

Some geometric applications of Hodge theory

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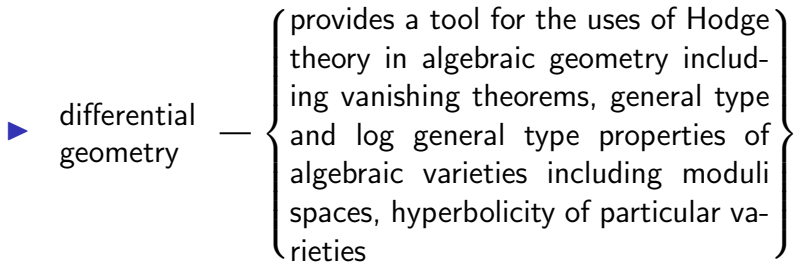
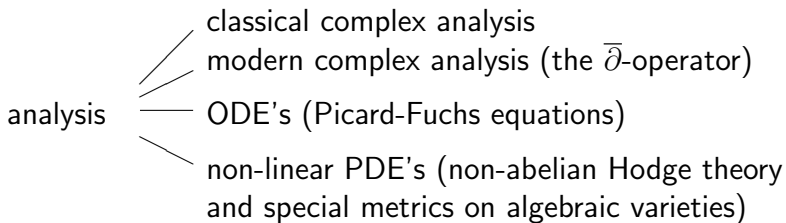
Outline

- I. Introduction
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- III. Objects of Hodge theory
- IV. Geometric applications of Hodge theory
 - ▶ classical case
 - ▶ Torelli theorems
 - ▶ topology of algebraic varieties
 - ▶ vanishing theorems

I. Introduction

- ▶ Modern Hodge theory is both a subject of study in its own right and a subject that is used in many areas of current mathematical research, especially in but no means restricted to algebraic geometry. This talk will be an informal and partial overview of some of its uses with emphasis on those in algebraic geometry. We will also discuss some of the historical development of the subject; how did it originate and how did it get to its current state? Here the emphasis will be on the period up until the time of Hodge and will only touch on a few of the major milestones.

- ▶ Hodge theory relates to many areas in mathematics



- ▶ topology
 - smooth varieties
 - $\left\{ \begin{array}{l} \text{general algebraic varieties,} \\ \text{singular and/or non-complete} \end{array} \right\}$
 - intersection cohomology
 - $\left\{ \begin{array}{l} \text{families of algebraic varieties; here all} \\ \text{of the above enter the picture} \end{array} \right\}$

- ▶ Lie theory — $\left\{ \begin{array}{l} \text{homogeneous complex manifolds, both} \\ \text{compact and open; period domains} \\ \text{and their compact duals (think of } \mathcal{H} \subset \\ \mathbb{P}^1 \text{); locally symmetric spaces and their} \\ \text{generalizations; representation theory} \end{array} \right\}$

and finally of course

- ▶ algebraic geometry; today we will only have time to discuss a very few aspects; will not be able to talk about the central problem in the subject (Hodge conjecture), mirror symmetry and other topics related to physics, character varieties, arithmetic algebraic geometry, . . .

- ▶ Specifically, the use of Hodge theory to study moduli requires using geometric constructions arising from Hodge theory, frequently some type of Torelli property, either for the variety itself or for the singular ones that appear on the boundary of moduli spaces. A Hodge structure and some of its generalizations are given by linear algebra data. In some cases from the linear algebra data one may construct a geometric object from which properties of the variety may be determined. Although very classical this original use of Hodge theory (elliptic functions and Riemann's theta function) continues to find applications to geometric questions including moduli.

II. Origins of Hodge theory

Although there is no single work that one can say “here is the beginning of Hodge theory,” an important part of its origins may be traced to the study of integrals

$$(II.1) \quad \int r(x, y(x)) dx$$

of algebraic functions. Here $y(x)$ is the “function” given by an equation

$$f(x, y(x)) = 0$$

where $f(x, y)$ is an irreducible polynomial and $r(x, y) = p(x, y)/q(x, y)$ is a rational function. Such integrals arise in geometry, e.g. as arclength

$\int \sqrt{dx^2 + dy^2} = \int \sqrt{1 + (y'(x)^2)} dx$, or in mechanics where $\dot{y}(t)^2 = f(y(t))$ so that $y(t) = \int \sqrt{f(y(x))} dx$.

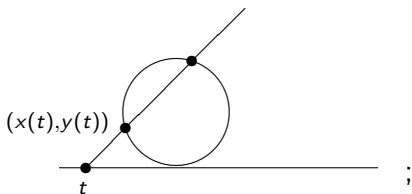
For illustrative purposes we will take

$$(II.2) \quad f(x, y) = y^2 - p(x)$$

where $p(x) = \prod_{i=1}^{2g+2} (x - a_i)$ is a polynomial with distinct roots. The integral (II.1) is understood to take place in the complex plane along a path γ avoiding the a_i and along which we have chosen a branch of $y(x) = \sqrt{p(x)}$. If $\deg f(x, y) = 2$, then such integrals can be evaluated in terms of “elementary functions,” e.g., for $f(x, y) = x^2 + y^2 - 1$ then

$$(II.3) \quad \int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dx}{y} = \arcsin.$$

The reason for $\deg f = 2$ is that $f(x, y) = 0$ is then a conic and can be parametrized by a line



substituting gives for the integral

$$\int r(t) dt$$

where $r(t)$ is a rational function and partial fractions may be used to evaluate it.

