POSITIVITY OF VECTOR BUNDLES AND HODGE THEORY

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I. Introduction and notations and terminology

I.A. **Introduction.** From S. S. Chern we learned the importance of curvature in geometry and its special features in the complex case. In this case there are significant geometric and analytic consequences of the curvature having a sign. Both positive and negative curvature have major implications in algebraic geometry and in holomorphic mappings between complex manifolds.

The vector bundles (Hodge bundles) and complex manifolds (period domains) that arise in Hodge theory have natural metrics and subsequent curvatures that through the work of very many people over an extended period of time have played a central role in the study of Hodge theory as a subject in its own right and in the applications of Hodge theory to algebraic geometry. Of particular importance are

(i) the sign properties of the curvature (positivity of the Hodge bundles and cotangent bundles of period domains);
(ii) the result that in the geometric case the non-degeneracy of curvature forms is an algebro-geometric property;
(iii) the singularity properties of the curvature, especially that of the Chern forms.

Regarding (iii) we note that the essential geometric fact that enables one to control the singularities is a curvature property of the bundles that arise in Hodge theory.

The primary purpose of this mainly expository paper is to present some (but not by any means all) of the fundamental concepts and to discuss a few of the basic results in this very active and now vast area of research.

A secondary purpose is to isolate a special feature, *norm positivity*, that is present for the vector bundles that arise in Hodge theory. To set a context, curvature Θ is defined using second derivatives and therefore is in general a second order invariant. However in certain circumstances it may be a first order invariant. An example is the curvature form Θ_H of the standard line bundle $H := \mathcal{O}_X(1)$ for $X \subset \mathbb{P}^N$ a smooth projective variety and where $\mathcal{O}_{\mathbb{P}^N}(1)$ has the standard metric. Then Θ_H is the metric induced on X by the Fubinin-Study metric on $\mathbb{P}^N$.

For the vector bundles $E \to X$ that arise in Hodge theory the curvature is also a first order invariant of the form

$$\Theta_E = (A, A) \tag{I.A.1}$$

This paper is partly based on joint work in progress with Radu Laza and Colleeen Robles (cf. [GGLR20]).

where $A \in \Omega^1_X(E)$ is a holomorphic matrix of $(1,0)$-forms and $(\ ,\ )$ is a C^∞ bilinear form constructed from the Hodge metrics. Then using notations to be explained below

$$\begin{aligned}&\Theta_E \geqq 0, \quad \text{and}\\ &\Theta_E = 0 \iff A = 0, \text{ which is a complex analytic condition.}\end{aligned}$$

The two central topics of this paper are positivity and singularities. These are discussed in Sections II, III respectively. For the first topic we have chosen to focus on *metric positivity* and *numerical positivity*. The first of these involves the positivity properties of the curvature form and resulting Chern forms, which express de Rham cohomology representatives of the Chern classes as polynomials in the entries of the curvature matrix. The second of these are topological expressions of positivity of certain polynomials in the Chern classes. In Hodge theory both types of positivity are present and ultimately reflect the norm positivity property of the curvature of the Hodge bundles. Thus for example, in the Hodge theoretic case if the positivity condition under consideration is expressed by $P \geqq 0$, then $P = 0$ imposes pointwise first order complex analytic conditions on the variation of Hodge structure. In the geometric case these become cohomologically expressed algebro-geometric conditions.

One reason for selecting numerical positivity is our feeling that in the geometric case the algebro-geometric implications of this property have only begun to be used (e.g., in the proof of the Iitaka conjecture). Even in the classical case of curves, abelian varieties, K3's, . . . the consequences of numerical positivity for moduli spaces seem yet to be more fully explored. And, as noted above, numerical positivity is a natural consequence of the norm positivity property of the curvature.

The main emphasis in Section II will, however, be on *metric positivity*, that being the various measures of positivity in holomorphic vector bundles, especially Hodge bundles and cotangent bundles, that arise from sign properties of expressions derived from the curvature. Most of the results we shall give are either standard ones or slight refinements of such.

One of the questions that we discuss is motivated by the observation that bundles that are semi positive but not positive, meaning that their curvature form $\Theta_E \geqq 0$ but not $\Theta_E > 0$, occur naturally. Examples include the universal quotient bundle of rank $\geqq 2$ over the Grassmannian and the Hodge bundle F^n when $h^{n,0} \geqq 2$. Moreover the "flatness" present in these bundles is frequently of geometric interest.

Aside from line bundles those that are positive, which implies ampleness, are much less common. For semi-positive bundles one would, however, like natural conditions that imply the existence of sections of symmetric powers $S^m E$. In Section II.C we give the following result along these lines. This result applies to the two bundles mentioned above, where we assume a local Torelli condition for the second:

> *If $\Theta_E \geqq 0$ and $\Theta_{\det E} > 0$, then $S^m E \to X$ is big for $m \geqq r$ where r is the rank of E.*

As mentioned above we are particularly interested in bundles that have the norm positivity property. As in (I.A.1) the curvature of such bundles is given by the norms of holomorphic bundle mappings

$$A : T \otimes E \to G. \tag{I.A.2}$$

For these maps we consider the conditions

(a) $T \to \operatorname{Hom}(E, G)$ is injective;

(b) for general $e \in E$ the mapping $A(e) : T \to G$ given for $\xi \in T$ by

$$A(e)(\xi) = A(e \otimes \xi)$$

is injective.

Then there is the result: *If on an open set in X we have either of the above conditions, then*

$$\begin{aligned} \text{(a)} &\implies \det E \text{ is big,} \\ \text{(b)} &\implies E \text{ is big.} \end{aligned}$$

Algebro-geometric and Hodge-theoretic consequences of these results are given in the text.

The other major topic in this paper is singularities, which is discussed in Section III. The structure of the singularities of a degenerating family of Hodge structures is of increasing importance, especially in the applications of Hodge theory to algebraic geometry. With notable exceptions such as the proof of the Iitaka conjecture ([Fuj78], [Vie83a], [Vie83b], [Kaw82], [Kaw85], [Kol87]) and the algebraicity of Hodge loci ([CDK95]), the analysis and application of singularities of degenerating Hodge structures has been primarily concerned with 1-parameter degenerations. More recently the detailed analysis of several variable degenerations is coming to play a central role, e.g., in the Satake-Baily-Borel and toroidal completions of period mappings used in the study of compactifications of moduli spaces ([GGLR20], [GGR21]). Moreover, the general structure of singularities continues to be used in establishing results about the hyperbolicity and log general type properties of parameter spaces of families of smooth algebraic varieties ([Den18], [VZ03]).

In the analysis of the asymptotics of several variable degenerations of Hodge structures two algebraic properties of monodromy cones $\sigma = \operatorname{span}_{\mathbb{Q}^+}\{N_1, \dots, N_k\}$ play a central role. Here, using the notations reviewed in Section I.B.4 the $N_i \in \operatorname{End}(V, Q)$ are commuting nilpotent elements. Each $N \in \sigma$ defines a *monodromy weight filtration* $W_0(N) \subset \cdots \subset W_{2n}(N) = V$, and the first property, due to Cattani-Kaplan, is

$$W_\bullet(N) \text{ is independent of } N \in \sigma. \tag{I.A.3}$$

The second is the *relative weight filtration* (RWFP) property, which for the case $k = 2$ where we have N_1, N_2 is the following: Since $[N_1, N_2] = 0$, the nilpotent transformation N_2 induces a nilpotent transformation $\overline{N}_2$ on $\operatorname{Gr}_\bullet^{W(N_2)}(V)$, and then $\overline{N}_2$ induces a monodromy weight filtration $W(\overline{N}_2)$ on $\operatorname{Gr}_\bullet^{W(N_1)}(V)$. On the other hand $N := N_1 + N_2$ induces a weight filtration $W(N)$ on V, and the RWFP implies that

$$\operatorname{Gr}_{k-m}^{W(\overline{N}_2)}\left(\operatorname{Gr}_m^{W(N_1)}(V)\right) \cong W_k(N) \cap \operatorname{Gr}_m^{W(N_1)}(V). \tag{I.A.4}$$

Here the right-hand side is the filtration induced on $\mathrm{Gr}^{W(N_1)}_\bullet(V)$ by $W_\bullet(N)$.[1]

Both (I.A.3) and (I.A.4) are highly non-generic properties of a set of commuting nilpotent operators. Although they are purely algebraic statements their current proofs are analytic using Hodge theory. One of the main purposes of the arguments in Section III.C is to clearly isolate the role of the RWFP in the analysis of the singularities of the Chern form of the augmented Hodge line bundle. This approach will enable us to avoid the rather complicated sectorial analysis in [CKS86] and [Kol87]. The problem to be dealt with is essentially to show the existence of a limit

$$\lim_{x_i \to 0} \frac{P(x)}{Q(x)}. \tag{I.A.5}$$

where $P(x), Q(x)$ are homogeneous polynomials of the same degree in real variables $x_1, \dots, x_k$ where $Q(x) > 0$ in the quadrant $x_i > 0$ for $i = 1, \dots, k$. Of course such limits do not exist in general; they depend on the path of approach of x to the origin. Here we are establish the desired limit by a direct computation of the Chern form. This computation turns out to make apparent the fact that the RWFP is the key general Hodge theoretic property behind the result.

I.B. **Notations and terminology.**

I.B.1. *General notations.*

- $X, Y, \dots$ will be compact, connected complex manifolds.

In practice they will be smooth, projective varieties.

- $E \to X$ is a holomorphic vector bundle with fibres E_x, $x \in X$ and rank $r = \dim E_x$;
- $A^{p,q}(X, E)$ denotes the global smooth E-valued (p, q) forms;
- we will not distinguish between a holomorphic bundle and its sheaf of holomorphic sections; the context should make the meaning clear;
- $L \to X$ will be a holomorphic line bundle.

Associated to $L \to X$ are the standard notions

(i) $\varphi_L : X \dashrightarrow \mathbb{P}H^0(X, L)^*$ is the rational mapping given for $x \in X$ by

$$\varphi_L(x) = [s_0(x), \dots, s_N(x)] \tag{I.B.1}$$

where $s_0, \dots, s_N$ is a basis for $H^0(X, L)$; in terms of a local holomorphic trivialization of $L \to X$ the $s_i(x)$ are given by holomorphic functions which are used to give the homogeneous coordinates on the right-hand side of (I.B.1);

(ii) the line bundle $L \to X$ is *big* if any one of the equivalent conditions
  - $h^0(X, L^m) = Cm^d + \cdots$ where $C > 0$, $\dim X = d$ and $\cdots$ are lower order terms;
  - $\dim \varphi_L(X) = \dim X$

  is satisfied;

(iii) $L \to X$ is *semi-ample* if any one of the equivalent conditions

[1] This is explained in more detail in Section III.C.

- for some $m > 0$, the evaluation maps

$$H^0(X, L^m) \to L^m_x \tag{I.B.2}$$

  are surjective for all $x \in X$;
- $\varphi_{mL}(x)$ is a morphism; i.e., for all $x \in X$ some $s_i(x) \neq 0$;
- the linear system $|mL| := \mathbb{P}H^0(X, L^m)$ is base point free for $m \gg 0$

is satisfied.

If the map (I.B.2) is only surjective for a general $x \in X$, we say that $L^m \to X$ is *generically globally generated.*

(iv) $L \to X$ is *nef* if $\deg \left(L\big|_C\right) \geqq 0$ for all curves $C \subset X$; here $L\big|_C = L \otimes_{\mathcal{O}_X} \mathcal{O}_C$ is the restriction of L to C;

(v) we will say that $L \to X$ is *strictly nef* if $\deg \left(L\big|_C\right) > 0$ for all curves $C \subset X$;

- for a vector bundle $E \to X$, we denote the k^{th} symmetric product by

$$S^k E := \operatorname{Sym}^k E;$$

- $\mathbb{P}E \xrightarrow{\pi} X$ is the projective bundle of 1-dimensional quotients of the fibres of $E \to X$; thus for $x \in X$

$$(\mathbb{P}E)_x = \mathbb{P}E^*_x;$$

- $\mathcal{O}_{\mathbb{P}E}(1) \to \mathbb{P}E$ is the tautological line bundle; then

$$\pi_* \mathcal{O}_{\mathbb{P}E}(m) = S^m E$$

  gives

$$H^0(X, S^m E) \cong H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(m))$$

  for all m.[2]

I.B.2. *Notations from complex differential geometry.* Given a Hermitian metric h in the fibres of a holomorphic vector bundle $E \to X$ there is a canonically associated Chern connection

$$D : A^0(X, E) \to A^1(X, E)$$

characterized by the properties ([Dem12a])

$$\begin{cases} D'' = \overline{\partial} \\ d(s, s') = (Ds, s') + (s, Ds') \end{cases} \tag{I.B.3}$$

where $s, s' \in A^0(X, E)$ and $(\ ,\ )$ denotes the Hermitian inner product in E. The curvature

$$\Theta_E := D^2$$

is linear over the functions; hence it is pointwise an algebraic operator. Using (I.B.3) it is given by a *curvature operator*

$$\Theta_E \in A^{1,1}(X, \operatorname{End} E)$$

which satisfies

$$(\Theta_E e, e') + (e, \Theta_E e') = 0$$

[2] The higher direct images $R^q_\pi \mathcal{O}_{\mathbb{P}E}(m) = 0$ for $q > 0$, $m \geqq -r$; we shall not make use of this.

where $e, e' \in E_x$. Relative to a local holomorphic frame $\{s_\alpha\}$, $h = \|h_{\alpha\bar{\beta}}\|$ is a Hermitian matrix and the corresponding connection and curvature matrices are given by

$$\begin{cases} \theta = h^{-1}\partial h \\ \Theta_E = \bar{\partial}(h^{-1}\partial h) = \left\| \sum_{\alpha,\beta,i,j} \Theta^{\alpha}_{\bar{\beta} i \bar{j}} s_\alpha \otimes s^*_\beta \otimes dz^i \wedge d\bar{z}^j \right\| \end{cases} .$$

For line bundles the connection and curvature matrices are respectively $\theta = \partial \log h$ and $\Theta_L = -\partial\bar{\partial}\log h$. If $h = e^{-\varphi}$, then

$$\Theta_L \geqq 0 \iff (i/2)\partial\bar{\partial}\varphi \geqq 0 \iff \varphi \text{ is plurisubharmonic.}$$

Definition: The *curvature form* is given for $x \in X$, $e \in E_x$ and $\xi \in T_x X$ by

$$\Theta_E(e, \xi) = \left\langle (\Theta_E(e), e), \xi \wedge \bar{\xi} \right\rangle . \tag{I.B.4}$$

When written out in terms of the curvature matrix $\Theta_E(e, \xi)$ is the bi-quadratic form

$$\sum_{\alpha,\beta,i,j} \Theta^{\alpha}_{\bar{\beta} i \bar{j}} e_\alpha \bar{e}_\beta \xi_i \bar{\xi}_j .$$

The bundle $E \to X$ is *positive*,[3] written $E_{\mathrm{met}} > 0$, if there exists a metric such that $\Theta_E(e, \xi) > 0$ for all non-zero e, ξ. For simplicity we will write $\Theta_E > 0$. If we have just $\Theta_E(e, \xi) \geqq 0$, then we shall say that $E \to X$ is *semi-positive* and write $E_{\mathrm{met}} \geqq 0$. It is *strongly semi-positive* if $E_{\mathrm{met}} \geqq 0$ and $(\det E)_{\mathrm{met}} > 0$ on an open set.

The bundle is *Nakano positive* if there exists a metric such that for all non-zero $\psi \in E_x \otimes T_x X$ we have

$$(\Theta_E(\psi), \psi) > 0. \tag{I.B.5}$$

The difference between positivity and Nakano positivity is that the former involves only the decomposable tensors in $E \otimes TX$ whereas the latter involves all tensors. In [Dem12a] there is the concept of m-positivity that involves the curvature acting on tensors of rank m and which interpolates between the two notions defined above.

Positivity and semi-positivity have functoriality properties ([Dem12a]). For our purposes the two most important are

(I.B.6) the tensor product of positive bundles is positive, and similarly for semi-positive;

(I.B.7) the quotient of a positive bundle is positive, and similarly for semi-positive.

The second follows from an important formula that we now recall (cf. [Dem12b]). If we have an exact sequence of holomorphic vector bundles

$$0 \to S \to E \to Q \to 0, \tag{I.B.8}$$

then a metric in E induces metrics in S, Q and there is a canonical *second fundamental form*

$$\beta \in A^{1,0}(X, \mathrm{Hom}(S, Q))$$

[3]We should say *metrically positive*, but since this is the main type of positivity used in this paper we shall drop the "metrically."

that measures the deviation from being holomorphic of the C^∞ splitting of (I.B.8) given by the metric. Equivalently it measures the failure of the Chern connection acting on $A^0(X,E)$ to map $A^0(X,S)$ to $A^1(X,S)$. For $j : Q \hookrightarrow E$ the inclusion given by the C^∞ splitting and $q \in Q_x, \xi \in T_x X$ the formula is (loc. cit.)

$$\Theta_Q(q\otimes\xi) = \Theta_E(j(q)\otimes\xi) + \|\beta^*(q)\otimes\xi\|^2 \tag{I.B.9}$$

where by definition the last term is $-\left\langle(\beta^*(q),\beta^*(q))_S, \xi\wedge\bar\xi\right\rangle$ and $(\ ,\ )_S$ is the induced metric in S. The minus sign is because the Hermitian adjoint β^* is of type (0,1).

Examples.

(i) The universal quotient bundle $Q \to G(k,n)$ with fibres $Q_\Lambda = \mathbb{C}^n/\Lambda$ over the Grassmannian $G(k,n)$ of k-planes $\Lambda \subset \mathbb{C}^n$ has a metric induced by that in $\mathbb{C}^n$, and with this metric

$$\Theta_Q \geqq 0 \ \text{ and }\ \Theta_Q > 0 \iff k = n-1.$$

Similarly, the dual $S^* \to G(k,n)$ of the universal sub-bundle has $\Theta_{S^*} \geqq 0$ and $\Theta_{S^*} > 0 \iff k = 1$.

Geometrically, for a k-plane $\Lambda \in \mathbb{C}^n$ we have the usual identification

$$T_\Lambda G(k,n) \cong \mathrm{Hom}(\Lambda, \mathbb{C}^n/\Lambda).$$

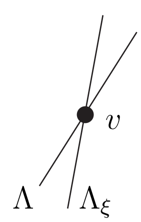
Then for $\xi \in \mathrm{Hom}(\Lambda,\mathbb{C}^n/\Lambda)$ and $v \in \Lambda$

$$\Theta_{S^*}(v,\xi) = 0 \iff \xi(v) = 0.$$

Here the RHS means that for the infinitesimal displacement Λ_ξ of Λ given by ξ we have

$$v \in \Lambda \cap \Lambda_\xi.$$

The picture for $G(2,4)$ viewed as the space of lines in $\mathbb{P}^3$ is



There are similar semi-positivity properties for any globally generated vector bundle, since such bundles are induced from holomorphic mappings to a Grassmannian, and positivity and semi-positivity have the obvious functoriality properties.

(ii) Below we will recall and establish notations for the standard concepts in Hodge theory, including the Hodge vector bundles $F^p \to B$ over the parameter space B of a variation of Hodge structure. The *geometric case* will be when the variation of Hodge structure arises from the cohomology along the fibres of a family $\mathcal{X} \xrightarrow{\pi} B$ of smooth, projective varieties $X_b = \pi(b)$. Here $\mathcal{X}$ and B are complex manifolds and π is a proper submersion. We will use the standard notation $\omega_{\mathcal{X}/B}$ for the relative dualizing sheaf. It is a line bundle over $\mathcal{X}$ whose fibre at $x \in X_b$ is the canonical bundle $K_{X_{b,x}} = \det T^*_x X_b$.

The Hodge bundle $F^n \to B$ with the metric given by the Hodge-Riemann bilinear relation satisfies $\Theta_F \geqq 0$ [Gri70], but unless $h^{n,0} = 1$ very seldom do we have $\Theta_F > 0$.

These examples will be further discussed in Section III.A.

(iii) In the geometric case of a family $\mathcal{X} \xrightarrow{\pi} B$ with smooth fibres, the Narashimhan-Simka [NS68] Finsler type metrics in $\pi_* \omega^m_{\mathcal{X}/B}$ have a Hodge theoretic interpretation. As a consequence of this

There is a metric h_m *in* $\mathcal{O}_{\mathbb{P}f_*\omega^m_{X/Y}}(1)$ *whose Chern form* $\omega_m \geqq 0$.

Some care must both be taken here as although h_m is continuous it is not smooth and so $\omega_m = (i/2)\overline{\partial}\partial \log h_m$ and the inequality $\omega_m \geqq 0$ must be taken in the sense of currents (cf. [Dem12b] and [Pă16]). Metrics of this sort were used in [Kaw82] and have been the subject of numerous recent works, including [Ber09], [BPă12], [PT14], [MT07], [MT08], and also [Pă16] where a survey of the literature and further references are given.

