COMPLETIONS OF PERIOD MAPPINGS

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Abstract. This work initiates a *global* study of period mappings at infinity, that is motivated by challenges arising when considering the questions

(i) What are natural completions of a period mapping?
(ii) What geometric applications do they have?

in the case that the period domain is not Hermitian. The approach is *relative* in the sense that the work depends on both the period map $\Phi : B \to \Gamma\backslash D$, and a choice of smooth projective variety $\overline{B}$ with reduced normal crossing divisor Z such that $B = \overline{B}\backslash Z$. The main results include:

(a) a toroidal-like completion of a finite cover of Φ;
(b) conditions under which the log canonical bundle $K_{\overline{B}} + [Z]$ is free and ample;
(c) a relationship between the conormal bundle $\mathcal{N}^*_{Z/\overline{B}}$ and ample "theta" line bundles over (level one) extension data of limiting mixed Hodge structures;
(d) illustration of how the geometric interpretation of the extension data in a limiting mixed Hodge structure may suggest how to desingularize boundary components of moduli.

Many conjectures are made.

Contents

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1. Introduction

This paper is part of a project, initiated in [GGLR20], that is motivated by two questions:

1: *What are natural completions of a period mapping?*

2: *What geometric applications do they have?*

A great deal is known in the classical case that the period domain is Hermitian and the monodromy group is arithmetic[1] (see §1.2.1 for some discussion), while relatively little is known in the nonclassical case. Here we are primarily concerned with the latter.

1.1. **Setting.** The data of a period mapping (to be completed) will consist of a triple $(\overline{B}, Z; \Phi)$ where

(i) $(\overline{B}, Z)$ is a pair consisting of smooth projective variety $\overline{B}$ and a reduced normal crossing divisor $Z \subset \overline{B}$; and where

(ii) the complement
$$B \;=\; \overline{B}\backslash Z$$
has a variation of (pure) polarized Hodge structure

$$\begin{array}{c} \mathcal{F}^p \subset \mathcal{V} = \tilde{B} \times_{\pi_1(B)} V \\ \downarrow \\ B \end{array} \tag{1.1a}$$

inducing a period map

$$\Phi : B \;\to\; \Gamma\backslash D\,. \tag{1.1b}$$

Here D is a period domain parameterizing weight n, Q–polarized Hodge structures on the vector space V (with fixed Hodge numbers), and $\pi_1(B) \twoheadrightarrow \Gamma \subset \mathrm{Aut}(V, Q)$ is the monodromy representation. Let
$$\wp \;=\; \Phi(B) \;\subset\; \Gamma\backslash D$$
denote the image.

[1] This is the situation arising when studying moduli of principally polarized abelian varieties and K3 surfaces.

Remark 1.2. We assume that the generators of Γ around the irreducible components Z_i of $Z = Z_1 \cup \cdots \cup Z_\nu$ are unipotent.[2]

Remark 1.3. Of particular interest is the case when $\overline{B}$ is the desingularization $\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}$ of a KSBA compactification $\overline{\mathcal{M}}$ of a moduli space $\mathcal{M}$. The understanding of the period mapping at infinity developed here may provide some guidance on how to desingularize the compactifications of some KSBA moduli spaces, which in contrast for the curve case are generally quite singular along the boundary; a preliminary illustration of this application may be found in §A. This is among the topics of the planned [FGG$^+$20].

1.2. **Question 1: Completions.** We may assume without loss of generality that the period map $\Phi : B \to \Gamma\backslash D$ is proper [Gri70, §9]. Then the proper mapping theorem implies that $\wp = \Phi(B)$ is a complex analytic variety.

The (*augmented*) *Hodge line bundle* is

$$\begin{array}{l} \Lambda \;=\; \det(\mathcal{F}^n) \otimes \det(\mathcal{F}^{n-1}) \otimes \cdots \otimes \det(\mathcal{F}^{\lceil (n+1)/2 \rceil}) \\ \downarrow \\ B \end{array}$$

Let

$$\begin{array}{rl} \mathcal{F}^p_{\mathrm{e}} \subset & \mathcal{V}_{\mathrm{e}} \\ & \downarrow \\ & \overline{B} \end{array} \tag{1.4}$$

denote Deligne's extension of the Hodge vector bundles (1.1a), and let

$$\begin{array}{l} \Lambda_{\mathrm{e}} \;=\; \det(\mathcal{F}^n_{\mathrm{e}}) \otimes \det(\mathcal{F}^{n-1}_{\mathrm{e}}) \otimes \cdots \otimes \det(\mathcal{F}^{\lceil (n+1)/2 \rceil}_{\mathrm{e}}) \\ \downarrow \\ \overline{B} \end{array}$$

denote the extended Hodge line bundle.

Theorem 1.5 (Bakker–Brunebarbe–Tsimerman [BBT18])**.** *The image $\wp = \Phi(B)$ is quasi-projective and the Hodge line bundle Λ descends to an ample bundle on $\wp$. In fact, one may projectively embed $\wp$ with sections of $\Lambda_{\mathrm{e}}^{\otimes m}$ that vanish along Z.*

Our goal is to construct projective compactifications of the quasi-projective $\wp$ and extensions $\overline{B} \to \overline{\wp}$ of the period map (1.1b). To that end, we define two canonical completions

$$\Phi^{\mathsf{T}} : \overline{B} \;\to\; \overline{\wp}^{\mathsf{T}} \tag{1.6a}$$

$$\Phi^{\mathsf{S}} : \overline{B} \;\to\; \overline{\wp}^{\mathsf{S}}\,. \tag{1.6b}$$

[2] This assumption is not necessary, but is made for convenience of exposition. The case in which the semi-simple part of monodromy is non-trivial is of central geometric interest [Gri18, Gri19, Gri20].

As a set $\overline{\wp}^{\mathsf{T}}$ parameterizes Γ–equivalence classes of limiting mixed Hodge structures $(W, F; \sigma)$, and as such encodes the maximal amount of Hodge theoretic information associated with $\overline{B}$. The superscript "T" stands for "toroidal" as $\overline{\wp}^{\mathsf{T}}$ will be a possible candidate for a toroidal-esque compactification of $\wp$. (This possibility is suggested by the Hodge theoretic interpretation of the toroidal $\overline{\mathcal{A}}_g$ in [CCK80], cf. Remark 1.13.)

At the other extreme, the second (1.6b) will be a minimal completion, and the superscript "S" stands for "Satake–Baily–Borel" since, in the classical case that $B = \Gamma\backslash D$ is locally Hermitian symmetric with Γ arithmetic, $\overline{\wp}^{\mathsf{S}}$ is the Satake–Baily– Borel compactification of $\Gamma\backslash D$ (§1.2.1). The set $\overline{\wp}^{\mathsf{S}}$ parameterizes Γ–equivalence classes of polarized Hodge structures on the associated graded $\mathrm{Gr}^W_\bullet$. The map $\overline{\wp}^{\mathsf{T}} \twoheadrightarrow \overline{\wp}^{\mathsf{S}}$ quotients-out the extension data of the limiting mixed Hodge structures parameterized by $\overline{\wp}^{\mathsf{T}}$. It is in this sense that $\overline{\wp}^{\mathsf{S}}$ retains the minimal amount of meaningful Hodge theoretic information.

Remark 1.7 (The constructions are relative)*.* We emphasize that the construction of (1.6) is *relative*; given the triple $(\overline{B}, Z; \Phi)$ one produces the completions. (In particular, $\overline{\wp}^{\mathsf{T}}$ may be viewed as a relative analog of [KU09], but without the assumption that Γ is arithmetic or the construction of a fan.) This is in contrast to classical case where one constructs a projective compactification $\overline{\Gamma\backslash D}$ of $\Gamma\backslash D$ (§1.2.1).

The construction (1.6) has functoriality properties that will be taken up elsewhere.

Remark 1.8 (The case that $\wp$ is a curve)*.* If $\wp = \Phi(B)$ is a curve, then [Som73] and [CDK95] imply that $\overline{\wp}^{\mathsf{S}}$ is algebraic. One may then show that $\Lambda_{\mathrm{e}} \to \overline{\wp}^{\mathsf{S}}$ is ample (cf. the argument of [GGLR20, §6]).

1.2.1. *The classical case that D is Hermitian and Γ arithmetic.* In general, $\Gamma\backslash D$ admits no algebraic structure itself [CT14, GRT14]. However, in the case that D is Hermitian and Γ is arithmetic, the locally Hermitian symmetric space $\Gamma\backslash D$ is quasi-projective, and admits a slew of projective compactifications $\overline{\Gamma\backslash D}$, including:

(i) the Satake–Baily–Borel compactification $\overline{\Gamma\backslash D}^{\mathsf{S}}$, [BB66];
(ii) the toroidal normalizations $\overline{\Gamma\backslash D}^{\mathsf{T}}$ of $\overline{\Gamma\backslash D}^{\mathsf{S}}$, [AMRT75, Mum75, Sat73];
(iii) and, in the case that D admits complex totally geodesic hypersurfaces (balls and type IV domains), the semi-toric compactifications of Looijenga, [Loo03a, Loo03b].

In any of these cases we may take $\overline{\wp}$ to be the closure of $\wp$ in $\overline{\Gamma\backslash D}$. In the case of the Satake–Baily–Borel compactification $\overline{\wp}^{\mathsf{S}} \subset \overline{\Gamma\backslash D}^{\mathsf{S}}$ we additionally have $\overline{\wp}^{\mathsf{S}} = \mathrm{Proj}\, R(\overline{B}, \Lambda_{\mathrm{e}})$, [BB66], and an extension $\Phi^{\mathsf{S}} : \overline{B} \to \overline{\Gamma\backslash D}^{\mathsf{S}}$ of the period map [Bor72]. In the case of the toroidal compactification, the existence of the extension $\Phi^{\mathsf{T}} : \overline{B} \to \overline{\Gamma\backslash D}^{\mathsf{T}}$ is a more subtle business [FS86, CMGHL17]. Hodge theoretic interpretations of the toroidal constructions include [CCK80].

1.2.2. *The toroidal candidate.* The definition of $\overline{\wp}^{\mathsf{T}}$ is conjectural. What is defined is a finite cover $\hat{\wp}^{\mathsf{T}} \twoheadrightarrow \overline{\wp}^{\mathsf{T}}$. More precisely, let

$$\begin{tikzcd} B \arrow[r, "\hat{\Phi}"'] \arrow[rr, bend left, "\Phi"] & \hat{\wp} \arrow[r] & \wp \end{tikzcd} \tag{1.9}$$

be the Stein factorization of the period map (1.1b); the fibres of $\hat{\Phi}$ are connected, the fibres of $\hat{\wp} \to \wp$ are finite, and $\hat{\wp}$ is normal.

Theorem 1.10. *The complex analytic variety $\hat{\wp}$ is Zariski open in a compact, complex analytic variety $\hat{\wp}^{\mathsf{T}}$, and the map $\hat{\Phi} : B \to \hat{\wp}$ admits a proper holomorphic completion $\hat{\Phi}^{\mathsf{T}} : \overline{B} \to \hat{\wp}^{\mathsf{T}}$.*

Outline of proof. The set

$$\Gamma(\hat{\Phi}) \;=\; \{(b_1, b_2) \in B \times B \mid \hat{\Phi}(b_1) = \hat{\Phi}(b_2)\}$$

defines an equivalence relation on B with the property that $\hat{\Phi} : B \to \hat{\wp}$ is the quotient map. It follows from [CDK95] that there exists a projective subvariety $\hat{X} \subset \overline{B} \times \overline{B}$ with the property that

$$\Gamma(\hat{\Phi}) \;=\; \hat{X} \cap (B \times B)\,.$$

If $\hat{X}$ defines an equivalence relation on $\overline{B}$, then [Gra83] asserts that the quotient map

$$\hat{\Phi}^{\mathsf{T}} : \overline{B} \;\to\; \hat{\wp}^{\mathsf{T}}$$

has the desired properties.

So the essential problem is to show that $\hat{X}$ is defines an equivalence relation.[3] For this, it suffices to show that every point $b \in \overline{B}$ admits a neighborhood $\overline{\mathcal{O}}^1 \subset \overline{B}$ with the properties:

(i) The restriction $\Phi|_{\mathcal{O}^1}$, with $\mathcal{O}^1 = B \cap \overline{\mathcal{O}}^1$, is proper (Corollary 5.4).
(ii) There exists a proper holomorphic map $\hat{f} : \overline{\mathcal{O}}^1 \to \hat{\mathcal{O}}^1$ whose fibres coincide over $\mathcal{O}^1$ with those of $\hat{\Phi}\Big|_{\mathcal{O}^1}$. As discussed in §7.2, this is an immediate consequence of Proposition 7.13.

□

Theorem 1.10 does not assert that $\hat{\wp}^{\mathsf{T}}$ is algebraic. We conjecture that $\hat{\wp}^{\mathsf{T}}$ does admit an ample line bundle; in particular,

Conjecture 1.11. *The complex analytic variety $\hat{\wp}^{\mathsf{T}}$ is a projective completion of $\hat{\wp}$.*

[3]The third author thanks Gregory Pearlstein for sharing the analogous observation about $\Gamma(\Phi)$.

Define

$$\Gamma(\Phi) \;=\; \{(b_1, b_2) \in B \times B \mid \Phi(b_1) = \Phi(b_2)\}\,.$$

As above, $\Gamma(\Phi)$ defines an equivalence relation on B, and $\Phi : B \to \wp$ is the quotient map. Likewise [CDK95] implies there is a projective subvariety $X \subset \overline{B} \times \overline{B}$ with the property that

$$\Gamma(\Phi) \;=\; X \cap (B \times B)\,.$$

Conjecture 1.12. *The subvariety X defines an equivalence relation on $\overline{B}$.*

Assuming the conjecture, we define the extension $\Phi^{\mathsf{T}} : \overline{B} \to \overline{\wp}^{\mathsf{T}}$ of (1.1b) to be the resulting quotient map. Then $\hat{X} \subset X$ factors Φ^{T} as the composition of $\hat{\Phi}^{\mathsf{T}}$ with the finite $\hat{\wp}^{\mathsf{T}} \twoheadrightarrow \overline{\wp}^{\mathsf{T}}$.

Remark 1.13. The period map over $\mathcal{O}^1$ can be represented by a sort of generalized period matrix. The holomorphic map $\hat{f}$ is constructed from the horizontal entries. This means that the function $\hat{f}$ essentially encodes the full variation of mixed Hodge structure over $Z_I^* \cap \overline{\mathcal{O}}^1$ (up to constants of integration). This gives $\overline{\wp}^{\mathsf{T}}$ the interpretation as a relative analog of the Kato–Usui construction [KU09]. Keeping in mind the Hodge theoretic interpretation of the toroidal $\overline{\mathcal{A}}_g$ in [CCK80], it also supports the expectation that $\overline{\wp}^{\mathsf{T}}$ be a candidate for a toroidal-esque compactification.

1.2.3. *Generalizing Satake–Baily–Borel.* Early efforts to generalize SBB to non-Hermitian D include the work of Cattani and Kaplan for weight $n = 2$ Hodge structures [Cat74, CK77]. There one considers a partial compactification of $\Gamma\backslash D$; in particular, this construction is not relative in the sense that it is independent of any choice of period map.

In contrast, the *topological* extension

$$\begin{tikzcd} \overline{B} \arrow[r, "\hat{\Phi}^{\mathsf{S}}"'] \arrow[rr, bend left, "\Phi^{\mathsf{S}}"] & \hat{\wp}^{\mathsf{S}} \arrow[r] & \overline{\wp}^{\mathsf{S}} \end{tikzcd}$$

of the Stein factorization (1.9) defined in [GGLR20] (see §2.2 for a review) depends on the choice of period map (1.1b). Here $\hat{\wp}^{\mathsf{S}}$ and $\overline{\wp}^{\mathsf{S}}$ are compact Hausdorff topological spaces, and the maps $\hat{\Phi}^{\mathsf{S}}$ and Φ^{S} are continuous and proper, and $\hat{\wp}^{\mathsf{S}} \twoheadrightarrow \overline{\wp}^{\mathsf{S}}$ is finite. The normal crossing divisor Z induces a stratification $\overline{B} = \cup Z_I^*$ with quasi-projective strata. The restriction $\Phi^{\mathsf{S}}\big|_{Z_I^*}$ is a period map

$$\Phi_I^0 : Z_I^* \;\to\; \Gamma_I\backslash D_I^0\,. \tag{1.14}$$

This map is defined by taking the variation of limiting mixed Hodge structure along Z_I^* (which is induced by (1.1)), and passing to the graded quotient ("forgetting" the extension data).

Conjecture 1.15 ([GGLR20]). *The topological space $\hat{\wp}^{\mathsf{S}}$ admits the structure of a normal complex analytic variety and the extension $\hat{\Phi}^{\mathsf{S}} : \overline{B} \to \hat{\wp}^{\mathsf{S}}$ of $\hat{\Phi} : B \to \hat{\wp}$ is analytic.*

The Hodge line bundle $\Lambda_{\mathrm{e}} \to \overline{B}$ is trivial on the connected fibres of $\hat{\Phi}^{\mathsf{S}}$, and so descends to $\Lambda_{\mathrm{e}} \to \hat{\wp}^{\mathsf{S}}$.

Theorem 1.16 ([GGLR20]). *If Conjecture 1.15 holds, then $\Lambda_{\mathrm{e}} \to \hat{\wp}^{\mathsf{S}}$ is holomorphic and ample. In particular, $\hat{\wp}^{\mathsf{S}} = \mathrm{Proj}\, R(\hat{\wp}^{\mathsf{S}}, \Lambda_{\mathrm{e}})$.*

Remark 1.17. In the classical case that D is Hermitian and Γ arithmetic (§1.2.1), the Hodge line bundle $\Lambda_{\mathrm{e}} \to \overline{\Gamma\backslash D}^{\mathsf{S}}$ is ample, [BB66].

Theorem 1.18 ([GGLR20]). *Conjecture 1.15 holds in the case that* $\dim B \le 2$*.*

Remark 1.19. If Conjecture 1.15 holds, then there are number of interesting questions one may ask. For example: *If $\overline{\wp}^0$ normal? If yes, is it $\mathbb{Q}$–Gorenstein. If so under what conditions are the singularities log canonical?* If $\overline{\wp}^0$ is $\mathbb{Q}$–Gorenstein, then we may assume that $(\overline{B}, Z)$ is a log resolution of $\overline{\wp}^0$. For simplicity also assume that $\overline{\wp}^0$ is smooth. Then, with $(\overline{B}, Z)$ a log resolution, we have an equation of $\mathbb{Q}$–line bundles

$$K_{\overline{B}} \ + \ \sum a_i[Z_i] \ = \ (\Phi^0)^*(K_{\overline{\wp}^0})\,.$$

Do the a_i have Hodge theoretic interpretations? For example, are they expressible in terms of the Chern classes of Hodge bundles over the strata of Z?

Example 1.20. Suppose that $\dim B = 2$, Z is irreducible, and that Φ satisfies local Torelli. If the local monodromy about Z is not finite, then Z is contracted by Φ^{S} to a singular point $p \in \overline{\wp}^{\mathsf{S}}$. Letting $d = -Z^2 > 0$ one may show that the genus of Z is $g \ge 1$ and that

$$K_{\overline{B}} \ + \ \frac{1}{d}\,(d + 2g - 2)\ [Z] \ = \ (\Phi^0)^* K_{\overline{\wp}^{\mathsf{S}}}\,.$$

Thus, $\overline{\wp}^{\mathsf{S}}$ is log canonical with one elliptic singularity only if $g = 1$.

To Conjectures 1.12 and 1.15 we add

Conjecture 1.21. *The map $\hat{\wp}^{\mathsf{T}} \twoheadrightarrow \hat{\wp}^{\mathsf{S}}$ is a morphism of algebraic varieties.*

1.3. **Question 2: geometric properties.** Given a line bundle $L \to X$ over a arbitrary compact complex manifold we consider two groups of properties:

- Numerical: L is nef; L is big.
- Geometric: L is free; L is ample.

The Hodge line bundle $\Lambda \to B$ is nef [Gri70, Proposition 7.15].[4] Together with [GGLR20, Theorem 1.4.1] this implies that that canonically extended Hodge line bundle $\Lambda_{\mathrm{e}} \to \overline{B}$

[4]See [Gol19] for nef-ness in characteristic p.

is nef. Under a general local Torelli assumption there is an extensive body of literature establishing various numerical properties such as: the Hodge line bundle $\Lambda \to B$ is big if and only if Φ satisfies generic local Torelli (§9.1); and the pair $(\overline{B}, Z)$ is of log general type, $K_{\overline{B}} + [Z]$ is big [Zuo00]. There are also a number of results on the hyperbolicity of B, including [BB20, Bru20, DLSZ19].

Here we are predominately interested in establishing conditions under which natural line bundles on $\overline{B}$ are free and ample. For the triple $(\overline{B}, Z; \Phi)$ natural line bundles include the Hodge line bundle Λ_{e}, the normal bundles $[Z_i] = \mathcal{N}_{Z_i/\overline{B}}$, and the log canonical bundle $K_{\overline{B}} + [Z]$.

1.3.1. *Hodge line bundle.* Results of this type include

Theorem 1.22 ([GGLR20]). *Suppose* $\dim B = 2$. *Assume that the differential of* Φ *is everywhere injective on* B*, and that* $0 \le -K_{\overline{B}} \cdot C$ *whenever the curve* $C \subset \overline{B}$ *lies in a* Φ^0*–fibre. Then* $\Lambda_{\mathrm{e}} \to \overline{B}$ *is free, and* $\overline{\wp}^0 = \operatorname{Proj} R(\overline{B}, \Lambda_{\mathrm{e}})$.

We conjecture that Theorems 1.5, 1.16 and 1.22 admit the following refinement.

Conjecture 1.23. (a) *If the differential of the period map* $\Phi : B \to \Gamma\backslash D$ *is generically injective, then there are* $0 \le a_i \in \mathbb{Q}$ *so that the* $\mathbb{Q}$ *line bundle* $\Lambda_{\mathrm{e}} - \sum a_i Z_i$ *is free.*

(b) *Under suitable local Torelli-type assumptions, there exist integers* $0 \le a_i \in \mathbb{Z}$ *and* m_0 *so that* $m\Lambda_{\mathrm{e}} - \sum a_i[Z_i]$ *is ample for* $m \ge m_0$.

Part (b) of the conjecture is known to hold in two cases: when $Z = Z_1$ is irreducible (Proposition 2.32); and when B is a surface we have the stronger Theorem 1.24. For the latter index the irreducible components Z_i of $Z = Z_1 \cup \cdots \cup Z_\nu$ so that $\Phi^0(Z_i)$ is a point if and only if $i \le \mu \le \nu$.

Theorem 1.24 (Theorem 8.1). *Suppose that* $\dim B = 2$ *and assume that the differential of* $\Phi : B \to \Gamma\backslash D$ *is everywhere injective. Then there exists* $a_i \ge 0$ *so that the line bundle*

$$\Pi \;=\; m\Lambda_{\mathrm{e}} \;-\; \sum_{i=1}^{\mu} a_i[Z_i]$$

is ample for $m \gg 0$.

1.3.2. *Local Torelli for* $(\overline{B}, Z; \Phi)$. Conjecture 1.23(b) alludes to "suitable local Torelli-type assumptions". What we have in mind here is a local Torelli condition for the triple $(\overline{B}, Z; \Phi)$. Specifically (and with the notations to be explained later) the Gauss–Manin connection ∇ on $\mathcal{V}$ induces a natural map

$$\Psi : T_{\overline{B}}(-\log Z) \;\to\; \mathrm{Gr}^{-1}_{\mathcal{F}_{\mathrm{e}}}\mathrm{End}(\mathcal{E}_{\mathrm{e}})\,. \tag{1.25}$$

We say that the triple $(\overline{B}, Z; \Phi)$ satisfies the *local Torelli property* if (1.25) is injective. Local Torelli for the triple $(\overline{B}, Z; \Phi)$ implies local Torelli for the period map: the differential of $\Phi : B \to \Gamma\backslash D$ is everywhere injective.

1.3.3. *The log canonical bundle.*

Theorem 1.26 (Zuo [Zuo00])**.** *If the differential of the period map $\Phi : B \to \Gamma\backslash D$ is injective at some point (generic local Torelli holds), then $K_{\overline{B}} + [Z]$ is big.*

Conjecture 1.27. *If the differential of the period map $\Phi : B \to \Gamma\backslash D$ is injective at some point (generic local Torelli holds), then $K_{\overline{B}} + [Z]$ is nef and big. If the differential of the period map is injective everywhere (local Torelli holds), then $K_{\overline{B}} + [Z]$ is free.*

The Base Point Free Theorem [Kol93] implies

Lemma 1.28. *If $K_{\overline{B}+[Z]}$ is both nef and big, then $2(\dim B + 2)!\,(\dim B + 2)\,(K_{\overline{B}} + [Z])$ is free.*

Theorem 1.29 (Theorem 9.1)**.** *Assume that the local Torelli condition holds for $(\overline{B}, Z; \Phi)$: the bundle map $\Psi : T_{\overline{B}}(-\log Z) \to \mathrm{Gr}^{-1}_{\mathcal{F}_\mathrm{e}}\mathrm{End}(\mathcal{E}_\mathrm{e})$ is injective. Then the line bundle $K_{\overline{B}}+[Z]$ is nef and big.*

The local Torelli condition for $(\overline{B}, Z; \Phi)$ enables us to realize $K_{\overline{B}} + [Z]$ as a line subbundle of the pull-back $\mathcal{H} = \Phi^*(\mathbf{H})$ of a homogenous subbundle $\mathbf{H} \to D$ (which descends to $\Gamma\backslash D$). Curvature properties of $\mathbf{H}$ imply that $c_1(\mathcal{H})$ is essentially equivalent to $c_1(\Lambda_\mathrm{e})$. The theorem is then deduced by considering the second fundamental form of $K_{\overline{B}} + [Z] \hookrightarrow \mathcal{H}$.

The fibres $A \subset \overline{B}$ of Φ^S are compact and parameterize mixed Hodge structures (W, F) with fixed associated graded. Here the weight filtration W is fixed, and the Hodge filtration F varies along A, but the induced Hodge filtration $F(\mathrm{Gr}^W_\bullet)$ on the graded quotient is fixed. As F varies, it defines an extension

$$0 \;\to\; \mathrm{Gr}^W_{\ell-1} \;\to\; W_\ell/W_{\ell-2} \;\to\; \mathrm{Gr}^W_\ell \;\to\; 0\,.$$

So along A we have a map Φ^1 to level 1 extension data (to be discussed below). Let A^0 be a connected component of A, and let $A^1 \subset A^0$ be a connected component of a Φ^1–fibre.

Theorem 1.30 (Theorem 9.2)**.** *Assume that the local Torelli condition holds for $(\overline{B}, Z; \Phi)$: the bundle map $\Psi : T_{\overline{B}}(-\log Z) \to \mathrm{Gr}^{-1}_{\mathcal{F}_\mathrm{e}}\mathrm{End}(\mathcal{E}_\mathrm{e})$ is injective. Suppose in addition that the effective cone $\mathrm{Eff}^1(\overline{B})$ is finitely generated and that the period maps $\Phi^0_W : Z_W \to \Gamma_W\backslash D^0_W$ have constant rank. Then there is a well-defined Gauss map $\mathcal{G}(\,\Phi^1\big|_{A^0}) : A^0 \to \mathrm{Gr}(r_W, \mathbb{C}^{d_W})$. The line bundle $K_{\overline{B}}+[Z]$ is ample if and only if the Gauss map $\mathcal{G}(\,\Phi^1\big|_{A^0})$ is locally injective.*

The maps Φ^0_W are proper extensions of the maps (1.14), see §2.2.1.

Remark 1.31. Both the assumption that the effective cone $\mathrm{Eff}^1(\overline{B})$ is finitely generated (appearing in several statements), and the hypothesis that the period maps Φ^0_W have constant rank are expected to be unnecessary. Our goal here is not to prove the optimal results, but to highlight the key geometric ideas underlying the arguments.

1.4. **Perspective: global study of the period mapping at infinity.** The *global* properties of the period mapping $\Phi : B \to \Gamma\backslash D$ are a classical and much studied subject [Gri70, CMSP17].[5] The *local* properties along Z of the canonically extended Hodge bundles have also been a topic of considerable interest (beginning with the works [Sch73, CKS86]), with significant applications including the Iitaka conjecture [Vie83a, Vie83b, Kol87] and the arithmeticity of Hodge loci [CDK95]. It became clear over the course of the work [GGLR20] that in order to satisfactorily address Question 1 (page 2), we need a better understanding of the *global* properties at infinity. (Indeed, the existence of the neighborhood $\overline{\mathcal{O}}^1$ in the proof of Theorem 1.10 is an example of one such property. Likewise, the role of the Gauss map in Theorem 1.30 hints at the importance of the global properties in establishing geometric properties.) To that end, the main thrust of this work is to begin the *global* analysis of the extended period mapping along Z; if one likes, the global study of the period mapping at infinity. The guiding question is: *What is the global geometry of a fibre A^* of* (1.14)*?*

1.4.1. *Role of limiting mixed Hodge structures.* One of the central points of this work is that the extension data associated to a *limiting* mixed Hodge structure (§3.2.4) encodes a rich geometric structure beyond that carried by a *graded-polarizable* mixed Hodge structure. (The extensive literature on mixed Hodge structures includes [PS08, CEZGT14].) For a mixed Hodge structure (W, F) with weight filtration

$$\{0\} \;\subset\; W_1 \;\subset \cdots \subset\; W_m \;=\; V\,,$$

one defines the *level* $\leq a$ *extension data* to consist of the at most a–fold extensions extracted from $W_\bullet$. For example, if $a = 1$ we have the extensions

$$0 \;\to\; \mathrm{Gr}^W_{\ell-1} \;\to\; W_\ell/W_{\ell-2} \;\to\; \mathrm{Gr}^W_\ell \;\to\; 0\,.$$

A fibre A^* of (1.14) parameterizes limiting mixed Hodge structures with a fixed associated graded. It turns out that for the set of limiting mixed Hodge structures having a fixed associated graded and polarizing monodromy cone σ, the level one extension data admits a family $\{L_M\}$ of line bundles, that are positive over the image of a variation of limiting mixed Hodge structure (Theorem 2.27).[6] These line bundles are analogs of the classical theta line bundles (Example 2.37).

[5] These do not involve the global geometry of the normal crossing divisor Z.

[6] Related results have recently been obtained by Bakker–Brunebarbe–Tsimerman [BBT20], and used to establish the quasi-projectivity of the image of mixed period maps. That work is of a somewhat different character: they study variations of *graded-polarizable* mixed Hodge structures. There is a global study

The fibres $A \subset \overline{B}$ of the map Φ^{S} are compact complex analytic varieties, that contain the fibres A^* of (1.14) as Zariski open subsets. As one moves along A one has a variation of limiting mixed Hodge structure with varying extension data (but fixed associated graded). This variation is captured by the sections of the restrictions $L_M|_A$. One of the main results (2.28) relates the line bundles $L_M|_A$ to the restrictions of the normal bundles $\mathcal{N}_{Z_i/\overline{B}}$ to A. This relates the geometry *along* A to the geometry *normal* to $A \subset \overline{B}$. It is perhaps the *fundamental geometric property* of the period mapping at infinity (§2.5). In particular, the positivity of L_M over the image $\Phi^{\mathsf{T}}(A) \subset \overline{\wp}^{\mathsf{T}}$ implies some negativity of $\mathcal{N}_{Z_i/\overline{B}}\big|_A$; this is what one expects since Φ^{S} contracts A to a point.

1.4.2. *Role of Lie theory.* The formal structures underlying Hodge theory are representations and homogeneous spaces of $\mathbb{Q}$–algebraic groups (specifically, Hodge and Mumford–Tate groups). And as one might expect the constructions and arguments here are primarily Lie theoretic. One of the key observations is a relationship between lifts of the period map at infinity and open Schubert cells in the compact dual of the period domain.

In the classical case, the Hermitian D admits various bounded and unbounded realizations as a symmetric domain in a complex affine space. Each of these is equivalent to the containment of D in an open Schubert cell $\mathcal{S} \subset \check{D}$ in the compact dual. This containment is an essential ingredient in many results, for example [BB66]. In general there is no such containment, and this has been an obstacle to generalizing some classical results. One of the essential observations of this paper (Corollary 5.9), is that for each connected component $A^0 \subset A$ there exists an open Schubert cell $\mathcal{S} \subset \check{D}$ and a neighborhood $A^0 \subset \overline{\mathcal{O}}^0 \subset \overline{B}$ with properties:

(i) The restriction of the local system $\mathcal{V}$ to $\mathcal{O}^0 = B \cap \overline{\mathcal{O}}$ has monodromy Γ_{A^0} that preserves $\mathcal{S}$.

(ii) The induced period map $\Phi_{A^0} : \mathcal{O}^0 \to \Gamma_{A^0} \backslash D$ takes value in $\Gamma_{A^0} \backslash (D \cap \mathcal{S})$.

This is the key observation to establishing the fundamental relationship (2.28) between the line bundles L_M and the normal bundles $\mathcal{N}_{Z_i/\overline{B}}$.

The fact that Φ_{A^0} takes value in $\Gamma_{A^0} \backslash (D \cap \mathcal{S})$ may be interpreted as saying that Φ_{A^0} may be represented as a period matrix whose entries are well-defined up to an action of the discrete group Γ_{A^0}. This is essentially the analog (for the punctured neighborhood $\mathcal{O}^0$) of the familiar fact that the period map Φ may be locally represented at infinity by a period matrix who entries are polynomials in $\log t_i$, with holomorphic entries, where the

of such structures (see [PS08] and the references therein). Here, we consider variations of *limiting* mixed Hodge structures that are *induced* along strata $Z_I^* \subset Z$ *by the period map* $\Phi : B \to \Gamma \backslash D$. One might think of these *limiting* mixed Hodge structures as (equivalence classes of) mixed Hodge structures together with a first-order smoothing.

$\{t_i = 0 \mid i \in I\}$ are the local defining equations of Z (Remark 2.21). In the case that D is Hermitian these polynomials are of degree ≤ 1. More generally, the horizontal matrix entries are always of degree ≤ 1 in the $\log t_i$. (The infinitesimal period relation implies that the period matrix is determined by the horizontal entries up to some discrete data.)

Furthermore a connected component $A^1 \subset A^0$ of the subset where both the associated graded *and* the level one extension data are fixed admits a neighborhood $A^1 \subset \overline{\mathcal{O}}^1 \subset \overline{B}$ with the property that the induced period map Φ_{A^1} on $\mathcal{O}^1 = B \cap \overline{\mathcal{O}}^1$ can be represented by a period matrix whose horizontal entries are linear in $\log \tau_\mu$, again with coefficients in $\mathcal{O}(\overline{\mathcal{O}}^1)$, where the $\tau_\mu \in \mathcal{O}(\overline{\mathcal{O}}^1)$ are the defining functions of $Z \cap \overline{\mathcal{O}}^1$. This essentially globalizes to a neighborhood of A^1 the familiar local expression at infinity for the period map. One obtains the analytic structure on $\hat{\wp}^{\mathsf{T}}$ as a corollary (Theorem 1.10).

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2. Global study of period mappings at infinity: overview

Write

$$Z \;=\; Z_1 \cup Z_2 \cup \cdots \cup Z_\nu\,,$$

with smooth irreducible components Z_i. We denote by

$$Z_I \;=\; \bigcap_{i \in I} Z_i$$

the closed strata, and $Z_I^* \subset Z_I$ the Zariski open smooth locus. (Notation will be defined with greater precision in subsequent sections.)

The restriction $\mathcal{V}_{\mathrm{e}}|_{Z_I^*}$ of Deligne's canonical extension (1.4) admits a weight filtration $\mathcal{W}^I$, and the pair $(\mathcal{W}^I, \mathcal{F}_{\mathrm{e}})$ defines a variation of mixed Hodge structure (VLMHS) over Z_I^*. (See §C for a brief review of these constructions.) The induced Hodge filtrations $\mathcal{F}_{\mathrm{e}}^p(\mathrm{Gr}_a^{\mathcal{W}^I})$ define a polarized variation of Hodge structure over Z_I^*, and in this way we obtain period maps

$$\Phi_I^0 : Z_I^* \;\to\; \Gamma_I \backslash D_I^0\,. \tag{2.1}$$

The question under consideration here is:

(2.2) *What is the global geometry of a fibre A^* of* (2.1)*?*

The answer is summarized in §2.5.

2.1. **A tower of maps.** What varies along the fibre is the extension data of the VLMHS $(\mathcal{W}^I, \mathcal{F}_{\mathrm{e}}|_{Z_I^*})$. This is precisely the information that is lost when we pass to the graded quotients in order to define Φ_I^0; here it is important to keep in mind that the associated weight-graded VHS is locally constant along A^*. There will be global monodromy arising from the action of $\pi_1(A^*)$ that will preserve the weight filtration along the connected components of A^* and that will act by a finite group on the associated weight-graded quotients. As a first step this invites the study of the geometry of the set of extension data associated to the set of LMHS with the same associated graded.[7] This is done by realizing (2.1) as one map in a tower

$$\begin{tikzcd}
Z_I^* \arrow[r, "\Phi_I"] \arrow[dr, bend right, "\Phi_I^a"'] \arrow[ddr, bend right, "\Phi_I^2"'] \arrow[dddr, bend right, "\Phi_I^1"'] \arrow[ddddr, bend right, "\Phi_I^0"'] & (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I \arrow[d, two heads] \\
& (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I^a \arrow[d, two heads] \\
& (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I^2 \arrow[d, two heads] \\
& \Gamma_I\backslash D_I^1 \arrow[d, two heads] \\
& \Gamma_I\backslash D_I^0
\end{tikzcd} \tag{2.3}$$

that is defined as follows. Given a MHS (W, F_0), define Hodge numbers $f_\ell^p := \dim F_0^p(\mathrm{Gr}_\ell^W)$, and set

$$D_W \ = \ \{F \in \check{D} \mid (W,F) \text{ is a MHS}, \dim F^p(\mathrm{Gr}_\ell^W) = f_\ell^p\}\,,$$

with $\check{D}$ the compact dual of D. Let $P_W \subset \mathrm{Aut}(V,Q)$ be the $\mathbb{Q}$–algebraic group stabilizing the weight filtration. Given any $g \in P_W$, there is an induced action on the quotients $W_\ell/W_{\ell-a}$. The normal subgroups

$$P_W^{-a} \ = \ \{g \in P_W \mid g \text{ acts trivially on } W_\ell/W_{\ell-a}\ \forall\ \ell\}$$

define a filtration $P_W = P_W^0 \supset P_W^{-1} \supset \cdots$. The group

$$G_W \ = \ (P_{W,\mathbb{R}}/P_{W,\mathbb{R}}^{-1}) \ltimes P_{W,\mathbb{C}}^{-1} \tag{2.4}$$

acts transitively D_W [KP16]. Let

$$D_W^a \ = \ P_{W,\mathbb{C}}^{-a-1}\backslash D_W$$

be the quotient. This yields a tower of fibre bundles

$$D_W \ \twoheadrightarrow \ D_W^a \ \twoheadrightarrow \ D_W^0\,.$$

[7] It is important here that we are working *polarized* MHS, rather than the larger class of MHS. The structure of the extension data of LMHS is richer than that for MHS, cf. §A.

Set

$$\Gamma_W \;=\; \Gamma \cap P^{-1}_{W,\mathbb{Q}}\,. \tag{2.5}$$

Definition 2.6 (Extension data of MHS). If $\delta_W = \delta_{W,F} \subset D_W$ is the fibre of the surjection $D_W \to D^0_W$, then $\Gamma_W \backslash \delta_{W,F}$ is the *extension data of the MHS* (W, F). The image $\delta^a_W = \delta^a_{W,F}$ of δ_W under the projection $D_W \to D^a_W$ is also a fibre of $D^a_W \twoheadrightarrow D^0_W$, and we say that $\Gamma_W \backslash \delta^a_{W,F}$ is the *extension data of level* $\le a$.

Much of our treatment of extension data will focus on its Lie theoretic properties as a locally homogeneous space (§1.4.2). For a Hodge-theoretic and geometric perspective see §A.

Now suppose that the MHS (W, F_0) is polarized by a nilpotent cone

$$\sigma_I \;=\; \mathrm{span}_{\mathbb{R}_{>0}}\{N_i \mid i \in I\} \;\subset\; \mathrm{End}(V_{\mathbb{R}}, Q)$$

of commuting logarithms of monodromy. (Here $\exp(N_i)$ is a local monodromy operator about Z^*_i.) Define

$$D_I \;=\; \{F \in D_W \mid (W, F) \text{ is polarized by } \sigma_I\}\,.$$

Then $W = W(\sigma_I)$ implies that the $\mathbb{Q}$–algebraic group $C_I \subset \mathrm{Aut}(V, Q)$ centralizing the cone σ_I is a subgroup of P_W. Note that this centralizer also admits a filtration $C_I = C^0_I \supset C^{-1}_I \supset \cdots$ by normal subgroups

$$C^{-a}_I \;=\; C_I \cap P^{-a}_W\,.$$

The group

$$G_I \;=\; (C_{I,\mathbb{R}}/C^{-1}_{I,\mathbb{R}}) \ltimes C^{-1}_{I,\mathbb{C}}$$

acts transitively D_I [KP16]. Let

$$\Gamma_I \;=\; \Gamma \cap C_{I,\mathbb{Q}}\,.$$

The VLMHS along Z^*_I induces the map Φ_I of (2.3); the maps Φ^a_I are defined by passing to the quotient spaces $D^a_I = C^{-a-1}_{I,\mathbb{C}} \backslash D_I$. We say that the quotient D^a_I has automorphism group $G^a_I = G_I/C^{-a-1}_{I,\mathbb{C}}$ to indicate that G_I acts on D^a_I, with the normal subgroup C^{-a-1}_I acting trivially. The base space D^0_I is a Mumford–Tate domain with Mumford–Tate group G^0_I. Again we have a tower of fibre bundles

$$D_I \;\twoheadrightarrow\; D^a_I \;\twoheadrightarrow\; D^0_I\,.$$

Definition 2.7 (Extension data of LMHS). If $\delta_I = \delta_{I,F} = \delta_{W,F} \cap D_I$ is the fibre of the surjection $D_I \to D^0_I$, then $\Gamma_I \backslash \delta_{I,F}$ is the *(polarized) extension data of the LMHS* (W, F). The image $\delta^a_I = \delta^a_{W,I}$ of δ_I under the projection $D_I \to D^a_I$ is also a fibre of $D^a_I \twoheadrightarrow D^0_I$, and we say that $\Gamma_I \backslash \delta^a_{I,F}$ is the *(polarized) extension data of level* $\le a$.

Recall the tower (2.3) and define $\wp_I^a = \Phi_I^a(Z_I^*)$. We have natural surjections $\wp_I^{a+1} \twoheadrightarrow \wp_I^a$. Proposition 7.1(c) implies

Theorem 2.8. *The maps $\wp_I^{a+1} \twoheadrightarrow \wp_I^a$ are finite to one for all $a \geq 2$.*

Remark 2.9. Theorem 2.8 asserts that the level two extension data map Φ_I^2 determines the full extension data map Φ_I up to constants of integration. Additionally the level 2 extension data is discrete. (The data not given by constants of integration is given by sections of line bundles with fixed divisor, Proposition 7.1(c) and Remark 7.2.) So it is then not surprising that we will see that the answer to the question (2.2) is to be found in studying the restriction of Φ^1 to A^*. This restriction takes value in some $\Gamma_I\backslash\delta^1_{I,F}$. The spaces $\Gamma_I\backslash\delta^1_{I,F}$ and $\Gamma\backslash\delta^1_{W,F}$ of level one extension data carry rich geometric structure. As observed by Carlson, these spaces are tori, and $\Gamma_I\backslash\delta^1_{I,F}$ is an abelian subvariety when $F^p(\mathrm{Gr}^W_{-1})$ defines a level one Hodge structure [Car87]. To this we add Theorem 2.27, and the corollary (2.28) that encodes the central geometric information that arises when considering the VLMHS along A^*.

Remark 2.10. In the classical case that D is Hermitian $\Phi_I = \Phi_I^2$, and there are no integration constants. In the non-classical case the integration constants are of a "polylogarithmic character" (§A.6), and thus have an arithmetic aspect not present in the classical case.[8]

Along each strata Z_I^* there is a well-defined Γ–congruence class $[W^I]$ of weight filtrations. Let

$$Z_W \;=\; \bigcup_{[W^I]=[W]} Z_I^*$$

be the union of those strata with the same "weight class." The intersection $Z_I \cap Z_W$ is the *weight-closure* of Z_I^*. There is a subset $I_W \supset I$ with the property that

$$Z_I \,\cap\, Z_W \;=\; \bigcup_{I\subset J\subset I_W} Z_J^*$$

(Corollary 4.6). The maps Φ_I^0 and Φ_I^1 in the tower (2.3) extend to the weight-closure (Lemma 4.10); in particular, we have a commutative diagram

$$\begin{tikzcd} Z_I^* \arrow[r, hook] & Z_I \cap Z_W \arrow[r, "\Phi_I^1"'] \arrow[rr, bend left, "\Phi_I^0"] & \Gamma_I\backslash D_I^1 \arrow[r, two heads] & \Gamma_I\backslash D_I^0 \,. \end{tikzcd}$$

In general, the maps Φ_I^a do not extend when $a \geq 3$ (the case $a = 2$ is subtle, cf. §4.3.1).

