# Limits in Hodge Theory*

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*Lecture series at the University of Miami, Spring 2022

## Abstract

Almost all of the deep results in Hodge theory and its applications to algebraic geometry require understanding the limits in a family of Hodge structures. In the literature the proofs of these results frequently use the consequences of the analysis of the singularities acquired in a degenerating family of Hodge structures; that analysis itself is treated as a "black box." In these lectures an attempt will be made to give an informal introduction to the subject of limits of Hodge structures and to explain some of the essential ideas of the proofs.

One additional topic not yet in the literature that we will discuss is the geometric interpretation of the extension data in limiting mixed Hodge structures and its use in moduli questions.

# Outline of the lectures

I. *Review of basic definitions; examples*
   - *Polarized Hodge structures*
   - *Mixed Hodge structures*
   - *Limiting mixed Hodge structures*

II. *Period mappings and the first limit theorem*
   - *Period domain $D$ and its compact dual $\check{D}$*
   - *Period mapping $\Phi : B \to \Gamma\backslash D$*
   - *Nilpotent orbit theorem*

III. *The second limit theorem*
   - $\mathrm{sl}_2$*-orbit theorem in 1 variable*
   - *Some applications*
   - *Extension data in the limit mixed Hodge structure*
   - *Chern forms of extended Hodge bundles*

References[†]

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[†]We will give only a few general references that may serve as a guide to the literature and in which there are further references to the original papers.

# I. Review of basic definitions; examples[‡]

## I.A. Hodge structures

A *polarized Hodge structure* (PHS) of weight $n$ is $(V, Q, F)$

- $V = \mathbb{Q}$-vector space with a lattice $V_{\mathbb{Z}} \subset V$;
- $Q : V \otimes V \to \mathbb{Q}$, $Q(u, v) = (-1)^n Q(v, u)$;
- $F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}$ with

$$(\mathrm{I}.1) \qquad F^p \oplus \overline{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}} \qquad 0 \leqq p \leqq n;$$

- *Hodge-Riemann* I, II (HRI, HRII) are satisfied.

For

$$(\mathrm{I}.2) \qquad V^{p,q} = F^p \cap \overline{F}^q,$$

$$(\mathrm{I}.1) \iff V_{\mathbb{C}} = \overset{p+q=n}{\oplus} V^{p,q}, \qquad \overline{V}^{p,q} = V^{q,p}.$$

$\{F^p\}$ is the *Hodge filtration* and (I.2) is the *Hodge decomposition*.

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[†]For general Hodge theory [CM-SP] is an excellent source.

(HRI) $Q(F^p, F^{n-p+1}) = 0 \iff Q(V^{p,q}, \overline{V}^{p',q'}) = 0$ for $p' \neq n-p$

(HRII) $(-1)^p Q(V^{p,q}, \overline{V}^{p,q}) > 0$.

The alternating of signs is important.[§]

Without the $Q$ we have just a *Hodge structure* (HS) *of weight $n$*

Example: $X =$ compact Kähler manifold or a smooth complete complex algebraic variety

- $H^n(X, \mathbb{Q})$ has a Hodge structure of weight $n$.

Example: $\omega = c_1(L)$ for $L \to X^d$ an ample line bundle

$$\text{Hard Lefschetz} \quad \omega^m : H^{d-m}(X, \mathbb{Q}) \xrightarrow{\sim} H^{d+m}(X, \mathbb{Q})$$

$$\implies H^{d-m}(X)_{\mathrm{prim}} := \ker\left\{\omega^{m+1} : H^{d-m}(X) \to H^{d-m+1}(X)\right\}$$

is a PHS where $Q(u, v) = \int_X \omega^m \wedge u \wedge v$.

[§] We omit powers of $i$.

Example:

- $\mathbb{Q}(-1) = H^2(\mathbb{P}^1, \mathbb{Q}) =$ *Tate HS* (weigh 2 and Hodge type (1,1));
- $\mathbb{Q}(-k) = \overset{k}{\otimes}\,\mathbb{Q}(-1)$.

Hodge structures are functorial for morphisms (may have to use Tate twists to adjust weights).

Example: $Y \subset X$ a smooth subvariety of codimension $k$

$\implies H^m(Y)(-k) \xrightarrow{\mathrm{Gy}} H^{m+2k}(X)$, $\mathrm{Gy} =$ Gysin

is a morphism of HS's each of weight $m + 2k$.

**Example:** $Y$ connected and $m = 0$ gives the fundamental class mapping

$$\begin{tikzcd} H^0(Y)(-k) \arrow[r, "\mathrm{cl}"] & H^{2k}(X) \\ \wr\| & \cup \\ \mathbb{Q}(-k) \arrow[r] & H^{2k}(X,\mathbb{Q}) \cap H^{k,k}(X) \end{tikzcd} .$$

The map $\mathrm{cl}$ can be defined for any irreducible $Y$.

- $\mathrm{Hg}\, V = V \cap V^{n,n}$ (*Hodge classes*);
- For the codimension $k$ algebraic cycles $Z^k(X) := \{Y = \Sigma n_i Y_i,\ n_i \in \mathbb{Q}\}$ we have

$$\mathrm{cl} : Z^k(X) \to \mathrm{Hg}^k(X)$$

Hodge conjecture (HC) is that map is surjective (would be an existence theorem).

Example: $n = 2m + 1$.

- $J = V_{\mathbb{C}}/F^{m+1} + V_{\mathbb{Z}}$ is a compact, complex torus or abelian complex Lie group (*intermediate Jacobian*).
- The complex tangent space at the identity

$$T_e J \cong V_{\mathbb{C}}/F^{m+1} = \underbrace{H^{m,m+1}} \oplus \cdots \oplus H^{0,2m+1}.$$

- $J_{\mathrm{ab}} \subset J$ is largest sub-torus such that

$$T_e J_{\mathrm{ab}} \to H^{m,m+1}.$$

- Below we will define $\mathrm{AJ}_X : \ker(\mathrm{cl}) \to J^k(X)$ $(V = H^{2k-1}(X))$.
- $\mathrm{AJ}_{X,*} : TZ^k(X) \to J^k(X)_{\mathrm{ab}}$.
  HC $\iff$ this map is surjective (also existence theorem).

# I.B Mixed Hodge structures

A *mixed Hodge structure* (MHS) is $(V, W, F)$ where

- $V, F$ are as above;
- $W_\ell \subset \cdots \subset W_{k-1} \subset W_k \subset \cdots \subset W_m = V$ is the *weight filtration* with *associated graded* $\mathrm{Gr}_k^W(V) = W_k/W_{k-1}$;[¶]
- the Hodge filtration induces a Hodge structure of weight $k$ on $\mathrm{Gr}_k^W V$ where

$$F^p \,\mathrm{Gr}_k^W(V) = F^p \cap W_k/W_{k-1}.$$

As with Hodge structures MHS's are closed under all the standard operations $\oplus, \otimes, \mathrm{Hom} \ldots$ of linear algebra.

