Hodge theory and Moduli

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References

I. Introduction

- Moduli is a topic of central interest in algebraic geometry. The theory roughly organizes into three areas:
  - varieties X of general type ($\kappa(X) = \dim X, K_X > 0$);
  - Calabi-Yau varieties ($\kappa(X) = 0$, $K_X = 0$);
  - Fano varietes ($\kappa(X) = -\infty$, $K_X < 0$).

  This lecture is mainly concerned with the first type.
- The techniques for studying moduli also roughly divide into three types:
  - algebraic (birational geometry, singularity theory, geometric invariant theory (GIT), etc.);
  - Hodge theoretic (topological and geometric);
  - analytic (L^2-$\overline{\partial}$ techniques, construction and properties of special metrics).

  The algebraic techniques are currently the main ones. The three methods also of course interact; e.g., complex analysis plays a central role in Hodge theory.

- For varieties of general type, drawing on ideas from the minimal model program Kollár–Shepherd-Baron–Alexeev (KSBA) proved the existence of a moduli space $\mathcal{M}$ having a canonical completion, later proved to be projective (cf. [KSB], the survey paper [K] and the references cited therein).
- For this talk the general motivating question is

What is the structure of $\overline{\mathcal{M}}$*?*

By structure, informally stated we mean the stratification of $\overline{\mathcal{M}}$ where the strata correspond to varieties of the same deformation type (equisingular deformations). We also include the incidence relations among the strata. For surfaces where there is a classification, we include which surfaces in the classification occur in a stratum.

The method we will discuss to study this question is to use Hodge theory. With notation to be more fully explained later, we denote by $\mathcal{P}$ the image of the period mapping $\Phi : \mathcal{M} \to \Gamma \backslash D$ and by $\overline{\mathcal{P}}$ the canonical minimal completion of $\mathcal{P}$.[†] To help understand the structure there are two basic types of subvarieties of $\overline{\mathcal{P}}$ and then there is the amalgam of these. The first type is the stratification associated to the boundary components given by limiting mixed Hodge structures $(V, Q, W(N), F_{\lim})$[‡]. Lie theory provides a classification of how this may happen [KR].

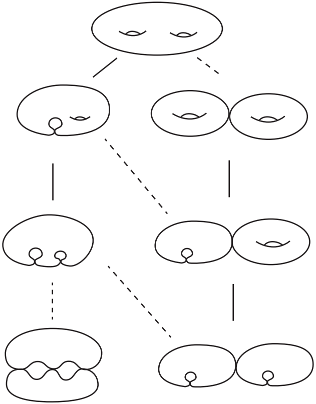
[†] Notably, in the non-classical case $\overline{\mathcal{P}}$ is not a subset of a completion of $\Gamma \backslash D$; it is a relative construction associated to period mapping.

[‡] The notations and terminology will be recalled below. For general background in Hodge theory we refer to [GGLR] and [GG].

The other type of subvarieties of $\overline{\mathcal{P}}$ is that corresponding to the Mumford-Tate sub-domains $D' \subset D$. Associated to a polarized Hodge structure (V, Q, F) is the algebra $T(F)$ of Hodge tensors in the tensor algebra of V, and D' is the orbit in D of F under the Lie subgroup $G' \subset G$ preserving that algebra. Geometrically, for algebraic surfaces in first approximation one thinks of those X's having additional Hodge classes in $H^2(X)$.

Our objective is use the structure of $\overline{\mathcal{P}}$ to help understand and organize the structure of $\overline{\mathcal{M}}$. One of the simplest illustrations of this is given by the following

Model Example: For algebraic curves the structure of $\overline{\mathcal{M}}_g$ is a much studied and very beautiful subject. For the first case $g = 2$ the picture of the stratification is



The results we shall discuss about algebraic surfaces are of the following two types.

1. General results valid for any KSBA moduli space of general type surfaces.
2. Results about I surfaces, defined to be smooth surfaces X with $q(X) = 0$, $p_g(X) = 2$ and K_X ample.

Informally stated we shall see there are three results about the completed moduli space $\overline{\mathcal{M}}_I$:

(a) for the part $\overline{\mathcal{M}}_I^G$ of Gorenstein degenerations there is an analogous picture to the solid line part of the one above for $g = 2$ curves; the stratification of $\overline{\mathcal{M}}^G$ is faithfully captured by the extended period mapping

$$\overline{\Phi} : \overline{\mathcal{M}}^G \to \overline{\mathcal{P}};$$

(b) in a phenomenon not present in the curve case, Hodge theory provides a guide as to how to desingularize a general point of the boundary $\overline{\mathcal{M}}_I^G \backslash \mathcal{M}_I$;

(c) for the part $\overline{\mathcal{M}}_I^{NG}$ of normal, non-Gorenstein degenerations these correspond to the dotted lines in the above figure and there are partial results, an interesting example, and a question/conjecture about what the general story might be.

