

# DIFFERENTIAL OF A PERIOD MAPPING AT A SINGULARITY

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*To Herb Clemens. Lefschetz wrote that he put the harpoon of topology into the whale of algebraic geometry. Herb put the harpoon of topology into the whale of Hodge theory.*

## ABSTRACT

Period mappings, or equivalently variations of Hodge structure, have been used both to study families of algebraic varieties and as a subject in its own right. Among the aspects of a period mapping  $\Phi$  that have received special attention are

- (i) singularities: the analysis of the behavior of  $\Phi$  along a lower dimensional subvariety outside of which  $\Phi$  is defined, and
- (ii) infinitesimal properties: the study of the geometry reflected in the differential of  $\Phi$ .

In this paper we will seek to combine these two.

The local 1-parameter situation is given by a period mapping

$$\Phi : \Delta^* \rightarrow \Gamma \backslash \mathcal{D}$$

where  $\Gamma$  is generated by the monodromy operator  $T$ . If  $T = T_s T_u$  is the Jordan decomposition into its semi-simple and unipotent factors where  $N$  is the nilpotent logarithm of  $T$ , our study will divide into two general areas:

- (1)  $N = 0$ , then  $\Phi$  extends to  $\Phi: \Delta \rightarrow \Gamma \backslash \mathcal{D}$  where the image  $\Phi(0)$  gives a polarized Hodge structure having an action of  $T_s$ .

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Our general references for the topics discussed in this paper are [CM-SP], [G1], and [CKS]. The first one treats period mappings and variations of Hodge structure, the second discusses infinitesimal methods in Hodge theory, and the third is a basic reference for singularities of period mappings. We shall generally use the notations in [GGR].

- (2)  $N \neq 0$ , in which case we will assume that  $T_s = I$ . It is conjectured [GGR] that  $\Phi$  extends to a mapping to a torodial like completion where  $\Phi(0)$  is a limiting mixed Hodge structure.

In case (1) the objective is define a generalized differential  $\delta\Phi$  that detects the first non-trivial part of the variation of  $\Phi$  around  $\Phi(0)$ . Our results will give a geometric interpretation of this in several cases. A guiding question (informally stated) is:

*Let  $\Phi: \mathcal{M} \rightarrow \Gamma \backslash D$  be the period mapping for a KSBA moduli space of general type varieties. Let  $\mathcal{M}_f \subset \overline{\mathcal{M}}$  be the subvariety of the canonical completion  $\overline{\mathcal{M}}$  corresponding to singular varieties around which the local monodromy of a smoothing is finite. Then there is an extension*

$$\Phi: \mathcal{M}_f \rightarrow \Gamma \backslash D$$

*and one would like to define subvarieties of  $\Gamma \backslash D$  whose intersections with  $\Phi(\mathcal{M}_f)$  define “special” subvarieties of  $\mathcal{M}_f$ .<sup>1</sup> For such subvarieties one could like to be able to say something about their structure, e.g., their expected codimension and tangent space.*

In case (2) the question is somewhat different. Here the leading part of the differential is  $N$ , which gives rise to the nilpotent orbit that approximates  $\Phi$ . The weight filtration for the limiting mixed Hodge structure defines a filtration on the differential and the successive quotients encode geometric information. One application is using versions of Torelli at the boundary to try to infer versions of Torelli in the interior.<sup>2</sup>

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<sup>1</sup>We will not formally define “special.” It could mean, e.g., points of  $\mathcal{M}_f$  corresponding to varieties that have additional algebraic cycles or have a particular singularity type.

<sup>2</sup>Related to (2) is the computation and examples of  $\delta\Phi$  for a family of mixed Hodge structures. The theory for these is essentially the same as for the  $N = 0$  part of a limiting mixed Hodge structure and that will be their main role in this paper.

## OUTLINE

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## I. INTRODUCTION

**I.A. General introduction.** The uses of Hodge theory in the study of the geometry of algebraic varieties include

- the topology of algebraic varieties. This is where Hodge theory began. It includes the topological properties of individual varieties, including singular and non-compact ones, and of families of varieties;
- geometric constructions that arise from the Hodge structure or mixed Hodge structure on the cohomology (a transcendental invariant), or from the first or higher order variations of the Hodge structure or mixed Hodge structure (an algebraic invariant).

In the second case the study has been largely confined to the first variation of the Hodge structure on the cohomology of a smooth variety.<sup>3</sup> In this paper we will seek to extend this to the situation where the

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<sup>3</sup>See the chapter on IVHS in [CM-SP].

variety may be singular or where the first variation may vanish and one has to go to higher order to extract geometric information.

We shall usually restrict to the 1-parameter case of a period mapping

$$(I.A.1) \quad \Phi : \Delta^* \rightarrow \Gamma \backslash D$$

where  $\Gamma = \{T^k : k \in \mathbb{Z}\}$  is generated by the monodromy operator. In the Jordan decomposition  $T = T_s T_u$  the semi-simple factor  $T_s$  is of finite order (its characteristic polynomial is a product of cyclotomic polynomials), and the unipotent factor  $T_u = e^N$ .<sup>4</sup> This work will separate into two parts:

- (a)  $N = 0$ ,
- (b)  $N \neq 0$ .

In the first case,  $\Gamma \backslash D$  is an analytic variety with quotient singularities by the action of a finite group and (I.A.1) extends to a mapping of analytic varieties

$$\Phi : \Delta \rightarrow \Gamma \backslash D$$

where  $\Phi(0) \in D$  is a polarized Hodge structure having an action of  $\Gamma$ .<sup>5</sup>

In the second case we shall assume that  $T_s = \text{Id}$ . Then  $\Phi(0) \in \exp(\mathbb{C}N) \backslash \check{D}$  where  $\check{D}$  is the compact dual of  $D$ ; it is given by an equivalence class of limiting mixed Hodge structures.

In both cases we shall define a differential  $\delta\Phi$  that is an invariant of the suitably interpreted first non-zero term in the expansion of  $\Phi(t)$  about  $t = 0$ . Some general properties of  $\delta\Phi$  will be given; however, the main part of this paper is in the examples.

One general guiding geometric question that motivated much of this study and that in special cases will be illustrated below is the following:

*Let  $\overline{\mathcal{M}}$  be the canonical completion of the KSBA moduli space  $\mathcal{M}$  of a class of varieties of general type (cf. [K]).*

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<sup>4</sup>As will be explained below, the case when  $\Phi$  depends on several parameters may be done by suitably restricting to 1-parameter sub-families.

<sup>5</sup>Here, with slight abuse of notation  $\Phi(0)$  is the lift to  $D$  of its image in  $\Gamma \backslash D$ .

*How much of the stratification of  $\overline{\mathcal{M}}$  by the singularity type of the corresponding varieties or by the presence of “additional” algebraic subvarieties is reflected in the stratification of the image of  $\overline{\mathcal{M}}$  by a canonically extended period mapping?<sup>6</sup> How can Hodge theory be used to help understand the geometry of the stratification of  $\overline{\mathcal{M}}$ ?*

For the case of varieties whose period domain is Hermitian symmetric (algebraic curves, abelian varieties, K3’s, hyperKähler varieties, cubic threefolds and fourfolds, . . . ) this question is classical and is the subject of ongoing work. We are particularly interested in the non-classical case when  $D$  is not Hermitian symmetric. Here there is evidence that at least in particular cases this general question may have an interesting and occasionally surprising answer, e.g. as provided by the example of  $I$ -surfaces [FPR], [G1], [G2].

Another type of motivating question is to establish Torelli results on boundary components of a moduli space, and from these deduce Torelli results in the interior. The point here is that although they are singular the varieties that correspond to boundary points may be easier to prove Torelli-type results about, and then these can be used to infer similar results in the interior. A prototype here is [F2] and recent related work is in [PZ].

We conclude this part of the introduction with some notations and terminology. The *geometric case* is when we start with a family  $\mathcal{X}'^* \rightarrow \Delta'^*$  of smooth varieties whose associated period mapping  $\Delta'^* \rightarrow \Gamma \backslash D$  has monodromy  $T = T_s T_u$ . By semi-stable reduction, or by some other construction (see Example II.B.4 below), we can obtain a family  $\mathcal{X} \xrightarrow{\pi} \Delta$  where the  $X_t = \pi^{-1}(t)$  are smooth for  $t \neq 0$ ,  $X_0$  is a reduced normal crossing divisor and the monodromy is unipotent. We can and will assume that the finite group  $\{T_s^k : k \in \mathbb{Z}\}$  acts equivariantly on  $\mathcal{X} \rightarrow \Delta$

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<sup>6</sup>Informally stated, “how much of the geometry of  $\overline{\mathcal{M}}$  can be described Hodge theoretically?”

and that the induced action on  $\Delta$  has the origin as an isolated fixed point.

We set  $X = X_0$  and shall use the standard identification

$$T \operatorname{Def}(X) = \mathbb{E}xt_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X).$$

The local to global spectral sequence of  $\mathbb{E}xt$ 's gives

(I.A.2)

$$\begin{array}{ccccc} H^1(\mathcal{E}xt_{\mathcal{O}_X}^0(\Omega_X^1, \mathcal{O}_X)) & \longrightarrow & \mathbb{E}xt_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) & \longrightarrow & H^0(\mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X)) \\ \parallel & & \parallel & & \parallel \\ T \operatorname{Def}^{es}(X) & \longrightarrow & T \operatorname{Def}(X) & \longrightarrow & T \operatorname{Def}^{sm}(X) \end{array}$$

where the left-hand term is the first order *equisingular deformations* of  $X$  and where the right-hand term (a quotient space) are the equivalence classes of *partially smoothing deformations*. In this paper we will be interested in both equisingular families and in families whose tangent in  $T \operatorname{Def}^{sm}(X)$  is completely smoothing (cf. the appendix for further explanation).

The analysis of  $\delta\Phi$  can roughly be organized into the following cases:

$N = 0$ :

- (i)  $X_0 = X$  is smooth,  $\Phi(0) = H^n(X)$  lies in a Mumford-Tate subdomain  $D' \subset D$ , and the condition  $\delta\Phi \neq 0$  means that  $\Phi(X_t)$  leaves  $D'$ ;
- (ii)  $X_0$  is singular and  $\delta\Phi$  measures either the variation of the Hodge structure in the equisingular directions or the Hodge theoretic properties of the smoothing of  $X_0$ .

$N \neq 0$ : Then the family will either be equisingular or smoothing and the successive terms in the associated graded to the weight filtration of  $\delta\Phi$  describe Hodge theoretic properties of either the variation of the mixed Hodge structure in the  $\ker N$  part of the limiting mixed Hodge structure, or of the smoothing itself.

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and [G2]). We wish to express our appreciation for the many discussions we have had with them on the general topic of Hodge theory and moduli.

**I.B. Introduction to Section II.** In the first part of Section II.A we give the definition of the differential  $\delta\Phi$  of the extension to

$$\Phi : \Delta \rightarrow \Gamma \backslash D$$

of a 1-parameter period mapping  $\Phi : \Delta^* \rightarrow \Gamma \backslash D$  where  $\Gamma = \{T^k : k \in \mathbb{Z}\}$  is a finite group. The definition uses a base change  $t = \tilde{t}^m$  to have a lifting

$$(I.B.1) \quad \begin{array}{ccc} \tilde{\Delta} & \xrightarrow{\tilde{\Phi}} & D \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{\Phi} & \Gamma \backslash D. \end{array}$$

In Section II we will drop the  $\tilde{\phantom{x}}$  and simply work with a period mapping

$$(I.B.2) \quad \Phi : \Delta \rightarrow D$$

where  $\Phi(0)$  is the pair consisting of a polarized Hodge structure together with an equivariant action of  $\Gamma$  on (I.B.2) such that the origin is an isolated fixed point of that action on  $\Delta$ . Then  $\Gamma$  acts on the polarized Hodge structure  $\Phi(0)$ . The property  $\delta\Phi \neq 0$  is independent of the base change. This is important because in the geometric case the degree of the base change is not well defined; all one can say is that it is a multiple of the order of  $\Gamma$ .

In Section II.A we give two geometric examples of a family of curves where the action of  $\Gamma$  on  $\Phi(0)$  is trivial. For the first we compute  $\delta\Phi$  in the smoothing direction for a compact stable curve. The point here is to illustrate in a very simple case one of the basic points that arise in general calculation of  $\delta\Phi$ ; namely how the BF condition [F1] for first

order smoothing leads to the intertwining of components in a semi-stable reduction in the formula for  $\delta\Phi$ .<sup>7 8</sup> The second example is when there is a subfamily of a family consisting of varieties having a non-trivial symmetry group whose induced action on the period domain is trivial. A classical example here, treated algebro-geometrically in [OS] and differential geometrically in [CPT] and [Gh], is hyperelliptic curves of genus  $g \geq 3$ . Here the basic result is that the usual differential vanishes in the normal directions to the hyperelliptic locus, whereas the second order behavior in those directions is given by the full system of quadrics through the canonical curve. After setting a general context we will discuss this topic from an alternative perspective.

The examples in Section II.C are normal surfaces  $X$  that arise on the boundary of a KSBA moduli space  $\mathcal{M}$  ([K] and [KS-B]). The first example, which is really an illustration, is the standard  $A_1$  singularity. The differential  $\delta\Phi$  is computed using semi-stable reduction and, although it is elementary and confirms the expected result, it illustrates the general method of how one uses the first order smoothing condition for a normal crossing variety in the calculation of  $\delta\Phi$ .

The next examples are the  $\mathbb{Q}$ -Gorenstein smoothable semi-log-canonical normal singularities whose smoothing has finite monodromy. These singularities are classified ([KS-B] and [K]): they are rational, and aside from two exceptional cases, are either ADE or quotient singularities of the  $\frac{1}{d^n}(1, dna - 1)$  type. They have the following special property that holds for any  $\mathbb{Q}$ -Gorenstein smoothable rational singularity: Given a local 1-parameter family such that after a base change the varieties over the punctured disc  $\Delta^*$  are smooth and have trivial monodromy, then the period mapping  $\Phi : \Delta^* \rightarrow D$  extends across the origin to give

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<sup>7</sup>Below we shall in this example illustrate a cohomological mechanism to address the question: Is semi-stable-reduction really necessary, or can one just fill in with a desingularized central fibre? There are Hodge theoretic obstructions to doing this; one such is the example of compact singular curves given in Section II.A.

<sup>8</sup>We shall not give detailed proofs of this computation or of several others in this paper. The point is that once one knows the answer the arguments to justify them are of a standard type.



a polarized Hodge structure  $\Phi(0)$ . In general this does not imply that we can then fill in the origin with a smooth variety. However in the above situation this is possible.

As mentioned an underlying geometric question is whether the subvariety of  $\overline{\mathcal{M}}$  consisting of surfaces  $(X, p)$  with a singularity of one of the above types is defined by Hodge theoretic conditions. Our main result gives in variational form sufficient conditions for this in terms of the conditions imposed by the singularity  $p$  on the canonical system  $|K_X|$ . For the case of canonical singularities,  $K_X$  is a line bundle and  $|K_X(-p)|$  has the usual meaning. For the non-Gorenstein quotient singularities the situation is more involved and the conditions are expressed in terms of  $|K_{\hat{X}}(-E)|$  where  $(\hat{X}, E) \rightarrow (X, p)$  is the resolution of the singularity.

In Section II.D we discuss the Wahl singularity  $\frac{1}{4}(1, 1)$  on an  $I$ -surface  $X$  (cf. [FPR], [H]). Such surfaces define a divisor in  $\overline{\mathcal{M}}_I$ , and we verify that this divisor is reduced by showing that at a general point on it  $\delta\Phi \neq 0$  in the normal (smoothing) direction. The desingularization  $\hat{X}$  of such surfaces is an elliptic surface with a bi-section being a  $-4$  curve that contracts to the singular point of  $X$ . It is classical that generic local Torelli holds for  $\hat{X}$ . Coupled with  $\delta\Phi \neq 0$  this implies that generic local Torelli holds for smooth  $I$ -surfaces.<sup>9</sup>

**I.C. The case of infinite monodromy.** When  $N \neq 0$  the leading term of  $\Phi(t)$  is a nilpotent orbit  $\exp(\mathbb{C}N) \cdot F_{\text{lim}}$ . By definition  $\delta\Phi$

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<sup>9</sup>One may also give an alternative argument for this generic local Torelli result using singular  $I$ -surfaces with a degree 2 elliptic singularity. This argument is based on the generic global Torelli result (III.A.12) for pairs  $(X, C)$  where  $X$  is a K3 surface of degree 2 and  $C$  is a smooth section of the polarizing line bundle. If this argument could be extended to the case where  $C$  has a node it would give a proof of generic global Torelli for  $I$ -surfaces. In this regard see [PZ].

It is due to Chakiris [C] that global Torelli holds for a class of regular elliptic surfaces with a section, and if one were able to extend this to elliptic surfaces with a bi-section such as  $\hat{X}$ , then also generic global Torelli for  $I$ -surfaces would be a consequence.

The recent papers [S-B1], [S-B2] give generic global Torelli for a large class of elliptic surfaces having a section. The methods use an analogue of a  $\delta\Phi$  that is adapted to the particular geometry of these surfaces.

is the next order term in the expansion of  $\Phi(t)$ . It is an element of  $\mathfrak{g}_{\mathbb{C}} \subset \text{End}(V_{\mathbb{C}})$  and measures the deviation of  $\Phi(t)$  from being a nilpotent orbit. One objective of Section III.B is to give general structural properties of  $\delta\Phi$  arising from the filtration on  $\mathfrak{g}_{\mathbb{C}}$  induced by the weight filtration  $W(N)$  on  $V$ . The weights  $w$  of  $\delta\Phi$  satisfy

$$-(n+1) \leq w \leq +(n-1)$$

and we will be concerned with the properties of  $\delta\Phi$  on the associated graded to the weight filtration. In the classical case  $n = 1$  and the associated graded terms  $\delta\Phi_w$  are in the range  $-2 \leq w \leq 0$ . One structural result is that in general for any weight  $n$

$$\delta\Phi_w \neq 0 \text{ only for } -2 \leq w \leq n-1.$$

This is a Hodge theoretic result that may be expressed informally by saying that, up to integration constants, the extension data in the limiting mixed Hodge structure is determined by that data of levels  $\leq 2$ . When  $n = 2$  the level  $+1$  part of  $\delta\Phi$  only arises in the non-classical case and measures the failure of the period matrix in a family of surfaces that are smoothings of surfaces with a double curve to remain in the image of a single Schubert cell. This seems to be the main new Lie theoretic property of period mappings that is encountered in the non-classical case (cf. the discussion in [GGR]).

The term  $\delta\Phi_0$  is the differential of the traditional period mapping for the graded pure Hodge structures associated to a family of limiting mixed Hodge structures. The next term  $\delta\Phi_{-1}$  measures how the level 1 extension data varies. It is an Abel-Jacobi type mapping, one part of which has a classical algebro-geometric construction and one part that does not. Roughly speaking if we have a family of smoothable normal crossing varieties, then  $\delta\Phi_0$  measures the variation of the Hodge structure of the individual pieces and  $\delta\Phi_{-1}$  measures the variation of how they pairwise fit together. The next term  $\delta\Phi_{-2}$  has a discrete part and a variable part that is given by a “secondary” Abel-Jacobi mapping

(like a cross-ratio). Finally as mentioned above the  $\delta\Phi_w$  vanish for  $w \leq -3$ .

Another objective of Section III.B is to illustrate by example how for a family of generically smooth algebraic varieties Torelli type results on the boundary may be used to infer Torelli type results in the interior. This requires formulating and establishing local Torelli type results for pairs  $(Y, Z)$  where it is the variation of both the algebraic part of the extension data, arising when  $Y$  is a surface and  $Z$  is a curve from the Abel-Jacobi image of  $\ker\{\text{Pic}Y \rightarrow \text{Pic}(Z)\}$ , and from the variation of the transcendental part (membrane integrals) of the extension data. In the example given there it is shown how generically  $(Y, Z)$  may be constructed from its mixed Hodge structure together with the algebraic information in  $\delta\Phi$ .

The diagram (III.B.7) gives a schematic for defining and interpreting the mappings  $\delta\Phi_w$ . To compute them cohomologically in the geometric case a convenient device is to use the schematic B.3 in part B of the appendix for the computation of the associated graded to a limiting mixed Hodge structure.

In concluding the introduction we emphasize that the purposes of this paper are

- to define the differential  $\delta\Phi$  of a period mapping, especially where  $\Phi$  may have degeneracies or singularities;
- to formulate some general properties of  $\delta\Phi$ , especially those related to the weight filtration in singular cases;
- to give techniques for computing  $\delta\Phi$  cohomologically in the geometric case,

and perhaps most importantly

- to illustrate in examples how these computations may be carried out and what their geometric consequences are.

II. THE CASE OF FINITE MONODROMY ( $N = 0$ )

II.A. **Definition of  $\delta\Phi$  in the  $N = 0$  case.** We assume given the data  $(D, \Gamma, \Phi)$  where

- (i)  $D = G_{\mathbb{R}}/H$  is a period domain and  $\Gamma \subset G_H$  is a finite group that acts on  $D$  and on the unit disc  $\Delta$ ;
- (ii)  $\Phi : \Delta \rightarrow D$  is a period mapping that is equivariant with respect to the action of  $\Gamma$ .

We denote by  $D_{\Gamma}$  and  $\Delta_{\Gamma}$  the fixed point sets of the action and assume that

- (iii)  $\Phi(0) \in D_{\Gamma}$  and  $\Delta_{\Gamma} = \{0\}$  is the origin.