[¶] The $\ell$'s can be negative.

- a morphism (of weight $\ell$, which also can be negative) $(V, W, F) \xrightarrow{\varphi} (V', W', F')$ is given by

$$\begin{cases} \varphi : V \to V' \text{ where} \\ \varphi : W_k \to W'_{k+2\ell} \\ \varphi : F^p \to F'^{p+\ell}; \end{cases}$$

- *the* basic propery is *strictness*

$$\begin{cases} \varphi(W) \cap W'_{k+2\ell} = \varphi(W_k) \\ \varphi(V) \cap F'^{p+\ell} = \varphi(F^p). \end{cases}$$

Using this one shows that MHS's form an abelian category.[‖]

Example: $X$ = complete complex algebraic variety
$\implies H^n(X)$ has a MHS with $W_0 \subset \cdots \subset W_n$.

Example: $X$ = affine complex algebraic variety
$\implies H^n(X)$ has a MHS with $W_n \subset \cdots \subset W_{2n}$.

[‖] PHS's form a semi-simple category.

Example: For $Y \subset X$ the $H^n(X,Y)$, $H^n(X\backslash Y)$, $H^n_Y(X)$, and dually the homology groups have MHS's; the usual exact sequences (e.g., Mayer-Vietoris) are exact sequences of MHS's.

Example: $\mathrm{Hom}(V,V')$ is a MHS where

$$W_k \,\mathrm{Hom}(V,V') = \{A : V \to V', A(W_\ell(V)) \subseteq W_{\ell+k}(V')\},$$
$$F^p \,\mathrm{Hom}(V,V') = \{A : V_{\mathbb{C}} \to V'_{\mathbb{C}} : A(F^q) \subseteq F'^{q+p}\}.$$

- Given a MHS $(V,W,F)$ set

$$I^{p,q} = F^p \cap \left( W_{p+q} \cap \overline{F}^q + W_{p+q-1} \cap \overline{F}^{q+1} + \cdots \right).$$

Then

$$W_k = \bigoplus_{p+q \leqq k} I^{p,q}, \qquad F^p = \bigoplus_{\begin{cases} p' \geqq p \\ q \end{cases}} I^{p',q},$$

$$\overline{I}^{p,q} \equiv I^{q,p} \bmod \Bigg( \bigoplus_{\begin{cases} a<p \\ b<q \end{cases}} I^{a,b} \Bigg),$$

$$\implies \overline{I}^{p,q} \equiv I^{q,p} \bmod W_{p+q-2,\mathbb{C}}.$$

The $I^{p,q}$'s give the unique decomposition of the MHS satisfying these conditions.

The MHS is $\mathbb{R}$-*split* if $\bar{I}^{p,q} = I^{q,p}$. Associated to any MHS there is a unique $\mathbb{R}$-split one $(V, W, F')$.**

Example: For $(V, F)$ and $(V', F')$ Hodge structures of weights $n, n'$ with $n > n'$

$$\text{(I.3)} \quad \mathrm{Ext}^1_{\mathrm{MHS}}(V, V') = \frac{\mathrm{Hom}_{\mathbb{C}}(V, V')}{F^0 \,\mathrm{Hom}_{\mathbb{C}}(V, V') + \mathrm{Hom}_{\mathbb{Z}}(V, V')}$$

is an abelian complex Lie group $\mathbb{C}^m/\Lambda$ where $\Lambda$ is a discrete subgroup (thus it is a composite of $\mathbb{C}^k$'s, $\mathbb{C}^{*\ell}$'s and compact tori).

**This is an algebraic, not a geometric (or motivic) construction.

If $V, V'$ have weights $k, k-1$, $\mathrm{Ext}^1_{\mathrm{MHS}}(V, V')$ is a compact torus whose complex tangent space at the identity is a HS of weight $-1$ that may be pictured

$$\underbrace{(k-1,-k)\oplus\cdots\oplus(0,-1)}_{F^0}\oplus\underbrace{\underbrace{(-1,0)}\oplus\cdots\oplus(-k,k-1)}_{T\,\mathrm{Ext}^1_{\mathrm{MHS}}(V,V')\cong\overline{F}^0}.$$

The tangent space to $\mathrm{Ext}^1_{\mathrm{MHS}}(V, V')_{ab}$ is contained in the part over the red bracket (it corresponds to the maximal $\mathbb{Q}$-subspace of $(0,-1)\oplus(-1,0)$).

Example: For $Y = \Sigma n_i Y_i$ a codimension $k$ algebraic cycle in $X$ with support $|Y| = \cup Y_i$ and with $\mathrm{cl}(Y) = 0$, setting $\dim_{\mathbb{C}} X = k + \ell$ the dual of

$$\cdots \to H_{2\ell+1}(X) \to H_{2\ell+1}(X, |Y|) \to H_{2\ell}(|Y|) \to H_{2\ell}(X) \to \cdots$$

gives a class $\mathrm{AJ}_X(Y)$ in

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(-k), H^{2k-1}(X)) \cong J^k(X).$$

- $\mathrm{Ext}^q_{\mathrm{MHS}}(\bullet, \bullet) = 0$ for $q \geqq 2$; there are no higher $\mathrm{Ext}$'s in mixed Hodge theory.

- In (I.3) if $k = k' + 2$, then the tangent space at the identity to $\mathrm{Ext}^1_{\mathrm{MHS}}(V, V')$ is the quotient of a Hodge structure of weight $-2$ by $F^0$ of that HS

$$\underbrace{(k', -k) \oplus \cdots \oplus (0, -2)}_{F^0} \oplus \underbrace{\underbrace{(-1, -1)} \oplus (-2, 0) \oplus \cdots \oplus (-k, k')}_{T\,\mathrm{Ext}^1_{\mathrm{MHS}}(V, V')}.$$

The connected analytic subgroup $S$ whose tangent space is the maximal $\mathbb{Q}$-subspace contained in the red is

$$S = (\mathbb{C}^*)^k.$$

- In (I.3) if $k = k' + 3$, the picture is

$$\underbrace{(k', -k) \oplus \cdots \oplus}_{F^0} \underbrace{\underbrace{(-1, -2) \oplus (-2, -1)} \oplus (-3, 0) \oplus \cdots \oplus (-k, k')}_{T\,\mathrm{Ext}^1_{\mathrm{MHS}}(V, V')} .$$

There is no non-trivial complex subgroup of $\mathrm{Ext}^1_{\mathrm{MHS}}(V, V')$ whose complex tangent space lies in the red.[††]

[††]The complexification of the real tangent space would have to be closed under conjugation.