Summarizing:

- For curves, $\overline{\mathcal{M}} = \overline{\mathcal{M}}_g$ is much studied and much is known about its rich structure.
- For $\dim X \geqq 2$, so far as I am aware only a few examples have been partially worked out (cf. [FPR], [H]). In this lecture we will use Hodge theory as a guide to help give some answers to the above question for what is in some sense the "first" general type surface one comes to (an analogue of $g = 2$ curves).
- An invariant of $\mathcal{M}$ is given by the period mapping

$$\Phi : \mathcal{M} \to \mathcal{P} \subset \Gamma \backslash D.$$

Here D is the period domain parametrizing polarized Hodge structures (V, Q, F) of a given weight and type, Γ is a discrete subgruop of $\mathrm{Aut}(V_{\mathbb{Z}}, G)$ that contains the monodromy group (discussed further below).[§]

- There is a Hodge-theoretically constructed minimal canonical completion $\overline{\mathcal{P}}$ of $\mathcal{P}$ and an extension

$$\overline{\Phi} : \overline{\mathcal{M}} \to \overline{\mathcal{P}}$$

of the period mapping. Our objective is to use the known structure of $\overline{\mathcal{P}}$ together with general algebro-geometric methods to infer properties of $\overline{\mathcal{M}}$.

[§] It has recently been proved in [BBT] that when Γ is arithmetic $\mathcal{P}$ is an algebraic variety over which the augmented Hodge line bundle $\Lambda \to \mathcal{P}$ is ample.

II. Background and general results

A. Moduli theory (informal account)

- We will consider moduli spaces $\mathcal{M}$ whose points correspond to (equivalence classes of) varieties X that have the property
  - X is smooth or has canonical singularities;
  - K_X is ample.

  The first condition means that the Weil canonical divisor class K_X is a line bundle and that for a minimal disingularization $\widetilde{X} \to X$ we have $f^*K_X = K_{\widetilde{X}}$.
- For this talk there are two main points:
  (i) the canonical completion $\overline{\mathcal{M}}$ exists (we will recall its definition);
  (ii) in the case when $\dim X = 1, 2$ there is a classification of the singularities of the curves and surfaces corresponding to points of $\partial\mathcal{M} = \overline{\mathcal{M}} \backslash \mathcal{M}$.

For (i) we use the valuative criterion: Given a family

$$\mathcal{X}^* \xrightarrow{\pi} \Delta^*$$

of smooth general type varieties $X_t = \pi^{-1}(t)$, possibly after a base change we want to define a unique limit $\lim_{t\to 0} X_t = X_0$. The conditions are

(a) mK_{X_0} should be a line bundle for some m;

(b) X_0 has semi-log-canonical (slc) singularities;

(c) K_{X_0} is ample.

Condition (b) is local along X_0; (c) is global.

To explain (b) we consider a minimal completion

$$\begin{array}{ccc} \mathcal{X}^* & \subset & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \Delta^* & \subset & \Delta. \end{array}$$

Then in 1^{st} approximation (b) means that

$$\mathcal{X} \text{ should have canonical singularities.}$$

Varieties X_0 with these properties are said to be *stable*. For a discussion of highly non-trivial technical issues we refer to [K], [H] and the references cited there.

- The second point is that for curves and surfaces the singularities of a stable X_0 have been classified.

For simplicity of notation we shall simply use X instead of X_0. For curves, the singularities of X consist of nodes. For surfaces it will be convenient to use a rough organization of the singularity type given by the table

X	normal singularities	non-normal singularities
K_X	G	NG

where G stands for Gorenstein and NG stands for non-Gorenstein.

In the K_X-NG spot, by definition there is smallest integer, the *index* $m \geqq 2$ of X, such that mK_X is a line bundle.[¶] The entries in the first row mean that the singularities of X could be isolated (i.e., points), or could occur along curves. In the K_X-G spot, $K_X = \omega_X$ is the dualizing sheaf and is a line bundle.

In this talk we shall be particularly interested in the case when X has normal singularities; we shall denote by (X, p) the pair given by a stable surface X and a normal (and hence isolated) singular point p. Then the classification breaks into 2-types.

[¶] A significant issue is to give a good bound on the index. Here we refer to [RU] for interesting recent work.

K_X-G: These include the *canonical singularities*, concerning which there is a rich and vast literature (e.g., Chapter 4 in [R]). They are also referred to as Du Val or ADE singularities and are locally analytically equivalent to isolated hypersurface singularities $f(x_1, x_2, x_3) = 0$ in $\mathcal{U} \subset \mathbb{C}^3$. For example, A_n is given by

$$x_1^2 + x_2^2 + x_3^{n+1} = 0.$$

For $n = 1$ there is the standard resolution $(\widetilde{X}, \widetilde{C}) \to (X, p)$ where $\widetilde{C} \cong \mathbb{P}^1$ is a -2 curve (i.e., $\widetilde{C}^2 = -2$). In general the $\widetilde{C}$ is a configuration of -2 rational curves corresponding to the nodes in a Dynkin diagram.

The remaining singularities are the simple elliptic singularities and the cusps. An important non-trivial constraint in the moduli theory of surfaces considered in this talk is that the isolated singularities (normal case) should be smoothable. For simple elliptic singularities this means that the degree $d = -\widetilde{C}^2$ should satisfy $1 \leqq d \leqq 9$.

For the next type we shall use the singularity theorists' notation

$$\frac{1}{n}(1, r), \qquad \gcd(n, r) = 1$$

for the quotient $\mathbb{C}^2 / \left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta^r \end{smallmatrix}\right)$ where $\zeta = e^{2\pi i/n}$ is a primitive n^{th} root of unity.