Then  $\Phi(0) = \{V, Q, F; \Gamma\}$  is a polarized Hodge structure on which the group  $\Gamma$  acts.

**Example:**  $\Phi_1 : \Delta_1^* \rightarrow \Gamma \backslash D$  arises from a variation of Hodge structure over the punctured disc  $\Delta_1^* = \{0 < |t_1| < 1\}$  and where the monodromy  $\Gamma = \{T_s^k : k \in \mathbb{Z}\}$  is finite. Using the base change  $\Delta \rightarrow \Delta_1$  given by  $t_1 = t^d$  where  $d$  is a multiple of the order of  $T_s$  there is a diagram

$$\begin{array}{ccc} \Delta^* & \xrightarrow{\Phi} & D \\ \downarrow & & \downarrow \\ \Delta_1^* & \xrightarrow{\Phi_1} & \Gamma \backslash D, \end{array}$$

and  $\Phi$  extends across  $t = 0$  to give a period mapping  $\Phi : \Delta \rightarrow D$  satisfying the conditions (i), (ii), (iii) above. In particular  $\Phi(0)$  is a polarized Hodge structure  $(V, Q, F; \Gamma)$ .

In the geometric case when one does equivariant semi-stable reduction choices are made that may affect the degree of the base change.

Given a complex manifold  $M$  and a non-constant holomorphic mapping  $f : \Delta \rightarrow M$  with  $f(0) = p \in T_p M$ , in local coordinates there is a smallest integer  $k$  such that the  $k^{\text{th}}$  derivative  $f^{(k)}(0) \neq 0$  is non-zero. Then there is an induced map

$$f^{(k)} : \mathbb{C} \rightarrow T_p M$$

that is homogeneous of degree  $k$ .

**Definition II.A.1:** Given the data  $(D, \Gamma; \Phi)$  we denote by

$$\delta\Phi : T_{\{0\}}\Delta \rightarrow T_{\Phi(0)}D$$

the map given by first non-zero derivative of  $\Phi$ .

Choosing a coordinate  $t$  for  $\Delta$  and making the usual identification of  $TD$  with a sub-bundle of  $\oplus \text{Hom}(F^p, V_{\mathbb{C}}/F^p)$ , we have

$$(II.A.2) \quad \delta\Phi : \mathbb{C} \rightarrow \frac{F^{-1} \text{End}(V_{\mathbb{C}})}{F^0 \text{End}(V_{\mathbb{Q}})}$$

Even in “standard” geometric situations  $\delta\Phi$  may not be the usual differential of  $\Phi$ . We note that the condition  $\delta\Phi \neq 0$  is invariant under base change.

**Note:** In some examples we will discuss situations where we want to define  $\delta\Phi$  on a vector space  $W$  that maps to the tangent space  $TB$  to a parameter space  $B$  for a variation of Hodge structure, and in this case we will define  $\delta$  on  $W$  by taking discs  $\Delta \subset W$ . For example if in the geometric case we have a smooth family  $\mathcal{X} \xrightarrow{\pi} B$  of projective varieties  $X_b = \pi^{-1}(b)$  and  $\Phi : B \rightarrow \Gamma \backslash D$  is the corresponding period mapping, then  $W = H^1(\Theta_{X_b})$  and  $W \rightarrow T_{\Phi(b)}(\Gamma \backslash D)$  is the composition of the Kodaira-Spencer map and the usual differential of a locally liftable map to  $\Gamma \backslash D$ .

A second example is one where we have a sub-manifold  $A \subset B$  such that for  $b \in A$  there is a natural splitting of the normal sequence

$$0 \longrightarrow T_b A \longrightarrow T_b B \overset{\leftarrow}{\dashrightarrow} N_{A/B,b} \longrightarrow 0$$

giving  $N_{A/B,b} \subset T_b B$ . In this case for  $W = N_{A/B,b}$  we will take a disc  $\Delta \subset B$  with  $T_{\{0\}}\Delta \subset N_{A/B,b}$ .

**Remark:** For a holomorphic mapping  $f : B \rightarrow \mathbb{P}^m$  of a complex manifold  $B$  to a projective space, there are associated fundamental forms I, II, III, . . . (cf. [La]). For a period mapping  $\Phi : B \rightarrow D$  we may use a Plücker embedding to define the fundamental forms associated to  $\Phi$ . These reflect the higher order behavior of  $\Phi$ . A sample of literature on this is in [CPT], [FP] and [Gh].

**Example:** Suppose that  $\mathcal{X} \xrightarrow{\text{tr}} \Delta$  is a smooth family of smooth curves  $X_t = \pi^{-1}(t)$  where  $X_0$  is hyperelliptic but  $X_t$  is non-hyperelliptic for  $t \neq 0$ . More precisely, we assume that  $\Delta$  is a disc in the Kuranishi space  $\text{Def}(X_0) \subset \mathbb{C}^{3g-3}$  that meets the hyperelliptic locus transversely at the origin and is invariant under the involution acting on  $\text{Def}(X_0)$ . Then it is due to Oort-Steenbrink [OS], and will be discussed below, that

$$\Phi'(0) = 0, \text{ but } \Phi''(0) \neq 0.$$

This phenomenon is general: If  $\gamma \in \Gamma$  acts trivially on the period domain  $D$ , then differentiating

$$\gamma\Phi(t) = \Phi(\gamma t)$$

at  $t = 0$  gives

$$\gamma\Phi'(0) = \Phi'(0),$$

and this may force  $\Phi'(0) = 0$ , as in this example. If  $\Phi$  is non-constant, then some  $\Phi^{(k)}(0) \neq 0$ . More generally we have the

## II.B. Examples when $N = 0$ and the action of $\Gamma$ on $D$ is trivial.

Suppose that we have

- a period mapping  $\Phi : B \rightarrow \Gamma \backslash D$ ;
- a finite group  $\Lambda$  acting on  $B$  with fixed point set a submanifold  $A \subset B$ ;
- a representation  $\rho : \Lambda \rightarrow \Gamma$  such that for  $\lambda \in \Lambda$  and  $b \in B$

$$\Phi(\lambda b) = \rho(\lambda)\Phi(b).$$

Then we have

$$(II.B.1) \quad \Phi_*\lambda_* = \rho(\lambda)\Phi_*.$$

This general relation imposes constraints on  $\Phi_*$ . For example, suppose that we assume

- $\rho(\Lambda)$  acts trivially on  $D$ .

Then there is an induced set-theoretic mapping

$$(II.B.2) \quad \Phi : B/\Lambda \rightarrow D.$$

Along  $A$  the relation (II.B.1) is trivial on  $TA \subset TB$  but may be non-trivial in the normal space  $N_{A/B}$ , which we assume to be a direct summand of  $TB|_A$ . Then (II.B.1) gives

- $\rho(\lambda)\Phi_* = \Phi_*$  in  $N_{A/B}$ .

If for example  $\rho(\Lambda)$  has no fixed vectors in  $N_{A/B}$ ,

$$\Phi_*|_{N_{A/B}} = 0$$

and we have to go to higher order to define  $\delta\Phi$  in the normal directions.

How high an order do we expect to have to go to to have  $(d^k\Phi/dt^k)(0) \neq 0$ ? The analytic variety  $B/\Lambda$  has quotient singularities along  $A$ . These will have a multiplicity  $\mu$  and a reasonable expectation is that  $\mu$  is related to the  $k$  above. With the precise formulation and details to be given elsewhere it may be shown that

(II.B.3) *Suppose that*

- (i)  $\Phi$  in (II.B.2) is 1-1;
- (ii)  $\Phi_*$  is 1-1 on  $B \setminus A$ , and
- (iii)  $(\Phi|_A)_*$  is 1-1.

*Then on any normal disc  $\Delta$  to  $A$  in  $B$ ,  $\delta\Phi \neq 0$  and the order of vanishing of  $\Phi|_\Delta$  is equal to the multiplicity of  $\Phi(\Delta)$  at  $\Delta \cap A$ .*

**Example II.B.4:** Continuing an example mentioned above, let  $C$  be a hyperelliptic curve of genus  $g \geq 3$  with hyperelliptic involution  $j$ . Denote by

$$B \subset H^1(\Theta_C) \cong \mathbb{C}^{3g-3}$$

the Kuranishi space. There is a smooth family of genus  $g$  curves parametrized by  $B$ . The involution  $j$  acting on  $H^1(\Theta_C)$  may be assumed to preserve  $B$  and the parameter space  $A$  for the hyperelliptic curves is given by the fixed point set of  $j$ . Writing

$$\begin{array}{ccc} H^1(\Theta_C) & = & H^1(\Theta_C)^- \quad \oplus \quad H^1(\Theta_C)^+ \\ & & \wr \parallel \qquad \qquad \qquad \wr \parallel \\ & & (H^0(2K_C)^-)^* \quad \quad (H^0(2K_C)^+)^* \\ & & \wr \parallel \qquad \qquad \qquad \wr \parallel \\ & & \mathbb{C}^{g-2} \qquad \qquad \qquad \mathbb{C}^{2g-1} \end{array}$$

we have

$$A = B \cap H^1(\Theta_C)^+.$$

The tangent space to  $B$  at the origin is the direct sum of the  $\pm 1$  eigenspaces of the induced action of  $j$ . At the point in question the  $+1$  eigenspace corresponds to  $TA \subset TB$  and the  $-1$  eigenspace may be identified with the normal space  $N_{A/B}$  to  $A$  or  $B$ , which is then a direct summand of  $TB$ . In particular

$$(II.B.5) \quad H^0(2K_C)^- \text{ may be identified with the co-normal space } N_{A/B}^* \text{ at the origin.}^{10}$$

Identifying  $H^1(C, \mathbb{C}) \cong \mathbb{C}^{2g}$  with the standard alternating form  $Q$ , the period domain  $D$  is an open domain in the homogeneous algebraic variety  $\check{D}$  of  $Q$ -isotropic  $g$ -planes in  $\mathbb{C}^{2g}$ . At a point  $F \in D$ ,

$$T_F D \cong \text{Hom}^s(F, \mathbb{C}^{2g}/F) \cong \text{Sym}^2 F^*$$

where we have used  $Q$  to identify  $\mathbb{C}^{2g}/F$  with  $F^*$  and  $\text{Hom}^s$  are the symmetric maps. If

$$\Phi : B \rightarrow D$$

is the period mapping with

$$\Phi(0) = F = H^0(K_C),$$

then

$$(II.B.6) \quad \Phi^* : \text{Sym}^2 H^0(K_C) \rightarrow H^0(2K_C) = H^0(2K_C)^+ \oplus H^0(2K_C)^-.$$

It is well known that this mapping surjects onto  $H^0(2K_C)^+$ ; thus

$$(II.B.7) \quad \begin{array}{l} \text{the differential of the period mapping is 1-1 on the sub-} \\ \text{space } TB \subset TB \text{ and vanishes on the normal space} \\ N_{A/B} \subset TA. \end{array}$$

The period mapping factors

$$\begin{array}{ccc} B & \longrightarrow & D \\ \downarrow & \nearrow & \\ B/j & & \end{array}$$

<sup>10</sup>Cf. [Gh] and the references cited there.



where the quotient  $B/j$  looks like  $\mathbb{C}^{3g-3} = \mathbb{C}^{g-2} \oplus \mathbb{C}^{2g-1}$  factored by  $(u, v) \rightarrow (-u, v)$ . This is a  $(2g-1)$ -parameter family of quotient singularities of quadratic type, i.e., the invariants are generated by degree 2 polynomials in the coordinates  $u_i$  of  $u$ . Taking (II.B.3) into account this suggests that quadrics in  $\text{Sym}^2 H^0(2K_C)^-$  may be related to  $\delta\Phi$  in the normal directions to  $A$  in  $B$ . Where might such quadrics come from?

For any smooth curve  $C$  the cotangent space to  $D$  at  $\Phi(0)$  is naturally identified with  $\text{Sym}^2 H^0(K_C)$ . For the canonical mapping

$$\varphi_{K_C} : C \rightarrow \mathbb{P}^{g-1} = \mathbb{P}H^0(K_C)^*$$

$\text{Sym}^2 H^0(K_C)$  are the quadrics in the space of the canonical curve. The co-differential of  $\Phi$  is the map

$$(II.B.8) \quad 0 \rightarrow I_2(\varphi_{K_C}(C)) \rightarrow \text{Sym}^2 H^0(K_C) \xrightarrow{\Phi^*} H^0(2K_C)$$

where the kernel are the quadrics containing the canonical curve. For  $C$  non-hyperelliptic the map  $\Phi^*$  is surjective and

$I_2(\varphi_{K_C}(C))$  is the co-normal space to the image of the period mapping.

For  $C$  hyperelliptic the image of  $\Phi^*$  in (II.B.8) is  $H^0(2K_C)^+$ ; thus

the cokernel of  $\Phi^*$  in (II.B.8) is  $H^0(2K_C)^-$ .

**PROPOSITION II.B.9:** *There is an isomorphism, natural up to scaling,<sup>11</sup>*

$$\begin{array}{ccc} \delta\Phi^* : \text{Sym}^2(H^0(2K_C)^-) & \xrightarrow{\sim} & I_2(\varphi_{K_C}(C)) \\ \parallel & & \\ & & N_{A/B}^* \end{array}$$

**Explanation:** Using the above identifications

$$\delta\Phi : N_{A/B} \rightarrow TD = \text{Sym}^2 H^0(K_C)^*$$

is a quadratic map. The dual is a map

$$\delta\Phi^* : \text{Sym}^2 H^0(K_C) \rightarrow \text{Sym}^2(H^0(2K_C)^-).$$

<sup>11</sup>In a different but equivalent form this result is in [OS]. This says that the period mapping in the normal directions to the hyperelliptic locus is analytically equivalent to the mapping  $(\dots, u_i, \dots) \rightarrow (\dots, u_i u_j, \dots)$ ,  $i \leq j$ .

Using the eigenspace decomposition of the action of  $j$  the sequence (II.B.8) splits; thus there is a natural direct sum decomposition

$$\mathrm{Sym}^2 H^0(K_C) = I_2(\varphi_{K_C}(C)) \oplus H^0(2K_C)^+.$$

Then  $\delta\Phi^*$  is zero on the second summand on the right and is an isomorphism on the first.<sup>12</sup>

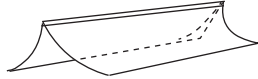
The scaling factor referred to in the statement of the theorem will be given by a choice of isomorphisms

$$\wedge^2 H^0(\mathcal{O}_C(1)) \cong \mathbb{C}$$

where  $\mathcal{O}_C(1)$  is a line bundle giving the 2:1 map  $C \rightarrow \mathbb{P}^1$ .

**Remark:** Geometrically the picture is this: If  $C'$  is a non-hyperelliptic curve with a specialization  $C' \rightarrow C$  to a hyperelliptic curve, then the canonical curve  $\varphi_{K_{C'}}(C')$  is a curve of degree  $2g - 2$  in a  $\mathbb{P}^{g-1}$ . As  $C' \rightarrow C$  it specializes to  $2 \times$  (rational normal curve in  $\mathbb{P}^{g-1}$ ).

For the period mapping on  $B$ , the kernel of the co-differential at  $C'$  is  $I_2(\varphi_{K_{C'}}(C'))$ , and as  $C' \rightarrow C$  this jumps up to  $I_2(\varphi_{K_C}(C))$ , which is an additional  $\mathrm{Sym}^2(H^0(2K_C)^-)$ 's worth of quadrics. This space is naturally identified with the co-normal space of  $A$  in  $B$ . The picture of the image  $\Phi(B)$  is something like



where the double lines are quotient singularities  $\mathbb{C}^{g-2}/(\pm \mathrm{Id})$ .

**Proof of Propostion II.B.9:** The argument will use standard representation theory. For the two-sheeted cyclic covering  $C \xrightarrow{\pi} \mathbb{P}^2$  with

<sup>12</sup>We note that

$$\begin{aligned} \dim I_2(\varphi_{K_C}(C)) &= \frac{g(g+1)}{2} - (2g-1) = \frac{g^2 - 3g + 2}{2} \\ \dim \mathrm{Sym}^2(H^0(2K_C)^-) &= \binom{g-2}{2} = \frac{g^2 - 3g + 2}{2}. \end{aligned}$$

It follows that along the hyperelliptic locus the embedding dimension of the image of the period map is  $g(g+1)/2 = \dim D$ . It also follows that along the hyperelliptic locus the second fundamental form of the period map is given by the quadrics containing the canonical curve, and the higher fundamental forms are all equal to zero.

$\mathcal{O}_C(1) = \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$  we set

$$U = H^0(\mathcal{O}_C(1)).$$

There is a  $\mathbb{Z}_2$ -action on  $U$  and hence on any direct summand  $R(U)$  in the tensor algebra of  $U$  and its dual. We will identify the vector spaces  $H^0(kK_C)$ ,  $H^0(\ell K_C)^*$  and their symmetric products in terms of the  $\pm$  eigenspaces  $R(U)^\pm$  of the  $\mathbb{Z}_2$ -action on the  $\mathrm{GL}(U)$ -module  $R(U)$ .<sup>13</sup> When this is done there will be a  $\mathrm{GL}(U)$  map, equivariant up to scaling,

$$\mathrm{Sym}^2(H^0(2K_C)^-) \rightarrow I_2(\varphi_{K_C}(C))$$

which will be the map in the statement of the theorem.

In coordinates if we realize  $C \subset \mathbb{P}(1, 1, g-1)$  as a non-singular curve  $y^2 = f_{2g+2}(x_0, x_1)$  where  $(x_0, x_1; y)$  are homogeneous coordinates with weights  $1, 1, g-1$  and where the involution acts by  $x_0 \rightarrow x_0, x_1 \rightarrow x_1, y \rightarrow -y$ , then

$$\begin{aligned} H^0(K_C) &\cong \left\{ \frac{a_{g-1}(x_0, x_1)}{y} \right\} = H^0(K_C)^-, \\ H^0(2K_C) &\cong \left\{ \frac{b_{g-3}(x_0, x_1)y}{y^2} \right\} \oplus \left\{ \frac{c_{2g-2}(x_0, x_1)}{y^2} \right\}. \end{aligned}$$

The canonical curve is given by  $\{a_{g-1}(x_0, x_1)\}$ , i.e., by  $S^{g-1}U$ . Thus

$$I_2(\varphi_{K_C}(C)) = \ker \{S^2(S^{g-1}U) \rightarrow S^{2g-2}U\}.$$

For any  $d$  we define

$$(II.B.10) \quad S^2U^{d-2} \rightarrow \ker\{S^2U^d - U^{2d}\}$$

by

$$(II.B.11) \quad P \circ Q \rightarrow x^2P \circ y^2Q - 2xyP \circ xyQ + y^2P \circ x^2Q.$$

<sup>13</sup>For instance, using the notation  $S^m = \mathrm{Sym}^m$

$$\begin{array}{ccc} H^0(K_C) & = & S^{g-1}U \\ H^0(2K_C) & = & S^{g-3}U \oplus S^{2g-2}U \\ & & \parallel \quad \parallel \\ & & H^0(2K_C)^- \quad H^0(2K_C)^+ \end{array}$$

The right-hand side is clearly in  $\ker\{S^2U^d \rightarrow U^{2d}\}$ , and we will prove the

LEMMA II.B.12: *Any element in  $\ker\{S^2U^2 \rightarrow U^{2d}\}$  is of this form.*

This will show that (II.B.10) is surjective and then by a dimension count it will be the isomorphism in Proposition II.B.9.

**Proof of Lemma II.B.12:** If we have

$$M_{ij} := x^i y^{d-i} \circ x^j y^{d-j}, \quad i \geq j,$$

then if  $i \geq 2$  and  $j \leq d-2$ ,

$$M_{ij} \equiv 2x^{i-1}y^{d-i+1} \circ x^{j-1}y^{d-j-1} - x^{i-2}y^{d-i+2} \circ x^{j+2}y^{d-j-2}$$

where the congruence is modulo terms on the right-hand side of (II.B.11).

If  $i \geq j+3$ ,

$$M_{ij} \equiv 2M_{i-1,j+1} - M_{i-2,j+2}.$$

If  $i = j+2$ ,

$$M_{i,i-2} \equiv 2M_{i-1,i-1} - M_{i,i-2} \equiv M_{i-1,i-1}$$

where the second step uses

$$\begin{aligned} x^{i-2} \circ y^{d-i+2} \circ x^{j+2} y^{d-j-1} &= x^{i-2} y^{d-i-2} \circ x^i y^{d-i} \\ &= x^{i-2} y^{d-i} \circ x^{i-2} y^{d-i+2} = M_{i,i-2}. \end{aligned}$$

Thus we can use the right-hand side of (II.B.11) to have only terms

$$M_{i,j} \text{ where } i \leq 1, j \geq d-1 \text{ or } i \leq j+1.$$

This is because  $(i, j) = (1, 1), (1, 0), (0, 0)$  so  $j = 1$  or  $j = i-1$ , and likewise  $i \geq j \geq d-1$  gives  $(i, j) = (d, d), (d, d-1), (d-1, d-1)$  so  $j = i$  or  $j = i-1$ . In all cases either  $j = i$  or  $j = i-1$ .