# I.C Limiting mixed Hodge structures

- $\mathrm{sl}_2$ has basis $\{N^+, H, N\}$ where

$$\begin{aligned} [N^+, N] &= H & H &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \\ [H, N^+] &= 2N^+ & N^+ &= \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \\ [H, N] &= -2N & N &= \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \end{aligned}$$

- irreducible $\mathrm{sl}_2$ module $V^{(k)} = \mathrm{Sym}^k(\mathbb{Q}^2)$ has basis $x^i y^{k-i}$ where

$$\begin{cases} N^+ = y\partial_x \\ N = x\partial_y \\ H = k - 2i \qquad 0 \leqq i \leqq k \end{cases}$$

- the eigenvalues (weights) of $H$ are $-k, \ldots, 0, \ldots, +k$; $N$ decreases weights by 2.

- Any $\mathrm{sl}_2$-module

$$V = \oplus\, m_k V^{(k)}.$$

- Let $V = \bigoplus_{j=-n}^{n} V_j$ be a graded vector space
  $N : V_j \to V_{j-2}$
  $N^k : V_k \xrightarrow{\sim} V_{-k}$ (Hard Lefschetz property).

(Jacobson-Morosov-Kostant). There is a unique $\mathrm{sl}_2 = \{N^+, H, N\}$ acting on $V$.

## Example:

$X = n$-dimensional compact Kähler manifold

$V = H^*(X)[-n]$

$N^+ = \omega = L$

$\Longrightarrow$ there is $\{L, H, \Lambda\}$ acting on $H^*(X)$.

Direct sums of the irreducible $\mathrm{sl}_2$-factors give the $H^k(X)_{\mathrm{prim}}$.

- Suppose now we just have a nilpotent $N : V \to V$ with $N^{n+1} = 0,\ N^n \neq 0$

$\implies$ there exists unique weight filtration (now centered at $n$)
$W_0 \subset \cdots \subset W_{2n} = V$

$$\begin{cases} N : W_k \to W_{k-2} \\ N^k : W_{n+k} \xrightarrow{\sim} W_{n-k}. \end{cases}$$

We write $W_k(N)$, or just $W(N)$.

Jacobson-Morosov: There exists a *non-unique* $\mathrm{sl}_2 = \{N^+, H, N\}$, $W(N)$ is the weight filtration for any such $\mathrm{sl}_2$.

- Given $(V, N)$

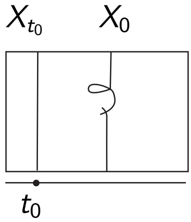
$$\mathrm{sl}_2\text{-actions} \iff V \cong \mathrm{Gr}_\bullet^{W(N)}(V).$$

- A *limiting mixed Hodge structure* (LMHS) is a MHS $(V, Q, W(N), F)$ where $N \in \mathrm{End}_Q(V), N^{n+1} = 0$, and

$$N : F^p \subset F^{p-1}.$$

**Example:** Associated to $(V, Q, W(N), F)$ is a unique $\mathbb{R}$-split LMHS $(V, Q, W(N), F')$, $V \cong \oplus \mathrm{Gr}_k^{W(N)}(V)$. Then there is a unique $\mathrm{sl}_2$ acting on $V$ with $N(F'^p) \subset F'^{p-1}$.

**Example:** $\mathcal{X}^* \xrightarrow{\pi} \Delta^* = \{0 < |t| < 1\}$ is a smooth family of Kähler manifolds $X_t = \pi^{-1}(t)$, $t \neq 0$, with $X_0 =$ generally singular analytic variety



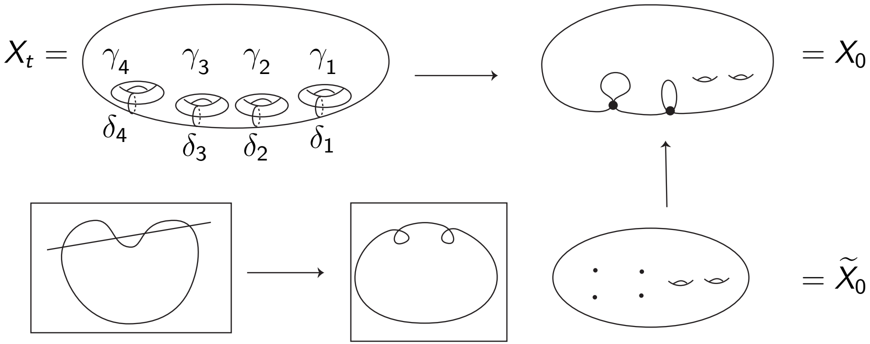

$T : H^n(X_{t_0}) \to H^n(X_{t_0})$ is monodromy

$T = T_s T_u$ Jordan normal form

**Monodromy theorem:** *$T_s^m = \mathrm{Id}$ (eigenvalues of $T$ are roots of unity) and $T_u^{n+1} = 0$.*
Will sketch proof later.

**Theorem (Schmid):** $\lim_{t \to 0} H^n(X_t) = H^n_{\lim}$ exists as a LMHS.

**Example:**





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$$\begin{cases} N\delta_i = 0 \qquad i = 1,2,3,4 \\ N\gamma_1 = N\gamma_2 = 0 \\ N\gamma_3 = \delta_3, N\gamma_4 = \delta_4 \end{cases}$$

$$N^2 = 0 \qquad \overbrace{\overbrace{W_0}^{\mathrm{Im}\,N} \subset W_1}^{\ker N} \subset W_2$$

$$\begin{array}{ccc} W_0 & & W_1 \\ \| & & \| \\ \{\delta_1, \delta_2\} & & \{\delta_i, \gamma_1, \gamma_2\} \end{array}$$

$$\begin{array}{ll} \mathrm{Gr}_2^W H^1_{\lim} \cong \mathbb{Q}(-1)^2 & \text{type } (1,1) \\ \mathrm{Gr}_1^W H^1_{\lim} \cong H^1(\widetilde{X}_0) & \text{type } (1,0)+(0,1) \\ \mathrm{Gr}_0^W H^1_{\lim} \cong \mathbb{Q}^2 & \text{type } (0,0) \end{array}$$

In general on $\mathrm{Gr}^W H^n_{\lim}$ the $N$ has type $(-1,-1)$.

**Basic estimate:** Suppose we have a family of PHS's over $\Delta^*$. For each $t \in \Delta^*$ we have a PHS $(V, Q, F_t)$. For $v \in V_{\mathbb{C}}$ we have

$$v = \sum_{p+q=n} v_{p,q}(t)$$

and the *Hodge norm*

$$|v|^2_{t,H} = \sum_{p+q=n} (-1)^p Q(v_{p,q}(t), \overline{v_{p,q}(t)}).$$

**Theorem:** $v \in W_k \iff |v|^2_{t,H} = 0\left((-\log|t|)^{k-n}\right)$.

**Corollary:** $Nv = 0 \implies |v|^2_{t,H} = 0(1)$.

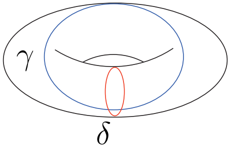
This result, which relates topology and analysis, is one of the deepest results in Hodge theory. In the next two lectures I will discuss how one goes about proving it and some of its consequences.

