\
+ \frac{2}{\pi} \sum \sum \left\{ \sum \sum \left[ \frac{(x-s)(y-t)(\xi-s)(\eta-t)}{(x-s)^2 + (y-t)^2} \right] \right\} \frac{1}{(x-s)^2 + (y-t)^2} \frac{d\phi(x,y)}{d\theta(x,y)} \frac{d\phi(s,t)}{d\theta(s,t)}
$$

for every circle $C(P, r)$. Hence

$$
\sum \left\{ \sum \left[ \frac{(x-s)(y-t)(\xi-s)(\eta-t)}{(x-s)^2 + (y-t)^2} \right] \right\} \frac{1}{(x-s)^2 + (y-t)^2} \frac{d\phi(x,y)}{d\theta(x,y)} \frac{d\phi(s,t)}{d\theta(s,t)} \\
\leq \frac{1}{4\pi^2} \sum \sum \left\{ \sum \sum \left[ \frac{(x-s)(y-t)(\xi-s)(\eta-t)}{(x-s)^2 + (y-t)^2} \right] \right\} \frac{1}{(x-s)^2 + (y-t)^2} \frac{d\phi(x,y)}{d\theta(x,y)} \frac{d\phi(s,t)}{d\theta(s,t)}
$$

for every circle $C(P, r)$, all the $6$-dimensional integrals existing as Lebesgue Stieltjes integrals. Thus (29) follows.

Now

$$
|V(x, y)| = \left| \frac{1}{4\pi} \sum \left[ \frac{1}{\sqrt{(x-s)^2 + (y-t)^2}} \right] \phi(s,t) \frac{1}{\sqrt{(x-s)^2 + (y-t)^2}} \frac{dH(x,y)}{d\theta(x,y)} \right| \\
= \frac{1}{2\pi} \left[ R \phi \frac{1}{\sqrt{R^2}} + \int_0^R \frac{1}{\sqrt{R^2 - r^2}} d\theta \right] \frac{dH(x,y)}{d\theta(x,y)} \\
= \frac{1}{2\pi} \left[ R \phi \frac{1}{\sqrt{R^2}} + \int_0^R \frac{1}{\sqrt{R^2 - r^2}} d\theta \right] \frac{dH(x,y)}{d\theta(x,y)}
$$

for every $(x, y)$ in the plane, $R$ being chosen so large that $C(x, y; R) \ni G$.

Thus (31) follows.
Finally

\[ |V_\alpha(x, y) - V_\gamma(x, y)| = \left| \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{z - \epsilon} \left[ \left( \frac{x - x'}{\epsilon} \right)^2 + \left( \frac{y - y'}{\epsilon} \right)^2 \right] \frac{1}{\epsilon^2} \right| \]

\[ \leq \frac{1}{2\pi} \int_{0}^{2\pi} \left| \frac{H(x, y; s)}{s} \right| ds \leq \frac{M}{2\pi L} \leq \lambda, \]

which proves (30).

To prove (32), we observe that the functions

\[ B = \frac{(x - \epsilon) \xi_x + (y - \epsilon) \xi_y}{(x - \epsilon)^2 + (y - \epsilon)^2}, \]

\[ \frac{1}{2} \left( \xi_x^2 + \xi_y^2 \right), \]

\[ C = \xi_x^2 + \xi_y^2, \]

\[ D = (x - \epsilon)^2 + (y - \epsilon)^2, \]

are measurable with respect to the completely additive set functions in

\((x, y; x, y, \gamma)\)-space generated by \(m(e, \gamma)\) and \(\phi(e, \gamma)\) or \(m(e, \gamma)\) and \(V_\phi(e, \gamma)\).

Now, define

\[ Z(x, y, \gamma) = \int_{\mathbb{R}} (\xi_x^2 + \xi_y^2) \, d\gamma \]

Then

\[ |Z(x, y, \gamma)| \leq \left( \frac{\pi D_2(\xi, \gamma)}{2} \right)^{\frac{1}{2}}. \]

Then, for each \(r > 0\),

\[ \left\{ \left\{ \left( \frac{\pi D_2(\xi, \gamma)}{2} \right)^{\frac{1}{2}} \right\} \right\} \leq \left\{ \left\{ \left( \frac{\pi D_2(\xi, \gamma)}{2} \right)^{\frac{1}{2}} \right\} \right\} \leq \left\{ \left\{ \left( \frac{\pi D_2(\xi, \gamma)}{2} \right)^{\frac{1}{2}} \right\} \right\} = \frac{\pi D_2(\xi, \gamma)}{2}. \]
Thus the four-dimensional Lebesgue-Stieltjes integral

\[
\iint B \, d\gamma \, d\Phi(c_{xy})
\]

exists and may be evaluated by repeated double integrals in either order.

Hence, by Theorem 1,

\[
- \oint \left( \oint _{\gamma \, \delta} \, d\gamma \right) \, d\Phi(c_{xy}) = -\frac{1}{2\pi} \iint B \, d\gamma \, d\Phi(c_{xy}) = \iint B \, d\gamma \, d\Phi(c_{xy})
\]

since \( \oint \) exists and is given by the inside bracket whenever the inside bracket exists on a Lebesgue integral.

Theorem 6: Let \( \Phi_n(x) \) be completely additive set-functions defined on the bounded region \( G \) which tend weakly to the set function \( \Phi(x) \) on \( G \) and satisfy (28) uniformly. Then \( \Phi(x) \) satisfies (28), the corresponding potential functions \( V_n \) and \( V \) satisfy (29), (30), and (31) uniformly and

\[
\lim_{n \to \infty} D_2(V_n - V_t) = 0
\]

Proof: From Theorem 5, the \( V_n \) and \( V \) are equicontinuous and

\[
|V_n(x, y) - V_n(x', y')| \leq \frac{1}{2\pi\lambda} \, t^4, \quad |V(x, y) - V(x', y')| \leq \frac{1}{2\pi\lambda} \, t^4
\]

the right sides being independent of \( n \). From the definition and standard theorems on weak convergence, it follows that \( V_n \to V_t \) on each bounded portion of the plane. Thus \( V_n \to V \) on each bounded portion of the plane.

Now \( D_2(V_n^2) \) and \( D_2(V_t^2) \) are uniformly bounded. Let \( U_n = V_n - V \);

\[
\psi_n(x) = \Phi_n(x) - \Phi(e^x) \quad \text{and we see that} \quad U_n \to 0 \text{ on } C(G) \text{ and } \psi_n(x) \to 0
\]

weakly on \( C(G) \) which circle is chosen to include \( G \) in its interior. Now let \( H_n \) be the function harmonic on \( C(G) \) and coinciding with \( U_n \) on \( C(G) \), and let

\[
W_n = U_n - H_n.
\]

Now
for \((x, y)\) on \(C^*(0, R)\) so that \(\frac{\partial}{\partial \theta} U_{a}(R \cos \theta, R \sin \theta)\) tends uniformly to zero, so that \(D_{\theta} [H_n, C(0, R)] \to 0\). Also, by Theorem 5,

\[
\int W_n \left( \frac{\partial}{\partial \theta} U_{a} + \frac{\partial}{\partial \phi} U_{b} \right) dx \, dy = D_{\Delta} \left[ U_{a}, C(0, R) \right] + \int \left( \frac{\partial}{\partial \theta} U_{a} + \frac{\partial}{\partial \phi} U_{b} \right) dx \, dy \bigg|_{C(0, R)}
\]

since \(W_n \to 0\) on \(C^*(0, R)\) and \(H_n\) is harmonic on \(C^*(0, R)\). Thus

\[
\lim_{n \to \infty} D_{\Delta} \left[ U_{a}, C(0, R) \right] = \lim_{n \to \infty} \left( D_{\Delta} \left[ H_n, C(0, R) \right] + D_{\Delta} \left[ U_{a}, C(0, R) \right] \right) = 0
\]

Then it is easy to see that \(v_0(x, y)\) is measurable and summable in \(x\) for each \(y\).

### §7. Haar's Lemmas

**Definition 1.** By "almost all rectangles" in a region \(G\) we mean all rectangles \((a, c; b, d)\) in \(G\) where \((a, c; b, d)\) is not in a four-dimensional set for every \((x, y; a, b, c, d)\) and also that \(v_0(x, y)\) is measurable and summable in \(y\) for each \(x\) in \(G\) such that

\[
A(x, y) \quad \text{and} \quad B(x, y)
\]

be of class \(L_\infty ( \geq 1)\) over the simply connected region \(G\) and suppose that

\[
\int_{K} \left( A \, dx + B \, dy \right) = 0
\]

for almost all rectangles in \(G\). Then there exists a function \(v\) of class \(D_\infty^\#\) on \(G\) such that

\[
D_x v = A, \quad D_y v = B
\]

almost everywhere on \(G\). The function is uniquely determined up to an additive constant plus a null function.

**Proof:** Let \(D_s\) \(= \{ \alpha; \beta; \gamma; \delta \}\) be a cell interior to \(G\). If \(a,\beta,\gamma,\delta\) are all \(\leq 0\), then \(\alpha = \beta,\gamma,\delta\) and the second for a cell \(a,\beta,\gamma,\delta\) and the second for a cell \(a,\beta,\gamma,\delta\) are the two functions of class \(D_\infty^\#\) satisfying the above conditions on \(G\) and \(H_n\) is almost everywhere on \(G\). The function is uniquely determined up to an additive constant plus a null function.
A(x,c) is summable for \( \alpha < x < \beta \). Also if \((a,c)\) is not in a set \(Z\) of measure

then \(Z\) is measurable on \((a,b)\) and suppose that

c = \int \int_B 2 \cdot 2 \cdot y \cdot \left[ A(x, y) - B(x, y) \right] dx \, dy \int \int_A 2 \cdot 2 \cdot x \cdot \left[ A(x, y) - B(x, y) \right] dx \, dy

for every bounded measurable function \(\varphi(x)\) for which

fails to exist and be zero is of measure zero.

Hence, suppose a not in \(X\), c not in \(Y\), \((a,c)\) not in \(Z\), and define

\[ V_1(x, y) = \int \int A(x, y) \, dy + \int \int B(x, y) \, dy \]

for every \((a,c); b,d)\) and also that \(V_2(x, y)\) is summable and summable in \(y\) for each \(x\) in \(D\) such that

\[ \int \int [V_1(x, y) - V_1(x, c)] dy \cdot \frac{dy}{dx} = \int \int [A(x, y) - A(x, c)] \, dx \, dy \]

for every \((a,c); b,d)\) on \(D\). Moreover \(V_1\) and \(V_2\) coincide except possibly on a set of measure zero so that either one satisfies the condition of the theorem.

Now if \(V_1\) and \(V_2\) are any two functions of class \(D_1\) satisfying the above conditions, then \(V = V_1 - V_2\) os si., abj. pm \(D\) and has the property that

\[ \int \int [V(x, y) - V(x, c)] \, dx \, dy = 0 \]

the first for \(c\) and \(d\) not in \(Z\), and any \((a,b)\), and the second for \(a\) and \(b\) not in \(Z\) in any \((c,d)\). Thus our theorem follows for the cell \(D\). It is clear how to extend the function to the whole of \(G\).
Lemma 2: Let \( f(x) \) be summable on \((a, b)\) and suppose that
\[
\int_a^b f(x) \gamma(x) \, dx = 0
\]
for every bounded measurable function \( \gamma(x) \) for which
\[
\int_a^b \gamma(x) \, dx = 0
\]
Then \( f \) is constant, except possibly for a set of measure zero. If (1) holds for every bounded measurable \( \gamma(x) \), then \( f(x) = 0 \) except possibly for a set of measure zero.

Proof: The second statement is obvious, for we may take \( \gamma(x) = 0 \) wherever \( f(x) = 0 \) and
\[
\gamma(x) = \frac{f(x)}{|f(x)|}
\]
wherever \( f(x) \neq 0 \).

Now any bounded measurable function \( \gamma(x) \) may be written as
\[
\gamma(x) = \gamma^+(x) + \frac{1}{b-a} \int_a^b \gamma(x) \, dx
\]
and
\[
\int_a^b \gamma^+(x) \, dx = 0
\]
Thus
\[
\int_a^b \gamma^+(x) \, dx = 0 = \int_a^b \{(f(x) - \frac{1}{b-a} \int_a^b f(x) \, dx) \gamma^-(x) \, dx
\]
for every bounded measurable \( \gamma \) so that
\[
f(x) = \frac{1}{b-a} \int_a^b f(x) \, dx
\]
almost everywhere.

Lemma 3: Let \( u(x,y) \) and \( v(x,y) \) be of class \( D^p \) (\( p \) \(\geq 2\)) and \( D^q \) respectively on a cell \( R : (a, c; b, d) \) with \( p^{-1} + q^{-1} = 1 \); we include the case \( p = 1, q = \infty \) for almost every rectangle \( R \) in \( \mathbb{R}^2 \). If \( \Omega \) is simply connected, there exists a
(or \( p = \infty \), \( q = 1 \)) by interpreting this to mean that \( v \) (or \( u \) in the second case) satisfies a uniform Lipschitz condition on \( R \). Suppose that \( v \) is absolutely continuous everywhere, \( \int |v|^q \) is summable for every \( f \) which satisfies a uniform Lipschitz condition on \( R \), \( s \) denoting arc length. Then

\[
\int \left( \frac{u}{v} \frac{\partial v}{\partial y} - \frac{v}{u} \frac{\partial u}{\partial y} \right) \, dx \, dy = \int \frac{v}{u} \, ds
\]

Proof: First, we may extend the definitions of \( u \) and \( v \) so that they are of class \( D^1 \) and \( D^1 \) respectively over the whole plane. Now let \( u_n \) and \( v_n \) be the usual average functions. Then

\[
\int \left( \frac{u}{v} \frac{\partial v}{\partial y} - \frac{v}{u} \frac{\partial u}{\partial y} \right) \, dx \, dy = \int u_n \, ds \, dy = - \int v_n \, dx \, ds
\]

Now we know that

\[
\lim_{h \to 0} \int \left( |u_n - u| + |u| + |v_n - v| \right) \, dx \, dy = \lim_{h \to 0} \int (|v_n - v| + |v| + |v_n - v|) \, dx \, dy = 0
\]

Also there exist subsequences \( h_n \) and \( k_n \) tending to zero so that

\[
\lim_{n \to \infty} \int |v_n(x, y_0) - v(x, y_0)| \, dx = \lim_{n \to \infty} \int |v_n(x_0, y) - v(x_0, y)| \, dy = 0
\]

\[
\lim_{n \to \infty} \int |u_n(x, y_0) - u(x, y_0)| \, dx = \lim_{n \to \infty} \int |u_n(x_0, y) - u(x_0, y)| \, dy = 0
\]

uniformly in \((x_0, y_0)\). Thus in (3) we may choose \( k_n \) and let \( n \to \infty \) obtaining

\[
\int \left( \frac{u_n}{v} \frac{\partial v}{\partial y} - \frac{v_n}{u} \frac{\partial u}{\partial y} \right) \, dx \, dy = \int \frac{v_n}{u} \, ds \, dy = \int u_n \, ds \, dy
\]

In this we may take \( h = h_n \) and let \( n \to \infty \) and we obtain our result.

Theorem 1: Let \( G \) be a bounded region and let \( A \) and \( B \) be functions of class \( L^\infty \) on \( G \) (\( \alpha \geq 1 \)). Suppose that

\[
\int \left( \frac{A}{B} \frac{\partial B}{\partial y} + \frac{B}{A} \frac{\partial A}{\partial y} \right) \, dx \, dy = 0
\]

for every \( f \) satisfying a uniform Lipschitz condition over \( G \) and vanishing on \( G^* \).

Then

\[
\int A \, dy - \partial \, dx = 0
\]

for almost every rectangle \( R \) in \( G \). If \( G \) is simply connected, there exists a
function \( v \) of class \( D^\alpha \) on \( G \) such that

\[
(6) \quad \overline{v} = D_y v = \alpha, \quad \overline{v}_x = D_x v = -B
\]

almost everywhere. If (3) is satisfied for every \( \xi \) which satisfies a uniform Lipschitz condition on \( G \) and if \( G \) is also of class \( L \), \( v \) may be chosen to vanish on \( G^* \).

Proof: Choose \((x_0, y_0)\) in \( G \) and choose \( \alpha > 0 \) and \( h > 0 \) so that the rectangle \((x_0 - \alpha, y_0 - h\alpha; x_0 + \alpha, y_0 + h\alpha)\) is interior to \( G \). In this rectangle we choose coordinates \((r, s)\) as follows: Given \((x_1, y_1)\), \( r_1 \) is the value of \( r \) for which the rectangle \( R_r: (x_0 - r, y_0 - hr; x_0 + r, y_0 + hr) \) contains \((x_1, y_1)\) on its boundary, and \( s_1 \) is the arc length along \( R_r^* \) measured in the positive direction from the point \((x_0 + r, y_0 - hr)\). Thus our rectangle contains all points \((r, s)\) for which \( 0 \leq r \leq \alpha \) and \( 0 \leq s \leq 4r(1 + h) \). Now, if we let \( \xi \) be a function of \( r \) alone defined and satisfying a uniform Lipschitz condition for all \( r \) and vanishing for \( r \geq \alpha \), we see that \( \xi \) satisfies a uniform Lipschitz condition in \((x, y)\) on \( G \) and vanishes on \( G^* \). Our integral (4) takes the form

\[
\int \int (A \xi_x + B \xi_y) \, dx \, dy = \int \int \xi \, \gamma(r) \, dr, \quad \gamma(r) = \int A \, dy - B \, dx \quad \text{on} \quad K^*.
\]

and \( \gamma(r) \) is certainly summable. Since \( \xi(0) \) may have any value, \( \xi_r \) is perfectly arbitrary, and it follows from Lemma 1 that \( \gamma(r) = 0 \) for almost all \( r \), \( 0 \leq r \leq \alpha \).

Now if we determine a rectangle \( R \) in \( G \) as above by the numbers \((x_0, y_0, r, h)\) we see first that

\[
\Psi(x_1, y_1, r, h) = \int_{K^*} A \, dy - B \, dx
\]

is measurable in \((x_0, y_0, r, h)\) and hence also in \((a, b, c, d)\) since we have

\[
a = x_0 - r, \quad b = x_0 + r, \quad c = y_0 - hr, \quad d = y_0 + hr, \quad x_0 = \frac{a + b}{2}, \quad y_0 = \frac{c + d}{2}, \quad r = \frac{b - a}{2}, \quad h = \frac{d - c}{2}.
\]

Also, for each fixed \((x_0, y_0, r, h)\), we have seen above that the set of values of \( r \) we have
where \( \psi \) is not defined or is different from zero is of measure zero. Hence except for a four-dimensional set of \((a, c; b, d)\) of measure zero, we see that (5) is defined and equal to zero. If \( G \) is simply connected, the existence of the function \( v \) satisfying (6) follows from Lemma 1.

To prove the last statement, let \( G \) be also of class \( L \), let \( v \) be a function of class \( D^*_x \) on \( G \) satisfying (6), and suppose that (4) is zero for every \( \xi \) satisfying a uniform Lipschitz condition on \( \overline{G} \). Let \( x = x(\xi, \gamma) \), \( y = y(\xi, \gamma) \) be a regular representation of class \( L \) of a portion of \( \overline{G} \) on a cell \( R; a \leq \xi \leq b, 0 \leq \gamma \leq d \) such that the points \( \gamma = 0 \) correspond to points of \( G^* \).