If

$$\sum_i a_i M_{i,i} + \sum_i b_i M_{i,i-1} \in \ker\{S^2U^d \rightarrow U^{2d}\},$$

then

$$\sum_{i=0}^d a_i x^{2i} y^{2d-2i} + \sum_{i=1}^d b_i x^{2i-1} y^{2d-2i+1} = 0.$$

But the terms are all distinct, hence all  $a_i, b_i$  are zero.  $\square$

This completes the construction of the map in Proposition II.B.9. The equivalence up to scaling using the standard notation in representation appears already when  $d = 2$ . Then

$$S^2U^2 = U^4 \oplus U^{2,2}$$

where  $U^{2,2} = (\wedge^2 U)^{\otimes 2}$  and the scaling factor arises from the choice of an isomorphism  $\wedge^2 U \cong \mathbb{C}$ .

It remains to prove that  $\delta\Phi \neq 0$  in the normal directions to  $A$  in  $B$ . In period matrix terms, for a normal disc  $A \subset B$  with  $\Delta \cap A = \{0\}$

$$(II.B.13) \quad \left. \frac{d^2\Phi(t)}{dt^2} \right|_{t=0} \neq 0.$$

This is true and is proved by an explicit coordinate calculation in [OS].

Assuming the global Torelli theorem that the mapping

$$\begin{array}{ccc} \mathcal{M}_g & \longrightarrow & \mathcal{A}_g \\ \Psi & & \Psi \\ C & \longrightarrow & J(C) \end{array}$$

is 1-1, another argument for (II.B.13) runs as follows: We may assume that  $\Delta \subset B$  is invariant under the involution  $j$ ; in fact we may take  $j(t) = -t$ . Referring to (II.B.1) we have a holomorphic mapping

$$f : \Delta \rightarrow \mathbb{C}$$

such that

- $f(-t) = f(t)$  (this implies  $f'(0) = 0$ );
- $f$  is 1-1 on  $\Delta/\mathbb{Z}_2$ .

Then it is an elementary complex variable fact that

$$f''(0) \neq 0.$$

**Remark:** In [OS] the moduli spaces  $\mathcal{M}_g$  and  $\mathcal{A}_g = \mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathcal{H}_g$  are rigidified to  $\mathcal{M}_g^{(n)}$  and  $\mathcal{A}_g^{(n)}$  by adding level  $n$  structures given by the  $n$ -division points in  $J(C)$  where  $C \in \mathcal{M}_g$  and in  $J(C) \in \mathcal{A}_g$ . The Abel-Jacobi map

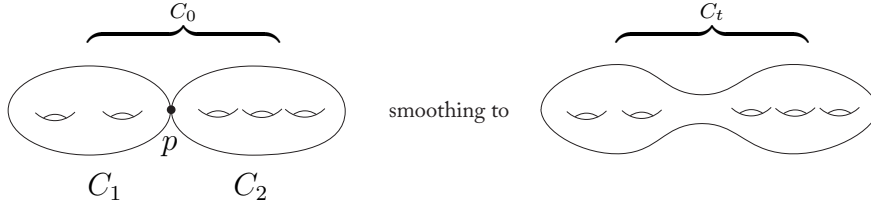
$$AJ : C \rightarrow \mathrm{Pic}^1(C)$$

induces a morphism of complex analytic varieties

$$\mathcal{M}_g^{(n)} \rightarrow \mathcal{A}_g^{(n)}.$$

This provides additional geometric information beyond the usual Torelli period matrix mapping, information that is an integral part of the arguments in [OS].

**Example II.B.14:** This is an example of family of curves  $\mathcal{X} \rightarrow \Delta$  when  $N = 0$  and  $T = \text{Id}$ . The picture is



where  $g(C_1) = g_1$ ,  $g(C_2) = g_2$  and  $g_1 + g_2 = g(C_t)$ . For the period mapping we have

- $D \supset D_1 \times D_2$  which are the principal polarized Hodge structures that are direct sums of principally polarized sub-Hodge structures;
- $\Phi : \Delta \rightarrow D$  where  $\Phi(0)$  corresponds to  $H^1(C_0)$  and  $\Phi(t)$  for  $t \neq 0$  corresponds to  $H^1(C_t)$ .

PROPOSITION II.B.15:<sup>14</sup> For  $C_1, C_2$  non-hyperelliptic,  $\delta\Phi$  is the usual differential  $\Phi_*$  and

$$(II.B.16) \quad \Phi_*(d/dt) \neq 0 \text{ in } N_{D_1 \times D_2 / D}$$

*Proof.* We let  $\tilde{C}_0 = C_1 \amalg C_2$  be the normalization of  $C_0$  and  $p_i \in C_i$  with  $C_0$  defined by the identification  $p_1 = p_2$ . The infinitesimal deformation sequence (I.A.2) is

$$(II.B.17) \quad \begin{array}{ccccccc} 0 \rightarrow H^1 \left( \mathcal{E}xt_{\mathcal{O}_{C_0}}^0 \left( \Omega_{C_0}^1, \mathcal{O}_{C_0} \right) \right) & \rightarrow & T \text{Def}(C_0) & \rightarrow & H^0 \left( \mathcal{E}xt_{\mathcal{O}_{C_0}}^1 \left( \Omega_{C_0}^1, \mathcal{O}_{C_0} \right) \right) & \rightarrow & 0 \\ & & \parallel & & \parallel & & \\ & & T \text{Def}^{es}(C_0) & & T \text{Def}^{sm}(C_0) & & \end{array}$$

<sup>14</sup>This is the well-known local Torelli theorem for stable compact curves. The point here is to illustrate a computation of  $\delta\Phi$  in a simple situation as a prelude to similar but more involved examples discussed below.

At the point in question and with the notations  $V = V_1 \oplus V_2$ ,  $Q = Q_1 \oplus Q_2$  and  $F = F_2 \oplus F_2$  we have from  $T_{\Phi(0)}D = \text{Hom}^s(F, V/F)$  that at  $F$  the normal space

$$(II.B.18) \quad N_{D_1 \times D_2 / D} \cong \text{Hom}(F_1, V_2/F_2) \oplus \text{Hom}(F_2, V_1/F_1).$$

We note that  $\Phi_*(0)$  induces a map

$$(II.B.19) \quad T \text{Def}(C_0) / T \text{Def}^{es}(C_0) \rightarrow N_{D_1 \times D_2 / D}.^{15}$$

Using the local Torelli theorem for the non-hyperelliptic curves  $C_1$  and  $C_2$ , we want to show that this map is non-zero.

For varying points  $p_1 \in C_1$  and  $p_2 \in C_2$  on fixed curves  $C_1$  and  $C_2$ , we join  $C_1$  and  $C_2$  at these points to obtain  $C_1 \cup_p C_2$ . Since the identified points  $p_1, p_2$  move at the same speed the left-hand side of (II.B.19) is naturally isomorphic to  $\Theta_{C_1, p_1} \otimes \Theta_{C_2, p_2}$ .

For the right-hand side of (II.B.19) using  $Q_1$  and  $Q_2$  we obtain

$$(H^0(K_{C_1})^* \otimes H^0(K_{C_2})^*) \oplus (H^0(K_{C_2})^* \otimes H^0(K_{C_1})^*).$$

Using the evident symmetry the dual of (II.B.19) is a map

$$H^0(K_{C_1}) \otimes H^0(K_{C_2}) \rightarrow K_{C_1, p_1} \otimes K_{C_2, p_2}$$

which is the evaluation mapping

$$\omega_1 \otimes \omega_2 \rightarrow \omega_1(p_1) \otimes \omega_2(p_2).$$

---

<sup>15</sup>Note that using the direct sum decompositions,  $N_{D_1 \times D_2 / D}$  is actually a subspace of  $T_F D$  and  $T \text{Def}^{es}(C_0)$  maps via  $\Phi_1 \times \Phi_2$  into the complementary subspace.

This may be verified by writing out the maps

$$\begin{array}{ccc}
\Theta_{C_1, p_1} \otimes \Theta_{C_2, p_2} & \longrightarrow & H^1 \left( \mathcal{E}xt_{\mathcal{O}_{C_0}}^0(\Omega_{C_0}^1, \mathcal{O}_{C_0}) \right) \longrightarrow \mathbb{E}xt_{\mathcal{O}_{C_0}}^1(\Omega_{C_0}^1, \mathcal{O}_{C_0}) \\
& & \parallel \qquad \qquad \qquad \parallel \\
& & T \text{Def}^{es}(C_0) \qquad \qquad T \text{Def}(C_0) \\
T \text{Def}(C_0) & \longrightarrow & H^0(\mathcal{E}xt_{\mathcal{O}_{C_0}}^1(\Omega_{C_0}^1, \mathcal{O}_{C_0})) \\
& & \parallel \\
& & T^{sm}(C_0) \\
T \text{Def}(C_0) & \longrightarrow & T_{\Phi(0)} D \cong \text{Hom}^s(F, V/F)
\end{array}$$

and tracing through their dualizations. For this it is convenient to use a covering by two open sets, one of which is a neighborhood of  $p$  and the other being the complement of a smaller closed neighborhood of  $p$ .  $\square$

**II.C. Examples when  $N = 0$  and  $\Gamma$  is non-trivial.** These will be examples arising from smoothing a normal surface singularity  $(X, p)$ . There are two methods that we shall use.

(i) This one works in general. Given a 1-parameter smoothing deformation of a variety  $X$  one may use semi-stable reduction to have a family

$$\mathcal{X} \xrightarrow{\pi} \Delta$$

of varieties  $X_t = \pi^{-1}(t)$  where  $\mathcal{X}$  is smooth,  $\mathcal{X}^* \rightarrow \Delta^*$  is a smooth fibration,  $X_0 = \cup X_i$  is a reduced normal crossing divisor and, when  $X$  is irreducible, one component of  $X_0$  is a desingularization  $\hat{X}$  of  $X$ . This process requires a base change after which the semi-simple part of the monodromy of the family  $\mathcal{X}^* \rightarrow \Delta^*$  becomes trivial.

The notations, basic definitions and results concerning deformations and smoothing of normal crossing varieties and a schematic for calculating the associated graded to the corresponding limiting mixed Hodge structure are collected in the appendix.



(ii) For the special case of a KSBA smoothing (defined below) of a normal surface singularity when  $N = 0$  there is an alternate perhaps better method. It is based on the result that semi-log-canonical  $\mathbb{Q}$ -Gorenstein smoothable normal surface singularities are rational and for such singularities there is the Artin component in the versal deformation space (cf. [St]). For this component, after base change the family over  $\Delta^*$  may be filled in over the origin by simply inserting  $\hat{X}$ . Thus, after we eliminate monodromy by a base change not only does the Hodge structure fill in over  $t = 0$  but the family of smooth surfaces does also. Within this component, but in general not equal to it, there is the subvariety of  $\mathbb{Q}$ -Gorenstein smoothings. It is this space that we shall be concerned with here.

Before turning to this we will first use semi-stable reduction to show how to compute  $\delta\Phi$  for the smoothing of a node ( $A_1$  singularity) on a surface. The results are well known; the interest is in illustrating some of the computational techniques in a simple but non-trivial case. These techniques will extend to the general case

We first give a general result and then apply it to the  $A_1$ -singularity case. Let  $X_1, X_2$  be smooth surfaces and  $C$  an irreducible smooth curve with embeddings  $C \hookrightarrow C_i \subset X_i$  for  $i = 1, 2$ . Denote by

$$X = X_1 \cup_C X_2$$

the normal crossing surface obtained by joining  $X_1$  and  $X_2$  along  $C_1$  and  $C_2$ . The BF condition [F1] that  $X$  have a first order smoothing is

$$(II.C.1) \quad N_{C_1/X_1} \otimes N_{C_2/X_2} \cong \mathcal{O}_C.$$

Assuming this the choice of a parameter  $\epsilon$  for the first order smoothing  $\mathcal{X}_\epsilon \rightarrow \Delta(\epsilon)$  of  $X$  is given by choosing an identification  $H^0(\mathcal{O}_C) \cong \mathbb{C}$ .

Associated to  $\mathcal{X}_\epsilon \rightarrow \Delta(\epsilon)$  there is a limiting mixed Hodge structure with unipotent monodromy. Assuming  $N = 0$  we obtain an ordinary weight 2 Hodge structure  $H^2 = \oplus H^{p,q}$  together with a first order variation of that Hodge structure. This is given by

$$(II.C.2) \quad \delta\Phi(d/dt) \in \text{Hom}(H^{2,0}, H^{1,1})$$

and we will give a recipe for computing the interesting part of this.<sup>16</sup> Here we will do this under the additional assumption that the genus  $g(C) = 0$ , as this will be satisfied in the application we shall give. Later on this method will be extended to the case when  $g(C) \neq 0$ .

PROPOSITION II.C.3: *There are natural isomorphisms*

$$\begin{cases} H^{2,0} \cong H^0(\Omega_{X_1}^2) \oplus H^0(\Omega_{X_2}^2), \\ H^{1,1} \cong ([C_1] + [C_2])^\perp / ([C_1] - [C_2]), \end{cases}$$

and maps (here  $i \in \{1, 2\}$  and  $\hat{i} = \{1, 2\} \setminus \{i\}$ )

$$(II.C.4) \quad H^0(\Omega_{X_i}^2) \rightarrow H^1(\Omega_{X_{\hat{i}}}^1)$$

that give  $\delta\Phi(d/dt)$ .

In (II.C.4) the intertwining of  $X_1$  and  $X_2$  will be a consequence of (II.C.1). For the formula for  $\Phi(0)$ , since  $N = 0$  the limiting mixed Hodge structure associated to the first order family  $\mathcal{X} \rightarrow \Delta(\epsilon)$  is a pure weight 2 Hodge structure; it is given by the cohomology of the complex

$$\begin{array}{ccccc} & & H^2(X_1) & & \\ & \nearrow G & & \searrow r & \\ H^0(C)(-1) & & \oplus & & H^2(C) \\ & \searrow G & & \nearrow r & \\ & & H^2(X_2) & & \end{array}$$

where the first arrows represent the Gysin map

$$(II.C.5) \quad 1_C \rightarrow [C_1] \oplus [C_2]$$

and the second arrows are signed restriction maps. The composition is

$$1_C \rightarrow (C_1^2 + C_2^2)[C]$$

which is zero as a consequence of the BF condition. The first part of the proposition follows from (II.C.5).

<sup>16</sup>By interesting part we mean that summand of  $\text{Hom}(H^{2,0}, H^{1,1})$  that involves the intertwining of  $X_1$  and  $X_2$ . There are other parts that involve either  $X_1$  or  $X_2$  alone. These are of standard IVHS maps and we will not discuss them.

For the second part we shall use the cohomology sequences associated to two exact sheaf diagrams, the first of which is

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Omega_{X_1}^2(-C_1) & \longrightarrow & \Omega_{X_1}^2 & \longrightarrow & \Omega_{C_1}^1 \otimes N_{C_1/X_1}^* \longrightarrow 0 \\
 & & & & & & \cong \\
 & & & & & & \Omega_{C_2}^1 \otimes N_{C_2/X_2}
 \end{array}$$

where the last term in the top sequence is adjunction and the vertical isomorphism, which is the key step, uses the BF condition. The second is the diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & \Omega_{X_2}^1 & \longrightarrow & \Omega_{X_2}^1(\log C_2) & \longrightarrow & \Omega_{X_2}^1(\log C_2)|_{C_2} \longrightarrow 0 \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & \Omega_{X_2}^1 & \longrightarrow & \Omega_{X_2}^1([C_2]) & \longrightarrow & \Omega_{X_2}^1([C_2])|_{C_2} \longrightarrow 0 \\
 & & & & \downarrow & & \\
 & & & & \Omega_{C_2}^1([C_2]) & & \\
 & & & & \downarrow & & \\
 & & & & 0 & & 
 \end{array}$$

Using  $\Omega_{X_2}^1(\log C_2)|_{C_2} \cong \mathcal{O}_{C_2}(-1)$ , since  $H^1(\mathcal{O}_{C_2})(-1) = 0$ , combining the cohomology diagrams we have induced maps

$$\begin{aligned}
 H^0(K_{C_1}) &\rightarrow H^0(K_{C_2} \otimes N_{C_2/X_2}), \\
 H^0(K_{C_2} \otimes N_{C_2/X_2}) &\rightarrow H^1(\Omega_{X_2}^1)/H^0(\mathcal{O}_{C_2})(-1).
 \end{aligned}$$

Here we have used that the composition

$$H^0(\mathcal{O}_{C_2})(-1) \rightarrow H^1\left(\Omega_{X_2}^1([C_2])|_{C_2}\right) \rightarrow H^1(\Omega_{X_2}^1)$$

gives the fundamental class of  $C_2$ . The issue is now to show that the map

$$H^0(K_{X_1}) \rightarrow H^1(\Omega_{X_2}^1)/H^1(\mathcal{O}_{C_2})(-1)$$

actually computes  $\Phi(d/dt)$ . The idea is

- to take the argument where you have a smooth family  $\mathcal{X} \xrightarrow{\pi} \Delta$  of smooth varieties and where  $\Phi(d/dt)$  is the connecting map in the long exact hypercohomology sequence of

$$0 \rightarrow \pi^* \Omega_{\Delta}^1 \otimes \Omega_{\mathcal{X}/1}^1 \rightarrow \Omega_{\mathcal{X}}^2 \rightarrow \Omega_{\mathcal{X}/\Delta}^2 \rightarrow 0$$

and modify it using

$$0 \rightarrow \pi^* \Omega_{\Delta}^1(\log\{0\}) \otimes \Omega_{\mathcal{X}/\Delta}^1(\log X_0) \rightarrow \Omega_{\mathcal{X}}^2(\log X_0) \rightarrow \Omega_{\mathcal{X}/\Delta}^2(\log X_0) \rightarrow 0;$$

- next trace through the same calculation in the case when  $X_t$  is smooth for  $t \neq 0$  and where  $X_0 = X_1 \cup_C X_2$  with the local smoothing  $x_1 x_2 = t$  in  $(x_1, x_2, x_3, t)$  space;
- in carrying this out one may use a covering by open sets like the one that one used in the previous example but jacked up with the additional parameter  $x_2$ ;
- in the resulting computation the interesting step is in the passage from  $X_1$  to  $X_2$  using BF conditions to have the replacement

$$\Omega_{C_1}^1 \otimes N_{C_1/X_1}^* \xrightarrow{\sim} \Omega_{C_2}^1 \otimes N_{C_2/X_2}.$$

**Example II.C.6:** Let  $\mathcal{X}' \xrightarrow{\pi} \Delta'$  be a family of surfaces  $X'_s$  acquiring a node  $p \in X'_0$ . Away from the node we have a smooth fibration, and in a neighborhood  $U \subset \mathcal{X}'_0$  of  $p$  we have

$$\begin{cases} U = \{x, y, z, s\} : x^2 + y^2 + z^2 = s \\ \pi(x, y, z, s) = s. \end{cases}$$

There is one vanishing cycle  $\Delta \in H_2(X'_s)$ , and local monodromy around  $s = 0$  is a  $\mathbb{Z}_2$  acting by  $\Delta \rightarrow -\Delta$ . After semi-stable reduction, retaining the notations in Proposition II.C.3 we obtain a new family  $\mathcal{X} \rightarrow \Delta$  where  $\Delta$  has coordinate  $t$  with  $s = t^2$  and  $X'_0$  is replaced by

$$X_0 = X_1 \cup_C X_2$$

where

$$\begin{array}{ccc} X_1 & \longrightarrow & X'_0 \\ \cup & & \cup \\ C_1 & \longrightarrow & p, \end{array}$$

with  $C_1 \cong \mathbb{P}^1$ ,  $C_1^2 = -2$  being the desingularization of  $X'_0$ , and  $X_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$  with  $C_2 \cong \mathbb{P}^1$  is the diagonal and  $C_2^2 = +2$ . Denoting by  $L_1, L_2$  the classes in  $H^2(X_2)$  generated by the two factors in the product,

$$\begin{aligned} [C_2] &= L_1 + L_2 \\ L_1 - L_2 &\in ([C_1] + [C_2])^\perp. \end{aligned} \quad {}^{17}$$

From the proposition we have that for  $\omega \in H^0(K_{X_1})$

$$(II.C.7) \quad \delta\Phi(d/dt)(\omega) = \omega(p)(L_1 - L_2).$$

More precisely, we identify  $H^0(\mathcal{O}_C) = \mathbb{C}$ . Then from

$$K_{X_1}|_{C_1} = N_{C_1/X_1} \otimes K_{C_1} = \mathcal{O}_{C_1}$$

we have

$$\omega(p) := \omega|_{C_1} \in \mathbb{C}.$$

**Remark:** The main purpose of this paper is to define and illustrate the use of the differential of the period mapping at the locus of singular varieties in a family  $\{X_b\}_{b \in B}$  of generically smooth algebraic varieties. Assume  $B$  is smooth and that the singular  $X_b$  are nodal surfaces that occur along a subvariety  $A \subset B$ . We will say that *the family satisfies local Torelli* if for any disc  $\Delta \subset B$  we have  $\delta\Phi \neq 0$  at the origin. If  $\Delta$  meets  $A$  only at the origin, then  $\delta\Phi$  is defined above. If  $\Delta \subset A$ , then  $\delta\Phi \neq 0$  means that the differential of the period mapping for the desingularized surfaces  $X_1$  is non-zero. In various forms the following is well known.

**PROPOSITION II.C.8:** *If each canonical series  $|K_{X_b}|$  is base point free and if local Torelli is satisfied, then  $A$  is a smooth, reduced divisor in  $B$ . Moreover,  $A$  may be defined Hodge theoretically.*

*Proof.* What the above computation shows is that the locus where a given cohomology class is a Hodge class is a smooth reduced divisor  $A \subset B$ . What we have to show is that if we have a family  $\{X_t\}_{t \in \Delta}$  of smooth surfaces and a Hodge class  $\gamma_t \in \text{Hg}^1(X_t)$  such that  $\gamma_0$  is the

<sup>17</sup>Using the notation from Proposition II.C.3,  $(L_1 - L_2) \rightarrow ((L_1 - L_2) \cdot C_2)[C] = 0$ . Thus  $L_1 - L_2 \in H^2$  and monodromy acts by  $L_1 - L_2 \rightarrow L_2 - L_1$ .

class of a  $-2$  curve  $C_0 \subset X_0$ , then  $C_0$  deforms to  $-2$  curves  $C_t \subset X_t$ . This is also well known and we shall only indicate why it is true.