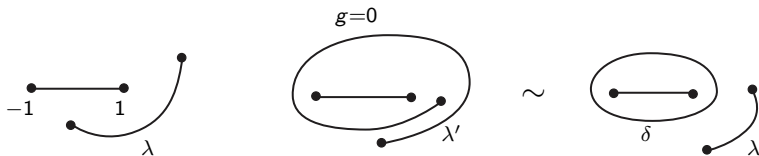
As soon as $\deg f \geq 3$ the integral (II.1) can no longer be understood in terms of elementary functions. For example

$$(II.4) \quad \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad k \neq \pm 1$$

turns up in the arc length of an ellipse.

Before continuing we want to make an observation that has both topological and analytic meaning.

For $f(x, y) = y^2 - p(x)$ as in (II.2) and $\omega = dx/y$ we have



$$\int_{\lambda'} \omega = \int_{\lambda} \omega + \int_{\delta} \omega = \int_{\lambda} \omega + c$$

where $c = \int \omega$ is a period of the integral.

If we invert the integral by setting

$$(II.5) \quad u = \int_{(x_0, y_0)}^{(x(u), y(u))} \omega$$

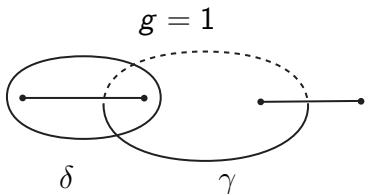
then

$$x(u + c) = x(u), \quad y(u + c) = y(u).$$

In fact, in (II.3) we have $c = 2\pi$ and this shows that defining $x(u), y(u)$ by (II.5) parametrizes the circle by arclength and gives the periodicity of $(\sin u, \cos u)$. Note that taking the derivative of (II.5) gives

$$x'(u) = y(u).$$

If we try to do the same thing with (II.4), then we arrive at a picture



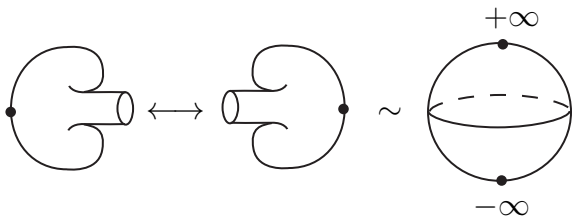
where inversion of the integral leads to doubly periodic functions $x(u), y(u)$.

Analytically

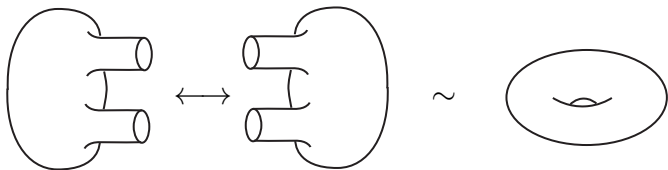
- ▶ for $g = 0$ the integral $\int_{\gamma} \omega \rightarrow \pm\infty$ as $\partial\gamma \rightarrow \infty$ (in fact, $\omega \sim \pm(dt/t^{1/2})$ for $x = 1/t$);
- ▶ for $g = 1$ the integral $\int_{\gamma} \omega < \infty$ for any γ (in fact, for $k \geq g - 1$ we have $\int_{\gamma} \frac{x^k dx}{y} < \infty$)

Topologically

- ▶ for $g = 0$ we have



- ▶ for $g = 1$ we have



What about a general integral (II.1)? Let $h(x, y, t)$ depend rationally on t (e.g., $h(x, y, t) = ax + by + ct$), and

$$(f(x, y) = 0) \cap (h(x, y, t) = 0) = \sum_i (x_i(t), y_i(t)) = \sum_i p_i(t).$$

The decisive result was given by

Abel ([A]): $\frac{d}{dt} \left(\sum_i \int_{p_0}^{p_i(t)} \omega \right) = r(t)$ is a rational function of t .

Thus the *abelian sum* $\sum_i \int_{p_0}^{p_i(t)} \omega$ is an elementary function. If we define $x(u), y(u)$ by inversion

$$u = \int_{(x_0, y_0)}^{(x(u), y(u))} \omega,$$

then it turns out that one can express $x(u + c), y(u + c)$ in terms of $x(u), y(u)$. Moreover, for $f(x, y) = y^2 - p(x)$ we have $x'(u) = y(u)$. For $g = 0, 1$ we have the periodicity or double periodicity of $x(u), y(u)$.