K_X-**NG:** These are required to be $\mathbb{Q}$-Gorenstein smoothable, meaning that there should be a local smoothing whose relative dualizing sheaf is $\mathbb{Q}$-Cartier (cf. [H]). Then there are two types of such singularities:

(i) the $\frac{1}{dn^2}(1, dna-1)$ singularities; for $d=1$ these are called *Wahl singularities*. Again for these there is an extensive literature (cf. [H] and the references cited therein);

(ii) the $\mathbb{Z}_2$-quotients of simple elliptic or cusp singularities (cf. (3.24)(c) in [K]).

The non-isolated KSBA singularities are given by pairs (X, C) where C is a (possibly reducible) double curve having isolated pinch points and nodes. Typically there is a resolution

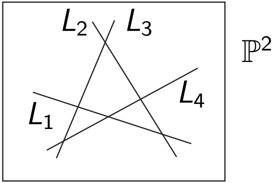
$$(\widetilde{X}, \widetilde{C}) \to (X, C)$$

where $\widetilde{X}$ is smooth, $\widetilde{C} \subset \widetilde{X}$ is a possibly reducible nodal curve with an involution

$$\tau : \widetilde{C} \to \widetilde{C},$$

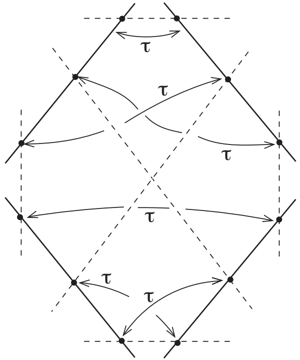
and (X, C) is the quotient of $(\widetilde{X}, \widetilde{C})$ by the involution τ where we identify $p \in \widetilde{C}$ with $\tau(p) \in \widetilde{C}$.

A particularly interesting example of this is due to Liu-Rollenske [LR]. Here $\widetilde{X}$ will be a blow up of $\mathbb{P}^2$, and the initial picture is



where we identify L_1 and L_2 by $\left\{\begin{matrix} 12 \longleftrightarrow 21 \\ 13 \longleftrightarrow 24 \\ 14 \longleftrightarrow 23 \end{matrix}\right\}$ and similarly for L_3 and L_4.

To define τ we must blow up the intersection points $E_{ij} = L_i \cap L_j$



The LR choice of τ is drawn in. Dotted lines are exceptional divisors E_{ij}. Here $\widetilde{C}$ is a reducible nodal curve. The surface X is an example of Kollár's gluing construction.

- Finally we shall be particularly interested in two subvarieties of $\overline{\mathcal{M}}$:
  - $\mathcal{M}_f \subset \overline{\mathcal{M}}$, defined to be the points corresponding to surfaces that have a smoothing with finite monodromy;
  - $\overline{\mathcal{M}}^G \subset \overline{\mathcal{M}}$, defined to be the points corresponding to surfaces X whose singularities are Gorenstein. In this case $K_X = \omega_X$ is a line bundle, and duality and Riemann-Roch hold as if X were smooth.

B. Hodge theory

- Roughly speaking Hodge theory has at least the following four interrelated aspects:

(1) *Topological*: the deeper topological properties of complex algebraic varieties X arise from the functorial Hodge structure, or mixed Hodge structure, on the cohomology $H^*(X)$.

(2) *Analytical*: associated to a family H^n_t of Hodge structures on the punctured disc Δ^* there is an essentially unique limiting mixed Hodge structure H^n_{lim}; the topological properties of the family are governed by the monodromy

$$T = T_s T_u \qquad \text{(Jordan decomposition)}$$

where the semi-simple part T_s is of finite order (the eigenvalues are roots of unity), and the unipotent part $T_u = e^N$ where $N^{n+1} = 1$.

Lie theory and complex analysis combine to give the subtle analytic properties of $H^m_{\lim}$ (cf. [S] for the 1-parameter case and [CKS] for the several parameter case over $(\Delta^*)^k$).

(3) *Geometric*: associated to a Hodge structure or to a 1^{st} order variation of such there may be algebro-geometric objects. Classically there is Riemann's theta divisor in the Jacobian variety of the curve. In the non-classical case (D is not a Hermitian symmetric domain) one construction is the infinitesimal variation of the Hodge structure. More recently the extension data associated to a limiting mixed Hodge structure has become of importance in studying $\partial\mathcal{M} = \overline{\mathcal{M}}\backslash\mathcal{M}$.

(3) *Non-abelian Hodge theory*: this is the study initiated by Simpson of the fundamental groups of algebraic varieties via their linear representation, especially those that arise from variations of Hodge structures. We shall not be able to discuss this very interesting area in this talk.

Stratification of $\overline{\mathcal{P}}$: As mentioned above, there are two basic types of subvarieties of $\overline{\mathcal{P}}$ and the resulting amalgam of these. The first type is the stratification associated to the boundary components given by the types of limiting mixed Hodge structures that occur when the polarized Hodge structures degenerate. Lie theory provides a classification of how this may happen (cf. [KR]). The second type is given by Mumford-Tate sub-domains.
A simple type of these occurs when the PHS decomposes non-trivially into a direct sum of PHS's; these correspond to projection operators in $\mathrm{End}(V) \subset T(F)$.