[8]There is extensive literature relating Hodge–Tate mixed Hodge structures to integrals of polylogarithmic type. So far as we are aware, the special properties that may arise when restricting to limiting mixed Hodge structures of Hodge–Tate type have not been discussed.

2.2. **Two topological completions.** The maps Φ_I^0 and Φ_I^1 are distinguished in the sense that they can be patched together to define proper analytic maps on Z_W.[9] This is important because it makes it possible for us to realize A^* as a Zariski open subset of a *compact* fibre A.

2.2.1. *Patching for compactness.* The inclusions $D_I \hookrightarrow D_W$ and $\Gamma_I \subset \Gamma_W$ induce

$$\Gamma_I \backslash D_I^a \ \to \ \Gamma_W \backslash D_W^a \,.$$

The maps Φ_W^0 and Φ_W^1 defined by the diagram

$$\begin{tikzcd} Z_I^* \arrow[r, hook] \arrow[d, "\Phi_I^1"] \arrow[dd, bend right=60, "\Phi_I^0"'] & Z_W \arrow[d, "\Phi_W^1"'] \arrow[dd, bend left=60, "\Phi_W^0"] \\ \Gamma_I \backslash D_I^1 \arrow[r] \arrow[d, two heads] & \Gamma_W \backslash D_W^1 \arrow[d, two heads] \\ \Gamma_I \backslash D_I^0 \arrow[r] & \Gamma_W \backslash D_W^0 \end{tikzcd} \tag{2.11}$$

are proper and analytic (Lemma 4.1).

Remark 2.12. The fibres A of Φ_W^0 are compact analytic subvarieties of $\overline{B}$. And given $Z_I^* \subset Z_W$, the intersection $A \cap Z_I^*$ is the fibre A^* of Question (2.2). The answer is given by studying the global geometry of A (which as the advantage over A^* of being compact).

The proper mapping theorem implies that the the images

$$\wp_W^0 \ = \ \Phi_W^0(Z_W) \quad \text{and} \quad \wp_W^1 \ = \ \Phi_W^1(Z_W)$$

are complex analytic spaces (Corollary 4.2). We define two set-theoretic "completions"

$$\overline{\wp}^{\mathsf{S}} \ = \ \overline{\wp}^0 \ = \ \bigcup_W \wp_W^0 \quad \text{and} \quad \overline{\wp}^1 \ = \ \bigcup_W \wp_W^1$$

(with the finite unions taken over a single representative $W \in [W]$) of $\wp = \Phi(B)$, and maps

$$\begin{tikzcd} \overline{B} \arrow[r, "\Phi^1"'] \arrow[rr, bend left, "\Phi^0"] & \overline{\wp}^1 \arrow[r, two heads] & \overline{\wp}^0 \end{tikzcd} \tag{2.13}$$

defined by specifying $\Phi^0\big|_{Z_W} = \Phi_W^0$ and $\Phi^1\big|_{Z_W} = \Phi_W^1$.

[9]This is essentially Griffiths's extension of Φ across points of finite monodromy to obtain a proper period map [Gri70, §9].

2.2.2. *Topology.* Fix a Riemannian metric on on $\overline{B}$. Let $\overline{\wp}^{\mathrm{e}}$ either of $\overline{\wp}^0$ or $\overline{\wp}^1$, let Φ^{e}_W denote Φ^0_W or Φ^1_W, respectively, and let Φ^{e} denote the corresponding Φ^0 or Φ^1. Since the fibres of Φ^{e} are compact, there is an induced metric on $\overline{\wp}^{\mathrm{e}}$. Endow $\overline{\wp}^{\mathrm{e}}$ with the metric topology. Then $\Phi^{\mathrm{e}} : \overline{B} \to \overline{\wp}^{\mathrm{e}}$ is continuous, so that $\overline{\wp}^{\mathrm{e}}$ is compact. This topology is Hausdorff, and the induced subspace topology on $\wp^{\mathrm{e}}_W = \Phi^{\mathrm{e}}_W(Z_W) \subset \overline{\wp}^{\mathrm{e}}$ coincides with the existing topology on $\wp^{\mathrm{e}}_W$ as a complex analytic space (Proposition 5.1).

2.2.3. *A "Stein factorization" of* Φ^{e}*.* Let

$$\begin{tikzcd} Z_W \arrow[r, "\hat\Phi^{\mathrm{e}}_W"'] \arrow[rr, bend left, "\Phi^{\mathrm{e}}_W"] & \hat\wp^{\mathrm{e}}_W \arrow[r] & \wp^{\mathrm{e}}_W \end{tikzcd} \tag{2.14}$$

be the Stein factorization of Φ^{e}_W; the fibres of $\hat\Phi^{\mathrm{e}}_W$ are connected, the fibres of $\hat\wp^{\mathrm{e}}_W \to \wp^{\mathrm{e}}_W$ are finite, and $\hat\wp^{\mathrm{e}}_W$ is normal. Set

$$\hat\wp^{\mathrm{e}} \ = \ \bigcup \hat\wp^{\mathrm{e}}_I \,,$$

and define maps

$$\begin{tikzcd} \overline{B} \arrow[r, "\hat\Phi^{\mathrm{e}}"'] \arrow[rr, bend left, "\Phi^{\mathrm{e}}"] & \hat\wp^{\mathrm{e}} \arrow[r] & \overline{\wp}^{\mathrm{e}} \end{tikzcd} \tag{2.15}$$

by specifying that the restriction of (2.15) to Z_W coincides with (2.14).

Remark 2.16*.* In contrast to Conjecture 1.15 ($\overline{\wp}^{\mathsf{S}} = \overline{\wp}^0$), the topological space $\overline{\wp}^1$ will not admit a compatible complex analytic structure: the fibre dimensions of $\overline{\wp}^1 \to \overline{\wp}^0$ may *drop* on proper subvarieties. For example, if the variation of limiting mixed Hodge structures is Hodge–Tate over Z^*_I, then it is Hodge–Tate over Z_I and the fibres of $\overline{\wp}^1 \to \overline{\wp}^0$ over $\Phi^0(Z_I)$ are *finite.*

2.3. Neighborhood of a Φ^0–fibre.

Let A be a (compact) fibre of Φ^0. Then

$$A \ \subset \ Z_{W^A}$$

is a fibre of some $\Phi^0_{W^A}$. Let A^0 be a (compact) connected component of A. We begin our study of the geometry of A^0 with the observation that this (connected component of a) fibre admits a neighborhood $\overline{\mathcal{O}}^0 \subset \overline{B}$ with special properties. First, we may assume that $\Phi^0\big|_{\overline{\mathcal{O}}^0}$ is proper (Corollary 5.4). Second the neighborhood admits a large class of line bundles with explicit, meromorphic sections that capture certain linear combinations of the divisors $Z_i \cap \overline{\mathcal{O}}^0$.

Theorem 2.17 (Corollary 5.25)**.** (a) *There is a family $L_M \to \overline{\mathcal{O}}^0$ of line bundles, $M \in \mathbf{N}^*$, and integers $\kappa(M,N_i)$ associated with each Z_i, so that*

$$L_M \ = \ \sum \kappa(M,N_i)\ [Z_i]|_{\overline{\mathcal{O}}^0} \ = \ \sum \kappa(M,N_i)\ \mathcal{N}_{Z_i/\overline{B}}\Big|_{\overline{\mathcal{O}}^0} \,.$$

(b) *There exists an explicit, canonical meromorphic section* $s_M : \overline{\mathcal{O}}^0 \to L_M$ *with divisor*

$$(s_M) \;=\; \sum \kappa(M,N_i)\,(Z_i \cap \overline{\mathcal{O}}^0)\,. \tag{2.18}$$

The set $\mathbf{N}^*$ is defined in (5.22). The key observation behind Theorems 2.17 is that we may also assume that $\overline{\mathcal{O}}^0$ has the properties:

(i) The restriction $\mathcal{V}|_{\mathcal{O}^0} = \tilde{\mathcal{O}}^0 \times_{\pi_1(\mathcal{O}^0)} V$ of the VHS to

$$\mathcal{O}^0 \;=\; \overline{\mathcal{O}}^0 \cap B$$

has monodromy

$$\Gamma_{A^0} \;\subset\; P_{W^A,\mathbb{Q}} \tag{2.19}$$

preserving an open Schubert cell $\mathcal{S} \subset \check{D}$ (Lemma 5.8).

(ii) Let

$$\Phi_{A^0} : \mathcal{O}^0 \;\to\; \Gamma_{A^0}\backslash D \tag{2.20}$$

be the induced period map. The lift of Φ_{A^0} to the universal cover $\tilde{\mathcal{O}}^0 \to \mathcal{O}^0$

$$\begin{array}{ccc} \tilde{\mathcal{O}}^0 & \xrightarrow{\tilde{\Phi}_{A^0}} & \mathcal{S} \cap D \\ \downarrow & & \downarrow \\ \mathcal{O}^0 & \xrightarrow{\Phi_{A^0}} & \Gamma_{A^0}\backslash D \end{array}$$

takes value in a Schubert cell $\mathcal{S} \subset \check{D}$ (Corollary 5.9).

We first construct line bundles $\mathcal{L}_M \to \Gamma_{A^0}\backslash\mathcal{S}$ and canonical, nowhere vanishing sections s_M (§5.4.1). The period map Φ_{A^0} pulls the bundle and section back to $\mathcal{O}^0$. We then verify that the bundle and section extend to $\overline{\mathcal{O}}^0$ (§5.4.3).

Remark 2.21. If we think of the period map as given by a period matrix (as in §3.4, then one may informally think of the M as parameterizing the matrix entries, and the section s_M as the associated matrix entry. In Theorem 2.17, the set $\mathbf{N}^*$ is parameterizing those matrix entries with the property that s_M extends across the singularity.

Remark 2.22. By definition (5.22) the set $\mathbf{N}^*$ parameterizing the line bundles L_M is a subgroup of the vector space $\mathfrak{g}_\mathbb{C} = \mathrm{End}(V_\mathbb{C}, Q)$, and $L_{M_1+M_2} = L_{M_1} + L_{M_2}$.

Recall that Deligne's extension (1.4) of the Hodge vector bundles is given by exhibiting a framing/trivialization over local coordinate neighborhoods $\overline{\mathcal{U}} \subset \overline{B}$ (§C). Under a mild assumption on Γ_{A^0} (Remark 5.12), the properties (i)–(ii) above enable us to extend this construction to $\overline{\mathcal{O}}^0$ and prove:

(a) The determinant $\det(\mathcal{F}^p_\mathrm{e})$ is trivial over $\overline{\mathcal{O}}^0$ (Theorem 5.15). In particular, the Hodge line bundle Λ_e is trivial over $\overline{\mathcal{O}}^0$.

(b) There is a well-defined weight filtration $\mathcal{W}^A$ of $\mathcal{V}_{\mathrm{e}}|_{Z_{W^A}}$. (This statement does not require the assumption on Γ_{A^0}.) The induced Hodge filtrations $\mathcal{F}^p_{\mathrm{e}}(\mathrm{Gr}^{\mathcal{W}^A}_a)$ on the associated graded $\mathrm{Gr}^{\mathcal{W}^A}_a = \mathcal{W}^A_a/\mathcal{W}^A_{a-1}$ are trivial over $\overline{\mathcal{O}}^0 \cap Z_{W^A}$ (Theorem 5.16).

2.4. **Restriction of Φ^1 to a Φ^0–fibre.** The period map $\Phi_{A^0} : \mathcal{O}^0 \to \Gamma_{A^0}\backslash D$ of (2.20) admits obvious analogs

$$\begin{tikzcd} Z_W \cap \overline{\mathcal{O}}^0 \arrow[r, "\Phi^1_{A^0,W}"'] \arrow[rr, bend left, "\Phi^0_{A^0,W}"] & \Gamma_{A^0,W}\backslash D^1_W \arrow[r, two heads] & \Gamma_{A^0,W}\backslash D^0_W \end{tikzcd} \tag{2.23}$$

and

$$\begin{tikzcd} \overline{\mathcal{O}}^0 \arrow[r, "\Phi^1_{A^0}"'] \arrow[rr, bend left, "\Phi^0_{A^0}"] & \overline{\wp}^1_{A^0} \arrow[r, two heads] & \overline{\wp}^0_{A^0} \end{tikzcd} \tag{2.24}$$

of (2.11) and (2.13).

Remark 2.25. It follows from the properness of Φ^1 and Φ^0, and Corollary 5.4 that the fibres of the obvious maps $\overline{\wp}^1_{A^0} \to \overline{\wp}^1$ and $\overline{\wp}^0_{A^0} \to \overline{\wp}^0$ are finite. So, modulo finite data, (2.24) carries the same information as the restriction of (2.13) to $\overline{\mathcal{O}}^0$.

In this section we wish to discuss (2.23) in the case that

$$W \;=\; W^A\,,$$

so that we may study Φ^1 along the strata Z_W containing A. In this case, (2.19) implies $\Gamma_{A^0,W} = \Gamma_{A^0}$ and $\Gamma_{A^0}\backslash D^0_W = D^0_W$, and (2.23) specializes to

$$\begin{tikzcd} Z_W \cap \overline{\mathcal{O}}^0 \arrow[r, "\Phi^1_{A^0,W}"'] \arrow[rr, bend left, "\Phi^0_{A^0,W}"] & \Gamma_{A^0}\backslash D^1_W \arrow[r, two heads, "\pi^1_W"'] & D^0_W\,. \end{tikzcd} \tag{2.26}$$

We will see that much of the global geometry of A^0 is encoded by the the restriction of $\Phi^1_{A^0}$ to $A^0 \subset Z_W$. This restriction takes value in a π^1_W–fibre.

Given $Z^*_I \subset Z_W$, we have inclusions $D^a_I \hookrightarrow D^a_W$. In particular, $D^1_I \twoheadrightarrow D^0_I$ determines a sub-bundle of the restriction π^1_I of π^1_W to $\Gamma_{A^0}\backslash D^1_I$

$$\begin{tikzcd} \Gamma_{A^0}\backslash D^1_I \arrow[r, hook] \arrow[d, two heads, "\pi^1_I"'] & \Gamma_{A^0}\backslash D^1_W \arrow[d, two heads, "\pi^1_W"] \\ D^0_I \arrow[r, hook] & D^0_W\,. \end{tikzcd}$$

The restriction of Φ^1 to $A^0 \cap Z_I$ takes value in a fibre of π^1_I.

Under the mild assumption that the monodromy Γ_{A^0} if unipotent (Remark 5.12) we have

Theorem 2.27 (Theorem 6.3)**.** *Set $W = W^A$, and suppose $Z_I^* \subset Z_W$.*

(a) *The bundle $\pi_W^1 : \Gamma_{A^0}\backslash D_W^1 \twoheadrightarrow D_W^0$ admits a subbundle*

$$\begin{array}{ccccc} T_W & \hookrightarrow & \mathcal{T}_W & \subset & \Gamma_W\backslash D_W^1 \\ & & \Big\downarrow{\scriptstyle \pi_W^1} & & \\ & & D_W^0 & & \end{array}$$

that is fibered by compact tori T_W. The restriction $\Phi^1\big|_{A^0}$ takes value in T_W.

(b) *The bundle $\pi_I^1 : \Gamma_{A^0}\backslash D_I^1 \twoheadrightarrow D_I^0$ admits a subbundle*

$$\begin{array}{ccccc} J_I & \hookrightarrow & \mathcal{J}_I & \subset & \Gamma_I\backslash D_I^1 \\ & & \Big\downarrow{\scriptstyle \pi_I^1} & & \\ & & D_I^0 & & \end{array}$$

that is fibered by abelian varieties J_I. The restriction $\Phi^1\big|_{A^0\cap Z_I}$ takes value in J_I.

(c) *There is a subset $\mathbf{N}^1 \subset \mathbf{N}^*$ parameterizing line bundles* (both denoted)

$$\begin{array}{ccc} \mathcal{L}_M & & \mathcal{L}_M \\ \downarrow & & \downarrow \\ J_I & \longrightarrow & T_W \end{array}$$

over the level one extension data, with the property that

$$L_M\big|_{A^0} \;=\; (\Phi^1\big|_{A^1})^*(\mathcal{L}_M)\,.$$

(d) *There is a subset $\mathbf{N}_I^{\mathfrak{sl}_2} \subset \mathbf{N}^1$ with the property that the abelian variety J_I is polarized by the $\mathcal{L}_M^*$ with $M \in \mathbf{N}_I^{\mathfrak{sl}_2}$.*

(e) *The set $\mathbf{N}_I^{\mathfrak{sl}_2,+} \;=\; \{M \in \mathbf{N}_I^{\mathfrak{sl}_2} \mid \kappa(M,N_i) > 0\,,\ \forall\ i \in I\}$ is nonempty. Indeed the dimension of the* (not necessarily convex) *cone $\{yM \mid M \in \mathbf{N}_I^{\mathfrak{sl}_2,+}\,,\ 0 < y \in \mathbb{R}\}$ is equal to $\dim \sigma_I$.*

A detailed outline of the proof of Theorem 2.27 follows the statement of Theorem 6.3.

2.5. **Geometry at infinity.** Together Theorems 2.17 and 2.27 imply that

$$(\Phi^1\big|_{A^0})^*(\mathcal{L}_M) \;=\; \sum \kappa(M,N_i)[Z_i]\big|_{A^0} \;=\; \sum \kappa(M,N_i)\,\mathcal{N}_{Z_i/\overline{B}}\Big|_{A^0}\,. \tag{2.28}$$

(The sum is over all i such that $Z_i^* \cap A^0 \neq \emptyset$, which is necessarily a subset of I_W.) It follows from Proposition 7.1 that *this is the central geometric information that arises when considering the VLMHS $(\mathcal{W},\mathcal{F}_{\mathrm{e}})$ along A^0.*

Example 2.29. Suppose that $A^0 \subset Z_i^*$ and $N_i \neq 0$. Taking $I = \{i\}$, we may choose $M \in \mathbf{N}_i^{\mathfrak{sl}_2,+}$, so that $\mathcal{L}_M^* \to J_i$ is ample and $\kappa(M, N_i) > 0$. Then $\mathcal{N}^*_{Z_i/\overline{B}}\big|_{A^0}$ is ample if the differential of $\Phi^1\big|_{A^0}$ is injective.

More generally, we have

Corollary 2.30. *Suppose the differential* $\Phi^1\big|_{A^0 \cap Z_I}$ *is injective. Then the line bundle* $\sum \kappa(M, N_i)\, \mathcal{N}^*_{Z_i/\overline{B}}\Big|_{A^0 \cap Z_I}$ *is ample.*

Remark 2.31. The sum in Corollary 2.30 is over those j with $Z_j \cap (A^0 \cap Z_I)$ nonempty. Theorem 2.27(e) asserts that we may choose M so that the integers $\kappa(M, N_j)$ are positive when $j \in I$; we are not able to say the same when $j \notin I$.

Applications of (2.28) include Propositions 2.32 and 2.33. The first is a special case of Conjecture 1.23(b).

Proposition 2.32. *Suppose that* $Z = Z_1$ *consists of a single irreducible component, and* $\mathrm{d}\Phi^1_W$ *is injective on* Φ^0*–fibres. Assume also that the effective cone* $\mathrm{Eff}^1(\overline{B})$ *of 1–cycles is finitely generated. Then the line bundle* $\Pi = m\Lambda_{\mathrm{e}} - [Z]$ *is ample for* $m \geq m_0$*.*

The second application is a constraint on the variations of limiting mixed Hodge structure that may arise along the divisor Z when $\dim B = 2$.

Proposition 2.33. *Assume that* $\dim B = 2$ *and that* $\Phi : B \to \Gamma\backslash D$ *satisfies generic local Torelli (equivalently,* Φ_* *is injective at some point* $b \in B$*, so that* $\dim \wp = 2$*). Then* Φ^1 *is necessarily non-constant on some irreducible component* Z_i *of* Z*.*

Definition 2.34. The variation $(\mathcal{W}^I, \mathcal{F}_{\mathrm{e}}|_{Z_I^*})$ of limiting mixed Hodge structure along Z_I^* is of *Hodge–Tate type*, if the associated graded variation $\mathcal{F}^p_{\mathrm{e}}(\mathrm{Gr}^{\mathcal{W}^I}_a)$ of Hodge structure is Hodge–Tate.

Remark 2.35. When the variation is of Hodge–Tate type, both the period map Φ^0 and the level one extension data map Φ^1 of (2.1) and (2.3) are locally constant along Z_I^*.

Corollary 2.36. *Suppose that* B *is a surface and that the LMHS along all of* Z *is of Hodge–Tate type. Then* $\dim \wp \leq 1$*.*

The propositions are proved in §9.2 and §8.3, respectively. See §A.6 for further discussion of the Hodge–Tate case.

Example 2.37. We would like to informally discuss (2.28) in a familiar example. Consider the case that D is the period domain parameterizing pure, weight $n = 1$, Q–polarized

Hodge structures on V with Hodge numbers $\mathbf{h} = (2,2)$. Suppose that Q is represented by the matrix

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

with respect to a basis $\{v_1, v_2, v_3, v_4\}$. Suppose that $\dim B = 2$, and fix local coordinates $(t,w) \in \Delta^2$ on $\overline{B}$ centered at a point $b \in Z$ so that $Z = \{t = 0\}$. Suppose the normalized period matrix takes the form

$$\widetilde{\Phi}(t,w) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha(t,w) & \lambda(t,w) \\ \hat{\nu}(t,w) & \alpha(t,w) \end{bmatrix},$$

with $\alpha(t,w)$, $\lambda(t,w)$, $\nu(t,w) = \hat{\nu}(t,w) - \log(t)/2\pi\mathbf{i}$ holomorphic functions on Δ^2. The nilpotent logarithm of monodromy about $\{t = 0\}$ is

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The associated weight filtration $W_0 \subset W_1 \subset W_2 = V$ is given by $\{v_4\} \subset \{v_4, v_3, v_2\}$. The function $\alpha(t,w)$ encodes the level one extension data, and the function $\hat{\nu}(t,w)$ encodes the level two extension data. If we write $m = \kappa(M,N) \in \mathbb{Z}$, then locally the line bundle L_M admits a trivialization with respect to which the canonical section s_M is given by the function

$$\tau_M(t,w) = t^m \exp(2\pi\mathbf{i}\, m\, \nu(t,w)).$$

While $\tau_M(t,w)$ is invariant under the local monodromy $\exp(N)$, we must also account for the monodromy about the fibre A^0. In general, monodromy will take the form

$$\gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathsf{a} & 1 & 0 & 0 \\ \mathsf{b} & 0 & 1 & 0 \\ \mathsf{c} & \mathsf{b} & -\mathsf{a} & 1 \end{bmatrix},$$

with $\mathsf{a},\mathsf{b},\mathsf{c}\in\mathbb{Z}$. Then the period matrix $\widetilde{\Phi}(t,w)$ transforms as

$$\gamma\cdot\widetilde{\Phi}(t,w) \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha(t,w)+\mathsf{b}-\mathsf{a}\lambda(t,w) & \lambda(t,w) \\ \hat{\nu}(t,w)+\mathsf{c}-\mathsf{ab}+2\mathsf{a}\alpha(t,w)+\mathsf{a}^2\lambda(t,w) & \alpha(t,w)+\mathsf{b}-\mathsf{a}\lambda(t,w) \end{bmatrix}$$

Under this action, $\nu(t,w)$ transforms as

$$\nu(t,w) \;\mapsto\; \nu(t,w)+\mathsf{c}-\mathsf{ab}+2\mathsf{a}\alpha(t,w)+\mathsf{a}^2\lambda(t,w)\,,$$

so that $\tau_M(t,w)$ transforms as

$$\begin{aligned} \tau_M(t,w) &\;\mapsto\; t^m \exp 2\pi\mathbf{i}\, m\left(\varepsilon(t,w)+\mathsf{a}^2\lambda(t,w)-2\mathsf{a}\alpha(t,w)\right) \\ &\;=\; \tau_M(t,w)\exp 2\pi\mathbf{i}\, m\left(\mathsf{a}^2\lambda(t,w)-2\mathsf{a}\alpha(t,w)\right). \end{aligned}$$

This is the functional equation for the classical θ-function. We may normalize our choice of coordinates (t,w) so that $\nu(t,w)=0$. Then, taking $m=1$, this computation implies that $t\cdot\vartheta$, with ϑ a section of the dual to the θ-line bundle, is globally well-defined along A^0; that is, the pull-back of the θ-line bundle on J_I under $\Phi^1_{A^0}$ is the conormal bundle.

Remark 2.38 (Interpretation of (2.28)). Given a VHS over the punctured disc Δ^* with unipotent monodromy $T=\exp(N)$, it is well-known that there is a well-defined *equivalence class* of LMHS (W,F,N) at the origin $t=0$. The equivalence relation is $(W,F,N)\sim(W,\tilde{F},N)$ if and only if $\tilde{F}\in\exp(\mathbb{C}N)\cdot F$. A unique representative of the equivalence class is determined by specifying a nonzero $v\in T_0^*\Delta$. If, for simplicity of discussion, we assume that $A^0\subset Z_i^*$ is contained in an open codimension one strata, then for each point $b\in A^0$ there is a well-defined equivalence class $[W,F,N]_b$ of LMHS. (The F in these LMNS all induce the same Hodge filtration on the $\mathrm{Gr}^W_\bullet$.) What (2.28) does is give a line bundle $(\Phi^1_{A^0})^*(\mathcal{L}_M)\to A^0$ with the property that nonzero elements of $\lambda\in(\Phi^1_{A^0})^*(\mathcal{L}_M)_b$ determine a unique representative of $[W,F,N]_b$.

2.6. **Summary of notation.**

- period domain D parameterizing pure, Q–polarized HS on V of weight n
- compact dual $\check{D}$
- algebraic automorphism group $G=\mathrm{Aut}(V,Q)$, with Lie algebra $\mathfrak{g}=\mathrm{End}(V,Q)$
- smooth projective $\overline{B}$ with reduced normal crossing divisor $Z\subset\overline{B}$
- polarized variation of Hodge structure $\mathcal{V}=\tilde{B}\times_{\pi_1(B)}V$ over $B=\overline{B}\backslash Z$ with monodromy representation $\pi_1(B)\twoheadrightarrow\Gamma\subset G$
- the induced period map $\Phi:B\to\Gamma\backslash D$

- Hodge filtration $\mathcal{F}^p \subset \mathcal{V}$ and Hodge line bundle

$$\Lambda \;=\; \det(\mathcal{F}^n) \otimes \det(\mathcal{F}^{n-1}) \otimes \cdots \otimes \det(\mathcal{F}^{\lceil (n+1)/2 \rceil})$$

- extensions $\mathcal{F}^p_{\mathrm{e}} \subset \mathcal{V}_{\mathrm{e}}$ and Λ_{e} to $\overline{B}$
- graded quotients $\mathcal{E}^p_{\mathrm{e}} = \mathcal{F}^p_{\mathrm{e}}/\mathcal{F}^{p+1}_{\mathrm{e}} = \mathrm{Gr}^p_{\mathcal{F}_{\mathrm{e}}}$, and $\mathcal{E}_{\mathrm{e}} = \oplus \mathcal{E}^p_{\mathrm{e}}$
- bundle map $\Psi : T_{\overline{B}}(-\log Z) \to \mathrm{Gr}^{-1}_{\mathcal{F}_{\mathrm{e}}}(\mathrm{End}(\mathcal{E}_{\mathrm{e}}))$ induced by flat connection
- algebraic subgroup $P_W \subset G$ stabilizing weight filtration W, filtered by normal subgroups

$$P^{-a}_W \;:=\; \{g \in P_W \mid g \text{ acts trivially on } W_\ell/W_{\ell-a}\ \forall\ \ell\}\,,$$

  $a \ge 0$, $P_W = P^0_W$
- algebraic subgroup $C_I \subset P_W$ centralizing cone $\sigma_I \subset \mathfrak{g}$, with $W = W^I = W(\sigma_I)$, filtered by normal subgroups

$$C^{-a}_I \;:=\; C_I \,\cap\, P^{-a}_W\,.$$

- reference filtration $F^\bullet_0 \in \check D$, Hodge numbers $f^p_\ell := \dim F^p_0(\mathrm{Gr}^W_\ell)$
- $D_W \;:=\; \{F \in \check D \mid (W,F) \text{ is a MHS, } \dim F^p(\mathrm{Gr}^W_\ell) = f^p_\ell\}\,.$
- $\Gamma_W = \Gamma \cap P^{-1}_{W,\mathbb{Q}}$
- $G_W := (P_{W,\mathbb{R}}/P^{-1}_{W,\mathbb{R}}) \ltimes P^{-1}_{W,\mathbb{C}}$ acts transitively on D_W
- $D^a_W := P^{-a-1}_{I,\mathbb{C}} \backslash W_I$ with automorphism group $G^a_W := G_W/P^{-a-1}_{W,\mathbb{C}}$. [10]
- projections $D_W \twoheadrightarrow D^a_W \twoheadrightarrow D^0_W$ and $G_W \twoheadrightarrow G^a_W \twoheadrightarrow G^0_W$
- $D_I \;:=\; \{F \in D_W \mid (W,F) \text{ is polarized by } \sigma_I\}$
- $G_I \;:=\; (C_{I,\mathbb{R}}/C^{-1}_{I,\mathbb{R}}) \ltimes C^{-1}_{I,\mathbb{C}}$ acts transitively on D_I
- $D^a_I := C^{-a-1}_{I,\mathbb{C}} \backslash D_I$ with automorphism group $G^a_I := G_I/C^{-a-1}_{I,\mathbb{C}}$. [11]
- projections $D_I \twoheadrightarrow D^a_I \twoheadrightarrow D^0_I$ and $G_I \twoheadrightarrow G^a_I \twoheadrightarrow G^0_I$
- D^0_I is a Mumford–Tate domain with Mumford–Tate group $G^0_I = C_I/C^{-1}_I$. [12]
- $\Gamma_I = \Gamma \cap C_{I,\mathbb{Q}}$

[10] We think of this as indicating that G_W acts on D^a_W, with the normal subgroup $P^{-a-1}_{W,\mathbb{C}}$ acting trivially.

[11] We think of this as indicating that G_I acts on D^a_I, with the normal subgroup $C^{-a-1}_{I,\mathbb{C}}$ acting trivially.

[12] One may define analogous spaces D^a_W for MHS. In the absence of the polarization, these spaces have less structure. For example, the analog D^0_W of D^0_I is a flag domain in Wolf's sense [FHW06, Wol69], but not a Mumford–Tate domain – the isotropy group is not compact.

- period map to various quotients of LMHS

$$\begin{tikzcd}
Z_I^* \arrow[r, "\Phi_I"] \arrow[dr, bend right, "\Phi_I^a"] \arrow[ddr, bend right, "\Phi_I^2"] \arrow[dddr, bend right, "\Phi_I^1"] \arrow[ddddr, bend right, "\Phi_I^0"'] & (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I \arrow[d, two heads] \\
& (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I^a \arrow[d, two heads] \\
& (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I^2 \arrow[d, two heads] \\
& \Gamma_I\backslash D_I^1 \arrow[d, two heads] \\
& \Gamma_I\backslash D_I^0
\end{tikzcd}$$

- $D_I^a \hookrightarrow D_W^a$ and $\Gamma_I \subset \Gamma_W$ induce $\Gamma_I\backslash D_I^a \ \to \ \Gamma_W\backslash D_W^a$
- *weight-strata* $Z_W \ = \ \bigcup\limits_{W^I=W} Z_I^*$
- $Z_I \cap Z_W$ is the *weight-closure* of Z_I^*
- unique maximal I_W such that $W^{I_W} = W$
- if $Z_J^* \subset Z_I \cap Z_W$, then $D_J \hookrightarrow D_I$ and $\Gamma_J \subset \Gamma_I$ induce $\Gamma_J\backslash D \to \Gamma_I\backslash D_I$
- Φ_I^0 and Φ_I^1 extend to proper holomorphic maps on the weight-closure, and the extensions are compatible with Φ_J^a on $Z_J^* \subset Z_I \cap Z_W$

$$\begin{tikzcd}
Z_J^* \arrow[r, hook] \arrow[d, "\Phi_J^1"] \arrow[dd, bend right=60, "\Phi_J^0"'] & Z_I \cap Z_W \arrow[d, "\Phi_I^1"'] \arrow[dd, bend left=60, "\Phi_I^0"] \\
\Gamma_J\backslash D_J^1 \arrow[r] \arrow[d, two heads] & \Gamma_I\backslash D_I^1 \arrow[d, two heads] \\
\Gamma_J\backslash D_J^0 \arrow[r] & \Gamma_I\backslash D_I^0
\end{tikzcd}$$

- $\Phi_W \in \{\Phi_W^0, \Phi_W^1\}$ defined by

$$\begin{tikzcd}
Z_I^* \arrow[r, hook] \arrow[d, "\Phi_I^1"] \arrow[dd, bend right=60, "\Phi_I^0"'] & Z_W \arrow[d, "\Phi_W^1"'] \arrow[dd, bend left=60, "\Phi_W^0"] \\
\Gamma_I\backslash D_I^1 \arrow[r] \arrow[d, two heads] & \Gamma_W\backslash D_W^1 \arrow[d, two heads] \\
\Gamma_I\backslash D_I^0 \arrow[r] & \Gamma_W\backslash D_W^0
\end{tikzcd}$$

- $\wp_W^0 = \Phi_W^0(Z_W) \subset \Gamma_W\backslash D_W^0$ and $\wp_W^1 = \Phi_W^1(Z_W) \subset \Gamma_W\backslash D_W^1$,

$$\overline{\wp}^0 \ = \ \bigcup \wp_W^0 \quad \text{and} \quad \overline{\wp}^1 \ = \ \bigcup \wp_W^1 \,,$$

- two topological extensions of $\Phi : B \to \Gamma\backslash D$ defined strata-wise

$$\begin{tikzcd} \overline{B} \arrow[r, "\Phi^1"'] \arrow[rr, bend left, "\Phi^0"] & \overline{\wp}^1 \arrow[r, two heads] & \overline{\wp}^0 \end{tikzcd}$$

- $A \subset Z_W$ compact Φ^0–fibre
- $A^0 \subset A$ (compact) connected component of A
- $A^1 \subset A^0$ (compact) connected component of Φ^1–fibre
- $\delta_W \subset D_W$ is the preimage of $\Phi^0(A^0) \in \Gamma_W\backslash D^0_W$ under the projection $D_W \twoheadrightarrow \Gamma_W\backslash D^0_W$; these are pairs (W, F) with the same $F(\mathrm{Gr}^W)$
- $\delta_I \subset D_I$ is the preimage of $\Phi^0(A^0 \cap Z_I) \in \Gamma_I\backslash D^0_I$ under the projection $D_I \twoheadrightarrow \Gamma_I\backslash D^0_I$; these are pairs (W, F) that are polarized by σ_I and with the same $F(\mathrm{Gr}^W)$.
- neighborhood $A^0 \subset \overline{\mathcal{O}}^0 \subset \overline{B}$, Schubert cell $\mathcal{S} \subset \check{D}$, period map

$$\Phi_{A^0} : B \cap \overline{\mathcal{O}}^0 \ \to\ \Gamma_{A^0}\backslash(D \cap \mathcal{S})$$

- $\Gamma_{A^0} \subset \Gamma \cap P_W$ monodromy about A^0, $\Gamma^{-1}_{A^0} = \Gamma_{A^0} \cap P_W$ monodromy acting trivially on Gr^W, *finite* quotient $\Gamma_{A^0}/\Gamma^{-1}_{A^0}$ acting on Gr^W is contained in $G_{I(A^0)} = C^0_{I(A^0)}/C^{-1}_{I(A^0)}$, $I(A^0) = \{i \mid A^0 \cap Z^*_i \neq 0\}$
- $A^1 \subset A^0$ (compact) connected component of Φ^1–fibre, neighborhood $A^1 \subset \overline{\mathcal{O}}^1 \subset \overline{\mathcal{O}}^0 \subset \overline{B}$, period map

$$\Phi_{A^1} : B \cap \overline{\mathcal{O}}^1 \ \to\ \Gamma_{A^1}\backslash(D \cap \mathcal{S})$$

- $\Gamma_{A^1} \subset \Gamma \cap P_W$ monodromy about A^0, $\Gamma^{-1}_{A^0} = \Gamma_{A^0} \cap P^{-1}_W$ monodromy acting trivially on Gr^W, *finite* quotient $\Gamma_{A^0}/\Gamma^{-1}_{A^0}$ acting on Gr^W

3. Asymptotics of period maps: review of local properties

Here we set notation and review well-known properties of period maps and their local behavior at infinity. Good references for this material include [CMSP17, CKS86, GGK12, GS69, PS08, Sch73].

3.1. **Notation.**

3.1.1. *Groups.* Given a $\mathbb{Q}$–algebraic group G, the Lie groups of real and complex points will be denoted by $G_\mathbb{R}$ and $G_\mathbb{C}$, respectively. The associated Lie algebras are denoted $\mathfrak{g}_\mathbb{R}$ and $\mathfrak{g}_\mathbb{C}$, respectively.

Let $V = V_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$ is a rational vector space, with underlying lattice $V_\mathbb{Z}$. Let $\mathrm{End}(V) = V \otimes V^*$ denote the Lie algebra of linear maps $V \to V$, and let $\mathrm{Aut}(V) \subset \mathrm{End}(V)$ denote the $\mathbb{Q}$–algebraic group of invertible linear maps.

Fix $n \in \mathbb{Z}$, and suppose that , and $Q : V \times V \to \mathbb{Q}$ is a nondegenerate (skew-)symmetric bilinear form satisfying

$$Q(u,v) \;=\; (-1)^n Q(v,u)\,, \quad \text{for all} \quad u,v \in V\,.$$

From this point on, G will denote the $\mathbb{Q}$–algebraic group

$$G \;=\; \mathrm{Aut}(V,Q) \;=\; \{g \in \mathrm{Aut}(V) \mid Q(gu,gv) = Q(u,v)\,,\ \forall\, u,v \in V\}\,.$$

with Lie algebra

$$\mathfrak{g} \;=\; \mathrm{End}(V,Q) \;=\; \{X \in \mathrm{End}(V) \mid 0 = Q(Xu,v) + Q(u,Xv)\,,\ \forall\, u,v \in V\}\,.$$

3.1.2. *Period domains.* Let $D = G_\mathbb{R}/K^0$ be the period domain parameterizing effective weight $n > 0$, Q–polarized Hodge structures on V with Hodge numbers $\mathbf{h} = (h^{n,0}, \ldots, h^{0,n})$. Given $\varphi \in D$, let

$$V_\mathbb{C} \;=\; \bigoplus_{p+q=n} V^{p,q}_\varphi$$

be the Hodge decomposition; let

$$F^n_\varphi \;\subset\; F^{n-1}_\varphi \;\subset \cdots \subset\; F^1_\varphi \;\subset\; F^0_\varphi \;=\; V_\mathbb{C}$$

be the Hodge filtration. The weight zero Hodge decomposition

$$\mathfrak{g}_\mathbb{C} \;=\; \oplus\, \mathfrak{g}^{p,-p}_\varphi \tag{3.1}$$

induced by φ, is polarized by $-\kappa$, where $\kappa \in \mathrm{Sym}^2\mathfrak{g}^*_\mathbb{C}$ is the Killing form. The isotropy group $K^0 = \mathrm{Stab}_G(\varphi)$ stabilizing $\varphi \in D$ is compact, with complexified Lie algebra

$$\mathfrak{k}^0_\mathbb{C} \;=\; \mathfrak{k}^0_\mathbb{R} \otimes \mathbb{C} \;=\; \mathfrak{g}^{0,0}_\varphi\,.$$

Let $\check{D} = G_\mathbb{C}/P_\varphi$ denote the compact dual of D. Here P_φ is the complex parabolic stabilizer of the Hodge filtration F_φ, and has Lie algebra $\mathfrak{p}_\varphi = \oplus_{p \ge 0}\, \mathfrak{g}^{p,-p}_\varphi$.

3.2. **Variations of Hodge structure and period maps.**

3.2.1. Unit disc

$$\Delta \;=\; \{t \in \mathbb{C} \mid |t| < 1\}$$

and punctured unit disc

$$\Delta^* \;=\; \{t \in \mathbb{C} \mid 0 < |t| < 1\}\,.$$

Upper half plane

$$\mathcal{H} \;=\; \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$$

and covering map

$$\mathcal{H} \;\to\; \Delta^* \quad \text{sending} \quad z \;\mapsto\; t = e^{2\pi \mathbf{i} z}\,.$$

Multivalued inverse

$$\ell(t) \;=\; \frac{\log t}{2\pi\mathbf{i}}\,,$$

and (well-defined) differential $\mathrm{d}\ell \;=\; \dfrac{\mathrm{d}t}{2\pi\mathbf{i}\,t}$.

3.2.2. Fix a point $b \in Z_I^* \subset \overline{B}$. Choose a coordinate chart

$$(t,w) : \overline{\mathcal{U}} \,\subset\, \overline{B} \;\xrightarrow{\simeq}\; \Delta^{k+r}$$

centered at a point b with

$$(t,w) : \mathcal{U} \,=\, B \cap \overline{\mathcal{U}} \;\xrightarrow{\simeq}\; (\Delta^*)^k \times \Delta^r\,.$$

Reindexing the Z_i if necessary, we may assume that

$$\overline{\mathcal{U}} \cap Z_i \;=\; \{t_i = 0\}\,, \quad \text{for all} \quad 1 \le i \le k\,,$$

and $\overline{\mathcal{U}} \,\cap\, Z_\mu \,=\, \emptyset$ for all $k+1 \,\le\, \mu \,\le\, \nu$. (We are assuming, as we may by shrining $\overline{\mathcal{U}}$ if necessary, that $\overline{\mathcal{U}} \cap Z_I = \overline{\mathcal{U}} \cap Z_I^*$.)

3.2.3. The counter-clockwise generator $\alpha_i \in \pi_1(\Delta^*) \hookrightarrow \pi_1((\Delta^*)^k) = \pi_1(\mathcal{U})$ induces a quasi-unipotent monodromy operator $\gamma_i \in \mathrm{Aut}(V,Q)$, $1 \le i \le k$ [Sch73]. Passing to a finite cover of B if necessary, we may assume without loss of generality that γ_i is unipotent; let

$$N_i \;=\; \log \gamma_i \;\in\; \mathfrak{g}$$

be the nilpotent logarithm of monodromy, and

$$\sigma \;=\; \mathrm{span}_{\mathbb{R}_{>0}}\{N_1, \ldots, N_k\} \;\subset\; \mathfrak{g}_\mathbb{R} \;=\; \mathrm{Aut}(V_\mathbb{R}, Q)\,,$$

the *monodromy cone (for the coordinate chart centered at b)*.

3.2.4. The universal cover of $\mathcal{U}$ is

$$\widetilde{\mathcal{U}} \;=\; \mathcal{H}^k \,\times\, \Delta^r\,.$$

The local lift

$$\widetilde{\Phi} : \widetilde{\mathcal{U}} \;\to\; D$$

of $\Phi|_\mathcal{U}$ is of the form

$$\widetilde{\Phi}(t,w) \;=\; \exp(\textstyle\sum \ell(t_i) N_i)\xi(t,w) \cdot F\,. \tag{3.2}$$

Here, $F \in \check{D}$,

$$\xi : \overline{\mathcal{U}} \;\to\; G_\mathbb{C}$$

is a holomorphic map, and we abuse notation by regarding the multi-valued $\ell(t_i)$ as giving coordinates on $\mathcal{H}$. Additionally, if $F(w) = \xi(0,w) \cdot F$, then $(W, F(w))$, is a mixed Hodge structure (MHS) polarized by the local monodromy cone

$$\sigma \;= \mathrm{span}_{\mathbb{R}_{>0}}\{N_1, \ldots, N_k\}\,.$$

We say (W, F, σ) is a *limiting mixed Hodge structure* (LMHS).

3.3. **Local VLMHS over $Z_I^* \cap \overline{\mathcal{U}}$.**

3.3.1. Given a mixed Hodge structure (W,F) on (V,Q), we have a Deligne splitting

$$V_{\mathbb{C}} \;=\; \oplus V^{p,q}_{W,F}$$

satisfying

$$W_\ell \;=\; \bigoplus_{p+q\le\ell} V^{p,q}_{W,F} \quad\text{and}\quad F^k \;=\; \bigoplus_{p\ge k} V^{p,q}_{W,F}\,.$$

The induced splitting

$$\mathfrak{g}_{\mathbb{C}} \;=\; \oplus\, \mathfrak{g}^{p,q}_{W,F}\,, \tag{3.3a}$$

of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is defined by

$$\mathfrak{g}^{p,q}_{W,F} \;=\; \{x\in\mathfrak{g}_{\mathbb{C}} \mid x(V^{r,s}_{W,F}) \subset V^{p+r,q+s}_{W,F}\,,\ \forall\ r,s\}\,, \tag{3.3b}$$

satisfies

$$\kappa(\mathfrak{g}^{p,q}_{W,F}\,,\,\mathfrak{g}^{r,s}_{W,F}) \;=\; 0 \quad\text{if}\quad (p,q)+(r,s)\neq(0,0)\,, \tag{3.3c}$$

and is compatible with the Lie bracket in the sense that

$$[\mathfrak{g}^{p,q}_{W,F}\,,\,\mathfrak{g}^{r,s}_{W,F}] \;\subset\; \mathfrak{g}^{p+r,q+s}_{W,F}\,. \tag{3.3d}$$

Remark 3.4. The obvious analogs of (3.3) hold with $\mathrm{End}(V_{\mathbb{C}})$ in place of $\mathfrak{g}_{\mathbb{C}}$. Given $X\in\mathrm{End}(V_{\mathbb{C}})$, let $X^{p,q}$ denote the component taking value in $\mathrm{End}(V_{\mathbb{C}})^{p,q}_{W,F}$.