I.B.3. *Interpretation of the curvature form.* Given a holomorphic vector bundle $E \to X$ there is the associated projective bundle $\mathbb{P}E \xrightarrow{\pi} X$ of 1-dimensional quotients of the fibres of E; thus $(\mathbb{P}E)_x = \mathbb{P}E^*_x$. Over $\mathbb{P}E$ there is the tautological line bundle $\mathcal{O}_{\mathbb{P}E}(1)$. A metric in $E \to X$ induces one in $\mathcal{O}_{\mathbb{P}E}(1) \to \mathbb{P}E$, and we denote by ω_E the corresponding curvature form. Then $\Omega_E := (i/2\pi)\omega_E$ represents the Chern class $c_1(\mathcal{O}_{\mathbb{P}E}(1))$ in $H^2(\mathbb{P}E)$.

Since $\omega_E\big|_{(\mathbb{P}E)_x}$ is a positive $(1,1)$ form, the vertical sub-bundle

$$V := \ker \pi_* : T\mathbb{P}E \to TX$$

to the fibration $\mathbb{P}E \to X$ has a C^∞ horizontal complement H. Thus as C^∞ bundles

$$\begin{cases} T\mathbb{P}E & \cong V \oplus H, \text{ and} \\ \pi_* : H \xrightarrow{\sim} \pi^* TX. \end{cases}$$

In more detail, using the metric we have a complex conjugate linear identification $E^*_x \cong E_x$, and using this we shall write points in $\mathbb{P}E$ as $(x, [e])$ where $e \in E_x$ is a non-zero vector. Then we have an isomorphism

$$\pi_* : H_{(x,[e])} \xrightarrow{\sim} T_x X. \tag{I.B.10}$$

Using this identification and normalizing to have $\|e\| = 1$, the interpretation of the curvature form is given by the equation

$$\Theta_E(e, \xi) = \left\langle \omega_E, \xi \wedge \bar{\xi} \right\rangle =: \omega_E(\xi) \tag{I.B.11}$$

where $\xi \in T_x X \cong H_{(x,[e])}$ and the RHS is evaluated at $(x, [e])$. Thus

$$\Theta_E > 0 \iff \omega_E > 0, \tag{I.B.12}$$

and similarly for $\geqq 0$. There are the evident extensions of (I.B.12) to open sets in $\mathbb{P}E$ lying over open sets in X. For semi-positive vector bundles we summarize by saying that *the curvature form* Θ_E *measures the degree of positivity of* ω_E *in the horizontal directions.*

For later use we conclude with the observation that using $\mathcal{O}_X(E) \cong \pi_* \mathcal{O}_{\mathbb{P}E}(1)$, given $s \in \mathcal{O}_{X,x}(E)$ there is the identification of (1,1) forms

$$(-\partial\overline{\partial} \log \|s\|^2)(x) = \omega_E(x, [s(x)]) \tag{I.B.13}$$

where the RHS is the (1,1) form ω_E evaluated at the point $(x, [s(x)]) \in \mathbb{P}E$ in the total tangent space (both vertical and horizontal directions).

I.B.4. *Hodge theory.* We shall follow the generally standard notations and conventions as given in [CMSP17][4] and are used in [GGLR20] and [GGR21]. Further details concerning the structure of limiting mixed Hodge structures (LMHS) will be given in Section III.D.

- B will denote a smooth quasi-projective variety;
- a *variation of Hodge structure* (VHS) parametrized by B will be given by the equivalent data
  (a) a *period mapping*
  $$\Phi : B \to \Gamma \backslash D$$
  where D is the period domain of weight n polarized Hodge structures $(V, Q, F^\bullet)$ with fixed Hodge numbers $h^{p,q}$, and where the infinitesimal period relation (IPR)
  $$\Phi_* : TB \to I \subset T(\Gamma \backslash D)$$
  is satisfied;
  (b) $(\mathbb{V}, F^\bullet, \nabla; B)$ where $\mathbb{V} \to B$ is a local system with Gauss-Manin connection
  $$\nabla : \mathcal{O}_B(\mathbb{V}) \to \Omega^1_B(\mathbb{V})$$
  and $F^\bullet = \{F^n \subset F^{n-1} \subset \cdots \subset F^0\}$ is a filtration of $\mathcal{O}(\mathbb{V})$ by holomorphic sub-bundles satisfying the IPR in the form
  $$\nabla F^p \subset \Omega^1_B(F^{p-1}),$$
  and where at each point $b \in B$ the data $(\mathbb{V}_b, F^\bullet_b)$ defines a polarized Hodge structure (PHS) of weight n.[5]

  In the background in both (a) and (b) is a bilinear form Q that polarizes the Hodge structures; we shall suppress the notation for it when it is not being explicitly used.
- The parameter space B will have a smooth projective completion $\overline{B}$ with the properties
  $Z := \overline{B} \backslash B$ is a reduced normal crossing divisor $Z = \bigcup Z_i$ having strata $Z_I := \bigcap_{i \in I} Z_i$ with $Z_I^* \subset Z_I$ denoting the non-singular points $Z^*_{I,\mathrm{reg}}$ of Z_I, and where the local monodromies T_i around the irreducible branches Z_i of Z are unipotent with logarithms N_i;[6]
- the *Hodge vector bundle* $F \to B$ has fibres $F_b := F^n_b$;

[4] This will be our main reference for Hodge theory.

[5] More precise notation would be that $\mathbb{V}$ is a local system of $\mathbb{Q}$-vector spaces with complexification $\mathbb{V}_\mathbb{C} = \mathbb{C} \otimes_\mathbb{Q} \mathbb{V}$. Then the Gauss-Manin connection is $\nabla : \mathcal{O}_B(\mathbb{V}_\mathbb{C}) \to \Omega^1_B \otimes \mathcal{O}_B(\mathbb{V}_\mathbb{C})$.

[6] To achieve this we may have to pass to a covering space of $\overline{B}$ branched along the Z_i's.

- the *Hodge line bundle* $\Lambda := \det F = \wedge^{h^{n,0}} F$;
- the polarizing forms induce Hermitian metrics in F and Λ;
- the differential of the period mapping is
$$\Phi_* : TB \to \bigoplus_{p \geqq \left[\frac{n}{2}\right]} \mathrm{Hom}(F^p, F^{p-1}/F^p)$$
where $F^p \to B$ denotes the Hodge filtration bundles;
- setting $F = F^n$ and $G = F^{n-1}/F^n$ the *end piece* of Φ_* is
$$\Phi_{*,n} : TB \to \mathrm{Hom}(F, G);$$
- the Hodge filtration bundles have canonical Deligne extensions
$$F_e^p \to \overline{B};$$
using the Hodge metrics on B, the holomorphic sections of $F_e^p \to \overline{B}$ are those whose Hodge norms have at most logarithmic growth along Z;
- equivalently, in a neighborhood $\overline{\mathcal{U}} \subset \overline{B}$ of a point $b \in \overline{B} \backslash B$, a holomorphic frame v_j for $\mathcal{F}_e^0 \to \overline{\mathcal{U}}$ is given by a holomorphic frame over $\mathcal{U} = \overline{\mathcal{U}} \cap B$ that when expressed in terms of a horizontal or flat frame is given by a matrix whose entries are polynomials in $\log t_1, \dots, \log t_k$ with coefficients that are holomorphic functions in $\overline{\mathcal{U}}$ and where the Gauss-Manin connection has regular singular points with residues $N_i := \log T_i$ around Z_i;
- the *geometric case* is when the VHS arises from the cohomology along the fibres in a smooth projective family
$$\mathcal{X} \xrightarrow{f} B;$$
- such a family has a completion to
$$\overline{\mathcal{X}} \xrightarrow{\bar{f}} \overline{B}$$
where this map has the Abramovich-Karu ([ATW20]) form of semi-stable reduction (cf. Section 4 in [GGLR20] for more details and for the notations to be used here); then the canonical extension of the Hodge vector bundle is given by
$$F_e = \bar{f}_* \omega_{\overline{\mathcal{X}}/\overline{B}}.$$

We conclude this introduction with an observation and a question. For line bundles $L \to X$ over a smooth projective variety X of dimension d, there are three important properties:

(i) L if nef;
(ii) L is big;
(iii) L if semi-ample.

Clearly (iii) $\Longrightarrow$ (i) and (ii), (iii) $\Longrightarrow$ $|mL|$ gives a birational morphism for $m \gg 0$.

In this paper we will consider the case where $L \to X$ has a metric h that may be singular with Chern ω that defines a $(1,1)$ current representing $c_1(L)$. The singularities of h are generally of the following types:

- h vanishes along a proper subvariety of X;

- h becomes infinite, either logarithmically or analytically (in the sense explained below) along a proper subvariety of X.

We note that if ω is the Chern form associated to a smooth metric, then $\omega \geqq 0 \implies$ (i), and that in general in a Zariski open $X^0 \subset X$ where h is a smooth metric

$$\omega^d > 0 \implies \text{(ii)}$$

(cf. (II.C.6) below and [Dem12a]). The Kodaira theorem states that if $X^0 = X$, then h is ample. There is a general conjecture [GGLR20] that the *augmented Hodge line bundle* $L_e = \det(F_e^n) \otimes \cdots \otimes \det(F^{((n+1)/2)}$ is semi-ample. For a number of purposes, including applications to Hodge theory, it would be desirable to have conditions on ω that imply (iii).

In more detail, if Φ is a proper immersion, then ω defines a complete Kähler metric on B. This metric has singularities along Z that reflect the geometry of the period mapping at infinity. It is conjectured, but not proved, that the special structure of this situation imply that $L_e \to \overline{B}$ is semi-ample. As explained in [GGR21] this will involve the geometry along Z of the singular Kähler metric ω_e.

In [BBT18] and [BKT18] the new technique of *o*-minimality was introduced that proves that the image of the period mapping defined on B is an algebraic variety on which $L \to B$ descends to give an ample line bundle. This result settled a long standing question and uses methods that are sure to lead to further results.

II. Positivity

II.A. **Iitaka dimension and numerical dimension.** In algebraic geometry positivity traditionally suggests "sections," and one standard measure of the amount of sections of a line bundle $L \to X$ is given by its *Kodaira-Iitaka dimension* $\kappa(L)$. This is defined by

$$\kappa(L) = \max_m \dim \varphi_{mL}(X) \tag{II.A.1}$$

where

$$\varphi_{mL} : X \dashrightarrow \mathbb{P}H^0(mL)^*$$

is the rational mapping given by the linear system $|mL|$. If $h^0(mL) = 0$ for all m we set $\kappa(L) = -\infty$. From [Dem12a] we have

$$h^0(mL) \leqq O(m^{\kappa(L)}) \qquad \text{for } m \geqq 1, \tag{II.A.2}$$

and $\kappa(L)$ is the smallest exponent for which this estimate holds.[7] We will sometimes write (II.A.2) as

$$h^0(mL) \sim Cm^{\kappa(L)}, \quad C > 0.$$

We note that

$$\kappa(L) = \dim X \iff L \to X \text{ is big.}$$

[7] Including $\kappa(L) = -\infty$ where we set $m^{-\infty} = 0$ for $m > 0$.

Numerical dimension. Let $L \to X$ be a line bundle with $L_{\mathrm{num}} \geqq 0$; i.e., L is nef.

Definition (cf. [Dem12a])**:** The *numerical dimension* is the largest integer $n(L)$ such that

$$c_1(L)^{n(L)} \neq 0.$$

In practice in this paper there will be a semi-positive (1,1) form ω such that $[(i/2\pi)\omega] = c_1(L)$, and then $n(L)$ is the largest integer such that $\omega^{n(L)+1} \equiv 0$ but

$$\omega^{n(L)} \neq 0$$

on an open set.

Relating the Kodaira-Iitaka and numerical dimensions, from [Dem12a] we have

$$\kappa(L) \leqq n(L) \tag{II.A.3}$$

with equality if $n(L) = \dim X$, but where equality may not hold if $n(L) < \dim X$.

Example (loc. cit.)**:** Let C be an elliptic curve and $p, q \in C$ points such that $p - q$ is not a torsion point. Then the line bundle $[p - q]$ has a flat unitary metric, but $h^0(C, m[p - q]) = 0$ for all $m > 0$. For any nef line bundle $L' \to X'$ we set

$$X = C \times X', \qquad L = [p - q] \boxtimes L'.$$

Then $\kappa(X) = -\infty$ while $n(L)$ may be any integer with $n(L) \leqq \dim X - 1$.

The reason we may have the strict inequality $\kappa(L) < n(L)$ seems to involve some sort of flatness as in the above example.[8]

For a vector bundle $E \to X$ with $E_{\mathrm{num}} \geqq 0$ we have the

Definition: The *numerical dimension* $n(E)$ of the vector bundle $E \to X$ is given by $n(\mathcal{O}_{\mathbb{P}E}(1))$.

Since $\mathcal{O}_{\mathbb{P}E}(1)$ is positive on the fibers of $\mathbb{P}E \to X$ we have

$$r - 1 \leqq n(E) \leqq \dim \mathbb{P}E = \dim X + r - 1.$$

Conjecture II.D.24 below suggests conditions under which equality will hold.

From (II.B.4) below we have

$$n(E) \textit{ is the largest integer with } S_{n(E)-r+1}(E) \neq 0, \tag{II.A.4}$$

which serves to define the numerical dimension of a semi-positive vector bundle in terms of its Segre classes that are defined in Section II.B below. We remark that as noted above

$$\mathcal{O}_{\mathbb{P}E}(1)_{\mathrm{num}} \geqq 0 \implies S_q(E) \geqq 0 \text{ for any } q \geqq 0; \tag{II.A.5}$$

we are not aware of results concerning the converse implication.

[8]Cf. [CD14] and [CD17] for a related discussion involving certain Hodge bundles.

II.B. **Numerical positivity.** [9]

In this section we shall discuss various measures of numerical positivity, one main point being that these will apply to bundles arising from Hodge theory. The basic reference here is [Laz04]. A conclusion will be that the Hodge vector bundle F is numerically semi-positive; i.e., $F_{\rm num} \geqq 0$ in the notation to be introduced below.

II.B.1. *Definition of numerical positivity.* We first recall the definition of the cone $\mathcal{C} = \oplus \mathcal{C}_d$ of positive polynomials $P(c_1, \dots, c_r)$ where c_i has weighted degree i. For this we consider partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ with $0 \leqq \lambda_n \leqq \lambda_{n-1} \leqq \cdots \leqq \lambda_1 \leqq r$, $\Sigma\lambda_i = n$, of $n = \dim X$. For each such λ the *Schur polynomial* s_λ is defined by the determinant

$$s_\lambda = \begin{vmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdot & \cdot & c_{\lambda_1+n-1} \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \\ c_{\lambda_n-n+1} & \cdot & \cdot & \cdot & c_{\lambda_n} \end{vmatrix}. \tag{II.B.1}$$

Then ([Laz04]) $\mathcal{C}$ is generated over $\mathbb{Q}^{>0}$ by the s_λ. It contains the Chern monomials

$$c_1^{i_1} \cdots c_r^{i_r}, \qquad i_1 + 2i_2 + \cdots + ri_r \leqq n$$

as well as some combinations of these with negative coefficients, the first of which is $c_1^2 - c_2$.

For each $P \in \mathcal{C}_d$ and d-dimensional subvariety $Y \subset X$ we consider

$$\int_Y P(c_1(E), \dots, c_r(E)) = P(c_1(E), \dots, c_r(E))[Y] \tag{II.B.2}$$

where the RHS is the value of the cohomology class $P(c_1(E), \dots, c_r(E)) \in H^{2d}(X)$ on the fundamental class $[Y] \in H_{2d}(X)$.

Definition: $E \to X$ is *numerically positive*, written $E_{\rm num} > 0$, if (II.B.2) is positive for all $P \in \mathcal{C}_d$ and subvarieties $Y \subset X$.

We may similarly define $E_{\rm num} \geqq 0$.

A non-obvious result ([Laz04]) is

(II.B.3) *For line bundles $L \to X$, we have*

$$L_{\rm num} \geqq 0 \iff L \textit{ is nef.}$$

The essential content of this statement is

> $c_1(L)[C] \geqq 0$ *for all curves* $C \implies c_1(L)^d[Y] \geqq 0$ *for all d-dimensional subvarieties* $Y \subset X$.[10]

This is frequently formulated as saying that if L is nef then it is in the closure of the ample cone.

---

[9] We are including a discussion of numerical positivity because this property will hold for the Hodge bundle but to our knowledge has yet to be generally applied in the application of Hodge theory to algebraic geometry.

[10] Here $c_1(L)[C] := \deg\big(L\big|_C\big)$, and similarly for $c_1(L)^d[Y]$.

II.B.2. *Relation between $E_{\text{num}} > 0$ and $\mathcal{O}_{\mathbb{P}E}(1)_{\text{num}} > 0$.*

For the fibration $\mathbb{P}E \xrightarrow{\pi} X$ there is a Gysin or *integration over the fibre* map

$$\pi_* : H^{2(d-r+1)}(\mathbb{P}E) \to H^{2d}(X).$$

It is defined by moving cohomology to homology via Poincaré duality, taking the induced map on homology and then again using Poincaré duality. In de Rham cohomology the mapping is given by what the name suggests. The d^{th} *Segre polynomial* is defined by

$$S_d(E) = \pi_* \left(c_1 \left(\mathcal{O}_{\mathbb{P}E}(1) \right)^{d-r+1} \right). \tag{II.B.4}$$

Then ([Laz04]): (i) $S_d(E)$ is a polynomial in the Chern classes $c_1(E), \dots, c_r(E)$, and (ii) $S_d(c_1, \dots, c_r) \in \mathcal{C}_d$. That it is a polynomial in the Chern classes is a consequence of the *Grothendieck relation*

$$c_1 \left(\mathcal{O}_{\mathbb{P}E}(1) \right)^r - c_1 \left(\mathcal{O}_{\mathbb{P}E}(1) \right)^{r-1} \pi^* e_1(E) + \cdots + (-1)^r \pi^* c_r(E) = 0. \tag{II.B.5}$$

The first few Segre polynomials are

$$\begin{cases} S_1 = c_1 \\ S_2 = c_1^2 - c_2 \\ S_3 = c_1 c_2 \\ S_4 = c_1^4 - 2c_1^2 c_2 + c_1 c_3 - c_4. \end{cases}$$

An important implication is

$$\mathcal{O}_{\mathbb{P}E}(1)_{\text{num}} > 0 \implies E_{\text{num}} > 0. \tag{II.B.6}$$

Proof. By Nakai-Moishezon (cf. (II.C.8) below), $\mathcal{O}_{\mathbb{P}E}(1)$ and hence E are ample. Then $E_{\text{num}} > 0$ by the theorem of Bloch-Gieseker ([Laz04]). $\square$

As will be noted below, the converse implication is not valid.

II.C. **Metric positivity.** Because of its general importance in complex differential geometry we begin with a discussion of

Tangent bundle. When $E = TX$ and the metric on $E \to X$ is given by a Kähler metric on X the curvature form has the interpretation

$$\Theta_{TX}(\xi, \eta) = \left\{ \begin{matrix} \text{holomorphic bi-sectional curvature} \\ \text{in the complex 2-plane } \xi \wedge \eta \\ \text{spanned by } \xi, \eta \in T_x X \end{matrix} \right\}. \tag{II.C.1}$$

When $\xi = \eta$ we have

$$\Theta_{TX}(\xi, \xi) = \left\{ \begin{matrix} \text{holomorphic sectional curvature in} \\ \text{the complex line spanned by } \xi \end{matrix} \right\}. \tag{II.C.2}$$

Of particular interest and importance in Hodge theory and in other aspects of algebraic geometry is the case when TX has some form of negative curvature.

Proposition II.C.3 ([BKT13]): *Assume there is $c>0$ such that* (i) $\Theta_{TX}(\xi,\xi)\leqq -c$ *for all* ξ, *and* (ii) $\Theta_{TX}(\lambda,\eta)\leqq 0$ *for all* λ,η. *Then there exists* ξ *such that* $\Theta_{TX}(\xi,\eta)\leqq -c/2$ *for all* η.

This implies that if the holomorphic sectional curvatures are negative and the holomorphic bi-sectional curvatures are non-positive, then they are negative on an open set in $G(2,TX)$, the Grassmann bundle of 2-planes in TX, and this open set maps onto X. Noting that $\mathrm{Gr}(2,TX)$ maps to an open subset of the horizontal sub-bundle in the fibration $\mathbb{P}TX\to X$, from (II.C.6) below we have the

Corollary II.C.4: *If the assumptions in II.C.3 are satisfied, then T^*X is big.*

We will see that these assumptions are satisfied for period mappings where Φ is an immersion.

First implications. In this section for easy reference we will summarize the first implications of the two types of positivity on the Kodaira-Iitaka dimension and numerical dimension. These are either well known or easily inferred from what is known.

Case of a line bundle $L\to X$

$$L_{\mathrm{met}}>0\implies L \text{ ample.} \tag{II.C.5}$$

This is the Kodaira theorem which initiated the relation between metric positivity and sections.