If $r(t)dt$ has no residues so that $\int r(t)dt$ is a rational function, then $x_i(u+c) = R_i(x_1(u), x'_1(u), \dots, x_d(u), x'_d(u))$. Taking $h(x, y, t)$ to be linear as above one may obtain the addition theorem

$$x(u + \tilde{u}) = R(x(u), x'(u), x(\tilde{u}), x'(\tilde{u})).$$

- ▶ Abel ([A]) defined the *genus* of $X := \{f(x, y) = 0\}$ as the dimension of the space $H^0(\Omega_X^1)$ of ω 's with $\int \omega < \infty$. Then Riemann ([R]) proved
 - ▶ $g = \left(\frac{1}{2}\right) b_1(X) = \frac{1}{2} (\dim H_1(X, \mathbb{Z}))$
 - ▶ $H^1(X, \mathbb{C}) \cong H^0(\Omega_X^1) \oplus \overline{H^0(\Omega_X^1)}$.

His argument for the second used $\int \omega \wedge \omega = 0$ (because $dx \wedge dx = 0$) and $i \int \omega \wedge \bar{\omega} > 0$ (because $i dx \wedge d\bar{x} > 0$). This was the beginning of Hodge theory.

- ▶ Finally a few words about Picard (cf. [PS]). He studied algebraic surfaces

$$X = \{f(x, y, z) = 0\}$$

by looking at the pencil of algebraic curves

$$Y_z = \{f(x, y, z) = 0, z = \text{constant}\}.$$

Then he proved

- ▶ Y_z is connected ($H_0(Y_z, \mathbb{Z}) \xrightarrow{\sim} H_0(X, \mathbb{Z})$)
- ▶ $H_1(Y_z, \mathbb{Z}) \rightarrow H_1(X, \mathbb{Z})$.

On X he considered differentials

- ▶ $\varphi = p(x, y, z)dx + q(x, y, z)dy, \quad \int \varphi < \infty$
- ▶ $\omega = r(x, y, z)dx \wedge dy, \quad \iint \omega < \infty$.

For the first Picard showed that $d\varphi = 0$ and from this inferred

- ▶ $H^1(Y, \mathbb{C}) \cong H^0(\Omega_X^1) \oplus \overline{H^0(\Omega_X^1)}$

This is because if $d\varphi \neq 0$, then

$$0 \neq \int d\varphi \wedge \overline{d\varphi} = \int d(\varphi \wedge \overline{d\varphi}) = 0$$

and similarly we cannot have $\varphi = d\eta$. For 2-forms he defined what we now call the *Picard number* ρ and showed that

$$(II.6) \quad \blacktriangleright \quad H^2(X, \mathbb{C}) \supset H^0(\Omega_X^2) \oplus \overline{H^0(\Omega_X^2)} \oplus (\mathbb{C}^\rho).$$

For the third term here the Lefschetz (1,1) theorem ([L]) gives

$$H_2(X, \mathbb{Z}) \cap H^0(\Omega_X^2)^\perp = NS(X).$$

The full identification of the term to make \supset into $=$ using $H^{1,1}(X)$ had to await the work of Hodge ([H]).



III. Objects of Hodge theory (cf. [CM-SP], [V1], [V2])

- ▶ *Polarized Hodge structure* (PHS): (V, Q, F)
 - ▶ $V = \mathbb{Q}$ -vector space
 - ▶ $Q : V \otimes V \rightarrow \mathbb{Q}$
 - ▶ $F^n \subset F^{n-1} \subset \dots \subset F^0 = V_{\mathbb{C}}$

where

$$F^p \cap \overline{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}}, \quad 0 \leq p \leq n.$$

For $V^{p,q} = F^p \cap \overline{F}^q$ we have the Hodge decomposition

- ▶ $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}, \quad \overline{V}^{p,q} = V^{q,p}$
- ▶ $F^p = \bigoplus_{p' \geq p} V^{p',q}$

This defines a Hodge structure of weight n . The polarization arises from the first and second Hodge-Riemann bilinear relations, which are generalizations of the ones given above for algebraic curves and which we don't need to make explicit here. Most of the deeper results in Hodge theory require the existence of a polarization.

Example: For X a smooth algebraic variety, $H^n(X, \mathbb{Q})$ has a Hodge structure of weight n . If $X \subset \mathbb{P}^N$, so that $L = \mathcal{O}_X(1)$ is an ample line bundle, there is a polarization on $H^n(X, \mathbb{Q})$.

▶ *Mixed Hodge structure (MHS):* (V, W, F)

- ▶ $V = \mathbb{Q}$ -vector space
- ▶ $W_0 \subset \dots \subset W_m = V$
- ▶ $F^n \subset F^{n-1} \subset \dots \subset F^0 = V_{\mathbb{C}}$

where F induces a weight k Hodge structure on

$$\mathrm{Gr}_k^W(V) = W_k/W_{k-1} := H^k.$$

Although $V_{\mathbb{C}}$ does not have a Hodge decomposition, there is the canonical Deligne decomposition

$$V_{\mathbb{C}} = \bigoplus_{p+q \leq m} I^{p,q}$$

where

$$\begin{cases} W_{k,\mathbb{C}} = \bigoplus_{p+q \leq k} I^{p,q} \\ \bar{I}^{p,q} \equiv I^{q,p} \pmod{W_{p+q-2,\mathbb{C}}} \end{cases}$$

which gives that

$$H^k \cong \bigoplus_{p+q=k} I^{p,q}.$$

There is a definition of a graded polarized MHS. Again this property is frequently present in applications of mixed Hodge theory.

