Extended Period Mappings*

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Abstract

This lecture will discuss the global structure of period mappings (variation of Hodge structure) defined over complete, 2-dimensional algebraic varieties. Some applications to moduli of general type algebraic surfaces will also be presented.

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I. Introduction
II. Construction and properties of extended period mappings
III. Geometry of extension data
IV. Basic formula
V. Applications to moduli of general type algebraic surfaces[†]
   A. Infinite monodromy
   B. Finite monodromy

[†] This section is based in part on joint work with Radu Laza and on the work of and discussion with Marco Franciosi, Rita Pardini and Sönke Rollenske.

I. Introduction

- Given $(\overline{B}, Z; \Phi)$ where
  - — $\overline{B}$ is a smooth projective variety, $Z = \cup Z_i$ is a normal crossing divisor and $B = \overline{B} \backslash Z$;
  - — $\Phi : B \to \Gamma \backslash D$ is a *period mapping* where $D = G_{\mathbb{R}}/H$, $\rho : \pi_1(B) \to \Gamma \subset G_{\mathbb{Z}}$ is monodromy.

  In the extensive literature there are
  - — global results on B (theorem of the fixed part, image $P \subset \Gamma \backslash D$ is an algebraic variety over which the Hodge line bundle $\overset{p}{\otimes} \det F^p := L \to P$ is ample, algebraicity of Hodge loci)
  - — local results on neighborhoods $\Delta^{*k} \times \Delta^{\ell}$ in B of points in Z (nilpotent and sl_2-orbit theorems, existence and properties of several variable limiting mixed Hodge structures, Chern forms of the extended Hodge bundles).

  This talk will be concerned with global results on $\overline{B}$

— extensions of Φ

$$\begin{array}{ccc} & \overset{\Phi_T}{\dashrightarrow} & \overline{P}_T \\ \overline{B} & & \downarrow f \\ & \underset{\Phi_S}{\searrow} & \overline{P}_S \end{array}$$

— properties of $\overline{P}_T, \overline{P}_S$ (e.g., ample line bundles)
— geometry of the fibres of f is of particular interest (variational properties of extension data)
— mostly restrict to the case $\dim B = 2$ and will then assume $\dim \Phi(B) = 2$.[‡]

Will also discuss some applications to moduli of general type algebraic surfaces, emphasizing one particular surface. Main emphasis will be on extending Φ across subvarieties in Z with infinite monodromy; will also briefly discuss extensions across subvarieties in B in the finite monodromy case.

[‡]A fundamental invariant of any VHS is monodromy that lives on a general 2-dimensional section of the parameter space.

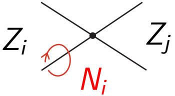
II. Construction and properties of extended period mappings[§]

- Given V, Q
  - — polarized Hodge structure (PHS) is $(V, F), F = \{F^p\}$
  - — mixed Hodge structure (MHS) is $(V, W, F), W = \{W_k\}$
  - — limiting mixed Hodge structure (LMHS) is $(V, W(N), F)$ where $N \in \mathrm{End}_Q(V)$ is a nilpotent operator with $N : F^p \to F^{p-1}$

$$\begin{cases} N : W_k(N) \to W_{k-2}(N) \\ N^k : W_{n+k}(N) \xrightarrow{\sim} W_{n-k}(N). \end{cases}$$

[§] A general reference for Hodge theory is [CM-SP]. For limits of Hodge structures see [CK] and the references cited therein.

The Q will be understood for MHS's and LMHS's. Will also have $(V, W(\sigma), F)$ where $\sigma = \mathrm{span}_{\mathbb{Q}^+}\{N_1, \ldots, N_k\}$ is a monodromy cone. When $\dim B = 2$ we have



$$\sigma = \begin{cases} \sigma_i \\ \sigma_{ij} \end{cases}$$

— equivalence class $[V, W(\sigma), F] := \mathcal{L}$ where

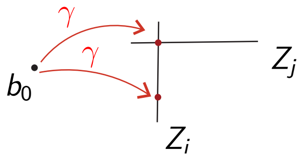
$$F \sim \exp(\lambda N)F, \quad \lambda \in \mathbb{C} \text{ and } N \in \sigma$$

— assuming T_i unipotent[¶] there are canonical extensions $F_e^p \to \overline{B}$.