Let \( w(\xi, \gamma) \) be the transform of \( v \) and let \( \varphi(\xi, \gamma) \) be that of a given \( \psi \).

Now, let \( \varphi(\xi, \gamma) \) be any function satisfying a uniform Lipschitz condition on \( R \) and zero on the whole boundary except on the points \( \gamma = 0, a < \xi < b \), where it may be arbitrary. Evidently the function \( \xi(x, y) \) defined as the transform of \( \varphi \) on the transform of \( \overline{R} \) and zero elsewhere satisfies a uniform Lipschitz condition on \( G \). Our integral (4) now takes the form

\[
\int \left( A_\xi x + B_\xi \xi \right) dx \times dy = \int \left( \xi \frac{\partial \varphi}{\partial x} - \xi \frac{\partial \varphi}{\partial y} \right) dx \times dy = \int \left( \xi_\gamma \overline{v}_\gamma - \xi_\gamma \overline{v}_\gamma \right) dx \times dy = - \int \overline{v}_\gamma \gamma \, ds = - \int \overline{v}_\gamma \gamma \, ds = 0.
\]

for every bounded measurable function \( \psi(\xi, 0) \) such that

\[
\int_{a}^{b} \psi_\gamma(x, 0) \, dx = \psi(b, 0) - \psi(a, 0) = 0.
\]

Thus \( \overline{w}(\xi, 0) \) is a constant almost everywhere. It follows that \( v \) takes on constant boundary values on \( G^* \).

Theorem 2: Let \( G \) be a bounded region, let \( \varphi(e) \) be a completely additive set function on \( G \), and let \( A \) and \( B \) be summable on \( G \). Suppose that, for each \( \xi \) which satisfies a uniform Lipschitz condition on \( \overline{G} \) and vanishes on \( G^* \), we have
(7) \[ \iint (A x + B y) \, dx \, dy + \iint \phi(x) \, dx \iint = 0 \]

Then, for almost all rectangles \( R \) in \( G \), we have

(8) \[ \int_R A \, dy - B \, dx = \phi(R) \]

Proof: Let \( V \) be the potential of \( \phi(x) \). Then

\[ \iint \phi(x) \, dx \iint = \iint (V_x x + V_y y) \, dx \, dy \]

for every \( \frac{\partial}{\partial} \) satisfying a uniform Lipschitz condition on \( G \) and vanishing on \( G^* \).

Thus for such \( \frac{\partial}{\partial} \)

\[ \iint [(A - V_x) x + (B - V_y) y] \, dx \, dy = 0 \]

Thus, for almost all rectangles \( R \), we have

\[ \int_R A \, dy - B \, dx = \int_V \phi(x) \, dx - \int_V V_y \, dx = \phi(R) \]

Theorem 3: Let \( G \) be a region of class \( C^2 \), let \( A \) and \( B \) be of class \( L_2 \) on \( G \), let \( \phi(x) \) be completely additive on \( G \), and suppose that \( \phi \) satisfies the condition

\[ V \left[ C(P, R) \right] \leq \Lambda \log \Lambda, \quad \Lambda > 0 \]

where \( V \left[ \phi \right] \) denotes the variation of \( \phi \) over \( E \). Suppose that (8) holds for almost every \( R \) in \( G \). Then (7) holds for every \( \frac{\partial}{\partial} \) of class \( D_2 \) on \( G \) which vanishes on \( G^* \).

Proof: By Lemma 3, \( \frac{\partial}{\partial} \), we can find a sequence \( \{ \frac{\partial}{\partial_n} \} \) of class \( C^2 \) on \( G \), zero near \( G^* \) and such that \( D_2 (\frac{\partial}{\partial_n} - \frac{\partial}{\partial}, G) \rightarrow 0 \). Let \( V \) be the potential of \( \phi(x) \). Then \( V \) is of class \( D_2 \) everywhere and

\[ \int_R (A - V_x) \, dy - (B - V_y) \, dx = 0 \]

for almost every \( R \) in \( G \). Thus if \( \frac{\partial}{\partial} \) is of class \( D_2 \) on \( G \) with \( D_2 (\frac{\partial}{\partial}, G) \) finite,
exists as a Lebesgue integral. Hence if a sequence $\mathcal{S}_n \to \mathcal{S}$ as above, then (9) will be the limit of the corresponding integrals for $\mathcal{S}_n$. Moreover, by Theorem 5, §6, we see that

$$\int \int \left( (A - \nu_x) S_x + (B - \nu_y) S_y \right) dx \, dy = \int \int (A S_x + B S_y) dx \, dy + \int \int \mathcal{S} \, d\phi$$

Now we can find a region $G_n$ of class $C^1$ for each $n$ with $\mathcal{G}_n \subset G$ such that $\mathcal{S}_n = 0$ on $G^*$ and

$$\int \int \left( (A - \nu_x) S_x + (B - \nu_y) S_y \right) dx \, dy = \int \int (A S_x + B S_y) dx \, dy$$

Now if $\Sigma$ is any simply connected subregion of $G$, there exists a function $v$ of class $D^2$ in $\Sigma$ such that

$$\bar{v}_x = -(B - \nu_y)$$
$$\bar{v}_y = A - \nu_x$$

almost everywhere in $\Sigma$. Hence if we define $A_{h-x}$ and $B_{h-y}$ as the iterated average functions for $(A - \nu_x)$ and $(B - \nu_y)$, we see that they are of class $C^1$ on $\bar{G}_n$ (if $h$ is small enough) with $A_{h-x} + B_{h-y} = 0$. Thus

$$\int \int \left( A_{h-x} S_x + B_{h-y} S_y \right) dx \, dy = - \int \int S_n \left( A_{h-x} + B_{h-y} \right) dx \, dy = 0$$

By letting $h \to 0$ it follows that (11) is zero for $\mathcal{S}_n$ and hence (10) is zero or (25) holds.

§8. Differentiability properties of functions minimizing a double integral

In this section we consider the functions $\mathcal{S}$ which minimize an integral

$$I(x, y) = \int \int f(\mathcal{S}_n, x^1, \ldots, x^n, p_1, \ldots, p_n) \, dx \, dy$$

among all functions of class $D^2$ which take on given boundary values on $G^*$, $G^*$ being assumed to be of class $C^2$. In case the given boundary values are continu
uous and there exist numbers \( m \) and \( M \) with \( 0 < m \leq M < \infty \) which are independent of \( \Gamma \), \( \gamma' \), for each \( (x,y,p,q) \) except possibly where \( q_i = 0 \) if \( 1 \leq i \leq n \); all functions \( f \) of the kind (\ref{1}) and hence satisfies all of one condition if it satisfies (\ref{1}).

and if \( z^* \) is of class \( D_2 \) with \( D_2(z,G) \) finite and takes on these boundary values, we have seen that a function \( z \) exists which minimizes \( I(z,G) \) among all functions of class \( D_2 \) on \( G \) and coinciding with \( z^* \) on \( G^* \), and that \( z \) is continuous on \( G \) and satisfies conditions \( A(\lambda, \overline{M}) \) and \( B(\lambda/2, \overline{N}) \) on \( G \), where \( \lambda = m/N \), \( \overline{M} = D_2(z,G) \). The latter conditions are satisfied in \( G \) whether the boundary values are continuous or not. It is assumed also that \( f \) is continuous in its arguments and convex in \((p,q)\) for each fixed \((x,y,z)\).

For the purposes of this paper we shall assume in addition to the above that \( f \) is of class \( C^* \) everywhere and of class \( C^n \), except possibly where \( p^i = q^i = 0 \), \( i = 1, \ldots, n \), where the second derivatives need not be continuous, and that there exist functions \( \varphi_1(R) \) and \( \varphi_2(R) \) defined and positive for all values of \( R \) such that

\[
\sum_{i=1}^{n} (p^i z^i + q^i z^i) \leq \frac{\varphi_1(R)}{2} \left( \sum_{i=1}^{n} (p_{i1} x_{i1} + q_{i1} y_{i1}) \right) \leq \frac{\varphi_2(R)}{2} \left( \sum_{i=1}^{n} (p_{i2} x_{i2} + q_{i2} y_{i2}) \right)
\]

for all \( x, y, z \) with \( p_i, q_i \) and for all \( x, y, z \) with \( x^2 + y^2 + \sum_{i=1}^{n} z_{i1}^2 \leq \overline{r}^2 \).

Clearly any function \( f \) which is of class \( C^n \) in all its arguments except possibly where \( p^i = q^i = 0 \), which is homogeneous of degree 2 in \((p,q)\) for each fixed \((x,y,z)\) and regular (i.e., with

\[
f_{p_i} = p_{i1} f_{p,1} + p_{i2} f_{p,2} \]

that each of its first partial derivatives satisfies the conditions of the \( \gamma \) of Lemma 1. Then, if \( z^* = f \) and \( x^i, i = 1, \ldots, n \), are not simultaneously zero, then
a positive definite quadratic form in $\langle \cdot \rangle$ for each $(x, y, z, p, q)$ except possibly where $p^i = q^i = 0$) satisfies (2) automatically and hence satisfies all of our conditions if it satisfies (1).

Lemma 1: Let $f(x^1, \ldots, x^n)$ be continuous for all values of $x$ and of class $C^1$ except possibly where $x^{k+1} = \ldots = x^n = 0$, in the neighborhood of which points the first derivatives of $f$ remain bounded provided $(x^1, \ldots, x^k)$ are in a bounded region of $k$-space. Then if $x^i + f^i \neq x^i = 0$, $i = k+1, \ldots, n$, are not simultaneously zero, we have

$$f(x^1 + f^1, \ldots, x^n + f^n) = f(x^1, \ldots, x^n) + A_1(x, f) f^k$$

where

$$A_1(x, f) = \int_0^1 \frac{f(x^1 + tf^1, \ldots, x^n + tf^n) - f(x^1, \ldots, x^n)}{t} dt, \quad i = 1, \ldots, N$$

the function of $t$ under the integral sign being continuous except for at most one value of $t$ in the interval and bounded throughout. Moreover $A_1(x, f)$ is continuous except possibly at a point $(x, f)$ such that $x^i + f^i = x^i = 0$, $i = k+1, \ldots, n$.

Proof: Let $f(t)$ satisfy a uniform Lipschitz condition on $(0, 1)$ and suppose $f(t)$ is continuous except possibly at one point in this interval. Then

$$f(1) = f(0) + \int_0^1 f'(t) dt.$$

Hence, let

$$f'(t) = \int f(x^1 + tf^1, \ldots, x^n + tf^n)$$

and we see that the above conditions on $f(t)$ are fulfilled with

$$f'(t) = \int \frac{f(x^1 + tf^1, \ldots, x^n + tf^n) - f(x^1, \ldots, x^n)}{t} dt$$

The stated continuity properties of the $A_1(x, f)$ are evident from its form.

Lemma 2: Let $f(x^1, \ldots, x^n)$ be of class $C^1$ everywhere and suppose that each of its first partial derivatives satisfies the conditions of the $f$ of Lemma 1. Then, if $x^i + f^i \neq x^i = 0$, $i = k+1, \ldots, n$, are not simultaneously zero, then
Proof: Let \( \gamma(t) \) be of class \( C^1 \) on \((0, 1)\) with \( \gamma'(t) \) satisfying a uniform Lipschitz condition and \( \gamma'(t) \) continuous except possibly at one point. Then we verify immediately that

\[
\gamma(s) = \gamma(0) + \int_0^s \gamma'(t) \, dt = \gamma(0) + s\gamma'(0) + \int_0^s (s-t) \gamma''(t) \, dt
\]

for

\[
\gamma'(s) = \gamma'(0) + \int_0^s \gamma''(t) \, dt, \quad \gamma(s) = \gamma(0) + \int_0^s \gamma'(t) \, dt.
\]

Thus, if we let

\[
\gamma(t) = \int \big( x^i + t \xi^i, \ldots, x^n + t \xi^n \big)
\]

we see that \( \gamma \) satisfies the above conditions with

\[
\gamma'(t) = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x^i} \gamma(x), \quad \gamma''(t) = \sum_{i=1}^n \xi_i \frac{\partial^2}{\partial x^i \partial x^j} \gamma(x).
\]

The stated continuity properties of the \( A_{ij} \) are evident from their forms.

**Theorem 1:** Let \( z \) be of class \( D_2 \) on a region \( G \) of class \( C_2 \) with \( D_2(z, G) \) finite, and suppose that it minimizes \( I(z, G) \) among all such functions which take on the same boundary values. Then \( z \) satisfies Haar's Equations

\[
\int_{\partial R} f_{\rho} \, d\gamma - \int_{\partial R} f_{\psi} \, d\chi = \int_{\partial R} f_{2i} \, d\chi \, d\gamma, \quad i = 1, \ldots, N,
\]

on almost all rectangles \( R \) in \( G \).
Proof: Let $\zeta$ satisfy a uniform Lipschitz condition of $\bar{G}$, vanishing on $C$. Then

$$
\gamma(\lambda) = \int \{z + \lambda \xi, \eta\} = \int \int (x, y, z + \lambda \xi, \rho + \lambda \eta, \sigma + \lambda \omega, \lambda + \lambda \zeta) \, dx \, dy
$$

is a minimum for $\lambda = 0$ so that $\nu(\zeta)$ must be zero for every $\zeta$. Thus we see that $\nu(C)$ exists and

$$
\int \int \left[ f_{2w}(x, y, z, \rho, \sigma) + \int \left( \int \frac{\partial f_{2w}}{\partial x}(x, y, z, \rho, \sigma) \, dx \right) \right] \, dx \, dy
$$

which we have admitted. From this, the theorem follows, using Morse's general

$$
\int \int \left[ \frac{1}{2} \left( A_{\beta j} (x, y, A, \zeta) \cdot (x, y, A, \zeta) \right) + B_{\alpha j} (x, y, A, \zeta) \cdot (x, y, A, \zeta) + C \right] \, dx \, dy
$$

where we define

$$
A_{\alpha \beta} = \int \left( \frac{1}{2} \frac{\partial f_{\rho}}{\partial x} \rho \rho \left[ f_{x x} + f_{y y} + f_{x y} + f_{y x} \right] \right) \, dx \, dy,
$$

$$
B_{\alpha \beta} = \int \left( \frac{1}{2} \int \rho \rho \left[ f_{x x} + f_{y y} + f_{x y} + f_{y x} \right] \right) \, dx \, dy,
$$

$$
C_{\alpha \beta} = \int \left( \frac{1}{2} \int \rho \rho \left[ f_{x x} + f_{y y} + f_{x y} + f_{y x} \right] \right) \, dx \, dy,
$$

$$
D_{\alpha \beta} = \int \left( \frac{1}{2} \int \rho \rho \left[ f_{x x} + f_{y y} + f_{x y} + f_{y x} \right] \right) \, dx \, dy,
$$

$$
F_{\alpha \beta} = \int \left( \frac{1}{2} \int \rho \rho \left[ f_{x x} + f_{y y} + f_{x y} + f_{y x} \right] \right) \, dx \, dy.
$$

at those points $(x, y)$ where we do not have $p^1(x, y) = q^1(x, y) = \tilde{p}(x, y) = \tilde{q}(x, y) = 0$

or where these functions are not defined $[p^1(x, y)$ and $\xi^1(x, y)$ are continuous]

all such points constituting a measurable set; at these points, we define

$$
A_{ij} = C_{ij} = B_{ij} = D_{ij} = E_{ij} = F_{ij} = 0. \quad \text{We then define}
$$

$$
B_{ij}^2 (x, y) = B_{ij}^1 (x, y), \quad D_{ij}^2 (x, y) = D_{ij}^1 (x, y), \quad E_{ij}^2 (x, y) = E_{ij}^1 (x, y),
$$

throughout $G$. Clearly all these functions as well as $f_{p^1}, f_{q^1},$ and $f_{\tilde{p}}, f_{\tilde{q}}$ are measurable functions of $(x, y, \lambda)$, and since $p$ and $q$ are of class $L^2$, $f_{\tilde{p}}$ is summable and $f_{\tilde{q}}$ are each of class $L^2$ in $(x, y)$. Also $A_{ij}, B_{ij}, B_{ij},$ and $C_{ij}$ are uniformly bounded, and

$$
\left( \int \int \left[ \frac{\partial f_{2w}}{\partial x} \, dx \, dy \right] \right) \, dx \, dy
$$

is uniformly bounded if $|\lambda| \leq 1$. Thus

$$
\left( \int \left( (x + \lambda \xi, y + \lambda \eta) - \int (x, y) \right) \, dx \, dy \right) \, dx \, dy \leq K \lambda^2
$$
for all $\lambda$ with $|\lambda| \leq 1$. Thus we see that $\gamma'(0)$ exists and

$$
\gamma'(0) = \sum \left( \int \int f_{\lambda}(x, y) \rho(x) \frac{dx}{\rho(x)} \right) dx dy
$$

But $\gamma(\lambda)$ has a minimum for $\lambda = 0$ so that $\gamma'(0)$ must be zero for every 5

which we have admitted. From this, the theorem follows, using Haar's second

lemma.

(b) $\int \int f_{\lambda}(x, y) \rho(x) \frac{dx}{\rho(x)} = \sum \left( \int \int f_{\lambda}(x, y) \rho(x) \frac{dx}{\rho(x)} \right) dx dy$

hold simultaneously for almost all rectangles $R$ in $S$, (6) and (5) being der

ved from (3) merely by translation.

Next, define

$$
\rho_h(x, y) = \frac{\rho(x, y) - \rho(x, y)}{h}, \quad \varphi_h(x, y) = \frac{\varphi(x, y) - \varphi(x, y)}{h},
$$

Then, if $h$ is sufficiently small but positive, we see that $\rho_h$ and $\varphi_h$ are

continuous on $R$ and of class $C^1$ on $R$ with

$$
\rho_h = \frac{\rho(x, y) - \rho(x, y)}{h}, \quad \varphi_h = \frac{\varphi(x, y) - \varphi(x, y)}{h}
$$

almost everywhere. If we now subtract (3) from (4) and (5) in turn, dividing

by $h$ each time, we see, using Lemma I and (6), that $\rho_h$ and $\varphi_h$ satisfy (7)

and (6) below, respectively.

$$
\sum \left( a_{1,1} \rho_{h1} + d_{1,1} \rho_{x1} + d_{1,1} \rho_{y1} \right) dx dy = \sum \left( a_{2,1} \rho_{h2} + d_{2,1} \rho_{x2} + d_{2,1} \rho_{y2} \right) dx dy
$$

$$
\sum \left( a_{3,1} \rho_{h3} + d_{3,1} \rho_{x3} + d_{3,1} \rho_{y3} \right) dx dy = \sum \left( a_{4,1} \rho_{h4} + d_{4,1} \rho_{x4} + d_{4,1} \rho_{y4} \right) dx dy
$$
Now let $H$ be a region of class $C_2$ with $H \subset G$. Then if $h$ is a sufficiently small positive number, we see from (3) that

\[ \int_R \left( \frac{x+y-z(x+y)}{h^2} \right) \, dx \, dy = \int_R \left( \frac{x+y-z(x+y)}{h^2} \right) \, dx \, dy = \int_R \left( \frac{x+y-z(x+y)}{h^2} \right) \, dx \, dy \]

hold simultaneously for almost all rectangles $R$ in $H$, (4) and (5) being derived from (3) merely by translation.