In practice we will have a lattice $V_{\mathbb{Z}} \subset V$ with induced lattices on the H^k 's. Thus a MHS is a successive extension of Hodge structures with the first level being a direct sum of terms

$$\mathrm{Ext}_{\mathrm{MHS}}^1(H^k, H^{k-1}) = \frac{\mathrm{Hom}(H^k, H^{k-1})}{F^0 \mathrm{Hom}(H^k, H^{k-1}) + \mathrm{Hom}_{\mathbb{Z}}(H^k, H^{k-1})}.$$

In general the set \mathcal{E}_k of at most k -fold iterated extensions of HS's with fixed H^k 's gives a sequence of fibrations $\mathcal{E}_k \rightarrow \mathcal{E}_{k-1}$ with typical fibre the direct sum of

$$\mathrm{Ext}_{\mathrm{MHS}}^1(H^\ell, H^{\ell-k})\text{'s.}$$

- ▶ *Variation of Hodge structure (VHS):* $(\mathbb{V}, \mathcal{F}, \nabla; B)$
 - ▶ $\mathbb{V} = \mathbb{V}_{\mathbb{Z}} \otimes \mathbb{Q}$ is a local system over a quasi-projective variety B
 - ▶ \mathcal{F} is a filtration $\mathcal{F}^n \subset \mathcal{F}^{n-1} \subset \dots \subset \mathcal{F}^0 = \mathcal{V} = \mathbb{V} \otimes \mathcal{O}_B$ by sub-bundles that induces a Hodge structure on each fibre
 - ▶ $\nabla : \mathcal{F} \rightarrow \mathcal{F} \otimes \Omega_B^1$ satisfies $\nabla \mathbb{F} = 0$ and the infinitesimal period relation (IPR)

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega_B^1.$$

It is understood that there is a horizontal bilinear form $Q : \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{Q}$ that polarizes the Hodge structure on each fibre.

Example: $\mathcal{X} \xrightarrow{\pi} B$ is a smooth family of projective algebraic varieties X_b and

- ▶ $\mathbb{V} = R^n \pi_* \mathbb{Q}_{\mathcal{X}}$
- ▶ \mathcal{F}_b gives the Hodge structure on $H^n(X_b, \mathbb{C})$
- ▶ ∇ is the Gauss-Manin connection; $\nabla^2 = 0$.
- ▶ *Infinitesimal variation of Hodge structure (IVHS):*
 (E, T, θ)
 - ▶ $E = \bigoplus E^p$
 - ▶ $\theta : E^p \rightarrow E^{p-1} \otimes T^*$
 - ▶ $\theta \wedge \theta = 0$.

Thus E is a Sym T -module.

Example: $E^p = \text{Gr}_{F_b}^p(\mathcal{V}_b)$, $T = T_b B$ and θ is induced by ∇ . It is essentially given by the differential of the period mapping (which we won't define here).

- ▶ *Limiting mixed Hodge structure (LMHS):* $(V, W(N), F)$
 - ▶ $N \in \text{End}(V)$ satisfying $N^{m+1} = 0$ gives a unique weight filtration $W_k(N)$ satisfying

$$\begin{aligned}
 N &: W_k(N) \rightarrow W_{k-2}(N) \\
 N^k &: \text{Gr}_{m+k}^{W(N)}(V) \cong \text{Gr}_{m-k}^{W(N)}(V) \\
 N &: F^p \rightarrow F^{p-1}.
 \end{aligned}$$

Again we assume there exists a $Q : V \otimes V \rightarrow \mathbb{Q}$ and $N \in \text{End}_{\mathbb{Q}}(V)$. Then there are induced bilinear forms

$$Q_k : \text{Gr}_k^{W(N)}(V) \otimes \text{Gr}_k^{W(N)}(V) \rightarrow \mathbb{Q}$$

and these are assumed to polarize the Hodge structures on the $\text{Gr}_k^{W(N)}(V)$.

Example: For $\Delta^* = \{0 < |t| < 1\}$ suppose we have a VHS $(\mathbb{V}, \mathcal{F}, \nabla; \Delta^*)$ over the punctured disc. For $V = \mathbb{V}_{t_0}$ it is known that the monodromy

$$T : V \rightarrow V$$

is quasi-unipotent; i.e., $T = T_s T_u$ where $T_s^\ell = I$ and $T_u = e^N$ with $N^{m+1} = 0$ for some $m \leq n$. Then Schmid proved that there is a LMHS H_{lim}^n (actually an equivalence class of such parametrized by $T_{\{0\}}^* \Delta$).