¶This assumption is not essential.

6 / 33

Definition: $\overline{P}_T$ = quotient by Γ of $\{(\gamma, [V, W(\sigma), F_b)]\}$
$= \{\gamma, \mathcal{L}_b\}$ where $\gamma = \overline{b_o b}$



- Given a MHS (V, W, F) the associated graded is a direct sum of PHS's. For $\mathcal{L} = [V, W(\sigma), F]$, $\mathrm{Gr}(\mathcal{L})$ is well-defined.

Definition: $\overline{P}_S$ = quotient by Γ of $\{\gamma, \mathrm{Gr}(\mathcal{L}_b)\}$.

In the following we assume $\dim B = 2 = \dim \Phi(B) = 2$.

Theorem([GGLR]): (i) *$\overline{P}_S$ is a compact analytic surface.*
(ii) *The Hodge line bundle descends to an ample line bundle on $\overline{P}_S$.*

- Regarding (i) essential case is $Z = \cup Z_i$ where $L \cdot Z_i = 0$. Then

$$\dim B = 2 \implies \|Z_i \cdot Z_j\| \leqq 0 \qquad \text{(Hodge index theorem)}$$
$$\implies Z \text{ contracts to a normal singular point (Grauert).}$$

- Regarding (ii), even if we know that $L\big|_Z \cong \mathcal{O}_Z$ there are generally non-trivial obstructions to trivialize L in a neighborhood of Z. Proof involves
  - new ingredient in Hodge theory (semi-global representations of Φ by period matrices)
  - observation that $L =$ pullback of $\mathcal{O}(1)$ under the Plücker embedding

$$D \subset \prod^{p} \mathbb{P}(\wedge^{h_p} F^p), \quad h_p = \text{rank } F_p,$$

    this applied to the maps $\mathrm{Gr}(\mathcal{L}) \to \{\text{Mumford-Tate domain}\}$.

Theorem: (i) $\overline{P}_T$ *is a compact analytic variety.* (ii) *Assuming* $\Phi : B \to P$ *does not contract any curve,*[‖] *there exists* m_0 *and* $a_i > 0$ *such that*

$$L_m := mL - \sum_i a_i Z_i$$

is ample for $m > m_0$*.*

Regarding the proof of (ii), the a_i are chosen so that for each j

$$Z_j \cdot \sum_i a_i Z_i > 0.$$

That this is possible is a property of negative definite symmetric matrices. The a_i reflect the nature of the singularity to which Z contracts.

In summary

- $\overline{P}_S = \text{Proj}\,(L)$
- $\overline{P}_T = \text{Proj}\,(L_m)$

[‖] This assumption can be removed with a slightly more elaborate statement of the result.

III. Geometry of extension data

Still assuming that $\dim B = \dim \Phi(B) = 2$, in the diagram

$$\begin{array}{ccc} & & \overline{P}_T \\ & \overset{\Phi_T}{\nearrow} & \\ \overline{B} & & \downarrow f \\ & \underset{\Phi_S}{\searrow} & \\ & & \overline{P}_S \end{array}$$

Z_i not a fibre of $\Phi_S \implies \Phi_T\big|_{Z_i^*}$ is like a usual period mapping

Z_i is a fibre of $\Phi_S \iff \mathcal{L}\big|_{Z_i}$ has locally constant $\mathrm{Gr}(\mathcal{L})$.

Assume along Z_i^* have VLMHS $\mathcal{L}$ where $\mathrm{Gr}(\mathcal{L}) = \{H^0, \ldots, H^m\}$ is constant.