Next, define

\[ p_k(x, y) = \frac{z(x+y)-z(x,y)}{h}, \quad q_k(x, y) = \frac{z(x+y)-z(x,y)}{h}, \quad (k = 1, \ldots, N) \]

Then, if $h$ is sufficiently small but positive, we see that $p_k$ and $q_k$ are continuous on $H$ and of class $C_2$ on $H$ with

\[ p_{k,x} = \frac{p(x+h, y) - p(x,y)}{h}, \quad p_{k,y} = \frac{q(x+h, y) - q(x,y)}{h} \]

almost everywhere. If we now subtract (3) from (4) and (5) in turn, dividing by $h$ each time, we see, using Lemma 1 and (6), that $p_k$ and $q_k$ satisfy (7) and (8) below, respectively:

\[ \int_R p_k(x, y) \, dx \, dy + \int_R q_k(x, y) \, dx \, dy = \int_R \left( \frac{z(x+y)-z(x,y)}{h^2} \right) \, dx \, dy \]

and

\[ \int_R p_k(x, y) \, dx \, dy + \int_R q_k(x, y) \, dx \, dy = \int_R \left( \frac{z(x+y)-z(x,y)}{h^2} \right) \, dx \, dy \]
on almost all rectangles in $H$ (if $h$ is small enough), where we may define

$$a_{ij}(x, y) = \int_0^1 p_{ij}^i (x + t, y; z, z_y) dt + t \left\{ p_{ij}^i (x + t, y; z, z_y) - p_{ij}^i (x, y; z, z_y) \right\} dt + t \left\{ p_{ij} (x, y; z, z_y) - p_{ij} (x, y; z, z_y) \right\} dt$$

at all points not of the measurable set $S_h$ of $H$ where we have $p_i^i (x + h, y) = p_i^i (x, y) = q_i^i (x + h, y) = q_i^i (x, y) = 0$, $i = 1, \ldots, n$, or at least one of these is not defined. At points of $S_h$ we may define

$$a_{ij}^{(h)} = c_{ij}^{(h)} = \frac{1}{n + m} \sum_i c_{ij}^{(h)}, \quad b_{ij}^{(h)} = d_{ij}^{(h)} = c_{ij}^{(h)} = f_{ij}^{(h)} = g_{ij}^{(h)} = k_{ij}^{(h)} = e_{ij}^{(h)} = 0$$

We then define

$$a_{ij}^{(h)} = b_{ij}^{(h)} = \cdots = e_{ij}^{(h)}$$

all over $H$. Clearly the $a_{ij}$, etc., may be defined analogously.

It is clear, first, that all of these functions are measurable.

Next, let $\gamma$, $\ldots$, $\gamma^N$, $\gamma'$, $\ldots$, $\gamma' \ldots$ be any values, and let $(x, y)$ be any point of $H - S_h$. Then $p_i^i (x + h, y)$, $p_i^i (x, y)$, $q_i^i (x + h, y)$, and $q_i^i (x, y)$ are all defined but not all zero. Then we see that

$$\lim_{i=1}^N (\gamma, \delta) = \int_0^1 \left( \int_0^1 \delta_{i=1}^N (\gamma, \delta, x, y) \right) dt$$

and since $S_h$ satisfies condition (10) holds. Similarly (12)

$$\int_0^1 \delta_{i=1}^N (\gamma, \delta, x, y) \right) dt$$

Since $\int_0^1 \delta_{i=1}^N (\gamma, \delta, x, y) \right) dt$ not since a condition A [\frac{1}{2}, 1]
the argument in the $f^{2}_{\mu}$, etc., in the integral being that occurring in (9). If $(x, y)$ is in $S_{h}$, it is obvious that (10) holds. Similarly (10)
holds also for the $a_{i,j}$, etc. We can choose $h$ and $M_{h}$ independently of
$h$ and everything else occurring in (10) as $z$ is continuous and hence
where $x$ is the distance of $z$ from $0$. Thus, we see that $V$ is of class
bounded in a region $D$ containing $H$, and, of course, $x$ and $y$ are in the $D_{0}$ all over $x$ and satisfies a condition $\Delta_{x}^{1} \Delta_{y}^{1}$ all over $G$. Then,
bounded region $G$. For the same reason, it is easy to see that we may
it follows from (5) and theorem 1, e.g., that
find an $H_{h}$ independent of $x, y, h$ such that
\[
\sum_{i=1}^{N} \sum_{j=1}^{M} \left[ (a_{i,j}^{2})^{2} + (b_{i,j}^{2}) + (c_{i,j}^{2}) + (d_{i,j}^{2}) + (e_{i,j}^{2}) + (f_{i,j}^{2}) \right] + \sum_{i=1}^{N} \left[ (g_{i}^{2}) + (h_{i}^{2}) + (i_{i}^{2}) \right]
\leq \frac{1}{n} \sum_{i=1}^{N} \left[ \left( 1 + e^{x+y} \right) + \left( 1 + e^{x+y} \right) \right] + \sum_{i=1}^{N} \left[ (c_{i}^{2}) + (d_{i}^{2}) + (e_{i}^{2}) \right] \right]^{2} d\lambda
\]
which is of class $1$ and is easily seen to satisfy a condition $\Delta_{x}^{1} \Delta_{y}^{1}$
(11)
\[
\leq \frac{1}{n} \sum_{i=1}^{N} \left[ \left( 1 + e^{x+y} \right) + \left( 1 + e^{x+y} \right) \right]
\]
\[
(12)
\]
\[
+ \sum_{i=1}^{N} \left[ (g_{i}^{2}) + (h_{i}^{2}) + (i_{i}^{2}) \right] \right] d\lambda \leq K \lambda
\]
Obviously, inequalities analogous to (11) and (12) hold for the $a_{i,j}$, etc.

Thus, for $0 < \epsilon$, sufficiently small, we see that there exists a $K$ and a
$\lambda = 2/n$, such that
\[
\sum_{i=1}^{N} \sum_{j=1}^{M} \left[ (a_{i,j}^{2})^{2} + (b_{i,j}^{2}) + (c_{i,j}^{2}) + (d_{i,j}^{2}) + (e_{i,j}^{2}) + (f_{i,j}^{2}) \right] + \sum_{i=1}^{N} \left[ (g_{i}^{2}) + (h_{i}^{2}) + (i_{i}^{2}) \right] \right]^{2} d\lambda \leq K \lambda
\]
\[
(13)
\]
\[
+ \sum_{i=1}^{N} \left[ (g_{i}^{2}) + (h_{i}^{2}) + (i_{i}^{2}) \right] \right] d\lambda \leq K \lambda
\]

Now, suppose that $H$ is also simply connected. Let $V^{(x, y)}$ be the
potential of the set function
\[
\phi_{1}(x) = \sum_{i=1}^{N} \int_{D} f_{2,i}^{(x, y)}(z, \rho_{i}, \sigma_{i}) d\lambda d\lambda
\]
Since $\int_{D} f_{2,i}^{(x, y)}(z, \rho_{i}, \sigma_{i}) d\lambda d\lambda$ and since $z$ satisfies condition $A \left[ \Delta_{x}^{1} \Delta_{y}^{1} \right]$. Now, if we translate (13) to $(x, y, \rho_{i}, \sigma_{i})$ and $(x, y, \rho_{i}, \sigma_{i})$
on $G$, we see that
\[ V_{\Phi} \left[ C (P, r) \right] \leq \frac{1}{k} \left( \frac{r}{a} \right)^{\frac{\lambda}{m}}, \quad \lambda = \frac{\mu}{n} \quad \sigma \leq r \leq a, \quad P \in \xi \]

where $a$ is the distance of $P$ from $G$. Thus, we see that $V$ is of class $D$ all over $G$ and satisfies a condition $[\tilde{A}, \lambda, M]$ all over $G$. Then, it follows from (3) and theorem 1, § 6, that
\[ \int_{\xi^*} (f_{x'} - V_{x'}) \, dy - (f_{y'} - V_{y'}) \, dx = 0 \]

around almost all rectangles $R$ in $G$ so that there exists a function $v$ which is of class $D$ and is easily seen to satisfy a condition $[\tilde{A}, \lambda, M]$ on $G$ such that
\[ \frac{\partial v}{\partial x} + v' = \int f_{x'}, \quad \frac{\partial v}{\partial y} - v' = \int f_{y'} \]

(13)

almost everywhere. Now if we let
\[ \Pi (x', y) = \frac{v'_{x}(x + h, y) - v'(x, y)}{h}, \quad \Pi (x', y) = \frac{v'_{y}(x + h, y) - v'(x, y)}{h}, \quad \Pi (x', y) = \frac{v'_{x+y}(x, y) - v'(x, y)}{h}, \quad \Pi (x', y) = \frac{v'_{x+y}(x, y) - v'(x, y)}{h} \]

we see that all of these functions are of class $D$ and continuous on $H$ if $h$ is small enough and their derivatives are given almost everywhere by formulas analogous to (6). Now, if we translate (13) to $(x+h, y)$ and $(x, y+h)$ and subtract (13) from each of these and divide by $h$, we see that
\[ \frac{\partial v}{\partial x} + \Pi (x', y) = \frac{\partial v}{\partial h} + \frac{\partial v}{\partial h}, \quad \frac{\partial v}{\partial y} - \Pi (x', y) = \frac{\partial v}{\partial h} + \frac{\partial v}{\partial h}, \quad \frac{\partial v}{\partial x} - \Pi (x', y) = \frac{\partial v}{\partial h} + \frac{\partial v}{\partial h}, \quad \frac{\partial v}{\partial y} + \Pi (x', y) = \frac{\partial v}{\partial h} + \frac{\partial v}{\partial h} \]

(14)
almost everywhere, etc. It is also important to notice that there exists an $L$ such that

$$\sum_{i=1}^{N} \left[ \frac{1}{N} \left( P_{i}^{e} \right)^{2} + \left( Q_{i}^{e} \right)^{2} + \left( \frac{\partial Q_{i}^{e}}{\partial y} \right)^{2} \right] \leq \lambda L^2$$

where $\lambda = \frac{m}{M}$.

In the sequel, we shall prove that a family of vector functions which satisfy (16) uniformly and satisfy a set of equations of the type of (14) and (16) or (15) and (17) where the coefficients are all measurable and satisfy (10) and (12) uniformly on a simply connected region $H$ of class $C$, also satisfy a condition $A \left[ \mu', M(a,d) \right]$ uniformly on $H$ (i.e. the same $\mu$ and $M(a,d)$ holds for all the functions) where $\mu$ is less than the smaller of $M/H$ and $m/H$ (from (10) and $M(a,d)$ depends only on $a,d$ as in definition 1, 5), $M,m,M',M,K$, and $L$ (the last two depending on the first four). From this and theorem 2, 5, it follows that $p(x,y)$ and $q(x,y)$ are also of class $D'$ and satisfy uniform Holder conditions on each closed subregion of $G$.

It is shown further concerning the solutions of the above systems that if the $a_i$, etc. also tend to limit functions
a_j (x, y) almost everywhere and if the solutions \( p' \) and \( q' \) tend on
subregions uniformly to limit functions \( p' \) and \( q' \), there the limiting
functions satisfy the limiting equations. Now, suppose that we assume,
in addition to what we have assumed about \( f(x, y, z, p, q) \) that there is a
point \((x_0, y_0, z_0, p_0, q_0)\), where \((x_0, y_0) \in G \) and \( z_0 = z(x_0, y_0)\).
\( p_0 = p(x_0, y_0), q_0 = q(x_0, y_0) \) for our solution, in the neighborhood of
in a neighborhood \( U \) such that there exist numbers \( a \) and \( B \) such that
which the second derivatives of \( f \) satisfy uniform Holder conditions.

Then we see that, in (7) and (8), \( p_0 \Rightarrow p' \) and \( q_0 \Rightarrow q' \) on each closed
subregion of \( G \) sufficiently near \((x_0, y_0) \) and the \( a_j \), etc., converge
uniformly on such sets to the functions \( a_j (x, y) \) given by

\[
\begin{align*}
a_{ij} &= f_{\rho_{ij}} (x, y, z, p, q) \quad \rho_{ij} = f_{\rho_{ij}} (x, y, z, p, q) \\
\varepsilon_{ij} &= f_{\varepsilon_{ij}} (x, y, z, p, q) \quad \varepsilon_{ij} = f_{\varepsilon_{ij}} (x, y, z, p, q) \\
\\theta_{ij} &= f_{\theta_{ij}} (x, y, z, p, q) \quad \theta_{ij} = f_{\theta_{ij}} (x, y, z, p, q) \\
\end{align*}
\]

and the \( \bar{a} \) are defined analogously and all satisfy uniform Holder conditions
on some small circle with center at \((x_0, y_0)\). It is also shown in the
sequel that any solution of a system of equations of the type (7) in
which the coefficients satisfy uniform Holder condition must be of
class \( C^r \) with its first derivatives satisfying uniform Holder conditions
on interior regions. From this it follows that the first derivatives of
furthermore if \( A_{ij} \) and \( B_{ij} \) are measurable functions of parameters \((x, y)\),
\( p' \) and \( q' \), i.e. the second derivatives of \( z' \), satisfy uniform Holder
conditions near the point \((x_0, y_0)\). In this case, it is clear that the
\( z' \) satisfy the Euler differential equations for a minimizing function \( z \)
on (1) in (1) we let \( x = y + z \), (1) becomes

\[
\begin{align*}
\frac{\partial^2 z}{\partial t^2} &= \frac{\partial^2 f}{\partial t^2} (x, y, z, p, q) \\
\frac{\partial^2 z}{\partial t^2} &= \frac{\partial^2 f}{\partial t^2} (x, y, z, p, q) \\
\end{align*}
\]

Let us choose \( p' \) and \( q' \) successively to satisfy

9 Some Preliminary Existence Theorems.

and we see that (1) reduces to

\[ y = A_{\alpha \beta} x^\alpha x^\beta + 2 B_{i} x^i \]

Clearly \( y \) satisfies (1) and the \( x \) satisfy

be a quadratic function such that there exist numbers \( m \) and \( M \) such that

\[ m \sum_{i=1}^{n} x_i^2 \leq A_{\alpha \beta} x^\alpha x^\beta \leq M \sum_{i=1}^{n} x_i^2, \quad 0 \leq m \leq M \]

Next we observe first, by letting \( x_i = 1 \), \( x_i = 0 \), let us take

Then there exist numbers \( k(m, M) \) and \( K(m, M) \), a number \( C \), and numbers

\[ a_{ij}, \quad b_i \] such that

\[ 1 \leq k(m, M), \quad k(m, M) \sum_{i=1}^{n} b_{i} \leq C \leq K(m, M) \sum_{i=1}^{n} b_{i} \]

and hence

\[ A_{\alpha \beta} x^\alpha x^\beta + 2 B_{i} x^i \]

and such that

\[ A_{\alpha \beta} x^\alpha x^\beta + 2 B_{i} x^i + C = \sum_{i=1}^{n} (\alpha_{i} x^i + c_i)^2 \]

Furthermore if \( A_{\alpha \beta} \) and \( B_{i} \) are measurable functions of parameters \((s, t)\),

\( C, a_{ij} \) and \( b_i \) may be chosen to be measurable functions of \((s, t)\) over

the same domain of definition.

Proof: If, in (1) we let \( x_i = y_i + h_i \), (1) becomes

\[ A_{\alpha \beta} y_i^\alpha y_i^\beta + 2 (A_{\alpha \beta} h_i^\alpha + b_i) y_i^\alpha + A_{\alpha \beta} h_i^\alpha h_i^\beta + \sum_{i=1}^{n} (\alpha_{i} x^i + c_i)^2 \]

Let us choose \( h_i \) and \( C \) successively to satisfy
and we see that (1) reduces to

\[ A_{\alpha \beta} \gamma^\alpha \gamma^\beta = \gamma + C \]

Clearly \( C \) satisfies (3) and the \( \gamma^i \) satisfy

\[ k \left( \sum_{i=1}^{m} \gamma_i^2 \right) \leq \sum_{i=1}^{n} \gamma_i^2 \leq K \left( \sum_{i=1}^{m} \gamma_i^2 \right) \]

for some \( k \) and \( K \).

Next we observe first, by letting \( x^i = 1, x^j = 0, j \neq i \), that

\[ m \leq A_{ii} \leq M. \]

Next, by letting \( x^i = 0 \) if \( k + 1 \) or \( j \), then

\[ m (x^i + x^j) \leq A_{ij} x^i \leq M (x^i + x^j) \]

and hence

\[ A_{ij} \leq M. \]

Now, define

\[ \gamma_i, (\gamma) = A_{\alpha \beta} \gamma^\alpha \gamma^\beta - \frac{(A_\alpha \gamma^\alpha)^2}{A_\alpha} \]

and one sees that \( \gamma_i \) is independent of \( \gamma \) as the above coefficient is zero if either \( \alpha \) or \( \beta = 1 \). Hence, if we define

\[ \gamma' = (A_{i\alpha})^{-\frac{1}{2}} (A_{\alpha \gamma} \gamma^\alpha), \quad \gamma^\varepsilon = \gamma', \quad \varepsilon = 2, \ldots, n \]
we see that
\[ \gamma + c = (\gamma')^2 + \gamma_1 (\gamma_1, \ldots, \gamma_n) \]
where \( \gamma_1 \) has to be positive definite in \((\gamma_2, \ldots, \gamma_n)\) as the normal form of \( \gamma \) is a sum of squares.

Now, first, we see that each coefficient of \( \gamma_1 \) is in absolute value \( \leq 2M^2/n \), and hence there exists a number \( M(\frac{M}{M}) \geq 0 \) such that
\[ \gamma_1 (\gamma_1, \ldots, \gamma_n) \leq M \sum_{i=2}^n (\gamma_i)^2 \]
Now for \((\gamma_1)^2 + \ldots + (\gamma_n)^2 = 1\), each \( \gamma_1 \) arising as above from a \( \gamma + c \) has a minimum \( d \) which is \( > 0 \). Let us choose a sequence \( \gamma_0 \) such that
\[ m \sum_{i=1}^n (\gamma_i)^2 \leq \gamma_0 (\gamma) + c_0 \leq M \sum_{i=2}^n (\gamma_i)^2 \]
and a sequence of points \( \gamma_0 \) such that \( \gamma_0 (\gamma_0) \) tends to the greatest lower bound of all the minima \( d \). We may choose a subsequence so that the points \( \gamma_0 \to \gamma_0 \) on the unit sphere \((\gamma_0)^2 + \ldots + (\gamma_0)^2 = 1\) and such that all the coefficients of \( \gamma_0 \) and \( \gamma_0 \) tend to limits, those for \( \gamma_0 \) being derived from the limiting form \( \gamma_0 \to C_0 \), say. But \( C_0 \to C_0 \) again satisfies (9) so that \( \gamma_0 (\gamma) \) must be \( > 0 \). In other words there exists a number \( n(\frac{M}{M}) > 0 \) such that
\[ \gamma_1 (\gamma_1, \ldots, \gamma_n) \geq m (m, \frac{M}{M}) \sum_{i=2}^n (\gamma_i)^2 \]
also, we have

Evidently, we may repeat the process starting with \( \gamma_0 \) and our lemma follows in a finite number of steps. The last statement is obvious from the above specified method given for obtaining the functions in question.