Generalized stratification of $\overline{\mathcal{P}}$: By generalized stratification we shall mean a set of subvarieties, not necessarily disjoint but whose union is all of $\overline{\mathcal{P}}$, and which satisfy certain conditions that will not be spelled out here. The generalized strata will be of the two types discussed above. The first type will be referred to as boundary components; these refer to the LMHS's that appear in $\overline{\mathcal{P}}$. The second will be called Mumford-Tate loci.

Boundary components: Very roughly speaking there are two types of boundary components; viz. over $\mathbb{Q}$ and over $\mathbb{Z}$. There is yet to be a formal definition of the latter, which in this talk will be taken to be the $\mathbb{Q}$-boundary component together with the $G_{\mathbb{Z}}$ conjugacy class of T_s (which is closely related to the *spectrum* in the case of isolated hypersurface singularities). For the purposes of the talk, for the former we use the conjugacy class of N.

For $n = 1$ since $N^2 = 0$ this is determined by rank N.

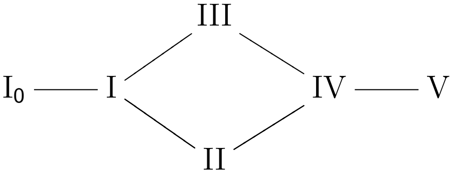
For $n = 2$ one has the classification

- $N^2 = 0$; then we have rank N.
- $N^2 \neq 0$; then we have rank N and rank N^2.

One may picture the $\mathbb{Q}$-boundary structure by a diagram in which the conjugacy classes and possible degenerations are represented. For $n = 1$ and $h^{1,0} = g$ the diagram is

$$\mathrm{I}_0 \text{ —— } \mathrm{I}_1 \text{ —— } \cdots \text{ —— } \mathrm{I}_g .$$

For $n = 2$ and $h^{2,0} = 2$ the diagram is



This diagram will be refined when we discuss I-surfaces. References to these diagrams are given in [KR].

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C. Some general results:

- Let $\mathcal{M}$ be a KSBA moduli space for a class of surfaces of general type and with canonical completion $\overline{\mathcal{M}}$. The first general result concerns the period mappings of $\mathcal{M}$ and $\overline{\mathcal{M}}$. It is known (Vakil) that the structure of $\overline{\mathcal{M}}$ may be arbitrarily nasty and the exact technical conditions under which the following results will hold have not been worked out. We do assume that each component of $\overline{\mathcal{M}}$ is generically reduced and that a general point corresponds to a smooth surface. Then there is a holomorphic period mapping

$$\Phi : \mathcal{M} \to \mathcal{P} \subset \Gamma \backslash D$$

whose image $\mathcal{P}$ is a locally closed analytic subvariety.

From [BBT] it follows that when Γ is arithmetic the closure of $\mathcal{P}$ in $\Gamma\backslash D$ is a quasi-projective algebraic variety over which the Hodge line bundle is ample.[‖]

- The first hoped for result is that the above period mapping extends to

$$\overline{\Phi} : \overline{\mathcal{M}} \to \overline{\mathcal{P}} \tag{1.1}$$

and that over $\overline{\mathcal{P}}$ the extended Hodge line bundle is ample. This has been established only in special cases. What is known [GGLR] is that $\overline{\mathcal{P}}$ exists as a compact Hausdorff space with a stratification by complex analytic subvarieties and that $\overline{\Phi}$ is defined and is a continuous proper mapping.

[‖] The interesting work [BBT] uses *o*-minimal structures (arising initially from model theory) to put an algebraic structure on $\mathcal{P}$. The techniques introduced there and in the references to that work seem certain to have further applications to Hodge theory.

The structure sheaf $\mathcal{O}_{\overline{\mathcal{P}}}$ is defined to be the sheaf of continuous functions that restrict to be holomorphic on strata. What remains to be proved is that $\mathcal{O}_{\overline{\mathcal{P}}}$ has enough local functions. This is a global problem along the compact fibres of $\overline{\Phi}$.

As a set $\overline{\mathcal{P}}$ consists of the associated graded PHS's to the equivalence classes of LMHS's obtained from families $\mathcal{X}^* \to \Delta^*$ of smooth surfaces parametrized by discs $g : \Delta^* \to \mathcal{M}$. The essential geometric content of the above assertion about $\overline{\Phi}$ is that

$$\mathrm{Gr}\left(\lim_{t\to 0} H^2(X_t)\right)$$

depends only on the limit surface X_0 and not on the $\overline{g} : \Delta \to \overline{\mathcal{M}}$ extending g above with $\overline{g}(0)$ corresponding to X_0. That is, the associated graded to the LMHS does not depend on the particular smoothing of X_0 (there may be several components of such).

In the example of I-surfaces discussed one may see this directly.

- A general result relating Hodge theory and moduli is that slc singularities are Du Bois ([K]). A Hodge-theoretic consequence of this is the surjectivity of the natural maps

$$H^p(X, \mathbb{C}) \to H^p(X, \mathcal{O}_X)$$

induced from the inclusion of sheaves $\mathbb{C}_X \hookrightarrow \mathcal{O}_X$. Among the applications is that numbers such as $h^q(\mathcal{O}_{X_s})$ are constant in flat families $\mathcal{X} \xrightarrow{f} S$ whose fibres have slc singularities, and the existence of relative dualizing sheaves for such families. In addition the part

$$I_{\mathrm{lim}}^{p,0} = I^{p,0}(X_0)$$

of the local invariant cycle theorem holds. One may infer a number of properties of KSBA families with going through the full semi-stable-reduction (SSR) process.