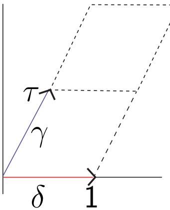
# II. Period mappings and the first limit theorem

## II.A. Basic model

- *toplological, analytic, algebro-geometric*

$w = u + iv$ plane

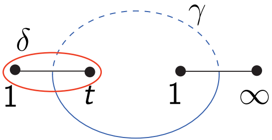
$X =$ 


$\mathrm{Im}\,\tau > 0$

$= \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$

$= y^2 = x(x - t)(x - 1)$





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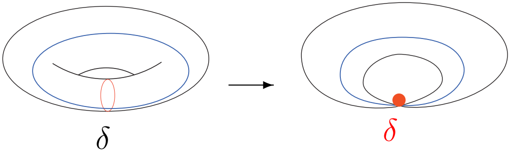
As $t \to 0$ we have




$\delta =$ vanishing cycle

As $t$ turns around $t = 0$ we have monodromy

$$(*) \qquad \begin{aligned} T\delta &= \delta \\ T\gamma &= \gamma + \delta \end{aligned} \qquad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- *Hodge theoretic*
  - $H^{1,0}(X)$ has basis $\omega$ given by
    - $\omega = dw$ (analytic realization)
    - $\omega = \frac{dx}{\sqrt{x(x-t)(x-1)}}$ (algebraic realization)

– *period matrix*

$$\Phi = \begin{pmatrix} \int_\delta \omega \\ \int_\gamma \omega \end{pmatrix} = \begin{pmatrix} 1 \\ \tau \end{pmatrix} \quad \text{where} \quad \operatorname{Im}\tau > 0.$$

As $t$ turns around $t = 0$, by $(*)$

$$\Phi \to \Phi + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = T\Phi.$$

- *period mapping:* $\mathcal{H} = \{z = x + iy : y > 0\} = \begin{bmatrix} 1 \\ z \end{bmatrix} \in \mathbb{P}^1$

$$\begin{array}{ccc} \Phi : \Delta^* & \longrightarrow & \{T^k\}\backslash\mathcal{H} \\ \Cup & & \Cup \\ t & \longrightarrow & \begin{bmatrix} 1 \\ z = \tau \end{bmatrix}, \end{array}$$

- *analysis:* as $t$ turns around $t = 0$

$$\int_\gamma \omega \to \int_\gamma \omega + 1 \implies \int_\gamma \omega - \frac{\log t}{2\pi i} = h(t)$$

is single-valued on $\Delta^*$; then by analysis

$$h(t) \text{ bounded} \implies h(t) \text{ holomorphic in } \Delta$$

$$(**) \qquad \implies \Phi(t) \sim \exp\left(\left(\frac{\log t}{2\pi i}\right) N\right) \begin{bmatrix} 1 \\ h(0) \end{bmatrix}$$

- period mapping is approximately a *nilpotent orbit*.

– *Lie theoretic formulation:* $\ell(t) := \log t/2\pi i$

$$(**) \qquad \Longrightarrow \; \Phi(t) \sim \exp(\ell(t)N)F, \quad F = \begin{bmatrix} 1 \\ h(0) \end{bmatrix} \in \mathbb{P}^1$$

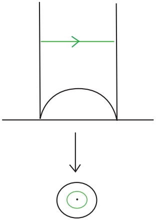
What does "$\sim$" mean?

– $\mathcal{H} = \mathrm{SL}(2,\mathbb{R})/U(1)$ has invariant metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ and $(**)$ means

$$(***) \qquad d_{\mathcal{H}}(\Phi(t)), (\exp(\ell(t)N)F) \leq y^{\beta} e^{-2\pi y}.$$

**Note:** Relation between $t \in \Delta^*$ and $z \in \mathcal{H}$

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \Delta^* \\ \Cup & & \Cup \\ z & \longrightarrow & t = e^{2\pi i z}, \end{array}$$



$$|t| = e^{-y} \implies \text{RHS of } (***) \text{ is } O(|t|^{1-\epsilon}), \text{any } \epsilon > 0.$$

**Conclusion:** *Above family of HS's over $\Delta^*$ is approximated by a nilpotent orbit arising from monodromy.*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# II.B. Monodromy and nilpotent orbit theorem in general

- *Period domain:* Given $(V, Q, f^p)$

$$\begin{aligned} D &= \text{set of PHS's } (V, Q, F : \dim F^p = f^p), \\ G &= \mathrm{Aut}(V, Q) = \mathbb{Q}\text{-algebraic group}, \\ G_{\mathbb{R}} &= \text{associated real Lie group, acts on} \\ &\qquad D \text{ by } F \to gF, g \in G_{\mathbb{R}} \\ G_{\mathbb{Z}} &= \mathrm{Aut}(V_{\mathbb{Z}}, Q) \\ D &= G_{\mathbb{R}}/H, \quad H = \textit{compact} \text{ subgroup.} \end{aligned}$$

- *Compact dual*

$$\check{D} = \{(V, Q, F) : \dim F = f^p \text{ and } Q(F^p, F^{n-p+1}) = 0\}$$
$$\check{D} = G_{\mathbb{C}}/P, \quad P = \text{parabolic subgroup}$$
$$D \subset \check{D} \text{ is open } G_{\mathbb{R}}\text{-orbit.}$$

## Example:

$D = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) = \mathcal{H} \subset \mathbb{P}^1 = \check{D} = \mathrm{SL}(2,\mathbb{C})/P$.

Here $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For $F = \begin{bmatrix} 1 \\ z \end{bmatrix} \in \mathbb{P}^1$ we have

$$Q(F,F) = 0, \quad iQ(F,F) = -i(z-\bar{z}) = 2y.$$

- $$\begin{array}{ccc} T\check{D} & \subset & \oplus \mathrm{Hom}(F^p, V_{\mathbb{C}}/F^p) \\ \cup & & \cup \\ \{F_t^p\} & \longrightarrow & \{dF_t^p/dt \in V_{\mathbb{C}}/F_0^p\} \end{array}$$
- Horizontal sub-bundle $I \subset T\check{D} = \{\dot{F}^p \subseteq F^{p-1}/F^p\}$
- Classical case: $D =$ Hermitian symmetric domain $\iff I = T\check{D}$.

Example (non-classical): $n=0,\ h^{2,0}=a, h^{1,1}=b$.

- $D=\{F^2 : \dim F^2 = a,\ Q(F^2,F^2)=0,\ Q(F^2,\overline{F}^2)>0\}$;
- $\Omega = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$, HRI, II are ${}^t\Omega Q \Omega = 0$, ${}^t\Omega Q \overline{\Omega} > 0$ where

  $-Q = \begin{pmatrix} 0 & 0 & I_a \\ 0 & -I_b & 0 \\ I_a & 0 & 0; \end{pmatrix}$
- $I$ is ${}^t d\Omega Q \Omega = 0$ (only for $a=1$ does it follow from HRI);
- for $a=2$ it is a contact system.