- $(V, F), (V', F)$ Hodge structures of weights $k > k'$

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathrm{MHS}}(V, V') & = & \frac{\mathrm{Hom}_{\mathbb{C}}(V,V')}{F^0 \mathrm{Hom}_{\mathbb{C}}(V,V') + \mathrm{Hom}_{\mathbb{Z}}(V,V')} \\ \| & & \\ E & \cong & \mathbb{C}^m/\Lambda, \quad \Lambda \text{ discrete} \end{array}$$

10 / 33

- $k' = k - 1$ gives

$$\underbrace{(k-1,-k) \oplus \cdots \oplus (0,-1)}_{F^0} \oplus \underbrace{\underbrace{(-1,0)} \oplus \cdots \oplus (-k,k-1)}_{T_eE}$$

E = compact complex torus with $E \supset E_{ab}$ where $T_eE_{ab} \subset \underbrace{(-1,0)}$

- $k' = k - 2$ gives

$$\underbrace{(k-2,-k) \oplus \cdots \oplus (0,-2)}_{F^0}$$

$$\oplus \underbrace{\underbrace{(-1,-1)} \oplus (-2,0) \oplus \cdots \oplus (-k,k-2)}_{T_eE}$$

connected analytic subgroup S where T_eS over $\underbrace{\quad}$ is a $\mathbb{C}^{*k}$.

- $k' = k - 3$ gives

$$\underbrace{(k-3,-k)\oplus\cdots\oplus}_{F^0}\underbrace{\underbrace{(-1,-2)\otimes(-2,-1)}\otimes\cdots\otimes(-k,k-3)}_{T_eE}$$

no non-trivial connected complex analytic subgroup with tangent space over $\underbrace{}$.
Need only consider $\mathrm{Ext}^1_{\mathrm{MHS}}$'s as the higher $\mathrm{Ext}^q_{\mathrm{MHS}}$'s $= 0$ for $q \geqq 2$.

Remark: For a VMHS of Hodge-Tate type (the $H^{2p} = \oplus\mathbb{Q}(-p)$'s) the

— level 1 extension data is trivial
— level 2 extension data given by $\log t_i$'s
— level 3 extension data given by $li_2 t_\alpha$'s
— d (level 3) $\in$ level 2 $\implies$ ODE expressing $li_2 t_\alpha$ in terms of $\log t_i$'s, etc.

- Along Z_i^* level 1 extension data gives

$$\Phi_1 : Z_i^* \to E_{ab} \subset E \qquad \text{(actually } E_{ab} + c\text{)}$$

$$\rightsquigarrow \begin{array}{ccc} Z_i & \longrightarrow & E_{ab} \\ & \searrow \quad \nearrow & \\ & \mathrm{Alb}\, Z_i & \end{array}$$

- Φ_1 (locally) constant $\rightsquigarrow \Phi_2 : Z_i^* \to \mathbb{C}^{*k}$.
- Then we have (up to a translation) the level 2 extension data mapping

$$\Phi_2 : Z_i^* \to \mathbb{C}^{*m_i}.$$

- If Φ_1, Φ_2 are both constant along Z_i^*, then $\Phi_3 = \Phi_4 = \cdots =$ constant along Z_i.
- At a point of $Z_i \cap Z_j$ if N_i, N_j are linearly independent, then Φ_2 extends by filling in the origin to some of the $\mathbb{C}^*$'s; essentially $\Delta^* \times \Delta^*$ completes to $\Delta \times \Delta$.

- If N_i, N_j are linearly dependent, then $\Delta^* \times \Delta^*$ fills in to $\Delta \times \Delta$ with the axes contracted to points.**
- The Albanese $\mathrm{Alb}(Z_i^*)$ is a semi-abelian variety S_i with

$$0 \to \mathbb{C}^{*m_i} \to S_i \to A_i \to 0$$

and Φ_1, Φ_2 combined give

$$\begin{array}{ccc} Z^{*_i} & \longrightarrow & S_i \\ & \searrow \quad \swarrow & \\ & \left\{ \begin{array}{c} \text{extension data} \\ \text{of levels} \leqq 2 \end{array} \right\} & \end{array}$$

**The general version of this case involves a somewhat subtle analysis of the relations among the N_i in a nilpotent orbit

$$\exp\left(\sum_i \left(\frac{\log t_i}{2\pi\sqrt{-1}}\right)\right) \cdot F.$$

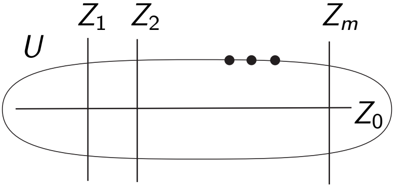
14 / 33

Theorem: $\dim \Phi(B) = 2 \implies$ *the map to extension data is non-constant.*

Corollary: *In general,* $\dim \Phi(B) = 2 \implies \Phi_T$ *contracts no curves in* Z.