\(^1\) For a direct proof, see the author's paper 0.4, pp. 129 - 131.

Lemma 2. Let \( \gamma_n (x,y) \), \( n = 1,2, \ldots \), and \( \gamma (x,y) \) be of class \( L^p \) on a rectangle \( R: (a,0b,d) \), with \( \gamma (x,y) \) be of class \( C \) on the bounded region \( S \) with \( \gamma (x,y) \) uniformly bounded, and suppose \( \gamma \to \gamma \) (in the previous sense), or let the function \( \gamma_n (x,y) \) and \( \gamma (x,y) \) be independent of \( n \). Let \( G \) be a strongly in \( L^p \) to the function \( a_n (x,y) \) and \( b_n (x,y) \). Suppose that the functions \( a_n (x,y) \) and \( b_n (x,y) \) are of class \( L^p \) on \( R \), with \( q = p/(p-1) \), and that

\[
\lim_{n \to \infty} \int \int |a_n - A|^q d\Omega = 0
\]

for each cell \( D \) in \( R \). Suppose that \( A(x,y) \) and \( \{A_n (x,y)\} \), \( n=1,2, \ldots \), are of class \( L^p \) on \( R \), with \( q = p/(p-1) \), and that

\[
\lim_{n \to \infty} \int \int |A_n - A| d\Omega = 0
\]

Then

\[
\lim_{n \to \infty} \int \int A_n \gamma_n d\Omega = \int \int A \gamma d\Omega
\]

Proof: From the definition of weak convergence in the space \( L^p \), and from a well known theorem characterizing this type of convergence, it follows that hypotheses (11) and (12) are equivalent to weak convergence.

Hence

\[
\lim_{n \to \infty} \int \int A_n \varphi d\Omega = \int \int A \varphi d\Omega
\]

Also, we have

\[
\int \int |A_n - A|^q d\Omega \leq \left( \int \int |\varphi|^q d\Omega \right)^{\frac{p}{q}} \left( \int \int |A_n - A|^q d\Omega \right)^{\frac{q}{p}}
\]

*) For a direct proof, see the author's paper D.E., pp. 129 - 131.

which tends to zero as \( n \to \infty \). From this the lemma follows.

We now examine the integrals \( I_{u(x,y)} \) and \( I(u,y) \) for functions of class \( L^2 \) by

Lemma 3: Let \( \{ u_i(x,y) \} \) be of class \( L^2 \) on the bounded region \( G \) with \( D_i u \) uniformly bounded, and suppose \( u_i \to u \) (in our previous sense) on \( G \). Let the functions \( a_i(x,y) \) and \( b_i(x,y) \) be of class \( L^2 \) on \( G \) and converge strongly in \( L^2 \) to the functions \( a_i(x,y) \) and \( b_i(x,y) \). Suppose that the functions \( c_i(x,y) \) are of class \( L^2 \) on \( G \) and converge weakly in \( L^2 \) to \( c_i(x,y) \). Then

\[
\int_G \sum_{i=1}^{\infty} \left( a_i(x,y) + b_i(x,y) \right) \, dx \, dy \leq \int_G \sum_{i=1}^{\infty} \left( a_i(x,y) + b_i(x,y) \right) \, dx \, dy \leq \int_G \sum_{i=1}^{\infty} \left( a_i(x,y) + b_i(x,y) \right) \, dx \, dy.
\]

Now, let \( a_i(x,y) \), \( b_i(x,y) \), \( c_i(x,y) \) be uniformly bounded and measurable on the region \( G \) of class \( C \) and suppose that there exist numbers \( m \) and \( M \), independent of \( (x,y) \) such that

\[
\sum_{i=1}^{\infty} (\xi_i + \eta_i)^2 \leq a_i + b_i + c_i \leq \sum_{i=1}^{\infty} (\xi_i + \eta_i)^2
\]

for all \( (\xi_i, \eta_i) \) and almost every \( (x,y) \) in \( G \). Let \( d_i(x,y) \) and \( e_i(x,y) \) be of class \( L^2 \) on \( G \) and let \( f_i(x,y) \) and \( A_i(x,y) \) and \( B_i(x,y) \) and \( C_i(x,y) \)

be determined, as in lemma 1 so that

\[
\sum_{i=1}^{\infty} (\xi_i + \eta_i)^2 \leq f_i \leq \sum_{i=1}^{\infty} (\xi_i + \eta_i)^2
\]

the identity being true for \( (x,y, \xi_i, \eta_i) \). Then the \( A_i \) and \( B_i \) are bounded and measurable, the \( C_i \) are of class \( L^2 \), and \( f \) is summable over \( G \) and we have two numbers \( k(m, M) \) and \( K(m, M) \) such that

\[
k(m, M) \sum_{i=1}^{\infty} (\xi_i + \eta_i)^2 \leq f \leq K(m, M) \sum_{i=1}^{\infty} (\xi_i + \eta_i)^2
\]

(14)

\[
k(m, M) \sum_{i=1}^{\infty} (\xi_i + \eta_i)^2 \leq \sum_{i=1}^{\infty} C_i \leq K(m, M) \sum_{i=1}^{\infty} (\xi_i + \eta_i)^2
\]

(14')

---

See D.E., p. 132.
We now define the integrals $I(u, g)$ and $J(u, g)$ for functions of class $D^2$ by exist unique functions $u$ and $g$ which are of class $D^2$ on $G$, coincide

\[ I(u, g) = \iint (a_{xy} u_{x} u_{y} + 2 b_{xy} u_{x} u_{y} + c_{xy} u_{x} u_{y}) \, dx \, dy \]

\[ J(u, g) = \iint (a_{xy} u_{x} u_{y} + 2 b_{xy} u_{x} u_{y} + c_{xy} u_{x} u_{y}) \, dx \, dy \]

Then it is immediately clear that $I(u, g) = I(u, u; g)$ and that

\[ I(u + \alpha u, g) = \lambda I(u, g) + 2 \lambda \mu I(u, u; g) + \lambda^2 I(u, u; g) \]

\[ J(u + \alpha u, g) = J(u, g) + 2 \lambda J(u, u; g) + \lambda^2 J(u, u; g) \]

Proof: The existence of $u$ and $g$ and the fact that $u$ satisfies

Using (14), we see first that

Secondly, it follows from (14) and lemma 3 that $I(u, g)$ and $J(u, g)$ are lower semi-continuous in $u$ among all functions $u$ of class $D^2$ on $G$ with convergence defined as in § 3.
Theorem 1: Let \( u^* \) be of class \( D^1 \) on \( G \) with \( D^1(u^*, G) \) finite. Then there exist unique functions \( u^1 \) and \( u^2 \) which are of class \( D^1 \) on \( G \), coincide with \( u^* \) on \( G^* \), and minimize \( I(u, G) \) and \( J(u, G) \) respectively, among all such functions. If \( \xi \) is any function of class \( D^1 \) on \( G \), with \( D^2(\xi, G) \) finite, which vanishes on \( G^* \), there
\[
(17) \quad \tilde{I}(u, \tilde{\xi}, 5, \tilde{\xi}) = \tilde{I}(u, \tilde{\xi}, 5) = 0, \quad \tilde{J}(u, \tilde{\xi}, 5) = \tilde{I}(u, \tilde{\xi}, 5) + I(\tilde{\xi}, 5),
\]
Hence \( u^1 \) and \( u^2 \) satisfy
\[
(18) \quad \int_R \left( \lambda_1 \rho \alpha u_{1x} + \lambda_2 \rho \alpha u_{1y} + c_1 \rho \alpha u_1^0 \right) dx - \left( \lambda_1 \rho \alpha u_{2x} + \lambda_2 \rho \alpha u_{2y} + c_1 \rho \alpha u_2^0 \right) dx = 0
\]
on almost all rectangles of \( G \). The function \( u^1 \) satisfies conditions
\[
A \{ A, \tilde{M} \} \quad \text{and} \quad B \{ \tilde{M}, \tilde{M} \} \quad \text{on} \quad G \quad \text{with} \quad \lambda = m/M \quad \text{and} \quad \tilde{M} = D^2(u_1, G), \quad \text{and is continuous on} \ G \quad \text{if the boundary values are continuous.}
\]
Proof: The existence of \( u^1 \) and \( u^2 \) and the fact that \( u^1 \) satisfies conditions \( A \{ A, \tilde{M} \} \) and \( B \{ \tilde{M}, \tilde{M} \} \) as stated above follows immediately from the fact that \( I \) and \( J \) are lower semi-continuous in \( u \), and from the fact that \( \tilde{I} \) and \( \tilde{J} \) are lower semi-continuous in \( u \). It follows immediately from theorems of 5 and 6 that \( \tilde{I}(\tilde{\xi}, 5) \) and equation (14), this part of the proof and that of the continuity of \( u^1 \) on \( \tilde{G} \) whenever its boundary values are continuous parallels the proof of theorem 5, 1.

Now, let \( \xi \) be any function of class \( D^1 \) on \( G \) with \( D^2(\xi, G) \) finite and zero on \( G^* \). There \( u^1 + \xi \) and \( u^2 + \xi \) are admissible functions for the integrals \( I \) and \( J \), and we have...
\[
I(u_r + \lambda \xi, \xi) = I(u_r, \xi) + 2 \lambda I(u_r, \xi, \xi) + \lambda^2 I(\xi, \xi)
\]

(20)

\[
J(u_r + \lambda \xi, \xi) = J(u_r, \xi) + 2 \lambda J(u_r, \xi, \xi) + \lambda^2 J(\xi, \xi)
\]

(22)

where \( \lambda \) is the distance of \( P \) from \( G \), in the distance of \( G \) and \( \lambda \) in the all the integrals being finite. Since \( u_r \) and \( u_j \) minimize \( I \) and \( J \), the middle terms in (20) on the right must vanish. This gives us formulas (17); and formulas (18) and (19) follow from (17) on account of theorem 2.

7. (Haar's second lemma).

Theorem 2: Let \( u_j \) be any function of class \( D \) on \( G \) (of class \( C \)) with \( D(u_j, G) \) finite, which satisfies (19) on almost all rectangles of \( G \). Then \( u_j \) minimizes \( J(u_j, G) \) among all functions \( u \) of class \( D \) on \( G \) and coinciding with \( u_j \) on \( G \). Clearly the analogous statement holds only for \( u_j \) and equation (18) holds, being a special case of the above.

(24)

Proof: It follows immediately from theorem 3, § 7 (the converse of Haar's lemma with \( \phi(\xi) \equiv 0 \)) that \( J(u_j, \xi, \xi) = 0 \) for every \( \xi \) which is of class \( D \) on \( G \) and zero on \( G \). There, from equations (20) with \( \lambda = 1 \), we obtain

\[
J(u_j + \xi, \xi) = J(u_j, \xi) + I(\xi, \xi)
\]

for each \( \xi \) of class \( D \) on \( G \) and zero on the boundary. But \( I(\xi, \xi) \geq 0 \) for every such \( \xi \) which is not essentially zero. Thus \( u_j \) has the stated minimum property.

Theorem 3: If, in theorem 1, the \( d \) and \( e \), satisfy

(21)

\[
\int \int (d^2 + e^2) \, dx \, dy = \int \int \frac{\xi^2}{(1 - \rho^2) \xi^2} (d^2 + e^2) \, dx \, dy \leq Q + R, \quad 0 < \rho < 1 = \frac{m}{n}
\]

for every \( P \) and \( \rho \), then \( u_j \) satisfies conditions \( A \left[ \frac{m}{n}, \frac{m}{n} \right] \) and \( B \left[ \frac{m}{n}, \frac{m}{n} \right] \).
on $G$; in fact

(22) \[ D_2 \left[ u_1, C(P, r) \right] = \frac{4KQ}{m^2 \lambda^2} \left( \sqrt{r^2 + \frac{a^2}{m^2}} \right)^{\mu} \left( \frac{\nu}{a} \right)^{\nu} \]

where $a$ is the distance of $P$ from $G^*$, $r$ is the diameter of $G$ and $K$ is the $K(m, M)$ of equation (14'). If the given boundary values are continuous,

\[ u \] is continuous on $G$.

If (21) holds only for circles with center at a fixed point $P_0$,

interior to $G$, then

(23) \[ 1 u_0 \left( P_0 \right) = \left[ \frac{4\pi K Q}{m^2 \lambda^2} \left( r^2 + \frac{a^2}{m^2} \right) \right]^{1/2} \left( 1 + \frac{a^2}{m^2} + \frac{1}{2} \log \frac{r}{a} \right) \]

where $u_0$ is the solution of (19) which vanishes on $G^*$.

To prove (24), we use equations (13), (15), and (16) as follows:

If we assume merely that $d$ and $e$ are of class $L^2$ on $G$,

(24) \[ D_2 \left[ u_0, \delta \right] = \frac{4K}{m^2 \lambda^2} \int \int (d^2 + e^2) \, dx \, dy, \quad \int \int u_0 \, dx \, dy \leq \int \int D_2 \left[ u_0, \delta \right] \]

where $u_0$ is the solution of (19) which vanishes on $G^*$.

Proof: We first observe that, for each $R$ of class $C^1$ in $G$, we have

(24') \[ u_1(x, y) = u_0(x, y) + u(R, x, y), \quad (x, y) \in R \]

where $u_0$ is the solution defined in $R$ and vanishing on $R^*$ and $u$ is the $u_0$ defined in $R$ and coinciding with $u_0$ on $R^*$. Then from (16) and (17)

it follows that

\[ I(u, R) = I(u_{IR}, R) + I(u_{IR}, R) + I(u_{IR}, R) + \int \int H(u, R) \, dx \, dy \]

\[ \leq M D_2 \left[ H(u, R), R \right] + 4 \int \int (d^2 + e^2) \, dx \, dy \]

\[ \leq M D_2 \left[ H(u, R), R \right] + 4 \int \int (d^2 + e^2) \, dx \, dy \]

$H(\gamma, R)$, having its significance in $\S 5$, and $K$ being the $K(m, M)$ of equation
Thus

(25) \[ D_2 (u_j, R) = \frac{H}{\mu} D_2 \left[ H(u_j, R, R) \right] + \frac{4K}{\mu} \int \int (d^2 z) \, dx \, dy \]

Evidently (22) follows from theorem 1.5. Now suppose the boundary values are continuous. Let \( H \) be a region of class \( C \) including \( G \) in its interior and let \( b_{ij} = d_i = e_i = 0 \) in \( H - G \) and \( a_{ij} = c_i = \frac{M_{ij}}{2} \) in \( H - G \). Then the function \( u_j \) for \( H \) which vanishes on \( H^* \) is continuous on \( \tilde{G} \) so that the function \( u_j \) which coincides on \( G^* \) with the given boundary values minus those of \( u_j \) is continuous on \( \tilde{G} \). Thus \( u_j = u_j^H + u_j^F \) is continuous on \( G \).

To prove (24), we use equations (13), (14'), and (19) as follows:

\[ m + D_2 (u_j, g) \leq T (u_j, f) = 2 \int (u_j, g) + 2 \int \int \left( \int \frac{D_2 (u_j, f)}{dx} \, dy \right) \, dx \, dy \leq 4K \int \int (d^2 z) \, dx \, dy \]

Next, let \( P_0 \) be a point of \( G \) and \( (r, \theta) \) be polar coordinates with pole at \( P_0 \). Then define \( u = u_j \) in \( G \) and \( u = 0 \) outside \( G \). We then define

\[ H(r) = \int 0 2\pi \int u_j^2 (r, \theta) \, d\theta \, dr \]

and \( H(r) \) is A.C. with \( H'(r) \geq 0 \) and \( H(r) \leq D_2 (u_j, G) \). Then

\[ \int \int u_j^2 \, dx \, dy \geq \int \int u_j^2 (r, \theta) \, dr \, d\theta = \int \int \left( -u_j (r, \theta) - u_j (r, \theta) \right)^2 \, dr \, d\theta \]

\[ \leq \int \int \left( \int u_j^2 (r, \theta) \, d\theta \right)^2 \, dr \, d\theta \]

which proves (21).

Theorem to Let \( u \) and \( v \) be of class \( C \) on \( G \) (of class \( C^{1,1} \)) and let

(26) \[ \int 0 2\pi \int u_j (r, \theta) \, d\theta \, dr \leq \int [H(r) - H(r)] \, dr \leq \int D_2 (u_j, G) \]

To prove (25), we observe that
\[
D_2 [u, C(\rho_0, r)] \leq K \left( \frac{r}{\rho_0} \right)^{2\alpha}, \quad 0 \leq r < a
\]

\[
K = \left( \frac{4K_4}{m^4 + a^4} \right)
\]

where \( u \) will hereafter denote \( u \). Define \( u = 0 \) outside \( G \) and let

\[
H(v) = \int_0^{2\pi} \int_0^r \rho v r \rho_{\nu}(\rho, \varphi) d\rho d\varphi, \quad h(v) = \int_0^{2\pi} \int_0\rho v r \rho_{\nu}(\rho, \varphi) d\rho d\varphi
\]

\[
H(v) \leq K \left( \frac{r}{\rho_0} \right)^{2\alpha}, \quad 0 \leq r < a
\]

Hence, \( u \) being the solution of \((27)\) for \( a \), we see that \( u \) also satisfies \((27)\).

Now, we know from theorem \( 2 \) that a function \( \tilde{u}(0, \varphi) \) exists so that

\[
\tilde{u}(0, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}(0, \varphi) d\varphi
\]

Then, we see that

\[
|\tilde{u}(0, \varphi)| \leq \int_0^{2\pi} \left| \int_0^r \rho v r \rho_{\nu}(\rho, \varphi) d\rho d\varphi \right| d\varphi = \int_0^{2\pi} \left| \int_0 r \rho v r \rho_{\nu}(\rho, \varphi) d\rho d\varphi \right| d\varphi
\]

\[
\leq \left( \frac{2\pi K}{2\pi} \right)^{\frac{1}{2}} \left( 2\pi K \right)^{\frac{1}{2}} \left( \frac{2\pi}{a} \right)^{\frac{1}{2}} \int_0^{\frac{\rho_0}{2}} \rho^{\frac{1}{2}} d\rho + \frac{1}{2} (2\pi K)^{\frac{1}{2}} \int_0^{\frac{\rho_0}{2}} \rho^{\frac{1}{2}} d\rho
\]

which proves \((23)\).