3.3.2. It follows that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{f}\oplus\mathfrak{f}^\perp$ with

$$\mathfrak{f} \;=\; \oplus_{p\ge0}\,\mathfrak{g}^{p,q}_{W,F}$$

the parabolic Lie algebra of the stabilizer $\mathrm{Stab}_{G_{\mathbb{C}}}(F)$ of F, and

$$\mathfrak{f}^\perp \;=\; \oplus_{p<0}\,\mathfrak{g}^{p,q}_{W,F} \tag{3.5}$$

a nilpotent subalgebra of $\mathfrak{g}_{\mathbb{C}}$. The holomorphic $\xi:\overline{\mathcal{U}}\to G_{\mathbb{C}}$ is uniquely determined by the property

$$\xi(t,w) \;\in\; \exp(\mathfrak{f}^\perp)\,.$$

Without loss of generality, we may assume that (W,F) is $\mathbb{R}$–split

$$\overline{V^{p,q}_{W,F}} \;=\; V^{q,p}_{W,F}\,,$$

which implies

$$\overline{\mathfrak{g}^{p,q}_{W,F}} \;=\; \mathfrak{g}^{q,p}_{W,F}\,.$$

Then $\xi(0,0)\in P^{-2}_{W,\mathbb{C}}$.

3.3.3. Given $I \subset \{1, \ldots, k\}$, the IPR implies that the restriction $\xi_I = \xi|_{\overline{\mathcal{U}} \cap Z_I^*}$ takes value in the centralizer

$$C_{I,\mathbb{C}} \;=\; \{g \in G_{\mathbb{C}} \mid \mathrm{Ad}_g N = N\,,\ \forall\ N \in \sigma_I\}$$

of the nilpotent cone

$$\sigma_I \;=\; \mathrm{span}_{\mathbb{R}_{>0}}\{N_i \mid i \in I\} \;\subset\; \mathfrak{g}^{-1,-1}_{W,F}\,.$$

3.3.4. In the case that $I = \{1, \ldots, k\}$, the map

$$F_I : Z_I^* \cap \overline{\mathcal{U}} \;\to\; D_I\,, \quad w \mapsto F_I(w) = \xi(0, w) \cdot F \tag{3.6}$$

defines a variation of LMHS $(W, F_I(w), \sigma_I)$ over $Z_I^* \cap \overline{\mathcal{U}} = \{t_1, \ldots, t_k = 0\}$. Let

$$\mathrm{Gr}^W_\ell \;=\; W_\ell / W_{\ell-1}$$

denote the associated weight-graded quotients. Given $N \in \sigma_I$, we have induced operators $N : \mathrm{Gr}^W_\ell \to \mathrm{Gr}^W_{\ell-2}$ with the property that $N^a : \mathrm{Gr}^W_{n+a} \to \mathrm{Gr}^W_{n-a}$ is an isomorphism. Then F induces a pure Hodge structure of weight $n + a$ on Gr^W_{n+a}, with a Hodge substructure

$$H^{n-a}(-a) \;=\; \bigcap_{N \in \sigma} \ker\{N^{a+1} : \mathrm{Gr}^W_{n+a} \to \mathrm{Gr}^W_{n-a-2}\}\,.$$

that is polarized by $Q(\cdot, N^a \cdot)$ for all $N \in \sigma_I$. (While the Hodge structure on Gr^W_b is not polarized, Gr^W_b is a direct sum of polarized Hodge structures.) The Mumford–Tate domain D_I^0 parametrizes these polarized Hodge structures.

The upshot is that the map (3.6) induces a period map

$$\Phi_I^0 : Z_I^* \;\to\; \Gamma_I \backslash D_I^0 \tag{3.7}$$

that is locally defined by composing (3.6) with the projection $D_I \twoheadrightarrow D_I^0$.

3.3.5. There is a subtle point here: the map (3.6) does *not* induce a well-defined $Z_I^* \to \Gamma_I \backslash D_I$. This is because (3.6) depends on our choice of local coordinates. What is well-defined is the composition

$$\nu_I \circ F_I : Z_I^* \cap \overline{\mathcal{U}} \;\to\; \exp(\mathbb{C}\sigma_I) \backslash D_I$$

of F_I with the quotient

$$\nu_I : D_I \;\twoheadrightarrow\; \exp(\mathbb{C}\sigma_I) \backslash D_I\,.$$

In this way we obtain the tower (2.3). The fact that $\exp(\mathbb{C}\sigma_I) \subset P^{-2}_{W,\mathbb{C}}$ implies that $(\exp(\mathbb{C}\sigma_I)\Gamma_I) \backslash D_I^a = \Gamma_I \backslash D_I^a$ for $a = 0, 1$.

3.3.6. Finally we note that the IPR implies that the map (3.6) satisfies

$$\mathrm{d}F_I^p \ \subset \ F_I^{p-1} \otimes \Omega^1(Z_I^* \cap \overline{\mathfrak{U}})\,.$$

Equivalently, the pull-back $\xi_I^{-1}\,\mathrm{d}\xi_I$ of the Maurer-Cartan form on $\exp(\mathfrak{c}_{I,\mathbb{C}} \cap \mathfrak{f}^\perp)$ under the map

$$\xi_I \ = \ \xi|_{Z_I^* \cap \overline{\mathfrak{U}}} \quad \text{sending} \quad w \ \mapsto \ \xi_I(w) = \xi(0,w) \tag{3.8}$$

takes value in $\mathfrak{c}_{I,\mathbb{C}} \cap (\oplus_p\, \mathfrak{g}_{W,F}^{-1,q})$. Since the centralizer inherits the Deligne splitting

$$\mathfrak{c}_{I,\mathbb{C}} \ = \ \bigoplus_{p+q\le 0} \mathfrak{c}_{I,F}^{p,q}\,, \quad \text{with} \quad \mathfrak{c}_{I,F}^{p,q} \ = \ \mathfrak{c}_{\sigma_I,\mathbb{C}} \cap \mathfrak{g}_{W,F}^{p,q}\,,$$

we may write this as

$$\xi_I^{-1}\,\mathrm{d}\xi_I \ \in \ \Omega^1(Z_I^* \cap \overline{\mathfrak{U}}\,,\, \mathfrak{c}_{I,F}^{-1,\bullet})\,. \tag{3.9}$$

3.3.7. The fibre $\delta_I = \delta_{I,F}$ of $D_I \twoheadrightarrow D_I^0$ through $F \in D_I$ is the set of $\tilde F \in D_I$ inducing the same pure, weight ℓ Hodge filtrations on the $H^{n-a}(-a)$ as F. It is a complex affine space. To see this, first note that $\delta_{I,F}^1 = C_{I,\mathbb{C}}^{-1}\cdot F$. As a unipotent group $C_{I,\mathbb{C}}^{-1} = \exp(\mathfrak{c}_{I,\mathbb{C}}^{-1})$ is biholomorphic to its Lie algebra $\mathfrak{c}_{I,\mathbb{C}}^{-1}$. The Lie algebra of $C_{I,\mathbb{C}}^{-a}$ is

$$\mathfrak{c}_{I,\mathbb{C}}^{-a} \ = \ \bigoplus_{p+q\le -a} \mathfrak{c}_{I,F}^{p,q}\,. \tag{3.10}$$

Since

$$\mathfrak{c}_{I,\mathbb{C}}^{-1} \ = \ \left(\mathfrak{c}_{I,\mathbb{C}}^{-1} \cap \mathfrak{f}\right) \ \oplus \ \left(\mathfrak{c}_{I,\mathbb{C}}^{-1} \cap \mathfrak{f}^\perp\right)$$

with

$$\mathfrak{c}_{I,\mathbb{C}}^{-1} \cap \mathfrak{f} \ = \ \bigoplus_{\substack{p\,\ge\, 0\\ p+q\,\le\, -1}} \mathfrak{c}_{I,F}^{p,q}\,.$$

the stabilizer F in $\mathfrak{c}_{I,\mathbb{C}}^{-1}$ and

$$\mathfrak{c}_{I,\mathbb{C}}^{-1} \cap \mathfrak{f}^\perp \ = \ \bigoplus_{\substack{p\,<\,0\\ p+q\,\le\, -1}} \mathfrak{c}_{I,F}^{p,q}\,,$$

we see that

$$\delta_{I,F}^1 \ = \ \exp(\mathfrak{c}_{\sigma,\mathbb{C}}^{-1} \cap \mathfrak{f}^\perp)\cdot F\,,$$

and the map $\mathfrak{c}_{\sigma,\mathbb{C}}^{-1} \cap \mathfrak{f}^\perp \to \delta_{I,F}^1$ is a biholomorphism.

Likewise, $\mathbb{C}\sigma_I \subset \mathfrak{g}_{W,F}^{-1,-1}$ is an abelian ideal of the nilpotent algebra $\mathfrak{c}_{I,\mathbb{C}}^{-1} \cap \mathfrak{f}^\perp$, and we have a well-defined induced biholomorphism

$$\frac{\mathfrak{c}_{I,\mathbb{C}}^{-1} \cap \mathfrak{f}^\perp}{\mathbb{C}\sigma_I} \ \xrightarrow{\simeq} \ \exp(\mathbb{C}\sigma_I)\backslash\delta_{I,F}\,.$$

An identical argument establishes analogous statements for the fibre $\delta_W = \delta_{W,F}$ of $D_W \twoheadrightarrow D_W^0$ through $F \in D_W$.

3.4. **Period matrices on a Schubert cell.** Since the *period matrix*

$$\exp(\textstyle\sum \ell(t_i) N_i)\xi(t,w)$$

of the local lift (3.2) takes value in $\exp(\mathfrak{f}^\perp)$ (cf. §3.3.2 and §3.3.3), the local lift $\widetilde{\Phi}(t,w)$ takes value in the open Schubert cell

$$\mathcal{S} \ = \ \exp(\mathfrak{f}^\perp)\cdot F \ = \ \left\{ \tilde F \in \check D \mid \dim(\tilde F^a \cap \overline{F^b_\infty}) = \dim(F^a \cap \overline{F^b_\infty}) \,, \ \forall\, a,b \right\} , \tag{3.11}$$

defined by

$$\overline{F^b_\infty} \ = \ \bigoplus_{c\le n-b} V^{c,a}_{W,F} \,.$$

The map $\mathfrak{f}^\perp \to \mathcal{S}$ sending $X \mapsto \exp(X)\cdot F$ is a biholomorphism. Let

$$X : \mathcal{S} \ \xrightarrow{\simeq} \ \mathfrak{f}^\perp \,. \tag{3.12}$$

denote the inverse.

Recalling the notation of Remark 3.4, we have

$$(\log \xi(t,w))^{-1,q} \ = \ \xi(t,w)^{-1,q} \,,$$

and

$$\begin{aligned} (X\circ\widetilde{\Phi}_{A^0})(t,w)^{-1,-1} &\ = \ \textstyle\sum_{i=1}^k \ell(t_i)N_i \ + \ \xi(t,w)^{-1,-1} \\ (X\circ\widetilde{\Phi}_{A^0})(t,w)^{-1,q} &\ = \ \xi(t,w)^{-1,q} \,, \quad q\neq -1 \,. \end{aligned}$$

We say

$$(X\circ\widetilde{\Phi}_{A^0})^{-1,\bullet} \ = \ \textstyle\sum (X\circ\widetilde{\Phi}_{A^0})^{-1,q}$$

is the *horizontal component of the* (logarithm of the) *period matrix.*

In general, the function $\tilde X : \tilde{\mathcal{U}} \to \mathfrak{f}^\perp$ defined by

$$\tilde X(t,w) \ = \ X\circ\widetilde{\Phi}_{A^0}(t,w) \ - \ \textstyle\sum \ell(t_i)\, N_i$$

is well-defined on $\widetilde{\mathcal{U}}$, but multi-valued over $\mathcal{U}$. But the discussion above implies

$$\tilde X^{-1,\bullet}(t,w) \ \in \ \mathcal{O}(\overline{\mathcal{U}}) \,. \tag{3.13}$$

4. Compatibility of weight closures

The purpose of this section is to establish various compatibility properties between the weight filtrations $W^I = W(\sigma_I)$ that will be used throughout the paper. Our first application of the compatibility properties is to establish the extension results below for the maps $\Phi^0_I : Z^*_I \to \Gamma_I\backslash D^0_I$ and $\Phi^1_I : Z^*_I \to \Gamma_I\backslash D^1_I$ in (2.3).

Lemma 4.1. *The maps Φ^0_W and Φ^1_W defined by* (2.11) *are proper and holomorphic.*

The lemma is a corollary of Lemma 4.17.

The proper mapping theorem yields

Corollary 4.2. *The images* $\wp^0_W = \Phi^0_W(Z_W)$ *and* $\wp^1_W = \Phi^1_W(Z_W)$ *are complex analytic spaces.*

4.1. **The commuting $\mathfrak{sl}(2)$'s.** Our constructions are defined over the open strata Z_I^*. We will need to see that these strata-wise constructions satisfying certain compatibility conditions in order to obtain the properties asserted in the lemmas above. The key technical result here is the SL(2) orbit theorem [CKS86]. We briefly review the theorem, and then discuss consequences.

Suppose that $Z_J \subset Z_I$; equivalently, $I \subset J$. To begin we assume that we have a local coordinate chart centered at $b \in Z_J^*$ with local monodromy cone $\sigma = \sigma_J$ generated by $N_1, \ldots, N_k$ as in §3.2. Given $I \subset J = \{1, \ldots, k\}$, let σ_I be the face of σ_J generated by the N_i, with $i \in I$. Define

$$N_I \ = \ \sum_{i\in I} N_i \quad \text{and} \quad N_J \ = \ \sum_{j \in J} N_j\,.$$

Given this data, the SL(2) orbit theorem [CKS86] produces two pairs

$$N_I, Y_I\,;\ \hat N_J, \hat Y_J \ \in \ \mathfrak{g}_\mathbb{R}$$

with the following properties: N_I and Y_I commute with $\hat N_J$ and $\hat Y_J$; and there is a $(Y_I, \hat Y_J)$–eigenspace decomposition $\mathfrak{g}_\mathbb{C} = \oplus\, \mathfrak{g}_{a,b}$,

$$\mathfrak{g}_{a,b} \ = \ \{\xi \in \mathfrak{g}_\mathbb{C} \mid [Y_I, \xi] = a\xi\,,\ [\hat Y_J, \xi] = b\xi\}\,,$$

with integer eigenvalues a, b that splits the weight filtrations

$$W^I_\ell(\mathfrak{g}_\mathbb{C}) \ = \ \bigoplus_{a\le\ell} \mathfrak{g}_{a,b} \quad \text{and} \quad W^{N^J}_\ell(\mathfrak{g}_\mathbb{C}) \ = \ \bigoplus_{a+b\le\ell} \mathfrak{g}_{a,b}\,. \tag{4.3}$$

We have

$$N_I \ \in \ \mathfrak{g}_{-2,0}$$

and

$$N_J \ \in \ \bigoplus_{a\le 0} \mathfrak{g}_{a,-a-2}\,.$$

If we write

$$N_J \ = \ \sum_{a\le 0} N_{J,a}\,, \tag{4.4a}$$

with $N_{J,a} \in \mathfrak{g}_{a,-a-2}$, then

$$N_{J,0} \ = \ \hat N_J\,. \tag{4.4b}$$

4.2. **When weight filtrations coincide.** The properties (4.3) and (4.4b) yield

Lemma 4.5. *Suppose that $I \subset J$. The following are equivalent:*

(i) *The weight filtrations coincide $W^I = W^J$.*

(ii) *We have $\hat{Y}_J = 0$.*

(iii) *We have $\hat{N}_J = 0$.*

(iv) *The cone $\sigma_J \subset \mathfrak{c}_I^{-1}$.*

Corollary 4.6. (a) *If $I \subset I' \subset J$ and $W^I = W^J$, then $W^I = W^{I'} = W^J$.*

(b) *If $W^{I_1} = W^{I_2}$, then $W^{I_i} = W^{I_1 \cup I_2}$.*

(c) *The union*

$$I_W \;=\; \bigcup_{W^I = W} I$$

is the unique maximal set I_W such that $W = W^{I_W}$.

If $W^I = W^J$, then $\mathfrak{g}_{a,\bullet} = \mathfrak{g}_{a,0}$ implies

$$\mathfrak{c}_J^{-a} \;\subset\; \mathfrak{c}_I^{-a}\,, \tag{4.7a}$$

and

$$\frac{\mathfrak{c}_J^{-a}}{\mathfrak{c}_J^{-a-1}} \;\hookrightarrow\; \frac{\mathfrak{c}_I^{-a}}{\mathfrak{c}_I^{-a-1}}\,. \tag{4.7b}$$

In the case $a = 1$, the inclusion (4.7a) yields the striking implication (known to the experts)

Lemma 4.8. *If $\sigma_J \subset \mathfrak{c}_I^{-1}$, then $\sigma_J \subset \mathfrak{c}_I^{-2}$.*

Corollary 4.9. *We have* $\exp(\mathbb{C}\sigma_{I_W}) \subset C_{I,\mathbb{C}}^{-2}$.

4.3. **Consequences for LMHS.** Note that $Z_J^* \subset Z_I$ if and only if $I \subset J$. In this case, $\Gamma_J \subset \Gamma_I$. We will also see that $D_J \subset D_I$, cf. (4.16). In particular, we have an induced $\Gamma_J\backslash D_J \to \Gamma_I\backslash D_I$. When $W^I = W^J$ (equivalently, $Z_J^* \subset Z_I \cap Z_W$), then this map descends to $\Gamma_J\backslash D_J^a \to \Gamma_I\backslash D_I^a$.

Lemma 4.10. *The maps Φ_I^0 and Φ_I^1 of* (2.3) *extend to proper holomorphic maps on $Z_I \cap Z_W$. These extensions are compatible with the Φ_J^0 and Φ_J^1 on $Z_J^* \subset Z_I \cap Z_W$ in the sense that the we have a commutative diagram*

$$\begin{tikzcd}
Z_J^* \arrow[r, hook] \arrow[d, "\Phi_J^1"] \arrow[dd, bend right=60, "\Phi_J^0"'] & Z_I \cap Z_W \arrow[d, "\Phi_I^1"'] \arrow[dd, bend left=60, "\Phi_I^0"] \\
\Gamma_J\backslash D_J^1 \arrow[r] \arrow[d, two heads] & \Gamma_I\backslash D_I^1 \arrow[d, two heads] \\
\Gamma_J\backslash D_J^0 \arrow[r] & \Gamma_I\backslash D_I^0\,.
\end{tikzcd} \tag{4.11}$$

Lemma 4.10 is a corollary of Lemma 4.13.

Recall (§3.3.5) that the local lift of $\Phi_I : Z_I \to (\Gamma_I \exp(\mathbb{C}\sigma_I))\backslash D_I$ is

$$\nu_I \circ F_I : Z_I^* \cap \overline{\mathfrak{U}} \to \exp(\mathbb{C}\sigma_I)\backslash D_I\,. \tag{4.12}$$

Lemma 4.13. *There is a well-defined holomorphic map*

$$\widetilde{\Phi}_I : Z_I \cap Z_W \cap \overline{\mathfrak{U}} \;\to\; \exp(\mathbb{C}\sigma_{I_W})\backslash D_I \tag{4.14}$$

that, when restricted to $Z_J^* \subset Z_I \cap Z_W$, coincides with the composition $\nu_{I_W} \circ F_J$.

Proof of Lemma 4.10. Given $a = 0, 1$, Corollary 4.9 implies that

$$(\exp(\mathbb{C}\sigma_{I_W})C^{-a-1}_{I,\mathbb{C}})\backslash D_I \;=\; C^{-a-1}_{I,\mathbb{C}}\backslash D_I \;=\; D_I^a\,.$$

So the composition

$$Z_I \cap Z_W \cap \overline{\mathfrak{U}} \xrightarrow{\widetilde{\Phi}_I} \exp(\mathbb{C}\sigma_{I_W})\backslash D_I \longrightarrow\!\!\!\!\!\rightarrow (\exp(\mathbb{C}\sigma_{I_W})C^{-a-1}_{I,\mathbb{C}})\backslash D_I \;=\; D_I^a$$

is the local coordinate expression for the extension $\Phi_I^a : Z_I \cap Z_W \to \Gamma_I\backslash D_I^a$ of (4.11). Thus Lemma 4.10 follows directly from Lemma 4.13. $\square$

Proof of Lemma 4.13. Suppose that $I \subset J$ and $W^I = W^J$. Consider a local lift $\widetilde{\Phi}(t, w)$ over a chart $\overline{\mathfrak{U}}$ centered at $b \in Z_J^*$ (as in §3.2). Along

$$Z_J \cap \overline{\mathfrak{U}} \;=\; \{t_j = 0 \;\forall\, j \in J\} \;=\; \{0\} \times \Delta^r \;\ni\; (0, w)$$

we have the map $F_J : Z_J^* \cap \overline{\mathfrak{U}} \to D_J$ of (3.6)

$$F_J(w) \;=\; \xi(0, w) \cdot F\,. \tag{4.15a}$$

Along $Z_I^* \cap \overline{\mathfrak{U}} = \{t_i = 0 \text{ iff } i \in I\}$ we may choose a well-defined branch of $\ell(t_j)$ for all $j \in J\backslash I$. Then the map $F_I : Z_I^* \cap \overline{\mathfrak{U}} \to D_I$ is given by

$$F_I(t, w) \;=\; \exp\Bigg(\sum_{j \in J\backslash I} \ell(t_j) N_j\Bigg)\xi(t, w) \cdot F\,. \tag{4.15b}$$

Comparing the expressions (4.15) for F_J and F_I, and keeping $C_J \subset C_I$ and (4.7a) in mind, we see that

$$F \;\in\; D_J \;\subset\; D_I \tag{4.16}$$

and F_J takes value in D_I. (Note that the containment $F \in D_I$ is nontrivial, as F arises from the LMHS along Z_J^*.) It follows from (4.15) and (4.16) that

$$\nu_J \circ F_J : Z_J^* \cap \overline{\mathfrak{U}} \;\to\; \exp(\mathbb{C}\sigma_J)\backslash D_I$$

also takes value in (a quotient of) D_I. The lemma now follows from (4.15). $\square$

If follows from Corollary 4.6(c) and (4.16) that the orbit

$$D_W \;=\; P_{W,\mathbb{C}} \cdot F \;\supset\; D_I$$

is independent of our choice of D_I and $F \in D_I$ so long as $W^I = W$. It is straightforward to verify

Lemma 4.17. *There is a well-defined holomorphic map*

$$\widetilde{\Phi}_W : Z_W \cap \overline{\mathfrak{U}} \;\to\; \exp(\mathbb{C}\sigma_{I_W})\backslash D_W \tag{4.18}$$

that, when restricted to Z_I^*, coincides with $\nu_{I_W} \circ F_I$.

Proof of Lemma 4.1. By essentially the same argument as given for Φ_I^a in the proof of Lemma 4.10, the composition

$$Z_W \cap \overline{\mathfrak{U}} \xrightarrow{\widetilde{\Phi}_I} \exp(\mathbb{C}\sigma_{I_W})\backslash D_W \twoheadrightarrow D_W^a$$

is the local coordinate expression for Φ_W^a. So it follows immediately that Φ_W^a is holomorphic.

To see that Φ_W^1 is proper, it suffices to show that Φ_W^0 is proper. And to see that Φ_W^0 is proper, it suffices to show that the extension $\Phi_I^0 : Z_I \cap Z_W \to \Gamma_I\backslash D^0$ of (4.11) is proper. The latter is due to [Gri70, §9]. □

4.3.1. *Remark on the extension question.* Given Lemmas 4.10 and 4.13, it is natural to ask if the extension (4.14) is global; that is, does there exist an extension of $\Phi_I : Z_I^* \to (\Gamma_I \exp(\mathbb{C}\sigma_I))\backslash D_I$ to the weight closure $Z_I \cap Z_W$? The answer in general is no, because the action of $\exp(\mathbb{C}\sigma_{I_W})$ on D_I does not descend to a well-defined action on $\Gamma_I\backslash D_I$. (Likewise, while the quotient $\exp(\mathbb{C}_{I_W})\backslash D_I$ is well-defined, the action of Γ_I on D_I does not descend to the quotient.) In general, to obtain such an extension, one would need at the very least for $\Gamma_I \exp(\mathbb{C}\sigma_{I_W}) \subset G_I$ to be a subgroup. (In general it is not. The product $\Gamma_I \exp(\mathbb{C}\sigma_I)$ is a subgroup because $\Gamma_I \subset C_I$ centralizes σ_I.) The ideal circumstance here would be for Γ_I to centralize the larger cone σ_{I_W}. If it is the case that the image of $\Gamma_I \exp(\mathbb{C}\sigma_{I_W})$ under the projection $G_I \twoheadrightarrow G_I^a$ is a subgroup, then one does obtain an extension of Φ_I^a. For example, since $\exp(\mathbb{C}\sigma_{I_W}) \subset C_{I,\mathbb{C}}^{-2}$, and the C_I^{-a} are normal subgroups of C_I, the image is always a subgroup when $a = 0, 1, 2$. In particular, in the case $a = 2$, we have

$$\begin{array}{ccc}
Z_I^* & \xrightarrow{\Phi_I^2} & (\exp(\mathbb{C}\sigma_I)\Gamma_I)\backslash D_I^2 \\
\downarrow & & \downarrow \\
Z_I \cap Z_W & \longrightarrow & (\exp(\mathbb{C}\sigma_{I_W})\Gamma_I)\backslash D_I^2 \\
\downarrow & & \downarrow \\
Z_W & \xrightarrow{\Phi_W^2} & (\exp(\mathbb{C}\sigma_{I_W})\Gamma_W)\backslash D_W^2 \,.
\end{array}$$

4.4. **Implications for polarizations.** We close §4 with two results on polarizations. These are consequences of: (i) the fact that $W(N)$ is independent of our choice of $N \in \sigma_{I_W}$ [CK82], and (ii) the classification of $\mathrm{Ad}(G_\mathbb{R})$–orbits of nilpotent $N \in \mathfrak{g}_\mathbb{R}$ [CM93] (not the SL(2) orbit theorem).

Lemma 4.19. *Suppose that (W, F) is a MHS and $W = W^I = W^J = W^{I\cup J}$. If (W, F) is polarized by both σ_I and σ_J, then the MHS is also polarized by $\sigma_{I\cup J}$. In particular, $D_I \cap D_J \subset D_{I\cup J}$.*

Proof. Let

$$\overline{\sigma}^W_{I\cup J} \ = \bigcup_{\substack{W = W^K \\ K \subset I\cup J}} \sigma_K$$

denote the "weight-closure" of $\sigma_{I\cup J}$; note that each of the σ_I, σ_J and $\sigma_{I\cup J}$ is contained in $\overline{\sigma}^W_{I\cup J}$. Suppose that $N \in \sigma_{I\cup J}$. The definition of $W = W(N)$ implies that $N^k : \mathrm{Gr}^W_{n+k} \to \mathrm{Gr}^W_{n-k}$ is an isomorphism. Standard $\mathfrak{sl}(2)$–representation theory implies that

$$Q^N_{n+k} \ = \ Q(\cdot, N^k\cdot)$$

defines a nondegenerate, $(-1)^{n+k}$–symmetric bilinear form on Gr^W_{n+k}, and that the restriction of this bilinear form to

$$\mathrm{Prim}^N_{n+k} \ = \ \ker\{N^{k+1} : \mathrm{Gr}^W_{n+k} \to \mathrm{Gr}^W_{n-k-2}\}$$

is also nondegenerate. The mixed Hodge structure (W, F) is polarized by N if and only if the Hodge–Riemann bilinear relations are satisfied by the Hodge filtration $F(\mathrm{Prim}^N_{n+k})$ and Q^N_{n+k}. The first Hodge–Riemann bilinear relation follows directly from $\overline{\sigma}^W_{I\cup J} \subset \mathfrak{g}^{-1,-1}_{W,F}$ and the fact that $Q(V^{p,q}_{W,F}, V^{r,s}_{W,F}) = 0$ unless $(p+q)+(r+s) = 2n$ and $p - q = s - r$.

Consider the adjoint action of G on $\mathfrak{g}$, and let $G^{0,0}_\mathbb{R} \subset G_\mathbb{R}$ be the subgroup preserving the Deligne splitting $\mathfrak{g}_\mathbb{C} = \oplus\, \mathfrak{g}^{p,q}_{W,F}$. The weight-closure $\overline{\sigma}^W_{I\cup J} \subset \mathfrak{g}^{-1,-1}_{W,F}$ is contained in a $G^{0,0}_\mathbb{R}$–orbit [BPR17, Lemma 3.5]. The second Hodge–Riemann bilinear relation is then a consequence of the representation theoretic classification [CM93] of $\mathrm{Ad}(G_\mathbb{R})$–orbits of nilpotent $N \in \mathfrak{g}_\mathbb{R}$ and the discussion of [BPR17, §2.5]. □

Lemma 4.20. *Suppose that (W, F_1) and (W, F_2) are MHS polarized by σ_{I_1} and σ_{I_2}, respectively, and that $F_1(\mathrm{Gr}^W) = F_2(\mathrm{Gr}^W)$. Set $J = I_1 \cup I_2$. Given $N \in \sigma_J$, the bilinear form Q^N_{n+k} is nondegenerate on Gr^W_{n+k}, and the restriction to Prim^N_{n+k} polarizes the Hodge structure defined by $F_1(\mathrm{Prim}^N_{n+k}) = F_2(\mathrm{Prim}^N_{n+k})$.*

Remark 4.21. Note that the lemma does not assert that σ_J *polarizes* the MHS (W, F_a), $a = 1, 2$: a priori, it need not be the case that $N(F^p_a) \subset F^{p-1}_a$. So, given the hypothesis of Lemma 4.20, it would be interesting to know if there exists a MHS (W, F) that is polarized by σ_J and such that $F(\mathrm{Gr}^W) = F_a(\mathrm{Gr}^W)$, $a = 1, 2$? Equivalently, are the $F_a(\mathrm{Gr}^W) \in D^0_J$?

Proof. Corollary 4.6 asserts that $W = W^J$. As in the proof of Lemma 4.19, the fact that $W(N)$ is independent of the choice of $N \in \overline{\sigma}_J^W$ implies that $\overline{\sigma}_J^W$ is contained in an $\mathrm{Ad}(G_\mathbb{R})$–orbit. Additionally, $\sigma_{I_a} \subset \mathfrak{g}^{-1,-1}_{W,F_a} \subset W_{-2}(\mathfrak{g}_\mathbb{C})$ and $F_1(\mathrm{Gr}^W) = F_2(\mathrm{Gr}^W)$ imply that

$$\overline{\sigma}_J \;\subset\; \mathfrak{g}^{-1,-1}_{W,F_a} \quad \text{modulo} \quad W_{-3}(\mathfrak{g}_\mathbb{C})\,, \quad a = 1,2\,.$$

The lemma then follows from the representation theoretic classification [CM93] of $\mathrm{Ad}(G_\mathbb{R})$–orbits of nilpotent $N \in \mathfrak{g}_\mathbb{R}$ and the discussion of [BPR17, §2.5]. □

5. Neighborhood of a fibre

5.1. **Topology.** Recall the notation of §2.2.2.

Proposition 5.1. *The topology on $\overline{\wp}^{\mathrm{e}}$ is Hausdorff. The induced subspace topology on $\wp^{\mathrm{e}}_W = \Phi^{\mathrm{e}}_W(Z_W) \subset \overline{\wp}^{\mathrm{e}}$ coincides with the existing topology on $\wp_W$ as a complex analytic space.*

Remark 5.2. In the case that D is Hermitian, $\overline{\wp}^0$ is the closure of $\wp \subset \Gamma\backslash D$ in the Satake-Baily-Borel compactification $(\Gamma\backslash D)^*$.

Proof. Suppose that $b_i \in \overline{B}$ is a sequence of points converging to $b_\infty \in \overline{B}$. Let A_i and A_∞ be the fibres of Φ^{e} through b_i and b_∞, respectively. Now let $b_i' \in A_i$. Since $\overline{B}$ is compact, $\{b_i'\}$ contains a convergent subsequence; abusing notation, let $\{b_i'\}$ denote that convergent subsequence with limit b_∞'. The essential point is to prove that

$$b_\infty' \;=\; \lim_{i\to\infty} b_i' \;\in\; A_\infty\,. \tag{5.3}$$

In formally this says

$$\lim_{i\to\infty} A_i \;\subset\; A_\infty\,.$$

Fix two coordinate charts $\overline{\mathfrak{U}}$ and $\overline{\mathfrak{U}}'$ centered at b_∞ and b_∞' respectively, and local lifts $\widetilde{\Phi}(t,w)$ and $\widetilde{\Phi}'(t,w)$. Without loss of generality, $b_i \in \overline{\mathfrak{U}}$ and $b_i' \in \overline{\mathfrak{U}}'$. Since $b_i, b_i' \in A_i$, there exists $\gamma_i \in \Gamma$ so that $\widetilde{\Phi}'(b_i') = \gamma_i \cdot \widetilde{\Phi}(b_i)$.

Shrinking $\overline{\mathfrak{U}}$ if necessary, there exists a finite union $\mathfrak{D} \subset D$ of Siegel sets so that $\widetilde{\Phi}(\widetilde{\mathfrak{U}}) \subset \mathfrak{D}$. (In the case of one-variable degenerations this is a corollary of Schmid's SL(2) orbit theorem [Sch73, (5.26)]. In the general case, this is [BKT18, Theorem 1.5], and is key to the Bakker–Klingler–Tsimerman result that period maps are $\mathbb{R}_{\mathrm{an,exp}}$–definable.) Likewise, we have a finite union $\mathfrak{D}' \subset D$ of Siegel sets so that $\widetilde{\Phi}'(\widetilde{\mathfrak{U}}') \subset \mathfrak{D}'$. It follows that there are only finitely many distinct γ_i. Restricting to a subsequence with all $\gamma_i = \gamma$ equal, we have $\widetilde{\Phi}'(b_i') = \gamma \cdot \widetilde{\Phi}(b_i)$. Since we may replace the local lift $\widetilde{\Phi}'$ with $\gamma^{-1}\widetilde{\Phi}'$, this forces b_∞ and b_∞' to lie in the same Φ^{e}–fibre, and (5.3) is proved. □

Recall the "Stein factorization" (2.15) of $\overline{\wp}^{\mathrm{e}}$. By the same construction/argument as for $\overline{\wp}^{\mathrm{e}}$, the set $\hat{\wp}^{\mathrm{e}}$ admits a topology with the all properties of §2.2.2 and Proposition 5.1.

Corollary 5.4. *Let $\hat{A} \subset \overline{B}$ be a fibre of $\hat{\Phi}^{\mathrm{e}}$.* (Equivalently, $\hat{A}$ is a connected component of a Φ^{e}–fibre.) *Fix a neighborhood $\hat{\mathcal{O}} \subset \hat{\wp}^{\mathrm{e}}$ of $\hat{\Phi}^{\mathrm{e}}(\hat{A}) \in \hat{\wp}$. Then $\overline{\mathcal{O}} = (\hat{\Phi})^{-1}(\hat{\mathcal{O}}) \subset \overline{B}$ is a neighborhood of $\hat{A}$ with the property that $\Phi|_{B\cap\overline{\mathcal{O}}}$ is proper.*

5.2. **Monodromy along the fibre.** Now take the case $\Phi^{\mathrm{e}} = \Phi^0$. Let A^0 be the fibre $\hat{A}$ of Corollary 5.4. The restriction $\mathcal{V}|_{\mathcal{O}^0} = \tilde{\mathcal{O}}^0 \times_{\pi_1(\mathcal{O}^0)} V$ of the VHS over B to $\mathcal{O}^0 = \overline{\mathcal{O}}^0 \cap B$ induces a period map

$$\Phi_{A^0} : \mathcal{O}^0 \ \to \ \Gamma_{A^0}\backslash D \tag{5.5}$$

with monodromy $\Gamma_{A^0} \subset \Gamma$.

Let (W, F, σ_I) be any LMHS arising along A^0 (as in §3.2.4). Let

$$I(A^0) \ = \ \{i \mid A^0 \cap Z_i^* \neq \emptyset\}\,.$$

By definition of Φ^0, $W = W^I$ is independent of I. Then Corollary 4.6 implies that

$$I \ \subset \ I(A^0) \ \subset \ I_W$$

and $W = W^{I(A^0)}$. We have $C_{I(A^0)} \subset P_W$, and $G_{I(A^0)} = \mathrm{Aut}(D^0_{I(A^0)}) = C_{I(A^0)}/C^{-1}_{I(A^0)}$ (§2.1).

Lemma 5.6. *We may choose the neighborhood $\overline{\mathcal{O}}^0$ of Corollary 5.4 so that*

$$\Gamma_{A^0} \ \subset \ G_{I(A^0),\mathbb{Q}} \ltimes P^{-1}_{W,\mathbb{Q}}\,,$$

and the image $\Gamma^0_{A^0} \subset G^0_{I(A^0)} = \mathrm{Aut}(D^0_{I(A^0)})$ of Γ_{A^0} under the quotient $G_{I(A^0)} \ltimes P^{-1}_W \twoheadrightarrow G_{I(A^0)}$ is finite and stabilizes $F(\mathrm{Gr}^W)$.

Proof. The weight filtration W is independent of our choice of LMHS (W, F, σ_I) along A^0. So we may choose the neighborhood $\overline{\mathcal{O}}^0$ so that $\Gamma_{A^0} \subset P_{W,\mathbb{Q}}$. Likewise, the Hodge structure $F(\mathrm{Gr}^W) \in D^0_W$ is independent of the choice of LMHS. So we may further assume that Γ_{A^0} fixes $F(\mathrm{Gr}^W)$; equivalently, the discrete quotient $\Gamma_{A^0}/(\Gamma_{A^0} \cap P^{-1}_W)$ stabilizes $F(\mathrm{Gr}^W)$.

Given $N \in \sigma_{I(A^0)}$, Lemma 4.20 asserts that $Q_{n+k} = Q(\cdot, N^k\cdot)$ polarizes the Hodge structure $F(\mathrm{Prim}^N_{n+k}) \subset F(\mathrm{Gr}^W_{n+k})$. So we may also choose the neighborhood $\overline{\mathcal{O}}^0$ so that Prim^N_{k+k} and Q_{n+k} are invariant under Γ_{A^0}. This implies $\Gamma_{A^0}/(\Gamma_{A^0} \cap P^{-1}_W) \subset G_{I(A^0)}$. And since Γ_{A^0} stabilizes the Hodge filtration $F(\mathrm{Gr}^W_{n+k})$, this forces the discrete $\Gamma_{A^0}/(\Gamma_{A^0} \cap P^{-1}_W)$ to be finite. □

Lemma 5.6 can be further strengthened. Without loss of generality $I = \{1, \ldots, k\}$. Let $\mathrm{Stab}_{G_\mathbb{C}}(F_\infty)$ denote the stabilizer in $G_\mathbb{C}$ of the reduced period limit filtration $F_\infty \in \check{D}$ defined by

$$F_\infty \ = \ \lim_{y\to\infty} \exp(\mathbf{i}yN) \cdot F\,.$$

This filtration is independent of the choice of $N \in \sigma_{I_W}$, and is related to the Deligne splitting (§3.3.1) by

$$F^q_\infty \;=\; \bigoplus_{b \le n-q} V^{a,b}_{W,F}\,.$$

Lemma 5.7. *We may choose the neighborhood $\overline{\mathcal{O}}^0$ so that $\Gamma_{A^0} \subset P_{W,\mathbb{Q}} \cap \mathrm{Stab}_{G_\mathbb{C}}(F_\infty)$.*

Proof. The IPR forces a very close relationship between Φ^0 and the reduced period limit map (Proposition B.6): the reduced period limit is locally constant on Φ^0–fibres. On strata $A^0 \cap Z^*_I \cap \overline{\mathcal{U}}$ this implies Corollary B.8. Over $A^0 \cap \overline{\mathcal{U}}$ this implies that the map $\widetilde{\Phi}_W$ of (4.18) takes value in $\exp(\mathbb{C}\sigma_{I_W})\backslash(\mathrm{Stab}_{G_\mathbb{C}}(F_\infty) \cap P^{-1}_{W,\mathbb{C}}) \cdot F \subset \exp(\mathbb{C}\sigma_{I_W})\backslash \delta_W$. (We have $\exp(\mathbb{C}\sigma_{I_W}) \subset \mathrm{Stab}_{G_\mathbb{C}}(F_\infty) \cap P^{-1}_{W,\mathbb{C}}$.) □

Lemma 5.7 has some strong consequences for Φ_{A^0}. Recall the Schubert cell $\mathcal{S} \subset \check{D}$ (§3.4).

Lemma 5.8. *The action of Γ_{A^0} on $\check{D}$ preserves the cell $\mathcal{S} \subset \check{D}$.*

Corollary 5.9. *Every local lift of Φ_{A^0} over a chart $\overline{\mathcal{U}}$ centered at a point $b \in A^0$ takes value in $\mathcal{S}$. In particular, the lift of Φ_{A^0} to the universal cover $\tilde{\mathcal{O}}^0 \to \mathcal{O}^0$ takes value in the Schubert cell:*

$$\begin{array}{ccc}
\tilde{\mathcal{O}}^0 & \xrightarrow{\widetilde{\Phi}_{A^0}} & \mathcal{S} \cap D \\
\downarrow & & \downarrow \\
\mathcal{O}^0 & \xrightarrow{\Phi_{A^0}} & \Gamma_{A^0}\backslash D\,.
\end{array}$$

First proof of Lemma 5.8. Since Γ_{A^0} is both real and stabilizes F_∞, it follows that Γ_{A^0} stabilizes $\overline{F_\infty}$; that is,

$$\Gamma_{A^0} \;\subset\; \mathrm{Stab}_{G_\mathbb{C}}(F_\infty, \overline{F}_\infty) \;=\; \mathrm{Stab}_{G_\mathbb{C}}(F_\infty) \cap \mathrm{Stab}_{G_\mathbb{C}}(\overline{F}_\infty)\,. \tag{5.10}$$

Since $\mathcal{S}$ is by definition those filtrations $\tilde{F} \in \check{D}$ intersecting $\overline{F_\infty}$ generically, it follows that $\mathcal{S}$ is preserved by Γ_{A^0}. □

It is instructive to consider a second proof.

Second proof of Lemma 5.8. The essential point is to note that the Lie algebra of $\mathrm{Stab}_{G_\mathbb{C}}(F_\infty, \overline{F}_\infty)$ is

$$\mathfrak{m} \;=\; \bigoplus_{p,q \le 0} \mathfrak{g}^{p,q}_{W,F}\,. \tag{5.11}$$

It follows from (3.3d) that $\mathfrak{f}^\perp + \mathfrak{m}$ is a nilpotent subalgebra of $\mathfrak{g}_\mathbb{C}$, and $\mathfrak{f}^\perp$ is an ideal of $\mathfrak{f}^\perp + \mathfrak{m}$. This implies that the action of $\mathrm{Stab}_{G_\mathbb{C}}(F_\infty, \overline{F}_\infty)$ on $\check{D}$ preserves $\mathcal{S}$. □

Remark 5.12 (Assume unipotent monodromy about A^0). It will be convenient at times to assume that the action of Γ_{A^0} on Gr^W is trivial; equivalently, the monodromy group

$$\Gamma_{A^0} \ \subset \ P^{-1}_{W,\mathbb{Q}} \tag{5.13}$$

is unipotent, and the quotient group $\Gamma^0_{A^0}$ is trivial.[13] Note that:

(i) The punctured neighborhood $\mathcal{O}^0$ always admits a finite cover with this property (Lemma 5.6).

(ii) Because Theorem 1.10 is a statement about the "normal cover" $\hat{\wp}^{\mathsf{T}}$, there is no loss of generality there if we assume (5.13).

When (5.13) holds, the fact that $P^{-1}_{W,\mathbb{C}}$ is unipotent implies that there is a well-defined logarithm

$$\log \Gamma_{A^0} \ \subset \ \mathfrak{m}^{-1} \ = \ \bigoplus_{\substack{p,\,q \le 0 \\ (p,\,q) \neq (0,0)}} \mathfrak{g}^{p,q}_{W,F}\,, \tag{5.14}$$

and the map $\Gamma_{A^0} \to \log \Gamma_{A^0}$ is a bijection.