IV. Basic formula

- Relates the geometry *along* Z_i to geometry *normal* to it



$U^* = U \backslash (Z \cap U)$

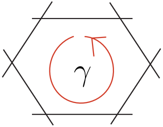
- Assume $\Phi_S(Z_0^*) =$ point, thus $\mathcal{L}$ locally constant along $Z_0^* \implies \pi_1(U^*)$ acts as finite group on $\mathrm{Gr}(\mathcal{L})$; assume this group is trivial; then
  - $W(N_0) = W(N_i)$, $\mathrm{Gr}^W(V)$ is a fixed vector space;
  - $N_0, N_i \in \mathrm{Gr}^W_{-2}\,\mathrm{End}(V)$, gives a cone $\sigma \subset \mathrm{Gr}^W_{-2}\,\mathrm{End}(V)$;
  - $\mathrm{Gr}^W_{+2}\,\mathrm{End}(V) \cong \mathrm{Gr}^W_{-2}\,\mathrm{End}(V)^*$ (uses Q);
  - $M \in \mathrm{Gr}^W_{+2}\,\mathrm{End}(V)$ gives $L_M \to E$ and $M \in \check{\sigma} \implies L_m \to E_{\mathrm{ab}}$ ample.

Theorem (basic formula): *$\Phi_1 : Z_i \to E_{ab}$ and in $\mathrm{Pic}(Z_0)$ we have*

$$(*) \qquad -\Phi_1^*(L_M) = \left\{ \sum_{i=0}^{m} \langle M, N_0 \rangle \, [Z_i] \right\} \Bigg|_{Z_0} .$$

Corollary: $-\deg \Phi_1^*(L_M) = \langle M, N_0 \rangle \, Z_0^2 + \sum_{i=1}^{m} \langle M, N_i \rangle$.

- RHS is $\left\langle M, \substack{\text{row corresponding to } Z_0 \\ \text{in the intersection matrix}} \right\rangle$. $(*)$ tells us how negative that row is in terms of the variation of the level 1 extension data.
- Special case: Z is a *cycle*

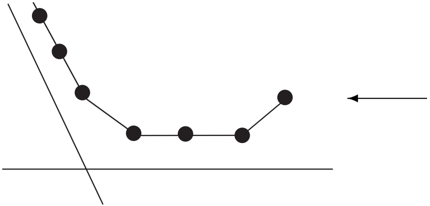


$Z_i^* = \mathbb{C}^*$

— $\langle M, Z_i \rangle \, Z_i^2 = \langle M, Z_{i-1} \rangle + \langle M, Z_{i+1} \rangle$, plus terms from going around the cycle.

- monodromy 

gives a *circuit* — then
  - γ acting on $N_1, \ldots, N_m$ spans a 2-plane in $\mathrm{Gr}^W_{-2}\,\mathrm{End}(V)$, and in this plane there is a sector such that the $\gamma^k N_i$ give in the sector a convex figure where $\gamma =$ translation by m



$\left\{ \begin{array}{c} \text{dual graph of} \\ \text{the universal} \\ \text{covering of } Z \end{array} \right\}$

  - from the basic formula $(*)$ we infer that

$$\begin{cases} \text{straight line at } Z_i \leftrightarrow Z_i^2 = -2 \\ \text{bend at } Z_i \leftrightarrow Z_i^2 \leqq -3. \end{cases}$$

Hilbert modular surface picture is general.