Theorem 4: Let \( d \) and \( e \) be of class \( L_2 \) on \( G \) (of class \( C_2 \)) and let

\[
\phi (a) \text{ be completely additive on } G \text{ satisfying}
\]

\[
\phi \left[ C(\rho, \rho) \right] \leq T \rho^{\mu}, \quad \mu > 0
\]
Let \( u^* \) be of class \( D^2 \) on \( G \) with \( D^2 (u^*, G) \) finite. Then there exists a unique function \( u \) of class \( D^1 \) on \( G \) and coinciding with \( u^* \) on \( G^* \) which satisfies

\[
\int_{\mathbb{R}^2} \left( a_{\alpha x} u^\alpha_x + b_{\alpha y} u^\alpha_y + d_{\alpha} \right) \, dx = \Phi_i (R)
\]

for almost all rectangles \( R \) of \( G \). If \( d \) and \( e \) tend strongly in \( L^\infty \) to \( d \) and \( e \), respectively. The result follows if \( u^* \) is the solution of (27) for \( d \), \( e \), and \( \phi^* (e) \), which coincides with \( u^* \) on \( G^* \).

In this section we shall develop a general existence theory for circular region \( G \) of radius \( a \). Greater generality may be obtained but in the general theory for a general linear system.

Proof: To demonstrate the first statement, let \( V(x, y) \) be the potential of \( \phi (e) \). Then \( V \) is of class \( D^2 \) on \( G \) and

\[
\int_{\mathbb{R}^2} \sqrt{x} \, dx = \Phi (R)
\]

for almost every \( R \). We then choose a function \( u \) of class \( D^1 \) on \( G \) to coincide with \( u^* \) on \( G^* \) and to satisfy

\[
\int_{\mathbb{R}^2} \left( a_{\alpha x} u^\alpha_x + b_{\alpha y} u^\alpha_y + d_{\alpha} (V_x) - \phi^* (e) \right) \, dx = 0
\]

on almost all rectangles and \( u \) is a desired solution. To see that \( u \) is unique, suppose \( u_i \) is another solution of (27) of class \( D^1 \) on \( G \) with \( D^2 (u_i, G) \) finite, and coinciding with \( u^* \) on \( G^* \). If \( U = u_i - u \), we see
that $U$ is of class $D_2$ on $G$, is zero on $G^*$, and satisfies (18) on almost
rectangles. Thus by theorem 2, $U$ is the unique minimizing function for
$I(u, G)$ which vanishes on $G^*$, and is therefore essentially zero.

To prove the second statement, we may assume $u^* = 0$. By theorem 6, it
follows that $D_2 (V - V, G) = 0$. Thus $d - V_x$ and $e - V_y$ tend
strongly in $L^2$ to $d - V_x$ and $e - V_y$, respectively. The result follows
immediately from this and theorem 3.

10 The existence theory for a general linear system.

In this section, we shall develop our general existence theory for
circular regions $Z_A$ of radius $A$. Greater generality may be attained but
this is sufficient for our purposes. We shall consider the system

$$
\sum_{i,j} \left( \int_{Z_{A_i}} \left( \frac{\partial u^j}{\partial x} + \frac{\partial u^i}{\partial y} \right) \, dx \right) = \int_{Z_{A_i}} \left( \int_{Z_{A_i}} \left( \frac{\partial^2 u^j}{\partial x^2} + \frac{\partial^2 u^j}{\partial y^2} \right) \, dx \right) \, dy
$$

to be satisfied on almost all $R$ in $Z_A$. We shall assume that $d_{ij}, e_{ij}, g_{ij},$ and $k_{ij}$ are of class $L^2$ on $Z_A$, that $f_{ij}$ and $l_{ij}$ are summable on $Z_A$

that

$$
\sum_{i,j} \int_{Z_A} \left( \int_{Z_{A_i}} \int_{Z_{A_j}} \left| f_{ij} \right|^2 \, dx \right) \, dy \leq M_{ij} \, \mu \, \lambda
$$

and that the $a_{ij}, b_{ij},$ and $c_{ij}$ satisfy the hypotheses of the last
section. Further restrictions on $g_{ij}$ and $k_{ij}$ will be added as desired.

For this section, we shall define

$$
\overline{D}_2 (u, Z_A) = D_2 (u, Z_A) + \frac{1}{\pi} \int_{Z_A} u^2 \, dx \, dy
$$
Lemma 1: Let \( u \) be of class \( D^1 \) on \( \mathbb{Z}_A \) with \( D_2(u, \mathbb{Z}_A) \) finite. Then, we may extend \( u \) to be of class \( D^1 \) on \( \mathbb{Z}_{3A} \) and vanish on \( \mathbb{Z}_{3A}^* \), and so that

\[
D_2(u, \mathbb{Z}_{3A}) \leq \left( 2 + \log 3 + 10\pi \right) D_2(u, \mathbb{Z}_A)
\]

Proof: Since \( D_2(u, \mathbb{Z}_A) \) is finite and

\[
D_2(u, \mathbb{Z}_A) = \frac{\pi}{\lambda} \int_0^{2\pi} \frac{\tilde{u}(\gamma, J) + \tilde{u}^*(\gamma, J)}{\lambda A^2} \, d\gamma \, dJ = K_2
\]

say, we see that we can find an \( \bar{r} \) with \( A/2 < \bar{r} < A \) such that \( \bar{u}(\bar{r}, J) \) is \( A, C \) in \( J \) and

\[
\int_0^{2\pi} \tilde{u}(\bar{r}, J) \, dJ \leq 4\pi K_2, \quad \int_0^{2\pi} \tilde{u}^*(\bar{r}, J) \, dJ \leq 2K_2
\]

We then extend \( u \) to \( \mathbb{Z}_{3A} \) as follows:

\[
\tilde{u}(\gamma, J) = \frac{3A - r}{3A - \frac{A^2}{\bar{r}}} \tilde{u}(\bar{r}, J), \quad A \leq \gamma \leq \frac{A^2}{\bar{r}}
\]

and the extended \( u \) is of class \( D^1 \) on \( \mathbb{Z}_{3A} \) and \( \mathbb{Z}_{3A}^* \). Moreover,

\[
\tilde{u}(\gamma, J) = \frac{3A - r}{3A - \frac{A^2}{\bar{r}}} \tilde{u}(\bar{r}, J), \quad A \leq \gamma \leq \frac{A^2}{\bar{r}}
\]

Thus

\[
D_2(u, \mathbb{Z}_{3A}) = D_2(u, \mathbb{Z}_A) + D_2(u, \mathbb{Z}^{*}_A - \mathbb{Z}_A) + D_2(u, \mathbb{Z}_{3A}^* - \mathbb{Z}_{3A})
\]

For all \( A \) we have that (using the above)

\[
\leq 2K_2 + \left( \frac{3A - \frac{A^2}{\bar{r}}}{\bar{r}} \right) \int_0^{2\pi} \left[ \tilde{u}^2(\gamma, J) \, dJ \right] d\gamma + \int_0^{2\pi} \left[ \frac{3A - \frac{A^2}{\bar{r}}}{\lambda A^2} \right] \left[ \tilde{u}^2(\gamma, J) \, dJ \right] d\gamma
\]

\[
\leq 2K_2 + 10\pi K_2 + \epsilon K_2 \leq 3
\]
Lemma 2: Let \( \rho(x, y) \) be summable on \( \Sigma_A \) and satisfy

\[
(3) \quad \sum_{\sigma \in \Sigma_A} \rho(x, y) \leq k(\tau), \quad \sum_{\sigma \in \Sigma_A} \rho^{-1}(y, x) = k(\tau) < \infty
\]

for every \( G(\Sigma_A) \) where \( k(\tau) \) is monotone non-decreasing. Let

\[
(4) \quad V_\tau(x, y; x_0, y_0) = \frac{1}{4\pi} \int \int \log \left( \sqrt{(x-x')^2 + (y-y')^2} \right) \gamma(x, y) \, d\sigma(x) \, d\sigma(y), \quad 0 < \tau < 1
\]

Then

\[
(5) \quad V_\tau(x, y; x_0, y_0) \leq k(\tau) \left( \frac{2\sqrt{2}}{\pi} + \log \frac{A}{\tau^2} \right) + \frac{9}{3\pi} k(\tau) k(\tau)
\]

Proof: To prove this lemma, we simply rewrite the first part of the proof of theorem 5, 6. We define

\[
y(x, y) = y(x, y), \quad (x, y) \in \Sigma_A \quad \text{and} \quad H_\tau(s, t) = \int \int \rho(x, y) \, d\sigma(x) \, d\sigma(y)
\]

\[
y(x, y) = 0, \quad (x, y) \notin \Sigma_A
\]

\[
\delta_A(s, t) = \begin{cases} 
\frac{4\pi \frac{A}{\alpha}}{2\sqrt{2}} + \frac{8\pi \log \frac{A}{\alpha}}{3}, & \alpha \leq \alpha_A \left[ \frac{2}{(s-t)^2 + (y-t)^2} \right] \\
\frac{4\pi \frac{A}{\alpha}}{2\sqrt{2}}, & \alpha = 0
\end{cases}
\]

When we are finally, that (3) follows.

Lemma 5: Let \( \gamma(x, y) \) be summable on \( \Sigma_A \) and satisfy (3) for every

It is then clear that

\[
H_\tau(s, t) \leq k(\tau), \quad H_\tau(s, t) \leq k(\tau)
\]

for all \( s \), and it follows from lemma 4, 5, 6 that

\[
\sum_{\sigma \in \Sigma_A} \mathbb{E} \sigma \leq \lambda_A(s, t)
\]

for all \( (s, t) \). Then we see that (using theorem 5, 6)
\[
\sum \int \int \frac{1}{4\pi^2} \int \int \left[ \int \int E \left( \gamma(s, t) \left| \gamma(s, t) \right| \right) dS \right] dS dtds
\]

where \( E \) is the constant. If \( k(r) \) is uniformly bounded for \( r \in \mathbb{R} \), we may choose \( \gamma \) to converge to a function \( u \) which is identically zero in \( I \) to get

\[
\int \int E \left( \gamma(s, t) \left| \gamma(s, t) \right| \right) dS dS dtds
\]

We find that

\[
\int \int \frac{1}{4\pi^2} \int \int \left[ \int \int E \left( \gamma(s, t) \left| \gamma(s, t) \right| \right) dS \right] dS dtds
\]

To prove (6), let \( u \) be any function of class \( D^1 \) on \( \Sigma \), and

\[
\int \int \frac{1}{4\pi^2} \int \int \left[ \int \int E \left( \gamma(s, t) \left| \gamma(s, t) \right| \right) dS \right] dS dtds
\]

Thus we see, finally, that (5) follows.

**Lemma 3:** Let \( \gamma(x, y) \) be summable on \( \Sigma \) and satisfy (2) for every \( C(P, r) \), where \( k(r) \) satisfies the hypotheses of lemma 2. Let \( \{u_n\} \) be a sequence of functions of class \( D^1 \) on \( \Sigma \) with \( D^1 \) uniformly bounded. Then all the integrals exist and we have

\[
\int \int \gamma u_n dxdy \leq (k(\epsilon))^\frac{1}{2} \left( \frac{2\epsilon}{\pi} + \frac{3\lambda}{\pi} \right)^\frac{1}{2} k(\epsilon) + 2k(\epsilon) \]

for \( \epsilon > 0 \).
where \( K \) is the constant of lemma 1 and \( L \) is a uniform bound for \( D_2(u, \Sigma) \).

Moreover, a subsequence \( \{ u_n \} \) may be chosen to converge to a function \( u \),
which is of class \( D^1 \) on \( \Sigma \) with \( D_2(u, \Sigma) \leq L \) and satisfies (6),
so that \( \gamma u_n \) converges weakly in \( L \) to \( \gamma u \).

Proof: According to lemma 1, we may extend each \( u_n \) to \( \Sigma \) so that
\[ u_n = 0 \text{ on } \Sigma^* \]
and
\[ D_2(u_n, \Sigma) = (2 + C \gamma 3 + 10\pi) D_2(u_n, \Sigma) \leq (2 + C \gamma 3 + 10\pi) \Sigma \]

Now, evidently \( u_n \) is of class \( D^1 \) on \( \Sigma \) with
\[ D_2(u_n, \Sigma) = D_2(u_n, \Sigma) \]

To prove (6), let \( u \) simply denote any function of class \( D^1 \) on \( \Sigma \), and
zero on \( \Sigma^* \) with \( D_2(u, \Sigma) \leq K, L \) where \( K \) is the constant of lemma 1 and \( L \) is the uniform bound for \( D_2(u, \Sigma) \). Clearly, \( K, L \) is a uniform bound for \( D_2(u, \Sigma) \). We let \( \gamma \) satisfy (3) and define
\[ V(x, y) = \frac{1}{t} \int \frac{\log \sigma}{1 + r} (x, y) \, dx \, dy \]
\[ V(x, y) = \frac{1}{t} \int \frac{\log \sigma}{1 + r} (x, y) \, dx \, dy \]

Then, by lemma 2, and theorem 5, 6, the integrals below exist and are
given by
\[ \int_{\Sigma} |V(x, y)| \left| \frac{\partial}{\partial x} |V(x, y)| \right| \, dx \, dy = -\int_{\Sigma} \left( \frac{\partial}{\partial x} V + \frac{\partial}{\partial y} V \right) \, dx \, dy \]
\[ \leq \left[ D_2(V, \Sigma) \right]^{\frac{1}{2}} \left[ D_2(u_n, \Sigma) \right]^{\frac{1}{2}} \left[ (K, L) \frac{1}{2} \left( \frac{2 K(3A)}{T} \right)^{\frac{1}{2}} \right] \]
Also, it has been found legitimate in §6 to proceed as follows:

\[
\int_{\Sigma_A} \int_{\Sigma_B} \frac{1}{r(x,y)} \, dx \, dy = -\frac{1}{2\pi} \int_{\Sigma_A} \int_{\Sigma_B} \frac{f(x,y)}{r(x,y)} \, dx \, dy
\]

Then a subsequence of \( \{ u_n \} \) may be chosen to converge to a function \( u \) on \( \Sigma_{3A} \).

\[
\int_{\Sigma_{3A}} \int_{\Sigma_{3B}} \phi(x,y) \, dx \, dy = \int_{\Sigma_{3A}} \int_{\Sigma_{3B}} \phi(x,y) \, dx \, dy
\]

which demonstrates (6).

Now, let \( \{ u_n \} \) be chosen to converge to \( u \) on \( \Sigma_{3A} \). To show that \( u_n \) tends weakly in \( L^1 \) to \( u \), we need only to show that, for each bounded function \( \psi \), we have

\[
\lim_{k \to \infty} \int_{\Sigma_{3A}} \int_{\Sigma_{3B}} \psi(x,y) u_k(x,y) \, dx \, dy = \int_{\Sigma_{3A}} \int_{\Sigma_{3B}} \psi(x,y) u(x,y) \, dx \, dy
\]

Obviously \( \psi \) is just another function \( \phi \) if \( |\psi| < 1 \), so we need to show only that

\[
\lim_{k \to \infty} \int_{\Sigma_{3A}} \int_{\Sigma_{3B}} \phi(x,y) \, dx \, dy = \int_{\Sigma_{3A}} \int_{\Sigma_{3B}} \phi(x,y) \, dx \, dy
\]

Evidently we may proceed as above and find that

\[
\int_{\Sigma_{3A}} \int_{\Sigma_{3B}} \phi(x,y) u_k(x,y) \, dx \, dy = -\int_{\Sigma_{3A}} \int_{\Sigma_{3B}} \phi(x,y) u_k(x,y) \, dx \, dy
\]

Obviously the \( \overline{u} \) and \( \overline{u} \) tend weakly in \( L^1 \) to \( \overline{u} \) and \( \overline{u} \) on \( \Sigma_{3A} \), which means that the right sides in (7) for \( \overline{u} \) tend to the right side for \( \overline{u} \). This proves the lemma.
Lemma 4: Let \( \{ \{u_n\} \) be a sequence of functions of class \( D^1_2 \) on \( \Sigma_A \) with \( D^2_2(u_n, \Sigma_A) \) uniformly bounded (KL). Let \( d \) be of class \( L_2 \) on \( \Sigma_A \) with \( \|d\| \) small in \( K \).

Theorem 5: Let \( u \) be the unique solution of
\[
\int_{C(0, r)} d \phi \, dx \, dy = \int_{C(0, r)} \phi \, dx \, dy, \quad r > 0
\]
(8)

Then a subsequence of \( \{u_n\} \) may be chosen to converge to a function \( u \) of class \( D^1_2 \) on \( \Sigma_A \) so that the functions \( d u_n \) converge strongly in \( L_2 \) to \( d u \). Furthermore the \( u_n \) and \( u \) satisfy
\[
\int_{C(0, r)} d^2 u_n \, dx \, dy \leq K \int_{C(0, r)} \left( \frac{2r^2}{\pi} + \frac{2}{3 n^2} + \log \frac{3A}{r} \right) \left( \frac{3A}{2 A^2} + \frac{2}{3 n^2} + \log \frac{3A}{r} \right) n \, r \, \rho \, \tau
\]
(9)

where \( K \) is the constant of lemma 1.

Proof: Let each \( u_n \) be extended as in lemma 1 to \( \Sigma_3 A \) and let \( d = 0 \) in \( \Sigma_3 A \). Then the functions \( d u_n \) are all summable on \( \Sigma_3 A \) and satisfy (6) uniformly with \( k(r) = M_r \). We choose a subsequence \( u_{n_k} \) so that \( u_{n_k} \) converges to a function \( u \) on \( \Sigma_3 A \) and then \( d u_{n_k} \) converges weakly on \( \Sigma_3 A \) to \( d u \). Evidently the functions \( d u_{n_k} \) are of the type of the \( \phi \) of lemma 3 with \( k_n(r) \leq (K_1 \rho) \left( \frac{1}{2n^2} + \frac{2}{3 n^2} + \log \frac{3A}{r} \right) \). Hence the potentials \( V_{u_{n_k}} \) of \( d u_{n_k} \) are all of class \( D^1_2 \) on \( \Sigma_3 A \), and by theorem 6, \( \rho \), we see that \( D_2(V - V_{u_{n_k}}, \Sigma_3 A) \rightarrow 0 \). Also it is clear that the functions \( d^2 u_{n_k} = (d u_{n_k}) u_{n_k} \) are all summable and we see by theorem 5, \( \rho \), that

\[
\int_{\Sigma_3 A} d^2(u_{n_k} - u) \, dx \, dy = \int_{\Sigma_3 A} d^2(u_{n_k} - u) \, dx \, dy = -\int_{\Sigma_3 A} (V_{u_{n_k} - V_{u_k}})(u_{n_k} - u) \, dx \, dy + \int_{\Sigma_3 A} (V_{u_{n_k}} - V_{u_k})(u_{n_k} - u) \, dx \, dy
\]

which is easily seen by the H"older inequality to converge to zero. Formula (8) follows from an application of (6) with \( \rho = d u_{n_k} \) and theorem 5, \( \rho \).

We now define
\[
\|u\| = \left[ \int_{\Sigma_2} (u_{n_k}^2 \Sigma_A) \right]^{1/2}
\]
for functions \( u \) of class \( D^r \) on \( \mathbb{Z}_x \) with \( D(\mu, \mathbb{Z}_x) \) finite. Evidently the space of such functions with this norm is a (complete) Banach space, which we shall call \( B \).