**Period mapping:** $B =$ complex manifold, $\Gamma \subset G_{\mathbb{Z}}$

$\Phi : B \to \Gamma \backslash D$ locally liftable, holomorphic and $\Phi_* : TB \to I$.

**Example:** $B = \Delta^*$, $\Gamma = \{T^k\}$ where $T \in G_{\mathbb{Z}}$.

**Monodromy theorem** $\implies T = T_s T_u,\ T_s^m = I,\ T_u = e^N$.

Passing to $\widetilde{\Delta} \to \Delta$, $t = \tilde{t}^m$ we may assume $m = 1$.

**Nilpotent orbit theorem:** *There exists $F \in \check{D}$ such that*

- $t \to \exp(\ell(t)N) \cdot F$ *is a period mapping* $\mathcal{O}(t)$;
- $d_D(\Phi(t), \mathcal{O}(t)) = O(|t|^{1-\epsilon})$ *for any* $\epsilon > 0$.

# II.C. Estimate using differential geometry/complex function theory

- $E \to M$ is a holomorphic vector bundle over a complex manifold. Given a Hermitian metric $h$ in $E \to M$ there is a canonical *Chern connection* with *curvature form*

$$\Theta_E(e,\xi) = \sum \Theta_{\alpha\bar{\beta}i\bar{j}} e_\alpha \bar{e}_\beta \xi_i \bar{\xi}_j$$

  where $e \in E_p$, $\xi \in T_pM$.
- For $E = TM$ the holomorphic sectional curvature is

$$\Theta_M(\xi) = \Theta_{TM}(\xi,\xi).$$

**Ahlfors' lemma:** *If* $\Theta_M(\xi) \leqq -1$ *for* $|\xi| = 1$, *then for* $\Delta$ *with the Poincaré metric where* $\Theta_\Delta(\xi) = -1$ *a holomorphic mapping*

$$\Phi : \Delta \to M$$

*is distance decreasing*

$$d_M(\Phi(t), \Phi(t')) \leqq d_\Delta(t, t').$$

**Example:** $M = \Delta$; then this is the *Schwarz lemma* from complex function theory.

- Using HRI, HRII the *Hodge bundles*

$$F^p \to D$$

  have $G_{\mathbb{R}}$-invariant metrics. Their curvature forms have special sign properties that have had many applications in algebraic geometry (cf. [CM-SP] and [GG]).

Example: The Hodge line bundle

$$L := \overset{p}{\otimes} \det F^p$$

has curvature form $\omega$ with

$$\omega > 0 \ \text{ on } \ I \subset TD.$$

For period mappings

$$\omega(\xi) = \|\Phi_*(\xi)\|^2.$$

Example: Using $TD \subset \oplus \operatorname{Hom}(F^p, V_{\mathbb{C}}/F^p)$ there is a $G_{\mathbb{R}}$-invariant metric on $TD$ and there is $c > 0$ such that for $\xi \in I$ and $|\xi| = 1$

$$(\natural\natural) \qquad \Theta(\xi) \leqq -c.$$

Corollary: *A period mapping*

$$\Phi : \Delta^* \to \{T^k\}\backslash D$$

*is distance decreasing.*

## Proof of the monodromy theorem:

- Since $T \in \mathrm{Aut}(V_{\mathbb{Z}})$, $\det(T - \lambda I) = P(\lambda) \in \mathbb{Z}[\lambda]$;
- if an eigenvalue $P(\lambda) = 0$ has $|\lambda| = 1$, then by Kronecker some $\lambda^m = 1$.

  On $\Delta^*$ the Poincaré metric is

$$\frac{dt\, d\bar{t}}{|t|^2|\ell(t)|^2} = \frac{dr\, dt}{r(\log r)^2} = d\left(\frac{1}{(-\log r)}\right) d\theta$$

$$\Longrightarrow \text{length}(|t| = \epsilon) \to 0 \text{ as } \epsilon \to 0.$$

  In fact

$$\text{length}(|t| = \epsilon) = \frac{2\pi}{(-\log|\epsilon|)}.$$

– $t_n \to 0$, $D = G_{\mathbb{R}}/H$ implies that $\Phi(t_n) = g_n \cdot x_0$ where $g_n \in G_{\mathbb{R}}$

$$\begin{aligned} d_D(\Phi(t_n), T\Phi(t_n)) &= d_D(g_n \cdot x_0, Tg_n \cdot x_0) \\ &= d_D(x_0, g_n^{-1} Tg_n \cdot x_0) \\ \implies g_n^{-1} Tg_n \to H \ &\text{where } H = \text{product of } \mathcal{O}(k), U(\ell)\text{'s} \end{aligned}$$

which gives the result.

## Setting up the proof of the nilpotent orbit theorem:

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\widetilde{\Phi}"] \arrow[d] & D \arrow[d] \\ \Delta^* \arrow[r, "\Phi"] & \{T^k\}\backslash D \end{tikzcd}$$

$$\widetilde{\Phi}(z+1) = \exp N \cdot \widetilde{\Phi}(z);$$

- $\Psi(z) = \exp(-zN)\widetilde{\Phi}(z) \in \check{D}$ (unwinding monodromy);
- $\Psi(z+1) = \Psi(z)$ implies $\Psi$ is induced from

$$\Psi : \Delta^* \to \check{D} \subset \mathbb{P}^m.$$

For any rational function $f$ on $\mathbb{P}^m$

$$\Psi^*(f) = f \circ \Psi = \text{meromorphic function on } \Delta^*.$$

Need to show: $\Psi^*(f)$ has no essential singularity at $t = 0$; i.e., $(\Psi^*(f)(t))$ has polynomial growth in $\frac{1}{|t|}$.

- From this we obtain

$$F = \Psi(0) \in \check{D}$$

and then

$$\Phi(t) \sim \exp(\ell(t)N)F$$

where $(V, Q, W(N), F)$ will be proved to be a LMHS.

- The idea is to show $\Psi^*(f)$ meromorphic is to
  - use the distance decreasing property and $d_{\Delta^*}(t_0, t) = -\log|t| + C$ as $t \to 0$;
  - use the comparison between the metrics in $D$ and $\check{D}$ near $\partial D$; for $D = \Delta$ and $\check{D} = \mathbb{P}^1$

$$\frac{dz\, d\bar{z}}{(1-|z|^2)^2} \text{ on } \Delta, \quad \frac{dz\, d\bar{z}}{(1+|z|^2)^2} \text{ on } \mathbb{P}^1$$

    $\implies$ as $t \to 0$ in $\Delta^*$ the metrics are comparable up to a polynomial in $-\log|t|$.
  - Let $f \in \mathbb{C}(\check{D})$ be regular near $\lim_{t\to 0} \Phi(t) \in \partial D$. Then arguments extending the above special case show that $\Phi^*(f)$ has a most moderate growth (i.e., $O(|t|^{-k})$) as $t \to 0$.