The next is the Grauert-Riemenschneider conjecture, established by Siu and Demailly (cf. [Dem12a] and the references there):

$$L^0_{\mathrm{met}}>0\implies \kappa(L)=\dim X. \tag{II.C.6}$$

Here we recall that $L^0_{\mathrm{met}}>0$ means that there is a metric in $L\to X$ whose curvature form $\omega\geqq 0$ and where $\omega>0$ on an open set. The result may be phrased as

$$L^0_{\mathrm{met}}>0\implies L \text{ is big.}$$

For Hodge theory this variant of the Kodaira theorem plays a central role as bundles constructed from the extended Hodge vector bundle tend to be big and perhaps semi-ample,[11] but just exactly what their "Proj" is seems to be an interesting issue. Because of this for later use, in the case when X is projective we now give a

Proof of (II.C.6). Let $H\to X$ be a very ample line bundle chosen so that $H-K_X$ is ample. Setting $F=L+H$ we have $F_{\mathrm{met}}>0$. For $D\in|F|$ smooth using the Kodaira vanishing theorem we have

$$\begin{cases} h^q(X,mF)=0, & q>0,\\ h^q(D,mF\big|_D)=0, & q>0.\end{cases}$$

[11]The issue of additional conditions that will imply that an L which is nef and big is also semi-ample is a central one in birational geometry (cf. [Ko-Mo]). The results there seem to involve assumptions on $mL-K_X$. In Hodge theory K_X is frequently one of the things that one wishes to establish properties of, so that at least thus far the base-point-free theorem has not seemed to be useful.

We note that the vanishing theorems will remain true if we replace L by a positive multiple.

Let $D, \dots, D_m \in |H|$ be distinct smooth divisors. From the exact sequence $0 \to m(F-H) \to mF \to \overset{m}{\underset{j=1}{\oplus}} mF\big|_{D_j}$ we have

$$0 \to H^0(X, m(F-H)) \to H^0(X, mF) \to \overset{m}{\underset{j=1}{\oplus}} H^0(D_j, mF).$$

This gives

$$h^0(X, mL) = h^0(X, m(F-H)) \geqq h^0(X, mF) - mh^0(D, mF).$$

Using the above vanishing results

$$h^0(X, mL) \geqq \chi(X, mF) - m\chi(D, mF).$$

For $d = \dim X$ and letting $\sim$ denote modulo lower order terms, from the Riemann-Roch theorem we have

$$\begin{aligned} \chi(X, mF) &\sim \frac{m^d}{d!} F^d \\ m\chi(D, mF) &\sim \frac{m^d}{d!} (dF^{d-1} \cdot H). \end{aligned}$$

For $m \gg 0$ this gives

$$h^0(X, mL) \geqq \frac{m^d}{d!} (F^d - dF^{d-1} \cdot H) + o(m^d).$$

From

$$\begin{aligned} F^d - dF^{d-1} \cdot H &= (L+H)^d - d(L+H)^{d-1} \cdot H \\ &= (L+H)^{d-1} \cdot (L - (d-1)H) \end{aligned}$$

replacing L by a multiple we may make this expression positive.[12] $\square$

The next result is

(II.C.7) $\quad L_{\mathrm{met}} > 0 \implies L_{\mathrm{num}} > 0,$ and similarly for $\geqq 0$ $\qquad$ (obvious).

The inequality in (II.C.7) is sometimes phrased as

$$L_{\mathrm{met}} \geqq 0 \implies L \text{ is nef.}$$

The theorem of Nakai-Moishezon is

(II.C.8) $\quad L_{\mathrm{num}} > 0 \iff L$ is ample.

The next inequality was noted above:

(II.C.9) $\quad \kappa(L) \leqq n(L),$ with equality if $n(L) = \dim X$;

Case of a vector bundle $E \to X$

(II.C.10) $\quad E_{\mathrm{met}} > 0 \implies E$ ample, and $E^0_{\mathrm{met}} > 0 \implies \kappa(E) = \dim X + r - 1.$

[12]This argument is due to Catanese; cf. [Dem12a].

In words, $E^0_{\mathrm{met}} > 0$ implies that E is big.

We next have

(II.C.11) $$E_{\mathrm{met}} > 0 \implies E_{\mathrm{num}} > 0.$$

This result may be found in [Laz04]; the proof is not obvious from the definition. We are not aware of any implication along the lines of

$$E_{\mathrm{met}} \geqq 0 \implies E_{\mathrm{num}} \geqq 0.$$

Next we have

(II.C.12) $\kappa(E) \leqq n(E)$ with equality if $n(E) = \dim X + r - 1$ (this follows from (II.C.9)).

This leads to

(II.C.13) $$E_{\mathrm{met}} \geqq 0 \text{ and } E_{\mathrm{num}} > 0 \implies \kappa(E) = \dim X + r - 1.$$

This follows from (II.A.4) and (II.C.12).

It is *not* the case that

$$E_{\mathrm{num}} > 0 \implies E \text{ ample;}$$

there is an example due to Fulton of a numerically positive vector bundle over a curve that is not ample (cf. [Laz04]).

We will conclude this section with a sampling of well-known results whose proofs illustrate some of the traditional uses of positivity.

PROPOSITION II.C.14: *If $E \to X$ is a Hermitian vector bundle with $\Theta_E > 0$, then $H^0(X, E^*) = 0$.*

Proof. Using (I.B.13) applied to $E^* \to X$, if $s \in H^0(X, E^*)$ then when we evaluate $\partial\overline{\partial} \log \|s\|^2$ at a strict maximum point where the Hessian is definite we obtain a contradiction. If the maximum is not strict then the usual perturbation argument may be used. □

PROPOSITION II.C.15: *If $E \to X$ is a Hermitian vector bundle of rank $r \leqq \dim X$ and with $\Theta_E > 0$, then every section $s \in H^0(X, E)$ has a zero.*

Proof. The argument is similar to the preceding proposition, only this time we assume that s has no zero and evaluate $\partial\overline{\partial} \log \|s\|^2$ at a minimum. Here we are viewing s as a section of $\mathcal{O}_{\mathbb{P}E}(1)$. This (1,1) form is positive in the pullback to X of the vertical tangent space, and it is negative in the pullback of the horizontal tangent space. The assumption $r \leqq \dim X$ then guarantees that it has at least one negative eigenvalue. □

PROPOSITION II.C.16: *If $E \to X$ is a Hermitian vector bundle with $\Theta_E \geqq 0$ and $s \in H^0(X, E)$ satisfies $\Theta_E(s) = 0$, then $Ds = 0$.*

Proof. Let ω be a Kähler form on X. Then

$$\partial\overline{\partial}(s, s) = (Ds, Ds) + (s, \Theta_E(s)) = (Ds, Ds) \geqq 0,$$

and if $\dim X = d$ using Stokes theorem we have

$$0 = \int_X \omega^{d-1} \wedge (i/2)\partial\overline{\partial}\|s\|^2 = \int_X \omega^{d-1} \wedge (i/2)(Ds, Ds)$$

which gives the result. □

A further result.

As discussed above, for many purposes positivity is too strong (see the examples in Section II.B) and semi-positivity is too weak (adding the trivial bundle to a semi-positive bundle gives one that is semi-positive). One desires a more subtle notion than just $\Theta_E \geqq 0$. With this in mind, a specific guiding question is

QUESTION: *Suppose that one has $\Theta_E \geqq 0$ and for $\wedge^r E = \det E$ we have $\Theta_{\det E} > 0$ on an open set; that is, E is strongly semi-positive. Does this enable one to produce sections of $\mathrm{Sym}^m E$ for $m \gg 0$?*

The following is a response to this question:

THEOREM II.C.17: *Suppose that $E \to X$ is a Hermitian vector bundle of rank r that is strongly semi-positive. Then $\mathrm{Sym}^m E \to X$ is big for any $m \geqq r$.*

Proof. Setting $S^r E = \mathrm{Sym}^r E$, we have $\Theta_{S^r E} \geqq 0$. Let ω_r be the curvature form for $\mathcal{O}_{\mathbb{P}S^r(E)}(1)$. Then $\omega_r \geqq 0$, and we will show that

$$\omega_r > 0 \text{ on an open set.} \tag{II.C.18}$$

For this it will suffice to find one point $p = (x, [e_1 \cdots e_r]) \in \mathbb{P}S^r(E)_x$ where (II.C.18) holds. Let $x \in X$ be a point where $(\operatorname{Tr}\Theta_E)(x) > 0$ and let $e_1, \dots, e_r$ be a unitary basis for E_x. Then some

$$\left\langle (\Theta_E(e_i), e_i)\, , \xi \wedge \overline{\xi} \right\rangle > 0.$$

We may assume that the Hermitian matrix

$$\left\langle (\Theta_E(e_i), e_j)\, , \xi \wedge \overline{\xi} \right\rangle = \delta_{ij}\lambda_i, \quad \lambda_i \geqq 0$$

is diagonalized. Then

$$\begin{aligned} \left\langle (\Theta_{S^r E}(e_1 \cdots e_r), e_1 \cdots e_r)\, , \xi \wedge \overline{\xi} \right\rangle &= \sum_i \left\langle \Big(\Theta_{S^r E}(e_i) e_1 \overset{\overset{i}{\wedge}}{\cdots} e_r, e_1 \cdots e_r\Big), \xi \wedge \overline{\xi} \right\rangle \\ &= \sum_i \lambda_i > 0. \end{aligned}$$

The same argument works for any $m \geqq r$. □

Using this we are reduced to proving the

LEMMA II.C.19: *Let $F \to X$ be a Hermitian vector bundle with $\Theta_F \geqq 0$, and where this is a point $x \in X$ and $f \in F_x$ such that the $(1,1)$ form $\Theta_F(f, \cdot)$ is positive. Then for the curvature form ω_F of the line bundle $\mathcal{O}_{\mathbb{P}F}(1) \to \mathbb{P}F$, we have $\omega_F > 0$ at the point $(x, [f]) \in \mathbb{P}F$.*

Proof. Let $f_1^*, \ldots, f_r^*$ be a local holomorphic frame for $F^* \to X$ and

$$\sigma = u([a_1, \ldots, a_r]; x) \sum_i a_i f_i^*$$

a local holomorphic section of $\mathcal{O}_{\mathbb{P}E}(1) \to \mathbb{P}F$, where $a_1, \ldots, a_r$ are variables defined modulo the scaling action $(a_1, \ldots, a_r) \to \lambda(a_1, \ldots, a_r)$ and $u([a_1, \ldots, a_r], x)$ is holomorphic. We have

$$\|\sigma\|^2 = |u|^2 \sum_{i,j} h_{i\bar{j}} \cdot a_i \bar{a}_j$$

where $h_{i\bar{j}} = (f_i^*, f_j^*)$ is the metric in $F \to X$. Up to a constant

$$\omega_F = \partial\overline{\partial} \log \|\sigma\|^2.$$

We may choose our frame and scaling parameter so that at the point $(x, [f])$

$$h_{i\bar{j}}(x) = \delta_{ij}, \qquad dh_{i\bar{j}}(x) = 0 \text{ and } \|\sigma(x, [f])\| = 1. \tag{II.C.20}$$

Computing $\partial\overline{\partial} \log \|\sigma\|^2$ and evaluating at the point where (II.C.20) holds any cross-terms involving $dh_{i\bar{j}}(x)$ drop out and we obtain

$$\omega_F = \sum_{ij} (\partial\overline{\partial} h_{i\bar{j}})(x) a_i \bar{a}_j + \left(\sum_{i,j} da_i \wedge \bar{d}a_i - \left(\sum_i a_i \cdot da_i \right) \wedge \left(\overline{\sum_j a_j da_j} \right) \right).$$

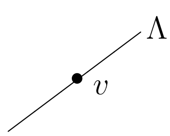
When we take the scaling action into account and use Cauchy-Schwarz it follows that that $\omega_F > 0$ in $T_{(x,[f])}\mathbb{P}F$. $\square$

Remark that the point $(x, [e_1 \cdots e_r])$ corresponding to a decomposable tensor in $S^r E_x$ is very special. Easy examples show that we do not expect to have $\omega_r > 0$ everywhere. In fact, the exterior differential system (EDS)

$$\omega_r = 0$$

is of interest and will be discussed in Section II.D in the situation when the curvature has the norm positivity property to be introduced in that section.

Example II.C.21: We will illustrate the mechanism of how passing to $S^r E$ increases the Kodaira-Iitaka dimension of the bundle. Let $E \to G(2,4)$ denote the dual of the universal sub-bundle. As above, points of $G(2,4)$ will be denoted by Λ and thought of as lines in $\mathbb{P}^3$. For $v \in \Lambda$ we denote by $[v]$ the corresponding line in $\mathbb{C}^4$. Points of $\mathbb{P}E$ will be (Λ, v)



and then the fibre of $\mathcal{O}_{\mathbb{P}E}(1)$ at (Λ, v) is $[v]$. The fibre $E_\Lambda \cong \Lambda^*$, and we have

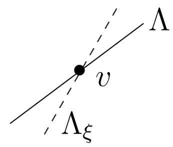
$$\begin{tikzcd}
H^0(G(2,4), E) \arrow[r] \arrow[d, equal] & E_\Lambda \arrow[r] \arrow[d, equal] & 0 \\
C^{4*} \arrow[r] & \Lambda^* \arrow[r] & 0.
\end{tikzcd}$$

The tangent space

$$T_\Lambda G(2,4) \cong \mathrm{Hom}(\Lambda, \mathbb{C}^4/\Lambda)$$

is isomorphic to the horizontal space $H_{(\Lambda,v)} \subset T_{(\Lambda,v)}\mathbb{P}E$. As previously noted, for $\xi \in T_\Lambda G(2,4)$ we have

$$\Theta_E(v,\xi) = \omega(\xi) = 0 \iff \xi(v) = 0 \tag{II.C.22}$$



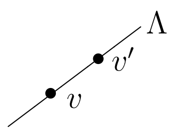
Here Λ_ξ is the infinitesimal displacement of Λ in the direction ξ.

We observe that

$$\varphi_{\mathcal{O}_{\mathbb{P}E}(1)} : \mathbb{P}E \to \mathbb{P}^3 \tag{II.C.23}$$

is the tautological map $(\Lambda, v) \to [v]$, and consequently the fibre of (II.C.23) through v is the $\mathbb{P}^2$ of lines in $\mathbb{P}^3$ through v. The tangent space to this fibre are the ξ's as pictured above. We note that $\dim \mathbb{P}E = 5$ while $\kappa(\mathcal{O}_{\mathbb{P}E}(1)) = n(\mathcal{O}_{\mathbb{P}E}(1)) = 3$.

Points of $\mathbb{P}S^2E$ are (Λ, v, v')



and unless $v = v'$ we have

$$\Theta_{S^2E}(v \cdot v', \xi) \neq 0$$

for any non-zero $\xi \in T_\Lambda G(2,4)$. Thus for ω_2 the curvature form of $\mathcal{O}_{\mathbb{P}S^2E}(1)$ we have

$$\omega_2 > 0 \ \text{ at } \ (\Lambda, v \cdot v')$$

unless $v = v'$; consequently S^2E is big.[13]

We shall give some further observations and remarks concerning the question posed at the beginning of this section.

Proposition II.C.24: *If $E \to X$ is generically globally generated and* $\det E$ *is big, then S^kE is big for some $k > 0$.*

Proof. By standard arguments passing to a blowup of X and pulling E back, we may reduce to the case where E is globally generated. Let $N = h^0(X, E)$ and denote by $Q \to G(N-r, N)$ the universal quotient bundle over the Grassmannian. We then

[13] In this example flatness occurs along a closed, proper algebraic subvariety. In Section II.D below we will discuss the general question/conjecture as to whether this phenomenon is general if the curvature has the norm positivity property.

have a diagram

(II.C.25)
$$\begin{tikzcd} \mathbb{P}E \arrow[r, "\alpha"] \arrow[d] & \mathbb{P}Q \arrow[r, "\beta"] \arrow[d] & \mathbb{P}^{N-1} \\ X \arrow[r, "f"] & G(N-r,N) & \end{tikzcd}$$

where

- $f^*Q = E$;
- $\beta \circ \alpha = \varphi_{\mathcal{O}_{\mathbb{P}E}}(1)$

where $\varphi_{\mathbb{P}E}(1)$ is the map induced by $H^0(\mathcal{O}_{\mathbb{P}E}(1))$. As a metric in $E \to X$ we use the one induced by the standard metric on $Q \to G(N-r,N)$. Then we claim that

- $\Theta_E \geqq 0$;
- $\operatorname{Tr}\Theta_E > 0$ on an open set.

The first of these is clear. For the second, $\det Q$ is an ample line bundle over $G(N-r,r)$ and $\det E = f^*\det Q$. Thus
$$\Theta_{\det E} = \operatorname{Tr}\Theta_E \geqq 0$$
and for $d = \dim X$
$$(\operatorname{Tr}\Theta_E)^d \equiv 0$$
contradicts the assumption that $\det E$ is big. The proposition now follows from Theorem II.C.17. $\square$

This proposition is frequently used in connection with the following well known

Proposition II.C.26: *If S^kE is big for some $k > 0$, then there exist arbitrarily large ℓ such that $S^{k\ell}E$ is generically globally generated.*

Proof. As a general comment, for any holomorphic vector bundle $F \to X$
$$\left\{ \begin{array}{c} F \text{ is generically} \\ \text{globally generated} \end{array} \right\} \iff \left\{ \begin{array}{c} \mathcal{O}_{\mathbb{P}F}(1) \to \mathbb{P}F \text{ is generically} \\ \text{globally generated} \end{array} \right\}.$$
Since sufficiently high powers of a big line bundle are generically globally generated, and since by definition $F \to X$ is big if the line bundle $\mathcal{O}_{\mathbb{P}F}(1)$ is big, we have
$$F \text{ big} \implies \text{ some } S^mF \to X \text{ is generically globally generated.}$$
Taking $F = S^kE$, our assumption then implies that there are arbitrarily large ℓ such that $S^\ell(S^kE)$ is generically globally generated, consequently the direct summand $S^{k\ell}E$ of $S^\ell(S^kE)$ is also generically globally generated. $\square$

Corollary II.C.27: *If $E \geqq 0$ and $\det E > 0$, then S^mE is big for arbitrarily large m.*

Proof. By Theorem II.C.17, S^mE is generically globally generated and $\det S^mE > 0$ so that (II.C.24) applies. $\square$

Remark II.C.28: For a line bundle $L \to X$ we consider the properties

(i) L is *nef*;
(ii) L is *big*;

(iii) L is *semi-ample*.

For vector bundles $E \to X$ one has the corresponding properties using $\mathcal{O}(1)$. In (ii) and (iii) one generally uses $\operatorname{Sym}^m E$ rather than just E itself.

Although all these properties are important, in some sense (iii) is the most subtle (note that (iii) $\Longrightarrow$ (i)). For a specific question, suppose that we have in $L \to X$ a Hermitian metric with Chern form ω such that

(II.C.29) $\quad \omega \geqq 0$ and $\omega > 0$ on $X \backslash Z$ where $Z \subset X$ is a normal crossing divisor.

Then ω defines a Kähler metric ω^* on $X^* := X \backslash Z$, and one may ask the

(II.C.30) **Question:** *Are there properties of the metric ω^*, especially those involving its curvature R_{ω^*}, that imply that $L \to X$ is semi-ample?*

A special case of this question was discussed above; it will be revisited in Section III below.

II.D. **Norm positivity.**

Definition and first properties.

As previously noted, for many purposes including those arising from Hodge theory strict positivity of a holomorphic vector bundle in the sense of (I.B.4) and (I.B.5) is too strong, whereas semi-positivity is too weak. The main observation of this section is that for many bundles that arise naturally in algebraic geometry the curvature has a special form, one that implies semi-positivity in both of the above senses and where in examples the special form has Hodge-theoretic and algebro-geometric interpretations.

Definition II.D.1: Let $E \to X$ be a Hermitian vector bundle with curvature Θ_E. Then Θ_E has the *norm positivity* property if there is a Hermitian vector bundle $G \to X$ and a holomorphic bundle mapping

(II.D.2)
$$A : TX \otimes E \to G$$

such that for $x \in X$ and $e \in E_x$, $\xi \in T_x X$

(II.D.3)
$$\Theta_E(e, \xi) = \|A(\xi \otimes e)\|^2_G.$$

Here we are identifying E_x with E_x^* using the metric, and $\| \;\; \|^2_G$ denotes the square norm in G. In matrix terms, relative to unitary frames in E and G there will be a matrix A of $(1,0)$ forms such that the curvature matrix is given by

(II.D.4)
$$\Theta_E = -{}^t\overline{A} \wedge A.$$

We note that (II.D.3) will hold for any tensors in $TX \otimes E$, not just decomposable ones. As a consequence $E \to X$ is semi-positive in both senses (I.B.4) and (I.B.5).