For KSBA degenerations there is an algorithmic SSR that is useful in examples to construct desingularization of moduli spaces.

- For the next general result we recall the notation

$$\mathcal{M}_f \subset \overline{\mathcal{M}}$$

for the subvariety of $\overline{\mathcal{M}}$ parametrizing singular surfaces X such that there exists a smoothing $\mathcal{X} \to \Delta$ of $X = X_0$ with finite monodromy. Then
The period mapping extends to $\Phi : \mathcal{M}_f \to \Gamma \backslash D$.

Moreover,

$$\overline{\mathcal{M}}^{\mathrm{NG}} \subset \mathcal{M}_f .$$

Here, $\overline{\mathcal{M}}^{\mathrm{NG}}$ denotes the subvariety of $\overline{\mathcal{M}}$ parametrizing *normal* surfaces X having non-Gorenstein singularities. Informally stated, to a normal and smoothable surface X having non-Gorenstein semi-log-canonical singularities one may associate a pure polarized Hodge structure $H^2_{\mathrm{lim}}(X)$.

This latter result is a consequence of the statements

- normal surfaces with rational singularities are parametrized by a subvariety of $\mathcal{M}_f$ (i.e., they have finite monodromy);

and from the above classification of normal, slc singularities

- normal, non-Gorenstein slc singularities are rational.

Canonical singularities are rational so they are also parametrized by a subvariety in $\mathcal{M}_f$. For I-surfaces thus far there is no example other than the above of singular surfaces with finite monodromy. It follows from the classification in [FPR] that if X is Gorenstein and non-normal, then the monodromy is infinite (i.e., $N \neq 0$). A natural question is whether non-normal I-surfaces always have infinite monodromy?

- The normal surfaces X with infinite monodromy are those with either simple elliptic singularities or cusps. For the following result we assume that a general smooth surface is regular and that the singular surface has either e simple elliptic singularities or c cusps but does not have some of each.* The statement is

$$\begin{cases} \operatorname{rank} N \leqq e \leqq p_g + 1 \\ \operatorname{rank} N^2 \leqq c \leqq p_g + 1. \end{cases}$$

The proofs are Hodge theoretic.

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*There is a general result without these assumptions, but it is more involved to formulate and the special cases given here capture the essential geometric content of the general result.

- Finally we will explain a general statement that might hold and that can be established in a couple of cases. The period mapping extends from $\mathcal{M}$ to give

$$\Phi : \mathcal{M}_f \to \Gamma\backslash D.$$

As noted earlier that the image $\mathcal{P}$ is a *closed* analytic subvariety and it follows from the results in [BBT] that $\mathcal{P}$ is quasi-projective.

Let $M \subset \mathcal{M}_f$ be an irreducible component of $\mathcal{M}_f$. The question that arises is

Does there exists a Γ-invariant Mumford-Tate subdomain $D' \subset D$ with Γ' the discrete group of automorphisms of D' induced by Γ such that

$$M = \Phi^{-1}(\mathcal{P} \cap \left(\Gamma' \backslash D'\right))?$$

Informally this means that these components of moduli can be detected Hodge theoretically.

How might one prove this, at least in some special cases? For those M such that the surfaces X parametrized by M are normal with either canonical or non-Gorenstein singularities, such singularities are rational and the resolution

$$(\widetilde{X}, \widetilde{C}) \to (X, p)$$

of a particular one has for $\widetilde{C}$ a configuration of $\mathbb{P}^1$'s.

Recalling that X gives a PHS $\Phi(X) \in \Gamma\backslash D$, the $\mathbb{P}^1$'s give Hodge classes that may not be present on a general point in D, and then D' could be the Mumford-Tate domain defined by PHS's having these additional Hodge classes.

One then hopes to use a variational argument to show that in $T\,\mathrm{Def}\,X$ the condition to retain these Hodge classes defines the tangent space to $M \subset \overline{\mathcal{M}}$. This argument has been carried out in the two cases

- an A_1-singularity that is not a base point of $|K_X|$;
- the $\frac{1}{4}(1,1)$-singularity on the general I-surface having that type of singularity.

In both use is made of differential of the period mapping at a singular surface.

III. *I*-surfaces

- Recall that an *I-surface* is defined to be a smooth (or having canonical singularities) surface X that satisfies
  - X is minimal of general type
  - $q(X) = 0$ and $p_g(X) = 2$
  - $K_X^2 = 1$

  It appears that regular surfaces for which Noether inequality

$$p_g(X) \leqq \frac{K_X^2}{2} + 2$$

  is close to equality seem to have favorable qualities, e.g. local Torelli holds, for the use of Hodge theory to study their moduli, and some of what follows has also been carried out for *H-surfaces* satisfying the first two conditions above together with $K_X^2 = 2$ (H stands for Horikawa [Ho] who made an extensive analysis of surfaces with small c_1^2).

Informally stated parts of the results that we shall discuss are

- $\overline{\Phi}, \overline{\mathcal{M}}_I^G \to \overline{\mathcal{P}}$ is a mapping of stratified varieties that is bijective on the set of components;
- the extension data in the limiting mixed Hodge structures over $\overline{\mathcal{P}} \backslash \mathcal{P}$ desingularizes $\overline{\mathcal{M}}_I^G$ at a general point over the boundary of $\mathcal{M}_I$.