18 / 33

V. Application to moduli of general type surfaces

A. Infinite monodromy

We begin with the question

- *What are the singularities of $\overline{P}_T$ and $\overline{P}_S$?*
  - — the singularities of $P = \Phi(B)$ arbitrary
  - — with our non-degeneracy assumption $\dim \Phi(B) = 2$ along a Z_i we cannot have $\Phi_1 =$ constant and $\Phi_2 =$ constant so that Φ_1 is finite-to-one; we will illustrate the general principle that the LMHS along Z helps determine the singularity type.

Example 1: Weight $n = 2m$

— $Z =$ smooth curve and $\Phi_S(Z) = p \in \overline{P}_S$;

— $N^2 = 0$, rank $N = 2$;

— $\mathrm{Gr}\,\mathcal{L} = \{H^{2m-1}, H^{2m}, H^{2m-1}(-1)\}$, with $N : H^{2m-1}(-1) \xrightarrow{\sim} H^{2m-1}$, $H^{2m-1} = H^1(C)(-(m-1))$ for an elliptic curve C;

— for simplicity assume rank $\mathrm{Hg}^m = 1$ and C is general;

$\implies E_{ab} = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathrm{Hg}^m, H^{2m-1}) \cong H^1(C)$;

— $\Phi_1 : Z \to C$ is a finite morphism;

— if Φ_1 is non-constant, then for U = neighborhood of Z in $\overline{B}$

$$\begin{array}{ccc} U & \xrightarrow{\Phi_S} & \overline{P} \\ \cup & & \Cup \\ Z & \xrightarrow{\Phi_1} & \{p\} \end{array}$$

gives a resolution of an elliptic singularity.[††]

Example 2: $n = 2m$

— Z = cycle;

— $N^2 \neq 0$, $N^3 = 0$ and rank $N = 1$.

Then by a similar analysis to the elliptic singularity case we find that $\Phi_S(Z)$ = cusp singularity.

———<∘>———

[††] In general $\Phi_1 : Z_a \to E_{ab}$ and the associated Gauss mapping enters into the geometry of the extension data.

- $\mathcal{M}$ = KSBA moduli space whose general point corresponds to a smooth general type surface.[‡‡]
- $\overline{\mathcal{M}}$ = canonical completion whose boundary points correspond to surfaces X_0 having slc-singularities.

Even if $\mathcal{M}$ is almost smooth,[†] in contrast to $\overline{\mathcal{M}}_g$ the boundary may be quite singular. There are geometric and Hodge theoretic reasons why this should be so.

Question: *How can Hodge theory help understand the geometry of $\mathcal{M}$ near $\partial\mathcal{M}$?*

‡‡[K] is a general reference for moduli.

†This means that locally $\mathcal{M}$ looks like the parameter space of a general smoothing of an ADE singularity.

- if a point of $\partial\mathcal{M}$ corresponds to a normal surface X_0 having a singular point p and where $N \neq 0$ for a general smoothing of X_0, then from the list in [K]
  p is either a simple elliptic singularity or a cusp.[‡]
- a general result, here stated informally, is that for a singular surface X_0 corresponding to a point x_0 of $\partial\mathcal{M}$, the associated graded to the LMHS$= \mathcal{L}$ for *any* smoothing X_t of X_0 the $\mathrm{Gr}(\mathcal{L})$ is the same.[§]
- above examples suggest that using the map

$$\overline{\mathcal{M}} \dashrightarrow \overline{P}_T$$

  may help resolve the singularities of $\overline{\mathcal{M}}$.

[‡]Interestingly if p is non-Gorenstein, then it is a rational singularity and consequently $N = 0$.

[§]More precisely the smoothings of X_0 may have several components and the $\mathrm{Gr}(\mathcal{L})$ depends only on the particular component. This result suggests why $\partial\mathcal{M}$ should be singular along components where $N \neq 0$. We will see below that we can obtain divisors in $\partial\mathcal{M} \subset \overline{\mathcal{M}}$ along certain components where $N = 0$.