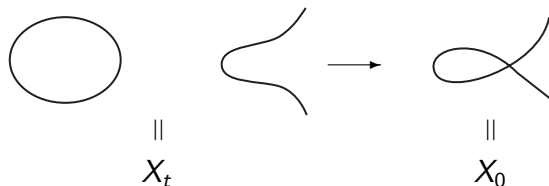
This example has been extended by Cattani-Kaplan-Schmid to the case when the parameter space is $\Delta^{*a} \times \Delta^b := B$. For our purposes it will be convenient to assume the local monodromies around the generators of $\pi_1(\Delta^{*a})$ are unipotent. In this case there is a canonical Deligne extension of the Hodge bundles to $\mathcal{F}_e^p \rightarrow \Delta^a \times \Delta^b := \overline{B}$ where the Gauss-Manin connection satisfies

$$\nabla : \mathcal{F}_e^p \rightarrow \mathcal{F}_e^{p-1} \otimes \Omega_{\overline{B}}^1(\log Z)$$

where $Z \subset \bar{B}$ is the normal crossing divisor with one of the coordinates in the Δ^a being set equal to 0:

$$\begin{array}{c}
 Z_1 \\
 | \\
 \hline \\
 | \\
 Z_2
 \end{array}
 \quad
 Z = Z_1 + Z_2$$

Example (continued): $\mathcal{X} \rightarrow \Delta$ is a projective family of varieties that is smooth over Δ^*



Then we have $\lim_{t \rightarrow 0} H^n(X_t) := H_{\text{lim}}^n$. The relation of H_{lim}^* to $H^n(X_0)$ is given by the Clemens-Schmid long exact sequence where X_0 is a NCD. When X_0 has slc singularities there is a variant of Clemens-Schmid discussed in [K].

Note that the left-hand side is independent of the particular singular fibre X_0 over the origin.

- ▶ In the global situation where one has a pair (\overline{B}, Z) consisting of a smooth projective variety \overline{B} with $Z \subset \overline{B}$ a reduced normal crossing divisor, given a VHS over B with unipotent monodromies around the irreducible branches of Z the above local discussion applies to give a canonical extension of the VHS to \overline{B} .*

*Again the assumption of unipotent monodromies is convenient for expository purposes; in interesting geometric situations it frequently does not occur (e.g., algebraic surfaces acquiring normal (and hence isolated) singularities).

IV. Some geometric applications of Hodge theory

- ▶ In some situations one may associate to a PHS, an IVHS or a LMHS a geometric construction, and when the Hodge theoretic object arises from geometry this construction may help understand the geometry.

The classical example: Associated to a weight $n = 1$ PHS (V, Q, F) is

- ▶ a compact complex torus

$$J = (F^1 \setminus V_{\mathbb{C}}) / V_{\mathbb{Z}};$$

- ▶ a line bundle $L \rightarrow J$ with $c_1(L) = Q \in (\Lambda^2 V_{\mathbb{Z}})^* \cong H^2(J, \mathbb{Z})$;
- ▶ it follows from the Hodge-Riemann bilinear relations that L is holomorphic and positive in the differential geometric sense, hence it is ample. If Q is unimodular, then $h^0(J, L) = 1$ and there is a non-zero section θ giving a divisor $\Theta \subset J$.

This is the case when the PHS arises from the H^1 of a smooth algebraic curve X . There is an Abel-Jacobi map

$$u : X \rightarrow J(X)$$

given for $p \in X$ by the linear function on $H^0(\Omega_X^1) \cong (F^1 \setminus V_{\mathbb{C}})^*$

$$\omega \rightarrow \int_{p_0}^p \omega \text{ modulo periods.}$$

Using Abelian sums as were encountered in Abel's theorem this mapping extends to

$$\begin{array}{ccc} X^{(d)} & \longrightarrow & J(X) \\ \Psi & & \Psi \\ \sum_i p_i & \longrightarrow & \sum_i u(p_i) = \sum_i \int_{p_0}^{p_i} \omega \end{array} .$$

Riemann ([R]) proved that, up to a translation,

$$u(X^{(g-1)}) = \Theta.$$

From this one may determine much of the geometry of X , culminating in Torelli's theorem that the PHS on $H^1(X)$ determines X .

In general it is not possible to associate a geometric object to a PHS. The reason is the differential constraint imposed on a VHS by the IPR. Only when the period domain is Hermitian symmetric, in which case the IPR is trivial and the PHS is a (Tate twist of) one of weight $n = 1$ or of weight $n = 2$ with Hodge number $h^{2,0} = 1$, can we construct naturally an algebro-geometric object from just a PHS. In the higher weight case one frequently may use an IVHS as a surrogate for the Θ -divisor. A recent case of this is given by the

Example (Shepherd-Barron):[†] If $X \rightarrow C$ is an elliptic surface with no multiple fibres and with

$$h^{2,0} \geq h^{1,0} + 3,$$

then the PHS on $H^2(X)$ generically determines X .
The IVHS used here is the first variation of the PHS on $H^2(X)$, which in a very interesting way is interpreted via the ramification of the j -function map $C \rightarrow \overline{\mathcal{M}}_1$.