The main implications of norm positivity will use the following observation:

(II.D.5) *If the curvatures of Hermitian bundles $E, E' \to X$ have the norm positivity property, then the same is true for $E \oplus E' \to X$ and $E \otimes E' \to X$, as well as Hermitian direct summands of these bundles.*

Proof. If we have $A : TX \otimes E \to G$ and $A' : TX \otimes E' \to G'$, then $\Theta_{E\otimes E'} = (\Theta_E \otimes \mathrm{Id}_{E'}) \oplus (\mathrm{Id}_E \otimes \Theta_{E'})$, and

$$(A \otimes \mathrm{Id}_{E'}) \oplus (\mathrm{Id}_E \otimes A') : TX \otimes E \otimes E' \to (G \otimes E') \oplus (E \otimes G')$$

leads to norm positivity for $\Theta_{E\otimes E'}$. The argument for $\oplus$ is evident. $\square$

A result using norm positivity.

The idea is this: For this discussion we abbreviate

$$T = T_x X, \quad E = E_x, \quad G = G_x$$

and identify $E \cong E^*$ using the metric. We have a linear mapping

$$A : T \otimes E \to G, \tag{II.D.6}$$

and using (II.D.3) non-degeneracy properties of this mapping will imply positivity properties of Θ_E. Moreover, in examples the mapping A will have algebro-geometric meaning so that algebro-geometric assumptions will lead to positivity properties of the curvature.

The simplest non-degeneracy property of (II.D.6) is that A is injective; this seems to infrequently happen in practice. The next simplest is that A has injectivity properties in each factor separately. Specifically we consider the two conditions

$$A : T \to \mathrm{Hom}(E, G) \text{ is injective;} \tag{II.D.7}$$

(II.D.8) for general $e \in E$, the mapping $A(e) : T \to G$ given by

$$A(e)(\xi) = A(\xi \otimes e), \quad \xi \in T$$

is injective.

The geometric meanings of these are:

(II.D.9) (II.D.7) *is equivalent to having*

$$\Theta_{\det E} = \mathrm{Tr}\ \Theta_E > 0$$

at x; and

(II.D.10) (II.D.8) *is equivalent to having*

$$\omega > 0$$

at $(x, [e]) \in (\mathbb{P}E)_x$*; here* ω *is the Chern form of* $\mathcal{O}_{\mathbb{P}E}(1)$*.*

This gives the

Proposition II.D.11: *If* $E \to X$ *has a metric whose curvature has the norm positivity property, then*

(i) (II.D.7) $\implies \det E$ *is big;*
(ii) (II.D.8) $\implies E$ *is big.*

A bit more subtle is the following result, which although it is a consequence of Theorem II.C.17 and (II.D.8), for later use we shall give another proof.

Theorem II.D.12: *If the rank r bundle $E \to X$ has a metric whose curvature has the norm positivity property, then*

$$\text{(I.B.4)} \implies S^r E \textit{ is big.}$$

Corollary II.D.13: *With the assumptions in* (II.D.12) *the evaluation map*

$$H^0(X, S^m(S^r E)) \to S^m(S^r E_x), \quad x \in X,$$

is generically surjective for $m \gg 0$.[14]

Proof of Theorem II.D.12. Keeping the above notations and working at a general point in $\mathbb{P}E$ over $x \in X$, given $\xi \in T_x X$ and a basis $e_1, \dots, e_r$ of E_x from (II.D.8) we have

$$\sum_{i=1}^{r} \|A(\xi \otimes e_i)\|_G^2 \neq 0. \tag{II.D.14}$$

Then using (II.D.5) for the induced map

$$A : T_x X \otimes S^r E_x \to S^{r-1} E_x \otimes G_x$$

from (II.D.14) for ω_r the canonical (1,1) form on $\mathbb{P}S^r E$ at the point $(x, [e_1 \cdots e_r])$

$$\left\langle \omega_r, \xi \wedge \bar{\xi} \right\rangle > 0. \qquad \square$$

Remark: Viehweg ([Vie83a]) introduced the notion of *weak positivity* for a coherent sheaf. For vector bundles this means that for any ample line bundle $L \to X$ there is a $k > 0$ such that the evaluation mapping

$$H^0(X, S^\ell(S^k E \otimes L)) \to S^\ell(S^k E \otimes L_x)$$

is generically surjective for $\ell \gg 0$. He then shows that for the particular bundles that arise in the proof of the Iitaka conjecture if one has $\det E > 0$ on an open set, then an intricate cohomological argument gives that E is weakly positive. One may show that (II.D.13) may be used to circumvent the need for weak positivity in this case.

We note that the ample line bundle $L \to X$ is not needed in (II.D.13). We also point out that

$$E_{\mathrm{met}} \geqq 0 \implies E \text{ is weakly positive (cf. [Pă16]).}$$

This is plausible since $S^k E \geqq 0$ and $L > 0 \implies S^k E \otimes L > 0$. In loc. cit. this result is extended to important situations where the metrics have certain types of singularities.

We conclude this section with a discussion of the Chern forms of bundles having the norm positivity property, including the Hodge vector bundle.

[14] Then this also implies that the map

$$H^0(X, S^m(S^r E)) \to S^{mr}$$

is generically surjective.

Proposition II.D.15: *The linear mapping A induces*

$$\wedge^q A : \wedge^q T \to \wedge^q G \otimes S^q E,$$

and up to a universal constant

$$c_q(\Theta) = \| \wedge^q A\|^2.$$

Proof. The notation means

$$\| \wedge^q A\|^2 = (\wedge^q A, \wedge^q A)$$

where in the inner product we use the Hermitian metrics in G and E, and we identify

$$\wedge^q T^* \otimes \overline{\wedge^q T^*} \cong (q,q)\text{-part of } \wedge^{2q} (T^* \otimes \overline{T}^*).$$

Then letting A^* denote the adjoint of A we have

$$\wedge^q \Theta = \wedge^q A \otimes \wedge^q A^* = \wedge^q A \otimes (\wedge^q A)^*$$

and

$$c_q(\Theta) = \operatorname{Tr} \wedge^q(\Theta) = (\wedge^q A, \wedge^q A). \qquad \square$$

In matrix terms, if

$$A = \dim G \times \dim T \text{ matrix with entries in } E$$

then

$$\wedge^q A = \left\{ \begin{array}{c} \text{matrix whose entries are the } q \times q \text{ minors of } A, \\ \text{where the terms of } E \text{ are multiplied as polynomials.} \end{array} \right\}$$

It follows that up to a universal constant for the Hodge vector bundle F

$$c_q(\Theta_F) = \sum_a \Psi_\alpha \wedge \overline{\Psi}_\alpha$$

where the Ψ_α are $(q,0)$ forms. In particular, any monomial $c_I(\Theta_F) \geqq 0$.

The vanishing of the matrix $\wedge^q \Phi_{*,n}$ is *not* the same as $\operatorname{rank} \Phi_{*,n} < q$. In fact,

$$\text{(II.D.16)} \qquad \operatorname{rank} \Phi_{*,n} < q \iff c_1(\Theta_F)^q = 0.$$

In general we have the

Proposition II.D.17: *If $E \to X$ has the norm positivity property, then $P(\Theta_E) \geqq 0$ for any $P \in \mathcal{C}$.*

A proof of this appears in [Gri69].

In the geometric case when we have a VHS arising from a family of smooth varieties we have the period mapping Φ with the end piece of differential being

$$\Phi_{*,n} : T_b B \to \operatorname{Hom}\left(H^0(\Omega^n_{X_b}), H^1(\Omega^{n-1}_{X_b})\right)$$

and the algebro-geometric interpretation of (II.D.16) is standard; e.g., $\Phi_{*,n}$ injective is equivalent to local Torelli holding for the $H^{n,0}$-part of the Hodge structure.

The following is a result that pertains to a question that was raised above.

Proposition II.D.18: *If $\Phi : B \to \Gamma\backslash D$ has no trivial factors, and if $h^{n,0} \leqq \dim B$ and $H^0(\overline{B}, F_e) \neq 0$, then*

$$c_{h^{n,0}}(F) \neq 0.$$

Proof. We will first prove the result when $B = \overline{B}$. We let $s \in H^0(F, B)$ and assume that $c_{h^{n,0}}(F) = 0$. Then s is everywhere non-zero and we may go to a minimum of $\|s\|^2$. From Proposition II.C.16 we have $D\sigma = 0$, which implies that the norm $\|s\|$ is constant and

$$\nabla s = 0$$

where ∇ is the Gauss-Manin connection. Using the arguments [Gri70] we may conclude that the variation of Hodge structure has a trivial factor.

If $B \neq \overline{B}$, the arguments given in Section III below may be adapted to show that the proof still goes through. The point is the equality of the distributional and formal derivatives that arise in integrating by parts. □

The exterior differential system defined by a Chern form. In this section we will discuss the exterior differential system

$$\omega = 0 \tag{II.D.19}$$

defined by the Chern form of the line bundle $\mathcal{O}_{\mathbb{P}E}(1)$ where $E \to X$ is a Hermitian vector bundle whose curvature has the norm positivity property (II.D.1). Without assuming the norm positivity property, this type of EDS has been previously studied in [BK77] and [Som59] and also appeared in [Kol87].

Here our motivation is the following question:

(II.D.20) *Under what conditions can one say that the Kodaira-Iitaka dimension of $E \to X$ is equal to its numerical dimension?*

Proposition II.D.21: *The exterior differential system* (II.D.19) *defines a foliation of $\mathbb{P}E$ by complex analytic subvarieties $W \subset \mathbb{P}E$ with the properties*

(i) W meets the fibres of $\mathbb{P}E \xrightarrow{\pi} X$ transversely; thus $W \to \pi(W)$ is an étalé map;

(ii) the restriction $E\big|_{\pi(W)}$ is flat.

Proof. Since $\omega > 0$ on the fibres of $\mathbb{P}E \to X$, the vectors $\xi \in T_{(x,[e])}\mathbb{P}E$ that satisfy $\omega(\xi) = 0$ project isomorphically to TX. The image of these vectors is the subspace (here identifying ξ with $\pi_*(\xi)$)

$$\{\xi \in T_x X : A(e \otimes \xi) = 0\}. \tag{II.D.22}$$

This is the same as the subspace of $T_x X$ defined by

$$\Theta_E(e \otimes \xi) = 0,$$

which implies that $E\big|_{\pi(w)}$ is flat. □

Remark: Given any holomorphic bundle map

$$A : TX \otimes E \to G, \tag{II.D.23}$$

if we have a metric in $E \to X$ we may use it to identify $E \cong E^*$ and then define the horizontal sub-bundle $H \subset T\mathcal{O}_{\mathbb{P}E}(1)$. It follows that (II.D.22) defines a C^∞ distribution (with jumping fibre dimensions) in $T\mathcal{O}_{\mathbb{P}E}(1)$, and when the map (II.D.23) arises from the curvature of the metric connection as in (II.D.3) this distribution is integrable and the maximal leaves of the corresponding foliation of $\mathbb{P}E$ by complex analytic subvarieties are described by Proposition II.D.21.

The restrictions $E\big|_{\pi(W)}$ being flat, the monodromy is discrete. Heuristic arguments suggest that the maximal leaves $W \subset \mathbb{P}E$ are *closed* analytic subvarieties.

Conjecture II.D.24: *Finite monodromy provides the necessary and sufficient condition to have the equality*

$$\kappa(E) = n(E)$$

of Kodaira-Iitaka and numerical dimensions of a holomorphic vector bundle having a Hermitian metric whose curvature satisfies the norm positivity condition.

The idea is that the quotient $\mathbb{P}E/\sim$, where $\sim$ is the equivalence relation given by the connected components of the foliation defined by (II.D.19), exists as a complex analytic variety of dimension equal to $n(E)$, and there is a meromorphic mapping

$$\mathbb{P}E \dashrightarrow \mathbb{P}E/\sim$$

together with an ample line bundle on $\mathbb{P}E/\sim$ that pulls back to $\mathcal{O}_{\mathbb{P}E}(1)$. The rather simple guiding model here is the dual of the universal sub-bundle over the Grassmannian that was discussed above. In fact, the conjecture holds if $E \to X$ is globally generated with metrics induced from the corresponding mapping to a Grassmannian.

We note that the foliation defined by the null space of the holomorphic bi-sectional curvature on quotients of bounded symmetric domains has been studied in [Mok87]. In this case the leaves are generally not closed.

Finally we point out the interesting papers [CD17] and [CD14]. In these papers the authors construct examples of smooth fibrations

$$f : X \to B$$

of a surface over a curve such that for $E = f_*\omega_{X/B}$ one has

$$E = A \oplus Q$$

where A is an ample vector bundle and Q is a flat $\mathcal{U}(m, \mathbb{C})$-bundle with infinite monodromy group.[15] In this case the leaves of the EDS (II.D.19) may be described as follows: For each $b \in B$ we have

$$\mathbb{P}Q_b^* \subset \mathbb{P}E_b^*$$

and using the flat connection on Q^* the parallel translate of any point in $\mathbb{P}Q_b^*$ defines an integral curve of the EDS.

II.E. **Cotangent bundle.**

(i) *Statement of results.* Let $\Phi : B \to \Gamma\backslash D$ be a period mapping with image a quasi-projective variety $P \subset \Gamma\backslash D$. The $G_{\mathbb{R}}$-invariant metric on D constructed from the Cartan-Killing form on $\mathfrak{g}_{\mathbb{R}}$ induces a Kähler metric on the Zariski open set P^0 of smooth points of P. We denote by $R(\eta, \xi)$ and $R(\xi)$ the holomorphic bi-sectional and holomorphic sectional curvatures respectively.

[15] We note that $Q \subset f_*\omega_{X/B} \subset R^1_f\mathbb{C}_X$ is *not* flat relative to the Gauss-Manin convention on $R^1_f\mathbb{C}_X$.

THEOREM II.E.1: [16] *There exists a constant $c>0$ such that*

(i) $R(\xi)\leqq -c$ *for all* $\xi\in TP^0$*;*
(ii) $R(\eta,\xi)\leqq 0$ *for all* $\eta,\xi\in TP^0\times_{P^0}TP^0$*;*
(iii) *For any* $b\in P_o$ *there exists a* $\xi\in T_bP^0$ *such that* $R(\eta,\xi)\leqq -c/2$ *for all* $\eta\in T_bP^0$*.*

Observe that using (II.C.7) from [BKT13] (iii) follows from (i) and (ii). As a corollary to (ii) we have

(iv) $R(\eta,\xi)\leqq -c/2$ is an open set in $TP^0\times_{P^0}TP^0$.

We note that (iii) implies that this open set projects onto each factor in $P^0\times P^0$.

As applications of the proof of Theorem II.E.1 and consideration of the singularity issues that arise we have the following results of Zuo [Zuo00] and others (cf. Chapter 13 in [CMSP17]):

$$P \text{ is of log-general type,} \tag{II.E.2}$$

$$\operatorname{Sym}^m\Omega^1_P(\log) \text{ is big for } m\geqq m_0. \tag{II.E.3}$$

The result in (II.E.2) means that for any desingularization $\overline{\widetilde{P}}$ of $\overline{P}$ with $\widetilde{P}$ lying over P and $\widetilde{Z}=\overline{\widetilde{P}}\backslash\widetilde{P}$, the Kodaira dimension

$$\kappa\left(K_{\overline{\widetilde{P}}}(\widetilde{Z})\right)=\dim P.$$

The result in (II.E.3) means that

$$\operatorname{Sym}^m\Omega^1_{\overline{\widetilde{M}}}(\log\widetilde{Z}) \text{ is big for } m\geqq m_0.$$

The proof will show that we may choose m_0 to depend only on the Hodge numbers for the original VHS.

The proof will also show that

$$P \text{ is of stratified-log-general type,} \tag{VI.B.2$_\mathrm{S}$}$$

$$\operatorname{Sym}^m\Omega^1_{\overline{P}}(\log) \text{ is stratified-big for } m\geqq m_0. \tag{VI.B.3$_\mathrm{S}$}$$

Here stratified-log-general type means that there is a canonical stratification $\{P_I^*\}$ of $\overline{P}$ such that each stratum P_I^* is of log-general type. There is the analogous definition for stratified big.

Without loss of generality, using the notations above we may take $\overline{\widetilde{\widetilde{M}}}=\overline{B}$, $\widetilde{P}=B$ and $\widetilde{Z}=Z$; we shall assume this to be the case.

Remark: The results of Zuo, Brunebarbe and others imply that $K_{\overline{P}}(\log)$ and $\Omega^1_P(\log)$ are weakly positive in the sense of Viehweg. This is also a consequence of (II.E.3).

The proof of Theorem II.E.1 will be done first in the case

$$B=\overline{B} \text{ and } \Phi_* \text{ is everywhere injective.} \tag{II.E.4}$$

It is here that the main ideas and calculations occur.

---
[16]This theorem is a variant of results in [CMSP17] and [Zuo00].

In general the singularities that arise are of the types

$$\text{(II.E.5)} \qquad \begin{cases} \text{(a)} & \text{where } \Phi_* \text{ fails to be injective (e.g., the inverse image of } P_{\text{sing}}), \\ \text{(b)} & \text{on } Z = \overline{B}\backslash B \text{ where the VHS has singularities,} \\ \text{(c)} & \text{the combination of (a) and (b).} \end{cases}$$

As will be seen below, there will be a coherent sheaf I with

$$\Phi_*(TB) \subset I \subset \Phi^*(T(\Gamma\backslash D)).^{17}$$

Denoting by I^o the Zariski open set where I is locally free, there is an induced metric and corresponding curvature form for I^o, and with the properties (i), (ii) in the theorem for I^o Theorem II.E.1 will follow from the curvature decreasing property of holomorphic sub-bundles, which gives

$$R(\eta,\xi) = \Theta_{TP^0}(\eta,\xi) \leqq \Theta_{I^o}(\eta,\xi).$$

As for the singularities, if we show that

$$\text{(II.E.6)} \qquad \kappa\,(\det I^o(\log)) = \dim B$$

$$\text{(II.E.7)} \qquad \operatorname{Sym}^m I^o(\log) \ \textit{ is big}$$

then (II.E.2) and (II.E.3) will follow from the general result: If over a projective variety Y we have line bundles L, L' and a morphism $L \to L'$ that is an inclusion over an open set, then

$$\text{(II.E.8)} \qquad L \to Y \ \textit{ big} \implies L' \to Y \ \textit{ is big.}$$

We will explain how (II.E.6) and (II.E.7) will follow from (II.E.8) for suitable choices of Y, L and L'.

(ii) *Basic calculation.* It is conventient to use Simpson's system of Higgs bundles framework (cf. [Sim92] and Chapter 13 in [CMSP17]) whereby a VHS gives a system of holomorphic vector bundles E^p, and maps

$$E^{p+1} \xrightarrow{\theta^{p+1}} E^p \otimes \Omega^1_B \xrightarrow{\theta^p} E^{p-1} \otimes \wedge^2\Omega^2_X$$

that satisfy

$$\text{(II.E.9)} \qquad \theta^p \wedge \theta^{p+1} = 0.$$

Thus there is induced

$$E^{p+1} \xrightarrow{\theta^{p+1}} E^p \otimes \Omega^1_B \xrightarrow{\theta^p} E^{p-1} \otimes \operatorname{Sym}^2 \Omega^1_B.$$

The data $(\bigoplus_p E^p \otimes \operatorname{Sym}^{k-p}\Omega^1_B, \bigoplus_p \theta^p)$ for any k with $k \geqq p$ is related to the notion of an infinitesimal variation of Hodge structure (IVHS) (cf. 5.5 ff. in [CMSP17]).

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[17] Here we are identifying a coherent sub-sheaf of a vector bundle with the corresponding family of linear subspaces in the fibres of the vector bundle. The coherent sheaf I will be a subsheaf of the pull-back $\Phi^*T(D\backslash\Gamma)_h$ of the horizontal tangent spaces to $\Gamma\backslash D$. The critical step in the calculation will be that it is integrable as a subsheaf $\Phi^*T(\Gamma\backslash D)_h$.

In our situation the vector bundles E^p will have Hermitian metrics with Chern connections D^p. The metrics define adjoints

$$\theta^{p^*} : E^p \to E^{p+1} \otimes \overline{\Omega^1_B},$$

and in the cases we shall consider if we take the direct sum over p we obtain

$$(E, \nabla = \theta^* + D + \theta), \quad \textit{with } \text{(II.E.9)} \textit{ equivalent to } \nabla^2 = 0.$$

The properties uniquely characterizing the Chern connection together with $\nabla^2 = 0$ give for the curvature matrix of E^p the expression

$$\Theta_{E^p} = \theta^{p+1} \wedge \theta^{p+1^*} + \theta^{p^*} \wedge \theta^p, \tag{II.E.10}$$

which is a difference of non-negative terms each of which has the norm positivity property (II.D.3) (cf. [Zuo00] and Chapter 13 in [CMSP17]).

For a PVHS (V, Q, ∇, F) we now set

$$E^p = \mathrm{Gr}^p_F \operatorname{Hom}_Q(V, V), \qquad -n \leqq p \leqq n$$

where Gr^p_F is relative to the filtration induced by F on $\operatorname{Hom}_Q(V, V)$. At each point b of B there is a weight zero PHS induced on $\operatorname{Hom}_Q(V, V) = \mathfrak{g}$ and

$$E^p_b = \mathfrak{g}^{p,-p}$$

with the bracket

$$[\ ,\ ] : E^p \otimes E^q \to E^{p+q}.$$

Thinking of θ as an element in $\mathfrak{g} \otimes \Omega^1_B$, the integrability condition II.E.9 translates into

$$[\theta, \theta] = 0. \tag{II.E.11}$$

We shall use the notation

$$\mathrm{Gr}^p = \mathrm{Gr}^p_F \operatorname{Hom}_Q(V, V)$$

rather than E^p for this example.