- *Removable singularity theorem:* If we have

$$\Phi : \Delta^* \to \{T^k\}\backslash D$$

where $T = T_s$ (i.e., $N = 0$), then

$$\Phi \quad \textit{extends across } t = 0 \textit{ to have } \Phi : \Delta \to \{T^k\}\backslash D.$$

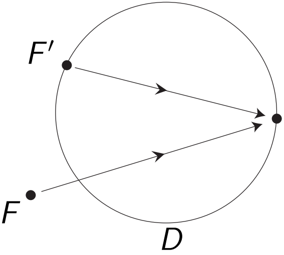
Proof: The images of the circles $|t| = \epsilon$ give closed curves $\gamma_\epsilon$ in $D$ whose length $\ell(\gamma_\epsilon) \to 0$. Moreover since the metric in $D$ is complete the $\gamma_\epsilon$ do not approach $\partial D$. Thus $\lim_{\epsilon \to 0} \gamma_\epsilon = F_0 \in D$ and $\Phi(0) = F_0$ gives the extension of $\Phi$. $\square$

- In the geometric case the variety $X_0$ usually has singularities, and the condition of finite monodromy is rather general (e.g., for normal surfaces $X_0$ can have rational singularities). Thus $H^n_{\lim}(X_t)$ is a pure Hodge structure — its relation to $H^n(X_0)$ is of significant interest.

**Preview of $\mathrm{sl}_2$-orbit theorem.** We want $\lim_{t\to 0}\Phi(t) = \mathrm{LMHS}\ (V, Q, W(N), F)$. If this is true, then there is an $\mathbb{R}$-split MHS $(V, Q, W(N), F')$ with $V \cong \mathrm{Gr}^{W(N)}(V)$ and $N$ uniquely completes to an $\mathrm{sl}_2$. This perhaps suggests looking to approximate the nilpotent orbit by an $\mathrm{sl}_2$-orbit.

**Relation to the $\mathbb{R}$-split MHS $(V, Q, W(N), F')$** Set

$$F_\infty = \lim_{t\to 0} \Phi(t)F \in \partial D \quad \text{(näive limit)}$$




$F_\infty$ = fixed point of $\exp(\mathbb{C}N)$ acting on $\check{D}$

$F$ *cannot* be on $\partial D$.

# III. The $\mathrm{sl}_2$-orbit theorem and applications ([CK1])

For applications a significant refinement of the nilpotent orbit theorem is needed. Just as a nilpotent operator $N \in \mathrm{End}(V)$ can be completed to an $\mathrm{sl}_2$, the $\mathrm{sl}_2$-orbit theorem gives an approximation to the nilpotent orbit and hence to a period mapping. The statement of the result is quite technical and will not be discussed. We will

(i) give a geometric interpretation of the result;
(ii) discuss one basic geometric idea behind the proof;
(iii) present some applications.

- For $D = \Delta \cong \mathcal{H}$ and $\check{D} = \mathbb{P}^1$ there is a standard variation of Hodge structure given by
  - for $z \in \mathcal{H}$ we take the weight $n = 1$ PHS $(V, Q, F)$ where $V = \mathbb{Q}^2$, $Q = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and

$$F^1 = \begin{bmatrix} 1 \\ z \end{bmatrix} \in \mathcal{H} \quad (\mathrm{Im}\, z > 0)$$

- the group $\mathrm{SL}(2,\mathbb{R})$ acting on $\mathcal{H}$ gives

$$\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2) = \mathcal{H} \subset \mathbb{P}^1 = \mathrm{SL}(2,\mathbb{C})/P;$$

- there is a standard linear fractional transformation (Cayley transform) on $\mathbb{P}^1$ that takes $\mathcal{H}$ to $\Delta$;
- the VHS over $\mathcal{H}$ is acted on by $\mathrm{SL}(2,\mathbb{R})$;

$$\text{for } T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \exp N \text{ where } N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

we have

$$\begin{array}{ccc} \{T^k\}\backslash\mathcal{H} & \cong & \Delta^* \\ \Cup & & \Cup \\ \begin{bmatrix} 1 \\ z \end{bmatrix} & \longrightarrow & t = e^{2\pi i z}. \end{array}$$

- An arbitrary period mapping $\Phi : \Delta^* \to \{T^k\}\backslash D$ is approximated by a nilpotent orbit

$$(*) \qquad \mathcal{O}(t) = \exp(\ell(t), N) \cdot F, \quad F \in \check{D}$$

where $T = \exp(N)$

$\mathrm{sl}_2$-orbit theorem: *The nilpotent orbit is approximated by an equivariant period mapping*

$$\begin{tikzcd} \mathcal{H} \arrow[r] \arrow[d] & D \arrow[d] \\ \Delta^* \arrow[r] & \{T^k\}\backslash D \end{tikzcd}$$

*induced by a homomorphism of Lie groups*

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{R}) & \xrightarrow{\rho} & G_{\mathbb{R}} \\ \cup & & \cup \\ \mathrm{SO}(2) & \longrightarrow & H \end{array}$$

*where*

$$\rho_* \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = T \quad (\rho_* = \textit{induced map on Lie algebras}).$$

- Implication:
  - any representation of $\mathrm{SL}(2,\mathbb{R})$ is a direct sum of the standard representations recalled in Lecture I.
  - the standard representation gives a weight $n$ VHS, hence *any* period mapping $\Phi : \Delta^* \to \{T^k\}\backslash D$ will have properties inferred from the standard ones.

This will then prove the existence of a LMHS associated to $\Phi : \Delta^* \to \{T^k\}\backslash D$.

**Example:** If $\Phi : \Delta^* \to \{T^k\}\backslash D$ is a period mapping of weight $n$, then

$$(N - I)^{m+1} = 0$$

for some $m \leqq n$. The reason is

$$\left\{\begin{matrix}\text{weight of a}\\ \text{direct sum}\end{matrix}\right\} = \left\{\begin{matrix}\text{maximum weight}\\ \text{of a summand}\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{weight of a}\\ \text{tensor product}\end{matrix}\right\} = \left\{\begin{matrix}\text{sum of the}\\ \text{weights of the factors}\end{matrix}\right\}.$$

This gives the index of unipotency part of the monodromy theorem.