As mentioned above, an extensive analysis of $\overline{\mathcal{M}}_I^G$ has been carried out by [FPR]; this will be summarized in a table below. Here we shall first give a Lie-theoretically constructed table of the possible types of LMHS's that could appear in $\overline{\mathcal{P}} \backslash \mathcal{P}$. We will then give the FPR table in the normal Gorenstein case and sketch how the Hodge-theoretic table suggests and interprets the algebro-geometric one.

- The principal properties of I-surfaces we shall use are
  - $h^1(mK_X) = 0$ for $m \geqq 0$ and

$$h^0(mK_X) = \begin{cases} 2 \text{ for } m = 1 \\ \frac{m(m-1)}{2} + 2, \quad m \geqq 2; \end{cases}$$

- using Kawamata-Viehweg vanishing one sees that these properties hold for any Gorenstein I-surface where the Weil canonical divisor class and the dualizing sheaf ω_X coincide as line bundles;

- in the Gorenstein case the pluri-canonical ring

$$R(X) = \overset{m}{\oplus} H^0(mK_X)$$

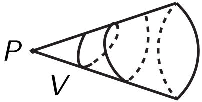
has the *postulated form*, meaning that generators and relations are added only when required by (**??**) (cf. [FPR] for a proof);
- classically from Castelnuovo-Enriques and since the work of Bonbieri and others one studies general type surfaces via their pluri-canonical maps ([BPVdV])

$$\varphi_{mK_X} : X \to \mathbb{P}H^0(mK_X)^* \cong \mathbb{P}^{P_m - 1};$$

- instead of these it is frequently more convenient to use weighted projective spaces corresponding to when new generators are added; thus

$$\begin{aligned} \varphi_{K_X} &: X \dashrightarrow \mathbb{P}^1 \ \ (|K_X| = \text{pencil of hyperelliptic curves}) \\ \varphi_{2K_X} &: X \to \mathbb{P}(1,1,2) \to \mathbb{P}^3 \quad (|2K_X| \text{ is base-point-free}) \\ &\vdots \\ \varphi_{5K_X} &: X \hookrightarrow \mathbb{P}(1,1,2,5) \hookrightarrow \mathbb{P}^{12} \quad (|5K_X| \text{ is very ample}). \end{aligned}$$

We denote by (x_0, x_1, y) weighted homogeneous coordinates in $\mathbb{P}(1,1,2)$, and by (x_0, x_1, y, z) those for $\mathbb{P}(1,1,2,5)$.

- Equations/picture
  - $z^2 = F_5(t_0, t_1, y)z + F_{10}(t_0, t_1, y)$
  - 

    $\mathbb{P}(1,1,2) \hookrightarrow \mathbb{P}^3$given by
    $[x_0, x_1, y] \to [x_0^2, x_0x_1, x_1^2, y]$

    $X =$ 2:1 map branched over P and $V \in |\mathcal{O}_{\mathbb{P}^3}(5)|$ where V does not pass through the vertex $P^\dagger$
- The moduli of smooth X's may be analyzed
  - from the above equation;
  - cohomologically using the Jacobian ideal formalism (cf. [Gr]) for weighted projective spaces.

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†*Any* Gorenstein X is irreducible ($K_X^2 = 1$) and is given by such a picture where V does not contain P.

Each has its advantages. The first is useful in studying degenerations, and also possibly in using GIT where in this case the group is non-reductive. From the second approach one has

- $\mathcal{M}$ is reduced and smooth of dimension 28;
- local Torelli holds.

More precisely, for smooth X the differential of the period mapping is 1-1. Versions of this have also been proved by Carlson-Toledo [CT] and Pearlstein-Zhang [(PZ)].

- We recall that for any non-classical period domain D there is a non-trivial invariant distribution $I \subset TD$ (the infinitesimal period relation or IPR) such that any period mapping
$$\Phi : B \to \Gamma\backslash D$$
satisfies
$$\Phi_* : TB \to I.$$

For polarized Hodge structures of weight 2 and with $h^{2,0} = 2$

$$\dim D = 2h^{1,1} + 1 \text{ and } I \text{ is a contact distribution.}$$

From Noether's formula

$$\chi(\mathcal{O}_X) = \frac{1}{12}(c_1^2 + c_2)$$

we may infer that $h^{1,1} = \dim H^{1,1}(X)_{\mathrm{prim}} = 28$.
From this it follows that

$$\Phi(\mathcal{M}) \subset \Gamma \backslash D \text{ is a contact submanifold.}$$

I know of no other such example.

- Remark that a cohomological analysis also gives
  - $h^0(T_X) = 0$ (since X is general type)
  - $\chi(T_X) = 28$ (Hirzebruch-Riemann-Roch)

$\implies$ $h^2(T_X) = 0$,

which again shows that for X smooth the Kuranishi space is smooth of dimension 28. We suspect that this still holds when X has canonical singularities, but this has not been checked.

- Finally the classical methods of Lefschetz ([L] and [B]) may be adapted to show that

  the monodromy group $\Gamma = \Phi_* \left(\pi_1(\mathcal{M}_{I,\mathrm{reg}})\right)$ is arithmetic.