A classical example of the use of IVHS is the theorem of Donagi-Green ([GMV]) concerning smooth hypersurfaces

$$\{F(x_0, x_1, \dots, x_{n+1}) = 0\} = X \subset \mathbb{P}^{n+1}.$$

[†](arXiv:2009.03683v, 9Sept2020)

In this case, for $n \geq 2$ the IVHS is given by a homogeneous subring $R \subset \mathbb{C}(x_0, \dots, x_{n+1})/J_F$ where $J_F = \{F_{x_0}, \dots, F_{x_{n+1}}\}$ is the Jacobian ideal of F . Except for a few cases of $\deg F$, it is shown that R determines J_F up to the action of PGL_{n+1} .

An even more classical example arises in the case of algebraic curves X of genus $g \geq 2$. In this case the IVHS turns out to be equivalent to the map

$$\mathrm{Sym}^2 H^0(K_X) \rightarrow H^0(2K_X).$$

For $g \geq 3$ and X non-hyperelliptic this mapping is surjective (Noether) (which then implies local Torelli) with kernel the space $I_2(X)$ of quadrics through the canonical curve $X \subset \mathbb{P}^{g-1}$. For $g \geq 5$, again by Noether

$$\bigcap_{Q \in I_2(X)} Q = X$$

and from this one infers generic global Torelli for $g \geq 5$.

In these examples involving full use of the two Hodge-Riemann bilinear relations is not made. In the following both the IVHS and the bilinear relations are used.

Example: A recent example that is in some ways reminiscent of the Θ -divisor arises from a pair (\bar{B}, Z) together with a VHS over $B = \bar{B} \setminus Z$ that has been canonically extended to \bar{B} . We have noted that at each point of Z the level 1 extension data \mathcal{E}_1 is a direct sum of compact, complex tori each of whose tangent spaces is a Hodge structure of weight-1 with Hodge decomposition

$$(III.1) \quad (k-1, -k) \oplus \cdots \oplus \underbrace{(0, -1) \oplus (-1, 0)} \oplus \cdots \oplus (-k, k-2).$$

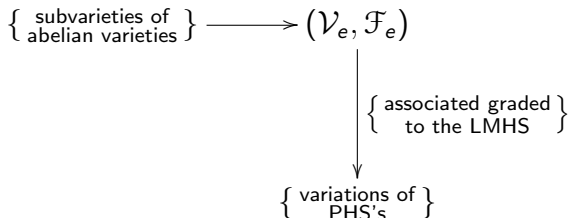
The level 2 extension data fibres over \mathcal{E}_1 with fibre a direct sum of $\text{Ext}_{\text{MHS}}^1(H^k, H^{k-2})$'s. Here the tangent space is a Hodge structure of weight -2 with Hodge decomposition

$$(k-2, -k) \oplus \cdots \oplus \underbrace{(-1, -1)} \oplus \cdots \oplus (-k, k-2).$$

Using $\text{Ext}_{\text{MHS}}^1(\mathbb{Q}(-1), \mathbb{Q}) \cong \mathbb{C}^*$ one sees that the Hodge part over the brackets gives line bundles $L_M \rightarrow E$ whose curvature is positive on the part of $T\mathcal{E}_1$ over the bracket in (III.1). As we vary along a subvariety $A \subset Z$ along which the associated graded $\bigoplus H^k$ to the limiting mixed Hodge structures remains constant, using the IPR we obtain an ample line bundle $L_M \rightarrow A$. A central geometric result is that (cf. [GGR])

$$L_M|_A \cong \bigoplus_i \langle M, N_i \rangle N_{Z_i/\bar{B}}^*|_A.$$

This gives the picture of the VHS at infinity



where using the above result in the box the co-normal bundle to the fibres in \overline{B} is expressed by Θ -line bundles along the fibres. We believe that the above picture serves as **the** governing property in the global understanding of the Hodge structure on the boundary of the canonically extended VHS ([GGR] and [G3]).

Geometric example: Related to the preceding, let \mathcal{M} be a KSBA moduli space of surfaces of general type and with canonical completion $\overline{\mathcal{M}}$ where the boundary $\partial\mathcal{M} = \overline{\mathcal{M}} \setminus \mathcal{M}$ parametrizes surfaces X with semi-log-canonical singularities (cf. [K]). A part \mathcal{N}_e of the boundary corresponds to normal Gorenstein X 's with simple elliptic singularities. In contrast to the case of algebraic curves where $\overline{\mathcal{M}}_g$ is essentially smooth, $\mathcal{N}_e \subset \partial\mathcal{M}$ is generally highly singular and consideration of the extension data in the LMHS suggests a natural partial desingularization of $\overline{\mathcal{M}}$ along \mathcal{N}_e . This is explained in the notes [G2], where it is also illustrated that this phenomenon may also extend to non-Gorenstein isolated singularities and to non-normal X 's as well.