**Theorem 1:** Let \( u^B \) and let \( U = Tu \) be the unique solution of

\[
\text{Proof: Let } \{ u^k \} \text{ be a sequence of functions in } B \text{ with } \| u^k \| \text{ uniformly bounded. From lemmas 2 and 3, a subsequence } u^k \rightarrow \text{ may be chosen to converge (in our previous sense, not according to our norm) to a function } u \text{ in } B \text{ with } \| u \| < G \text{ so that } d_{i,j}^k \text{ converges strongly in } L_2 \text{ to } d_{i,j} \text{ for each } i \text{ and } j, \text{ and sothat } f_{i,j}^k \text{ and also, clearly, } d_{i,j} \text{ and } e_{i,j}^k \text{ converge weakly in } L_2 \text{ to } f_{i,j} \text{ and } e_{i,j} \text{ respectively where we have (using (2) and (6))}
\]

\[
\text{Theorem 2: There exists a number } A_x > 0 \text{ which depends only on } M, \mu, \text{ and } \operatorname{const} \text{ of this section } K \text{ being the constant of lemma 1. Thus, from theorem 4, 8, 9, it follows such that } \| U - U^* \| \leq \frac{2}{\mu} \text{ and } \operatorname{const} \text{ of this section.}
\]

\[
\text{Hence, for each } \phi \text{, } c = 0, A_x = 0 \text{ is not a characteristic value.}
\]

**Theorem 2:** Let \( u^* \) be of class \( D^r \) on \( \mathbb{Z}_x \) with \( D(u^*, \mathbb{Z}_x) \) finite.

Then there exists a unique solution \( u \) in \( B \) of (1) which coincides with \( u^* \) and (2) and theorem 3, 1, 2, it follows that \( U = Tu \) is the solution of on \( \mathbb{Z}_x \) provided that \( f \) is not one of a denumerable isolated set \( \{ f_n \} \) of characteristic values. If \( f \neq f_n \) for some \( n \), there exist solutions of the
homogeneous equation \( (1) \) \( (g_i = k_i = l_i = 0) \) which vanish on \( \Sigma^*_k \) and are not essentially zero.

Proof: Let be the solution in \( B \) of

\[
\int_{\Sigma^*_k} \left( \alpha_i \frac{\partial}{\partial \xi_i} \psi + \beta_i \frac{\partial}{\partial \xi_i} \psi + \gamma_i \right) d\xi - \left( \int_{\Sigma} \frac{\partial}{\partial \xi_i} \psi \right) d\xi = \int_{\Sigma} \xi_i d\xi
\]

which coincides with \( u^* \) on \( \Sigma^*_k \). By a well known theorem of F. Riesz, the equation

\[
u - fTu = \psi
\]

has a unique solution \( u \) for each \( \psi \) in \( B \) provided that \( f \) is not one of a set discrete characteristic values. We observe that the function \( u \) so determined by our above \( \psi \) is the desired solution.

We now define \( ||T|| \) for linear transformations \( T \) defined from \( B \) to \( B \) as the greatest lower bounded of all numbers \( G \) such that

\[
||Tu|| \leq G ||u||
\]

for every \( u \) in \( B \). Concerning this concept, we now prove

Theorem 3: There exists a number \( \Lambda > 0 \) which depends only on \( m, M, M_1 \), and \( \mu \) (\( m \) and \( M \) from (13), \( \delta \), \( \beta \), and \( \mu \) from (2) of this section) such that \( ||T|| \leq 1/2 \) for each \( a, \beta \in \Lambda \), \( T \) being the transformation of theorem 1. Thus, for such \( \Lambda, \beta = 1 \) is not a characteristic value.

Proof: Let \( u \) be any function of \( B \) with \( ||u|| \leq 1 \). From inequalities (6) and (9) and theorem 5, \( \delta, \beta \), it follows that \( U = Tu \) is the solution of

\[
\int_{\Sigma^*_k} \left( \alpha_i \frac{\partial}{\partial \xi_i} W + \beta_i \frac{\partial}{\partial \xi_i} W + \gamma_i \right) d\xi - \left( \int_{\Sigma} \frac{\partial}{\partial \xi_i} W \right) d\xi = \int_{\Sigma} \xi_i d\xi
\]

where \( W \) is the potential of \( d\alpha_i \frac{\partial}{\partial \xi_i} u + e_i \frac{\partial}{\partial \xi_i} u + f_i \frac{\partial}{\partial \xi_i} u \) and
where $\overline{p}$ depends only on $N$, $m$, $A$, $\mu$, and any upper bound for $A$. Now, by theorem 9, 3, we have

$$D_c(U, \mathbf{z}_A) \leq \frac{4K(N,m)}{n} \left( \int \left( g^2 + k^2 \right) dx \, dy \right)^{\frac{1}{2}}$$

so that

$$\overline{D}_c(U, \mathbf{z}_A) \leq \overline{p}, \quad \frac{2^2}{A^2}$$

where $\overline{p}$ depends only on $N$, $m$, $A$, $\mu$, and any upper bound (say 1) for $A$.

Thus the existence of $A$ follows. Clearly, for $A \leq A_0$, we have

$$\frac{1}{2} \| u \|_1 \leq \| u - T_1 u \|_1 \leq \frac{3}{2} \| u \|_1$$

so that the transformation $u = Tu$ has an inverse whose norm is $\leq 2$, so that $f = 1$ is not a characteristic value.

Theorem 4: Let $A < A_0$, $l = 0$, and let $g$ and $k$ be of class $L^2$ on $\mathbf{z}_A$.

Let $u$ be the solution of (1) with $f = 1$ which vanishes on $\mathbf{z}_A$. Then

$$\| u \|_1^2 \leq \frac{4}{(1 + \frac{A}{2})^{\frac{4K}{n}}} \left( \int \left( g^2 + k^2 \right) dx \, dy \right)^{\frac{1}{2}}$$

the $K$ being the $K(N,m)$ of equation (14'), 3, 9. If $g$ and $k$ satisfy

$$\int \left( g^2 + k^2 \right) dx \, dy \leq \frac{1}{2} \quad 0 \leq y \leq \frac{1}{2}$$

for all circles with center at a fixed point $P_0$, then $u(P_0)$ exists and

$$\overline{u}(P_0) \leq P_2 \mathbf{z}_A^{-\frac{1}{2}} A^{\frac{K}{2}}$$

where $P_2$ depends only on $m$, $N$, $A$, $\mu$, and $a$, the distance of $P_0$ from $A$. Theorem 9, 3, we have

$$\overline{D}_c(U, \mathbf{z}_A) \leq \overline{p}, \quad \frac{2^2}{A^2}$$

where $A$ is the distance of $P$ from $A$. Now, let $u \in G$ with $u < \frac{1}{2}$, let $\mathbf{z}_A$ be the potential of $u$, $u = \mathbf{z}_A$, and we set

$$\frac{1}{2} \| u \|_1 \leq \| u - T_1 u \|_1 \leq \frac{3}{2} \| u \|_1$$

so that the transformation $u = Tu$ has an inverse whose norm is $\leq 2$, so that $f = 1$ is not a characteristic value.
from $\mathbb{Z}_A^k$. If $g$ and $k$ satisfy (13) for every $P_0$ and $r_0$, then $u$ is continuous on $\overline{\mathbb{A}}^k$ and satisfies conditions $A[v, M]$ and $B[v/2, M]$ on $\mathbb{A}$ where

$$\mathbb{A} = P_0, A, A^V$$

where $P$ depends only on $m$, $M$, $A$, $M$, and $N$.

**Proof.** Since $\|1/2\|$, it is clear that $\| (1 - T)^{1/2} \| \leq 2$ so that $\| u \| < \varepsilon$. $\gamma$, being the solution of (1) with $\int = 0$ and taking on the desired boundary values. From theorem 3. § 9, we see that

$$\mathcal{D}_2(\gamma, \mathbb{A}) \leq \frac{\| K \|}{\mathcal{M}} \int_{\mathbb{A}} \left( \frac{\kappa}{\pi} + \frac{1}{4} \right) \gamma^2 \, dx \, dy \leq \frac{\| K \|}{\mathcal{M}} \int_{\mathbb{A}} \gamma^2 \, dx \, dy \leq \frac{\| K \|}{\mathcal{M}} \int_{\mathbb{A}} \gamma^2 \, dx \, dy$$

so that

$$\| \gamma \| \leq D_2(\gamma, \mathbb{A}) + \frac{\| K \|}{\mathcal{M}} \int_{\mathbb{A}} \gamma^2 \, dx \, dy \leq \left( 1 + \frac{\| K \|}{\mathcal{M}} \right) \mathcal{D}_2(\gamma, \mathbb{A})$$

from which (12) follows.

Now, from theorem 3, § 9, and (12), we see that

$$\left( \mathcal{D}_2(\gamma, \mathbb{A}) \right)^{1/2} \leq \left( \frac{\| K \|}{\mathcal{M}} \right)^{1/2} \left( 1 + \frac{\| K \|}{\mathcal{M}} \right)^{1/2}$$

where $a$ is the distance of $P_0$ from $\mathbb{A}$. Now, let $u \in B$ with $\| u \| \leq \varepsilon$, let $V(x, y)$ be the potential of $u_x \cdot e_x + u_y \cdot e_y + f(x, y)$, and we see from (11) and theorem 5, § 6 that

$$\int_{\mathbb{A}} \left[ \int_{\mathbb{A}} \left( V_x^2 + V_y^2 \right) \, dx \, dy \right] \leq \left( \frac{\| K \|}{\mathcal{M}} \right)^{1/2} \left[ \frac{\| K \|}{\mathcal{M}} + \frac{\| K \|}{\mathcal{M}} \left( \frac{1}{2} + \frac{1}{2} \sigma \right) \right] \mathcal{D}_2(\gamma, \mathbb{A})^{1/2}$$

where $P$ depends only on $m$, $M$, $A$, $M$, and $N$. Moreover, by inspecting (9) we see that

$$\int_{\mathbb{A}} \left[ \int_{\mathbb{A}} \left( u_x^2 + u_y^2 \right) \, dx \, dy \right] \leq \left( \frac{\| K \|}{\mathcal{M}} \right)^{1/2} \left( \frac{\| K \|}{\mathcal{M}} + \frac{\| K \|}{\mathcal{M}} \left( \frac{1}{2} + \frac{1}{2} \sigma \right) \right) \mathcal{D}_2(\gamma, \mathbb{A})^{1/2}$$

where $P$ depends only on $m$, $M$, $A$, $M$, $\sigma$, and $N$, since $A < A_0(m, M, M, \sigma, N)$.  

[End of text]
K, being the constant of lemma 1. Finally, if we let

\[ y_1 = a \frac{\partial^{2} u}{\partial x^{2}} - v, \quad k_1 = c \frac{\partial^{2} u}{\partial y^{2}} - v \]

we see that

\[ \int C(P, y) \leq p \int y \frac{\partial^{2} y}{\partial x^{2}} \leq y \frac{\partial^{2} y}{\partial x^{2}} \]

where \( P \) depends only on \( m, M, M_1, c, \) and \( N \). Hence if \( U = T \), there \( U \)
is continuous on \( \Omega \) and satisfies a condition \( A \left[ \frac{\partial^{2} y}{\partial x^{2}}, \frac{\partial^{2} y}{\partial y^{2}} \right] \) on \( \Omega \), where

\( M \) depends only on \( m, M, M_1, c, \) and \( n \), being linear in \( G \). Furthermore, by theorem 3,

we shall derive the existence of a harmonic function \( f \). Thus can then

\[ \left. \frac{\partial^{2} y}{\partial x^{2}} \right| \leq \left[ \frac{8 \mu \lambda}{\mu_1} \left( \frac{\partial^{2} y}{\partial x^{2}} \right) + \frac{2 \mu_1}{\mu} \left( \frac{\partial^{2} y}{\partial y^{2}} \right) \right] \]

and certain conjugate functions \( v \) and \( y \) and \( v \) and \( y \) and \( v \) and \( y \). Thus, we see, by combining (16) and (17) and observing from (12) and (13) that

\[ \left. \frac{\partial^{2} y}{\partial x^{2}} \right| \leq \left[ \frac{4 \lambda}{\mu} \right] \left( \frac{\partial^{2} y}{\partial x^{2}} \right) \]

and we find that

\[ \left( \frac{\partial^{2} y}{\partial x^{2}} \right) \leq \frac{4}{\mu} \left( \frac{\partial^{2} y}{\partial x^{2}} \right) \]

In other words \( u(P_0) \) is a bounded and obviously linear functional on the space

where \( P \) depends only on \( m, M, c, \) and \( N \), and \( a(x, \lambda) \).

Finally, by theorem 3, we see that if \( g \) and \( k \) satisfy (13) for every \( P \) and \( \gamma \), we see that \( y \) is continuous on \( \Omega \) and satisfies a condition \( A \left[ \frac{\partial^{2} y}{\partial x^{2}}, \frac{\partial^{2} y}{\partial y^{2}} \right] \), where

\[ \frac{\partial^{2} y}{\partial x^{2}} \left( a \right) \leq \left( 2 \lambda \right)^{\lambda} \left( \frac{\partial^{2} y}{\partial x^{2}} \right) \left( \frac{\partial^{2} y}{\partial y^{2}} \right) \]

Combining the results of the previous paragraph with equations (13) and (19) we see that \( u \) satisfies a condition \( A \left[ \frac{\partial^{2} y}{\partial x^{2}}, \frac{\partial^{2} y}{\partial y^{2}} \right] \) where \( \overline{M} \) satisfies (15).
§11. The Green's Function

In this section we shall consider the equation (1) of §10 where we assume \( A \leq A_0 \), \( f = 1 \), \( f_0 = 0 \), and where the coefficients \( a_{ij} \), \( b_{ij} \), \( c_{ij} \), \( d_{ij} \), \( e_{ij} \), and \( f_{ij} \) satisfy the conditions there set forth. We shall assume also that \( g \) and \( k \) are of class \( L_2 \) and will add supplementary conditions on \( g \) and \( k \) as desired. We rewrite our equation as

\[
\int_{\mathbb{R}^n} \left( a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b_{ij} \frac{\partial u}{\partial x_j} + c_{ij} \frac{\partial u}{\partial x_i} \right) \, dx - \int_{\mathbb{R}^n} \left( d_{ij} \frac{\partial u}{\partial x_i} + e_{ij} \frac{\partial u}{\partial x_j} + f_{ij} \right) \, dx dy
\]

From an examination of the existence theory which we have developed for (1), we shall derive the existence of a Green's function for (1). We can then derive a double integral representation for \( u \) in terms of the values of \( u \) and certain "conjugate functions" \( v \) and \( V \) and \( G \) and its conjugate functions.

From this, the first main result stated in §8 will be derived.

Let \( g \) and \( k \) be of class \( L_p \) with \( p > 2 \) on \( \Sigma_A \). Then (15) of §10 holds for every \( P_0 \) and \( r \) for a certain \( M \) and \( \gamma(p) > 0 \). Then the solution \( u \) of (1) which vanishes on \( \Sigma_A \) is continuous on \( \Sigma_A \) and

\[
\int_{\Sigma_A} \left[ \frac{SS}{f + k} \right] \, dx dy
\]

In other words \( u(P_0) \) is a bounded and obviously linear functional on the space \( (\Sigma_A, k) \) for each \( p > 2 \). From this we may conclude *) that

\[
\tilde{u}(P_0) = SS \left[ \Delta \left( \frac{1}{i}, \frac{1}{j}, \frac{1}{k} \right) \delta \left( x - y, y \right) + \Delta \left( x, y, k \right) \delta \left( x - y, y \right) + \Delta \left( x, y, k \right) \delta \left( x - y, y \right) \right] \, dx dy
\]

*) See S. Banach, "Theorie des operations lineaires", Chapter 4.
where $\Delta_{1ij}$ and $\Delta_{2ij}$ are of class $L_1(p^{-1} + q^{-1} = 1)$ on $\Sigma_A$.

Now, let $P_0$ be any point interior to $\Sigma_A$, and let $p > 0$. Let $g$ and $k$ be of class $L_2 \in \Sigma_A - C(P_0, s)$, $\Sigma_A$ and zero in $C(P_0, s) \cdot \Sigma_A$. Then it is clear that $g$ and $k$ satisfy (13) of §10 for each $0 < \gamma < \lambda/2$, if we take

\[ M_3 = \int_{\Sigma_A} \int_{\Sigma_A - C(P_0, s)} (g^2 + k^2) \, d\gamma \, \, \, \text{for each } 0 < \gamma < \lambda/2. \]

In other words, $u^i(P_0)$ is a bounded linear functional over the space of functions $g$ and $k$ of class $L_2$ on $\Sigma_A - C(P_0, s) \cdot \Sigma_A$ and we have

\[ |\bar{u}^i(P_0)| \leq P_3 \left( \frac{2\lambda}{\gamma} \right) \frac{\lambda}{2}, \quad \gamma \geq \frac{3}{2} \| g, k \| \]

where $\| g, k \|$ is that on $\Sigma_A - C(P_0, s) \cdot \Sigma_A$, i.e.

\[ \| g, k \| = \left( \int_{\Sigma_A - C(P_0, s)} (g^2 + k^2) \, d\gamma \right)^{1/2} \]

and $P_3$ depends only on $m, n, \mu, M, \bar{M}, \beta, A, a, \lambda$. Thus, from the above chapter in Banach, we conclude that

\[ \int_{\Sigma_A} \int_{\Sigma_A - C(P_0, s)} (\Delta_{1ij} + \Delta_{2ij}) \, d\gamma \leq P_3 \left( \frac{2\lambda}{\gamma} \right)^{3/2}, \quad \gamma \geq \frac{3}{2} \|

for each $0 < \gamma < \lambda/2$. By investigating the function on the right, we see that

\[ \int_{\Sigma_A} \int_{\Sigma_A - C(P_0, s)} (\Delta_{1ij} + \Delta_{2ij}) \, d\gamma \leq \begin{cases} P_3 \left( \frac{2\lambda}{\gamma} \right)^{3/2} \left( \frac{\lambda}{2} \right)^2, & \gamma \geq 2 \lambda c^{-1/2} \\ P_3 \left( \frac{\lambda}{2} \right)^3 \left( \frac{\lambda}{2} \right)^3, & \gamma \leq 2 \lambda c^{-1/2} \end{cases} \]

We first prove:

**Theorem 1:** Let $D$ and $E$ be of class $L_2$ on $\Sigma_A$ and satisfy

\[ \int_{\Sigma_A - C(P, r)} (D^2 + E^2) \, d\gamma \leq M_3 \gamma^{1 - \gamma}, \quad 0 < \gamma < \frac{\lambda}{2} \]

for every $P$ and $r$. Then the integrals

\[ \int_{\Sigma_A} \int_{\Sigma_A - C(P, r)} (\Delta_{1ij} + \Delta_{2ij}) E^2 \, d\gamma, \quad i = 1, \ldots, N \]

exist as Lebesgue integrals and define functions $u^i(P, y)$, which are con-
timous on $\Sigma^* \chi$, and satisfy conditions $A[\chi, \bar{M}]$ and $B[\chi/2, \bar{N}]$ on $\Sigma^* \chi$, where $\bar{M}$
and $\bar{N}$ depend only on $m, M, \mu, M_1 \cdots, M_5$, and $A$. Moreover,

$$(3) \quad \int \Delta \chi \cdot \Delta \chi \cdot E^+ \chi \, dx \, dy \leq N_6 \, r \frac{3}{2} \left( \log \frac{\Delta}{r} \right)^{3/2}, \quad 0 \leq r \leq \Delta - \frac{6}{r^2}$$

for every $P$ and $r$, where $N_6$ depends only on $m, M, \mu, M_1 \cdots, M_5, A$, and $a$.