Example ([FPR]): The "first" non-classical general type surface with $p_g \neq 0$ is an I-surface X

$$p_g(X) = 2,\ q(X) = 0,\ K_X^2 = 1;$$

- well known classically, on the Noether line $p_g = [K_X^2/2 + 2]$;
- $\mathcal{M}_I$ is almost smooth, $\dim \mathcal{M}_I = 28$;
- $D = \mathrm{SO}(4,28)/U(2) \times SO(28)$, $\dim D = 57$;
- IPR is a contact system and $\Phi(\mathcal{M}_I)$ is a contact subvariety;

- FPR have determined the stratification of $\overline{\mathcal{M}}_I^{\mathrm{Gor}}$, and have almost determined that of $\overline{\mathcal{M}}_I$ (much more difficult because have to bound the index in the non-Gorenstein case);
- part of their table is¶

¶In general for a smoothable surface X_0 that is irreducible, regular and normal with k elliptic singularities $\implies k \leqq p_g + 1$.

stratum	dimension	minimal[‖] resolution $\widetilde{X}$	$\sum_{i=1}^{k}(9-d_i)$	k	codim in $\overline{\mathcal{M}}_I$
I_0	28	canonical singularities	0	0	0
I_2	20	blow up of a K3-surface	7	1	8
I_1	19	minimal elliptic surface with $\chi(\widetilde{X})=2$	8	1	9
$\mathrm{III}_{2,2}$	12	rational surface	14	2	16
$\mathrm{III}_{1,2}$	11	rational surface	15	2	17
$\mathrm{III}_{1,1,R}$	10	rational surface	16	2	18
$\mathrm{III}_{1,1,E}$	10	blow up of an Enriques surface	16	2	18
$\mathrm{III}_{1,1,2}$	2	ruled surface with $\chi(\widetilde{X})=0$	23	3	26
$\mathrm{III}_{1,1,1}$	1	ruled surface with $\chi(\widetilde{X})=0$	24	3	27

[‖] $\widetilde{X}\to X$ contracts k elliptic curves $\widetilde{C}_i$ with $\widetilde{C}_i^2=-d_i$.

- How can Hodge help understand the desingularization of $\overline{\mathcal{M}}_I$ along these components?

$$\begin{array}{ccc} & (\widetilde{X}, \widetilde{C}) & \\ \swarrow & & \searrow \\ (X_{\text{min}}, C) & & (X, p) \end{array}$$

Example: For I_2 the picture is

$$\begin{array}{ccc} & (\widetilde{X}, \widetilde{C}) & \\ \swarrow & & \searrow \\ (X_{\text{min}}, C) & & (X, p) \end{array}$$

26 / 33

Here, $p =$ isolated normal singular point on X, $\widetilde{C} =$ curve on $\widetilde{X}$ that contracts to p — the LMHS

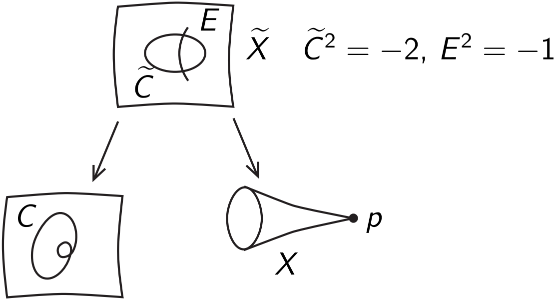
$$2 = p_g(\widetilde{X}) + g(\widetilde{C}) \text{ and } p_g(\widetilde{X}) = 1$$

gives $g(\widetilde{C}) = 1$ (simple elliptic singularity).**

- $\mathrm{Gr}(\text{LMHS})/\mathbb{Z}$ suggests that $\mathrm{Hg}^1(\widetilde{X})$ has a $\mathbb{Z}^2$ with intersection form
$$\begin{pmatrix} -2 & 2 \\ 2 & -1 \end{pmatrix}$$
for heuristic reasoning assume basis classes are effective.