If  $L_t \rightarrow X_t$  are line bundles with  $c_1(L_t) = \xi_t$  and  $L_0 = [C_0]$ , then it will suffice to show the vanishing of the obstruction space to first order deforming  $\xi_0$  along with  $L_0$ ; this is

$$H^1(L_0) = 0.$$

For simplicity we assume that  $X_0$  is regular; in general an additional argument is needed.<sup>18</sup> By duality  $h^1(L_0) = h^1(K_{X_0} - L_0) = h^1(K_{X_0}(-C_0))$ . Since  $C_0$  is a  $-2$  curve,  $K_{X_0}|_{C_0} \cong \mathcal{O}_{C_0}$  and we have

$$0 \rightarrow K_{X_0}(-C_0) \rightarrow K_{X_0} \rightarrow \mathcal{O}_{C_0} \rightarrow 0.$$

Using  $h^1(K_{X_0}) = 0$  this gives

$$0 \neq h^1(K_0(-C_0)) \iff H^0(K_{X_0}(-C_0)) \xrightarrow{\sim} H^0(K_{X_0}).$$

If the linear system  $|K_{X_0}|$  is base point free, then it follows that  $h^1(L_0) = 0$ .  $\square$

Our next example will be the computation for a KSBA family  $\mathcal{X} \rightarrow \Delta$  of surfaces where  $X_0$  has a  $\mathbb{Q}$ -Gorenstein smoothable normal singular point  $p$  and the  $X_t$  are smooth for  $t \neq 0$ . Here we will consider the case where the monodromy of  $\mathcal{X}^* \rightarrow \Delta^*$  is finite. The period mapping then extends to  $\Delta$  and we will give a method for computing  $\delta\Phi$ . Among the underlying geometric questions are

*What is the expected number of Hodge theoretic conditions imposed on moduli by having a singularity of the type of  $X_0$ ? In examples, such as the case of  $I$ -surfaces, are these conditions independent?*

Before turning to this we will give a quick

(II.C.9) *Review of  $\mathbb{Q}$ -Gorenstein smoothable isolated surface singularities.*<sup>14</sup>

<sup>18</sup>The additional argument entails modifying the  $L_t$  by elements in  $\text{Pic}^0(X_t)$  to kill the obstruction to deforming the section of  $L_0$  whose divisor is  $C_0$ . A standard cohomological argument shows that to first order this is possible, from which we may conclude the desired result.

The theory is local and we shall discuss the deformation theory of a germ  $(Y, p)$  of an isolated surface singularity.

- $K_Y$  denotes the Weil divisorial canonical sheaf; it is represented on  $Y^* := Y \setminus \{p\}$  by the linear equivalence class of the divisor of a differential  $\psi \in H^0(K_{Y^*})$ ;
- there is a unique minimal (no  $(-1)$  curves) resolution  $(\hat{Y}, E) \xrightarrow{f} (Y, p)$  of the singularity; here  $E = \sum_i E_i$  is a normal crossing divisor (the  $E_i$  are irreducible, smooth curves meeting transversely) and the intersection matrix  $\|E_i \cdot E_j\|$  is negative definite; we set  $E_i^2 = -d_i$  where  $d_i > 1$ ;
- the singularity is *rational* if all the  $E_i$ 's are  $\mathbb{P}^1$ 's and the graph of  $E$  contains no cycles; this is equivalent to  $R_f^1 \mathcal{O}_{\hat{Y}} = 0$ ;
- we shall assume that  $(Y, p)$  is a semi-log-canonical singularity (not defined here; cf. [K], [H] and [KS-B]), then there is a unique *index 1 cover*  $(Z, q) \rightarrow (Y, p)$  such that
  - $(Z, q)$  is Gorenstein, thus  $K_Z$  is a line bundle;
  - for  $m = \text{index}(Y, p)$  and  $\mu_m = m^{\text{th}}$  roots of unity, there is a  $\mu_m$  action on  $(Z, q)$  with  $q$  an isolated fixed point and  $(Y, p)$  is the quotient of  $(Z, q)$ ;
- by a  $\mathbb{Q}$ -Gorenstein smoothing of  $(W, p)$  we mean a smoothing  $Z_t$  of the index 1 cover  $Z = Z_0$  of  $Y$  such that  $\mu_m$  acts on the total space of the family  $\{Z_t\}$  fixing the fibre  $Z_0$  and where the quotient of  $\{Z_t\}_{t \in \Delta}$  is  $\{W_t\}_{t \in \Delta}$ ; this may be formalized by a diagram

$$\begin{array}{ccc}
 \mathcal{Z} & \longrightarrow & \mathcal{Y} \\
 \downarrow & & \downarrow \\
 \Delta & \longrightarrow & \Delta
 \end{array}$$

on which  $\mu_m$  acts with  $q \in Z_0$  as the unique fixed point (cf. [H]); an important point is that  $\omega_{\mathcal{Y}/\Delta}$  is  $\mathbb{Q}$ -Gorenstein with  $\mathcal{Z} \rightarrow \mathcal{Y}$  being the relative index 1 cover;

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<sup>14</sup>Cf. [H] and [St]. In Section II.D these definitions and facts will be illustrated in an example.

- For rational singularities  $(Y, p)$  there is the *Artin component*  $A \subset \text{Def}(Y)$  (cf. [St]); it has the property that any family pulled back from a finite map to  $A$  is, after a base change, a simultaneous resolution of the singularities; the Artin component has dimension  $h^1(\Theta_{\hat{Y}})$ .
- The  $\mathbb{Q}$ -Gorenstein smoothings of  $(Y, p)$  are, after base change, the quotients of the ordinary deformations of the index 1 cover in which the  $\mu_m$  action is preserved; they map to the Artin component.

The semi-log-canonical  $\mathbb{Q}$ -Gorenstein smoothable normal surface singularities have been classified (cf. [KS-B]). Those that are open sets in complete general type surfaces for which the monodromy of the smoothing has  $N = 0$  satisfy

- (a) they are rational;
- (b) there are two types
  - (i) ADE or Du Val singularities,
  - (ii) some quotient singularities.

The ADE singularities are hypersurface singularities. Hence they are Gorenstein and their versal deformation, which is equal to the Artin component, may be described explicitly by varying their defining equation (cf. [A]). The exceptional divisor of a minimal resolution is a tree of  $-2$  curves given by a Dynkin diagram.

For the second type of singularities, for  $a, d$  coprime integers with  $a < d$ , the singularity  $\frac{1}{d}(1, a)$  is the quotient of  $\mathbb{C}^2$  by the action

$$(u, v) \rightarrow (\zeta u, \zeta^a v)$$

where  $\zeta$  is a primitive  $d^{\text{th}}$  root of unity.

- The [KS-B] quotient singularities we shall consider are of type  $\frac{1}{d^2n}(1, dna - 1)$  where  $d, n, a$  are relatively prime. They are  $\mathbb{Z}/d\mathbb{Z}$  quotients of their index 1 cover which is an  $A_{dn-1}$  singularity

$$xy + z^{dn} = 0$$

and the action of  $\zeta = e^{2\pi i/d}$  is

$$x \rightarrow \zeta x, y \rightarrow \zeta^{dn-1} y, z \rightarrow \zeta^a z;$$



the  $\mathbb{Q}$ -Gorensetin smoothings are quotients of

$$(II.C.10) \quad xy + z^{dn} + a_1 t^d z^{d(n-1)} + a_2 t^{2d} z^{d(n-1)} + \dots + a_n t^{nd} = 0$$

under the action

$$x \rightarrow \xi x, y \rightarrow \zeta^{dn-1} y, z \rightarrow \zeta^a z, t \rightarrow t;$$

we note that implicit in (II.C.10) is a base change  $s = t^{nd}$  from the standard smoothing  $xy + z^{dn} + s = 0$  of the  $A_{dn-1}$  singularity.

**Remark:** For some circumstances it is preferable to write a general smoothing of an  $A_{d-1}$  singularity  $Y$  given by  $xy + z^d = 0$  as

$$(II.C.11) \quad Y_t = \left\{ xy + \prod_{i=1}^d (z - ta_i) = 0 \right\}$$

where the  $a_i$  are the ordered roots of the degree  $d$  homogeneous polynomial in  $z, t$  that gives the smoothing after base change. As explained in [A] the  $a_i$  correspond to the vanishing cycles by

$$\Delta_1 \leftrightarrow a_2 - a_1, \Delta_2 \leftrightarrow a_3 - a_2, \dots, \Delta_{d-1} \leftrightarrow a_d - a_{d-1}.$$

Replacing  $Y = Y_0$  by  $\hat{Y}$  gives a smoothing family  $\mathcal{Y} \rightarrow \Delta$  where  $\Delta$  maps as a  $d$ -sheeted branched covering over a general disc in  $\text{Def}(Y_0)$ .<sup>15</sup> The vanishing cycles are now invariant under monodromy and the corresponding homology classes are the  $[E_i] \in H_2(\hat{Y})$ .

Although not needed here, there are similar descriptions for all of the ADE singularities.

**Deformation theory** Let  $(\hat{Y}, E) \rightarrow (Y, p)$  denote the resolution of a germ of normal surface singularity. We are interested in

- (i)  $T \text{Def}(\hat{Y}; E_1, \dots, E_k) :=$  tangent space to the deformations of  $\hat{Y}$  along which all of the  $E_i$  deform;<sup>16</sup>

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<sup>15</sup>Here general means meeting the discriminant locus transversely at the origin.

<sup>16</sup>These correspond to the equisingular deformations of  $(Y, p)$ .

- (ii)  $T \text{Def}(\hat{Y}, E) :=$  tangent space to the deformations of  $\hat{Y}$  along which the curve  $E$  deforms, and

$\cap$

- (iii)  $T \text{Def}(\hat{Y}) :=$  tangent space to the deformations of  $\hat{Y}$ .

By abuse of notation we shall identify the tangent space to the Artin component of  $\text{Def}(Y)$  with  $T \text{Def}(\hat{Y}) \cong H^1(\Theta_{\hat{Y}})$ .

The normal sheaf to  $E$  in  $\hat{Y}$  is defined by

$$\mathcal{N}_{E/\hat{Y}} = \text{Hom}_{\mathcal{O}_E}(\mathcal{J}_E/\mathcal{J}_E^2, \mathcal{O}_E).$$

The following are cohomological expressions for each of the above.

PROPOSITION II.C.12: *There are natural identifications*

$$\begin{aligned} T \text{Def}(\hat{Y}) &\cong H^1(\Theta_{\hat{Y}}), \\ T \text{Def}(Y, E) &\cong \ker \left\{ H^1(\Theta_{\hat{Y}}) \rightarrow H^1(\mathcal{N}_{E/\hat{Y}}) \right\}, \\ T \text{Def}(Y, E_1, \dots, E_k) &\cong \ker \left\{ H^1(\Theta_{\hat{Y}}) \rightarrow \bigoplus_{i=1}^k H^1(\mathcal{N}_{E_i/\hat{Y}}) \right\}. \end{aligned}$$

*Proof.* The first of these is standard (cf. [St]). Because  $\dim \hat{Y} = 2$  and we are working with the germ of singularity,  $H^2(\Theta_{\hat{Y}}) = 0$  and consequently the first order deformations of  $\hat{Y}$  are unobstructed.

For the second another standard identification in deformation theory is

$$T \text{Def}(\hat{Y}, E) \cong H^1(\Theta_{\hat{Y}}(-\log E)).$$

From the exact cohomology sequence of

$$0 \rightarrow \Theta_{\hat{Y}}(-\log E) \rightarrow \Theta_{\hat{Y}} \rightarrow \mathcal{N}_{E/\hat{Y}} \rightarrow 0$$

we obtain the result. Finally, the third identification is obtained by applying the proof of the second one to each of the  $E_i$ .  $\square$

**Remark:** We may “explain” the Artin component as follows, here illustrated for the  $A_1$ -singularity discussed above (keeping the notations used there).

The tangent space to the deformation space of a pair  $C \subset \hat{X}$  is given by  $\mathbb{H}^1(\Theta_{\hat{X}} \rightarrow N_{C/\hat{X}})$ . Then to first order

- $\ker\{H^1(\Theta_{\hat{X}}) \rightarrow H^1(N_{C/\hat{X}})\} =$  tangent space to deformations where  $C$  deforms with  $\hat{X}$ ; and
- $\text{coker}\{H^1(\Theta_{\hat{X}}) \rightarrow H^1(N_{C/\hat{X}})\} =$  deformations of  $\hat{X}$  where  $C$  disappears (fails to deform to first order).

For the  $A_1$ -singularity where  $X_0 = X_1 \cup_C X_2$  with  $X_1 = \hat{X}$  and  $X_2 = \mathbb{P}^1 \times \mathbb{P}^1$ , as usual  $T \text{Def}(X_0) \cong \mathbb{E}\text{xt}_{\mathcal{O}_{X_0}}^1(\Omega_{X_0}^1, \mathcal{O}_{X_0})$  maps to  $T^{sm} \text{Def}(X_0) = H^0(\mathcal{E}\text{xt}_{\mathcal{O}_{X_0}}^1(\Omega_{X_0}^1, \mathcal{O}_{X_0}))$ . A computation using  $h^1(\Theta_{X_2}) = 0$  and  $h^1(\mathcal{O}_C) = 0$  gives a *surjective* map

$$\text{coker}\{H^1(\Theta_{\hat{X}}) \rightarrow H^1(N_{C/\hat{X}})\} \twoheadrightarrow T^{sm} \text{Def}(X_0).$$

This means that the smoothing deformations of  $(X, p)$  are obtained by deforming the desingularization  $\hat{X}$  in the directions where  $C$  disappears. In other words, in this example using semi-stable-reduction is not necessary; we may simply deform the desingularization of  $X$ .

**Remark:** Keeping these notations, suppose we have just one  $E \cong \mathbb{P}^{117}$  and a 1-parameter deformation of  $\hat{Y}$  given by a diagram

$$\begin{array}{ccccc} E & \subset & \hat{Y} & \subset & \hat{y} \\ & & \downarrow & & \downarrow \\ & & \{0\} & \in & \Delta. \end{array}$$

The exact sequence of normal bundles

$$\begin{array}{ccccccc} 0 & \longrightarrow & N_{E/\hat{Y}} & \longrightarrow & N_{E/\hat{y}} & \longrightarrow & \mathcal{O}_E \longrightarrow 0 \\ & & & & & & \wr \parallel \\ & & & & & & N_{\hat{Y}/\hat{y}}|_E \end{array}$$

has extension class  $e \in H^1(N_{E/\hat{Y}})$ . We also have the Kodaira-Spencer class  $d/dt \in H^1(\Theta_{\hat{Y}})$ . We will omit the straightforward proof of the following.

**PROPOSITION II.C.13:** (i)  $e$  is the image of  $d/dt$  in the map  $H^1(\Theta_{\hat{Y}}) \rightarrow H^1(N_{E/\hat{Y}})$ ;  
 (ii)  $e$  is non-zero if, and only if, to first order  $E$  disappears in the deformation  $Y_t$  of  $\hat{Y} = \hat{Y}_0$ ;

<sup>17</sup>Then  $\hat{Y}$  is a cone over a rational normal curve in  $\mathbb{P}^d$  where  $d = -E^2$ .

(iii) *in this case*

$$N_{E/\hat{Y}} \cong \mathcal{O}_E(-\alpha) \oplus \mathcal{O}_E(-\beta)$$

where  $\alpha, \beta > 0$ , and then  $E$  contracts to a terminal singularity in the deformation space  $\mathcal{Y}$  of  $Y$ .

PROPOSITION II.C.14: *There are maps of  $\mathcal{O}_E$ -modules*

$$\mathcal{J}_E/\mathcal{J}_E^2 \rightarrow N_{E_i/\hat{Y}}^* \otimes N_{E_{i+1}/\hat{Y}}^*$$

that upon dualizing lead to the exact sheaf sequence of  $\mathcal{O}_E$ -modules

$$(II.C.15) \quad 0 \rightarrow \bigoplus_{i=1}^k N_{E_i/\hat{Y}} \rightarrow \mathcal{N}_{E/\hat{Y}} \rightarrow \bigoplus_{i=1}^{k-1} N_{E_i/\hat{Y}} \otimes N_{E_{i+1}/\hat{Y}}|_{E_i \cap E_{i+1}} \rightarrow 0.$$

*Proof.* The issue is local around a point of  $E_i \cap E_{i+1}$  locally given by  $xy = 0$ . Then

$$\mathcal{J}_E/\mathcal{J}_E^2 = \frac{(xy)}{(x^2y^2)} \text{ has a } \mathbb{C}\text{-bases } 1 \cdot (xy), x^i(xy), y^j(xy) \text{ for } i > 0, y > 0,$$

$$N_{E_i/\hat{Y}}^* \cong \frac{(x)}{(x)^2} \text{ has a basis } 1 \cdot x, x^i \cdot x \text{ for } i > 0,$$

$$N_{E_{i+1}/\hat{Y}}^* \cong \frac{(y)}{(y)^2} \text{ has a basis } 1 \cdot y, y^j \cdot y \text{ for } j > 0,$$

$$yN_{E_i/\hat{Y}}^* \cong \frac{(xy)}{(x^2y)} \text{ and } xN_{E_{i+1}/\hat{Y}}^* \cong \frac{(xy)}{(xy^2)}.$$

The map

$$\mathcal{J}_E/\mathcal{J}_E^2 \rightarrow N_{E_i/\hat{Y}}^* \oplus N_{E_{i+1}/\hat{Y}}^*$$

given by

$$\frac{(xy)}{(x^2y^2)} \rightarrow \frac{(xy)}{(x^2y)} \oplus \frac{(xy)}{(xy^2)}$$

is well defined, and because  $(x^2y) \cap (xy^2) = (x^2, y^2)$  it is injective. Since  $1 \rightarrow 1 \cdot (xy) \oplus 1 \cdot (xy)$ , the maps

$$\frac{(xy)}{(x^2y)} \rightarrow \frac{(xy)}{(x^2y, xy^2)}, \quad \frac{(xy)}{(xy^2)} \rightarrow \frac{(xy)}{(x^2y, xy^2)}$$

are individually surjective. By taking the difference locally around  $xy = 0$  where  $E_i = (x = 0)$  and  $E_{i+1} = (y = 0)$  we obtain the exact

sequence

(II.C.16)

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{J}_E/\mathcal{J}_E^2 & \longrightarrow & N_{E_i/\hat{Y}}^* \oplus N_{E_{i+1}/\hat{Y}}^* & \longrightarrow & \mathcal{O}_{E_i \cap E_{i+1}} \longrightarrow 0 \\
& & \cup & & & & \\
& & \frac{(xy)}{(x^2y^2)} & \longrightarrow & \frac{(xy)}{(x^2y)} \oplus \frac{(xy)}{(xy^2)} & \longrightarrow & \frac{(xy)}{(x^2y, xy^2)} \cong \frac{(1)}{(xy)}
\end{array}$$

which globalizes to

$$0 \rightarrow \mathcal{J}_E/\mathcal{J}_E^2 \rightarrow \bigoplus_{i=1}^k N_{E_i/\hat{Y}}^* \rightarrow \bigoplus_{i=1}^{k-1} N_{E_i/\hat{Y}}^* \otimes N_{E_{i+1}/\hat{Y}}^* \rightarrow 0.$$

Dualizing gives the exact sequence (II.C.15).  $\square$

PROPOSITION II.C.17:  $H^0(\mathcal{N}_{E/\hat{Y}}) = 0$ , so that we have

$$0 \rightarrow \bigoplus_{i=1}^{k-1} H^0(N_{E_i/\hat{Y}} \otimes N_{E_{i+1}/\hat{Y}}) \rightarrow \bigoplus_{i=1}^k H^1(N_{E_i/\hat{Y}}) \rightarrow H^1(\mathcal{N}_{E/\hat{Y}}) \rightarrow 0.$$

*Proof.* Using adjunction and  $N_{E_i/\hat{Y}} \cong \mathcal{O}_{E_i}(-d_i)$ , in the exact cohomology sequence of (II.C.15) the map

$$\bigoplus_{i=1}^{k-1} H^0 \left( N_{E_i/\hat{Y}} \otimes N_{E_{i+1}/\hat{Y}} \Big|_{E_i \cap E_{i+1}} \right) \rightarrow \bigoplus_{i=1}^k H^1(N_{E_i/\hat{Y}})$$

is dual to

(II.C.18)

$$\bigoplus_{i=1}^k H^0(\mathcal{O}_{E_{i+1}}(d_i - 2)) \rightarrow \bigoplus_{i=1}^k H^0 \left( N_{E_i/\hat{Y}}^* \otimes N_{E_{i+1}/\hat{Y}}^* \Big|_{E_i \cap E_{i+1}} \right).$$

Since  $d_i \geq 2$ , this mapping is surjective and this implies the result.  $\square$

**Note II.C.19:** Below we will interpret (II.C.18) as giving a map

$$\bigoplus_{i=1}^{k-1} \left( H^0(K_{\hat{Y}}|_{E_i}) \rightarrow \bigoplus H^0(K_{\hat{Y}}|_{E_i \cap E_{i+1}}) \right)$$

whose kernel is equal to  $H^0(K_{\hat{Y}}|_E)$ .