  It is not known if $\Gamma = G_{\mathbb{Z}}$ is the full arithmetic group.‡

‡Note that since $K_X^2 = 1$, the primitive decomposition is

$$H^2(X,\mathbb{Z}) = \mathbb{Z}\cdot c_1(X) \oplus H^2(X,\mathbb{Z})_{\mathrm{prim}}$$

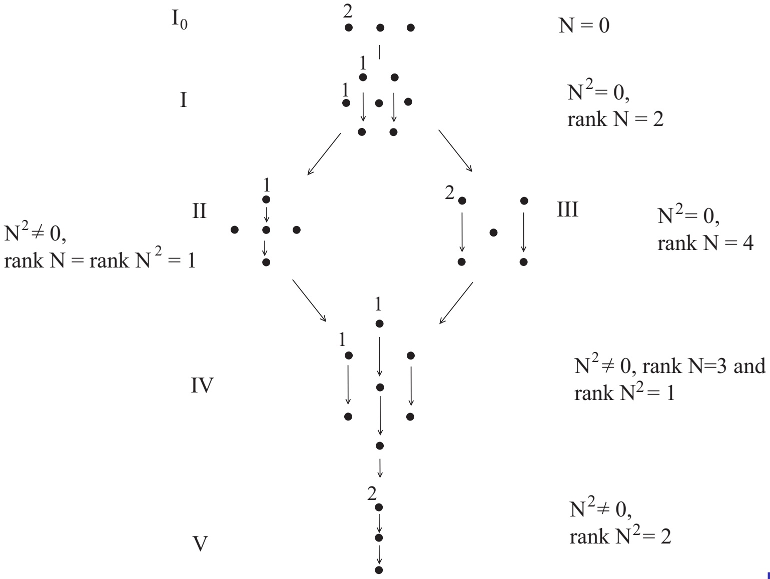
and the intersection form on the primitive part is unimodular. It is suspected that this intersection form is even and that it has the maximal number of E_8's allowed by the signature but this has not been verified.

Also outstanding are the questions

- Does generic global Torelli hold (i.e., does Φ have degree one)?
- Does global Torelli hold?

We expect that the first question may be addressed using the methods in [Gr] and/or by using a version of global Torelli on the boundary as in [F2].

- Turning to the classification we first give a refined picture of the Hodge-theoretic boundary components of $\overline{\mathcal{P}}$ for the case of surfaces where $p_g = 2$. For this we will use Hodge diamonds to depict the Gr(LMHS). The numbers above some of the entries are dimensions; from those that are depicted all of the remaining dimensions may be determined.



I_0
$N = 0$
I
$N^2 = 0$, rank $N = 2$
II
$N^2 \neq 0$, rank N = rank $N^2 = 1$
III
$N^2 = 0$, rank $N = 4$
IV
$N^2 \neq 0$, rank $N=3$ and rank $N^2 = 1$
V
$N^2 \neq 0$, rank $N^2 = 2$

To illustrate how one uses Hodge theory to suggest what to look for in analyzing the stratification of $\overline{\mathcal{M}}_I^G$, under a degeneration

$$I_0 \to I,$$

where a pure Hodge structure of weight 2 with Hodge numbers $(2, h^{1,1}, 2)$ degenerates into a limiting mixed Hodge structure with graded pieces

$$H^1_{\text{lim}}, H^2_{\text{lim}}, H^1_{\text{lim}}, (-1)$$

where H^1_{lim} has Hodge numbers $(1, 1)$ and H^2_{lim} has Hodge numbers $(1, h^{1,1}, 1)$. Algebro-geometrically one may use semi-stable reduction (SSR) to expect that the KSBA degeneration $X_t \to X_0$ becomes a family whose central fibre is a normal crossing surface $X = X_1 \cup \cdots \cup X_m$, one component of which is the minimal desingularization $\widetilde{X}$ of X_0.

Since $N^2 = 0$, one may also hope that X has only a smooth double curve C (no triple points). Moreover, since the p_g drops by one in the limit, one may reasonably expect that $\lim \omega_t = \widetilde{\omega}$ where $\omega_t \in H^0(\Omega^2_{X_t})$ and $\widetilde{\omega} \in H^0(\Omega^2_{\widetilde{X}}(\widetilde{C}))$ with $\mathrm{Res}_{\widetilde{C}}(\widetilde{\omega}) \in H^0(\Omega^1_{\widetilde{C}}) = H^{1,0}_{\mathrm{lim}}$. Thus we may expect that under degenerations of type I the limit surface X_0 has one simple elliptic singularity and the remaining surfaces are rational.

The following is a table of normal Gorenstein degenerations of I-surfaces with $N^2 = 0$. In it

- $k = \#$ of simple elliptic singularities;
- $d_i =$ degree of the elliptic singularity;
- the numbers in the subscripts on I, III are the degrees of the elliptic singularities.

stratum	dimension	minimal resolution $\widetilde{X}$	$\sum_{i=1}^{k}(9-d_i)$	k	codim in $\overline{\mathcal{M}}_I$
I_0	28	canonical singularities	0	0	0
I_2	20	blow up of a K3-surface	7	1	8
I_1	19	minimal elliptic surface with $\chi(\widetilde{X})=2$	8	1	9
$\mathrm{III}_{2,2}$	12	rational surface	14	2	16
$\mathrm{III}_{1,2}$	11	rational surface	15	2	17
$\mathrm{III}_{1,1,R}$	10	rational surface	16	2	18
$\mathrm{III}_{1,1,E}$	10	blow up of an Enriques surface	16	2	18
$\mathrm{III}_{1,1,2}$	2	ruled surface with $\chi(\widetilde{X})=0$	23	3	26
$\mathrm{III}_{1,1,1}$	1	ruled surface with $\chi(\widetilde{X})=0$	24	3	27

Note that the last column is the sum of the two columns preceding it. This will be illustrated below and explained elsewhere. The geometric point is that the $\sum_{i=1}^{k}(9 - d_i)$ term will be the number of parameters in the extension data of the LMHS; this extension data suggests how to blow up $\overline{\mathcal{M}}_I$ along the corresponding component to obtain a desingularization of $\overline{\mathcal{M}}_I$ along that component.