**Example:** $v \in W_k(N) \iff |v|_{t,H} = 0\left((-\log|t|)^{(k-n)/2}\right)$. In particular,

$$\ker N \subseteq W_n(N).$$

**Corollary:** *Any invariant vector has bounded Hodge length.* 47/64

- Now let $B$ be an algebraic variety, generally non-complete, and

$$\Phi : B \to \Gamma\backslash D$$

a period mapping with monodromy group

$$\Gamma = \Phi_*(\pi_1(B)).$$

**Theorem of the fixed part:** *If $v \in V^\Gamma$ is $\Gamma$-invariant, then the Hodge components $v_t^{p,q}$ are constant.*

**Proof:** Using the curvature properties of the Hodge bundles and the corollary

$$\begin{aligned} &\log |v^{n,0}|_{t,H} \text{ is pluri-subharmonic and bounded on } \overline{B} \\ \implies &|v^{n,0}|_{t,H} = \text{constant} \\ \implies &v_t^{n,0} \text{ is constant} \qquad (\text{uses form of } \Theta_{F^n}). \end{aligned}$$

Then $v - v^{n,0}$ is constant, and we apply the same argument to this vector to get $v_t^{n-1,1}$ is constant, etc.

## Application: $\Gamma$ *is semi-simple.*

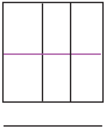
*Idea:* For an invariant subspace look at its Plücker coordinate in $\wedge^d V$ and show it is constant. From the theorem of the fixed part the Hodge $(p, q)$ components of the Plücker coordinate are constant $\implies$ invariant subspace is a sub-Hodge structure.

- *Where does the* $\mathrm{sl}_2$ *come from in the* $\mathrm{sl}_2$*-orbit theorem?*
  - nilpotent orbit gives $y \to \exp(iyN) \cdot F_0 \in D = G_{\mathbb{R}}/H$ for $y \gg 0$;
  - want to lift this to a map

$$y \to g(y) \in G_{\mathbb{R}};$$

  - any map to a Lie group is determined (up to fixed left-translation) by pulling back the $\mathfrak{g}$-valued Maurer-Cartan form $\omega = g^{-1}dg$;

- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$

$\mathfrak{h}^{\perp} =$ 

$$\begin{array}{c} G_{\mathbb{R}} \\ \downarrow \\ G_{\mathbb{R}}/H \end{array}$$

- apply this to $\exp(iy)N \cdot F$ to get $\omega_h(y)$
- $\omega_h$ satisfies three equations:
    (i) $\omega_h \in \mathfrak{h}^{\perp}$;
    (ii) $g(y)$ is a lift of $\exp(iy)N \cdot F_0$;
    (iii) $z \to \exp(zN) \cdot F_0$ is holomorphic.

Setting

$$\begin{aligned} A(y) &= -2g(y)^{-1}g'(y) = -2\omega_h \\ F(y) &= g(y)^{-1}Ng(y) \\ E(y) &= -JF(y), \qquad J^2 = \text{“}i\text{”} \text{ on } \mathfrak{h}^\perp \end{aligned}$$

the conditions (i)–(iii) lead to

$$\left.\begin{aligned} 2E'(y) &= [A(y), E(y)] \\ 2F'(y) &= [A(y), F(y)] \\ A'(y) &= -[E(y), F(y)] \end{aligned}\right. \quad \leftrightarrow \left\{ \begin{array}{c} \text{structure} \\ \text{equations for} \\ \mathrm{sl}_2 \end{array} \right\}$$

(These turn out to be the Nahn equations from mathematical physics)

The distance decreasing property gives

$$\|A(y)\| = O(y^{-1}) \text{ as } y \to \infty$$

and with "some work" there are similar estimates for $E(y), F(y)$. This leads to

$$A(y) = A_0 y^{-1} + A_1 y^{-1-1/2} + \cdots$$
$$E(y) = E_0 y^{-1} + E_1 y^{-1-1/2} + \cdots$$
$$F(y) = F_0 y^{-1} + F_1 y^{-1-1/2} + \cdots$$

where $A_0, E_0, F_0 \in \mathfrak{g}$ give an $\mathrm{sl}_2$!

Application [CK2]: Let $\Phi : B \to \Gamma\backslash D$ be a period mapping (weight $= 2n$) where $B$ is an algebraic variety. Then the *Hodge locus*

$$\begin{aligned}\mathcal{H} := \{(v, b, \gamma : v \in V_{\mathbb{Z}}, \gamma \\ &= \text{homotopy class of paths } \overline{b_0, b}, \gamma v \in \mathrm{Hg}^n(F_b)\}\end{aligned}$$

is a countable union of connected components each of which is proper and finite over $B$.

Corollary (informally stated): *The subset of $B$ where a given $v \in V_{\mathbb{Z}}$ is a Hodge class is an* algebraic *subvariety of $B$*

- Sketch of proof:

$$\begin{tikzcd} \widetilde{B} \arrow[r, "\widetilde{\Phi}"] \arrow[d] & D \arrow[d] \\ B \arrow[r, "\Phi"] & \Gamma\backslash D \end{tikzcd}$$

$$\begin{array}{l} \mathrm{Hg}(v) := \{\tilde{b} \in \widetilde{B},\ v \in \mathrm{Hg}^n(\widetilde{\Phi}(\tilde{b}))\} \\ \quad\ \ \downarrow \\ \quad\ \ B \end{array}$$

  - $V_{\mathbb{C}}/F^{n+1} \to \widetilde{B}$ is a holomorphic vector bundle $F^{\#}$;
  - $v \in V_{\mathbb{Z}}$ gives section $\sigma(v) \in H^0(\widetilde{B}, F^{\#})$;

– $\mathrm{Hg}(v) = \{\tilde{b} \in \widetilde{B} : \sigma(v)(\tilde{b}) = 0\}$

$\implies \mathrm{Hg}(v) = \sigma(v)^{-1}$ (zero-section)

$$Q(v, v) = \|v^{n,n}\|_H^2 + \sum_{p \neq 0} (-1)^p \|v^{n+p,n-p}\|_H^2 \quad \text{(topological quantity)}$$

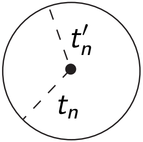
$$\|v\|_H^2 = \|v^{n,n}\|_H^2 + \sum_{p \neq 0} \|v^{p,n-p}\|_H^2 \quad \text{(analytic quantity)}$$

$\implies \|v\|_H^2$ bounded by $|Q(v, v)|$ and $\|v^{\#}\|_H^2$.

Proof: $|Q(v, v)| \leqq C$, $\|v\|_H^2 \leqq C'$, $v \in V_{\mathbb{Z}}$ implies at each point of $\widetilde{B}$ that $v$ is in a finite set; thus the map $\mathcal{H} \to B$ is *proper* and *finite*
($Q(v, v) =$ constant gives covering space of $B$ and $v \in V_{\mathbb{Z}}$, $\|v\|_H < C' \implies v$ in a finite set).

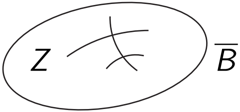
For $B$ complete this gives the result. In general we have to extend the Hodge bundles to $\overline{B}$ and analyze $\| \ \|_H$ along $Z = \overline{B} \backslash B$ (cf. [CK2]).