Proof: Define $\Delta_{11j} = \Delta_{21j} = \Delta_{11} = \Delta_{21} = 0$ outside $\Sigma \chi$ and choose

$$P \in \Sigma^* \chi \quad \text{Now, let } 0 < \rho < r, \text{ and we see that}$$

$$\int \left[ \sum_{j=1}^{N_6} (\Delta_{11j}^2 + \Delta_{21j}^2) \right] \left[ \sum_{j=1}^{N_6} (\Delta_{11j}^2 + \Delta_{21j}^2) \right] \, dx \, dy \leq C \left( \sum_{j=1}^{N_6} (\rho^2 + \rho^2) \right) \leq C \left( \sum_{j=1}^{N_6} (\rho^2 + \rho^2) \right)$$

Thus, if $0 < r \leq 2\rho_0$ we have

$$\int \left( \frac{\rho^2}{\rho_0} \right) \sum_{j=1}^{N_6} \left( \frac{\rho^2}{\rho_0} \right) \, dx \, dy \leq \frac{1}{\rho_0^2} \sum_{j=1}^{N_6} \left( \frac{\rho^2}{\rho_0} \right) \, dx \, dy \leq C \left( \sum_{j=1}^{N_6} (\rho^2 + \rho^2) \right) \leq C \left( \sum_{j=1}^{N_6} (\rho^2 + \rho^2) \right)$$

on almost all $r$. These functions $u^2$ are seen by Theorem 4, 10, to satisfy

the conditions $A$ and $B$ on $\Sigma^* \chi$, and $\Sigma \chi$ as next proves.

Theorem 2: There exist functions $\xi_{ij}$ which are of class $C^1$ on $\Sigma^* \chi$

for each $i, j$, and are of class $C^2$ on $\Sigma \chi$ such that

$$(11) \quad \Delta_{11} = \Delta_{21} = 0$$

Moreover, if $\xi_{ij}$ are continuous on $\Sigma^* \chi$ and satisfy

$$\int \left( \frac{\rho^2}{\rho_0} \right) \sum_{j=1}^{N_6} \left( \frac{\rho^2}{\rho_0} \right) \, dx \, dy \leq \frac{1}{\rho_0^2} \sum_{j=1}^{N_6} \left( \frac{\rho^2}{\rho_0} \right) \, dx \, dy \leq C \left( \sum_{j=1}^{N_6} (\rho^2 + \rho^2) \right) \leq C \left( \sum_{j=1}^{N_6} (\rho^2 + \rho^2) \right)$$
where \( v(x) = \sum_{k=1}^{\infty} k \frac{1}{k-1} \) and \( M^* \) depends only on \( m, M, \alpha \), \( M_1, \gamma, M_2, A \). (and \( a \). Thus all the integrals (7) exist as Lebesgue integrals.

Now let \( G(P, r) \) be any circle. If \( |PP_o| \leq 2r \), then

\[
\iint_{D} \Delta_i^{+} E^x dxdy = \iint_{D} \Delta_i^{+} E^x dxdy \leq C(P, r) \left( \log \frac{2A}{r} \right)^{\frac{3}{2}},
\]

where \( D = \{ x \in D \mid |PP_o| \leq 2r \} \).

All \( |PP_o| > 2r \), then

\[
\iint_{D} \Delta_i^{+} E^x dxdy \leq \int \int_{D} \left( \frac{1}{0^2 + E^2} \right) dxdy \leq \left( \log \frac{2A}{r} \right)^{\frac{3}{2}}.
\]

It is clear that (8) follows from (9) and (10).

Now, by a simple limiting process in (2), we see that (7) defines the functions \( U^1(x,y) \) which are of class \( D_2 \) on \( \Sigma A^* \) vanish on \( \Sigma A^* \), and satisfy

on almost all rectangles \( R \). Since \( u = 0 \) on \( \Sigma A^* \) and since the solution of

\[
\iint_{R^*} \left( \frac{1}{2} \left( \nabla u \cdot \nabla u + \nabla u : \nabla u \right) \right) dxdy = \iint_{R} \left( \frac{1}{2} \left( \nabla u \cdot \nabla u + \nabla u : \nabla u \right) \right) dxdy.
\]

on almost all \( R \). These functions \( U^1 \) are seen by Theorem 4, §10, to satisfy the conditions \( A \) and \( B \) as stated and to be continuous on \( \Sigma A^* \).

We next prove:

Theorem 2: There exist functions \( G_{ij} \) which are of class \( D_1 \) on \( \Sigma A^* \) and are of class \( D_2 \) on \( \Sigma A^* - P_0 \), such that

\[
\nabla \cdot G_{ij} = \xi_{ij} x, \quad \Delta \xi_{ij} = \xi_{ij} y
\]

Moreover, if \( F^1 \) are summable on \( \Sigma A^* \) and satisfy

\[
\iint_{\Sigma A^*} F^1 dxdy = \iint_{\Sigma A^*} F^1 dxdy, \quad i = 1, \ldots, N
\]
then the integrals

$$\mathop{\int}\limits_{\Sigma} \int_{\mathbb{R}^n} G(i) \varphi x \, dx \, dy \quad i = 1, \ldots, N,$$

exist as Lebesgue integrals, and we have

$$\mathop{\int}\limits_{\Sigma} \mathop{\int}\limits_{\mathbb{R}^n} \varphi x \, dx \, dy \leq P_6 \eta_1 r + P_7 \eta_2 r^{-3} \left( \log \frac{2A}{\epsilon} \right)^3$$

for every \(G(r, \theta)\) where \(P_6, P_7\) depend only on \(m, M, \frac{m}{M}, \frac{1}{\eta_1}, \frac{1}{\eta_2}, N, A, \) and \(a\).

Also, if \(V^i\) is the potential of \(F^i\), we have

$$\mathop{\int}\limits_{\Sigma} \int_{\mathbb{R}^n} \varphi x \, dx \, dy = - \mathop{\int}\limits_{\Sigma} \mathop{\int}\limits_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_j} V^i + \frac{\partial}{\partial y_j} V^j \right) \varphi x \, dx \, dy$$

Proof: Let \(V(x, y)\) be any function satisfying a uniform Lipschitz condition on \(\Sigma\) and define

$$g^i = v^i_y, \quad k^i = -v^i_x$$

and we see that

$$\int_{\mathbb{R}^n} (g^i \, dx - k^i \, dy) = 0$$

on almost all rectangles \(R\). Since \(u = 0\) on \(\Sigma^*\) and since the solution of (1) is unique and given by (2), it follows that

$$u(x, y) = \mathop{\int}\limits_{\Sigma} \mathop{\int}\limits_{\mathbb{R}^n} \left[ \Delta_i \varphi x (x, y) \frac{\partial}{\partial x_j} u(x, y) - \Delta_j \varphi x (x, y) \frac{\partial}{\partial y_j} u(x, y) \right] \varphi x \, dx \, dy = 0$$

for each such \(V\) and each \((x_0, y_0)\). From §7, Theorem 1, it follows that there exist functions \(G_{ij}(x_0, y_0; x, y)\) which are of class \(D^q\) in \((x, y)\) on \(\Sigma\) for each \(q < 2\), which vanish on \(\Sigma^*\) are of class \(D^q\) on \(\Sigma - P_0\) and satisfy (11) almost everywhere.
Now, let \((x, y)\) be a point where
\[
\sum_{(x, y) \in \mathcal{P}} \left( f_{ij}^2 + s_{ij}^2 \right) \, dx \, dy \leq M^2 \frac{r^2}{a^2} + \rho^2, \quad \rho > 0, \quad 0 < r < \frac{a}{2}, \quad \rho \geq \frac{a}{2}, \quad \ldots, \quad N
\]
this being true for every \((x, y)\) not in a certain set of measure zero. We know from Lemma 1, §6, that \(G_{ij}(x_0, y_0, z, w, x, y)\) exists and that there is a function \(\gamma(\theta)\) such that
\[
\sum_{x, y} \left( f_{ij}^2 + s_{ij}^2 \right) \int_{0}^{2\pi} \gamma(\theta) \, d\theta = \frac{1}{2\pi} \sum_{x, y} \int_{0}^{2\pi} \left( f_{ij}^2 + s_{ij}^2 \right) \, d\theta \int_{0}^{2\pi} \gamma(\theta) \, d\theta.
\]
Thus, define \(G_{ij} = 0\) for \((x, y)\) not in \(\mathcal{Z}_A\) and \(G_{ij}\) is of class \(\mathcal{D}^q\) in \(\mathcal{P}_0\) and \(\mathcal{D}^q\) in \(\mathcal{P}_0\) for each \(q < 2\). Thus
\[
\sum_{x, y} \left( f_{ij}^2 + s_{ij}^2 \right) \int_{0}^{2\pi} \gamma(\theta) \, d\theta = \int_{0}^{2\pi} \left( f_{ij}^2 + s_{ij}^2 \right) \, d\theta \int_{0}^{2\pi} \gamma(\theta) \, d\theta = \int_{0}^{2\pi} \left( f_{ij}^2 + s_{ij}^2 \right) \, d\theta \int_{0}^{2\pi} \gamma(\theta) \, d\theta.
\]
Almost everywhere.

Now, let \(G(x_0, y_0, z, w, x, y)\) be any function of class \(\mathcal{D}_2^q\) in \((x, y)\) for \((x, y) \in \mathcal{Z}_A - (x_0, y_0)\), and zero on \(\mathcal{Z}_A^*\), and let \(F(x, y)\) be any summable function on \(\mathcal{Z}_A^*\) where \(F\) and \(G\) satisfy
\[
\sum_{x, y} \left( f_{ij}^2 + s_{ij}^2 \right) \, dx \, dy \leq \left\{ \begin{array}{ll}
P_2 \left( \frac{r}{a} \right) \left( \frac{a}{r} \right)^3, & \text{if } r \leq 2A \varepsilon^{-\frac{a}{2}} \\
P_2 \left( \frac{r}{a} \right) \left( \frac{a}{r} \right)^3, & \text{if } r \leq 2A \varepsilon^{-\frac{a}{2}} 
\end{array} \right.
\]
for every \(P\) and \(r\), where \(P_2\) depends only on \(m, M, \mu, M_1, N, A, \) and \(a\). Let
\[ G = 0 \text{ outside } Z_A \text{ and define} \]
\[ H(x, y) = -\int\int (\frac{\partial}{\partial x} + \frac{\partial}{\partial y}) \, dx \, dy \]
\[ Z_A \cdot C(\rho, \gamma) \]

Then \( \mathbb{H}(2A) = 0 \) and \( \mathbb{H}(r) \) increases with \( r \). Now
\[
\int \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left[ 1 + \log \frac{2A}{\sqrt{1 - x^2 - y^2}} \right] \, dx \, dy \to 0 \quad \text{as} \quad r \to 0
\]
\[
= \lim_{r \to 0} \left[ -\left(1 + \log \frac{2A}{r}\right) \cdot H(r) - 4 \int \left(1 + \log \frac{2A}{r}\right) \cdot H(r) \, dr \right]
\]
\[
\leq \lim_{r \to 0} \left[ 4 \int \left(1 + \log \frac{2A}{r}\right) \frac{d}{dr} \left( \frac{2A}{r} \right)^{-5} \, dr \right] \leq \frac{4}{5}
\]

where \( P_5 \) depends only on \( M, M^2, M_1, N, A, \) and \( a \). Finally we see that, for each \( r > 0 \), we have
\[
\int \int \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \, dx \, dy = \int \int \left( \frac{d}{dr} \left( \frac{2A}{r} \right)^{-5} \right) \, dr \, dx \, dy
\]
\[
\leq \int \int \left( \frac{d}{dr} \left( \frac{2A}{r} \right)^{-5} \right) \left( \log \frac{2A}{r} \right) \, dr \, dx \, dy
\]

(16)

\[ \int \int \left( \frac{d}{dr} \left( \frac{2A}{r} \right)^{-5} \right) \left( \log \frac{2A}{r} \right) \, dr \, dx \, dy \]

\[ \leq P_5 \left( \frac{2}{5} \right) \int \int \left( \log \frac{2A}{r} \right) \, dr \, dx \, dy
\]

(19)
where we have put

\[ h(x) = \max_{(x, y) \in \mathcal{A}} \int_{\mathcal{A}} F(\alpha, \beta) d\alpha d\beta \]

The right side of (16) is thus evidently bounded independently of \( \mathcal{A} \). Hence the integral

\[ \sum_{\mathcal{A}} \sum_{\mathcal{B}} \int_{\mathcal{A}} \left( \int_{\mathcal{B}} \frac{F(\alpha, \beta)}{\mathcal{A}(\alpha, \beta)^2} \right) d\alpha d\beta \]

exists as a 4-dimensional Lebesgue integral. Moreover, if we let \( F_{\mathcal{A}, \mathcal{B}}(x, y) \) = \( F(x, y) \) if \( (x, y) \in \mathcal{A}(\mathcal{B}) \), and let it be zero otherwise, we see that

\[ h_{\mathcal{A}, \mathcal{B}}(x, y) \leq h_{\mathcal{A}, \mathcal{B}}(x, y) \]

for all \( r \). Then, in this case we see, by substituting in (16), that

\[ \sum_{\mathcal{A}} \sum_{\mathcal{B}} \int_{\mathcal{A}} \left( \int_{\mathcal{B}} \frac{F(\alpha, \beta)}{\mathcal{A}(\alpha, \beta)^2} \right) d\alpha d\beta \leq \mathcal{P}_{\mathcal{A}} \mathcal{P}_{\mathcal{B}} + \mathcal{P}_{\mathcal{A}} \mathcal{P}_{\mathcal{B}} + \mathcal{P}_{\mathcal{A}} \mathcal{P}_{\mathcal{B}} \mathcal{P}_{\mathcal{B}} \]

where \( \mathcal{P}_{\mathcal{A}} \) and \( \mathcal{P}_{\mathcal{B}} \) depend only on \( m, N, A \). Now evidently (14) follows immediately from (17). Moreover, the above shows that the following is legitimate:

\[ \sum_{\mathcal{A}} \sum_{\mathcal{B}} \int_{\mathcal{A}} \left( \int_{\mathcal{B}} \frac{F(\alpha, \beta)}{\mathcal{A}(\alpha, \beta)^2} \right) d\alpha d\beta \]

which proves (15).

**Theorem 3:** The functions \( \mathcal{G}_{j, i}(x, y) \) satisfy the equations

\[ \int_{\mathcal{R}} S_\mathcal{R}(x, y) \mathcal{G}_{j, i}(x, y) dy - \left( \int_{\mathcal{R}} \mathcal{G}_{j, i}(x, y) dy \right) \mathcal{R} = \left( \int_{\mathcal{R}} \mathcal{G}_{j, i}(x, y) dy \right) \mathcal{R} \]

on almost all \( x \), since
where
\[ \phi_0(x, y; z) = \begin{cases} 1 & \text{if } e \text{ contains } (x_0, y_0), \\ 0 & \text{if } e \text{ does not contain } (x_0, y_0). \end{cases} \]

and the double integral on the right exists as a Lebesgue integral. The functions \( G_{ij} \) satisfy conditions \( A[\gamma, M(a, d)] \) and \( B[\sqrt{2}, N(a, d)] \) on \( \Sigma_A - P_0 \).

where \( M(a, d) \) and \( N(a, d) \) depend only on \( a, d, \lambda, M, N, \tau, \) and \( A_s \). Moreover there exist functions \( K_{ij} \) and \( H_{ij} \) with these same properties such that

\[
K_{ij} = \frac{c_{ij}}{4\pi} \log \left( \frac{k}{r^2} \right) + (y - y')^2
\]

hold with the usual limitation.

Proof: Let \( V \) be any function which satisfies a uniform Lipschitz condition on \( \Sigma_A \) and let \( W^i \) be the potential function of \( d_{1x} V + e_{1y} V + f_{ix} V^a \).

Clearly, from (2) of \( \xi_0 \), it follows that

\[
\int_{A} \frac{c_{ij}}{4\pi} \log \left( \frac{k}{r^2} \right) + (y - y')^2 \, dx \, dy \leq K \sqrt{V}
\]

for each \( r > 0 \) and the whole plane and satisfies a condition \( A(p, \mu, \nu) \) there. We next define

\[
g^i = L^i - (a_{i}^{1x} V^x + c_{i}^{1y} V + f_{i}^{1x} V^a)
\]

and we see that \( g^i \) and \( k^i \) satisfy (6). We then see, by substitution in (1) that \( u \) satisfies

\[
\int_{A} \frac{c_{ij}}{4\pi} \log \left( \frac{k}{r^2} \right) + (y - y')^2 \, dx \, dy - \int \left( (a_{i}^{1x} V^x + c_{i}^{1y} V + f_{i}^{1x} V^a) \right) dx - \int \left( (a_{i}^{1x} V^x + c_{i}^{1y} V + f_{i}^{1x} V^a) \right) dy
\]

on almost all \( R \), since...
\[
\int_{\mathcal{A}} \sum_{\gamma \alpha} d\gamma - \int_{\mathcal{A}} \sum_{\gamma \alpha} d\gamma = \sum_{\alpha \beta} \left( \frac{\partial \alpha}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} + \frac{\partial \beta}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} + \frac{\partial \gamma}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} \right) d\gamma
\]
for almost all \( \mathcal{R} \). Since \( V = 0 \) on \( \mathcal{A} \) and since \( u \) is unique, it follows that
\[
u = V \quad \text{on} \quad \mathcal{A} \quad \text{.}
\]
Thus
\[
V(\mathcal{A}) = -\int_{\mathcal{A}} \sum_{\gamma \alpha} \left( \frac{\partial \alpha}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} + \frac{\partial \beta}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} + \frac{\partial \gamma}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} \right) d\gamma
\]
where the second integral on the right exists as a Lebesgue integral. We then see, using Theorem 2, and the fact that \( W \) satisfies a condition \( A(\nu, M^+) \) all over the plane, that
\[
\int_{\mathcal{A}} \sum_{\gamma \alpha} \left( \frac{\partial \alpha}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} + \frac{\partial \beta}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} + \frac{\partial \gamma}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} \right) d\gamma = -\int_{\mathcal{A}} \sum_{\gamma \alpha} \left( \frac{\partial \alpha}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} + \frac{\partial \beta}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} + \frac{\partial \gamma}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} \right) d\gamma
\]
so that
\[
\int_{\mathcal{A}} \sum_{\gamma \alpha} \left( \frac{\partial \alpha}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} + \frac{\partial \beta}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} + \frac{\partial \gamma}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} \right) d\gamma = \int_{\mathcal{A}} \sum_{\gamma \alpha} \left( \frac{\partial \alpha}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} + \frac{\partial \beta}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} + \frac{\partial \gamma}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} \right) d\gamma
\]
for each \( V \) satisfying a uniform Lipschitz condition on \( \mathcal{A} \) and zero on \( \mathcal{A}^+ \).

where
\[
\phi_{ij} (\mathcal{A}) = \int_{\mathcal{A}} \sum_{\gamma \alpha} \left( \frac{\partial \alpha}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} + \frac{\partial \beta}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} + \frac{\partial \gamma}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} \right) d\gamma
\]
where we see from Theorems 1 and 2 that \( \phi_{ij} (\mathcal{A}) \) is completely additive on \( \mathcal{A} \) and
\[
\int_{\mathcal{A}} \sum_{\gamma \alpha} \left( \frac{\partial \alpha}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} + \frac{\partial \beta}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} + \frac{\partial \gamma}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma} \right) d\gamma \leq k \int_{\mathcal{A}} V \quad \text{for every} \quad \nu > 0
\]
for every \( \nu \) in \( \mathcal{A} \). From (23) and Theorem 2, it follows that (18) holds.
Now, let \( K_{ij} \) be the potential of \( e^{\sum_{i \neq j} G_j x} + e^{\sum_{i \neq j} G_j y} + f_{i \rho} G_j \rho \)
and we see that \( K_{ij} \) satisfies a condition \( A [v^2, M] \) over the whole plane. Using
Theorems 1 and 2 several times, we see that \( G_{ij} \) satisfies

\[
\int_{K} \left( a \rho^{\alpha} \sum_{i \neq j} G_j x + e \rho^{\alpha} \sum_{i \neq j} G_j y + \gamma_{ij} \right) \ dx - \left( b \rho^{\alpha} \sum_{i \neq j} G_j x + e \rho^{\alpha} \sum_{i \neq j} G_j y + \gamma_{ij} \right) \ dx = 0
\]

where

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} (Y_{ij} + \delta_{ij}) \ dx \ dy \leq k_{x} \sqrt{r'}, \quad Y_{ij} = \phi_{i} \rho^{\alpha} G_{ij} + K_{ij} x, \quad \delta_{ij} = \phi_{i} \rho^{\alpha} G_{ij} + K_{ij} \rho
\]

so that \( G_{ij} \) satisfy conditions A and B as indicated. From (24) and (25) and
the fact that \( K_{ij} \) is the potential of

the equations (20) follow.