This is the first non-classical example I am aware of where an actual geometric object may be constructed from a (generalized) Hodge structure alone, in this case a LMHS. It is an IVHS because from the work of Cattani-Kaplan-Schmid a LMHS may be smoothed to a 1-parameter of ordinary PHS's.

Geometric example (continued): As above suppose we have a KSBA moduli space \mathcal{M} for general type surfaces and with canonical completion $\overline{\mathcal{M}}$. Assume for simplicity that \mathcal{M} is smooth and that a general point of \mathcal{M} corresponds to a smooth regular surface. One may ask

- ▶ can Hodge theory suggest what surfaces X appear on the boundary $\partial\mathcal{M} = \overline{\mathcal{M}} \setminus \mathcal{M}$?
- ▶ Can Hodge theory suggest how one might construct a desingularization $\widetilde{\overline{\mathcal{M}}}$ of $\overline{\mathcal{M}}$?

As discussed and illustrated in [G2] the answer to both questions is positive. For the first, a general limiting mixed Hodge structure has

$$N^2 = 0, \quad \text{rank } N = 2.$$

Thus the LMHS has associated graded $(H^1, H^2, H^1(-1))$ where $H^1 = H^1(C)$ for a smooth elliptic curve C . Given a KSBA degeneration $\mathcal{X} \rightarrow \Delta$ where X_t is smooth for $t \neq 0$ and X_0 is a normal surface corresponding to a point of $\partial\mathcal{M}$, what is suggested is that $X_0 = (X, p)$ where p is a simple elliptic singularity of a surface X and where the resolution of that singularity is $(\tilde{X}, C) \rightarrow (X, p)$ where \tilde{X} is smooth and $C \subset \tilde{X}$ is an elliptic curve.

For the desingularization of $\overline{\mathcal{M}}$ one needs to do a semi-stable reduction

$$\tilde{\mathcal{X}} \rightarrow \tilde{\Delta}$$

of the family $\mathcal{X} \rightarrow \Delta$. Since $N^2 = 0$, Clemens-Schmid suggests that the central fibre \tilde{X}_0 should have a double curve C and no triple points. The simplest possibility is that

$$\tilde{X}_0 = \tilde{X} \bigcup_C Y$$

where \tilde{X} is as above and Y is a smooth surface containing C . Since C is a smooth elliptic curve we might try a smooth cubic $C \subset \mathbb{P}^2$. The normal bundle $N_{C/\tilde{X}}$ has degree $d = -C^2$ while $N_{C/\mathbb{P}^2} \cong \mathcal{O}_C(3)$. To achieve the necessary condition

$$N_{C/\tilde{X}} \cong \check{N}_{C/Y}$$

for smoothability, we must blow up $9 - d$ points p_i on C . Since for a smoothable elliptic singularity we have

$$1 \leq d \leq 9$$

Y is a del Pezzo surface. Moreover from \tilde{X}_0 as above we can construct the potential limiting mixed Hodge structure, and a standard computation gives that

$$\mathrm{Ext}_{\mathrm{MHS}}^1(H^1(-1), H^2)$$

contains a factor constructed from the subspace $\mathrm{Hg}^1(Y, \mathbb{Z})$ in H^2 . It then follows that the information contained in the level 1 extension data in the LMHS is essentially the

$$\mathrm{AJ}_C(p_i - p_j) \in J(C).$$

This tells us which points p_i on $C \subset \mathbb{P}^2$ to blow up to construct Y .

Of course the above is heuristic, but hopefully it does the tight interplay between Hodge theory and geometry and illustrates how a geometric construction from a IVHS may be used to study moduli.

Topology: The above examples of uses of Hodge theory were to geometric constructions arising from linear algebra Hodge theoretic data. The original application of Hodge theory, meaning now the existence of a functorial Hodge structure on the cohomology of a smooth projective variety $X^n \subset \mathbb{P}^N$, were to topology. Letting $Y = \mathbb{P}^{N-1} \cap X$ be a general hyperplane section there is the

(III.2) **Lefschetz theorem ([L]):** *The induced mapping*

$$H^q(X, \mathbb{Z}) \rightarrow H^q(Y, \mathbb{Z})$$

is an isomorphism for $q \leq n - 2$ and is injective for $q = n - 1$.

This result was proved by Picard ([PS]) when $n = 2$ (algebraic surfaces), and Lefschetz's argument for the general case was an extension of that of Picard. It made use of Lefschetz pencils (Y_t) given by the \mathbb{P}^1 of hyperplane sections containing a general \mathbb{P}^{N-2} . There are finitely many singular Y_{t_α} 's where the hyperplane is tangent to X , and the monodromy around these (Picard-Lefschetz transformations) plays a central role in the analysis. The result is a topological one; no use is made of Hodge theory.