5.3. **Trivializations about the fibre.** Recall that Deligne's extension of the Hodge vector bundles $\mathcal{F}^p_{\mathrm{e}} \subset \mathcal{V}_{\mathrm{e}} \to \overline{B}$ are trivial over $\overline{\mathcal{U}}$ (§C). Together (5.14) and Corollary 5.9 make it possible to trivialize $\det(\mathcal{F}^p_{\mathrm{e}})$ in the neighborhood $\overline{\mathcal{O}}^0$ of the fibre.

Theorem 5.15. *If* (5.13) *holds, then the bundles* $\det(\mathcal{F}^p_{\mathrm{e}})$ *are trivial over* $\overline{\mathcal{O}}^0$.

There is a well-defined weight filtration $\mathcal{W}$ of $\mathcal{V}_{\mathrm{e}}|_{Z_W}$ (§C.3).

Theorem 5.16. *Assume* (5.13) *holds. Let* Z_W *be the weight strata containing* A^0*. The induced Hodge filtrations* $\mathcal{F}^p_{\mathrm{e}}(\mathrm{Gr}^{\mathcal{W}}_a)$ *on the associated graded* $\mathrm{Gr}^{\mathcal{W}}_a = \mathcal{W}_a/\mathcal{W}_{a-1}$ *are trivial over* $\overline{\mathcal{O}}^0 \cap Z_W$.

The theorems are proved in §§5.3.1–5.3.4.

5.3.1. *Preliminaries.* The obvious map $\exp(\mathfrak{f}^\perp) \to \exp(\mathfrak{f}^\perp)\cdot F = \mathcal{S}$ is a biholomorphism. So Corollary 5.9 implies that there is a uniquely determined holomorphic

$$g : \tilde{\mathcal{O}}^0 \ \to \ \exp(\mathfrak{f}^\perp)$$

so that

$$\tilde{\Phi}(\zeta) \ = \ g(\zeta)\cdot F\,.$$

[13] Under this assumption, Φ_{A^0} induces a *unipotent* variation of (limiting) mixed Hodge structure along A^0 (which is not necessarily smooth). "Good" unipotent variations of (graded-polarizable) mixed Hodge structure (over smooth a smooth base) have been classified via an equivalence with the category of mixed Hodge representations of the fundamental group [HZ87a, HZ87b] (conjectured by Deligne).

As in §C.1.1, we have

$$\tilde{\Phi}(\zeta\cdot\gamma) \;=\; \gamma^{-1}\cdot\tilde{\Phi}(\zeta)\,;$$

equivalently,

$$g(\zeta\cdot\gamma)\cdot F \;=\; \gamma^{-1}g(\zeta)\cdot F\,.$$

Remark 5.17. Were it the case that $\Gamma_{A^0}\subset\exp(\mathfrak{f}^\perp)$, then we would have $g(\zeta\cdot\gamma)=\gamma^{-1}g(\zeta)$. This would imply that the function $\tilde{\mathcal{O}}^0\to V$ sending $\zeta\mapsto g(\zeta)v$ defines a section of $\mathcal{V}\to\mathcal{O}^0$, and we would have a framing of $\mathcal{F}^p_{\mathrm{e}}$ over $\overline{\mathcal{O}}^0$.

However, while γ^{-1} preserves the Schubert cell $\mathcal{S}$, it need not an element of $\exp(\mathfrak{f}^\perp)$. So we can not assert that $g(\zeta\cdot\gamma)=\gamma^{-1}g(\zeta)$. In order to determine $g(\zeta\cdot\gamma)$ we must first factor the monodromy.

5.3.2. *Factorization of monodromy.* In order to explicitly describe the action of $\gamma\in\Gamma_{A^0}$ on $\delta_W\subset\mathcal{S}$ we first need to factor the monodromy group. Any element $\gamma\in\mathrm{Stab}_{G_\mathbb{C}}(F_\infty,\overline{F}_\infty)$ may be uniquely factored as

$$\begin{aligned}
\gamma &= \alpha\,\beta\,, \qquad \text{with}\\
\beta &\in \mathrm{Stab}_{G_\mathbb{C}}(F_\infty,\overline{F}_\infty,F) \qquad \text{and}\\
\alpha &\in \exp(\mathfrak{m}\cap\mathfrak{f}^\perp) \;=\; \exp(\mathfrak{f}^\perp)\,\cap\,\mathrm{Stab}_{G_\mathbb{C}}(F_\infty,\overline{F}_\infty)\,.
\end{aligned}$$

(The proof of [ČS09, Theorem 3.1.3] applies here.) Then the action of γ on $\xi\cdot F\in\mathcal{S}$ is given by

$$\gamma\xi\cdot F \;=\; \alpha\beta\xi\cdot F \;=\; \alpha\,\beta\xi\beta^{-1}\,\beta\cdot F \;=\; \alpha\,(\beta\xi\beta^{-1})\cdot F\,. \tag{5.18}$$

Note that $\mathfrak{m}\cap\mathfrak{f}^\perp=\mathfrak{m}^{-1}\cap\mathfrak{f}^\perp$. The fact that $\mathfrak{m}^{-1}$ is nilpotent implies that the exponential map $\mathfrak{m}^{-1}\to\exp(\mathfrak{m}^{-1})$ is a biholomorphism. So there exists a unique $a\in\mathfrak{m}\cap\mathfrak{f}^\perp$ such that

$$\alpha \;=\; e^a\,.$$

Likewise β admits a unique factorization as

$$\beta \;=\; \beta_0\,e^b\,,$$

with the adjoint action of $\beta_0\in G_\mathbb{C}$ on $\mathfrak{g}_\mathbb{C}$ preserving each $\mathfrak{g}^{p,q}_{W,F}$ and $b\in\mathfrak{m}^{-1}\cap\mathfrak{f}$ (again by [ČS09, Theorem 3.1.3]).

We have $\gamma\in P^{-1}_{W,\mathbb{C}}$ if and only if $\beta_0=1$ is the identity; equivalently β is unipotent. In this case there exists a unique $c\in\mathfrak{m}^{-1}$ so that $\gamma=e^c$.

5.3.3. *Proof of Theorem 5.15.* While we do not expect to have a framing of $\mathcal{F}^p_{\mathrm{e}}$ over $\overline{\mathcal{O}}^0$ (Remark 5.17), we do have a framing of $\det(\mathcal{F}^p_{\mathrm{e}})$ over $\overline{\mathcal{O}}^0$ when (5.13) holds. This is a consequence of the factorization in §5.3.2. We have

$$\gamma^{-1} \;=\; \alpha\beta\,,$$

with $\alpha \in \exp(\mathfrak{f}^\perp)$ and β unipotent and stabilizing F, and

$$\beta\exp(\mathfrak{f}^\perp)\beta^{-1} \;=\; \exp(\mathfrak{f}^\perp)\,.$$

This implies that

$$g(\zeta\cdot\gamma) \;=\; \alpha\,\beta g(\zeta)\beta^{-1}\,.$$

Since β stabilizes F, it preserves the line $\det(F^p) \subset \bigwedge^{d_k} V$, $d_p = \dim F^p$. Since β is unipotent (§5.3.2) it acts trivially on the line. Fix a nonzero $\mu \in \det(F^p)$. Then $\beta\cdot\mu \;=\; \mu$. So the function

$$f : \tilde{\mathcal{O}}^0 \;\to\; U\,,\quad f(\zeta) \;=\; g(\zeta)\cdot\lambda$$

satisfies

$$\begin{aligned} f(\zeta\cdot\gamma) &\;=\; g(\zeta\cdot\gamma)\lambda \;=\; \alpha\,\beta g(\zeta)\beta^{-1}\cdot\lambda \\ &\;=\; \alpha\,\beta g(\zeta)\cdot\lambda \;=\; \gamma^{-1}\cdot f(\zeta)\,, \end{aligned}$$

and so defines a section of $\det(\mathcal{F}^p) \to \mathcal{O}^0$. Now this section locally extends across infinity (by the constructions of §C), and so extends to a framing of $\det(\mathcal{F}^p_{\mathrm{e}})$ over $\overline{\mathcal{O}}^0$. □

5.3.4. *Proof of Theorem 5.16.* The fact that $\Gamma_{A^0} \subset P^{-1}_{W,\mathbb{C}}$ (Remark 5.12) implies that Γ_{A^0} acts trivially on Gr^W_a. Arguing as in §5.3.3, we conclude that $\mathcal{F}^p_{\mathrm{e}}(\mathrm{Gr}^W_a)$ is trivial over $\overline{\mathcal{O}}^0 \cap Z_W$. □

5.4. **Line bundles.**

5.4.1. *Line bundles over* $\Gamma_{A^0}\backslash\mathcal{S}$. We will construct line bundles over $\Gamma_{A^0}\backslash\mathcal{S}$ from the data:

- The left-action of Γ_{A^0} on $\mathcal{S}$ induces a right-action on the functions $f : \mathcal{S} \to \mathbb{C}$ by the prescription $(f\cdot\gamma)(\xi) = f(\gamma\cdot\xi)$.
- Let
  $$\mathfrak{f}^1 \;=\; F^1(\mathfrak{g}_\mathbb{C}) \;=\; \bigoplus_{p\geq 1} \mathfrak{g}^{p,q}_{W,F}$$
  be the nilpotent radical of the Lie algebra $\mathfrak{f}$ stabilizing F. The relation (3.3c) implies that the bilinear pairing
  $$\kappa : \mathfrak{f}^1 \times \mathfrak{f}^\perp \;\to\; \mathbb{C}$$
  is nondegenerate.

Recall the biholomorphism $X : \mathcal{S} \xrightarrow{\simeq} \mathfrak{f}^\perp$ of (3.12). Given $M \in \mathfrak{f}^1$, define

$$f_M : \mathcal{S} \to \mathbb{C} \quad \text{by} \quad f_M = \exp 2\pi\mathbf{i}\,\kappa(M,X)\,.$$

Given $\gamma \in \Gamma_{A^0}$, define a holomorphic function $e^M_\gamma : \mathcal{S} \to \mathbb{C}^*$ by

$$e^M_\gamma \;=\; \frac{f_M \cdot \gamma}{f_M} \;=\; \frac{\exp 2\pi\mathbf{i}\,\kappa(M, X\cdot\gamma)}{\exp 2\pi\mathbf{i}\,\kappa(M,X)}\,. \tag{5.19}$$

Then

$$e^M_{\gamma_1\gamma_2}(\xi) \;=\; e^M_{\gamma_1}(\gamma_2\cdot\xi)\, e^M_{\gamma_2}(\xi)\,.$$

so that

$$\gamma\cdot(z,\xi) \;=\; (z e^M_\gamma(\xi), \gamma\cdot\xi)$$

defines a left action of Γ_{A^0} on $\mathbb{C}\times\mathcal{S}$. Let

$$\begin{tikzcd} \mathcal{L}_M \arrow[d] \arrow[r, phantom, "=\!="] & (\mathbb{C}\times\mathcal{S})/\sim \\ \Gamma_{A^0}\backslash\mathcal{S} & \end{tikzcd}$$

be the associated line bundle over the quotient. Then f_M induces a section s_M

$$\begin{tikzcd} \mathcal{L}_M \arrow[d] \\ \Gamma_{A^0}\backslash\mathcal{S}\,. \arrow[u, bend left, "s_M"] \end{tikzcd}$$

5.4.2. *Line bundles over* $\mathcal{O}^0$. Pull the line bundle $\mathcal{L}_M$ back to the (punctured) neighborhood $\mathcal{O}^0$

$$\begin{tikzcd} (\Phi_{A^0})^*\mathcal{L}_M \arrow[d] & \mathcal{L}_M \arrow[d] \\ \mathcal{O}^0 \arrow[u, bend left, "\Phi^*_{A^0}(s_M)"] \arrow[r, "\Phi_{A^0}"] & \Gamma_{A^0}\backslash\mathcal{S}\,. \arrow[u, bend right, "s_M"'] \end{tikzcd}$$

The local expression for the pulled-back section $\Phi^*_{A^0}(s_M)$ is

$$\tau_M(t,w) \;=\; f_M \circ \widetilde{\Phi}_{A^0}(t,w) \;=\; \exp 2\pi\mathbf{i}\kappa(M, X\circ\widetilde{\Phi}_{A^0}(t,w))\,. \tag{5.20}$$

If $M \in \mathfrak{g}^{1,\bullet}_{W,F}$ and $\kappa(M,N_i) \in \mathbb{Z}$ for all $i \in I_W$, then (3.13) implies

$$\tau_M(t,w) \;=\; \exp 2\pi\mathbf{i}\kappa(M, \tilde{X}(t,w)) \prod t_i^{\kappa(M,N_i)} \tag{5.21}$$

is a well-defined holomorphic function on $\mathcal{U}$. If in addition $0 \le \kappa(M,N_i) \in \mathbb{Z}$ for all $i \in I_W$, then $\tau_M(t,w)$ is holomorphic on $\overline{\mathcal{U}}$. Additionally, $\tau_M(t,w)$ vanishes along $Z^*_I \cap \overline{\mathcal{U}}$ if and only if $\kappa(M,N_i) > 0$ for some $i \in I$.

5.4.3. *Extension to $\overline{\mathcal{O}}^0$.* Define

$$\mathbf{N}^* \;=\; \{M \in \mathfrak{g}^{1,\le 1}_{W,F} \mid \kappa(M,N_i) \in \mathbb{Z}\,,\ \forall\, i \in I_W\}\,. \tag{5.22}$$

Lemma 5.23. *If $M \in \mathbf{N}^*$, then the line bundle $(\Phi_{A^0})^*\mathcal{L}_M$ is the restriction to $\mathcal{O}^0$ of a holomorphic vector bundle $L_M \to \overline{\mathcal{O}}^0$. And $(\Phi_{A^0})^* s_M$ extends to a section of L_M* (which, in a minor abuse of notation, we also denote s_M).

$$\begin{tikzcd}
L_M \arrow[d] & (\Phi_{A^0})^*\mathcal{L}_M \arrow[d] & \mathcal{L}_M \arrow[d] \\
\overline{\mathcal{O}}^0 \arrow[u, bend left, "s_M"] & \tilde{\mathcal{O}}^0 \arrow[l, hook'] \arrow[r, "\Phi_{A^0}"'] \arrow[u, bend right, "(\Phi_{A^0})^* s_M"'] & \Gamma_{A^0}\backslash\mathcal{S}\,. \arrow[u, bend right, "s_M"']
\end{tikzcd}$$

Proof. Set

$$\tilde X_\gamma(t,w) \;=\; (X\cdot\gamma)\circ\widetilde\Phi_{A^0}(t,w) \;-\; \textstyle\sum \ell(t_i)\,N_i$$

Again, the key point is that it follows from (3.3d), (3.13), Lemma 5.6, (5.10), (5.11) and §5.3.2 that the component $\tilde X_\gamma^{-1,q}(t,w)$ taking value in $\mathfrak{g}^{-1,q}_{W,F}$ is a well-defined holomorphic function on $\overline{\mathcal{U}}$, so long as $q \ge -1$. So $\kappa(M,\tilde X_\gamma(t,w))$ is a holomorphic function on $\overline{\mathcal{U}}$, so long as $M \in \mathbf{N}^*$. Then

$$\begin{aligned}
(\widetilde\Phi_{A^0})^*(f_M\cdot\gamma)(t,w) &= (f_M\cdot\gamma)\circ\widetilde\Phi(t,w)\\
&= \exp 2\pi\mathbf{i}\kappa(M,\tilde X_\gamma(t,w)) \prod t_i^{\kappa(M,N_i)}\,,
\end{aligned}$$

and

$$\begin{aligned}
(\widetilde\Phi_{A^0})^*(e^M_\gamma)(t,w) &= \frac{(\widetilde\Phi_{A^0})^*(f_M\cdot\gamma)(t,w)}{(\widetilde\Phi_{A^0})^*(f_M)(t,w)}\\
&= \frac{\exp 2\pi\mathbf{i}\,\kappa(M,\tilde X_\gamma(t,w))}{\exp 2\pi\mathbf{i}\,\kappa(M,\tilde X(t,w))}
\end{aligned} \tag{5.24}$$

is a well-defined holomorphic function on $\overline{\mathcal{U}}$. □

From (5.21) we deduce that the divisor of the section $s_M \in H^0(\overline{\mathcal{O}}^0, L_M)$ is

$$(s_M) \;=\; \sum \kappa(M,N_i)\,(Z_i\cap\overline{\mathcal{O}}^0)\,.$$

Corollary 5.25. *The line bundle $L_M \to \overline{\mathcal{O}}^0$ is related to the divisor $Z\cap\overline{\mathcal{O}}^0$ by*

$$L_M \;=\; \sum \kappa(M,N_i)\,[Z_i]|_{\overline{\mathcal{O}}^0} \;=\; -\sum \kappa(M,N_i)\,\mathcal{N}^*_{Z_i/\overline{B}}\Big|_{\overline{\mathcal{O}}^0}\,.$$

6. Level one extension data

In this section we restrict to the punctured neighborhood $\mathcal{O}^0 = B \cap \overline{\mathcal{O}}^0$ of $A^0 \subset Z_W$, and work with the period map $\Phi_{A^0} : \mathcal{O}^0 \to \Gamma_{A^0} \backslash D$ of (5.5). Fix a LMHS (W, F, σ_I) of the period map along $A^0 \cap Z_W$. To this MHS we have associated two sets

$$\Gamma_{A^0,I} \backslash \delta_I \quad \text{and} \quad \Gamma_{A^0} \backslash \delta_W$$

of extension data (§2.1). The goal of this section is to study the level one extension data (Definitions 2.6 and 2.7) and the resulting implications for the fibre A^0. The principal tool in our study is Lie theory. Geometric interpretations of the extension data are discussed in §A.

6.1. **Description.** Recall the discussion of §4.3. The *level one extension data* of the MHS (W, F) is $\Gamma_{A^0} \backslash \delta^1_W$ (Definition 2.6). The *σ_I–polarized level one extension data* is $\Gamma_{A^0,I} \backslash \delta^1_I$ (Definition 2.7). The diagram (2.11) induces *level one extension data maps*

$$\begin{array}{ccc} A^0 \cap Z_I & \xrightarrow{\Phi^1_I} & \Gamma_{A^0,I} \backslash \delta_I \\ \downarrow & & \downarrow \\ A^0 & \xrightarrow{\Phi^1_W} & \Gamma_{A^0} \backslash \delta_W \,. \end{array} \tag{6.1}$$

Hodge theoretically, $\Gamma_{A^0} \backslash \delta^1_W$ is a direct sum

$$\begin{aligned} \Gamma_{A^0} \backslash \delta^1_W &= \bigoplus_{\ell=0}^{2n} \mathrm{Ext}^1_{\mathrm{MHS}}(\mathrm{Gr}^W_\ell, \mathrm{Gr}^W_{\ell-1}) \\ &= \bigoplus_{\ell=0}^{2n} \frac{\mathrm{Hom}_{\sigma,\mathbb{C}}(\mathrm{Gr}^W_\ell, \mathrm{Gr}^W_{\ell-1})}{F^0\mathrm{Hom}_{\sigma,\mathbb{C}}(\mathrm{Gr}^W_\ell, \mathrm{Gr}^W_{\ell-1}) \ + \ \Gamma_{A^0}} \end{aligned}$$

with $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathrm{Gr}^W_\ell, \mathrm{Gr}^W_{\ell-1})$ the set of extensions

$$0 \ \to \ \mathrm{Gr}^W_\ell \ \to \ W_{\ell+1}/W_{\ell-1} \ \to \ \mathrm{Gr}^W_{\ell+1} \ \to \ 0 \,.$$

6.2. **Structure.** Note that both $\delta_I \subset \delta_W$ are subsets of the Schubert cell $\mathcal{S}$ of (3.11). It follows that the quotients $\Gamma_{A^0,I} \backslash \delta_I$ and $\Gamma_{A^0} \backslash \delta_W$ inherit the line bundles $\mathcal{L}_M$ of Lemma 5.23,

$$\begin{array}{ccc} \mathcal{L}_M & & \mathcal{L}_M \\ \downarrow & & \downarrow \\ \Gamma_{A^0,I} \backslash \delta_I & \longrightarrow & \Gamma_{A^0} \backslash \delta_W \,. \end{array} \tag{6.2}$$

Theorem 6.3. *Set $W = W^A$, and suppose $Z^*_I \subset Z_W$. Assume that the monodromy $\Gamma_{A^0} \subset P^{-1}_{W,\mathbb{Q}}$ is unipotent* (Remark 5.12).

(a) *The bundle* $\pi^1_W : \Gamma_{A^0}\backslash D^1_W \twoheadrightarrow D^0_W$ *admits a subbundle*

$$\begin{array}{ccccc} T_W & \hookrightarrow & \mathcal{T}_W & \subset & \Gamma_W\backslash D^1_W \\ & & \big\downarrow\!\!\!\downarrow{\scriptstyle \pi^1_W} & & \\ & & \Gamma_W\backslash D^0_W & & \end{array}$$

that is fibered by compact tori $T_W \subset$. *The restriction* $\Phi^1\big|_{A^0}$ *takes value in* T_W.

(b) *The bundle* $\pi^1_I : \Gamma_{A^0,I}\backslash D^1_I \twoheadrightarrow D^0_I$ *admits a subbundle*

$$\begin{array}{ccccc} J_I & \hookrightarrow & \mathcal{J}_I & \subset & \Gamma_I\backslash D^1_I \\ & & \big\downarrow\!\!\!\downarrow{\scriptstyle \pi^1_I} & & \\ & & \Gamma_I\backslash D^0_I & & \end{array}$$

that is fibered by abelian varieties J_I. *The restriction* $\Phi^1\big|_{A^0\cap Z_I}$ *takes value in* J_I.

(c) *If* $M \in \mathfrak{g}^{1,1}_{W,F}$, *then the line bundles* (6.2) *descend*

$$\begin{array}{ccc} \mathcal{L}_M & & \mathcal{L}_M \\ \downarrow & & \downarrow \\ \Gamma_{A^0,I}\backslash \delta^1_I & \longrightarrow & \Gamma_{A^0}\backslash \delta^1_W \end{array}$$

to both $\Gamma_{A^0,I}\backslash\delta^1_I$ *and* $\Gamma_{A^0}\backslash\delta^1_W$. *In the case that* $M \in \mathbf{N}^* \cap \mathfrak{g}^{1,1}_{W,F}$, *we have*

$$L_M|_{A^0} \;=\; (\Phi^1\big|_{A^0})^*(\mathcal{L}_M) \quad \text{and} \quad L_M|_{A^0\cap Z_I} \;=\; (\Phi^1\big|_{A^0\cap Z_I})^*(\mathcal{L}_M)\,. \tag{6.4}$$

(d) *There is a nonempty subset* $\mathbf{N}^{\mathfrak{sl}_2}_I \subset \mathbf{N}^* \cap \mathfrak{g}^{1,1}_{W,F}$ *with the property that the abelian variety* J_I *is polarized by the* $\mathcal{L}^*_M$ *with* $M \in \mathbf{N}^{\mathfrak{sl}_2}_I$.

(e) *The set* $\mathbf{N}^{\mathfrak{sl}_2,+}_I = \{M \in \mathbf{N}^{\mathfrak{sl}_2}_I \mid \kappa(M, N_i) > 0\,,\ \forall\ i \in I\}$ *is nonempty. Indeed the dimension of the real span is* $\dim \sigma_I$.

The remainder of §6 is occupied with the proof of Theorem 6.3. In outline, the argument is as follows:

- To begin, we review the structure of $\Gamma_{A^0}\backslash\delta^1_W$ and $\Gamma_{A^0,I}\backslash\delta^1_I$ in §6.3. The compact torus $J_I \subset \Gamma_{A^0,I}\backslash\delta^1_I$ is identified in §6.5.
- The action of Γ_{A^0} on $\delta_W \subset \mathcal{S}$ was analyzed in §5.3.2. This action preserves δ_W, and the restricted action is further analyzed in §6.6.
- The line bundle $\mathcal{L}_M \to \Gamma_{A^0}\backslash\delta_W$ descends to $\Gamma_{A^0}\backslash\delta^1_W$ if and only if the functions e^M_γ of (5.19) are constant on the fibres of $\delta_W \twoheadrightarrow \delta^1_W$. In §6.7 it is shown that the bundles parameterized by $M \in \mathfrak{g}^{1,1}_{W,F}$ have this property. If, in addition, $M \in \mathbf{N}^* \cap \mathfrak{g}^{1,1}_{W,F}$ then we also have $L_M|_{A^0}$ (Lemma 5.23). In order to see that (6.4) holds, we must show that the associated systems of multipliers coincide.

- We then restrict to a subset $\mathbf{N}^1 \subset \mathfrak{g}^{1,1}_{W,F} \cap \mathbf{N}^*$ (which may be thought as imposing an integrality condition on M) and compute the Chern forms ω_M in §6.8.
- We restrict to a final subset $\mathbf{N}^{\mathfrak{sl}_2}_I \subset \mathbf{N}^1$ (which may be thought of as a positivity condition) and confirm that $-\omega_M$ is positive on J_I. It then follows that the line bundle $\mathcal{L}^*_M \to J_I$ is ample and J_I is an abelian variety.

6.3. **Lie theoretic description.** The level one extension data has the following structure. First note that $P^{-1}_{W,\mathbb{C}}/P^{-2}_{W,\mathbb{C}}$ is an abelian group. Since the exponential map $\exp : \mathfrak{p}_{W,\mathbb{C}} \to P_{W,\mathbb{C}}$ is a biholomorphism, and

$$\mathfrak{p}^{-a}_{W,\mathbb{C}} \;=\; \bigoplus_{p+q\le -a} \mathfrak{g}^{p,q}_{W,F}\,,$$

we see that there is a canonical identification

$$P^{-1}_{W,\mathbb{C}}/P^{-2}_{W,\mathbb{C}} \;\simeq\; \bigoplus_{p+q=-1} \mathfrak{g}^{p,q}_{W,F}\,.$$

Setting

$$\mathbb{L} \;=\; \bigoplus_{\substack{p+q=-1\\ p<0}} \mathfrak{g}^{p,q}_{W,F}\,,$$

we have

$$P^{-1}_{W,\mathbb{C}}/P^{-2}_{W,\mathbb{C}} \;\simeq\; \mathbb{L} \oplus \overline{\mathbb{L}}\,.$$

Additionally $\mathfrak{p}^{-a}_{W,\mathbb{C}} = (\mathfrak{f} \cap \mathfrak{p}^{-a}_{W,\mathbb{C}}) \oplus (\mathfrak{f}^\perp \cap \mathfrak{p}^{-a}_{W,\mathbb{C}})$, and the map $\mathfrak{f} \cap \mathfrak{p}^{-1}_{W,\mathbb{C}} \to \delta_W$ given by $x \mapsto \exp(x)\cdot F$ is a biholomorphism. It follows that we have a canonical identification

$$P^{-2}_{W,\mathbb{C}}\backslash(P^{-1}_{W,\mathbb{C}}\cdot F) \;=\; \mathbb{L}\,.$$

Taking Λ to be the discrete image of Γ_{A^0} under the projection $P^{-1}_{W,\mathbb{C}} \to \mathbb{L}$, we obtain

$$\Gamma_{A^0}\backslash\delta^1_W \;=\; \Lambda\backslash\mathbb{L} \tag{6.5a}$$

In particular,

$$\Gamma_{A^0}\backslash\delta^1_W \;=\; \mathbb{C}^{d_1} \,\times\, (\mathbb{C}^*)^{d_2} \,\times\, T_W \tag{6.5b}$$

is biholomorphic to the the product of an affine space $\mathbb{C}^{d_1}$ with a complex torus $(\mathbb{C}^*)^{d_2} \times T_W$ having compact factor T_W.

Setting

$$\mathbb{L}_I \;=\; \bigoplus_{\substack{p+q=-1\\ p<0}} \mathfrak{c}^{p,q}_{I,F}$$

and letting Λ_I be the discrete image of $\Gamma_{A^0,I}$ under the projection $C^{-1}_{\sigma_I} \to \mathbb{L}_I$, we have

$$\Gamma_{A^0,I}\backslash\delta^1_I \;=\; \Lambda_I\backslash\mathbb{L}_I \;=\; \mathbb{C}^{d_{I,1}} \,\times\, (\mathbb{C}^*)^{d_{I,2}} \,\times\, J_I\,, \tag{6.6}$$

with $(\mathbb{C}^*)^{d_{I,2}} \times J_I$ a complex torus having compact factor J_I. Note the obvious map

$$\Lambda_I \hookrightarrow \Lambda .$$

6.4. **The IPR along fibres.** Consider the restriction $F_{I,A} : A^0 \cap Z_I^* \cap \overline{\mathcal{U}} \to \delta_I$ of (3.6). Let

$$\xi_{I,\beta} = \xi|_{A^0 \cap Z_I^* \cap \overline{\mathcal{U}}} = \xi_I|_{A^0 \cap Z_I^* \cap \overline{\mathcal{U}}} .$$

The infinitesimal period relation (3.9) and the discussion of §3.3.7 imply that the Maurer-Cartan form

$$\xi_{I,\beta}^{-1} \, \mathrm{d}\xi_{I,\beta} \quad \textit{takes value in} \quad \bigoplus_{q \le 0} \mathfrak{c}_{\sigma,F}^{-1,q} . \tag{6.7}$$

We have a well-defined logarithm

$$\log \xi_{I,\beta} : A^0 \cap \overline{\mathcal{U}} \to \mathfrak{c}_{\sigma,\mathbb{C}}^{-1} \cap \mathfrak{f}^{\perp} .$$

Let $(\log \xi_{I,\phi}^a)^{p,q}$ denote the component taking value in $\mathfrak{c}_{\sigma,F}^{p,q}$. Then (6.7) implies

$$(\log \xi_{I,\beta})^{p,q} \quad \text{is locally constant for all} \quad p+q=-1 \,,\ p \le -2 . \tag{6.8}$$

6.5. **Compact torus: Proof of Theorem 6.3(a).** It follows from (5.11) that

$$\Lambda \subset \mathfrak{g}_{W,F}^{-1,0} \subset \mathbb{L} \quad \text{and} \quad \Lambda_I \subset \mathfrak{c}_{\sigma,F}^{-1,0} \subset \mathbb{L}_I .$$

In particular, the torus factor $(\mathbb{C}^*)^{d_2} \times T^{d_3}$ of $\Gamma_{A^0}\backslash\delta_W^1$ of (6.5) is contained in the image of $\mathfrak{g}_{W,F}^{-1,0} \to \Lambda\backslash\mathbb{L}$; likewise, the torus factor $(\mathbb{C}^*)^{d_{I,2}} \times J_I$ of $\Gamma_{A^0}\backslash\delta_I^1$ is contained in the image of $\mathfrak{c}_{I,F}^{-1,0} \to \Lambda_I\backslash\mathbb{L}_I$. It follows from the IPR (6.8) and the compactness of A^0 that the image of $\Phi_{A^0,W}^1 : A^0 \to \Gamma_{A^0}\backslash\delta_W^1$ is contained in the compact torus T^{d_3} of (6.5b). Likewise, the the image of $\Phi^1 : A^0 \cap Z_I \to \Gamma_{A^0}\backslash\delta_I^1$ is contained in the compact torus J_I of (6.6). We will show that J_I is abelian by exhibiting ample Lie bundles $\mathcal{L}_M \to J_I$.

6.6. **Action on LMHS of the fibre.** When restricted to $\delta_W \subset \mathcal{S}$, the map $X : \mathcal{S} \to \mathfrak{f}^{\perp}$ of §5.4.1 takes value in

$$X : \delta_W \to \mathfrak{p}_{W,\mathbb{C}}^{-1} \cap \mathfrak{f}^{\perp} .$$

Set $\xi = \exp(X)$, so that $X = \xi \cdot F = \exp(X) \cdot F$. In anticipation of the arguments to follow, it will be helpful to work out some formula. To begin, recall the Deligne splitting (§3.3.1) of $\mathfrak{g}_{\mathbb{C}}$. Given any $x \in \mathfrak{g}_{\mathbb{C}}$, there are unique $x^{p,q} \in \mathfrak{g}_{W,F}^{p,q}$ so that

$$x = \sum x^{p,q} .$$

Recall the notation and observations of §5.3.2. Given $\gamma = \alpha\beta \in \Gamma_{A^0} \subset P^{-1}_{W,\mathbb{Q}}$, one may verify that the logarithms satisfy

$$\begin{aligned} c^{-1,0} &= a^{-1,0} \\ c^{0,-1} &= b^{0,-1} \\ c^{-1,-1} &= a^{-1,-1} + \tfrac{1}{2}[a^{-1,0}, b^{0,-1}]\,. \end{aligned}$$

The action of γ on $\xi = \exp(X) \cdot F \in \delta_W$ satisfies

$$(\log \alpha\, \beta\xi\beta^{-1})^{-1,0} = X^{-1,0} + a^{-1,0} \tag{6.9a}$$

$$(\log \alpha\, \beta\xi\beta^{-1})^{-1,-1} = X^{-1,-1} + a^{-1,-1} + [b^{0,-1}, X^{-1,0}]\,. \tag{6.9b}$$

The containment (5.14) implies

$$(\log \alpha\, \beta\xi\beta^{-1})^{p,q} = X^{p,q}\,, \quad \forall\, p+q = -1 > p\,. \tag{6.9c}$$

Under the identifications of §6.3 we have

$$\lambda = a^{-1,0} \quad \text{and} \quad \overline{\lambda} = b^{0,-1}\,,$$

and $(X^{p,-1-p})_{p\le -1} = X^{-1,0} + X^{-2,1} + X^{-3,2} + \cdots$ parameterizes a point in $\mathbb{L}$. So (6.9) is describing the action of Λ on $\mathbb{L}$.

Consider $\gamma_i = \alpha_i\beta_i \in \Gamma_{A^0}$, with $\gamma_i = e^{c_i}$, $\alpha_i = e^{a_i}$ and $\beta_i = e^{b_i}$, as above. Suppose that $\gamma = \gamma_1\,\gamma_2$. Then one may verify that

$$\begin{aligned} a^{-1,0} &= a_1^{-1,0} + a_2^{-1,0} \\ b^{0,-1} &= b_1^{0,-1} + b_2^{0,-1} \\ c^{-1,-1} &= c_1^{-1,-1} + c_2^{-1,-1} + \tfrac{1}{2}[a_1^{-1,0}\,,\, b_2^{0,-1}] + \tfrac{1}{2}[b_1^{0,-1}\,,\, a_2^{-1,0}] \\ a^{-1,-1} &= a_1^{-1,-1} + a_2^{-1,-1} + [b_1^{0,-1}, a_2^{-1,0}]\,. \end{aligned}$$

6.7. **Proof of Theorem 6.3(c).** The line bundle $\mathcal{L}_M \to \Gamma_{A^0}\backslash\delta_W$ descends to $\Gamma_{A^0}\backslash\delta^1_W$ if and only if the functions e^M_γ of (5.19) are constant on the fibres of $\delta_W \twoheadrightarrow \delta^1_W$. If $M \in \mathfrak{g}^{1,1}_{W,F}$, then (3.3c), (5.18), (5.19), and (6.9) yield

$$e^M_\gamma(X) = \exp 2\pi\mathbf{i}\kappa(M, a^{-1,-1} + [b^{0,-1}, X^{-1,0}]) \tag{6.10}$$

on δ_W. These functions are constant on the fibres of $\delta_W \twoheadrightarrow \delta^1_W$, and so descend to well-defined on functions on δ^1_W. There they induce line bundles (also denoted)

$$\begin{array}{ccc} \mathcal{L}_M & & \mathcal{L}_M \\ \downarrow & & \downarrow \\ \Gamma_{A^0,I}\backslash\delta^1_I & \longrightarrow & \Gamma_{A^0}\backslash\delta^1_W \end{array}$$

over the level one extension data.

Additionally, if $M \in \mathbf{N}^* \cap \mathfrak{g}^{1,1}_{W,F}$, then (5.24), (6.9) and (6.10) yield

$$(\widetilde{\Phi}_{A^0})^*(e^M_\gamma)\Big|_{A^0} \;=\; (\Phi^1_{A^0,W})^* e^M_\gamma(X)\,;$$

establishing (6.4).

6.8. **Chern classes.** We now wish to compute the first Chern class $c_1(\mathcal{L}_M)$ of $\mathcal{L}_M \to \Gamma_{A^0}\backslash\delta^1_W = \Lambda\backslash\mathbb{L}$ for $M \in \mathfrak{g}^{1,1}_{W,F}$. We have

$$H^1(\Lambda\backslash\mathbb{L},\mathbb{C}) \;=\; (\mathbb{L}\oplus\overline{\mathbb{L}})^* \;\simeq\; \bigoplus_{p+q=-1} \mathfrak{g}^{p,q}_{W,F}\,,$$

and

$$\begin{aligned} H^2(\Lambda\backslash\mathbb{L},\mathbb{C}) \;&=\; \textstyle\bigwedge^2 H^1(\Lambda\backslash\mathbb{L},\mathbb{C}) \;=\; \bigwedge^2(\mathbb{L}\oplus\overline{\mathbb{L}})^*\,,\\ H^{1,1}(\Lambda\backslash\mathbb{L}) \;&=\; \mathbb{L}^*\otimes\overline{\mathbb{L}}^*\,. \end{aligned}$$

We have a map

$$\omega : \mathfrak{g}^{1,1}_{W,F} \;\hookrightarrow\; \mathbb{L}^*\otimes\overline{\mathbb{L}}^* \;\simeq\; H^{1,1}(\Lambda\backslash\mathbb{L})\,,$$

defined by sending $M \in \mathfrak{g}^{1,1}_{W,F}$ to the form $\omega_M \in H^{1,1}(\Lambda\backslash\mathbb{L})$ defined by

$$\omega_M(u,\bar v) \;:=\; \kappa(M,[u,\bar v]) \;=\; -\kappa(u,\mathrm{ad}_M(\bar v))$$

with $u,v \in \mathbb{L}$.

Recall the definition of $\mathbf{N}^*$ in (5.22) and consider the subset

$$\mathbf{N}^1 \;=\; \left\{ M \in \mathfrak{g}^{1,1}_{W,F} \;\middle|\; \begin{array}{l} \kappa(M\,,\,[a^{-1,0},b^{0,-1}]) \in \mathbb{Z}\,,\ \forall\,\gamma\in\Gamma_{A^0}\,;\\ \kappa(M,N_i)\in\mathbb{Z}\,,\ \forall\, i \in I_W \end{array} \right\}.$$

Remark 6.11. (i) When $\gamma = \exp(N_i)$, we have $a^{-1,-1} = N$ and $a^{-1,0}, b^{0,-1} = 0$.

(ii) The fact that κ is defined over $\mathbb{Q}$ implies that $\mathbf{N}^1$ is non-empty; in fact, $\mathbf{N}^1$ spans $\mathfrak{g}^{1,1}_{W,F}$.

Lemma 6.12. *If $M \in \mathbf{N}^1$, then the form ω_M represents the Chern class $c_1(\mathcal{L}_M)$.*

Proof. Define a smooth function $h_M : \mathbb{L} \to \mathbb{R}$ by

$$h_M(z) \;:=\; \exp 2\pi\mathbf{i}\,\kappa\,(M,[z,\bar z])\;.$$

With the formulæ of §§6.6–6.7, is straightforward to confirm

$$h_M(z+\lambda) \;=\; |e^M_\gamma(z)|^{-2}\,h_M(z)\,.$$

So h_M defines a metric on $\mathcal{L}_M \to \Lambda\backslash\mathbb{L}$ with curvature form $-\partial\bar\partial\log h_M$, cf. [GH94, p. 310–311]. It follows that the Chern form of $\mathcal{L}_M$ is

$$c_1(\mathcal{L}_M) \;=\; -\frac{\mathbf{i}}{2\pi}\partial\bar\partial\log h_M \;=\; \partial\bar\partial\kappa\,(M,[z,\bar z]) \;=\; \kappa\,(M,[\mathrm{d}z,\mathrm{d}\bar z]) \;=\; \omega_M\,.$$

$\square$

6.9. **$\mathfrak{sl}_2$–triples.** The ample line bundles $\mathcal{L}_M \to J_I$ are constructed from $\mathfrak{sl}_2$–triples $\{M, Y, N\}$ constructed from the data of a LMHS (W, F, N), $N \in \sigma_I$. Here we briefly review this well-known construction (see, for example, [CM93] or [Sch73]), and discuss those properties that we will use later.

Define $Y \in \mathrm{End}(\mathfrak{g}_\mathbb{C})$ by specifying that Y acts on $\mathfrak{g}^{p,q}_{W,F}$ by the eigenvalue $(p+q)$. Then $Y \in \mathfrak{g}^{0,0}_{W,F} \cap \mathfrak{g}_\mathbb{R}$, and

$$\mathrm{ad}_Y(N) \;=\; [Y, N] \;=\; -2\,N\,.$$

Notice that Y depends only on (W, F); in particular Y is independent of N. The pair $\{Y, N\}$ may be uniquely completed to a triple $\{M, Y, N\} \subset \mathfrak{g}_\mathbb{R}$ with the properties that

$$[M, N] \;=\; Y \quad \text{and} \quad [Y, M] \;=\; 2\,M\,; \tag{6.13}$$

In particular, $\{M, Y, N\}$ spans a subalgebra of $\mathfrak{g}_\mathbb{R}$ that is isomorphic to $\mathfrak{sl}_2\mathbb{R}$. We have

$$M \;\in\; \mathfrak{g}^{1,1}_{W,F} \cap \mathfrak{g}_\mathbb{R}\,.$$

From $[M, N] = Y$ and $\kappa(Y, Y) > 0$ it follows that

$$0 \;<\; \kappa(Y, Y) \;=\; \kappa([M, N], Y) \;=\; \kappa(M, [N, Y]) \;=\; 2\kappa(M, N)\,. \tag{6.14}$$

We regard (W, F), and hence Y, as fixed. And consider $M = M(N)$ as a function of $N \in \sigma_I$.

Remark 6.15. The map $N \mapsto M(N)$ is the restriction to σ_I of a diffeomorphism $M : \mathcal{N} \to \mathcal{M}$ from an open cone $\mathcal{N} \subset \mathfrak{g}^{-1,-1}_{W,F}$ onto an open cone $\mathcal{M} \subset \mathfrak{g}^{1,1}_{W,F}$. This is a well-known and classical result in the theory of nilpotent elements of semisimple Lie algebras, cf. [CM93] and the references therein, and is discussed in the context of Hodge theory and polarized mixed Hodge structures in [BPR17, §3.2]. In general the map is not linear; in particular, while the image $M(\sigma_I)$ is a cone, it need not be convex.

Notice that the first equation of (6.13) implies that

$$M(\lambda N) \;=\; \tfrac{1}{\lambda} M(N)\,, \tag{6.16}$$

for all $\lambda > 0$. We claim that

$$\mathrm{ad}_N^2(\mathrm{d}M) \;=\; 2\mathrm{d}N\,. \tag{6.17}$$

To see this note that the fact that $Y = [M, N]$ is constant implies

$$[N, \mathrm{d}M] \;=\; [M, \mathrm{d}N]\,.$$

Since elements of the vector subspace $\mathrm{span}_\mathbb{R}\, \sigma_I \subset \mathfrak{g}^{-1,-1}_{W,F} \cap \mathfrak{g}_\mathbb{R}$ commute, we also have

$$[N, \mathrm{d}N] \;=\; 0\,. \tag{6.18}$$

Thus

$$\mathrm{ad}_N^2(\mathrm{d}M) \;=\; [N\,,\,[M\,\mathrm{d}N]] \;=\; [\mathrm{d}N\,,\,[M,N]] \;=\; 2\,\mathrm{d}N\,.$$

In particular, the differential $\mathrm{d}M$ of $N \mapsto M(N)$ is injective.

Notice that (6.17) and (6.18) imply that

$$\mathrm{ad}_N^3(\mathrm{d}M) \;=\; 0\,.$$

Since $N \in \sigma_I$ polarizes the MHS (W,F) on $(\mathfrak{g}, -\kappa)$, we have

$$0 \;\le\; -\tfrac{1}{2}\kappa(\mathrm{d}M, \mathrm{ad}_N^2(\mathrm{d}M)) \;=\; -\kappa(\mathrm{d}M, \mathrm{d}N)\,, \tag{6.19}$$

with equality if and only if $\mathrm{d}N = 0$.

Lemma 6.20. *Fix $0 \neq N' \in \mathrm{span}_{\mathbb{R}}\,\sigma_I$. The set*

$$\sigma_0' \;=\; \{N \in \sigma_I \mid \kappa(M(N), N') = 0\}$$

is contained in the closure of

$$\sigma_+' \;=\; \{N \in \sigma_I \mid \kappa(M(N), N') > 0\}\,.$$

Proof. Suppose that $N \in \sigma_0'$. Fix a smooth curve $\nu(t)$ in σ_I with the property that $\nu(0) = N$ and $\nu'(0) = -N'$. Set $\mu(t) = M(\nu(t))$. Then (6.19) implies

$$0 \;<\; \kappa(\mu'(0), N')\,.$$

In particular, $\nu(t) \in \sigma_+'$ for small $t > 0$. □

6.10. **Ample line bundles.** Define

$$\mathbf{N}_I^{\mathfrak{sl}_2} \;=\; \{M \in \mathbf{N}^1 \mid M = M(N) \text{ for some } N \in \sigma_I\}\,.$$

The fact that both σ_I and κ are defined over $\mathbb{Q}$ implies that $\mathbf{N}_I^{\mathfrak{sl}_2}$ is nonempty.