The differential of Φ gives a map

$$\Phi_* : TB \to \mathrm{Gr}^{-1}.$$

Definition: $I \subset \mathrm{Gr}^{-1}$ is the coherent subsheaf generated by the sections of Gr^{-1} that are locally in the image of Φ_* over the Zariski open set where Φ_* is injective.

For ξ a section of I we denote by ad_ξ the corresponding section of Gr^{-1}. The integrability condition (II.E.11) then translates into the first part of the

PROPOSITION II.E.12: *I is a sheaf of abelian Lie sub-algebras of $\bigoplus_p \mathrm{Gr}^p$. For η, ξ sections of I*

$$\Theta_{\mathrm{Gr}^{-1}}(\eta, \xi) = -\| \operatorname{ad}^*_\xi(\eta)\|^2.$$

Proof. For $\eta, \xi \in \mathrm{Gr}^{-1}$ the curvature formula (II.E.10) is

$$\Theta_{\mathrm{Gr}^{-1}}(\eta, \xi) = \| \operatorname{ad}_\xi(\eta)\|^2 - \| \operatorname{ad}^*_\xi(\eta)\|^2.$$

The result then follows from $\operatorname{ad}_\xi(\eta) = [\xi, \eta] = 0$ for $\eta, \xi \in I$. ☐

On the open set where I^o is a vector bundle with metric induced from that on Gr^{-1} we have

$$\Theta_{I^o}(\eta,\xi) \leqq \Theta_{\mathrm{Gr}^{-1}}(\eta,\xi) \leqq 0.$$

The first term is the holomorphic bi-sectional curvature for the indued metric on $\Phi(B)$.

To complete the proof of Theorem II.E.10 we need to show the existence of $c > 0$ such that for all ξ of unit length

$$\|\operatorname{ad}^*_\xi(\xi)\| \geqq c. \tag{II.E.13}$$

The linear algebra situation is this: At a point of B we have

$$V = \bigoplus_{p+q=n} V^{p,q}$$

and ξ is given by maps

$$A_p : V^{p,q} \to V^{p-1,q+1}, \qquad \left\lfloor \frac{n+1}{2} \right\rfloor \leqq p \leqq n.$$

In general a linear map

$$A : E \to F$$

between unitary vector spaces has *principal values* λ_i defined by

$$Ae_i = \lambda_i f_i, \qquad \lambda_i \text{ real and non-zero}$$

where e_i is a unitary basis for $(\ker A)^\perp$ and f_i is a unitary basis for $\operatorname{Im} A$. The square norm is

$$\|A\|^2 = \operatorname{Tr} A^*A = \sum_i \lambda_i^2.$$

We denote by $\lambda_{p,i}$ the principal values of A_p. The $\lambda_{p,i}$ depend on ξ, and the square norm of ξ as a vector in $T_pB \subset T_{\Phi(p)}(\Gamma\backslash D)$ is

$$\|\xi\|^2 = \sum_p \sum_i \lambda_{p,i}^2.$$

In the above we now replace V by $\operatorname{Hom}_Q(V,V)$ and use linear algebra to determine the principal values of ad^*_ξ. These will be quadratic in the $\lambda_{p,i}$'s, and then

$$\|\operatorname{ad}_{\xi^*}(\xi)\|^2$$

will be quartic in the $\lambda_{p,i}$. A calculation gives

$$\|\operatorname{ad}_{\xi^*}(\xi)\|^2 = \sum_p \left(\frac{\sum_i a_p \lambda_{p,i}^4}{\left(\sum_i \lambda_{p,i}^2\right)^2} \right) \tag{II.E.14}$$

where the a_p are non-negative integers that are positive if $A_p \neq 0$, and from this by an elementary algebra argument we may infer the existence of the $c > 0$ in Theorem II.E.1. $\square$

At this point we have proved the theorem. The basic idea is simple:

> *For a VHS the curvature* (II.E.10) *of the Hodge bundles is a difference of non-negative terms, each of which is of norm positivity type where the "A" in Definition II.D.3 is a Kodaira-Spencer map or its adjoint. For the* $\mathrm{Hom}_Q(V,V)$ *variation of Hodge structure,* $A(\xi)(\eta) = [\xi, \eta] = 0$ *by integrability. Consequently the curvature form has a sign, and a linear algebra calculation gives the strict negativity* $\Theta_I(\xi,\xi) \leqq -c\|\xi\|^4$ *for some* $c > 0$.[18]

The central point here is the observation in [Zuo00] that curvatures have a sign on kernels of Kodaira-Spencer mappings.

(iii) *Singularities.* The singularity issues were identified in (II.E.5), and we shall state a result that addresses them. The proof of this result follows from the results in [CKS86] as extended in [GGLR20], [Kol87] and the arguments in [Zuo00].[19]

Using the notations introduced in (ii) above, a key observation is that the differential

$$\Phi_* : TB \to \mathrm{Gr}^{-1}$$

extends to

$$\Phi_* : T\overline{B}\,\langle -Z\rangle \to \mathrm{Gr}_e^{-1}$$

where $T\overline{B}\,\langle -Z\rangle = \Omega^1_{\overline{B}}(\log Z)^*$ and Gr_e^{-1} is the canonical extension to $\overline{B}$ of $\mathrm{Gr}^{-1} \to B$. This is just a reformulation of the general result (cf. [CMSP17]) that for all p, $\theta^p : E^p \to E^p \otimes \Omega^1_{\overline{B}}$ extends to

$$\theta^p_e : E^p_e \to E^{p-1}_e \otimes \Omega^1_{\overline{B}}(\log Z). \tag{II.E.15}$$

As noted above, the image $\Phi_* TB \subset \mathrm{Gr}^{-1}$ generates a coherent subsheaf $I \subset \mathrm{Gr}^{-1}$ and from (II.E.15) we may infer that I extends to a coherent subsheaf $I_e \subset \mathrm{Gr}_e^{-1}$. As in [Zuo00] we now blow up $\overline{B}$ to obtain a vector sub-bundle of the pullback of Gr^{-1} and note that $I_e \subset \mathrm{Gr}_e^{-1}$ *will be an integrable sub-bundle.*

The metric on Gr^{-1} induces a metric in I and we use the notations

- φ = Chern form of $\det I^{o*}$;
- ω = Chern form of $\mathcal{O}_{\mathbb{P}I^{o*}}(1)$.

THEOREM II.E.16: *Both* φ *and* ω *extend to closed,* $(1,1)$ *currents* φ_e *and* ω_e *on* $\overline{B}$ *and* $\mathbb{P}I_e^*$ *that respectively represent* $c_1(\det I_e^*)$ *and* $c_1(\mathcal{O}_{\mathbb{P}I_e^*})(1)$. *They have mild logarithmic singularities*[20] *and satisfy*

- $\varphi_e \geqq 0$ *and* $\varphi_e > 0$ *on an open set;*
- $\omega_e \geqq 0$ *and* $\omega_e > 0$ *on an open set.*

With one extra step this result follows from singularity considerations similar to those in Section III below. The extra step is that

---

[18]This first proof of the result that appeared in the literature was Lie-theoretic where the metric on $\mathfrak{g}$ was given by the Cartan-Killing form. As will be illustrated below the above direct algebra argument is perhaps more amenable to computation in examples.

[19]These arguments have been amplified at a number of places in the literature; cf. [VZ03] and [Pă16].

[20]These are defined in Section III below.

I_e is not a Hodge bundle, but rather it is the kernel of the map $\theta^{-1}: \mathrm{Gr}_e^{-1} \to \mathrm{Gr}_e^{-2} \otimes \Omega^1_{\overline{B}}(\log Z)$.

As was noted in [Zuo00], either directly or from (5.20) in [Kol87], using the definition and properties of mild logarithmic singularities as defined in Section III.A below, we may infer the stated properties of φ_e and ω_e. $\square$

Remark: It is almost certainly not the case that any sub-bundle $G \subset \mathrm{Gr}_e^{-1}$ will have Chern forms with mild logarithmic singularities. The bundle I_e is special in that it is the kernel of the map $\mathrm{Gr}_e^{-1} \to \mathrm{Gr}_e^{-2} \otimes \Omega^1_{\overline{B}}(\log Z)$. Although we have not computed the 2^{nd} fundamental form of $I_e \subset \mathrm{Gr}_e^{-1}$, for reasons to be discussed below it is reasonable to expect it to also have good properties.

Note added in proof: This has now been done in [GGR21]. The level 1 extension data along the fibres of Φ_e maps to compact complex tori and the 2^{nd} fundamental form referred to above is expressed in terms of the associated *Gauss mapping.* This is a part of the rich geometry underlying the period mapping at infinity.

The issue of the curvature form of the induced metric on the image $P = \Phi(B) \subset \Gamma\backslash D$ seems likely to be interesting. Since the metric on the smooth points $P^0 \subset P$ is the Kähler metric given by the Chern form of the augmented Hodge line bundle, the curvature matrix of TP^0 is computed from a positive (1,1) form that is itself the curvature of a singular metric. In the 1-parameter case the dominant term in ω is the Poincaré metric $\mathrm{PM} = dt \otimes d\bar{t}/|t|^2(-\log|t|)^2$, and the curvature of the PM is a positive constant times $-\mathrm{PM}$. One may again suspect that the contributions of the lower order terms in ω are less singular than PM.

(iv) *Examples.* On the smooth points of P^0 of the image of a period mapping the holomorphic bi-sectional curvature satisfies

$$R(\eta, \xi) \leqq 0, \tag{II.E.17}$$

and for η, ξ in an open set in $TP^0 \times_{P^0} TP^0$ it is strictly negative. This raises the interesting question of the degree of flatness of T^*P^0. In the classical case when D is a Hermitian symmetric domain and $B = \Gamma\backslash D$ is compact this question has been studied by Mok [Mok87] and others. In case B is a Shimura variety the related question of the degree of flatness of the extended Hodge bundle F_e over a toriodal compactification of $\Gamma\backslash D$ is one of current interest (cf. [Bru16a], [Bru16b] and the references cited there).

Here we shall discuss the equation

$$\Theta_{I^0}(\eta, \xi) = 0$$

over the smooth locus P^0 of P. In view of (II.E.10) this equation is equivalent to

$$\mathrm{ad}_{\xi^*}(\eta) = 0, \qquad \eta \in I. \tag{II.E.18}$$

To compute the dimension of the solution space to this equation, we use the duality

$$\ker(\mathrm{ad}^*_\xi) = (\mathrm{Im}(\mathrm{ad}_\xi))^\perp$$

to have

$$\dim\ker(\mathrm{ad}^*_\xi) = \dim\left(\mathrm{coker}\left(\mathrm{Im}\{\mathrm{ad}_\xi : \mathrm{Gr}^0 \to \mathrm{Gr}^{-1}\}\right)\right).$$

Since I depends on the particular VHS, at least as a first step it is easier to study the equation

$$\mathrm{Ad}_{\xi^*}(\eta) = 0, \qquad \eta \in \mathrm{Gr}^{-1}\,. \tag{II.E.19}$$

Because the curvature form decreases on the sub-bundle $I \subset \mathrm{Gr}^{-1}$, over P^0 we have

$$\text{(II.E.18)} \Longrightarrow \text{(II.E.19)}$$

but in general not conversely.

Example 1: For weight $n = 1$ with $h^{1,0} = g$, with a suitable choice of coordinates the tangent vector ξ is given by $g \times g$ symmetric matrix A, and on Gr^{-1} we have

$$\dim\ker(\mathrm{ad}^*_\xi) = \binom{g - \text{rank } A + 1}{2} \tag{II.E.20}$$

Proof. At a point we may choose a basis for that $Q = \left(\begin{smallmatrix} 0 & Ig \\ -Ig & 0 \end{smallmatrix}\right)$ and

$$\begin{aligned}
F^1 \text{ is given by } &\begin{pmatrix} \Omega \\ I_g \end{pmatrix}, \quad \mathrm{Im}\,\Omega > 0 \\
\xi \in \mathrm{Gr}^{-1} \text{ is given by } &\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad A = {}^tA \\
\eta \in \mathrm{Gr}^{0} \text{ is given by } &\begin{pmatrix} C & 0 \\ 0 & -{}^tC \end{pmatrix}.
\end{aligned}$$

Then

$$[\xi, \eta] = \begin{pmatrix} 0 & AC + {}^tCA \\ 0 & 0 \end{pmatrix}.$$

Diagonalizing A and using (II.E.18) we obtain (II.E.20).

Example 2: For weight $n = 2$, ξ is given by

$$A = h^{2,0} \times h^{1,1} \text{ matrix.}$$

We will show that on Gr^{-1}

$$\dim\ker(\mathrm{ad}^*_\xi) = (h^{2,0}\text{-rank } A)(h^{1,1}\text{-rank } A). \tag{II.E.21}$$

Proof. We may choose bases so that $Q = \operatorname{diag}(I_{h^{2,0}}, -I_{h^{1,1}}, I_{h^{2,0}})$ and

$$F^2 \text{ is given by } \begin{pmatrix} \Omega \\ 0 \\ i\Omega \end{pmatrix}, \quad \Omega \text{ non-singular,}$$

$$\xi \text{ is given by } \begin{pmatrix} 0 & A & 0 \\ 0 & 0 & {}^tA \\ 0 & 0 & 0 \end{pmatrix},$$

$$\eta \text{ is given by } \begin{pmatrix} C & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & -{}^tC \end{pmatrix}.$$

Then

$$[\xi, \eta] = \begin{pmatrix} 0 & AC - DA & 0 \\ 0 & 0 & {}^tAD + {}^t(AC) \\ 0 & 0 & 0 \end{pmatrix}.$$

Choosing bases so that $A = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{smallmatrix}\right)$ and $D = \left(\begin{smallmatrix} D_{11} & D_{12} \\ -{}^tD_{12} & D_{22} \end{smallmatrix}\right)$, we have

$$AC - DA = \begin{pmatrix} C_{11} & -D_{11} & D_{12} \\ 0 & -{}^tD_{12} & 0 \end{pmatrix}.$$

Setting $\operatorname{rk}(E) = \operatorname{rank} E$ for a matrix E, this gives

$$\begin{array}{r c} & \begin{array}{cc} \operatorname{rk} A & h^{2,0}\text{-}\operatorname{rk} A \end{array} \\ \begin{array}{r} \operatorname{rk} A \\ h^{1,1}\text{-}\operatorname{rk} A \end{array} & \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \end{array}$$

where the $*$'s are arbitrary. □

As in the $n = 1$ case we note that

(II.E.22) $$A \text{ of maximal rank} \iff \ker(\operatorname{ad}^*_\xi) = 0.$$

Example 3: Associated to a several parameter nilpotent orbit[21]

$$\exp\left(\sum_i \ell(t_i) N_i\right) \cdot F$$

is a nilpotent cone $\sigma = \{N_\lambda = \sum \lambda_i N_i, \lambda_i > 0\}$ and the weight filtration $W(N)$ is independent of $N \in \sigma$. As discussed in Section 2 of [GGLR20], without loss of generality in what follows here we may assume that the LMHS associated to $N \in \sigma$ is $\mathbb{R}$-split. Thus there is a single $Y \in \operatorname{Gr}^0 \operatorname{Hom}_Q(V, V)$ such that for any $N \in \sigma$

$$[Y, N] = -2N,$$

[21]Here $F \in \check{D}$, the compact dual to the period domain.

and using the Hard Leftschetz Property $N^k : \mathrm{Gr}^{W(N)}_{n+k}(V) \xrightarrow{\sim} \mathrm{Gr}^{W(N)}_{n-k}(V)$ we may uniquely complete Y, N to an $\mathrm{sl}_2\,\{N, Y, N^+\}$. Let $\mathfrak{g}_\sigma \subset \mathrm{End}(\mathrm{Gr}^{W(N)}_\bullet V)$ be the Lie algebra generated by the N, Y, N^+'s as N varies over σ. The properties of this important Lie algebra introduced by Looijenga-Lunts will be discussed elsewhere; here we only note that $\mathfrak{g}_\sigma$ is semi-simple and that the nilpotent orbit gives a period mapping

$$\Delta^{*k} \xrightarrow{\Phi_\sigma} \Gamma_{\mathrm{loc}}\backslash D_\sigma$$

where $D_\sigma = G_{\sigma,\mathbb{R}}/H_\sigma$ is a Mumford-Tate sub-domain of D. Of interest are the holomorphic bi-sectional curvatures of $\Phi_\sigma(\Delta^{*k})$. We shall not completely answer this, but shall give a proof of the

Proposition II.E.23: *$\Theta_I(\eta, N) = 0$ for all $N \in \sigma$, if and only if, $\eta \in \mathcal{Z}(\mathfrak{g}_\sigma)$.*

Proof. We denote by $\mathfrak{g}_\mathbb{C} = \bigoplus_p \mathfrak{g}^{p,-p}$ the Hodge decomposition on the associated graded to the limiting mixed Hodge structure defined by σ. The Hodge metric is given on $\mathfrak{g}_\mathbb{C}$ by the Cartan-Killing form, and its restriction to $\mathfrak{g}^{-1,-1}$ is non-degenerate.[22] The decomposition of $\mathfrak{g}_\mathbb{C}$ into N-strings for the sl_2 given by $\{N, Y, N^+\}$ is orthogonal with respect to the Hodge metric, from which we may infer that the adjoint ad_{N^*} acts separately on each N-string. The picture is something like

$$\eta\circ \underset{N^*}{\overset{\eta}{\rightleftarrows}} \circ \underset{N^*}{\overset{\eta}{\rightleftarrows}} \circ \underset{N^*}{\overset{\eta}{\rightleftarrows}} \circ.$$

Because N is an isomorphism the same is true of N^*; consequently

$$\Theta_I(\eta, N) = 0 \iff \eta \text{ belongs to an } N\text{-string of length } 1,$$

and this implies that $[\eta, Y] = [\eta, N^+] = 0$. By varying N over σ we may conclude the proposition. ☐

Example 4: One of the earliest examples of the positivity of the Hodge line bundle arose in the work of Arakelov ([Ara71]). For 1-parameter families it gives an *upper* bound on the degree of the Hodge line bundle in terms of the degree of the logarithmic canonical bundle of the parameter spaces.[23] This result has been extended in a number of directions; we refer to [CMSP17], Section 13.4 for further general discussion and references to the literature.

One such extension is due to [Zuo00], [VZ03] and [VZ06]. This proof of that result centers around the above observation that the curvature of Hodge bundles has a sign on the kernels of Kodaira-Spencer mappings. There is a new ingredient in the argument that will be useful in other contexts and we shall now explain this. As above there are singularity issues that arise where the differential of Φ fails to be injective. These may be treated in a similar manner to what was done above, and for simplicity of exposition and to get at the essential new point we shall assume

[22] The decomposition of $\mathfrak{g}_\mathbb{C}$ into the primitive sub-spaces and their images under powers of N depends on the particular N. The Hodge metric on $\mathfrak{g}^{-1,-1}$ is only definite on the subspaces arising from the primitive decomposition for such an N.

[23] Using the above notations, the logarithmic canonical bundle of the parameter space is $K_{\overline{B}}(Z)$.

that Φ_* is everywhere injective and that the relevant Kodaira-Spencer mappings have constant rank.

The basic Arakelov-type inequality then exists at the curvature level. For a variation of Hodge structure (V, Q, ∇, F) over B with a completion to $\overline{B}$ with $Z = \overline{B} \backslash B$ a reduced normal crossing divisor, the inequality is

$$\left(\begin{matrix}\text{curvature of} \\ \det \operatorname{Gr}^p V\end{matrix}\right) \leqq C_p \left(\begin{matrix}\text{curvature of} \\ \det \Omega^1_{\overline{B}}(\log Z)\end{matrix}\right) \tag{II.E.24}$$

where C_p is a positive constant that depends on the ranks of the Kodaira-Spencer mappings. Here we will continue using the notations

$$\begin{cases}\operatorname{Gr}^p V = F^p V / F^{p+1} V, \\ \operatorname{Gr}^p V \xrightarrow{\theta} \operatorname{Gr}^{p-1} V \otimes \Omega^1_{\overline{B}}(\log Z).\end{cases} \tag{II.E.25}$$

The second of these was denoted by θ^p above; we shall drop the "p" here but note that below we shall in this argument alone use θ^ℓ to denote the ℓ^{th} iterate of θ.