The other result of Lefschetz concerns the mapping

$$(III.3) \quad h^k : H^{n-k}(X, \mathbb{Q}) \rightarrow H^{n+k}(X, \mathbb{Q})$$

where $h \in H^2(X, \mathbb{Q})$ is dual to the homology class defined by Y .

(III.4) **Hard Lefschetz theorem ([L] and [H]):**
The mapping h^k in (III.3) is an isomorphism.

The reason for the “hard” is that in many senses this result is deeper than (III.2). In fact, Lefschetz’s original topological argument was incomplete. It was partly in seeking to give a proof of (III.4) that Hodge [H] originated Hodge theory; the result itself is a Hodge-theoretic, not a topological, one. Interestingly Picard stated and gave a correct proof of (III.4) when $n = 2$. His argument used the Poincaré complete reducibility theorem ([P]) for abelian varieties, applied here to the family of Jacobian varieties $J(Y_t)$. The complete reducibility theorem is basically a Hodge-theoretic result and its higher dimensional analogue had to await the construction of PHS’s on the $H^{n-1}(Y_t)$ for $n \geq 3$.

Vanishing theorems

The Italian algebraic geometers had extraordinary geometric intuition and deep knowledge of examples. In its more mature period there were things they intuitively suspected but were not able to fully prove, and even some mistakes appeared. It has been said that what they were missing was H^1 . The meaning of this is illustrated by the

Example: Let $L \rightarrow X$ be a line bundle over a smooth algebraic surface and $C \subset X$ a curve. Various forms of the issue of showing that all of $H^0(C, L)$ comes by restriction from $H^0(X, L)$ kept arising. The Italians knew that this is not true in general. With the advent of sheaf theory and the long exact cohomology sequence we now know this will be OK if one has the vanishing theorem $H^1(X, L(-C)) = 0$.[‡]

[‡]This issue does not arise as much for points on a curve because in the case the dual of H^1 is an H^0 .

Thus one needs (a) the formalism of sheaf cohomology, and (b) vanishing theorems. The thesis of this last part of the talk is

The Lefschetz theorem (III.2) plus the existence of Hodge structures on ordinary cohomology imply vanishing theorems.

More precisely, one needs the existence of the Hodge structure

$$(III.5) \quad H^r(X, \mathbb{C}) \cong \bigoplus^{p+q=r} H^{p,q}(X)$$

plus the isomorphism

$$(III.6) \quad H^{p,q}(X) \cong H^q(\Omega_X^p).$$

This principle is well known since the 1950's and over the years has attained a much extended and more widely applicable version (cf. [SS], [EV] and the lecture notes [G2]). Here we will simply illustrate it in a very special case.

Let $X \subset \mathbb{P}^N$ and $Y = P^{N-1} \cap X$ be as above and $L = \mathcal{O}_X(1)$ the restriction to X of the standard hyperplane bundle. Then the Kodaira vanishing theorem is

$$(III.7) \quad H^q(X, \check{L}) = 0 \text{ for } q < n = \dim X.$$

Proof: If $s \in H^0(X, L)$ is the restriction to X of the linear form that defines $Y \subset X$, we have

$$(III.8) \quad 0 \rightarrow \check{L} \xrightarrow{S} \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0.$$

Then by (III.5) and (III.6)

$$\begin{array}{ccccc} H^q(\mathcal{O}_X) & \cong & H^{0,q}(X) & \subset & H^q(X, \mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow \\ H^q(\mathcal{O}_Y) & \cong & H^{0,q}(Y) & \subset & H^q(Y, \mathbb{C}). \end{array}$$

The right-hand arrow is injective for $q \leq n - 1$ and then (III.7) follows from the long exact cohomology sequence of (III.8).

This talk began with some historical discussion and we would like to conclude with a final historical point. One of the main classical results established by Picard and the Italian algebraic geometers concerned a general hyperplane section $C \subset X$ of a smooth surface X . The adjunction formula is

$$K_X + C|_C \cong K_C$$

and the result is

The cokernel of the map $H^0(K_X + C) \rightarrow H^0(K_C)$ has dimension equal to the irregularity $q := h^0(\Omega_X^1)$ of X .

This follows from the exact cohomology sequence of

$$0 \rightarrow K_X \rightarrow K_X + C \rightarrow K_C \rightarrow 0$$

once one we know that

- ▶ $h^1(K_X + C) = 0$
- ▶ $h^1(K_X) = h^0(\Omega_X^1)$.

For $L = [C]$ as above, the first of these follows from the above vanishing theorem and Kodaira-Serre duality

$$H^1(K_X + C) \cong H^1(L)^\vee.$$

The second follows from Hodge theory and Hard Lefschetz

$$H^0(\Omega_X^1) \xrightarrow{h} H^1(\Omega_X^2).$$

The dual of this map is part of the map

$H_3(X) \xrightarrow{\check{h}} H_1(C) \rightarrow H_1(X)$ where the image of \check{h} is the space of invariant cycles as C varies in a general pencil.

I think that once one has the Poincaré complete reducibility theorem this argument, which is given in [PS], may be basically correct.

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