Example:




$$\Phi(t_n) = \gamma_n \Phi(t'_n) \ \text{ for } \ \gamma_n \in \Gamma \subset G_{\mathbb{Z}}$$
$$\implies \ \gamma_n = \gamma \ \text{ for } \ n \gg 0.$$

- What do the limit theorems tell us about the structure of $\Phi$ at infinity?
  Here $Z = \cup Z_i$ is a normal crossing divisor

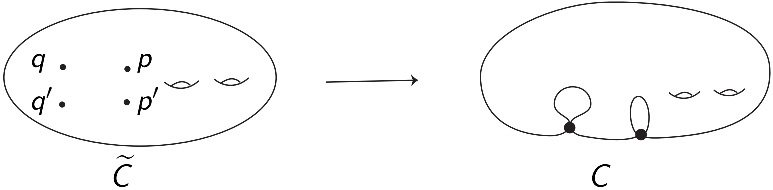



  Locally $Z = (\Delta^*)^k \times \Delta^\ell$ and the limit theorems may be (very non-trivially) extended to this case (cf. [CK1])
- For $b \in Z_I^*$ the map

$$\Phi_0(b) = \{\text{associated graded to } H^n_{\lim}(b)\}$$

  is like the earlier period mapping where $B = Z_I^*$. Hence we are interested in a fibre of $\Phi_0$; i.e., in a family of LMHS's where the associated graded PHS's don't vary.

## Example:




$$(\mathrm{AJ}_{\widetilde{C}}(p-p'), \mathrm{AJ}_{\widetilde{C}}(q-q')) \in J(\widetilde{C}) \oplus J(\widetilde{C})$$

- the level 1 extension data gives a map

$$\begin{tikzcd} Z_I^* \arrow[rr] \arrow[dr] & & \mathrm{Alb}_{Z_I^*} \arrow[dl] \\ & J_{ab} & \end{tikzcd}$$

which extends to $Z_I$

$$\begin{tikzcd} Z_I \arrow[rr] \arrow[dr] & & \mathrm{Alb} Z_I \arrow[dl] \\ & J_{ab} & \end{tikzcd}$$

Interpretation: $\mathrm{Alb} Z_I^*$ is a semi-abelian variety

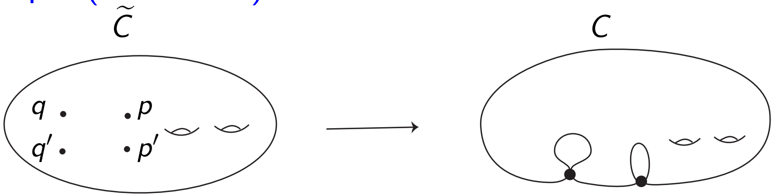
$$\begin{tikzcd} 0 \arrow[r] & T_I \arrow[r] \arrow[d, phantom, "\wr\|"] & S_I \arrow[r] & A_I \arrow[r] \arrow[d, phantom, "\|"] & 0 \\ & (\mathbb{C}^*)^m & & \text{abelian variety} & \end{tikzcd}$$

and any morphism $S_I \to J_{ab}$ factors

$$\begin{tikzcd} S_I \arrow[r] & J_{ab} \\ Z_I^* \arrow[u] \arrow[ur, "\Phi_1"] \arrow[r] & A_I \arrow[u, dashed] \end{tikzcd}$$

- The fibres of $\Phi_0$ and $\Phi_1$ (Gr(LMHS)= constant and level 1 extension data = constant) are $(\mathbb{C}^*)^m$'s (algebraic tori)

## Example (continued):




There exists $f$ with divisor $(f) = p + q - p' - q'$. Using differentials of the third kind there exists a cross-ratio $\{p, p'; q, q'\} \in \mathbb{C}^*$ that gives the level 2 extension data.

Open question: Define the completion $\Phi_2$ as a map of $Z_I$ to a toric-like variety such that we have compatibility on $Z_I \cap Z_J$'s.

- *What about extension data of levels $\geqq 3$?*

Here as a result of the differential constraint imposed by $I \subset TD$

> *the extension data of level $\geqq 3$ is determined up to integration constants by $\Phi_0, \Phi_1, \Phi_2$.*

Example: The period matrices for VHS's of Hodge-Tate type over $\mathbb{P}^1 \backslash \{0, 1, 0\}$ have as entries the higher logarithms $\ell i_k(z)$ (these give the extension data in this example). The above result becomes the classical result that the $\ell i_k(z)$ are iterated integrals of $\log z$.

- Finally we discuss the Chern forms $P_k(\Theta_{F^p})$ of the canonically extended Hodge bundles.

Given the data $(\overline{B}, Z; \Phi)$ where $\overline{B}$ is a smooth projective variety, $Z = \cup Z_i$ is a reduced NCD, $B = \overline{B}/Z$ and

$$\Phi : B \to \Gamma\backslash D$$

is a period mapping there are canonical extensions

$$F_e^p \to \overline{B}$$

of the Hodge vector bundles over $\overline{B}$. Using HRI, HRII over $B$ these bundles have Hermitian metrics with curvature matrices $\Theta_{F^p}$ and Chern forms $c_k(\Theta_{F^p})$ defined by

$$\det(I - \left(\frac{\sqrt{-1}}{2\pi}\right)\Theta_{F^p}) = \sum c_k(\Theta_{F^p}).$$

## Applications:

- If $\dim \Phi(B) = \dim B$, then $(\overline{B}, Z)$ is of log general type;
- if $\Phi_*$ is injective, then $B$ is complete hyperbolic.

A subtle analysis of the singularities of the Hodge metrics and resulting curvatures give that

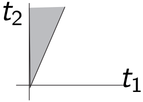
> The $c_k(\Theta_{F^p})$ are the restrictions to $B$ of closed currents whose Lelong numbers are all zero and that represent $c_k(F_e^p) \in H^{2p}(\overline{B})$.

The proof[‡‡] essentially shows that

- the metrics along $Z$ of the canonically extended Hodge bundles have logarithmic singularities (norms of holomorphic sections are $O(-\log|t|)$);
- the Chern forms have at most Poincaré singularities (they are $O\left(\frac{dt\wedge\overline{d}t}{|t|^2(-\log|t|)^2}\right)$.

[‡‡]Cf. [GG].

Especially subtle is that these bounds are not uniform in sectors



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where the "$O$" becomes infinite but at a slower rate than the width of the sectors; the Chern forms will be in $L^1$.
Using this further arguments due to Vieweg and completed by Kollár gives the

- *Iitaka conjecture:* If $f : \overline{\mathcal{X}} \to \overline{B}$ is a family where a general $X_b$ is of general type ($\kappa(X_b) = \dim X_b$), and where $\mathrm{Var}(f) = \dim B$, then $\kappa(\overline{\mathcal{X}}) \geqq \kappa(X_b) + \kappa(\overline{B})$).

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