The following summarizes the above:

$$(II.C.20) \quad \begin{array}{ccc} & \text{quotient is } \oplus H^1(N_{E_i/\hat{Y}}) & \\ & \xrightarrow{\hspace{10em}} & \\ T \text{ Def}(\hat{Y}; E_1, \dots, E_k) \subsetneq & T \text{ Def}(\hat{Y}, E) \subsetneq & \text{the Artin component in } T \text{ Def}(\hat{Y}) \\ & \xrightarrow{\hspace{10em}} & \\ & \text{quotient is } \oplus_{i=1}^{k-1} \Theta_{\hat{Y}}|_{E_i \cap E_{i+1}} & \text{quotient is } H^1(\mathcal{N}_{E/\hat{Y}}) \end{array}$$

**The canonical bundle.**

Setting  $Y^* = Y \setminus \{p\}$  with  $j : Y^* \hookrightarrow Y$  the inclusion, the canonical Weil divisorial sheaf

$$K_Y = j_* K_{Y^*}$$

is the linear equivalence class of the divisor of any differential  $\psi \in H^0(\Omega_{Y^*}^2)$ . Since  $K_Y$  is only  $\mathbb{Q}$ -Cartier we cannot evaluate  $\psi \in H^0(K_Y)$  at the singular point  $p$ . Rather we shall use the identification

$$H^0(K_Y) \cong H^0(K_{\hat{Y}})$$

to define evaluation of  $\psi$  at  $p$  by considering  $\psi \in H^0(K_{\hat{Y}}|_E)$ . The basic observation is that by adjunction

$$(II.C.21) \quad K_{\hat{Y}}|_{E_i} \cong N_{E_i/\hat{Y}}^* \otimes K_{E_i} \cong \mathcal{O}_{E_i}(d_i - 2).$$

We are now ready to put things together. The usual form of the differential of a period mapping for weight 2 is

$$T \rightarrow \text{Hom}(H^{2,0}, H^{1,1}),$$

or equivalently

$$(II.C.22) \quad T \otimes H^{2,0} \rightarrow H^{1,1}.$$

In the geometric case

$$\begin{cases} T = H^1(\Theta), \\ H^{2,0} = H^0(\Omega^2), \\ H^{1,1} = H^1(\Omega^1). \end{cases}$$

In the local version we use the cohomology of  $\hat{Y}$ , and the commutative diagram

$$(II.C.23) \quad \begin{array}{ccc} H^1(\Theta_{\hat{Y}}) \otimes H^0(K_{\hat{Y}}) & \longrightarrow & H^1(\Omega_{\hat{Y}}^1) \\ \downarrow & & \downarrow \\ \oplus H^1(N_{E_i/\hat{Y}}) \otimes H^0(K_{\hat{Y}}) & \longrightarrow & \oplus H^1(\Omega_{E_i}^1). \end{array}$$

From Propositions II.C.12 and II.C.14 we have the

**THEOREM II.C.24:** *If the mappings*

$$\begin{array}{ccc} H^1(\Theta_{\hat{Y}}) & \longrightarrow & \bigoplus_{i=1}^k H^1(N_{E_i/\hat{Y}}) \\ H^0(K_{\hat{Y}}) & \longrightarrow & H^0(K_{\hat{Y}}|_E) \end{array}$$

are surjective, then for a general  $\xi \in H^1(\Theta_{\hat{Y}})$  and any  $E_i$ , there is a  $\psi \in H^0(K_{\hat{Y}})$  such that the image of  $\xi \otimes \psi$  in  $H^1(\Omega_{E_i}^1)$  is non-zero.

This implies that all images  $\xi_i$  of  $\xi$  in  $H^1(N_{E_i/\hat{Y}})$  are non-zero; thus  $\xi$  maps to a non-zero element in  $T \text{Def}(\hat{Y})/T \text{Def}(Y, E)$  under which to first order all of the  $E_i$  disappear. Thus  $\xi$  gives a non-zero element in  $T \text{Def}^{sm}(X)$ . This is the local picture.

For the global case we assume that we have a surface  $(X, p)$  with an isolated semi-log-canonical singularity of the above type and with canonical resolution  $(\hat{X}, E) \rightarrow (X, p)$ . Denote by  $(Y, p)$  the germ of neighborhood of  $p$  in  $X$  and by  $(\hat{Y}, E) \rightarrow (Y, p)$  the resolution. For simplicity of exposition we assume that  $H^1(\Theta_{\hat{X}})$  is unobstructed. There

are maps

$$\begin{array}{ccc}
 & T \text{Def}(\hat{X}) & \\
 & \downarrow \rho & \searrow \alpha \\
 \text{(a)} & H^1(\Theta_{\hat{X}}) & \longrightarrow H^1(\Theta_{\hat{Y}}) \longrightarrow \bigoplus_{i=1}^k H^1(N_{E_i/\hat{Y}}) \\
 \\
 \text{(II.C.25)} & \text{(b)} & \\
 & H^0(K_{\hat{X}}) & \xrightarrow{\beta} H^0(K_{\hat{Y}}) \longrightarrow \bigoplus_{i=1}^k H^0(K_{\hat{Y}}|_{E_i}) \\
 & \parallel & \\
 & H^0(K_X) & \\
 \\
 \text{(c)} & & \\
 & H^1(\Omega_{\hat{X}}^1) & \longrightarrow H^1(\Omega_{\hat{Y}}^1) \longrightarrow \bigoplus_{i=1}^k H^1(K_{E_i}).
 \end{array}$$

The differential of the period mapping at the point corresponding to the singular surface  $(X, p)$  may be identified with

$$\delta\Phi : H^1(\Theta_{\hat{X}}) \rightarrow \text{Hom}(H^0(K_{\hat{X}}), H^1(\Omega_{\hat{X}}^1)).^{18}$$

The following is by far not the strongest possible result. It is given to illustrate how via the extended period mapping Hodge theory enters into the geometry of completed moduli spaces, in this case the locus of surfaces having a particular type of singularity.

**THEOREM II.C.26:** *A sufficient condition that  $\xi \in T \text{Def}(\hat{X})$  map to a smoothing direction in  $T \text{Def}(X)$  is that for each  $i$  there exists  $\psi_i \in H^0(K_X)$  such that the composite map  $\lambda$  in*

$$\begin{array}{ccc}
 \alpha\rho(\xi) \otimes \beta(\psi_i) \in H^1(N_{E_i/\hat{X}}) \otimes H^0(K_{\hat{X}}|_{E_i}) & & \\
 \searrow & \downarrow & \\
 & H^1(K_{E_i}) &
 \end{array}$$

*be non-zero.*

<sup>18</sup>Here we recall our convention that  $\delta\Phi$  on  $H^1(\Theta_{\hat{X}})$  is defined by composing the Kodaira-Spencer map on discs  $\Delta \subset H^1(\Theta_{\hat{X}})$  with  $\delta\Phi$  as formally defined in the 1-parameter case.



Indeed, by the analysis above this is sufficient to insure that to first order every  $E_i$  does not move when we deform  $X$  in the direction  $\xi$ .

There are similar results expressing when a subset of the  $E_i$  does not deform, so that the direction  $\xi$  will give at least partially smoothing deformation of  $X$ .

**Summary:** Stripped to its essentials the above argument is the following:

- $(\hat{X}, E) \rightarrow (X, p)$  is the resolution of a rational,  $\mathbb{Q}$ -Gorenstein smoothable singular surface;
- the [KS-B] smoothing deformations of  $X$  are  $\mathbb{Q}$ -Gorenstein smoothings in  $\text{Def}(X)$  and are the images of deformations of  $\hat{X}$  under which the normal crossing curve  $E = \cup E_i$  disappears;
- we ask whether this is the same condition as that the Hodge classes  $[E_i] \in \text{Hg}^1(\hat{X})$  cease to be Hodge classes;
- the result is that the conditions in the second and third bullets are to first order equivalent if, and only if,  $p$  is not a base point of the canonical series  $|K_X|$ ;<sup>19</sup>
- to express this condition cohomologically, first taking the case where there is one  $E_i := E$  we have the commutative diagram

$$\begin{array}{ccc}
 H^1(\Theta_{\hat{X}}) \otimes H^0(K_{\hat{X}}) & \longrightarrow & H^1(\Omega_{\hat{X}}^1) \\
 \downarrow & & \downarrow \\
 H^1(N_{E/\hat{X}}) \otimes H^0(K_{\hat{X}}|_E) & \longrightarrow & H^1(\Omega_E^1);
 \end{array}$$

- for  $\xi \in H^1(\Theta_{\hat{X}})$  the condition that to first order the Hodge class given by  $E$  not deform to a Hodge class is that  $\xi \otimes H^0(K_{\hat{X}})$  maps non-zero to  $H^1(\Omega_E^1)$ ;<sup>20</sup>

---

<sup>19</sup>This means that there is  $\psi \in H^0(K_X)$  such that  $\psi_i := \text{image of } \psi \text{ in } H^0(K_{\hat{X}}|_{E_i})$  for all  $i$  when  $X$  is Gorenstein so that  $K_X$  is a line bundle this is the same as  $\psi(p) \neq 0$ ; when  $K_X$  is only  $\mathbb{Q}$ -Cartier the condition is more subtle.

<sup>20</sup>The condition that to first order  $E$  not deform along with  $\hat{X}$  is that  $\xi$  maps to a non-zero element in  $H^1(N_{E/\hat{X}})$ . We observe that to first order deforming preserving  $E$  and preserving the Hodge class  $[E] \in H^1(\Omega_{\hat{X}}^1)$  are equivalent.

- for the case when  $E = \cup E_i$  we use the normal sheaf  $\mathcal{N}_{E/\hat{X}} := \text{Hom}_{\mathcal{O}_E}(\mathcal{J}_E/\mathcal{J}_E^2, \mathcal{O}_E)$  to obtain the same result where now the map must map non-zero onto each  $H^0(K_{E_i})$ .

#### II.D. Wahl singularity for $I$ -surfaces.

**Local theory:** We recall that any  $\frac{1}{d}(1, a)$  quotient singularity  $(Y, p)$  has an index 1 cover  $(\hat{Y}, \hat{p})$  with an action of  $\mu_m$  with fixed point  $\hat{p}$  and with quotient  $(Y, p)$  (cf. [H]). For the  $\frac{1}{d^2n}(1, dna - 1)$  singularity we have

$$(\hat{Y}, \hat{p}) \rightarrow (Y, p)$$

where  $(\hat{Y}, \hat{p})$  is an  $A_{dn-1}$  singularity with a  $\mu_d$  action. The *Wahl singularity* is the first case  $d = 2, n = a = 1$  giving a  $\frac{1}{4}(1, 1)$  singularity with index 1 cover the  $\frac{1}{2}(1, 1)$  singularity  $A_1$  having an equation and  $\mathbb{Z}_2$ -action

$$\begin{cases} xy = z^2 \\ x \rightarrow -x, y \rightarrow -y, z \rightarrow -z. \end{cases}$$

The smoothing  $xy = z^2 + s$  gives, after a base change  $s = t^2$  with  $t \rightarrow -t$  and then taking the  $\mathbb{Z}_2$ -quotient, a  $\mathbb{Q}$ -Gorenstein smoothing of it. The Wahl singularity is of historical significance: it is the first non-Gorenstein cyclic quotient singularity, and its smoothing has trivial monodromy.

Before illustrating Theorem (II.C.24) and the above construction we shall briefly describe some geometric interpretations of the Wahl singularity and its  $\mathbb{Q}$ -Gorenstein smoothing.

- a  $\frac{1}{d}(1, 1)$  singularity is realized as the cone over the rational normal curve of degree  $d$  in  $\mathbb{P}^d$ ; equivalently, it is the section  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus 0)$  of the  $\mathbb{P}^1$ -bundle  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-d))$  over  $\mathbb{P}^1$ ;
- for a smooth non-degenerate surface  $S$  of minimal degree  $d$  in  $\mathbb{P}^{d+1}$ , a hyperplane through the vertex of the cone over  $S$  gives a realization of this singularity, and deforming the hyperplane to not pass through vertex gives a smoothing;

- for the Wahl singularity when  $d = 4$  there are two choices for  $S$ , the rational normal scroll and the Veronese surfaces  $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$  embedded by  $|\mathcal{O}_{\mathbb{P}^2}(2)|$ ; only the second of these gives a  $\mathbb{Q}$ -Gorenstein smoothing.

For the desingularization  $(\hat{Y}^\#, E^\#) \xrightarrow{f^\#} (Y^\#, p^\#)$  of the node, since the singularity is canonical,

$$f^{\#\ast} K_{Y^\#} |_{E^\#} = K_{\hat{Y}^\#} |_{E^\#} = \mathcal{O}_{E^\#}.$$

We note that

$$H^1(N_{E^\#/\hat{Y}^\#}) \cong H^1(\mathcal{O}_{\mathbb{P}^2}(-2)) \cong \mathbb{C}$$

and the map

$$H^1(\Theta_{\hat{Y}^\#}) \rightarrow H^1(N_{E^\#/\hat{Y}^\#})$$

is non-zero; thus by the analysis at the end of Section II.C the  $-2$  curve  $E^\#$  disappears (in fact already to first order).

For the desingularization  $(\hat{Y}, E) \xrightarrow{f} (Y, p)$  of the Wahl singularity we have the equations of  $\mathbb{Q}$ -line bundles

$$\begin{aligned} f^\ast K_Y &= K_{\hat{Y}} + \frac{1}{2}[E], \\ K_{\hat{Y}} |_E &\cong \mathcal{O}_E(2). \end{aligned}$$

The localized derivative of the period mapping s

$$\begin{aligned} (II.D.1) \quad & H^0(K_{\hat{Y}} |_E) \otimes H^1(N_{E/\hat{Y}}) \longrightarrow H^1(N_{E/\hat{Y}}) \\ & \quad \quad \quad \wr \quad \quad \quad \wr \quad \quad \quad \wr \\ & H^0(\mathcal{O}_E(2)) \otimes H^1(\mathcal{O}_E(-4)) \rightarrow H^1(\mathcal{O}_E(-2)). \end{aligned}$$

Thus under the smoothing direction of the Wahl singularity the class  $[E] \in H^1(\Omega_E^1)$  disappears.

**Global theory:** *Example of I-surfaces.* An  $I$ -surface is a smooth, minimal general type surface  $X$  with

$$p_g(X) = 2, \quad q(X) = 0, \quad K_X^2 = 1.$$

These surfaces were studied classically; recently they have emerged as among the first examples where the completed KSBA moduli space

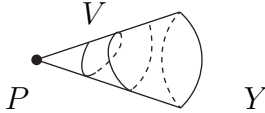
$\overline{\mathcal{M}}_I$  has been analyzed.<sup>21</sup> From the cited works there is one divisor in  $\overline{\mathcal{M}}_I$  whose general point corresponds to an  $I$ -surface  $(X, p)$  with a single Wahl singularity. Here we shall describe these surfaces and, with details to be provided elsewhere, will discuss how this divisor may be detected Hodge theoretically under the extended period mapping

$$\Phi : \mathcal{M}_{I,f} \rightarrow \Gamma \backslash D$$

where  $\mathcal{M}_{I,f} \subset \overline{\mathcal{M}}_I$  parametrizes the possibly singular  $I$ -surfaces that have a  $\mathbb{Q}$ -Gorenstein smoothing with finite monodromy.<sup>22</sup>

Let  $X$  be an  $I$ -surface with a  $\frac{1}{4}(1, 1)$  singularity  $p$ . There are several ways to construct  $(X, p)$ , and to describe one of these we use the notations

- $\mathbb{P}(1, 1, 2)$  has coordinates  $(x_1, x_2, y)$  and is embedded in  $\mathbb{P}^3$  as a quadric cone with vertex  $P$



- $\mathbb{P}(1, 1, 2, 5)$  has coordinates  $(x_1, x_2, y, z)$ ; any Gorenstein  $I$ -surface  $Y$  has the equation

$$z^2 = a_0 y^5 + a_2(x_1, x_2) y^4 + \cdots + a_{10}(x_1, x_2)$$

where  $a_{2k}(x_1, x_2)$  is homogeneous of degree  $2k$  and  $a_0 \neq 0$  (cf. [FPR]).

- $Y \rightarrow \mathbb{P}(1, 1, 2) \subset \mathbb{P}^3$  is branched over a quintic  $V \in |\mathcal{O}_{\mathbb{P}^3}(5)|$  where  $P \notin V$ .

<sup>21</sup>See [FPR] for a detailed analysis of the Gorenstein components of  $\overline{\mathcal{M}}_I$  and also for the elusive non-Gorenstein divisorial components of that space. See also [H] for the Wahl singularity case. Much of the following discussion is based on notes by them that are a sequel to [FPR] and on discussions with them and Radu Laza and Colleen Robles.

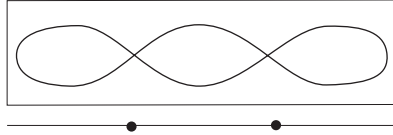
<sup>22</sup>The period domain  $D$  has dimension 57 and  $H \subset TD$  is a contact structure. The differential  $\Phi_* : \mathcal{M}_f \rightarrow \Gamma \backslash D$  is injective at all smooth surfaces  $X$ , and the image  $\Phi(\mathcal{M}_f)$  is a contact subvariety. The monodromy group  $\Gamma$  is arithmetic; it is not known if it is the full  $G_{\mathbb{Z}}$ .

- As  $a_0 \rightarrow 0$  the quintic  $V$  passes through the vertex  $P = (0, 0, 1, 0)$  and the limit surface  $X$  ceases to be Gorenstein but acquires a  $\frac{1}{4}(1, 1)$  singularity over  $P$ .
- For  $a_0 = 0$  and  $(x_1, x_2) \neq (0, 0)$  over the fibre of projection  $\mathbb{P}(1, 1, 2) \rightarrow \mathbb{P}^1$  we have a double cover of a quartic intersection with the line over  $(x_1, x_2)$ . The discriminant of the quartic has terms like  $a_2^3 a_{10}^3, a_4^2 a_6^2 a_0^2, \dots$ ; they all have the same degree  $3 \cdot 2 + 3 \cdot 10, 2 \cdot 4 + 2 \cdot 6 + 2 \cdot 8, \dots = 36$ . Hence, when  $a_0 = 0$  outside of what happens at  $P$  we have a 2:1 covering of  $\mathbb{P}(1, 1, 2) \setminus P$  with four branch points over the intersection of  $V$  with the rulings of the quadric. In general there are 36 such rulings where two of the branch points come together; i.e., the corresponding curves acquire nodes.
- Over  $x_1 = x_2 = 0$  we have  $z = 0$  which is the singular point  $(0, 0, 1, 0)$  on  $X$ . Here  $\mathbb{P}(1, 1, 2, 5)$  has a  $\frac{1}{2}(1, 1, 1)$  singularity of index 2. A weighted blowup inserts a Veronese surface  $X_2$  under the map

$$(x_1, x_2, 0, z) \rightarrow \{\text{all quadratic monomials in } x_1, x_2, z\};$$

the blown up surface  $X_1$  intersects the Veronese  $X_2$  in the conic hyperplane section  $E$  given by  $z^2 - a_2(x_1, x_2) = 0$ . Here we assume  $a_2$  is non-degenerate. The conic  $E$  maps 2:1 to the  $\mathbb{P}^1$  coming from  $x_1, x_2$ ; i.e., we have  $z = \pm \sqrt{a_2(x_1, x_2)}$ .

- Thus  $X_1 \xrightarrow{f} \mathbb{P}^1$  is an elliptic surface having a bi-section



with branch points at the zeroes of  $a_2(x_1, x_2)$ .

- For the numerology we have for  $E_i := E \subset X_i$ 
  - $\begin{cases} E_1^2 = -4 \\ E_2^2 = 4 \end{cases}$
  - $K_{X_1} \cdot E_{X_2} = 2$  (thus  $K_{X_1} \cdot E_{X_1} + E_{X_1}^2 = -2$ )
  - $f_* \omega_{X_1/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ .

Since the monodromy  $T = I$ , from Hodge theory we have  $p_g(X_1) = 2$ .

We now denote by  $E$  the  $\mathbb{P}^1$  in either  $X_1$  or  $X_2$  and consider the surface  $X_0 = X_1 \cup_E X_2$ . Then  $H_{\text{lim}}^2$  is the cohomology of

$$H^0(E)(-1) \xrightarrow{G} H^2(X_1) \oplus H^2(X_2) \xrightarrow{R} H^2(E)$$

where  $G$  is the direct sum of Gysin maps and  $R$  is the sum of signed restriction maps. Thus in the diagram (II.D.1) the image of the map to  $H^2(X_1)$  is spanned by

$$[E]^\perp \in H^1(\Omega_{X_1}^1) \cong H^2(\text{Veronese}) \cong \mathbb{C}.$$

This has dimension 1, and from a work in preparation on moduli of  $I$ -surfaces and Hodge theory we will provide details of the argument to show

*For the moduli of  $I$ -surfaces there is 1-condition to have a  $\frac{1}{4}(1, 1)$  singularity. This condition is detected Hodge theoretically by the presence of an additional Hodge class in  $\Phi(0) = H_{\text{lim}}^2$ .<sup>23</sup>*

**Note:** There is an interesting dimension count going on here. Namely,

- elliptic surfaces  $Y$  with  $q(Y) = 0$ ,  $p_g(Y) = 2$  have  $30 = 10 \cdot \chi(\mathcal{O}_Y)$  moduli. Specifically, in the case at hand

$$h^1(\Theta_Y) = 30 \text{ and } h^0(\Theta_Y) = 0;$$

- the “expected codimension” in moduli for  $Y$  to have a line bundle  $L \rightarrow Y$  is  $p_g(Y) = 2$ ;
  - the  $L$  we are interested in has  $L^2 = -4$ ,  $L \cdot K_Y = 2$ ;
  - if  $h^0(L) \neq 0$ , then  $h^0(L) = 1$  and there is a unique curve  $E \in |L|$ ;
- from

$$E^2 = -4, \quad E \cdot K_Y = 2$$

we infer that the arithmetic genus  $p_a(E) = 0$ , and from the above local picture generically  $E = \mathbb{P}^1$ ;

---

<sup>23</sup>It is due to FPR that there is another divisor in  $\mathcal{M}_{I,f}$  given by  $I$ -surfaces having a  $\frac{1}{25}(1, 14)$  singularity. We do not know if this one may be detected Hodge-theoretically. It is work in progress to develop a good formalism for dealing with the calculation of general such examples.