Proof of (II.E.24). Using the integrability condition (II.E.9) the iterates of (II.E.25) give

$$\operatorname{Gr}^p V \xrightarrow{\theta^\ell} \operatorname{Gr}^{p-\ell} V \otimes \operatorname{Sym}^\ell \Omega^1_{\overline{B}}(\log Z)$$

We use the natural inclusion $\operatorname{Sym}^\ell \Omega^1_{\overline{B}}(\log Z) \subset \overset{\ell}{\otimes} \Omega^1_{\overline{B}}(\log Z)$ and consider this map as giving

$$\operatorname{Gr}^p V \xrightarrow{\theta^\ell} \operatorname{Gr}^{p-\ell} V \otimes \left(\overset{\ell}{\otimes} \Omega^1_{\overline{B}}(\log Z)\right). \tag{II.E.26}$$

There is a filtration

$$\ker \theta \subset \ker(\theta^2) \subseteq \cdots \subseteq \ker \theta^{p+1} = \operatorname{Gr}^p V$$

and $\operatorname{Gr}^p V$ has graded quotients

$$\ker \theta, \frac{\ker \theta^2}{\ker \theta}, \ldots, \frac{\operatorname{Gr}^p V}{\ker \theta^p}.$$

The crucial observation due to Zuo ([Zuo00]) (and what motivates the above use of $\overset{\ell}{\otimes}$ rather than Sym^ℓ), is

$$\begin{gathered}\frac{\ker \theta^\ell}{\ker \theta^{\ell+1}} \hookrightarrow \operatorname{Gr}^{p-\ell+1} V \otimes \left(\overset{\ell}{\otimes} \Omega^1_{\overline{B}}(\log Z)\right) \\ \textit{lies in } K^{p-\ell+1} \otimes \left(\overset{\ell-1}{\otimes} \Omega^1_{\overline{B}}(\log Z)\right) \quad \textit{where} \\ K^{p-\ell+1} = \ker\left\{\operatorname{Gr}^{p-\ell+1} V \xrightarrow{\theta} \operatorname{Gr}^{p-\ell} V \otimes \Omega^1_{\overline{B}}(\log Z)\right\}.\end{gathered} \tag{II.E.27}$$

From this we infer that

(i) $K^p, K^{p-1}, \ldots, K^o$ all have negative semi-definite curvature forms;

(ii) $\frac{\ker\theta^\ell}{\ker\theta^{\ell-1}} \hookrightarrow K^{p-\ell+1}\otimes\left(\overset{\ell-1}{\otimes}\,\Omega^1_{\overline{B}}(\log Z)\right)$

which gives

(iii) $\det\left(\frac{\ker\theta^\ell}{\ker\theta^{\ell-1}}\right) \hookrightarrow \wedge^{d_{p,\ell}}\left(K^{p-\ell+1}\otimes\left(\overset{\ell-1}{\otimes}\,\Omega^1_{\overline{B}}(\log Z)\right)\right)$.

Using

(iv) $\det \operatorname{Gr}^p V \cong \bigotimes_{\ell=1}^{p+1} \det\left(\frac{\ker\theta^\ell}{\ker\theta^{\ell-1}}\right)$

and combining (iv), (iii) and (ii) at the level of curvatures gives (II.E.24). $\square$

Note: In [GGK08] there are results that in the 1-parameter case express the "error term" in the Arakelov inequality by quantities involving the ranks of the Kodaira-Spencer maps and structure of the monodromy at the singular points.

III. Singularities

III.A. **Logarithmic and mild singularities.**

As our main applications will be to Hodge theory, in this section we will use the notations from Section I.C. We recall from that section that

- B is a smooth quasi-projective variety;
- $\overline{B}$ is a smooth projective completion of B;
- $Z = \overline{B}\backslash B$ is a divisor with normal crossings

$$Z = \cup Z_i$$

  where $Z_I := \bigcap_{i\in I} Z_i$ is a stratum of Z and $Z_I^* = Z_{I,\mathrm{reg}}$ are the smooth points of Z_I;
- $E \to \overline{B}$ is a holomorphic vector bundle.

A neighborhood $\mathcal{U}$ in $\overline{B}$ of a point $p\in Z$ will be

$$\mathcal{U} \cong \Delta^{*k}\times\Delta^\ell$$

with coordinates $(t,w) = (t_1,\dots,t_k;w_1,\dots,w_\ell)$.

We now introduce the co-frame in terms of which we shall express the curvature forms in $\mathcal{U}$. The Poincaré metric in $\Delta^* = \{0<|t|<1\}$ is given by the (1,1) form

$$\omega_{\mathrm{PM}} = (i/2)\frac{dt\wedge d\bar{t}}{|t|^2(-\log|t|)^2}.$$

We are writing $-\log|t|$ instead of just $\log|t|$ because we will want to have positive quantities in the computations below. As a check on signs and constants we note the formula

$$\text{(III.A.1)}\qquad (i/2)\partial\overline{\partial}\big(-\log(-\log|t|)\big) = (1/4)\omega_{\mathrm{PM}}.$$

The inner minus sign is to have $-\log|t|>0$ so that $\log(-\log|t|)$ is defined. The outer one is to have the expression in parentheses equal to $-\infty$ at $t=0$ so that we have a plurisubharmonic function. For

$$\varphi = \log(-\log|t|)$$

the curvature form in the trivial bundle over Δ with the singular metric given by $e^{-\varphi}$ has curvature form

$$(i/2)\overline{\partial}\partial \log(e^{-\varphi}) = (1/4)\omega_{\mathrm{PM}}. \tag{III.A.2}$$

Remark: The functions that appear as coefficients in formally computing (III.A.1) using the rules of calculus are all in L^1_{loc} and therefore define distributions. We may then compute ∂ and $\overline{\partial}$ either in the sense of currents or formally using the rules of calculus. An important observation is

(III.A.3) *these two methods of computing $\overline{\partial}\partial\varphi$ give the same result.*

This is in contrast with the situation when we take

$$\varphi = \log |t|$$

in which case we have in the sense of currents the Poincaré-Lelong formula

$$(i/\pi)\partial\overline{\partial} \log |t| = \delta_0 \tag{III.A.4}$$

where δ_0 is the Dirac δ-function at the origin. Anticipating the discussion below, a charateristic feature of the metrics that arise in Hodge theory will be that the principle (III.A.3) will hold.

Definition: The *Poincaré coframe* has as basis the (1,0) forms

$$\frac{dt_i}{t_i(-\log |t_i|)},\ dw_\alpha$$

and their conjugates.

Definition: A metric in the holomorphic vector bundle $E \to B$ is said to have *logarithmic singularities* along the divisor $Z = \overline{B}\backslash B$ if locally in an open set $\mathcal{U}$ as above and in terms of a holomorphic frame for the bundles and the Poincaré coframe the metric h, the connection matrix $\theta = h^{-1}\partial h$, and the curvature matrix $\Theta_E = \overline{\partial}(h^{-1}\partial h)$ have entries that are Laurent polynomials in the $\log |t_i|$ with coefficients that are real analytic functions in $\mathcal{U}$.

Proposition III.A.5: *The Hodge metrics in the Hodge bundles $F^p \to B$ have logarithmic singularities relative to the canonically extended Hodge bundles $F^p_e \to \overline{B}$.*

In the geometric case this result may be inferred from the theorem on regular singular points of the Gauss-Manin connection ([Del70]). In the general case it is a consequence of the several variable nilpotent orbit theorem ([CKS86]). More subtle is the behavior of the coefficients of the various quantities, especially the Chern polynomials $P(\Theta_{F^p})$, when they are expressed in terms of the Poincaré frame, a topic analyzed in [CKS86] and where the analysis is refined in [Kol87], and to which we now turn.

We recall that a distribution Ψ on a manifold M has a *singular support* $\Psi_{\mathrm{sing}} \subset M$ defined by the property that on any open set $W \subset M\backslash\Psi_{\mathrm{sing}}$ in the complement the restriction $\Psi\big|_W$ is given by a smooth volume form. A finer invariant of the singularities of Ψ is given by its *wave front set*[24]

$$WF(\Psi) \subset T^*M.$$

Among other things the wave front set was introduced to help deal with two classical problems concerning distributions:

(III.A.6) (a) distributions cannot in general be multiplied;
(b) in general distributions cannot be restricted to submanifolds $N \subset M$.

For (a) the wave front sets should be transverse, and for (b) to define $\Psi\big|_N$ it suffices to have $TN \subset WF(\Psi)^\perp$.

In the case of currents represented as differential forms with distribution coefficients, multiplication should be expressed in terms of the usual wedge product of forms. For restriction, if N is locally given by $f_1 = \cdots = f_m = 0$, then for a current Ψ we first set $df_i = 0$; i.e., we cross out any terms with a df_i. Then the issue is to restrict the distribution coefficients of the remaining terms to N. Thus the notion of the wave front set for a current Ψ involves both the differential form terms appearing in Ψ as well as the distribution coefficients of those terms.

Definition: The holomorphic bundle $E \to B$ has *mild logarithmic singularities* in case it has a metric with logarithmic singularities and the following conditions are satisfied:

(i) the Chern polynomials $P(\Theta_E)$ are closed currents given by differential forms with L^1_{loc} coefficients and which represent $P(c_1(E), \dots, c_r(E))$ in $H^*_{\mathrm{DR}}(\overline{B})$;
(ii) the products $P(\Theta_E) \cdot Q(\Theta_E)$ may be defined by formally multiplying them as L^1_{loc}-valued differential forms, and when this is done we obtain a representative in cohomology of the products of the polynomials in the Chern classes;
(iii) the restrictions $P(\Theta_E)\big|_{Z_I^*}$ are defined and represent $P\left(c_1\left(E\big|_{Z_I^*}\right), \dots, c_r\left(E\big|_{Z_I^*}\right)\right)$.

We note the opposite aspects of analytic singularities[25] and mild logarithmic singularities: In the former one wants the singularities to create behavior different from that of smooth metrics, either with regard to the functions that are in L^2 with respect to the singular metric, or to create non-zero Lelong numbers in the currents that arise from their curvatures. In the case of mild logarithmic singularities, basically one may work with them as if there were no singularities at all. An important additional point to be explained in more detail below is that the presence of singularities *increases* the positivity of the Chern forms of the Hodge bundles, so that in this sense one

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[24]A good discussion of wave front sets and references to the literature is given in Wikipedia. We will not use them in a technical sense but rather as a suggestion of an important aspect to be analyzed for the Chern polynomials of the Hodge bundles. The term here is being used in a somewhat different context than that used in the theory of D-modules.

[25]These are singularities that are in L^2 relative to weight functions $e^{-\varphi}$ where φ is of the form $\log(|f_1|^{k_1} + \cdots + |f_m|^{k_m})^\ell$ for analytic functions f_i (cf. [Dem12b]).

uses singularities to positive effect. Of course one has to be careful: The singularities increase positivity so long as they are not bad enough to make the Chern forms non-integrable, in which case they cease to represent the Chern classes.

The main result, stated below and which will be discussed in the next section, is that the Hodge bundles have mild logarithmic singularities. This would follow if one could show that

(III.A.7) *When expressed in terms of the Poincaré frame the polynomials $P(\Theta_E)$ have bounded coefficients.*

This is true when Z is a smooth divisor, but when Z is not a smooth divisor this is not the case and the issue is more subtle.

Main result.

THEOREM III.A.8 ([CKS86], with amplifications in [Kol87], [GGLR20]): *The Hodge bundles have mild logarithmic singularities.*

The general issues (III.A.6)(a), (III.A.6)(b) concerning distributions were raised above. Since currents are differential forms with distribution coefficients, these issues are also present for currents, where as noted above the restriction issue (III.A.6)(b) involves both the differential form aspect and the distribution aspect of currents. This is part (iii) of the definition and is the property of the Chern polynomials that appears in [GGLR20].

A complete proof of Theorem III.A.8 is given in Section 5 of [GGLR20]. In the next section we shall give the argument for the Chern form $\Omega = c_1(\Theta_{\det F})$ of the Hodge line bundle and in the special case when the localized VHS is a nilpotent orbit. The computation will be explicit; the intent is to provide a perspective on some of the background subtleties in the general argument, one of which we now explain. This special case in fact captures the essence of the general situation.

We restrict to the case when $\mathcal{U} \cong \Delta^{*k} = \{(t_1, \dots, t_k) : 0 < |t_j| < 1\}$, and setting $\ell(t_j) = \log t_j / 2\pi i$ and $x_j = -\log |t_j|$ consider a nilpotent orbit

$$\Phi(t) = \exp\left(\sum_{j=1}^{k} \ell(t_j) N_j\right) \cdot F_0.$$

Following explicit computations of the Chern form Ω and of the Chern form Ω_I for the restriction of the Hodge line bundle to Z_I^*, the desired result comes down to showing that a limit

$$\lim_{x_j \to 0} \frac{Q(x)}{P(x)} \tag{III.A.9}$$

exists where $Q(x), P(x)$ are particular homogeneous polynomials of the same degree with $P(x) > 0$ if $x_j > 0$. Limits such as (III.A.9) certainly do not exist in general, and the issue to be understood is how in the case at hand the very special properties of several parameter limiting mixed Hodge structures imply the existence of the limit.

In more detail, a general VHS may be locally written as a nilpotent orbit times a holomorphic expression (cf. (5.2.3a) in [GGLR20]). The singularities of the Chern forms arise from the nilpotent orbit factors (cf. loc. cit. for the computations that establish this). The major issue is to pass from the nilpotent orbit to the product of SL_2-orbits. For 1-parameter degenerations this is already quite subtle. In the several parameter case the level of subtlety greatly increases. This is where the sectorial analysis in [CKS86] enters. An exposition of this is given in section (5.3) and (5.4) in [GGLR20]. Part of the point in the present discussion is to replace the sectorial analysis by a conjugation whose central point is the conceptual one provided by RWFP.

As an application of Theorems II.D.12 and III.A.8, using the notations from Section I.C we consider a VHS given by a period mapping

$$\Phi : B \to \Gamma\backslash D.$$

Denoting by

$$F_e \to \overline{B}$$

the canonically extended Hodge vector bundle we have

THEOREM III.A.10:

(i) *The Kodaira-Iitaka dimension*

$$\kappa(F_e) \leqq 2h^{n,0} - 1.$$

(ii) *Assuming the injectivity of the end piece $\Phi_{*,n}$ of the differential of Φ,*

$$\kappa(S^{h^{n,0}} F_e) = \dim \mathbb{P} S^{h^{n,0}} F_e;$$

i.e., $S^{h^{n,0}} F_e \to \overline{B}$ is big.

Proof. It is well known [Gri70], [CMSP17] that the curvature of the Hodge vector bundle has the norm positivity property. In fact, the curvature form is given by

$$\Theta_F(e, \xi) = \|\Phi_{*,n}(\xi)(e)\|^2. \tag{III.A.11}$$

Concerning the singularities that arise along $\overline{B}\backslash B$, it follows from Theorem III.A.8 that we may treat the Chern form ω of $\mathcal{O}_{\mathbb{P}F_e}(1) \to \overline{B}$ as if the singularities were not present.

The linear algebra situation is

$$T \otimes F \to G \tag{III.A.12}$$

where $\dim T = \dim B$, $\dim F = h^{n,0}$ and $\dim G = h^{n-1,1}$. By (III.A.11) condition (II.D.7) is equivalent to the injectivity of $\Phi_{*,n}$, and Theorem III.A.10 is then a consequence of Theorem II.D.12. □

This result gives one answer to the question

The Hodge vector bundle is somewhat positive. Just how positive is it?

Since in the geometric case the linear algebra underlying the map (III.A.12) is expressed cohomologically, in particular cases the result (i) in (III.A.10) can be considerably sharpened. For example, in the weight $n = 1$ case the method of proof of the theorem gives the

Proposition III.A.13 ([Bru16a]): *In weight $n=1$,* (i) $\kappa(F_e) \leqq 2g-1$, *and* (ii) $S^2F_e \to \overline{B}$ *is big.*

Proof. In this case $D \subset \mathcal{H}_g$ where $g = h^{1,0}$ and $\mathcal{H}_g$ is the Siegel generalized upper-half-plane. We then have

- $T \subset S^2V^*$;
- $G = V^*$;
- $T \otimes V \to G$ is induced by the natural contraction map $S^2V^* \otimes V \to V^*$.

For any $v \in V$ the last map has image of dimension $\leqq g$, and therefore the kernel has dimension $\geqq \dim T - g$. This gives (i) in the proposition.

For (ii) we have

$$\begin{tikzcd} T \otimes S^2V \arrow[r] \arrow[d, phantom, "\cap"] & V^* \otimes V \\ S^2V^* \otimes S^2V. \arrow[ur, "\rfloor"'] & \end{tikzcd}$$

For a general $q \in S^2V$ the contraction mapping $\rfloor$ is injective, and this implies (ii). $\square$

At the other extreme we have the

Proposition III.A.14: *Let $\mathcal{M}_{d,n}$ denote the moduli space of smooth hypersurfaces $Y \subset \mathbb{P}^{n+1}$ of degree $d = 2n+4$, $n \geqq 3$. Then the Hodge vector bundle $F \to \mathcal{M}_{d,n}$ is big.*

Proof. Set $V = \mathbb{C}^{n+2*}$ and let $P \in V^{(d)}$ be a homogeneous form of degree d that defines Y. Denote by $J_P \subset \bigoplus_{k \geqq d-1} V^{(k)}$ the Jacobian ideal. Then (cf. Section 5 in [CMSP17])

- $T_Y\mathcal{M}_{d,n} \cong V^{(d)}/J_P^{(d)}$;
- $F_Y = H^{n,0}(Y) \cong V^{(d-n-2)}$;
- $G_Y = H^{n-1,0}(Y) \cong V^{(2d-n-2)}/J_P^{(2d-n-2)}$.

It will suffice to show

(III.A.15) *For general P and general $Q \in V^{(d-n-2)}$ the mapping*

$$V^{(d)}/J_P^{(d)} \xrightarrow{Q} V^{(2d-n-2)}/J_P^{(2d-n-2)}$$

is injective.

Noting that $d-n-2 = n+2$ and that it will suffice to prove the statement for one P and Q, we take

$$\begin{aligned} Q &= x_0 \cdots x_{n+1}, \\ P &= x_0^d + \cdots + x_{n+1}^d. \end{aligned}$$

Then $J_P = \{x_0^{2n+3}, \dots, x_{n+1}^{2n+3}\}$ and a combinatorial argument gives (III.A.15). $\square$

Remark III.A.16: The general principle that Proposition III.A.14 illustrates is this: Let $L \to$ be an ample line bundle. Then both for general smooth sections $Y \in |mL|$ and for cyclic coverings $\widetilde{X}_Y \to X$ branched over a smooth Y, as m increases the

Hodge vector bundle $F \to |mL|^0$ over the open set of smooth Y's becomes increasingly positive in the sense that the k such that $S^k F$ is big decreases, and for $m \gg 0$ F itself is big.

III.B. **Formulation of the result.** We consider a variation of Hodge structure given by a period mapping

$$\Phi : \Delta^{*k} \to \Gamma_{\text{loc}} \backslash D.$$

Here we assume that the monodromy generators $T_i \in \operatorname{Aut}_Q(V)$ are unipotent with logarithms $N_i \in \operatorname{End}_Q(V)$; Γ_{loc} is the local monodromy group generated by the T_i.

For $I \subset \{1, \cdots, k\}$ with complement $I^c = \{1, \ldots, k\} \backslash I$ we set

$$\Delta_I^* = \{(t_1, \ldots, t_k) : t_i = 0 \text{ for } i \in I \text{ and } t_j \neq 0 \text{ for } j \in I^c\}.$$

From the work of Cattani-Kaplan-Schmid [CKS86] the limit $\lim_{t \to \Delta_I^*} \Phi(t)$ is defined as a polarized variation of limiting mixed Hodge structures on Δ_I^*. Passing to the primitive parts of the associated graded polarized Hodge structures gives a period mapping

$$\Phi_I : \Delta_I^* \to \Gamma_{\text{loc},I} \backslash D_I$$

where D_I is a product of period domains and $\Gamma_{\text{loc},I}$ is generated by the T_j for $j \in I^c$. This may be suggestively expressed by writting

$$\lim_{t \to t_I} \Phi(t) = \Phi_I(t_I).$$

However caution must be taken in interpreting the limit, as the "rate of convergence" is not uniform but depends on the sector in which the limit is taken in the manner explained in [CKS86].

We denote by $\Lambda \to \Delta^{*k}$ and $\Lambda_I \to \Delta_I^*$ the Hodge line bundles. The Hodge-Riemann bilinear relations give metrics in these bundles and we denote by Ω and Ω_I the respective Chern forms. The result to be proved is

$$\lim_{t \to \Delta_I} \Omega = \Omega_I, \tag{III.B.1}$$

where again care must be taken in interpreting this equation. In more detail, this means: In Ω set $dt_i = d\bar{t}_i = 0$ for $i \in I$. Then the limit, in the usual sense, as $t \to \Delta_I$ of the remaining terms exists and is equal to Ω_I. We will write (III.B.1) as

$$\Omega\big|_{\Delta_I^*} = \Omega_I. \tag{III.B.2}$$

The proof of (III.B.1) that we shall give can easily be adapted to the case when the period mapping depends on parameters.