We have $NMu = u$ for all $u \in \mathfrak{c}_{I,F}^{p,q}$ with $p+q = -1$. The fact that that $N \in \sigma_I$ polarizes the MHS (W,F) on $(\mathfrak{g}, -\kappa)$ implies that

$$\begin{aligned}
-\mathbf{i}\,\omega_M(u,\bar u) \;&=\; -\mathbf{i}\,\kappa(M,[u,\bar u]) \;=\; \mathbf{i}\,\kappa(u, \mathrm{ad}_M \bar u)\\
&=\; \mathbf{i}\,\kappa(\mathrm{ad}_N\mathrm{ad}_M u\,,\,\mathrm{ad}_M\bar u) \;=\; -\mathbf{i}\,\kappa(\mathrm{ad}_M u\,,\,\mathrm{ad}_N\mathrm{ad}_M\bar u) \;<\; 0
\end{aligned}$$

for all $0 \neq u \in \mathfrak{c}_{I,F}^{-1,0} \subset \mathbb{L}_I$. It follows that the line bundle $\mathcal{L}_M^* \to \Gamma_{A^0,I}\backslash\delta_I^1$ has positive Chern form $-\omega_M$ for every $M \in \mathbf{N}_I^{\mathfrak{sl}_2}$ (Lemma 6.12). Thus $\mathcal{L}_M^* \to J_I$ is ample.

6.11. **Positivity.** It remains to establish Theorem 6.3(e); this is a consequence of Remark 6.15 and Lemma 6.21.

Lemma 6.21. *The cone*

$$\sigma_I^+ \;=\; \{N \in \sigma_I \mid \kappa(M(N), N_i) > 0\,,\ \forall\, i \in I\}$$

is open and nonempty.

Proof. In the case that $\dim \sigma_I = 1$, (6.16) and (6.22) yield $\sigma_I^+ = \sigma_I$.

For the general case $\dim \sigma_I \geq 1$, with $I = \{1, \ldots, k\}$, set

$$\mathbb{R}^k_+ \;=\; \{y = (y^1, \ldots, y^k) \in \mathbb{R}^k \mid y^i > 0\}$$

so that

$$\sigma_I \;=\; \{N(y) = y^i N_i \mid y \in \mathbb{R}^k_+\}\,.$$

Set $M(y) = M(N(y))$ and $\kappa_i(y) = \kappa(M(y), N_i)$. Then it suffices to show that the cone

$$S^+ \;=\; \{y = (y^1, \ldots, y^k) \in \mathbb{R}^k_+ \mid \ \kappa_i(y) > 0\}$$

is open. From (6.13) and (6.14) we see that

$$0 \;<\; \kappa(M(y), N(y)) \;=\; y^i\, \kappa_i(y)\,. \tag{6.22}$$

Since the y^i are all positive, this forces some $\kappa_i(y)$ to be positive (with i depending on y).

Decompose

$$\mathbb{R}^k_+ \;=\; S \cap S' \cap S''$$

with

$$\begin{aligned} S_1 \;&=\; \left\{ y \in \mathbb{R}^k_+ \;\middle|\; \kappa_1(y) \geq 0\,,\ \textstyle\sum_{i=2}^k y^i \kappa_i(y) \geq 0 \right\} \\ S_1' \;&=\; \left\{ y \in \mathbb{R}^k_+ \mid \kappa_1(y) < 0 \right\} \\ S_1'' \;&=\; \left\{ y \in \mathbb{R}^k_+ \;\middle|\; \textstyle\sum_{i=2}^k y^i \kappa_i(y) < 0 \right\}. \end{aligned}$$

The inequality (6.22) forces the open sets S_1' and S_1'' to be disjoint. Since $\mathbb{R}^k_+$ is open and connected, this in turn forces S to be nonempty. Then Lemma 6.20 implies that the cone

$$S_1^+ \;=\; \left\{ y \in \mathbb{R}^k_+ \;\middle|\; \kappa_1(y) > 0\,,\ \textstyle\sum_{i=2}^k y^i \kappa_i(y) > 0 \right\} \;\subset\; S$$

is nonempty and open in $\mathbb{R}^k_+$. This proves Theorem 6.3(e) in the case that $|I| \leq 2$.

For the general case $|I| = k$ we induct. Assume that the cone

$$S_a^+ \;=\; \left\{ y \in \mathbb{R}^k_+ \;\middle|\; \kappa_i(y) > 0\,,\ 1 \leq i \leq a\,;\quad \textstyle\sum_{i=a+1}^k y^i \kappa_i(y) > 0 \right\}$$

is nonempty (and therefore open) for some $1 \leq a \leq k-1$. Define a decomposition

$$S_a^+ \;=\; S_{a+1} \cup S_{a+1}' \cup S_{a+1}''$$

by

$$\begin{aligned} S_{a+1} &= \left\{ y \in S_a^+ \;\middle|\; \kappa_{a+1}(y) \geq 0\,,\; \textstyle\sum_{i=a+2}^k y^i \kappa_i(y) \geq 0 \right\} \\ S'_{a+1} &= \left\{ y \in S_a^+ \;\mid\; \kappa_{a+1}(y) < 0 \right\} \\ S''_{a+1} &= \left\{ y \in S_a^+ \;\middle|\; \textstyle\sum_{i=a+2}^k y^i \kappa_i(y) < 0 \right\}\,. \end{aligned}$$

The definition of S_a^+ forces the open sets S'_{a+1} and S''_{a+1} to be disjoint. Since S_a^+ is open, every connected component of S_a^+ must have nonempty intersection with S_{a+1}. Then Lemma 6.20 implies that the cone S_{a+1}^+ is nonempty and open in $\mathbb{R}^k_+$. This completes the inductive step. $\square$

7. Higher level extension data

The goal here is to study the higher level extension data along a connected component

$$A^1 \;\subset\; A^0$$

of a $\Phi^1_{A^0,W}$–fibre. We will see that the monodromy around A^1 takes value in $\exp(\mathbb{C}\sigma_{I(A^1)}) \cap P_{W,\mathbb{Q}}$ (Proposition 7.1). This is the essential structural result that will be used to prove Theorem 1.10 (§7.2).

7.1. Extension data along Φ^1–fibres.

Set

$$I(A^1) \;=\; \{ i \mid Z_i^* \cap A^1 \neq \emptyset \}\,.$$

Consider the period map

$$\Phi_{A^1} : \mathcal{O}^1 \;\to\; \Gamma_{A^1} \backslash D$$

induced by $\mathcal{V}|_{\mathcal{O}^1}$. Set $W = W^{I(A^1)}$, so that

$$A^1 \;\subset\; A^0 \;\subset\; Z_W\,.$$

Given $Z_I^* \subset Z_W$, let

$$\Phi_{A^1,I} : Z_I^* \cap \overline{\mathcal{O}}^1 \;\to\; (\exp(\mathbb{C}\sigma_I)\Gamma_{A^1}) \backslash D_I$$

be the map induced by Φ_{A^1} (as Φ_I in (2.3) is induced by Φ). While Φ_I does not in general extend to the weight closure $Z_I \cap Z_W$ (§4.3.1), the map $\Phi_{A^1,I}$ does admit an extension if we replace the quotient of $\exp(\mathbb{C}\sigma_I)$ with the quotient by the larger $\exp(\mathbb{C}\sigma_{I(A^1)})$.

Proposition 7.1. (a) *The neighborhood $A^1 \subset \overline{\mathcal{O}}^1 \subset \overline{B}$ of Corollary 5.4 may be chosen so that the restriction of $\mathcal{V}$ to $\mathcal{O}^1 = \overline{\mathcal{O}}^1 \cap B$ has monodromy $\Gamma_{A^1} \subset \Gamma_{A^0}$ with unipotent radical $\Gamma_{A^1} \cap P_W^{-1} \subset \exp(\mathbb{C}\sigma_{I(A^1)}) \subset P_W^{-2}$. In particular,*

$$\Gamma_{A^1} \;\subset\; G_{I(A^0)} \ltimes \exp(\mathbb{C}\sigma_{I(A^1)}) \;\subset\; C_{I(A^0)}\,.$$

(b) *There is a well-defined holomorphic map*

$$\Phi'_{A^1,W} : Z_W \cap \overline{\mathcal{O}}^1 \;\to\; (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash D_W\,,$$

and commutative diagram

$$\begin{array}{ccc} Z_I^* \cap \overline{\mathcal{O}}^1 & \xrightarrow{\Phi_{A^1,I}} & (\exp(\mathbb{C}\sigma_I)\Gamma_{A^1})\backslash D_I \\ \big\downarrow & & \big\downarrow \\ Z_W \cap \overline{\mathcal{O}}^1 & \xrightarrow{\Phi'_{A^1,I}} & (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash D_W\,. \end{array}$$

(c) *The map $\Phi'_{A^1,W}$ is locally constant on the fibres of Φ^1.*

Remark 7.2. The information contained in $\exp(\mathbb{C}\sigma_{I(A^1)})$ is level two extension data. So the content of Proposition 7.1(c) is that *the full extension data is determined by the level ≤ 2 extension data*, up to constants of integration.[14] The level 2 extension data contained in $\exp(\mathbb{C}\sigma_{I(A^1)})$ is not truly lost; it is encoded in the sections $s_M \in H^0(\overline{\mathcal{O}}^0, L_M)$, with $M \in \mathfrak{g}^{1,1}_{W,F}$, of Theorem 2.17. These sections are essentially discrete data as their restriction to the Φ^0–fibres is determined up to a constant factor by (2.18).

Subject to the assumption of Remark 5.12, we have

Corollary 7.3. *Assume that the monodromy Γ_{A^1} is unipotent; that is, $\Gamma_{A^1} \subset \exp(\mathbb{C}\sigma_{I(A^1)})$* (Remark 5.12)*. Then the Hodge filtrations $\mathcal{F}^p_{\mathrm{e}}$ are trivial over $\overline{\mathcal{O}}^1$.*

Proof of Corollary 7.3. Let $W = W^A$ be the weight filtration of $A^0 \supset A^1$. Since $\sigma_{I(A^1)} \subset \mathfrak{g}^{-1,-1}_{W,F} \subset \mathfrak{f}^\perp$, the proposition implies $\Gamma_{A^1} \subset \exp(\mathfrak{f}^\perp)$. The theorem now follows from Remark 5.17. ∎

7.1.1. *Outline of the proof of Proposition 7.1.* The proposition is proved by an inductive analysis of the higher level extension data along A^1. We begin with the level ≤ 2 extension data. Applying the discussion of §4.3.1 to the period map Φ_{A^0} yields a commutative diagram

$$\begin{tikzcd} Z_I^* \cap \overline{\mathcal{O}}^0 \arrow[r, "\Phi^2_{A^0,I}"] \arrow[d, hook] & (\exp(\mathbb{C}\sigma_I)\Gamma_{A^0,I})\backslash D_I^2 \arrow[d, two heads] \\ Z_W \cap \overline{\mathcal{O}}^0 \arrow[r, "\Phi^2_{A^0,W}"] \arrow[dr, "\Phi^1_{A^0,W}"'] & (\exp(\mathbb{C}\sigma_{I(A^0)})\Gamma_{A^0})\backslash D_W^2 \arrow[d, two heads] \\ & \Gamma_{A^0}\backslash D_W^1\,. \end{tikzcd}$$

Lemma 7.4. *The map $\Phi^2_{A^0,W}$ is locally constant on $\Phi^1_{A^0,W}$–fibres.*

[14]In the case that D is Hermitian, all extension data is level ≤ 2; that is, $D_W = D_W^2$. So here we find here another example of the ansatz that horizontality (the IPR) forces period maps and their images to behave "as if they were Hermitian".

A straightforward modification of the proof of Lemma 5.6 establishes

Corollary 7.5. *There is a neighborhood $\overline{\mathcal{O}}^1$ of A^1 in $\overline{B}$ with the property that the restriction of $\mathcal{V}$ to $\mathcal{O}^1 = \overline{\mathcal{O}}^1 \cap B$ has monodromy $\Gamma_{A^1} \subset \Gamma_{A^0}$ taking value in $G_{I(A^0)} \ltimes (\exp(\mathbb{C}\sigma_{I(A^1)})P_W^{-3})$.*

It then follows that the action of $\exp(\mathbb{C}\sigma_{I(A^1)})$ on D_W^3 does descend to $\Gamma_{A^1}\backslash D_W^3$ (§4.3.1), yielding a well-defined map

$$\Phi^3_{A^1,W} : Z_W \cap \overline{\mathcal{O}}^1 \ \to \ (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash D_W^3 \,.$$

The inductive step for $a \geq 3$ is

Lemma 7.6. *If the monodromy Γ_{A^1} about A^1 takes value in $G_{I(A^0)} \ltimes (\exp(\mathbb{C}\sigma_{I(A^1)})P_W^{-a})$, then the action $\exp(\mathbb{C}\sigma_{I(A^1)})$ on D_W^a does descend to the $\Gamma_{A^1}\backslash D_W^a$, yielding a well-defined map*

$$\Phi^a_{A^1,W} : Z_W \cap \overline{\mathcal{O}}^1 \ \to \ (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash \delta_W^a \,.$$

This map is constant on A^1, implying $\Gamma_{A^1} \subset G_{I(A^0)} \ltimes (\exp(\mathbb{C}\sigma_{I(A^1)})P_W^{-a-1})$.

Note that Proposition 7.1 follows directly from Lemma 7.6. The remainder of §7 is occupied with the proof of Lemma 7.6 (which subsumes Lemma 7.4 and Corollary 7.5).

7.1.2. *Lie theoretic description.* Fix $a \geq 2$. The fibres of $\Gamma_{A^1}\backslash\delta_W^a \twoheadrightarrow \Gamma_{A^1}\backslash\delta_W^{a-1}$ are the *level a extension data* (Definition 2.6). We begin by observing that these fibres are biholomorphic to the quotient $\Lambda^a\backslash\mathbb{L}^a$ of a vector space $\mathbb{L}^a$ by a discrete subgroup $\Lambda^a \subset \mathbb{L}^a$. To see this, first note that the fibre is

$$\begin{array}{ccc} \dfrac{P^{-a}_{W,\mathbb{C}} \cdot F}{(\Gamma_{A^1} \cap P^{-a}_{W,\mathbb{C}}) \cdot P^{-a-1}_{W,\mathbb{C}}} & \hookrightarrow & \Gamma_{A^1}\backslash\delta_W^a \\ & & \downarrow\!\!\downarrow \\ & & \Gamma_{A^1}\backslash\delta_W^{a-1} \,. \end{array}$$

We have

$$\begin{aligned} P^{-a-1}_{W,\mathbb{C}}\backslash P^{-a}_{W,\mathbb{C}} &\simeq \bigoplus_{p+q=-a} \mathfrak{g}^{p,q}_{W,F} \\ P^{-a-1}_{W,\mathbb{C}}\backslash(P^{-a}_{W,\mathbb{C}} \cdot F) &\simeq \bigoplus_{\substack{p+q=-a \\ p<0}} \mathfrak{g}^{p,q}_{W,F} \ = \ \mathbb{L}^a \,. \end{aligned}$$

The latter is an abelian group, with discrete subgroup

$$\Lambda^a \ = \ \frac{P^{-a}_{W,\mathbb{C}} \cap \Gamma_{A^1}}{P^{-a-1}_{W,\mathbb{C}} \cap \Gamma_{A^1}} \,.$$

We now see that the level a extension data of (W,F) is biholomorphic to the the product

$$\Lambda^a\backslash\mathbb{L}^a \ \simeq \ \mathbb{C}^{d_1} \times (\mathbb{C}^*)^{d_2} \times \mathbb{T}^{d_3} \tag{7.7}$$

of an affine space $\mathbb{C}^{d_1}$ with a complex torus $(\mathbb{C}^*)^{d_2} \times \mathbb{T}^{d_3}$ having compact factor $\mathbb{T}^{d_3}$. (The dimensions d_i depend on a.)

Since $\sigma_{I(A^1)} \subset \mathfrak{g}_{W,F}^{-1,-1}$, it then follows that the fibres of

$$(\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash\delta_W^a \ \twoheadrightarrow\ (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash\delta_W^{a-1}$$

are, for $a = 2$:

$$\begin{array}{ccc} (\Lambda^2 \cdot \sigma_{I(A^1)})\backslash\mathbb{L}^2 & \hookrightarrow & (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash\delta_W^2 \\ & & \downarrow \\ & & \Gamma_{A^1}\backslash\delta_W^1\,, \end{array}$$

and, for $a \geq 3$:

$$\begin{array}{ccc} \Lambda^a\backslash\mathbb{L}^a & \longrightarrow & (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash\delta_W^a \\ & & \downarrow \\ & & (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash\delta_W^{a-1}\,. \end{array}$$

Note that $(\Lambda^2 \cdot \sigma_{I(A^1)})\backslash\mathbb{L}^2$ inherits (7.7) in the sense that it is also biholomorphic to the product

$$(\Lambda^2 \cdot \sigma_{I(A^1)})\backslash\mathbb{L}^2 \ \simeq\ \mathbb{C}^{d_1} \times (\mathbb{C}^*)^{d_2} \times \mathbb{T}^{d_3} \tag{7.8}$$

of an affine space $\mathbb{C}^{d_1}$ with a complex torus $(\mathbb{C}^*)^{d_2} \times \mathbb{T}^{d_3}$ having compact factor $\mathbb{T}^{d_3}$. (We abuse notation by continuing to denote the dimensions by d_i.)

7.1.3. *The IPR along fibres.* The local version of $\Phi^a_{A^1,W}$ is the map

$$\widetilde{\Phi}^a_W : Z_W \cap \overline{\mathfrak{U}} \ \to\ \exp(\mathbb{C}\sigma_{I(A^1)})\backslash D^a_W$$

defined by (4.18). If the level $\leq a$ extension data map $\Phi^a_{A^0,W}$ is constant along A^1, then the restriction

$$\xi^a_{A^1} \ =\ \xi\big|_{A^1\cap\overline{\mathfrak{U}}}$$

takes value in the affine space

$$\delta^a_W \ =\ P^{-a-1}_{W,\mathbb{C}} \cdot F \ \simeq\ \exp(\mathfrak{p}^{-a-1}_{W,\mathbb{C}} \cap \mathfrak{f}^\perp) \ \simeq\ \mathfrak{p}^{-a-1}_{W,\mathbb{C}} \cap \mathfrak{f}^\perp \ =\ \bigoplus_{b\geq a} \mathbb{L}^{b+1}\,.$$

Recall the discussion of the IPR in §6.4, and note that (6.7) implies

$$(\xi^a_{A^1})^{-1}\,\mathrm{d}\xi^a_{A^1} \ \ \textit{takes value in} \ \ \bigoplus_{b\geq a} \mathfrak{g}^{-1,-b}_{W,F} \ \subset\ \bigoplus_{b\geq a} \mathbb{L}^{b+1}\,. \tag{7.9}$$

Additionally, we have well-defined logs

$$\log \xi^a_{A^1} : A^1 \cap \overline{\mathfrak{U}} \ \to\ \mathfrak{p}^{-a-1}_{W,\mathbb{C}} \cap \mathfrak{f}^\perp\,.$$

Let $(\log \xi^a_{A^1})^{p,q}$ denote the component taking value in $\mathfrak{g}^{p,q}_{W,F}$. Then (7.9) implies

$$(\log \xi^a_{A^1})^{p,q} \quad \text{is locally constant for all} \quad p+q = -a-1\,,\ p \leq -2\,. \tag{7.10}$$

7.1.4. *Proof of Lemma 7.6.* The argument is inductive. Assume that $a \geq 1$ and that we have a well-defined

$$\Phi^{a+1}_{A^1,W} : Z_W \cap \overline{\mathcal{U}} \;\to\; (\exp(\mathbb{C}\sigma_{I(A^1)})\Gamma_{A^1})\backslash D^a_W\,.$$

We will show that $\Phi^{a+1}_{A^0,W}$ is constant along A^1.

Recalling (7.9), let η^a be the component of the Maurer-Cartan form $(\xi^a_{A^1})^{-1}\,\mathrm{d}\xi^a_{A^1}$ taking value in

$$\mathfrak{g}^{-1,-a}_{W,F} \;\hookrightarrow\; \mathbb{L}^{a+1} \;\simeq\; \mathcal{P}^{-a-2}_W\backslash\left(P^{-a-1}_{W,\mathbb{C}}\cdot F\right)\,. \tag{7.11}$$

Then fixing a point $z_0 \in A^1$ we may define a holomorphic map

$$A^1 \;\to\; \begin{cases} (\Lambda^2\cdot\sigma_{I_W})\backslash\mathbb{L}^2\,, & a=1\,,\\ \Lambda^{a+1}\backslash\mathbb{L}^{a+1}\,, & a\geq 2\,, \end{cases} \tag{7.12a}$$

by integration

$$z \;\mapsto\; \int_{z_0}^{z}\eta^a \tag{7.12b}$$

along a curve $\delta : [0,1]\to A^1$ joining $z_0=\delta(0)$ and $z=\delta(1)$.

The key point is that when $b\geq 2$, the complex conjugate

$$\overline{\mathfrak{g}^{-1,-b}_{W,F}} \;=\; \mathfrak{g}^{-b,-1}_{W,F}$$

is contained in $\mathfrak{p}^{-b-1}_{W,\mathbb{C}}\cap\mathfrak{f}^{\perp}$ and has trivial intersection with $\mathfrak{g}^{-1,-b}_{W,F}$. This implies that the image of $\mathfrak{g}^{-1,-b}_{W,F}$ under the composition of (7.11) with the projection

$$\mathbb{L}^{a+1} \;\twoheadrightarrow\; \begin{cases} (\Lambda^2\cdot\sigma_{I_W})\backslash\mathbb{L}^2\,, & a=1\,,\\ \Lambda^{a+1}\backslash\mathbb{L}^{a+1}\,, & a\geq 2\,, \end{cases}$$

lies in the noncompact factors $\mathbb{C}^{d_1}\times(\mathbb{C}^*)^{d_2}$ of (7.7) and (7.8). Since η^a takes value in $\mathfrak{g}^{-1,a}_{W,F}$, it follows that (7.12) defines a holomorphic map $A^1\to\mathbb{C}^{d_2}\times(\mathbb{C}^*)^{d_3}$. Since A^1 is compact, this map must be locally constant. This forces $\eta^a=0$. Equivalently, the Maurer–Cartan form $(\xi^a_{A^1})^{-1}\,\mathrm{d}\xi^a_{A^1}$ takes value in $\mathfrak{p}^{-a-2}_{W,\mathbb{C}}$ along A^1. This is precisely the statement that $\Phi^{a+1}_{A^0,W}$ is locally constant along A^1. □

7.2. **Proof of Theorem 1.10.** It suffices to prove

Proposition 7.13. *There exists a proper holomorphic map $f:\overline{\mathcal{O}}^1\to\mathbb{C}^d$ with the following properties:*

(a) *The map $f|_{\mathcal{O}^1}$ is constant on the fibres of $\Phi|_{\mathcal{O}^1}$.*

(b) *Conversely, $\Phi|_{\mathcal{O}^1}$ is locally constant on the fibres of $f|_{\mathcal{O}^1}$.*

The proposition is proved in §§7.2.1–7.2.3. Assume for the moment that Proposition 7.13 holds. Let

$$\begin{tikzcd} \overline{\mathcal{O}}^1 \arrow[r, "\hat f"'] \arrow[rr, bend left, "f"] & \hat{\mathcal{O}}^1 \arrow[r] & \mathbb{C}^m \end{tikzcd}$$

be the Stein factorization. This completes the proof of Theorem 1.10 as outlined in §1.2.2.

7.2.1. *Preliminaries.* Consider the lift

$$\begin{array}{ccc} \tilde{\mathcal{O}}^1 & \xrightarrow{\widetilde{\Phi}_{A^1}} & \mathcal{S} \cap D \\ \downarrow & & \downarrow \\ \mathcal{O}^1 & \xrightarrow{\Phi_{A^1}} & \Gamma_{A^1} \backslash D\,. \end{array} \tag{7.14}$$

Recall notations of §3.4. Let W index the weight strata Z_W containing A^1, so that

$$\mathfrak{f}^\perp \;=\; \bigoplus_{p<0} \mathfrak{g}^{p,q}_{W,F}\,,$$

and set

$$I(A^1) \;=\; \{i \mid A^1 \cap Z_i^* \neq \emptyset\} \;=\; \cup\{I \mid A^1 \cap Z_I^* \neq \emptyset\} \;\subset\; I_W\,.$$

7.2.2. *Horizontal entries of the period matrix.* Fix a basis $\{M_\mu\}$ of $\mathfrak{g}^{1,\bullet}_{W,F} = \oplus_q \, \mathfrak{g}^{1,q}_{W,F}$. Keeping (3.3c) in mind, the

$$\varepsilon_\mu \;=\; \kappa(X \circ \widetilde{\Phi}_{A^1}, M_\mu) : \tilde{\mathcal{O}}^1 \;\to\; \mathbb{C} \tag{7.15}$$

are the *horizontal coefficients of the period matrix.* We may choose the basis $\{M_\mu\}$ so that for each $I \subset I(A^1)$ there is a disjoint union $\{M_\mu\} = \mathbf{N}_I^* \cup \mathbf{N}_I^\perp$ so that

$$\mathrm{span}_{\mathbb{C}}\{M_\mu \in \mathbf{N}_I^\perp\} \;=\; \mathrm{Ann}(\sigma_I) \;\subset\; \mathfrak{g}^{1,\bullet}_{W,F}\,.$$

It follows that

$$\varepsilon_\mu \;\in\; \mathcal{O}(\overline{\mathcal{O}}^1)\,, \quad \forall\; M_\mu \in \mathbf{N}^\perp_{I(A^1)}\,.$$

We think of the ε_μ indexed by $M_\mu \in \mathbf{N}^\perp_{I(A^1)}$ as the *smooth horizontal coefficients of the period matrix.*

We may additionally suppose the basis $\{M_\mu\}$ is chosen so that

$$0 \;\le\; \kappa(N_i, M_\mu) \;\in\; \mathbb{Z}\,, \quad \forall\; i \in I(A^1)\,. \tag{7.16}$$

Then, for the $M_\mu \in \mathbf{N}^*_{I(A^1)} \supset \mathbf{N}^*_I$, we have

$$\tau_\mu \;=\; \exp(2\pi\mathbf{i}\,\varepsilon_\mu) \;\in\; \mathcal{O}(\overline{\mathcal{O}}^1)\,.$$

We think of the ε_μ indexed by $M_\mu \in \mathbf{N}_{I(A^1)}$ as the *logarithmic horizontal coefficients of the period matrix.*

7.2.3. *Proof of Proposition 7.13.* Let $f : \overline{\mathcal{O}}^1 \to \mathbb{C}^m$ be the holomorphic map defined by the $\{\varepsilon_\mu\}_{M_\mu \in \mathbf{N}^\perp_{I(A^1)}}$ and $\{\tau_\mu\}_{M_\mu \in \mathbf{N}^*_{I(A^1)}}$. The IPR implies f has the desired properties.

7.2.4. *Zero locus of the τ_μ.* Note that (5.21) is the local expression for τ_μ. In particular,

$$A_I \;=\; Z_I \,\cap\, \overline{\mathcal{O}}^1 \;\subset\; \{\tau_\mu = 0\} \quad \text{if and only if} \quad M_\mu \in \mathbf{N}^*_I\,.$$

Reciprocally, τ_μ is nowhere vanishing on $A^*_I = Z^*_I \cap \overline{\mathcal{O}}^1$ if and only if $M_\mu \in \mathbf{N}^\perp_I$. More generally, the function τ_μ is nowhere vanishing on the weight strata $Z_W \cap \overline{\mathcal{O}}^1$ if and only if $M_\mu \in \mathbf{N}^\perp_{I_W}$.

Suppose that $j \notin I_W$, and set $J = I_W \cup \{j\}$. Then $W \neq W^J$ (Corollary 4.6). So there exists $M_{\mu_j} \in \mathbf{N}^\perp_{I_W}$ such that $\kappa(M_{\mu_j}, N_j) > 0$. The associated τ_{μ_j} is nowhere vanishing on $Z_W \cap \overline{\mathcal{O}}^1$, but vanishes along $Z_j \cap \overline{\mathcal{O}}^1$. Whence

$$\tau_W \;=\; \prod_{j \notin I_W} \tau_{\mu_j}\,.$$

is nowhere vanishing on $Z_W \cap \overline{\mathcal{O}}^1$, but vanishes on every $j \notin I_W$. In particular, τ_W vanishes along every $Z_J \cap \overline{\mathcal{O}}^1$ with $J \not\subset I_W$.

In the case that $I = \emptyset$ (the weight filtration $W^\emptyset$ is trivial and), we have

$$Z \,\cap\, \overline{\mathcal{O}}^1 \;=\; \{\tau_{W^\emptyset} = 0\}\,.$$

7.3. **Logarithmic differentials and a local Torelli condition.**

7.3.1. *Logarithmic differentials on $(\overline{\mathcal{O}}^1, Z \cap \overline{\mathcal{O}}^1)$.* In §9.3.1 we will discuss a map $\Psi : T_{\overline{B}}(-\log Z) \to F^{-1}\mathrm{End}(\mathcal{E}_\mathrm{e})$ that is induced by the Gauss–Manin connection on $\mathcal{V} \to B$. In anticipation of that discussion it is convenient to close §7.2 with a discussion of the algebra $\Omega^\bullet_{\overline{\mathcal{O}}^1}(Z \cap \overline{\mathcal{O}}^1)$ of logarithmic differentials on $(\overline{\mathcal{O}}^1, Z \cap \overline{\mathcal{O}}^1)$. It is evident from the discussions of §7.2.2 and §7.2.4 that

$$\mathrm{d}\varepsilon_\mu \;\in\; \Omega^1_{\overline{\mathcal{O}}^1}(\log Z \cap \overline{\mathcal{O}}^1)\,, \tag{7.17}$$

and

$$\{\mathrm{d}\varepsilon_\mu \mid M_\mu \in \mathbf{N}^\perp_{I(A^1)}\} \;\subset\; \Omega^1_{\overline{\mathcal{O}}^1}\,.$$

The differentials define a map

$$\Psi_1 : T_{\overline{\mathcal{O}}^1}(-\log Z \cap \overline{\mathcal{O}}^1) \;\to\; \overline{\mathcal{O}}^1 \,\times\, \mathbb{C}^m \tag{7.18}$$

by mapping $v \in T_{\overline{\mathcal{O}}^1}(-\log Z \cap \overline{\mathcal{O}}^1)$ to $(\mathrm{d}\varepsilon_\mu(v)) \in \mathbb{C}^m$. (Here we suppress the base point $b \in \overline{\mathcal{O}}^1$ of v.)

7.3.2. *Local Torelli condition for* $(\overline{\mathcal{O}}^1, Z\cap\overline{\mathcal{O}}^1;\Phi_{A^1})$. Since the coordinates of $\Psi_1|_{\mathcal{O}^1}$ are the horizontal period matrix entries of $\widetilde{\Phi}_{A^1}$, we see that the differential of $\Phi|_{\mathcal{O}^1}$ is injective if and only if the differential of $\Psi_1|_{\mathcal{O}^1}$ is injective. More generally, we have

Lemma 7.19. *The sheaf map* Ψ_1 *is injective at points* $b\in\overline{\mathcal{O}}^1\cap Z_I^*$ *if and only if*

(i) *The differential* $\mathrm{d}\Phi^1_{A^1,I} : T(Z_I^*\cap\overline{\mathcal{O}}^1)\to T(\Gamma_{A^1,I}\backslash D^1_I)$ *is injective.*

(ii) *The* $\{N_i \mid i\in I\}$ *are linearly independent.*

Proof. It will be convenient to write

$$\Psi_1 = (\Psi_1^{\mathrm{hol}}, \Psi_1^{\log})$$

with

$$\Psi_1^{\mathrm{hol}}(v) \ = \ (\mathrm{d}\varepsilon_\mu(v))_{M_\mu\in\mathbf{N}^\perp_{I(A^1)}}$$

given by the holomorphic differentials, and

$$\Psi_1^{\log}(v) \ = \ (\mathrm{d}\varepsilon_\mu(v))_{M_\mu\in\mathbf{N}^*_{I(A^1)}}$$

given by the log differentials. It follows from the IPR and Remark 7.2 that the following are equivalent:

(a) The restriction of Ψ_1^{hol} to $T_b(Z_I^*\cap\overline{\mathcal{O}}^1)$ is injective.

(b) The restriction of $\mathrm{d}\Phi^1_{A^1,I}$ to $T_b(Z_I^*\cap\overline{\mathcal{O}}^1)$ is injective.

Fix a coordinate chart $(t,w)\in\overline{\mathcal{U}}\subset\overline{\mathcal{O}}^1$ centered at a point $b\in Z_I^*\cap\overline{\mathcal{O}}^1$, as in §3.2.2. Then

$$\{\mathrm{d}\log t_i\,,\,\mathrm{d}w_a\}$$

is a local framing of $\Omega^1_{\overline{B}}(\log Z)$ over $\overline{\mathcal{U}}$,

$$\{t_i\partial_{t_i}\,,\,\partial_{w_a}\}$$

is a local framing of $T_{\overline{B}}(-\log Z)$ over $\overline{\mathcal{U}}$, and $\{\partial_{w_a}\}$ is a local framing of $T(Z_I^*\cap\overline{\mathcal{O}}^1)$ over $\overline{\mathcal{U}}\cap Z_I^* = \{t=0\}$. We have

$$\Psi_1^{\mathrm{hol}}(t_i\partial_{t_i})\Big|_{t_i=0} \ = \ 0\,.$$

Following the notation of (5.21), the logarithmic differentials are

$$\mathrm{d}\varepsilon_\mu \ = \ \frac{\mathrm{d}\tau_\mu}{2\pi\mathbf{i}\tau_\mu} \ = \ \kappa(M_\mu,\mathrm{d}\tilde{X}(t,w)) \ + \ \sum\frac{\kappa(M_\mu,N_i)\mathrm{d}t_i}{2\pi\mathbf{i}\,t_i}\,,\quad M_\mu\in\mathbf{N}^*_{I(A^1)}\,.$$

Recalling that $\tilde{X}(t,w)$ is holomorphic on $\overline{\mathcal{U}}$, we see that

$$\mathrm{d}\varepsilon_\mu\,(t_i\partial_{t_i})|_{t=0} \ = \ \frac{\kappa(M_\mu,N_i)}{2\pi\mathbf{i}}\,. \tag{7.20}$$

□

Informally we express (7.20) as

$$\Psi_1^{\log}\left(t_i\partial_{t_i}\big|_{t=0}\right) \;=\; 2\pi\mathbf{i}\,N_i\,.$$

8. The case that $\dim B = 2$

Consider the case that B is a surface. We assume that $\wp$ is a surface; equivalently, Φ_* is injective at some point $b \in B$. (See Remark 1.8 for the case that $\wp$ is a curve.) Index the irreducible components Z_i of $Z = Z_1 \cup \cdots \cup Z_\nu$ so that $\Phi^0(Z_i)$ is a point if and only if $i \le \mu \le \nu$.

8.1. An ample bundle on $\overline{B}$.

Theorem 8.1. *Assume that the differential of $\Phi : B \to \Gamma\backslash D$ is everywhere injective. Then there exists $a_i \ge 0$ so that the line bundle*

$$\Pi \;=\; m\Lambda_{\mathrm{e}} \;-\; \sum_{i=1}^{\mu} a_i[Z_i]$$

is ample for $m \gg 0$.

Proof. It follows from [GGLR20, Lemma 5.4.20] that it suffices to prove the following: given a curve $C \subset \overline{B}$, we have

$$\Pi \cdot C \;=\; \deg\, \Pi\big|_C \;>\; 0\,. \tag{8.2}$$

Without loss of generality C is irreducible and there are three cases to consider:

(a) The intersection $C \cap B$ is Zariski open in C. (In which case $C \cap Z$ is a finite set of points.)

(b) $C = Z_i$ for some $i > \mu$, and $\Phi^0(C)$ is a curve.

(c) $C = Z_i$ for some $i \le \mu$, and $\Phi^0(C)$ is a point.

In cases (a) and (b), we have $\Lambda_{\mathrm{e}} \cdot C > 0$, cf. §9.1. We will see that the a_i are determined by the intersection matrix

$$A \;=\; (A_{ij}) \;=\; \|Z_i \cdot Z_j\|_{i,j=1}^{\mu}\,. \tag{8.3}$$

Then (8.2) will follow for $m \gg 0$.

In the case (c) we have $\Lambda_{\mathrm{e}} \cdot C = 0$. The Hodge Index Theorem implies that A is negative definite [GGLR20, Lemma 3.1.1].

Lemma 8.4. *Let A be any integral, negative definite symmetric matrix with the property that $A_{ij} \ge 0$ when $i \ne j$. Then A has an eigenvector $a = {}^t(a_1, \ldots, a_\mu)$ with $a_i > 0$.*

The lemma is proved below. Assuming the lemma for the moment, let $\alpha < 0$ denote the eigenvalue of a. Then

$$\sum_{j=1}^{\mu} a_j Z_j \cdot Z_i \;=\; \alpha\, a_i \;<\; 0\,, \quad i \le \mu\,.$$

The desired (8.2) now follows. $\square$

Remark 8.5. The point here is that a simpler result, such as $m\Lambda_{\mathrm{e}} - \sum_i [Z_i]$ is ample, does not hold. The coefficients are necessary; they reflect a property of the singularity that Z_i is contracted to.

Proof of Lemma 8.4. Suppose that $a = (a_1, \dots, a_\mu)$ is an eigenvalue with maximal eigenvalue $\alpha < 0$. We claim that $\hat{a} = {}^t(|a_1|, \dots, |a_\mu|)$ is also an eigenvector with maximal eigenvalue α. To see this note that

$$\begin{aligned} \sum_{i,j=1}^{\mu} |a_i|\, A_{ij}\, |a_j| \;&=\; \sum_i |a_i|\, A_{ii}\, |a_i| \;+\; \sum_{i \neq j} |a_i|\, A_{ij}\, |a_j| \\ &\ge\; \sum_i A_{ii}\, (a_i)^2 \;+\; \sum_{i\neq j} a_i\, A_{ij}\, a_j \;=\; \alpha\, \|a\|^2\,. \end{aligned}$$

So without loss of generality we may suppose that $a_i \ge 0$.

We further claim that $a_i > 0$ for all $1 \le i \le \mu$. Suppose that some $a_j = 0$. Set $a_\epsilon = (a_1 + \epsilon\delta_{1j}, \dots, a_\mu + \epsilon\delta_{\mu j})$. Then

$${}^t a_\epsilon\, A\, a_\epsilon \;=\; {}^t a\, A\, a \;+\; 2\epsilon \sum_{i=1}^{\mu} A_{ij}\, a_i \;+\; \epsilon^2 A_{jj}\,.$$

Since $a_j = 0$ we have $\|a_\epsilon\|^2 = \|a\|^2 + \epsilon^2$. This implies

$$\frac{{}^t a_\epsilon\, A\, a_\epsilon}{\|a_\epsilon\|^2} \;\ge\; \frac{{}^t a\, A\, a}{\|a\|^2} \;+\; \frac{2\epsilon \sum_{i=1}^{\mu} A_{ij}\, a_i}{\|a\|^2} \;+\; \frac{\epsilon^2 A_{jj}}{\|a\|^2}\,.$$

The connectedness of Z implies that some $A_{ij}\, a_i > 0$. So for $0 < \epsilon \ll 1$ we have

$$\frac{{}^t a_\epsilon\, A\, a_\epsilon}{\|a_\epsilon\|^2} \;>\; \frac{{}^t a\, A\, a}{\|a\|^2}\,,$$

contradicting the maximality of $\alpha < 0$. Thus, $a_i > 0$ for all $1 \le i \le \mu$. $\square$

8.2. **Remark on negative definiteness of A.** In the proof of Theorem 8.1 we invoked the Hodge Index Theorem to conclude that the matrix (8.3) is negative definite. Alternatively, one may show (without the Hodge Index Theorem) that (8.3) is negative definite if the conormal bundle is ample. To be precise, suppose that X is a surface and that $Z_i \subset X$ are smooth curves forming a normal crossing divisor $Z = Z_1 \cup \cdots \cup Z_\mu$

Lemma 8.6. *If $\mathcal{N}^*_{Z/X} \to Z$ is ample, then the intersection matrix $A = \|Z_i \cdot Z_j\|_{i,j=1}^{\mu}$ is negative definite.*

Proof. Let α be a maximal eigenvector; we wish to show that $\alpha < 0$. The argument of Lemma 8.4 applies here to give us an eigenvector $(a_1, \ldots, a_\mu)$ with $a_i > 0$.

The condition that $\mathcal{N}^*_{Z/X} \to Z$ be ample is

$$\sum_{i=1}^{\mu} Z_i \cdot Z_j \ < \ 0 \quad \forall\, 1 \le j \le \mu\,.$$

We have

$$\sum_{i=1}^{\mu} a_i\, Z_i \cdot Z_j \ = \ \alpha\, a_j \quad \forall\, 1 \le j \le \mu\,.$$

Without loss of generality the Z_i are indexed so that $a_1 \ge a_i$. Then

$$\sum_{i=1}^{\mu} a_i Z_i \ = \ a_1 \sum_{i=1}^{\mu} Z_i \ - \ \sum_{i=2}^{\mu} (a_1 - a_i) Z_i\,,$$

so that

$$\alpha\, a_1 \ = \ \sum_{i=1}^{\mu} a_i\, Z_i \cdot Z_1 \ = \ a_1 \sum_{i=1}^{\mu} Z_i \cdot Z_1 \ - \ \sum_{i=2}^{\mu} (a_1 - a_i) Z_i \cdot Z_1\,.$$

Since $Z_i \cdot Z_1 \ge 0$ for all $i \ge 2$, our ampleness hypothesis implies that $\alpha\, a_1 < 0$. □

Remark 8.7. The converse to Lemma 8.6 does not hold. For example, consider

$$A \ = \ \begin{bmatrix} -5 & 2 \\ 2 & -1 \end{bmatrix}.$$

8.3. **Proof of Proposition 2.33.** We argue by contradiction. Suppose that Φ^1 is constant along all of Z. Then Φ^0 is necessarily constant along all of Z; that is, $Z = A^0$, and $\mu = \nu$. Since $\Phi^1(Z) = \Phi^1(A^0)$ is a point in the compact torus T_W of Theorem 6.3, it follows from (2.28) that

$$(\Phi^1\big|_Z)^*(\mathcal{L}_M) \ = \ \sum_{i=1}^{\nu} \kappa(M, N_i) [Z_i]\big|_Z \quad \text{is trivial.}$$

So

$$0 \ = \ \left(\sum_{i=1}^{\nu} \kappa(M, N_i) [Z_i] \right)^2 \ = \ \sum_{i,j=1}^{\nu} \kappa(M, N_i) \kappa(M, N_j)\, Z_i \cdot Z_j$$

The negative definiteness of (8.3) forces $\kappa(M, N_i) = 0$ for all i. As M is arbitrary, this contradicts (6.14). □

9. Geometric properties of $K_{\overline{B}} + [Z]$

The purpose of this section is to establish conditions under which $K_{\overline{B}} + [Z]$ is free and ample. The principle assumption is a local Torelli condition on the triple $(\overline{B}, Z; \Phi)$ (Definition 9.11).

Theorem 9.1. *Assume that the local Torelli condition holds for* $(\overline{B}, Z; \Phi)$*: the bundle map* $\Psi : T_{\overline{B}}(-\log Z) \to \mathrm{Gr}^{-1}_{\mathcal{F}_\mathrm{e}}\mathrm{End}(\mathcal{E}_\mathrm{e})$ *is injective. Then the line bundle* $K_{\overline{B}} + [Z]$ *is nef and big.*

Outline of proof.

1. The Hodge line bundle Λ_e is nef (9.7). And it is big when the period map Φ satisfies generic local Torelli (§7.3.2).
2. Local Torelli for $(\overline{B}, Z; \Phi)$ implies local Torelli for Φ (Lemma 9.12).
3. Lemma 9.18 which asserts that there exists a positive constant ϵ so that $c_1(K_{\overline{B}}+[Z]) \geq \epsilon c_1(\Lambda_\mathrm{e})$.

The local Torelli condition for $(\overline{B}, Z; \Phi)$ enables us to realize $K_{\overline{B}} + [Z]$ as a line subbundle of the pull-back $\mathcal{H} = \Phi^*(\mathbf{H})$ of a homogenous subbundle $\mathbf{H} \to D$ (which descends to $\Gamma\backslash D$). Curvature properties of $\mathbf{H}$ imply that $c_1(\mathcal{H})$ is essentially equivalent to $c_1(\Lambda_\mathrm{e})$ (Lemma 9.17). Lemma 9.18 is then deduced by considering the second fundamental form of $K_{\overline{B}} + [Z] \hookrightarrow \mathcal{H}$. □

Theorem 9.2. *Assume that the local Torelli condition holds for* $(\overline{B}, Z; \Phi)$*: the bundle map* $\Psi : T_{\overline{B}}(-\log Z) \to \mathrm{Gr}^{-1}_{\mathcal{F}_\mathrm{e}}\mathrm{End}(\mathcal{E}_\mathrm{e})$ *is injective. Suppose in addition that the effective cone* $\mathrm{Eff}^1(\overline{B})$ *is finitely generated that the period maps* $\Phi^0_W : Z_W \to \Gamma_W\backslash D^0_W$ *have constant rank. Then there is a well-defined Gauss map* $\mathcal{G}(\Phi^1\big|_{A^0}) : A^0 \to \mathrm{Gr}(r_W, \mathbb{C}^{d_W})$*. The line bundle* $K_{\overline{B}} + [Z]$ *is ample if and only if the Gauss map* $\mathcal{G}(\Phi^1\big|_{A^0})$ *is locally injective.*

The Gauss map is defined in (9.23).