One may also illustrate how Hodge theory and standard algebro-geometric techniques may be combined to identify which K3 surface, elliptic surface, rational and ruled surfaces appear in the table. For example, for I_2 the K3 will have a degree 2 polarization and $\widetilde{C}$ will be the normalization of a tangent to the sextic in $\mathbb{P}^2$ that is the branch curve for the double covering of the K3. For I_1 the elliptic surface will be an elliptic pencil with $p_g = 1$ and having a bi-section.
We note that the I-surface X with the $\frac{1}{4}(1,1)$ Wahl singularity over the vertex P of the quadric cone has desingularization $\widetilde{X}$ that is also an elliptic surface, this time with $p_g = 2$, and which also has a bi-section.

Referring to the above diagram we note that

$$7 = \left\{\begin{array}{l}\text{number of points to blow up on a}\\ \text{cubic curve } C \text{ in } \mathbb{P}^2 \text{ to obtain a Del}\\ \text{Pezzo } \widetilde{\mathbb{P}}^2 \text{ containing } \widetilde{C} \text{ with } \widetilde{C}^2 = 2\end{array}\right\}$$
$$= 1 + \text{codimension in moduli of } I_2.$$

This suggests that to desingularize $\overline{\mathcal{M}}_I$ along I_2 we insert normal crossing surfaces $\widetilde{X} \cup_{\widetilde{C}} \widetilde{\mathbb{P}}^2$.

The central point is that these 7 points also give the extension data in the LMHS. Of course this heuristic dimension count must must be made precise, which can be done. The point here is to suggest how the Hodge theory and geometry interact. So far this example seems to indicate a general pattern.

We conclude with a brief remark on the non-normal Liu-Rollenske example mentioned earlier. It turns out that there are three irreducible components of $\overline{\mathcal{M}}_I^G$ whose general member is non-normal and obtained from $(\widetilde{X}, \widetilde{C}, \tau)$ by passing to the quotient. For one of these, say $\mathcal{R}$, we have

- $\widetilde{X} = \mathbb{P}^2$;
- $\widetilde{C}$ = plane quartic;
- for general $\widetilde{C}$ we have $\widetilde{C}/\tau$ is an elliptic curve;
- $\dim \mathcal{R} = 3$.

The LMHS's are of types IV and V. Those of type V are Hodge-Tate and the summand on which N^2 is an isomorphism has graded pieces

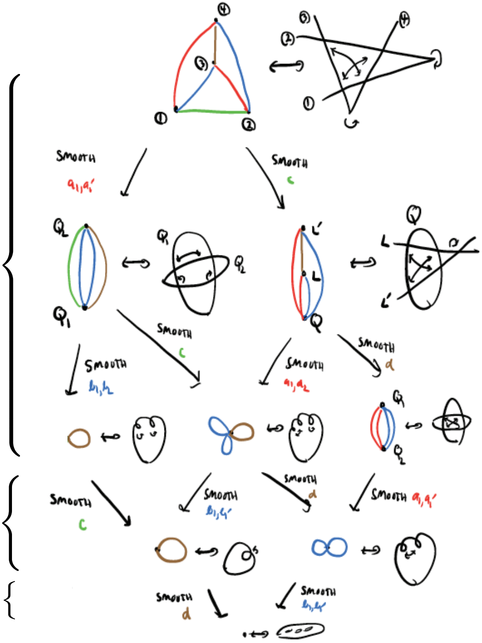
$$H^0_{\lim}, H^0_{\lim}(-2)$$

where $h^0 = 2$. The picture of the boundary component is

V

IV

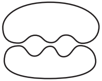
0


SMOOTH
a_1, a_1'
SMOOTH
c
SMOOTH
c
SMOOTH
b_1, b_2
SMOOTH
a_1, a_2
SMOOTH
d
SMOOTH
c
SMOOTH
b_1, b_1'
SMOOTH
d
SMOOTH a_1, a_1'
SMOOTH
d
SMOOTH
b_1, b_1'

There are three parameters in $\mathrm{Ext}^1_{\mathrm{MHS}}(H^0_{\mathrm{lim}}, H^0_{\mathrm{lim}}(-2))$ and in this case the extension data determines the point of $\mathcal{R}$. The extreme case is the aforementioned LR surface

$$z^2 = y(x_0^2 - y)^2(x_1^2 - y)^2$$

obtained by identifying opposite sides of a quadrilateral in $\mathbb{P}^2$. In this case the LMHS is split. It is a rigid I-surface whose monodromy cone has the maximal dimension 28 and is a surface analogue of the dollar bill curve



with dual graph \$.

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