The limit can also be reduced to the case when Φ is a *nilpotent orbit.* This means that

$$\Phi(t) = \exp\left(\sum_{i=1}^{k} \ell(t_i) N_i\right) F \tag{III.B.3}$$

where $F \in \check{D}$ and the conditions

(i) $N : F^p \to F^{p-1}$,
(ii) $\Phi(t) \in D$ for $0 < |t| < \epsilon$

are satisfied. This reduction is non-trivial and is given in Section 5 of [GGLR20].

The main points in the proof of Theorem IV.B.8 in the nilpotent orbit case are as follows:

(a) without changing the associated graded's to Φ and Φ_I we may replace the F in (III.B.3) by an F_0 such that the limiting mixed Hodge structure is $\mathbb{R}$-split;
(b) in this case N_I can be completed to an sl_2 which we denote by $\{N_I^+, Y_I, N_I\}$;[26]
(c) the Y_I-weight decomposition of N_{I^c} is

$$N_{I^c} = N_{I^c,0} + N_{I^c,-1} + N_{I^c,-2} + \cdots$$

where $N_{I^c,-m}$ has Y_I-weight $-m$, $m \geqq 0$;
(d) if all the $N_{I^c,-m} = 0$ for $m > 0$, then there is an $\mathrm{sl}_2^c = \{N_{I^c}^+, Y_{I^c}, N_{I^c}\}$ that commutes with the previous sl_2, and the result (III.B.1) is immediate;
(e) in general, by direct computation we have

$$\Omega_I \equiv \Omega + R \mod dt_i, d\bar{t}_i \text{ for } i \in I$$

where the remainder term R consists of expressions $Q(x)/P(x)$ as in (III.A.9), and then direct computation using the relative filtration property and the fact that for $m > 0$ the $N_{I^c,-m}$ have negative Y_I-weights gives the result.

III.C. **Proof of the result.**

Weight filtrations, representations of sl_2 and limiting mixed Hodge structures.

The proof of (III.B.1) will be computational, using only that $\Phi(t)$ is a nilpotent orbit (III.B.3) and that the commuting $N_i \in \mathrm{End}_Q(V)$ have the *relative weight filtration property* (RWFP), which will be reviewed below.[27] The computation will be facilitated by using the representation theory of sl_2 adapted to the Hodge theoretic situation at hand. The non-standard but hopefully suggestive notations for doing this will now be explained.

(i) Given a nilpotent transformation $N \in \mathrm{End}_Q(V)$ with $N^{n+1} = 0$ there is a unique increasing weight filtration $W(N)$ given by subspaces

$$V_k^{W(N)} := W_k(N)V \tag{III.C.1}$$

satisfying the conditions

- $N : V_k^{W(N)} \to V_{k-2}^{W(N)}$,

and with the notation to be explained just below

- $N^k : \mathrm{Gr}_{n+k}^{W(N)}(V) \xrightarrow{\sim} \mathrm{Gr}_{n-k}^{W(N)}(V)$ (Hard Lefschetz property).

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[26] There are two ways of doing this—one is the method in [CKS86] and the other one, which is purely linear algebra, is due to Deligne.

[27] The proof in [GGLR20] uses the detailed analysis of limiting mixed Hodge structures from [CKS86], of which the RWFP is one consequence. Part of the point for the argument given here is to isolate the central role played by that property.

The *associated graded* to the weight filtation is the direct sum of the

$$\mathrm{Gr}_\ell^{W(N)} V := V_\ell^{W(N)}/V_{\ell-1}^{W(N)},$$

and the *primitive subspaces* are defined for $\ell \geqq n$ by

$$\mathrm{Gr}^{W(N)}_{n+k,\mathrm{prim}} V = \ker \left\{ N^{k+1} : \mathrm{Gr}^{W(N)}_{n+k} V \to \mathrm{Gr}^{W(N)}_{n-k-2} V \right\}.$$

Remark that the two standard choices for the ranges of indices in (III.C.1) are

$$\begin{cases} 0 \leqq k \leqq 2n & \text{(Hodge theoretic)} \\ -n \leqq k \leqq n & \text{(representation theoretic).}^{28} \end{cases}$$

We will use the first of these.

The weight filtration is self-dual in the sense that using the bilinear form Q

$$V_k^{W(N)\perp} = V^{W(N)}_{2n-k-1} \tag{III.C.2}$$

which gives

$$V_k^{W(N)*} \cong V/V^{W(N)}_{2n-k-1}.$$

(ii) A *grading element* for $W(N)$ is given by a semi-simple $Y \in \mathrm{End}_Q(V)$ with integral eigenvalues $0, 1, \dots, 2n$, weight spaces $V_k = V_k^{W(N)}$ for the eigenvalue k, and where the induced maps

$$V_k \xrightarrow{\sim} \mathrm{Gr}_k^{W(N)} V$$

are isomorphisms. Grading elements always exists, and for any one such Y we have

- $[Y, N] = -2N$;
- there is a unique $N^+ \in \mathrm{End}_Q(V)$ such that $\{N^+, Y, N\}$ is an sl_2-triple.

The proof of the second of these uses the first together with the Hard Lefschetz property of $W(N)$.

We denote by $\mathcal{U}$ the standard representation of sl_2 with weights $0, 1, 2$. Thinking of $\mathcal{U}$ as degree 2 homogeneous polynomials in x, y we have

- weight $x^a y^b = 2$, $a + b = 2$;
- $N = \partial_x$ and $N^+ = \partial_y$.

We denote by

$$\mathcal{U}_i = \mathrm{Sym}^i\, \mathcal{U} \cong \left\{ \begin{matrix} \text{homogeneous polynomials} \\ \text{in } x, y \text{ of degree } i+1 \end{matrix} \right\}$$

the standard $(i+1)$-dimensional irreducible representation of sl_2. The N-*string* associated to $\mathcal{U}_i$ is

$$\{x^{i+1}\} \to \{x^i y\} \to \cdots \to \{y^{i+1}\}$$

where $N = \partial_x$. The top of the N-string is the primitive space.

Given (V, Q, N) as above and a choice of a grading element Y, for the sl_2-module

$$V_{\mathrm{gr}} = \overset{2n}{\underset{k=0}{\oplus}} V_k$$

[28] Thus $W_\ell(N)\,\mathrm{End}(V) = \{\varphi \in \mathrm{End}(V) : \varphi : V_k^{W(N)} \to V_{k+\ell}^{W(N)}\}$.

we have a unique identification

$$V_{\mathrm{gr}} \cong \bigoplus_{i=0}^{n} H^{n-i} \otimes \mathcal{U}_i \tag{III.C.3}$$

for vector spaces H^{n-i}. The notation is chosen for Hodge-theoretic purposes. The N-string associated to $H^{n-i} \otimes \mathcal{U}_i$ will be denoted by

$$H^{n-i}(-i) \xrightarrow{N} H^{n-i}(-(i-1)) \xrightarrow{N} \cdots \xrightarrow{N} H^{n-i} \tag{III.C.4}$$

and we define

(III.C.5) *the Hodge-theoretic weight of* $H^{n-i}(-j)$ *is* $n-i+2j$.

The representation-theoretic weight of $H^{n-i}(-j)$ is $2j$. It follows that $H^{n-i}(-i)$ is the primitive part of the $\mathcal{U}_i$-component of V_{gr}.

Relative to Q the decomposition (III.C.3) is orthogonal and the pairing

$$Q_i : H^{n-i}(-i) \otimes H^{n-i}(-i) \to \mathbb{Q}$$

given by

$$Q_i(u,v) = Q(N^i u, v) \tag{III.C.6}$$

is non-degenerate.

(iii) We recall the

Definition: A *limiting mixed Hodge structure* (LMHS) is a mixed Hodge structure $(V, Q, W(N), F)$ with weight filtration $W(N)$ defined by a nilpotent $N \in \mathrm{End}_Q(V)$ and Hodge filtration F which satisfies the conditions

(a) $N : F^p \to F^{p-1}$;

(b) the form Q_i in (III.C.6) polarizes $\mathrm{Gr}^{W(N)}_{n+k,\mathrm{prim}} V \cong H^{n-k}(-k)$.

The MHS on V induces one on $\mathrm{End}_Q(V)$, and (a) is equivalent to

$$N \in F^{-1}\,\mathrm{End}_Q(V).$$

We denote by

$$V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$$

the unique Deligne decomposition of $V_{\mathbb{C}}$ that satisfies

- $W_k(N)V = \bigoplus_{p+q\leqq k} I^{p,q}$;
- $F^p V = \bigoplus_{\substack{p'\geqq p \\ q}} I^{p',q}$;
- $\overline{I}^{q,p} \equiv I^{p,q} \bmod W_{p+q-2}(N)V$.

The LMHS is $\mathbb{R}$-*split* if $\overline{I}^{p,q} = I^{q,p}$. Canonically associated to a LMHS is an $\mathbb{R}$-split one $(V, Q, W(N), F_0)$ where

$$F_0 = e^{-\delta} F$$

for a canonical $\delta \in I^{-1,-1}\,\mathrm{End}_Q(V_R)$. For this $\mathbb{R}$-split LMHS there is an evident grading element $Y \in I^{0,0}(\mathrm{End}_Q(V_{\mathbb{R}}))$.

Given a LMHS $(V, Q, W(N), F)$ there is an associated nilpotent orbit

$$\begin{array}{ccc} \Delta^* & \longrightarrow & \Gamma_T \backslash D \\ \cup & & \cup \\ t & \longrightarrow & \exp(\ell(t)N)F \end{array}$$

where $\ell(t) = \log t / 2\pi i$ and $\Gamma_T = \{T^{\mathbb{Z}}\}$. Conversely, given a 1-variable nilpotent orbit as described above there is a LMHS. We shall use consistently the bijective correspondence

$$\text{LMHS's} \iff \text{1-parameter nilpotent orbits.}$$

Since

$$\det F_0 = \det F$$

without loss of generality for the purposes of this paper we will assume that our LMHS's are $\mathbb{R}$-split and therefore have canonical grading elements.

(iv) Let $N_1, N_2 \in \operatorname{End}_Q(V)$ be commuting nilpotent transformations and set $N = N_1 + N_2$. Then there are two generally different filtrations defined on the vector space $\operatorname{Gr}^{W(N_1)} V$:

(A) the weight filtration $W(N)V$ induces a filtration on any sub-quotient space of V, and hence induces a filtration on $\operatorname{Gr}_\bullet^{W(N_1)} V$;

(B) N induces a nilpotent map $\overline{N} : \operatorname{Gr}_\bullet^{W(N_1)} V \to \operatorname{Gr}_\bullet^{W(N_1)} V$,[29] and consequently there is an associated weight filtration $W(\overline{N}) \operatorname{Gr}_\bullet^{W(N_1)} V$ on $\operatorname{Gr}_\bullet^{W(N_1)} V$.

Definition: The *relative weight filtration property* (RWFP) is that these two filtrations coincide:

$$W(N) \cap \operatorname{Gr}_\bullet^{W(N_1)} V = W(\overline{N}) \operatorname{Gr}_\bullet^{W(N_1)} V.^{30} \tag{III.C.7}$$

We note that $\overline{N}$ is the same as the map induced by N_2 on $\operatorname{Gr}_\bullet^{W(N_1)} V$, so that (III.C.7) may be perhaps more suggestively written as

$$W(N) \cap \operatorname{Gr}_\bullet^{W(N_1)} V = W(N_2) \operatorname{Gr}_\bullet^{W(N_1)} V. \tag{III.C.8}$$

In more detail this is

$$\frac{W_k(N)(V) \cap W_m(N_1)(V)}{W_k(N)(V) \cap W_{m-1}(N_1)(V)} = W_{k-m}(\overline{N})(\operatorname{Gr}_m^{W(N_1)}(V)).$$

The RWFP is a highly non-generic condition on a pair of commuting nilpotent transformation, one that will be satisfied in our Hodge-theoretic context.

(v) Suppose now that Y_1 is a grading element for N_1 so that the corresponding $\mathrm{sl}_2 = \{N_1^+, Y_1, N_1\}$ acts on V and hence on $\operatorname{End}_Q(V)$. We observe that the Y_1-eigenspace deomposition of N_2 is of the form

$$N_2 = N_{2,0} + N_{2,-1} + \cdots + N_{2,-m}, \qquad m > 0 \tag{III.C.9}$$

[29] $\overline{N}$ was denoted by $\overline{N}_2$ in the introduction.

[30] There is a shift in indices that will not be needed here.

where $[Y, N_{2,-m}] = -mN_{2,-m}$. The reason for this is that

$$[N_1, N_2] = 0 \implies \left\{ \begin{matrix} N_2 \text{ is at the bottom of the } N_1\text{-strings} \\ \text{for } N_1 \text{ acting on } \mathrm{End}_Q(V) \end{matrix} \right\}.$$

It can be shown that there is an $\mathrm{sl}_2' = \{N_2^+, Y_2, N_{2,0}\}$ that commutes with the sl_2 above. Thus

(III.C.10) Given N_1, N_2 as above, there are commuting sl_2's with N_1 and $N_{2,0}$ as nil-negative elements. Moreover, $N_2 = N_{2,0}+$(terms of *strictly negative* weights) relative to $\{N_1^+, Y_1, N_1\}$.

The $N_{2,0}$ here is the same as the $\overline{N}$ above. It is the "strictly negative" that will be the essential ingredient needed to establish that the limit exists in the main result.

III.D. **Calculation of the Chern forms Ω and Ω_I.**

Step 1: For a nilpotent orbit (III.B.3) holomorphic sections of the canonically extended VHS over Δ are given by

$$\exp\left(\sum_{j=1}^{k} \ell(t_j)N_j\right) v, \qquad v \in V_{\mathbb{C}}.$$

Up to non-zero constants the Hodge metric is

$$\begin{aligned}(u,v) &= Q\left(\exp\left(\sum_j \ell(t_j)N_j\right) u, \exp\left(\sum_j \ell(t_j)N\right)\bar{v}\right) \\ &= Q\left(\exp\left(\sum_j \log|t_j|^2 N_j\right) u, \bar{v}\right).\end{aligned}$$

Using the notation (III.C.3) the associated graded to the LMHS as $t \to 0$ will be written as

(III.D.1) $$V_{\mathrm{gr}} = \overset{n}{\underset{i=0}{\oplus}} H^{n-i} \otimes \mathfrak{U}_i$$

and

$$F^n = \overset{n}{\underset{i=0}{\oplus}} H^{n-i,0}.$$

For $u \in H^{n-i,0}(-i)$ and $v \in H^{n-i,0}$

$$Q\left(\exp\left(\sum_j \log|t_j|^2 N_j\right) u, \bar{v}\right) = \left(\frac{1}{i!}\right) Q\left(\left(\sum_j \log|t_j|^2 N_j\right)^i u, \bar{v}\right).$$

Setting

$$x_j = -\log|t_j|$$

the metric on the canonically extended line bundle is a non-zero constant times

$$P(x) = \prod_{i=0}^{n} \det\left(\left(\sum_j x_j N_j\big|_{H^{n-i,0}}\right)^i\right). \tag{III.D.2}$$

Here to define "det" we set $N = \sum N_j$ and are identifying $H^{n-i}(-i)$ with H^{n-i} using N^i. Note that the homogenous polynomial $P(x)$ is positive in the quadrant $x_j > 0$. The Chern form is

$$\Omega = \partial\overline{\partial} \log P(x). \tag{III.D.3}$$

Step 2: Define

$$\begin{cases} N_I = \sum_{i\in I} x_i N_i,\ N_{I^c} = \sum_{j\notin I} x_j N_j \\ N = \sum_{i=1}^{k} x_i N_i = N_I + N_{I^c} \end{cases}$$

and set

$$P = \prod_{i=0}^{n} \det\left(N\big|_{H^{n-i,0}(-i)}\right)^i.$$

Denoting by

$$V_{\mathrm{gr},I} = \overset{n}{\underset{i=0}{\oplus}} H_I^{n-i,i} \otimes \mathfrak{U}_i$$

the associated graded to the LMHS as $t \to \Delta_I^*$, we define

$$P_I = \prod_{i=0}^{n} \det\left(N_I\big|_{H_I^{n-i,0}(-i)}\right)^i.$$

Taking $N_I = N_1$ and $N_{I^c} = N_2$ in (iv) in Section III.C, we have

$$N_{I^c,0} = \text{ weight zero component of } N_{I^c}$$

where weights are relative to the grading element Y_I for N_I. Decomposing the RHS of (III.D.1) using the $\mathrm{sl}_2 \times \mathrm{sl}_2'$ corresponding to N_I and $N_{I^c,0}$ by (III.A.10) we obtain

$$V_{\mathrm{gr}} \cong \underset{i,j}{\oplus} H_{i,j}^{n-i-j} \otimes \mathfrak{U}_i \otimes \mathfrak{U}_j \tag{III.D.4}$$

where $H_{i,j}^{n-i-j}$ is a polarized Hodge structure of weight $n-i-j$. Note that this decomposition depends on I. On $H_{i,j}^{n-i-j} \otimes \mathfrak{U}_i \otimes \mathfrak{U}_j$ we have a commutative square

$$\begin{tikzcd} H^{n-i-j}(-i-j) \arrow[r, "N_I^i"] \arrow[d, "N_{I^c,0}^j"'] & H_{i,j}^{n-i-j}(-j) \arrow[d, "N_{I^c,0}^j"] \\ H_{i,j}^{n-i-j}(-i) \arrow[r, "N_I^i"] & H_{i,j}^{n-i-j}. \end{tikzcd}$$

Using (III.D.4) this gives

$$P = \prod_{i,j} \det\left(N_I^i N_{I^c,0}^j \big|_{H_{i,j}^{n-i-j}(-i-j)} \right) + R$$

where the remainder term R involves the $N_{I^c,-m}$'s for $m > 0$. We may factor the RHS to have

$$P = \prod_{i,j} \det\left(N_I^i \big|_{H_{i,j}^{n-i-j}(-i-j)} \right) \prod_{i,j} \det\left(N_{I^c,0}^j \big|_{H_{i,j}^{n-i-j}(-i-j)} \right) + R$$

which we write as

(III.D.5) $$P = P_I \cdot P_{I^c} + R$$

where P_I and P_{I^c} are the two $\prod_{i,j}$ factors. We note that

(III.D.6) *the remainder term $R = 0$ if we have commuting* sl_2*'s.*[31]

We next have the important observation

LEMMA III.D.7: *P_{I^c} is the Hodge metric in the line bundle $\Lambda_I \to Z_I^*$.*

Proof. This is a consequence of the RWFP (III.C.7) applied to the situation at hand when we take $N_1 = N_I$ and $N_2 = N_{I^c}$. □

By (III.D.6), if we have commuting sl_2's, then $R = 0$

$$\begin{aligned} \Omega = -\partial\overline{\partial} \log P &= -\partial\overline{\partial} \log P_I - \partial\overline{\partial} R_{I^c} \\ &\equiv -\partial\overline{\partial} \log P_{I^c} \ \text{ modulo } \ dt_i, d\bar{t}_i \ \text{ for } \ i \in I \\ &\equiv \Omega_I \end{aligned}$$

and we are done.

In general, we have

(III.D.8) $$\Omega \equiv \Omega_I + S_1 + S_2$$

where

(III.D.9) $$\begin{cases} S_1 = \dfrac{\partial P_{I^c} \wedge \overline{\partial} R + \partial R \wedge \overline{\partial} P_{I^c} - P_{I^c} \partial\overline{\partial} R}{P_I P_{I^c}} \\ S_2 = \dfrac{\partial R \wedge \overline{\partial} R}{P_I^2 P_{I^c}^2}. \end{cases}$$

Step 3: We will now use specific calculations to analyze the correction terms S_1, S_2. The key point will be to use that

$$N = N_I + N_{I^c} = \underbrace{N_I + N_{I^c,0}} + \underbrace{\sum_{m \geqq 1} N_{I^c,-m}}$$

where the terms over the first brackets may be thought of as "the commuting sl_2-part of N_I, N_{I^c}" and the correction term over the second bracket has negative Y_I-weights.

[31]To have commuting sl_2's means that $N_{2,-m} = 0$ for $m > 0$.

We set

$$h_I^{n-i,0} = \dim H_I^{n-i,0}$$

and for a monomial $P = x_1^{\ell_1} \cdots x_k^{\ell_k}$ we define

$$\deg_I P = \sum_{i \in I} \ell_i.$$

Lemma III.D.10:

(i) *For any monomial P appearing in P*

$$\deg_I P \leqq n h_I^0 + (n-1) h_I^{1,0} + \cdots + h_I^{n-i,0} = \sum_{i=1}^n i h_I^{n-i,0}.$$

(ii) *If π is any permutation of $1, \dots, k$ and*

$$\ell_{\pi,i} = \sum_{j=1}^n j \left(h^{n-j,0}_{\{\pi(1),\dots,\pi(i)\}} - h^{n-j,0}_{\{\pi(1),\dots,\pi(i-1)\}} \right)$$

then

$$P_\pi := x_{\pi(1)}^{\ell_{\pi,1}} \cdot x_{\pi(2)}^{\ell_{\pi,2}} \cdots x_{\pi(k)}^{\ell_{\pi,k}} = x_1^{\ell_{\pi,\pi^{-1}(k)}} \cdots x_k^{\ell_{\pi,\pi^{-1}(k)}}$$

appears with a non-zero coefficient in P.