Remark 9.3. The assumption that $\mathrm{Eff}^1(\overline{B})$ is finitely generated is probably unnecessary (Remark 1.31). Here it is a technical convenience: we will show that for each curve $C \subset \overline{B}$, there exists $m_0(C)$ so that $(m\Lambda_\mathrm{e} - [Z]) \cdot C > 0$ for all $m \geq m_0(C)$. Finite generation of $\mathrm{Eff}^1(\overline{B})$ allows us to assume that $m_0(C)$ is independent of C. This will suffice to establish ampleness.

Outline of proof. The proof of Theorem 9.2 takes off from the end of that for Theorem 9.1. What remains is to show that $c_1(K_{\overline{B}} + [Z])$ is positive if and only if the differential of the Gauss map is injective; this is (9.21) and Lemma 9.24. See §9.4.2. □

There is an interesting subtlety in Theorem 9.1, as illustrated by the following[15]

[15] We are indebted to Kang Zuo for bringing Example 9.4 to our attention.

Example 9.4. Let $\overline{\mathcal{A}}_g$ be a toroidal compactification of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties. Then $K_{\overline{B}} + [Z]$ is not ample.

Proof. Following [Mum77], we assume that $\overline{\mathcal{A}}_g$ is smooth, and that $Z = \overline{\mathcal{A}}_g \backslash \mathcal{A}_g$ is a local normal crossing divisor.[16] If $\mathcal{A}_g^*$ is the Satake–Baily–Borel compactification of $\mathcal{A}_g$, then the natural map

$$\pi : \overline{\mathcal{A}}_g \ \to \ \mathcal{A}_g^* \tag{9.5}$$

is a resolution of the singularities of $\mathcal{A}_g^*$. There is an ample line bundle $\mathcal{O}_{\mathcal{A}_g^*}(1) \to \mathcal{A}_g^*$ satisfying $K_{\overline{\mathcal{A}}_g} + Z = \pi^*(\mathcal{O}_{\mathcal{A}_g^*}(1))$ [Mum77, (3.4)]. □

Remark 9.6. The fibres of (9.5) can be identified with the abelian varieties J_I of Theorem 2.27; consequently, the Gauss map (9.23) of Φ^1 on these fibres is constant.

9.1. **Local Torelli for the period map and the Chern form $c_1(\Lambda_{\mathrm{e}})$.** The Chern form $c_1(\Lambda) \in \mathcal{A}^{1,1}_{\overline{B}}$ of the Hodge line bundle $\Lambda \to B$ has the property that

$$c_1(\Lambda)(v, \bar{v}) \ = \ \|\Phi_*(v)\|^2 \,, \tag{9.7}$$

for all $v \in TB$, [Gri70, Proposition 7.15]. The product $\omega^{\dim B}$ is non-negative, and positive at those points $b \in B$ where $\mathrm{d}\Phi_b$ is injective. At infinity $c_1(\Lambda)$ extends to a $(1,1)$–current $c_1(\Lambda_{\mathrm{e}})$ on $\overline{B}$ where it represents the Chern class of the extended $\Lambda_{\mathrm{e}} \to \overline{B}$ [CKS86]. In particular, the line bundle Λ_{e} is big if and only if the period map $\Phi : B \to \Gamma\backslash D$ satisfies generic local Torelli.

The restriction of $c_1(\Lambda_{\mathrm{e}})$ to Z_I^* is well-defined, and as an element of $\mathcal{A}^{1,1}_{Z_I^*}$ and represents the Chern class of the Hodge line bundle $\Lambda_I = \Lambda_{\mathrm{e}}|_{Z_I^*}$ of the induced polarized VHS (3.7), [GGLR20, Theorem 1.4.1]. (More generally, one may show that ω is well-defined on Z_W, [GG20].) If $v \in T_b Z_I^*$, then (9.7) implies

$$c_1(\Lambda_{\mathrm{e}})(v, \bar{v}) \ = \ \|\Phi^0_{I,*}(v)\|^2 \,. \tag{9.8}$$

In particular, Λ_{e} is nef. And the differential of the period map $\Phi_I : Z_I^* \to \Gamma_I\backslash D$ is every where injective if and only if $c_1(\Lambda_{\mathrm{e}})|_{Z_I^*}$ is positive. Thus, if $C \subset \overline{B}$ is an irreducible curve and $\Phi^0(C)$ is not a point, then

$$\Lambda_{\mathrm{e}} \cdot C \ = \ \deg \Lambda_{\mathrm{e}}|_C \ = \ \int_C c_1(\Lambda_{\mathrm{e}}) \ > 0 \,.$$

[16] The actual singularities of $\overline{\mathcal{A}}_g$ are mild quotient singularities that do not affect the argument. Similarly, the fact that Z is only a local, rather than a global, normal crossing divisor makes no essential difference.

9.2. **Proof of Proposition 2.32.** It suffices to show that there exists m_0 so that $(\Lambda_{\mathrm{e}} - [Z]) \cdot C > 0$ for all curves $C \subset \overline{B}$ and $m \geq m_0$. Without loss of generality, we may assume that C is an irreducible curve.

If the image $\Phi^0(C)$ is also a curve, then §9.1 implies that $\Lambda_{\mathrm{e}} \cdot C > 0$. So we will have $(m\Lambda_{\mathrm{e}} - [Z]) \cdot C > 0$ when $m \gg 0$. Now suppose that $C \subset A^0$ is contained in a Φ^0–fibre. Again, §9.1 implies that $\Lambda_{\mathrm{e}} \cdot C = 0$. However, the hypothesis that $\mathrm{d}\Phi^1_W$ is injective and §2.5 imply that $\mathcal{N}^*_{Z/\overline{B}}\big|_C$ is ample. In particular, $-[Z] \cdot C > 0$. The proposition now follows from Remark 9.3. $\square$

9.3. **A local Torelli condition for** $(\overline{B}, Z; \Phi)$**.** One of the technical hypothesis needed for our applications is a local Torelli condition for $(\overline{B}, Z; \Phi)$. The condition is expressed in terms of the extension of the Gauss–Manin connection (Lemma 9.12).

Deligne's extension

$$\begin{array}{ccc} \mathcal{F}^p_{\mathrm{e}} & \subset & \mathcal{V}_{\mathrm{e}} \\ & & \downarrow \\ & & \overline{B} \end{array}$$

of the Hodge bundles (1.1a) is reviewed in §C. Let

$$\mathcal{E}^p_{\mathrm{e}} \ = \ \mathcal{F}^p_{\mathrm{e}}/\mathcal{F}^{p+1}_{\mathrm{e}} \ = \ \mathrm{Gr}^p_{\mathcal{F}_{\mathrm{e}}} ,$$

and consider the associated graded vector space

$$\mathcal{E}_{\mathrm{e}} \ = \ \oplus\, \mathcal{E}^p_{\mathrm{e}} .$$

The Gauss–Manin connection induces a bundle map

$$\Psi : T_{\overline{B}}(-\log Z) \ \to \ \mathrm{Gr}^{-1}_{\mathcal{F}_{\mathrm{e}}}(\mathrm{End}(\mathcal{E}_{\mathrm{e}})) . \tag{9.9}$$

We review the definition of Ψ in §§9.3.1–9.3.2.

Remark 9.10. At points $b \in B$ the map Ψ is the differential of the period map

$$\mathrm{d}\Phi_b \ = \ \Psi|_{T_{B,b}} .$$

Definition 9.11. We say that *local Torelli condition holds for* $(\overline{B}, Z; \Phi)$ when (9.9) is injective.

Lemma 9.12 is a generalization of Remark 9.10.

Lemma 9.12. *The local Torelli condition holds for* $(\overline{B}, Z; \Phi)$ *if and only if*

(i) *The differential* $\mathrm{d}\Phi^1_I : T(Z^*_I) \to T(\Gamma_I \backslash D^1_I)$ *is injective for all* I*.*

(ii) *The* $\{N_i \mid i \in I\}$ *are linearly independent for all* I*.*

The lemma is proved in §9.3.4.

9.3.1. *Maurer–Cartan form.* The composition of the lift (7.14) with map (3.12) defines

$$X \circ \widetilde{\Phi}_{A^1} : \tilde{\mathcal{O}}^1 \ \to \ \mathfrak{f}^\perp .$$

It will be convenient to set

$$\xi = \exp(X \circ \widetilde{\Phi}_{A^1}) : \mathcal{S} \ \to \ \exp(\mathfrak{f}^\perp) .$$

Keep in mind that, since $\mathfrak{f}^\perp$ is a nilpotent subalgebra, the exponential defines a biholomorphism $\mathfrak{f}^\perp \simeq \exp(\mathfrak{f}^\perp)$ so that $X \circ \widetilde{\Phi}_{A^1}$ and ξ carry the equivalent information.

Fix a MHS (W,F) arising along A^1. Fix a basis $\{v_j\}$ of V so that every v_j is contained in some $V^{p_j,q_j}_{W,F}$; equivalently, $v_j \in F^{p_j}$ but $v_j \notin F^{p_j+1}$, and $v_j \in W_{p_j+q_j}$ but $v_j \notin W_{p_j+q_j-1}$. Then

$$\phi_j \ = \ \xi \cdot v_j$$

defines a framing of $\mathcal{V}_\mathrm{e}$ that is adapted to the Hodge filtration $\mathcal{F}^p_\mathrm{e} \subset \mathcal{V}_\mathrm{e}$. The key point here is that Proposition 7.1 implies that the familiar construction of §C.2.2 applies to this slightly more general setting. So we may identify the $\{\phi_j \mid p_j = p\}$ with a holomorphic framing of $\mathcal{E}^p_\mathrm{e}$.

The pullback

$$\theta \ = \ \xi^{-1}\mathrm{d}\xi$$

under ξ of the Maurer–Cartan form on $\exp(\mathfrak{f}^\perp) \subset G_\mathbb{C}$ is $\mathfrak{f}^\perp$–valued. Let ξ^i_j be the matrix coefficients of ξ with respect to the basis $\{v_j\}$; that is, $\xi \cdot v_j = \xi^i_j v_i$. Likewise, let $\theta^i_j = (\xi^{-1})^i_k \mathrm{d}\xi^k_j$ be the matrix entries of

$$\theta = \theta^i_j \, v_i \otimes v^j .$$

The Gauss–Manin connection satisfies

$$\nabla \phi_j \ = \ \theta^i_j \otimes \phi_i \, ,$$

and

$$\Psi|_{\overline{\mathcal{O}}^1} \ = \ \theta^i_j \, \phi_i \otimes \phi^j . \tag{9.13}$$

It is instructive to review the proof of the well-known

Lemma 9.14. *The pull-back of the Maurer-Cartan form is a log 1-form; that is,*

$$\theta^i_j \in \Omega^1_{\overline{\mathcal{O}}^1}(\log Z \cap \overline{\mathcal{O}}^1) .$$

This is done in §9.3.3. First we review the IPR in this context.

9.3.2. *Horizontality.* Since θ takes value in $\mathfrak{f}^\perp = \mathfrak{g}^{-,\bullet}_{W,F}$, we have

$$\theta^i_j \;=\; 0 \quad \forall \quad p_i - p_j > 0\,.$$

Horizontality asserts that

$$\theta^i_j \;=\; 0 \quad \forall \quad p_i - p_j \neq -1\,. \tag{9.15a}$$

And this implies

$$\theta^i_j \;=\; \mathrm{d}(X \circ \widetilde{\Phi}_{A^1})^i_j \quad \forall \quad p_i - p_j = -1\,, \tag{9.15b}$$

with X^i_j the matrix entries of X with respect to the basis $\{v_j\}$ (defined by $Xv_j = X^i_j v_i$); in the notation of Remark 3.4, this is equivalently the statement that

$$\theta \;=\; \mathrm{d}(X \circ \widetilde{\Phi}_{A^1})^{-1,\bullet}\,. \tag{9.15c}$$

Letting $\theta^{-1,q}$ denote the component of θ taking value in $\mathfrak{g}^{-1,q}_{W,F}$, we have

$$\theta \;=\; \theta^i_j\, v_i \otimes v^j \;=\; \textstyle\sum \theta^{-1,q}\,,$$

and

$$\theta^{-1,q} \;=\; \sum_{\substack{p_i - p_j = -1\\ q_i - q_j = q}} \theta^i_j\, v_i \otimes v^j\,.$$

As a consequence we obtain (9.9).

9.3.3. *Proof of Lemma 9.14.* It follows from (9.15) that it suffices to prove

$$\mathrm{d}(X \circ \widetilde{\Phi}_{A^1})^{-1,\bullet} \;\in\; \Omega^1_{\overline{\mathcal{O}}^1}(\log Z \cap \overline{\mathcal{O}}^1)\,. \tag{9.16}$$

The 1-form $\mathrm{d}(X \circ \widetilde{\Phi}_{A^1})^{-1,\bullet}$ is the differential of the horizontal component of the period matrix. In particular, the $\mathrm{d}\varepsilon_\mu$ of §7.3.1 are the matrix entries of $\mathrm{d}(X \circ \widetilde{\Phi}_{A^1})^{-1,\bullet}$. So (9.16) is equivalent to (7.17). $\square$

9.3.4. *Proof of Lemma 9.12.* The coordinates $\mathrm{d}\varepsilon_\mu$ of the map Ψ_1 defined in (7.18) are the horizontal coordinates of $\Psi|_{\overline{\mathcal{O}}^1}$. In particular, Lemma 7.19 is equivalent to Lemma 9.12. $\square$

9.4. **Geometric properties of $K_{\overline{B}} + [Z]$.**

9.4.1. *Geometric implications of local Torelli for* $(\overline{B}, Z; \Phi)$. Suppose that the local Torelli condition holds for $(\overline{B}, Z; \Phi)$ (Definition 9.11). Then we may identify $T_{\overline{B}}(-\log Z)$ with a subbundle of $\mathrm{Gr}^{-1}_{\mathcal{F}_{\mathrm{e}}}\mathrm{End}(\mathcal{E}_{\mathrm{e}})$, and

$$(K_{\overline{B}} + [Z])^* \;=\; \textstyle\bigwedge^{\dim B} T_{\overline{B}}(-\log Z)$$

with subbundle of

$$\mathcal{H} \;:=\; \textstyle\bigwedge^{\dim B} \mathrm{Gr}^{-1}_{\mathcal{F}_{\mathrm{e}}}\mathrm{End}(\mathcal{E}_{\mathrm{e}})\,.$$

The singular metric on the Hodge bundles $\mathcal{E}^p_{\mathrm{e}} \to \overline{B}$ induces singular metrics on $\mathrm{Gr}^{-1}_{\mathcal{F}_{\mathrm{e}}}\mathrm{End}(\mathcal{E}_{\mathrm{e}})$ and $\mathcal{H}$. The singularities, curvatures and Chern forms of these metrics are much studied [CKS86, Kol87, GG20, GGLR20]. Over B the metrics and Chern forms are smooth; the Chern forms extend to currents on $\overline{B}$ where they represent the extended vector bundles. (As already discussed in §9.1, the analogous statements hold for the Hodge line bundle.)

Let $\Theta_{\mathcal{H}}$ denote the curvature matrix of $\mathcal{H}$, and

$$c_1(\mathcal{H}) \;=\; \tfrac{\mathbf{i}}{2\pi}\mathrm{tr}\,\Theta_{\mathcal{H}}$$

the first Chern form. Let

$$c_1(\Lambda_{\mathrm{e}}) \;=\; \tfrac{\mathbf{i}}{2\pi}\Theta_{\Lambda_{\mathrm{e}}}$$

denote the Chern form of the Hodge line bundle $\Lambda_{\mathrm{e}} \to \overline{B}$ (§9.1).

The line bundle $(K_{\overline{B}} + [Z])^*$ inherits a singular metric from its containment in $\mathcal{H}$. Let

$$c_1(K_{\overline{B}} + [Z]) \;=\; \tfrac{\mathbf{i}}{2\pi}\Theta_{K_{\overline{B}}+[Z]}$$

denote the Chern form of $K_{\overline{B}} + [Z]$. We will see that $c_1((K_{\overline{B}} + [Z])^*)$ is related to $c_1(\mathcal{H})$ by the second fundamental form of $(K_{\overline{B}} + [Z])^* \hookrightarrow \mathcal{H}$ (Lemma 9.18), and this will give us control over the $c_1(K_{\overline{B}} + [Z])$.

Lemma 9.17. *The curvature form $\Theta_{\mathcal{H}}$ is nonpositive and there exist positive constants ϵ, ϵ' so that*

$$\epsilon\, c_1(\Lambda_{\mathrm{e}}) \;\le\; -c_1(\mathcal{H}) \;\le\; \epsilon'\, c_1(\Lambda_{\mathrm{e}})\,.$$

The lemma is proved in §D.4.1.

Lemma 9.18. *There exists a non-negative $\tau \in \mathcal{A}^{1,1}_B$ that extends to a current on $\overline{B}$, and a positive constant ϵ so that*

$$\epsilon\, c_1(\Lambda_{\mathrm{e}}) \,+\, \tau \;\le\; c_1(K_{\overline{B}} + [Z])\,.$$

Sketch of Proof. The lemma is proved in §D.4.2. Here we outline the underlying geometric ideas. Over B the curvature forms $\Theta_{\mathcal{H}}$ and $\Theta_{(K_{\overline{B}}+[Z])^*}$ are related by the second fundamental form

$$\Upsilon : \mathcal{A}^0_B((K_{\overline{B}} + [Z])^*) \;\to\; \mathcal{A}^{1,0}_B(\mathcal{H}/(K_{\overline{B}} + [Z])^*)\,, \tag{9.19}$$

which measures the failure of the Chern connection on $\mathcal{H}$ to preserve the subbundle $(K_{\overline{B}}+[Z])^*$. We have

$$\Theta_{\mathcal{H}}\big|_{(K_{\overline{B}}+[Z])^*} \;=\; \Theta_{(K_{\overline{B}}+[Z])^*} \;+\; (\Upsilon,\Upsilon)\,, \tag{9.20}$$

with (Υ,Υ) a $(1,1)$–form constructed from Υ and the Hermitian metric. Setting

$$\tau \;=\; \tfrac{\mathbf{i}}{2\pi}(\Upsilon,\Upsilon)\,,$$

we have

$$\tfrac{\mathbf{i}}{2\pi}\,\Theta_{\mathcal{H}}\big|_{(K_{\overline{B}}+[Z])^*} \;=\; -c_1(K_{\overline{B}}+[Z]) \,+\, \tau\,. \tag{9.21}$$

It then remains to show that there exists a positive constant ϵ so that

$$\epsilon\, c_1(\Lambda_{\mathrm{e}}) \;\le\; -\tfrac{\mathbf{i}}{2\pi}\,\Theta_{\mathcal{H}}\big|_{(K_{\overline{B}}+[Z])^*}\,.$$

We will see that this is a consequence of homogeneity, the structure of the curvature of vector bundles on D and the IPR. $\square$

Note that (9.8), Lemma 9.17 and (9.21) imply

$$c_1(K_{\overline{B}}+[Z])\big|_{A^0} \;=\; \tau\big|_{A^0}\,. \tag{9.22}$$

So to establish the ampleness of $K_{\overline{B}}+[Z]$ we will need to show that $\tau|_{A^0}$ is positive. For this we make the simplifying assumption (expected to be unnecessary, Remark 1.31) that the map $\Phi^0_W : Z_W \to \wp^0_W \subset \Gamma_W\backslash D^0_W$ has constant rank. Then $\Phi^0_W : Z_W \to \wp^0_W$ is a fibration with fibre A^0. Recall that the restriction $\Phi^1\big|_{A^0}$ takes value in an compact torus T_W (Theorem 2.27). The local Torelli condition for $(\overline{B},Z;\Phi)$ implies that the differential of $\Phi^1\big|_{A^0}$ is injective (Lemma 9.12). So we have a Gauss map

$$A^0 \;\to\; \mathrm{Gr}(r_W, T(T_W))\,.$$

Since T_W is a torus, we may translate each tangent space $T_x(T_W)$ to a fixed $T_e(T_W)=\mathbb{C}^{d_W}$. In this way we obtain a Gauss map

$$\mathcal{G}(\Phi^1\big|_{A^0}) : A^0 \;\to\; \mathrm{Gr}(r_W,\mathbb{C}^{d_W}) \tag{9.23}$$

to a fixed Grassmannian.

Lemma 9.24. *Suppose that the map $\Phi^0_W : Z_W \to \wp^0_W \subset \Gamma_W\backslash D^0_W$ has constant rank. Then the restriction $\Upsilon|_{A^0}$ may be identified with the differential of the Gauss map* (9.23)*. In particular, the restriction $c_1(K_{\overline{B}}+[Z])\big|_{A^0}$ of the Chern form to A^0 is well defined, and it is positive if and only if the differential of the Gauss map $\mathcal{G}(\Phi^1\big|_{A^0})$ is injective (equivalently, the Gauss map is finite to one).*

Sketch of Proof. The lemma is proved in §D.4.3. Given (9.22), the essential content of the argument is that the restriction of Υ to the the fibre A^0 may be identified with the differential of the Gauss map (9.23). □

9.4.2. *Proof of Theorem 9.2.* Local Torelli for $(\overline{B}, Z; \Phi)$ implies generic local Torelli for Φ (Lemma 9.12). It follows from §9.1 and Lemma 9.18 that $K_{\overline{B}} + [Z]$ is nef and big.

It remains to establish ampleness. Following Remark 9.3, we need to show that

$$C \cdot (K_{\overline{B}} + [Z]) \;>\; 0\,, \tag{9.25}$$

for every irreducible curve $C \subset \overline{B}$. There are two cases to consider.

Case 1: The image $\Phi^0(C)$ is a curve. Lemma 9.18 and §9.1 imply

$$C \cdot (K_{\overline{B}} + [Z]) \;=\; \int_{C \cap B} c_1(K_{\overline{B}} + [Z]) \;\geq\; \epsilon \int_C c_1(\Lambda_{\mathrm{e}}) \;>\; 0\,,$$

yielding the desired (9.25).

Case 2: The image $\Phi^0(C)$ is a point. This is the most interesting case. We necessarily have $C \subset A^0 \subset Z$. Then Lemma 9.24 yields

$$C \cdot (K_{\overline{B}} + [Z]) \;=\; \int_C c_1(K_{\overline{B}} + [Z]) \;>\; 0\,,$$

establishing the desired (9.25). □

Appendix A. Geometric interpretation of some extension data

Thus far our study of extension data has focused on its Lie theoretic structure (as the discrete quotient of a homogeneous space, §2.1). However for applications to moduli one is particularly interested in geometric interpretations. (See [Car87] for a very nice overview of both geometric interpretations and applications of extension data.) Here we discuss geometric interpretations of extension data in the cases of a nodal curves (§A.3), and surfaces with a double curve (§§A.4–A.5). Higher level extension data in the Hodge–Tate case is discussed in §A.6. We begin with a brief summary of the extension data for mixed Hodge structures (§A.1), and limiting mixed Hodge structures (§A.2).

A.1. **Extension data for a mixed Hodge structure.** We consider extension data for MHS (V, W, F), where we assume the existence of a lattice $V_{\mathbb{Z}}$ in the $\mathbb{Q}$–vector space V. The weight filtration will be

$$\{0\} \;\subset\; W_0 \;\subset\; W_1 \;\subset \cdots \subset\; W_{2n} \;=\; V\,.$$

Let

$$H^a \;=\; F^\bullet(\mathrm{Gr}^W_a) \;\in\; D^0_W$$

denote the graded quotient $\mathrm{Gr}^W_a = W_a/W_{a-1}$ equipped with the pure, weight a Hodge structure induced by F; we assume throughout §A that this structure is fixed. Recall (2.5) and Definition 2.6.[17] As discussed in §2.1, we have tower of holomorphic maps

$$\Gamma_W\backslash\delta_{W,F} \;=\; \Gamma_W\backslash\delta^{2n}_{W,F} \;\twoheadrightarrow\; \Gamma_W\backslash\delta^{2n-1}_{W,F} \;\twoheadrightarrow \cdots \twoheadrightarrow\; \Gamma_W\backslash\delta^1_{W,F} \;\twoheadrightarrow\; \delta^0_{W,F}\,,$$

and the fibre of $\Gamma_W\backslash\delta^a_{W,F} \twoheadrightarrow \Gamma_W\backslash\delta^{a-1}_{W,F}$ is the *level a extension data*. Here $\delta^0_{W,F}$ is identified with the point $H^\bullet \in D^0_W$ (upon which Γ_W acts trivially by definition), so that $\Gamma_W\backslash\delta^1_{W,F}$ is the *level one extension data*. The latter is precisely the set of extensions

$$0 \;\to\; H^a \;\to\; W_{a+1}/W_{a-1} \;\to\; H^{a+1} \;\to\; 0\,;$$

that is,

$$\begin{aligned}\Gamma_W\backslash\delta^1_{W,F} \;&=\; \bigoplus_{a=1}^{2n} \mathrm{Ext}^1_{\mathrm{MHS}}(H^a,H^{a-1})\\ &=\; \bigoplus_{a=1}^{2n} \frac{\mathrm{Hom}_{\mathbb{C}}(H^a,H^{a-1})}{F^0\mathrm{Hom}_{\mathbb{C}}(H^a,H^{a-1}) \,+\, \mathrm{Hom}_{\mathbb{Z}}(H^a,H^{a-1})}\,.\end{aligned} \tag{A.1}$$

The level ≤ 2 extension data $\Gamma_W\backslash\delta^2_{W,F}$ consists[18] of level one extension data, plus the level two extension data

$$0 \;\to\; H^{a-2} \;\to\; W_a/W_{a-3} \;\to\; W_a/W_{a-2} \;\to\; 0\,.$$

It fibres over the level one extension data with fibre

$$\bigoplus_{a=2}^{2n} \frac{\mathrm{Hom}_{\mathbb{C}}(H^a,H^{a-2})}{F^0\mathrm{Hom}_{\mathbb{C}}(H^a,H^{a-2}) \,+\, \mathrm{Hom}_{\mathbb{Z}}(H^a,H^{a-2})}$$

And so on ...

A.1.1. As observed in §6.3 and §7.1.2, the fibre of the quotient map from the level $\le a$ extension data to the level $\le a-1$ extension data is an abelian complex Lie group $\Lambda^a\backslash\mathbb{L}^a$ arising as the quotient of a complex vector space

$$\mathbb{L}^a \;=\; P^{-a-1}_W\backslash(P^{-a}_W\cdot F) \;\simeq\; \frac{W_{-a}\mathrm{End}(V_{\mathbb{C}})}{W_{-a-1}\mathrm{End}(V_{\mathbb{C}}) + F^0W_{-a}(\mathrm{End}(V_{\mathbb{C}})}$$

by a discrete subgroup

$$\Lambda^a \;=\; \frac{\Gamma_W \,\cap\, P^{-a}_W}{\Gamma_W \,\cap\, P^{-a-1}_W}\,.$$

[17]The discussion that follows may equally well apply to any subgroup of Γ_W; such as Γ_{A^0}, as in §6, and Γ_{A^1}, as in §7.

[18]There is no analog of (A.1) for extension data of level $a \ge 2$: given any two MHS M_1, M_2, we have $\mathrm{Ext}_{\mathrm{MHS}}(M_1,M_2) =$ for all $a \ge 2$.

If $\mathrm{End}(V_{\mathbb{C}}) = \oplus \mathrm{End}(V)^{p,q}_{W,F}$ is the Deligne splitting induced by the MHS, then

$$\mathbb{L}^a \simeq \bigoplus_{p<0} \mathrm{End}(V)^{p,-a-p}_{W,F} .$$

In particular,

$$\mathrm{fibre}\{\Gamma_W \backslash \delta^a_{W,F} \twoheadrightarrow \Gamma_W \backslash \delta^{a-1}_{W,F}\} = \Lambda^a \backslash \mathbb{L}^a \simeq \mathbb{T}^a_W \times (\mathbb{C}^*)^d ,$$

with $\mathbb{T}^a_W$ a compact complex torus.[19]

A.1.2. In the case of the level one extension data $\Gamma_W \backslash \delta^1_{W,F} = \Lambda^1 \backslash \mathbb{L}^1$, we have

$$\mathbb{L}^1 \simeq \mathrm{End}(V)^{-1,0}_{W,F} \oplus \mathrm{End}(V)^{-2,1}_{W,F} \oplus \cdots \oplus \mathrm{End}(V)^{-n,n-1}_{W,F} .$$

The factor $(\mathbb{C}^*)^d$ is trivial, and we have

$$\Gamma_W \backslash \delta^1_{W,F} = \Lambda^1 \backslash \mathbb{L}^1 = \mathbb{T}^1_W .$$

Let $\mathbb{I}_{W,F} \subset \Lambda^1 \backslash \mathbb{L}^1$ be the image of $\mathrm{End}(V)^{-1,0}_{W,F}$ under the projection $\mathbb{L}^1 \twoheadrightarrow \Lambda^1 \backslash \mathbb{L}^1$. Then

$$\mathbb{I}_{W,F} = \mathbb{T}^{-1,0}_{W,F} \times \mathbb{C}^a \times (\mathbb{C}^*)^b ,$$

with $\mathbb{T}^{-1,0}_{W,F}$ a compact torus (possibly trivial).

Remark A.2. In the case that D is Hermitian, we have $\mathbb{I}_{W,F} = \mathbb{T}^1_W$.

Suppose we have a VMHS defined over a connected complex variety A. If the associated graded quotient of the variation is constant, then we have a map $A \to \Lambda^1 \backslash \mathbb{L}^1$ to the level one extension data. The infinitesimal period relation implies that the map takes value in a translate of $\mathbb{I}_{W,F}$. Redefining F is necessary, we may assume that the map takes value in $\mathbb{I}_{W,F}$. If in addition A is compact, then the map takes value in a translate of $\mathbb{T}^{-1,0}_{W,F}$.

A.1.3. Topological line bundles over $\mathbb{T}^1_W = \Lambda^1 \backslash \mathbb{L}^1$ are uniquely specified by their Chern classes. Noting that $\Lambda^1 = \mathrm{Gr}^W_{-1}\mathrm{End}_{\mathbb{Z}}(V)$, we see that these line bundles are indexed by $\bigwedge^2 \mathrm{Gr}^W_{-1}\mathrm{End}_{\mathbb{Z}}(V)^*$. The Lie bracket defines a map

$$\mathrm{Gr}^W_{-1}\mathrm{End}_{\mathbb{Z}}(V) \otimes \mathrm{Gr}^W_{-1}\mathrm{End}_{\mathbb{Z}}(V) \to \mathrm{Gr}^W_{-2}\mathrm{End}_{\mathbb{Z}}(V)$$

induces

$$\mathrm{Gr}^W_{-2}\mathrm{End}_{\mathbb{Z}}(V)^* \to \textstyle\bigwedge^2 \mathrm{Gr}^W_{-1}\mathrm{End}_{\mathbb{Z}}(V)^* \simeq H^2(\mathbb{T}^1_W, \mathbb{Z}) . \tag{A.3}$$

The elements in $\mathrm{Gr}^W_{-2}\mathrm{End}_{\mathbb{Z}}(V)^*$ that map to $\mathrm{Hg}^1(\mathbb{T}^1_W) = H^{1,1}(\mathbb{T}^1_W) \cap H^2(\mathbb{T}^1_W, \mathbb{Z})$ give rise to holomorphic line bundles on $\mathbb{T}^1_W$ that are well-defined up to translation. (The line bundles $\mathcal{L}_M \to T_W$ of Theorem 2.27 are examples of such.)

[19] When we replace Γ_W with a subgroup there may also be a $\mathbb{C}^{d_1}$ factor, cf. §§6, 7.

Notice that $\mathrm{Gr}^W_{-2}\mathrm{End}_{\mathbb{Z}}(V)^*$ is naturally isomorphic with $(\Lambda^2)^* = H^1(\Lambda^2\backslash\mathbb{L}^2,\mathbb{Z})$. The map (A.3) may be identified with the the transgression mapping

$$H^0\left(\mathbb{T}^1_W\,,\, H^1(\Lambda^2\backslash\mathbb{L}^2)\right) \xrightarrow{d_2} H^2(\mathbb{T}^1_W)$$

in the Leray spectral sequence for the fibre bundle

$$\begin{array}{ccl} \Lambda^2\backslash\mathbb{L}^2 \hookrightarrow & \Gamma_W\backslash\delta^2_{W,F} & \\ & \downarrow & \\ & \Gamma_W\backslash\delta^1_{W,F} & = \mathbb{T}^1_W\,. \end{array}$$

A.2. **Extension data for a limiting mixed Hodge structure.** Fix a LMHS (V,Q,W,F,N) as in §3.2.4.

A.2.1. We have $N : W_a \to W_{a-1}$ and

$$N^a : \mathrm{Gr}^W_{n+a} \xrightarrow{\simeq} \mathrm{Gr}^W_{n-a}$$

is an isomorphism of vector spaces. The weight filtration is Q–isotropic

$$Q(W_a, W_{2n-1-a}) \ = \ 0\,.$$

It follows that Q induces a nondegenerate bilinear pairing $\mathrm{Gr}^W_{n+a}\otimes\mathrm{Gr}^W_{n-a}\to\mathbb{Q}$. This in turn defines an isomorphism

$$H^a \ \simeq \ H^a(a-n)^*\,.$$

Then

$$N^{n-a} \ \in \ H^a(a-n)^* \otimes H^a \ \simeq \ H^a\otimes H^a\,.$$

In fact,

$$N^{n-a} \ \in \ \mathrm{Sym}^2 H^a \ \simeq \ \mathrm{Sym}^2 H^a(a-n)^*\,.$$

A.2.2. Regard N as an element of $\mathrm{End}(\oplus_a \mathrm{Gr}^W_a)$. Define $Y\in\mathrm{End}(\oplus_a\mathrm{Gr}^W_a)$ by specifying that Y act on Gr^W_a by the scalar $a-n$. There is a unique $M\in\mathrm{End}(\oplus_a\mathrm{Gr}^W_a)$ so that

$$[M,N] \ = \ Y\,,\quad [Y,M] \ = \ 2M\,,\quad [Y,N] \ = \ -2Y\,;$$

in particular, $\{M,Y,N\}$ spans a subalgebra $\mathfrak{sl}_2\subset\mathrm{End}(\oplus_a\mathrm{Gr}^W_a)$, [Kos59, Corollary 3.5].

We may decompose $\oplus_a\mathrm{Gr}^W_a$ in to a direct sum of irreducible $\mathfrak{sl}_2$–modules. The summands are "N-strings". This decomposition gives Gr^W_a the structure of a direct sum of polarized Hodge structures.

A.2.3. The action of N on V induces an action on $\mathrm{End}(V)$ mapping

$$N : \mathrm{End}(V)^{p,q}_{W,F} \ \to \ \mathrm{End}(V)^{p-1,q-1}_{W,F}$$

and

$$N : \mathrm{Gr}^W_a \mathrm{End}(V) \ \to \ \mathrm{Gr}^W_{a-2}\mathrm{End}(V)\,.$$

Recall the notations of §A.1.2, and let $\mathbb{I}_{N,F} \subset \mathbb{I}_{W,F}$ the image of $\ker\{N : \mathrm{End}(V)^{-1,0}_{W,F} \to \mathrm{End}(V)^{-2,-1}_{W,F}\}$ under the projection $\mathbb{L}^1 \twoheadrightarrow \Lambda^1\backslash\mathbb{L}^1$. Then

$$\mathbb{I}_{N,F} \ = \ \mathbb{J}^{-1,0}_{N,F} \times \mathbb{C}^\alpha \times (\mathbb{C}^*)^\beta\,,$$

with $\mathbb{J}^{-1,0}_{N,F} \subset \mathbb{T}^{-1,0}_{W,F}$ a compact torus. As noted in §6.10, the fact that the MHS (W,F) is polarized by N implies that $\mathbb{J}^{-1,0}_{N,F}$ is an abelian variety.

Remark A.4. If D is Hermitian symmetric, then $\mathbb{J}^{-1,0}_{N,F} = \mathbb{T}_W$ is the full level one extension data $\Lambda^1\backslash\mathbb{L}^1 = \Gamma_W\backslash\delta^1_{W,F}$.

If the VMHS over A discussed in §A.1.2 is a variation of *limiting* mixed Hodge structure, then the level one extension data map $A \to \Lambda^1\backslash\mathbb{L}^1$ takes value in $\mathbb{I}_{N,F} \subset \mathbb{I}_{W,F}$. And if A is compact, then the map takes value in a translate of the abelian variety $\mathbb{J}^{-1,0}_{N,F} \subset \mathbb{T}^{-1,0}_{W,F}$.

A.3. **The $n = 1$ case of nodal curves.** Let C be an irreducible curve with h^0 nodes $\{r_i\}_{i=1}^{h^0}$. Let $\pi : \widehat{C} \to C$ denote the normalization, and let $\pi^{-1}(r_i) = \{p_i, q_i\}$ denote the preimages of the nodes. A smoothing deformation of C produces a LMHS (V,Q,W,F,N), where

$$N \ = \ N_1 + \cdots + N_{h^0}\,,$$

and N_i corresponds to smoothing the i-th node $r_i \in C$.

We fix an ordering of $\{p_i, q_i\} \subset \widehat{C}$.[20] The associated $H^a = \mathrm{Gr}^W_a$, $a = 0,1,2$, are

$$H^0_{\mathbb{Z}} \ \simeq \ \mathbb{Z}^{h^0}\,, \quad H^1_{\mathbb{Z}} \ \simeq \ H^1(\widehat{C},\mathbb{Z})\,, \quad H^2_{\mathbb{Z}} \ \simeq \ H^0_{\mathbb{Z}}(-1) \ \simeq \ \mathbb{Z}^{h^0}\,.$$

We may fix a basis of $\oplus_a \mathrm{Gr}^W_a = H^2 \oplus H^1 \oplus H^0$ that respects this direct sum, and with respect to which

$$Q \ = \ \left[\begin{array}{ccc} 0 & 0 & I_{h^0} \\ 0 & \hat{Q} & 0 \\ -I_{h^0} & 0 & 0 \end{array}\right],$$

with $\hat{Q}$ the intersection form on $H^1_{\mathbb{Z}}$, and

$$N_i \ = \ \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \nu_i & 0 & 0 \end{array}\right],$$

[20] This may require that we take a branched cover for the family of curves.

where ν_i is the square matrix whose only nonzero entry is the i-th diagonal entry. We have

$$W_0 \simeq \frac{H^0(\{r_i\})}{H^0(\widehat{C})},$$

with $H^0(\widehat{C}) \to H^0(\{r_i\})$ the signed restriction map,

$$W_1/W_0 \simeq H^1(\widehat{C}) \quad \text{and} \quad W_2/W_1 \simeq \ker\{H^0(\{r_i\})(-1) \to H^2(\widehat{C})\}.$$

A neighborhood of C in the corresponding stratum of moduli is swept out by varying $\widehat{C}$ and the $\{p_i, q_i\}$. Restricting to a Φ^0–fibre A corresponds to fixing $\widehat{C}$. On that fibre, the level one extension data is $\mathrm{Ext}^1_{\mathrm{MHS}}(H^1, H^0) \oplus \mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-1), H^1)$. Setting $D = \cup\{p_i, q_i\} \subset \widehat{C}$, the group $\mathrm{Ext}^1_{\mathrm{MHS}}(H^1, H^0)$ parameterizes the extension data in the sequence

$$\begin{tikzcd} 0 \arrow[r] & W_0 \arrow[r] \arrow[d, hook] & W_1 \arrow[r] \arrow[d, equal] & \mathrm{Gr}^W_1 \arrow[r] \arrow[d, equal] & 0 \\ & H^0(D) \arrow[r] & H^1(\widehat{C}, D) \arrow[r] & H^1(\widehat{C}) \arrow[r] & 0\,. \end{tikzcd} \tag{A.5}$$

It is

$$\frac{\mathrm{Hom}_{\mathbb{C}}(H^1, H^0)}{F^0\mathrm{Hom}_{\mathbb{C}}(H^1, H^0) + \mathrm{Hom}_{\mathbb{Z}}(H^1, H^0)} \simeq (H^{0,1}/H^1_{\mathbb{Z}}) \otimes H^0_{\mathbb{Z}} \simeq \bigoplus_1^{h^0} J(\widehat{C}),$$

where $J(\widehat{C})$ is the Jacobian variety of $\widehat{C}$ and h^0 is the rank of $H^0_{\mathbb{Z}}$. Then

$$\Phi^1(C) = \sum_i \mathrm{AJ}_{\widehat{C}}(p_i - q_i) \in \bigoplus_1^{h^0} J(\widehat{C}).$$

In the classical formulation using differential forms, we have

$$J(\widehat{C}) = H^{0,1}/H^1_{\mathbb{Z}} \simeq H^0(\Omega^1_{\widehat{C}})^*/H_1(\widehat{C}, \mathbb{Z}).$$

Given $\omega \in H^0(\Omega^1_{\widehat{C}})$ we choose a path γ with $\partial\gamma = \sum p_i - q_i$. Then Φ^1 is given by the map $\omega \mapsto \int_\gamma \omega$ modulo periods.

The group $\mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-1), H^1)$ parameterizes the extension data in the sequence

$$\begin{tikzcd} 0 \arrow[r] & W_1/W_0 \arrow[r] \arrow[d, equal] & W_2/W_0 \arrow[r] \arrow[d, equal] & \mathrm{Gr}^W_2 \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & H^1(\widehat{C}) \arrow[r] & H^1(\widehat{C}\backslash D) \arrow[r] & H^0(D)(-1)\,. \end{tikzcd}$$

We have

$$\mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-1), H^1) = \frac{\mathrm{Hom}_{\mathbb{C}}(H^0(-1), H^1)}{F^0\mathrm{Hom}_{\mathbb{C}}(H^0(-1), H^1) + \mathrm{Hom}_{\mathbb{Z}}(H^0(-1), H^1)}.$$

For each $\{p_i, q_i\}$ we choose $\eta_i \in H^0(\Omega^1_{\widehat{C}}(\log D))$ with $\mathrm{Res}_{p_i}\eta_i = 1$ and $\mathrm{Res}_{q_i}\eta_i = -1$, and $\mathrm{Res}_{p_j}\eta_i = 0 = \mathrm{Res}_{q_n}\eta_i$ for all $i \neq j$. Then

$$\eta \;=\; \sum_i \eta_i \;\in\; H^0(\Omega^1_{\widehat{C}}(\log D)) \;\subset\; \mathbb{H}^1(\Omega^\bullet_{\widehat{C}}(\log D)) \;\simeq\; H^1(\widehat{C}\backslash D)$$

lifts $\sum p_i - q_i \in H^0(D)(-1)$ and is well-defined modulo $H^0(\Omega^1_{\widehat{C}})$.

The above is standard and is just a matter of tracing through the definitions. Perhaps less familiar is the geometric expression of the level two extension data in terms of differential forms and integrals. We are considering equivalence classes of LMHS with monodromy weight filtrations $\{0\} \subset W_0 \subset W_1 \subset W_2 = V$, and where the Hodge structures $\oplus_{a=0}^2 H^a$ and the level one extensions of MHS

$$\begin{aligned} &0 \longrightarrow H^0 \longrightarrow W_1 \longrightarrow H^1 \longrightarrow 0\,, \\ &0 \longrightarrow H^1 \longrightarrow W_2/W_0 \longrightarrow H^0(-1) \longrightarrow 0 \end{aligned} \tag{A.6}$$

are fixed. It follows from the special properties of the extension data for limiting mixed Hodge structures (§A.2) that these are given the symmetric part of

$$\frac{\mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-1), H^0)}{\exp(\mathbb{C}\sigma)} \;\simeq\; \frac{\mathrm{Hom}^{\mathrm{sym}}_{\mathbb{Z}}(H^0(-1), H^0)}{\mathrm{span}_{\mathbb{Z}}\{N_1, \ldots, N_{h^0}\}} \otimes \mathbb{C}^*\,.$$

Using the identification above, this data is represented by the off-diagonal terms in $h^0 \times h^0$ symmetric matrices whose entries are in $\mathbb{C}/2\pi\mathbf{i}\mathbb{Z}$. Those off-diagonal entries are obtained as follows. For each i, we choose a path γ_i with $\partial\gamma_i = p_i - q_i$. Then for $i \neq j$, the bilinear relations for differentials of the third kind give the classical

$$\int_{\gamma_i} \eta_j \;\equiv\; \int_{\gamma_j} \eta_i \quad \text{modulo periods.}$$

In more detail, for general mixed Hodge structures with weight filtration $0 \subset W_0 \subset W_1 \subset W_2 = V$ and fixed associated graded $H^\bullet$, the level one information consists of the pair (A.6) of extensions of mixed Hodge structures. For limiting mixed Hodge structures, the level two extension data can be formulated as the equivalence classes of diagrams

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & H^0 \arrow[r] \arrow[d, equal] & W_1 \arrow[r] \arrow[d] & H^1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^0 \arrow[r] & W_2 \arrow[r] \arrow[d] & W_2/W_0 \arrow[r] \arrow[d] & 0 \\ & & H^0(-1) \arrow[r, equal] \arrow[d] & H^0(-1) \arrow[d] & \\ & & 0 & 0 & \end{tikzcd} \tag{A.7}$$

In the geometric example at hand we have $W_1 \simeq H^1(\widehat{C}, D)$ and $W_2/W_0 \simeq H^1(\widehat{C}\backslash D)$, as noted in (A.5). The information in (A.7) beyond that in (A.6) is given by the symmetric matrix $\left[\int_{q_j}^{p_j} \eta_i\right]$ modulo periods.