- for  $(Y, E)$  as above we may contract  $E$  to give an  $I$ -surface with a  $\frac{1}{4}(1, 1)$  singularity.

With the details to be given elsewhere,

- $\dim\{\text{pairs } (Y, L) \text{ as above}\} = 30 - 2 = 28$ ;
- among such pairs  $(Y, L)$  it is one condition to have  $h^0(L) \neq 0$ ;
- the space of pairs  $(Y, E)$  has dimension 27;
- since  $\dim \mathcal{M}_I = 28$ , the above explains why imposing a  $\frac{1}{4}(1, 1)$  singularity is one condition in moduli of  $I$ -surfaces.

### III. THE CASE OF INFINITE MONODROMY ( $N \neq 0$ )

In this section we will define and give examples of  $\delta\Phi$  in the situation where a semi-stable reduction gives a family  $\mathcal{X} \rightarrow \Delta$  where  $X_0 := X$  is a reduced normal crossing divisor. We will break the discussion into two parts.<sup>24</sup>

- (i)  $T \text{Def}^{es}(X)$ ,
- (ii)  $T \text{Def}^{sm}(X)$ .

**III.A. General structure and examples of  $\delta\Phi$  in the equisingular in case.** This is essentially the case of  $\delta\Phi$  for a family of mixed Hodge structures ([SZ]). The assumption that these are *limiting* mixed Hodge structures does not enter.

There is a structural result concerning the differential  $\delta\Phi$ . Following some preliminary discussion this result is stated as Proposition III.A.3 below. The weight filtration is preserved in a variation of mixed Hodge structure, and in the non-classical case the interactions among the associated graded pure Hodge structures are reflected by how the extension data in the mixed Hodge structures is constrained. Basically up to integration constants all of the extension data is determined by that of level  $\leq 2$ .

---

<sup>24</sup>We refer to the appendix for the notation and summary of deformation theory and Hodge theory for normal crossing varieties.

The space of graded polarized mixed Hodge structures with given Hodge-Deligne numbers forms a period domain  $D$  [PP]. It is a homogeneous complex manifold consisting of filtrations  $\{F^p\}$  on a complex vector space  $V_{\mathbb{C}}$ , and the tangent bundle to  $D$  has a horizontal subbundle  $H \subset TD$  defined by the condition

$$(III.A.1) \quad \dot{F}^p \subseteq F^{p-1}.$$

There is a weight filtration  $W$  on  $V$  such that for each point  $F \in D$  the filtration induced by  $F$  on each  $\text{Gr}_m^W(V_{\mathbb{C}})$  defines a pure Hodge structure of weight  $m$ . Finally there are bilinear forms that polarize the  $\text{Gr}_m^W(V)$ 's.

The period mappings we consider will give holomorphic mappings  $\Phi : \Delta \rightarrow D$  that satisfy (III.A.1); i.e.,

$$\Phi_*(T\Delta) \subseteq H,$$

and will induce variations of pure Hodge structure on  $\text{Gr}_k^W(V) := H^k$ . Moreover, the variations of mixed Hodge structure we will consider over discs  $\Delta$  will be restrictions to  $\Delta$  of global admissible ones over possibly non-complete algebraic curves  $C$  with  $\Delta \subset C$ . The reason is that  $\Phi$  will give the restriction to  $\Delta$  of a morphism  $C \rightarrow J$ , where  $J$  is an Abelian variety, and the condition that the induced map  $H_1(C) \rightarrow H_1(J)$  be a morphism of mixed Hodge structures is used in the proof of Proposition III.A.3.

The period domain  $D$  is an iterated analytic fibration with successive fibres isomorphic to the space

$$(III.A.2) \quad E_\ell := \text{Ext}_{\text{MHS}}^1(H^{k+\ell}, H^k)$$

of level  $\ell$  extensions.  $E_\ell$  is an abelian complex Lie group that is a quotient of a complex Euclidean space by a discrete subgroup. Using the notations from [GGR] it is shown there that

- (i) for  $\ell = 1$ , the Lie algebra of  $E_\ell$  has a Hodge structure whose Hodge decomposition looks like

$$(2k+1, 0) + \cdots + \underbrace{(k+1, k) \oplus (k, k+1)} + \cdots + (0, 2k+1)$$



and a holomorphic mapping of a possibly non-complete algebraic variety to  $E_k$  has differential mapping to the complexification of the largest sub-Hodge structure over the bracket;

- (ii) for  $\ell = 2$  the differential is a mapping to the complexification of the largest  $\mathbb{Q}$ -subspace, which is then a sub-Hodge structure, over the bracket in

$$(2k, 0) + \cdots + \underbrace{(k, k)} + \cdots + (0, 2k);$$

- (iii) for  $\ell \geq 3$  the differential is zero.<sup>25</sup>

In summary we have the

**PROPOSITION III.A.3:** *For an equisingular deformation of  $X$ , the differential of the period mapping to the mixed Hodge structures is determined by the differentials of the map to the associated graded pure Hodge structures and to the extension data of levels  $\leq 2$ .*

**Example III.A.4 (general):** Let  $X_1, X_2$  be smooth surfaces,  $C$  a smooth irreducible curve with embeddings

$$C \hookrightarrow C_i \subset X_i, \quad i = 1, 2,$$

and

$$(III.A.5) \quad X = X_1 \cup_C X_2$$

the normal crossing surface obtained by gluing  $X_1$  and  $X_2$  along  $C_1$  and  $C_2$ . With the further assumptions that the  $X_i$  are regular so that

$$\text{Pic}(X_i) = \text{Hg}^1(X_i)$$

and notation

$$\begin{aligned} \text{Pic}^0(X_i, C_i) &= \{L_i \in \text{Pic}(X_i) : \deg(L_i|_{C_i}) = 0\} \\ &\cong \ker\{\text{Hg}^1(X_i) \rightarrow H^2(C_i)\} \end{aligned}$$

we want to interpret  $\delta\Phi$  on  $T \text{Def}^{es}(X)$ .

---

<sup>25</sup>This does not mean that the higher level extension data is uninteresting. For instance if the  $\text{Gr}_m^W(V)$  are Hodge-Tate, then entries in the period matrix representation of the higher level extension data is given by polylogarithms and the constants of integration give functional equations for them.

Assuming that  $[C_i] \neq 0$  in  $H^2(X_i)$ , Mayer-Vietoris gives the exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(C) & \longrightarrow & H^2(X) & \longrightarrow & \text{Ker}\{H^2(X_1) \oplus H^2(X_2) \rightarrow H^2(C)\} \longrightarrow 0 \\ & & \wr \parallel & & & & \wr \parallel \\ & & \text{Gr}_1^W H^2(X) & & & & \text{Gr}_2^W H^2(X). \end{array}$$

Part of  $\delta\Phi$  detects the differential of the period mappings on  $\text{Gr}_2^W H^2(X)$  and on  $\text{Gr}_1^W H^2(X)$ . These are standard mappings, so for the purpose of illustration we shall assume that both of them are constant on  $\Delta \subset T\text{Def}^{es}(X)$ . Then what is varying is the extension data in

$$(III.A.6) \quad 0 \rightarrow H^1(C) \rightarrow H^2(X, C) \rightarrow \ker\{H^2(X) \rightarrow H^2(C)\} \rightarrow 0.$$

From this and setting we may infer the

PROPOSITION III.A.7: *This extension data is a sum of pieces in*

- (i)  $\text{Ext}_{\text{MHS}}^1(\text{Pic}^0(X_i, C_i), H^1(C_i)) \cong \text{Ext}_{\text{MHS}}^2(\oplus \mathbb{Q}(-1), H^1(C_i));$
- (ii)  $\text{Ext}_{\text{MHS}}^1(H^2(X_i)_{\text{tr}}, H^1(C_i)).$ <sup>26</sup>

For the geometric interpretation of (i), we let  $D_{i,t}$  be a family of divisors on  $X_i$  such that  $\deg(D_{i,t} \cdot C_i) = 0$ . Then  $AJ_C(D_{i,t} \cdot C_i)$  is a curve in  $J(C)$  and the corresponding part of  $\delta\Phi(d/dt)$  is the tangent to this curve.<sup>27</sup>

The term (ii) is represented by membrane integrals, which are transcendental invariants. Their derivative is however algebraic and we shall give an explicit example to illustrate how this may contain useful geometric information.

PROPOSITION III.A.8: *If  $T\text{Def}^{es}(X)$  is unobstructed, assuming generic local Torelli holds for each of  $\text{Def}^{es}(X_1, C_1)$  and  $\text{Def}^{es}(X_2, C_2)$ , it then holds for  $\text{Def}^{es}(X)$  as well.*

We will not give a formal proof of this result; it is a straightforward consequence of deformation theory, Hodge theory and the period mapping for normal crossing divisors.

<sup>26</sup> $H^2(X_i)_{\text{tr}} = \text{Pic}^0(X_i)^\perp$  is the transcendental part of  $H^2(X_i)$ .

<sup>27</sup>This is due to Carlson (cf. [CM-SP] for an exposition).

The point of this proposition will be to be able to use generic local Torelli on a boundary component in moduli to infer generic local Torelli in the interior.

A typical example we have in mind is given by the following well-known construction:

- $(X_1, C_1)$  is the desingularization of a surface  $(X_0, p)$  with a simple elliptic singularity of degree  $d = -C_1^2$ ; we assume that  $T \text{Def}(X_1, C_1)$  is unobstructed;
- $(X_2, C_2)$  is a del Pezzo surface given by realizing  $C_2$  as a cubic curve isomorphic to  $C_1$  and blown up at  $9 - d$  points  $\{q_i\}$  so as to have

$$N_{C_1/X_1} \otimes N_{C_2/X_2} \cong \mathcal{O}_C;$$

- then  $X = X_1 \cup_C X_2$  is to first order smoothable, and the limiting mixed Hodge structure  $H_{\text{lim}}^2$  locally in  $T \text{Def}^{\text{es}}(X)$  determines the pair  $(X_2, C_2)$ ; this is because  $\text{Gr}_1^W H_{\text{lim}}^2$  determines  $C_2$  and the level 1 extension data determines the pair  $(X_2, C_2)$ ;
- $\text{Gr}_2^W H_{\text{lim}}^2$  locally determines  $X_1$  and then the extension data in  $H^2(X_1, C_1)$  locally determines the pair  $(X_1, C_1)$ ;

(III.A.9)

- when we discuss  $\delta\Phi$  in smoothing directions it may be shown that if  $T \text{Def}(X)$  is unobstructed, so that the Kuranishi space of  $X$  is smooth, and that if the conditions in Proposition III.A.8 are satisfied, generic local Torelli will hold for the KSBA moduli space of a smoothing of  $X_1$ .

**Example III.A.10** (specific): Suppose that  $X_1$  is a desingularization of an  $I$ - or  $H$ -surface  $X'$ . This means that  $X'$  is a smooth, minimal, regular general type surface with  $K_{X'}^2 = 1$  ( $I$ -surface) or 2 ( $H$ -surface). Then the above conditions are satisfied ([FPR], [G1], [G2], [CT]). From this we may infer that

*generic local Torelli holds for  $I$ - and  $H$ -surfaces.*

**Example III.A.11:** We will now illustrate how  $\delta\Phi$  may be used to prove generic *global* Torelli results.<sup>28</sup> The specific example will be the mixed Hodge structure on a pair  $(X, C)$  where  $C \subset X$  is a smooth curve in a smooth surface. The idea is that  $\delta\Phi$  is an *algebraic invariant* of the pair  $(X, C)$ , and in some circumstances it contains sufficient information to be able to determine  $(X, C)$ . This example arose in the study of the boundary of moduli of  $I$ -surfaces

- $D \subset \mathbb{P}^2$  is a general plane sextic defined by an equation  $\hat{F}(x, y, z) = 0$ ;
- $X \subset \mathbb{P}(1, 1, 1, 2)$  is a 2-sheeted covering of  $\mathbb{P}^2$  branched along  $D$ ;  $X$  is given by

$$F(x, y, z, w) = w^2 - \hat{F}(x, y, z) = 0;$$

- $\ell \subset \mathbb{P}^2$  is a line, not tangent to  $D$ ; we may take  $x = 0$  to be the equation of  $\ell$ ;
- $C \subset X$  is the 2-sheeted covering of  $\ell$  branched along  $\ell \cap D$ .

Then  $X$  is a K3 surface of degree 2, i.e., with an ample line bundle  $L \rightarrow X$  with  $L^2 = 2$ , and  $C \in |L|$  is a general section. We note that  $(X, C)$  uniquely determines the pair  $(D, \ell)$ .<sup>29</sup>

PROPOSITION III.A.12: *The mapping*

$$(D, \ell) \rightarrow H^2(X, C)$$

*has degree 1.*

What will be shown is if  $(D, \ell)$  is general, then up to a projective transformation the differential of the period mapping sending  $(D, \ell)$  to the mixed Hodge structure  $H^2(X, C)$  is injective and the algebraic expression for it uniquely determines  $F, \ell$ . This is the analogous situation to Donagi's proof of generic global Torelli for most hypersurfaces in  $\mathbb{P}^{n+1}$ ,  $n \geq 2$  (cf. [DG] and [G1]).

<sup>28</sup>In [PZ] there is a global Torelli theorem for a particular class of  $I$ -surfaces. It is deduced from a global Torelli result for lattice polarized K3's.

<sup>29</sup>What this means is that given  $(X, C)$  we may uniquely determine  $(D, \ell)$  up to a projective transformation.

We set

$$W = H^0(\mathcal{O}_X(1)), \quad W^k = \text{Sym}^k W,$$

$$\overline{W} = W/(\ell).$$

The following are done by similar calculations to those in [G1] and we shall just give the basic steps. The intermediate local Torelli step in the proof is summarized in (III.A.15) below.

Using

$$H^2(X \setminus C) \cong \mathbb{H}^*(\Omega_X^\bullet(\log C))$$

we have

$$F^2 H^2(X \setminus C) \cong H^0(K_X(C)) \cong H^0(\mathcal{O}_X(1)) \cong W.$$

This fits in the standard exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{2,0}(X) & \longrightarrow & F^2 H^2(X \setminus C) & \longrightarrow & H^{1,0}(C) \longrightarrow 0 \\ & & \wr \parallel & & \wr \parallel & & \wr \parallel \\ & & \mathbb{C} & & W & & \overline{W}. \end{array}$$

A further calculation gives

$$F^1 H^2(X \setminus C) / F^2 H^2(X \setminus C) \cong W^7 / (F, F_y, F_z).$$

Denoting by  $\mathcal{J}_F$  and  $J_F$  the Jacobian ideals the extension exact sequences are

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{2,0}(X) & \longrightarrow & F^2 H^2(X \setminus C) & \longrightarrow & H^{1,0}(C)(-1) \longrightarrow 0 \\ & & \wr \parallel & & \wr \parallel & & \wr \parallel \\ 0 & \longrightarrow & \mathbb{C} & \xrightarrow{\ell} & W & \longrightarrow & \overline{W} \longrightarrow 0; \\ \\ 0 & \longrightarrow & H^{1,1}(X)_{\text{pr}} & \longrightarrow & \frac{F^1 H^2(X \setminus C)}{F^2 H^2(X \setminus C)} & \longrightarrow & H^{0,1}(C)(-1) \longrightarrow 0 \\ & & \wr \parallel & & \wr \parallel & & \wr \parallel \\ 0 & \longrightarrow & W^6 / J_F & \xrightarrow{x} & \frac{W^7}{(F, F_y, F_z)} & \longrightarrow & \frac{\overline{W}^7}{J_F} \longrightarrow 0 \end{array}$$

and

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^{1,0}(C) & \longrightarrow & \frac{F^1 H^2(X, C)}{F^2 H^2(X, C)} & \longrightarrow & H^{1,1}(X)_{\text{pr}} \longrightarrow 0 \\
& & \wr \parallel & & \wr \parallel & & \wr \parallel \\
0 & \longrightarrow & \overline{W}^4 / J_F & \longrightarrow & \frac{W^6}{(F, F_y, F_z)} & \longrightarrow & \frac{W^6}{J_F} \longrightarrow 0.
\end{array}$$

Here the map on the bottom left is

$$u \rightarrow \tilde{u} F_x$$

where  $\tilde{u}$  is a lift of  $u$  to  $W^4$ ; the map on the bottom right is projection using  $J_F \supseteq (F, F_y, F_z)$ .

Next, for the differential of the period mapping we have

$$\begin{array}{ccccccc}
0 & \longrightarrow & T\{\ell\} & \longrightarrow & T\{(D, \ell)\} & \longrightarrow & T\{D\} \longrightarrow 0 \\
& & \wr \parallel & & \wr \parallel & & \wr \parallel \\
0 & \longrightarrow & \overline{W} & \longrightarrow & W^6 / (F, F_y, F_z) & \longrightarrow & W^6 / J_F \longrightarrow 0
\end{array}$$

where again the map on the left is  $u \rightarrow \tilde{u} F_x$ . Then

$$\begin{array}{ccc}
T_{\{D, \ell\}} \otimes F^2 H^2(X \setminus C) & \longrightarrow & \frac{F^1 H^2(X \setminus C)}{F^2 H^2(X \setminus C)} \\
\wr \parallel & & \wr \parallel \\
W^6 / (F, F_y, F_z) \otimes W & \longrightarrow & W^7 / (F, F_y, F_z)
\end{array}
\tag{III.A.13}$$

is multiplication, as is (using  $W^0 = \mathbb{C}$ )

$$\begin{array}{ccc}
T_{\{D, \ell\}} \otimes F^2 H^2(X, C) & \longrightarrow & F^1 H^2(X, C) / F^2 H^2(X, C) \\
\wr \parallel & & \wr \parallel \\
W^6 / (F, F_y, F_z) \otimes \mathbb{C} & \longrightarrow & W^6 / (F, F_y, F_z).
\end{array}
\tag{III.A.14}$$

From Macaulay's theorem

$$W^6 / (F, F_y, F_z) \cong (W^7 / (F, F_y, F_z))^*,$$

and surjectivity of the multiplication of polynomials gives the

(III.A.15) **Conclusion:** *Local Torelli holds for  $H^2(X \setminus C)$  and  $H^2(X, C)$ .*

As in the proof of generic global Torelli for hypersurfaces ([DG] and [G1]), the final step is to deduce the polynomial structure on the  $W^k$

appearing in (III.A.13) and (III.A.14). Setting

$$R^k = W^k/I_k, \quad I_k \supseteq (F, F_y, F_z)_k$$

the period mapping is

$$W \otimes R^6 \xrightarrow{\delta} R^7.$$

There is a Koszul complex

$$(III.A.16) \quad R^2 \xrightarrow{\alpha} W^* \otimes R^6 \xrightarrow{\beta} \wedge^2 W^* \otimes R^7$$

where  $\beta$  can be deduced from  $\delta$  and we need that (III.A.16) is exact.

Using Macaulay's theorem in this situation, the dual of (III.A.16) is

$$\wedge^2 W \otimes R^6 \rightarrow W \otimes R^7 \rightarrow R^8.$$

Then using ([G2], [G3])

$$\wedge^2 W \otimes W^6 \rightarrow W \otimes W^7 \rightarrow W \otimes W^8$$

is exact at the middle term. Also

$$W \otimes I_7 \twoheadrightarrow I_8$$

is surjective, so that we may infer that (III.A.16) is exact and that  $\alpha$  is injective. Thus we have

$$R^5 \rightarrow W^* \otimes R^6.$$

By duality

$$R^4 \rightarrow W^* \otimes R^5 \rightarrow \wedge^2 W^* \otimes R^6$$

is exact at the middle so that we recover

$$R^4 \rightarrow W^* \otimes R^5.$$

Continuing in this way we obtain

$$R^3 \rightarrow W^* \otimes R^4.$$

But  $R^3 \cong W^3$  and  $R^4 \cong W^4$  so that we have recovered the multiplicative structure on  $R^3, R^4$ . From this we are able to recover  $(F, F_y, F_z)$  in all degrees. It remains to show that

We can recover  $F, \ell$  from  $(F, F_y, F_z)$ .

We have  $F_y, F_z$  and this gives  $F = G(x)$  for some  $G(x)$ . If

$$(F + G(x), F_y, F_z) = (F, F_y, F_z),$$

then  $G(x) \in (F_y, F_z)$ . For a general  $F$  there is no function of  $x$  above in  $(F_y, F_z)_6$ .