**Theorem 4:** Let \( u, v, \) and \( V \) be of class \( D_2 \) on \( \Sigma = C(P, A) \)

\[
V'_x + V'_y = a \rho^{\alpha} u^\rho_x + e \rho^{\alpha} u^\rho_y + f_{i \rho} u^\rho _i, \quad V'_y - V'_x = b \rho^{\alpha} u^\rho_x + e \rho^{\alpha} u^\rho_y + g_{i \rho} u^\rho _i
\]

(26)

\[
\int_{K} V'_x \ dx - V'_y \ dy = \int_{K} \left( d_{i \rho} \rho^{\alpha} u^\rho_x + e_{i \rho} \rho^{\alpha} u^\rho_y + f_{i \rho} \rho^{\alpha} u^\rho _i \right) \ dx \ dy
\]

Then \( u(x, y) \) is given in \( C(P, A) \) by

\[
u(i \times y') = K_{1} \int_{\Sigma}^{A} \int_{\Sigma}^{A} \left( \int_{\Sigma}^{A} \int_{\Sigma}^{A} \right) \frac{u^\rho_{x}(x, y) \rho^{\alpha} G_{ij} x + G_{ij} y + d_{i \rho} \rho^{\alpha} u^\rho x + e_{i \rho} \rho^{\alpha} u^\rho y + f_{i \rho} \rho^{\alpha} u^\rho _i}{(x - x') + (y - y')^2} \ dx \ dy \]

(27)

\[
- K_{2} \sum_{i \neq j} \sum_{j \neq k} V^\rho x \ dx \ dy + K_{3} \sum_{i \neq j} V^\rho x \ dx \ dy
\]

\[
c(P, A) - c(P, \frac{3A}{\pi}) - c(P, \frac{A}{\pi})
\]

\[
K_{1} = \frac{128}{\pi A^2 \sigma}, \quad K_{2} = \frac{28}{\pi A^2 \sigma}, \quad K_{3} = \frac{28}{\pi A^2 \sigma}, \quad \sigma = 49 \log 2 - 27 \log 3
\]
Proof: By the methods of §§1 and 6, it may be shown that there exist sets $Z_1$ and $Z_2$ of linear measure zero such that if $r$ is not in $Z_1$, and is not in $Z_2$, and $C(P_0, r) < C(P_1, r)$, we have

$$\int u^2 (k_{ij} x + k_{11} y + k_{12} y^2) dx dy - \int u^2 (k_{ij} x + k_{11} y + k_{12} y^2) dx dy = \int u^2 (k_{ij} x + k_{11} y + k_{12} y^2) dx dy$$

$$\int u^2 (k_{ij} x + k_{11} y + k_{12} y^2) dx dy.$$

Adding all of the above equations and using (20) and (26), we see that

$$\int (u^2 k_{ij} x + u^2 k_{11} y + u^2 k_{12} y^2) dx dy = \int (u^2 k_{ij} x + u^2 k_{11} y + u^2 k_{12} y^2) dx dy$$

if $r$ is not in $Z_1$ and $s$ is not in $Z_2$. We may prove also that for almost all we have
and the right side evidently tends to zero with \( s \); we shall call this

\[ \iota_{ij}(x_o, y_o; s) \]

We then see that if \( r \) is not in \( Z_1 \) and \( j \) is not in \( Z_2 \), we have

\[
\int \left( u_k K_{kj} r + u_k K_{kj} r - \frac{s}{r} \right) + u_k H_{kj} r - \frac{s}{r} V_{kj} r \, ds - \iota_{ij}(x_o, y_o; s) \, ds_{0x, y_o}
\]

where we have set \( u_0 = u(x_o, y_o) \).

Next we observe that

\[
S \int \left( \frac{u - u_0}{\sqrt{d_0}} \right) \left( (x - x_0)(K_{ij} x + H_{ij} y) + (y - y_0)(K_{ij} y - H_{ij} x) \right) - \frac{s}{r} \left( \frac{u - u_0}{\sqrt{d_0}} \right) \left( V_{ij} x + W_{ij} y \right) \, ds - \iota_{ij}(x_o, y_o; s) \, ds_{0x, y_o}
\]

\[
= \int \frac{u - u_0}{\sqrt{d_0}} \left( (x - x_0)(K_{ij} x + H_{ij} y) + (y - y_0)(K_{ij} y - H_{ij} x) \right) - \frac{s}{r} \left( \frac{u - u_0}{\sqrt{d_0}} \right) \left( V_{ij} x + W_{ij} y \right) \, ds - \iota_{ij}(x_o, y_o; s) \, ds_{0x, y_o}
\]
where the $M$ is our original bound $M$. If we remember that $u$ satisfies conditions A[√, $\tilde{M}$] and B[√, $\tilde{M}$], and choose $p$ properly and slightly > 2 with $p^{-1} + q^{-1} = 1$, and remember the fact, sketched in the proof of Theorem 3, that $d^{2}G_{jk}$ is summable under our hypotheses on $d$, and if we apply Theorems 1 and 2 several times, we see that the above tends to zero more rapidly than $f_{0}$.

Hence, for each $r$ not in $Z_{1}$, we can pick a sequence of values of $r$ tending to zero so that the integral over $C^{1}(P_{0}, f)$ in (29) tends to zero.

We therefore see that, for each $P_{0}$ in $C^{1} \left( P_{0} ; \frac{1}{2} \right)$ say and each $r$ not in a set $Z_{1}(P_{0})$ of measure zero with $r > \frac{A}{2}$, we have

\[ u^{f}(x_{0}, y_{0}) = \frac{1}{2\pi} \int_{C^{1}(P_{0}, f)} \left[ \begin{array}{c} \tau(u^{k_{xj}}_{lajr} + u^{k_{xj}}_{lajr} - \frac{\partial}{\partial x_{j}} V_{r}^{k}) + u^{k_{xj}}_{lajr} - \frac{\partial}{\partial x_{j}} V_{r}^{k} \end{array} \right] dx \]

where the coefficients $a$, $b$, $c$, $d$, $e$, and $f$ satisfy their previous conditions.

\[ \frac{1}{2\pi} \int_{C^{1}(P_{0}, f)} \left[ \begin{array}{c} \tau(u^{k_{xj}}_{lajr} + u^{k_{xj}}_{lajr} - \frac{\partial}{\partial x_{j}} V_{r}^{k}) + u^{k_{xj}}_{lajr} - \frac{\partial}{\partial x_{j}} V_{r}^{k} \end{array} \right] dx \]
Theorem 6: Let $u, v, w$ and $y$ be of class $D$, on a region $G$ and satisfy their previous conditions on $G$, and where

\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2},
\]

Then (27) follows by multiplying (31) by $r^2$ and integrating with respect to $r$ and $\theta$, and we obtain

\[
\frac{1}{r} \int_{0}^{2\pi} \int_{0}^{\infty} \frac{1}{r} \frac{\partial}{\partial r} (r^2 u) \, dr \, d\theta = \frac{1}{r} \int_{0}^{\infty} \int_{0}^{2\pi} \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} (r^2 u) \, dr \, d\theta.
\]

We then choose $r_1$ and $r_2$ with $\frac{\partial}{\partial r} r_1 \leq \frac{\partial}{\partial r} r_2$ and we obtain
dr from (30) by $r$ and integrate between $r_1$ and $r_2$, and we obtain
Then the functions \( u, v, \) and \( V \) satisfy conditions \( A_\mathbf{t}, M(a, d) \) and
\( B(\mathbf{t}/2, N(a, d)) \) on \( G \) where the choice of \( \mathbf{t}, M(a, d), \) and \( N(a, d) \) depends only
on \( m, M, \mathbf{r} = \mathbf{r}_1 \vee \mathbf{r}_2, M_3, \mathbf{r}_3, M_4, N, \) and \( N. \) In particular, \( u, v, \) and \( V \) satisfy
a uniform H\"older condition on each bounded closed subregion \( \mathbf{g} \) of \( G \) where this
H\"older condition depends only on \( m, M, \mathbf{r}, M_1 \vee \mathbf{r}_2, M_3, \mathbf{r}_3, M_4, N, \) and the dis-
tance of \( H \) from \( G. \)

Proof: Let \( P_1 \) be any point of \( G, \) and let its distance from \( G^* \) by
\( A(\mathbf{S}, A, A^*). \) Let \( u_0 \) be the solution of \( (1) \) which vanishes on \( \Sigma \), let \( v_0 \)
be the potential of \( J \), for \( i_{ox} + \eta_x \), \( f_{ox} \), \( f_{oy} \), and then choose \( v_0 \) to
satisfy \( (32) \) with \( u_0 \) and \( v_0. \) From our previous theorems, \( v_0 \), \( v_0^* \), and \( v_0 \) sat-
ify conditions \( A_{\mathbf{t}}(\mathbf{t}/M) \) and \( B(\mathbf{t}/2, \mathbf{N}) \) on \( \Sigma \) and satisfy conditions like
\( (33) \) on \( \Sigma \) where the \( \mathbf{t}, \mathbf{N}, \mathbf{N}, \) and corresponding \( M^* \) of \( (33) \) depend only
on \( m, M, \mathbf{r}, M_1 \vee \mathbf{r}_2, M_3, \mathbf{r}_3, M_4, N, \) and \( A(\mathbf{S}, A, A^*). \)

Thus, if we let \( u_1 = u - u_0, v_1 = v - v_0, V_1 = V - V_0 \), we see that
\( u_1, v_1, \) and \( V_1 \) satisfy \( (28) \) almost everywhere on \( \Sigma \) and \( u_1, v_1, \) and \( V_1 \)
satisfy \( (33) \) with a new \( M^* \) which depends on the indicated quantities. Thus
a representation of \( u_1 \) on \( C(P_1, \Sigma) \) by \( (27) \) is valid. We observe that this
representation gives
\[
\int u_1(x, y, \omega) \frac{d\omega}{4\pi} = \int \sum_{\mathbf{r}} \left( \frac{\phi_{\mathbf{r}} + 1}{\mathbf{r}!} \right) \psi_{\mathbf{r}}(x, y) \frac{d\mathbf{r}}{\mathbf{r}!} \frac{d\omega}{4\pi}
\]
where \( G^*, K^*, \) and \( L^* \) are so related to \( u_1, v_1, \) and \( V_1, \) that we have
\[
\int \sum_{\mathbf{r}} \left( \frac{\phi_{\mathbf{r}} + 1}{\mathbf{r}!} \right) \psi_{\mathbf{r}}(x, y) \frac{d\mathbf{r}}{\mathbf{r}!} \frac{d\omega}{4\pi} \leq M_{10} \frac{\mathbf{r}}{\mathbf{r}!} \frac{d\omega}{4\pi}
\]
where \( M_{10} \) depends on \( A \) as well as the other quantities and \( \mathbf{r}, \) depends on
By letting $W^\alpha$ be the potential of $L^\alpha$, we may obtain

$$u^{\alpha}_1(I_1,\gamma_0) = \sum \{ \frac{\partial}{\partial \alpha} x \frac{\partial}{\partial \gamma} y \} \alpha \times \alpha \gamma$$

where

$$\sum \{ \frac{\partial}{\partial \alpha} x \frac{\partial}{\partial \gamma} y \} \alpha \times \alpha \gamma \leq \sqrt{\int_{\gamma_0}^{\gamma_0} \left( \frac{\partial}{\partial \gamma} \right) \cdot \left( \frac{\partial}{\partial \alpha} \right) \}$$

Thus, by Theorem 1, $u_1$ satisfies conditions A and B as indicated. Since $V_1$ is the potential of $d^{\beta}_i u^{\beta}_1 + \epsilon^{\beta}_i k^{\beta}_1 y + f^{\beta}_1 u^{\beta}_1$ plus a harmonic function on $\Sigma_A$, the same holds for $V_1$ on $C(P_1^\alpha, A_1^\alpha)$, and hence also for $v_1$. This proves the theorem.

Theorem 6: Let $u_n, v_n$, and $V_n$ satisfy a sequence of equations of the type of (32) on $G$, and suppose the coefficients $a_n, b_n, c_n, d_n, e_n, f_n$ satisfy our previous conditions uniformly, and suppose $g_n, k_n, \ell_n, u_n, v_n$ and $V_n$ satisfy (33) uniformly. Suppose also that $u_n, v_n$, and $V_n$ converge uniformly on each bounded closed subset of $G$ to functions $u, v$, and $V$, and suppose that the $a_n, b_n, c_n$ converge almost everywhere to $a, b, c$, that $d_n, e_n, f_n$, and $k_n$ converge strongly in $L^2$ to $d, e, g$, and $k$, and that $f_n$ and $\ell_n$ converge strongly in $L^1$ to $f$ and $\ell$. Then our coefficients satisfy the same conditions as the $a_n$, etc., the $u, v, and V$ are of class $D_2$ on $G$ and satisfy the limiting equations (32) almost everywhere.

Proof: Since the $u_n, v_n$, and $V_n$ all satisfy certain conditions $A[ \gamma, M(a, d)]$ and $B[ \gamma/2, N(a, d)]$ uniformly, it is clear that the $u, v$, and $V$ will satisfy the same conditions. That the limiting coefficients satisfy the conditions of boundedness satisfied uniformly by the $a_n$, etc., is obvious. It is easily seen further that the first derivatives of $u_n, v_n$, and $V_n$ converge
which shows that the limiting equations \( (2) \) hold on \( D' \). But Theorem 2.5 tells us that

\[
\text{weakly in } L^2 \text{ to those of } u, v, \text{ and } V, \text{ respectively, on any bounded closed subset of } G.
\]

Thus the functions

\[
d^{(n)}_{\beta\rho} u_{n\alpha}^\beta + e^{(n)}_{\beta\rho} u_{n\gamma}^\beta + f^{(n)}_{\beta\rho} u_{n\gamma}^\beta
\]

tend weakly in \( L^1 \) to

\[
d_{\beta\rho} u_{\alpha}^\beta + e_{\beta\rho} u_{\gamma}^\beta + f_{\beta\rho} u_{\gamma}^\beta
\]
on any such bounded closed subset of \( G \).

Now let \( D \) be a circle with \( \overline{D} \subset G \). On \( D, V_n^i = H_n^i + W_n^i \), where \( W_n^i \) is the potential of

\[
d^{(n)}_{\beta\rho} u_{n\alpha}^\beta + e^{(n)}_{\beta\rho} u_{n\gamma}^\beta + f^{(n)}_{\beta\rho} u_{n\gamma}^\beta
\]

on \( D \) only.

From Theorem 5, \( \S 6, W_n \) tends uniformly to \( W \) on \( D \) and \( \lim D_2(W_n - W, D) = 0 \).

Thus, clearly, \( H_n^i \) tends uniformly on \( \overline{D} \) to \( H^i \). Thus, if \( D' \) is a circle with

\( D' \subset D, D_2(V_n - V, D') \rightarrow 0 \). Thus, we see that on \( D' \) we have

\[
\exists \text{ a proof here.}
\]

\[
U^i = a^i_{\beta\rho} u_{\alpha}^\beta + e^i_{\beta\rho} u_{\gamma}^\beta + f^i_{\beta\rho} u_{\gamma}^\beta
\]

was made in \( \S 3 \) concerning the solutions of equations of the type of \( \{2\}\).

\[
\begin{cases}
\xi^i = \theta^i_{\beta\rho} u_{\alpha}^\beta + \eta^i_{\beta\rho} u_{\gamma}^\beta - \gamma^i
\quad \xi^i = \delta^i_{\beta\rho} u_{\alpha}^\beta + \xi^i
\quad \xi^i = \epsilon^i_{\beta\rho} u_{\alpha}^\beta + \zeta^i
\end{cases}
\]

where

\[
K^i = \epsilon^i_{\beta\rho} u_{\alpha}^\beta + \zeta^i - \nu^i
\]

and

\[
\lim_{n \to \infty} \int_D \left( (\psi_n - \psi)^2 + (k_n - k)^2 \right) d\alpha d\gamma = 0
\]

Then, from Lemma 3, \( \S 9 \), it follows that

\[
\leq \lim_{n \to \infty} \int_D \left[ (\psi_n^i - \xi^i)^2 + (\phi_n^i + \phi^i - \xi^i)^2 \right] d\alpha d\gamma
\]

\[
= 0
\]
which shows that the limiting equations (32) hold on \( D' \). The theorem follows easily from this.

**Theorem 7:** Let \( u, v, \) and \( V \) be of class \( D_2 \) on a region \( G \) and satisfy (32) almost everywhere on \( G \). Suppose, in addition to the previous conditions, the coefficients all satisfy uniform Hölder conditions on each bounded closed subset of \( G \). Then \( u, v, \) and \( V \) are of class \( C^1 \) in \( G \) and the first partial derivative satisfy uniform Hölder conditions on each bounded closed subset of \( G \).

**Proof:** This theorem has been proved in essence by E. Hopf \(^\ast\) and we shall not include a proof here.


Theorems 5, 6, and 7 give a complete demonstration of the statements made in §8 concerning the solutions of equations of the type of (32).