Corollary: *The monomials appearing in P are in the convex hull of the monomials P_π.*

Proof. For $V_{\mathrm{gr},I} = \mathrm{Gr}^{W(N_I)} V$ we have as $\{N_I^+, Y_I, N_I\}$-modules

$$V_{\mathrm{gr},I} \cong \bigoplus_{i=0}^{n} H_I^{n-i} \otimes \mathfrak{U}_i.$$

Decomposing the RHS as sl_2'-modules we have

$$V_{\mathrm{gr},I} \cong \bigoplus_{i=0}^{n} H_{a,i-a}^{n-i} \otimes \mathfrak{U}_a \otimes \mathfrak{U}'_{i-a}$$

where the $H_{a,i-a}^{n-i}$ depend on I. The map

$$N^i : H^{n-i,0}(-i) \to H^{n-i}$$

gives

$$\bigoplus_{a=0}^{i} H_{a,i-a}^{n-i,0}(-i) \to \bigoplus_{a=0}^{i} H_{a,i-a}^{n-i,0}.$$

The Y_I-weights of vectors in $H_{a,i-a}^{n-i,0}$ are equal to a, and thus $\wedge^{h^{n-i,0}} \left(\bigoplus_{a=0}^{i} H_{a,i-a}^{n-i,0}(-i) \right)$ has weight $\sum a h_{a,i-a}^{n-i,0}$ and $\wedge^{h^{n-i,0}} \left(\bigoplus_{a=0}^{i} H_{a,i-a}^{n-i,0} \right)$ has weight $-\sum a h_{a,i-a}^{n-i,0}$. As a consequence

$$\textit{any monomial in } \det \left(N^i \big|_{H^{n-i,0}(-i)} \right) \textit{ drops weights by } 2 \sum h_{a,i-a}^{n-i,0}.$$

We have

$$\text{(III.D.11)} \qquad \det\left(\left(N\big|_{H^{n-i,0}(-i)}\right)^i\right) = \det\left(\left((N_I + N_{I^c,0})\,\big|_{H^{n-i,0}(-i)}\right)^i\right) + T$$

where T = terms involving $N_{I^c,\mathrm{neg}}$. For any monomial P in a minor involving the $N_{I^c,\mathrm{neg}}$ of total weight $-d$,

$$2\deg_I P + d = 2\sum_{a=0}^{i} i h^{n-i,0}_{a,i-a}, \qquad d > 0$$

and so

$$\deg_I P < \sum_{a=0}^{i} a h^{n-i,0}_{a,i-a}.$$

Putting everything together, we have (III.D.10) where

$$\text{(III.D.12)} \qquad T = \begin{Bmatrix} \text{linear combinations of monomials } P \\ \text{satisfying } \deg_I P < \sum_{i=0}^n \sum_{a=0}^i a h^{n-i,0}_{a,i-a} \end{Bmatrix}.$$

Using the bookkeeping formula $h_I^{n-i,0} = \sum_{i=a}^n h^{n-i,0}_{a,i-a}$ we obtain

$$\sum_{i=0}^{n}\sum_{a=0}^{i} a h^{n-i,0}_{a,i-a} = \sum_{a=0}^{i}\sum_{i=a}^{n} h^{n-i,0}_{a,i-a} = \sum_{a=0}^{n} a h_I^{n-a,0}$$

which gives

$$P = P_I P_{I^c} + \left(\text{correction term with } \deg_I < \sum_{a=0}^{n} a h_I^{n-a,0}\right)$$

where $\deg_I P_I = \sum_{a=0}^n a h_I^{n-a,0}$ and $\deg_I P_{I^c} = 0$, giving (i) in (III.D.10).

A parallel argument shows that for $I \cap J = \emptyset$

$$D_{I\cup J} := \prod_{i=0}^{n} \det\left(\left(N_I + N_{I,0}\big|_{H^{n-i,0}_{I\cup J}(-i)}\right)\right)^i$$
$$+ \text{ a correction with } \deg_I < \sum_{a=0}^{m} a h_I^{n-a,0}.$$

By the definition of $H^{n-i}_{I\cup J}$,

$$\det\left(\left(N_I + N_{J,0}\big|_{H^{n-i,0}_{I\cup J}(-i)}\right)^i\right) \neq 0$$

and

$$\deg_I\left(\det\left(\left(N_I + N_{J,0}\big|_{H^{n-i,0}_{I\cup J}(-i)}\right)^i\right)\right) = \sum_{a=0}^{n} a h_I^{n-a,0}$$

while automatically

$$\deg_{I\cup J}(\text{all terms of } D_{I\cup J}) = \sum_{a=0}^{n} a h_{I\cup J}^{n-a,o}.$$

Thus

$$\begin{aligned}
\deg_J \det\left(\left(N_I + N_{J,0}\big|_{H_{I\cup J}^{n-i,0}(-i)}\right)^i\right) &= \deg_{I\cup J} \det\left(\left(N_I + N_{J,0}\big|_{H_{I\cup J}^{n-i,0}(-i)}\right)^i\right)\\
&\quad - \deg_I \det\left(\left(N_I + N_{J,0}\big|_{H_{I\cup J}^{n-i,0}(-i)}\right)^i\right)\\
&= \sum_{a=0}^{n} a\left(h_{I\cup J}^{n-a,0} - h_I^{n-a,0}\right).
\end{aligned}$$

Proceeding inductively on $\{\pi(1)\} \subset \{\pi(1),\pi(2)\} \subset \cdots \subset \{\pi(1),\dots,\pi(k)\}$ we obtain, if $N_{\{\pi(1),\dots,\pi(\ell)\},0}$ = weight 0 piece of $N_{\{\pi(1),\dots,\pi(\ell)\}}$ with respect to $\mathrm{Gr}^{W(N)_{\{\pi(1),\dots,\pi(k)\}}}$ then

$$\prod_{i=0}^{n} \det\left(\left(N_{\{\pi(1)\}} + N_{\{\pi(1),\pi(2)\},0} + \cdots + N_{\{\pi(1),\dots,\pi(k)\},0}\big|_{H^{n-i,0}}\right)^i\right)$$

is a *non-zero* multiple of $x_{\pi(1)}^{\ell_1} x_{\pi(2)}^{\ell_2} \cdot \cdot x_{\pi(k)}^{\ell_k}$. This is our P_π. Tracking the correction terms we have

$$P = \sum_{\pi} C_\pi P_\pi + \text{ terms strictly in the convex hull of the } P_\pi$$

where $C_\pi \neq 0$ for all π. This proves (ii) in III.D.10. □

Step 4: Referring to (III.D.8) and (III.D.9), from Lemma III.D.10 we have:

(a) R_1 has $\deg_I R_1 < \deg_I P_I$, and all monomials satisfy (i) in III.D.10.
(b) R_2 is a sum of products of monomials $P_1 P_2$ where each $\deg_I P < \deg_I P_I$ and P_i satisfies (i) in III.D.10.

To complete the proof we have

Lemma III.D.13: *Given a monomial P in the I-variables satisfying* $\deg_I P < \deg_I P$ *and* (ii) *in Lemma III.D.10,*

$$\lim_{t\to\Delta_I^*} P/P_I = 0.$$

Proof. Implicit in the lemma is that the limit exists. We note that $t \to \Delta_I^*$ is the same as $x_i \to \infty$ for $i \in I$. We also observe that the assumptions in the lemma imply that there is a positive degree monomial P' with $\deg_I(P'P) = \deg_I P_I$ and where $P'P$ lies in the convex hull of the P_π's for P_I. Using this convex hull property we will show that

(III.D.14) $\qquad P'P/P_I$ is bounded as $x_i \to \infty$ for $i \in I$.

Since $\lim_{x_i\to\infty} P'(x) = \infty$, this will establish the lemma.

We now turn to the proof of (III.D.14). Because the numerator and denominator are homogeneous of the same degree, the ratio is the same for $(x_1, \dots, x_k)$ and $(\lambda x_1, \dots, \lambda x_k)$, $\lambda > 0$.

For simplicity, reindex so that $I = \{1, \dots, d\}$. Suppose that $x_\nu = (x_{\nu 1}, \dots, x_{\nu d})$ is a sequence of points in $(x_i > 0, i \in I)$ such that

$$\lim_{\nu \to \infty} \frac{P' P(x_\nu)}{D_I(x_\nu)} = \infty.$$

Consider a successive set of subsequences such that for all i, j, we have one of three possibilities:

(i) $\lim_{\nu \to \infty} x_{\nu i}/x_{\nu j} = \infty$;
(ii) $x_{\nu i}/x_{\nu j}$ is bounded above and below, which we write as $x_{\nu i} \equiv x_{\nu j}$;
(iii) $\lim_{\nu \to \infty} x_{\nu i}/x_{\nu j} = 0$.[32]

Now replace our sequence by this subsequence. Let $I_i, \dots, I_r$ be the partition of I such that

$$i \equiv j \iff \text{(ii) holds for } i, j$$

and order them so that (i) holds for $i, j \iff i \in I_{m_1}, j \in I_{m_2}$ and $m_1 < m_2$. We may thus find a $C > 0$ such that $\frac{1}{C} \leqq x_{\nu i}/x_{\nu j} \leqq C$ if i, j in same I_m, and for any $B > 0$

$$x_{\nu i}/x_{\nu j} > B^{m_2 - m_1} \text{ if } i \in I_{m_1}, j \in I_{m_2}, \nu \text{ sufficiently large.}$$

By compactness, we may pick a subsequence so that $\lim_{\nu \to \infty}(x_{\nu i}/x_{\nu j}) = C_{ij}$ if $i, j \in$ same I_m.

Now introduce variables $y_1, \dots, y_d$ and let

$$x_i = a_i y_m \text{ if } i \in I_m, \quad a_i/a_j = C_{ij}, a_i > 0.$$

We may restrict our cone by taking

$$\widetilde{N}_m = \sum_{i \in I_m} a_i N_i.$$

This reduces us to the case $|I_m| = 1$ for all m, i.e.,

$$\lim_{\nu \to \infty} x_{\nu i}/x_{\nu j} = \infty \text{ if } i < j.^{33}$$

Thus for any B,

$$x_{\nu i}/x_{\nu j} > B^{j-i} \text{ for } v \gg 0.$$

Now

$$\frac{x_{\nu_1}^{m_1} x_{\nu_2}^{m_2} \cdot \cdot x_{\nu_d}^{m_d}}{x_{\nu_1}^{\ell_1} x_{\nu_2}^{\ell_2} \cdot \cdot x_{\nu_d}^{\ell_d}} \to 0$$

if $m_2 + m_2 + \cdot \cdot + m_d = \ell_2 + \ell_2 + \cdot \cdot + \ell_d$ and $m_1 < \ell_1$, or $m_1 = \ell_1$ and $m_2 < \ell_2, \cdot \cdot$. Thus

$$P_I = c M_{\{1,2,\dots,d\}} + \text{terms of slower growth as } \nu \to \infty, c > 0,$$

---

[32] In effect we are doing a sectoral analysis in the co-normal bundle to the stratum $\Delta^* - I$, which explains the wave front set analogy mentioned above.

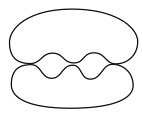
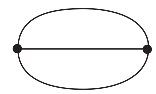
[33] In effect we are making a generalized base change $\Delta^{*d} \to \Delta^{*k}$ such that for the pullback to Δ^{*d} the coordinates y_m go to infinity at different rates.

i.e.,

$$(P_{\{1,2,\ldots,d\}}/\text{other terms})(x_\nu) > B.$$

Since $P'P$ belongs to the convex hull of the P_π, $(P'P/P_{\{1,2,\ldots,d\}})(x_\nu)$ is bounded as $\nu \to \infty$. This proves the claim. $\square$

Example III.D.15: An example that illustrates many of the essential points in the argument is provided by a neighborhood Δ^3 of the dollar bill curve[34]

\$ $\longleftrightarrow$  with the dual graph 

in $\overline{\mathcal{M}}_2$. The family may be pictured as follows:

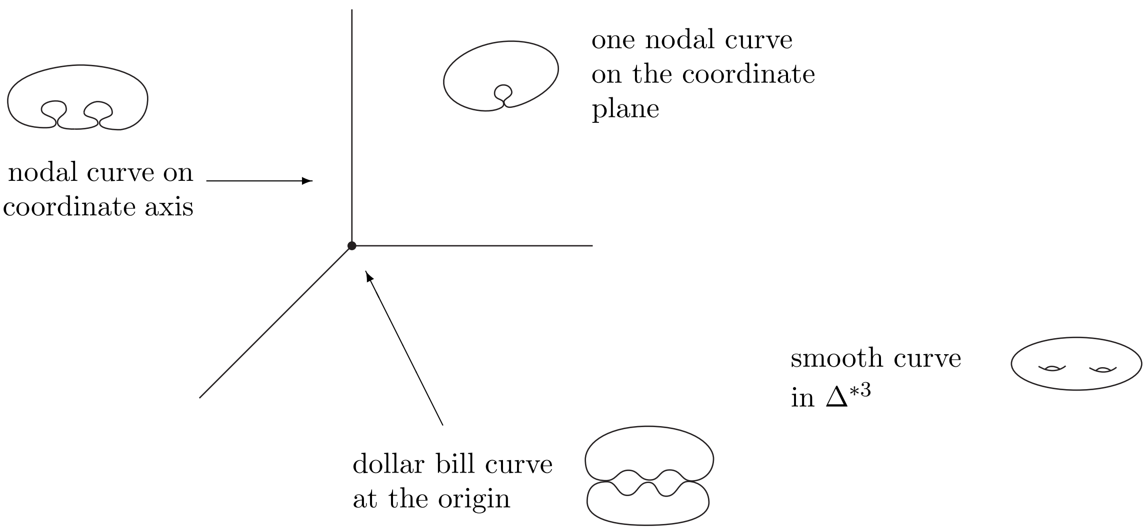
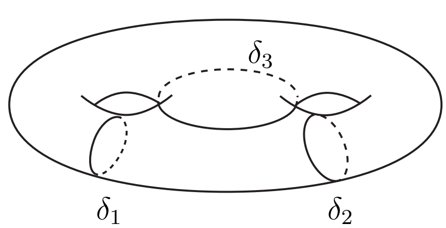


FIGURE 1

With the picture

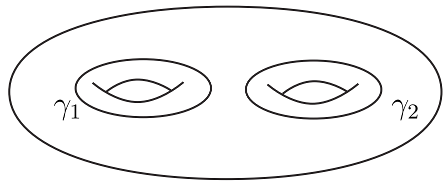


each of the coordinate planes outside the axes is a family of nodal curves where one of the vanishing cycles δ_i has shrunk to a point. Along each of the coordinate axes

[34]The name originates from symbolically drawing the curve as \$.

two of the three cycles have shrunk to a second node, and at the origin we have the dollar bill curve.

We complete the δ_i to a symplectic basis by adding cycles γ_i.[35]



The corresponding monodromies around the coordinate axes are Picard-Lefschetz transformations with logarithms

$$N_i(\mu) = (\mu, \delta_i)\delta_i$$

and with matrices

$$N_1 = \left(\begin{array}{c|c} 0 & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \\ \hline 0 & 0 \end{array}\right), \quad N_2 = \left(\begin{array}{c|c} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \\ \hline 0 & 0 \end{array}\right), \quad N_3 = \left(\begin{array}{c|c} 0 & \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \\ \hline 0 & 0 \end{array}\right).$$

Setting as usual $\ell(t) = \log t/2\pi i$, the normalized period matrix is

$$\Omega(t) = \begin{pmatrix} \ell(t_1) + \ell(t_3) & \ell(t_3) \\ \ell(t_3) & \ell(t_2) + \ell(t_3) \end{pmatrix} + \begin{pmatrix} \text{holomorphic} \\ \text{term} \end{pmatrix}.$$

The corresponding nilpotent orbit is obtained by taking the value at the $t_i = 0$ of the holomorphic term, and by rescaling this term may be eliminated.

Setting $L(t) = (-\log|t|)/4\pi^2$ and $\mathrm{PM}(t) = (i/2)\overline{\partial}\partial \log L(t)$, the metric in the canonically framed Hodge vector bundle is the 2×2 Hermitian matrix

$$H(t) = \begin{pmatrix} L(t_1) + L(t_3) & L(t_3) \\ L(t_3) & L(t_2) + L(t_3) \end{pmatrix};$$

in the Hodge line bundle the metric is

$$\begin{aligned} h(t) &= L(t_1)L(t_2) + (L(t_1) + L(t_2))L(t_3) \\ &= L(t_1)L(t_2) + L(t_1t_2)L(t_3). \end{aligned}$$

Setting

$$\begin{aligned} \omega &= \partial\overline{\partial} \log h(t) = \partial\overline{\partial}\big(\log(L(t_1)L(t_2) + L(t_1t_2)L(t_3))\big) \\ \omega_3 &= \partial\overline{\partial} \log L(t_1t_2) \end{aligned}$$

we will show that

(III.D.16) $\qquad \omega\big|_{t_3=0}$ *is defined and is equal to* ω_3.

[35]Here γ_3 is not drawn in.

Proof. Setting $\psi = \partial h/h$ and $\eta = \partial\overline{\partial}h/h$ we have

$$\omega = -\psi \wedge \overline{\psi} + \eta.$$

Now

$$\psi = \frac{\partial\left(L(t_1)L(t_2) + L(t_2t_2)L(t_3)\right)}{L(t_1)L(t_2) + L(t_1t_2)L(t_3)}.$$

Setting $dt_3 = 0$ the dominant term of what is left is the left-hand term in

$$\frac{\partial L(t_1t_2)}{\frac{L(t_1)L(t_2)}{L(t_3)} + L(t_1t_2)} \longrightarrow \frac{\partial L(t_1t_1)}{L(t_1t_2)},$$

and the arrow means that the limit as $t_3 \to 0$ exists and is equal to the term on the right.

For η, letting $\equiv$ denote modulo dt_3 and $\overline{dt_3}$ and taking the limit as above

$$\eta \equiv \frac{\partial\overline{\partial} L(t_1t_2)}{\frac{L(t_1)L(t_2)}{L(t_3)} + L(t_1t_2)} \longrightarrow \frac{\partial\overline{\partial} L(t_1t_2)}{L(t_1t_2)},$$

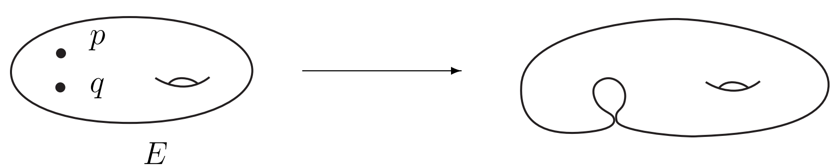
which gives the result. $\square$

We next observe that

(III.D.17) $\quad \omega_3\big|_{t_2=0}$ *is defined and is equal to zero.*

Proof. The computation is similar to but simpler than that in the proof of (III.D.16). $\square$

Interpretations: The curves pictured in Figure 1 map to an open set $\Delta^3 \subset \overline{\mathcal{M}}_2$. The PHS's of the smooth curves in Δ^{*3} vary with three parameters. Those on the codimension 1 strata such as $\Delta^{*2} \times \Delta$ vary in moduli with two parameters. Their normalizations are

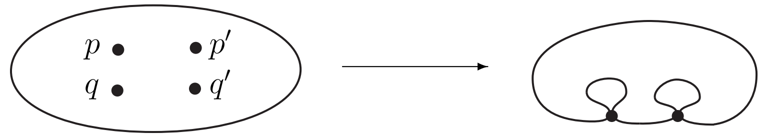


and their LMHS's vary with two parameters with

$$\begin{cases} \mathrm{Gr}_1(\mathrm{LMHS}) \cong H^1(E) \\ \mathrm{Gr}_0(\mathrm{LMHS}) \cong \mathbb{Q} \end{cases}$$

and where the extension data in the LMHS is locally given by $\mathrm{AJ}_E(p-q)$. Thus Gr(LMHS) varies with one parameter and for the approximating nilpotent orbit is constant along the curves $t_1t_2 = c$. This local fibre of the map $\overline{\mathcal{M}}_2 \to \overline{P}$ is part of the closed fibre parametrized by E.

Along the codimension 2 strata such as $\Delta^* \times \Delta^2$ the curves vary in moduli with 1-parameter. Their normalizations are



and the moduli parameter is locally the cross-ratio of $\{p, q; p', q'\}$. The LMHS's are purely Hodge-Tate and thus Gr(LMHS) has no continuous parameters. In summary

- Φ_e is locally 1-1 on Δ^{*3};
- $\Phi_{e,*}$ has rank 1 on $\Delta^{*2} \times \Delta$;
- Φ_e is locally constant on $\Delta^* \times \Delta^2$.

As $c \to 0$ the fibres of Φ_e on the $\Delta^{*2} \times \Delta$ tend to the coordinate axis $\Delta^* \times \Delta^2$ along which Φ_e is locally constant.

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