Having  $F$ , since  $F_x, F_y, F_z$  are linearly independent

$$W^* \rightarrow \delta^2 W$$

picks out  $(W/\ell)^*$  and hence  $\ell$ , which completes the proof.

### III.B.1. Definition and properties of $\delta\Phi$ in the smoothing case.

Given a period mapping

$$\Phi : \Delta^* \rightarrow \Gamma \backslash D$$

where  $\Gamma = \{T^k : k \in \mathbb{Z}\}$  with  $T = T_u = e^N$ ,  $N \neq 0$ , setting  $\Gamma(N) = \exp(\mathbb{C}N) \subset G_{\mathbb{C}}$  there is a set theoretic extension of this mapping to

$$(III.B.1) \quad \Phi : \Delta \rightarrow \Gamma(N) \backslash \check{D}$$

where  $\check{D}$  is the compact dual of  $D$ . The image of the origin

$$\Phi(0) := H_{\lim}^n = \{V, W(N), F_{\lim}\}$$

is an equivalence class of limiting mixed Hodge structures. There is an induced mixed Hodge structure on  $\text{End}(V)$ , and we shall define

$$(III.B.2) \quad \delta\Phi \in \frac{F^{-1} \text{End}(V)}{F^0 \text{End}(V) + \mathbb{C}N}.$$

The principal part of  $\Phi(t)$  is  $N$  and  $\delta\Phi$  will be the term of next order in the expansion of  $\Phi(t)$ .

To define  $\delta\Phi$  we denote by

$$V_{\mathbb{C}} = \oplus I^{p,q}$$

the Deligne decomposition of the mixed Hodge structure  $H_{\lim}^n$  and by

$$\mathfrak{g}_{\mathbb{C}} = \oplus \mathfrak{g}^{p,q}$$



the induced Deligne decomposition on  $\mathfrak{g} \subset \text{End}(V)$ . The Hodge and weight filtrations on  $\mathfrak{g}$  are

$$(III.B.3) \quad \begin{aligned} F^p \mathfrak{g}_{\mathbb{C}} &= \bigoplus_{\substack{p' \geq p \\ q}} \mathfrak{g}^{p',q} \\ W_m \mathfrak{g} &= \bigoplus_{p+q \leq m} \mathfrak{g}^{p,q}. \end{aligned}$$

The tangent and horizontal spaces are

$$\begin{aligned} T_F \check{D} &\cong \mathfrak{g}/F^0 \mathfrak{g} \\ &\cup \\ I_F \check{D} &\cong F^{-1} \mathfrak{g}/F^0 \mathfrak{g}. \end{aligned}$$

Setting  $\ell(t) = \log t/2\pi i$  the lift of  $\Phi$  in (II.B.1) is

$$(III.B.4) \quad \Phi(t) = \exp(\ell(t)N) \exp \xi(t) \cdot F_{\text{lim}}$$

where

$$\xi(t) \in \bigoplus_q \mathfrak{g}^{-1,q}.^{30}$$

**Definition III.B.5:**  $\delta\Phi = \xi'(0) \bmod \mathbb{C}N$ .

The interpretation of  $\delta\Phi$  is that  $\exp(\ell(t)N)F_{\text{lim}}$  is the nilpotent orbit approximating  $\Phi(t)$  and  $\delta\Phi$  measures the first order deviation from  $\Phi(t)$  actually being a nilpotent orbit.

The weight filtration on  $\mathfrak{g}$  induces one on  $\delta\Phi$ . For weight  $n$  the weights  $w$  satisfy  $-(n+1) \leq w \leq +(n-1)$ . The associated graded to the weight  $w$  part of the differential is given by maps

$$I^{p,q} \rightarrow I^{p-1,q+w+1}$$

in the Deligne decomposition of  $\mathfrak{g}$ . In the geometric case the  $I^{p,q}$  are interpreted cohomologically and this leads to a Kodaira-Spencer type interpretation of the graded pieces of the differential of the extended period mapping. An application of this will be a cohomological expression for the first order variation of the level  $k$  extension data when that data of levels less than  $k$  are held constant.

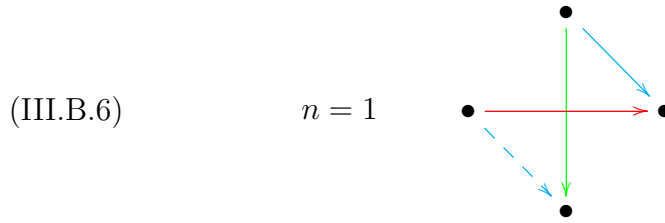
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<sup>30</sup>Here we are following the notations in [GGR].

For the cases  $n = 1, 2$  we shall give a schematic depicting this structure. We recall that for an ordinary period map to polarized Hodge structures of weight  $n$  when  $n = 1, 2$ , the differential is determined by the maps  $V^{n,0} \rightarrow V^{n-1,1}$ . Hence for the cases of curves and surfaces the differential will be determined by the maps

$$I^{n,q} \rightarrow I^{n-1,q+w+1}, \quad 0 \leq q \leq n$$

and so we shall only depict those. The first case is

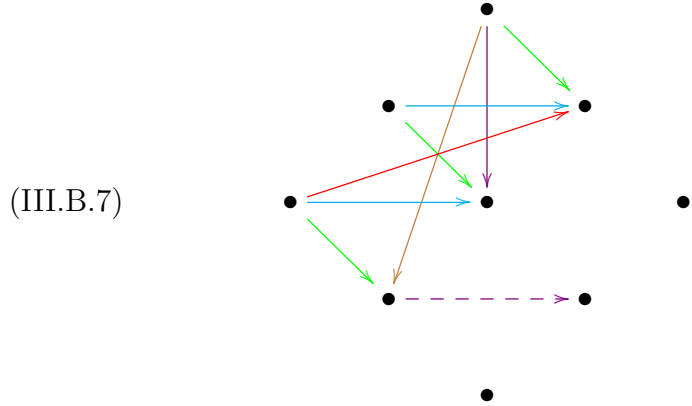


The interpretations are

- $\longrightarrow$  is weight 0 and reflects the first order variation of the associated graded to the limiting mixed Hodge structure;
- $\longrightarrow$  is weight  $-1$  and reflects the first variation in the level 1 extension data when the associated graded to the limiting mixed Hodge structures are held constant;
- $\dashrightarrow$  contains the same information, in dual form, as  $\searrow$ ;
- $\longrightarrow$  is weight  $-2$  and reflects the first variation in the level 2 extension data when the associated graded to the limiting mixed Hodge structures and the level 1 extension data are held constant. This is only well defined modulo the action of  $N$ .

We note that there are no elements of positive weight in (III.B.6) and that the lowest weight is  $-2$ . The first of these properties are special to the classical case.

We next turn to the  $n = 2$  case



The interpretations are

- $\rightarrow$  has weight  $+1$ . It is not present in the classical case, and has the Lie-theoretic interpretation that to first order the point  $\Phi(0) \in \check{D}$  moves out of the Schubert cycle in which it lies.<sup>31</sup>

The arrows  $\rightarrow$ ,  $\rightarrow$  and  $\rightarrow$  have the same interpretation as in the  $n = 1$  case. The  $\rightarrow$  has weight  $-3$  and reflects the first order variation of the level 3 extension data when to first order  $\Phi(t)$  remains in its Schubert cycle, and the associated graded to the limiting mixed Hodge structure together with the first two levels of extension data remain fixed. We have seen that under these conditions this map is zero.

In the geometric case the  $I^{p,q}$ 's are given by the algorithm represented pictorially in (B.3) in the appendix. In the geometric case the maps giving these arrows will then be expressed by multiplication by cohomology classes, and we now illustrate one of these.

**Example:** We will cohomologically interpret the arrows  $\rightarrow$ ,  $\rightarrow$  and  $\rightarrow$  in (III.B.7) in the case when

$$X = X_1 \cup_C X_2$$

---

<sup>31</sup>In the classical case the image of the lift of  $\Phi : \Delta^* \rightarrow \Gamma \backslash D$  to the universal cover  $\mathcal{H}$  of  $\Delta^*$  lies in a bounded domain in a  $\mathbb{C}^m$ , hence in a Schubert cell. This is no longer true in the non-classical case, and its failure is a significant property there. Geometric examples occur already for the smoothing of two smooth surfaces glued along a smooth double curve (cf. [GGR].) The proof of the main result in that paper is based on the fact that there is a lifting of  $\Phi$  to a Schubert cell in the neighborhood of a fibre of  $\Phi_1$ .

consists of two smooth surfaces  $X_1, X_2$  joined along a double curve  $C$ . We use  $C_i \subset X_i$  for the curve  $C$  in the surface  $X_i$  ( $i = 1, 2$ ). The BF condition is

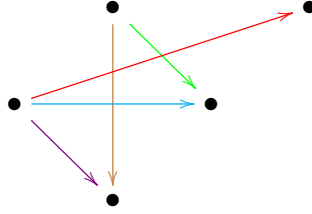
$$N_{C_1/X_1} \cong \check{N}_{C_2/X_2},$$

or equivalently

$$(III.B.8) \quad \mathcal{O}_C(C_1) \cong \mathcal{O}_C(-C_2).$$

The relevant parts of (III.B.7) are

(III.B.9)



The dots representing the  $I^{p,q}$ 's are given by the prescription in (B3). Using these a part of the red arrow will be a mapping<sup>32</sup>

$$(III.B.10) \quad H^0(\Omega_{X_1}^2) \rightarrow H^1(\mathcal{O}_{C_2})(-1)/H^1(\Omega_{X_2}^1).$$

Note the exchange between  $X_1$  and  $X_2$ ; this will be a reflection of (BF).

To describe the further maps in (III.B.9) we will use the cohomology mappings arising from the commutative diagram

$$\begin{array}{ccccccc}
 \Omega_{X_1}^2 & \rightarrow & \Omega_{X_1}^2|_{C_1} & \cong & \Omega_{C_1}^1(-C_1) & & \\
 0 & & & & & & \\
 \downarrow & & & & & & \cong \\
 \Omega_{X_2}^1 & & & & & & \\
 \downarrow & & & & & & \\
 0 \rightarrow \Omega_{X_2}^1(\log C_2) & \rightarrow & \Omega_{X_2}^1(C_2) & \rightarrow & \Omega_{C_2}^1(C_2) & \rightarrow & 0 \\
 \downarrow & & & & & & \\
 \mathcal{O}_{C_2}(-1) & & & & & & \\
 \downarrow & & & & & & \\
 0. & & & & & & 
 \end{array}$$

<sup>32</sup>This mapping was discussed in the special case when  $C \cong \mathbb{P}^1$  in II.C.3.

The horizontal isomorphism on the top is adjunction, and the vertical isomorphism on the right uses (BF). The composition of the maps on cohomology give the map

$$\begin{array}{ccc} H^0(\Omega_{X_1}^2) & \longrightarrow & \ker \{H^1(\mathcal{O}_{C_2})(-1) \rightarrow H^2(\Omega_{X_2}^1)\} \\ \cap & & \cap \\ I^{2,0} & & I^{1,2} \end{array}$$

An explicit example where this map is non-zero is given by an  $I$ -surface having a simple elliptic singularity.

We next turn to the mapping  $\longrightarrow$ . In general this mapping is defined only if the mapping  $\longrightarrow$  is zero, which will be the case if we are in the  $N = 0$  subspace of the LMHS. Thus one may think of  $X$  as giving an equi-singular deformation  $X_t = X_{1,t} \bigcup_{C_t} X_{2,t}$ . Then

$$\longrightarrow : H^0(\Omega_{X_1}^2) \rightarrow H^1(\Omega_{X_1}^1)$$

is the usual derivative of a period mapping. We note that the image of this mapping lies in

$$(C_1)^\perp \subset H^1(\Omega_{X_1}^1)$$

reflecting the assumption that  $C_1$  deforms along with  $X_1$ .

For the mapping

$$(III.B.11) \quad \searrow : I^{2,1} \rightarrow I^{1,2}$$

we first note that it is defined when both  $\longrightarrow$  and  $\longrightarrow$  are zero.<sup>33</sup> Geometrically we imagine a family

$$X_t = X_1 \bigcup_{t,C} X_2$$

where  $X_1, X_2, C$  are constant but the gluing of  $X_1$  and  $X_2$  along  $C$  varies with  $t$ . Now

$$\text{Ext}_{\text{MHS}}^1(H^1(-1), H^2) \cong \frac{H^1(\widetilde{-1}) \otimes H^2}{F^0(H^1(\widetilde{-1}) \otimes H^2) + (H^1(\widetilde{-1}) \otimes H^2)_{\mathbb{Z}}}.$$

<sup>33</sup>In a somewhat different form this description has been given above.

This is a compact complex torus having a summand that is an abelian variety  $J$  with tangent space

$$TJ \cong \text{Hom}(I^{0,1}, I_{\mathbb{Z}}^{2,2} \otimes \mathbb{C}).$$

Using the duality  $\check{I}^{0,1} \cong I^{2,1}$  we shall give the geometric interpretation of (III.B.11) under the simplifying assumption that the  $H^1(X_i) = 0$  for  $i = 1, 2$ . Then  $J = J(C)$  is the Jacobian variety of  $C$ . We set

$$(\text{Pic}X_1 \oplus \text{Pic}X_2)^0 = (C_1 \oplus C_2)^\perp.$$

For the family of embeddings  $j_t : C \hookrightarrow X_1 \times X_2$  there is a mapping

$$\alpha_t : (\text{Pic}X_1 \oplus \text{Pic}X_2)^0 \rightarrow J(C)$$

and unwinding the definitions the mapping  $\longrightarrow$  may be identified with the derivative of  $\alpha_t$ . In words

Fixing  $X_1, X_2, C$  and mapping the gluing of  $X_1, X_2$  along a family of different embeddings of  $C$  in these surfaces, a part of the variation in the first order extension data is measured by the variation when the  $\text{Pic}(X_i)$  map to  $J(C)$ .

To interpret the arrow  $\searrow$  we note that because we have a limiting mixed Hodge structure there is a duality between  $\text{Ext}_{\text{MHS}}^1(H^1(-1), H^2)$  and  $\text{Ext}_{\text{MHS}}^1(H^2, H^1)$ . Thus this arrow contains no new information beyond  $\searrow$ .

That leaves the interpretation of  $\downarrow$ . Here we just note that in [GGR] it is shown that the level 1 extension data gives a cone  $\sigma$  of line bundles over a compact complex torus  $T$  and that the fibres of the map to the associated graded maps to a sub-torus  $J$  of  $T$  over which the line bundles  $L \in \sigma$  are ample. The level 2 extension data then maps the fibres of  $\Phi_1$  to nowhere vanishing sections of these line bundles.

**Note:** In the geometric case of a family  $\mathcal{X} \xrightarrow{\pi} \Delta$  a natural option for the derivative of the period mapping at the origin is given by using

the canonically extended Gauss-Manin connection  $\nabla$ . As we will now explain,  $\delta$  corresponds to  $d/dt$  while  $\nabla$  corresponds to  $t d/dt$ .<sup>34</sup>

Given the standard geometric situation

$$\mathcal{X} \xrightarrow{\pi} \Delta$$

of a family  $X_t$  with  $X_t$  smooth for  $t \neq 0$  and  $X_0$  a reduced normal crossing divisor, for  $\mathcal{V}_e$  the canonical Deligne extension of the cohomology bundle of the smooth fibres there is the map

$$T_\Delta(-\log\{0\}) \rightarrow F^{-1} \text{End}(\mathcal{V}_e)$$

induced by  $\nabla$ . When written out in coordinates this corresponds to  $t d/dt$ .

The difference between the two may be illustrated by the period matrix of a family of genus 2 curves acquiring a node at  $t = 0$ . The picture is



and the normalized period matrix has the well-known form (cf. [GGR] for the notation)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & \lambda \\ b & a \end{pmatrix}$$

where  $a, \lambda$  are holomorphic with  $\text{Im } \lambda > 0$  and  $b = \ell(t) + b_0$  where  $b_0$  is holomorphic. Then

$$\delta\Phi = \begin{pmatrix} a' & \lambda' \\ 0 & a' \end{pmatrix}^{35}$$

while

$$t d/dt = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

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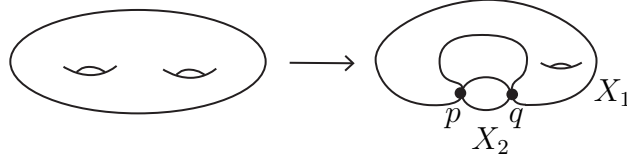
<sup>34</sup>More precisely, if we twist  $\Phi(t)$  by composing with  $\exp(-\ell(t)N)$ , then  $\delta\Phi$  corresponds to  $d/dt|_{t=0}$  of the twist.

<sup>35</sup>Recall the  $\delta\Phi$  is  $\Phi_*$  modulo  $\mathbb{C}N$ .

**Example III.B.13:** We will show that the example (III.B.12) is *not* a nilpotent orbit. This is interesting not so much in its statement, which is certainly what one expects, but in the method of proof. The result is pretty obvious, but how can one prove it?

**PROPOSITION III.B.14:** *The degeneration (III.B.12) is not a nilpotent orbit.*

*Proof.* To obtain for  $X_0$  a global normal crossing divisor we must blow up the node to have



where  $p, q$  are ordered. The relevant sheaf sequences are first

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K_{X_1}(-p-q) & \longrightarrow & K_{X_1} & \longrightarrow & K_{X_1,p} \oplus K_{X_1,q} \longrightarrow 0 \\
 & & & & & & \cong \\
 & & & & & & N_{p/X_1}^* \oplus N_{q/X_1}^* \\
 & & & & & & \cong \\
 & & & & & & N_{p,X_2} \oplus N_{q,X_2} \\
 & & & & & & \cong \\
 & & & & & & \Theta_{X_2,p} \oplus \Theta_{X_2,q}
 \end{array}$$

where the second isomorphism on the right uses the BF condition, and secondly

$$0 \rightarrow \Theta_{X_2}(p-q) \rightarrow \Theta_{X_2} \rightarrow \Theta_{X_2,p} \oplus \Theta_{X_2,q} \rightarrow 0.$$

The cohomology sequences lace together to give

$$\begin{array}{ccccc}
 H^0(K_{X_1}) & \longrightarrow & K_{X_1,p} \oplus K_{X_1,q} & & \\
 \searrow \alpha & & \downarrow & & \\
 H^0(\Theta_{X_1}) & \longrightarrow & \Theta_{X_2,p} \oplus \Theta_{X_2,q} & \longrightarrow & H^1(\Theta_{X_2}(-p-q)).
 \end{array}$$



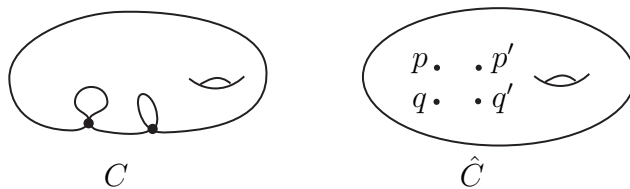
The map  $\alpha$  is non-zero provided that there is  $\omega \in H^0(K_{X_1})$  with  $\omega(p), \omega(q) \neq 0$  and  $H^1(\Theta_{X_2}(-p-q)) = 0$ . The latter happens when  $X_2 = \mathbb{P}^1$ , which is the case above.

For the former since in semi-stable reduction the points  $p, q$  are ordered,

$$AJ_{X_1}(p - q) \subset J(X_1)$$

is well defined. If it is non-zero, which may be assumed in the example at hand, then the interpretation of the blue arrow (representing extension data) in (III.B.6) gives that  $\alpha \neq 0$ . □

**Example III.B.15:** A related example that illustrates the weight filtration on  $\delta\Phi$  is given by semi-stable reduction applied to the family of curves



where the order pairs of points  $p, p'$  and  $q, q'$  on  $\hat{C}$  are identified to give the two nodes on  $C$ . Then

$$\delta\Phi \text{ has } Gr_0, Gr_{-1}, Gr_{-2}$$

pieces.

$Gr_0(\delta\Phi)$ : This is the differential of the period mapping  $\hat{\Phi}$  for the variable curve  $\hat{C}$ .

$Gr_{-1}(\delta\Phi)$ : If  $\hat{\Phi}$  is constant, then  $Gr_{-1}(\delta\Phi)$  is given by the pair  $\{AJ_{\hat{C}}(p-p'), AJ_{\hat{C}}(q-q')\}$ .

$Gr_{-2}(\delta\Phi)$ : Suppose first that  $\hat{C} = \mathbb{P}^1$ . Then the points  $\{p, p', q, q'\}$  have a cross-ratio, and as in [GGR]  $\delta\Phi$  is given by its variation.

For  $\hat{C}$  of any genus one may define a generalized cross-ratation using normalized differentials of the third kind.