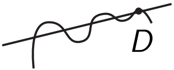
** LMHS has $\mathrm{Gr}_2 \cong H^2(X_{\mathrm{min}})_{\mathrm{prim}}$ and $\mathrm{Gr}_3 \cong H^1(\widetilde{C})(-1)$

- Hodge theory now suggests the picture



$$\begin{cases} X_{\min} = K3 \\ C^2 = 2 \end{cases}$$

$\Longrightarrow X_{\min} \xrightarrow{2:1} \mathbb{P}^2$ branched over D



- # of PHS's of type $\mathrm{Gr}_3 \oplus \mathrm{Gr}_2 = 19 + 1 = 20$ which suggests
  — codim $= 8$
- How to get this number? The fibre over origin in a SSR is blowing up p in $\mathcal{X}$ to have

$$\widetilde{X} \cup_{\widetilde{C}} \mathbb{P}^2$$

  where $\widetilde{C} \in |\mathcal{O}_{\mathbb{P}^2}(3)|$

- Now have to blow up $9 - (-\widetilde{C}^2) = 7$ points on $\widetilde{C}$ to obtain triviality of the infinitesimal normal bundle as a necessary condition for smoothability. Thus

Fibre over origin in Δ is given by blowing up seven points on $\widetilde{C}$, is a del Pezzo.

- $$(\#\widetilde{C}) + \dim(\textit{level } 1 \textit{ extension data}) = 8.$$

$$\underset{1}{\|} \qquad\qquad\qquad \underset{7}{\|}$$

(i.e. $\#\widetilde{C} = 1$, $\dim(\textit{level } 1 \textit{ extension data}) = 7$)

B. Finite monodromy

- for $\Phi : \Delta^* \to \{T^k\}\backslash D$ another classical type of extension is when $T = T_s$ is of finite order

$$\begin{array}{ccc} \widetilde{\Delta}^* & \xrightarrow{\widetilde{\Phi}} & D \\ \downarrow & & \downarrow \\ \Delta^* & \xrightarrow{\Phi} & \{T^k\}\backslash\Delta \end{array} \quad \rightsquigarrow \widetilde{\Phi} : \Delta \to D \text{ extends.}$$

- In geometric case X_0 will be singular and LMHS=PHS (but $\neq H^n(\widetilde{X}_0)$).
- Generally $\widetilde{\Phi}_* : T_{\{0\}}\widetilde{\Delta} \to TD$ is zero but can define $\delta\Phi$ that has geometric information.
- For KSBA moduli of surfaces on $\partial\mathcal{M}$
  - X_0 is non-Gorenstein
  - singularity is $\frac{1}{dn^2}(1, dn^2 - 1)$ quotient singularity
  - rational $\implies N = 0$ (resolution is a tree of $\mathbb{P}^1$'s).

30 / 33

- Extension of Φ from $\mathcal{M}$ to $\mathcal{M}_f$ gives

$$\Phi : \mathcal{M}_f \to \Gamma\backslash D.$$

- In contrast to the $N \neq 0$ singularity the presence of an $N = 0$ singularity may define a divisor in $\overline{\mathcal{M}}$. This happens in particular for the Wahl singularity $\frac{1}{4}(1,1)$, the quotient of $\mathbb{C}^2$ by $(u,v) \to (\zeta u, \zeta v)$ where $\zeta = e^{2\pi i/4}$. This singularity is of particular interest as the monodromy $T = \mathrm{Id}$.

Example ([FPR]): For $\overline{\mathcal{M}}_I$ there are two divisors in $\partial\mathcal{M}_I$: I-surfaces (X_0, p) with $\frac{1}{4}(1,1)$ or $\frac{1}{18}(1,5)$ singularity; denote first by $\mathcal{M}_{I,W}$.

- resolution of Wahl singularity is $(\widetilde{X}, E) \to (X, p)$ where X = elliptic surface with a bisection E, $E^2 = -4$;

- semi-stable-reduction has $\widetilde{X} \cup_E S$ where $S =$ Veronese surface ((X, p) looks locally like a plane section through the vertex of a cone over S);
- $\widetilde{\Phi}(0) =$ HS computed from $\widetilde{X} \cup_E S$.

Theorem: *$\mathcal{M}_{I,W} =$ component of $\Phi^{-1}(\Gamma'\backslash D')$ where $D' \subset D$ is a Mumford-Tate domain.*

- Proof uses computation of $\delta\Phi$ in $T\,\mathrm{Def}(X)$.

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