Example A.8. The simplest and most classical example is when $\widehat{C} = \mathbb{P}^1$ and $h^0 = 2$: in this case, the above construction produces the cross-ratio of 2 pairs of ordered distinct points in $\mathbb{P}^1$. (When C is not irreducible it is necessary to introduce combinatorial data arising from its dual graph. Since the purpose of this appendix is to illustrate some aspects of the geometric interpretation of the extension data we shall refrain from discussing this topic.)

A.4. **The $n = 2$ case of surfaces with a smooth double curve.** We first recall the levels one and two extensions data for a mixed Hodge structure with weight filtration

$$0 \;=\; W_0 \;\subset\; W_1 \;\subset\; W_2 \;\subset\; W_3 \;=\; W_4 \;=\; V\,.$$

A.4.1. *Level one extension data.* The level one (polarized) extension data is $\mathrm{Ext}^1_{\mathrm{MHS}}(H^2, H^1) = \mathbb{T}^1_W$ with tangent space $\mathrm{Hom}_{\mathbb{C}}(H^2, H^1)/F^0\mathrm{Hom}_{\mathbb{C}}(H^2, H^1)$. The numerator admits a weight -1 Hodge decomposition

$$\mathrm{Hom}_{\mathbb{C}}(H^2, H^1) \;\simeq\; \underbrace{\mathrm{End}(V)^{1,-2}_{W,F} \oplus \mathrm{End}(V)^{0,-1}_{W,F}}_{F^0\mathrm{Hom}_{\mathbb{C}}(H^2,H^1)} \oplus \underbrace{\mathrm{End}(V)^{-1,0}_{W,F} \oplus \mathrm{End}(V)^{-2,1}_{W,F}}_{\mathbb{L}^1}\,.$$

A.4.2. *Level two extension data.* Fixing the level one extension data, the level two extension data is

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{MHS}}(H^1(-1), H^1) \;&=\; \frac{\mathrm{Hom}_{\mathbb{C}}(H^1(-1), H^1)}{F^0\mathrm{Hom}_{\mathbb{C}}(H^1(-1), H^1) \,+\, \mathrm{Hom}_{\mathbb{Z}}(H^1(-1), H^1)}\\ &=\; \Lambda^2\backslash\mathbb{L}^2 \;=\; \mathbb{T}^2_W \,\times\, (\mathbb{C}^*)^d\,,\end{aligned}$$

with

$$\mathrm{Hom}_{\mathbb{C}}(H^1(-1), H^1) \;\simeq\; \underbrace{\mathrm{End}(V)^{0,2}_{W,F}}_{F^0\mathrm{Hom}_{\mathbb{C}}(H^1(-1),H^1)} \;\oplus\; \underbrace{\mathrm{End}(V)^{-1,-1}_{W,F} \;\oplus\; \mathrm{End}(V)^{-2,0}_{W,F}}_{\mathbb{L}^2}\,.$$

Under the projection $\mathbb{L}^2 \twoheadrightarrow \Lambda^2\backslash\mathbb{L}^2$ we have $\mathrm{End}(V)^{-1,-1}_{W,F} \twoheadrightarrow (\mathbb{C}^*)^d$. Again the infinitesimal period relation implies the differential of the level two extension data map takes value in $\mathrm{End}(V)^{-1,-1}_{W,F} \subset \mathbb{L}^2$. As in §A.1.2, redefining F is necessary, we may assume that the map itself takes value in $(\mathbb{C}^*)^d$.

In the case of a LMHS polarized by a cone σ, the subspace $\mathrm{End}(V)^{-1,-1}_{W,F}$ must be replaced with the quotient by $\mathbb{C}\sigma$.

A.4.3. *Extension data for $C \subset X$.* We next wish to recall the geometric interpretation of the extension data for a smooth, but possibly reducible, curve C on a smooth surface X. Then the relevant dual exact sequences are

$$0 \longrightarrow \frac{H^1(C)}{H^1(X)} \longrightarrow H^2(X,C) \longrightarrow \ker\left\{H^2(X) \to H^2(C)\right\} \longrightarrow 0$$

$$0 \longrightarrow \frac{H^2(X)}{\mathrm{Gy}H^0(C)} \longrightarrow H^2(X\backslash C) \longrightarrow \ker\left\{\mathrm{Gy} : H^1(C)(-1) \to H^3(X)\right\} \longrightarrow 0$$

with Gy the Gysin map. Assuming for simplicity that X is regular, the numerator of $\mathrm{Ext}^1_{\mathrm{MHS}}$ for the first sequence is

$$\mathrm{Hom}_{\mathbb{C}}\left(\ker\{H^2(X) \to H^2(C)\}\,,\ H^1(C)\right)\,.$$

It is convenient to write $H^2(X) = \mathrm{Hg}^1(X) \oplus H^2(X)_{\mathrm{trans}}$, with $H^2(X)_{\mathrm{trans}}$ the transcendental part of $H^2(X)$, and where the summands are orthogonal with respect to the intersection pairing. Assuming for the moment that C is irreducible, elements of $\ker\{\mathrm{Hg}^1(X) \to H^2(C)\}$ are given by divisors D on X such that $D \cdot C = 0$. Unwinding the definitions, we see that the extension class corresponding to $\mathrm{Hg}^1(X) \subset H^2(X)$ is given by the

$$D \ \mapsto\ \mathrm{AJ}_C(D \cdot C)\,.$$

For the transcendental part $H^2(X)_{\mathrm{trans}}$ of $H^2(X)$, after factoring by the F^0–part of the denominator, a typical element is $\xi = \alpha + \beta$ with $\alpha \in H^{2,0}(X) = H^0(\Omega^2_X)$ and $\beta \in H^{1,1}(X)$ with $\beta|_C = \mathrm{d}\gamma$, with γ a $(1,0)$–form on C that is orthogonal to the harmonic forms $H^{1,0}(C)$. Given $\delta \in H_1(C,\mathbb{Z})$ we have $\delta = \partial\Delta$ for a 2-chain Δ in X. The transcendental part of the extension class is then given by

$$\langle \xi, \delta\rangle \ =\ \int_\Delta \alpha \ -\ \int_\delta \gamma\,,$$

modulo periods. The term $\int_\Delta \alpha$ is a *membrane integral*; we will encounter a variant of this below (§§A.5.4–A.5.5). (For more on membrane integrals, see [KLMS06] and the expository [Lew06].

The above can be extended to the case that C is reducible. There are also interpretations, which we will not discuss, using the dual form of the MHS.

A.4.4. *Extension data for a pair of surfaces glued together along a curve.* Consider two smooth surfaces X_1 and X_2 with smooth curves $C_1 \subset X_1$ and $C_2 \subset X_2$ together with an isomorphism $C_1 \simeq C_2$. Let X be the surface obtained by gluing X_1 and X_2 together along the curves (via the isomorphism). For simplicity of notation, we identify the curves and denote them by C. Then

$$X \ =\ X_1 \cup_C X_2\,.$$

A necessary condition [Fri83] for X to be smoothable is

$$N_{C/X_1} \simeq N^*_{C/X_2}\,. \tag{A.9}$$

In this case, there is a well-defined equivalence class of limiting mixed Hodge structures [PS08, Ste76]. Again for simplicity we assume that X_1 and X_2 are regular. Then the limiting mixed Hodge structure has

$$H^1 \simeq H^1(C)\,.$$

To describe H^2, consider the complex

$$H^0(C)(-1) \xrightarrow{\alpha} H^2(X_1)\oplus H^2(X_2) \xrightarrow{\beta} H^2(C)\,;$$

here α is the direct sum of the Gysin maps and β is the difference of the restriction mappings. The smoothing condition (A.9) implies $C_1^2 + C_2^2 = 0$, as line bundles, so that

$$\beta\circ\alpha = 0\,.$$

Then

$$H^2 = \frac{\ker\beta}{\operatorname{im}\alpha}$$

is the cohomology of this complex.

A.5. **I-surfaces.** For a specific illustration of §A.4 we consider the Kollár–Shepherd-Barron–Alexeev (KSBA) moduli space $\mathcal{M}_I$ of smooth, minimal, regular ($q(X)=0$), general type surfaces X with $K_X^2 = 1$ and $p_g(X) = 2$.[21] These surfaces are in many ways the analog of genus two curves. The moduli space $\mathcal{M}_I$ is essentially smooth and of dimension 28.[22] The period domain D is of dimension 57 and the IPR is a contact structure on D. The period mapping

$$\Phi : \mathcal{M}_I \to \Gamma\backslash D$$

is locally injective, and the image $\Phi(\mathcal{M}_I)$ is a contact submanifold.[23]

[21] The discussion that follows is cursory. Much of this is discussed in more detail in [Gri19, Gri18, Gri20]. A reader with a working knowledge of surface theory and mixed Hodge theory, and the papers [FPR15a, FPR15b, FPR17] will be able to to fill in the details.

[22] This means that $H^1(TX)$ is unobstructed. In particular, the Kuranishi space is smooth and $\mathcal{M}_I$ is locally the quotient of an open set in $\mathbb{C}^{28}$ by a finite group.

[23] The monodromy group Γ is of finite index in $\mathrm{Aut}(V_{\mathbb{Z}}, Q)$. Since $K_X^2 = 1$, the intersection form is unimodular on the primitive cohomology. The ideal situation would be that $\Gamma = \mathrm{Aut}(V_{\mathbb{Z}}, Q)$ and that global Torelli holds; but this is not known.

A.5.1. *The KSBA compactification.* The moduli space admits a canonical projective compactification $\overline{\mathcal{M}}_I$ that has been extensively studied by Franciosi–Pardini–Rollenske [FPR15a, FPR15b, FPR17]. Unlike $\overline{\mathcal{M}}_2$, the space $\overline{\mathcal{M}}_I$ is highly singular along the boundary. It is exactly the extension data in the LMHS that may guide a desingularization of the boundary (Remark A.10).

The surfaces X_0 parameterized by the boundary $\partial\mathcal{M}_I = \overline{\mathcal{M}}_I \backslash \mathcal{M}_I$ have $\mathbb{Q}$-Gorenstein canonical divisor class K_{X_0} and semi-log-canonical (slc) singularities. These slc singularities have been classified [Kol13]. In the case that X_0 is normal, and $p \in X_0$ is a singular point:

(i) If X_0 is Gorenstein, then p is either simple elliptic or a cusp.

(ii) If X_0 is non-Gorenstein, then p is a rational singularity.

The period map $\Phi : \mathcal{M}_I \to \Gamma\backslash D$ admits an extension $\Phi^0 : \overline{\mathcal{M}}_I \to \overline{\wp}^0$, ibid. The monodromy about points of type (ii) is finite. The monodromy about points of type (i) is infinite and there is a nontrivial LMHS (W, F, σ) associated with a degeneration $X \to X_0$.

A.5.2. *The stratum* $\mathcal{N}_2$. There is a 20-dimensional boundary component $\mathcal{N}_2 \subset \overline{\mathcal{M}}_I$ whose general point parameterizes a singular I-surface X_0 that is normal, Gorenstein and with a simple elliptic singularity of degree 2.[24] The resolution $(\widetilde{X}, \widetilde{C}) \to (X_0, p)$ of this singularity is a smooth surface $\widetilde{X}$, whose minimal model is a K3 surface X, with an elliptic curve $\widetilde{C} \subset \widetilde{X}$ of self-intersection $\widetilde{C}^2 = -2$. The map $\widetilde{X} \to X$ contracts a -1 curve E with $E \cdot \widetilde{C} = 2$. In particular, the image $C \subset X$ of $\widetilde{C}$ is a curve with one node and self-intersection $C^2 = 2$. From this it follows that X is a 2:1 cover of $\mathbb{P}^2$ branched over a sextic curve C', and that C is a double cover of a tangent line ℓ to C'.

A.5.3. *The period map* $\Phi^0\big|_{\mathcal{N}_2}$. The LMHS corresponding to X_0 has associated graded described in terms of $H^2(X)$, $H^1(C)$ and $H^2(X, C)$. It depends on 20 parameters, and determines the pair (X, C) up to finite data; that is we have a local Torelli theorem for the boundary component $\mathcal{N}_2$.

A.5.4. *Level one extension data along* $\mathcal{N}_2$. Given $X_0 \in \mathcal{N}_2$ consider a one-parameter degeneration $X_t \to X_0$ and do a semi-stable reduction to have a smooth total space with normal crossing divisor $\widetilde{X}_0$ over the origin. From the Clemens–Schmid exact sequence [Mor84] the simplest possibility is that $\widetilde{X}_0$ has a double curve isomorphic to $\widetilde{C}$; that is,

$$\widetilde{X}_0 \;=\; \widetilde{X} \,\cup_{\widetilde{C}}\, Y\,,$$

[24] For us this example arose in the September 2017 meeting at Duke with Radu Laza, Marco Franciosi, Rita Pardini and Sönke Rollenske and was instrumental in suggesting the use in general of extension data to study the period mapping at infinity.

with $Y \supset \widetilde{C}$ a smooth surface. Friedman's smoothability condition (A.9) implies

$$N^*_{\widetilde{C}/\widetilde{X}} \simeq N_{\widetilde{C}/Y} .$$

The line bundle $N^*_{\widetilde{C}/\widetilde{X}}$ has degree 2. And if we think of Y as obtained from a smooth cubic $\widetilde{C}$ in $\mathbb{P}^2$ by blowing up points $\{q_i\}$ on the cubic, then there must be seven points q_i in order to have $\deg N_{\widetilde{C}/Y} = 2$. The level one extension data in the LMHS is the Jacobian $J = \mathrm{Jac}(\widetilde{C})$, and the seven parameters in the extension data may be seen to correspond to the points in $\mathrm{Jac}(\widetilde{C})$ given by the seven $\{q_i\}$.

Remark A.10 (A heuristic to desingularize). In contrast to the case of algebraic curves, where $\overline{\mathcal{M}}_g$ is essentially smooth, $\overline{\mathcal{M}}_I$ is singular along the boundary component $\mathcal{N}_2 \subset \partial\mathcal{M}_I$.[25] Given a KSBA degeneration $\mathfrak{X} \to \Delta$ where X_t is smooth for $t \neq 0$, and X_0 is a normal surface corresponding to a point of $\mathcal{N}_2$. For the desingularization of $\overline{\mathcal{M}}_I$ one needs to do a semi-stable reduction $\widetilde{\mathfrak{X}} \to \widetilde{\Delta}$. The discussion above suggests a candidate for $\widetilde{X}_0$.

The nilpotent logarithm N of monodromy for this degeneration has rank 2 and satisfies $N^2 = 0$. In the notation of §3.3.4 we have

$$\begin{aligned} H^1 &= \mathrm{Gr}_1^W \\ H^2 &= \mathrm{Gr}_2^W \\ H^1(-1) &= H^1 \otimes \mathbb{Q}(-1) = \mathrm{Gr}_3^W . \end{aligned} \tag{A.11}$$

In $\mathrm{Ext}^1_{\mathrm{MHS}}(H^1(-1), H^2)$ we have the subgroup $\mathrm{Ext}^1_{\mathrm{MHS}}(H^1(-1), \mathrm{Hg}^1)$. The group $\mathrm{Hg}^1 = H^{1,1} \cap H^2_{\mathbb{Z}}$ contains those classes in $\mathrm{Hg}^1(Y) = H^{1,1}(Y) \cap H^2(Y, \mathbb{Z})$ arising from blowing up the points p_i on $\widetilde{C} \subset Y$. It is this part of $\mathrm{Ext}^1_{\mathrm{MHS}}(H^1(-1), H^1)$ that the Abel-Jacobi maps correspond to.

The other part of the level one extension data corresponds to the transcendental part H^2_{tr} of H^2. Geometrically, as discussed above this is given in part by *membrane integrals*. Keeping in mind that $H_2(\widetilde{X}, \mathbb{Z}) = 0$, if $\gamma \in H_1(\widetilde{C}, \mathbb{Z})$ and $\omega \in H^0(\Omega^2_{\widetilde{X}}) \subset H^2_{\mathrm{tr}}$, then $\gamma = \partial\Gamma$ for a 2-chain $\Gamma \subset X$. The membrane integral is $\int_\Gamma \omega$. Using the cup-product on $H^2(\widetilde{X})$ we may view this integral as an element of

$$\frac{\mathrm{Hom}(H^1(C), H^2(X))_{\mathrm{tr}}}{F^0\mathrm{Hom}(H^1(C), H^2(X))_{\mathrm{tr}} + \mathrm{Hom}_{\mathbb{Z}}(H^1(C), H^2(X))}$$

A.5.5. *Level two extension data.* We will illustrate a membrane-integral-type geometric interpretation of the level two extension data. This is of a rather different character than the cross-ratio-type level two extension data of Example A.8.

[25] This seems likely to be a general phenomenon for any normal Gorenstein slc singularity.

Consider now an I-surface (X, p_1, p_2) having two elliptic singularities and with resolution $(\widetilde{X}, \widetilde{C}_1 \cup \widetilde{C}_2)$, cf. the works of Franciosi–Pardini–Rollenske. We may then form the smoothable normal crossing surface

$$\widetilde{X} \cup_{\widetilde{C}_1 \cup \widetilde{C}_2} Y \,.$$

Fixing the level one extension data fixes Y.

Again $N^2 = 0$ and (A.11) holds. The level two extension data is $\mathrm{Ext}^1_{\mathrm{MHS}}(H^1(-1), H^1)$ and contains

$$\frac{\mathrm{Hom}_{\mathbb{C}}(H^1(\widetilde{C}_1)(-1), H^1(\widetilde{C}_2))}{F^0\mathrm{Hom}_{\mathbb{C}}(H^1(\widetilde{C}_1)(-1), H^1(\widetilde{C}_2)) \,+\, \mathrm{Hom}_{\mathbb{Z}}(H^1(\widetilde{C}_1)(-1), H^1(\widetilde{C}_2))} \,.$$

The numerator $\mathrm{Hom}_{\mathbb{C}}(H^1(\widetilde{C}_1)(-1), H^1(\widetilde{C}_2))$ maps to $\mathrm{Hom}_{\mathbb{C}}(H^{1,0}(\widetilde{C}_1), H^1(\widetilde{C}_2))$. We may geometrically produce an element of the latter as follows. Utilizing the residue map

$$\mathrm{Res} : H^0(\Omega^2_{\widetilde{X}}(\log \widetilde{C}_1)) \;\to\; H^0(\Omega^1_{\widetilde{C}_1})\,,$$

given $\omega \in H^0(\Omega^2_{\widetilde{X}}(\log \widetilde{C}_2))$ and $\gamma \in H_1(\widetilde{C}_2, \mathbb{Z}) \simeq H^1(\widetilde{C}_1, \mathbb{Z})$, to describe an element of the extension data we need to produce the value $\langle \omega, \gamma \rangle$. Write $\gamma = \partial\Gamma$ with $\Gamma \subset \widetilde{X}$ a 2-chain. We claim that

$$\langle \omega, \gamma \rangle \;=\; \int_\Gamma \omega \,.$$

Suppose that $\Delta \subset \widetilde{C}_2$ is a 2-chain, so that $\gamma = \partial(\Gamma + \Delta)$. Then $\int_\Delta \omega = 0$.

Suppose that $\Sigma \subset \widetilde{X}\backslash\widetilde{C}_1$ is a 2-cycle, and $\omega' \in H^0(\Omega^2_{\widetilde{X}})$; then

$$\begin{aligned} \int_{\Gamma+\Delta} \omega &\;=\; \int_\Gamma \omega \,+\, \int_\Delta \omega \,, \\ \int_\Gamma (\omega + \omega') &\;=\; \int_\Gamma \omega \,+\, \int_\Gamma \omega' \,. \end{aligned}$$

Now, and this is the key point, both $\int_\Sigma \omega$ and $\int_\Gamma \omega'$ are part of the extension data for $H^2(\widetilde{X}, \widetilde{C}_1)$ and $H^2(\widetilde{X}, \widetilde{C}_2)$, respectively; and this is level one extension data. Thus $\langle \omega, \gamma \rangle$ transforms like level two extension data. More specifically, referring to the diagram (A.7) one may verify that $\langle \omega, \gamma \rangle$ transforms like the information needed to complete the extensions given by the top row and right-hand column in (A.7) to the full diagram.

A.6. **The Hodge–Tate case.** As noted in Remark 7.2, the extension data of level ≤ 2 determines, up to constants of integration, the full extension data. We wish to discuss this in the special case that the limiting mixed Hodge structure (W, F, σ) is Hodge–Tate type, cf. Definition 2.34 and Remark 2.35.

There is extensive literature on variations of *graded polarized* mixed Hodge structures of Hodge–Tate type; see [Bro14, Gon01, Hai94] and the references therein. One may anticipate that those arising in this paper, as variations of *limiting* mixed Hodge structures, will have special properties.

A.6.1. *Level two extension data for Hodge–Tate LMHS.* Among VLMHS are those arising from the data $(\overline{B}, Z; \Phi)$ as in this paper. As will now be illustrated we may think of these as *VLMHS that satisfy possible Schottky relations.* We will consider LMHS of Hodge–Tate type (§A.6.2) with $\dim H^0 = g$.[26] The extension data is all of level two, and upon a choice of basis, the set of extension data will be given by $g \times g$ symmetric matrices all of whose entries are nonzero.

Example A.12. The LMHS corresponds to a semi-abelian variety with no abelian part. If we think of it as corresponding to a point of a toroidal compactification $\overline{\mathcal{A}}_g^{\mathsf{T}}$, then the extension data is a general symmetric matrix is nonzero entries [CCK80].

Example A.13. In $\mathbb{P}^1$ we choose g distinct pairs of points (p_i, q_i). For each i we choose $t_i \neq 0$ which gives an identification $T_{p_i}\mathbb{P}^1 \otimes T^*_{q_i}\mathbb{P}^1 \simeq \mathbb{C}$. It is standard, and will be explained in more detail in the current context in ([Gri18] or [FGG$^+$20]), that is data gives a first-order smoothing of the g–nodal curve obtained by identifying p_i and q_i. In particular, there is a well-defined LMHS. If N_i is the logarithm of monodromy corresponding to smoothing the i–th node, then $N = N_1 + \cdots + N_g$. The diagonal entries of the symmetric matrix are the t_i. The off diagonal entries are the exponentials of

$$\int_{q_j}^{p_j} \eta_i \;\equiv\; \int_{q_i}^{p_i} \eta_j \,, \tag{A.14}$$

modulo periods, where η_i is the unique differential on $\mathbb{P}^1$ with poles at p_i, q_i and normalized to be $\mathrm{d}\log t_i$ (modulo a holomorphic 1-form) near p_i. Then (A.14) is the logarithm of the cross ratio $(p_i, q_i; p_j, q_j)$, cf. §A.3.

The number of parameters of the p_i, q_i is $2g - 3$. There are g of the t_i's, giving the total number of parameters $3g - 3$. On the other hand, as noted above there are $g(g+1)/2$ parameters in the extension data for a general LMHS. For $g = 2, 3$, the numbers are equal, but for $g \geq 4$ we have $g(g+1)/2 > 3g - 3$, so that there are algebraic Schottky relations among the cross ratios $(p_i, q_i; p_j, q_j)$.[27]

Example A.15. For non-classical Hodge–Tate VLMHS there will be universal infinitesimal Schottky relations imposed by the IPR. In general, there will be additional ones as happens above for $g \geq 4$. In this regard an interesting question arises concerning the I–surface

[26] These may be thought of as "classical"; the IPR is trivial.

[27] There is extensive literature, both classical current, concerning Schottky relations. The papers [SB19, SB20] are particularly relevant here as they involve interesting Hodge theoretic considerations. The smoothing of nodes process is also discussed.

(cf. [FPR15a, FPR15b, FPR17] and [Gri18]). In this case the KSBA moduli space $\mathcal{M}_I$ is smooth at all points corresponding to smooth surfaces X. The IPR is a contact structure, local Torelli holds at any such X and the image of the period map hear X is a contact submanifold. That is, there are no Schottky–type relations beyond the IPR. Since there are boundary components (coming from non-normal Gorenstein degenerations) whose LMHS are Hodge–Tate (loc. cit.), one may ask if on those components there are no additional Schottky-type relations, as is the case for curves when $g = 2, 3$?

Finally we note that for the variations of graded polarized MHS of Hodge–Tate type and that arise from arithmetic considerations the finite Schottky relations correspond to identities among polylogarithms [Bro14, Gon01, Hai94].

For more on the role of the IPR in the language of period matrices, see the discussion of the classical and non-classical cases $n = 1$ and $n = 2$, respectively below (§§A.6.3–A.6.4).

A.6.2. *The Hodge–Tate case: notations.* In the Hodge–Tate case we have $V^{p,q}_{W,F} = 0$ for all $p \neq q$; equivalently,

$$H^{2p} \;=\; \mathrm{Gr}^W_{2p} \;\simeq\; V^{p,p}_{W,F} \quad \text{and} \quad H^{2p+1} \;=\; \mathrm{Gr}^W_{2p+1} \;=\; 0\,.$$

Using the notation of §§3.2.2–3.2.3, we will consider a local family over $\mathcal{U} \subset \overline{\mathcal{U}}$ with monodromy cone

$$\sigma \;=\; \mathrm{span}_{\mathbb{R}_{>0}}\{N_1, \ldots, N_k\}\,.$$

We also assume that the LMHS has the property that

$$N : H^{2p} \;\xrightarrow{\simeq}\; H^{2p-2}$$

is an isomorphism for all $N \in \sigma$ and for all $1 \leq p \leq n$; equivalently,

$$H^{2p} \;=\; H^0(-p)\,, \quad \forall\; 0 \leq p \leq n\,.$$

As in Example 2.37, we will express the local lift $\widetilde{\Phi}(t, w)$ in (3.2) as a period matrix. We will also suppress the holomorphic parameter w, and write

$$\widetilde{\Phi}(t, w) \;=\; \widetilde{\Phi}(t) \;=\; \exp(\textstyle\sum \ell(t_i) N_i)\xi(t) \cdot F\,.$$

Recall the discussion of §7.2.2: the killing form κ defines a pairing between the horizontal entries of the period matrix and $\mathfrak{g}^{1,\bullet}_{W,F} \subset \mathfrak{g}_{\mathbb{C}}$. We fix a basis $\{M_\mu\} = \mathbf{N}^*_\sigma \cup \mathbf{N}^\perp_\sigma$ of $\mathfrak{g}^{1,\bullet}_{W,F}$ with the properties

$$\mathrm{span}_{\mathbb{C}}\{M_\mu \in \mathbf{N}^\perp_\sigma\} \;=\; \mathrm{Ann}(\sigma) \;\subset\; \mathfrak{g}^{1,\bullet}_{W,F}$$

and

$$0 \;\leq\; \kappa(N_i, M_\mu) \;\in\; \mathbb{Z}\,, \quad \forall\; 1 \leq i \leq k\,;$$

the

$$\varepsilon_\mu(t) \;=\; \kappa\left(\exp(\textstyle\sum \ell(t_i) N_i)\xi(t)\,,\, M_\mu\right)$$

are the *horizontal* coefficients of the period matrix; the $\varepsilon_\mu(t)$ with $\mu \in \mathbf{N}_\sigma^\perp$ are holomorphic, and the $\tau_\mu(t) = \exp(2\pi\mathbf{i}\varepsilon_\mu(t))$ are holomorphic for all μ, and nonvanishing on $\mathcal{U}$.

Finally we note that while we are working locally on $\overline{\mathcal{U}}$, in the global setting these functions are well-defined on a neighborhood $\overline{\mathcal{O}}^1 \subset \overline{B}$ of a (connected component of a) Φ^1–fibre A^1 (§7.2.2). Without loss of generality, each M_μ is contained in some $\mathfrak{g}^{1,q_\mu}_{W,F}$, so that $\mathbf{N}^*_\sigma \subset \mathfrak{g}^{1,1}_{W,F}$, and $\varepsilon_\mu(t)$ is level $q_\mu + 1$ extension data.

A.6.3. *The Hodge–Tate case with $n = 1$.* Fix a basis of $V_\mathbb{C}$ that is adapted to the Hodge filtration $F^1 \subset F^0 = V_\mathbb{C}$ and with respect to which the intersection matrix and monodromy operators are given by

$$Q \;=\; \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \quad \text{and} \quad N_i \;=\; \begin{bmatrix} 0 & 0 \\ \nu_i & 0 \end{bmatrix},$$

with ν_i an integral symmetric matrix. The period matrix is

$$F^1_t \;=\; \exp(\textstyle\sum \ell(t_i) N_i)\xi(t) \cdot F^1 \;=\; \begin{bmatrix} I \\ A(t) \end{bmatrix},$$

with

$$A(t) \;=\; \sum_{i=1}^k \ell(t_i)\nu_i \;+\; \alpha(t)\,.$$

The matrix $A(t)$ encodes the level two extension data along A^1. Using the first Hodge–Riemann bilinear relation, the period matrix for F^0_t/F^1_t is

$$F^0_t/F^1_t \;=\; \begin{bmatrix} I \\ A(t) \end{bmatrix}.$$

The the $\varepsilon_\mu(t)$ with $\mu \in \mathbf{N}_\sigma^\perp$ are essentially the matrix entries of the holomorphic $\alpha(t)$.

The level two extension data along A^1 is given by

$$\mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-1), H^0)/\exp(\mathbb{C}\sigma)\,,$$

and is encoded in the logarithmic $\varepsilon_\mu(t)$ with $M_\mu \in \mathbf{N}^*_\sigma \subset \mathfrak{g}^{1,1}_{W,F}$. The corresponding $\tau_\mu(t) = \exp(2\pi\mathbf{i}\varepsilon_\mu(t))$ is holomorphic and may be thought of as a generalized cross-ratio, cf. Example A.8.

A.6.4. *The Hodge–Tate case with $n = 2$.* Fix a basis of $V_\mathbb{C}$ that is adapted to the Hodge filtration $F^2 \subset F^1 \subset F^0 = V_\mathbb{C}$ and with respect to which the intersection matrix and monodromy operators are given by

$$Q \;=\; \begin{bmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{bmatrix} \quad \text{and} \quad N_i \;=\; \begin{bmatrix} 0 & 0 & 0 \\ \nu_i & 0 & 0 \\ 0 & {}^t\nu_i & 0 \end{bmatrix}.$$

The period matrix is

$$\left[\begin{array}{cc} F^2_t & F^1_t/F^2_t \end{array}\right] \;=\; \left[\begin{array}{cc} I & 0 \\ A(t) & I \\ B(t) & {}^t A(t) \end{array}\right],$$

with $A(t)$ linear in the $\ell(t_i)$, and $B(t)$ quadratic in the $\ell(t_i)$. Again, the matrix $A(t)$ encodes the level two extension data along A^1; the matrix $B(t)$ encodes level four data. The first Hodge–Riemann bilinear relation yields

$$B + {}^tB \;=\; {}^tAA\,. \tag{A.16}$$

The infinitesimal period relation $Q(\mathrm{d}F^2, F^1) = 0$ implies

$$\mathrm{d}B \;=\; {}^tA\,\mathrm{d}A\,, \tag{A.17}$$

so that the level four extension data $B(t)$ is determined (up to constants of integration) by the level two extension data $A(t)$. The vanishing of the coefficients of the $\mathrm{d}\ell(t_i) \wedge \mathrm{d}\ell(t_j)$ in $0 = \mathrm{d}^2B = {}^t\mathrm{d}A \wedge \mathrm{d}A$ is equivalent the the commutativity relations $[N_i, N_j] = 0$. (While (A.17) implies that B contains no new information, modulo constants of integration, the existence of this level four extension data implies ${}^t\mathrm{d}A \wedge \mathrm{d}A$.)

As noted in (A.16), the symmetric part of B is determined algebraically from A. The skew-symmetric part satisfies

$$\begin{aligned} \mathrm{d}(B - {}^tB) \;&=\; {}^tA \cdot \mathrm{d}A \,-\, \mathrm{d}{}^tA \cdot A \\ &=\; \sum_{i,j} {}^t\nu_i\,\nu_j \left(\ell(t_i)\mathrm{d}\ell(t_j) - \ell(t_j)\mathrm{d}\ell(t_i)\right)\,. \end{aligned}$$

Recalling the dilogarithm function

$$\mathrm{Li}_2(z) \;=\; \int_0^z \ell(t)\mathrm{d}\ell(t)\,,$$

and noting that $\ell(t)$ is transcendental, this suggests that (in contrast to the level two extension data), the higher level (level four in this case) data is of a transcendental nature. In fact the differential equations

$$z\,\frac{\mathrm{d}\,\mathrm{Li}_m(z)}{\mathrm{d}z} \;=\; \mathrm{Li}_{m-1}\,, \quad m \geq 2\,,$$

for the polylogarithms suggest that for $m \geq 1$, the extension data of level $\leq m{+}2$ should be expressible in terms of these functions. (For more on the connections, both established and conjectural, between polylogarithms and Hodge–Tate structures see, for example, [Gon01] and the references therein.)

Appendix B. Relationship to the reduced limit period map

Here we describe the relationship between Φ^0 and the reduced period limit map $\Phi_\infty : \overline{B} \to \overline{\wp}_\infty$ (defined in §B.1). For our purposes the key observation here is that Φ_∞ is locally constant on Φ^0 fibres (Proposition B.6); in particular, Φ^0 factors through Φ_∞, and the fibres of $\overline{\wp}_\infty \to \overline{\wp}^0$ are finite. From Corollary B.8 we are able to deduce that Γ_{A^0} is subject to the constraint (5.10).

B.1. **Definition.** Fix a local lift $\widetilde{\Phi}(t,w)$, and let (W,F,σ) be the associated LMHS (§3.2.4). The *reduced limit period*

$$F_\infty(w) \;=\; \lim_{y\to\infty} \widetilde{\Phi}(z,w) \;=\; \lim_{y\to\infty} \exp(\mathbf{i}yN)\xi(0,w)\cdot F \;\in\; \overline{D} \tag{B.1}$$

is independent of our choice of $N \in \sigma$ (and the limit is understood to be taken with x bounded), [GGK13, KP14, GGR17]. The two filtrations F and $F_\infty(0)$ are related by the Deligne splitting (§3.3.1)

$$F^p \;=\; \bigoplus_{a\geq p} V^{a,b}_{W,F} \quad\text{and}\quad F^p_\infty(0) \;=\; \bigoplus_{b\leq n-p} V^{a,b}_{W,F}\,.$$

In particular, the Lie algebra $\mathfrak{f}_\infty$ of the stabilizer $\mathrm{Stab}_{G_\mathbb{C}}(F_\infty(0))$ is

$$\mathfrak{f}_\infty \;=\; \bigoplus_{q\leq 0} \mathfrak{g}^{p,q}_{W,F}\,.$$

Recalling that the map $\xi(0,w)$ takes value in $C_{I,\mathbb{C}}$ (§3.3.3), we see that

$$F_\infty(w) \;=\; \xi(0,w)\cdot F_\infty(0)\,. \tag{B.2}$$

In particular, the map $F_\infty : \{0\}\times\Delta^r \to \check{D}$ is holomorphic, and takes value in the $C_{I,\mathbb{C}}$–orbit of $F_\infty(0)$. What is less obvious is that: (i) The holomorphic $F_\infty(0,w)$ takes value in the real orbit

$$\mathcal{O}_I \;=\; C_{I,\mathbb{R}}\cdot F_\infty(0) \;\subset\; \check{D}\,.$$

(ii) The real orbit $\mathcal{O}_I$ is open in the (complex) orbit $C_{I,\mathbb{C}}\cdot F_\infty(0)$, and so is a complex submanifold of $\check{D}$. (In fact, $\mathcal{O}_I$ is a CR–submanifold of the real orbit $G_\mathbb{R}\cdot F_\infty(0) \subset \partial D$). Note that F_∞ is independent of the local coordinates (t,w) expressing $\widetilde{\Phi}$. So the reduced period limit induces a well-defined holomorphic map

$$\Phi^\infty_I : Z^*_I \;\to\; \Gamma_I\backslash\mathcal{O}_I\,. \tag{B.3}$$

Let

$$\wp^\infty_I \;=\; \Phi^\infty_I(Z^*_I) \;\subset\; \Gamma_I\backslash\mathcal{O}_I$$

denote the image.

B.2. Φ_I^0 **versus** Φ_I^∞**.** Recall the period map $\Phi_I^0 : Z_I^* \to \Gamma_I\backslash D_I^0$ of (**??**). We claim that Φ_I^0 factors through Φ_I^∞. To see this, observe that there is a natural identification

$$D_I^0 \;\simeq\; C_{I,\mathbb{R}}^{-1}\backslash\mathcal{O}_I\,.$$

This identification induces

$$\pi_I : \Gamma_I\backslash\mathcal{O}_I \;\to\; \Gamma_I\backslash D_I^0\,. \tag{B.4}$$

We have

$$\Phi_I^0 \;=\; \pi_I \circ \Phi_I^\infty\,. \tag{B.5}$$

In particular, $\pi_I : \wp_I^\infty \twoheadrightarrow \wp_I$.

Proposition B.6. *The map Φ_I^∞ is locally constant on Φ_I^0–fibres. In particular, the map $\pi_I : \wp_I^\infty \to \wp_I$ is finite.*

Remark B.7. When D is Hermitian the map (B.4) is an isomorphism and $\Phi_I^0 = \Phi_I^\infty$.

Let $C_{I,\infty,\mathbb{C}}^{-1}$ denote the stabilizer in $C_{I,\mathbb{C}}^{-1}$ of the filtration $F_\infty(0) \in \check{D}$.

Corollary B.8. *The map $\Phi_{A^0,I}^0 : A^0 \cap Z_I \to \delta_I$ takes value in*

$$C_{\sigma,\infty,\mathbb{C}}^{-1}\cdot F \;\subset\; C_{I,\mathbb{C}}^{-1}\cdot F \;=\; \delta_I\,.$$

B.3. **Proof of Proposition B.6.** It is enough to show that $F_\infty(w)$ is constant along the Φ^0–fibres in $\{0\}\times\Delta^r$. This is a consequence of the IPR. The essential point is that the map

$$w \;\mapsto\; \xi(0,w)\cdot F \quad \text{is horizontal.} \tag{B.9}$$

B.3.1. *Formulation of the argument.* Recall that $\xi(t,w)$ takes value in $\exp(\mathfrak{f}^\perp)$, and $\xi(0,w)$ takes value in $\exp(\mathfrak{c}_{I,\mathbb{C}})$, cf. §3.3.1 and §3.3.3. We have

$$\mathfrak{f}^\perp \,\cap\, \mathfrak{c}_{I,\mathbb{C}} \;=\; \bigoplus_{\substack{p<0\\ p+q\le 0}} \mathfrak{c}_{I,F}^{p,q}\,.$$

Note that

$$\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}} \cap \mathfrak{f}_\infty \;=\; \bigoplus_{\substack{p\le 0\\ q\le 0}} \mathfrak{c}_{I,F}^{p,q}\,,$$

and consider the decomposition

$$\mathfrak{f}^\perp \,\cap\, \mathfrak{c}_{I,\mathbb{C}} \;=\; \mathfrak{d} \;\oplus\; \mathfrak{e} \;\oplus\; (\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}} \cap \mathfrak{f}_\infty)$$

defined by

$$\mathfrak{d} \;=\; \bigoplus_{\substack{p<0\\ p+q=0}} \mathfrak{c}_{I,F}^{p,q} \quad\text{and}\quad \mathfrak{e} \;=\; \bigoplus_{\substack{p<0<q\\ p+q<0}} \mathfrak{c}_{I,F}^{p,q}\,.$$

Each of these three summands is a Lie subalgebra of $\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}}$.

Since $\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}}$ is nilpotent, the function $\xi(0,w)$ may be uniquely decomposed as

$$\xi(0,w) \;=\; e(w)f(w)s(w)$$

with $f(w) \in \exp(\mathfrak{d})$, $e(w) \in \exp(\mathfrak{e})$ and $s(w) \in \exp(\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}} \cap \mathfrak{f}_\infty)$. Since $\xi(0,w) = e(w)\,f(w)s(w)f(w)^{-1}\,f(w)$, and both $e(w)$ and $f(w)s(w)f(w)^{-1}$ take value in the unipotent radical $C_{I,\mathbb{C}}^{-1}$, we may

$$\text{identify } \Phi_I^0(0,w) \text{ with } f(w).$$

Furthermore, since $\mathfrak{f}_\infty$ is the stabilizer of $F_\infty(0)$ in $\mathfrak{f}$, (B.2) implies we may

$$\text{identify } F_\infty(w) \text{ with } e(w)f(w).$$

So to prove the lemma, it suffices to show that

$$e(w) \text{ is locally constant along } f\text{–fibres.}$$

So we assume

$$\mathrm{d}f = 0\,, \tag{B.10a}$$

and will show that $\mathrm{d}e = 0$; equivalently,

$$e^{-1}\mathrm{d}e \;=\; 0\,. \tag{B.10b}$$

B.3.2. *Horizontality.* Horizontality is the condition

$$(\xi^{-1}\mathrm{d}\xi)^{p,q} \;=\; 0\,, \quad \forall\, p \le -2\,, \tag{B.11}$$

with $(\xi^{-1}\mathrm{d}\xi)^{p,q}$ the component of the $\mathfrak{f}^\perp$–valued $\xi^{-1}\mathrm{d}\xi$ taking value in $\mathfrak{g}^{p,q}_{W,F}$, cf. Remark 3.4 and §3.3.6. At $(0,w)$ we have

$$\begin{aligned} \xi^{-1}\mathrm{d}\xi \quad &= \quad (efs)^{-1}\mathrm{d}(efs) \\ &= \quad \mathrm{Ad}_{fs}^{-1}(e^{-1}\mathrm{d}e) \;+\; \mathrm{Ad}_s^{-1}(f^{-1}\mathrm{d}f) \;+\; s^{-1}\mathrm{d}s \\ &\overset{\text{(B.10a)}}{=} \quad \mathrm{Ad}_{fs}^{-1}(e^{-1}\mathrm{d}e) \;+\; s^{-1}\mathrm{d}s\,. \end{aligned} \tag{B.12}$$

Note that $e^{-1}\mathrm{d}e$ and $s^{-1}\mathrm{d}s$ take value in $\mathfrak{e}$ and $\mathfrak{f}_\infty$, respectively. Furthermore, (3.3d) and $fs \in \exp(\mathfrak{f}^\perp \cap \mathfrak{c}_{I,\mathbb{C}})$ imply that

$$e^{-1}\mathrm{d}e \;=\; 0 \quad \text{if and only if} \quad \left(\mathrm{Ad}_{fs}^{-1}(e^{-1}\mathrm{d}e)\right)^{p,q} \;=\; 0$$

for all $q > 0$ and $p + q < 0$. At the same time (3.3d), (B.11) and (B.12) imply that

$$0 \;=\; (\xi^{-1}\mathrm{d}\xi)^{p,q} \;=\; \left(\mathrm{Ad}_{fs}^{-1}(e^{-1}\mathrm{d}e)\right)^{p,q}$$

for all $q > 0$ and $p + q < 0$. The desired (B.10b) now follows, completing the proof of Proposition B.6.

Appendix C. Deligne's extension

Deligne's extension of the local system $\mathcal{V}$ and Hodge bundles $\mathcal{F}^p \subset \mathcal{V}$ to $\overline{B}$ is defined by constructing trivializations of the bundles over $\overline{\mathcal{U}}$. It will be helpful in §5.3 and §9.3 to review this construction.

C.1. **Preliminaries.**

C.1.1. We have a right-action on $\widetilde{B} \times V$ given by

$$(\zeta, v) \cdot \gamma \;=\; (\zeta \cdot \gamma, \gamma^{-1} v)\,,$$

and $\mathcal{V} = B \times_{\pi_1(B)} V$ is the quotient. A section $\varsigma : B \to \mathcal{V}$ is equivalent to a function $f : \tilde{B} \to V$ with the property that $f(\zeta \cdot \gamma) = \gamma^{-1} \cdot f(\zeta)$.