APPENDIX: SUMMARY OF NOTATIONS  
AND OF PROPERTIES OF NORMAL CROSSING VARIETIES

**Notations:** In general we shall follow those in [F1].

- $X = \cup X_i$ ;
- locally  $X$  is isomorphic to  $x_1 \cdots x_k = 0$  in  $\mathbb{C}^{n+1}$ ;
- $\tilde{X} = \coprod_i X_i$  is the desingularization of  $X$ ;
- $X^{[k]} = \coprod_{|I|=k} X_{i_1} \cap \cdots \cap X_{i_k}$ ,  $I = (i_1, \dots, i_k)$ ;
- $a_k : X^{[k]} \hookrightarrow X$ ,  $a_1 = a$ ;
- $D = \cup_{i < j} X_i \cap X_j = X_{\text{sing}}$ ;
- $\Omega_X^1 =$  sheaf of Kähler differentials with

$$0 \rightarrow \tau_X \rightarrow \Omega_X^1 \rightarrow a_* \Omega_{X^{[1]}}^1$$

where  $\tau_X$  is the torsion subsheaf of  $\Omega_X^1$ ;

- $\tau_X$  is locally generated by  $\varphi_i = x_1 \cdots \hat{x}_i \cdots x_k dx_i$ ;
- $\mathcal{O}_D([X]) = \text{Ext}_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) \cong \tau_X$  is the infinitesimal normal bundle;
- if  $X$  is embedded in a smooth  $(n+1)$ -dimensional variety  $Y$ , then

$$\mathcal{O}_D([X]) = \mathcal{O}_X([X])|_D = \mathcal{O}_Y([X])|_D$$

where the middle term is the usual normal bundle of  $X$  in  $Y$ ;

- the *BF condition* is

$$(BF) \quad \mathcal{O}_D([X]) \cong \mathcal{O}_D.$$

$$\text{-----} \langle \circ \rangle \text{-----}$$

**A. Deformation theory** [F1]. We shall make the usual identification

$$T \text{Def}(X) = \mathbb{E}xt_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X).$$

The local to global spectral sequence of Ext gives

$$(A.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^1(\mathcal{E}xt_{\mathcal{O}_X}^0(\Omega_X^1, \mathcal{O}_X)) & \longrightarrow & \mathbb{E}xt_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) & \longrightarrow & H^0(\mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X)) \longrightarrow \\ & & \parallel & & \parallel & & \parallel \\ & & T \text{Def}^{es}(X) & & T \text{Def}(X) & & H^0(\mathcal{O}_D([X])) \end{array}$$

The first term is the first order equisingular deformations of  $X$ .

For the third term a *smoothing* of  $X$  is given by a proper holomorphic fibration  $\mathcal{X} \xrightarrow{\pi} \Delta$  where  $\mathcal{X}$  is smooth and, setting  $X_t = \pi^{-1}(t)$ ,  $X_0 \cong X$ . A *first order smoothing* is given by similar data  $\mathcal{X}_\epsilon \rightarrow \Delta_\epsilon$  where  $\Delta_\epsilon = \text{Spec} \mathbb{C}[\epsilon]$  with  $\epsilon^2 = 0$ . The necessary and sufficient condition that there be a first order smoothing is that (BF) hold. Assuming this we denote by

$$T \text{Def}^{sm}(X) \subset T \text{Def}(X)$$

the open set of first order smoothings of  $X$  given by the  $\xi \in \mathbb{E}xt_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X)$  that map to a global section of  $\mathcal{O}_D$  that is non-vanishing on every connected component of  $D$ . Given  $\xi \in T \text{Def}^{sm}(X)$  we denote by

$$\mathcal{X}_\xi \rightarrow \Delta_\epsilon$$

the corresponding first order smoothing.

In the use of the period mapping to moduli two types of deformations of normal crossing varieties are particularly relevant:

- (i) equisingular deformations,
- (ii) smoothing deformations.

In (i) it is natural to assume the BF condition; and the deformations should be constrained to those where the BF condition remains satisfied.

We define a sheaf  $\Theta_X(-\log X)$  of  $\mathcal{O}_X$ -modules by prescribing local generators

$$\begin{cases} x_i \partial / \partial x_i & 2 \leq i \leq k \\ \partial / \partial x_j & k+1 \leq j \leq n+1. \end{cases}$$

Then

$$(A.2) \quad \text{Ext}_{\mathcal{O}_X}^0(\Omega_X^1, \mathcal{O}_X) \cong \Theta_X(-\log X)$$

and the first order equisingular deformations of  $X$  are given by

$$(A.3) \quad T \text{Def}^{es}(X) \cong H^1(\Theta_X(-\log X)).$$

We note that if we have  $X \subset Y$  where  $Y$  is a smooth  $(n+1)$ -fold, then  $\Theta_Y(-\log X)$  is the standard sheaf dual to  $\Omega_Y^1(\log X)$ . As with  $\mathcal{O}_D([X])$  we may define  $\Theta_X(-\log X)$  without knowing  $Y$ .

Finally, we shall collect some standard facts [St] about the local deformation theory of normal crossing varieties that have been used in the computation of examples. Here  $\{0\} \in X \subset \mathbb{C}^N$  is a germ of an analytic variety with ideal sheaf  $I_X \subset \mathcal{O}_{X^0}$ .<sup>36</sup> The normal sheaf is

$$\mathcal{N}_{X/\mathbb{C}^N} = \text{Hom}(I_X/I_X^2, \mathcal{O}_X).$$

There is a natural mapping  $\Theta_{\mathbb{C}^N}|_X \rightarrow \mathcal{N}_{X/\mathbb{C}^N}$ , and the space of isomorphism classes of deformations of the germ  $X \subset \mathbb{C}^N$  is

$$T_X^1 = \text{coker} \{ \Theta_{\mathbb{C}^N} \rightarrow \mathcal{N}_X \} \cong \text{Ext}_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X).$$

We shall illustrate this with the cases of a normal crossing divisor  $X = \bigcup_{i=1}^k X_i$  when  $k = 2, 3$ . From these cases the general pattern should be clear.

$k = 2$ :  $X$  is given by  $xy = 0$  and from

$$(A.4) \quad \frac{(xy)}{(x^2y^2)} = a + \sum_{i>0} b_i x^i + \sum_{j>0} c_j y^j$$

we have

$$(A.5) \quad 0 \rightarrow N_{X_1/\mathbb{C}^N} \otimes N_{X_2/\mathbb{C}^N} \rightarrow \mathcal{N}_{X/\mathbb{C}^N} \rightarrow N_{X_{12}/X_1} \oplus N_{X_{12}/X_2} \rightarrow 0.$$

$k = 3$ :  $X$  is given by  $xyz = 0$ , and expanding  $(xyz)/(x^2y^2z^2)$  as in (A.4) above gives a filtration of  $\mathcal{N}_{X/\mathbb{C}^N}$  whose graded pieces with their interpretations are

$$(A.6) \quad \underbrace{\oplus N_{X_i/\mathbb{C}^N}}_{\text{equisingular}}, \underbrace{\oplus N_{X_{ij}/X_i} \otimes N_{X_{ij}/X_j}}_{\text{smooths some double curves}}, \underbrace{N_{X_{123}/X_1} \otimes N_{X_{123}/X_2} \otimes N_{X_{123}/X_3}}_{\text{smooths the triple point locus}}.$$

The map  $\Theta_{\mathbb{C}^N}|_X \rightarrow \mathcal{N}_{X/\mathbb{C}^N}$  surjects onto the first piece in (A.6) and misses the other pieces. Thus we have

$$k = 2 \quad T_X^1 \cong \bigotimes_{i=1}^2 N_{X_{12}/X_i},$$

$$k = 3 \quad 0 \rightarrow \bigoplus_{i<j} N_{X_{ij}/X_i} \oplus N_{X_{ij}/X_j} \rightarrow T_X^1 \rightarrow \bigotimes_{i=1}^3 N_{X_{123}/X_i} \rightarrow 0.$$

Finally we observe that

<sup>36</sup>As we are considering germs of analytic varieties all sheaves are to be understood as the corresponding stalk at the origin.

(A.7) Denoting by  $D = X_{\text{sing}} = \bigcup_{i < j} X_{ij}$  the double locus of  $X$ , we have

$$T_X^1 \cong \mathcal{O}_D([X]).$$

In particular the BF condition is equivalent to

$$T_X^1 \cong \mathcal{O}_D.$$

One may determine the subspace of  $T_X^1$  corresponding to first order deformations that preserve this condition. Since we will not use this, the derivation will not be given here.

**B. Hodge theory.** We shall give cohomological expressions for the differential of the period mapping associated to equisingular and to first order smoothing deformations of  $X$ . More precisely, in the first case the weight filtration is preserved and we shall give a cohomological expression for the differential of the mapping to the associated graded. From this one may work out an expression for the differential of the map to the level 1 extension data along the fibres of the map to the associated graded.

In the smoothing case the limiting mixed Hodge structure  $H_{\text{lim}}^n(X)$ , of which by Clemens-Schmid the image of  $H^n(X)$  in  $H_{\text{lim}}^n$  is the  $N = 0$  part, jumps to a pure Hodge structure and the weight filtration induces a filtration on what one might call the first order smoothing of the limiting mixed Hodge structure to a pure Hodge structure. We will give cohomological expressions for the mixed Hodge structure on  $H^n(X)$  and for  $H_{\text{lim}}^n(X)$ .

For the first case we have on  $X$  a double complex of sheaves

$$\Omega_{X^{[1]}}^* = \bigoplus_{k,p} (a_k)_* \Omega_{X^{[k]}}^p$$

and the formula is

$$(B.1) \quad H^*(X) \cong \mathbb{H}^*(\Omega_{X^{[1]}}^*).$$

In the double complex the differentials are the usual  $d$  and signed restriction. The Hodge and weight filtration on cohomology are induced

by

$$F^p \Omega_{X^{[0]}}^\bullet = \bigoplus_{\substack{p' \geq p \\ q}} \Omega_{X^{[0]}}^{p',q}$$

$$W_k \Omega_{X^{[1]}}^\bullet = \bigoplus_{k' \geq k} (a_{k'})_* \Omega_{X^{[k']}}^\bullet$$

(see [F1] for details).

For the limiting mixed Hodge structure associated to  $\xi \in T \text{Def}^{sm}(X)$   $\mathcal{X}_\xi \rightarrow \Delta_\epsilon$  one uses the complex

$$\Omega_{X,\text{lim}}^\bullet := \Omega_{\mathcal{X}_\xi/\Delta_\epsilon}^\bullet(\log X) \otimes \mathcal{O}_X.$$

Then the limiting mixed Hodge structure is

$$(B.2) \quad H_{\text{lim}}^*(X, \xi) = \mathbb{H}(\Omega_{X,\text{lim}}^\bullet).$$

Abusing notation we shall simply write the left-hand side as  $H_{\text{lim}}^*$ .

The Hodge filtration on  $H_{\text{lim}}^*$  is induced by the usual bête filtration on  $\Omega^\bullet$ . The weight filtration is more subtle. If we are considering  $H_{\text{lim}}^n$ , the weight filtration is  $W_0 \subset W_1 \subset \cdots \subset W_{2n}$ . Thus  $\text{Gr}^W H_{\text{lim}}^n$  has  $2n + 1$  pieces. We shall describe a set of  $2n + 1$  complexes whose cohomology gives the Hodge structure on the corresponding term in  $\text{Gr}^W H_{\text{lim}}^n$ .

To do this we shall give a useful computational device that is for computing the limiting mixed Hodge structure in examples. Given a normal crossing variety  $X = \cup X_i$  one may define a precomplex whose terms are the groups  $H^a(X^{[b]})(-c)$ ,  $0 \leq c \leq b-1$  and whose differentials are composed of signed restriction and Gysin mappings. Then the condition that this precomplex be a complex, i.e., that  $d^2 = 0$ , is that the topological consequence  $c_1(\mathcal{O}_D([X]) = 0$  of the BF condition hold.<sup>37</sup> If this is the case, then the cohomology of the complex gives the associated graded to the limiting mixed Hodge structure. Moreover the  $N$  is induced by the identity map, tensored with a Tate twist, wherever the resulting terms in the cohomology of the complex are non-zero.

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<sup>37</sup>Computing the extension data in the LMHS requires use of the full BF condition  $\mathcal{O}_D([X]) \cong \mathcal{O}_D$ .

We shall give the general schematic in the case  $n = 3$ :

$$\begin{array}{l}
 \text{Gr}_0^W \quad \underbrace{H^0(X^{[1]})}_{\text{red}} \rightarrow \underbrace{H^0(X^{[2]})}_{\text{purple}} \rightarrow \underbrace{H^0(X^{[3]})}_{\text{blue}} \\
 \\
 \text{Gr}_1^W \quad \underbrace{H^1(X^{[1]})}_{\text{purple}} \rightarrow \underbrace{H^1(X^{[2]})}_{\text{blue}} \\
 \\
 \text{(B.3) Gr}_2^W \quad \underbrace{H^0(X^{[2]})(-1)}_{\text{purple}} \rightarrow \underbrace{H^2(X^{[1]}) \oplus H^0(X^{[3]})(-1)}_{\text{blue}} \rightarrow \underbrace{H^0(X^{[2]})}_{\text{green}} \\
 \\
 \text{Gr}_3^W \quad \underbrace{H^1(X^{[2]})(-1)}_{\text{blue}} \rightarrow \underbrace{H^3(X^{[1]})}_{\text{green}} \\
 \\
 \text{Gr}_4^W \quad \underbrace{H^0(X^{[3]})(-2)}_{\text{blue}} \rightarrow \underbrace{H^2(X^{[2]})(-1)}_{\text{green}} \rightarrow \underbrace{H^4(X^{[1]})}_{\text{orange}} \\
 \\
 \underbrace{\hspace{1.5cm}}_{\text{red}} = H_{\text{lim}}^0 \quad \underbrace{\hspace{1.5cm}}_{\text{green}} = H_{\text{lim}}^3 \\
 \underbrace{\hspace{1.5cm}}_{\text{purple}} = H_{\text{lim}}^1 \quad \underbrace{\hspace{1.5cm}}_{\text{orange}} = H_{\text{lim}}^4 \\
 \underbrace{\hspace{1.5cm}}_{\text{blue}} = H_{\text{lim}}^2
 \end{array}$$

The  $N$  maps are given by all arrows  $H^a(X^{[b]})(-c-1) \rightarrow H^a(X^{[b]})(-c)$  that can be drawn and that are between non-zero groups. The rules are

- the horizontal rows form complexes where the maps are either “ $R =$  signed restriction” or “ $G =$  Gysin,” whichever makes sense at a particular spot.

**Example:** The top row is all  $R$ ’s and the bottom row is all  $G$ ’s.

**Example:**

$$\text{(B.4) } \begin{array}{ccccc}
 & & H^2(X^{[1]}) & & \\
 & \nearrow G & & \searrow R & \\
 H^0(X^{[2]})(-1) & & \oplus & & H^2(X^{[2]}) \\
 & \searrow R & & \nearrow G & \\
 & & H^0(X^{[3]})(-1) & & 
 \end{array}$$

where

$$H^0(X^{[2]})(-1) \ni \alpha \rightarrow \begin{pmatrix} G\alpha \\ R\alpha \end{pmatrix}$$

and

$$\begin{array}{c} H^2(X^{[1]}) \\ \oplus \\ H^0(X^{[3]})(-1) \end{array} \ni \begin{pmatrix} \beta' \\ \beta'' \end{pmatrix} \rightarrow R\beta' + G\beta''.$$

The cohomology of these complexes at the appropriate color gives the associated gradeds of the corresponding  $H_{\text{lim}}^q$ 's.

**Example:**

$$\text{Gr}_0^W H_{\text{lim}}^0 = \ker\{H^0(X^{[2]}) \rightarrow H^0(X^{[2]})\},$$

$$\text{Gr}_0^W H_{\text{lim}}^1 = \ker\{H^0(X^{[2]}) \rightarrow H^0(X^{[3]})\}/\text{im}\{H^0(X^{[2]}) \rightarrow H^0(X^{[2]})\},$$

$$\text{Gr}_0^W H_{\text{lim}}^2 = H^0(X^{[3]})/\text{im}\{H^0(X^{[2]}) \rightarrow H^0(X^{[3]})\}$$

$$\text{Gr}_2^W H_{\text{lim}}^2 = \{\text{cohomology at the middle spot of (B.4)}\}.$$

- The map  $N$  is given by the process described above where all non-zero maps  $H^a(X^{[b]})(-c-1) \rightarrow H^a(X^{[b]})(-c)$  are used.

As a final note we remark that when  $X = X_1 \cup X_2 \cup X_3$  the condition that (B.4) be a complex reduces to the triple point formula.

## REFERENCES

- [A] V. I. Arnold, *Singularity Theory I*, 1998, Springer-Verlag.
- [C] F. M. E. Catanese, Infinitesimal Torelli theorems and counterexamples to Torelli problems, in *Topics in Transcendental Algebraic Geometry* (Princeton, NJ, 1981/1982), 143–156, *Ann. of Math. Stud.* **106**, Princeton Univ. Press, Princeton, NJ, 1984.
- [CKS] E. Cattani, A. Kaplan and W. Schmid, Degenerations of Hodge structures, *Ann. of Math.* **123** (1986), 457–535.
- [CM-SP] J. Carlson, S. Müller-Stach, and C. Peters, *Period Mappings and Period Domains*, Second edition, *Cambridge Stud. Adv. Math.* **168**, Cambridge Univ. Press, Cambridge, 2017.
- [CGGH] J. Carlson, M. Green, P. Griffiths, and J. Harris, Infinitesimal variations of Hodge structure. I. *Compositio Math.* **50(2-3)** (1983), 109–205.
- [CT] J. Carlson and D. Toledo, unpublished work.
- [Ch] K. Chakiris, The Torelli problem for elliptic pencils, in *Topics in Transcendental Algebraic Geometry* (Princeton, NJ, 1981/1982), 157–181, *Ann. of Math. Stud.* **106**, Princeton Univ. Press, Princeton, NJ, 1984.
- [CPT] E. Colombo, G. P. Priola, and A. Tortora, Hodge-Gaussian maps, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **30**, no. 1 (2001), 125–146.
- [DG] R. Donagi and M. L. Green, A new proof of the symmetrizer lemma and a stronger weak Torelli theorem for projective hypersurfaces, *J. Differential Geom.* **20** (1984), no. 2, 459–461.



- [F1] R. Friedman, Global smoothings of varieties with normal crossings, *Ann. of Math.* **118** (1983), 75–114.
- [F2] R. Friedman, The period map at the boundary of moduli, in *Topics in Transcendental Algebraic Geometry* (Princeton, NJ, 1981/1982), 183–208, *Ann. of Math. Stud.* **106**, Princeton Univ. Press, Princeton, NJ, 1984.
- [FP] P. Frediani and G. P. Pirol, On the geometry of the second fundamental form of the Torelli map, 2020. arXiv:1907.11407
- [FPR] M. Franciosi, R. Pardini, and S. Rollenske, Gorenstein stable surfaces with  $K_X^2 = 1$  and  $p_g > 0$ , *Math. Nachr.* **290**(5-6) (2017), 794–814. arXiv:1511.03238
- [Gh] A. Ghigi, On some differential-geometric aspects of the Torelli map, *Boll. Unione Math. Italiana* **12** (2019), 133–144. <https://doi.org/10.1007/s40574-018-0171-3>
- [G1] M. L. Green, Infinitesimal methods in Hodge theory, in *Algebraic Cycles and Hodge Theory* (Torino, 1993), pp. 1–91, (A. Albano and F. Bardelli, eds.), Springer-Verlag.
- [G2] M. L. Green, Koszul cohomology and the geometry of projective varieties, *J. Differential Geom.* **19** (1984), no. 1, 125–171.
- [G3] M. L. Green, Koszul cohomology and the geometry of projective varieties II., *J. Differential Geom.* **19** (1984), no. 1, 279–289.
- [GGLR] M. Green, P. Griffiths, R. Laza and C. Robles, Period mappings and ampleness of the Hodge line bundle, 2020. arXiv:1708.09523v3
- [GGR] M. Green, P. Griffiths, and C. Robles, Completions of period mappings: Progress Report, 2021. arXiv:2106.04691.
- [G1] P. Griffiths, Some geometric applications of Hodge theory (talks in Chile and Montreal), 2020. <https://hdl.handle.net/20.500.12111/7923>
- [G2] P. Griffiths, Hodge Theory and moduli, 2020. <https://hdl.handle.net/20.500.12111/7885>
- [H] P. Hacking, Compact moduli spaces of surfaces and exceptional vector bundles, in *Compactifying Moduli Spaces CRM*, Birkhäuser.
- [K] J. Kollár, Moduli of varieties of general type, in *Handbook of Moduli. Vol. II, Adv. Lectures in Math.* **25** (Internat. Press, Somerville, MA, 2013), 131–157. <http://www.ams.org/mathscinet-getitem?mr=1MR3184176>  
<http://www.zentralblatt-math.org/zmath/en/search/?q=an:1Zbl1322.14006>
- [KS-B] J. Kollár and N. Shepherd-Barron, Threefolds and deformations of surface singularities, *Inv. Math.* **91** (1988), 299–338.
- [La] J. M. Landsberg, On second fundamental forms of projective varieties, *Invent. Math.* **117** (1994), no. 2, 303–315.
- [OS] F. Oort and J. Steenbrink, The local Torelli problem for algebraic curves, *J. Geom. Alg. d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pp. 157–204, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, MD, 1980.
- [PP] G. Pearlstein and C. Peters, Differential geometry of the mixed Hodge metric, arXiv:1407.4082v2 [math.AG] 12Apr20017

- [PZ] G. Pearlstein and Z. Zhang, A generic global Torelli theorem for certain Horikawa surfaces, *Algebr. Geom.* **6** (2019), no. 2, 132–147.
- [S-B1] N. Shepherd-Barron, Generic Torelli and local Schottky theorems for Jacobian elliptic surfaces, 2021. arXiv:2009.03633
- [S-B2] N. Shepherd-Barron, Asymptotic period relations for Jacobian elliptic surfaces, *Proc. L.M.S.*, to appear. arXiv:1907.11407
- [SZ] J. Steenbrink and S. Zucker, Variation of Hodge structure I. *Invent. Math.* **80** (1985), 489–542.
- [St] J. Stevens, *Deformations of Singularities, Lecture Notes in Math.* **1811**, 2003.