C.1.2. The flat connection ∇ on $\mathcal{V}$ is induced by the canonical flat connection on $\tilde{B} \times V$ as follows. Fix $v \in V$. Then $\zeta \mapsto (\zeta, v)$ is a parallel global section of the trivial bundle $\tilde{B} \times V$. Set $b = p(\zeta) \in B$. Identify $\mathcal{V}_b$ with V. Let $\tilde{\gamma}$ be a curve in $\widetilde{B}$ joining $\zeta = \tilde{\gamma}(0)$ and $\zeta \cdot \gamma = \tilde{\gamma}(1)$. Then $p(\tilde{\gamma})$ is a loop in B, based at b and representing $\gamma \in \pi_1(B)$. Parallel transport of $v \in V \simeq \mathcal{V}_b$ along $p(\gamma)$ is determined by observing that the equivalence relation identifies $(\zeta \cdot \gamma, v)$ with $(\zeta, \gamma \cdot v)$. The upshot is the anticipated compatibility statement: The monodromy action of $\gamma \in \pi_1(B)$ on $\mathcal{V}_b = V$ is by $\gamma \in \mathrm{Aut}(V)$.

C.2. **The extension.**

C.2.1. To define the extension $\mathcal{V}_{\mathrm{e}} \to \overline{B}$ we work locally. For simplicity, assume B is a curve and consider our usual local coordinates at infinity $\mathcal{U} \simeq \Delta^* \subset \Delta$. We continue to conflate $\ell(t)$ with coordinates on $\mathcal{H}$. Suppose that $\gamma \in \pi_1(\Delta^*)$ acts on $\mathcal{H}$ by $z \mapsto z+1$, and on V by $\exp(-N)$. Fix $v \in V$. The map

$$f : \mathcal{H} \;\to\; V \quad \text{sending} \quad z \mapsto \exp(\ell(t)N) \cdot v$$

satisfies

$$f(\ell(t) \cdot \gamma) \;=\; f(z+1) \;=\; \exp(N) f(\ell(t)) \;=\; \gamma^{-1} \cdot f(\ell(t))\,,$$

and so defines a section

$$\begin{array}{c} \mathcal{V} \\ {\scriptstyle \varsigma_v}\!\uparrow\!\downarrow \\ \Delta^* \end{array} \qquad\qquad \varsigma_v(t) \;=\; \exp\left(\frac{\log t}{2\pi \mathbf{i}} N\right) v\,.$$

The section is parallel with respect to the flat connection

$$\nabla^t \;:=\; \nabla \;-\; \frac{1}{2\pi\mathbf{i}} N \otimes \mathrm{d}\log t\,.$$

The connection ∇^t has no monodromy, making $\mathcal{V} \to \Delta^*$ a constant bundle. The upshot is: *The canonical extension $\mathcal{V}_{\rm e} \to \Delta$ is obtained by extending $\mathcal{V}$ as a constant bundle.* If $\{v_j\}$ is a basis of V, then

$$\varsigma_j \;=\; \varsigma_{v_j} \;=\; \exp\left(\frac{\log t}{2\pi\mathbf{i}}N\right) v_j$$

is a framing of $\mathcal{V}_{\rm e} \to \overline{\mathcal{U}} \simeq \Delta$. While ∇^t depends on the choice of coordinates $t \in \Delta$, the extension $\mathcal{V}_{\rm e}$ does not.

C.2.2. While the extension $\mathcal{V}_{\rm e}$ of $\mathcal{V}$ is relatively straightforward, the extension of the Hodge bundles $\mathcal{F}^p \subset \mathcal{V}$ is nontrivial: it is a consequence of Schmid's nilpotent orbit theorem. Continuing with the local curve case $\mathcal{U} \simeq \Delta^*$, the theorem expresses the lift as $\tilde\Phi(\ell(t)) = \exp(\ell(t)N)\xi(t)\cdot F$, with $\xi : \Delta \to \exp(\mathfrak{f}^\perp)$ holomorphic, §3.2.4.

Fix $v \in V$. The map

$$f : \mathcal{H} \to V \quad \text{sending} \quad f(\ell(t)) = \exp(\ell(t)N)\xi(t)\cdot v$$

satisfies $f(\gamma\cdot\ell(t)) = f(\ell(t)+1) = \gamma\cdot f(\ell(t))$, and so defines a section

$$\begin{tikzcd} \mathcal{V} \arrow[d] \\ \Delta^* \arrow[u, bend left, "\phi_v"] \end{tikzcd}$$

If $v \in F^p$, then the section ϕ_v takes value in $\mathcal{F}^p$. Fix a basis $\{v_j\}$ of V. Without loss of generality, the basis $\{v_j\}$ is adapted to F, so that the framing $\{\phi_j = \phi_{v_j}\}$ of $\mathcal{V}$ is adapted to the Hodge bundles $\mathcal{F}^p \subset \mathcal{V}$. Let $\xi(t)\cdot v_j = \xi^i_j(t)v_i$ denote the matrix coefficients of $\xi(t)$ with respect to this basis. Then

$$f_j(\ell(t)) \;:=\; \exp(\ell(t)N)\xi(t)\cdot v_j \;=\; \xi^i_j(t)\exp(\ell(t)N)\cdot v_j\,.$$

Whence the

$$\phi_j \;=\; \xi^i_j(t)\varsigma_j\,.$$

are sections of $\mathcal{V}_{\rm e}$ that are adapted to the Hodge filtration $\mathcal{F}^p$ over $\mathcal{U} \simeq \Delta^*$. So we may extend $\mathcal{F}^p$ by taking $\mathcal{F}^p_{\rm e} \subset \mathcal{V}_{\rm e}$ to be the subbundle framed by the $\{\phi_j \mid v_j \in F^p\}$.

C.2.3. So far we've discussed the local curve case: $\mathcal{U} \simeq \Delta^*$. The generalization to VHS over $\mathcal{U} \simeq (\Delta^*)^k \times \Delta^r$ is straightforward. Here we note only that we use

$$\nabla^t \;=\; \nabla - \frac{1}{2\pi\mathbf{i}}\sum N_j \otimes \mathrm{d}\log t_j\,.$$

And the local framing of the Hodge vector bundles $\overline{\mathcal{F}}^p$ over $\overline{\mathcal{U}} = \Delta^{k+\ell}$ is given by

$$\phi_j \;=\; \exp\left(\textstyle\sum z_i N_i\right)\xi(t,w)\cdot v_j\,.$$

C.3. **Weight filtration.** The restriction $\mathcal{V}_{\mathrm{e}}|_{Z_W}$ of $\mathcal{V}_{\mathrm{e}}$ to Z_W admits a weight filtration $\mathcal{W}_\bullet$ defined as follows. Fix a coordinate chart $\overline{\mathcal{U}}$ centered at a point of Z_W. Suppose that we have chosen the basis $\{v_j\}$ so that each v_j is contained in some summand $V^{p,q}_{W,F}$ of the Deligne splitting (§3.3.1). That is, the basis is adapted to both W and F. Over $Z_W \cap \overline{\mathcal{U}} \simeq \{0\} \times \Delta^r$, we take $\mathcal{W}_a$ to be the subbundle framed by $\{\phi_j \mid v_j \in W_a\}$. While the ϕ_j depend on our choice of coordinates, the subbundle $\mathcal{W}_a$ does not: this is a consequence of §3.3.3 and the fact that $C_{I,\mathbb{C}} \subset P_{W,\mathbb{C}}$ whenever $W^I = W$.

In this way $(\mathcal{W}^I, \mathcal{F}^p_{\mathrm{e}}|_{Z_W})$ defines a variation of MHS over Z_W.

Finally, we note that the local framing $\{\phi_j\}$ defines isomorphisms

$$\mathcal{W}^I_a\big|_{Z_W \cap \overline{\mathcal{U}}} \;\simeq\; (Z_W \cap \overline{\mathcal{U}}) \times W_a \quad \text{and} \quad \mathcal{F}^p_{\mathrm{e}}\big|_{Z_W \cap \overline{\mathcal{U}}} \;\simeq\; (Z_W \cap \overline{\mathcal{U}}) \times F^p .$$

The isomorphisms make it possible to identify the two definitions (2.1) and (3.7) of Φ^0_I.

Appendix D. Curvature in Hodge theory

D.1. **Tangent bundle.** Recall the Killing form κ of $\mathfrak{g}$, §3.1.2. Define a Hermitian inner product h_φ on $T_\varphi D \simeq \oplus_{p>0}\, \mathfrak{g}^{-p,p}_\varphi$ by $h_\varphi(x,y) = -\kappa(\varphi(\mathbf{i})x, \overline{y})$. This Hermitian inner product is K^0–invariant, and so determines a $G_\mathbb{R}$–invariant Hermitian metric h on TD; that is, we have a homogeneous, Hermitian, holomorphic vector bundle

$$(TD, h) \;=\; G_\mathbb{R} \times_{K^0} (T_\varphi D, h_\varphi) .$$

The curvature 2-form $\Theta_D \in \mathcal{A}^{1,1}(D, \mathfrak{k}^0_\mathbb{C})$ of this metric is of the form

$$\Theta_D \;=\; -\sum_{0>p\,\mathrm{odd}} A_p \wedge {}^t\overline{A_p} \;+\; \sum_{0>p\,\mathrm{even}} A_p \wedge {}^t\overline{A_p} \tag{D.1}$$

for some matrices A_p of holomorphic 1-forms with the property that $A_p(\varphi)$ vanishes on every $\mathfrak{g}^{q,-q}_\varphi$ with $p \neq q$ [GS69, Theorem 4.13]. We say that Θ_D is the difference of disjoint positive $(1,1)$–forms. We briefly review the construction of A_p. Since these forms are homogeneous, it suffices to determine A_p at the point φ. The key observations that are applied below are that (i) the Hodge decomposition (3.1) is polarized by $-\kappa$, and (ii) the identity

$$\kappa([x,y]\,,\, z) \;=\; \kappa(x\,,\,[y,z])\,,$$

for all $x,y,z \in \mathfrak{g}_\mathbb{C}$.

- We may choose a compact Cartan subalgebra $\mathfrak{t}_\mathbb{R} \subset \mathfrak{k}^0_\mathbb{R}$ of $\mathfrak{g}_\mathbb{R}$. Let $\mathcal{R} \subset \mathfrak{t}^*_\mathbb{C}$ denote the roots of $\mathfrak{g}_\mathbb{C}$. Given a root $\alpha \in \mathcal{R}$, let $\mathfrak{g}_\alpha \subset \mathfrak{g}_\mathbb{C}$, be the corresponding root space. If $x_\alpha \in \mathfrak{g}_\alpha$, then $\overline{x_\alpha} \in \mathfrak{g}_{-\alpha}$. So we define $\bar\alpha = -\alpha$. Fix root vectors x_α so that $\overline{x_\alpha} = x_{-\alpha}$. We may scale the x_α so that $-\kappa(x_\alpha, x_{-\beta}) = (-1)^{p_\alpha}\,\delta_{\alpha\beta}$, where $p_\alpha \in \mathbb{Z}$ is defined by $\mathfrak{g}_\alpha \subset \mathfrak{g}^{p_\alpha,-p_\alpha}$.

- Let $\vartheta \in \Omega^1(G_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ be the component of the left-invariant Maurer-Cartan form taking value in $\oplus_{p<0}\, \mathfrak{g}_\varphi^{p,-p} \simeq T_\varphi D$. At the point $\varphi \in D$,
$$\Theta_D \;=\; -[\vartheta, \overline{\vartheta}]_{\mathfrak{k}^0_{\mathbb{C}}} \;\in\; \textstyle\bigwedge^{1,1}(T_\varphi D \oplus \overline{T_\varphi D})^* \,\otimes\, \mathfrak{k}^0_{\mathbb{C}}\,.$$
- We may define $\vartheta^\alpha \in \Omega^1(G_{\mathbb{C}})$ by
$$\sum_{p_\alpha<0} \vartheta^\alpha\, x_\alpha \;=\; \vartheta\,.$$
Employing the identification
$$T_\varphi D \;=\; T_\varphi \check D \;=\; \mathfrak{g}_{\mathbb{C}}/\mathfrak{p}_\varphi \;\simeq\; \oplus_{p>0}\, \mathfrak{g}_\varphi^{-p,p}\,,$$
the forms $\{\vartheta^\alpha \mid p_\alpha < 0\}$ are a basis of the holomorphic cotangent space $T^*_\varphi D$. Likewise, $\{\vartheta^{-\alpha} = \overline{\vartheta^\alpha} \mid \alpha \in \mathcal{R}_p\,,\ p<0\}$ is a basis of the conjugate $\overline{T_\varphi D}^*$.
- Let $\mathcal{R}_p \subset \mathcal{R}$ denote those roots α with $p = p_\alpha$. Write $\vartheta = \sum_{p<0} \vartheta_p$ with $\vartheta_p = \sum_{p_\alpha = p} \vartheta^\alpha\, x_\alpha$ the component of the Maurer–Cartan form taking value in $\mathfrak{g}^{p,-p}$. Then

(D.2) $$\Theta_D \;=\; -\sum_{p<0} [\vartheta_p, \overline{\vartheta_p}]\,.$$

- We have chosen root vectors x_α so that $-\kappa(x_\alpha, \overline{x_\beta}) = (-1)^{p_\alpha}\delta_{\alpha\beta}$. Fix a basis $\{x_1, \ldots, x_r\}$ of $\mathfrak{t}_{\mathbb{R}}$ so that $-\kappa(x_i, x_j) = -\kappa(x_i, \overline{x_j}) = \delta_{ij}$, and set $p_i = 0$ and $\bar{\bar{i}} = i$. Then $\{x_\mu\} = \{x_\alpha\}_{\alpha\in\mathcal{R}} \cup \{x_i\}_{i=1}^r$ is a basis of $\mathfrak{g}_{\mathbb{C}}$ satisfying $-\kappa(x_\mu, \overline{x_\nu}) = (-1)^{p_\mu}\delta_{\mu\nu}$. Then the matrix representation $M_\alpha = (M^\nu_{\alpha\mu})$ of $\mathrm{ad}(x_\alpha) : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ with respect to this basis is defined by $\mathrm{ad}(x_\alpha)x_\mu = M^\nu_{\alpha\mu} x_\nu$, and satisfies ${}^t\overline{M_\alpha} = (-1)^{p_\alpha+1} M_{\overline{\alpha}}$. (The conjugate transpose ${}^t\overline{M_\alpha}$ is defined with respect to h_φ.) So if we identify ϑ_p with the matrix of one forms $A_p = \sum_{p_\alpha=p} \vartheta^\alpha\, M_\alpha$, then (D.1) holds.

D.2. **Hodge bundles.** Many of the vector bundles considered over B are the pullbacks (under the period map Φ) of homogeneous holomorphic vector bundles defined on $\check D \supset D$.

D.2.1. For example, $\check D$ parameterizes filtrations $F^\bullet$ of $V_{\mathbb{C}}$, the trivial bundle $\check D \times V_{\mathbb{C}}$ admits a canonical filtration
$$\mathbf{F}^n \;\subset \ldots \subset\; \mathbf{F}^1 \;\subset \mathbf{F}^0$$
by homogeneous holomorphic vector bundles

(D.3) $$\begin{array}{l} \mathbf{F}^p \,=\, G_{\mathbb{C}} \times_P F^p \\ \downarrow \\ \check D\,. \end{array}$$

and $\mathcal{F}^p = \Phi^*(\mathbf{F}^p)$. (Here, $P \subset G_{\mathbb{C}}$ is the stabilizer of a flag $F^\bullet \in \check D$. We may assume with out loss of generality that $F = \varphi \in D$.) Likewise, the Hodge line bundle $\Lambda = \Phi^*(\mathbf{L})$ with
$$\mathbf{L} \;=\; \det(\mathbf{F}^n) \otimes \det(\mathbf{F}^{n-1}) \otimes \cdots \otimes \det(\mathbf{F}^{\lceil (n+1)/2 \rceil})\,.$$

Define

$$\begin{array}{l} \mathbf{E}^p \;=\; \mathbf{F}^p/\mathbf{F}^{p+1} \;=\; G_{\mathbb{C}} \times_P (F^p/F^{p+1}) \\ \downarrow \\ \check{D}\,. \end{array}$$

Both $\mathbf{L}|_D$ and $\mathbf{E}^p|_D$ admits Hermitian metrics, via the identification (as smooth vector bundles)

$$\begin{array}{rcl} \mathbf{E}^p \;\cong & \mathbf{V}^{p,q} & =\; G_{\mathbb{R}} \times_{K^0} V^{p,q}_{\varphi} \\ & \downarrow & \\ & D\,. & \end{array}$$

By definition $h_\varphi(u,v) = Q(\varphi(\mathbf{i})u, \overline{v})$ defines a Hermitian inner-product on the Hodge summand $V^{p,q}_\varphi$. This inner-product is invariant under the action of $K^0 \subset \mathrm{Aut}(V^{p,q})$, and so determines a homogeneous Hermitian vector bundle

$$(\mathbf{V}^{p,q}, h) \;=\; G_{\mathbb{R}} \times_{K^0} (V^{p,q}_\varphi, h_\varphi)\,.$$

D.2.2. The curvature $\Theta_{\mathbf{E}^p}$ is also given by an expression similar to (D.1), [Gri70, (5.3)]. In fact, the curvature forms Θ_D and $\Theta_{\mathbf{E}^p}$ are even more closely related than this might suggest.

In general, the bundles $\mathbf{U} \to D$ considered in Hodge theory are all of the following type: the are homogeneous, Hermitian vector bundles

$$\begin{array}{l} (\mathbf{U}, h) \;=\; G_{\mathbb{R}} \times_{K^0} (U, h_\varphi) \\ \quad\downarrow \\ \quad D\,, \end{array}$$

with U a K^0 submodule of some Hodge representation $G \to \mathrm{Aut}(\tilde{U}, Q)$, and a Hermitian inner product h_φ induced by a polarization (on $\tilde{U}$ and then restricted to U). They are also the restriction to D of homogeneous, holomorphic vector bundles

$$\begin{array}{l} \mathbf{U} = G_{\mathbb{C}} \times_P U \\ \downarrow \\ \check{D}\,. \end{array}$$

In each of these cases the resulting curvature form is

$$\Theta_{\mathbf{U}} \;=\; -[\vartheta, \overline{\vartheta}]_{\mathfrak{u}} \;=\; -\rho_U([\vartheta, \overline{\vartheta}]_{\mathfrak{k}^0_{\mathbb{C}}}) \;=\; \rho_U(\Theta_D)\,, \tag{D.4}$$

where $\mathfrak{u}$ is the image of the Lie algebra representation $\rho_U : \mathfrak{k}^0_{\mathbb{C}} \to \mathrm{End}(U)$. That is, one may think of the various Θ_U as different matrix representations of the same underlying, endomorphism valued 2-form Θ_D. (Some care must be taken when ρ_U is not faithful.)

D.3. **Chern forms.** As (D.4) suggests, there is a certain sense in which the associated first Chern forms

$$c_1(\mathbf{U}, h) \;=\; \tfrac{\mathbf{i}}{2\pi}\,\mathrm{tr}\,\Theta_{\mathbf{U}} \;\in\; \mathcal{A}^{1,1}_D$$

are all the same; at least when the representation ρ_U is faithful. This is made precise in the following lemma. Essentially they agree up to a constant when restricted to an irreducible invariant subbundle of TD. The key point is to observe that the $c_1(\mathbf{U}, h)$ are all $G_\mathbb{R}$–invariant. So their value at an arbitrary $\tilde\varphi \in D$ is determined by their value at a fixed $\varphi \in D$. So the issue is to show that they agree up to a constant when restricted to an irreducible K^0 submodule $\mathfrak{v} \subset T^\mathbb{R}_\varphi D$ of the (real) tangent space at φ. That restriction $c_1(\mathbf{U}, h)|_\mathfrak{v}$ is K^0 invariant. The same is true of the nondegenerate

$$c_1(TD) \;=\; \tfrac{\mathbf{i}}{2\pi}\mathrm{tr}\,\Theta_D\,.$$

Lemma D.5. *Let $\mathfrak{v} \subset T^\mathbb{R}_\varphi D$ be any irreducible K^0–submodule. Then there exists a constant $\epsilon(U, \mathfrak{v}) \in \mathbb{R}$ so that the restrictions satisfy*

$$c_1(\mathbf{U}, h)|_\mathfrak{v} \;=\; \epsilon(U, \mathfrak{v})\;\, c_1(TD)|_\mathfrak{v}\;.$$

Proof. This is a consequence of Schur's lemma and the nondegeneracy of $c_1(TD)$, [CMSP17, Ch. 13]. □

Remark D.6. We note that when the weight $n = 1, 2$, then the horizontal subspace

$$\mathbf{I}_\varphi \;= \mathfrak{g}^{-1,1}_\varphi \;\subset\; T_\varphi D$$

is an irreducible K^0–module. So in this case Lemma D.5 asserts that any $c_1(\mathbf{U}, h)$ arising naturally in Hodge theory is a multiple of the Kähler form when restricted to the horizontal subspace.

D.4. **Curvature forms under the IPR.** Over the compact dual, the horizontal (homogeneous holomorphic) sub-bundle is

$$\begin{array}{l} \mathbf{I} = G_\mathbb{C} \times_P (F^{-1}\mathfrak{g}_\mathbb{C}/F^0\mathfrak{g}_\mathbb{C}) \;\subset\; T\check D \\ \downarrow \\ \check D. \end{array}$$

When restricting to D we may identify this with

$$\begin{array}{l} \mathbf{I} = G_\mathbb{R} \times_{K_0} (\mathfrak{g}^{-1,1}_\varphi) \;\subset\; TD \\ \downarrow \\ D, \end{array}$$

and we have

$$\Theta_D|_\mathbf{I} \;=\; -A_1 \wedge {}^t\overline{A_1} \;=\; -[\vartheta_1, \overline\vartheta_1]\,. \tag{D.7}$$

This two form takes value in

$$[\mathfrak{g}_\varphi^{-1,1}, \mathfrak{g}_\varphi^{1,-1}] \;\subset\; \mathfrak{g}_\varphi^{0,0} \;=\; \mathfrak{k}_{\mathbb{C}}^0 \,.$$

The holomorphic sectional curvature of the period domain is negative and bounded away from zero in the horizontal directions [CMSP17, Theorem 13.6.3]. Likewise, there exists $\epsilon > 0$ so that

$$\operatorname{tr}\Theta_D(v,\bar v) \;=\; -\operatorname{tr}\left(A_1(v)\,{}^t\overline{A_1(v)}\right) \;<\; -\epsilon\, h(v)\,, \quad \forall\, v \in \mathbf{I}|_D \,. \tag{D.8}$$

Lemma D.5 implies there exist positive constants ϵ, ϵ' so that

$$-\epsilon\ c_1(TD)|_{\mathbf{I}} \;\le\; c_1(\mathbf{L})|_{\mathbf{I}} \;\le\; -\epsilon'\ c_1(TD)|_{\mathbf{I}} \,. \tag{D.9}$$

D.4.1. *Proof of Lemma 9.17.* The key observation is that the restriction

$$\mathrm{Gr}^{-1}_{\mathcal{F}_{\mathrm{e}}}\mathrm{End}(\mathcal{E}_{\mathrm{e}})\big|_B \;\simeq\; \Phi^*(\mathbf{I})\,.$$

Set

$$\mathbf{H} \;=\; \textstyle\bigwedge^{\dim B}\mathbf{I}\,,$$

so that

$$\mathcal{H} \;=\; \Phi^*(\mathbf{H})\,. \tag{D.10}$$

Since Θ_D and $\Theta_{\mathbf{H}}$ are homogeneous, Schur's lemma implies there exist positive constants ϵ_1, ϵ_2 so that

$$\epsilon_1 \operatorname{tr}\ \Theta_D|_{\mathbf{I}} \;\le\; \operatorname{tr}\ \Theta_{\mathbf{H}}|_{\mathbf{I}} \;\le\; \epsilon_2 \operatorname{tr}\ \Theta_D|_{\mathbf{I}} \,. \tag{D.11}$$

The lemma now follows from (D.9) and (D.10). □

Remark D.12. The inequalities (D.11) may also be deduced from Lemma D.5. Indeed, one may show that there exist positive constants ϵ, ϵ' so that the the Chern forms satisfy

$$\begin{array}{rcccl} \epsilon\ c_1(\mathbf{L})|_{\mathbf{I}} & \le & -\,c_1(TD)|_{\mathbf{I}} & \le & \epsilon'\ c_1(\mathbf{L})|_{\mathbf{I}}\,, \\ \epsilon\ c_1(\mathbf{L})|_{\mathbf{I}} & \le & -\,c_1(\mathbf{H})|_{\mathbf{I}} & \le & \epsilon'\ c_1(\mathbf{L})|_{\mathbf{I}}\,. \end{array}$$

D.4.2. *Proof of Lemma 9.18.* The outline of the proof is sketched on page 71; here we verify the details.

Fix a smooth local framing $\{\eta_j\}$ of

$$\mathcal{H} \;=\; \Phi^*(\mathbf{H})$$

over B so that η_1 spans $(K_{\overline{B}} + [Z])^* = \bigwedge^{\dim B} T_{\overline{B}}(-\log Z)$, and is orthogonal to the $\{\eta_a\}_{a\ge 2}$ with respect to the Hermitian form. Let $\nabla \eta_j = \theta^i_j \otimes \eta_i$ denote the local connection 1-forms, and

$$\Theta_{\mathcal{H}} \;=\; \Theta^i_j\, \eta_i \otimes \eta^j$$

the local curvature $(1,1)$–forms. Then

$$\Theta_{\mathcal{H}}\big|_{(K_{\overline{B}}+[Z])^*} \;=\; \Theta^1_1 \quad\text{and}\quad \Theta_{(K_{\overline{B}}+[Z])^*} \;=\; \mathrm{d}\theta^1_1\,,$$

are related by

$$\Theta^1_1 \;=\; \mathrm{d}\theta^1_1 \;+\; \theta^1_j\wedge\theta^j_1\,.$$

The fact that η_1 is h–orthogonal to $\{\eta_a\}_{a\geq 2}$ implies that

$$h_{11}\,\theta^1_a \;+\; h_{ab}\,\overline{\theta^b_1} \;=\; 0\,,$$

with $h_{11}=h(\eta_1,\eta_1)$ and $h_{ab}=h(\eta_a,\eta_b)$. Since $(h_{ab})_{a,b\geq 2}$ is positive definite, we see that

$$\frac{\mathbf{i}}{2\pi}\,\theta^1_j\wedge\theta^j_1 \;=\; \frac{\mathbf{i}h_{ab}}{2\pi h_{11}}\,\theta^a_1\wedge\overline{\theta^b_1}$$

is a non-negative $(1,1)$–form. Setting

$$(\Upsilon,\Upsilon) \;=\; \frac{1}{h_{11}}h_{ab}\,\theta^a_1\wedge\overline{\theta^b_1} \tag{D.13}$$

yields (9.20).

Remark D.14. The endomorphism valued $(1,0)$–form

$$\Upsilon \;=\; \theta^b_1\,\eta_b\otimes\eta^1$$

is the second fundamental form (9.19) of $(K_{\overline{B}}+[Z])^*\subset\mathcal{H}$, [Gri70, §4].

It remains to show that $-\frac{\mathbf{i}}{2\pi}h(\Theta\cdot\eta_1,\eta_1)$ is positive. This is a consequence of the structure (D.4) and (D.7) of the curvature, and the IPR. Given $b\in B$, the injectivity of the differential $\mathrm{d}\Phi$ and the IPR allow us to identify identify T_bB with an abelian subalgebra $\mathfrak{b}\subset\mathfrak{g}^{-1,1}_\varphi$, $\dim\mathfrak{b}=\dim B$. So for the purpose of this algebraic computation we may work point-wise and identify $\Theta_{\mathcal{H}}$ with $\Theta=-[\vartheta_1,\overline{\vartheta}_1]$, with ϑ_1 taking value $\mathfrak{b}$, and η_1 with an element of the line $\bigwedge^{\dim B}\mathfrak{b}\subset\bigwedge^{\dim B}\mathfrak{g}^{-1,1}_\varphi$. The adjoint action $\mathrm{ad}:\mathfrak{g}_\mathbb{C}\to\mathrm{End}(\mathfrak{g}_\mathbb{C})$ induces an action of $\mathfrak{g}_\mathbb{C}$ on $\bigwedge^{\dim B}\mathfrak{g}_\mathbb{C}$. The fact that $\mathfrak{b}$ is abelian implies $\vartheta_1\cdot\eta_1=0$, so that $[\vartheta_1,\overline{\vartheta}_1]\cdot\eta_1=\vartheta_1\overline{\vartheta}_1\cdot\eta_1$ and

$$\begin{aligned}
h(\Theta\cdot\eta_1,\eta_1) &= -h\left([\vartheta_1,\overline{\vartheta}_1]\cdot\eta_1\,,\,\eta_1\right)\\
&= -h\left(\vartheta_1\,\overline{\vartheta}_1\cdot\eta_1\,,\,\eta_1\right)\\
&= -h(\overline{\vartheta}_1\cdot\eta_1\,,\,\overline{\vartheta}_1\cdot\eta_1)\,.
\end{aligned}$$

The final equality is the due to the fact that the $\overline{\vartheta}_1$ is the h–conjugate transpose of ϑ_1, [CMSP17, Corollary 12.6.3].

To see that $\overline{\vartheta}_1 \cdot \eta_1 \in \bigwedge^{\dim B} \mathfrak{g}_{\mathbb{C}}$ is nonzero, recall that ϑ_1 is nonzero (the Torelli hypothesis) and takes value in $\mathfrak{b}$. We can complete $\mathfrak{b}$ to a basis $\{\vartheta_1, \xi_2, \ldots, \xi_d\}$ so that $\eta_1 = \vartheta_1 \wedge \xi_2 \wedge \cdots \wedge \xi_d$. Keeping in mind that $[\overline{\vartheta}_1, \vartheta_1]$ is nonzero [CMSP17, Corollary 12.6.3], we see that

$$\overline{\vartheta}_1 \cdot \eta_1 \;=\; [\overline{\vartheta}_1, \vartheta_1] \wedge \xi_2 \wedge \cdots \wedge \xi_d \;+\; \sum_{j=2}^{d} \vartheta_1 \wedge \xi_2 \wedge \cdots \wedge [\overline{\vartheta}_1, \xi_j] \wedge \cdots \wedge \xi_d$$

is nonzero. This establishes the positivity of $h(\Theta \cdot \eta_1, \eta_1)$.

It now follows from the local Torelli assumption (§9.1 and Lemma 9.12) that there exists $\epsilon > 0$ so that $\epsilon\, c_1(\Lambda_{\mathrm{e}}) \leq -\frac{\mathbf{i}}{2\pi} h(\Theta \cdot \eta_1, \eta_1)$. A priori this ϵ depends on our choice of $b \in B$. However, homogeneity under the action of $G_{\mathbb{R}}$ and the fact that the grassmannian $\mathrm{Gr}(\dim B, \mathfrak{g}_{\varphi}^{-1,1}) \ni \mathfrak{b}$ is compact imply that we can find $\epsilon > 0$ that works for all $b \in B$. □

D.4.3. *Proof of Lemma 9.24.* As indicated in the sketch of the proof (page 73) it suffices to show that $\Upsilon|_{A^0}$ may be identified with the differential of the Gauss map $\mathcal{G}(\Phi^1\big|_{A^0})$. To see this recall the set-up of §D.4.2. The local section η_1 of $(K_{\overline{B}} + [Z])^*$ defines (globally) a map

$$[\eta_1] : \overline{B} \;\to\; \mathbb{P}(\textstyle\bigwedge^{\dim B} T_{\overline{B}}(-\log Z))\,.$$

The derivative of $[\eta_1]$ is

$$\begin{aligned} \mathrm{d}[\eta_1] \;&=\; \nabla \eta_1 \mod (K_{\overline{B}} + [Z])^* \\ &=\; \sum_{a \geq 2} \theta_1^a\, \eta_a \mod (K_{\overline{B}} + [Z])^* \end{aligned}$$

and may be identified with Υ (defined in Remark D.14). We may then identify the differential $\mathrm{d}\mathcal{G}(\Phi^1\big|_{A^0})$ with the restriction of $(\theta_1^a)_{a\geq 2}$ to A^0. It then follows from (D.13) that $\tau|_{A^0}$ is positive if and only if $\mathrm{d}\mathcal{G}(\Phi^1\big|_{A^0})$ is injective. □

Remark D.15. It follows from Remark 7.2 and Lemma 9.12 that $[\eta_1]$ may be thought of as a Gauss map for Φ^{T}.

References

[AMRT75] A. Ash, D. Mumford, M. Rapoport, and Y. Tai. *Smooth compactification of locally symmetric varieties*. Math. Sci. Press, Brookline, Mass., 1975. Lie Groups: History, Frontiers and Applications, Vol. IV.

[BB66] W. L. Baily, Jr. and A. Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Ann. of Math. (2)*, 84:442–528, 1966.

[BB20] Damian Brotbek and Yohan Brunebarbe. Arakelov-Nevanlinna inequalities for variations of Hodge structures and applications. arXiv:2007.12957, 2020.

[BBT18] Benjamin Bakker, Yohan Brunebarbe, and Jacob Tsimerman. o-minimal GAGA and Hodge Theory. arXiv:1811.12230, 2018.

[BBT20] Benjamin Bakker, Yohan Brunebarbe, and Jacob Tsimerman. Quasiprojectivity of images of mixed period maps. arXiv:2006.13709, June 2020.

[BKT18] Benjamin Bakker, Bruno Klingler, and Jacob Tsimerman. Tame topology of arithmetic quotients and algebraicity of Hodge loci. arXiv:1810.04801, 2018.

[Bor72] Armand Borel. Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem. *J. Differential Geometry*, 6:543–560, 1972. Collection of articles dedicated to S. S. Chern and D. C. Spencer on their sixtieth birthdays.

[BPR17] P. Brosnan, G. Pearlstein, and C. Robles. Nilpotent cones and their representation theory. In Lizhen Ji, editor, *Hodge theory and L^2-analysis*, volume 39 of *Adv. Lect. Math. (ALM)*, pages 151–205. Int. Press, Somerville, MA, 2017. arXiv:1602.00249.

[Bro14] Francis Brown. Motivic periods and $\mathbb{P}^1\backslash\{0,1,\infty\}$. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pages 295–318. Kyung Moon Sa, Seoul, 2014.

[Bru20] Yohan Brunebarbe. Increasing hyperbolicity of varieties supporting a variation of hodge structures with level structures. arXiv:2007.12965, 2020.

[Car87] James A. Carlson. The geometry of the extension class of a mixed Hodge structure. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46, pages 199–222. Amer. Math. Soc., Providence, RI, 1987.

[Cat74] Eduardo Cattani. On the partial compactification of the arithmetic quotient of a period matrix domain. *Bull. Amer. Math. Soc.*, 80:330–333, 1974.

[CCK80] James A. Carlson, Eduardo H. Cattani, and Aroldo G. Kaplan. Mixed Hodge structures and compactifications of Siegel's space (preliminary report). In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979.* 77–105, 1980.

[CDK95] Eduardo Cattani, Pierre Deligne, and Aroldo Kaplan. On the locus of Hodge classes. *J. Amer. Math. Soc.*, 8(2):483–506, 1995.

[CEZGT14] Eduardo Cattani, Fouad El Zein, Phillip A. Griffiths, and Lê Dũng Tráng, editors. *Hodge theory*, volume 49 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 2014.

[CK77] Eduardo Cattani and Aroldo Kaplan. Extension of period mappings for Hodge structures of weight two. *Duke Math. J.*, 44(1):1–43, 1977.

[CK82] Eduardo Cattani and Aroldo Kaplan. Polarized mixed Hodge structures and the local monodromy of a variation of Hodge structure. *Invent. Math.*, 67(1):101–115, 1982.

[CKS86] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid. Degeneration of Hodge structures. *Ann. of Math. (2)*, 123(3):457–535, 1986.

[CM93] David H. Collingwood and William M. McGovern. *Nilpotent orbits in semisimple Lie algebras.* Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993.

[CMGHL17] Sebastian Casalaina-Martin, Samuel Grushevsky, Klaus Hulek, and Radu Laza. Extending the Prym map to toroidal compactifications of the moduli space of abelian varieties. *J. Eur. Math. Soc. (JEMS)*, 19(3):659–723, 2017.

[CMSP17] James Carlson, Stefan Müller-Stach, and Chris Peters. *Period mappings and period domains*, volume 168 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017.

[ČS09] Andreas Čap and Jan Slovák. *Parabolic geometries. I*, volume 154 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2009. Background and general theory.

[CT14] James A. Carlson and Domingo Toledo. Compact quotients of non-classical domains are not Kähler. In *Hodge theory, complex geometry, and representation theory*, volume 608 of *Contemp. Math.*, pages 51–57. Amer. Math. Soc., Providence, RI, 2014.

[DLSZ19] Ya Deng, Steven Lu, Ruiran Sun, and Kang Zuo. Picard theorems for moduli spaces of polarized varieties. arXiv:1911.02973, 2019.

[FGG$^+$20] Marco Franciosi, Mark Green, Phillip Griffiths, Radu Laza, Rita Pardini, Sönke Rollenske, and Colleen Robles. Monograph on modui spaces of algebraic surfaces and Hodge theory. In preparation, 2020.

[FHW06] Gregor Fels, Alan Huckleberry, and Joseph A. Wolf. *Cycle spaces of flag domains*, volume 245 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2006. A complex geometric viewpoint.

[FPR15a] Marco Franciosi, Rita Pardini, and Sönke Rollenske. Computing invariants of semi-log-canonical surfaces. *Math. Z.*, 280(3-4):1107–1123, 2015.

[FPR15b] Marco Franciosi, Rita Pardini, and Sönke Rollenske. Log-canonical pairs and Gorenstein stable surfaces with $K_X^2 = 1$. *Compos. Math.*, 151(8):1529–1542, 2015.

[FPR17] Marco Franciosi, Rita Pardini, and Sönke Rollenske. Gorenstein stable surfaces with $K_X^2 = 1$ and $p_g > 0$. *Math. Nachr.*, 290(5-6):794–814, 2017. arXiv:1511.03238.

[Fri83] Robert Friedman. Global smoothings of varieties with normal crossings. *Ann. of Math. (2)*, 118(1):75–114, 1983.

[FS86] Robert Friedman and Roy Smith. Degenerations of Prym varieties and intersections of three quadrics. *Invent. Math.*, 85(3):615–635, 1986.

[GG20] Mark Green and Phillip Griffiths. Positivity of vector bundles and Hodge theory. *Adv. Math.*, 2020. to appear, arXiv:1803.07405.

[GGK12] Mark Green, Phillip Griffiths, and Matt Kerr. *Mumford-Tate groups and domains: their geometry and arithmetic*, volume 183 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2012.

[GGK13] Mark Green, Phillip Griffiths, and Matt Kerr. *Hodge theory, complex geometry, and representation theory*, volume 118 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2013.

[GGLR20] M. Green, P. Griffiths, R. Laza, and C. Robles. Period mappings and properties of the Hodge line bundle. arXiv:1708.09523, 2020.

[GGR17] Mark Green, Phillip Griffiths, and Colleen Robles. Extremal degenerations of polarized Hodge structures. In Lizhen Ji, editor, *Hodge theory and L^2-analysis*, volume 39 of *Adv. Lect. Math. (ALM)*, pages 321–376. Int. Press, Somerville, MA, 2017. arXiv:1403.0646.

[GH94] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[Gol19] Wushi Goldring. The Griffiths bundle is generated by groups. *Math. Ann.*, 375(3-4):1283–1305, 2019.

[Gon01] A.B. Goncharov. Multiple polylogarithms and mixed tate motives. arXiv:0103059, May 2001.

[Gra83] Hans Grauert. Set theoretic complex equivalence relations. *Math. Ann.*, 265(2):137–148, 1983.

[Gri70] Phillip A. Griffiths. Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping. *Inst. Hautes Études Sci. Publ. Math.*, pages 125–180, 1970.

[Gri18] Phillip Griffiths. Hodge Theory and Moduli (St Petersburg). https://albert.ias.edu/handle/20.500.12111/7864, May 2018.

[Gri19] Phillip Griffiths. Using Hodge theory to detect the structure of a compactified moduli space. https://hdl.handle.net/20.500.12111/7873, November 2019.

[Gri20] Phillip Griffiths. Hodge Theory and Moduli (Clay). https://albert.ias.edu/handle/20.500.12111/7885, 2020.

[GRT14] Phillip Griffiths, Colleen Robles, and Domingo Toledo. Quotients of non-classical flag domains are not algebraic. *Algebr. Geom.*, 1(1):1–13, 2014.

[GS69] Phillip Griffiths and Wilfried Schmid. Locally homogeneous complex manifolds. *Acta Math.*, 123:253–302, 1969.

[Hai94] Richard M. Hain. Classical polylogarithms. In *Motives (Seattle, WA, 1991)*, volume 55, pages 3–42. Amer. Math. Soc., Providence, RI, 1994.

[HZ87a] Richard M. Hain and Steven Zucker. A guide to unipotent variations of mixed Hodge structure. In *Hodge theory (Sant Cugat, 1985)*, volume 1246 of *Lecture Notes in Math.*, pages 92–106. Springer, Berlin, 1987.

[HZ87b] Richard M. Hain and Steven Zucker. Unipotent variations of mixed Hodge structure. *Invent. Math.*, 88(1):83–124, 1987.

[KLMS06] Matt Kerr, James D. Lewis, and Stefan Müller-Stach. The Abel-Jacobi map for higher Chow groups. *Compos. Math.*, 142(2):374–396, 2006.

[Kol87] János Kollár. Subadditivity of the Kodaira dimension: fibers of general type. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 361–398. North-Holland, Amsterdam, 1987.

[Kol93] János Kollár. Effective base point freeness. *Math. Ann.*, 296(4):595–605, 1993.

[Kol13] János Kollár. Moduli of varieties of general type. In *Handbook of moduli. Vol. II*, volume 25 of *Adv. Lect. Math. (ALM)*, pages 131–157. Int. Press, Somerville, MA, 2013.

[Kos59] Bertram Kostant. The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group. *Amer. J. Math.*, 81:973–1032, 1959.

[KP14] Matt Kerr and Gregory Pearlstein. Naive boundary strata and nilpotent orbits. *Ann. Inst. Fourier (Grenoble)*, 64(6):2659–2714, 2014.

[KP16] Matt Kerr and Gregory Pearlstein. Boundary components of Mumford-Tate domains. *Duke Math. J.*, 165(4):661–721, 2016.

[KU09] Kazuya Kato and Sampei Usui. *Classifying spaces of degenerating polarized Hodge structures*, volume 169 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Lew06] James D. Lewis. Abel–jacobi maps associated to algebraic cycles. Leiden seminar, www.math.ualberta.ca/people/Faculty/Lewis,%20James/Leiden.pdf, 2006.

[Loo03a] Eduard Looijenga. Compactifications defined by arrangements. I. The ball quotient case. *Duke Math. J.*, 118(1):151–187, 2003.

[Loo03b] Eduard Looijenga. Compactifications defined by arrangements. II. Locally symmetric varieties of type IV. *Duke Math. J.*, 119(3):527–588, 2003.

[Mor84] David R. Morrison. The Clemens-Schmid exact sequence and applications. In *Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982)*, volume 106 of *Ann. of Math. Stud.*, pages 101–119. Princeton Univ. Press, Princeton, NJ, 1984.

[Mum75] David Mumford. A new approach to compactifying locally symmetric varieties. In *Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973)*, pages 211–224. 1975.

[Mum77] D. Mumford. Hirzebruch's proportionality theorem in the noncompact case. *Invent. Math.*, 42:239–272, 1977.

[PS08] Chris A. M. Peters and Joseph H. M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2008.

[Sat73] I. Satake. On the arithmetic of tube domains (blowing-up of the point at infinity). *Bull. Amer. Math. Soc.*, 79:1076–1094, 1973.

[SB19] N.I. Shepherd-Barron. Asymptotic period relations for jacobian elliptic surfaces. arXiv:1904.13344, 2019.

[SB20] N.I. Shepherd-Barron. Effective generic torelli theorems for elliptic surfaces. arXiv:2009.03633, 2020.

[Sch73] Wilfried Schmid. Variation of Hodge structure: the singularities of the period mapping. *Invent. Math.*, 22:211–319, 1973.

[Som73] Andrew J. Sommese. Some algebraic properties of the image of the period mapping. *Rice Univ. Studies*, 59(2):123–128, 1973. Complex analysis, 1972 (Proc. Conf., Rice Univ., Houston, Tex., 1972),Vol. II: Analysis on singularities.

[Ste76] Joseph Steenbrink. Limits of Hodge structures. *Invent. Math.*, 31(3):229–257, 1975/76.

[Vie83a] Eckart Viehweg. Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, volume 1 of *Adv. Stud. Pure Math.*, pages 329–353. North-Holland, Amsterdam, 1983.

[Vie83b] Eckart Viehweg. Weak positivity and the additivity of the Kodaira dimension. II. The local Torelli map. In *Classification of algebraic and analytic manifolds (Katata, 1982)*, volume 39 of *Progr. Math.*, pages 567–589. Birkhäuser Boston, Boston, MA, 1983.

[Wol69] Joseph A. Wolf. The action of a real semisimple group on a complex flag manifold. I. Orbit structure and holomorphic arc components. *Bull. Amer. Math. Soc.*, 75:1121–1237, 1969.

[Zuo00] Kang Zuo. On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications. *Asian J. Math.*, 4(1):279–301, 2000. Kodaira's issue.

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